# Stoked out About Stokes Groupoids (Orig: Get Stoked About Stokes Groupoids)

GST: April 23, 2014

Reference: M. Gualtieri, Songhao Li, Brent Pym 1305.7288

## Motivation and Overview

Riemann-Hilbert Correspondence: $X$ a (Complex) manifold

(holomorphic) Flat Connections $\qquad\qquad$ Parallel Transports

$$\{\mathcal{E},\ \nabla:\mathcal{E}\to\Omega^1_X(\mathcal{E})\} \underset{\text{DIFF.}}{\overset{\text{integrate}}{\rightleftarrows}} \{P_\gamma : \mathcal{E}_{\gamma(0)} \longrightarrow \mathcal{E}_{\gamma(1)},\ \gamma \text{ a path}\}$$

Fancier Language:

Flatness $\Leftrightarrow$ $\nabla_{[\xi,\eta]} = [\nabla_\xi, \nabla_\eta]$

$\Rightarrow$ Rep$^{\underline{n}}$ of Tangent "Algebroid"

$$\mathcal{T}_X \longrightarrow \mathrm{Der}(\mathcal{E})$$

Rep$^{\underline{n}}$ of Fundamental Groupoid

$$\Pi_{\leq 1}(X) \longrightarrow \mathrm{Aut}(\mathcal{E})$$

So

$$\mathrm{Rep}(\mathcal{T}_X) \simeq \mathrm{Rep}(\Pi_{\leq 1}(X))$$

## A Fancier Problem: Let Us raise our pinkies high (Sip tea and raise our pinkies high)

- $X$ a smooth Complex Curve
- $D$ an effective divisor on $X$: $D = \sum_{i=1}^{n} \nu(p_i)\, p_i$, $\nu(p_i) \in \mathbb{Z}_{>0}$, $p_i \in X$.

Want to Study Connections on $X$ with Singularities bounded by $D$:

$z$ a local Coord around $p_i \in D$, $(\nabla, \mathcal{E})$ Flat bundle w/ local Frame:

$$\nabla = d + A(z)\, z^{-k}\, dz \quad , \quad A: \text{Holomorphic Matrix-Valued Function}$$

At worst Sing. of order $K$,

"bounded above by $K$"

Naive RH Correspondence:

$$\underset{\substack{\text{Contains Connections}\\ \text{w/ essential Sing}\\ \text{on } D.}}{\mathrm{Rep}(\mathcal{T}_{X\setminus D})} \simeq \underset{\substack{\text{Lose all local data}\\ \text{around Sing.}}}{\mathrm{Rep}(\Pi_{\leq 1}(X,D))}$$

Do not rewrite, reuse previous RH statement

Appropriate Refinement:

$$\mathrm{Rep}(\mathcal{T}_X(-D)) \simeq \mathrm{Rep}(\Pi_{\leq 1}(X,D))$$

Sheaf of v.F. w/ Zeros bounded below by $D$

(locally $\langle z^k \frac{\partial}{\partial z} \rangle_{\mathcal{O}_{U\subset X}}$)

$\Rightarrow$ $\nabla_\xi : \mathcal{E} \to \mathcal{E}$ non-Singular if $\xi \in \mathcal{T}_X(-D)$ and $\nabla$ has Sing bounded by $D$

$$\Pi_{\leq 1}(X,D)|_{X\setminus D} = \underbrace{\Pi_{\leq 1}(X\setminus D)}_{\shortparallel} \cup_\varphi \coprod_{p\in D} \mathrm{Sto}_{\nu(p)}|_{\mathbb{D}\ni p}$$

"Preserves local data at $D$".

Claim. By appropriate pullbacks to the (Lie Groupoid) $\Pi_{\leq 1}(X,D)$ we can take Fundamental Solutions/ Parallel Transports of a diagonal Connection Formally equiv to another (non-diag. Conn.) to actual Sol$^n$s.

IF $(\mathcal{E}_1, \nabla_1)$, $(\mathcal{E}_2, \nabla_2)$ are Formally equivalent, then by Pullbacks to $\Pi_{\leq 1}(X,D)$ we can determine $P_2$ From $P_1$: The Formal Sol$^n$ $\hat{P}_2$ Converges.

(Holomorphic) Lie Groupoids : Groupoids whose arrows and objects are Complex mans.

Def

A Groupoid $G \underset{t}{\overset{s}{\rightrightarrows}} X$ is a hol. Lie Groupoid iF

1) $G$ (arrows), $X$ (objects) are $\mathbb{C}$-maniFolds [$G$ possibly non-HausdorFF]

2) $s, t : G \longrightarrow X$ (source/target) are hol. Submersions

3) $m : G \,{}_s\!\times_t G \longrightarrow G$ is holomorphic

4) $id : X \hookrightarrow G$ (embedding oF identity arrows) is a Closed embedding

Ex:

1) 

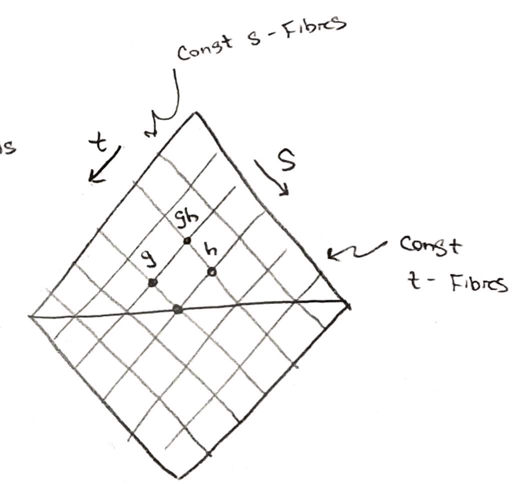
2) $X$ a $\mathbb{C}$-man.

Projections onto 1st and 2nd coords

$$\text{Pair}(X) = X \times X \underset{s = p_2}{\overset{s = p_1}{\rightrightarrows}} X$$

$$(x, y) \cdot (y, z) := (x, z)$$

$$id = \Delta : X \hookrightarrow X \times X$$
$$x \longmapsto (x, x)$$




3) Gauge Groupoid: $\mathcal{E}$ a locally Free SheaF (Vector bun.), $\mathcal{E}_p$ = Fibre over $p$

$$\text{Aut}(\mathcal{E}) = \{ \mathcal{E}_p \xrightarrow{\sim} \mathcal{E}_q \ \ \mathbb{C}\text{-linear iso's For } p, q \in X \}$$

4) $\Pi_{\leq 1}(X) = \{ [\gamma] : \gamma \text{ a path on } X \}$, $s(\gamma) = \gamma(0)$, $t(\gamma) = \gamma(1)$.

Note: S-Fibres $s^{-1}(x_0)$ are the usual construction oF the universal Cover oF $X$ using paths based at $X$.

$(s^{-1}(x_0) \cong \tilde{X})$.

## Lie Algebroids

**Def**

$$\mathrm{Lie}(G) := N_G X \cong \mathrm{Ker}(s_*)$$

canonical splitting using $s_*$

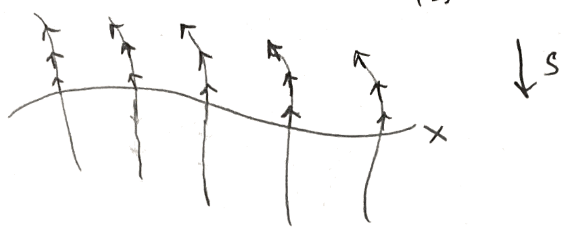
image of id: $X \hookrightarrow G$

is a vector bundle over $X$, equipped w/ :

- $[\cdot,\cdot]$ : Extend sections of normal bundle to Right invar V.F. on $G$ tangent to $s$ fibres (use isotropy Groups)

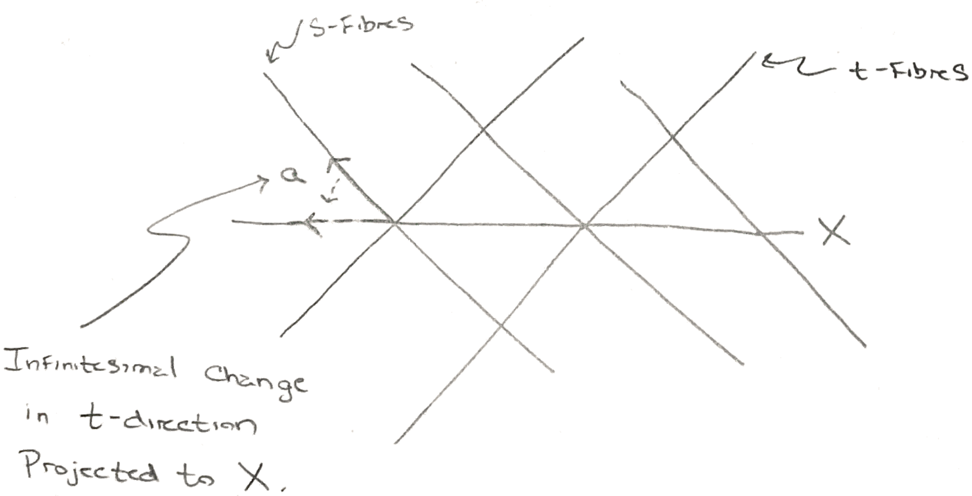
under groupoid action on itself

Section of normal bundle $\updownarrow$ Element of $\mathrm{Lie}(G)$




- Anchor map $a : \mathrm{Lie}(G) \longrightarrow T_X$ given by $t_* = dt|_{\mathrm{Ker}(s_*)} : \mathrm{Ker}(s_*) \to T_X$.

**Ex:** • $\mathrm{Lie}(\mathrm{Pair}(X)) \cong T_X$ via $a$.



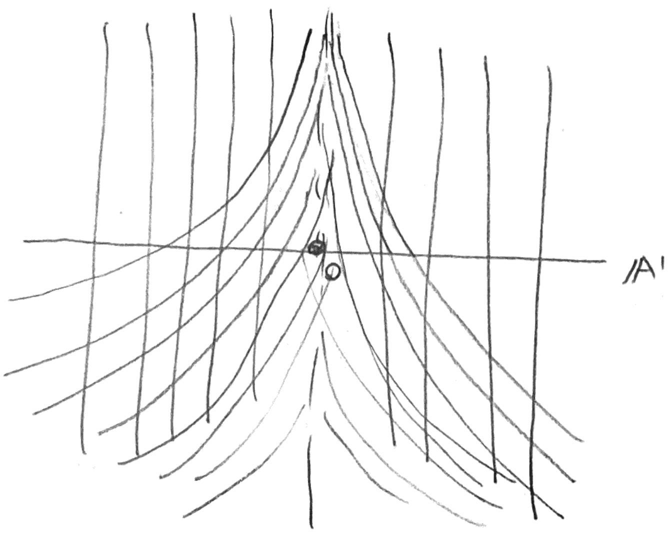

- $\mathrm{Lie}(\Pi_{\leq 1}(X)) \cong T_X$ (Same local picture)

See Version 1 For rest of Lecture (Contains more detail than Should be Said)
Page 6

Ex: "Twisting" Groupoids along a Divisor.

- $\mathrm{Lie}(\mathrm{Pair}(X, D)) \simeq \mathrm{Lie}(\Pi_{\leq 1}(X, D)) \simeq \mathcal{T}_X(-D) \xrightarrow{a} \mathcal{T}_X$

$$\mathrm{Sto}_1 := \Pi_{\leq 1}(\mathbb{A}^1, 1 \cdot 0) \quad (\cong \mathrm{Pair}(\mathbb{A}^1, 1 \cdot 0)).$$




$\downarrow s$ Fibres

Curves: t-Fibres

Isotropy Grp @ p $= s^{-1}(pt) \cap t^{-1}(pt) = \begin{cases} * \cong \pi_1(\mathbb{A}^1, pt.) & \text{if } pt \neq 0 \\ \mathbb{G}_a \cong T_{pt}\mathbb{A}^1 & \text{if } pt = 0 \end{cases}$

$\mathbb{G}_a \cong \mathbb{C}$.

## Integrations

### Def

An integration of an algebroid $(A, [\cdot,\cdot], a : A \to \mathcal{T}_X)$ is a pair $(G, \phi)$ with $G$ a groupoid and $\phi : \mathrm{Lie}(G) \xrightarrow{\sim} A$.

### Rmk / Def$^{\underline{n}}$ of $\Pi_{\leq 1}(X, D)$

- The Set of Integrations Forms a Category

- When $A = \mathcal{T}_X(-D)$   $\mathrm{Pair}(X, D)$ is Final in this Category $\leftarrow$ Fibres $s^{-1}(pt.) \cong X$ For pt. generic

  $\Pi_{\leq 1}(X, D)$ is initial $\leftarrow$ Source Simply Connected Fibres $\Rightarrow$ unique up to iso

- $\mathrm{Pair}(X,D)$ and $\Pi_{\leq 1}(X,D)$ can be constructed via "blowups" on $\mathrm{Pair}(X)$ and $\Pi_{\leq 1}(X)$ or alternatively via glueing in copies of $\mathrm{Sto}_K := \Pi_{\leq 1}(\mathbb{A}^1, K\cdot 0) = \mathrm{Pair}(\mathbb{A}^1, K\cdot 0)$.

<u>Thm</u>

Let $X$ be a complex curve, $D \subseteq X$ a divisor $D = \sum_{p\in X} \nu(p) D$; $U = X \setminus D$

1) $\Pi_{\leq 1}(X,D)|_U = \Pi_{\leq 1}(X,D) \setminus (s^{-1}(U) \cup t^{-1}(U)) \cong \Pi_{\leq 1}(U)$

2)

$$\Pi_{\leq 1}(X,D) \simeq \Pi_1(U) \cup \coprod_{p\in D} \mathrm{Sto}_{\nu(p)}|_{\mathbb{D}}$$

Gluing map different for $\mathbb{P}^1$, $D = K\cdot pt$.

If $U$ is non-contractible, then the resulting space is Hausdorff; (gluing is via the map $\varphi : \Pi_{\leq 1}(U \cap V) \longrightarrow \coprod_{p\in D} \mathrm{Sto}_{\nu(p)}|_{\mathbb{D}^*}$.

<u>Extension of Solutions over Singularities</u>

<u>Thm</u>

$$\mathrm{Rep}(\mathcal{T}_X(-D)) \cong \mathrm{Rep}(\Pi_{\leq 1}(X,D))$$

Objects: $(\mathcal{E}, \nabla)$, $\nabla$ has sing. bounded by $D$.

$\updownarrow$

$\mathcal{T}_X(-D) \longrightarrow \mathrm{Der}(\mathcal{E})$

$\xi \longmapsto \nabla_\xi : \mathcal{E} \to \mathcal{E}$

$(\nabla : \mathcal{E} \to (\mathcal{T}_X(-D))^\vee \otimes \mathcal{E})$.

G-equivariant

Obj: $(\mathcal{E}, P)$, $P : s^*\mathcal{E} \xrightarrow{\sim} t^*\mathcal{E}$

$$\begin{array}{ccc} s^*\mathcal{E} & \xrightarrow{\sim} & t^*\mathcal{E} \\ \downarrow & & \downarrow \\ \Pi_{\leq 1}(X,D) & \xrightarrow{id} & \Pi_{\leq 1}(X,D) \end{array}$$

$\updownarrow$

$P : \Pi_{\leq 1}(X,D) \longrightarrow \mathrm{Aut}(\mathcal{E})$

$\gamma \longmapsto P|_\gamma : \mathcal{E}_{s(\gamma)} \to \mathcal{E}_{t(\gamma)}$

"Parallel transport."

$s(\gamma) = \gamma(0)$
$t(\gamma) = \gamma(1)$
if $\gamma$ is a path, i.e. in $\Pi_{\leq 1}(X \setminus D)$.

## Constructing $\mathcal{P}$ in practice: $\mathcal{P}$ From Crap

Find Fundamental Solutions: $\{S_i \in \mathcal{E}\}_{i=1}^{r = \mathrm{Rank}\, \mathcal{E}}$

$$\nabla S_i = 0$$

"Stokes Phenomena"

$S_i$ have varying asymptotics toward the divisor

$\Rightarrow$ Framing

$$\psi : (\mathcal{O}_X^r, d) \xrightarrow{\sim} (\mathcal{E}, \nabla) \qquad (*)$$

$$(f_1, \dots, f_r) \longmapsto f_1 S_1 + \cdots + f_n S_n$$

Unless $\mathcal{E}$ is trivial and $D$ is trivial

(a) $\psi$ is multivalued $\longrightarrow$ Monodromy Matrices

(b) $\psi$ is singular along $D$ (w/ Stokes-type asymptotics) $\rightsquigarrow$ Stokes Matrices

(me "Speculating")

(a) $\longleftrightarrow$ $P_\gamma$ For $\gamma \in \Pi_{\leq 1}(X \setminus D) = \Pi_{\leq 1}(X, D)|_U \cong G_a$

(b) $\longleftrightarrow$ $P_\gamma$ For $\gamma \in \Pi_{\leq 1}(X, D)|_D \cong \coprod_{p \in D} (T_p^* X)^{\otimes (\gamma(p)-1)}$ Isotropy Groups

$\mathrm{iso}_p(\nabla) \in (T_p X)^{\otimes K} \otimes \mathrm{End}(\mathcal{E}_p)$

K-matrices attached to K-directions

**Proposition**

For any Fundamental Solution $\psi$ as in $(*)$, the expression

$$\delta\psi = t^* \psi \circ (s^* \psi)^{-1}$$

extends holomorphically to $\Pi_{\leq 1}(X, D)$ and coincides w/ $\mathcal{P}$. Multi-valuedness is removed by requiring $\delta\psi|_X = 1$ (over identity bisection).

Ex: (only if there is enough time).

Rank 1 Rep$^{n}$ for $\Upsilon_{A'}(-k\cdot p)$ : $(\mathcal{O}_{A'}, \nabla)$ w/

$$\nabla = d + a z^{-k} dz$$

we have multi-valued fundamental Sol$^{n}$s:

$$\psi_1 = z^{-a} \quad , k=1$$
$$\psi_k = \exp\left\{ \frac{a z^{-(k-1)}}{k-1} \right\} \quad , k>1$$

Which give ($(z,u)$ coords on $Sto_k$)

$$P_1\Big|_{(z,u)} = e^{-au}$$

$$P_2\Big|_{(z,u)} = e^{-a S_k} \quad , \quad S_k = \frac{1 - e^{-u(k-1)} z^{k-1}}{(k-1) z^{k-1}}$$

---

## Summation of Divergent Series

### Motivation

Fundamental Solutions for Diagonal Connections are easy, Want $g$ a hol. gauge transf. ($\in \mathrm{Autbun}(\mathcal{E})$). s.t.

Semi-simple (diagonalizable) Matrix

$$\nabla = d + \left( \frac{T_k}{z^k} + \cdots + \frac{T_1}{z} \right) dz \xrightarrow{g^*} g^*\nabla = d + \left( \frac{T_k^{diag}}{z^k} + \cdots + \frac{T_1^{diag}}{z} \right) dz$$

(ignore hol. stuff)

Can find order by order : Caveat: Most of the time $g$ is a Formal power series, i.e. has zero radius of convergence.

## Theorem / Observation

Let $\hat{g}$ be a Formal iso between $\mathcal{T}_X(-D)$ reps $((\mathcal{E}_1,\nabla_1) \,\&\, (\mathcal{E}_2,\nabla_2))$

$$\begin{tikzcd}
(\hat{\mathcal{E}}_1, \hat{\nabla}_1) \arrow[r, "g"] \arrow[d] & (\hat{\mathcal{E}}_2, \hat{\nabla}_2) \arrow[d] \\
\hat{X} \arrow[r, "id"] & \hat{X}
\end{tikzcd}$$

$\hat{X}$ = Formal nbhd of $D$

(Formal Completion of $D$)

and $P_i : s^*\mathcal{E}_1 \longrightarrow t^*\mathcal{E}_2$ the corresponding $\Pi_{\leq 1}(X,D)$ reps define $\hat{L}$ via:

Formal

Parallel transport

$$\begin{tikzcd}
s^*\hat{\mathcal{E}}_1 \arrow[r, "\hat{P}_1 = P|_{\hat{\mathcal{E}}_1}"] \arrow[d, "s^*\hat{g}"'] \arrow[dr, phantom, "\hat{L}"] & t^*\hat{\mathcal{E}}_1 \arrow[d, "t^*\hat{g}"] \\
s^*\hat{\mathcal{E}}_2 \arrow[r, "t^*\hat{g} \circ \hat{P}_1 \circ (s^*\hat{g})"] & t^*\hat{\mathcal{E}}_2
\end{tikzcd}$$

Then $\hat{L} = \hat{P}|_2$, i.e. $\hat{L}$ extends to a holomorphic/convergent parallel transport op. $P$.

( PF: Trivial, $\hat{P}_2 = P|_{\hat{\mathcal{E}}_2}$ is the unique operator that fits into the bottom arrow above assuming Formal $\mathcal{T}_X(-D)$ reps are in 1:1 correspondence with Formal $\Pi_{\leq 1}(X,D)$ - reps. )

**Ex:** Rep$^{\underline{\text{ns}}}$ of $\mathcal{T}_{\mathbb{A}^1}(-2\cdot 0)$

Let

$$\nabla_1 = d + \begin{pmatrix} -1 & \\ & 0 \end{pmatrix} z^{-2}dz, \qquad \mathcal{E}_2 = d + \begin{pmatrix} -1 & z \\ & 0 \end{pmatrix} z^{-2}dz$$

then

$$g \circ \nabla_1 = \nabla_2 \circ g \iff z^2 g' = g - z$$

Actual Sol$^{\underline{\text{n}}}$ is $C^\infty$ but not hol. at $z=0$

$\exists$ a Formal Solution:

$$\hat{g}(z) = \sum_{n=0}^{\infty} n!\, z^{n+1}$$

$P_1$ is a parallel transport op on $\mathrm{Sto}_z = \Pi_{\leq 1}(A', z\cdot 0)$ ← (use $\mathrm{Pair}(A', z\cdot 0)$ instead)

$$P_1 = \begin{pmatrix} e^{u(1+zu)^{-1}} & \\ & 1 \end{pmatrix}$$

$$P_2 = \begin{pmatrix} 1 & t^* g \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} e^{u(1+zu)^{-1}} & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & -s^* g \\ & 1 \end{pmatrix}$$

$$= \begin{pmatrix} e^{u(1+zu)^{-1}} & s^* g - e^{u(1+zu)^{-1}} s^* g \\ & 1 \end{pmatrix}$$

where

$$s(z, \mu) = z \qquad , \quad \mu = u(1+zu)^{-1}$$
$$t(z, \mu) = z(1 - z\mu)$$

Then we find

$$s^* \hat{g} - e^{\mu} s^* \hat{g} = - \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{z^{i+1} \mu^{i+j+1}}{(i+1)(i+2)\cdots(i+j+1)}$$

a convergent power series for

$$g(z; \mu) = e^{\frac{z\mu - 1}{z}} \left( Ei\left(\frac{1 - z\mu}{z}\right) - Ei\left(\frac{1}{z}\right)\right)$$