Homological Toolkit For the Quantum Mechanic
Part II

Recap: ρ_P a multipartite density state; we motivated:

$$
\begin{tikzcd}
& \chi_{\alpha,q,r}(\rho_P) = \sum_{\emptyset\subseteq T\subseteq P} (-1)^{|T|} \dim(\mathcal{H}_T)^{\alpha}\, \mathrm{Tr}[\rho_T^{\,q}]^{r} \arrow[dl, "\frac{d}{dq}\big|_{q=1}"'] \arrow[dr, "q\to 0"] & \\
\text{Mutual Inf.} & & \sum_{\emptyset\subseteq T\subseteq P} (-1)^{|T|} \dim(\mathcal{H}_T)^{\alpha}\, \mathrm{rank}(\rho_T)^{r} \in \mathbb{Z} \quad (\alpha, r\in\mathbb{Z})
\end{tikzcd}
$$

$\Rightarrow$ is $\dim(\mathcal{H}_T)^{\alpha}\,\mathrm{rank}(\rho_T)^{r}$ the dimension of some vector space?

Thm:

A state $\psi \in \mathcal{H}_A \otimes \mathcal{H}_B$ is factorizable $\Longleftrightarrow$ $\exists$ (for $X = A, B$)

- Left $B\mathcal{H}_X$-modules M_X
- Distinguished points $m_X \in M_X$
- Equivariant maps $\mu_X : M_X \longrightarrow \mathcal{H}_A \otimes \mathcal{H}_B$

s.t.

- $B\mathcal{H}_X \cdot m_X = M_X$ (Cyclic)
- $\mu_X(m_X) = \psi$
- $0 \longrightarrow \mathbb{C} \xrightarrow{\lambda \mapsto (\lambda m_A, \lambda m_B)} M_A \times M_B \xrightarrow{d^0 = pr_A^* \mu_A - pr_B^* \mu_B} \mathcal{H}_A \otimes \mathcal{H}_B = \mathcal{M}$

is exact ($\mathrm{Ker}\, d^0 = \mathrm{Span}_{\mathbb{C}}(m_A, m_B)$).

$$H^0(\mathcal{M}) = 0$$

Thus, given $\psi \in \mathcal{H}_A \otimes \mathcal{H}_B$ we want to find (M_X, m_X, r_X) w/ smallest cohomology.

The GNS-module (Gelfand - Neumark - Segal)

Quasi-Defs:

- A C^*-algebra is a $\mathbb{C}$-V.S. A w/
  - Norm $\|-\|$
  - $m: A \times A \longrightarrow A$ associative mult.
  - $*: A \longrightarrow A$ an anti-linear involution

  s.t.
  - A is complete
  - $*$ plays nicely w/ m
  - $\|x^* x\| = \|x\|^2$ (C^*-Cond^{n})

Ex: $B\mathcal{H}$, $C^\circ(X)$ ← commutative, $\prod_{i=1}^{L} M_{n_i}\mathbb{C}$ ← All f.d. C^*-algs $\cong$ to this.

- A W^*-algebra is a C^*-algebra w/ a predual: A_*

$$\iota : (A_*)^\vee \xrightarrow{\sim} A$$

"Normal linear functionals / density states)

Ex:
- $(B_1 \mathcal{H})^\vee \cong B\mathcal{H}$
- $(L^1(X, \mu))^\vee \cong L^\infty(X, \mu)$.
- Every f.d. C^*-algebra is W^*.

<u>Def</u>: • A State on a C^*-algebra A is a $\underbrace{\text{positive linear Functional}}_{\rho(a^*a) \geq 0}$

$$\rho : A \longrightarrow \mathbb{C}$$

(No normalization in this talk : i.e. $\rho(1)$ not nec. 1).

• A normal State on a W^*-algebra is an element of $(A_*)_+$

<u>Why?</u> $(A_*)^\vee \cong A \Rightarrow A_* \longrightarrow (A_*{}^\vee)^\vee \xrightarrow{\sim} A^\vee$

<u>Ex</u>: • $\mathrm{Dens}(\mathcal{H}) \subseteq (B\mathcal{H})_+ \qquad A = B\mathcal{H}$

$$\hat{\rho} \rightsquigarrow \rho(-) = \mathrm{Tr}[\hat{\rho}(-)] : A \longrightarrow \mathbb{C}$$

• $L^1(X) \ni f \rightsquigarrow \mu_f : \underset{L^\infty(X)}{a} \longmapsto \int_X a \, df$

<u>The GNS-module</u>

A a C^*-alg. w/ unit ; $\rho : A \longrightarrow \mathbb{C}$ a State. Define:

$$\mathcal{I}_\rho = \{ a \in A : \rho(a^*a) = 0 \}$$

$$\overset{\text{Cauchy-Schwarz}}{=} \{ a \in A : \rho(x^*a) = 0 \ \ \forall x \in A \}$$

$$\mathrm{GNS}(\rho) = A/\mathcal{I}_\rho \ , \quad g_\rho = 1_A + \mathcal{I}_\rho$$

<u>Claim</u>: $\mathcal{I}_\rho$ is a left ideal $\Rightarrow$ $\mathrm{GNS}(\rho)$ is a left A-module.

$r \cdot a \in \mathcal{I}_\rho \ \ \forall r \in A$, $a \in \mathcal{I}_\rho$.

Interpretation:

- $GNS(\rho) =$ Right Essential equivalence classes of operators

$$a \overset{r.e.c.}{\sim} b \iff \rho(x^* a) = \rho(x^* b) \quad \forall x \in A.$$

Non-Commutative analog of a.e. equiv Functions
(take $a, b \in L^\infty(\mathcal{X})$).

Rmk: • $GNS(\rho)$ + Inner-prod $(a,b) \mapsto \rho(a^* b)$ + Completion $\rightsquigarrow$ GNS - rep$^{\underline{n}}$ ($L^2(\mathcal{X}, \mu)$ in Commutative theory)

- Can recover ρ From GNS-rep$^{\underline{n}}$: $\rho(x) = \langle g_\rho, x \cdot g_\rho \rangle$

- $A = B\mathcal{H}$ $\Rightarrow$ Fits into a Family of $L^{1/q}$ - norms
$x \longmapsto Tr\left[|x \hat{\rho}^{1/q}|^q \right]^{1/q}$.

Submodules and Support Projections

Def: The Support proj of a W^*-alg normal State $\rho : A \longrightarrow \mathbb{C}$ is the Smallest Self-adj. proj S_ρ s.t.

$$\rho(x S_\rho) = \rho(x) \quad \forall x \in A$$

(equiv. $\rho(S_\rho x) = \rho(x)$)

Ex: • $\hat{\rho} \in Dens(\mathcal{H})$; S_ρ = Proj onto $Image(\hat{\rho})$

- μ : a measure on $\mathcal{X}$; " $S_\rho = 1_{\mu \neq 0}$ "

Claim: • $\mathcal{I}_\rho = A(1 - S_\rho)$

- $$\begin{aligned} AS_\rho &\xrightarrow{\sim} GNS(\rho) \\ a &\longmapsto a + \mathcal{I}_\rho \end{aligned}$$

Cor: $\hat{\rho} \in \mathrm{Dens}(\mathcal{H}) \rightsquigarrow \mathrm{GNS}(\rho) \cong \mathcal{B}\mathcal{H}\, S_\rho$

$\rho = \mathrm{Tr}[\hat{\rho}\,(-)]$

$$\cong \mathrm{Hom}^b(\mathrm{Image}(\hat{\rho}), \mathcal{H})$$

$$\underset{\substack{\text{Finite}\\ \text{dims}}}{\cong} \underset{\substack{\circlearrowleft \\ \mathcal{B}\mathcal{H}}}{\mathcal{H}} \otimes \mathrm{Image}(\hat{\rho})^\vee$$

$$\left(\underset{\text{Basis}}{\cong} \mathcal{H}^{\oplus \operatorname{rank}(\hat{\rho})} \right)$$

Pf that GNS-module is the smallest pointed module:

- Show $\mathrm{GNS}(\rho_x) \cong \mathcal{H}_A \otimes \mathcal{H}_B \big/ \underbrace{\mathrm{Ann}_{\mathcal{B}\mathcal{H}_x}(\psi)}_{= \{a \in \mathcal{B}\mathcal{H}_x \,:\, a \cdot \psi = 0\}}$

- Look @ short exact sequence of complexes

$$\underset{\text{kernel}}{K} \hookrightarrow \underset{\text{arbitrary}}{M} \longrightarrow \underset{\text{Built From } (\mathrm{GNS}(\rho_x),\, g_{\rho_x})}{\mathbb{C}\psi}$$

- Show that every morphism of pointed modules

$$f : (M, m) \longrightarrow (N, n)$$

Factors through the "cyclification" of (N, n) uniquely

$$\begin{tikzcd} & (N/\mathrm{Ann}_A(n),\, n) \cong (A \cdot n,\, n) \arrow[d] \\ (M, m) \arrow[ur, dashed, "\exists!"] \arrow[r, "f"] & (N, n) \end{tikzcd}$$

- Show !-map is surjective

- Take $N \rightsquigarrow \mathcal{H}_A \otimes \mathcal{H}_B$

  $A \rightsquigarrow \mathcal{B}\mathcal{H}_X$

- Show $\mathrm{GNS}(\rho_x) \cong \mathcal{H}_A \otimes \mathcal{H}_B \big/ \mathrm{Ann}_{\mathcal{B}\mathcal{H}_x}(\psi)$

- Look @ SES of Complexes ← Arbitrary

$$0 \to K \longrightarrow \mathcal{M} \longrightarrow G_\psi \longrightarrow 0$$

Kernel (via Componentwise Surjectivity) Built w/ GNS-module

- Show $H^1(K) = 0$

$$\Rightarrow \quad H^0(\mathcal{M}) \cong H^0(G) \oplus H^0(K)$$

Constructing Multipartite Complexes

Multipartite State over Set of $\otimes$-Factors P:

$$\underline{\rho_P} = \left((A_s)_{s \in P},\ \rho_P : \bigotimes_{s \in P} A_s \longrightarrow \mathbb{C} \right)$$

Claim: $\underline{\rho_P} \overset{(A)}{\rightsquigarrow}$ Presheaf of $\mathbb{C}$-vector Spaces over P $\overset{\text{Čech}}{\underset{(B)}{\rightsquigarrow}}$ Cohomology

(A): Define

$$G = G(\underline{\rho_P}) : \mathrm{Open}(P) \longrightarrow \mathrm{Vect}_{\mathbb{C}}$$

$$T \longmapsto \mathrm{GNS}(\rho_T)$$

$\rho_T := \iota_T^* \rho : X \to \mathbb{C}$

$\iota_T : \bigotimes_{t \in T} A_t \to \bigotimes_{p \in P} A_p$

$$T \hookrightarrow V \longmapsto \mathrm{GNS}(\rho_T) \longrightarrow \mathrm{GNS}(\rho_V)$$

$$a \longmapsto \text{"}(a \otimes S_{V \setminus T})\text{"} S_V$$

Claim: This is a Functor:

$$G(T \hookrightarrow W \hookrightarrow V) = G(W \hookrightarrow V) \circ G(T \hookrightarrow V)$$

Want a map

$$T \hookrightarrow V \longmapsto \mathrm{GNS}(\rho_T) \longrightarrow \mathrm{GNS}(\rho_V)$$