RESEARCH STATEMENT

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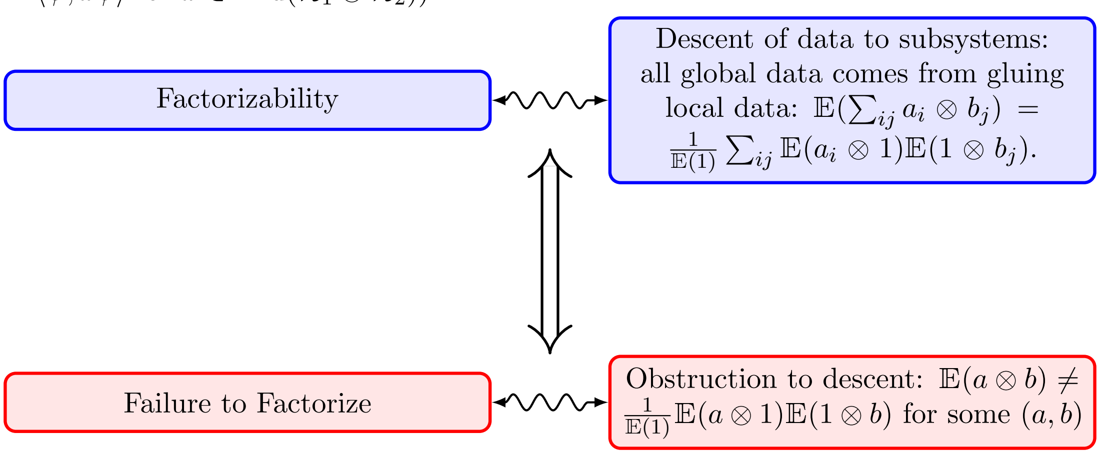
1. Overview

My interests reside near the vast range of topics at the intersection of quantum field theory/string theory, quantum information, and geometry. My recent focus is on the use of homological/homotopical techniques to understand entanglement, with an eye toward applications to quantum information and gauge/gravity duality. My time as a graduate student and early postdoc was spent thinking about four-dimensional $\mathcal{N} = 2$ supersymmetric field theories.

2. Entanglement, Cohomology, and the Categorification of Mutual Information

Given a tensor product of Hilbert spaces $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$, the set of entangled states—states that *fail* to factorize as $\psi_1 \otimes \cdots \otimes \psi_n$ for some $\psi_i \in \mathcal{H}_i$—play a fundamental role in quantum information/computation [1, 2]. The study of entangled states has recently spilled over into high-energy physics, particularly after the work of Ryu-Takayanagi [3], relating entanglement entropy (a numerical measure of entanglement) of states in a conformal field theory to the area of minimal surfaces in a gravity dual theory. Some of these ideas have discrete, computationally accessible versions [4, 5, 6] using the techniques of tensor networks [7, 8, 9] and error correcting codes.

My recent focus is on a cohomological measure of entanglement: to every multipartite state there is a graded vector space—resulting from the cohomology of a cochain complex—that provides a measure of how badly a state fails to factorize. The intuition behind the existence of such cohomologies for a bipartite state $\psi \in \mathcal{H}_1 \otimes \mathcal{H}_2$ is provided by the following diagram (where $\mathbb{E}(x) := \langle \psi, x\psi \rangle$ for $x \in \mathrm{End}(\mathcal{H}_1 \otimes \mathcal{H}_2)$).



Cohomology is precisely formulated to detect obstructions of this form. I was able to realize the above idea in [10]: given an arbitrary multipartite state, I was able to construct cochain complexes whose associated cohomology outputs tuples of multi-body operators that are "non-locally" correlated: a phenomenon related to entangled states and made famous by accounts of the Einstein-Podolsky-Rosen paradox. This machinery, however, is not restricted purely to the quantum regime: one can also construct cochain complexes measuring how badly a probability measure on a product of measurable spaces fails to split as a product measure: i.e. the failure of independence of commutative random variables distributed by a joint probability distribution. From the perspective of non-commutative measure theory, this classical probabilistic question, and the quantum mechanical question about factorizability of a state are special cases of a question regarding the factorizability of a multipartite positive linear functional—the "expectation value"—on W^*-algebras: special types of C^*-algebras that abstract von Neumann algebras, and play the role of algebras of non-commutative random variables. The cochain complexes alluded to above are constructed most naturally in this language.[1]

Moreover, these cochain complexes can be thought of as a step toward a categorification of *multivariate mutual informations*: $\mathbb{R}$-valued quantities defined as an alternating sum of entropies that quantify the information shared among all subsystems [11]. The multivariate mutual information associated to a multipartite expectation value vanishes if the expectation value is factorizable, making it a good indicator of factorizability. Bivariate mutual information is particularly famous among high energy theorists, having important finiteness properties for field-theoretic states [12]. Given a multipartite expectation value, I define an associated three-parameter invariant[2]—called the *state index*—which behaves like an "Euler characteristic" in the sense that it behaves well under tensor products of expectation values and a certain notion of "gluing" of expectation values. Moreover, the state index is holomorphic in each of its parameters; by taking certain limits we see that it interpolates between multipartite mutual information, Tsallis/Renyí deformations [13, 14, 15, 16] of mutual information [10, §8.5], and the integer-valued Euler characteristics of our complexes.

In a more sophisticated treatment, one should be able to recover the full three-parameter state index from cohomological techniques. But even for the complexes of vector spaces developed so far, cohomology gives more than a numerical measure of how much information is shared between subsystems: elements of cohomology specify *what* information is shared among subsystems in the form of tuples of random variables. Moreover, there are entangled states for which multivariate mutual information vanishes, but have non-vanishing cohomology (see §**??**). This is analogous to how the Euler characteristic of any orientable compact manifold of odd-dimension is vanishing, making it useless for detecting the difference between two such manifolds (as opposed to the calculation of homology).

This work is closely related to that of Baudot and Bennequin [17] (J.P. Vigneaux provides an excellent exposition in [18]), who independently construct chain complexes of functions on spaces of probability measures such that mutual informations (and their Tsallis q-deformations) arise as generators of the first cohomology group. My work differs from theirs in the sense that I have a cohomology theory associated to a fixed measure (rather than the space of measures), allowing the mutual informations to arise as Euler characteristics rather than cocycles; however, the two theories are undoubtedly intimately related. I am hoping to elucidate this relationship, perhaps by carefully phrasing their construction in the language of obstruction theory and classifying stacks.

2.1. **Goals and Future Work.** A few interrelated goals of this research are as follows:

(1) New invariants constructed from topological field theories: e.g. link invariants in Chern Simons theory.
(2) The development of measures of shared information both quantum mechanically, and classically: the quantum mechanical case being relevant to entanglement in quantum information theory and the latter possibly relevant to the analysis of large data sets;

[1] Using ideas behind the Gelfand-Neumark-Segal construction of $*$-representations of C^*-algebras, given a multipartite state one constructs a presheaf of vector spaces over the set of tensor factors; cochain complexes then arise as Čech complexes associated to this presheaf.

[2] Here "invariant" means that this function is invariant under automorphisms of random variables that decompose as a tensor product of automorphisms on each tensor factor.

(3) An understanding of the role of homotopy theory/homological algebra in *non-commutative probability theory*: the study of expectation value functions on (possibly) non-commutative algebras;
(4) Developing rigorous techniques for exploring the connection between entanglement and non-trivial geometries or topologies (e.g. the "gauge/gravity" or "AdS/CFT" correspondence).

With regard to (1): One can use Chern Simons theory [19, 20] to assign N-partite state to complements of (framed) N-component links in the three sphere.[3] By taking Poincaré polynomials of the cohomologies associated to such multipartite states, one produces link invariants. The relation of these link invariants to known link invariants remains unknown.

With regard to (3): the machinery behind these complexes suggests that multipartite states behave like spaces, and there is a whole homotopy theory associated to them. Indeed, to every multipartite expectation value, there is a geometric object: formally given by a simplicial object in an appropriate "category of states". Our cohomological techniques are simple probes of this geometric object.

Moreover, there are certainly connections with the work of Drummond-Cole, Park, and Terilla [24, 25, 26, 27] who approach non-commutative probability theory from an A_∞/L_∞-perspective. One sophisticated version of the chain complexes above admits the structure of a differential graded module for a differential graded algebra; the resulting cohomology is an A_∞-module of some A_∞-algebra. Very little is known about these higher algebraic structures at the moment but, drawing vague analogies with the way that Massey products can identify subtle linkage properties of knot complements in the three sphere, my hope is that the higher structures can detect the Borromean-like entanglement properties of the tripartite Greenberger-Horne-Zeilinger (GHZ) state [28, 29].

I also believe that a homological/homotopical approach to (non-commutative) probability theory might lead to a geometrically inspired proof of strong subadditivity: a relation between entropies on a tripartite system, expressible as a simple inequality between bipartite mutual informations. Strong subaddivity of quantum entropies is a non-trivial statement, first proven by Leib and Ruskai [30]. However, Ryu-Takayanagi formulae lead to a very simple geometric interpretation of quantum strong subadditivity; the caveat is that these formulae only hold for small class of states that have special field theoretic descriptions. The homological/homotopical techniques mentioned above are not bound to this limitation.

With regard to (4): as discussed above, there is a deep duality between the classical geometry of general relativity and entangled states: e.g. the Ryu-Takayanagi formula, or even more vague statements such as EPR (Einstein-Podolsky-Rosen) = ER (Einstein-Rosen) [31]. Currently, numerical quantities such as mutual informations or entanglement entropies are used to study this duality; however, by their very nature as methods for studying space, cohomological techniques are likely to shed more light. Furthermore, it would be interesting if there were a connection with other perspectives on entanglement which draw on ideas of the linkage between entanglement and topology by associating entangled states to braids [32].

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