RESEARCH STATEMENT

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1. Overview

My interests mostly reside near the vast range of topics at the intersection of quantum field theory and geometry (in the broad senses of those words). My Ph.D. is in physics and I have experience in high-energy theory; on the other hand, the majority of my time is spent interacting with mathematicians and some of my motivations are skewed toward questions of geometry, with physics as an inspiration.

2. Entanglement, Cohomology, and the Categorification of Mutual Information

Let us begin with two questions of similar flavour:

(C) Suppose we are handed a probability measure μ on a product of measurable spaces (sets each equipped with a σ-algebra of "measurable subsets") $\mathbb{X}_1 \times \mathbb{X}_2 \cdots \times \mathbb{X}_n$, is μ a product measure?

(Q) Suppose we are handed a state (vector) ψ in a tensor product of Hilbert spaces $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$, does $\psi = \psi_1 \otimes \cdots \otimes \psi_n$ for some $\psi_i \in \mathcal{H}_i$?

The first question is one of classical probability theory: one is asking if random variables associated to $\mathbb{X}_i$—e.g. measurable functions $f : \mathbb{X}_i \to (\mathbb{C}, \mathrm{Borel}(\mathbb{C}))$—are independent of random variables associated to $\mathbb{X}_j$ for $j \neq i$. On the other hand, the second question is quantum mechanical in nature: states that fail to factorize in such a way are called *entangled*, and play a fundamental role in quantum information/computation [1, 2]. The study of entangled states has also found its way into high-energy physics, particularly after the work of Ryu-Takayanagi [3], relating entanglement entropy (a numerical measure of entanglement in a field-theoretic state) of states in a conformal field theory to the area of minimal surfaces in a gravity dual theory. Some of these ideas have discrete, computationally accessible versions [4, 5, 6] using the techniques of tensor networks [7, 8, 9] and error correcting codes.

My recent focus is on a project that initially began as a search for a (co)homological measure of failure to question (Q), providing a measure of entanglement with more information than a numerical quantity (in much the same way that homology of spaces provides more information than the Euler characteristic). Given an arbitrary vector in a tensor product Hilbert space, I was able to construct cochain complexes whose associated cohomology outputs tuples of multi-body operators that are "non-locally" correlated: a phenomenon related to entangled vectors and made famous by accounts of the Einstein-Podolsky-Rosen paradox. However, from the perspective of non-commutative measure theory questions (C) and (Q) are special cases of a question regarding the factorizability of a positive linear functional (the "expectation value") on W^*-algebras—special types of C^*-algebras that abstract von Neumann algebras; indeed, the cochain complexes alluded to above are constructed most naturally in this language, helping to answer (C), (Q), and quantum-classical mixtures.

Moreover, there are strong hints that these cochain complexes provide a categorification of multivariate mutual informations—$\mathbb{R}$-valued quantities defined as an alternating sum of entropies [10]. Bivariate mutual information is particularly famous among high energy theorists as the bivariate mutual information associated to disjoint regions in space, and in the presence of a field theoretic state, is a finite quantity [11] (unlike entanglement entropy). Our chain complexes should categorify multivariate mutual information in the sense that their Euler-characteristics[1] are a q-deformation of multivariate mutual information. In particular, these q-deformations are proportional to alternating sums of Tsallis entropies; the $q \to 1$ limit recovers the usual multivariate mutual information. On the other hand, in finite-dimensional quantum mechanics or probability theory on finite sets, the $q \to 0$ limit produces a $\mathbb{Z}$-valued quantity which has the general interpretation as a signed count of irreducible modules of matrix algebras (finite dimensional W^*-algebras): specifically, it is a signed count of submodules of the Gelfand-Naimark-Segal representations associated to the reduced states on tensor factors; concretely, it is easily computable as the signed count of ranks of density states (quantum mechanically) or a signed count of points with non-vanishing probability (for probability theory on finite sets).

This work is very closely related to that of Baudot and Bennequin [12] (J.P Vigneaux provides an excellent detailed exposition in [13]), who independently constructed chain complexes of functions on spaces of probability measures such that mutual informations (and their Tsallis q-deformations) arise as generators of the first cohomology group. My work differs from theirs in the sense that I have a cohomology theory associated to a fixed measure (rather than the space of measures), allowing the mutual informations arise as Euler characteristics rather than cocycles; however, the two theories are undoubtedly intimately related. I am hoping to elucidate this relationship, perhaps by carefully phrasing their construction in the language of obstruction theory and classifying stacks.

2.1. **Goals and Future Work.** A few interrelated goals of this research are as follows:

1. The development of multivariate measures of shared information both quantum mechanically, and classically: the quantum mechanical case being relevant to entanglement in quantum information theory and the latter possibly relevant to the analysis of large data sets;
2. An understanding of the role of homotopy theory/homological algebra in (non-commutative) probability theory;
3. Developing rigorous techniques for exploring the connection between entanglement and non-trivial geometries or topologies.

With regard to (2): there are certainly connections with the work of Drummond-Cole, Park, and Terilla [14, 15, 16, 17] who approach (non-commutative) probability theory from an A_∞/L_∞-perspective. In fact, one sophisticated version of the chain complexes above actually admits the structure of a differential graded module for a differential graded algebra; hence, the resulting cohomology should be an A_∞-module of some A_∞-algebra. Very little is known about these higher algebraic structures at the moment, but—drawing vague analogies with the way that Massey products can identify subtle linkage properties of knots complements in the three sphere—my hope is that the higher structures can detect the Borromean-like entanglement properties of the famous tripartite Greenberger-Horne-Zeilinger (GHZ) state:

$$\frac{1}{\sqrt{2}}\,(e \otimes e \otimes e + f \otimes f \otimes f) \in (\mathrm{span}_{\mathbb{C}}\{e, f\})^{\otimes 3}$$

(see e.g. [18, 19]), and distinguish it from the tripartite W-state:

$$\frac{1}{\sqrt{3}}\,(f \otimes e \otimes e + e \otimes f \otimes e + e \otimes e \otimes f) \in (\mathrm{span}_{\mathbb{C}}\{e, f\})^{\otimes 3}.$$

The W-state and the GHZ state have the same cohomology at the level of graded vector spaces, but are known to be part of different LOCC (Local Operations and Classical Communications) equivalence classes of entangled states.

[1] Or, depending on the perspective, Lefschetz indices of distinguished chain-automorphisms.

With regard to (3): as discussed above, there is a deep duality between the classical geometry of general relativity and entangled states: e.g. the Ryu-Takayanagi formula, or even more vague statements such as EPR (Einstein-Podolsky-Rosen) = ER (Einstein-Rosen) [20]. At the moment, numerical quantities such as mutual informations or entanglement entropies are used to study this duality; however, by their very nature as methods for studying space, cohomological techniques are likely to shed even more light. Moreover, it would be interesting if there were a connection with other perspectives on entanglement which draw on ideas of the linkage between entanglement and topology by associating entangled states to braids [21].

2.2. **A Sample of some Concrete Results.** As a small taste of the cohomological techniques above, we provide a theorem for bipartite pure states. In the following: if $\mathcal{H}$ is a Hilbert space $\mathrm{End}(\mathcal{H})$ denotes its space of (bounded) endomorphisms.

Theorem. *Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be Hilbert spaces and $\psi \in \mathcal{H}_A \otimes \mathcal{H}_B$. Then there is a chain complex of $\mathbb{C}$-vector spaces*

$$0 \to \mathbb{C} \xrightarrow{d_{\mathrm{aug}}} G_A^{\psi} \times G_B^{\psi} \xrightarrow{d_0} \mathcal{H}_A \otimes \mathcal{H}_B \to 0$$

such that

(1) $H^0(\psi) := \ker(d_0)/\mathrm{Im}(d_{\mathrm{aug}})$ *is naturally identifiable with a quotient of the space of pairs $(a, b) \in \mathrm{End}(\mathcal{H}_A) \times \mathrm{End}(\mathcal{H}_B)$ that are "maximally correlated" in a precise sense (in particular, $\langle \psi, (a \otimes 1_B - 1_A \otimes b)^*(a \otimes 1_B - 1_A \otimes b)\psi\rangle = 0$);*
(2) $\dim_{\mathbb{C}} H^0(\psi) = r^2 - 1$, *where r is the "Schmidt rank" of ψ; in particular ψ is entangled if and only if $H^0 \not\equiv 0$. Moreover, one can use the Schmidt decomposition of ψ to generate a basis for H^0.*

As an example, if $\mathcal{H}_X$ is the orthonormal $\mathbb{C}$-span of e_X and f_X for $X \in \{A, B\}$, and we take ψ to be the Bell-state $\frac{1}{\sqrt{2}}(e_A \otimes e_B + f_A \otimes f_B)$, then

$$H^0(\psi) \cong \mathrm{span}_{\mathbb{C}} \left\{ (x_A \otimes y_A^{\vee}, x_B \otimes y_B^{\vee}) : x, y \in \{e, f\} \right\} / \mathrm{span}_{\mathbb{C}} \left\{ (1_A, 1_B) \right\};$$

where 1_X denotes the identity operator on $\mathcal{H}_X$, and $x_X^{\vee} := \langle x, - \rangle_{\mathcal{H}_X}$. This can be thought of as a quotient of the vector space of all non-trivial "non-locally" correlated one-body local pairs of operators ("non-commutative random-variables") by the one-dimensional subspace spanned by the "trivial" correlated operators (i.e. identity operators).

The vector spaces G_X^{ψ}, $X \in \{A, B\}$ in the theorem are actually left $\mathrm{End}(\mathcal{H}_X)$-modules equipped with inner products; their completions are the Gelfand-Naimark-Segal (GNS) representations associated to the reduced density states on $\mathcal{H}_A$ and $\mathcal{H}_B$. The most general construction of such complexes for a multivariate system proceeds by constructing a presheaf of GNS representations over the set of tensor factors (equipped with the discrete topology); one then applies Čech-techniques with respect to a particular cover to produce chain complexes with respect to a given cover of the set of tensor factors; e.g. for tripartite systems (either quantum or classical) one can recover an (augmented) semi-simplicial object of the form

$$\mathbb{C} \longrightarrow G_C \times G_B \times G_A \rightrightarrows G_{BC} \times G_{AC} \times G_{AB} \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} G_{ABC}$$

where the G_X are GNS representations associated to the reduced state/measure on the factor X. One can reduce to cochain complexes and compute cohomology using e.g. associated alternating chain complex, or the normalized Moore complex. I wrote software in both Mathematica and Octave that aids in the computation of cohomology associated to multipartite systems; using this software one can see, e.g. that despite the fact that the GHZ state and W state have vanishing Tsallis entropies (in particular, vanishing mutual informations), the cohomology groups H^0 and

H^1 are non-trivial (and equal in dimension). In this sense, the cohomology provides a more refined way to detect the multipartite entanglement.

3. DT-invariants and Spectral Networks

Donaldson-Thomas (DT) Invariants [22, 23, 24, 25, 26] are (virtual) counts of (semi-stable) objects in a 3-Calabi-Yau (CY) category C; specifically, DT-invariants arise as counts of:

(1) Special Lagrangians in some CY threefold ("A-branes"): $\mathsf{C} = \mathsf{Fuk}(X)$ for some CY threefold X;
(2) Coherent sheaves ("B-branes") on a CY threefold $X^\vee$: $\mathsf{C} = \mathsf{D^bCoh}(X^\vee)$ (the mirror-dual description to 1);
(3) Moduli of complex quiver representations (e.g. $\mathsf{C} = \mathsf{D^bRep}_{\mathbb{C}}(Q)$ for some quiver Q without loops);

In physics, the quantities listed above are geometric manifestations of *BPS states*: states in a supersymmetric field theory that are, roughly-speaking, stable under small deformations of the parameters defining the theory [27]; sometimes equivalent (or dual) descriptions of the same theory allow all three geometric manifestations to play a role (e.g. Denef [28] describes a physical derivation of the relationship between special Lagrangians and quivers).

My thesis work was focused on calculating DT-invariants utilizing techniques developed by Gaiotto-Moore-Neitzke (GMN) [29, 30, 31, 32] in the physics literature – where DT-invariants are conjectured to coincide with "BPS-indices": a weighted-count of BPS states [33]. Specifically, the main theme of my thesis pertains to the algebraicity over $\mathbb{Q}(x)$– hereafter referred to just as *algebraicity* – of (appropriately defined) generating functions for DT-invariants; my main results are:

(1) explicit algebraic relations obeyed by DT-invariant generating functions;
(2) constraints on the asymptotics of DT-invariants/BPS-indices due to algebraicity.

Details can be found in [34] and my earlier work with Galakhov, Longhi, Moore, and Neitzke [35].

Algebraicity has implications for cluster varieties – varieties constructed by gluing together algebraic tori via certain birational maps– as the generating functions for DT-invariants (associated to quivers) appear explicitly in such gluing transformations. Cluster varieties are themselves interesting objects: they sometimes arise as the total space of an integrable system (e.g. the Hitchin integrable system)– a subject that has a large overlap with supersymmetric field theory– and offer explicit geometric realizations of mirror symmetry. On the other hand, in physics, the asymptotics induced by algebraicity imply that, in some supersymmetric field theories, the number of states of mass $\leq M$ grows exponentially with M – indicating the presence of "maximum" temperatures (so-called Hagedorn-temperatures) or possible phase transitions for such theories.

Kontsevich and Soibelman have an understanding of algebraicity for a large class of 3-CY categories (the precise statement hopefully appearing in the literature at some time)[36]; however they claim that their proof uses rather indirect methods. Using machinery developed by GMN called *spectral networks* [32], one can see algebraicity directly, and algorithmically construct explicit algebraic relations obeyed by generating functions. Roughly speaking, a spectral network is a directed, decorated graph associated to a $\mathbb{Z}$-family of BPS states; algebraic relations for generating functions of the associated BPS-indices follows by a system of algebraic relations determined by the edges and vertices of this graph.

3.1. **Future Directions.** As mentioned above, algebraicity has implications for those studying cluster-varieties. A related consequence is a need to carefully define GMN's non-abelianization map [32, §10]: a map that (conjecturally) supplies cluster coordinates to a moduli space of $\mathrm{GL}_K\mathbb{C}$-local systems (with flag data) on a fixed (real, smooth) punctured surface (with fixed monodromy around punctures), arising as a symplectic leaf of (one of) Fock and Goncharov's $\mathcal{X}$-space(s) [37]. As a graduate student and an early postdoc, I partially sketched a definition of a functorial version of the non-abelianization map where algebraicity plays a central role, and I plan on returning to these ideas. Related to this functorial construction: I would also like to explore possible derivations of (generalizations of) the spectral network formalism from a purely representation theoretic perspective: one can view the pushforward of local systems via a covering map as the process of applying the induced representations functor associated to a subgroup(oid) of the fundamental group(oid); the data of a spectral network allows one to define an induced representation-like functor in the situation of *branched* covers (which can be formulated in the language of groupoids).

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