**LES BEAUX BUISSONS DE DELZANT: UN TOUR DE TROIS HEURE**

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# 1. Introduction

1.1. **Motivation.** The celebrities of this paper are symplectic manifolds $X$ equipped with a Hamiltonian $T$-action where $T$ is a compact, connected, abelian Lie group: i.e. $T$ is a torus group $\mathbb{T}^k$ for some $k$. If the action of $T$ is effective, then one can show $\dim(T) \leq \frac{1}{2}\dim(X)$ so in some sense $\dim(T) = \frac{1}{2}\dim(X)$ is the "critical case." But there is also practical reason for interest in such spaces: they arise in the study of integrable Hamiltonian systems. In physics language, $X$ is the "phase space" for a mechanical system equipped with a Hamiltonian $H : X \to \mathbb{R}$. If there are $n = \frac{1}{2}\dim(X)$ independent commuting real-valued functions $H = f_1, \cdots, f_n$ then we say the system is integrable. The $f_i$ generate $n$-commuting Hamiltonian vector fields corresponding to the action of a torus group $\mathbb{T}^n$ [1] and our phase space is foliated by tori. It goes without saying such systems have been quite popular in physics for the past few centuries as they lead to plenty of explicitly solvable models. But even in mathematics they arise in interesting ways despite being far from the generic case of a symplectic group action.

Moreover, there is now a well-understood classification theorem for the case $\dim(T) = \frac{1}{2}\dim(X)$ which can be stated roughly as

Title: See [1].
[1]if the $f_i$ are proper

**Theorem 1.1** (Delzant, 1988)**.**

$$\frac{(X^{2n},\omega)\ \textit{with Ham. effective } \mathbb{T}^n \textit{ action}}{\mathbb{T}^n\textit{-equivariant symplectomorphisms}} \longleftrightarrow \frac{\textit{Convex (Delzant) Polytopes}}{\textit{translations}}.$$

The arrow $\rightarrow$ was given in 1982 courtesy of Atiyah [2] as well as Guillemin and Sternberg [3] in two nearly simultaneous papers proving the convexity of the image of the moment map. A few years later in 1988 the arrow $\leftarrow$ was discovered by T. Delzant [5] via the construction of a symplectic manifold $X_\Delta$ from special convex polytopes now referred to as Delzant polytopes.

There is even a name to measure how far we are from the Delzant situation: the *complexity* of a Hamiltonian $T$-manifold is defined as $\frac{1}{2}\dim(X) - \dim(G)$ [6]. Complexity zero yields the classification above, whereas in 2003 Y. Karshon and S. Tolman classified complexity one spaces.

The goal of this paper is to explore the complexity zero case: the Delzant situation, and make the correspondence listed above explicit and precise. We begin by outlining the Atiyah-Guillemin-Sternberg convexity theorem to secure the $\rightarrow$ part of our theorem: we start with general compact lie groups $G$ and make our way to the $n = \dim(X)/2$ dimensional torus. The paper then moves onto the definition of Delzant polytopes $\Delta$ and the construction of the space $X_\Delta$, some much needed examples are presented at the end of this section. The details of Delzant's theorem are then explored and, finally, we complete our journey by blowing-up symplectic manifolds. Of course, not every detail is presented in this paper and some proofs are relegated to the original papers and other sources. Furthermore, as usual there are many different viewpoints and statements of the same material: this is just but one of many and we take a relatively low-brow approach to make the constructions explicit. On the other hand, the paper serves to bring together various results in the references listed.

1.2. **Setup and Preliminaries.** Let $G$ be a compact $d$-dimensional Lie group, $\mathfrak{g}$ its Lie algebra, and $\mathfrak{g}^*$ the dual of its Lie algebra. Now equip a $2n$-dimensional compact connected symplectic manifold $(X,\omega)$ with a $G$-action $\tau : G \to \mathrm{Diff}(X)$ that preserves the symplectic form $\omega$:

$$\tau(g)^*\omega = \omega,\ \forall g \in G.$$

This action gives rise to an infinitesimal action of $\mathfrak{g}$ that associates to every $\xi \in \mathfrak{g}$ a vector field $\xi^\#$. Indeed, let $\rho_t = \tau\left[\exp(t\xi)\right]$, then $\xi^\# : \mathbb{C}^\infty(X) \to \mathbb{C}^\infty(X)$ is defined by

$$(\xi^\# f)(x) = \left[\frac{d}{dt} f(\rho_t(x))\right]_{t=0},\ \forall f \in C^\infty(X).$$

As our action is symplectic, $\xi^\#$ is locally Hamiltonian. In this paper, we will assume $\xi^\#$ is globally Hamiltonian, i.e. $\tau$ defines a Hamiltonian $G$-action on $X$. In other words, for every $\xi \in \mathfrak{g}$ there exists a smooth function $\phi^\xi : X \to \mathbb{R}$ such that

$$\iota_{\xi^\#}\omega = -d\phi^\xi, \tag{1}$$

which determines $\phi^\xi$ up to an additive constant. Alternatively we have a map

$$\begin{aligned} \phi : \mathfrak{g} &\to C^\infty(X) \\ \xi &\mapsto \phi^\xi. \end{aligned}$$

It follows from (1) that $\phi$ depends linearly on $\xi$ and $\phi([\xi,\eta]) = \{\phi^\xi, \phi^\eta\}$ where $\{\cdot,\cdot\}$ is the Poisson bracket on $C^\infty(X)$; hence, $\phi$ is a morphism of Lie algebras. A dual version of $\phi$ is the mapping

$$\Phi : M \to \mathfrak{g}^* \tag{2}$$

defined by

$$\langle \Phi(x), \xi \rangle = \phi^\xi(x) \tag{3}$$

for all $x \in X$ and all $\xi \in \mathfrak{g}$. Because $\phi^\xi$ is determined up to an additive constant, $\Phi$ is determined up to a constant $c \in \mathfrak{g}^*$.

**Definition** The map $\Phi : M \to \mathfrak{g}^*$ is called a *moment map* associated to the Hamiltonian $G$-action on $(X,\omega)$.

As an aside, we quickly define the term *G-equivariant* for smooth manifolds. The definition obviously generalizes to more general categories of objects, but to suggest we require such machinery for this paper would require such divine comedic intervention as to render the remainder of this paper comparatively lackluster.

**Definition** Let $M$ and $N$ be manifolds equipped with smooth $G$-actions $\alpha : G \to \mathrm{Diff}(M)$ and $\beta : G \to \mathrm{Diff}(N)$. A smooth map $\Phi : M \to N$ is $G$-equivariant if $\Phi(g \cdot m) = g \cdot \Phi(m)$ – more precisely if the following diagram commutes

$$\begin{tikzcd} M \arrow[r, "\Phi"] \arrow[d, "\alpha(g)"'] & N \arrow[d, "\beta(g)"] \\ M \arrow[r, "\Phi"] & N. \end{tikzcd}$$

For all $g \in G$. If $\Phi(g \cdot m) = \Phi(m)$, i.e., $\beta \equiv Id_M$ then we say $\Phi$ is $G$-invariant.

Let $\xi^{\#}$ and $\eta^{\#}$ be Hamiltonian vector fields induced via $\xi, \eta \in \mathfrak{g}$. Then via (3) and the fact that $\phi$ is a Lie algebra morphism we have

$$\omega(\xi^{\#}, \eta^{\#}) = \left\{\phi^{\xi}, \phi^{\eta}\right\} = \phi^{[\xi,\eta]},$$

so $\Phi$ satisfies

$$\omega_x(\xi^{\#}, \eta^{\#}) = \langle \Phi(x), [\xi, \eta] \rangle. \tag{4}$$

Using (4) we can show that $\Phi$ is a $G$-equivariant map $X \to \mathfrak{g}^*$, i.e. it intertwines the action of $G$ on $M$ and the coadjoint action of $G$ on $\mathfrak{g}^*$:

$$\Phi(\tau(g)(x)) = \mathrm{Ad}^{\#}\Phi(x). \tag{5}$$

In this paper we will be primarily concerned with the case $G = T$ a compact connected abelian Lie group, i.e. a torus. In this case the coadjoint action is trivial and so Eq. (5) implies $\Phi$ is $T$-invariant. To prevent confusion, we provide one more important definition of terminology used in this paper.

**Definition** Let $(X_1, \omega_1), (X_2, \omega_2)$ be symplectic manifolds. A *symplectomorphism* $\psi : (X_1, \omega_1) \to (X_2, \omega_2)$ is a diffeomorphism such that $\psi^*\omega_2 = \omega_1$. A *local symplectomorphism* is a map $(X_1, \omega_1) \to (X_2, \omega_2)$ which restricts to a symplectomorphism on sufficiently small neighborhoods of $X_1$ and $X_2$.

**Remark** When there is only one manifold involved a (local) symplectomorphism of $(X, \omega)$ is a (local) diffeomorphism $\psi : X \to X$ such that $\psi^*\omega = \omega$. If $X$ is equipped with two symplectic forms $\omega_1$ and $\omega_2$, to prevent confusion we will say $\psi : X \to X$ is a diffeomorphism such that $\psi^*\omega_2 = \omega_1$ instead of calling it a symplectomorphism.

## 2. The image of the moment map is a convex polytope

We begin with the convexity theorem of Atiyah, Guillemin, and Sternberg.

**Theorem 2.1** (Atiyah-Guillemin-Sternberg convexity theorem)**.** *Equip $(X, \omega)$ with a Hamiltonian $T$-action for $T$ a $d$-dimensional torus. Then the image of the associated moment map $\Phi : X \to \mathfrak{t}^*$ is a convex polytope: the convex hull of $\Phi(M^T)$ where $M^T$ denotes the fixed point set of the action of $T$.*

To prove this theorem we first bring the moment map to a canonical form in the neighborhood of a fixed point of $T$ through use of an equivariant version of the Darboux theorem. This allows us to show that the image of the moment map is convex in a neighborhood around a fixed point. Global convexity then follows by use of a lemma regarding the uniqueness of local maxima of the moment map whose proof relies on Morse-Bott theory.

**Theorem 2.2** (Equivariant-Darboux Theorem)**.** *Let $G$ be a compact Lie group, and $\omega_0$ and $\omega_1$ be $G$ invariant symplectic forms on $X$ with $\omega_0 = \omega_1$ at $x \in X^G$. Then there exists a $G$-invariant neighborhood $U$ of $x$ and a $G$-equivariant (local) diffeomorphism $\Psi : (U, x) \to (X, x)$ such that $\Psi^*\omega_1 = \omega_0$.*

*Proof.* A proof can be provided by an extension of the standard Moser-Weinstein argument along with a group-averaging technique. The result is a family of $G$-equivariant flows $\rho_{tt'}$ defined in a neighborhood of $x$ and such that $\rho_{01} = \Psi$. We relegate the full proof to the appendix. $\square$

Let $(X, \omega)$ be a symplectic manifold equipped with a Hamiltonian $G$-action and associated moment map $\Phi : X \to \mathfrak{g}^*$. Then at a fixed point $x \in X^G$, the $G$-action induces a linear action on $T_xX$. Moreover, setting $\omega^1 = \omega_x$, we actually have a symplectic vector space $(T_xX, \omega^1)$. By definition the linear induced $G$-action is symplectic and, hence, Hamiltonian as $T_xX$ is contractible. Let the associated moment map be $\Phi_0 : T_xX \mapsto \mathfrak{g}$. Not surprisingly, our equivariant version of the Darboux theorem implies as Hamiltonian $G$-spaces $(X, \omega)$ and $(T_xX, \omega)$ are locally "isomorphic".

**Corollary 2.3.** *There exists a $G$-equivariant local symplectomorphism $(T_xX, 0, \omega^1) \to (X, x, \omega)$.*

*Proof.* We proceed similarly to an argument in [3] to show this. Equip $X$ with a $G$-invariant Riemannian metric; such a metric exists by equipping $X$ with a a Riemannian metric and using the standard "averaging procedure" over $G$. Let

$$\exp : T_xX \to X$$

be the exponential map defined by this metric. Because our metric is $G$-invariant, this map intertwines the action of $G$ on $X$ and the linear action of $G$ on $T_xX$ (it is $G$-equivariant) and diffeomorphically maps $U_0$, a $G$-invariant neighborhood of 0, to a $G$-invariant neighborhood $U$ of $x$. Let $\omega^0 = \exp^* \omega$ which defines another symplectic form on $X$ that agrees with $\omega^1$ at 0. By Theorem 2.2 if $U_0$ is sufficiently small then there exists a $G$-equivariant local diffeomorphism $\Psi : (U_0, 0) \to (T_xX, 0)$ such that $\Psi^*\omega^1 = \omega^0$. Finally, as the exponential map is a local $G$-equivariant diffeomorphism, we have a $G$-equivariant symplectomorphism $\exp \circ \Psi : (U_0, 0) \mapsto (U, x)$. $\square$

The following corollary is immediate from the Equivariant-Darboux theorem and the fact the moment map is only determined up to a constant. However, we include the proof for sake of completeness in the appendix.

**Corollary 2.4.** *Let $\Phi : X \to \mathfrak{g}^*$ be the moment map associated to the $G$-action on $X$ and $\Phi_0 : T_xX \mapsto \mathfrak{g}^*$ the moment map associated to the linearized action at $x \in X^G$. Define $\Phi_1 = \Phi \circ \exp$ the pullback of $\Phi$ to $T_xX$. Then for $U_0$ and $U_1$ sufficiently small $G$-invariant neighborhoods of 0: $\Phi_0 : U_0 \to \mathfrak{g}^*$ and $\Phi_1 : U_1 \to \mathfrak{g}^*$ differ by a translation in $\mathfrak{g}^*$.*

2.1. **Representations of the Torus.** From now on we specialize to the case $G = T$ a $d$-dimensional torus. Via the above propositions, understanding the moment map in a neighborhood of a fixed point reduces down to understanding the linearized action of $T$ on the vector space $T_xX$.

**Proposition 2.5.** *Let $V$ be a real $2n$-dimensional vector space and $\rho : T \mapsto GL(V)$ a representation of a $d$-dimensional torus $T$ on $V$ which leaves no vector fixed except 0. Let $\xi \cdot v = d\rho(\xi)(v)$ indicate the induced action of the Lie algebra $\mathfrak{t}^*$. Then there is a unique decomposition of $V$ (the weight decomposition)*

$$V = \bigoplus_{k=1}^{n} V_k,$$

*where the $V_k$ are 2-dimensional subspaces,*

$$V_k = \{v \in V : \forall \xi \in \mathfrak{t}, \, \xi \cdot v = \alpha_k(\xi) v\}$$

*for $\alpha_k \in (\mathbb{Z}^d)^* \subset \mathfrak{t}^*$, i.e. we are identifying $\mathfrak{t}^*$ with $(\mathbb{R}^d)^*$ so $\alpha_k$ is in the dual integral lattice.*

*Sketch of proof.* $T$ is just the direct sum of $d$ copies of $\mathbb{T}$ which all commute with one another. For notational convenience identify $V$ with $\mathbb{C}^n$. Hence, the induced representations on each $\mathbb{T}$ component yields a simultaneous eigenspace decomposition of $V$ into $n$ one-dimensional (complex) subspaces $V_k$ $(k = 1, \cdots, n)$. On the $l$th factor of $\mathbb{T}$ in $T$ acts on $V_k$ as $\exp(im_{lk}t_{lk})$ for some $m_{lk} \in \mathbb{Z}$. Under this decomposition $\rho$, viewed as a matrix representation, takes the diagonal form

$$\rho(e^{it_1}, e^{it_2}, \cdots, e^{it_d}) = \begin{pmatrix} e^{i\sum_{l=1}^{d} m_{1l}t_l} & 0 & \cdots & 0 \\ 0 & e^{i\sum_{l=1}^{d} m_{2l}t_l} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{i\sum_{l=1}^{d} m_{nl}t_l} \end{pmatrix}$$

Identify $\mathfrak{t}$ with $\mathbb{R}^d$; let $\xi_l \in \mathbb{Z}$ generate the $l$th factor of $\mathbb{T}$ in $T$ and $\eta^l \in \mathbb{Z}^*$ be their duals, then we have the weights

$$\alpha_k = \sum_{l=1}^{d} m_{kl}\eta^l \in \mathfrak{t}^*,\ k = 1, \cdots, n.$$

Because $m_{kl} \in \mathbb{Z}$, then $\alpha_k \in (\mathbb{Z}^d)^*$; furthermore, $\alpha_k \neq 0$ for each $k$ as the representation leaves no non-zero vectors fixed: thus, for each $k$ we must have $m_{kl} \neq 0$ for some $l$. □

Note that we can extend the above proposition to the case where the representation of $T$ has non-zero kernel $V_0$. In such a case $V/V_0$ forms a representation of $T$; it is necessarily even-dimensional as irreps of $\mathbb{T}$ are even-dimensional and it satisfies the assumptions of the proposition. Hence,

$$V = V_0 \oplus \bigoplus_{k=1}^{n-r} V_k$$

where the $V_k$ are 2-dimensional and $V_0$ is $2r$ dimensional. The representation acts trivially on $V_0$ (for which we can associate $r$ identically zero weights) and non-trivially on each $V_k$ via non-vanishing integral weights. Moreover, we can always split the $V_0$ term arbitrarily into 2-dimensional subspaces (in which case the decomposition would be unique up to a splitting of $V_0$).

2.2. **Normal form of the Moment map around a fixed point.** As usual, consider a $2n$-dimensional Hamiltonian $T$-space $X$, an associated moment map $\Phi : X \to \mathfrak{t}^*$ and a fixed point $x \in X^T$. Equipped with the proposition from the previous section, the induced linear representation on $T_xX$ is classified by the weights $\alpha_{1,x}, \cdots, \alpha_{1,x} \in \mathfrak{t}^*$ which lie in the integral lattice $\mathbb{Z}^*$ when $\mathfrak{t}^*$ is identified with $(\mathbb{R}^{2d})^*$.

**Lemma 2.6.** *There exist coordinates $p_1, \cdots p_n, q_1, \cdots, q_n$ in a neighborhood $U$ containing $x \in X^T$ so that the moment map takes the form*

$$\Phi(p, q) = \Phi(x) + \sum_{k=1}^{n} \frac{1}{2}\alpha_{k,x}\left(p_n^2 + q_n^2\right).$$

*Proof.* By Cor.2.4 we need only consider the tangent space $T_xX$ equipped with the induced linear $T$ representation and its associated moment map $\Phi_0 : T_xX \to \mathfrak{t}^*$. Identify $T_xX$ with $\mathbb{C}^n$. Then via Prop. 2.5 $T_xX$ splits into a direct sum

$$T_xX = \bigoplus_{k=1}^{\infty} V_k,$$

where the $T$ action on $z = (z_1, \cdots, z_n)$ with $z_k \in V_k$ is given by

$$e^{i\xi t} \cdot z = (e^{i\alpha_{1,x}(\xi)t}z_1, \cdots, e^{i\alpha_{n,x}(\xi)t}z_n),$$

for $\xi \in \mathfrak{t}$. This action generates a Hamiltonian vector field $\xi^{\#}$ defined on any $f \in C^{\infty}(X)$ via

$$\begin{aligned}
\xi^{\#} f(z, \overline{z}) &= \left[\frac{d}{dt} f\left(e^{i\xi t} \cdot z, e^{-i\xi t} \cdot \overline{z}\right)\right]_{t=0} \\
&= \left[\frac{d}{dt} f(e^{i\alpha_{1,x}(\xi)t} z_1, \cdots, e^{i\alpha_{n,x}(\xi)t} z_n, e^{-i\alpha_{1,x}(\xi)t} \overline{z}_1, \cdots, e^{-i\alpha_{n,x}(\xi)t} \overline{z}_n)\right]_{t=0} \\
&= \sum_{k=1}^{n} \left[\frac{\partial f}{\partial z_k} \frac{\partial}{\partial t}\left(e^{i\alpha_{k,x}(\xi)t} z_k\right) + \frac{\partial f}{\partial \overline{z}_k} \frac{\partial}{\partial t}\left(e^{-i\alpha_{k,x}(\xi)t} \overline{z}_k\right)\right]_{t=0} \\
&= \left(\sum_{k=1}^{n} i\alpha_{k,x}(\xi) z_k \frac{\partial}{\partial z_k} - \sum_{k=1}^{n} i\alpha_{k,x}(\xi) \overline{z}_k \frac{\partial}{\partial \overline{z}_k}\right) f.
\end{aligned}$$

We claim that we can choose the coordinates $z_k \in V_k$ such that symplectic form $\Omega = \omega_x$ on $T_x X$ can be written as

$$\Omega = \frac{i}{2} \sum_{k=1}^{n} dz_k \wedge d\overline{z}_k.$$

Indeed, this quantity is $T$-invariant; hence, the claim follows by the Equivariant-Darboux theorem (Thm. 2.2). Via the definition of the moment map

$$\begin{aligned}
d\phi_0^{\xi} &= -\,\iota_{\xi^{\#}} \Omega \\
&= \frac{1}{2} \sum_{k=1}^{n} \alpha_{k,x}(\xi) \left(z_k d\overline{z}_k + \overline{z}_k dz_k\right)
\end{aligned}$$

where $\phi_0^{\xi} = \langle \Phi_0, \xi \rangle$. So we have the associated moment map

$$\Phi_0(z, \overline{z}) = \frac{1}{2} \sum_{k=1}^{n} \alpha_{k,x} |z_k|^2.$$

where $\Phi(0)$ is the constant we wish to assign to $\Phi_0$ at the origin. $\square$

2.3. **Local Convexity.** For notational convenience, let

$$S(\alpha_1, \cdots, \alpha_n) = \left\{\sum_{k=1}^{n} s_k \alpha_k : s_k \geq 0\right\} \subset \mathfrak{t}^*$$

be the cone [2] defined by $\alpha_k \in \mathfrak{t}^*$.

**Theorem 2.7** (Local Convexity Theorem)**.** *There exists a neighborhood $U'$ of $x \in X^T$ and a convex neighborhood $V'$ of $\Phi(x) \in \mathfrak{t}^*$ such that*

$$\Phi(U') = V' \cap \ (\Phi(x) + S(\alpha_{1,x}, \cdots, \alpha_{n,x})) \subset \mathfrak{t}^*.$$

*Proof.* Intuitively, we should be able to find a neighborhood of $x$ such that the image of $\Phi$ is a truncated cone. Let $U$ be a neighborhood of $x$ equipped with coordinates $z_1 = p_1 + iq_1, \cdots, z_n = p_n + iq_n$ such that $\Phi$ has the normal form of Lemma 2.6. Let $a = \min_{k \in \{1, \cdots, n\}} \sup\{|z_k| : z \in U\}$ and define

$$U' = \{(z_1, \cdots, z_n) \in U : |z_k| < a, \forall k = 1, \cdots, d\} \subset U,$$

which is an open neighborhood of $x$. Then,

$$V' = \Phi(x) + \left\{\eta \in \mathfrak{t}^* : \eta(\xi) < \sum_{k=1}^{d} \alpha_{k,x}(\xi) a^2, \forall \xi \in \mathfrak{t}\right\}.$$

$\square$

[2] $S(\alpha_1, \cdots, \alpha_n)$ describes the region around a vertex of a polytope, i.e. it is a "polyhedral" cone.

Hence, local convexity of the image of $\Phi$ is immediate around fixed points. However, we would like to show local convexity around any point $y \in X$ not necessarily fixed. Let $H \leq T$ be the stabilizer subgroup of such a point (another torus group which may be trivial), i.e. $y \in X^H$, and let $\iota : H \hookrightarrow T$ be the inclusion. On the lie algebra level there is an induced inclusion map which we will sadistically also denote $\iota : \mathfrak{h} \hookrightarrow \mathfrak{t}$, and a corresponding dual (restriction) map $\iota^* : \mathfrak{t}^* \to \mathfrak{h}^*$. The Hamiltonian $T$-action on $X$ restricts to $H$ and the induced moment map is just

$$\Phi_H = \iota^* \circ \Phi : X \to \mathfrak{t}^* \to \mathfrak{h}^*$$

Let $\beta_{1,y}, \ldots, \beta_{n,y} \in \mathfrak{h}^*$ be the weights of the linear representation of $H$ induced at $T_yX$. Then by the local convexity theorem we can find an open set $U$ such that we have the convex set

$$\Phi_H(U) = V \cap (\Phi_H(y) + S_H(\beta_{1,y}, \cdots, \beta_{n,y})).$$

Define

$$S'(\beta_{1,y}, \cdots, \beta_{n,y}) = (\iota^*)^{-1}[S_H(\beta_{1,y}, \cdots, \beta_{n,y})].$$

The following relative version of the above theorem then holds

**Theorem 2.8.** *There exists a neighborhood $U$ of $y \in X^H$ and a neighborhood $V$ of $\Phi(y)$ in $\mathfrak{t}^*$ such that*

$$\Phi(U) = V \cap \ (\Phi(x) + S'(\beta_{1,y}, \cdots, \beta_{n,y})) \subset \mathfrak{t}^*$$

*Proof.* By the equivariant Darboux theorem we may instead focus on the linear space $V = T_yX$ and the moment map $\Phi_H^0 : V \to \mathfrak{h}^*$ induced by the linear $H$ action on $V$. $\square$

2.4. **Global Convexity.** Global convexity of the moment map image follows almost immediately from the results above along with the following lemma.

**Lemma 2.9.** *For any $\xi \in \mathfrak{t}$, the function $\langle \Phi, \xi \rangle = \phi^\xi : X \to \mathbb{R}$ has a unique local maximum.*

The proof involves showing the function $\phi^\xi : X \to \mathbb{R}$ is Morse-Bott and, furthermore, the number of negative eigenvalues of $Hess_x\phi^\xi$ around all critical points $x$ is even. A result of Morse theory is that all such functions have unique local maxima. We defer the full proof to [3], but we will reap its benefits by completing the proof of Thm. 2.1.

Let $\eta \in \mathfrak{t}^*$ be a point on the boundary of the image of the moment map, $x \in \Phi^{-1}(\eta)$, $H \subset T$ the stabilizer group of $x$ and $\alpha_{1,x}, \cdots, \alpha_{n,x} \in \mathfrak{h}^*$ the corresponding weights of the induced linear representation at $T_yX$. By Thm. 2.8 we can find neighborhoods $U$ of $y \in X^H$ and $V$ of $\eta$ such that

$$\Phi(U) = V \cap \ (\eta + S'(\alpha_{1,x}, \cdots, \alpha_{n,x})) \subset \mathfrak{t}^*$$

Let $S_j$ be a boundary component of $S'(\alpha_{1,x}, \cdots, \alpha_{n,x})$. As $S_j$ is at least codimension 1 we can choose $\xi \in \mathfrak{g}$ such that the corresponding linear functional $l_\xi : \mathfrak{t}^* \to \mathbb{R}$ (given by $l_\xi(\nu) = \langle \xi, \nu \rangle$) satisfies $l_\xi \equiv 0$ on $S_j$ and $l_\xi < 0$ on the interior of $S'(\alpha_{1,x}, \cdots, \alpha_{n,x})$. Then if $l_\xi(\nu) = a$, for all $y \in U$ we have

$$\phi^\xi(y) = \langle \xi, \Phi(y) \rangle = (l_\xi \circ \Phi)(y) \leq a$$

Hence, $a$ is a local maximum of $\phi^\xi$. By the lemma above it is an absolute maximum so $\Phi(X) \leq a$. Repeating this argument for all faces of $S'(\alpha_{1,x}, \cdots, \alpha_{n,x})$ it follows that the full image lies in the cone

$$\Phi(X) \subset \eta + S'(\alpha_{1,x}, \cdots, \alpha_{n,x}).$$

Hence, as $\eta$ is an arbitrary boundary point, $\Phi(M)$ behaves like a convex set relative to its boundary and so is convex. For a more detailed explanation of this last statement (dealing with the details of polyhedral cones) see [4].

## 3. The Delzant Construction

**Definition** A convex polytope $\Delta$ in $(\mathbb{R}^n)^*$ is Delzant if

(1) (Simplicity) There are $n$ edges meeting in each vertex $p$.
(2) (Rationality) The edges meeting in the vertex $p$ are rational, i.e., each edge is of the form $p + tv_i$, $0 \leq t \leq \infty$ where $v_i \in (\mathbb{Z}^n)^*$.
(3) (Smoothness) The $v_1, \cdots, v_n$ in (2) can be chosen to be a basis of $(\mathbb{Z}^n)^*$.

There is an equivalent formulation of this definition. Let

$$\mathbb{R}_+ = \{(\xi_1, \cdots, \xi_n) \in \mathbb{R}^n : \xi_i \geq 0, \forall i = 1, \cdots, n\}$$

and

$$(\mathbb{R}_+)^* = \{x \in (\mathbb{R}^n)^* : \xi(x) > 0, \forall x \in \mathbb{R}^n_+\}.$$

Then (1), (2), and (3) are equivalent to the statement that for every vertex $p \in \Delta$ there exists a matrix $A$ with integer coefficients and determinant $\pm 1$ such that the map

$$\mathbb{R}^* \ni x \mapsto Ax - p$$

maps a neighborhood of $p \in \Delta$ to a neighborhood of 0 in $(\mathbb{R}^n_+)^*$. The equivalence is readily seen by noticing $A$ must be a matrix consisting of the $v_i \in (\mathbb{Z}^n)^*$, $i = 1, \cdots, n$.

**Remarks**

- This definition can be extended to arbitrary $n$-dimensional vector spaces $V$ and lattices $L$ with the the obvious replacements $(\mathbb{R}^n)^* \rightsquigarrow V^*$ and $(\mathbb{Z}^n)^* \rightsquigarrow L^*$, but for our concerns this does not add any additional enlightenment.
- Condition (3) can be relaxed to

  $(3)^*$: The $v_1, \cdots, v_n$ in (2) can be chosen to be a basis of $(\mathbb{R}^n)^*$

  The result will be that the Delzant space $X_\Delta$ we will construct below will not be a smooth manifold (arising as a quotient space of a manifold with group that acts freely), but an orbifold: a smooth manifold with mild isolated singularities (arising as the quotient space of a manifold with a compact group that acts *locally* freely).

3.1. **Construction of the Moment Map.** Let $\Delta$ have $d$ faces. We begin by noting we can describe these codimension 1 faces via linear equations of the form

$$\langle u_i, x\rangle = \lambda_i, \; i = 1, \cdots, d \tag{6}$$

where $u_i \in \mathbb{Z}^n$ and $\lambda_i \in \mathbb{R}$ are fixed such that $x \in (\mathbb{R}^n)^*$ is constrained to move on one of the faces of $\Delta$. Geometrically, the $u_i$ are normal vectors to the $(n-1)$-dimensional faces of $\Delta$; and, in fact wlog we can take them to be *primitive* (unit normal vectors), i.e. the $u_i$ are not of the form $u_i = ku_i'$ for $k \neq \pm 1$. Furthermore, to insure uniqueness of the $u_i$ describing the above equations we take them to be inward pointing normal vectors. [3] More precisely, we assume they are oriented in such a way that $\Delta$ is the intersection of the half-spaces

$$H_i = \{x \in (\mathbb{R}^n)^* : \langle u_i, x\rangle \geq \lambda_i\}, \; i = 1, \cdots, d. \tag{7}$$

Now let $e_1, \cdots, e_d$ be the standard basis vectors of $\mathbb{R}^d$ and consider the map

$$\pi : \mathbb{Z}^d \to \mathbb{Z}^n$$
$$e_i \mapsto u_i$$

which extends linearly to a map $\pi : \mathbb{R}^d \to \mathbb{R}^n$. Hence, there is an induced quotient map

$$\Pi : T^d \mapsto T^n$$

Now let $N = \ker(\Pi)$ (which is also a torus group), so we have an exact sequence

$$0 \longrightarrow N \longrightarrow T^d \xrightarrow{\Pi} T^n \longrightarrow 0.$$

[3] These statements are immediate from basic linear algebra. Choose the standard basis $e_1, \cdots, e_n \in \mathbb{Z}^n \subset \mathbb{R}^n$ of $\mathbb{R}^n$ and the corresponding dual basis $e_1^*, \cdots, e_n^*$ of $(\mathbb{R}^n)^*$ with $e_k^*(e_l) = \delta^{kl}$. Then we can write any arbitrary $x \in (\mathbb{R}^n)^*$ as $x = \sum_{k=1}^n x_k e_k^*$ for $x_k \in \mathbb{R}$. A face of $\Delta$ is a codimension 1 affine hyperplane consisting of $x$ whose coefficients satisfy a linear equation of the form

$$a_1 x_1 + \cdots a_n x_n = \lambda_i$$

where the $a_k \in \mathbb{Q}^n$ by condition (2); by an overall rescaling of the equation we can take the $a_k \in \mathbb{Z}^n$ such that $u_i = \sum_{k=1}^n a^k e_k$ is a primitive normal vector to our hyperplane.

Take the standard (unit weight) linear action $\rho$ of $T^d$ on $\mathbb{C}^d$, given by

$$\rho(x)z = \left(e^{i\theta_1}z_1, \cdots, e^{i\theta_d}z_d\right),$$

which preserves the symplectic form

$$\omega = \frac{i}{2}\sum_{j=1}^{d} dz_j \wedge d\overline{z}_j.$$

As noted in the proof of Lemma 2.6, we can choose the associated moment map $\Phi : \mathbb{C}^d \to (\mathbb{R}^d)^*$ to be

$$\Phi(z) = \frac{1}{2}\sum_{j=1}^{d} |z_j|^2 e_j^* + c.$$

For some constant $c \in (\mathbb{R}^d)^*$. For reasons which will become clear below, a convenient choice for $c$ happens to be

$$c = \lambda := \sum_{j=1}^{d} \lambda_j e_j^* \in (\mathbb{R}^d)^*,$$

i.e. we take

$$\Phi(z) = \frac{1}{2}\sum_{j=1}^{d}(|z_j|^2 + 2\lambda_j)e_j^*. \tag{8}$$

Now the inclusion map $N \hookrightarrow T^d$ induces an inclusion map at the Lie algebra level, $\iota : \mathfrak{n} \hookrightarrow \mathbb{R}^d$, and its corresponding restriction on duals $\iota^* : (\mathbb{R}^d)^* \to \mathfrak{n}^*$. As mentioned previously in the proof of Thm. 2.7 (the local convexity theorem), the moment map associated to the restricted action of $\rho$ to $N$ is

$$\Phi_N = \iota^* \circ \Phi : \mathbb{C}^d \to \mathfrak{n}^*.$$

3.2. **N acts freely on the pre-image of 0.** We wish to construct a symplectic manifold $X_\Delta$ to $\Delta$ via symplectic reduction on an appropriate pre-image of $\Phi_N$. The constant $\lambda$ in $\Phi(z)$ was chosen above such that our pre-image of interest is, in fact, $\Phi_N^{-1}(0)$. In order that the corresponding symplectically reduced space $\Phi_N^{-1}(0)/N$ be a compact manifold, we require the following theorem.

**Theorem 3.1.** *$\Phi_N^{-1}(0)$ is a compact subset of $\mathbb{C}^d$ and $N$ acts freely on this set.*

We begin the proof by noting taking note of the exact sequence

$$0 \longrightarrow \mathfrak{n} \xrightarrow{\iota} \mathbb{R}^d \xrightarrow{\pi} \mathbb{R}^n \longrightarrow 0$$

and its dual

$$0 \longrightarrow (\mathbb{R}^n)^* \xrightarrow{\pi^*} (\mathbb{R}^d)^* \xrightarrow{\iota^*} \mathfrak{n}^* \longrightarrow 0.$$

Now let $\Delta' = \pi^*\Delta$ be the pullback of our polytope via the injective map $\pi^*$.

**Lemma 3.2.**

$$\Phi_N^{-1}(0) = (\iota^* \circ \Phi)^{-1}(0) = \Phi^{-1}(\Delta'). \tag{9}$$

*Proof.* As $|z_j|^2 \geq 0$, the image of $\Phi$ given in (8) is

$$\Phi(\mathbb{C}^n) = \left\{x \in (\mathbb{R}^d)^* : \langle e_i, x\rangle \geq \lambda_i, \forall i = 1, \cdots, d\right\}.$$

Let $x \in (\iota^* \circ \Phi)^{-1}(0)$, then $\iota^*(x) = 0$ and $x \in \Phi(\mathbb{C}^n)$, but by the exact sequence above $\iota^*(x) = 0 \Leftrightarrow x = \pi^*(y)$ for some $y \in (\mathbb{R}^n)^*$. Hence,

$$\langle e_i, \pi^*(y)\rangle \geq \lambda_i,\ \forall i.$$

So by definition of $\pi^*$,

$$\langle \pi(e_i), y\rangle = \langle u_i, y\rangle \geq \lambda_i,\ \forall i;$$

thus, $y \in \Delta$. □

**Definition** Let $z = (z_1, \cdots, z_n) \in \mathbb{C}^d$ and

$$I_z = \{i : z_i = 0\}.$$

Define

$$\mathbb{R}^I_z = \left\{(x_1, \cdots, x_d) \in \mathbb{R}^d : x_i = 0 \text{ for } i \notin I_z\right\},$$

and let $T^I$ be the image of $R^I$ in $T^d$, i.e.

$$T^I_z = \left\{(e^{i\theta_1}, \cdots, e^{i\theta_n}) : e^{i\theta_i} = 1 \text{ for } i \notin I_z\right\}$$

The following is then immediate.

**Lemma 3.3.** *The stabilizer group of $z$ in $\mathbb{C}^d$ is $T^I_z$.*

Now, via lemma 3.2, $z \in \Phi_N^{-1}(0) \Leftrightarrow \Phi(z) = \pi^*(p)$ for some $p \in \Delta$. In order to understand the action of $N \subset T^d$ on $\Phi_N^{-1}(0)$, we first study the action of the full $T^d$. The following lemma classifies this action depending based on the location of $p \in \Delta$. The "worst" possible case is when $p$ is a vertex, and this is where the Delzant conditions (1)-(3) are utilized.

**Lemma 3.4.**

(1) *If $p$ is in the interior of $\Delta \subset (\mathbb{R}^n)^*$ then $T^d$ acts freely on $\Phi^{-1}(\pi^*(p)) \subset \mathbb{C}^n$.*

(2) *If $p$ is on the boundary of $\Delta$ and lies on the intersection of the $(n-1)$-dimensional faces*

$$\langle u_i, p\rangle = \lambda_i,\ i \in I_z,$$

*then the stabilizer group of $z \in \Phi^{-1}(\pi^*(p))$ is $T^I_z$ and $N$ acts freely at $z$.*

(3) *As a special case of (2), if $p$ is a vertex of $\Delta$ then $T^I_z \overset{\Pi}{\cong} \mathbb{T}^n$ is the stabilizer of $z$, defined as the quotient of*

$$\mathbb{R}^I = \left\{x \in \mathbb{R}^d : x = \sum_{i \le n} c_i e_i\right\}$$

*and $N$ acts freely at $z \in \Phi^{-1}(\pi^*(p))$.*

*Proof. (1): Interior Point.*
Let $x = \pi^*(p) = J(z)$. Then $p$ is an interior point of $\Delta$ iff

$$\langle u_i, p\rangle > \lambda_i,\ \forall i = 1, \cdots, d.$$

But as $\pi(u_i) = e_i$ we have

$$\langle x, e_i\rangle > \lambda_i,\ \forall i = 1, \cdots, d.$$

Writing $x = \sum_{j=1}^d x_j e_j^*$ and noting that $x = \Phi(z)$, via (8) we have $x_i = |z_i|^2/2 + \lambda_i > \lambda_i$ so $z_i \neq 0$ for all $i$. Thus, by Lemma 3.3, $T^d$ acts freely at $z$.
*(2): Boundary Point.*
The situation differs only slightly from interior scenario. Indeed, let $\pi^*(p) = J(z)$ so

$$\langle u_i, p\rangle = \lambda_i,\ i \in I_z$$
$$\langle u_i, p\rangle > \lambda_i,\ i \notin I_z$$

From which it follows that $z_i = 0 \Leftrightarrow i \in I_z$; hence the stabilizer group of $z$ is $T^I$. To show that $N$ acts freely at $z$ we must show that $T^I \cap N = \{0\}$ in $T^d$. The critical scenario is then when $p$ is a vertex so $T^I$ is as large as possible; so we show trivial intersection in this case first.
*(3): Vertex.*
Without loss of generality take $p = 0$ so that the faces at $p$ can be taken to be hyperplanes passing through 0 and, thus, are described by equations where we can take the $\lambda_i$ to vanish:

$$\langle u_i, x\rangle = 0,\ i = 1, \cdots, n.$$

We now make use of conditions $(1)-(3)$, i.e. via a $GL(n,\mathbb{Z})$ transformation we can take $u_1,\cdots,u_n$ to be the standard basis of $\mathbb{R}^n$. Thus,

$$\pi : e_i \mapsto u_i,\ i = 1,\cdots,d$$

maps

$$\mathbb{R}^I_z = \left\{ x \in \mathbb{R}^d : \sum_{i \le n} c_i e_i \right\} \cong \mathbb{R}^n$$

bijectively onto $\mathbb{R}^n$; so the corresponding quotient map $\Pi : T^I_z \mapsto T^n$ is a bijection. But $N = \ker(\Pi)$; hence, $T^I_z \cap N = \{0\}$ the trivial subgroup of $T^d$. However, $T^I_z$ is the stabilizer group of $z \in \Phi^{-1}(\pi^*(p))$ so $N$ must act freely at $z$.

When $p$ is a boundary point that is not a vertex, then via similar arguments the stabilizer group of $z$ is bijectively mapped to a subgroup of $T^n$ so $N$ intersects the stabilizer trivially. □

The lemma above shows that $N$ acts freely on all of $\Phi_N^{-1}(0) = \Phi^{-1}(\Delta')$ (and, furthermore, classifies the full action of $T^d$ on this pre-image). All that remains to show to prove Theorem 3.1 is that $\Phi_N^{-1}(0)$ is compact. But this is immediate as $\Phi$, defined by (8) is proper and $\Delta' = \pi^*\Delta$ is compact as $\pi^*$ is injective. Thus, the proof of the theorem is complete.

3.3. **The space $X_\Delta$.** By Theorem 3.1 we can reduce $\mathbb{C}^d$ with respect to the action of $N$ to obtain the space

$$X_\Delta = \Phi_N^{-1}(0)/N$$

by the Marsden-Weinstein reduction theorem (see Thm. 6.1 in the appendix) $X_\Delta$ is a symplectic manifold and its dimension is [4]

$$\dim_{\mathbb{R}}(X_\Delta) = \dim_{\mathbb{R}}(\mathbb{C}^d) - 2\dim_{\mathbb{R}}(N) = 2d - 2(d-n) = 2n.$$

Now, via Lemma 3.3, if $p \in \Delta$ is a vertex and $z \in \Phi^{-1}(\pi^*(p))$, the stabilizer subgroup $T^I_z$ of $z$ is a complementary subgroup of $N$ in $\mathbb{T}^d$; furthermore, this subgroup is mapped bijectively via $\Pi : T^d \to T^n$ onto $T^n$. Thus, we have an embedding

$$j : T^n \hookrightarrow T^d \tag{10}$$

with

$$\Pi \circ j = Id_{\mathbb{T}^n} \tag{11}$$

depending on the choice of vertex $p$ (more precisely $j : T^n \to T^I_z$ bijectively). With this identification, as $N$ is complementary to the stabilizer, the action of $T^n$ on $\mathbb{C}^d$ commutes with the action of $N$, so there is an induced Hamiltonian action of $T^n$ on $X_\Delta$ with an associated moment map

$$\Psi : X_\Delta \to (\mathbb{R}^n)^*$$

that fits into the following diagram

$$\begin{tikzcd}
& \mathbb{C}^d \arrow[dr, "\Phi"] & \\
\Phi_N^{-1}(0) \arrow[ur, hook] \arrow[rr, dotted] \arrow[d, "q"] & & (\mathbb{R}^d)^* \arrow[d, "j^*"] \\
X_\Delta \arrow[rr, "\Psi"'] & & (\mathbb{R}^n)^*
\end{tikzcd}$$

[4] We utilize $\mathrm{codim}(\Phi^{-1}(0)) = \dim(N)$ and so $\mathrm{codim}(\Phi^{-1}(0)/N) = 2\dim(N)$.

where $q : \Phi_N^{-1}(0) \to X_\Delta$ is the quotient map. and $\Phi_N^{-1}(0) \hookrightarrow \mathbb{C}^d$ the natural embedding. In other words,

$$\Psi \circ q = j^* \circ \Phi|_{\Phi_N^{-1}(0)}. \tag{12}$$

Note furthermore, the $T^n$ action is effective on $\Delta$ as via Lemma 3.3 $T^d$, and thus, $T^n$ acts freely on the open dense subset $\Phi^{-1}(\pi^*\mathrm{Int}(\Delta))$.

**Theorem 3.5.** *The image of $\Psi$ is $\Delta$.*

*Proof.* Using surjectivity of $q$ and equations (9), (11), and (12).

$$\begin{aligned}
\Psi(X_\Delta) =& \Psi \circ q(\Phi_N^{-1}(0)) \\
=& j^* \circ \Phi(\Phi_N^{-1}(0)) \\
=& j^* \circ \Phi((\iota^* \circ \Phi)^{-1}(0)) \\
=& j^* \circ \Phi(\Phi^{-1}(\pi^*\Delta)) \\
=& j^*(\pi^*\Delta) \\
=& (\pi \circ j)^*(\Delta) \\
=& \Delta.
\end{aligned}$$

□

**Definition** The space $X_\Delta$ constructed above is called the Delzant space associated with the polytope $\Delta$. We have proved that $X_\Delta$ is a compact Hamiltonian-$T^n$ space on which $T^n$ acts effectively.

**Remark** We should note that the embedding $j : T^n \hookrightarrow T^d$ implicitly depends on the choice of vertex $p$ used in its construction; so it may appear that the induced $T^n$ action and associated moment map $\Psi$ depend on this choice. However, as we will see below, the resulting actions from any two choices of $p$ are related via a $T^n$-equivariant symplectomorphism.

The information above gives us an explicit technique for computing $X_\Delta$ and the moment map $\Psi : X_\Delta \to (\mathbb{R}^n)^*$ from the polytope $\Delta \subset (\mathbb{R}^n)^*$ with $d$ faces:

(1) Determine $u_i \in \mathbb{Z}^n$, $i = 1, \cdots d$ (the "inward pointing normal vectors" to the faces of $\Delta$) and the corresponding $\lambda_i \in \mathbb{R}$ in equations (6) and (7) that uniquely determine $\Delta$.
(2) Write out the linear extension of the map $\pi : \mathbb{Z}^d \to \mathbb{Z}^n$, $e_i \mapsto u_i$, $i = 1, \cdots, d$ and calculate

$$N = \ker(\Pi : T^d \mapsto T^n) = \exp\left(i \ker(\pi)\right).$$

(3) Determine the pullback map $\pi^* : (\mathbb{R}^n)^* \to (\mathbb{R})^d$ and calculate $\Delta' = \pi^*\Delta$
(4) Write out the moment map $\Phi : \mathbb{C}^d \to (\mathbb{R}^d)^*$ given by

$$\Phi(z) = \frac{1}{2}\sum_{i=1}^{d} |z_i|^2 e_i^* + \lambda_i e_i^*$$

and determine $\Phi_N^{-1}(0) = \Phi^{-1}(\Delta')$
(5) Construct the quotient $X_\Delta = \Phi^{-1}(\Delta')/N$.
(6) To construct the moment map $\Psi : X_\Delta \to (\mathbb{R}^n)^*$, choose a vertex $p \in \Delta$ and determine the stabilizer subgroup $T_z^I$ of points $z \in K = \Phi^{-1}(\pi^*(p))$. This gives an $n$-dimensional torus with the embedding $j : T^n \hookrightarrow T^d$ $(j(T^n) = T_z^I)$.
(7) Determine $j^* : (\mathbb{R}^d)^* \to (\mathbb{R}^n)^*$ and define the action of $\Psi$ via $\Psi \circ q = j^* \circ \Phi|_K$ for $q : \Phi^{-1}(\Delta') \to X_\Delta$ the quotient map.

3.4. **Examples of Delzant Spaces.** We now provide some examples.

**Example** Let $\Delta$ be the interval $[-1, 1]$. More precisely, let $f \subset \mathbb{R}$ be a length 1-vector in $\mathbb{R}$ and $f^*$ its dual, then $\Delta$ is the set

$$\Delta = \{x = af^* : a \in [-1, 1]\}.$$

We will show $X_\Delta = \mathbb{C}P^1$. Our first step is to describe the faces of $\Delta$ by linear equations of the form (6). Because $\Delta$ is one-dimensional, the faces are just the 0-dimensional vertices whose inward pointing normal vectors are $u_1 = f$ at the left $(-1)$ vertex and $u_2 = -f$ at the right $(+1)$ vertex. Indeed, the equations

$$\begin{aligned}\langle u_1, x\rangle &\geq -1\\ \langle u_2, x\rangle &\geq -1\end{aligned}$$

constrain $x = af^*$ such that $a \geq -1$ via the first equation and $a \leq 1$ via the second equation so we recover $\Delta$. The map $\pi : \mathbb{Z}^2 \to \mathbb{Z}$ is then

$$\begin{aligned}\pi : e_1 &\mapsto f\\ e_2 &\mapsto -f\end{aligned}$$

extended to $\mathbb{R}^2 \to \mathbb{R}$ then

$$\pi(ae_1 + be_2) = (a-b)f.$$

with quotient $\mathbb{T}^2 \to \mathbb{T}$

$$\Pi\left[(e^{i\theta_1}, e^{i\theta_2})\right] = e^{i(\theta_1-\theta_2)}$$

so

$$N = \ker(\Pi) = \left\{(e^{i\theta}, e^{i\theta})\right\} \cong \mathbb{T}.$$

As a matrix $\pi = (1, -1)$; so $\pi^* = (1, -1)^T$, i.e.

$$\pi^*(af^*) = ae_1^* - ae_2^*.$$

Thus,

$$\Delta' = \pi^*\Delta = \{a(e_1^* - e_2^*) : a \in [-1, 1]\}$$

As $\lambda_1 = \lambda_2 = -1$ for this example, we have the moment map

$$\Phi(z) = \frac{1}{2}\left(|z_1|^2 e_1^* + |z_2|^2 e_2^*\right) - e_1^* - e_2^*.$$

So $\Phi^{-1}(\Delta')$ is given by

$$\begin{aligned}\Phi^{-1}(\Delta') &= \left\{(z_1, z_2) \in \mathbb{C}^2 : \frac{1}{2}|z_1|^2 - 1 = a = -\frac{1}{2}|z_2|^2 + 1 : a \in [-1, 1]\right\}\\ &= \left\{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 4\right\}\\ &\cong S^3.\end{aligned}$$

i.e. $\Phi^{-1}(\Delta')$ is just the 3-sphere of radius 2. Now $N$ acts on $\mathbb{C}^2$ as

$$(e^{i\theta}, e^{i\theta}) \cdot (z_1, z_2) = (e^{i\theta} z_1, e^{i\theta} z_2)$$

so the quotient map $\Phi^{-1}(\Delta') \to \Phi^{-1}(\Delta')/N$ is the map $(z_1, z_2) \mapsto [z_1 : z_2]$, i.e.

$$X_\Delta = \Phi^{-1}(\Delta')/N \cong \mathbb{C}P^1.$$

There is a residual $\mathbb{T}^2/N \cong \mathbb{T}$ action on $X_\Delta$ with moment map $\Psi : X_\Delta \to (\mathbb{R})^*$ satisfying equation (11). We will construct this in the manner described above. First, choose the right $f^*$ vertex of $\Delta$, then

$$\begin{aligned}\Phi^{-1}(\pi^*(f^*)) =& \Phi^{-1}(e_1^* - e_2^*)\\ &= \left\{(z_1, z_2) : |z_1|^2 = 4,\ |z_2|^2 = 0\right\}\\ &= \left\{2(e^{i\theta}, 0)\right\}.\end{aligned}$$

The stabilizer group of such a point is $\{(0, e^{i\theta})\}$. So our embedding $j : \mathbb{T} \hookrightarrow \mathbb{T}^2$ is given by

$$j(e^{i\theta}) = (0, e^{i\theta}).$$

The corresponding induced map on dual Lie algebra $j^* : (\mathbb{R}^2)^* \to (\mathbb{R})^*$ is

$$j^*(ae_1^* + be_2^*) = bf.$$

Hence,

$$\begin{aligned}\Psi([z_1 : z_2]) =& j^* \circ \Phi(z_1, z_2)\\ =& \left(\frac{1}{2}|z_2|^2 - 1\right) f.\end{aligned}$$

Alternatively, if we took the right $-f^*$ vertex of $\Delta$ the induced $\Phi$ would have been the same as above with $z_2$ replaced by $z_1$.

**Example** Let $\Delta$ be the isoceles right triangle with vertices $(0,0)$, $(0,1)$ and $(1,0)$. We will show that $X_\Delta = \mathbb{C}P^2$. Let $f_1, f_2$ be the standard basis of $\mathbb{R}^2$ and $f_1^*$, $f_2^*$ the corresponding dual basis. As a subset of $(\mathbb{R}^2)^*$, $\Delta$ is given by

$$\Delta = \left\{x = af_1^* + bf_2^* : a, b \geq 0,\, a + b \leq 1\right\}.$$

with faces $a = 0$, $b = 0$, $a + b = 1$. The "inward normal" vectors to these faces are $u_1 = f_1$, $u_2 = f_2$ and $u_3 = -(f_1 + f_2)$ and, indeed, $\Delta$ is given by the equations

$$\begin{aligned}\langle u_1, x\rangle &= a \geq 0\\ \langle u_2, x\rangle &= b \geq 0\\ \langle u_3, x\rangle &= -(a + b) \geq -1\end{aligned}$$

The corresponding map $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ acts as

$$\pi(ae_1 + be_2 + ce_3) = au_1 + bu_2 + cu_3 = (a - c)e_1 + (b - c)e_2.$$

so

$$\Pi\left[(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3})\right] = \left(e^{i(\theta_1-\theta_3)}, e^{i(\theta_2-\theta_3)}\right)$$

and

$$N = \ker(\Pi) = \left\{(e^{i\theta}, e^{i\theta}, e^{i\theta})\right\} \cong \mathbb{T}.$$

Furthermore, as matrices

$$\pi = \begin{pmatrix} 1 & 0 & -1\\ 0 & 1 & -1\end{pmatrix};\ \pi^* = \pi^T = \begin{pmatrix} 1 & 0\\ 0 & 1\\ -1 & -1\end{pmatrix}.$$

In other words

$$\pi^*(af_1^* + bf_2^*) = ae_1^* + be_2^* - (a + b)e_3^*.$$

Thus,

$$\Delta' = \pi^*\Delta = \left\{ae_1^* + be_2^* - (a + b)e_3^* : a, b \geq 0, a + b \leq 1\right\}.$$

The moment map is

$$\Phi(z) = \frac{1}{2}\sum_{i=1}^{3}|z_i|^2 e_i^* - e_3^*.$$

and

$$\Phi^{-1}(\Delta') = \left\{(z_1, z_2, z_3) : |z_1|^2 + |z_2|^2 + |z_3|^2 = 2\right\} \cong S^5.$$

Once again, quotienting by $N$, $(z_1, z_2, z_3) \mapsto [z_1 : z_2 : z_3]$ so we have

$$X_\Delta = \Phi^{-1}(\Delta')/N \cong \mathbb{C}P^2.$$

We defer the calculation of the associated moment map for the remaining $T^2$ effective action to the more general example below.

**Example** To complete the pattern set by the previous two examples, let $\Delta$ be the $n$-simplex spanned by the origin and the vectors $f_1^*, f_2^*, \cdots, f_n^*$ (where as usual the $f_i^*$ are dual vectors to the standard basis $f_i$ in $\mathbb{R}^n$). Then $X_\Delta = \mathbb{C}P^n$. $\Delta$ is given by

$$\Delta = \left\{ x = \sum_{i=1}^{n} a_i f_i^* : a_i \geq 0,\, \forall i = 1, \cdots, n \text{ and } \sum_{i=1}^{n} a_i \leq 1 \right\}$$

So our inward "normal vectors" are

$$u_i = f_i,\, i = 1, \cdots n$$
$$u_{n+1} = -\sum_{i=1}^{n} f_i.$$

And $\Delta$ is described by the equations

$$\langle u_i, x\rangle \geq 0,\, i = 1, \cdots, n$$
$$\langle u_{n+1}, x\rangle \geq -1.$$

The first equation describes the $f_1^* \cdots \widehat{f_i^*} \cdots, f_n^*$ face and the latter the remaining face not contained in any coordinate plane. Now

$$\pi\left(\sum_{i=1}^{n+1} a_i e_i\right) = \sum_{i=1}^{n} (a_i - a_{n+1}) f_i$$
$$\pi^*\left(\sum_{i=1}^{n} b_i f_i^*\right) = \sum_{i=1}^{n} b_i e_i^* - \left(\sum_{i=1}^{n} b_i\right) e_{n+1}^*,$$

and

$$N = \ker(\Pi) = \exp(i \ker(\pi)) = \left\{ (e^{i\theta}, \cdots, e^{i\theta}) \in \mathbb{T}^n \right\} \cong \mathbb{T}.$$

Further

$$\Delta' = \pi^*\Delta = \left\{ \sum_{i=1}^{n+1} a_i e_i^* : a_i \geq 0,\, a_{n+1} = -\sum_{i=1}^{n} a_i \right\}.$$

The relevant moment map and corresponding pre-image are

$$\Phi(z) = \frac{1}{2} \sum_{i=1}^{n+1} |z_i|^2 e_i^* - e_{n+1}^*$$
$$\Phi^{-1}(\Delta') = \left\{ (z_1, \cdots, z_{n+1}) : \sum_{i=1}^{n+1} |z_i|^2 = 2 \right\} \cong S^{2n-1};$$

so

$$X_\Delta = \Phi^{-1}(\Delta')/N \cong \mathbb{C}P^n.$$

We now calculate a moment map $\Psi : X_\Delta \to (\mathbb{R}^n)^*$ on $X_\Delta$ describing the effective $\mathbb{T}^n$ action. To do such, following the method outlined above we first choose a point $p \in \Delta$; choose the easiest point: the zero vertex $0 \in \Delta$. Then $\Phi^{-1}(\pi^*(0)) = \Phi^{-1}(0)$ which is given as

$$\Phi^{-1}(0) = \{(z_1, \cdots, z_{n+1}) : z_i = 0, \forall i = 1, \cdots, n \text{ and } z_{n+1} = 2\}$$

The stabilizer subgroup for $z \in \Phi^{-1}(0)$ is then automatically

$$T_z^I = \left\{ (e^{i\theta_1}, \cdots, e^{i\theta_n}, 0) \right\} \leq T^{n+1}.$$

Hence, at the Lie algebra level, the dual $j^* : (\mathbb{R}^{n+1})^* \to (\mathbb{R}^n)^*$ of the embedding $j : \mathbb{R}^n \hookrightarrow \mathbb{R}^d$ is the "restriction" map

$$j^*\left(\sum_{i=1}^{n+1} a_i e_i^*\right) = \sum_{i=1}^{n} a_n f_n^*.$$

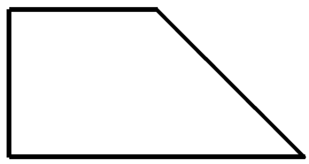

Figure 1: The trapezoid of the last example.

Giving (from $\Psi \circ q = j^* \circ \Phi$)

$$\Psi([z_1 : \cdots : z_n]) = \frac{1}{2} \sum_{i=1}^{n} |z_i|^2 f_i^*.$$

**Example** Let us look at a slightly more complicated example. Let $\Delta \subset (\mathbb{R}^2)^*$ be the trapezoid formed by joining the unit square to a right triangle with hypotenuse of slope $1/m$, $m \in \mathbb{Z}$. Letting $x = af_1^* + bf_2^*$ the faces of $\Delta$ are given by $\{a=0\}, \{b=0\}, \{b=1\}, \{a+mb=1\}$; so

$$\Delta = \{x = af_1^* + bf_2^* : a \geq 0,\, 0 \leq b \leq 1,\, a + mb \leq 1\} \subset (\mathbb{R})^2$$

and

$$\begin{aligned} u_1 =& f_1 \\ u_2 =& f_2 \\ u_3 =& -f_2 \\ u_4 =& -(f_1 + mf_2) \end{aligned}$$

And the equations describing $\Delta$ are

$$\begin{aligned} \langle u_1, x \rangle &\geq 0 \\ \langle u_2, x \rangle &\geq 0 \\ \langle u_3, x \rangle &\geq -1 \\ \langle u_4, x \rangle &\geq -1. \end{aligned}$$

The map $\pi : \mathbb{R}^4 \to \mathbb{R}^2$, $e_i \mapsto u_i$ is given by

$$\pi(ae_1 + be_2 + de_3 + ce_4) = (a-c)f_1 + (b - mc - d)f_2.$$

With kernel given by the equations $a - c = 0$, $b - mc - d = 0$; hence,

$$\begin{aligned} N =& \ker(\Pi) \\ =& \exp(i \ker(\pi)) \\ =& \left\{ (e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_1}, e^{i(\theta_2 - m\theta_1)}) \right\}. \end{aligned}$$

Now, as a matrix in the $e_i$ and $f_i$,

$$\pi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -m \\ 0 & -1 \end{pmatrix} \Rightarrow \pi^* = \pi^T = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -m & -1 \end{pmatrix}.$$

In other words

$$\pi^* \left( af_1^* + bf_2^* \right) = ae_1^* + be_2^* - (a + bm)e_3^* - be_4^*.$$

Hence,

$$\Delta' = \pi^* \Delta = \left\{ ae_1^* + be_2^* - (a + mb)e_3^* - be_4^* : a \geq 0,\, 0 \leq b \leq 1,\, a + mb \leq 1 \right\}.$$

Our moment map associated to the $\mathbb{T}^4$ action is

$$\Phi(z) = \frac{1}{2}\left[|z_1|^2 e_1^* + |z_2|^2 e_2^* + (|z_3|^2 - 2)e_3^* + (|z_4|^2 - 2)e_4^*\right].$$

The pre-image $\Phi^{-1}(\Delta')$ is described by the equations

$$\begin{aligned} |z_1|^2 =& 2a \\ |z_2|^2 =& 2b \\ |z_3|^2 =& -2(a+mb)+2 \\ |z_4|^2 =& -2b+2. \end{aligned}$$

So

$$\Phi^{-1}(\Delta') = \left\{(z_1, \cdots, z_4) \in \mathbb{C}^4 : |z_1|^2 + m|z_2|^2 + |z_3|^2 = 2,\ |z_2|^2 + |z_4|^2 = 2\right\}.$$

The resulting four (real) dimensional space $X_\Delta = \Phi^{-1}(\Delta')/N$ is called a Hirzebruch surface [10]. In fact, this surface turns out to be the symplectic blow-up of $\mathbb{C}P^2$: see section 5.

To calculate the moment map $\Psi : X_\Delta \to (\mathbb{R})^2$ for the associated $T^2$ action we again choose the "easy vertex" $0 \in \Delta$. Then, similar to the previous example we end up with $\pi^*(0) = 0 \Rightarrow$ we are concerned with $z \in \Phi^{-1}(0)$. Here

$$\Phi^{-1}(0) = \left\{(z_1, z_2, z_3, z_4) : z_1 = z_2 = 0 \text{ and } |z_3|^2 = |z_4|^2 = 2\right\}.$$

The stabilizer group of such points $z$ is

$$T_z^I = \left\{(e^{i\theta_1}, e^{i\theta_2}, 0, 0)\right\} \le T^4$$

and once again, the corresponding $j^* : (\mathbb{R}^4)^* \to (\mathbb{R}^2)^*$ acts like a restriction map

$$j^*(ae_1^* + be_2^* + ce_3^* + de_4) = af_1^* + bf_2^*.$$

Hence, denoting $[z] = [z_1, z_2, z_3, z_4]$ as the equivalence class of $z \in \Phi^{-1}(\Delta')$ under the map $q : \Phi^{-1}(\Delta') \to \Phi^{-1}(\Delta')/N = X_\Delta$, we have the simple expression

$$\Psi([z]) = \frac{1}{2}\left(|z_1|^2 f_1^* + |z_2|^2 f_2^*\right).$$

## 4. Delzant's Theorem

For the sake of conciseness and common decency, it serves us well to temporarily expand our vocabulary.

**Definition** A compact connected symplectic $2n$-dimensional manifold $(X, \omega)$ is toric if it admits an effective Hamiltonian $\mathbb{T}^n$-action. By an isomorphism between toric manifolds $(X_1, \omega_1)$ and $(X_2, \omega_2)$ we mean a $\mathbb{T}^n$-equivariant symplectomorphism $\varphi : (X_1, \omega_1) \to (X_2, \omega_2)$.

Now that we have constructed toric manifolds $X_\Delta$ from Delzant polytopes $\Delta$, the question then arises "how many toric manifolds arise this way?" The answer is a surprising "all of them." More precisely,

**Theorem 4.1.**

$$\frac{\textit{Symplectic Toric Manifolds}}{\mathbb{T}^n\textit{-equivariant symplectomorphisms}} \longleftrightarrow \frac{\textit{Delzant Polytopes}}{\textit{translations}}.$$

Following the outline in [1], the proof is organized into several steps. Here $T = \mathbb{T}^n$ and $X$ is a $2n$-dimensional symplectic manifold.

(1) $X$ is toric with associated moment map $\Phi \Rightarrow \Phi(X)$ is Delzant.
(2) $\Delta$ is Delzant $\rightsquigarrow$ construct a compact, connected, symplectic manifold $X_\Delta$. (Done)
(3) Check $X_\Delta$ is toric with associated moment map $\Psi : X_\Delta \to \mathfrak{t}^*$. (Done)
(4) $X_1$ isomorphic to $X_2 \Rightarrow \Psi(X_1)$ is equivalent to $\Psi(X_2)$ up to translation (i.e. "isomorphisms" of convex polytopes)
(5) $\Phi(X_1) = \Phi(X_2) \Rightarrow \Delta_1$ is equivalent to $\Delta_2$ up to translation.

As a corollary of the unproven steps (1), (4), and (5), the following theorem is also sometimes mentioned.

**Theorem 4.2.** *Let $X$ be toric with associated moment map $\Phi : X \to \mathfrak{t}^*$. Then the image $\Delta$ of $\Phi$ is a Delzant polytope and $X$ is isomorphic to $X_\Delta$.*

To show step (1) we will need the following proposition.

**Proposition 4.3.** *Let $X$ be a symplectic manifold equipped with a symplectic $G$-action, then there is an $G$-invariant open dense set on which the action of $G$ is free.*

*Proof.* See [9], chapter 27. □

**Lemma 4.4.** *Let $X$ be toric with associated moment map $\Phi : X \to \mathfrak{t}^*$, then $\Phi(X)$ is a Delzant polytope.*

*Proof.* By the convexity theorem (theorem 2.1), $\Delta = \Phi(X)$ is a convex polytope. It remains to show $\Delta$ satisfies the Delzant conditions which we list again for convenience.

(1) (Simplicity) There are $n$ edges meeting in each vertex $p$.
(2) (Rationality) The edges meeting in the vertex $p$ are rational, i.e., each edge is of the form $p+tv_i$, $0 \leq t \leq \infty$ where $v_i \in (\mathbb{Z}^n)^*$.
(3) (Smoothness) The $v_1, \cdots, v_n$ in (2) can be chosen to be a basis of $(\mathbb{Z}^n)^*$.

Let $x \in X^T$, then by the local convexity theorem (Thm. 2.7) $p = \Phi(x)$ is a vertex of $\Delta$ and the moment polytope in a neighborhood $U$ of $p$ is

$$\Phi(U) = V \cap \left\{ p + \sum_{k=1}^{n} s_k \alpha_k : s_k \geq 0 \right\} \subset \mathfrak{t}^*$$

where the $\alpha_k$ are the weights of the linear representation of $T$ induced on $T_x X$. Identifying $\mathfrak{t}^*$ with $(\mathbb{R}^n)^*$, then $\alpha_k \in (\mathbb{Z}^n)^*$. Now via Lemma 2.6, these $\alpha_k$ describe the $T$-action in an open neighborhood of our fixed point. Hence, as the action of $T$ is effective, by Prop. 4.3 the $\alpha_k$ must form a basis of $(\mathbb{R}^n)^*$. Thus, conditions (1) and (2) are satisfied. Suppose $\Delta$ does not satisfy (3). Let $A$ be the $n \times n$ matrix consisting of the $\alpha_k$'s, thought of as a matrix $\mathfrak{t} \to \mathfrak{t}$. Then $A$ is invertible as an $\mathbb{R}$-matrix but not a $\mathbb{Z}$-matrix. Hence, we can find $\tau \in A^{-1}(\mathbb{Z}^n)$ such that $\tau \notin (\mathbb{Z}) \Rightarrow \exp(\tau) \in T$ is not the identity. Now $A\tau \in (\mathbb{Z}^n)$ so $\langle \alpha_k, \tau \rangle \in \mathbb{Z} \Rightarrow \exp(\tau)$ acts trivially on a neighborhood of $x$, but via Prop. 4.3 this is a contradiction. □

Following nearly the same proof for 2.4, step (4) is immediate.

**Lemma 4.5.** *If $X_1$ and $X_2$ are isomorphic then, letting $\Phi_i : X_i \to \mathfrak{t}^*$ be associated moment maps ($i = 1, 2$), $\Phi_1(X_1) \cong \Phi_2(X_2)$, i.e. the images of the moment maps are equivalent up to translation in $\mathfrak{t}^*$.*

Step (5) listed above is one of the most remarkable parts of the proof of Delzant's theorem; we restate it below as it appears in Delzant's paper [5].

**Theorem 4.6.** *Let $(X_1, \omega_1)$ and $(X_2, \omega_2)$ be toric manifolds with associated moment maps $\Phi_i : X_i \to \mathfrak{t}^*$ ($i = 1, 2$). Assume that $\Phi_1(X_1) = \Phi_2(X_2)$, then there exists a $T$-equivariant symplectomorphism $\varphi : X_1 \to X_2$ fitting into the commutative diagram*

$$\begin{array}{ccc} X_1 & \xrightarrow{\varphi} & X_2 \\ \Big\downarrow{\scriptstyle \Phi_1} & & \Big\downarrow{\scriptstyle \Phi_2} \\ \Phi_1(X_1) & \overset{Id}{=} & \Phi_2(X_2) \end{array}$$

*Proof.* We defer the reader to Delzant's paper [5], Theorem 2.1, for the full proof. A repeat of part of this proof is also present in [10]. □

[5] Translated into English.

## 5. Symplectic Blow-Ups

Let $[p] \in \mathbb{C}P^{n-1}$ be the equivalence class of $p \in \mathbb{C}^{\times}$ and $V_p = \{\lambda p : \lambda \in \mathbb{C}\} \subset \mathbb{C}^n$. Then there is a natural map $[p] \to V_p$. Define the space

$$B = \left\{([p], v) : [p] \in \mathbb{C}P^{n-1},\, v \in V_p\right\}$$

equipped with

$$\begin{aligned} \iota : \mathbb{C}P^{n-1} &\to B \\ [p] &\mapsto ([p], 0) \end{aligned}$$

and

$$\begin{aligned} \beta : B &\to \mathbb{C}^n \\ ([p], v) &\mapsto v \end{aligned}$$

**Definition** $B$ is the *blow-up* of $\mathbb{C}^n$ at the origin and the image of $\iota$ is its singular locus (also called the *exceptional divisor*). The holomorphic map $\beta : B \to \mathbb{C}^n$ is the *blow-down* map and is bijective off the image of $\iota$.

The terminology "blow-up" here is referring to how the space $X$ is obtained: we take the origin of $\mathbb{C}^n$ and expand it as if we are blowing up up a balloon. Better yet, we can think of replacing the origin with the projectivized tangent plane at the origin (i.e. all of the independent directions emanating from the point).

We note that the group $U(n)$ acts naturally on both $\mathbb{C}P^n$ and $\mathbb{C}^n$; this extends to an action on $B$ [6]. Furthermore, via inspection it is immediate that the maps $\iota$ and $\beta$ are equivariant with respect to this action. Now equip $\mathbb{C}^n$ with the standard $U(n)$-invariant symplectic form

$$\omega_0 = \frac{i}{2}\sum_{k=1}^{n} dz_k \wedge d\overline{z}_k$$

We would like to define what it means to be a *symplectic* blow-up. In order to do such we must also "blow-up" the symplectic form $\omega_0$ to the total space $B$ in a nice manner.

**Definition** A symplectic form $\omega$ on $B$ is a *blow-up* of $\omega_0$ if it is $U(n)$ invariant and $\omega - \beta^*\omega_0$ is compactly supported.

Of course, it would be nice to to have a notion of equivalence of such forms.

**Definition** $\omega_1$ and $\omega_2$ symplectic blow-up forms on $B$ are equivalent if there exists a $U(n)$-equivariant diffeomomorphism $\varphi : B \to B$ such that $\varphi^*\omega_2 = \omega_1$.

Guillemin and Sternberg have shown that symplectic blow-up forms are in fact equivalent iff their restrictions to the exceptional divisor $\mathbb{C}P^{n-1}$ are equal.

**Theorem 5.1.** *$\omega_1$ and $\omega_2$ are equivalent iff $\iota^*\omega_1 = \iota^*\omega_2$*

We will be concerned with a particular class of symplectic blow-ups. Let $\epsilon_{FS}$ be the Fubini-study form on $\mathbb{C}P^{n-1}$.

**Definition** Let $\epsilon > 0$. An $\epsilon$-blow-up of $\mathbb{C}^n$ at the origin is a pair $(B, \omega)$ with $\omega$ a blow-up of $\omega_0$ and $\iota^*\omega = \epsilon\omega$.

We can then think of such an $\epsilon$-blow-up $(B, \omega)$ as constructed by deleting the origin and replacing it by a projective space $\mathbb{C}P^{n-1}$ with the symplectic form $\epsilon\omega_{FS}$; for small $\epsilon$ this can be thought of as a mild deformation of $\mathbb{C}^n$ near the origin.

Now let $X$ be any $2n$-dimensional (compact, connected) Hamiltonian $G$-space. As $G$ is a compact Lie group we can find a unitary representation of $G$, i.e. we can think of $G$ as a subgroup of $U(n)$. With this identification, $\mathbb{C}^n$ can be thought of as a Hamiltonian $G$-space. Let $p \in X^G$ be a fixed point of our $G$-action on $X$. Then by the equivariant-Darboux theorem for some sufficiently small neighborhood of $U$ of $p$, there exists a $G$-invariant local symplectomorphism $\varphi : (U, p) \to (\mathbb{C}^n, 0)$. In other words, around a fixed point $(X, \omega)$ and are locally isomorphic as Hamiltonian $G$-spaces. Hence, it makes sense to perform an $\epsilon$ blow-up

[6]Indeed $v \in V_p \Rightarrow Uv \in UV_p = V_{Up}$ for $U \in U(n)$

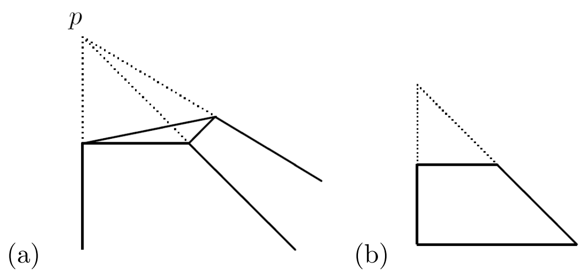


Figure 2: (a): Picture describing the operation in Theorem 5.3. We remove a vertex $p$ and replace it with the $n$ vertices $p + \epsilon v_i$ where the $v_i, i = 1, \cdots n$ were the edges emanating from $p$, i.e. we chop off a neighborhood of the vertex $p$. The corresponding operation for symplectic manifolds a la Delzant is an $\epsilon$ blow-up. (b):The trapezoid in the last example can be obtained from a simplex by chopping off a neighborhood of one of its vertices. Hence, the corresponding Delzant space was a blow-up of $\mathbb{C}P^2$.

of $X$ at $p \in X^G$ fixed.[7] Furthermore, as $G$ is identified with a unitary subgroup, the blowup of $X$ at $p$ is equipped with a Hamiltonian $G$-action.

Now consider the special case of a symplectic toric manifold $X$. According to Delzant's theorem (4.1), we can associate a polytope $\Delta$ to $X$. However, via the discussion above, the $\epsilon$ blow-up of a symplectic toric manifold around a fixed point is also a symplectic toric manifold. Hence, via the correspondence (4.1) there should be a Delzant polytope $\Delta_\epsilon$ associated to the blow-up as well. The natural question then arises, how exactly does the symplectic blow-up modify the original Delzant polytope $\Delta$ or, equivalently, how can we construct $\Delta_\epsilon$ from $\Delta$? The result is actually quite intuitive. We first mention the following lemma that follows from the Delzant construction.

**Lemma 5.2.** *The moment map $\Psi : X_\Delta \to \Delta \subset (\mathbb{R}^n)^*$ maps the fixed points of $T^n$ bijectively onto the vertices of $\Delta$.*

We can now state our theorem.

**Theorem 5.3.** *Let $X_\Delta$ be the Delzant space constructed from $\Delta$. Take $p$ a vertex of $\Delta$, with the rays*

$$p + tv_i,\ i = 1, \cdots, n$$

*(for $t > 0$) forming the edges of $\Delta$ through $p$. By the lemma, $p$ is the image of some fixed point $q \in X_\Delta$ under the moment map $\Psi : X_\Delta \to (\mathbb{R}^n)^*$. Then if we blow up $X_\Delta$ an $\epsilon$ amount at the fixed point $q$ we obtain the Delzant space $X_\epsilon$ associated with the polytope $\Delta_\epsilon$. Where $\Delta_\epsilon$ is formed by replacing the vertex $p$ by the $n$ vertices $p + \epsilon v_i$, $i = 1, \cdots n$.*

*Proof.* (Sketch) Let $\Psi_\epsilon : X_\epsilon \to \Delta_\epsilon$ be the moment map for the $\epsilon$ blow-up. Now $X_\epsilon$ is obtained from $X$ by deleting $q$ and replacing it with $(\mathbb{C}P^{n-1}, \epsilon\omega_{FS})$. In terms of the image of the moment map this corresponds to removing the cone emanating from the vertex $p$ (this follows from the local convexity theorem). Let $\iota : (\mathbb{C}P^{n-1}, \epsilon\omega_{FS}) \hookrightarrow (X_\epsilon, \omega)$ be our embedding of $\mathbb{C}P^{n-1}$ into the blow-up. Then the moment map restricts to the image of the embedding: $\Psi_\epsilon \circ \iota : \mathbb{C}P^{n-1} \hookrightarrow (\mathbb{R}^n)^*$. But we calculated in an example above such a moment map on $\mathbb{C}P^{n-1}$ has Delzant polytope the $(n-1)$ simplex. Thus, via Delzant's theorem $\Psi_\epsilon$ restricted to the $\mathbb{C}P^{n-1}$ sitting inside of $X_\epsilon$ has an image the $(n-1)$ simplex. Hence, the Delzant polytope for $X_\epsilon$ is obtained by removing the vertex $p$ and replacing it by an $(n-1)$ simplex. □

As an application of this theorem we note that the the "$1/m$-trapezoid" considered in the last of our examples can be formed by removing the vertex from a simplex and replacing it with the $n$-vertices of the form described in the theorem above (alternatively we "chop" off a neighborhood around a vertex of the simplex). Hence, the Delzant spaces obtained for the $1/m$-trapezoids, the so-called Hirzebruch surfaces, are symplectic blow-ups of $\mathbb{C}P^2$.

[7] If we wish to bask in the aura of preciseness the blow-up of $X$ can be performed by a pullback via $\varphi : U \to \mathbb{C}^n$ then extending it to the entire space $X$.

## 6. Appendix

6.1. **Complex Coordinates.** Given a real vector space $V$ of dimension $2n$ and Darboux-coordinates $p_1, \cdots, p_n, q_1, \cdots, q_n$ we will often find it convenient to identify it with $\mathbb{C}^n$ equipped with complex coordinates $z_k = p_k + iq_k$. To be more precise, given $V$ and the coordinates $(p, q)$ we construct an imbedding $(p = (p_1, \cdots, p_n),\ q = (q_1, \cdots, q_n))$

$$\begin{aligned} V &\to V \otimes \mathbb{C} \\ (p, q) &\mapsto (p + iq, p - iq) = (z, \overline{z}). \end{aligned}$$

Furthermore, via the Equivariant-Darboux theorem, a sufficiently small neighborhood of $x \in X$, a Hamiltonian $G$-space, is locally isomorphic as a Hamiltonian $G$-space to the vector space $V = T_x X$; so it makes sense to choose local complex coordinates $z = p + iq, z = p - iq$ in a neighborhood around $x$. We define the exterior derivatives on these coordinates as

$$\begin{aligned} dz_k &= dp_k + idq_k \\ d\overline{z}_i &= dp_i - idq_i \end{aligned}$$

and the vector fields $\partial/\partial z_i$ such that $dz_k(\partial/\partial z_j) = \delta_{kj}$; so,

$$\begin{aligned} \frac{\partial}{\partial z} &= \frac{1}{2}\left(\frac{\partial}{\partial p_i} - i\frac{\partial}{\partial q_i}\right) \\ \frac{\partial}{\partial \overline{z}} &= \frac{1}{2}\left(\frac{\partial}{\partial p_i} + i\frac{\partial}{\partial q_i}\right). \end{aligned}$$

The standard symplectic form can be rewritten in complex coordinates as

$$\omega = \sum_{k=1}^{n} dp_k \wedge dq_j \longmapsto \frac{i}{2}\sum_{k=1}^{n} dz_k \wedge d\overline{z}_k.$$

We should note that the imbedding $V \to V \otimes \mathbb{C}$ is generally not natural, i.e. it will depend on the choice of coordinates $(p, q)$.

6.2. **Marsden-Weinstein Reduction.** Here we state the Marsden-Weinstein reduction theorem. A proof can be found in almost any introductory text on symplectic geometry but one particular method using equivariant cohomology can be found in [8]; the precise statement here is taken from this source. For the setup let $(M, \omega)$ be a symplectic manifold with a $G$-action that is both free and Hamiltonian. Let $\Phi : X \to \mathfrak{g}^*$ an associated moment map. As $\Phi$ is $G$-equivariant the zero level-set

$$M_0 = \Phi^{-1}(0)$$

is a $G$-invariant submanifold of $X$; as $G$ acts freely then the quotient

$$X_0 = M_0/G$$

is a manifold. We have the inclusion

$$\iota_0 : M_0 \hookrightarrow M$$

and the quotient map

$$\pi_0 : M_0 \to X_0$$

**Theorem 6.1** (Marsden-Weinstein)**.** *There exists a unique symplectic form $\nu_0$ on $X_0$ with the property*

$$\pi_0^* \nu_0 = \iota_0^* \omega.$$

There is an extension of the Marsden-Weinstein theorem to quotients of pre-images which are *not* derived from zero-level sets of $\Phi$. However, because such manifolds are not $G$-invariant, constructing a proper quotient manifold requires a bit more finesse: the proper notion turns out to be quotients via co-adjoint orbits. However, if $G$ is abelian, then the moment map is, in fact, $G$-invariant so the submanifold

$$M_a = \Phi^{-1}(a)$$

for some $a \in \mathfrak{g}^*$, is $G$-invariant and the quotient

$$X_a = M_a/G$$

is a well-defined submanifold. The Marsden-Weinstein theorem then holds for $X_a$ with the obvious replacements.

**Remark** If $M$ is also equipped with a Hamiltonian $H$-action that commutes with our $G$-action, then there is an induced action of $H$ on the reduction $M_a$. Furthermore, this action is also Hamiltonian. In fact, if

$$\Phi_{M_a} : M_a \to \mathfrak{h}^*$$

and

$$\Phi_M : M \to \mathfrak{h}^*$$

are teh two moment maps, then they are related by

$$\Phi_M \circ \iota_a = \Phi_{M_a} \circ \pi_a.$$

6.3. **Proof of the Equivariant Darboux Theorem.** We supply the proof of Theorem 2.2, the Equivariant-Darboux theorem.

*Proof.* As $d(\omega^1 - \omega^0) = 0$, there is a 1-form $\alpha$ on some contractible neighborhood $W$ of $x \in M^G$ such that

$$d\alpha = \omega^1 - \omega^0; \tag{13}$$

wlog as $x \in M^G$ we can find $W$ small enough so that it is $G$-invariant. By adding an exact form to $\alpha$ (the exterior derivative of a 0-form), we can insure that $\alpha(x) = 0$. Furthermore, by averaging $\alpha$ over the group $G$ we can insure that $\alpha$ is $G$-invariant. More precisely let

$$\alpha' = \int_G (\tau^*_{g^{-1}}\alpha) dg$$

where $\tau_{g^{-1}} : X \to X$ is the symplectomorphism induced by the action of $g^{-1} \in G$ and $dg$ is the right-invariant Haar measure on $G$. Then $\alpha'$ is right $G$-invariant ($\tau^*_{g^{-1}}\alpha'_p = \alpha'_{\tau_g(p)}$) and satisfies (13) so wlog we can take $\alpha$ to be $G$-invariant. Define the time-dependent 2-form

$$\Omega^t = \omega^0 + t(\omega^1 - \omega^0)$$

which is non-degenerate at $x$ as $\Omega^t_x = \omega^0_x$; hence, we can replace $W$ by a $G$-invariant neighborhood $V \subset W$ so that $\Omega^t$ is non-degenerate throughout $V$. We then define the time-dependent vector field $X$ by

$$\iota_X \Omega^t + \alpha = 0$$

which is well defined on $U$. Then

$$\begin{aligned} L_X \Omega^t =& d(\iota_X \Omega^t) + \iota_X d\Omega^t + \partial_t \Omega \\ =& - d\alpha + \omega^1 - \omega^0 \\ =& 0. \end{aligned}$$

So if $\rho_{tt'}$ is the flow of $X$ from time $t$ to time $t'$, we have $\rho^*_{tt'}\Omega^{t'} = \Omega^t$. So as $\Omega^0 = \omega^0$ and $\Omega^1 = \omega^1$, if we take $\Psi = \rho_{01}$ we have $\Psi^*\omega^1 = \omega^0$. Moreover $\Psi(x) = x$ as $X(x) = 0$ for all $t$ and $\Psi$ is $G$-equivariant as $X$ is $G$-invariant. Now the integral curves of $X$ may leave $U$; so $\Psi$ may not be defined on all of $V$; however, $\Psi(x) = x$ so we can find a sufficiently small $G$-invariant neighborhood $U \subset V$ such that $\Psi(U) \subset V$. □

6.4. **Proof of Corollary 2.4.** The following is a careful proof of Cor. 2.4 for the wary reader.

*Proof.* Let $\omega = \omega_x$ be the symplectic form induced on $T_xX$ and $\Omega = \exp^* \omega_x$, defined on some neighborhood of $0 \in T_xX$. For $\xi \in \mathfrak{g}$ the $G$ action on $X$ generates a Hamiltonian vector field $\xi^\#$. Let $\xi_0^\#$ be the Hamiltonian vector field induced by the linearized action and $\xi_1^\# = (\exp^{-1})_* \xi^\#$ be the "pullback" of $\xi^\#$ to some neighborhood of $0 \in T_xX$. Letting $\phi_i^\xi(v) = \langle \Phi_i(v), \xi \rangle$ for $i = 1, 2$, then on a sufficiently small neighborhood of 0

$$\iota_{\xi_0^\#}\omega = \phi_0^\xi$$
$$\iota_{\xi_1^\#}\Omega = \phi_1^\xi.$$

By the equivariant-Darboux theorem there is a $G$-equivariant diffeomorphism $\psi : (U_0, 0) \to (U_1, 0)$, on some sufficiently small neighborhoods of 0, and $\psi^*\Omega = \omega$. Via $G$-equivariance, $\psi_*\xi_0^\# = \xi_1^\#$. Hence, for every vector field $v$ on $T_xX$ (where $\Omega$ is defined),

$$\begin{aligned}
\psi^* \left(\iota_{\xi_1^\#}\Omega\right)(v) =& \Omega(\xi_1^\#, \psi_* v) \\
=& \Omega(\psi_*\xi_0^\#, \psi_* v) \\
=& (\psi^*\Omega)(\xi_0^\#, v) \\
=& (\iota_{\xi_0^\#}\omega)(v).
\end{aligned}$$

So as $\psi^* d\phi_1 = d(\phi_1 \circ \psi)$ we have

$$\iota_{\xi_0^\#}\omega = d(\phi_1^\xi \circ \psi).$$

Thus, $\phi_1^\xi \circ \psi : U_0 \to \mathbb{R}$ and $\phi_0^\xi : U_0 \to \mathbb{R}$ are moment maps associated to the same Hamiltonian $G$-action $\Rightarrow$ they are equivalent up to a constant $\Rightarrow \Phi_1 \circ \psi$ and $\Phi_0$ are equivalent up to a constant in $\mathfrak{g}^* \Rightarrow$ the image of $\Phi_1 : U_1 \to \mathfrak{g}^*$ and the image of $\Phi_0 : U_0 \to \mathfrak{g}^*$ are equivalent up to translation. ☐

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