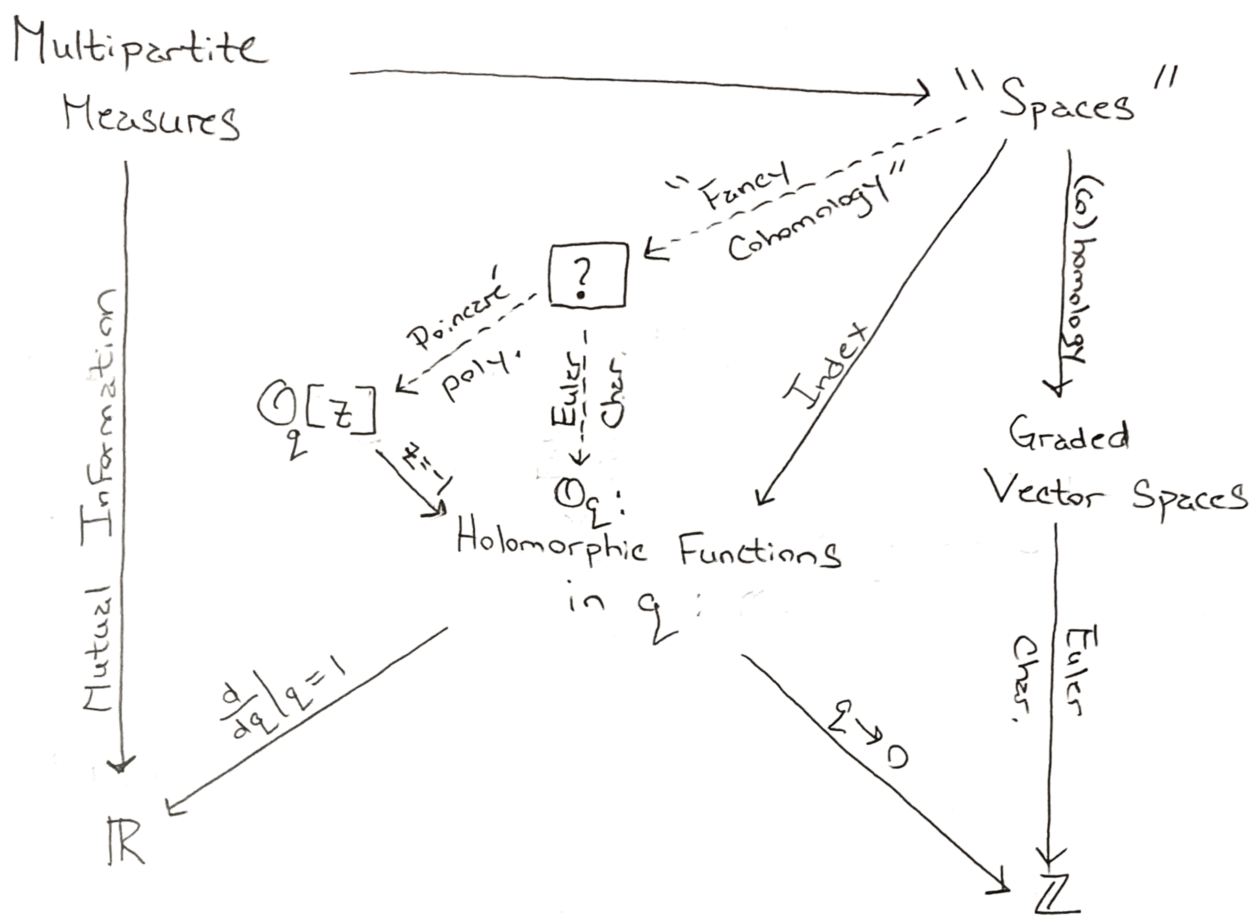
Open Problems in Higher Information



Open Question: What is $\boxed{?}$ and the Corresponding dotted arrows?

Why?

Fancier measures of Shared information that:

- May help Classify entangled quantum states up to Unitary transformations
- May Provide a deeper Understanding of entropy inequalities (e.g. Strong Subadditivity: $S_{ABC} + S_{B} \leq S_{AB} + S_{BC}$)
- Will provide new Link invariants upon application to States in Chern-Simons theory

What is "Higher Information"?

1-Partite understanding: (Higher Entropy)

How much ignorance do I have $\rightsquigarrow$ What is it that I don't know?

Entropy — (Linear) Spaces of Random Variables

$$\hat{\mu} : \Omega \to \mathbb{R}_{\geq 0} \overset{GNS_q}{\rightsquigarrow} L^q(\Omega_{\hat{\mu} \neq 0}) + \text{distinguished point}$$

$$q \in \mathbb{C}_{\mathrm{Re}\, q \geq 1}$$

homotopical understanding: multipartite systems, e.g. $\mu_{AB} : \Omega_A \times \Omega_B \to \mathbb{R}_{\geq 0}$

How much info. is shared $\rightsquigarrow$ What info. is shared?

Mutual info:

$$I(\mu_{AB}, \{A,B\}) = S_A + S_B - S_{AB}$$

"Spaces"/Co-Simplicial Objects

$$\mu_\emptyset \leftarrow \mu_A \boxplus \mu_B \leftleftarrows \mu_{AB}$$

$$\Big\downarrow GNS_q$$

$$\Big\uparrow \frac{d}{dq}\Big|_{q=1}$$

$$\chi_q := 1 - \dim_q \mu_A - \dim_q \mu_B + \dim_q \mu_{AB} \xleftarrow{\text{"Euler Char"}} \left(\mathbb{C} \longrightarrow L^q(\Omega_A) \times L^q(\Omega_B) \rightrightarrows L^q(\Omega_{AB})\right)$$

$$\Big\downarrow q = 0$$

$$\chi_0 \in \mathbb{Z} \xleftarrow[\text{char}]{\text{Euler}} \mathbb{C} \longrightarrow \mathbb{C}[\Omega_A^{\neq 0}] \times \mathbb{C}[\Omega_B^{\neq 0}] \rightrightarrows \mathbb{C}[\Omega_{AB}^{\neq 0}]$$

(Alternating sum of counts of points with measure > 0)

A Good Category of Measures

Note:

$\mu(a^*a) \geq 0$

Measures on $(\mathcal{X}, \Sigma_{\mathcal{X}})$ (σ-alg.) $\hat{\mu} : \Sigma_X \longrightarrow \mathbb{R}_{\geq 0}$	Expectation Values $\longrightarrow$ $\longleftarrow$ Restrict to Projections	Positive linear Functionals $\mu : \underbrace{L^\infty(\mathcal{X})}_{w^*\text{-algebra}} \longrightarrow \mathbb{C}$

Def: The Category Meas has

- Objects "measures" $\mu := (A, \mu : A \longrightarrow \mathbb{C})$

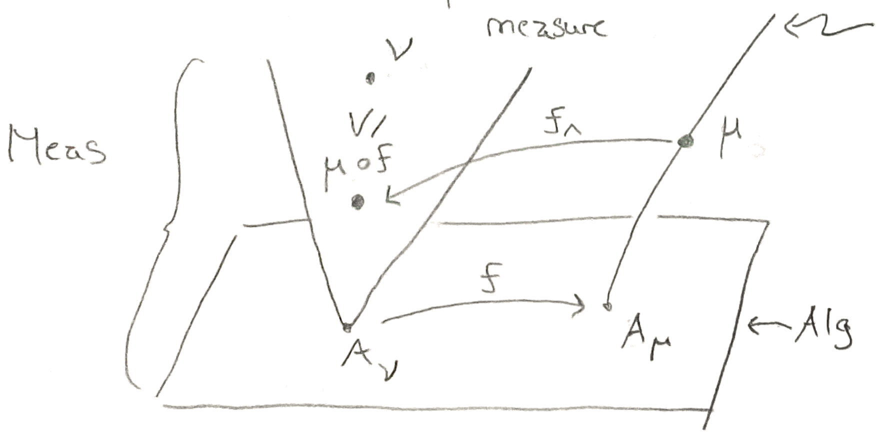
  A: w^*-alg; μ: Positive linear Func.

- Morphisms $\mu \xrightarrow{f_\wedge} \nu$ given by

  $$f : A_\nu \longrightarrow A_\mu \quad (*\text{-hom.})$$

  Such that

  $$\underbrace{\mu \circ f}_{\text{"pushForward" measure}} \leq \nu$$



Each "Fiber" is a poset under $\leq$

- Meas has Coproducts:

$$\mu \boxplus \nu : A_\mu \times A_\nu \longrightarrow \mathbb{C}$$
$$(a, b) \longmapsto \mu(a) + \nu(b)$$

- Meas has $\otimes$:

$$\mu \otimes \nu : A \otimes B \longrightarrow \mathbb{C}$$
$$a \otimes b \longmapsto \mu(a)\,\nu(b)$$

(• Meas is an $\mathbb{R}_{\geq 0}$ Category: We can rescale measures
$\Rightarrow \exists$ a Functor $\mathbb{R}_{\geq 0} \longrightarrow \mathrm{End}(\mathrm{Meas})$)

- For "Finite" measures there exists a homomorphism
F.d. algebra

$$\dim_q : K_0(\mathrm{Meas}^{\mathrm{F.d.}}) \longrightarrow \mathbb{C}$$
$$\text{"}\ \hat{\mu} \longmapsto \sum_{\omega \in \Omega} \hat{\mu}_\omega^{\,q}\ \text{"}$$

For $A = L^\infty(\Omega)$

Quasi-Open Questions:

- Understand properties of

$$\text{"}\ \dim_q : \mathrm{Arrow}(\mathrm{Meas}) \longrightarrow \mathbb{R}_{\geq 0}\ \text{"}$$
$$\mu \leq \varphi \longmapsto \varphi\left(\left[\frac{D\mu}{D\varphi}\right]^q\right)$$

RN-derivative;

$$\mu = \varphi\left[\left(\frac{D\mu}{D\varphi}\right)^{1/2} - \left(\frac{D\mu}{D\varphi}\right)^{1/2}\right]$$

For $q \geq 1$

- $q \in \mathrm{Rep}_{\mathbb{C}}(\mathbb{R}_{\geq 0}^{\times})$?

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$$\begin{tikzcd} \text{GNS}: \text{Meas}^{op} \arrow[r] \arrow[dr, "\simeq"'] & \text{Rep}^{\bullet} \arrow[d, shift left] \\ & \text{Rep}^{\odot} \arrow[u, shift left] \end{tikzcd}$$

$\text{Rep}^{\bullet} \rightsquigarrow \left(\begin{array}{l} \text{Rep}^{\bullet}_{A} = \\ \text{Pointed representations} \\ + \text{Point-preserving} \\ \text{morphisms} \end{array} \right)$

" $\mu \longmapsto \left(L^2(\Omega_{\mu \neq 0}, \mu), \omega \mapsto ? \right)$
$\overline{\left(A/\eta_{\mu=0}, [1] \right)}^{q}$ in general

• " $\text{GNS}(\mu \leq \varphi) = \sqrt{\frac{D\mu}{D\varphi}}$ "

Measure Families:

Plain old measures on $(\mathbb{X}, \Sigma_X)$ $\rightsquigarrow$ Functors $M : \Sigma_X \longrightarrow \mathbb{R}_{\geq 0}$ + Cosheaf Condition

Def:

- A measure Family $(\mathbb{X}_M, M) =: M$ is a Functor

$$M : \Sigma_X \longrightarrow \text{Meas}$$

- $\text{MeasFam} = \text{Functors}\left(\begin{array}{c}\text{Measurable}\\ \text{Spaces}\end{array}, \text{Meas} \right)$
  - Has $\boxplus$: $\mathbb{X}_{M \boxplus N} = \mathbb{X}_M \amalg \mathbb{X}_N$
    $M \boxplus N(T \amalg V) = M(T) \boxplus M(V)$
  - Has $\otimes$: $\mathbb{X}_{M \otimes N} = \mathbb{X}_M \times \mathbb{X}_N$
    $M \otimes N(T \times V) = M(T) \otimes M(V)$

Example: • Let μ be a measure on Ω, then we have a measure family

$$F_\mu : \text{Subset}(\Omega) \longrightarrow \text{Meas}$$
$$T \longmapsto \mu|_T$$
$$T \subseteq V \longmapsto \mu|_T \xrightarrow{\iota_{T,V}} \mu|_V$$

Inclusion of algebras

• Let μ be a "multipartite measure":

$$\mu : \bigotimes_{p \in P} A_p \longrightarrow \mathbb{C}$$

$$\rightsquigarrow$$

$$M_\mu : \text{Subset}(P) \longrightarrow \mathbb{C}$$
$$T \longmapsto \mu_{T^c} : \bigotimes_{t \in T} A_t \longrightarrow \mathbb{C}$$
$$T \subseteq V \longmapsto \mu_{T^c} \longrightarrow \mu_{V^c}$$

Why Complements?

$$T \subseteq V \Rightarrow A_T \longrightarrow A_V$$
$$a \longmapsto a \otimes 1_{V \setminus T}$$

$\Rightarrow$ Morphism $\mu_V \longrightarrow \mu_T$

but we want a <u>Covariant</u> Functor.

For Simplicity: $\mathfrak{X} = P$, $|P| < \infty$

- Čech Object with respect to the Cover $\mathcal{U} = \{\{p\}^c\}_{p \in P}$

$$M(\phi^c) \leftarrow \underset{|T|=1}{\boxplus} M(T^c) \leftleftarrows \cdots \underbrace{\overset{\leftarrow}{\underset{\leftarrow}{\vdots}}}_{n-1} \underset{|T|=n-1}{\boxplus} M(T^c) \underbrace{\overset{\leftarrow}{\underset{\leftarrow}{\vdots}}}_{n} M(P^c)$$

$M_\mu(T^c) = \mu_T$

For a multipartite State

- Index:

$$\chi(M)_q = \sum_{T \subseteq P} (-1)^{|T|} \dim_q M(T^c)$$

$($= "Euler-Char of Čech object"$)$

- $\chi(M \boxplus N) = \chi(M) + \chi(N)$
- $\chi(M \otimes N) = \chi(M)\,\chi(N)$

Prop: If M is a Z-measure, then $\chi(M) = 0$.

Prop: μ multipartite, then $\frac{d}{dq}\Big|_{q=1} \chi(M_\mu)$ = mutual info.

& μ Factorizable $\Rightarrow$ mutual info $= 0$.

(($\Leftarrow\!\!\!\!\!/$ For 3-partite).

Question: What are the "right" notions of quotients and Kernels in Meas to make a Chain Complex out of the Čech object and pass to Cohomology.

Note

$$\mathrm{GNS}\left(\text{Simplicial Measure}\right) = \text{"Pointed" Co-Simplicial representations}$$

$$\begin{tikzcd}
\text{"Pointed" Co-Simplicial representations} \arrow[d, "\text{Forget}"] \\
\text{Co-Simplicial Vector Space} \arrow[d] \\
\text{Cochain Complex} \arrow[d, "\text{Cohomology}"] \\
\text{Graded V.S.} \arrow[d] \\
H^k = \text{"Non-trivial Correlations between Subsystems of Size } k+1\text{"}
\end{tikzcd}$$

Ex: $H^0\left(\hat{\mu}_{AB} : \Omega_A \times \Omega_B \to \mathbb{R}_{\geq 0}\right)$

$$= \left\{ (r_A, r_B) \in \mathbb{C}[\Omega_A^{\mu \neq 0}] \times \mathbb{C}[\Omega_B^{\mu \neq 0}] : r_A \otimes 1 \overset{a.e.}{=} 1 \otimes r_B \right\} \Big/ \langle (1,1) \rangle$$

Almost everywhere equal

The Riemann Zeta Function and Indices

In QM: $\hat{\mu} \in \mathrm{Dens}(\mathcal{H})$, multipartite $\hat{\mu}_P \in \mathrm{Dens}(\bigotimes_{p\in P} \mathcal{H}_p)$

Density States

$$\mathcal{Z}_q(\hat{\mu}_P) = \sum_{T\subseteq P} (-1)^{|T|} \mathrm{Tr}[\hat{\mu}_T^{\,q}]$$

$\hat{\mu}_T = \mathrm{Tr}_{\mathcal{H}_{T^c}} \hat{\mu}_P$

$$= \sum_{T\subseteq P} (-1)^{|T|} \mathrm{Tr}[e^{-q \log \hat{\mu}_T}]$$

$$\equiv \mathrm{Tr}_{\mathcal{F}}[(-1)^F e^{-\beta \hat{H}}]$$

Fermionic Partition Function

Where
- $\mathcal{F} = \bigoplus_{T\subseteq V} \mathcal{H}_T$, $\mathcal{H}_T := \bigotimes_{t\in T} \mathcal{H}_T$

  $\cong \Lambda^\bullet(\bigoplus_p \mathcal{H}_p)$
- $\hat{H} = \bigoplus_T \log \hat{\mu}_T$ $\quad \bigoplus_{|T|=n} \hat{\mu}_T$ describes $|T|$-body interactions
- $(-1)^F = (-1)^{|T|}$

Example: $P =$ Set of primes

$\mathcal{H} = \bigotimes_{p\in P} \mathbb{C}^2_p$, $\hat{H}_p = \log p \,|1\rangle + 0\, |0\rangle$

$\hat{H} = \bigotimes \hat{H}_p$

$\hat{\mu} = e^{-\beta \hat{H}} = \bigotimes_p \mu_p$

States in $\mathcal{H}$ $\leftrightsquigarrow$ Square-Free integers

$$\mathcal{Z}(\hat{\mu}) = \sum_{n=0}^{\infty} \mu(n)\cdot n^{-q\beta} = \prod_{p\in P}(1-p^{-q\beta}) = \prod_p \mathcal{Z}(\mu_p)$$