Foundations of Math

Week 4 - Boolean algebra: Logic gates and switching functions

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Agenda for today

- Solutions for homework from last week
  - Exercises 36.1 – 36.16 (submit one `.pdf`)
- Homework for next week.

Plan for the rest of the term (tentative):

- Chapter 1: Standard functions and techniques
- Chapters 2-4: Differentiation
- Chapters 14-15: Integration
- Chapters 7–10: Linear Algebra
- Chapters 39-40: Probability and statistics

Self-tests

- It is strongly recommend to do them.
- A variant usually appears in the exercises.
- Sometimes they are the key to solving exercises.
- If you have problems with the self-test, you are not ready to move ahead.

Exercise 36.1:

Read through Example 36.1. Now prove the other absorption law:

$$a * (a \oplus b) = a$$

(Example 36.1 and this result illustrate the duality principle, which states that any theorem which can be proved in Boolean algebra implies another theorem with $*$ and $\oplus$ interchanged for the same elements.)

Idea was not to use laws in box 36.3.

For all $a, b \in B$

$$
\begin{aligned}
a * (a \oplus b) &= (a \oplus 0) * (a \oplus b) && \text{(Identity law)} \\
&= a \oplus (0 * b) && \text{(Distributive law)}
\end{aligned}
$$

Now

$$
\begin{aligned}
0 * b &= \overline{b} * b * b && \text{(Complement law)} \\
&= \overline{b} * b && \text{(Definition of } *\text{)} \\
&= 0 && \text{(Complement law)}
\end{aligned}
$$

Finally

$$
\begin{aligned}
a * (a \oplus b) &= a \oplus 0 \\
&= a && \text{(Identity law)}
\end{aligned}
$$

Exercise 36.2:

(Section 36.1). Prove the de Morgan result

$$\overline{a \oplus b} = \overline{a} * \overline{b},$$

by showing that $(a \oplus b) \oplus (\overline{a} * \overline{b}) = 1$. Explain how the duality result (Problem 36.1) gives the other de Morgan theorem.

First of all:

$$\begin{aligned} \overline{a \oplus b} &= \overline{a} * \overline{b} && \text{(Add } a \oplus b \text{ on both sides.)} \\ (a \oplus b) \oplus \overline{a \oplus b} &= (a \oplus b) \oplus \overline{a} * \overline{b} \\ 1 &= (a \oplus b) \oplus \overline{a} * \overline{b} && \text{(Complement law)} \end{aligned}$$

$$
\begin{aligned}
a \oplus b \oplus \overline{a} * \overline{b} &= b \oplus (a \oplus \overline{a}) * (a \oplus \overline{b}) && \text{(Distr.)} \\
&= b \oplus (a \oplus \overline{b}) && \text{(Identity)} \\
&= (b \oplus \overline{b}) \oplus a && \text{(Comm., Assoc.)} \\
&= 1 \oplus a && \text{(Identity)} \\
&= 1 && \text{(Identity)}
\end{aligned}
$$

Exercise 36.3:

(Section 36.1). Let B be the Boolean algebra with the two elements 0 and 1. For arbitrary $a, b \in B$, prove the following:

- $a * (\overline{a} \oplus b) = a * b$

$$\begin{aligned} a * (\overline{a} \oplus b) &= (a * \overline{a}) \oplus (a * b) && \text{(Distributive law)} \\ &= 1 \oplus a * b && \text{(Complement law)} \\ &= a * b && \text{(Identity law)} \end{aligned}$$

- $(a \oplus b) * (a \oplus \overline{b}) = a$

$$\begin{aligned}(a \oplus b) * (a \oplus \overline{b}) &= a \oplus b * \overline{b} \\ &= a \oplus 0 \\ &= a\end{aligned}$$

- $(a \oplus b) * \overline{a} * \overline{b} = 0$

$$\begin{aligned}(a \oplus b) * \overline{a} * \overline{b} &= (\overline{a} * a \oplus \overline{a} * b) * \overline{b} \\ &= \overline{a} * b * \overline{b} \\ &= \overline{a} * 0 \\ &= 0\end{aligned}$$

Exercise 36.4:

(Section 36.1). Using the laws of Boolean algebra for the set with two elements 0 and 1, show that:

- $a * b \oplus a * \overline{b} = a$
- $a \oplus \overline{a} * \overline{b} * c = a \oplus \overline{b} * c$

Use the result to obtain the truth tables in each case.

$$
\begin{aligned}
a * b \oplus a * \overline{b} &= a * (b \oplus \overline{b}) \\
&= a * 1 \\
&= a
\end{aligned}
$$

Truth table:

a	b	$\overline{b}$	$a * b$	$a * \overline{b}$	$a * b \oplus a * \overline{b}$
0	0	1	0	0	0
0	1	0	0	0	0
1	0	1	0	1	1
1	1	0	1	0	1

$$
\begin{aligned}
a \oplus \overline{a} * \overline{b} * c &= a \oplus (\overline{a} * \overline{b} * c) \\
&= (a \oplus \overline{a}) * (a \oplus \overline{b}) * (a \oplus c) \\
&= (a \oplus \overline{b}) * (a \oplus c) \\
&= a \oplus (\overline{b} * c) \\
&= a \oplus \overline{b} * c
\end{aligned}
$$

How do we know that the distributive generalizes to products with more than two factors?

Truth table:

a	b	c	$\overline{b} * c$	$a \oplus \overline{b} * c$
0	0	0	0	0
0	0	1	1	1
0	1	0	0	0
0	1	1	0	0
1	0	0	0	1
1	0	1	1	1
1	1	0	0	1
1	1	1	0	1

Exercise 36.5:

(Section 36.4). In Problem 36.4b, it is shown that

$$a \oplus \overline{a} * \overline{b} * c = a \oplus \overline{b} * c.$$

Design two sequences of gates which give the same output for the inputs a, b, and c. The resultant gates are said to be logically equivalent.

Exercise 36.6:

(Section 36.4). Design a circuit of gates to produce the output

$$(a \oplus \overline{b}) * (a \oplus \overline{c}).$$

Construct the truth table for this Boolean expression.

Exercise 36.7:

(Section 36.1). Show that the Boolean expression $(a \oplus b) * (\overline{a} \oplus b) \oplus a$ and $a \oplus b$ are equivalent.

$$
\begin{aligned}
(a \oplus b) * (\overline{a} \oplus b) \oplus a &= (a \oplus a \oplus b) * (a \oplus \overline{a} \oplus b) \\
&= (a \oplus b) * (a \oplus \overline{a} \oplus b) \\
&= (a \oplus b) * (1 \oplus b) \\
&= (a \oplus b) * 1 \\
&= a \oplus b
\end{aligned}
$$

Exercise 36.8:

(Section 36.1). Show that the following Boolean expression are equivalent:

- $a \oplus b$
- $a \oplus b * b$

$$\begin{aligned} a \oplus b * b &= a \oplus (b * b) \\ &= a \oplus b \end{aligned}$$

But how do we know that $b * b = b$?

- b could be 1, then $b * 1 = b$.
- b could be 0, then $b * 0 = 0 = b$.

Exercise 36.9:

(Section 36.3). Find a Boolean expression f which corresponds to the truth table shown in Table 36.13.

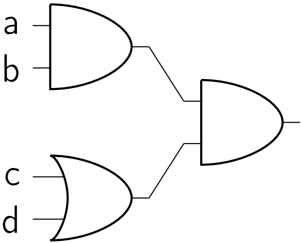
a	b	c	f
0	0	0	1
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	0

$$(\overline{a} * \overline{b}) \oplus (a * \overline{b}) = (\overline{a} \oplus a) * \overline{b} = \overline{b}$$

Exercise 36.10:

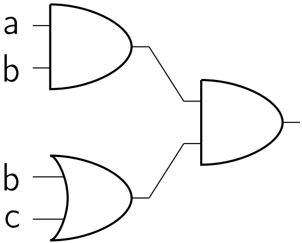
(Section 36.3). Construct Boolean expression for the output f in the devices shown in Figs. 36.17a–d. Construct the truth tables in each case.

(a)



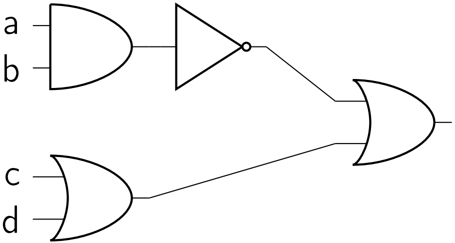
- $a * b * (c \oplus d)$

(b)



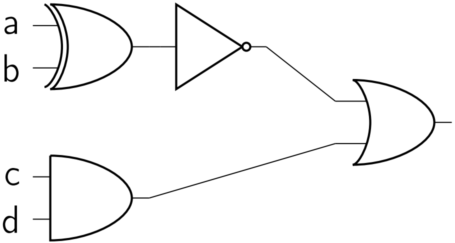
- $a * b * (b \oplus c) = a * b$

(c)



- $\overline{a * b} \oplus (c \oplus d) = \overline{a} \oplus \overline{b} \oplus c \oplus d$

(d)



- $\overline{\overline{a} * b \oplus a * \overline{b}} \oplus c * d = a * b \oplus \overline{a} * \overline{b} \oplus c * d$

Exercise 36.11:

Find the output f and g in the logic circuits shown in Fig. 36.18. This device can represent binary addition in which g is the 'carry' in the binary table shown in Table 36.14. The output g gives the '1' in the '10' in the binary sum
$1 + 1 = 10$.

- $g = a * b$
- $f = \overline{a * b} * (a \oplus b)$

$x = a$	$y = b$	$x + y$	g	f
0	0	0	0	0
0	1	1	0	1
1	0	1	0	1
1	1	10	1	0

Exercise 36.12:

(Section 36.3). Reproduce the logic gate in Fig. 36.6 using just the NOR gate. (Circuit is $a * b \oplus c \oplus d$).

We need to build $*$ and $\oplus$ using just NOR ($\downarrow$):

- $a * b = \overline{a} \downarrow \overline{b}$
- $\overline{a} = a \downarrow a$
- Therefore: $a * b = (a \downarrow a) \downarrow (b \downarrow b)$
- $a \oplus b = \overline{a \downarrow b} = (a \downarrow b) \downarrow (a \downarrow b)$
- All taken together:

$$\begin{aligned} a * b \oplus c \oplus d = {} & (((a \downarrow a) \downarrow (b \downarrow b)) \downarrow ((c \downarrow d) \downarrow (c \downarrow d))) \downarrow \\ & (((a \downarrow a) \downarrow (b \downarrow b)) \downarrow ((c \downarrow d) \downarrow (c \downarrow d))) \\ = {} & (a \downarrow c \downarrow d) \downarrow (b \downarrow c \downarrow d) \end{aligned}$$

Scratchpad:

Exercise 36.13:

(Section 36.4). Using the disjunctive normal form, construct a Boolean expression f for the truth tables given in Tables 36.15 and 36.16.

Table: 36.15

a	b	f
0	0	0
0	1	1
1	0	1
1	1	1

$$f = \overline{a} * b \oplus$$
$$a * \overline{b} \oplus$$
$$a * b = a \oplus b$$

Table: 36.16

a	b	c	f
0	0	0	1
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	0

$$f = \overline{a} * \overline{b} * \overline{c} \oplus$$
$$\overline{a} * b * c \oplus$$
$$a * \overline{b} * \overline{c} \oplus$$
$$a * b * \overline{c}$$

Exercise 36.14:

(Section 36.3). Show that any Boolean expression can be modelled using just a NAND gate. (Hint: use a method similar to that explained in Example 36.4.)

a	b	$a \uparrow b$	$\overline{a}$	$a * b$	$a \oplus b$
0	0	1	1	0	0
0	1	1	1	0	1
1	0	1	0	0	1
1	1	0	0	1	1

We have to show that $*$ (conjunction), $\oplus$ (disjunction), and $\overline{a}$ (negation) can be expressed in terms of just NAND:

- $\overline{a} = a \uparrow a$
- $a * b = \overline{a \uparrow b} = (a \uparrow b) \uparrow (a \uparrow b)$
- $a \oplus b = \overline{a} \uparrow \overline{b} = (a \uparrow a) \uparrow (b \uparrow b)$

Exercise 36.15:

(Section 36.4). Find switching functions for the switching circuits shown in Figs 36.19a,b.

1. $a_1 \oplus a_2 \oplus ((a_3 \oplus a_4) * a_5)$
2. $a_1 \oplus ((a_2 * a_3 \oplus a_4) * a_5)$

Exercise 36.16:

A lecture theatre has three entrances and the lighting can be controlled from each entrance; that is, it can be switched on or off independently. The light is 'on' if the output f equals 1 and 'off' if $f = 0$. Let $a_i = 1$ $(i = 1, 2, 3)$ when switch i is up, and let $a_i = 0$ $(i = 1, 2, 3)$ when it is down. Construct a truth table for the state of the lighting for all states of the switches. Also specify a Boolean expression which will control the lighting.

a_1	a_2	a_3	f
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1

$$\begin{aligned} f = \ & (\overline{a_1} * \overline{a_2} * a_3) \oplus \\ & (\overline{a_1} * a_2 * \overline{a_3}) \oplus \\ & (a_1 * \overline{a_2} * \overline{a_3}) \oplus \\ & (a_1 * a_2 * a_3) \end{aligned}$$

Is there another solution?
Yes, the complement, $\overline{f}$!

Homework for next week

- Read sections 1.1–1.9 in Chapter 1 (27 pages).
- Do exercises: 1.1 – 1.14, 1.18, 1.20, 1.21.

End of file.