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Controlling dynamical behavior of drive-response system through linear augmentation

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Abstract. Unidirectionally coupled chaotic systems give rise to driver induced bistability in response system under certain parameters setting. Such a system is studied here with augmented dynamics. A linear augmentation provides a controlled dynamical behavior of response system in two different ways: augmented drive system brings the stabilization of the steady state where as augmented response system is able to control the bistability. We present a detailed analysis of Lorenz–Rössler system with linear augmentation for controlled dynamical behavior.

1 Introduction

It is shown that the coupled oscillators are able to mimic the functioning of abundant variety of physical, chemical, biological and engineering systems [1,2]. Along with the fixed point behavior, these systems elucidate several other behaviors such as limit cycle, chaos and even multistability. These assorted behavior arises due to the nature of coupling. Even though these systems show a variety of different dynamical behavior, however, for many practical purposes a desired controlled behavior of such systems may be required. Consequently, the topic of controlling the dynamical behavior in such extended nonlinear systems has been a central issue for researchers to explore over the last few decades. These control methods [3–13] claim to achieve various controlled behavior in the systems with complex dynamics.

Recently, a one–way coupled system, i.e. drive–response system, is investigated [14] where driver induced bistability is observed in the response system. The motivation of the present work is to study the augmented dynamics of such drive–response system where the control is either applied to the drive or to the response system. For this purpose, we use the recently proposed linear control method [15,16]. This control scheme works differently when the linear system is coupled with a monostable or a bistable system. Here, due to the dissimilar nature of drive and response systems, we expect different results based on the design of coupling. We observe that augmented–drive system stabilizes the unstable steady state while the augmented–response system annihilates one of the coexisting attractor.

The paper is organized as follows: in the next Section, we review the control methods and outline the basic idea of the control scheme based on linear augmentation of the nonlinear system. In Section 3 we demonstrate that how this control method works in drive–response system when the linear system is either coupled to the drive or to the response system. This is followed by the discussion and summary in Section 4.

2 Control through linear augmentation: A review

Some control methods have been proposed for the stabilization of a unstable periodic orbit [3–7] and a few attempts are also directed towards the control of multistability as well [8–13]. Most of these existing methods [3–7,17] of stabilization of the fixed point are successful with the accessibility of the internal parameters of the system. However, for many real world problems when the internal parameters of the system are not approachable, stabilization of a fixed point is achieved by the phenomenon of amplitude death (AD) [18] that occurs when nonlinear systems are subjected to various type of interactions [19–24]. Moreover, many multistability control methods [8–13] work on the principle of annihilation of all other attractors by leaving the desired one. In most of the existing methods, [8–13] control is applied to one of the parameters of the system. Broadly speaking, modification of the internal parameters of the system is not always possible to get a desired control behavior. Therefore, an external control which works independent of the parameters would be a better choice.

With this aim in mind, recently a control scheme named *linear augmentation* [15,16] is proposed which is adequate to control the dynamical behavior of the system without disturbing the system's parameter. In this scheme a nonlinear system is coupled with the linear system. This is done for two main objectives: (i) for the stabilization of the steady state in nonlinear oscillators [15]. (ii) To control the bistability of a system with coexisting attractors [16]. Moreover, the applicability of the linear augmentation scheme is also elicited for logical computation [25].

Consider a general nonlinear oscillator, $\dot{X} = F(X)$, where X is the m-dimensional vector of dynamical variables and $F(X)$ is the vector field coupled with the linear system as given below:

$$\begin{aligned} \dot{X} &= F(X) + \epsilon U, \\ \dot{U} &= -kU - \epsilon(X - B). \end{aligned} \tag{1}$$

The variable U of m-dimension describes the dynamics of the linear system, ($\dot{U} = -kU$) where k is its decay parameter [15]. The linear system is incapable of having sustained oscillations without feedback from the nonlinear system. Here, ϵ describes the strength of feedback between the oscillator and the linear system. B is the control parameter of the augmented system. In order to stabilize a particular fixed point [15] or to control the bistability of a system with coexisting attractors [16], B can be taken as one of the unstable fixed point of the system.

Here, the state of X can be monostable or bistable one. Therefore, the control scheme works out in two different ways as follows: if the nonlinear system is in monostable state, the stabilization of fixed point is achieved by linear augmentation [15]. In contrast, in a system with coexisting attractors, transition from bi–stability to mono–stability takes place [16]. In the following section we show how this scheme works in drive–response system.

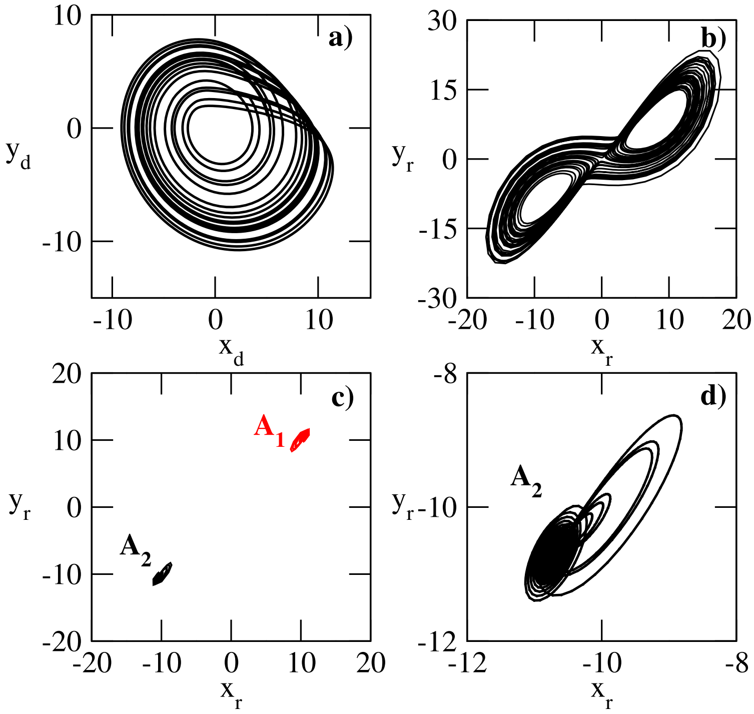

Fig. 1. Attractors of (a) the drive (Rössler) and (b) the response (Lorenz) system at $\epsilon_1 = 0$ (without interaction). (c) Two coexisting attractors A_1 and A_2 of the response system and (d) expanded view of one of the coexisting attractor A_2 at $\epsilon_1 = 1.59$.

3 Drive-response system

Here, we study the unidirectionally coupled (master–slave or drive–response configuration) chaotic oscillators, where the evolution of driver is unaltered by the coupling, while dynamics of response system is influenced by the driver. Let's consider the Lorenz system driven by the chaotic Rössler system given by [14]

$$
\begin{aligned}
\dot{x_d} &= -y_d - z_d \\
\dot{y_d} &= x_d + \eta_1 y_d \\
\dot{z_d} &= \eta_2 + z_d(x_d - \eta_3) \\
\dot{x_r} &= \sigma(y_r - x_r) \\
\dot{y_r} &= r x_r - y_r - x_r z_r \\
\dot{z_r} &= x_r y_r - \beta z_r + \epsilon_1(z_d - z_r).
\end{aligned} \tag{2}
$$

The coupling between the drive and the response system is designed in such a way that the symmetry of the response system (Lorenz) in the x_r-y_r plane is preserved. We set the parameter values of both the systems in chaotic region as: $\eta_1 = 0.2$, $\eta_2 = 0.2$, $\eta_3 = 5.7$, $\sigma = 10$, $r = 28$ and $\beta = 8/3$. Here, ϵ_1 is the coupling parameter. In the absence of coupling i.e $\epsilon_1 = 0$ dynamics of the drive and the response system is shown in Fig. 1a and b respectively.

At the coupling parameter $\epsilon_1 = 1.59$, two co–existing attractors of the response system are shown in Fig. 1c for the system Eq. (2). These two attractors

A_1 and A_2 are situated around two unstable fixed points and obtained from the two different initial conditions $X(0) = [0.01, 0.1, 0.45, 0.3, 0.3, 0.23]^T$ and $X(0) = [0.01, 0.1, 0.45, -0.3, -0.3, 0.23]^T$ respectively. Figure 1d is the expanded view of attractor A_2.

3.1 Linear system coupled to Drive system

Let us first consider the case when the linear system is coupled to the drive system (Rössler) in the following manner

$$\begin{aligned}
\dot{x_d} &= -y_d - z_d + \epsilon_2 u \\
\dot{y_d} &= x_d + \eta_1 y_d \\
\dot{z_d} &= \eta_2 + z_d(x_d - \eta_3) \\
\dot{x_r} &= \sigma(y_r - x_r) \\
\dot{y_r} &= r x_r - y_r - x_r z_r \\
\dot{z_r} &= x_r y_r - \beta z_r + \epsilon_1 (z_d - z_r) \\
\dot{u} &= -ku - \epsilon_2 (x_d - b)
\end{aligned} \tag{3}$$

with $U = [u, 0, 0....]^T$ and $B = [b, 0, 0.....]$, where T is the transpose. Here, ϵ_2 describes the interaction between the linear and the response systems. Except b and ϵ_2, other parameters are fixed at the same value as mentioned above with $k = 1$. We set the control parameter ($b = x^* = 0.007$), where x^* is the x component of one of the unstable fixed point of the drive system. We plot the two largest Lyapunov exponents (solid and dashed lines) and the real part of the largest eigenvalue (open circles) as a function of the coupling strength ϵ_2 in Fig. 2a. These curves depict that with increasing coupling strength, chaotic motion of the drive–response system goes to periodic state after $\epsilon_2 \sim 0.3$ as shown by black curves in Fig. 2b and Fig. 2c respectively. At a critical value of the coupling strength $\epsilon_{2c} \sim 0.6$ the largest Lyapunov exponent becomes negative, showing transition of the system from quasi–periodic (QP) to the fixed point state (AD). The fixed point solution of the drive and the response systems at $\epsilon_2 = 0.8$ is portrayed by red dots as shown in Fig. 2b and Fig. 2c respectively. Here, as the drive system has mono–stability therefore, independent of the initial conditions it is stabilized to the corresponding unstable fixed point. However, depending upon the choice of the control parameter b, the bi–stable oscillator (response system) can be stabilized to the corresponding unstable fixed point. This controlled augmented dynamics is enlighten in Fig. 3 where we plot the bifurcation diagram with 1000 points (after removing the sufficient transients) of variable x of drive and response systems as a function of the coupling strength ϵ_2. For the initial condition $x_{r0} = 0.3, y_{r0} = 0.3$, Fig. 3a evinces the dynamics of the drive system where it is stabilized to one of its unstable fixed point as shown in Fig. 2b, while Fig. 3b exhibits the dynamics of the response system which is stabilized to -10.647 as shown in Fig. 2c. For the another initial condition $x_{r0} = -0.3, y_{r0}, = -0.3$ dynamics of the drive system remains the same as in Fig. 3a however, the response system is stabilized to the another fixed point 10.647 as shown in Fig. 3c.

3.2 Linear system coupled to Response system

Now we discuss the effect of the linear system, when it is coupled to the response system (Lorenz) as follows:

$$\begin{aligned}
\dot{x_d} &= -y_d - z_d \\
\dot{y_d} &= x_d + \eta_1 y_d \\
\dot{z_d} &= \eta_2 + z(x_d - \eta_3)
\end{aligned}$$

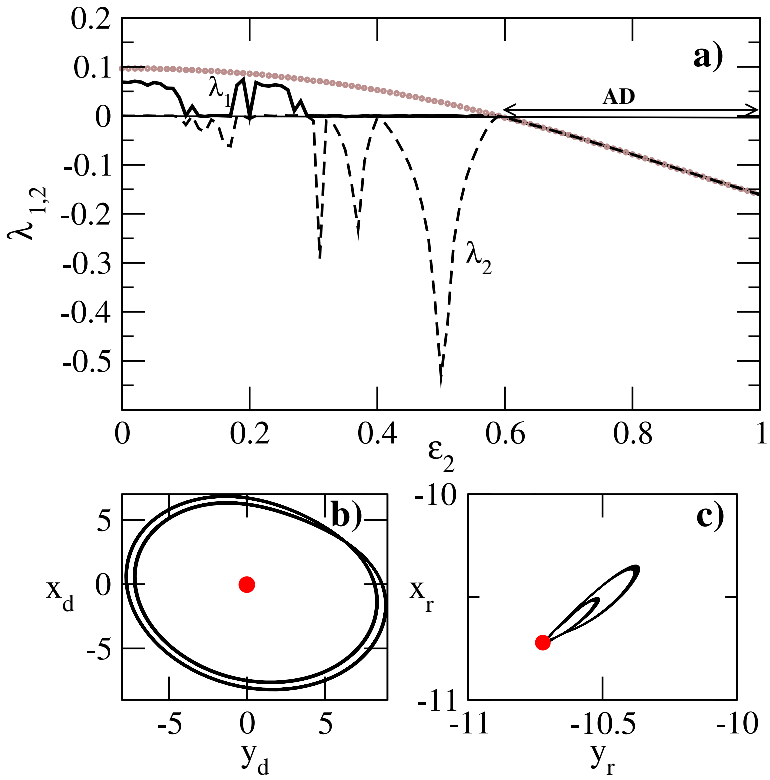

Fig. 2. (a) The two largest Lyapunov exponents (solid and dashed black lines) and the real part of the largest eigenvalue (open brown circle) as a function of the coupling strength ϵ_2. At $\epsilon_2 = 0.4$ periodic trajectory of (b) the drive and (c) the response systems. The red dots depict the unstable fixed points of the respective systems.

$$
\begin{aligned}
\dot{x_r} &= \sigma(y_r - x_r) + \epsilon_2 u \\
\dot{y_r} &= r x_r - y_r - x_r z_r \\
\dot{z_r} &= x_r y_r - \beta z_r + \epsilon_1 (z_d - z_r) \\
\dot{u} &= -ku - \epsilon_2 (x_r - b).
\end{aligned} \tag{4}
$$

Here, ϵ_2 describes the interaction between the linear and the response systems. When $\epsilon_2 = 0$ the drive system have a single attractor, while the response system have bistability in the form of two co–existing attractors A_1 and A_2 as shown in Fig. 4 [14]. These two co–existing attractors are located around the two unstable fixed points of coupled system $x^*_{c1} = 8.369$, $x^*_{c2} = -8.396$ respectively. The aim of this type of interaction is to control the bistability of the response system through linear augmentation. A scheme for bistability control is explained in detail [16] (ct. Sect. 2) through linear augmentation where unstable fixed points play an important role to target the desired attractor. Depending upon the proper choice of the control parameter b, any one of these two attractors can be annihilated and the system can be stabilized to a desired monostable state. Suppose, one would like to annihilate the attractor A_2, for that we set $b = x^*_{c1} = 8.396$ (see [16] for explanation). With increasing coupling ϵ_2, attractor A_2 shrinks in size and moves towards the attractor A_1 of the uncoupled system and the resulting attractor due to coupling is shown in Fig. 4c. At this parameter value there exists only one attractor, near A_1, to which all the initial conditions

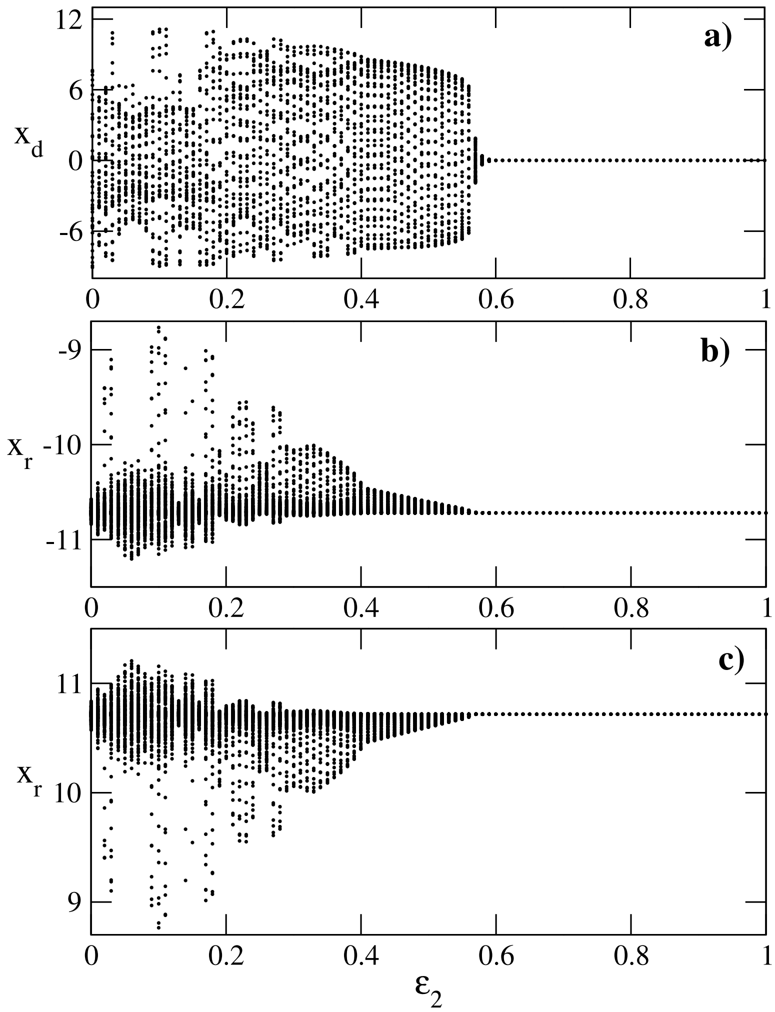

Fig. 3. The bifurcation diagrams for (a) the drive and (b, c) the response systems as a function of the coupling strength ϵ_2. The (b) and (c) are drawn for the two different sets of initial conditions (see text for details).

converge. This clearly indicates that the transition from bistability to monostability takes place due to the coupling of the response system with the linear system.

To look into the mechanism of transition to monostability, we consider the relative basin size as a measure. It can be estimated as the fraction of the initial conditions converging to an attractor divided by the total number of initial conditions taken. Here, the fraction is calculated with 10^4 initial conditions chosen over a mesh on $(x(0), y(0)) \in [12, -12] \times [12, -12]$. Initially, in the paucity of coupling i.e. at $\epsilon_2 = 0$, some of the initial conditions converge to the basin of attractor A_1 (black solid line), while rest go to the attractor A_2 (red dashed line) as rendered in Fig. 5a. With the increment of the coupling strength ϵ_2, fraction of initial conditions, moving towards the attractor A_1 increases (approaches one), while the fraction of initial conditions going to the attractor A_2 decreases to zero. It is clear from Fig. 5a that the attractor A_2 disappears completely at the critical coupling strength $\epsilon_{2c} = 0.8$ and beyond that only attractor A_1 survive. This transition can be explained by the boundary

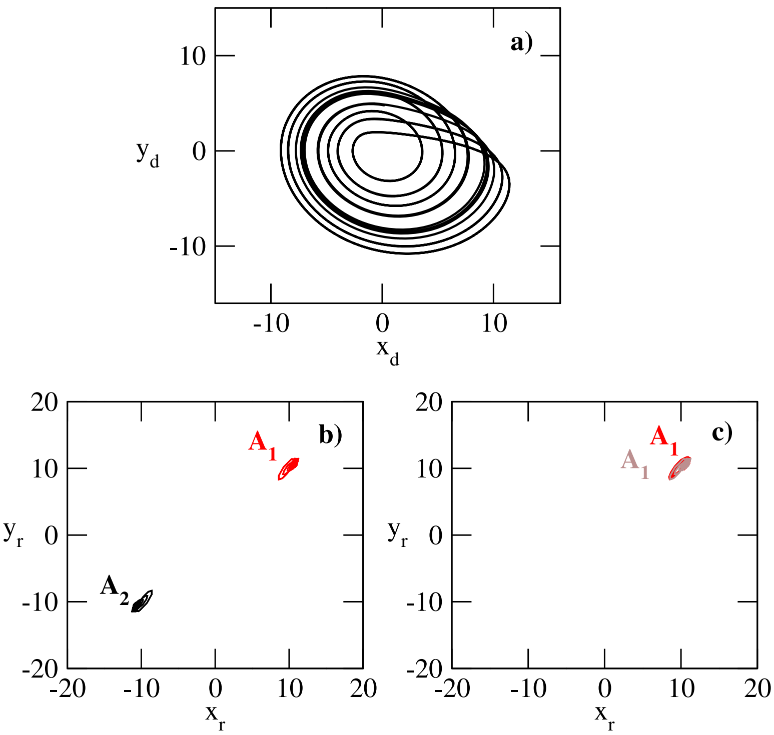

Fig. 4. (a) Attractor of the drive system and (b) the two coexisting attractors A_1 (red) and A_2 (black) of the response system at $\epsilon_2 = 0$. (c) One of the coexisting attractor A_1 (red) and the resulting attractor A_1 (brown) of the response system due to interaction with the linear system at $\epsilon_2 = 0.87$. Initial conditions for the two attractors A_1 and A_2 are $(x_{r0} = 0.3, y_{r0} = 0.3)$ and $(x_{r0} = -0.3, y_{r0} = -0.3)$ respectively.

crisis [16]. With increasing coupling strength the attractor A_2 is strongly influenced by the linear system, it shrinks in size and moves in the state space towards the unstable fixed point $\mathbf{x}^*_{c3}$. Finally the attractor A_2 collides with that fixed point x^*_{c3} and its stable manifolds, give rise to a boundary crises of the attractor A_2. A further increase of the coupling strength leads to merging of the two unstable fixed points $\mathbf{x}^*_{\mathbf{c4}}$ and $\mathbf{x}^*_{\mathbf{c6}}$ and then $\mathbf{x}^*_{\mathbf{c2}}$ and $\mathbf{x}^*_{\mathbf{c5}}$ as shown in Fig. 5b.

4 Discussion and summary

To conclude, in this paper we have presented a simple scheme of linear augmentation [15,16] of drive–response system to engineer its controlled dynamics in a desired way i.e., either suppression of oscillations or annihilation of one of the coexisting attractor. From the results, it is evident that the effect of the control scheme is different based on its coupling to the drive or to the response system. A significant consideration is that a single control scheme is adequate to stabilize the fixed point of the system or to control the bistability in a higher dimension nonlinear system without disturbing the internal parameters of the system. Thus, our control scheme is simple, multipurpose and easy to apply. Occurrence of the higher dimensional system is very common in many ecological, electronic and engineering systems, and sometimes their desired controlled behavior, i.e. targeting steady state or controlling bistability, is necessary, in that case the proposed scheme can be of significant use. In chaos based communication,

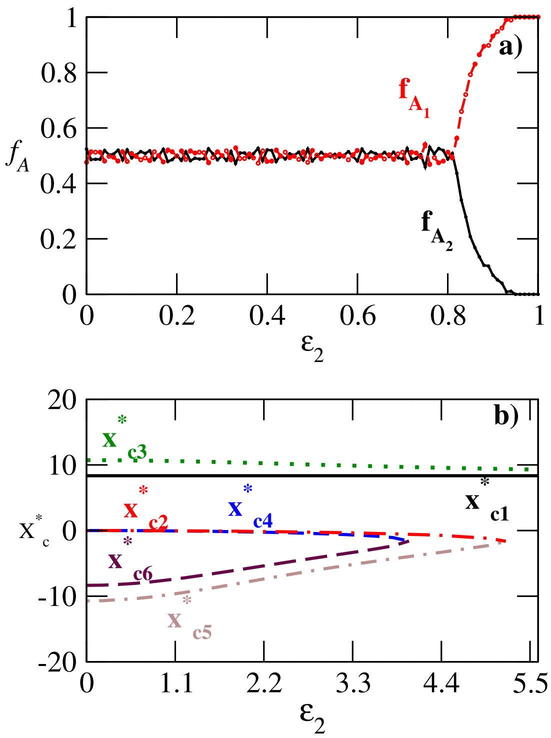

Fig. 5. The variation of (a) the relative basin sizes f_{A1} and f_{A2} for the coexisting attractors A_1 and A_2 respectively. (b) The fixed points x^*_{c1}, x^*_{c2}, x^*_{c3}, x^*_{c4}, x^*_{c5} and x^*_{c6} of the augmented drive–response system as a function of the coupling strength ϵ_2 at $b = x^*_{c1} = 8.369$.

particularly, in a system with two coexisting attractors, the linear system can act as a common controller to stabilize the two systems to one of the desired chaotic state by the proper choice of the control parameter for synchronization [26,27].

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