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Complicated basins and the phenomenon of amplitude death in coupled hidden attractors

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ABSTRACT

Understanding hidden attractors, whose basins of attraction do not contain the neighborhood of equilibrium of the system, are important in many physical applications. We observe riddled-like complicated basins of coexisting hidden attractors both in coupled and uncoupled systems. Amplitude death is observed in coupled hidden attractors with no fixed point using nonlinear interaction. A new route to amplitude death is observed in time-delay coupled hidden attractors. Numerical results are presented for systems with no or one stable fixed point. The applications are highlighted.

1. Introduction

Attractors are termed as self-excited attractors if their basins intersect with the neighborhood of equilibria present in the system. Such attractors in various systems, e.g. Lorenz, Rossler, Chua, etc., have been studied in detail [1–3]. Very recently, a new type of attractors called hidden attractors, that don't intersect with the neighborhood of any equilibrium, have been reported [4–6]. Due to absence of unstable equilibrium in its neighborhood these type of attractors are less traceable. Therefore, it is also difficult to understand their characteristic behavior [4–11]. Even to locate the attractors in a given system requires proper search methods [8–11]. In recent years various attempts have been made to understand such attractors. Various physical as well as mathematical models have been explored with possibility of finding hidden oscillating attractors in systems having no [12–15], one [12,16], and more [17–20] stable fixed point. Understanding the properties of such attractors are important as they are observed in various systems e.g. Chua, Electrical Machines, drilling system, etc. [4,5,7,12,19]

Systems with coexisting attractors have a complex dynamical behavior due to the extreme sensitivity towards initial conditions, system parameter, and noise [1,21,22]. The presence of coexisting attractors in a system creates a dilemma in deciding the final asymptotic state of the system. In many practical situations, particularly from an engineering point of view, understanding the structures of the basins of such attractors are essential [23,24]. In this paper we study coexisting hidden attractors which have riddled-like complicated basins.

Natural systems are rarely isolated, and hence the interaction between such systems have been extensively studied for self-excited systems, both from theoretical and experimental points of view. Several interesting new phenomena have been observed in such interacting systems [3,25,26]. One such phenomena, amplitude death (AD) [27], is important because it can also occur in coupled nonlinear oscillators. It occurs when interaction causes the fixed points to become stable and attracting. Since no fixed point or a set of stable fixed points exist in systems with hidden attractors, understanding the nature and consequences of coupling in such systems is equally important. For a hidden attractor having no-stable point, AD can be induced only by creating new fixed points in the coupled system. However, for the case of hidden attractors having stable fixed points, AD, can be observed using appropriate interactions. In this paper we show that AD can be achieved using nonlinear interaction in systems having no-fixed points while in systems having stable fixed points this can be achieved using time-delay interaction. A new route to AD is observed for the case of stable fixed point systems that is very different from existing routes to AD in self-excited attractors [27].

This paper is organized as follows. In Section 2 we study individual systems with no or one stable fixed point. The riddled-like basin for coexisting hidden attractors are presented in this section. The phenomena of AD due to nonlinear and time-delayed interactions in coupled hidden attractors are also demonstrated. In Section 3 summary, conclusions are presented.

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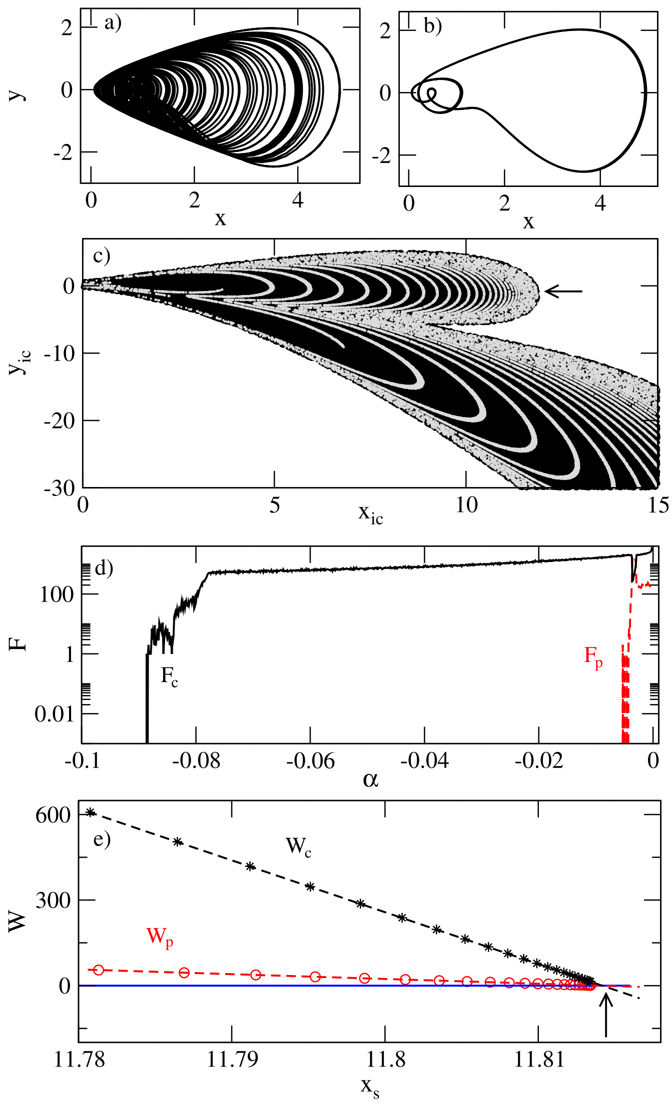

Fig. 1. (Color online.) The trajectories of (a) chaotic and (b) periodic attractors, and (c) the corresponding basins (chaotic – black and periodic – grey colors) at $\alpha = -0.001$. (d) The fractions of initial conditions which go to chaotic (F_c, solid black line) and periodic (F_p, dashed red line) attractors as a function of parameter α. (e) The width of the strips of period and chaotic basins as a function starting positions x_s of the strips, along the marked arrow in (c).

2. Results and discussions

In this section, we study a particular class of systems having either no or one fixed point. However, similar results are observed in systems having two or more stable fixed points as well [28].

2.1. System with no fixed point

We first consider a system that has no fixed point and provides hidden attractors [12],

$$\dot{x} = y,$$

$$\dot{y} = z,$$

$$\dot{z} = -y + 3y^2 - x^2 - xz + \alpha. \tag{1}$$

This system doesn't have any fixed point for parameter $\alpha < 0$ [12]. This system has been studied earlier where the existence of a single chaotic attractor has been demonstrated [12]. Such a typical chaotic attractor is shown in Fig. 1(a) for $\alpha = -0.001$. Apart from this attractor, a new periodic attractor is also observed at the same parameter as shown in Fig. 1(b). Because there is no fixed point in this system at this parameter, these coexisting attractors are termed as hidden ones. Since these coexisting attractors depend on initial conditions, we explore their corresponding basins. These are shown in Fig. 1(c). The dark and brown strips (within dark region) correspond to the chaotic and periodic motions respectively. The blank (white) regions correspond to the initial conditions which don't have bound solutions, i.e., system goes to infinity. These basins are generated from 10^6 random initial conditions in the range of $x_{ic} \in (0, 15)$, $y_{ic} \in (-30, 40)$ and $z_{ic} = 0$. The basins of chaotic and periodic motions are determined from the largest Lyapunov exponents (LE) [29] i.e., for LE > 0.05 corresponds to the chaotic attractor (criteria for systems having Perron effect – see Refs. [30,31]) otherwise motion is considered as periodic one. Apart from these coexisting attractors no other attractor was found in this range of the initial conditions.

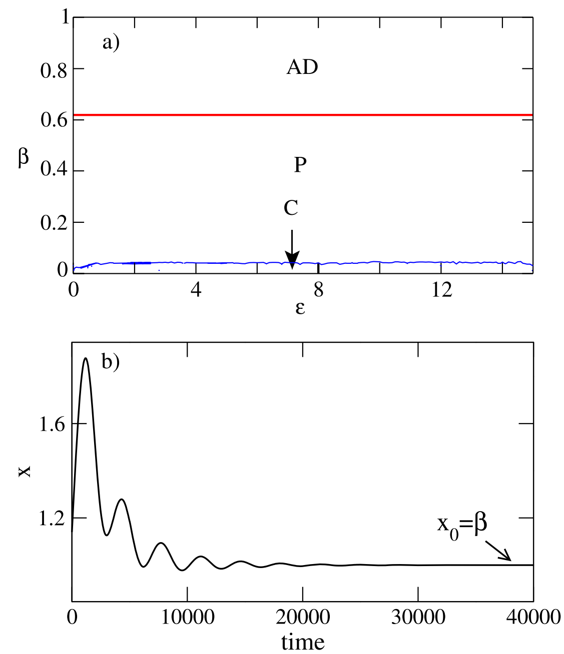

Fig. 2. (Color online.) (a) The schematic phase diagram in parameters ϵ and β of Eq. (2). (b) The transient trajectory for targeted fixed point solution in AD region of (a).

In order to show how the basins of these attractors change as a function of the parameter α, shown in Fig. 1(d) are the plot of fractions (F) of initial conditions which go to either chaotic (solid line) or periodic (dashed line) attractors. Here the fractions are calculated by considering 10^4 random initial conditions (in the range used for Fig. 1(b)) and averaged over the number of initial conditions which go to the oscillatory motion (chaotic or periodic). This figure shows that there is no bound solution for $\alpha \lesssim -0.09$. However as α is increased, the chaotic attractor appears. As α is further increased, for $\alpha \gtrsim -0.005$, both the chaotic as well as the periodic attractors coexist (Fig. 1(b)). From these analysis it is concluded that the systems with hidden attractors can have different types of oscillating dynamics as well as complicated basins. Therefore, such systems need to be extensively studied for a complete understanding of its characteristic behavior.

As we see in Fig. 1(c), the strips corresponding to the periodic or chaotic regime are becoming thinner and thinner near the boundary of oscillatory and unbounded solutions. In order to see how the widths (W) of these strips are changing, we consider the initial conditions on a fixed line at $y_{ic} = 0$ (indicated by arrow in Fig. 1(c)). We calculate the width of the each strip which start at x_s. Shown in Fig. 1(e) are the widths of periodic (W_p) and chaotic (W_c) regimes as a function of x_s, drawn with symbols circle and star respectively. Fit to the data (dashed lines) show that these are varying linearly with different slopes. An important observation here is that these two fitted lines meet exactly at zero width (solid line) that is near the basin boundary of unstable solution. This suggests that near to the boundary any change in initial conditions can lead to periodic or chaotic motions. Therefore, predicting the type of motion near to the boundary is difficult as widths of basins are almost zero. This means that the basins near to the boundary are quite complicated similar to the riddled basin (discussed latter) [32–34].

In coupled self-excited oscillators a host of new phenomena including amplitude death [27] are possible [3,25,26]. In order to demonstrate the phenomenon of AD, we have considered only two hidden attractors. However, the results are similar for the large number of oscillators as well [28]. Since there is no fixed point in the individual oscillator, Eq. (1), the possibility of AD doesn't arise. For example, let us consider a diffusive interaction, say $\epsilon(x_i - x_j)$ with coupling strength ϵ. Here AD is not possible because the required condition $x_i = x_j$ (independent identical systems) lead to no fixed point in the system.

However, if we consider an appropriate coupling function that creates new fixed points, then possibility of AD can be explored. Consider a nonlinear interaction that was proposed previously to create a new targeted fixed point [35]

$$\begin{aligned}
\dot{x}_i &= y_i - \epsilon(x_i - \beta)\exp(x_j - 10) - y_0, \\
\dot{y}_i &= z_i, \\
\dot{z}_i &= -y_i + 3y_i^2 - x_i^2 - x_i z_i + \alpha,
\end{aligned} \tag{2}$$

where β is a parameter and $i, j = 1, 2$. y_0 is the y-component of the newly created fixed point ($x_0 = \beta$, $y_0 = (1 - \sqrt{1 - 12(\alpha - \beta^2)})/6$, $z_0 = 0$). In Fig. 2 (a) the schematic phase diagram is presented in parameters ϵ and β. Here C, P and AD regions correspond to the chaotic, periodic and fixed-point motion respectively. Note that the individual systems have no fixed point. However as interaction is switched, a new fixed point is created. In the regime of AD, for a given β, x-component of the fixed point will be $x_0 = \beta$, i.e., a targeted new fixed point is created as shown in Fig. 2(b) for $\beta = 1.3$. This result shows that even if there is no fixed point in the individual system, the targeted new fixed points can be created, and hence the phenomenon of AD can be achieved. This has important application in avoiding unwanted fluctuations (oscillatory motion) in systems with no fixed point.

2.2. System with one stable fixed point

In this section we consider a system which has only one stable fixed point [12],

$$\begin{aligned}
\dot{x} &= yz + \alpha, \\
\dot{y} &= x^2 - y, \\
\dot{z} &= 1 - 4x.
\end{aligned} \tag{3}$$

This system has only one fixed point at $(x_0, y_0, z_0) = (1/4, 1/16, -16\alpha)$ that is stable for the range of $\alpha > 0$. Apart from this stable attractor, there coexists other oscillating attractors as well. Shown in Fig. 3(a) is a chaotic attractor at $\alpha = 0.01$ which has been observed earlier [12]. We also observe another periodic coexisting attractor in this system as shown in Fig. 3(b). Since these oscillating attractors are away from their stable fixed point and their basins don't contain the stable fixed point, these are coexisting hidden attractors. One must explore the basin structures of these hidden attractors, shown in Fig. 3(c). These basins are generated

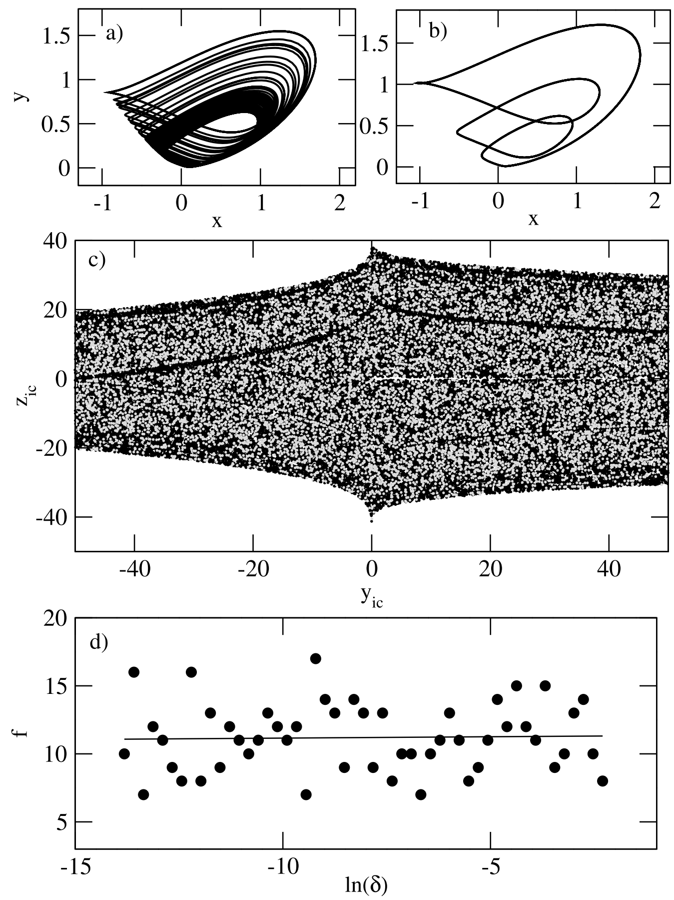

Fig. 3. The trajectories of (a) chaotic and (b) periodic attractors and their (c) basins (chaotic – black and periodic – grey colors) at $\alpha = 0.01$ of Eq. (3). (d) The fraction, f as a function δ. The fraction if calculated along the line $z_{ic} = 0$ within the range of $y_{ic} \in (-20, 20)$.

from 10^5 initial conditions in $y_{ic} \in (-50, 100)$, $z_{ic} \in (-50, 100)$ with fixed $x_{ic} = 0$. Blank region corresponds to the initial conditions which go to stable fixed point. In the shaded region, initial condition with black dots go to chaotic while grey dots go to periodic attractors. The basins of the chaotic and periodic attractors show that these complicated basins are riddled-like [32–34]. Here riddling means a pair of random initial conditions of separation δ may lead to coexistence of both type of attractors. In order to quantify this, we randomly choose a pair of initial conditions δ-distance apart. We consider 50 pairs of such initial conditions which are δ-distance apart and find the fraction of pairs, f, which give both type of attractors. This fraction is plotted in Fig. 3(d) as a function of δ. A linear fit to the data indicates that slope is very small and hence f does not appear to decrease appreciably. This is typical for of riddled or riddled-like basins [32–34]. The practical implication of complex basins is that the asymptotic attractor cannot be predicted, i.e., no matter how small the phase-space region is chosen to be, there are always two classes of initial conditions mixed in a riddled-like manner which go to the two different attractors. Fig. 3(d) is thus a clear demonstration of the phenomenon of riddled-like basins.

In order to see the phenomenon of amplitude death we consider two time-delay coupled systems [36,37] as

$$\begin{aligned}
\dot{x}_i &= y_i z_i + \alpha + \epsilon\big(x_j(t - \tau) - x_i\big), \\
\dot{y}_i &= x_i^2 - y_i, \\
\dot{z}_i &= 1 - 4x_i,
\end{aligned} \tag{4}$$

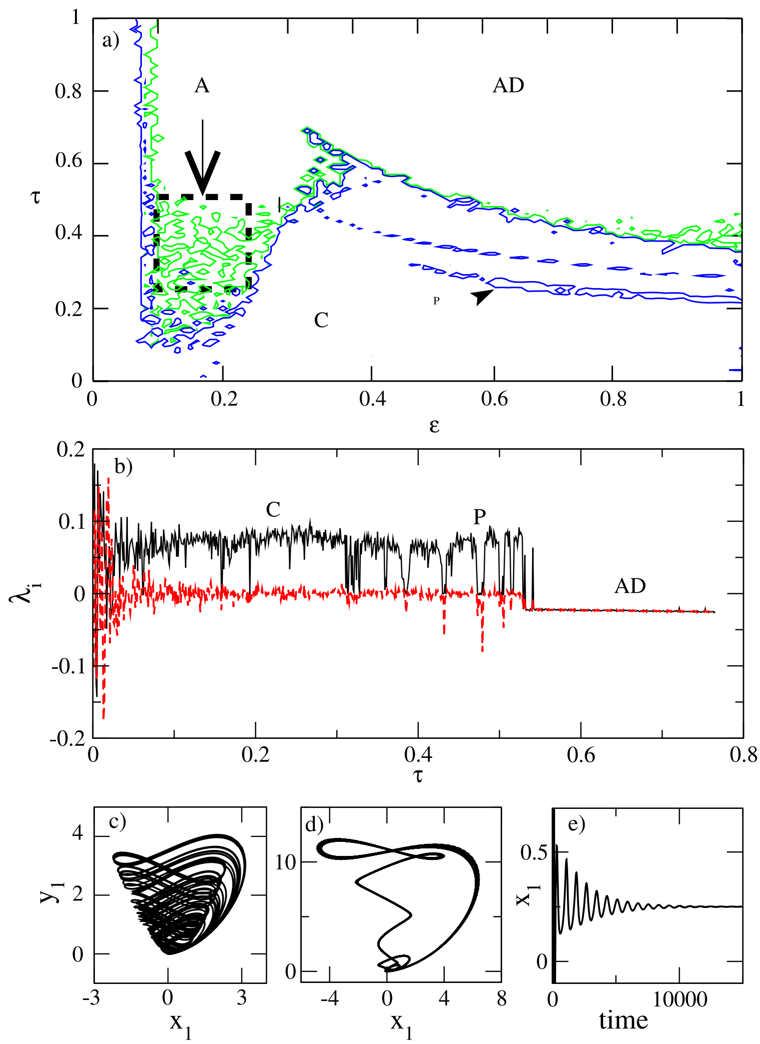

Fig. 4. (a) Schematic phase diagram in coupling parameters, strength ϵ and time delay τ of Eq. (4). (b) The largest two Lyapunov exponents as a function of delay τ at fixed coupling strength $\epsilon = 0.5$. The trajectories of (c) chaotic and (d) periodic attractors at $\tau = 2$ and 0.475 respectively in x_1–y_1 plane. (e) The variable x_1 as a function time in AD region at $\tau = 0.7$.

where $i, j = 1, 2$. The parameters ϵ and τ are the coupling strength and the time-delay respectively. Here we consider only time-delay interaction as it is most general and widely used interaction for getting AD in self-excited attractors [27,36–38]. Shown in Fig. 4(a) is the schematic diagram in the parameter space, ϵ and τ. This indicates the possibility of different type motions, chaotic, periodic, and stable fixed point, in different regions. This diagram is generated using spectrum of Lyapunov exponents [29]. The largest two LE are shown in Fig. 4(b) for fixed coupling strength $\epsilon = 0.5$. The typical trajectories for chaotic and periodic attractors are shown in Fig. 4(c) and (d) at $\tau = 0.2$ and $\tau = 0.47$ respectively.

A trajectory in AD regime is shown in Fig. 4(e) at $\tau = 0.7$ which clearly indicates the damping amplitude at fixed point of the uncoupled system. Note that in the previously reported example of AD in self-excited attractors, either the unstable fixed points of the uncoupled systems or newly created fixed point(s) were get stabilized [27]. However in the present case of hidden attractors, where already one fixed point is stable, the oscillating attractors get destroyed and become the stable same fixed point. This suggests that AD occurs due to the change in the size of the basins of attractors. In order to see how this happens shown in Fig. 5 (a) are the fractions, F, of 1000 initial conditions which go to the oscillatory ($F_{C,P}$: chaotic or periodic, solid line) and fixed point (F_{FP}: dashed line) solutions as a delay, τ at fixed coupling strength $\epsilon = 0.5$. These curves show that as delay is increased beyond $\tau \sim 0.2$, fraction of fixed point motion starts increasing and gets saturated to $f = 1$ for $\tau \gtrsim 0.6$, i.e., after certain values of delay only fixed point solutions exist. This variation of fraction suggests that the size of basin of oscillating attractors, which is riddled, decreases slowly and vanishes for $\tau > 0.6$. Therefore, this destruction or disappearance of size of basins is a route to AD which has not been observed so far in self-excited oscillators [27] (mostly Hopf, saddle-node, direct transitions have been observed in self-excited attractors). Therefore this route to AD in hidden oscillators is quite new.

The comparison of Figs. 4(c) and (d) suggests that the size of the chaotic attractor gets smaller than that of the periodic attractor as delay is increased. In order to see the variation of size of the attractors in phase space we calculate the averaged maximum distance $D = \langle y_{max} - y_{min} \rangle$ of y-variable. Here average is taken over 10^5 initial conditions. Note that this distance D doesn't give exact size of the attractor but it can represent an approximate size of the attractor. Shown in Fig. 5(b) is the variation of size D as a function of delay. This shows that average size of the oscillating attractors increases and then decreases to zero in the AD regime. This increase of size of attractor before becoming to zero looks counter intuitive. This behavior may be termed as *anomalous to AD*, similar to the definition used for the anomalous phase synchrony [39,40] for coupled self-excited attractors.

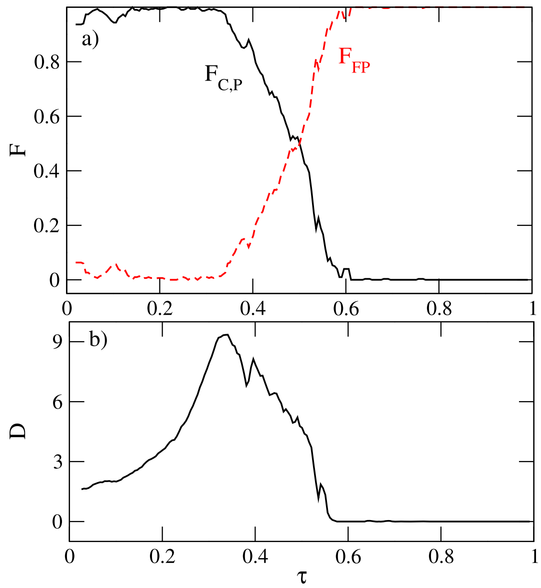

Fig. 5. The variation of (a) fraction of initial conditions which go to the oscillatory motion ($F_{C,P}$: chaotic or periodic, solid line) and fixed point solutions (F_{FP}: dashed line) and (b) the averaged distance, representing the size of attractors, as a function of delay τ at fixed coupling strength $\epsilon = 0.5$.

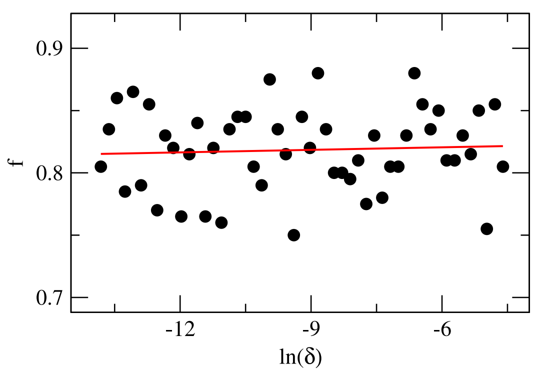

Fig. 6. Fraction f of uncertain parameter pairs (out of 100 parameter pairs) versus parameter perturbation δ for the parameter, τ in the marked region A of (a).

Fig. 4(a) is generated with fixed initial conditions. If we change the initial conditions then the loci of bifurcation curves change marginally. Depending on the initial conditions, both the oscillatory (periodic or chaotic) and stable fixed point solutions are possible in the marked region A (dashed box) of Fig. 4(a). The presence of such multistability usually gets reflected with the exhibition of wild fluctuations in the measured Lyapunov exponents as shown in Fig. 4(b). This sensitivity of the oscillation on parameter variations can be conveniently quantified by the uncertainty exponent [32–34]. We fix a small perturbation δ and randomly choose a pair of parameters with δ-distance apart in region A of Fig. 4(a). The two parameters are uncertain with respect to the perturbation δ if they result in different attractors. In this case the fraction of uncertain parameter pairs f typically decreases with δ and scales as $f \sim \delta^{\beta}$, where $\beta > 0$, and is defined as the uncertainty exponent [32–34]. Fig. 6 shows a typical plot of f (symbols) as a function of δ. Here as δ is reduced, f does not appear to decrease appreciably. A linear fit (solid line) to the data gives $\beta = 0.0006 \pm 0.001$, indicating that β cannot be distinguished from zero and hence the basins are riddled or riddled-like [32–34,41]. This shows that even after coupling the possibility of riddled-type basins still exists.

3. Summary

In this work we have studied both the coupled and uncoupled hidden attractors having either no or one stable fixed point. We observed chaotic as well periodic coexisting hidden attractors in individual systems. The riddled or riddled-like basins are observed in either individual or coupled hidden attractors. The practical implication of these complicated interwoven basins is that the asymptotic attractor cannot be predicted if there is an uncertainty in the specification of either parameters or initial conditions. These analysis of basins are important for practical applications where targeted dynamics are required.

The phenomenon of AD is observed in time-delayed coupled attractors having stable fixed points. However, the route to AD is different from that observed in self-excited attractors. Here, the basins of oscillating attractors disappear slowly just before the amplitude death. This suggests a new route to AD in coupled systems. AD is also observed in systems having no fixed point by creating a new one by nonlinear interaction.

Although this work focuses on either no or one stable fixed point only, we obtain similar results [28] for two [12] and a line (infinite [19]) stable fixed points as well. We also observe similar results for AD in a network of 10 globally coupled hidden attractors [28]. Therefore, the results presented here are quite general and can be found in other systems as well.

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References

[1] E. Ott, Chaos in Dynamical Systems, Cambridge University Press, Cambridge, 1993.
[2] S.H. Strogatz, Nonlinear Dynamics and Chaos, Westview Press, New York, 1994.
[3] A. Pikovsky, M. Rosenblum, J. Kurths, Synchronization: A Universal Concept in Nonlinear Sciences, Cambridge University Press, Cambridge, 2003.
[4] N.V. Kuznetsov, G.A. Leonov, V.I. Vagaitsev, IFAC Proc. 4 (2010) 29.
[5] G.A. Leonov, V.I. Vagaitsev, N.V. Kuznetsov, Phys. Lett. A 375 (2011) 2230.
[6] G.A. Leonov, N.V. Kuznetsov, V.I. Vagaitsev, Physica D 241 (2012) 1482.
[7] G.A. Leonov, N.V. Kuznetsov, Int. J. Bifurc. Chaos 23 (2013) 1330002.
[8] G.A. Leonov, V.I. Vagaitsev, N.V. Kuznetsov, Dokl. Math. 82 (2010) 663.
[9] G.A. Leonov, N.V. Kuznetsov, Dokl. Math. 84 (2011) 475.
[10] V.O. Bragin, V.I. Vagaitsev, N.V. Kuznetsov, G.A. Leonov, J. Comput. Syst. Sci. Int. 50 (2011) 511.
[11] G.A. Leonov, N.V. Kuznetsov, IFAC Proc. 18 (2011) 2494.
[12] S. Wang, G. Chen, Nonlinear Dyn. 71 (2013) 249.
[13] S. Jafari, J.C. Sprott, S.M.R.H. Golpayegani, Phys. Lett. A 377 (2013) 699.
[14] Z. Wei, Phys. Lett. A 376 (2011) 102.
[15] Z. Wang, S. Cang, E.O. Ochola, Y. Sun, Nonlinear Dyn. 69 (2012) 531.
[16] X. Wang, G.A. Chen, Commun. Nonlinear Sci. Numer. Simul. 17 (2012) 1264.
[17] Z. Wei, Comput. Math. Appl. 63 (2011) 728.
[18] Z. Wei, Q. Yang, Appl. Math. Comput. 217 (2010) 422.
[19] S. Jafari, J.C. Sprott, http://sprott.physics.wisc.edu/chaos../jafari/.
[20] Q. Yang, Z. Wei, G. Chen, Int. J. Bifurc. Chaos 20 (2010) 1061.
[21] E. Ott, Scholarpedia 1 (2006) 1701.
[22] U. Feudel, Int. J. Bifurc. Chaos 18 (2008) 1607.
[23] P.R. Sharma, M.D. Shrimali, A. Prasad, U. Feudel, Phys. Lett. A 377 (2013) 2329.
[24] G.A. Leonov, B.R. Andrievskii, N.V. Kuznetsov, A.Yu. Pogromskii, Differ. Equ. 48 (2012) 1700.
[25] K. Kaneko, Theory and Applications of Coupled Map Lattices, John Wiley and Sons, New York, 1993.
[26] A. Prasad, L.D. Iasemmidis, S. Sabesan, K. Tsakalis, Pramāna 64 (2005) 513.
[27] G. Saxena, A. Prasad, R. Ramaswamy, Phys. Rep. 521 (2012) 205.
[28] U. Chaudhuri, unpublished.
[29] G. Benettin, L. Galgani, J.-M. Strelcyn, Phys. Rev. A 14 (1976) 2338.
[30] G.A. Leonov, N. Kuznetsov, Int. J. Bifurc. Chaos 17 (2007) 1079.
[31] N.V. Kuznetsov, T.N. Mokaev, P.A. Vasilyev, Nonlinear Sci. Numer. Simul. 19 (2014) 1027.
[32] C. Grebogi, S.W. McDonald, E. Ott, J.A. Yorke, Phys. Lett. A 99 (1983) 415.

[33] S.W. McDonald, C. Grebogi, E. Ott, J.A. Yorke, Physica D 17 (1985) 125.
[34] Y.-C. Lai, R.L. Winslow, Phys. Rev. Lett. 72 (1994) 1640.
[35] A. Prasad, M. Dhamala, B.M. Adhikari, R. Ramaswamy, Phys. Rev. E 82 (2010) 027201.
[36] D.V.R. Reddy, A. Sen, G.L. Johnston, Phys. Rev. Lett. 80 (1998) 5109.
[37] A. Prasad, Phys. Rev. E 72 (2005) 056204.
[38] K. Bar-Eli, Physica D 14 (1985) 242.
[39] B. Blasius, E. Montbrio, J. Kurths, Phys. Rev. E 67 (2003) 035204.
[40] A. Prasad, J. Kurths, R. Ramaswamy, Phys. Lett. A 372 (2008) 6150.
[41] A. Prasad, Y.-C. Lai, A. Gavrielides, V. Kovanis, Phys. Lett. A 314 (2003) 44.