Hopf Bifurcation Analysis of Coupled Oscillators

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1 The Problem

Given an optoelectronic loop as shown in figure 1, described by parameters τ_{sf} (the self-feedback delay), τ_{c} (the coupling delay), γ_{sf} (the self-feedback strength), γ_{c} (the coupling strength), as well as the behavior of the band-pass filter F and nonlinear modulator, we want to predict whether the system will exhibit oscillation, or damp any oscillation over time.

This is obviously a very difficult problem to solve, with four free parameters and tight coupling between the loops. To simplify the problem, we will begin by setting $\gamma_{\text{c}} = 0$, so that the loops are uncoupled and can evolve independently.

2 Uncoupled Oscillator Analysis

Since the loops are identical, solving for the behavior of one is sufficient in the uncoupled case, since the second will behave identically. We can write down a pair of coupled differential equations that describe the uncoupled loop:

$$\begin{aligned} \dot{x} &= -x + y - \gamma_{11} f(x^{\tau_{11}}) \\ \dot{y} &= \epsilon x \end{aligned} \tag{1}$$

In this equation, x is proportional to the signal strength, and y proportional to the attenuation of the bandpass filter. ϵ described the bandpass filter, and is given by $\epsilon = \omega_0^2$, where ω_0 is the center of the passband. γ_{11} is a dimensionless quantity representing the strength of the self-coupling (though not equal to γ_{sf} as described above!). f is a function representing the behavior of the nonlinearity (in our case, a $\cos^2$ type relationship), and $x^{\tau_{11}}$ is shorthand for $x(t - \tau_{11})$.

Assume the solution to these differential equations is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} e^{\lambda t} \tag{2}$$

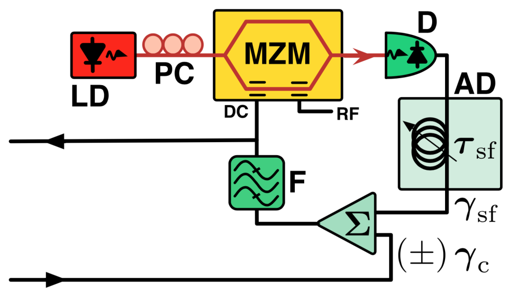

Figure 1: The optoelectronic loop whose behavior we are hoping to characterize

Plugging this solution in and linearizing the equation gives that

$$\begin{aligned}\lambda x_0 e^{\lambda t} &= -x_0 e^{\lambda t} + y_0 e^{\lambda t} - \gamma_{11} f'(0) x_0 (e^{\lambda t} e^{-\lambda \tau_{11}}) \\ \lambda y_0 &= \epsilon x_0\end{aligned}$$

Factor out $e^{\lambda t}$ from the first equation, and multiply both sides by λ to give

$$\lambda^2 x_0 = -\lambda x_0 + \lambda y_0 - \lambda \gamma_{11} f'(0) x_0 e^{-\lambda \tau_{11}} \tag{3}$$

$$\lambda y_0 = \epsilon x_0 \tag{4}$$

We can then substitute eq. 4 into eq. 3, giving

$$\begin{aligned}\lambda^2 x_0 &= -\lambda x_0 + \epsilon x_0 - \lambda \gamma_{11} f'(0) x_0 e^{-\lambda \tau_{11}} \\ \implies \lambda^2 &= -\lambda + \epsilon - \lambda \underbrace{\gamma_{11} f'(0)}_{=\gamma_{\mathrm{sf}}} e^{-\lambda \tau_{11}}\end{aligned}$$

$$\implies 0 = \lambda^2 + \lambda - \epsilon + \lambda \gamma_{\mathrm{sf}} e^{-\lambda \tau_{11}} \tag{5}$$

Note that we now have the (physically measurable) γ_{sf} parameter.

Recall that we are interested in finding the critical curves along which the system transitions from damping to growing behavior. If $\mathrm{Re}(\lambda) > 0$, we will see divergent behavior, while $\mathrm{Re}(\lambda) < 0$ gives damping behavior—therefore, at points along the critical curve, $\mathrm{Re}(\lambda) = 0$. We can enforce this by saying $\lambda = i\Omega$, for real Ω.

Plugging this definition into eq. 5, and separating the real and imaginary components yields two equations:

$$\begin{aligned}-\Omega^2 - \epsilon + \gamma_{\mathrm{sf}} \Omega \sin(\Omega \tau_{11}) &= 0 \\ \Omega + \gamma_{\mathrm{sf}} \Omega \cos(\Omega \tau_{11}) &= 0\end{aligned}$$

Manipulating the second equation gives

$$\begin{aligned}0 &= 1 + \gamma_{\mathrm{sf}} \cos(\Omega \tau_{11}) \\ \gamma_{\mathrm{sf}} &= -\frac{1}{\cos(\Omega \tau_{11})}\end{aligned}$$

so

$$0 = \Omega^2 + \epsilon + \Omega \frac{\sin(\Omega \tau_{11})}{\cos(\Omega \tau_{11})} = \Omega^2 + \epsilon + \Omega \tan(\Omega \tau_{11})$$

For a given τ, then, we can solve for all valid values of Ω, and use those to find the associated γ. (Typically, we draw the bifurcation plot with τ on the horizontal axis and γ on the vertical, but this is just convention.)

But wait! Note that each τ has an infinite number of γs for which there is a valid solution (and v.v.), due to the periodicity of the trigonometric functions—giving us an infinite number of critical curves, as shown in figure 2. How do we determine which curve is the physical critical curve, where our actual, physical oscillator transitions from damping to growth?

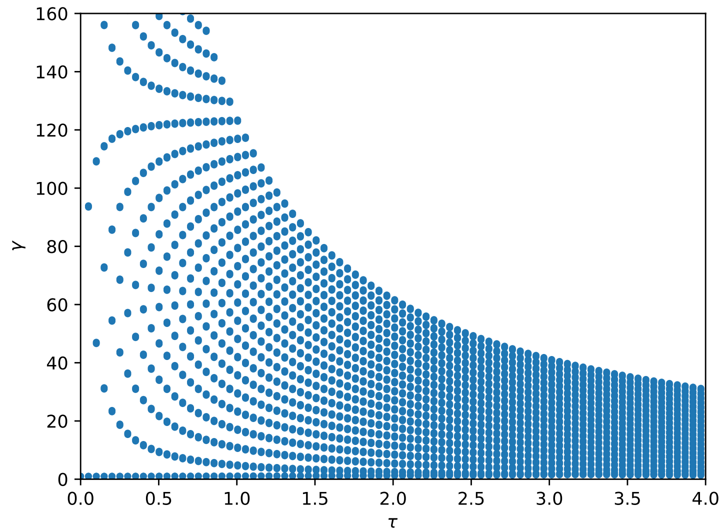

Figure 2: A subset of the (infinite) valid critical curves.

Well, we know that $(\tau, \gamma) = (0, 0)$ must exhibit damping, since (without any feedback) there can be no oscillation. We also don't care about solutions with negative τ, since we can't look into the future. Therefore, the damping region consists of every point that can be reached from the origin without crossing a critical curve, and the physical critical curve consists of the smallest positive curve at each γ.

If we do this, we get something resembling figure 3. Of course, that begs the question of how we *know* that we have the lowest critical curve at each point—after all, with infinite curves, we need some heuristic to figure out whether we have the right curve. However, that's a question about numerical analysis, covered in a later section.

3 Coupled Oscillator Analysis

The differential equations for the coupled case look very similar to those analyzed above—only now, there are four of them (two per oscillator), and there is an additional coupling term involved. With u taking the place of x above, and v taking the place of y (to avoid confusion), these equations can

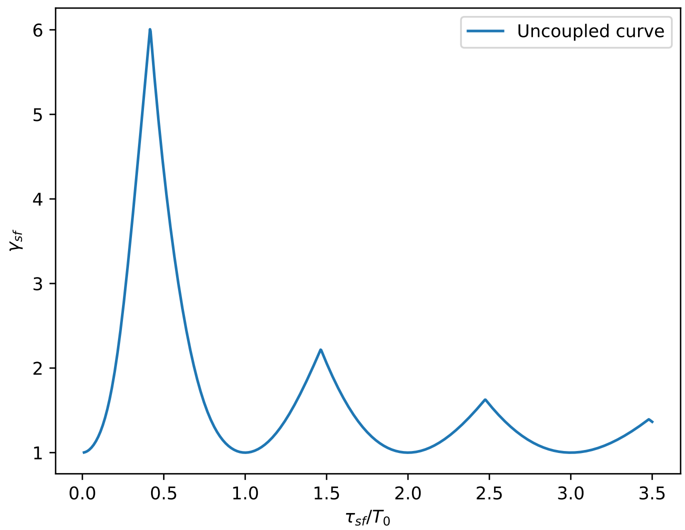

Figure 3: The physical curve, composed of the lowest critical curve at each point, stitched together

be written

$$\begin{aligned}
\dot{u}_1 &= -u_1 + v_1 - \gamma_{11} f(u_1^{\tau_{11}}) + \gamma_{12} f(u_2^{\tau_{12}}) \\
\dot{v}_1 &= \epsilon u_1 \\
\dot{u}_2 &= -u_2 + v_2 - \gamma_{22} f(u_1^{\tau_{11}}) - \gamma_{21} f(u_1^{\tau_{21}}) \\
\dot{v}_2 &= \epsilon u_2
\end{aligned}$$

Note the opposite signs on the coupling terms.

With this many variables floating around, writing the system as a matrix simplifies things. Let

$$\mathbf{x} = \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{pmatrix}$$

and again assume $\mathbf{x} = \mathbf{x_0} e^{\lambda t}$. Performing the same linear approximation as above, we get that the system can be written as

$$\lambda \mathbf{x}_0 = \begin{pmatrix} -1 - \gamma_{\text{sf}} e^{-\lambda \tau_{\text{sf}}} & 1 & \gamma_{\text{c}} e^{-\lambda \tau_{\text{c}}} & 0 \\ \epsilon & 0 & 0 & 0 \\ -\gamma_{\text{c}} e^{-\lambda \tau_{\text{c}}} & 0 & -1 - \gamma_{\text{sf}} e^{-\lambda \tau_{\text{sf}}} & 1 \\ 0 & 0 & \epsilon & 0 \end{pmatrix} \mathbf{x}_0 \equiv \mathbf{A} \mathbf{x}_0 \tag{6}$$

where $\gamma_{\text{sf}} = \gamma_{11} f'(0) = \gamma_{22} f'(0)$ and $\gamma_{\text{c}} = \gamma_{12} f'(0) = \gamma_{21} f'(0)$.

This is a transcendental eigenvalue problem, so let's try to solve it:

$$\begin{aligned}
0 &= \det(\mathbf{A} - \lambda \mathbf{I}) \\
&= \left(1 + \lambda + \gamma_{\text{sf}} e^{-\lambda \tau_{\text{sf}}}\right) \left(\lambda \left[\lambda \left(1 + \gamma_{\text{sf}} e^{-\lambda \tau_{\text{sf}}} + \lambda\right) - \epsilon\right]\right) - \epsilon \left[\lambda \left(1 + \gamma_{\text{sf}} e^{-\lambda \tau_{\text{sf}}} + \lambda\right) - \epsilon\right] + \gamma_{\text{c}}^2 \lambda^2 e^{-2\lambda \tau_{\text{c}}}
\end{aligned}$$

Typically, we would solve this characteristic equation for λ at this point. However, this isn't a nice, solvable polynomial like normal—the exponentials mean that there are infinite possible solutions. Instead, we can make assumptions about the value of λ, and use that to make inferences about the relationship between the other parameters.

Just as in the uncoupled case, we know that λ is purely imaginary on the boundary between damping and oscillatory behavior—that is, we can set $\lambda = i\Omega$ as before. Plugging that in, expanding everything out, and separating out the real and imaginary components, we get that

Re:

$$\begin{aligned}0 &= -2\Omega^2 - 2\Omega^2\gamma_{\mathrm{sf}}\cos(\Omega\tau_{\mathrm{sf}}) + 2\Omega\gamma_{\mathrm{sf}}\sin(\Omega\tau_{\mathrm{sf}}) + \Omega\gamma_{\mathrm{sf}}^2\sin(2\Omega\tau_{\mathrm{sf}})\\ &\quad + 2\Omega\epsilon\gamma_{\mathrm{sf}}\sin(\Omega\tau_{\mathrm{sf}}) - 2\Omega^2\epsilon + \epsilon^2 - \gamma_{\mathrm{c}}^2\Omega^2\cos(2\Omega\tau_{\mathrm{c}})\end{aligned}$$

Im:

$$\begin{aligned}0 &= 1 - 2\Omega\gamma_{\mathrm{sf}}\sin(\Omega\tau_{\mathrm{sf}}) + 2\gamma_{\mathrm{sf}}\cos(\Omega\tau_{\mathrm{sf}}) + \gamma_{\mathrm{sf}}^2\cos(2\Omega\tau_{\mathrm{sf}})\\ &\quad - \Omega^2 + 2\epsilon + 2\epsilon\gamma_{\mathrm{sf}}\cos(\Omega\tau_{\mathrm{sf}}) + \gamma_{\mathrm{c}}^2\Omega\sin(2\Omega\tau_{\mathrm{c}})\end{aligned}$$

If we combine these two equations to eliminate γ_{sf} and normalize, we get that, for points on the critical curve,

$$0 = \Omega^2\cos(\Omega\tau_{\mathrm{sf}}) + \Omega\left[1 - \gamma_{\mathrm{c}}\sin(\Omega\tau_{\mathrm{c}})\right]\sin(\Omega\tau_{\mathrm{sf}}) - \epsilon + \gamma_{\mathrm{c}}\Omega\cos(\Omega\tau_{\mathrm{c}})\cos(\Omega\tau_{\mathrm{sf}}) \tag{7}$$

Then, for a particular value of Ω, the associated γ_{sf} is

$$\gamma_{\mathrm{sf}} = \frac{1 - \gamma_{\mathrm{c}}\sin(\Omega\tau_{\mathrm{c}})}{\cos(\Omega\tau_{\mathrm{sf}})} \tag{8}$$

4 Numerical Analysis

We now have equations that can tell us, both in the uncoupled and coupled cases, whether a given Ω lies on the critical curve at some τ, and the γ associated with that Ω. However, as described at the end of section 2, we only care about the *lowest* critical point for each τ—and there is no easy way to determine whether a given critial point is actually the lowest.

4.1 The Naive Problem

At first, this problem seems relatively simple. We know the combined critical curve (fig. 3) must be continuous—otherwise, it would be nonphysical and lead to some strange behavior. Given a point γ_n on the curve at τ_n, we could carefully search the interval

$$[\gamma_n - \varepsilon, \gamma + \varepsilon]$$

for a valid solution at τ_{n+1}. If more than one valid solution is found in this interval, we (of course) prefer the smaller one—allowing us to detect the point where curves cross. With careful choice of τ spacing and ε, we should be able to follow the critical curve.

Unfortunately, things are never that simple. Given a gamma, there is *no way* to check whether it lies on a critical curve. As seen in equations 7 and 8, both τ and γ are parameterized by Ω, and neither of these equations has an inverse (at least in the coupled case). Determining whether (τ, γ) lies on a critical curve requires determining whether there is any Ω (which, again, could be any real number) that satisfies both equations 7 and 8.

4.2 Bounding Ω