Box Num. 33
Problem Set 32
April 18, 2018

1. (a) There are $2^N$ ways to arrange this lattice.

   (b) Define the fully demagnetized state to be when exactly half the spins are pointed up. The number of microstates that satisfy this criterion is

$$\binom{N}{N/2} = \frac{N!}{2(N/2)!} = \frac{N!}{((N/2)!)^2}$$

Since there are 2 ways for the bar to be fully magnetized, the entropy of this state is

$$S_m = k_B \log 2$$

and the entropy of the fully demagnetized state is

$$\begin{aligned} S_d &= k_B \log \frac{N!}{((N/2)!)^2} = k_B \left(\log N! - \log((N/2)!)^2\right) \\ &= k_B \left(\log N! - 2\log(N/2)!\right) \\ &\approx k_B \left((N \log N - N) - (N \log N/2 - N)\right) \\ &= k_B N(\log N - \log N/2) \\ &= k_B N \log 2N \end{aligned}$$

The difference in entropy is therefore

$$\Delta S = k_B(N \log 2N - \log 2)$$

2. (a) Starting with

$$\begin{aligned} \langle E \rangle &= -\frac{d}{d\beta} \log Z \\ &= -\frac{d}{dT}\frac{dT}{d\beta} \log(z) \\ &= \frac{d}{dT} \log ZkT^2 \\ &= 2 \log ZkT \end{aligned}$$

3. The Fermi energy for a particle in this gas is (by definition)

$$E_F = \frac{h^2}{2m}\left(\frac{3N}{8\pi V}\right)^{2/3}$$

The mean energy of such a gas is then given by

$$\begin{aligned} \langle E \rangle &= \frac{1}{N}\int_0^\infty EN(E)dE \\ &= \frac{3h^3}{16\pi V\sqrt{2}m^{3/2}E_F^{3/2}} \int_0^\infty EN(E)dE \end{aligned}$$

However, because the Fermi-Dirac distribution function is 1 below $E_F$ and 0 above,

$$= \frac{3h^3}{16\pi V\sqrt{2}m^{3/2}E_F^{3/2}} \frac{8\pi V\sqrt{2}m^{3/2}}{h^3} \int_0^{E_f} E^{3/2} = \frac{3}{5}E_F$$

4. (a) The total energy of the initial kaon is $E_k = m_k c^2$. Then the total energy of each of the pions must be $E_\pi = E_k/2 = m_k c^2/2$. If the rest mass of each pion is $m_\pi$, its rest energy is $m_\pi c^2$. Then

$$E^2 = (pc)^2 + (mc^2)^2 \implies (m_k c^2/2)^2 = (pc)^2 + (m_\pi c^2)^2 \implies |p| = \frac{1}{2}c\sqrt{m_k^2 - 4m_\pi^2}$$

and, since

$$\frac{v}{c} = \frac{pc}{E},$$

$$v = \frac{pc^2}{E} = \frac{\sqrt{m_k^2 - 4m_\pi^2}}{m_k}c$$

(b) The kaon has a mass of 497.6 MeV/c$^2$, and the pion has a mass of 139.6 MeV/c$^2$. Plugging these values into the equations above gives

$$\begin{aligned} E_\pi &= 248.8\,\mathrm{MeV} \\ p_\pi &= 205.9\,\mathrm{MeV/c} \\ v_\pi &= 0.827\,\mathrm{c} \end{aligned}$$