ID #33
Problem Set 13
Physics 202
February 21, 2018

1. (a) As showed last semester, the differential equation for this situation is

$$\ddot{x}(t) = \frac{k}{m}x(t)$$

with solution

$$x(t) = x_i \cos(\omega t) = x_i \cos\left(\sqrt{k/m t}\right)$$

Then, taking the derivative, the velocity is

$$v_x(t) = -\omega x_i \sin(\omega t)$$

(b) The probability that the mass can be found at a point x is inversely proportional to the velocity of the mass at that point, since a higher velocity means the mass spends less time at that point. The velocity of the mass at a given point can be found through energy, as follows:

$$E_{\text{tot}} = \frac{1}{2}kx_i^2$$

$$U(x) = \frac{1}{2}kx^2 \implies K(x) = E_{\text{tot}} - U(x) = \frac{1}{2}kx_i^2 - \frac{1}{2}kx^2 = \frac{1}{2}k(x_i^2 - x^2)$$

Then, solving for $v(x)$,

$$\frac{1}{2}mv(x)^2 = \frac{1}{2}k(x_i^2 - x^2) \implies v(x) = \sqrt{\frac{k}{m}(x_i^2 - x^2)}$$

Since the mass cannot be beyond the point x_i (there is not enough energy for it to reach that point),

$$P(x)dx = \begin{cases} \frac{A}{\sqrt{x_i^2 - x^2}}dx & -x_i \leq x \leq x_i \\ 0 & \text{otherwise} \end{cases}$$

(c) A probability distribution function must be normalized, so we need to find A s.t.

$$\int_{-\infty}^{\infty} \frac{A}{\sqrt{x_i^2 - x^2}}dx = \int_{-x_i}^{x_i} \frac{A}{\sqrt{x_i^2 - x^2}}dx = 1$$

Evaluating this integral gives that

$$\int_{-x_i}^{x_i} \frac{A}{\sqrt{x_i^2 - x^2}}dx = -A\arctan\left(\frac{x\sqrt{x_i^2 - x^2}}{x^2 - x_i^2}\right)\Bigg|_{x=-x_i}^{x_i} = A\pi \implies A = \frac{1}{\pi}$$

(d) Since the PDF is symmetric and centered at 0, the block's position has an expected value of $x = 0$.

$$\langle x \rangle = \int_{\infty}^{\infty} xP(x)dx = 0$$

(e) The probability density function for the displacement is given by

$$P_{\text{disp}}(x)dx = \begin{cases} \frac{2/\pi}{\sqrt{x_i^2 - x^2}}dx & 0 \leq x \leq x_i \\ 0 & \text{otherwise} \end{cases}$$

Then the expected value of the displacement is

$$\langle |x| \rangle = \int_0^{x_i} x\frac{2/\pi}{\sqrt{x_i^2 - x^2}}dx = -\frac{2\sqrt{x_i^2 - x^2}}{\pi}\Bigg|_{x=0}^{x_i} = \frac{2x_i}{\pi}$$

2. (a) Deriving the speed distribution for the 3d case is similar to deriving it for the 2d case. Start by imagining a 3D plot of the velocity components. For every gas particle, $v_x \approx v_y \approx v_z$. Then imagine a cylinder C in that space. Because $g(v_x)dv_x \propto dv_x$ (and similar for the other two directions), switching to cylindrical coordinates gives that

$$g(v)dv = \int_C v dv = \int_{\theta=0}^{2\pi} \int_{z=0}^{v} v\, dv\, dz\, d\theta = 2\pi v^2 dv$$

Then, as shown in class, the speed distribution function must be

$$P(v)dv = \left(\frac{m}{2\pi k_B T}\right)^{\frac{3}{2}} 2\pi v^2 e^{-\frac{mv^2}{2k_B T}} dv$$

(b) The average speed of an ideal gas particle is

$$\langle v \rangle = \int_{-\infty}^{\infty} vP(v)dv = \int_{0}^{\infty} v \left(\frac{m}{2\pi k_B T}\right)^{\frac{3}{2}} 2\pi v^2 e^{-\frac{mv^2}{2k_B T}} dv = \frac{\sqrt{\frac{2}{\pi}} T^2 k_B^2 \left(\frac{m}{Tk_B}\right)^{3/2}}{m^2}$$

(c) The distribution function of v^2 needs to be multiplied by the constant

$$\frac{8\sqrt{\pi} T^3 k_B^3 \left(\frac{m}{Tk_B}\right)^{5/2}}{3m^3}$$

to be normalized. Then

$$\langle v^2 \rangle = \int_{-\infty}^{\infty} \frac{8\sqrt{\pi} T^3 k_B^3 \left(\frac{m}{Tk_B}\right)^{5/2}}{3m^3} \left(\frac{\sqrt{\frac{2}{\pi}} T^2 k_B^2 \left(\frac{m}{Tk_B}\right)^{3/2}}{m^2}\right)^2 = \frac{4T^3 k_B^3 \left(\frac{m}{Tk_B}\right)^{5/2}}{3\sqrt{\pi} m^3}$$

and

$$\sqrt{\langle v^2 \rangle} = \frac{2}{\sqrt{3}\sqrt[4]{\pi}\sqrt[4]{\frac{m}{Tk_B}}}$$