

1. Since the amount of gas does not change, Nk_B is constant. Then the initial volume is given by

$$V_i = \frac{Nk_B T}{P_1} = \frac{Nk_B (626 \text{ K})}{1.7 \times 10^7 \text{ Pa}}$$

and the output volume is

$$V_o = \frac{Nk_B T}{P_2} = \frac{Nk_B (323 \text{ K})}{10\,000 \text{ Pa}}$$

The ratio of the volumes is therefore

$$\frac{V_o}{V_i} = \frac{\frac{Nk_B (626 \text{ K})}{1.7 \times 10^7 \text{ Pa}}}{\frac{Nk_B (323 \text{ K})}{10\,000 \text{ Pa}}} = 0.00114$$

The total amount of gas per second is $400 \text{ kg/s} \times 1000 \text{ g/kg} / 18.01 \text{ g/mol} = 22\,209 \text{ mol/s}$. Then

$$E_{th,i} = 3Nk_B T = 3(8.314 \text{ J/molK})(22\,209 \text{ mol/s})(626 \text{ K}) = 3.47 \times 10^8 \text{ J/s}$$

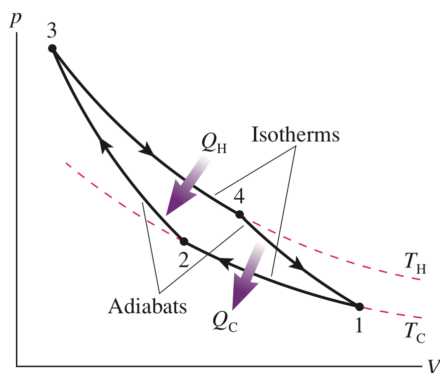
and

$$E_{th,f} = 3Nk_B T = 3(8.314 \text{ J/molK})(22\,209 \text{ mol/s})(323 \text{ K}) = 1.79 \times 10^8 \text{ J/s}$$

If all the thermal energy lost is converted into work, the plant outputs

$$3.47 \times 10^8 \text{ J/s} - 1.79 \times 10^8 \text{ J/s} = 1.79 \times 10^8 \text{ J/s}$$

2. Like with the heat engine example in class or in the textbook, the most efficient possible refrigerator uses the Carnot cycle. The work input is part of the two isothermal processes.



The cycle diagram for the most efficient refrigerator is like this one, but with the arrows reversed (diagram from the Knight textbook).

Since the thermal efficiency of any refrigerator is

$$K = \frac{Q_H}{Q_H - Q_C}$$

we can figure out the optimal case by calculating the heat transfer for the isothermal processes in the diagram above. Going from position 2 to 1, the heat transfer is

$$Q_C = |Nk_B T_C \log(V_1/V_2)|$$

The heat transfer from 4 to 3 is similarly

$$Q_H = |Nk_B T_H \log(V_4/V_3)|$$

so therefore

$$K = \frac{Nk_B T_H \log(V_4/V_3)}{Nk_B T_H \log(V_4/V_3) - Nk_B T_C \log(V_1/V_2)} = \frac{T_H \log(V_4/V_3)}{T_H \log(V_4/V_3) - T_C \log(V_1/V_2)}$$

However, for the adiabatic processes,

$$V_2 = V_3 \left(\frac{T_H}{T_C} \right)^{1/(\gamma-1)} \quad V_1 = V_4 \left(\frac{T_H}{T_C} \right)^{1/(\gamma-1)}$$

so that

$$\frac{V_1}{V_2} = \frac{V_4}{V_3}$$

and

$$K_{\max} = \frac{T_H}{T_H - T_C}$$

3. (a) This cycle can be used as a heat engine, but not as a refrigerator because a refrigerator requires adiabatic compression and expansion steps, and this cycle only has an adiabatic expansion.
- (b) The work output of one cycle is equal to the area inside the cycle. To calculate that, we first need functions for each of the three processes:

$$\begin{aligned} P_{1-2}(V) &= P_1 \\ P_{2-3}(V) &= \frac{C_1}{V^\gamma} = \frac{P_1 V_2^\gamma}{V^\gamma} \\ P_{3-1}(V) &= \frac{C_2}{V} = \frac{P_1 V_1}{V} \end{aligned}$$

The last component we need is the volume at point 3. Setting the adiabat and isotherm equal and solving for V gives

$$V_3 = \left(\frac{V_2^\gamma}{V_1} \right)^{1/(\gamma-1)}$$

Then the work is

$$\begin{aligned}
W &= \int_{V_1}^{V_2} P_{1-2}(V)dV + \int_{V_2}^{V_3} P_{2-3}(V)dV + \int_{V_3}^{V_1} P_{3-1}(V)dV \\
&= \int_{V_1}^{V_2} P_1 dV + \int_{V_2}^{\left(\frac{V_2^\gamma}{V_1}\right)^{1/(\gamma-1)}} \frac{P_1 V_2^\gamma}{V^\gamma} dV + \int_{\left(\frac{V_2^\gamma}{V_1}\right)^{1/(\gamma-1)}}^{V_1} \frac{P_1 V_1}{V} dV \\
&= \left(PV \Big|_{V=V_1}^{V_2} \right) + \left(\frac{P_1 V^{1-\lambda} V_2^\lambda}{1-\lambda} \Big|_{V=V_2}^{\left(\frac{V_2^\gamma}{V_1}\right)^{1/(\gamma-1)}} \right) + \left(P_1 V_1 \log(V) \Big|_{V=\left(\frac{V_2^\gamma}{V_1}\right)^{1/(\gamma-1)}}^{V_1} \right) \\
&= P_1 \left(-\frac{V_2^\lambda \left(\left(\frac{V_2^\lambda}{V_1} \right)^{\frac{1}{\lambda-1}} \right)^{1-\lambda}}{\lambda-1} - V_1 \log \left(\left(\frac{V_2^\lambda}{V_1} \right)^{\frac{1}{\lambda-1}} \right) + \frac{V_2}{\lambda-1} - V_1 + V_1 \log(V_1) + V_2 \right)
\end{aligned}$$

(c) The efficiency is given by

$$\eta = \frac{W}{Q_H} = \frac{P_1 \left(-\frac{V_2^\lambda \left(\left(\frac{V_2^\lambda}{V_1} \right)^{\frac{1}{\lambda-1}} \right)^{1-\lambda}}{\lambda-1} - V_1 \log \left(\left(\frac{V_2^\lambda}{V_1} \right)^{\frac{1}{\lambda-1}} \right) + \frac{V_2}{\lambda-1} - V_1 + V_1 \log(V_1) + V_2 \right)}{nC_P \log(V_2/V_1)}$$

4. (a) Rewriting van der Waals' equation as a function of V gives that

$$P(V) = \frac{abn - aV + bn^3 - n^2V + nRTV^2}{V^2(V - bn)}$$

The work done on a gas undergoing a volume change from V_1 to V_2 is

$$\int_{V_1}^{V_2} \frac{abn - aV + bn^3 - n^2V + nRTV^2}{V^2(V - bn)} dV = nRT \log((V_2 - bn)/(V_1 - bn)) + \frac{an^2(V_1 - V_2)}{V_1 V_2}$$

When $a = 0, b = 0$,

$$W = nRT \log \left(\frac{V_2}{V_1} \right)$$

(b) In the ideal gas case, the work done is

$$W_{\text{ideal}} = nRT \log(V_2/V_1) = (1 \text{ mol})(8.314 \text{ J/molK})(293 \text{ K}) \log(0.5) = 1688 \text{ J}$$

In the nonideal case, the work is

$$\begin{aligned}
W_{vdW} &= nRT \log((V_2 - bn)/(V_1 - bn)) + \frac{an^2(V_1 - V_2)}{V_1 V_2} \\
&= (1 \text{ mol})(8.314 \text{ J/molK})(293 \text{ K}) \log \left(\frac{0.0112 \text{ m}^3 - 23.71 \times 10^{-6} \text{ m}^3}{0.0224 \text{ m}^3 - 23.71 \times 10^{-6} \text{ m}^3} \right) \\
&\quad + \frac{((3.46 \times 10^{-3} \text{ Pam}^6)(0.0122 \text{ m}^3))}{0.000251 \text{ m}^6} = 1691 \text{ J}
\end{aligned}$$

Then

$$\Delta W = 3 \text{ J}, \quad \frac{\Delta W}{W_{\text{ideal}}} = 0.00178 = 0.178\%$$