Box Num. 33
Problem Set 21
March 21, 2018

1. The wavefunction for the ground state of the simple harmonic oscillator is the Gaussian, normalized by the constant A:
$$\psi(x) = Ae^{-(\sqrt{km}/2\hbar)x^2}$$
Since the Gaussian is symmetric, x_{av} is 0. The average squared distance, $(x_{av})^2$, is given by the standard deviation of the Gaussian, as described below. Therefore,
$$\sigma^2 = \frac{\sqrt{km}}{\hbar}$$
From section 4.4 of Krane, the positional uncertainty of a Gaussian distribution like this is
$$\Delta x = \sigma_x = \sqrt{(x^2)_{av} - (x_{av})^2}$$

2. The simple harmonic oscillator system has a potential energy function of $U = \frac{1}{2}kx^2$, so the Schrödinger equation for this system is (from Krane)
$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2}kx^2\psi = E\psi$$
Then, since we know ψ is of the form $\psi(x) = Axe^{-ax^2}$,
$$\frac{d\psi}{dx} = Ae^{-ax^2} - 2Aax^2e^{-ax^2} \text{ and } \frac{d^2\psi}{dx^2} = 4a^2Ae^{-ax^2}x^3 - 6aAe^{-ax^2}x = Ae^{-ax^2}\left(4a^2x^3 - 6ax\right)$$
Then subbing this derivative into the Schrödinger equation gives
$$\begin{aligned} &-\frac{\hbar^2}{2m}Ae^{-ax^2}\left(4a^2x^3 - 6ax\right) + \frac{1}{2}kx^2Axe^{-ax^2} = EAxe^{-ax^2} \\ \implies &-\frac{\hbar^2}{2m}\left(4a^2x^3 - 6ax\right) + \frac{1}{2}kx^3 = Ex \\ \implies &\frac{1}{2}kx^2 + \frac{3a\hbar^2}{m} - \frac{2a^2x^2\hbar^2}{m} = E \end{aligned}$$
The only way for an equation like this to be true is for the coefficients of x^2 to sum to 0, and for the constant coefficients to do the same. This gives that
$$\begin{aligned} &\frac{1}{2}k = \frac{2a^2\hbar^2}{m} \text{ and } E = \frac{3a\hbar^2}{m} \\ &a = \frac{\sqrt{km}}{2\hbar} \text{ and } E = \frac{3}{2}\hbar\omega_0 \end{aligned}$$

3. Since the two points are at the same temperature, and change in entropy is path-independent, this transformation is equivalent to an isotherm. We know that the change in entropy along this isotherm is
$$\Delta S = Nk_B \log\frac{V_B}{V_A}$$