ID #33
Problem Set 15
Physics 202
February 28, 2018

1. (a) Planck's radiation intensity equation gives that

$$\begin{aligned} I(\lambda) &= \frac{2\pi hc^2}{\lambda^5}\frac{1}{e^{hc/\lambda kT}-1} \\ &\approx \frac{2\pi hc^2}{\lambda^5}\frac{1}{(1+hc/\lambda kT)-1} \\ &= \frac{2\pi hc^2}{\lambda^5}\frac{1}{hc/\lambda kT} \\ &= \frac{2\pi hc^2\lambda kT}{\lambda^5 hc} \\ &= \frac{2\pi ckT}{\lambda^4} \end{aligned}$$

which is the equation from classical wave theory. Therefore, Planck's theory is consistent with the observed results at small frequencies.

(b) As the frequency increases ($\lambda \to 0$), in the classical model, the intensity approaches infinity, since

$$\lim_{\lambda\to 0}\frac{2\pi ckT}{\lambda^4} = \lim_{\lambda\to 0}\frac{1}{\lambda^4} = \infty$$

However, for Planck's theory, as $\lambda \to 0$, the exponential term $e^{hc/\lambda kT}-1$ grows to infinity (exponentially, surprisingly) while the polynomial term λ^4 goes to 0. Since exponential growth always dominates polynomial growth,

$$\lim_{\lambda\to 0}\frac{2\pi hc^2}{\lambda^5}\frac{1}{e^{hc/\lambda kT}-1} = \lim_{\lambda\to 0}\frac{1}{\infty} = 0$$

2. The derivative of $I(\lambda)$ is

$$\begin{aligned} \frac{dI(\lambda)}{d\lambda} &= \frac{2\pi hc^2}{\lambda^5}\frac{1}{e^{hc/\lambda kT}-1}\frac{d}{d\lambda} \\ &= \frac{2\pi hc^2}{\lambda^5}\left(\frac{1}{e^{hc/\lambda kT}-1}\frac{d}{d\lambda}\right) + \frac{1}{e^{hc/\lambda kT}-1}\left(\frac{2\pi hc^2}{\lambda^5}\frac{d}{d\lambda}\right) \\ &= \frac{2\pi c^3h^2e^{-\frac{ch}{k\lambda T}}}{k\lambda^7 T} - \frac{10\pi c^2he^{-\frac{ch}{k\lambda T}}}{\lambda^6} \end{aligned}$$

which gives a maximum at

$$\lambda = \frac{ch}{5kT} \implies \lambda T = \frac{ch}{5k} = 2.72\times 10^{-3}\,\mathrm{m\,K}$$

which is very close to the expected value of $2.89\times 10^{-3}\,\mathrm{m\,K}$.

3. Integrating equation 3.41 gives

$$\begin{aligned} \int_0^\infty I(\lambda) &= \int \frac{2\pi hc^2}{\lambda^5}\frac{1}{e^{hc/\lambda kT}-1}d\lambda \\ &= \frac{12\pi k^4T^4}{c^2h^3} \end{aligned}$$

4. From problem 2, we know the most intense wavelength is at

$$\lambda = \frac{2.89\times 10^{-3}\,\mathrm{m\,K}}{6000\,\mathrm{K}} = 4.817\times 10^{-7}\,\mathrm{m} = 482\,\mathrm{nm}$$

This is in the visible light range, near the color green. According to [1], the human eye is most sensitive at wavelengths near 555 nm, so this maximum intensity is slightly below the region of maximum sensitivity (our eyes are still very good at recognizing it, though.)

[1] wikipedia.org/wiki/Color_vision