Numerical Analysis of Amplitude Death in Directly-Coupled Optoelectronic Oscillators

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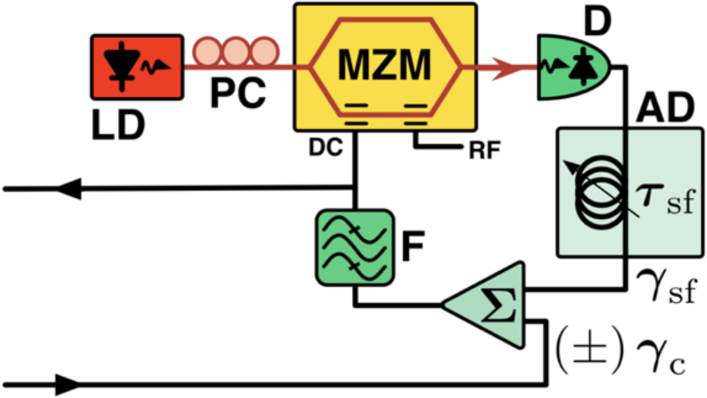
Amplitude death

- Many systems exhibit new behaviors when coupled to others
  - This is called **emergent behavior**
- A system with some kind of self-feedback often oscillates
  - Think of a mass on a spring
- If tuned right, an oscillating system will stop oscillating when coupled to another oscillator
- This is **amplitude death**
- Seems pretty easy to solve—just a pair of coupled ODEs
- Becomes more difficult when the oscillator isn't linear

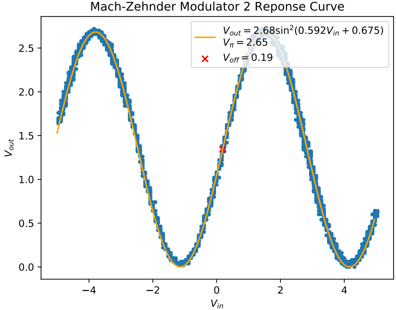
The Optoelectronic Oscillator



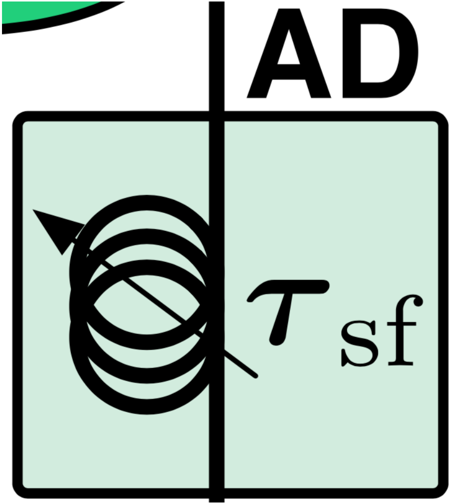
We want to show that this system can exhibit amplitude death

The Mach-Zehnder Modulator

- Modulates a light signal based on DC input
- Follows a $\cos^2$ relationship
- For best results, keep $f < 10\,\text{Hz}$
- To make analysis easier, we want to work in the linear regime (near $\times$)
    - Add an offset voltage!

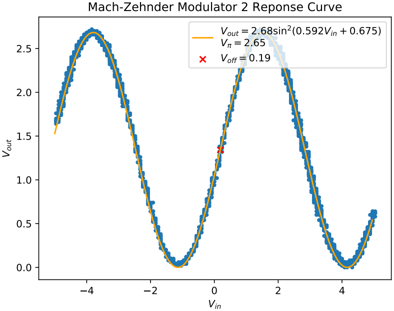


Audio Delay Module



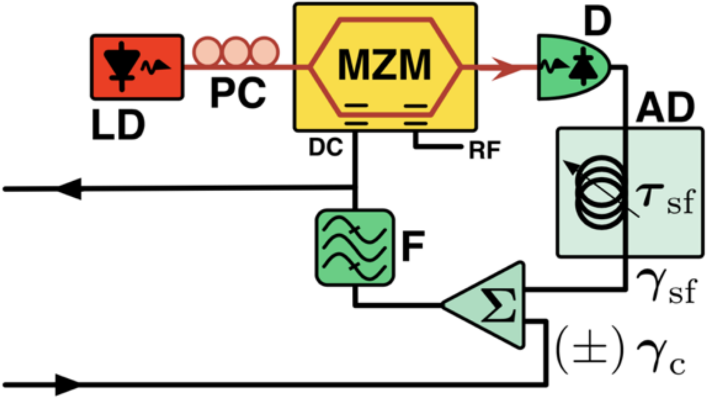
- Takes a signal and delays it by some programmable τ
- Two outputs, with separate τs
- Commercially available module for stadiums and concert halls
- Could be replaced with Alex's device

Why is this hard to analyze?



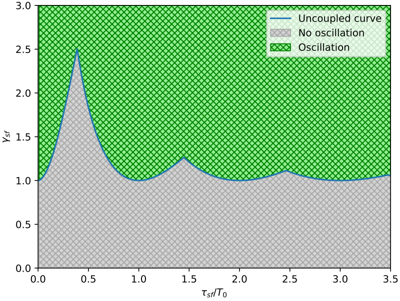
- Feedback follows a $\cos^2$ relationship—nonlinear!
- There is a time delay, turning our nice ODEs into DDEs
- The bandpass filter squelches most frequencies—how to account for this?

The Optoelectronic Oscillator



Hopf Bifurcation Analysis

- We want to determine parameters where the system oscillates, and where it doesn't
  - Both in the coupled and uncoupled cases
- Idea: Find the line where the system transitions between states
- This is a **Hopf Bifurcation Analysis**



Finding Critical Curves

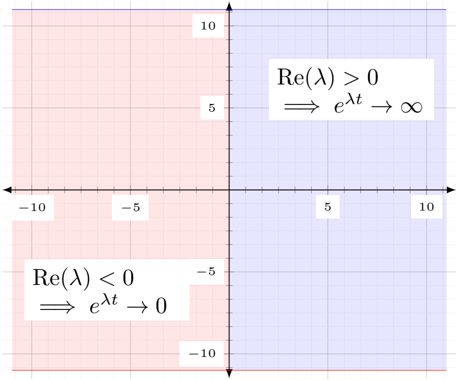
The uncoupled oscillator can be approximated by

$$\lambda \mathbf{x}_0 = \begin{pmatrix} -1 - \gamma_{\mathsf{sf}} e^{-\lambda \tau_{\mathsf{sf}}} & 1 \\ \epsilon & 0 \end{pmatrix} \mathbf{x}_0$$

(skipping the derivation here)
On the critical curve, $\lambda = i\Omega$.
What does this tell us about γ_{sf} and τ_{sf}?

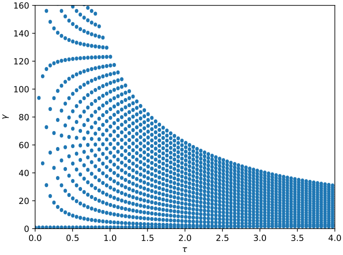
$$0 = \Omega^2 + \epsilon + \Omega \tan(\Omega \tau_{\mathsf{sf}})$$

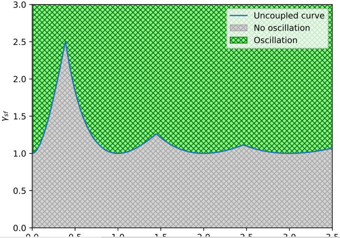
$$\gamma_{\mathsf{sf}} = -\frac{1}{\cos(\Omega \tau_{\mathsf{sf}})}$$



Critical Curves cont'd

- Because of the periodicity of tan and cos, we have infinite solutions
- No oscillations along $\gamma_{\mathsf{sf}} = 0$ (the x axis)
- The physical curve consists of the lowest solution at each τ.
- Stitch these together to get the physical curve





The Coupled Case

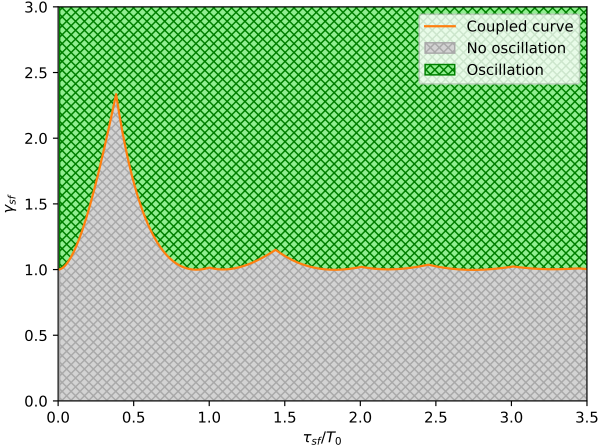
We have two oscillators now, each with two equations:

$$\lambda \mathbf{x}_0 = \begin{pmatrix} -1-\gamma_{\text{sf}}e^{-\lambda\tau_{\text{sf}}} & 1 & \gamma_{\text{c}}e^{-\lambda\tau_{\text{c}}} & 0 \\ \epsilon & 0 & 0 & 0 \\ -\gamma_{\text{c}}e^{-\lambda\tau_{\text{c}}} & 0 & -1-\gamma_{\text{sf}}e^{-\lambda\tau_{\text{sf}}} & 1 \\ 0 & 0 & \epsilon & 0 \end{pmatrix} \mathbf{x}_0 \equiv \mathbf{A}\mathbf{x}_0$$

which yields solutions

$$\begin{aligned} 0 &= \Omega^2 \cos(\Omega\tau_{\text{sf}}) + \Omega\left[1-\gamma_{\text{c}}\sin(\Omega\tau_{\text{c}})\right]\sin(\Omega\tau_{\text{sf}}) \\ &\quad -\epsilon + \gamma_{\text{c}}\Omega\cos(\Omega\tau_{\text{c}})\cos(\Omega\tau_{\text{sf}}) \\ \gamma_{\text{sf}} &= \frac{1-\gamma_{\text{c}}\sin(\Omega\tau_{\text{c}})}{\cos(\Omega\tau_{\text{sf}})} \end{aligned}$$

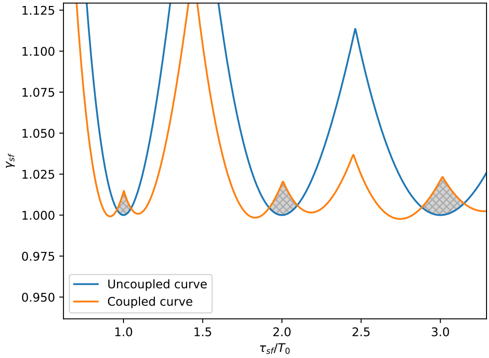
Coupled Critical Curve



Note the more complex shape of the curve

Putting the Two Together

- Overlay these critical curves
- Amplitude death requires us to be:
  - Below the coupled curve
  - Above the uncoupled curve
- The directly-coupled oscillator can exhibit amplitude death!



Acknowledgments

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