Abstract Algebra Chapter 2

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Important Statements:

Groups:

1. *Associativity:* $(ab)c = a(bc)\ \forall\ a, b, c \in G$
2. *Identity:*
3. *Inverses:*

Uniqueness of the Identity:

In a group G, there is only one identity element.

Cancellation:

In a group G, the right and left cancellation laws hold; that is,

$ba = ca \Rightarrow b = c$ and $ab = ac \Rightarrow b = c$

Uniqueness of Inverses:

For each element a in a group G, there is a unique element b in G such that $ab = ba = e$.

Socks-Shoes Principle:

For group elements a and b, $(ab)^{-1} = b^{-1}a^{-1}$.

End of Chapter Exercises

Question 1.

Give two reasons why the set of odd integers under addition is not a group.

1.1) If a and b are in such a group, then $a + b = 2n$ for some $n \in \mathbb{Z}$ which is not in the group. So the group lacks closure.

1.2) The identity element $\mathbf{0}$ is not in the group.

Question 2.

Referring to Example 13, verify the assertion that subtraction is not associative.

Let a, b, c be elements of G, then $-a, -b, -c$ are elements of G.

$$\begin{aligned}(a) - (b - c) &= (a - b) - (-c) \\ &= (a - b) + c \\ &\neq (a - b) - c \text{ in general}\end{aligned}$$

Question 3.

Show that $\{1, 2, 3\}$ under multiplication modulo 4 is not a group but that $\{1, 2, 3, 4\}$ under multiplication modulo 5 is a group.

The element 2 does not have an inverse modulo 4.

Question 4.

Show that the group $GL(2, \mathbb{R})$ of Example 9 is non-Abelian by exhibiting a pair of matrices A and B in $GL(2, \mathbb{R})$ such that $AB \neq BA$.

Question 5.

Find the inverse of the element a in $GL(2, \mathbb{Z}_{11})$.

Question 6.

Give an example of group elements a and b with the property that $a^{-1}ba \neq b$.

Question 7.

Translate each of the following multiplicative expressions into its additive counterpart. Assume that the operation is commutative.

(a) a^2b^3

(b) $a^{-2}(b^{-1}c)^2$

(c) $(ab^2)^{-3}c^2 = e$

Question 8.

Show that the set $\{5, 15, 25, 35\}$ is a group under multiplication modulo 40. What is the identity element of this group? Can you see any relationship between this group and $U(8)$?

Question 9.

Not Available

Question 10.

List the members of $H = \{x^2 \mid x \in D_4\}$ and $K = \{x \in D_4 \mid x^2 = e\}$.

Question 11.

Prove that the set of all 2×2 matrices with entries from $\mathbb{R}$ and determinant $+1$ is a group under matrix multiplication.

Question 12.

For any integer $n > 2$, show that there are at least two elements in $U(n)$ that satisfy $x^2 = 1$.

Question 13.

An abstract algebra teacher intended to give a typist a list of nine integers that form a group under multiplication modulo 91. Instead, one of the nine integers was inadvertently left out, so that the list appeares as $1, 9, 16, 22, 53, 74, 79, 81$. Which integer was left out? (This really happened!)

Question 14.

Let G be a group with the following property: Whenever a, b, and c belong to G and $ab = ca$, then $b = c$. Prove that G is Abelian. ("Cross cancellation" implies commutativity.)

Question 15.

(Law of Exponents for Abelian Groups) Let a and b be elements of an Abelian group and let n be any integer. Show that $(ab)^n = a^n b^n$. Is this also true for non-Abelian groups?

Question 16.

(Socks-Shoes Property) Draw an analogy between the statement $(ab)^{-1} = b^{-1}a^{-1}$ and the act of putting on and taking off your socks and shoes. Find an example that shows that in a group, it is possible to have $(ab)^{-2} \neq b^{-2}a^{-2}$. Find distinct nonidentity elements a and b from a non-Abelian group such that $(ab)^{-1} = a^{-1}b^{-1}$.

Question 17.

Prove that a group G is Abelian if and only if $(ab)^{-1} = a^{-1}b^{-1}$ for all a and b in G.

Question 18.

Prove that in a group, $(a^{-1})^{-1} = a$ for all a.

Question 19.

For any elements a and b from a group and any integer n, prove that $(a^{-1}ba)^n = a^{-1}b^n a$.

Question 20.

If $a_1, a_2, ..., a_n$ belong to a group, what is the inverse of $a_1 a_2 ... a_n$?

Question 21.

The integers 5 and 15 are among a collection of 12 integers that form a group under mutiplication modulo 56. List all 12.

Question 22.

Give an example of a group with 105 elements. Give two examples of groups with 44 elements.

Question 23.

Prove that every group table is a *Latin Square*; that is, each elemtn of the group appears exactly once in each row and each column.

Question 24.

Construct a Cayley table for $U(12)$.

Question 25.

Suppose the table below is a group table. Fill in the blank entries.

x	e	a	b	c	d
e	e	–	–	–	–
a	–	b	–	–	e
b	–	c	d	e	–
c	–	d	–	a	b
d	–	–	–	–	–

Question 26.

Prove that if $(ab)^2 = a^2b^2$ in a group G, then $ab = ba$.

Question 27.

Let a, b, and c be elements of a group. Solve the equation $axb = c$ for x. Solve $a^{-1}xa = c$ for x.

Question 28.

Prove that the set of all rational numbrs of the form 3^m6^n, where m and n are integers, is a group under mutiplication.

Question 29.

Let G be a finite group. Show that the number of elements x of G such that $x^3 = e$ is odd. Show that the number of elements x of G such that $x^2 \neq e$ is even.

Question 30.

Give an example of a group with elements a, b, c, d, and x such that

$axb = cxd$ but $ab \neq cd$. (Hence "middle cancellation" is not valid in groups.)

Question 31.

Let R be any rotation in some dihedral group and F any reflection in the same group. Prove that $RFR = F$.

Question 32.

Let R be any rotation in some dihedral group and F any reflection in the same group. Prove that $FRF = R^{-1}$ for all integers k.

I think there may be a typo in this question. (?)

Question 33.

Suppose that G is a group with the property that for every choice of elements in G, $axb = cxd$ implies $ab = cd$. Prove that G is Abelian. ("Middle cancellation" implies commutativity.)

Question 34.

In the dihedral group D_n, let $R = R_{360/n}$ and let F be any reflection. Write each of the following products in the form R^i or R^iF, where $0 \leq i < n$.

(a) In D_4, $FR^{-2}FR^5$

(b) In D_5, $R^{-3}FR^4FR^{-2}$

(c) In D_6, $FR^5FR^{-2}F$

Question 35.

Prove that if G is a group with the property that the square of every element is the identity, then G is Abelian.

Question 36.

Prove that the set of all 3×3 matrices with real entries of the form

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$$

is a group. (Multiplication is defined by

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & a' & b' \\ 0 & 1 & c' \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a+a' & b'+ac'+b \\ 0 & 1 & c+c' \\ 0 & 0 & 1 \end{bmatrix}$$

This group, sometimes called the *Heisenberg group* after the Nobel Prize-winning physicist Werner Heisenber, is intimately related to the Heisenberg Uncertainty Principle of quantum physics.)

Question 37.

Prove the assertion that the set $\{1, 2, ..., n-1\}$ is a group under multiplication modulo n if and only if n is prime.

Question 38.

In a finite group, show that the number of nonidentity elements that satisfy the equation $x^5 = e$ is a multiple of 4. If the stipluation that the group be finite is omitted, what can you say about the number of nonidentity elements that satisfy the equation $x^5 = e$?

Question 39.

Let $G = \left\{ \begin{bmatrix} a & a \\ a & a \end{bmatrix} | a \in \mathbb{R}, a \neq 0 \right\}$. Show that G is a group under matrix multiplication. Explain what each element of G has inverse even though the matrices have 0 determinant. (Compare with Example 10.)