Abstract Algebra Chapter 1

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End of Chapter Exercises

Question 1.

With words, describe each symmetry in D_3 (the set of symmetries of an equilateral triangle).

Let a, b, c be the points that form an equilateral triangle.

R_0: The identity element of the group; does nothing.

R_{120}: A clockwise rotation of 120^o about the center.

R_{240}: A clockwise rotation of 240^o about the center.

F_a: A reflection of the triangle keeping point a in place.

F_b: A reflection of the triangle keeping point b in place.

F_c: A reflection of the triangle keeping point c in place.

Question 2.

Write out a complete Cayley table for D_3.

D_3	R_0	R_{120}	R_{240}	F_a	F_b	F_c
R_0	R_0	R_{120}	R_{240}	F_a	F_b	F_c
R_{120}	R_{120}	R_{240}	R_0	F_c	F_a	F_b
R_{240}	R_{240}	R_0	R_{120}	F_b	F_c	F_a
F_a	F_a	F_b	F_c	R_0	R_{120}	R_{240}
F_b	F_b	F_c	F_a	R_{240}	R_0	R_{120}
F_c	F_c	F_a	F_b	R_{120}	R_{240}	R_0

Question 3.

Is D_3 Abelian?

No, D_3 is not Abelian (commutative).

An example of operations on D_3 which do not commute is F_a and F_b

Question 4.

Describe in words the elements of D_5 (symmetries of a regular pentagon).

Question 5.

For $n \geq 3$, describe the elements of D_n. *(Hint: Consider two cases - n even and n odd.)* How many elements does D_n have?

Note: differences are in the reflection of the n-gon. For n odd: each corner is assigned a reflection. For n even: half of the corners are assigned a reflection, and half of the sides are assigned a reflection.

$|D_n| = 2n$

Question 6.

In D_n, explain geometrically why a reflection followed by a reflection must be a rotation.

For the following questions concerning geometrical explanations of composing rotations and reflections, we interpret the object being manipulated as a 2-dimensional n-gon. In this fashion, we can assign a "front-side" and "back-side" to the object.

By performing two subsequent reflections. The resulting operation conserves the polarity (sidedness) of the object. Through this perspective, we can see that the resulting operation is not a reflection, and thus is a rotation (or the identity element).

Question 7.

In D_n, explain geometrically why a rotation followed by a rotation must be a rotation.

See explanation for Question 6.

Question 8.

In D_n, explain geometrically why a rotation and a reflection taken together in either order must be a reflection.

See explanation for Question 6.

Question 9.

Associate the number $+1$ with a rotation and the number -1 with a reflection. Describe an analogy between multiplying these two numbers and multiplying elements of D_n.

This is similar to the "sidedness" argument of Question 6 in which subsequent rotation operations are analagous to repetitive multiplications of -1. The resulting product of the associated numbers $(+1, -1)$ is a description of which side of the n-gon is facing the reader.

Question 10.

If r_1, r_2, and r_3 represent rotations from D_n and f_1, f_2, and f_3 represent reflections from D_n, determine whether $r_1 r_2 f_1 r_3 f_2 f_3 r_3$ is a rotation or a reflection.

Using the analogy from Question 9,

$$\begin{aligned} r_1 r_2 f_1 r_3 f_2 f_3 r_3 &\simeq (+1)(+1)(-1)(+1)(-1)(-1)(+1) \\ &\simeq (-1) \\ &\simeq \text{Reflection} \end{aligned}$$

Question 11.

Find elements A, B, and C in D_4 such that $AB = BC$ but $A \neq C$. (Thus, "cross cancellation" is not valid.)

Using the notation from this chapter:

$$HR_{90} = D' = R_{90}V$$
$$H \neq V$$

H: Horizontal reflection

V: Vertical reflection

R_{90}: 90^o rotation about the center

D': Diagonal reflection (SW&NE corners fixed)

Question 12.

Explain what the following diagram proves about the group D_n.

In general, D_n is not Abelian (commutative) because:

$$FR_{360/n} \neq R_{360/n}F$$

Question 13.

Describe the symmetries of a nonsquare rectangle. Construct the corresponding Cayley table.

The key difference between the symmetries of a nonsquare rectangle and D_4 is the absence of R_{90}, R_{270} and the two diagonal reflections D and D'.

x	R_0	R_{180}	H	V
R_0	R_0	R_{180}	H	V
R_{180}	R_{180}	R_0	V	H
H	H	V	R_0	R_{180}
V	V	H	R_{180}	R_0

Question 14.

Describe the symmetries of a parallelogram that is neither a rectangle nor a rhombus. Describe the symmetries of a rhombus that is not a rectangle.

The only symmetry of a nonrectangular-nonrhomboid parallelogram is $R_0 = R_{360}$.

By assigning each corner of the rhombus a corresponding edge of a nonsquare rectangle we can draw an analogy (isomorphism) between the symmetries of a rhombus and a nonsquare rectangle.

Question 15.

Describe the symmetries of a noncircular ellipse. Do the same for a hyperbola.

These are the same as the symmetries of the nonsquare rectangle.

Question 16.

Consider an infinitely long strip of equally spaced H's:

... H H H H ...

Describe the symmetries of this strip. Is the group of symmetries of the strip Abelian?

This is a cross of the group of integers under addition and the group of symmetries of a nonsquare rectangle.

Claim: If G_1 and G_2 are Abelian, then $G_1 \times G_2$ is also Abelian.

Question 17.

For each of the snowflakes in the figure, find the symmetry group and locate the axes of reflective symmetry.

Question 18.

Determine the symmetry group of the outer shell of the cross section of the human immunodeficiency virus (HIV).

Question 19.

Does an airplane propeller have a cyclic symmetry group or a dihedral symmetry group?

Question 20.

Bottle caps that are pried off typically have 22 ridges around the rim. Fine the symmetry group of such a cap.

Question 21.

What group theoretic property do upper-case letters F, G, J, K, L, P, Q, R have that is not shared by the remaining upper-case letters in the alphabet?

Question 22.

For each design below, determine the symmetry group.

Question 23.

What would the effect be if a six-bladed ceiling fan were designed so that the centerlines of two of the blades were at 70^o angle and all the other blades were set at 58^o angle?