

Abstract Algebra Chapter 2

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Important Statements:

Groups:

1. *Associativity:* $(ab)c = a(bc) \forall a, b, c \in G$
2. *Identity:*
3. *Inverses:*

Uniqueness of the Identity:

In a group G , there is only one identity element.

Cancellation:

In a group G , the right and left cancellation laws hold; that is,

$$ba = ca \Rightarrow b = c \text{ and } ab = ac \Rightarrow b = c$$

Uniqueness of Inverses:

For each element a in a group G , there is a unique element b in G such that $ab = ba = e$.

Socks-Shoes Principle:

For group elements a and b , $(ab)^{-1} = b^{-1}a^{-1}$.

End of Chapter Exercises

Question 1.

Give two reasons why the set of odd integers under addition is not a group.

Question 2.

Referring to Example 13, verify the assertion that subtraction is not associative.

Question 3.

Show that $\{1, 2, 3\}$ under multiplication modulo 4 is not a group but that $\{1, 2, 3, 4\}$ under multiplication modulo 5 is a group.

Question 4.

Show that the group $GL(2, \mathbb{R})$ of Example 9 is non-Abelian by exhibiting a pair of matrices A and B in $GL(2, \mathbb{R})$ such that $AB \neq BA$.

Question 5.

Find the inverse of the element a in $GL(2, \mathbb{Z}_{11})$.

Question 6.

Give an example of group elements a and b with the property that $a^{-1}ba \neq b$.

Question 7.

Translate each of the following multiplicative expressions into its additive counterpart. Assume that the operation is commutative.

(a) a^2b^3

(b) $a^{-2}(b^{-1}c)^2$

(c) $(ab^2)^{-3}c^2 = e$

Question 8.

Show that the set $\{5, 15, 25, 35\}$ is a group under multiplication modulo 40. What is the identity element of this group? Can you see any relationship between this group and $U(8)$?

Question 9.

Not Available

Question 10.

List the members of $H = \{x^2 \mid x \in D_4\}$ and $K = \{x \in D_4 \mid x^2 = e\}$.

Question 11.

Prove that the set of all 2×2 matrices with entries from \mathbb{R} and determinant $+1$ is a group under matrix multiplication.

Question 12.

For any integer $n > 2$, show that there are at least two elements in $U(n)$ that satisfy $x^2 = 1$.

Question 13.

An abstract algebra teacher intended to give a typist a list of nine integers that form a group under multiplication modulo 91. Instead, one of the nine integers was inadvertently left out, so that the list appears as 1, 9, 16, 22, 53, 74, 79, 81. Which integer was left out? (This really happened!)

Question 14.

Let G be a group with the following property: Whenever a, b , and c belong to G and $ab = ca$, then $b = c$. Prove that G is Abelian. ("Cross cancellation" implies commutativity.)

Question 15.

(Law of Exponents for Abelian Groups) Let a and b be elements of an Abelian group and let n be any integer. Show that $(ab)^n = a^n b^n$. Is this also true for non-Abelian groups?

Question 16.

(Socks-Shoes Property) Draw an analogy between the statement $(ab)^{-1} = b^{-1}a^{-1}$ and the act of putting on and taking off your socks and shoes. Find an example that shows that in a group, it is possible to have $(ab)^{-2} \neq b^{-2}a^{-2}$. Find distinct nonidentity elements a and b from a non-Abelian group such that $(ab)^{-1} = a^{-1}b^{-1}$.

Question 17.

Prove that a group G is Abelian if and only if $(ab)^{-1} = a^{-1}b^{-1}$ for all a and b in G .

Question 18.

Prove that in a group, $(a^{-1})^{-1} = a$ for all a .

Question 19.

For any elements a and b from a group and any integer n , prove that $(a^{-1}ba)^n = a^{-1}b^na$.

Question 20.

If a_1, a_2, \dots, a_n belong to a group, what is the inverse of $a_1a_2\dots a_n$?

Question 21.

The integers 5 and 15 are among a collection of 12 integers that form a group under multiplication modulo 56. List all 12.

Question 22.

Give an example of a group with 105 elements. Give two examples of groups with 44 elements.

Question 23.

Prove that every group table is a *Latin Square*; that is, each element of the group appears exactly once in each row and each column.

Question 24.

Construct a Cayley table for $U(12)$.

Question 25.

Suppose the table below is a group table. Fill in the blank entries.

x	e	a	b	c	d
e	e	—	—	—	—
a	—	b	—	—	e
b	—	c	d	e	—
c	—	d	—	a	b
d	—	—	—	—	—

Question 26.

Prove that if $(ab)^2 = a^2b^2$ in a group G , then $ab = ba$.

Question 27.

Let a, b , and c be elements of a group. Solve the equation $axb = c$ for x .
Solve $a^{-1}xa = c$ for x .

Question 28.

Prove that the set of all rational numbrs of the form 3^m6^n , where m and n are integers, is a group under mutiplication.

Question 29.

Let G be a finite group. Show that the number of elements x of G such that $x^3 = e$ is odd. Show that the number of elements x of G such that $x^2 \neq e$ is even.

Question 30.

Give an example of a group with elements a, b, c, d , and x such that $axb = cxd$ but $ab \neq cd$. (Hence "middle cancellation" is not valid in groups.)

Question 31.

Let R be any rotation in some dihedral group and F any reflection in the same group. Prove that $RFR = F$.

Question 32.

Let R be any rotation in some dihedral group and F any reflection in the

same group. Prove that $FRF = R^{-1}$ for all integers k .

I think there may be a typo in this question. (?)

Question 33.

Suppose that G is a group with the property that for every choice of elements in G , $axb = cxd$ implies $ab = cd$. Prove that G is Abelian. ("Middle cancellation" implies commutativity.)

Question 34.

In the dihedral group D_n , let $R = R_{360/n}$ and let F be any reflection. Write each of the following products in the form R^i or R^iF , where $0 \leq i < n$.

- (a) In D_4 , $FR^{-2}FR^5$
- (b) In D_5 , $R^{-3}FR^4FR^{-2}$
- (c) In D_6 , $FR^5FR^{-2}F$

Question 35.

Prove that if G is a group with the property that the square of every element is the identity, then G is Abelian.

Question 36.

Prove that the set of all 3×3 matrices with real entries of the form

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$$

is a group. (Multiplication is defined by

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & a' & b' \\ 0 & 1 & c' \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a + a' & b' + ac' + b \\ 0 & 1 & c + c' \\ 0 & 0 & 1 \end{bmatrix}$$

This group, sometimes called the *Heisenberg group* after the Nobel Prize-winning physicist Werner Heisenber, is intimately related to the Heisenberg Uncertainty Principle of quantum physics.)

Question 37.

Prove the assertion that the set $\{1, 2, \dots, n-1\}$ is a group under multiplication modulo n if and only if n is prime.

Question 38.

In a finite group, show that the number of nonidentity elements that satisfy the equation $x^5 = e$ is a multiple of 4. If the stipulation that the group be finite is omitted, what can you say about the number of nonidentity elements that satisfy the equation $x^5 = e$?

Question 39.

Let $G = \left\{ \begin{bmatrix} a & a \\ a & a \end{bmatrix} \mid a \in \mathbb{R}, a \neq 0 \right\}$. Show that G is a group under matrix multiplication. Explain what each element of G has inverse even though the matrices have 0 determinant. (Compare with Example 10.)