Linear Algebra Chapter 1

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End of Chapter Exercises

Section 1.1 Introduction

Question 1.

Determine whether the vectors emanating from the origin and terminating at the following pairs of points are parallel.

(a) (3,1,2) and (6,4,2)

(b) (-3,1,7) and (9,-3,-21)

(c) (5,-6,7) and (-5,6,-7)

(d) (2,0,-5) and (5,0,-2)

Question 2.

Find the equations of the lines through the following pairs of points in space.

(a) (3,-2,4) and (-5,7,1)

(b) (2,4,0) and (-3,-6,0)

(c) (3,7,2) and (3,7,-8)

(d) (-2,-1,5) and (3,9,7)

Question 3.

Find the equations of the planes containing the following points in space.

(a) (2,-5,-1), (0,4,6) and (-3,7,1)

(b) (3,-6,7), (-2,0,-4) and (5,-9,-2)

(c) (-8,2,0), (1,3,0) and (6,-5,0)

(d) (1,1,1), (5,5,5) and (-6,4,2)

Question 4.

What are the coordinates of the vector 0 in the Euclean plane that satisfies Property 3? (There exists a vector denoted 0 such that $x + 0 = x$ for each vector x.)

Question 5.

Prove that if the vector x emanates from the origin of the Euclidean plane and terminates at the point with coordinates (a_1, a_2), then the vector tx that emanates from the origin terminates at the point with coordinates (ta_1, ta_2).

Question 6.

Show that the midpoint of the line segment joining the points (a, b) and (c, d) is $((a + c)/2, (b + d)/2)$.

Question 7.

Prove that the diagonals of a parallelogram bisect each other.

Section 1.2 Vector Spaces

Question 1.

Label the following statements as True or False.

(a) Every vector space contains a zero vector.

(b) A vector space may have more than one zero vectos.

(c) In any vector space, $ax = bx$ implies that $a = b$.

(d) In any vector space, $ax = ay$ implies that $x = y$.

(e) A vector in $\mathbb{F}^n$ may be regarded as a matrix in $M_{n\times 1}(F)$.

(f) An $m \times n$ matrix has m columns and n rows.

(g) In $\mathbf{P}(F)$, only polynomials of the same degree may be added.

(h) If f and g are polynomicals of degree n, then $f + g$ is a polynomial of degree n.

(i) If f is a polynomial of degree n and c is a nonzero scalar, then cf is a polynomial of degree n.

(j) A nonzero scalar of F may be considered to be a polynomial in $\mathbf{P}(F)$ having degree zero.

(k) Two functions in $\mathcal{F}(S, F)$ are equal if and only if they have the same value at each element of S.

Question 2.

Write the zero vector of $M_{3\times 4}(F)$.

Question 3.

If

$$M = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

what are $M_{1,3}$, $M_{2,1}$, and $M_{2,2}$?

Question 4.

Perform the indicated operations.

(a) $\begin{pmatrix} 2 & 5 & -3 \\ 1 & 0 & 7 \end{pmatrix} + \begin{pmatrix} 4 & -2 & 5 \\ -5 & 3 & 2 \end{pmatrix}$

(b) $\begin{pmatrix} -6 & 4 \\ 3 & -2 \\ 1 & 8 \end{pmatrix} + \begin{pmatrix} 7 & -5 \\ 0 & -3 \\ 2 & 0 \end{pmatrix}$

(c) $4 \begin{pmatrix} 2 & 5 & -3 \\ 1 & 0 & 7 \end{pmatrix}$

(d) $-5\begin{pmatrix} -6 & 4 \\ 3 & -2 \\ 1 & 8 \end{pmatrix}$

(e) $(2x^4 - 7x^3 + 4x + 3) + (8x^3 + 2x^2 - 6x + 7)$

(f) $(-3x^3 + 7x^2 + 8x - 6) + (2x^3 - 8x + 10)$

(g) $5(2x^7 - 6x^4 + 8x^2 - 3x)$

(h) $3(x^5 - 2x^3 + 4x + 2)$

Question 5.

Question 6.

Question 7.

Let $S = \{0, 1\}$ and $\mathbb{F} = \mathbb{R}$. In $\mathcal{F}(S, R)$, show that $f = g$ and $f + g = h$, where $f(t) = 2t + 1$, $g(t) = 1 + 4t - 2t^2$, and $h(t) = 5^t + 1$.

Question 8.

In any vector space $\mathbf{V}$, show that $(a + b)(x + y) = ax + ay + bx + by$ for any $x, y \in \mathbf{V}$ and any $a, b \in \mathbb{F}$.

Question 9.

Prove Corollaries 1 and 2 of Theorem 1.1 and Theorem 1.2(c).

Question 10.

Let $\mathbf{V}$ denote the set of all differentiable real-valued functions defined on the real line. Prove that $\mathbf{V}$ is a vector space with the operations of addition and scalar multiplication defined in Example 3.

Question 11.

Let $\mathbf{V} = \{\mathbf{0}\}$ consist of a single vector $\mathbf{0}$ and define $\mathbf{0} + \mathbf{0} = \mathbf{0}$ and $c\mathbf{0} = \mathbf{0}$

for each scalar c in $\mathbb{F}$. Prove that $\mathbf{V}$ is a vector space over $\mathbb{F}$. ($\mathbf{V}$ is called the **zero vector space**.)

Question 12.

A real-valued function f defined on the real line is called an **even function** if $f(-t) = f(t)$ for each real number t. Prove that the set of even functions defined on the real line with the operations of addition and scalar multiplication defined in Example 3 is a vector space.

Question 13.

Let $\mathbf{V}$ denote the set of ordered pairs of real numbers. If (a_1, a_2) and (b_1, b_2) are elements of $\mathbf{V}$ and $c \in \mathbb{R}$, define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 b_2) \text{ and } c(a_1, a_2) = (ca_1, a_2)$$

Is $\mathbf{V}$ a vector space over $\mathbb{R}$ with these operations? Justify your answer.

Question 14.

Let $\mathbf{V} = \{(a_1, a_2, ..., a_n) \mid a_i \in \mathbb{C} \text{ for } i = 1, 2, ..., n\}$; so $\mathbf{V}$ is a vector space over $\mathbb{R}$ by Example 1. If $\mathbf{V}$ a vector space over the field of complex numbers with the operations of coordinatewise addition and multiplication?

Question 15.

Let $\mathbf{V} = \{(a_1, a_2, ..., a_n) \mid a_i \in \mathbb{R} \text{ for } i = 1, 2, ..., n\}$; so $\mathbf{V}$ is a vector space over $\mathbb{R}$ by Example 1. Is $\mathbf{V}$ a vecotr space over the field of complex numbers with the operations of coordinatewise addition and multiplication?

Question 16.

Let $\mathbf{V}$ denote the set of all $m \times n$ matrices with real entries; so $\mathbf{V}$ is a vector space over $\mathbb{R}$ by Example 2. Let $\mathbb{Q}$ be the field of rational numbers. Is $\mathbf{V}$ a vector space over $\mathbb{F}$ with the usual definitions of matrix addition and scalar multiplication?

Question 17.

Let $\mathbf{V} = \{(a_1, a_2) \mid a_1, a_2 \in \mathbb{F}\}$, where $\mathbb{F}$ is a field. Define addition of elements of $\mathbf{V}$ coordinatewise and for $c \in \mathbb{F}$ and $(a_1, a_2) \in \mathbf{V}$, define

$$c(a_1, a_2) = (a_1, 0).$$

Is $\mathbf{V}$ a vector space over $\mathbb{F}$ with these operations? Justify your answer.

Question 18.

Let $\mathbf{V} = \{(a_1, a_2) \mid a_1, a_2 \in \mathbb{R}\}$. For $(a_1, a_2), (b_1, b_2) \in \mathbf{V}$ and $c \in \mathbb{R}$, define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + 2b_1, a_2 + 3b_2) \text{ and } c(a_1, a_2) = (ca_1, ca_2).$$

Is $\mathbf{V}$ a vector space over $\mathbb{R}$ with these operations?

Question 19.

Let $\mathbf{V} = \{(a_1, a_2) \mid a_1, a_2 \in \mathbb{R}\}$. Define addition of elements of $\mathbf{V}$ coordinatewise, and for (a_1, a_2) in $\mathbf{V}$ and $c \in \mathbb{R}$, define

$$c(a_1, a_2) = \begin{cases} (0, 0) & \text{if } c = 0 \\ \left(ca_1, \frac{a_2}{c}\right) & \text{if } c \neq 0 \end{cases}$$

Is $\mathbf{V}$ a vector space over $\mathbb{R}$ with these operations?

Question 20.

Let $\mathbf{V}$ be the set of sequences $\{a_n\}$ of real numbers. For $\{a_n\}, \{b_n\} \in \mathbf{V}$ and any real number t, define

$$\{a_n\} + \{b_n\} = \{a_n + b + n\} \text{ and } t\{a_n\} = \{ta_n\}$$

Prove that, with these operations, $\mathbf{V}$ is a vector space over $\mathbb{R}$.

Question 21.

Let $\mathbf{V}$ and $\mathbf{W}$ be vector spaces over a field $\mathbb{F}$. Let

$$\mathbf{Z} = \{(v, w) \mid v \in \mathbf{V} \text{ and } w \in \mathbf{W}\}.$$

Prove that $\mathbf{Z}$ is a vector space over $\mathbb{F}$ with the operations

$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2) \text{ and } c(v_1, w_1) = (cv_1, cw_1).$$

Question 22.

How many matrices are there in the vector space $M_{m\times n}(\mathbb{Z}_2)$?

Section 1.3 Subspaces

Question 1.

Question 2.

Question 3.

Question 4.

Question 5.

Section 1.4 Linear Combinations and Systems of Linear Equations

Question 1.

Question 2.

Question 3.

Question 4.

Question 5.

Section 1.5 Linear Dependence and Linear Independence

Question 1.

Question 2.

Question 3.

Question 4.

Question 5.

Section 1.6 Bases and Dimension

Question 1.

Question 2.

Question 3.

Question 4.

Question 5.

Section 1.7 Maximal Linearly Independent Subsets

Question 1.

Question 2.

Question 3.

Question 4.

Question 5.

Question 6.

Question 7.

Question 8.

Question 9.

Question 10.