MAT 473: Intermediate Real Analysis II

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Problem 1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} \frac{|x_1|^a |x_2|^b}{\|x\|^c}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0, \end{cases}$$

where a, b, and c are positive real numbers. Prove that $\lim_{x\to 0} f(x)$ exists if and only if $a + b > c$.

Proof. $(\Longrightarrow)$: Suppose $a + b > c$. Then clearly $a + b - c > 0$. Then for all $x \in \mathbb{R}^2$, we have

$$\begin{aligned} \left| \frac{|x_1|^a |x_2|^b}{\|x\|^c} - 0 \right| &= \frac{|x_1|^a |x_2|^b}{\|x\|^c} \\ &\leq \frac{\|x\|^a \|x\|^b}{\|x\|^c} && \text{(By the fundamental inequalities)} \\ &= \|x\|^{a+b-c}. \end{aligned}$$

Since $a + b - c > 0$, $\|x\|^{a+b-c}$ converges to 0 as x converges to 0, and thus by the squeeze theorem, $|f(x) - 0| = \left| \frac{|x_1|^a |x_2|^b}{\|x\|^c} - 0 \right|$ converges to 0, and thus $\lim_{x\to 0} f(x) = 0$.

$(\Longleftarrow)$: Proof by contrapositive. We'll show that if $a + b \leq c$, then $\lim_{x\to 0} f(x)$ does not exist.

Case 1: Suppose $a + b < c$. Consider $Z = \{(t, t) : t \in \mathbb{R}^+\}$. Then,

$$\begin{aligned} \lim_{x\to 0} f\big|_Z(x) &= \lim_{t\to 0^+} \frac{|t|^a |t|^b}{\sqrt{t^2 + t^2}^c} \\ &= \lim_{t\to 0^+} \frac{t^a + b}{(\sqrt{2}t)^c} \\ &= \lim_{t\to 0^+} \frac{1}{(\sqrt{2}t)^{c-a-b}} \\ &= \infty. && \text{(Since } c - a - b > 0) \end{aligned}$$

Thus, $\lim_{x\to 0} f\big|_Z(x)$ does not exist, which implies $\lim_{x\to 0} f(x)$ does not exist.

Case 2: Suppose $a + b = c$. Consider $Z_1 = \{(t, 0) : t \in \mathbb{R}^+\}$. Then we have,

$$\begin{aligned} \lim_{x\to 0} f\big|_{Z_1}(x) &= \lim_{t\to 0^+} \frac{|t|^a |0|^b}{\sqrt{t^2 + 0}^c} \\ &= \lim_{t\to 0^+} \frac{0}{t^c} \\ &= 0. \end{aligned}$$

Now consider $Z_2 = \{(t,t) : t \in \mathbb{R}^+\}$. Then,

$$\begin{aligned}
\lim_{x\to 0} f\big|_{Z_2}(x) &= \lim_{t\to 0^+} \frac{|t|^a\,|t|^b}{\sqrt{t^2+0}^c} \\
&= \lim_{t\to 0^+} \frac{t^{a+b}}{\sqrt{2}^c t^c} \\
&= \lim_{t\to 0^+} \frac{t^{a+b-c}}{\sqrt{2}^c} \\
&= \lim_{t\to 0^+} \frac{t^0}{\sqrt{2}^c} && \text{(Since } a+b=c\text{)} \\
&= \frac{1}{\sqrt{2}^c}.
\end{aligned}$$

Thus, $\lim_{x\to 0} f\big|_{Z_1}(x) = 0 \neq \frac{1}{\sqrt{2}^c} = \lim_{x\to 0} f\big|_{Z_2}(x)$. Thus, $\lim_{x\to 0} f(x)$ does not exist. Therefore, in all cases, $\lim_{x\to 0} f(x)$ does not exist. □

Problem 2. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} \frac{x_1 x_2^3}{x_1^2 + x_2^6}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

Prove that $\lim_{x\to 0} f(x)$ does not exist.

Proof. Suppose $f : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$. Let $Z_1 = \{(u,0)^t : u \in \mathbb{R}\setminus\{0\}\}$ and $Z_2 = \{(u^3,u)^T : u \in \mathbb{R}\setminus\{0\}\}$. We have

$$\begin{aligned}
f(u,0) &= \frac{u \cdot 0^3}{u^2 + 0^6} = \frac{0}{u^2} = 0 \\
f(u^3,u) &= \frac{u^3 \cdot u^3}{u^6 + u^6} = \frac{u^6}{2u^6} = \frac{1}{2}.
\end{aligned}$$

Thus, $\lim_{x\to 0} f\big|_{Z_1}(x) = 0 \neq \frac{1}{2} = \lim_{x\to 0} f\big|_{Z_2}(x)$. Therefore, $\lim_{x\to 0} f(x)$ does not exist. □

Problem 3. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a linear function.

(a) Prove that $\frac{f(x)}{\|x\|}$ is a bounded function of x on $\mathbb{R}^n \setminus \{0\}$. (Hint: if f is represented by a matrix, then $f(x)$ equals a linear combination of the columns of that matrix.)

Proof. Define $g : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^m$ by $g(x) = \frac{f(x)}{\|x\|}$ for all $x \in \mathbb{R}^n \setminus \{0\}$. Let $x \in \mathbb{R}^n \setminus \{0\}$. Then,

$$\begin{aligned}
\left\| \frac{\sum_{i=1}^m \left(\sum_{j=1}^n f_i(e_j^{(n)}) x_j \right) e_i^{(m)}}{\|x\|} \right\| &\leq \sum_{i=1}^m \left| \frac{\left(\sum_{j=1}^n f_i(e_j^{(n)}) x_j \right)}{\|x\|} \right| \\
&\leq \frac{\sum_{i=1}^m \sum_{j=1}^n \left| f_i(e_j^{(n)}) \right| |x_j|}{\|x\|} \\
&\leq \frac{\sum_{i=1}^m \sum_{j=1}^n \left| f_i(e_j^{(n)}) \right| \|x\|}{\|x\|} \\
&= \sum_{i=1}^m \sum_{j=1}^n \left| f_i(e_j^{(n)}) \right| < \infty.
\end{aligned}$$

Thus, $\frac{f(x)}{\|x\|}$ is bounded. □

(b) Suppose that f is not the zero map. Prove that $\lim_{x\to 0} \frac{f(x)}{\|x\|}$ does not exist. (Hint: if $f(v) \neq 0$ consider $x = tv$ for $t \in \mathbb{R} \setminus \{0\}$.)

Proof. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a linear function that is not the zero map. Fix $v \in \mathbb{R}^n$ such that $f(v) \neq 0$. Let $Z_1 = \{kv : k \in \mathbb{R}^+\}$ and $Z_2 = \{kv : k \in \mathbb{R}^-\}$. Then,

$$\begin{aligned}
\lim_{x\to 0} \frac{f\big|_{Z_1}(x)}{\|x\|} &= \lim_{t\to 0^+} \frac{f\big|_{Z_1}(tv)}{\|tv\|} \\
&= \lim_{t\to 0^+} \frac{tf\big|_{Z_1}(v)}{|t|\,\|v\|} \\
&= \frac{f\big|_{Z_1}(v)}{\|v\|} \cdot \lim_{t\to 0^+} \frac{t}{|t|} \\
&= \frac{f\big|_{Z_1}(v)}{\|v\|} \\
&> 0
\end{aligned}$$

and

$$\begin{aligned}
\lim_{x\to 0}\frac{f\big|_{Z_2}(x)}{\|x\|} &= \lim_{t\to 0^-}\frac{f\big|_{Z_2}(tv)}{\|tv\|}\\
&= \lim_{t\to 0^-}\frac{tf\big|_{Z_2}(v)}{|t|\,\|v\|}\\
&= \frac{f\big|_{Z_2}(v)}{\|v\|}\cdot\lim_{t\to 0^-}\frac{t}{|t|}\\
&= \frac{-f\big|_{Z_2}(v)}{\|v\|}\\
&< 0.
\end{aligned}$$

Thus, the limit does not exist. □

Problem 4. Let V and W be normed vector spaces. Recall that $B(V,W)=\{T\in L(V,W) : \sup_{\|x\|\le 1}\|Tx\|<\infty\}$.

(a) Prove that $B(V,W)$ is a vector space.

Proof. Let V and W be normed vector spaces. First, we'll show that $L(V,W)$ is a vector space. Suppose $T_1,T_2,T_3\in L(V,W)$ and $\alpha,\beta\in\mathbb{K}$ and $x\in V$. Note, $L(V,W)$ is clearly closed from Definition 2.1. This leaves the following properties.

(1) Commutativity:

$$\begin{aligned}
(T_1 +_L T_2)(x) &= T_1x +_W T_2x && \text{(Addition on } L(V,W))\\
&= T_2x +_W T_1x && \text{(Commutativity of addition in } W)\\
&= (T_2 +_L T_1)(x). && \text{(Addition on } L(V,W))
\end{aligned}$$

(2) Associativity:

$$\begin{aligned}
(T_1 +_L (T_2 +_L T_3))(x) &= T_1x +_W (T_2 +_L T_3)(x) && \text{(Addition on } L(V,W))\\
&= T_1x +_W (T_2x +_W T_3x) && \text{(Addition on } L(V,W))\\
&= (T_1x +_W T_2x) +_W T_3x && \text{(Associativity in } W)\\
&= (T_1 +_L T_2)(x) +_W T_3x && \text{(Addition on } L(V,W))\\
&= ((T_1 +_L T_2) +_L T_3)(x). && \text{(Addition on } L(V,W))
\end{aligned}$$

(3) Zero: Let $0_L\in L(V,W)$ be the zero map.

$$\begin{aligned}
(T_1 +_L 0_L)(x) &= T_1x +_W 0_Lx && \text{(Addition on } L(V,W))\\
&= T_1x +_W 0_W && \text{(Definition of zero map)}\\
&= T_1x. && \text{(Addition of } 0_W)
\end{aligned}$$

(4) Additive inverse: Define $-T_1 : V \to W$ by $-T_1(v) = T_1(-v)$ for all $v \in V$. Then,

$$\begin{aligned}
(T_1 +_L -T_1)(x) &= T_1 x +_W -T_1 x \\
&= -T_1 x +_W T_1 - x \\
&= T_1(x +_V (-x)) && \text{(Linearity of } T_1\text{)} \\
&= T_1 \cdot 0_V && \text{(Addititive inverse in } V\text{)} \\
&= 0_W. && \text{(Linearity of } T_1\text{)}
\end{aligned}$$

Since $x \in V$ was arbitrary, this is true for all $x \in V$. Thus, $T_1 +_L -T_1 = 0_L$.

(5) Multiplication over $\mathbb{K}$:

$$\begin{aligned}
\alpha \cdot (\beta \cdot T_1(x)) &= \alpha \cdot (\beta T_1 x) && \text{(Definition of } T_1\text{)} \\
&= \alpha\beta(T_1 x) && \text{(Multiplication over } \mathbb{K} \text{ in } W\text{)} \\
&= ((\alpha\beta) \cdot T_1)(x). && \text{(Definition of } T_1\text{)}
\end{aligned}$$

(6) Unit of scalar multiplication:

$$\begin{aligned}
(1 \cdot T_1)(x) &= 1 \cdot T_1(x) && \text{(Linearity of scalar multiplication)} \\
&= T(x). && \text{(Definition of 1)}
\end{aligned}$$

(7) Distribution of scalar multiples:

$$\begin{aligned}
(\alpha(T_1 +_L T_2))(x) &= \alpha((T_1 +_L T_2)(x)) && \text{(Linearity of scalar multiplication)} \\
&= \alpha(T_1 x +_W T_2 x) && \text{(Addition on } L(V,W)\text{)} \\
&= \alpha T_1 x +_W \alpha T_2 x && \text{(Distribution of scalar multiples on } W\text{)} \\
(\alpha T_1 +_L \alpha T_2)(x) &= (\alpha T_1)x +_W (\alpha T_2)x && \text{(Addition on } L(V,W)\text{)} \\
&= \alpha T_1 x +_W \alpha T_2 x. && \text{(Scalar multiplication on } L(V,W)\text{)}
\end{aligned}$$

Therefore, $L(V,W)$ is a vector space.

Now, we'll show $B(V,W)$ is closed and thus a subspace of $L(V,W)$. Let $T_1, T_2 \in B(V,W)$ and $\alpha, \beta \in \mathbb{K}$ and $x \in V$. Then,

$$\begin{aligned}
\sup_{\|x\| \le 1} \|(\alpha T_1 +_B \beta T_2)(x)\| &\le \sup_{\|x\| \le 1} (|\alpha| \, \|T_1 x\| + |\beta| \, \|T_2 x\|) && \text{(By triangle inequality)} \\
&\le |\alpha| \sup_{\|x\| \le 1} \|T_1 x\| + |\beta| \sup_{\|x\| \le 1} \|T_2 x\| && \text{(Definition of supremum)} \\
&= |\alpha| \, \|T_1\| + |\beta| \, \|T_2\| && \text{(Definition of } \|T_1\| \text{ and } \|T_2\|\text{)} \\
&< \infty.
\end{aligned}$$

Thus, $(\alpha T_1 +_B \beta T_2) \in B(V,W)$, which implies $B(V,W)$ is a subspace of $L(V,W)$. Therefore, $B(V,W)$ is a vector space. $\square$

(b) For $T \in B(V, W)$, let $\|T\| = \sup_{\|x\| \leq 1} \|Tx\|$. Prove that $\|\cdot\|$ is a norm on $B(V, W)$.

Proof. Since $\|T_1 x\|_W \geq 0$ for all $x \in V$ by positivity of $\|\cdot\|_W$, we have $\sup_{\|x\| \leq 1} \|T_1 x\|_W \geq 0$. Now suppose $\|T_1\| = 0$. Then $\sup_{\|x\| \leq 1} \|T_1 x\| = 0$. Suppose to the contrary there exists $v \in V$ such that $\|T_1 v\| > 0$. Then since $\sup_{\|x\| \leq 1} \|T_1 x\| = 0$, $\|v\| > 1$. But then $\|\frac{v}{\|v\|}\| = 1$, so $\|T_1 \frac{v}{\|v\|}\| = 0$. But, then $\|T_1 \frac{v}{\|v\|}\| = \left|\frac{1}{\|v\|}\right| \|T_1 v\| = 0$, which implies $\|T_1 v\| = 0$. This is a contradiction. Therefore, T_1 is the zero map and we have positivity of $\|\cdot\|$.

Next,

$$\begin{aligned}
\|\alpha T_1\| &= \sup_{\|x\| \leq 1} \|\alpha T_1 x\| && \text{(Definition of } \|\alpha T_1\|) \\
&= |\alpha| \sup_{\|x\| \leq 1} \|T_1 x\| && \text{(Homogeneity of supremum norm)} \\
&= |\alpha| \, \|T_1\|. && \text{(Definition of } \|T_1\|)
\end{aligned}$$

Thus, we have homogeneity of $\|\cdot\|$.

Lastly,

$$\begin{aligned}
\|T_1 +_B T_2\| &= \sup_{\|x\| \leq 1} \|(T_1 +_B T_2)(x)\| && \text{(Definition of } \|T_1 +_B T_2\|) \\
&\leq \sup_{\|x\| \leq 1} (\|T_1 x\| + \|T_2 x\|) && \text{(Triangle inequality of the norm on } W) \\
&\leq \sup_{\|x\| \leq 1} \|T_1 x\| + \sup_{\|x\| \leq 1} \|T_2 x\| && \text{(Property of supremum norm)} \\
&= \|T_1\| + \|T_2\|. && \text{(Definition of } \|T_1\| \text{ and } \|T_2\|)
\end{aligned}$$

Thus, we have the triangle inequality of $\|\cdot\|$. Therefore, $\|\cdot\|$ is a norm on $B(V, W)$. □

Problem 5. Let $f : M_{m \times n} \to M_n$ be given by $f(A) = A^t A$. Prove that f is differentiable, and find a formula for $f'(A)$. (Hint: use facts about the operator norm and the transpose of a matrix.)

Proof. Let $f : M_{m \times n} \to M_n$ be given by $f(A) = A^t A$. Let $A \in M_{m \times n}$. Define $T \in B(M_{m \times n}, M_n)$ by $T(h) = A^t h + h^t A$. Also, define $\|\cdot\|_E$ as the Euclidean norm and $\|\cdot\|_O$ as the operator norm. Then

$$
\begin{aligned}
\lim_{h\to 0}\frac{\|f(A+h)-f(A)-T(h)\|_O}{\|h\|_O} &= \lim_{h\to 0}\frac{\|(A+h)^t(A+h)-A^tA-A^th-h^tA\|_O}{\|h\|_O} && \text{(Definition of } f \text{ and } T)\\
&= \lim_{h\to 0}\frac{\|(A+h)^tA+(A+h)^th-A^tA-A^th-h^tA\|_O}{\|h\|_O} && \text{(By distribution)}\\
&= \lim_{h\to 0}\frac{\|(A^t+h^t)A+(A^t+h^t)h-A^tA-A^th-h^tA\|_O}{\|h\|_O} && \text{(Property of transpose)}\\
&= \lim_{h\to 0}\frac{\|A^tA+h^tA+A^th+h^th-A^tA-A^th-h^tA\|_O}{\|h\|_O} && \text{(By distribution)}\\
&= \lim_{h\to 0}\frac{\|h^th\|_O}{\|h\|_O} && \text{(By subtraction)}\\
&\le \frac{\|h^t\|_O\cdot\|h\|_O}{\|h\|_O} && \text{(Property of operator norm)}\\
&= \lim_{h\to 0}\frac{\|h\|_O^2}{\|h\|_O} && \text{(Property of operator norm)}\\
&= \lim_{h\to 0}\|h\|_O\\
&= 0.
\end{aligned}
$$

Since $0 \le \lim_{h\to 0}\frac{\|f(A+h)-f(A)-T(h)\|_O}{\|h\|_O}$, then by squeeze theorem, $\lim_{h\to 0}\frac{\|f(A+h)-f(A)-T(h)\|_O}{\|h\|_O} = 0$. Also, we know that, $\lim_{h\to 0}\frac{f(A+h)-f(A)-T(h)}{\|h\|_E} = 0 \iff \lim_{h\to 0}\frac{\|f(A+h)-f(A)-T(h)\|_E}{\|h\|_E} = 0$. Thus, since $A \in M_{m\times n} = \mathbb{R}^{mn}$, then any two norms are comparable by Corollary 2.11, which implies there exists $k_1, k_2 \in \mathbb{R}$ such that for all $x \in \mathbb{R}$, $\|x\|_0 \le k_1 \cdot \|x\|_E$ and $\|x\|_E \le k_2 \cdot \|x\|_O$. Then, we get

$$
\begin{aligned}
\lim_{h\to 0}\frac{\|f(A+h)-f(A)-T(h)\|_E}{\|h\|_E} &\le \lim_{h\to 0}\frac{k_2\|f(A+h)-f(A)-T(h)\|_O}{\frac{1}{k_1}\|h\|_O} && \text{(By comparability of } \|\cdot\|_E \text{ and } \|\cdot\|_O)\\
&= k_1k_2\lim_{h\to 0}\frac{\|f(A+h)-f(A)-T(h)\|_O}{\|h\|_O} && \text{(Property of limits)}\\
&= k_1k_2\cdot 0\\
&= 0.
\end{aligned}
$$

Since, $0 \leq \lim_{h\to 0} \frac{\|f(A+h)-f(A)-T(h)\|_E}{\|h\|_E}$, then by squeeze theorem, $\lim_{h\to 0} \frac{\|f(A+h)-f(A)-T(h)\|_E}{\|h\|_E} = 0$. This implies, $\lim_{h\to 0} \frac{f(A+h)-f(A)-T(h)}{\|h\|_E} = 0$. Thus, f is differentiable at A. However, since A was arbitrary, then f is differentiable for all $A \in M_{m\times n}$. Therefore, $f'(A)(h) = A^t h + h^t A$. $\square$

Problem 6. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} \frac{x_1^2 x_2}{\|x\|^2}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

Prove that all directional derivatives of f exist at 0, and that $D_v f(0)$ is not a linear function of v.

Proof. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} \frac{x_1^2 x_2}{\|x\|^2}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

Let $v \in \mathbb{R}^2 \setminus (0,0)$. Then,

$$\begin{aligned} D_v f(0) &= \lim_{t\to 0} \frac{f(0+tv)-f(0)}{t} && \text{(Definition of } D_v f(x) \text{ in } \mathbb{R}^2) \\ &= \lim_{t\to 0} \frac{\frac{(tv_1)^2(tv_2)}{\|tv\|^2} - 0}{t} && \text{(Definition of } f \\ &= \lim_{t\to 0} \frac{t^2 v_1^2 t v_2}{t\sqrt{(tv_1)^2+(tv_2)^2}^2} && \text{(Definition of Euclidean norm)} \\ &= \lim_{t\to 0} \frac{t^2 v_1^2 v_2}{t^2 v_1^2 + t^2 v_2^2} \\ &= \lim_{t\to 0} \frac{t^2 v_1^2 v_2}{t^2(v_1^2+v_2^2)} && \text{(by factoring } t^2) \\ &= \lim_{t\to 0} \frac{v_1^2 v_2}{v_1^2+v_2^2}. \end{aligned}$$

Also, we have

$$\begin{aligned} D_{(0,0)} f(0) &= \lim_{t\to 0} \frac{f(0+t\cdot 0)-f(0)}{t} \\ &= \lim_{t\to 0} \frac{0}{t} \\ &= 0. \end{aligned}$$

So, we have $D_v f(0) : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$D_v f(0) = \begin{cases} \frac{v_1^2 v_2}{v_1^2 + v_2^2}, & \text{if } v \neq 0 \\ 0, & \text{if } v = 0. \end{cases}$$

Consider $p = (1, 0)$ and $q = (0, 1)$. Then, $D_p f(0) = \frac{1^2 \cdot 0}{1^2 + 0^2} = 0$ and $D_q f(0) = \frac{0^2 \cdot 1}{0^2 + 1^2} = 0$, and $D_{p+q} f(0) = \frac{1^2 \cdot 1}{1^2 + 1^2} = \frac{1}{2}$. Therefore, $D_v f(0)$ is not linear since $D_{p+q} f(0) = \frac{1}{2} \neq 0 = 0 + 0 = D_p f(0) + D_q f(0)$. $\square$

Problem 7. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} \frac{x_1 x_2^3}{x_1^2 + x_2^4}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

Prove that all directional derivatives of f exist at 0, that $D_v f(0)$ is a linear function of v, and that f is not differentiable at 0.

Proof. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} \frac{x_1 x_2^3}{x_1^2 + x_2^4}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

Let $v \in \mathbb{R}^2 \setminus (0, 0)$. Then, we have

$$\begin{aligned} D_v f(0) &= \lim_{t \to 0} \frac{f(0 + tv) - f(0)}{t} \\ &= \lim_{t \to 0} \frac{\frac{tv_1 (tv_2)^3 - 0}{(tv_1)^2 + (tv_2)^4}}{t} && \text{(Definition of } f) \\ &= \lim_{t \to 0} \frac{t^3 v_1 v_2}{t^2 v_1^2 + t^4 v_2^4} \\ &= \lim_{t \to 0} \frac{t v_1 v_2}{v_1^2 + t^2 v_2^4} \\ &= \frac{0}{v_1^2} \\ &= 0. \end{aligned}$$

Also note that

$$D_0 f(0) = \lim_{t \to 0} \frac{f(0 + t \cdot 0) - f(0)}{t} = \lim_{t \to 0} \frac{0}{t} = 0.$$

Thus, $D_v f(0) : \mathbb{R}^2 \to \mathbb{R}$ is defined by $D_v f(0) = 0$, which is linear since it is the zero map. Also, the Jacobian matrix of f evaluated at 0 is $(0, 0)$. To see if f is differentiable at 0, we have that

$$\lim_{h\to 0} \frac{f(0+h) - f(0) - (0,0)\cdot h}{\|h\|} = \lim_{h\to 0} \frac{\frac{h_1 h_2^3}{h_1^2 + h_2^4}}{\sqrt{h_1^2 + h_2^2}}.$$

Consider $Z_1 = \{(t^2, t) : t \in \mathbb{R}^+\}$. Then,

$$\begin{aligned}
\lim_{h\to 0} \left.\frac{h_1 h_2^3}{(h_1^2 + h_2^4)\sqrt{h_1^2 + h_2^2}}\right|_{Z_1} &= \lim_{t\to 0^+} \frac{t^2 t^3}{(t^4 + t^4)\sqrt{t^4 + t^2}} \\
&= \lim_{t\to 0^+} \frac{t}{2\sqrt{t^4 + t^2}} \\
&= \lim_{t\to 0} \sqrt{\frac{t^2}{4t^4 t^2}} \\
&= \sqrt{\frac{1}{4t^2 + 4}} \\
&= \frac{1}{2}.
\end{aligned}$$

This means that the limit is $\frac{1}{2}$ or does not exist. However, since it is not equal to 0 either way, the derivative does not exist at 0. □

Problem 8. Let $E = \{x \in \mathbb{R}^2 : x_1 > 0 \text{ and } 0 < x_2 < x_1^2\}$. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1, & \text{if } x \in E \\ 0, & \text{if } x \notin E. \end{cases}$$

(f is called the *characteristic function* of the set E.) Prove that all directional derivatives of f exist at 0, and equal 0, but that f is not differentiable at 0.

Proof. Let $E = \{x \in \mathbb{R}^2 : x_1 > 0 \text{ and } 0 < x_2 < x_1^2\}$. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1, & \text{if } x \in E \\ 0, & \text{if } x \notin E. \end{cases}$$

Define $A = \{x \in \mathbb{R}^2 : x_1 > 0 \text{ and } x_2 > 0 \text{ or } x_1 < 0 \text{ and } x_2 < 0\}$. Let $v \in A$ be arbitrary.

Case 1: Suppose $v_1 > 0$ and $v_2 > 0$.

Since $(tv_1) < 0$ for all $t < 0$, which implies $tv \notin E$, we have,

$$\lim_{t\to 0^-} \frac{f(0+tv) - f(0)}{t} = \lim_{t\to 0^-} \frac{f(tv)}{t} = \lim_{t\to 0^-} \frac{0}{t} = 0,$$

which implies $f(tv) = 0$. Let $\delta \in \mathbb{R}$ such that $0 < \delta < \frac{v_2}{v_1^2}$. By multiplication of $\delta v_1^2 > 0$, $0 < (\delta v_1)^2 < \delta v_2$. This means, $\delta v \notin E$. Thus, for all $0 < t \leq \frac{v_2}{v_1^2}$,

$$\lim_{t \to 0^+} \frac{f(0+tv) - f(0)}{t} = \lim_{t \to 0^+} \frac{f(tv)}{t} = \lim_{t \to 0^+} \frac{0}{t} = 0,$$

which implies $f(tv) = 0$.

<u>Case 2</u>: Suppose $v_1 < 0$ and $v_2 < 0$.

Since $(tv_1) < 0$ for all $t > 0$, which implies $tv \notin E$, we have,

$$\lim_{t \to 0^+} \frac{f(0+tv) - f(0)}{t} = \lim_{t \to 0^+} \frac{f(tv)}{t} = \lim_{t \to 0^+} \frac{0}{t} = 0,$$

which implies $f(tv) = 0$. Let $\phi \in \mathbb{R}$ such that $\frac{v_2}{v_1^2} < \phi < 0$. By multiplication of $\phi v_1^2 < 0$, $\phi v_2 > (\phi v_1)^2 > 0$, which implies $\phi v \notin E$. Then, for all $\frac{v_2}{v_1^2} \leq t < 0$, we have

$$\lim_{t \to 0^-} \frac{f(0+tv) - f(0)}{t} = \lim_{t \to 0^-} \frac{f(tv)}{t} = \lim_{t \to 0^-} \frac{0}{t} = 0,$$

which implies $f(tv) = 0$.

Therefore, since in all cases we have $\lim_{t \to 0^-} \frac{f(0+tv)-f(0)}{t} = \lim_{t \to 0^+} \frac{f(0+tv)-f(0)}{t}$, that implies $D_v f(0) = \lim_{t \to 0} \frac{f(0+tv)-f(0)}{t} = 0$.

Now, let $u \notin A$. Then, we have that $u_1 \geq 0$ and $u_2 \leq 0$ or $u_1 \leq 0$ and $u_2 \geq 0$. This implies that $u \notin E$. Moreover, for all $t \in \mathbb{R}$, we have $tu \notin E$ since we would still have the property mentioned. Thus, $f(tu) = 0$ for all $t \in \mathbb{R}$. This means that we have

$$\begin{aligned} D_u f(0) &= \lim_{t \to 0} \frac{f(0+tu) - f(0)}{t} \\ &= \lim_{t \to 0} \frac{f(tu)}{t} \\ &= \lim_{t \to 0} \frac{0}{t} \\ &= 0. \end{aligned}$$

Thus, for all $w \in \mathbb{R}^2$, $D_w f(0) = 0$. So, the Jacobian matrix of f evaluated at 0 is $(0,0)$. Consider $Z_1 = \{(t, t^3) : 0 < t < 1\}$. Then for all $x \in Z_1$, x is of the form (k, k^3) and $x_1 = k > 0$ and $x_1^2 = k^2 > k^3 = x_2 > 0$. This implies $x \in E$. Hence, for all $x \in Z_1$, $f(x) = 1$. Thus,

$$\lim_{h \to 0} \left. \frac{f(0+h) - f(0) - (0,0)h}{\|h\|} \right|_{Z_1} = \lim_{t \to 0^+} \frac{1}{\|(t, t^3)\|} = \infty.$$

Therefore, the derivative of f at 0 does not exist. $\square$

Problem 9. Let $U \subseteq \mathbb{R}^n$ be open, let $f : U \to \mathbb{R}^n$, and let $a \in U$. Suppose that f is differentiable at a, and that $f'(a)$ is a non-singular linear transformation. Prove that there is a number $r > 0$ such that for all $x \in U$, if $0 < \|x - a\| < r$ then $f(x) \neq f(a)$. (Hint: use the second version of differentiability.)

Proof. Let $U \subseteq \mathbb{R}^n$ be open. Let $f : U \to \mathbb{R}^n$. Let $a \in U$. Suppose f is differentiable at a and that $f'(a)$ is a nonsingular linear transformation. Then, there exists a function $\phi : B_r(0) \to \mathbb{R}^n$ for some $r > 0$ such that $\phi(0) = 0$, ϕ is continuous at 0, and $f(a + h) = f(a) + T(h) + \phi(h)\|h\|$, for $h \in B_r(0)$. Suppose for contradiction that for all $\delta > 0$, there exists $h \in U$ such that $0 < \|h - a\| < \delta$ and $f(x) = f(a)$. Define $(h_n - a)_{n\in\mathbb{N}}$ where $h_n - a$ satisfies $0 < \|h_n - a\| < \min\{\frac{1}{n+1}, r\}$ and $f(h_n) = f(a)$ for all $n \in \mathbb{N}$. Then, we have $f(a+h_n-a) = f(a)+f'(a)(h_n-a)+\phi(h_n-a)\|h_n-a\|$ from the properties of ϕ. Simplifying, we get $f(h_n) = f(a)+f'(a)(h_n-a)+\phi(h_n-a)\|h_n-a\|$. Then, $f'(a)(h_n-a) = -\phi(h_n-a)\|h_n-a\|$ since $f(a) = f(h_n)$. Since we supposed $f'(a)$ was non-singular, then $f'(a)^{-1}$ exists. Thus,

$$\begin{aligned} f'(a)^{-1}(f'(a)(h_n - a)) &= f'(a)^{-1}(-\phi(h_n - a)\|h_n - a\|) \\ h_n - a &= f'(a)^{-1}(-\phi(h_n - a)\|h_n - a\|) && \text{(Since } f'(a) \text{ bijective)} \\ &= -\|h_n - a\|f'(a)^{-1}(\phi(h_n - a)) && \text{(By linearity of } f'(a)^{-1}\text{)} \\ \|h_n - a\| &= \|-\|h_n - a\|f'(a)^{-1}(\phi(h_n - a))\| \\ &= \|h_n - a\| \cdot \|f'(a)^{-1}(\phi(h_n - a))\|. && \text{(Since } \|h_n - a\| \in \mathbb{R}\text{)} \end{aligned}$$

By division, $1 = \|f'(a)^{-1}(\phi(h_n - a))\|$. Then, as $n \to \infty$, then $h_n - a \to 0$. Since ϕ is continuous at 0 and $\phi(0) = 0$, $\phi(h_n - a) \to 0$ as $n \to \infty$. Then, $f'(a)^{-1}$ is continuous at 0 and $f'(a)^{-1}(0) = 0$ since $f'(a)^{-1}$ is a linear function on a finite vector space. Thus, $\|f'(a)^{-1}(\phi(h_n - a))\| \to 0$ as $n \to \infty$, contradiction. Therefore, there exists $\delta > 0$ such that for all $x \in U$, whenever $0 < \|x - a\| < \delta$, then $f(a) \neq f(a)$. □

Problem 10. Let $E \subseteq \mathbb{R}^n$ be an open set, and let $f : E \to \mathbb{R}$. Suppose that $\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}$ exist and are bounded in E. Prove that f is continuous in E. (Hint: imitate the proof of differentiability when the partial derivatives are continuous.)

Proof. Let $E \subseteq \mathbb{R}^n$ be an open set, and let $f : E \to \mathbb{R}$. Let $\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}$ exist and be bounded in E. Let $a \in E$ be arbitrary but fixed. Since E is open, there exists $r > 0$ such that $B_r(a) \subseteq E$ such that for all $h \in B_r(a)$, we have

$$\begin{aligned} f(a + h) - f(a) &= \sum_{j=1}^{n} f(a + h_1e_1 + \cdots + h_je_j) - f(a + h_1e_1 + \cdots + h_{j-1}e_{j-1}) \\ &= \sum_{j=1}^{n} f(a_1 + h_1, \ldots, a_j + h_j, a_{j+1}, \ldots, a_n) - f(a_1 + h_1, \ldots, a_{j-1} + h_{j-1}, a_j, \ldots, a_n). \end{aligned}$$

Since f is differentiable with respect to x_j in E for all $j \in \{1, \ldots, n\}$, then f is continuous with respect to x_j. By mean value theorem, we have that there exists $0 < \theta_j < 1$ such that for all $j \in \{1, \ldots, n\}$,

$$\begin{aligned} f(a_1 + h_1, \ldots, a_j + h_j, a_{j+1}, \ldots, a_n) - f(a_1 + h_1, \ldots, a_{j-1} + h_{j-1}, a_j, \ldots, a_n) \\ = h_j \cdot D_j f(a_1 + h_1, \ldots, a_j + \theta_j h_j, a_{j+1}, \ldots, a_n). \end{aligned}$$

This implies $f(a+h) - f(a) = \sum_{j=1}^n h_j D_j f(a_1 + h_1, \ldots, a_j + \theta_j, a_{j+1}, \ldots, a_n)$. By taking the norm and limit, we get

$$\begin{aligned} \lim_{h \to 0} \|f(a+h) - f(a)\| &= \lim_{h \to 0} \|\sum_{j=1}^n h_j D_j f(a_1 + h_1, \ldots, a_j + \theta_j, a_{j+1}, \ldots, a_n)\| \\ &\leq \lim_{h \to 0} \sum_{j=1}^n \|h\| \cdot \|D_j f(a_1 + h_1, \ldots, a_j + \theta_j, a_{j+1}, \ldots, a_n)\| \\ & \qquad \text{(By Cauchy-Schwartz)} \\ &= 0. \qquad \text{(Since } D_j f \text{ is bounded and } \lim_{h \to 0} \|h\| = 0) \end{aligned}$$

So, $\lim_{h \to 0} \|f(a+h) - f(a)\| = 0$. Therefore, f is continuous at a. Since $a \in E$ was arbitrary, then f is continuous. □

Problem 11. Let $f_1, f_2 : \mathbb{R}^2 \to \mathbb{R}$ be continuously differentiable functions, and suppose that $D_i f_j(x) = D_j f_i(x)$ for all i and j, and for all $x \in \mathbb{R}^2$. Prove that there exists a function $F : \mathbb{R}^2 \to \mathbb{R}$ such that $f_i = D_i F$ for all i. (Hint: fix a point $a \in \mathbb{R}^2$. Define F by

$$F(x) = \int_{a_1}^{x_1} f_1(t, a_2) dt + \int_{a_2}^{x_2} f_2(x_1, t) dt.$$

You may use the theorem on passing a derivative through an integral: if $f : \mathbb{R}^2 \to \mathbb{R}$, and if f and $D_2 f$ are continuous, then $\frac{d}{dt} \int_a^b f(s,t) ds = \int_a^b D_2 f(s,t) ds$.)

(This problem is still true for $f : \mathbb{R}^n \to \mathbb{R}$, and for extra credit (double) you can write out the statement in the general case (in addition to, or instead of) the case $n = 2$. Some more hints for the general case: it makes for easier bookkeeping to separate the calculation into terms of three types. When calculating $\frac{\partial F}{\partial x_i}$, there are n terms to differentiate. Consider the three possibilities: $\frac{\partial}{\partial x_i}$ of the jth term, where $j < i$, where $j = i$, where $j > i$. In the first case, you should get to 0, in the second, you can use the usual fundamental theorem of calculus, and in the third, you must pass the derivative under the integral, and then use the hypothesis of the problem. If you are really stuck, work the problem in the case $n = 3$. Then you should be able to see what is going on.)

Proof. Let $f_1, f_2 : \mathbb{R}^2 \to \mathbb{R}$ be continuously differentiable functions, and suppose that $D_i f_j(x) = D_j f_i(x)$ for all i and j, and for all $x \in \mathbb{R}^2$. Fix $a \in \mathbb{R}^2$. Define $F : \mathbb{R}^2 \to \mathbb{R}$ by

$$F(x) = \int_{a_1}^{x_1} f_1(t, a_2) dt + \int_{a_2}^{x_2} f_2(x_1, t) dt.$$

Then,

$$\begin{aligned}
D_1F(x) &= \frac{\partial}{\partial x_1}\left(\int_{a_1}^{x_1} f_1(t,a_2)dt + \int_{a_2}^{x_2} f_2(x_1,t)dt\right) \\
&= \frac{\partial}{\partial x_1}\int_{a_1}^{x_1} f_1(t,a_2)dt + \frac{\partial}{\partial x_1}\int_{a_2}^{x_2} f_2(x_1,t)dt \\
&= f_1(x_1,a_2) + \frac{\partial}{\partial x_1}\int_{a_2}^{x_2} f_2(x_1,t)dt && \text{(By fundamental theorem of calculus)} \\
&= f_1(x_1,a_2) + \int_{a_2}^{x_2} D_1f_2(x_1,t)dt && \text{(By theorem from the hint)} \\
&= f_1(x_1,a_2) + \int_{a_2}^{x_2} D_2f_1(x_1,t)dt && \text{(From assumption)} \\
&= f_1(x_1,a_2) + f_1(x_1,x_2) - f_1(x_1,a_2) && \text{(By fundamental theorem of calculus)} \\
&= f_1(x).
\end{aligned}$$

Also, we have

$$\begin{aligned}
D_2F(x) &= \frac{\partial}{\partial x_2}\left(\int_{a_1}^{x_1} f_1(t,a_2)dt\right) + \frac{\partial}{\partial x_2}\left(\int_{a_2}^{x_2} f_2(x_1,t)dt\right) \\
&= 0 + \frac{\partial}{\partial x_2}\left(\int_{a_2}^{x_2} f_2(x_1,t)dt\right) && \text{(Since } f_1(t,a_2) \text{ is constant with respect to } x_2\text{)} \\
&= f_2(x_1,x_2). && \text{(By fundamental theorem of calculus)}
\end{aligned}$$

Therefore, $f_1 = D_1F$ and $f_2 = D_2F$. □

Problem 12. Let $f_1, f_2 : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ be given by

$$f_1(x) = \frac{-x_2}{x_1^2 + x_2^2}, \qquad f_2(x) = \frac{x_1}{x_1^2 + x_2^2}.$$

(a) Prove that $D_1f_2 = D_2f_1$ on $\mathbb{R}^2 \setminus \{0\}$.

Proof. Let $x \in \mathbb{R}^2 \setminus \{0\}$. Then,

$$\begin{aligned}
D_1f_2(x) &= \frac{(x_1^2 + x_2^2)\cdot 1 - x_1(2x_1)}{(x_1^2 + x_2^2)^2} && \text{(By quotient rule)} \\
&= \frac{-x_1^2 + x_2^2}{(x_1^2 + x_2^2)^2}
\end{aligned}$$

Also,

$$\begin{aligned} D_2 f_1(x) &= \frac{(x_1^2+x_2^2)(-1)-(-x_2)(2x_2)}{(x_1^2+x_2^2)^2} && \text{(By quotient rule)} \\ &= \frac{-x_1^2+x_2^2}{(x_1^2+x_2^2)^2}. \end{aligned}$$

Thus, $D_1 f_2(x) = \frac{-x_1^2+x_2^2}{(x_1^2+x_2^2)^2} = D_2 f_1(x)$. Since $x \in \mathbb{R}^2 \setminus \{0\}$ was arbitrary, then $D_1 f_2 = D_2 f_1$. $\square$

(b) Prove that there does not exist a continuously differentiable function $F : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ such that $f_i = D_i F$ for $i = 1, 2$. (Hint: Let $g : [0, 2\pi] \to \mathbb{R}^2 \setminus \{0\}$ be given by $g(t) = (\cos t, \sin t)$. Apply the mean value theorem to $F(g(t))$.)

Proof. Suppose, for a contradiction, that there exists a continuously differentiable function $F : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ such that $f_i = D_i F$ for $i = 1, 2$. Let $g : [0, 2\pi] \to \mathbb{R}^2 \setminus \{0\}$ be defined by $g(t) = (\cos t, \sin t)$. Then, $F \circ g$ is differentiable on $(0, 2\pi)$ since the composition of differentiable functions is differentiable. Also, $F \circ g$ is continuous on $[0, 2\pi]$ since the composition of continuous functions is continuous.

Let $x \in (0, 2\pi)$. Then,

$$\begin{aligned} D(F \circ g)(x) &= \sum_{k=1}^{2} D_k F(g(x)) \cdot D_{g_k}(x) && \text{(By chain rule)} \\ &= f_1(g(x)) \cdot D_{g_1}(x) + f_2(g(x)) \cdot D_{g_2}(x) && \text{(From assumption)} \\ &= \frac{-\sin(x)}{\sin^2(x)+\cos^2(x)} \cdot (-\sin(x)) + \frac{\cos(x)}{\sin^2(x)+\cos^2(x)} \cdot \cos(x) \\ & && \text{(By definition of } f_1 \text{ and } f_2) \\ &= \frac{\sin^2(x)}{\sin^2(x)+\cos^2(x)} + \frac{\cos^2(x)}{\sin^2(x)+\cos^2(x)} \\ &= 1. \end{aligned}$$

Note, $F \circ g(0) = F(g(0)) = F(\cos(0), \sin(0)) = F(1, 0) = F(\cos(2\pi), \sin(2\pi)) = F(g(2\pi)) = F \circ g(2\pi)$. By Rolle's theorem, there exists $c \in (0, 2\pi)$ such that $D(F \circ g)(c) = 0$. However, $D(F \circ g)(x) = 1$ for all $x \in (0, 2\pi)$, contradiction. Therefore, there does not exist a function $F : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ such that $f_i = D_i f$ for $i = 1, 2$. $\square$

Problem 13. Define the function $r : \mathbb{R}^p \to \mathbb{R}$ by $r(x) = \sqrt{x_1^2 + \cdots + x_p^2}$.

(a) Prove that $\frac{\partial r}{\partial x_i} = \frac{x_i}{r}$.

Proof. Define the function $r : \mathbb{R}^p \to \mathbb{R}$ by $r(x) = \sqrt{x_1^2 + \cdots + x_p^2}$. Let $U \subseteq \mathbb{R}^p$ be open and r be differentiable at some $x \in U$ and fix $i \in \{1, \ldots, p\}$. Then,

$$\begin{aligned}
\frac{\partial r}{\partial x_i}(x) &= \frac{\partial}{\partial x_i}\sqrt{x_1^2 + \cdots + x_p^2} \\
&= \frac{1}{2}\sqrt{x_1^2 + \cdots + x_p^2}^{-1}(2x_i) && \text{(By chain rule)} \\
&= \frac{x_i}{\sqrt{x_1^2 + \cdots + x_p^2}} \\
&= \frac{x_i}{r(x)}.
\end{aligned}$$

□

(b) Prove that $\sum_{i=1}^p \frac{\partial^2 r}{\partial x_i^2} = \frac{p-1}{r}$.

Proof. Define the function $r : \mathbb{R}^p \to \mathbb{R}$ by $r(x) = \sqrt{x_1^2 + \cdots + x_p^2}$. Let $U \subseteq \mathbb{R}^p$ be open and r be twice differentiable at some $x \in U$ and fix $i \in \{1, \ldots, p\}$. Then,

$$\begin{aligned}
\frac{\partial^2 r}{\partial x_i^2}(x) &= \frac{\partial}{\partial x_i}\frac{x_i}{r(x)} \\
&= \frac{r(x) - (x_i)\frac{x_i}{r(x)}}{r(x)^2} && \text{(By quotient rule)} \\
&= \frac{r(x)^2 - x_i^2}{r(x)^3}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\sum_{i=1}^p \frac{\partial^2 r}{\partial x_i^2}(x) &= \sum_{i=1}^p \frac{r(x)^2 - x_i^2}{r(x)^3} \\
&= \left(\frac{x_1^2 + \cdots + x_p^2 - x_1^2}{r(x)^3}\right) + \left(\frac{x_1^2 + \cdots + x_p^2 - x_2^2}{r(x)^3}\right) + \cdots + \left(\frac{x_1^2 + \cdots + x_p^2 - x_p^2}{r(x)^3}\right) \\
&= \frac{(p-1)x_1^2 + \cdots + (p-1)x_p^2}{r(x)^3} \\
&= \frac{(p-1)r(x)^2}{r(x)^3} \\
&= \frac{p-1}{r(x)}.
\end{aligned}$$

□

(c) Prove that $\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} \frac{1}{r^{p-2}} = 0$.

Proof. Let $\frac{1}{r^{p-2}} : \mathbb{R}^p \setminus \{0\} \to \mathbb{R}$ be given by $\left(\frac{1}{r^{p-2}}\right)(x) = r(x)^{-p+2}$. Fix $i \in \{1, \dots p\}$ and $x \in \mathbb{R}^p \setminus \{0\}$. Then,

$$\begin{aligned}\left(\frac{\partial}{\partial x_i}\frac{1}{r^{p-2}}\right)(x) &= (-p+2)\cdot r(x)^{-p+1}\cdot \frac{\partial r}{\partial x_i}(x) && \text{(By chain rule)}\\ &= \frac{(-p+2)x_i}{r(x)^p}.\end{aligned}$$

Then,

$$\begin{aligned}\left(\frac{\partial^2}{\partial x_i^2}\frac{1}{r^{p-2}}\right)(x) &= \frac{r(x)^p(-p+2) - (-p+2)\cdot x_i(p\cdot r(x)^{p-1}\cdot \frac{\partial r}{\partial x_i}(x))}{r(x)^{2p}} \\ & && \text{(By quotient rule)}\\ &= (-p+2)\left(\frac{r(x)^p - x_i^2 p r(x)^{p-2}}{r(x)^{2p}}\right)\\ &= \frac{(-p+2)(r(x)^{p-2})}{r(x)^{2p}}(r(x)^2 - x_i^2 p).\end{aligned}$$

Then, we have

$$\begin{aligned}\sum_{i=1}^{p}\left(\frac{\partial^2}{\partial x_i^2}\frac{1}{r^{p-2}}\right)(x) &= \sum_{i=1}^{p}\frac{(-p+2)(r(x)^{p-2})}{r(x)^{2p}}(r(x)^2 - x_i^2 p)\\ &= \frac{(-p+2)(r(x)^{p-2})}{r(x)^{2p}}\sum_{i=1}^{p}(r(x)^2 - x_i^2 p) && \text{(Summation property)}\\ &= \frac{(-p+2)(r(x)^{p-2})}{r(x)^{2p}}(r(x)^2 - x_1^2 p + r(x)^2 - x_2^2 p + \dots + r(x)^2 - x_p^2 p)\\ &= \frac{(-p+2)(r(x)^{p-2})}{r(x)^{2p}}(r(x)^2 p - p(x_1^2 + \dots + x_p^2))\\ &= \frac{(-p+2)(r(x)^{p-2})}{r(x)^{2p}}(r(x)^2 p - p r(x)^2)\\ &= \frac{(-p+2)(r(x)^{p-2})}{r(x)^{2p}}\cdot 0\\ &= 0\end{aligned}$$

Therefore, $\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} \frac{1}{r^{p-2}} = 0$. □

Problem 14. Let $U \subseteq \mathbb{R}^n$ be an open set, and let $f : U \to \mathbb{R}$ be a differentiable function. Suppose that $f'(x) = 0$ for all $x \in U$.

(a) Prove that for each $a \in U$ there exists $r > 0$ such that f is constant in $B_r(a)$. (Here $B_r(a) = \{x \in \mathbb{R}^n : \|x - a\| < r\}$ is the open ball in $\mathbb{R}^n$ with a center a and radius r.)

Proof. Let $U \subseteq \mathbb{R}^n$ be an open set, and let $f : U \to \mathbb{R}$ be a differentiable function. Suppose that $f'(x) = 0$ for all $x \in U$. Since f is continuous on an open set, then there exists $r > 0$ arbitrary but fixed such that $B_r(x) \subset U$ for all $x \in U$. Let $y \in B_r(x)$ be arbitrary. Then, $[x, y] \subset U$. Since f is differentiable, then by the mean value theorem, there exists $c \in [x, y]$ such that $\|f(y) - f(x)\| \leq \|f'(c)(y - x)\|$. Since $f'(x) = 0$ for all $x \in U$, then we have $\|f(y) - f(x)\| \leq \|0 \cdot (y - x)\|$. Hence, $\|f(y) - f(x)\| = 0 \implies f(y) - f(x) = 0$. Thus, $f(x) = f(y)$. Therefore, since $y \in B_r(x)$ was arbitrary, f is constant in $B_r(x)$. □

(b) Suppose that U is connected. Prove that f is constant in U.

Proof. Let $U \subseteq \mathbb{R}^n$ be a connected open set, and let $f : U \to \mathbb{R}$ be a differentiable function. Suppose that $f'(x) = 0$ for all $x \in U$. Suppose f is not constant for contradiction. Let $y \in f(U)$. Let $S = f^{-1}(\{y\})$ and $T = f^{-1}(f(U) \setminus \{y\})$. We will show the four following properties of S and T.

(i) $S \cup T = U$:
Note, $\{y\} \cup f(U) \setminus \{y\} = f(U)$ and $f^{-1}(f(U)) = U$. Thus, $f^{-1}(\{y\} \cup f(U) \setminus \{y\}) = U$. Since $\{y\}$ and $f(U) \setminus \{y\}$ are disjoint, then $f^{-1}(\{y\}) \cup f^{-1}(f(U) \setminus \{y\}) = U$, which implies $S \cup T = U$.

(ii) $S \cap T = \emptyset$:
Suppose $S \cap T \neq \emptyset$. Then, there exists $x \in U$ such that $f(x) = y$ and $f(x) = z$ for some $z \in f(U) \setminus \{y\}$. Then, f is not a function. Therefore, $S \cap T = \emptyset$.

(iii) $S \neq \emptyset$ and $T \neq \emptyset$:
Note, $y \in f(U)$. That implies there exists some $z_1 \in U$ such that $z_1 \mapsto f(U) \setminus \{y\}$. Thus, $S \neq \emptyset$. Now, there must exist some $z_2 \in U$ such that $z_2 \mapsto f(U) \setminus \{y\}$. Therefore, $T \neq \emptyset$.

(iv) S and T open:
Let $x \in S$ be arbitrary. That is, $f(x) = y$. Since $S \subset U$, then from part (a), there exists some $r > 0$ such that f is constant in $B_r(x)$. So, $f(B_r(x)) = \{y\}$, which implies $B_r(x) \subset f^{-1}(\{y\})$. Therefore, for all $x \in S$, $B_r(x) \subset S$, which implies S is open. By a similar argument, T is open.

Therefore, there exist S and T nonempty open sets that are disjoint and whose union is U. Thus, U is not disconnected, contradiction. Therefore, f is constant. □

Problem 15. Recall that $GL := \{T \in M_n : T \text{ is invertible }\}$ is an open subset of M_n. Let $\text{inv} : GL_n \to GL_n \subseteq M_n$ be the inversion map: $\text{inv}(T) = T^{-1}$. Prove that inv is continuous on GL_n. (Hint: let $A \in GL_n$ and use the following outline to show that inv is continuous at A. Note that $T^{-1} - A^{-1} = T^{-1}(A-T)A^{-1}$. Apply the operator norm to both sides, then use the reverse triangle inequality to the left, and the operator norm inequality on the right. From the result you should be able to show that $\|T^{-1}\|$ is bounded in some ball centered at A. Then the righthand portion of the inequality work from before can be used to prove the continuity of inv at A.)

Proof. Let $\text{inv} : GL_n \to GL_n \subseteq M_n$ be the inversion map: $\text{inv}(T) = T^{-1}$. Let $A \in GL_n$. Define $\det : M_n \to \mathbb{R}$ to be

$$\det(a_{ij}) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) a_{1,\sigma(1)} \dots a_{n,\sigma(n)}$$

from Linear Algebra and Group Theory. Then, clearly det is continuous since it is a polynomial, which is always continuous. Now, for $A \in GL_n$, define $b_{ij}(A) = \det(A_{mk})_{m \neq j, k \neq i}$. Note, b_{ij} is also continuous. Then, by Cramer's rule,

$$(A^{-1})_{ij} = \frac{(-1)^{i+j}}{\det(A)} b_{ij}(A).$$

Therefore, since $\det(A)$ is a continuous polynomial and $b_{ij}(A)$ is a continuous polynomial, then $(A^{-1})_{ij}$ is continuous. Therefore, since each index of A^{-1} is continuous, then A^{-1} is continuous, which implies inv is continuous since $A \in GL_n$ was arbitrary. □

Problem 16. Continuing from the previous problem, prove that inv is differentiable on GL_n, and that $\text{inv}'(A)(H) = -A^{-1}HA^{-1}$. (Hint: investigate the difference $(A+H)^{-1}$ as a geometric series (for $\|H\|$ small enough).)

Proof. Let $\text{inv} : GL_n \to GL_n \subseteq M_n$ be the inversion map: $\text{inv}(T) = T^{-1}$. Let A in GL_n be arbitrary but fixed. Note, for $r = \frac{1}{\|A^{-1}\|}$ and $H \in B_r(o)$

$$\begin{aligned}
\|-A^{-1}H\| &\leq \|A^{-1}\| \cdot \|H\| && \text{(By triangle inequality)} \\
&< \frac{\|A^{-1}\|}{\|A^{-1}\|} && \text{(Since } H \text{ in } B_r(0)) \\
&= 1
\end{aligned}$$

Then, let $H \in M_n$ arbitrary.

$$
\begin{aligned}
\lim_{H\to 0} \frac{\|(A+H)^{-1} - A^{-1} - (A^{-1}HA^{-1}\|}{\|H\|} &= \lim_{H\to 0} \frac{\|(I + A^{-1}H)^{-1}A^{-1} - A^{-1} + A^{-1}HA^{-1}\|}{\|H\|} \\
&\qquad \text{(Multiplying the identity matrix in } (A+H)^{-1}) \\
&= \lim_{H\to 0} \frac{\|(I - (-A^{-1}H))^{-1}A^{-1} - A^{-1} + A^{-1}HA^{-1}\|}{\|H\|} \\
&= \lim_{H\to 0} \frac{\|\sum_{j=0}^{\infty}((-A^{-1}H)^j \cdot A^{-1}) - A^{-1} + A^{-1}HA^{-1}\|}{\|H\|} \\
&\qquad \text{(By geometric series)} \\
&= \lim_{H\to 0} \frac{\|\sum_{j=2}^{\infty}(((-A^{-1}H)^j)A^{-1}\|}{\|H\|} \qquad \text{(Simplification)} \\
&\le \lim_{H\to 0} \frac{\sum_{j=2}^{\infty}\|((-A^{-1}H)^j)\| \cdot \|A^{-1}\|}{\|H\|} \\
&\qquad \text{(By triangle inequality)} \\
&\le \lim_{H\to 0} \frac{\sum_{j=2}^{\infty}\|(-A^{-1}H)\|^j \cdot \|A^{-1}\|}{\|H\|} \\
&= \lim_{H\to 0} \frac{\left(\frac{1}{1-\|-A^{-1}H\|} - 1 - \|-A^{-1}H\|\right) \cdot \|A^{-1}\|}{\|H\|} \\
&\qquad \text{(From above, when } H \text{ is sufficiently small)} \\
&= \lim_{H\to 0} \frac{\left(\frac{1-1+\|-A^{-1}H\|-\|-A^{-1}H\|+\|-A^{-1}H\|^2}{1-\|-A^{-1}H\|}\right) \cdot \|A^{-1}\|}{\|H\|} \\
&\le \lim_{H\to 0} \frac{\left(\frac{\|-A^{-1}H\|^2}{1-\|-A^{-1}H\|}\right) \cdot \|A^{-1}\|}{\|H\|} \qquad \text{(Simplification)} \\
&= \lim_{H\to 0} \frac{\|-A^{-1}H\|^2 \cdot \|A^{-1}\|}{\|H\|(1 - \|A^{-1}H\|)} \\
&= \lim_{H\to 0} \frac{\|-A^{-1}\|^3 \cdot \|H\|}{1 - \|A^{-1}H\|} \qquad \text{(Simplification)} \\
&= \frac{0}{1} \\
&= 0.
\end{aligned}
$$

Therefore, $0 \le \lim_{H\to 0} \frac{\|(A+H)^{-1} - A^{-1} - (A^{-1}HA^{-1}\|}{\|H\|} \le 0$, which implies $\lim_{H\to 0} \frac{\|(A+H)^{-1} - A^{-1} - (A^{-1}HA^{-1}\|}{\|H\|} = 0$ by squeeze theorem. Thus, inv is differentiable at A. Since $A \in GL_n$ was arbitrary, then inv is differentiable on GL_n with $\text{inv}'(A)(H) = -A^{-1}HA^{-1}$.

$\square$

Problem 17. Prove that the following function $f : \mathbb{R}^2 \to \mathbb{R}$ is (once) continuously differentiable on $\mathbb{R}^2$, that all second-order partial derivatives of f exist at the origin, but that $D_1D_2f(0) \neq D_2D_1f(0)$:

$$f(x) = \begin{cases} \frac{x_1^3 x_2}{x_1^2+x_2^2}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

Proof. Let $f : U \subseteq \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \frac{x_1^3 x_2}{x_1^2+x_2^2}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

Then,

$$\begin{aligned} D_1f(0,0) &= \lim_{h\to 0} \frac{1}{h} \frac{h^3 \cdot 0}{h^2 + 0^2} \\ &\overset{\text{L'H}}{=} \lim_{h\to 0} \frac{0}{3h^2} \\ &\overset{\text{L'H}}{=} \lim_{h\to 0} \frac{0}{6h} \\ &\overset{\text{L'H}}{=} \lim_{h\to 0} \frac{0}{6} \\ &= 0. \end{aligned}$$

Also,

$$\begin{aligned} D_2f(0,0) &= \lim_{h\to 0} \frac{1}{h} \frac{0^3 \cdot h}{0^2 + h^2} \\ &\overset{\text{L'H}}{=} \lim_{h\to 0} \frac{0}{3h^2} \\ &\overset{\text{L'H}}{=} \lim_{h\to 0} \frac{0}{6h} \\ &\overset{\text{L'H}}{=} \lim_{h\to 0} \frac{0}{6} \\ &= 0. \end{aligned}$$

Thus, since all first-order partial derivatives of f exist and are continuous, then $f(x)$ is at least C^1. Note,

$$D_1f(x) = \begin{cases} \frac{(x_1^2+x_2^2)(3x_1^2x_2)-(x_1^3x_2)(2x_1)}{(x_1^2+x_2^2)^2}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

and

$$D_2f(x) = \begin{cases} \frac{(x_1^2+x_2^2)(x_1^3)-(x_1^3x_2)(2x_2)}{(x_1^2+x_2^2)^2}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

Now,

$$
\begin{aligned}
D_2 D_1 f(0,0) &= \lim_{h \to 0} \frac{1}{h} \frac{(h^2+0)(h^3) - (h^3 \cdot 0)(0)}{(h^2+0)^2} \\
&= \lim_{h \to 0} \frac{h^5}{h^5} \\
&= 1, \\
D_2 D_2 f(0,0) &= \lim_{h \to 0} \frac{1}{h} \frac{(0+h^2)(0^3) - (0 \cdot h)(2h)}{(0+h^2)^2} \\
& \lim_{h \to 0} \frac{0}{h^5} \\
&\overset{\text{L'H}}{=} \lim_{h \to 0} \frac{0}{120} \\
&= 0, \\
D_1 D_2 f(0,0) &= \lim_{h \to 0} \frac{1}{h} \frac{(0+h^2)(0 \cdot h) - (0 \cdot h)(0)}{(0+h^2)^2} \\
&= \lim_{h \to 0} \frac{0}{h^5} \\
&\overset{\text{L'H}}{=} \frac{0}{120} \\
&= 0, \\
D_1 D_1 f(0,0) &= \lim_{h \to 0} \frac{1}{h} \frac{(h^2+0)(3h^2 \cdot 0) - (h^3 \cdot 0)(2h)}{(h^2+0)^2} \\
&= \lim_{h \to 0} \frac{0}{h^5} \\
&\overset{\text{L'H}}{=} \frac{0}{120} \\
&= 0.
\end{aligned}
$$

Therefore, all second-order partial derivatives of f exist at the origin. But, $D_1 D_2 f(0,0) = 1 \neq 0 = D_2 D_1 f(0,0)$, which implies f is not C^2.

$\square$

Problem 18.

(a) Let (X, d) be a metric space, let $T : X \to M_n$ be a continuous function, and let $x_0 \in X$. Suppose that $T(x_0)$ has a positive eigenvalue and a negative eigenvalue. Prove that there are unit vectors v_+ and $v_- \in \mathbb{R}^n$ such that

$$\langle T(x)v_+, v_+ \rangle > 0, \qquad \langle T(x)v_-, v_- \rangle < 0$$

for all x in a neighborhood of x_0.

Proof. Let (X, d) be a metric space, let $T : X \to M_n$ be a continuous function, and let $x_0 \in X$. Suppose that $T(x_0)$ has a positive eigenvalue and a negative eigenvalue. Then there exists $u, v \in \mathbb{R}^n$, $\lambda_+, \lambda_- \in \mathbb{R}$ such that $\lambda_+ > 0$ and $\lambda_- < 0$ such that $T(x_0)u = \lambda_+ u$ and $T(x_0)v = \lambda_- v$ Let $u_= = \frac{u}{\|u\|}$ and $v_- = \frac{v_2}{\|v\|}$, which are unit vectors. Then, one can show that

$$\begin{aligned}\langle T(x_0)u_+, u_+\rangle &= \langle \frac{T(x_0)u}{\|u\|}, \frac{u}{\|u\|}\rangle \\ &= \frac{\langle \lambda_+ u, u\rangle}{\|u\|^2} \\ &= \frac{\lambda_+}{\|u\|^2} \cdot \langle u, u\rangle \\ &> 0.\end{aligned}$$

Similarly,

$$\langle T(x_0)v, v\rangle < 0.$$

Now, there exists $r_+, r_- \in \mathbb{R}$ such that for all $x_+ \in B_{r_+}(x_0)$ and for all $x_- \in B_{r_-}(x_0)$, then

$$\|T(x_+) - T(x_0)\| < \frac{\lambda_+}{\|u\|^2} \cdot \sum_{j=1}^{n} u_j^2, \quad \text{and} \quad \|T(x_-) - T(x_0)\| \left| < \frac{\lambda_v}{\|v\|^2} \cdot \sum_{j=1}^{n} v_j^2 \right|.$$

So, for all $x_+ \in B_{r_+}(x_0)$,

$$\begin{aligned}\langle T(x_+)u_+, u_+\rangle &= \langle T(x_+)u_+, u_+\rangle + \langle (T(x_+) - T(x_0))u_+, u_+\rangle \\ &\geq \langle T(x_+)u_+, u_+\rangle - \left|\langle (T(x_+) - T(x_0))u_+, u_+\rangle\right|.\end{aligned}$$

So, we get

$$\begin{aligned}\left|\langle (T(x_+) - T(x_0))u_+, u_+\rangle\right| &\leq \|T(x_+) - T(x_0)\| \cdot \|u_+\| \cdot \|u_+\| \quad \text{(By triangle inequality)} \\ &= \|T(x_+) - T(x_0)\| \\ &< \frac{\lambda_+}{\|u\|^2} \cdot \sum_{j=1}^{n} u_u^2.\end{aligned}$$

Thus, $\langle (T(x_+) - T(x_0))u_+, u_+\rangle > 0$ for all $x_+ \in B_{r_+}(x_0)$. Also, $\langle T(x_0)u, u\rangle = \frac{\lambda_+}{\|u\|^2} \cdot sum_{j=1}^{n} u_j^2$. Thus, $\langle (T(x_+) - T(x_0))u_+, u_+\rangle > 0$ for all $x_+ \in B_{r_+}(x_0)$. In a similar argument, one can show that $\langle T(x_-)v, v\rangle < 0$ for all $x_- \in B_{r_-}(x_0)$. Now, let $r = \min\{r_+, r_-\}$. Therefore, $\langle (T(x)u_+, u_+\rangle > 0$ and $\langle T(x_-)v, v\rangle < 0$ for all $x \in B_r(x_o)$ $\square$

(b) Let $U \subseteq \mathbb{R}^n$ be open, let $a \in U$, let $f : U \to \mathbb{R}$ be a C^2 function, and suppose that $f'(a) = 0$. Suppose further that $f''(a)$ is neither positive nor negative semidefinite. Prove that f does not have a local extremum at a.

Proof. Let $U \subseteq \mathbb{R}^n$ be open, let $a \in U$, let $f : U \to \mathbb{R}$ be a C^2 function, and suppose that $f'(a) = 0$. Suppose further that $f''(a)$ is neither positive nor negative semidefinite. So, $f''(a)$ has a positive and negative eigenvalue. Hence, by the previous part, there exists $u_+, v_- \in \mathbb{R}^n$ and $r > 0$ such that for all $x \in B_r(a)$, then $\langle f''(a), u_+, u_+\rangle > 0$ and $\langle f''(a)v_-, v_-\rangle$. Let $r_1 > 0$ be arbitrary but fixed. Then let $s = \min\{r, r_1\}$. So, $(a + \frac{v_-}{2s}), (a + \frac{v_-}{2s}) \in B_r(a)$ and $(a + \frac{v_-}{2s}), (a + \frac{v_-}{2s}) \in B_{r_1}(a)$. Then,

$$\begin{aligned} f(a + \frac{u_+}{2s} &= f(a) + f'(a)\frac{u_+}{2s} + \frac{1}{2}f''(a + \theta_+\frac{u_+}{2s})(\frac{u_+}{2s}, \frac{u_+}{2s}) && (0 < \theta_+ < 1) \\ &= f(a) + \frac{1}{2}f''(a + \theta_+\frac{u_+}{2s})(\frac{u_+}{2s}, \frac{u_+}{2s}) && (\text{Since we know } f'(a) = 0) \\ &= f(a) + \frac{1}{2}\langle f''(a + \theta_+\frac{u_+}{2s})\frac{u_+}{2s}, \frac{u_+}{2s}\rangle && (1) \\ &= f(a) + \frac{1}{8s^2}\langle f''(a + \theta_+\frac{u_+}{2s})u_+, u_+\rangle. && (2) \end{aligned}$$

Thus, $\langle f''(a + \theta_+\frac{u_+}{2s})u_+, u_+\rangle > 0$ since $(a + \theta_+\frac{u_+}{2s}) \in B_r(a)$. In a similar argument, one can show that $\langle f''(a+\theta_-\frac{v_-}{2s})v_-, v_-\rangle < 0$. So, $f(a+\frac{u_+}{2s}) = f(a)+\frac{1}{8s^2}\langle f''(a+\theta_+\frac{u_+}{2s})u_+, u_+\rangle$ and $f(a + \frac{v_-}{2s}) = f(a) + \frac{1}{8s^2}\langle f''(a + \theta_+\frac{v_-}{2s})v_-, v_-\rangle$, which implies $f(a + \frac{u_+}{2s}) > f(a)$ and $f(a + \frac{v_-}{2s}) < f(a)$. Therefore, there exists $x, y \in B_{r_1}(a)$ such that $f(x) > a$ and $f(y) < f(a)$. Therefore, f has no local extrema at a. □

Problem 19. Let $f(x, y) = \frac{1}{1-x-2y}$ for (x,y) in a neighborhood of 0 in $\mathbb{R}^2$.

(a) Find $D_i f(0, 0)$ and $D_{ij}f(0, 0)$ for $i, j = 1, 2$. Calculate $P_2(x, y)$ using the formula for the second order Taylor polynomial.

$$D_1f(0, 0) = D_2f(0, 0) = \frac{1}{9y^2 - 6y + 1}, \qquad D_1D_2f(0, 0) = \frac{1}{27y^2 - 27y + 9y - 1}$$

So, $P_2(x, y) = -\frac{1}{3y-1} + \frac{x}{9y^2-6y+1} + \frac{1}{9y^2-6y+1} + \frac{x^2}{27y^2-27y+9y-1} + \frac{y^2}{27y^2-27y+9y-1} + \frac{2xy}{27y^2-27y+9y-1}$

(b) Use the formula for a geometric series to calculate $P_2(x, y)$.

Problem 20. Let $0 < r < R$ and define $f : \mathbb{R}^2 \to \mathbb{R}^3$ by

$$f(\theta, \alpha) = ((R + r\cos\alpha)\cos\theta, (R + r\cos\alpha)\sin\theta, r\sin\alpha).$$

(The range, T, of f is a *torus*.)

(a) Find all points of the form $f(\theta, \alpha) \in T$ such that $Df_1(\theta, \alpha) = 0$. (Hint: your answer will be a finite subset of $\mathbb{R}^3$.)

Proof. Let $0 < r < R$ and define $f : \mathbb{R}^2 \to \mathbb{R}^3$ by

$$f(\theta, \alpha) = ((R + r\cos\alpha)\cos\theta, (R + r\cos\alpha)\sin\theta, r\sin\alpha).$$

Then,

$$\begin{aligned} D_\theta f_1 &= -r\cos\alpha\sin\theta = 0 \Longrightarrow \alpha = 0, \pi, \quad \theta = \frac{\pi}{2}, \frac{3\pi}{2} \\ D_\alpha f_1 &= -r\cos\theta\sin\alpha = 0 \Longrightarrow \theta = 0, \pi, \quad \alpha = \frac{\pi}{2}, \frac{3\pi}{2}. \end{aligned}$$

So, in $\mathbb{R}^3$, we have the set of critical values $\{(0, R+r, 0), (0, R-r, 0), (0, -R-r, 0), (0, r-R, 0), (R + r, 0, r), (R - r, 0, -r), (-R - r, 0, r), (r - R, 0, -r)\}$. □

(b) Show that one of the points in part (a) corresponds to a local maximum of f_1, one corresponds to a local minimum of f_1 and the others are neither local maxima nor local minima of f_1.

Problems 21 - 22 finish the proof of the implicit function theorem in two variables. Let $f : U \subseteq \mathbb{R}^2 \to \mathbb{R}$, $(a, b) \in U$, $c = f(a, b)$, and $g : (a - s, a + s) \to (b - r, b + r)$ be as in the implicit function theorem (Theorem 9.1 in the notes). It has been shown that g is continuous on $(a - s, a + s)$. Complete the proof of the theorem by showing that g is differentiable on $(a - s, a + s)$, with derivative $g'(x) = -D_1 f(x, g(x))/D_2 f(x, g(x))$, using the following outline.

Problem 21. First prove it for $x = a$ as follows. Let $A = D_1 f(a, b)$ and $B = D_2 f(a, b)$ (so that $f'(a, b)$ has a matrix $(A \;\; B) \in M_{1\times 2}$.) Let $x \in (a-s, a+s)$ and set $h = x-a$, $k = g(x)-b$.

(a) Prove that there are real-valued functions ψ_1 and ψ_2 defined in a neighborhood of 0 such that $\lim_{(x,y)\to 0} \phi_i(x, y) = 0$ for $i = 1, 2$, and such that

$$\frac{h}{k} + \frac{A}{B} + \frac{1}{B}\psi_1(h, k) + \frac{1}{B}\psi_2(h, k)\frac{k}{h} = 0.$$

(Hint: let $\phi(h, k)$ be as in the alternate version of differentiability of f (notes, Lemma 3.16), and write

$$\phi(h, k)\|(h, k)\| = \phi(h, k)\frac{\|(h, k)\|}{|h| + |k|}\left(\frac{|h|}{h}h + \frac{|k|}{k}k\right).)$$

Proof. Let $f : U \subseteq \mathbb{R}^2 \to \mathbb{R}$ be continuously differentiable, let $(a, b) \in U$ with $D_2 f(a, b) \neq 0$. Let $c = f(a, b)$. Let $g : B_s(a) \to B_r(b)$ with $f(x, , g(x)) = c$. Let $A =$

$D_1 f(a,b)$ and $B = D_2 f(a,b)$. Now, there exists $\phi : B_r(0) \to \mathbb{R}$, since f is differentiable, such that $\phi(0) = 0$, ϕ is continuous at 0, and $f(a+h) = f(a) + T(h) + \phi(h)\|h\|$. Define $\psi_1, \psi_2 : B_r(0) \to \mathbb{R}$ by

$$\psi_1(h,k) = \begin{cases} \frac{\phi(h,k)\|(h,k)\||h|}{(|h|+|k|)h}, & \text{if } h \neq 0 \\ 0, & \text{if } h = 0 \end{cases}$$

and

$$\psi_2(h,k) = \begin{cases} \frac{\phi(h,k)\|(h,k)\||k|}{(|h|+|k|)k}, & \text{if } k \neq 0 \\ 0, & \text{if } k = 0. \end{cases}$$

Note, $\lim_{(h,k)\to 0} \phi(h,k) = 0$ because of the definition of ϕ, which implies that $\lim_{(h,k)\to 0} \psi_1(h,k) = 0$ and $\lim_{(h,k)\to 0} \psi_2(h,k) = 0$. Let $x \in (a-s, a+s) \setminus \{a\}$ such that $(x-a, g(x)-b) \in B_r(0)$. Let $h = x - a$ and $k = g(x) - b$. Therefore, by definition of ϕ, $f((a,b)+(h,k)) = f(a,b) + f'(a,b)(h,k) + \phi(h,k)\|(h,k)\|$. So, $f((a,b)+(h,k)) = f((a,b)+(x-a, g(x)-b)) = f(x, g(x)) = c$ and $f(a,b) = c$, which gives us

$$\begin{aligned}
0 &= f'(a,b)(h,k) + \phi(h,k)\|(h,k)\| \\
&= (A \;\; B)(h,k) + \phi(h,k)\frac{\|(h,k)\|}{|h|+|k|}\left(\frac{|h|}{h}h + \frac{|k|}{k}k\right) \\
&= (A \;\; B)(h,k) + \frac{\phi(h,k)\|(h,k)\|\,|h|}{(|h|+|k|)} \cdot h + \frac{\phi(h,k)\|(h,k)\|\,|k|}{(|h|+|k|)} \cdot k \\
&= (A \;\; B)(h,k) + \psi_1(h,k)\cdot h + \psi_2(h,k)\cdot k && \text{(By definition of } \psi_1 \text{ and } \psi_2\text{)} \\
&= Ah + Bk + \psi_1(h,k)\cdot h + \psi_2(h,k)\cdot k \\
&= Bh\left(\frac{A}{B} + \frac{k}{h} + \frac{1}{B}\psi_1(h,k) + \frac{1}{b}\psi_2(h,k)\frac{k}{h}\right) \\
&= \frac{h}{k} + \frac{A}{B} + \frac{1}{B}\psi_1(h,k) + \frac{1}{B}\psi_2(h,k)\frac{k}{h}. && \text{(Since } B = D_2 f(a,b) \neq 0\text{)}
\end{aligned}$$

□

(b) Prove that $g'(a) = -A/B$. (Hint: solve for $\frac{h}{k}$ in part (a).)

Proof. Solving $\frac{h}{k} + \frac{A}{B} + \frac{1}{B}\psi_1(h,k) + \frac{1}{B}\psi_2(h,k)\frac{k}{h} = 0$ for $\frac{k}{h}$, then we get

$$\frac{k}{h} = \frac{-(A + \psi_1(h,k))}{B + \psi_2(h,k)}.$$

Then,

$$\begin{aligned}
\lim_{x\to a}\frac{g(a)-g(x)}{a-x} &= \lim_{x\to a}\frac{g(x)-g(a)}{x-a}\\
&= \lim_{x\to a}\frac{g(x)-b}{x-a}\\
&= \lim_{(h,k)\to 0}\frac{k}{h} && \text{(By definition of } h \text{ and } k\text{)}\\
&= \frac{-(A+\psi_1(h,k))}{B+\psi_2(h,k)}\\
&= \frac{-A}{B}. && \text{(Since } \lim_{(h,k)\to 0}\psi_{1,2}=0\text{)}
\end{aligned}$$

Therefore, $g'(a) = \frac{-A}{B}$. □

Problem 22. Finish the proof of the implicit function theorem. Also show that if f in the statement is C^k for $k > 1$ the g is also C^k.

Proof. Continuing, we must show that g is continuous at all $a \in B_s(a)$. Let $\epsilon > 0$ be arbitrary but fixed. Let $a' \in B_s(a)$ be arbitrary but fixed. Let $b' \in B_r(b)$ such that $g(a') = b'$. Let $Z = \{(x,y) \in \mathbb{R}^2 : x \in B_s(a), y \in B_r(b), |y-b'| < \epsilon\}$. Now, by construction of s, $D_2 f\big|_Z(a',b') \neq 0$. So, we now have a $B_s(a)', B_r(b)'$ such that $B_s(a)' \subseteq B_s(a)$ and $B_r(b)' \subseteq B_r(b)$ and g_1 such that $g_1 : B_s(a)' \to B_r(b)'$ such that for each $x \in B_s(a)$, $D_i(x, g_1(x)) = 0$ for $i = 1,2$ and $d(g_1(x), b') < \epsilon$. But, by uniqueness of $g(x)$, we get $g_1(x)$ for all $x \in B_s(a)$. Thus, for all $x \in B_s(a)$, $d(g(x), g_1(x)) < \epsilon$. Hence, g is continuous at a'. But, since a' was arbitrary in $B_s(a)$, then g is continuous in over $B_s(a)$.

Now, on showing f is C^k implies g is C^k, we have already proven the base case of if f is C^1, then g is C^1. So, we will continue with the inductive step. Suppose the theorem is true for some $k > 1$. So, when f is C^{k+1}, then g is C^k. This is because $A, B^{-1} \in C^k$ and g is a composition of the two. Therefore, $g' \in C^k$, which implies $g \in C^{k+1}$. Therefore, if f is C^k, then g is C^k. □

Problem 23. Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be given by $f(\rho,\phi,\theta) = (\rho\sin\phi\sin\theta, \rho\cos\phi)$.

(a) Use the implicit function theorem to show that the equation $f(\rho,\phi,\theta) = (1,1)$ can be solved for (ϕ,θ) as a function of ρ near the point $(\sqrt{3}, \tan^{-1}\sqrt{2}, \pi/4)$.

Proof. Consider the surface $S := \{(\rho,\phi,\theta) \in \mathbb{R}^3 : \rho\sin\phi\sin\theta = 1 \text{ and } \rho\cos\phi = 1\}$. This can be rewritten as $\{(\rho,\phi,\theta) \in \mathbb{R}^3 : f(\rho,\phi,\theta) = (0,0)\}$ where $f : \mathbb{R}^3 \to \mathbb{R}^2$ is given by $f(\rho,\phi,\theta) := (\rho\sin\phi\sin\theta - 1, \rho\cos\phi - 1)$. Then,

$$\begin{aligned}
f'(\rho,\phi,\theta) &= \begin{pmatrix} D_1 f_1 & D_2 f_1 & D_3 f_1 \\ D_1 f_2 & D_2 f_2 & D_3 f_2 \end{pmatrix}\\
&= \begin{pmatrix} \sin(\phi)\sin(\theta) & \rho\cos(\phi)\sin(\theta) & \rho\cos(\theta)\sin(\phi) \\ \cos(\phi) & -\rho\sin(\phi) & 0 \end{pmatrix}.
\end{aligned}$$

Now, at $(\sqrt{3}, \tan^{-1}\sqrt{2}, \pi/4)$, we have

$$f'(\sqrt{3}, \tan^{-1}\sqrt{2}, \pi/4) = \begin{pmatrix} \frac{\sqrt{3}}{3} & \frac{\sqrt{2}}{2} & 1 \\ \frac{\sqrt{3}}{3} & -\sqrt{2} & 0 \end{pmatrix}.$$

So, $\frac{\partial f}{\partial(\phi,\theta)}(\sqrt{3}, \tan^{-1}\sqrt{2}, \pi/4) = \begin{pmatrix} \frac{\sqrt{2}}{2} & 1 \\ -\sqrt{2} & 0 \end{pmatrix}$ and has a determinant $0 + \sqrt{2} = \sqrt{2} \neq 0$. Since the matrix is invertible, then by the implicit function theorem, there exists $r, s > 0$, and a unique function $g : (\sqrt{3} - s, \sqrt{3} + s) \to B_r\left((\tan^{-1}\sqrt{2}, \pi/4)\right)$, such that $f(\rho, g(\rho)) = (0, 0)$ for all $x \in (\sqrt{3} - s, \sqrt{3} + s)$. □

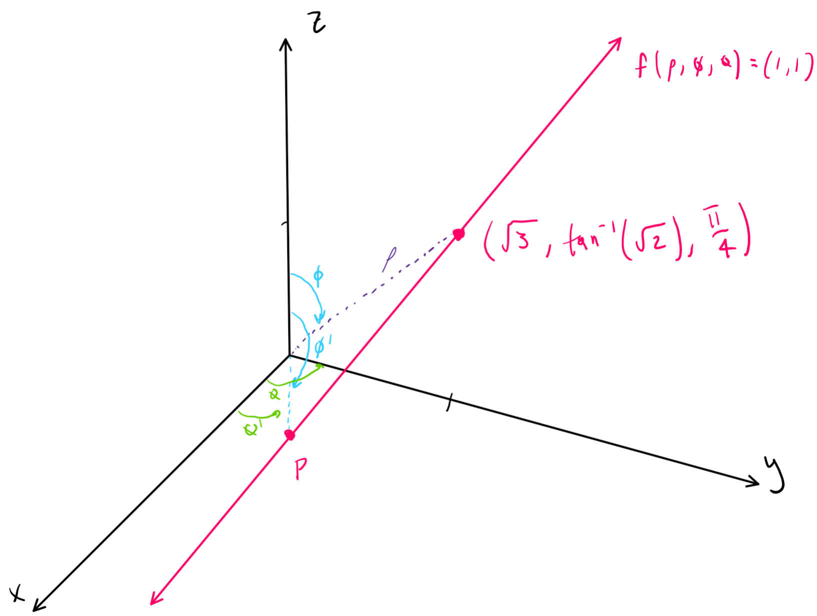
(b) Use the implicit function theorem to find $\phi'(\sqrt{3})$ and $\theta'(\sqrt{3})$.

Proof. We have that $\frac{\partial f}{\partial \theta}(\sqrt{3}, \tan^{-1}\sqrt{2}, \pi/4) = \begin{pmatrix} \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \end{pmatrix}$. So,

$$\begin{aligned} g'(\sqrt{3}) &= -\begin{pmatrix} \frac{\sqrt{2}}{2} & 1 \\ -\sqrt{2} & 0 \end{pmatrix}^{-1} \begin{pmatrix} \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \end{pmatrix} \\ &= -\left(\frac{1}{2}\right) \begin{pmatrix} 0 & -\sqrt{2} \\ 2 & 1 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \end{pmatrix} \\ &= \begin{pmatrix} \frac{\sqrt{6}}{6} \\ -\frac{\sqrt{3}}{2} \end{pmatrix}. \end{aligned}$$

Therefore, $\phi'(\sqrt{3}) = \frac{\sqrt{6}}{6}$ and $\theta'(\sqrt{3}) = -\frac{\sqrt{3}}{2}$. □

(c) Give a geometric description of the situation, and explain why the results are reasonable.



Note, the point P is after our change in x. So, $\phi'(\sqrt{3}) = \frac{\sqrt{6}}{6} > 0$ and $\theta'(\sqrt{3}) = -\frac{\sqrt{3}}{2} < 0$, which makes sense because when ρ increases, then θ decreases and ϕ increases, which is what the math tells us.

Problem 24. Consider the equation $xe^y + ye^x = 0$.

(a) Prove that this equation defines y as a C^∞ function of x in a neighborhood of $(0,0)$.

Proof. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = xe^y + ye^x$. Then, $D_1 f(x,y) = e^y + ye^x$, $D_{1,1} f(x,y) = ye^x$, $D_2 f(x,y) = xe^y + e^x$, $D_{2,2} f(x,y) = xe^y$, and $D_{1,2} f(x,y) = e^y + e^x$.

Let $P(n)$ be the statement "$\frac{\partial^n f}{\partial y^n}(x,y) = xe^y$".

Base Case: $\frac{\partial^2 f}{\partial y^2} = xe^y$. Thus, the base case is true.

Inductive Step: Suppose $P(k)$ is true for some $k \geq 2$. Then, $\frac{\partial^k f}{\partial y^k}(x,y) = xe^y$, which implies $\frac{\partial^{k+1} f}{\partial y^{k+1}}(x,y) = xe^y$. So, $P(k+1)$ is true.

Now, let $P(n)$ be the statement "$\frac{\partial^n f}{\partial x^n}(x,y) = ye^x$".

Base Case: $\frac{\partial^2 f}{\partial x^2}(x,y) = ye^x$. Thus, the base case is true.

Inductive Step: Suppose $P(k)$ is true for some $k \geq 2$. Then, $\frac{\partial^k f}{\partial x^k}(x,y) = ye^x$, which implies $\frac{\partial^{k+1} f}{\partial x^{k+1}}(x,y) = ye^x$. So, $P(k+1)$ is true. Let $m \geq 2$ be arbitrary but fixed. Then,

$\frac{\partial^m f}{\partial y^m}(x,y) = xe^y$ implies $\frac{\partial^{m+1} f}{\partial y^m x}(x,y) = e^y$. Also, $\frac{\partial^m f}{\partial x^m}(x,y) = ye^x$ implies $\frac{\partial^{m+1} f}{\partial x^m y}(x,y) = e^x$.

Lastly, let $P(n)$ now be the statement " $\frac{\partial^{m+n} f}{\partial x^m y^n}(x,y) = 0$.

Base Case: $\frac{\partial^{m+1} f}{\partial x^m y} = e^x$, which implies $\frac{\partial^{m+2} f}{\partial x^m y^2}(x,y) = 0$. Thus, the base case is true.

Inductive Step: Suppose $P(k)$ is true for some $k \geq 2$. Then, $\frac{\partial^{m+k} f}{\partial x^m y^k}(x,y) = 0$, which implies $\frac{\partial^{m+k+1} f}{\partial x^m y^{k+1}} = 0$. So, $P(k+1)$ is true.

In total, we have $\frac{\partial f}{\partial x}(x,y) = e^y + ye^x$, $\frac{\partial f}{\partial y}(x,y) = xe^y + e^x$, $\frac{\partial^n f}{\partial x^n}(x,y) = ye^x$, $\frac{\partial^{n+1} f}{\partial x^n}(x,y) = e^x$, $\frac{\partial^n f}{\partial y^n}(x,y) = xe^y$, $\frac{\partial^{n+1} f}{\partial y^n}(x,y) = e^y$, and $\frac{\partial^{m+n} f}{\partial x^m y^n}(x,y) = 0$ for all $n, m \geq 2$. Clearly, they are all continuous. Therefore, f is C^∞. Note, $f(0,0) = 0$ and $D_2 f(0,0) = 1 \neq 0$, so by the implicit function theorem, there exists $r, s > 0$ and $g : B_s(0) \to B_r(0)$ defined by $f(x, g(x)) = 0$. Therefore, f and g are C^∞ and $f(x,y) = 0$ defines y as a C^∞ function of x in a neighborhood of $(0,0)$. □

(b) Let $y = g(x)$ be this implicitly defined function. Find $g'(0)$ and $g''(0)$.

Proof. Note,

$$g'(x) = -\frac{D_1 f}{D_2 f} = -\frac{e^y + ye^x}{xe^y + e^x}$$

and

$$\begin{aligned} g''(x) &= \frac{-(\frac{\partial f}{\partial y})^2 \frac{\partial^2 f}{\partial x^2} + 2\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}\frac{\partial^2 f}{\partial xy} - (\frac{\partial f}{\partial x})^2 \frac{\partial^2 f}{\partial y^2}}{(\frac{\partial f}{\partial y})^3} \\ &= \frac{-(xe^y + e^x)^2(ye^x) + 2(e^y + ye^x)(xe^y + e^x)(e^y + e^x) - (e^y + ye^x)^2(xe^y)}{(xe^y + e^x)^3}. \end{aligned}$$

So, from evaluating, we get $g'(0) = -1$ and $g''(0) = -4$. □

(c) Use this information to explain the appearance of the curve $xe^y + ye^x = 0$ near $(0,0)$. As (x,y) approaches $(0,0)$, the slope is directed downward at a decreasing rate.

Problem 25. Let $p, q \geq 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

(a) Use the method of Lagrange multipliers to find the minimum of $\frac{1}{p}x^p + \frac{1}{q}y^q$ subject to the constraints $xy = 1$ and $x > 0$.

Proof. Taking each partial derivative, we must find a Lagrange multiplier λ such that

$$x^{p-1} - \lambda y = 0 \tag{3}$$
$$y^{q-1} - \lambda x = 0. \tag{4}$$

Note, $x = \frac{1}{y}$ and $y = \frac{1}{x}$. So, adding λy and multiplying by x, or $\frac{1}{y}$, to equation (1), we get $x^p = \lambda$. Similarly with equation (2), we get $y^q = \lambda$. Since $xy = 1$, then $x = y = 1$. Therefore, the minimum of $\frac{1}{p}x^p + \frac{1}{q}y^q$ subject to the constraints $xy = 1$ and $x > 0$ with $p, q \geq 1$ and $\frac{1}{p} + \frac{1}{q} = 1$ is $\frac{1}{p} + \frac{1}{q} = 1$. $\square$

(b) Prove that $\frac{1}{p}x^p + \frac{1}{q}y^q \geq xy$ for all $x, y \geq 0$.

Proof. Clearly, it is true if either x or y are zero, so suppose x and y are greater than zero. Note, if (x, y) satisfy the inequality, then all numbers of the form $xr^{\frac{1}{p}}$ and $yr^{\frac{1}{q}}$ are true for any $r \in \mathbb{R}^+$. Thus, we may restrict ourselves such that $xy = 1$, which implies we want to show that for all $x, y \in \mathbb{R}^+$ with $xy = 1$, we get

$$\frac{1}{p}x^p + \frac{1}{q}y^q \geq 1.$$

So, we must see if the minimum of $\frac{1}{p}x^p + \frac{1}{q}y^q$ subject to the constraints exists. This is exactly part (a). Thus, $\frac{1}{p}x^p + \frac{1}{q}y^q \geq xy$ for all $x, y \geq 0$. $\square$

(c) Prove Hölder's inequality: if $u_i, v_i \geq 0$ for $i = 1, \ldots, n$, then

$$\sum_{i=1}^n u_i v_i \leq \left(\sum_{i=1}^n u_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^n v_i^q\right)^{\frac{1}{q}}.$$

(Hints: for $u = (u_1, \ldots, u_n)$ let $\|u\|_p = (\sum_{i=1}^n u_i^p)^{\frac{1}{p}}$. If $\|u\|_p, \|v\|_q \neq 0$, let $x = \frac{u_i}{\|u\|_p}$ and $\frac{v_i}{\|v\|_q}$ in part (b).)

Proof. Let

$$u = \frac{u_i}{(\sum_{i=1}^n u_i^p)^{\frac{1}{p}}} \qquad \text{and} \qquad v = \frac{v_i}{(\sum_{i=1}^n v_i^q)^{\frac{1}{q}}},$$

such that $u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n \in \mathbb{R}^+$ and each component of (u, v) nonzero. Then, by Young's inequality, which is part (b), we get

$$\sum_{i=1}^n |u_i v_i| \leq \sum_{i=1}^n \left(\frac{u_i^p}{p} + \frac{v_i^q}{q}\right).$$

Using the fact that $\frac{1}{p} + \frac{1}{q} = 1$, we get $\sum_{i=1}^n |u_i v_i| \leq 1$. One can also show that $\sum_{i=1}^n u_i^p = 1$ and $\sum_{i=1}^n v_i^p = 1$. Therefore, we get Hölder's inequality

$$\sum_{i=1}^n u_i v_i \leq \left(\sum_{i=1}^n u_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^n v_i^q\right)^{\frac{1}{q}}.$$

$\square$

Problem 26. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f(x) = (e^{x_1} \cos x_2, e^{x_1} \sin x_2)$.

(a) Find (with proof) the range of f.

Proof. Consider the following sets: $Z_1 = \mathbb{R}_* \times \mathbb{R}$ and $Z_2 = \mathbb{R} \times \mathbb{R}_*$. For $a \in Z_1$ or $a \in Z_2$, let $g_1 : Z_1 \to \mathbb{R}^2_*$ be defined by $g_1(a) = f(b_1, b_2)$ and let $g_2 : Z_2 \to \mathbb{R}^2_*$ be defined by $g_2(a) = f(b_1, b_2)$, respectively such that $b_1 = \ln\left(\sqrt{a_1^2 + a_2^2}\right)$ and

$$b_2 = \begin{cases} \tan^{-1}\left(\frac{a_2}{a_1}\right), & \text{if } a_1 \neq 0 \\ \cot^{-1}\left(\frac{a_1}{a_2}\right), & \text{if } a_2 \neq 0. \end{cases}$$

Without loss of generality, since it can be shown similarly in both cases, let $a \in Z_1$. Then,

$$\begin{aligned} g_1(a) &= f(b_1, b_2) \\ &= \left(e^{\ln\left(\sqrt{a_1^2+a_2^2}\right)} \cos\left(\tan^{-1}\left(\frac{a_2}{a_1}\right)\right), e^{\ln\left(\sqrt{a_1^2+a_2^2}\right)} \sin\left(\tan^{-1}\left(\frac{a_2}{a_1}\right)\right)\right) \\ &= \left(\sqrt{a_1^2+a_2^2}\left(\frac{a_1}{\sqrt{a_1^2+a_2^2}}\right), \sqrt{a_1^2+a_2^2}\left(\frac{a_2}{\sqrt{a_1^2+a_2^2}}\right)\right) \\ &= (a_1, a_2). \end{aligned}$$

Thus, f is surjective if we consider the codomain to be $\mathbb{R}^2_*$. Also, clearly, we cannot have $0 \in \operatorname{ran} f$ since $e^{x_1} > 0$ for all $x_1 \in \mathbb{R}$ and if $\sin x_2 = 0$, then $\cos x_2 \neq 0$. Therefore, exhausting all possibilities, we have $\operatorname{ran} f = \mathbb{R}^2_*$. □

(b) Prove that $f'(x)$ is non-singular for every $x \in \mathbb{R}^2$, but that f is not one-to-one.

Proof. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f(x) = (e^{x_1} \cos x_2, e^{x_1} \sin x_2)$. Then,

$$f'(x) = \begin{pmatrix} e^{x_1} \cos x_2 & -e^{x_1} \sin x_2 \\ e^{x_1} \sin x_2 & e^{x_1} \cos x_2 \end{pmatrix}.$$

Now,

$$\begin{aligned} \left| \begin{pmatrix} e^{x_1} \cos x_2 & -e^{x_1} \sin x_2 \\ e^{x_1} \sin x_2 & e^{x_1} \cos x_2 \end{pmatrix} \right| &= e^{2x_1} \cos^2 x_2 + e^{2x_1} \sin^2 x_2 \\ &= e^{2x_1}. \end{aligned}$$

Since $e^{2x_1} > 0$ for all $x_1 \in \mathbb{R}$, then $f'(x)$ is always non-singular. Also, since neither cos nor sin are injective on $\mathbb{R}$, then f is not injective. □

(c) Let $U = \{x \in \mathbb{R}^2 : |x_2| < \pi\}$. Prove that $f\big|_U$ is one-to-one, and find $f(U)$. Also prove that for any open set V properly containing U, $f\big|_V$ is not one-to-one.

Proof. Let $U = \{x \in \mathbb{R}^2 : |x_2| < \pi\} = \{x \in \mathbb{R}^2 : x_2 \in (-\pi, \pi)\}$. Let $x, y \in U$ such that $f(x) = f(y)$. Then,

$$e^{x_1} \cos x_2 = e^{y_1} \cos y_2 \qquad e^{x_1} \sin x_2 = e^{y_1} \sin y_2$$
$$e^{2x_1} \cos^2 x_2 = e^{2y_1} \cos^2 y_2 \qquad e^{2x_1} \sin^2 x_2 = e^{2y_1} \sin^2 y_2. \quad \text{(By squaring both sides)}$$

By adding the two equations together, we get

$$e^{2x_1} \cos^2 x_2 + e^{2x_1} \sin^2 x_2 = e^{2y_1} \cos^2 y_2 + e^{2y_1} \sin^2 y_2,$$

which simplifies to $e^{2x_1} = e^{2y_1}$. These are injective functions, which implies $x_1 = y_1$. By substitution, we get $e^{x_1} \cos x_2 = e^{x_1} \cos y_2$ and $e^{x_1} \sin x_2 = e^{x_1} \sin y_2$. By division, we have $\cos x_2 = \cos y_2$ and $\sin x_2 = \sin y_2$. This is broken down in the following cases:

Case 1: $x_2 \in (-\pi, 0)$:

Case 1a: Let $y_2 \in (-\pi, 0)$. Note, cos is injective on this interval, so $\cos x_2 = \cos y_2$ implies $x_2 = y_2$.

Case 1b: Let $y_2 \in [0, \pi)$. Since $\sin x_2 < 0$ and $\sin y_2 \geq 0$, then $\sin x_2 \neq \sin y_2$, which is impossible.

Case 2: $x_2 \in [0, \pi)$:

Case 1a: Let $y_2 \in (-\pi, 0)$. Since $\sin x_2 \geq 0$ and $\sin y_2 < 0$, then $\sin x_2 \neq \sin y_2$, which is impossible.

Case 1b: Let $y_2 \in [0, \pi)$. Note, cos is injective on this interval, which implies $\cos x_2 = \cos y_2$, which implies $x_2 = y_2$.

So, in all possible cases, we have $x_2 = y_2$. Therefore, $x = y$. Thus, $f\big|_U$ is injective.

Now, let $a \in \mathbb{R}^2_*$. Also, let $b_1 = \ln\left(\sqrt{a_1^2 + a_2^2}\right)$ and

$$b_2 = \begin{cases} \tan^{-1}\left(\frac{a_2}{a_1}\right), & \text{if } a_1 \neq 0 \\ \cot^{-1}\left(\frac{a_1}{a_2}\right), & \text{if } a_2 \neq 0. \end{cases}$$

Note, $b \in U$ since the range of $\tan^{-1}$ and $\cot^{-1}$ is $(\frac{-\pi}{2}, \frac{\pi}{2})$. From (a), we know that $f(b) = a$. So, $\text{ran}\left(f\big|_U\right) \supseteq \mathbb{R}^2_*$.However, the cardinality of the range cannot be larger when the domain is restricted. Thus,

$$\text{ran}\left(f\big|_U\right) = \mathbb{R}^2_*.$$

Now, suppose $V \subseteq \mathbb{R}^2$ open such that $U \subset V$. Then, there exists $q \in V \setminus U$. Let

$$E = \begin{cases} \{q_2 + 2\pi k : k \in \mathbb{Z}, q_2 \geq q_2 + 2\pi k > -\pi\}, & \text{if } q_2 \geq \pi \\ \{q_2 + 2\pi k : k \in \mathbb{Z}, \pi > q_2 + 2\pi k \geq q_2\}, & \text{if } q_2 \leq -\pi. \end{cases}$$

Suppose, without loss of generality, that $q_2 \geq \pi$. Now, E is finite and $E \neq \emptyset$, since $q_2 \in E$. Let $m = \min E$. Then, if $m \notin U$, then $m = \pi$, else $m - 2\pi \in E$, which is a contradiction. If $m \in U$, let $u_2 = m$ with $u \in U$. Else, there exists $r > 0$ such that $B_r(q_1, q_2) \subseteq V$ since V is open.

Let $p = \min\{q_2 + \frac{r}{2}, q_2 + \frac{3\pi}{2}\}$. So, $(q_1, p) \in V$. Also, there exists $k \in \mathbb{Z}$ such that $q_2 + 2\pi k = m$ since $m \in E$, which implies

$$\pi = m = q_2 + 2\pi k < p + 2\pi k \leq q_2 + \frac{3\pi}{2} + 2\pi k = m + \frac{3\pi}{2} = \frac{5\pi}{2}.$$

By subtracting 2π, we get $-\pi < p + 2\pi(k-1) \leq \frac{3\pi}{2}$, which implies $(q_1, p + 2\pi(k-1)) \in U$. If this is the case, let $u_2 = p + 2\pi(k-1)$. Either way, we have $(q_1, u_2) \in V$ and $(q_1, u_2 + 2\pi\alpha) \in U \subset V$ for some $\alpha \in \mathbb{Z}$. So,

$$\begin{aligned} f(q_1, u_2) &= (e^{q_1} \cos u_2, e^{q_1} \sin u_2) \\ &= (e^{q_1} \cos(u_2 + 2\pi\alpha), e^{q_1} \sin(u_2 + 2\pi\alpha)) \\ &= f(q_1, u_2 + 2\pi\alpha). \end{aligned}$$

Therefore, $f\big|_V$ is not injective. □

Problem 27. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be continuously differentiable. In each part, the portion of the level set $\{x \in \mathbb{R}^2 : f(x) = 0\}$ is sketched near a point on the level set. What can you say about the derivatives $f'(A)$ and $f'(B)$? Justify your answers precisely.

Proof. Note, the important thing to consider with the two diagrams is whether, for $x = (x_1, x_2) \in \mathbb{R}^2$, x_1 can be represented as a function of x_2 or if x_2 can be represented as a function of x_1. Considering the diagram with point A, there clearly does not exist a function that can approximate one variable with the other. Thus, $D_1 f(A) = D_2 f(A) = 0$. As for the diagram with point B, x_1 can be represented as a function of x_2, but x_2 can not be represented as a function of x_1. Therefore, $D_2 f(B) = 0$. □

Problem 28. Let $f : U \subseteq \mathbb{R}^3 \to \mathbb{R}^2$ be continuously differentiable, let $a \in U$, and suppose that $\frac{\partial f}{\partial(x_2,x_3)}(a)$ is non-singular (as a 2×2 matrix). Prove that there are open subsets V and W of $\mathbb{R}^3$ with $a \in W$, and a C^1-diffeomorphism $h : V \to W$, such that $f \circ h(x) = (x_2, x_3)$ for all $x \in V$. (Hint: let $F(x) = (x_1, f(x))$ and use the inverse function theorem.)

Proof. Let $f : U \subseteq \mathbb{R}^3 \to \mathbb{R}^2$ be continuously differentiable, let $a \in U$, and suppose that $\frac{\partial f}{\partial(x_2,x_3)}(a) \in M_2$ is non-singular. Define $F : \mathbb{R}^3 \to \mathbb{R}^3$ by $F(x) = (x_1, f(x))$. Since $\frac{\delta f}{\delta(a_2,a_3)}$ is invertible, then by the implicit function theorem, there exists $r, s > 0$ such that $B_r(a_2, a_3) \subseteq U$, $\frac{\delta f}{\delta(x_2,x_3)}$ is invertible for all $(x_2, x_3) \in B_r(a_2, a_3)$, and for each $x_1 \in B_s(a_1)$, there exists a unique $h(x) \in B_r(a)$. ☐

Problem 29. Let $f : [a, b] \to \mathbb{R}$ and let $c \in [a, b]$. Recall that the *oscillation of f at c* is the quantity

$$\operatorname{osc}(f, c) = \lim_{r \to 0^+} \left(\sup_{x,y \in B_r(c) \cap [a,b]} |f(x) - f(y)| \right).$$

Prove that f is continuous at c if and only if $\operatorname{osc}(f, c) = 0$.

Proof. Let $f : [a, b] \to \mathbb{R}$ and $c \in (a, b)$.

($\Longrightarrow$) : Let f be continuous at c. Let $\epsilon > 0$ be arbitrary but fixed. Then, there exists $r > 0$ such that for $x, y \in B_r(c)$, we have $|f(x) - f(c)| < \frac{\epsilon}{2}$ and $|f(y) - f(c)| < \frac{\epsilon}{2}$, which from the triangle inequality implies

$$|f(x) - f(y)| \leq |f(x) - f(c)| + |f(y) - f(c)| < \epsilon.$$

Thus,

$$\lim_{r \to 0^+} \left(\sup_{x,y \in B_h(c) \cap [a,b]} |f(x) - f(y)| \right) \leq \epsilon \quad \text{if} \quad h < r.$$

This implies $\operatorname{osc}(f, c) = 0$.

($\Longleftarrow$) : Let $\operatorname{osc}(f, c) = 0$ and $\epsilon > 0$ be arbitrary but fixed. Then,

$$\lim_{r \to 0^+} \left(\sup_{x,y \in B_r(c) \cap [a,b]} |f(x) - f(y)| \right) < \epsilon$$

for some $r > 0$. So, $|f(x) - f(y)| < \epsilon$ if $x, y \in B_r(c)$. If $c = y$, then $|f(x) - f(c)| < \epsilon$. Therefore, f is continuous at c.

This argument is similar if $c = a$ or $c = d$. ☐

Problem 30. Let f be as in the previous problem, and let $L > 0$. Prove that the set $\{z \in [a, b] : \operatorname{osc}(f, z) \geq L\}$ is a closed set.

Proof. Let $f : [a,b] \to \mathbb{R}$ and $L > 0$. Define $A = \{z \in [a,b] : \operatorname{osc}(f,z) \geq L\}$. Notice that $A^c = (-\infty, a) \cup \{z \in [a,b] : \operatorname{osc}(f,z) < L\} \cup (b, \infty)$. Then we have the following cases.

<u>Case 1:</u> Let $c \in (-\infty, a) \cup (b, \infty)$. Then, since $(-\infty, a) \cup (b, \infty)$ is open, then there exists $r_0 > 0$ such that $B_{r_0}(c) \subseteq A^c$.

<u>Case 2:</u> Let $c \in \{z \in [a,b] : \operatorname{osc}(f,z) < L\}$. Then, there must exist $\delta > 0$ such that for all $r \in (0, \delta)$, then

$$\sup_{x,y \in B_r(c) \cap [a,b]} |f(x) - f(y)| < L.$$

Let $d \in [a,b]$. Fix $r \in (0, \delta)$. Let $d \in B_{\frac{r}{2}}(c)$. Clearly, if $d \notin [a,b]$, then we have $d \in (-\infty, a) \cup (b, \infty) \subseteq A^c$. Suppose $d \in [a,b]$. Let $x \in B_{\frac{r}{2}}(d) \cap [a,b]$. This gives us

$$\begin{aligned} |x - c| &\leq |x - d| + |c - d| && \text{(By triangle inequality)} \\ &< \frac{r}{2} + \frac{r}{2} \\ &= r. \end{aligned}$$

So, $x \in B_r(c)$, which implies $B_{\frac{r}{2}}(d) \subseteq B_r(c)$. Thus,

$$\sup_{x,y \in B_{\frac{r}{2}}(d) \cap [a,d]} |f(x) - f(y)| \leq \sup_{x,y \in B_r(c) \cap [a,b]} |f(x) - f(y)|. \tag{5}$$

We now have,

$$\begin{aligned} \operatorname{osc}(f,d) &= \lim_{r \to \infty} \left(\sup_{x,y \in B_{\frac{r}{2}}(d) \cap [a,d]} |f(x) - f(y)| \right) && \text{(Definition of oscillation of } f \text{ at } d\text{)} \\ &\leq \lim_{r \to \infty} \left(\sup_{x,y \in B_r(c) \cap [a,b]} |f(x) - f(y)| \right) && \text{(By (1))} \\ &= \operatorname{osc}(f,c) && \text{(Definition of oscillation of } f \text{ at } c\text{)} \\ &< L. \end{aligned}$$

Thus, $d \in A^c$, which implies $B_{\frac{r}{2}}(c) \subseteq A^c$.

Therefore, in both cases, we have that A^c is open, which implies that A is closed. $\square$

Problem 31. Let $[c,d]$ be a closed bounded interval, and let $(a_1, b_1), \ldots, (a_n, b_n)$ be open intervals such that $[c,d] \subseteq \cup_{i=1}^n (a_i, b_i)$. Prove that $d - c < \sum_{i=1}^n (b_i - a_i)$. (Hints: choose i_1 so that $c \in (a_{i_1}, b_{i_1})$. If $b_{i_1} \leq d$ choose i_2 so that $b_{i_1} \in (a_{i_2}, b_{i_2})$. Explain why in the continuation of this process there must be $k \leq n$ such that $d \in (a_{i_k}, b_{i_k})$.)

Proof. Let $C = [c, d]$ be a closed bounded interval and $A_1 = (a_1, b_1), \ldots, A_n = (a_n, b_n)$ be open intervals such that $[c, d] \subseteq \cup_{i=1}^{n} A_i$. Let $\epsilon > 0$ be arbitrary but fixed. Choose j such that $Q_j = (q_j, p_j) \supseteq A_j$ with $p_j - q_j \leq (1 + \epsilon) |b_j - a_j|$. Since $\cup_{j=1}^{\infty} Q_j$ is an open cover of the compact set $[c, d]$, there exists a finite subcover $[c, d] \subseteq \cup_{i=1}^{N} Q_j$. By taking the closure of each Q_j, we have $d - c \leq \sum_{j=1}^{N} (p_j - q_j)$. As a result, we have

$$d - c \leq (1 + \epsilon) \sum_{j=1}^{N} (b_j - a_j).$$

Since $\epsilon > 0$, then we have the strict inequality $d - c < \sum_{j=1}^{N} (b_j - a_j)$. □

Problem 32. For this exercise you must recall the definition and properties of *Lebesgue outer measure* from the notes. Let $A, B \subseteq \mathbb{R}$, and suppose that $m^*(A) = 0$. Prove that $m^*(A \cup B) = m^*(A) + m^*(B)$.

Proof. Let $A, B \subseteq \mathbb{R}$ and $m^*(A) = 0$. By Theorem 15.3 (2), we have $m^*(A \cup B) \leq m^*(A) + m^*(B)$. Also,

$$\begin{aligned} m^*(A) + m^*(B) &= 0 + m^*(B) && \text{(By monotonicity of the outer measure)} \\ &= m^*(B) \\ &\leq m^*(A \cup B). \end{aligned}$$

So, since $m^*(A) + m^*(B) \leq m^*(A \cup B)$ and $m^*(A \cup B) \leq m^*(A) + m^*(B)$, then we must have $m^*(A \cup B) = m^*(A) + m^*(B)$. □

Problem 33. Recall the *Borel σ-algebra* $\mathcal{B}_{\mathbb{R}}$ from the course notes. Prove that $\mathcal{B}_{\mathbb{R}}$ is generated as a σ-algebra by the collection of closed intervals $\{[a, \infty) : a \in \mathbb{R}\}$.

Proof. Let $\mathcal{O}$ denote the collection of all open intervals in $\mathbb{R}$. Since every open set in $\mathbb{R}$ is at most a countable union of open intervals, Then $\mathcal{M}(\mathcal{O}) = \mathcal{B}_{\mathbb{R}}$. Let $\mathcal{E}$ denote the collection of intervals of the form $[a, \infty)$ for all $a \in \mathbb{R}$. Let $(a, b) \in \mathcal{O}$ for some $a, b \in \mathbb{R}$ such that $b > a$. Let $a_n = a + \frac{1}{n}$ and $b_n = b - \frac{1}{n}$. Then,

$$(a, b) = \bigcup_{n=1}^{\infty} [a_n, b_n) = \bigcup_{n=1}^{\infty} \{[a_n, \infty) \cap [b_n, \infty)^c\},$$

which implies that $(a, b) \in \mathcal{M}(\mathcal{E})$. This means that $\mathcal{O} \subseteq \mathcal{M}(\mathcal{E})$, which implies $\mathcal{M}(\mathcal{O}) \subseteq \mathcal{M}(\mathcal{E})$. But, since every element of $\mathcal{E}$ is closed, then $\mathcal{M}(\mathcal{E}) \subseteq \mathcal{B}_{\mathbb{R}}$. This gives us

$$\mathcal{B}_{\mathbb{R}} = \mathcal{M}(\mathcal{O}) \subseteq \mathcal{M}(\mathcal{E}) \subseteq \mathcal{B}_{\mathbb{R}}.$$

Therefore, $\mathcal{M}(\mathcal{E}) = \mathcal{B}_{\mathbb{R}}$. □

Problem 34. Prove that for every subset $E \subseteq \mathbb{R}$ there is a G_δ-set A with $E \subseteq A$ and $m^*(E) = m^*(A)$.

Proof. Let $E \subseteq \mathbb{R}$ be arbitrary. By outer-measure, there exists a collection of open intervals $I_n \subseteq \mathbb{R}$ such that $E \subseteq \cup_{n=1}^{\infty} I_n$. From the result of Problem 31, this gives us

$$m^*(E) \leq \sum_{n=1}^{\infty} m(I_n) < m^*(E) + \frac{1}{n}.$$

Also, by countable subadditivity, $m^*(\cup_{n=1}^{\infty} I_n) \leq \sum_{n=1}^{\infty} m(I_n)$. Let $A = \cap_{n=1}^{\infty} \cup_{n=1}^{\infty} I_n$. Then, A is a G_δ-set. This gives us $E \subseteq A$. Then, for each n, we get

$$m^*(E) \leq m^*(A) \leq m^*(\cup_{n=1}^{\infty} I_n) \leq \sum_{n=1}^{\infty} m(I_n) < m^*(E) + \frac{1}{n}.$$

Therefore, by squeeze theorem, as n approaches ∞, then $m^*(E) = m^*(A)$. $\square$

Problem 35. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Let $A = \{x \in \mathbb{R} : f \text{ is continuous at } x\}$. Prove that A is a G_δ-set. (Hint: use the oscillation of f from homework 8.)

Proof. Let $A = \{x \in \mathbb{R} : f \text{ is continuous at } x\}$. Note, f is continuous at c if and only if $\operatorname{osc}(f, c) = 0$. That is, f is continuous at c if and only if

$$\limsup_{x \to c} f(x) = \liminf_{x \to c} f(x).$$

So, by looking at the complement of A, we get

$$\begin{aligned} A^c &= \left\{ x \in \mathbb{R} : \liminf_{x \to c} f(x) < \limsup_{x \to c} f(x) \right\} \\ &= \left\{ x \in \mathbb{R} : \exists a, b \in \mathbb{Q} \text{ s.t. } \liminf_{x \to c} f(x) \leq a < b \leq \limsup_{x \to c} f(x) \right\} \\ &= \bigcup_{a,b} \left(\left\{ x \in \mathbb{R} : \liminf_{x \to c} f(x) \leq a \right\} \bigcap \left\{ x \in \mathbb{R} : b \leq \limsup_{x \to c} f(x) \right\} \right). \qquad \text{(with } a < b\text{)} \end{aligned}$$

Note, if $\liminf_{x \to c} f(x) > a$, then there must exist $\epsilon > 0$ arbitrary but fixed such that $\inf_{|x-c|<\epsilon} f(x) > a$. Now, for each $x \in B_\epsilon(c)$, there is $\epsilon_0 > 0$ arbitrary but fixed such that $B_{\epsilon_0}(x) \subset B_\epsilon(c)$. So, we get

$$\inf_{|x-c'|<\epsilon_0} f(x) \geq \inf_{|x-c|<\epsilon} f(x) > a.$$

This means that $\left\{x \in \mathbb{R} : \liminf_{x\to c} f(x) > a\right\}$ is open, which implies $\left\{x \in \mathbb{R} : \liminf_{x\to c} f(x) \leq a\right\}$ is closed. We can similarly show that $\left\{x \in \mathbb{R} : b \leq \limsup_{x\to c} f(x)\right\}$ is closed since $\limsup_{x\to c} f(x) = -\left(\liminf_{x\to c}(-f(x))\right)$.

Finally, since each set in the pair of a countable union defined above is closed, then A^c is a F_σ-set, which implies A is a G_δ-set. □

Problem 36. For subsets $A, B \subseteq \mathbb{R}$ recall that the *distance* between A and B is defined to be $\text{dist}(A, B) = \inf\{|x - y| : x \in A, y \in B\}$. Let A and B be subsets of $\mathbb{R}$ such that $\text{dist}(A, B) > 0$. Prove that $m^*(A \cup B) = m^*(A) + m^*(B)$.

Proof. Let $A, B \subseteq \mathbb{R}$. Let $\epsilon > 0$ be arbitrary but fixed such that $\text{dist}(A, B) \geq \epsilon$. Let $E = \cup_{x\in A} B_{\frac{\epsilon}{2}}(x)$. Then, $A \subset E$. Also, since E used the ball of radius $\frac{\epsilon}{2}$, then $E \cap B = \emptyset$. Also, since E used a countable union of open balls, then E is measurable. This gives us $m^*(A\cup B) = m^*((A\cup B)\cap E) + m^*((A\cup B)\cap E^c)$ from Definition 17.1 in the notes. But, note that $(A\cup B)\cap E^c = B$ and $(A\cup B)\cap E = A$. Therefore, we have $m^*(A\cup B) = m^*(A) + m^*(B)$. □

Problem 37. Let $A_1, A_2, \ldots$ be measurable sets, and suppose that $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots$. Prove that $m(\cup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} m(A_n)$. (This is called *continuity from below* of Lebesgue measure.) (Hints: use Proposition 16.4 of the notes. It is useful also to remember that $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \sum_{i=1}^{n} a_i$.)

Proof. Let $A_1, A_2, \ldots$ be measurable sets, and suppose that $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots$. Suppose $\lim_{N\to\infty} \bigcup_{n=1}^{N} A_n = A$. From Proposition 16.4, we know

$$A = A_1 \cup \bigcup_{n=1}^{\infty} (A_{n+1} \setminus A_n).$$

Note, A_1 and $\bigcup_{n=1}^{\infty}(A_{n+1} \setminus A_n)$ are disjoint and A is a σ-algebra. Since they are disjoint and measurable, then we have

$$\begin{aligned} m(A) &= \sum_{n=1}^{\infty} m(A_n \setminus A_{n-1}) \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} m(A_i \setminus A_{i-1}) \\ &= \lim_{n\to\infty} m(A_n). \end{aligned}$$

□

Problem 38. Let $A_1, A_2, \dots$ be measurable sets, and suppose that $A_1 \supseteq A_2 \supseteq A_3 \supseteq \dots$. Suppose further that $m(A_1) < \infty$. Prove that $m(\cap_{n=1}^{\infty} A_n) = \lim_{n\to\infty} m(A_n)$. Be sure to indicate where the finiteness hypothesis is used. (This is called *continuity from above* of Lebesgue measure.) (Hints: as in the previous problem. Also, you will need to consider $B_\infty \coloneqq \cap_{n=1}^{\infty} A_n$.) Give an example of a decreasing sequence of measurable sets of infinite measure for which the above conclusion is false.

Proof. Let $A_1, A_2, \dots$ be measurable sets, and suppose that $A_1 \supseteq A_2 \supseteq A_3 \supseteq \dots$. Suppose further that $m(A_1) < \infty$ and $B_\infty \coloneqq \lim_{N\to\infty} \bigcap_{n=1}^{N} A_n$. Note, $m(A_1 \setminus B_\infty) = \lim_{n\to\infty} m(A_1 \setminus A_n)$. Also, $m(B_\infty) \le m(A_n) \le m(A_1) < \infty$. So, by Problem 37,

$$\begin{aligned} m(A_1) - m(B_\infty) &= m(A_1 \setminus B_\infty) \\ &= \lim_{n\to\infty} m(A_1 \setminus A_n) \\ &= m(A_1) - \lim_{n\to\infty} m(A_n). \end{aligned}$$

So, by subtracting the $m(A_1)$ terms from both sides, we get

$$m(B_\infty) = \lim_{n\to\infty} m(A_n).$$

□

Now, it is important that $A_k < \infty$ for some $k \in \mathbb{Z}$. For example, if not, suppose $A_n = (n, \infty)$. Then, $A_n \supseteq A_{n+1} \supseteq A_{n+2} \supseteq \dots$ and $m(A_n) = \infty$ for each n, but $\bigcap_{n=1}^{\infty} A_n = \emptyset$. So, we get

$$\infty = \lim_{n\to\infty} m(A_n) \ne m(\bigcap_{n=1}^{\infty} A_n) = 0.$$

Problem 39. Let E be a measurable set, and let $\epsilon > 0$. Prove that there are an open $U \supseteq E$ and a closed set $F \subseteq E$ such that $m(U \setminus F) < \epsilon$. Here is an outline.

(a) Suppose that $E \subseteq [a, b]$. Use the definition of outer measure to find an open set $U \supseteq E$ with $m(U \setminus E) < \epsilon$.

(b) Suppose that $E \subseteq [a, b]$. Apply the previous part to $[a, b] \setminus E$ to prove that there is a closed set $F \subseteq E$ with $m(E \setminus F) < \epsilon$.

(c) For the general case let $E_n = E \cap [n, n+1]$ for $n \in \mathbb{Z}$, and apply the previous two parts with $\epsilon 4^{-(|n|+1)}$. Use the fact that if $S_n \subseteq T_n$ then $(\cup_n T_n) \setminus (\cup_n S_n) \subseteq \cup_n (T_n \setminus S_n)$.

Proof. Let E be a measurable set with $m(E) < \infty$, and let $\epsilon > 0$ be arbitrary but fixed. Then, there are intervals (a_n, b_n) with $E \subseteq \bigcup_{n=1}^{\infty} (a_n, b_n)$ and $m(E) \le \sum_{n=1}^{\infty} m((a_n, b_n)) + \epsilon$. Define $U = \bigcup_{n=1}^{\infty} (a_n, b_n)$. Then, $U \supseteq E$ is open and $m(U) \le m(E) + \epsilon$. Now, since $m(U) < \infty$ and $m(E) < \infty$, then $m(U \setminus E) = m(U) - m(E) < \epsilon$.

Then, since E is measurable, then E^c is measurable. So, there exists $O \supseteq E^c$ open such that $m(O \setminus E^c) \leq \epsilon$. Let $F = O^c$, which implies F is closed and $F \subseteq E$. Then, $E \setminus F = O \setminus E^c$, which implies $m(E \setminus F) \leq \epsilon$.

Now, let $U \supseteq E$ open and $F \subseteq E$ be closed such that $m(U \setminus E) < \frac{\epsilon}{2}$ and $m(E \setminus F) < \frac{\epsilon}{2}$. Note,

$$\begin{aligned}
(U \setminus E) \cup (E \setminus F) &= (U \cap E^c) \cup (E \cap F^c) \\
&= [(U \cap E^c) \cup E] \cap [(U \cap E^c) \cup F^c] \\
&= [(U \cup E) \cap (E \cup E^c)] \cap [(U \cup F^c) \cap (E^c \cup F^c)] \\
&= (U \cup E) \cap [(F \setminus U)^c \cap F^c] \\
&= U \cap (\emptyset^c \cap F^c) \\
&= U \cap F^c \\
&= U \setminus F.
\end{aligned}$$

This gives us

$$m(U \setminus F) = m\left((U \setminus E) + m\left((E \setminus F)\right) < \epsilon.\right.$$

□

Problem 40. The *Cantor set*, C, is a subset of $[0,1]$ defined as follows. Let $F_0 = [0,1], F_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$, and in general, F_{n+1} is obtained from F_n by deleting the middle open third of each subinterval of F_n. (Thus $F_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$.) Then $C := \cap_{n=1}^{\infty} F_n$. Prove the following:

(a) F_n is the union of 2^n pairwise disjoint closed intervals each of length 3^{-n}.

Proof. Clearly, they are disjoint as you are removing the open middle third of each interval, essentialy doubling the amount of intervals each iteration. Thus, there are 2^n disjoint intervals for F_n. As for the length, F_0 has length $3^{-0} = 1$, and through induction, one can clearly see, without loss of generality, by taking the first of the 2^n intervals in F_n, call it A, that $\sup\{x - 0 : x \in A\} = 3^{-n}$. □

(b) $m(C) = 0$.

Proof. By continuity from above of Lebesgue measure, we know that

$$\begin{aligned}
m(C) &= m\left(\bigcap_{n=1}^{\infty} F_n\right) \\
&= \lim_{n\to\infty} m(F_n) \\
&= 0. && \text{(Since } m(F_n = \left(\tfrac{2}{3}\right)^n))
\end{aligned}$$

□

(c) C is a closed set, C has no isolated points, and the interior of C is empty.

Proof. Since C is a countable union of closed intervals, then C is closed. □

Problem 41. Let E be the nonmeasurable set desribed in section 18 of the notes. Prove that if $N \subseteq E$ and N is measurable, then $m(N) = 0$. (Hint: imitate the second part of the proof of Theorem 18.1.)

Proof. Let E be as described in section 18 of the notes. Let $N = \emptyset$. Then, $N \subseteq E$ and, trivially, $m(N) = 0$. So, there does exist a measurable subset of E. Now, suppose $N \subseteq E$ with $m(N) > 0$. Let $A = \mathbb{Q} \cap [0, 1]$. Then, for each $x_1, x_2 \in A$, with $x_1 \neq x_2$, we know

$$N + x_1 \bigcap N + x_2 = \emptyset.$$

Also, by translation invariance, $m(N) = m(N + x)$ for all $x \in A$. So, we have

$$\begin{aligned}\sum_{n=1}^{\infty} m(N) &= \sum_{n=1}^{\infty} m(N + x_n) && \text{(Each } x_n \in A\text{)} \\ &= m\left(\bigsqcup_{n=1}^{\infty} N + x_n\right) \\ &\leq m([0, 2]) \\ &= 2.\end{aligned}$$

Thus, by contradiction, since the first equivalence should clearly be infinity, we have $m(N) = 0$ for all measurable sets $N \subseteq E$. □

Problem 42. Let $A \subseteq \mathbb{R}$ be a measurable set with $m(A) > 0$. Prove that there exists a subset $B \subseteq A$ such that B is not measurable. (Hint: if E is the nonmeasurable set described in section 18 of the notes, then $A \subseteq \sqcup_{q\in\mathbb{Q}}(q + E)$.)

Proof. Let $A \subseteq \mathbb{R}$ with $m(A) > 0$. Without loss of generality, assume $A \subseteq [0, 1]$ since if not, there is some $n \in \mathbb{Z}$ such that $m(A \cap [n, n + 1]) > 0$ and by translation invariance, for $A' := \{x - n : x \in [n, n + 1]\}$, we have $A \cap A' \subseteq [0, 1]$ and $m(A \cap A') > 0$. So, if $B \subseteq A \cap A'$ is nonmeasurable, then $B + n \subseteq A \cap [n, n + 1] \subseteq A$ is nonmeasurable.

Now, we know that A is partitioned by the relation defined in section 18. By the axiom of choice, we can make a set $B \subseteq A$, which is the same as E defined in section 18, which is nonmeasurable. □

Problem 43. Let $\mathcal{E}$ be a collection of Borel sets that generates $\mathcal{B}_{\mathbb{R}}$ (i.e. such that $\mathcal{M}(\mathcal{E}) = \mathcal{B}_{\mathbb{R}}$). Let $f : \mathbb{R} \to \mathbb{R}$. Prove that f is measurable if and only if $f^{-1}(E)$ is measurable for all $E \in \mathcal{E}$. (Hint: show that $\{A \subseteq \mathbb{R} : f^{-1}(A)$ is measurable$\}$ is a σ-algebra.)

Proof. Suppose $\mathcal{E}$ is a collection of Borel sets that generates $\mathcal{B}_{\mathbb{R}}$.
($\Longrightarrow$): Suppose f is measurable for some $E \in \mathcal{E}$. Let

$$\mathcal{G} = \{A \subseteq \mathbb{R} : f^{-1}(A) \text{ is measurable}\}.$$

Then, $\mathcal{G}$ is a σ-algebra since

$$f^{-1}\left(\bigcup_{n\in\mathbb{N}} E_n\right) = \bigcup_{n\in\mathbb{N}} f^{-1}(E_n),$$
$$f^{-1}\left(\bigcap_{n\in\mathbb{N}} E_n\right) = \bigcap_{n\in\mathbb{N}} f^{-1}(E_n), \text{ and}$$
$$f^{-1}(E^c) = \left(f^{-1}(E)\right)^c.$$

Thus, $f^{-1}(E)$ is measurable for all $E \in \mathcal{E}$.
($\Longleftarrow$): Suppose $f^{-1}(E)$ is measurable for some $E \in \mathcal{E}$. Then, f is measurable by Definition 19.1. □

Problem 44. Let $f_1, f_2, \cdots : \mathbb{R} \to \mathbb{R}$ be measurable function, let $f : \mathbb{R} \to \mathbb{R}$, and suppose that $f_n \to f$ almost everywhere. Prove that f is measurable.

Proof. Since $\{x \in \mathbb{R} : \lim_{n\to\infty} |f_n(x) - f(x)| \geq \epsilon\ \forall \epsilon > 0\}$ is measurable with measure zero, then for $A \coloneqq \{x \in \mathbb{R} : \lim_{n\to\infty} |f_n(x) - f(x)| < \epsilon\ \forall \epsilon > 0\} \subseteq \mathbb{R}$, we have

$$\left\{ f_n\big|_A(x) \right\}_{n\in\mathbb{N}} \to f$$

pointwise, which implies f is measurable by Proposition 19.16. □

Problem 45. Recall that a function $f : \mathbb{R} \to \overline{\mathbb{R}}$ is *measurable* (or *Lebesgue measurable* if for every Borel set E in $\overline{\mathbb{R}}$, we have that $f^{-1}(E)$ is a (Lebesgue) measurable set (in $\mathbb{R}$).) We say that f is *Borel measurable* if for every Borel set $E \subseteq \overline{\mathbb{R}}$, $f^{-1}(E)$ is a Borel set.

Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \overline{\mathbb{R}}$. Prove the following.

(a) If f and g are both Borel measurable, then $g \circ f$ is Borel measurable.

Proof. Suppose $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \overline{\mathbb{R}}$ are both Borel measurable. Note, that for every $E \subseteq \overline{\mathbb{R}}$ Borel,

$$(g \circ f)^{-1}(E) = f^{-1}(g^{-1}(E)).$$

Since E is borel and g is Borel measurable, then $g^{-1}(E)$ is Borel. Since f is Borel measurable and $g^{-1}(E)$ is Borel, then $f^{-1}(g^{-1}(E))$ is Borel, which implies $(g \circ f)^{-1}(E)$ is Borel. Therefore, $g \circ f$ is Borel measurable. □

(b) If f is measurable and g is Borel measurable, then $g \circ f$ is measurable.

Proof. Suppose $f : \mathbb{R} \to \mathbb{R}$ is measurable and $g : \mathbb{R} \to \overline{\mathbb{R}}$ is Borel measurable. Similarly, for every $E \subseteq \overline{\mathbb{R}}$ Borel,

$$(g \circ f)^{-1}(E) = f^{-1}(g^{-1}(E)).$$

Since g is Borel measurable and E is Borel, then $g^{-1}(E)$ is Borel. Since f is measurable and $g^{-1}(E)$ is Borel, then $f^{-1}(g^{-1}(E))$ is measurable, which implies $(g \circ f)^{-1}(E)$ is measurable. Therefore, $g \circ f$ is measurable. □

(It is a fact that there exists examples of measurable functions f and g such that $g \circ f$ is not measurable.)

Problem 46. Let f be a nonnegative simple function. Define a function $\mu : \mathcal{L} \to [0, \infty]$ by $\mu(E) = \int (f \cdot \chi_E)$. Prove that μ is *countably additive*: if $E_1, E_2, \ldots$ are pairwise disjoint measurable sets, then $\mu(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(E_i)$.

Proof. For each $N > 0$, $\cup_{i=1}^{N} E_i \subset \cup_{i=1}^{\infty} E_i$, which implies $\mu\left(\bigcup_{j=1}^{N} E_i\right) \leq \mu\left(\bigcup_{i=1}^{\infty} E_i\right)$. So, since each E_i is disjoint and by finite subadditivity, we have $\sum_{i=1}^{N} \mu(E_i) \leq \mu\left(\bigcup_{i=1}^{\infty} E_i\right)$. Then, since N does not determine the inequality, we have $\lim_{N\to\infty} \sum_{i=1}^{N} \mu(E_i) = \sum_{i=1}^{\infty} \mu(E_i) \leq \mu\left(\bigcup_{i=1}^{\infty} E_i\right)$. Since countable subadditivity give the other direction of the inequality, we must have $\sum_{i=1}^{\infty} \mu(E_i) = \mu\left(\bigcup_{i=1}^{\infty} E_i\right)$, which implies μ is countably additive. □

Problem 47. Let f be a nonnegative simple function. Prove that the following conditions are equivalent:

(a) $\int f = 0$

(b) $f = 0$ a.e.

(c) Let $f = \sum_{i=1}^{n} a_i \chi_{A_i}$ be any representation of f with $a_i \geq 0$ for all i. For each i, if $a_i > 0$, then $m(A_i) = 0$.

Proof. (a)$\Longrightarrow$(b): Suppose for some $X \in \text{dom}(f)$, $\int_X f = 0$. Let $\{x \in X : f(x) > 0\} = \cup_{n\in\mathbb{N}}\{x \in X : f(x) > \frac{1}{n}\}$. Then,

$$\begin{aligned}
m(\{x \in X : f(x) > \frac{1}{n}\}) &= \int_{\{x\in X: f(x)>\frac{1}{n}\}} 1 \\
&= n \int_{\{x\in X: f(x)>\frac{1}{n}\}} \frac{1}{n} \\
&\leq n \int_{\{x\in X: f(x)>\frac{1}{n}\}} f \\
&\leq n \int_X f \\
&= 0.
\end{aligned}$$

Therefore, $m(\{x \in X : f(x) > \frac{1}{n}\}) = 0$, which implies $f = 0$ a.e.

(b)$\Longrightarrow$(a): Let $A = \{x : f(x) = 0\}$ and $m(A^c) = 0$. Then, for $X \in \text{dom}(f)$,

$$\begin{aligned}\int_X f &= \int_X f \cdot (\chi_A + \chi_{A^c}) \\ &= \int_X f \cdot \chi_A + \int_X f \cdot \chi_{A^c} && \text{(Since } A \cap A^c = \emptyset\text{)} \\ &= \int_A f + \int_{A^c} f \\ &= 0.\end{aligned}$$

(a)$\Longrightarrow$(c): Suppose that $\int_A f = 0$ where A is a collection of disjoint sets. Thus, since $\int_E f = \int a_1\chi_{A_1} + \cdots = 0$, then each term must be zero. That means that if $a_i > 0$ for some i, then $\chi_{A_i} = 0$, which implies $m(A_i) = 0$.

(c)$\Longrightarrow$(a): Suppose $f = \sum_{i=1}^n a_i\chi_{A_i}$ be any representation of f with $a_i \geq 0$ for all i. For each i, if $a_i > 0$, then $m(A_i) = 0$. Then, each term $a_i\chi_{A_i} = 0$ in the expansion of f, which implies $\int f = 0$. $\square$

Problem 48. For $f, g : \mathbb{R} \to \mathbb{R}$ the *join* of f and g is the function $f \vee g : \mathbb{R} \to \mathbb{R}$ defined by

$$(f \vee g)(x) = \max\{f(x), g(x)\}$$

(i.e. the pointwise maximum of the two functions). The *meet* is defined by

$$(f \wedge g)(x) = \min\{f(x), g(x)\}.$$

The *positive and negative parts* of f are defined by

$$f_+ = f \vee 0, \quad f_- = -(f \wedge 0).$$

Prove the following.

(i) If f and g are measurable then so are $f \vee g$ and $f \wedge g$.

Proof. Note, $\{x : (f \vee g)(x) > c\} = \{x : f(x) > c\} \cup \{x : g(x) > c\}$ and $\{x : (f \wedge g)(x) > c\} = \{x : f(x) > c\} \cap \{x : g(x) > c\}$. So, since the join and meet are a countable collection of union and intersected measurable sets, then the join and meet must also be measurable functions. $\square$

(ii) $f_+ \geq 0$, $f_- \geq 0$, and $f_+f_- = 0$.

Proof. There are the three following cases,

$$f > 0 \longrightarrow f_+ > 0 \text{ and } f_- = 0 \Longrightarrow f_+ f_- = 0,$$

$$f < 0 \Longrightarrow f_+ = 0 \text{ and } f_- > 0 \Longrightarrow f_+ f_- = 0,$$

$$f = 0 \Longrightarrow f_+ = 0 \text{ and } f_- = 0 \Longrightarrow f_+ f_- = 0.$$

Therefore, in all cases, $f_+ \geq 0$, $f_- \geq 0$, and $f_+ f_- = 0$. □

(iii) $f = f_+ - f_-$ and $|f| = f_+ + f_-$.

Proof. Focusing on the first part, if $f > 0$, then $f_+ - f_- = f - 0 = f$. If $f < 0$, then $f_+ - f_- = 0 - (-f) = f$. If $f = 0$, then $f_+ - f_- = 0 - 0 = 0 = f$. Therefore, $f = f_+ - f_-$.

Now, on the second part, if $f > 0$, then $f_+ + f_- = f + 0 = f = |f|$. If $f < 0$, then $f_+ + f_- = 0 - f = |f|$. If $f = 0$, then $f_+ + f_- = 0 + 0 = 0$. Therefore, $|f| = f_+ + f_-$. □

(iv) If $g, h \geq 0$ and $f = g - h$, then $g \geq f_+$ and $h \geq f_-$. Also, $g = f_+$ if and only if $h = f_-$, and this happens if and only if $gh = 0$.

Proof. From (iii), we know $f = f_+ - f_-$. Suppose $f = g - h$ and $g, h \geq 0$. So, $g - h = f_+ - f_-$. There are three cases.

One, if $f > 0$, then either $g > h$, which implies the difference $g - h$ and $f_+ - f_-$ must be the same, which implies $g > f_+$ and $h > f_-$ and $gh > 0$, or the differences are the same, which implies $g = f_+$ and $h = f_-$ and $gh = 0$.

Two, if $f < 0$ imples either $g < h$, which implies $g > f_+$ and $h > f_-$ to make up for the difference, which implies $gh > 0$, or, as in the last case, the differences are the same, which implies $g = f_+$ and $h = f_-$ and $gh = 0$.

Lastly, if $f = 0$, then $g = h = 0$ and $f_+ = f_- = 0$, which implies $g = f_+$ and $h = f_-$, which implies $gh = 0$.

Therefore, we have if $g, h \geq 0$ and $f = g - h$, then $g \geq f_+$ and $h \geq f_-$. Also, $g = f_+$ if and only if $h = f_-$, and this happens if and only if $gh = 0$. □