MAT 473: Intermediate Real Analysis II

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Problem 1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} \frac{|x_1|^a |x_2|^b}{\|x\|^c}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0, \end{cases}$$

where a, b, and c are positive real numbers. Prove that $\lim_{x\to 0} f(x)$ exists if and only if $a + b > c$.

Proof. $(\Longrightarrow)$: Suppose $a + b > c$. Then clearly $a + b - c > 0$. Then for all $x \in \mathbb{R}^2$, we have

$$\begin{aligned} \left| \frac{|x_1|^a |x_2|^b}{\|x\|^c} - 0 \right| &= \frac{|x_1|^a |x_2|^b}{\|x\|^c} \\ &\leq \frac{\|x\|^a \|x\|^b}{\|x\|^c} && \text{(By the fundamental inequalities)} \\ &= \|x\|^{a+b-c}. \end{aligned}$$

Since $a + b - c > 0$, $\|x\|^{a+b-c}$ converges to 0 as x converges to 0, and thus by the squeeze theorem, $|f(x) - 0| = \left| \frac{|x_1|^a |x_2|^b}{\|x\|^c} - 0 \right|$ converges to 0, and thus $\lim_{x\to 0} f(x) = 0$.

$(\Longleftarrow)$: Proof by contrapositive. We'll show that if $a + b \leq c$, then $\lim_{x\to 0} f(x)$ does not exist.

<u>Case 1</u>: Suppose $a + b < c$. Consider $Z = \{(t, t) : t \in \mathbb{R}^+\}$. Then,

$$\begin{aligned} \lim_{x\to 0} f\big|_Z(x) &= \lim_{t\to 0^+} \frac{|t|^a |t|^b}{\sqrt{t^2 + t^2}^c} \\ &= \lim_{t\to 0^+} \frac{t^a + b}{(\sqrt{2}t)^c} \\ &= \lim_{t\to 0^+} \frac{1}{(\sqrt{2}t)^{c-a-b}} \\ &= \infty. && \text{(Since } c - a - b > 0) \end{aligned}$$

Thus, $\lim_{x\to 0} f\big|_Z(x)$ does not exist, which implies $\lim_{x\to 0} f(x)$ does not exist.

<u>Case 2</u>: Suppose $a + b = c$. Consider $Z_1 = \{(t, 0) : t \in \mathbb{R}^+\}$. Then we have,

$$\begin{aligned} \lim_{x\to 0} f\big|_{Z_1}(x) &= \lim_{t\to 0^+} \frac{|t|^a |0|^b}{\sqrt{t^2 + 0}^c} \\ &= \lim_{t\to 0^+} \frac{0}{t^c} \\ &= 0. \end{aligned}$$

Now consider $Z_2 = \{(t,t) : t \in \mathbb{R}^+\}$. Then,

$$\begin{aligned}
\lim_{x\to 0} f\Big|_{Z_2}(x) &= \lim_{t\to 0^+} \frac{|t|^a\,|t|^b}{\sqrt{t^2+0}^c} \\
&= \lim_{t\to 0^+} \frac{t^{a+b}}{\sqrt{2}^c t^c} \\
&= \lim_{t\to 0^+} \frac{t^{a+b-c}}{\sqrt{2}^c} \\
&= \lim_{t\to 0^+} \frac{t^0}{\sqrt{2}^c} && \text{(Since } a+b=c\text{)} \\
&= \frac{1}{\sqrt{2}^c}.
\end{aligned}$$

Thus, $\lim_{x\to 0} f\Big|_{Z_1}(x) = 0 \neq \frac{1}{\sqrt{2}^c} = \lim_{x\to 0} f\Big|_{Z_2}(x)$. Thus, $\lim_{x\to 0} f(x)$ does not exist. Therefore, in all cases, $\lim_{x\to 0} f(x)$ does not exist. □

Problem 2. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} \frac{x_1 x_2^3}{x_1^2 + x_2^6}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

Prove that $\lim_{x\to 0} f(x)$ does not exist.

Proof. Suppose $f : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$. Let $Z_1 = \{(u,0)^t : u \in \mathbb{R}\setminus\{0\}\}$ and $Z_2 = \{(u^3,u)^T : u \in \mathbb{R}\setminus\{0\}\}$. We have

$$\begin{aligned}
f(u,0) &= \frac{u\cdot 0^3}{u^2+0^6} = \frac{0}{u^2} = 0 \\
f(u^3,u) &= \frac{u^3\cdot u^3}{u^6+u^6} = \frac{u^6}{2u^6} = \frac{1}{2}.
\end{aligned}$$

Thus, $\lim_{x\to 0} f\Big|_{Z_1}(x) = 0 \neq \frac{1}{2} = \lim_{x\to 0} f\Big|_{Z_2}(x)$. Therefore, $\lim_{x\to 0} f(x)$ does not exist. □

Problem 3. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a linear function.

(a) Prove that $\frac{f(x)}{\|x\|}$ is a bounded function of x on $\mathbb{R}^n \setminus \{0\}$. (Hint: if f is represented by a matrix, then $f(x)$ equals a linear combination of the columns of that matrix.)

Proof. Define $g : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^m$ by $g(x) = \frac{f(x)}{\|x\|}$ for all $x \in \mathbb{R}^n \setminus \{0\}$. Let $x \in \mathbb{R}^n \setminus \{0\}$. Then,

$$\begin{aligned}
\left\| \frac{\sum_{i=1}^m \left(\sum_{j=1}^n f_i(e_j^{(n)}) x_j \right) e_i^{(m)}}{\|x\|} \right\| &\le \sum_{i=1}^m \left| \frac{\left(\sum_{j=1}^n f_i(e_j^{(n)}) x_j \right)}{\|x\|} \right| \\
&\le \frac{\sum_{i=1}^m \sum_{j=1}^n \left| f_i(e_j^{(n)}) \right| |x_j|}{\|x\|} \\
&\le \frac{\sum_{i=1}^m \sum_{j=1}^n \left| f_i(e_j^{(n)}) \right| \|x\|}{\|x\|} \\
&= \sum_{i=1}^m \sum_{j=1}^n \left| f_i(e_j^{(n)}) \right| < \infty.
\end{aligned}$$

Thus, $\frac{f(x)}{\|x\|}$ is bounded. □

(b) Suppose that f is not the zero map. Prove that $\lim_{x\to 0} \frac{f(x)}{\|x\|}$ does not exist. (Hint: if $f(v) \neq 0$ consider $x = tv$ for $t \in \mathbb{R} \setminus \{0\}$.)

Proof. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a linear function that is not the zero map. Fix $v \in \mathbb{R}^n$ such that $f(v) \neq 0$. Let $Z_1 = \{kv : k \in \mathbb{R}^+\}$ and $Z_2 = \{kv : k \in \mathbb{R}^-\}$. Then,

$$\begin{aligned}
\lim_{x\to 0} \frac{f\big|_{Z_1}(x)}{\|x\|} &= \lim_{t\to 0^+} \frac{f\big|_{Z_1}(tv)}{\|tv\|} \\
&= \lim_{t\to 0^+} \frac{tf\big|_{Z_1}(v)}{|t|\,\|v\|} \\
&= \frac{f\big|_{Z_1}(v)}{\|v\|} \cdot \lim_{t\to 0^+} \frac{t}{|t|} \\
&= \frac{f\big|_{Z_1}(v)}{\|v\|} \\
&> 0
\end{aligned}$$

and

$$\begin{aligned}
\lim_{x\to 0} \frac{f\big|_{Z_2}(x)}{\|x\|} &= \lim_{t\to 0^-} \frac{f\big|_{Z_2}(tv)}{\|tv\|} \\
&= \lim_{t\to 0^-} \frac{tf\big|_{Z_2}(v)}{|t|\,\|v\|} \\
&= \frac{f\big|_{Z_2}(v)}{\|v\|} \cdot \lim_{t\to 0^-} \frac{t}{|t|} \\
&= \frac{-f\big|_{Z_2}(v)}{\|v\|} \\
&< 0.
\end{aligned}$$

Thus, the limit does not exist. □

Problem 4. Let V and W be normed vector spaces. Recall that $B(V,W) = \{T \in L(V,W) : \sup_{\|x\|\le 1}\|Tx\| < \infty\}$.

(a) Prove that $B(V,W)$ is a vector space.

Proof. Let V and W be normed vector spaces. First, we'll show that $L(V,W)$ is a vector space. Suppose $T_1, T_2, T_3 \in L(V,W)$ and $\alpha, \beta \in \mathbb{K}$ and $x \in V$. Note, $L(V,W)$ is clearly closed from Definition 2.1. This leaves the following properties.

(1) Commutativity:

$$\begin{aligned}
(T_1 +_L T_2)(x) &= T_1x +_W T_2x && \text{(Addition on } L(V,W))\\
&= T_2x +_W T_1x && \text{(Commutativity of addition in } W)\\
&= (T_2 +_L T_1)(x). && \text{(Addition on } L(V,W))
\end{aligned}$$

(2) Associativity:

$$\begin{aligned}
(T_1 +_L (T_2 +_L T_3))(x) &= T_1x +_W (T_2 +_L T_3)(x) && \text{(Addition on } L(V,W))\\
&= T_1x +_W (T_2x +_W T_3x) && \text{(Addition on } L(V,W))\\
&= (T_1x +_W T_2x) +_W T_3x && \text{(Associativity in } W)\\
&= (T_1 +_L T_2)(x) +_W T_3x && \text{(Addition on } L(V,W))\\
&= ((T_1 +_L T_2) +_L T_3)(x). && \text{(Addition on } L(V,W))
\end{aligned}$$

(3) Zero: Let $0_L \in L(V,W)$ be the zero map.

$$\begin{aligned}
(T_1 +_L 0_L)(x) &= T_1x +_W 0_Lx && \text{(Addition on } L(V,W))\\
&= T_1x +_W 0_W && \text{(Definition of zero map)}\\
&= T_1x. && \text{(Addition of } 0_W)
\end{aligned}$$

(4) Additive inverse: Define $-T_1 : V \to W$ by $-T_1(v) = T_1(-v)$ for all $v \in V$. Then,

$$\begin{aligned}
(T_1 +_L -T_1)(x) &= T_1x +_W -T_1x \\
&= -T_1x +_W T_1 - x \\
&= T_1(x +_V (-x)) && \text{(Linearity of } T_1\text{)} \\
&= T_1 \cdot 0_V && \text{(Addititive inverse in } V\text{)} \\
&= 0_W. && \text{(Linearity of } T_1\text{)}
\end{aligned}$$

Since $x \in V$ was arbitrary, this is true for all $x \in V$. Thus, $T_1 +_L -T_1 = 0_L$.

(5) Multiplication over $\mathbb{K}$:

$$\begin{aligned}
\alpha \cdot (\beta \cdot T_1(x)) &= \alpha \cdot (\beta T_1 x) && \text{(Definition of } T_1\text{)} \\
&= \alpha\beta(T_1x) && \text{(Multiplication over } \mathbb{K} \text{ in } W\text{)} \\
&= ((\alpha\beta) \cdot T_1)(x). && \text{(Definition of } T_1\text{)}
\end{aligned}$$

(6) Unit of scalar multiplication:

$$\begin{aligned}
(1 \cdot T_1)(x) &= 1 \cdot T_1(x) && \text{(Linearity of scalar multiplication)} \\
&= T(x). && \text{(Definition of 1)}
\end{aligned}$$

(7) Distribution of scalar multiples:

$$\begin{aligned}
(\alpha(T_1 +_L T_2))(x) &= \alpha((T_1 +_L T_2)(x)) && \text{(Linearity of scalar multiplication)} \\
&= \alpha(T_1x +_W T_2x) && \text{(Addition on } L(V,W)\text{)} \\
&= \alpha T_1x +_W \alpha T_2x && \text{(Distribution of scalar multiples on } W\text{)} \\
(\alpha T_1 +_L \alpha T_2)(x) &= (\alpha T_1)x +_W (\alpha T_2)x && \text{(Addition on } L(V,W)\text{)} \\
&= \alpha T_1x +_W \alpha T_2x. && \text{(Scalar multiplication on } L(V,W)\text{)}
\end{aligned}$$

Therefore, $L(V,W)$ is a vector space.

Now, we'll show $B(V,W)$ is closed and thus a subspace of $L(V,W)$. Let $T_1, T_2 \in B(V,W)$ and $\alpha, \beta \in \mathbb{K}$ and $x \in V$. Then,

$$\begin{aligned}
\sup_{\|x\| \le 1} \|(\alpha T_1 +_B \beta T_2)(x)\| &\le \sup_{\|x\| \le 1} (|\alpha| \, \|T_1x\| + |\beta| \, \|T_2x\|) && \text{(By triangle inequality)} \\
&\le |\alpha| \sup_{\|x\| \le 1} \|T_1x\| + |\beta| \sup_{\|x\| \le 1} \|T_2x\| && \text{(Definition of supremum)} \\
&= |\alpha| \, \|T_1\| + |\beta| \, \|T_2\| && \text{(Definition of } \|T_1\| \text{ and } \|T_2\|\text{)} \\
&< \infty.
\end{aligned}$$

Thus, $(\alpha T_1 +_B \beta T_2) \in B(V,W)$, which implies $B(V,W)$ is a subspace of $L(V,W)$. Therefore, $B(V,W)$ is a vector space. $\square$

(b) For $T \in B(V,W)$, let $\|T\| = \sup_{\|x\|\leq 1}\|Tx\|$. Prove that $\|\cdot\|$ is a norm on $B(V,W)$.

Proof. Since $\|T_1x\|_W \geq 0$ for all $x \in V$ by positivity of $\|\cdot\|_W$, we have $\sup_{\|x\|\leq 1}\|T_1x\|_W \geq 0$. Now suppose $\|T_1\| = 0$. Then $\sup_{\|x\|\leq 1}\|T_1x\| = 0$. Suppose to the contrary there exists $v \in V$ such that $\|T_1v\| > 0$. Then since $\sup_{\|x\|\leq 1}\|T_1x\| = 0$, $\|v\| > 1$. But then $\|\frac{v}{\|v\|}\| = 1$, so $\|T_1\frac{v}{\|v\|}\| = 0$. But, then $\|T_1\frac{v}{\|v\|}\| = \left|\frac{1}{\|v\|}\right| \|T_1v\| = 0$, which implies $\|T_1v\| = 0$. This is a contradiction. Therefore, T_1 is the zero map and we have positivity of $\|\cdot\|$.

Next,

$$\begin{aligned}
\|\alpha T_1\| &= \sup_{\|x\|\leq 1}\|\alpha T_1x\| && \text{(Definition of } \|\alpha T_1\|\text{)}\\
&= |\alpha| \sup_{\|x\|\leq 1}\|T_1x\| && \text{(Homogeneity of supremum norm)}\\
&= |\alpha|\, \|T_1\|. && \text{(Definition of } \|T_1\|\text{)}
\end{aligned}$$

Thus, we have homogeneity of $\|\cdot\|$.

Lastly,

$$\begin{aligned}
\|T_1 +_B T_2\| &= \sup_{\|x\|\leq 1}\|(T_1 +_B T_2)(x)\| && \text{(Definition of } \|T_1 +_B T_2\|\text{)}\\
&\leq \sup_{\|x\|\leq 1}(\|T_1x\| + \|T_2x\|) && \text{(Triangle inequality of the norm on } W\text{)}\\
&\leq \sup_{\|x\|\leq 1}\|T_1x\| + \sup_{\|x\|\leq 1}\|T_2x\| && \text{(Property of supremum norm)}\\
&= \|T_1\| + \|T_2\|. && \text{(Definition of } \|T_1\| \text{ and } \|T_2\|\text{)}
\end{aligned}$$

Thus, we have the triangle inequality of $\|\cdot\|$. Therefore, $\|\cdot\|$ is a norm on $B(V,W)$. □