MAT 473: Intermediate Real Analysis II

Trey Manuszak
Arizona State University

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Problem 37. Let $A_1, A_2, \dots$ be measurable sets, and suppose that $A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots$. Prove that $m(\cup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} m(A_n)$. (This is called *continuity from below* of Lebesgue measure.) (Hints: use Proposition 16.4 of the notes. It is useful also to remember that $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \sum_{i=1}^{n} a_i$.)

Proof. Let $A_1, A_2, \dots$ be measurable sets, and suppose that $A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots$. Suppose $\lim_{N\to\infty} \bigcup_{n=1}^{N} A_n = A$. From Proposition 16.4, we know

$$A = A_1 \cup \bigcup_{n=1}^{\infty} (A_{n+1} \setminus A_n).$$

Note, A_1 and $\bigcup_{n=1}^{\infty} (A_{n+1} \setminus A_n)$ are disjoint and A is a σ-algebra. Since they are disjoint and measurable, then we have

$$\begin{aligned} m(A) &= \sum_{n=1}^{\infty} m(A_n \setminus A_{n-1}) \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} m(A_i \setminus A_{i-1}) \\ &= \lim_{n\to\infty} m(A_n). \end{aligned}$$

□

Problem 38. Let $A_1, A_2, \dots$ be measurable sets, and suppose that $A_1 \supseteq A_2 \supseteq A_3 \supseteq \dots$. Suppose further that $m(A_1) < \infty$. Prove that $m(\cap_{n=1}^{\infty} A_n) = \lim_{n\to\infty} m(A_n)$. Be sure to indicate where the finiteness hypothesis is used. (This is called *continuity from above* of Lebesgue measure.) (Hints: as in the previous problem. Also, you will need to consider $B_\infty := \cap_{n=1}^{\infty} A_n$.) Give an example of a decreasing sequence of measurable sets of infinite measure for which the above conclusion is false.

Proof. Let $A_1, A_2, \dots$ be measurable sets, and suppose that $A_1 \supseteq A_2 \supseteq A_3 \supseteq \dots$. Suppose further that $m(A_1) < \infty$ and $B_\infty := \lim_{N\to\infty} \bigcap_{n=1}^{N} A_n$. Note, $m(A_1 \setminus B_\infty) = \lim_{n\to\infty} m(A_1 \setminus A_n)$. Also, $m(B_\infty) \le m(A_n) \le m(A_1) < \infty$. So, by Problem 37,

$$\begin{aligned} m(A_1) - m(B_\infty) &= m(A_1 \setminus B_\infty) \\ &= \lim_{n\to\infty} m(A_1 \setminus A_n) \\ &= m(A_1) - \lim_{n\to\infty} m(A_n). \end{aligned}$$

So, by subtracting the $m(A_1)$ terms from both sides, we get

$$m(B_\infty) = \lim_{n\to\infty} m(A_n).$$

□

Now, it is important that $A_k < \infty$ for some $k \in \mathbb{Z}$. For example, if not, suppose $A_n = (n, \infty)$. Then, $m(A_n) = \infty$ for each n, but $\bigcap_{n=1}^{\infty} A_n = \emptyset$. So, we get

$$\infty = \lim_{n\to\infty} m(A_n) \neq m(\bigcap_{n=1}^{\infty} A_n) = 0.$$

Problem 39. Let E be a measurable set, and let $\epsilon > 0$. Prove that there are an open $U \supseteq E$ and a closed set $F \subseteq E$ such that $m(U \setminus F) < \epsilon$. Here is an outline.

(a) Suppose that $E \supseteq [a, b]$. Use the definition of outer measure to find an open set $U \supseteq E$ with $m(U \setminus E) < \epsilon$.

(b) Suppose that $E \subseteq [a, b]$. Apply the previous part to $[a, b] \setminus E$ to prove that there is a closed set $F \subseteq E$ with $m(E \setminus F) < \epsilon$.

(c) For the general case let $E_n = E \cap [n, n+1]$ for $n \in \mathbb{Z}$, and apply the previous two parts with $\epsilon 4^{-(|n|+1)}$. Use the fact that if $S_n \subseteq T_n$ then $(\cup_n T_n) \setminus (\cup_n S_n) \subseteq \cup_n (T_n \setminus S_n)$.

Proof.

□

Problem 40. The *Cantor set*, C, is a subset of $[0, 1]$ defined as follows. Let $F_0 = [0, 1]$, $F_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$, and in general, F_{n+1} is obtained from F_n by deleting the middle open third of each subinterval of F_n. (Thus $F_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$.) Then $C := \cap_{n=1}^{\infty} F_n$. Prove the following:

(a) F_n is the union of 2^n pairwise disjoint closed intervals each of length 3^{-n}.

(b) $m(C) = 0$.

(c) C is a closed set, C has no isolated points, and the interior of C is empty.

Proof.

□