MAT 473: Intermediate Real Analysis II

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Problem 45. Recall that a function $f : \mathbb{R} \to \overline{\mathbb{R}}$ is *measurable* (or *Lebesgue measurable* if for every Borel set E in $\overline{\mathbb{R}}$, we have that $f^{-1}(E)$ is a (Lebesgue) measurable set (in $\mathbb{R}$).) We say that f is *Borel measurable* if for every Borel set $E \subseteq \overline{\mathbb{R}}$, $f^{-1}(E)$ is a Borel set.

Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \overline{\mathbb{R}}$. Prove the following.

(a) If f and g are both Borel measurable, then $g \circ f$ is Borel measurable.

(b) If f is measurable and g is Borel measurable, then $g \circ f$ is measurable.

(It is a fact that there exists examples of measurable functions f and g such that $g \circ f$ is not measurable.)

Problem 46. Let f be a nonnegative simple function. Define a function $\mu : \mathcal{L} \to [0, \infty]$ by $\mu(E) = \int (f \cdot \chi_E)$. Prove that μ is *countably additive*: if $E_1, E_2, \ldots$ are pairwise disjoint measurable sets, then $\mu(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} m(E_i)$.

Problem 47. Let f be a nonnegative simple function. Prove that the following conditions are equivalent:

(a) $\int f = 0$

(b) $f = 0$ a.e.

(c) Let $f = \sum_{i=1}^{n} a_i \chi A_i$ be any representation of f with $a_i \geq 0$ for all i. For each i, if $a_i > 0$, then $m(A_i) = 0$.

Problem 48. For $f, g : \mathbb{R} \to \mathbb{R}$ the *join* of f and g is the function $f \vee g : \mathbb{R} \to \mathbb{R}$ defined by

$$(f \vee g)(x) = \max\{f(x), g(x)\}$$

(i.e. the pointwise maximum of the two functions). The *meet* is defined by

$$(f \wedge g)(x) = \min\{f(x), g(x)\}.$$

The *positive and negative parts* of f are defined by

$$f_+ = f \vee 0, \quad f_- = -(f \wedge 0).$$

Prove the following.

(i) If f and g are measurable then so are $f \vee g$ and $f \wedge g$.

(ii) $f_+ \geq 0$, $f_- \geq 0$, and $f_+ f_- = 0$.

(iii) $f = f_+ - f_-$ and $|f| = f_+ + f_-$.

(iv) If $g, h \geq 0$ and $f = g - h$, then $g \geq f_+$ and $h \geq f_-$. Also, $g = f_+$ if and only if $h = f_-$, and this happens if and only if $gh = 0$.