MAT 473: Intermediate Real Analysis II

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Problem 49. Prove Theorem 22.5 parts (iii) and (iv): Let $f, g : \mathbb{R} \to \overline{\mathbb{R}}$ be integrable functions.

(iii) Suppose that $f \leq g$. Prove that $\int f \leq \int g$.

Proof. Let $f, g : \mathbb{R} \to \overline{\mathbb{R}}$ be integrable functions and $f \leq g$. Then, by linearity,

$$0 \leq \int (g - f) = \int g - \int f,$$

which implies $\int f \leq \int g$. □

(iv) Prove that $\left|\int f\right| \leq \int |f|$.

Proof. Let $f : \mathbb{R} \to \overline{\mathbb{R}}$ be integrable Since $|f|$ is measurable and bounded and $-|f| \leq f \leq |f|$, by linearity and monotonicty, we get

$$-\int |f| \leq \int f \leq \int |f|.$$

It follows that $\left|\int f\right| \leq \int |f|$. □

Problem 50. Compute the value of the limit

$$\lim_{n\to\infty} \int_0^\infty \left(1 + \frac{x}{n}\right)^{-n} \cos\frac{x}{n} dx.$$

Justify every step of your argument. (Hint: use the monotone convergence theorem, and the theorem on equality of the Reimann and Lebesgue integrals when both apply, to show that e^{-x} is integrable on $[0, \infty)$. Then use the dominated convergence theorem.)

Proof. Let $f_n = \left(1 + \frac{x}{n}\right)^{-n} \cos\frac{x}{n}$. Then, f_n is pointwise convergent to $\frac{1}{e^x}$. Since e^{-x} is integrable, then by the dominated convergence theorem, we have

$$\begin{aligned}
\lim_{n\to\infty} \int_0^\infty \left(1 + \frac{x}{n}\right)^{-n} \cos\frac{x}{n} dx &= \int_0^\infty \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^{-n} \cos\frac{x}{n} dx \\
&= \int_0^\infty e^{-x} dx \\
&= -e^{-x} + C.
\end{aligned}$$

□

(Couldn't figure out how to use monotone convergence theorem to show e^{-x} was integrable.)