

MAT 473: Intermediate Real Analysis II

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Problem 33. Recall the Borel σ -algebra $\mathcal{B}_{\mathbb{R}}$ from the course notes. Prove that $\mathcal{B}_{\mathbb{R}}$ is generated as a σ -algebra by the collection of closed intervals $\{[a, \infty) : a \in \mathbb{R}\}$.

Proof. Let \mathcal{O} denote the collection of all open intervals in \mathbb{R} . Since every open set in \mathbb{R} is at most a countable union of open intervals, Then $\mathcal{M}(\mathcal{O}) = \mathcal{B}_{\mathbb{R}}$. Let \mathcal{E} denote the collection of intervals of the form $[a, \infty)$ for all $a \in \mathbb{R}$. Let $(a, b) \in \mathcal{O}$ for some $a, b \in \mathbb{R}$ such that $b > a$. Let $a_n = a + \frac{1}{n}$ and $b_n = b - \frac{1}{n}$. Then,

$$(a, b) = \bigcup_{n=1}^{\infty} [a_n, b_n) = \bigcup_{n=1}^{\infty} \{[a_n, \infty) \cap [b_n, \infty)^c\},$$

which implies that $(a, b) \in \mathcal{M}(\mathcal{E})$. This means that $\mathcal{O} \subseteq \mathcal{M}(\mathcal{E})$, which implies $\mathcal{M}(\mathcal{O}) \subseteq \mathcal{M}(\mathcal{E})$. But, since every element of \mathcal{E} is closed, then $\mathcal{M}(\mathcal{E}) \subseteq \mathcal{B}_{\mathbb{R}}$. This gives us

$$\mathcal{B}_{\mathbb{R}} = \mathcal{M}(\mathcal{O}) \subseteq \mathcal{M}(\mathcal{E}) \subseteq \mathcal{B}_{\mathbb{R}}.$$

Therefore, $\mathcal{M}(\mathcal{E}) = \mathcal{B}_{\mathbb{R}}$. □

Problem 34. Prove that for every subset $E \subseteq \mathbb{R}$ there is a G_{δ} -set A with $E \subseteq A$ and $m^*(E) = m^*(A)$.

Proof. Let $E \subseteq \mathbb{R}$ be arbitrary. By outer-measure, there exists a collection of open intervals $I_n \subseteq \mathbb{R}$ such that $E \subseteq \bigcup_{n=1}^{\infty} I_n$. From the result of Problem 31, this gives us

$$m^*(E) \leq \sum_{n=1}^{\infty} m(I_n) < m^*(E) + \frac{1}{n}.$$

Also, by countable subadditivity, $m^*(\bigcup_{n=1}^{\infty} I_n) \leq \sum_{n=1}^{\infty} m(I_n)$. Let $A = \bigcap_{n=1}^{\infty} \bigcup_{n=1}^{\infty} I_n$. Then, A is a G_{δ} -set. This gives us $E \subseteq A$. Then, for each n , we get

$$m^*(E) \leq m^*(A) \leq m^*(\bigcup_{n=1}^{\infty} I_n) \leq \sum_{n=1}^{\infty} m(I_n) < m^*(E) + \frac{1}{n}.$$

Therefore, by squeeze theorem, as n approaches ∞ , then $m^*(E) = m^*(A)$. □

Problem 35. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function. Let $A = \{x \in \mathbb{R} : f \text{ is continuous at } x\}$. Prove that A is a G_{δ} -set. (Hint: use the oscillation of f from homework 8.)

Proof. Let $A = \{x \in \mathbb{R} : f \text{ is continuous at } x\}$. Note, f is continuous at c if and only if $\text{osc}(f, c) = 0$. That is, f is continuous at c if and only if

$$\limsup_{x \rightarrow c} f(x) = \liminf_{x \rightarrow c} f(x). \quad (\text{I like this definition provided by [?]})$$

So, by looking at the complement of A , we get

$$\begin{aligned} A^c &= \left\{ x \in \mathbb{R} : \liminf_{x \rightarrow c} f(x) < \limsup_{x \rightarrow c} f(x) \right\} \\ &= \left\{ x \in \mathbb{R} : \exists a, b \in \mathbb{Q} \text{ s.t. } \liminf_{x \rightarrow c} f(x) \leq a < b \leq \limsup_{x \rightarrow c} f(x) \right\} \\ &= \bigcup_{a, b} \left(\left\{ x \in \mathbb{R} : \liminf_{x \rightarrow c} f(x) \leq a \right\} \cap \left\{ x \in \mathbb{R} : b \leq \limsup_{x \rightarrow c} f(x) \right\} \right). \quad (\text{with } a < b) \end{aligned}$$

Note, if $\liminf_{x \rightarrow c} f(x) > a$, then there must exist $\epsilon > 0$ arbitrary but fixed such that $\inf_{|x-c|<\epsilon} f(x) > a$. Now, for each $x \in B_\epsilon(c)$, there is $\epsilon_0 > 0$ arbitrary but fixed such that $B_{\epsilon_0}(x) \subset B_\epsilon(c)$. So, we get

$$\inf_{|x-c'|<\epsilon_0} f(x) \geq \inf_{|x-c|<\epsilon} f(x) > a.$$

This means that $\left\{ x \in \mathbb{R} : \liminf_{x \rightarrow c} f(x) > a \right\}$ is open, which implies $\left\{ x \in \mathbb{R} : \liminf_{x \rightarrow c} f(x) \leq a \right\}$ is closed. We can similarly show that $\left\{ x \in \mathbb{R} : b \leq \limsup_{x \rightarrow c} f(x) \right\}$ is closed since $\limsup_{x \rightarrow c} f(x) = -\left(\liminf_{x \rightarrow c} (-f(x))\right)$.

Finally, since each set in the pair of a countable union defined above is closed, then A^c is a F_σ -set, which implies A is a G_δ -set. \square

Problem 36. For subsets $A, B \subseteq \mathbb{R}$ recall that the *distance* between A and B is defined to be $\text{dist}(A, B) = \inf\{|x - y| : x \in A, y \in B\}$. Let A and B be subsets of \mathbb{R} such that $\text{dist}(A, B) > 0$. Prove that $m^*(A \cup B) = m^*(A) + m^*(B)$.

Proof. Let $A, B \subseteq \mathbb{R}$. Let $\epsilon > 0$ be arbitrary but fixed such that $\text{dist}(A, B) \geq \epsilon$. Let $E = \bigcup_{x \in A} B_{\frac{\epsilon}{2}}(x)$. Then, $A \subset E$. Also, since E used the ball of radius $\frac{\epsilon}{2}$, then $E \cap B = \emptyset$. Also, since E used a countable union of open balls, then E is measurable. This gives us $m^*(A \cup B) = m^*((A \cup B) \cap E) + m^*((A \cup B) \cap E^c)$ from Definition 17.1 in the notes. But, note that $(A \cup B) \cap E^c = B$ and $(A \cup B) \cap E = A$. Therefore, we have $m^*(A \cup B) = m^*(A) + m^*(B)$. \square

References