MAT 473: Intermediate Real Analysis II

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Problem 33. Recall the *Borel σ-algebra* $\mathcal{B}_{\mathbb{R}}$ from the course notes. Prove that $\mathcal{B}_{\mathbb{R}}$ is generated as a σ-algebra by the collection of closed intervals $\{[a, \infty) : a \in \mathbb{R}\}$.

Proof. Let $\mathcal{O}$ denote the collection of all open intervals in $\mathbb{R}$. Since every open set in $\mathbb{R}$ is at most a countable union of open intervals, Then $\mathcal{M}(\mathcal{O}) = \mathcal{B}_{\mathbb{R}}$. Let $\mathcal{E}$ denote the collection of intervals of the form $[a, \infty)$ for all $a \in \mathbb{R}$. Let $(a, b) \in \mathcal{O}$ for some $a, b \in \mathbb{R}$ such that $b > a$. Let $a_n = a + \frac{1}{n}$ and $b_n = b - \frac{1}{n}$. Then,

$$(a, b) = \bigcup_{n=1}^{\infty} [a_n, b_n) = \bigcup_{n=1}^{\infty} \{[a_n, \infty) \cap [b_n, \infty)^c\},$$

which implies that $(a, b) \in \mathcal{M}(\mathcal{E})$. This means that $\mathcal{O} \subseteq \mathcal{M}(\mathcal{E})$, which implies $\mathcal{M}(\mathcal{O}) \subseteq \mathcal{M}(\mathcal{E})$. But, since every element of $\mathcal{E}$ is closed, then $\mathcal{M}(\mathcal{E}) \subseteq \mathcal{B}_{\mathbb{R}}$. This gives us

$$\mathcal{B}_{\mathbb{R}} = \mathcal{M}(\mathcal{O}) \subseteq \mathcal{M}(\mathcal{E}) \subseteq \mathcal{B}_{\mathbb{R}}.$$

Therefore, $\mathcal{M}(\mathcal{E}) = \mathcal{B}_{\mathbb{R}}$. □

Problem 34. Prove that for every subset $E \subseteq \mathbb{R}$ there is a G_δ-set A with $E \subseteq A$ and $m^*(E) = m^*(A)$.

Proof. Let $E \subseteq \mathbb{R}$ be arbitrary. By outer-measure, there exists a collection of open intervals $I_n \subseteq \mathbb{R}$ such that $E \subseteq \cup_{n=1}^{\infty} I_n$. From the result of Problem 31, this gives us

$$m^*(E) \leq \sum_{n=1}^{\infty} m(I_n) < m^*(E) + \frac{1}{n}.$$

Also, by countable subadditivity, $m^*(\cup_{n=1}^{\infty} I_n) \leq \sum_{n=1}^{\infty} m(I_n)$. Let $A = \cap_{n=1}^{\infty} \cup_{n=1}^{\infty} I_n$. Then, A is a G_δ-set. This gives us $E \subseteq A$. Then, for each n, we get

$$m^*(E) \leq m^*(A) \leq m^*(\cup_{n=1}^{\infty} I_n) \leq \sum_{n=1}^{\infty} m(I_n) < m^*(E) + \frac{1}{n}.$$

Therefore, by squeeze theorem, as n approaches ∞, then $m^*(E) = m^*(A)$. □

Problem 35. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Let $A = \{x \in \mathbb{R} : f \text{ is continuous at } x\}$. Prove that A is a G_δ-set. (Hint: use the oscillation of f from homework 8.)

Proof. Let $A = \{x \in \mathbb{R} : f \text{ is continuous at } x\}$. Note, f is continuous at c if and only if $\operatorname{osc}(f, c) = 0$. That is, f is continuous at c if and only if

$$\limsup_{x \to c} f(x) = \liminf_{x \to c} f(x).$$

So, by looking at the complement of A, we get

$$\begin{aligned}
A^c &= \left\{x \in \mathbb{R} : \liminf_{x\to c} f(x) < \limsup_{x\to c} f(x)\right\} \\
&= \left\{x \in \mathbb{R} : \exists a, b \in \mathbb{Q} \text{ s.t. } \liminf_{x\to c} f(x) \le a < b \le \limsup_{x\to c} f(x)\right\} \\
&= \bigcup_{a,b} \left(\left\{x \in \mathbb{R} : \liminf_{x\to c} f(x) \le a\right\} \bigcap \left\{x \in \mathbb{R} : b \le \limsup_{x\to c} f(x)\right\}\right). \qquad \text{(with } a < b\text{)}
\end{aligned}$$

Note, if $\liminf_{x\to c} f(x) > a$, then there must exist $\epsilon > 0$ arbitrary but fixed such that $\inf_{|x-c|<\epsilon} f(x) > a$. Now, for each $x \in B_\epsilon(c)$, there is $\epsilon_0 > 0$ arbitrary but fixed such that $B_{\epsilon_0}(x) \subset B_\epsilon(c)$. So, we get

$$\inf_{|x-c'|<\epsilon_0} f(x) \ge \inf_{|x-c|<\epsilon} f(x) > a.$$

This means that $\left\{x \in \mathbb{R} : \liminf_{x\to c} f(x) > a\right\}$ is open, which implies $\left\{x \in \mathbb{R} : \liminf_{x\to c} f(x) \le a\right\}$ is closed. We can similarly show that $\left\{x \in \mathbb{R} : b \le \limsup_{x\to c} f(x)\right\}$ is closed since $\limsup_{x\to c} f(x) = -\left(\liminf_{x\to c}(-f(x))\right)$.

Finally, since each set in the pair of a countable union defined above is closed, then A^c is a F_σ-set, which implies A is a G_δ-set. $\square$

Problem 36. For subsets $A, B \subseteq \mathbb{R}$ recall that the *distance* between A and B is defined to be $\text{dist}(A, B) = \inf\{|x-y| : x \in A, y \in B\}$. Let A and B be subsets of $\mathbb{R}$ such that $\text{dist}(A, B) > 0$. Prove that $m^*(A \cup B) = m^*(A) + m^*(B)$.

Proof. Let $A, B \subseteq \mathbb{R}$. Let $\epsilon > 0$ be arbitrary but fixed such that $\text{dist}(A, B) \ge \epsilon$. Let $E = \cup_{x\in A} B_{\frac{\epsilon}{2}}(x)$. Then, $A \subset E$. Also, since E used the ball of radius $\frac{\epsilon}{2}$, then $E \cap B = \emptyset$. Also, since E used a countable union of open balls, then E is measurable. This gives us $m^*(A \cup B) = m^*((A \cup B) \cap E) + m^*((A \cup B) \cap E^c)$ from Definition 17.1 in the notes. But, note that $(A \cup B) \cap E^c = B$ and $(A \cup B) \cap E = A$. Therefore, we have $m^*(A \cup B) = m^*(A) + m^*(B)$. $\square$