MAT 473: Intermediate Real Analysis II

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Problem 41. Let E be the nonmeasurable set desribed in section 18 of the notes. Prove that if $N \subseteq E$ and N is measurable, then $m(N) = 0$. (Hint: imitate the second part of the proof of Theorem 18.1.)

Proof. Let E be as described in section 18 of the notes. Let $N = \emptyset$. Then, $N \subseteq E$ and, trivially, $m(N) = 0$. So, there does exist a measurable subset of E. Now, suppose $N \subseteq E$ with $m(N) > 0$. Let $A = \mathbb{Q} \cap [0, 1]$. Then, for each $x_1, x_2 \in A$, with $x_1 \neq x_2$, we know

$$N + x_1 \bigcap N + x_2 = \emptyset.$$

Also, by translation invariance, $m(N) = m(N + x)$ for all $x \in A$. So, we have

$$\begin{aligned} \sum_{n=1}^{\infty} m(N) &= \sum_{n=1}^{\infty} m(N + x_n) && \text{(Each } x_n \in A) \\ &= m\left(\bigsqcup_{n=1}^{\infty} N + x_n\right) \\ &\leq m([0, 2]) \\ &= 2. \end{aligned}$$

Thus, by contradiction, since the first equivalence should clearly be infinity, we have $m(N) = 0$ for all measurable sets $N \subseteq E$. □

Problem 42. Let $A \subseteq \mathbb{R}$ be a measurable set with $m(A) > 0$. Prove that there exists a subset $B \subseteq A$ such that B is not measurable. (Hint: if E is the nonmeasurable set described in section 18 of the notes, then $A \subseteq \sqcup_{q \in \mathbb{Q}} (q + E)$.)

Proof. Let $A \subseteq \mathbb{R}$ with $m(A) > 0$. Without loss of generality, assume $A \subseteq [0, 1]$ since if not, there is some $n \in \mathbb{Z}$ such that $m(A \cap [n, n + 1]) > 0$ and by translation invariance, for $A' := \{x - n : x \in [n, n + 1]\}$, we have $A \cap A' \subseteq [0, 1]$ and $m(A \cap A') > 0$. So, if $B \subseteq A \cap A'$ is nonmeasurable, then $B + n \subseteq A \cap [n, n + 1] \subseteq A$ is nonmeasurable.

Now, we know that A is partitioned by the relation defined in section 18. By the axiom of choice, we can make a set $B \subseteq A$, which is the same as E defined in section 18, which is nonmeasurable. □

Problem 43. Let $\mathcal{E}$ be a collection of Borel sets that generates $\mathcal{B}_{\mathbb{R}}$ (i.e. such that $\mathcal{M}(\mathcal{E}) = \mathcal{B}_{\mathbb{R}}$). Let $f : \mathbb{R} \to \mathbb{R}$. Prove that f is measurable if and only if $f^{-1}(E)$ is measurable for all $E \in \mathcal{E}$. (Hint: show that $\{A \subseteq \mathbb{R} : f^{-1}(A) \text{ is measurable}\}$ is a σ-algebra.)

Proof. Suppose $\mathcal{E}$ is a collection of Borel sets that generates $\mathcal{B}_{\mathbb{R}}$.

($\Longrightarrow$): Suppose f is measurable for some $E \in \mathcal{E}$. Let

$$\mathcal{G} = \{A \subseteq \mathbb{R} : f^{-1}(A) \text{ is measurable}\}.$$

Then, $\mathcal{G}$ is a σ-algebra since

$$f^{-1}\left(\bigcup_{n\in\mathbb{N}} E_n\right) = \bigcup_{n\in\mathbb{N}} f^{-1}(E_n),$$
$$f^{-1}\left(\bigcap_{n\in\mathbb{N}} E_n\right) = \bigcap_{n\in\mathbb{N}} f^{-1}(E_n), \text{ and}$$
$$f^{-1}(E^c) = \left(f^{-1}(E)\right)^c.$$

Thus, $f^{-1}(E)$ is measurable for all $E \in \mathcal{E}$.

($\Longleftarrow$): Suppose $f^{-1}(E)$ is measurable for some $E \in \mathcal{E}$. Then, f is measurable by Definition 19.1. □

Problem 44. Let $f_1, f_2, \cdots : \mathbb{R} \to \mathbb{R}$ be measurable function, let $f : \mathbb{R} \to \mathbb{R}$, and suppose that $f_n \to f$ almost everywhere. Prove that f is measurable.

Proof. Since $\{x \in \mathbb{R} : \lim_{n\to\infty} |f_n(x) - f(x)| \geq \epsilon \ \forall \epsilon > 0\}$ is measurable with measure zero, then for $A \coloneqq \{x \in \mathbb{R} : \lim_{n\to\infty} |f_n(x) - f(x)| < \epsilon \ \forall \epsilon > 0\} \subseteq \mathbb{R}$, we have

$$\left\{f_n\big|_A(x)\right\}_{n\in\mathbb{N}} \to f$$

pointwise, which implies f is measurable by Proposition 19.16. □