MAT 473: Intermediate Real Analysis II

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April 24, 2020

Problem 45. Recall that a function $f : \mathbb{R} \to \overline{\mathbb{R}}$ is *measurable* (or *Lebesgue measurable* if for every Borel set E in $\overline{\mathbb{R}}$, we have that $f^{-1}(E)$ is a (Lebesgue) measurable set (in $\mathbb{R}$).) We say that f is *Borel measurable* if for every Borel set $E \subseteq \overline{\mathbb{R}}$, $f^{-1}(E)$ is a Borel set.

Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \overline{\mathbb{R}}$. Prove the following.

(a) If f and g are both Borel measurable, then $g \circ f$ is Borel measurable.

Proof. Suppose $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \overline{\mathbb{R}}$ are both Borel measurable. Note, that for every $E \subseteq \overline{\mathbb{R}}$ Borel,

$$(g \circ f)^{-1}(E) = f^{-1}(g^{-1}(E)).$$

Since E is borel and g is Borel measurable, then $g^{-1}(E)$ is Borel. Since f is Borel measurable and $g^{-1}(E)$ is Borel, then $f^{-1}(g^{-1}(E))$ is Borel, which implies $(g \circ f)^{-1}(E)$ is Borel. Therefore, $g \circ f$ is Borel measurable. □

(b) If f is measurable and g is Borel measurable, then $g \circ f$ is measurable.

Proof. Suppose $f : \mathbb{R} \to \mathbb{R}$ is measurable and $g : \mathbb{R} \to \overline{\mathbb{R}}$ is Borel measurable. Similarly, for every $E \subseteq \overline{\mathbb{R}}$ Borel,

$$(g \circ f)^{-1}(E) = f^{-1}(g^{-1}(E)).$$

Since g is Borel measurable and E is Borel, then $g^{-1}(E)$ is Borel. Since f is measurable and $g^{-1}(E)$ is Borel, then $f^{-1}(g^{-1}(E))$ is measurable, which implies $(g \circ f)^{-1}(E)$ is measurable. Therefore, $g \circ f$ is measurable. □

(It is a fact that there exists examples of measurable functions f and g such that $g \circ f$ is not measurable.)

Problem 46. Let f be a nonnegative simple function. Define a function $\mu : \mathcal{L} \to [0, \infty]$ by $\mu(E) = \int (f \cdot \chi_E)$. Prove that μ is *countably additive*: if $E_1, E_2, \ldots$ are pairwise disjoint measurable sets, then $\mu(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(E_i)$.

Proof. For each $N > 0$, $\cup_{i=1}^{N} E_i \subset \cup_{i=1}^{\infty} E_i$, which implies $\mu\left(\bigcup_{j=1}^{N} E_i\right) \leq \mu\left(\bigcup_{i=1}^{\infty} E_i\right)$. So, since each E_i is disjoint and by finite subadditivity, we have $\sum_{i=1}^{N} \mu(E_i) \leq \mu\left(\bigcup_{i=1}^{\infty} E_i\right)$. Then, since N does not determine the inequality, we have $\lim_{N\to\infty} \sum_{i=1}^{N} \mu(E_i) = \sum_{i=1}^{\infty} \mu(E_i) \leq \mu\left(\bigcup_{i=1}^{\infty} E_i\right)$. Since countable subadditivity give the other direction of the inequality, we must have $\sum_{i=1}^{\infty} \mu(E_i) = \mu\left(\bigcup_{i=1}^{\infty} E_i\right)$, which implies μ is countably additive. □

Problem 47. Let f be a nonnegative simple function. Prove that the following conditions are equivalent:

(a) $\int f = 0$

(b) $f = 0$ a.e.

(c) Let $f = \sum_{i=1}^{n} a_i \chi_{A_i}$ be any representation of f with $a_i \geq 0$ for all i. For each i, if $a_i > 0$, then $m(A_i) = 0$.

Proof. (a)$\Longrightarrow$(b): Suppose for some $X \in \text{dom}(f)$, $\int_X f = 0$. Let $\{x \in X : f(x) > 0\} = \cup_{n\in\mathbb{N}}\{x \in X : f(x) > \frac{1}{n}\}$. Then,

$$\begin{aligned} m(\{x \in X : f(x) > \frac{1}{n}\}) &= \int_{\{x\in X: f(x) > \frac{1}{n}\}} 1 \\ &= n \int_{\{x\in X: f(x) > \frac{1}{n}\}} \frac{1}{n} \\ &\leq n \int_{\{x\in X: f(x) > \frac{1}{n}\}} f \\ &\leq n \int_X f \\ &= 0. \end{aligned}$$

Therefore, $m(\{x \in X : f(x) > \frac{1}{n}\}) = 0$, which implies $f = 0$ a.e.

(b)$\Longrightarrow$(a): Let $A = \{x : f(x) = 0\}$ and $m(A^c) = 0$. Then, for $X \in \text{dom}(f)$,

$$\begin{aligned} \int_X f &= \int_X f \cdot (\chi_A + \chi_{A^c}) \\ &= \int_X f \cdot \chi_A + \int_X f \cdot \chi_{A^c} && \text{(Since } A \cap A^c = \emptyset\text{)} \\ &= \int_A f + \int_{A^c} f \\ &= 0. \end{aligned}$$

(a)$\Longrightarrow$(c): Suppose that $\int_A f = 0$ where A is a collection of disjoint sets. Thus, since $\int_E f = \int a_1\chi_{A_1} + \cdots = 0$, then each term must be zero. That means that if $a_i > 0$ for some i, then $\chi_{A_i} = 0$, which implies $m(A_i) = 0$.

(c)$\Longrightarrow$(a): Suppose $f = \sum_{i=1}^{n} a_i \chi_{A_i}$ be any representation of f with $a_i \geq 0$ for all i. For each i, if $a_i > 0$, then $m(A_i) = 0$. Then, each term $a_i\chi_{A_i} = 0$ in the expansion of f, which implies $\int f = 0$. $\square$

Problem 48. For $f, g : \mathbb{R} \to \mathbb{R}$ the *join* of f and g is the function $f \vee g : \mathbb{R} \to \mathbb{R}$ defined by

$$(f \vee g)(x) = \max\{f(x), g(x)\}$$

(i.e. the pointwise maximum of the two functions). The *meet* is defined by

$$(f \wedge g)(x) = \min\{f(x), g(x)\}.$$

The *positive and negative parts* of f are defined by

$$f_+ = f \vee 0, \quad f_- = -(f \wedge 0).$$

Prove the following.

(i) If f and g are measurable then so are $f \vee g$ and $f \wedge g$.

Proof. Note, $\{x : (f \vee g)(x) > c\} = \{x : f(x) > c\} \cup \{x : g(x) > c\}$ and $\{x : (f \wedge g)(x) > c\} = \{x : f(x) > c\} \cap \{x : g(x) > c\}$. So, since the join and meet are a countable collection of union and intersected measurable sets, then the join and meet must also be measurable functions. □

(ii) $f_+ \geq 0$, $f_- \geq 0$, and $f_+ f_- = 0$.

Proof. There are the three following cases,

$$f > 0 \longrightarrow f_+ > 0 \text{ and } f_- = 0 \Longrightarrow f_+ f_- = 0,$$

$$f < 0 \Longrightarrow f_+ = 0 \text{ and } f_- > 0 \Longrightarrow f_+ f_- = 0,$$

$$f = 0 \Longrightarrow f_+ = 0 \text{ and } f_- = 0 \Longrightarrow f_+ f_- = 0.$$

Therefore, in all cases, $f_+ \geq 0$, $f_- \geq 0$, and $f_+ f_- = 0$. □

(iii) $f = f_+ - f_-$ and $|f| = f_+ + f_-$.

Proof. Focusing on the first part, if $f > 0$, then $f_+ - f_- = f - 0 = f$. If $f < 0$, then $f_+ - f_- = 0 - (-f) = f$. If $f = 0$, then $f_+ - f_- = 0 - 0 = 0 = f$. Therefore, $f = f_+ - f_-$.

Now, on the second part, if $f > 0$, then $f_+ + f_- = f + 0 = f = |f|$. If $f < 0$, then $f_+ + f_- = 0 - f = |f|$. If $f = 0$, then $f_+ + f_- = 0 + 0 = 0$. Therefore, $|f| = f_+ + f_-$. □

(iv) If $g, h \geq 0$ and $f = g - h$, then $g \geq f_+$ and $h \geq f_-$. Also, $g = f_+$ if and only if $h = f_-$, and this happens if and only if $gh = 0$.

Proof. From (iii), we know $f = f_+ - f_-$. Suppose $f = g - h$ and $g, h \geq 0$. So, $g - h = f_+ - f_-$. There are three cases.

One, if $f > 0$, then either $g > h$, which implies the difference $g - h$ and $f_+ - f_-$ must be the same, which implies $g > f_+$ and $h > f_-$ and $gh > 0$, or the differences are the same, which implies $g = f_+$ and $h = f_-$ and $gh = 0$.

Two, if $f < 0$ imples either $g < h$, which implies $g > f_+$ and $h > f_-$ to make up for the difference, which implies $gh > 0$, or, as in the last case, the differences are the same, which implies $g = f_+$ and $h = f_-$ and $gh = 0$.

Lastly, if $f = 0$, then $g = h = 0$ and $f_+ = f_- = 0$, which implies $g = f_+$ and $h = f_-$, which implies $gh = 0$.

Therefore, we have if $g, h \geq 0$ and $f = g - h$, then $g \geq f_+$ and $h \geq f_-$. Also, $g = f_+$ if and only if $h = f_-$, and this happens if and only if $gh = 0$. $\square$