MAT 473: Intermediate Real Analysis II

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Problems 21 - 22 finish the proof of the implicit function theorem in two variables. Let $f : U \subseteq \mathbb{R}^2 \to \mathbb{R}$, $(a,b) \in U$, $c = f(a,b)$, and $g : (a-s, a+s) \to (b-r, b+r)$ be as in the implicit function theorem (Theorem 9.1 in the notes). It has been shown that g is continuous on $(a-s, a+s)$. Complete the proof of the theorem by showing that g is differentiable on $(a-s, a+s)$, with derivative $g'(x) = -D_1 f(x, g(x))/D_2 f(x, g(x))$, using the following outline.

Problem 21. First prove it for $x = a$ as follows. Let $A = D_1 f(a,b)$ and $B = D_2 f(a,b)$ (so that $f'(a,b)$ has a matrix $(A \;\; B) \in M_{1\times 2}$.) Let $x \in (a-s, a+s)$ and set $h = x-a$, $k = g(x)-b$.

(a) Prove that there are real-valued functions ψ_1 and ψ_2 defined in a neighborhood of 0 such that $\lim_{(x,y)\to 0} \phi_i(x,y) = 0$ for $i = 1, 2$, and such that

$$\frac{h}{k} + \frac{A}{B} + \frac{1}{B}\psi_1(h,k) + \frac{1}{B}\psi_2(h,k)\frac{k}{h} = 0.$$

(Hint: let $\phi(h,k)$ be as in the alternate version of differentiability of f (notes, Lemma 3.16), and write

$$\phi(h,k)\|(h,k)\| = \phi(h,k)\frac{\|(h,k)\|}{|h|+|k|}\left(\frac{|h|}{h}h + \frac{|k|}{k}k\right).)$$

Proof. Let $f : U \subseteq \mathbb{R}^2 \to \mathbb{R}$ be continuously differentiable, let $(a,b) \in U$ with $D_2 f(a,b) \neq 0$. Let $c = f(a,b)$. Let $g : B_s(a) \to B_r(b)$ with $f(x,, g(x)) = c$. Let $A = D_1 f(a,b)$ and $B = D_2 f(a,b)$. Now, there exists $\phi : B_r(0) \to \mathbb{R}$, since f is differentiable, such that $\phi(0) = 0$, ϕ is continuous at 0, and $f(a+h) = f(a) + T(h) + \phi(h)\|h\|$. Define $\psi_1, \psi_2 : B_r(0) \to \mathbb{R}$ by

$$\psi_1(h,k) = \begin{cases} \frac{\phi(h,k)\|(h,k)\||h|}{(|h|+|k|)h}, & \text{if } h \neq 0 \\ 0, & \text{if } h = 0 \end{cases}$$

and

$$\psi_2(h,k) = \begin{cases} \frac{\phi(h,k)\|(h,k)\||k|}{(|h|+|k|)k}, & \text{if } k \neq 0 \\ 0, & \text{if } k = 0. \end{cases}$$

Note, $\lim_{(h,k)\to 0} \phi(h,k) = 0$ because of the definition of ϕ, which implies that $\lim_{(h,k)\to 0} \psi_1(h,k) = 0$ and $\lim_{(h,k)\to 0} \psi_2(h,k) = 0$. Let $x \in (a-s, a+s) \setminus \{a\}$ such that $(x-a, g(x)-b) \in B_r(0)$. Let $h = x-a$ and $k = g(x)-b$. Therefore, by definition of ϕ, $f((a,b)+(h,k)) = f(a,b) + f'(a,b)(h,k) + \phi(h,k)\|(h,k)\|$. So, $f((a,b)+(h,k)) =$

$f((a,b)+(x-a,g(x)-b)) = f(x,g(x)) = c$ and $f(a,b)=c$, which gives us

$$\begin{aligned}
0 &= f'(a,b)(h,k) + \phi(h,k)\|(h,k)\| \\
&= (A\ \ B)(h,k) + \phi(h,k)\frac{\|(h,k)\|}{|h|+|k|}\left(\frac{|h|}{h}h + \frac{|k|}{k}k\right) \\
&= (A\ \ B)(h,k) + \frac{\phi(h,k)\|(h,k)\|\,|h|}{(|h|+|k|)}\cdot h + \frac{\phi(h,k)\|(h,k)\|\,|k|}{(|h|+|k|)}\cdot k \\
&= (A\ \ B)(h,k) + \psi_1(h,k)\cdot h + \psi_2(h,k)\cdot k && \text{(By definition of } \psi_1 \text{ and } \psi_2) \\
&= Ah + Bk + \psi_1(h,k)\cdot h + \psi_2(h,k)\cdot k \\
&= Bh\left(\frac{A}{B} + \frac{k}{h} + \frac{1}{B}\psi_1(h,k) + \frac{1}{b}\psi_2(h,k)\frac{k}{h}\right) \\
&= \frac{h}{k} + \frac{A}{B} + \frac{1}{B}\psi_1(h,k) + \frac{1}{B}\psi_2(h,k)\frac{k}{h}. && \text{(Since } B = D_2f(a,b) \neq 0)
\end{aligned}$$

□

(b) Prove that $g'(a) = -A/B$. (Hint: solve for $\frac{h}{k}$ in part (a).)

Proof. Solving $\frac{h}{k} + \frac{A}{B} + \frac{1}{B}\psi_1(h,k) + \frac{1}{B}\psi_2(h,k)\frac{k}{h} = 0$ for $\frac{k}{h}$, then we get

$$\frac{k}{h} = \frac{-(A+\psi_1(h,k))}{B+\psi_2(h,k)}.$$

Then,

$$\begin{aligned}
\lim_{x\to a}\frac{g(a)-g(x)}{a-x} &= \lim_{x\to a}\frac{g(x)-g(a)}{x-a} \\
&= \lim_{x\to a}\frac{g(x)-b}{x-a} \\
&= \lim_{(h,k)\to 0}\frac{k}{h} && \text{(By definition of } h \text{ and } k) \\
&= \frac{-(A+\psi_1(h,k))}{B+\psi_2(h,k)} \\
&= \frac{-A}{B}. && \text{(Since } \lim_{(h,k)\to 0}\psi_{1,2} = 0)
\end{aligned}$$

Therefore, $g'(a) = \frac{-A}{B}$. □

Problem 22. Finish the proof of the implicit function theorem. Also show that if f in the statement is C^k for $k > 1$ the g is also C^k.

Proof. Continuing, we must show that g is continuous at all $a \in B_s(a)$. Let $\epsilon > 0$ be arbitrary but fixed. Let $a' \in B_s(a)$ be arbitrary but fixed. Let $b' \in B_r(b)$ such that $g(a') = b'$.

Let $Z = \{(x,y) \in \mathbb{R}^2 : x \in B_s(a), y \in B_r(b), |y - b'| < \epsilon\}$. Now, by construction of s, $D_2 f\big|_Z(a', b') \neq 0$. So, we now have a $B_s(a)', B_r(b)'$ such that $B_s(a)' \subseteq B_s(a)$ and $B_r(b)' \subseteq B_r(b)$ and g_1 such that $g_1 : B_s(a)' \to B_r(b)'$ such that for each $x \in B_s(a)$, $D_i(x, g_1(x)) = 0$ for $i = 1, 2$ and $d(g_1(x), b') < \epsilon$. But, by uniqueness of $g(x)$, we get $g_1(x)$ for all $x \in B_s(a)$. Thus, for all $x \in B_s(a)$, $d(g(x), g_1(x)) < \epsilon$. Hence, g is continuous at a'. But, since a' was arbitrary in $B_s(a)$, then g is continuous in over $B_s(a)$.

Now, on showing f is C^k implies g is C^k, we have already proven the base case of if f is C^1, then g is C^1. So, we will continue with the inductive step. Suppose the theorem is true for some $k > 1$. So, when f is C^{k+1}, then g is C^k. This is because $A, B^{-1} \in C^k$ and g is a composition of the two. Therefore, $g' \in C^k$, which implies $g \in C^{k+1}$. Therefore, if f is C^k, then g is C^k. $\square$

Problem 23. Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be given by $f(\rho, \phi, \theta) = (\rho \sin\phi \sin\theta, \rho\cos\phi)$.

(a) Use the implicit function theorem to show that the equation $f(\rho, \phi, \theta) = (1, 1)$ can be solved for (ϕ, θ) as a function of ρ near the point $(\sqrt{3}, \tan^{-1}\sqrt{2}, \pi/4)$.

Proof. Consider the surface $S := \{(\rho, \phi, \theta) \in \mathbb{R}^3 : \rho\sin\phi\sin\theta = 1 \text{ and } \rho\cos\phi = 1\}$. This can be rewritten as $\{(\rho, \phi, \theta) \in \mathbb{R}^3 : f(\rho, \phi, \theta) = (0, 0)\}$ where $f : \mathbb{R}^3 \to \mathbb{R}^2$ is given by $f(\rho, \phi, \theta) := (\rho\sin\phi\sin\theta - 1, \rho\cos\phi - 1)$. Then,

$$\begin{aligned} f'(\rho, \phi, \theta) &= \begin{pmatrix} D_1 f_1 & D_2 f_1 & D_3 f_1 \\ D_1 f_2 & D_2 f_2 & D_3 f_2 \end{pmatrix} \\ &= \begin{pmatrix} \sin(\phi)\sin(\theta) & \rho\cos(\phi)\sin(\theta) & \rho\cos(\theta)\sin(\phi) \\ \cos(\phi) & -\rho\sin(\phi) & 0 \end{pmatrix}. \end{aligned}$$

Now, at $(\sqrt{3}, \tan^{-1}\sqrt{2}, \pi/4)$, we have

$$f'(\sqrt{3}, \tan^{-1}\sqrt{2}, \pi/4) = \begin{pmatrix} \frac{\sqrt{3}}{3} & \frac{\sqrt{2}}{2} & 1 \\ \frac{\sqrt{3}}{3} & -\sqrt{2} & 0 \end{pmatrix}.$$

So, $\frac{\partial f}{\partial(\phi,\theta)}(\sqrt{3}, \tan^{-1}\sqrt{2}, \pi/4) = \begin{pmatrix} \frac{\sqrt{2}}{2} & 1 \\ -\sqrt{2} & 0 \end{pmatrix}$ and has a determinant $0 + \sqrt{2} = \sqrt{2} \neq 0$. Since the matrix is invertible, then by the implicit function theorem, there exists $r, s > 0$, and a unique function $g : (\sqrt{3} - s, \sqrt{3} + s) \to B_r\left((\tan^{-1}\sqrt{2}, \pi/4)\right)$, such that $f(\rho, g(\rho)) = (0, 0)$ for all $x \in (\sqrt{3} - s, \sqrt{3} + s)$. $\square$

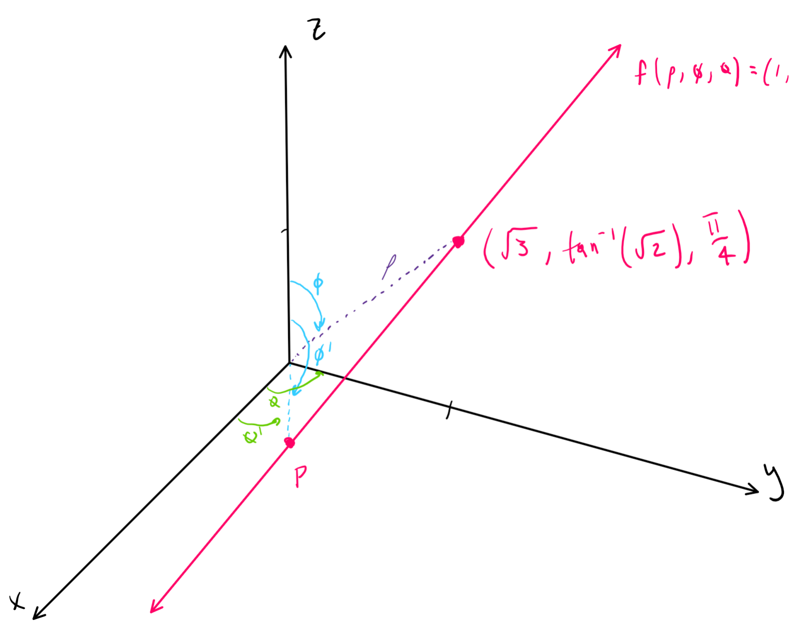
(b) Use the implicit function theorem to find $\phi'(\sqrt{3})$ and $\theta'(\sqrt{3})$.

Proof. We have that $\frac{\partial f}{\partial \theta}(\sqrt{3}, \tan^{-1}\sqrt{2}, \pi/4) = \begin{pmatrix} \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \end{pmatrix}$. So,

$$\begin{aligned} g'(\sqrt{3}) &= -\begin{pmatrix} \frac{\sqrt{2}}{2} & 1 \\ -\sqrt{2} & 0 \end{pmatrix}^{-1} \begin{pmatrix} \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \end{pmatrix} \\ &= -\left(\frac{1}{2}\right) \begin{pmatrix} 0 & -\sqrt{2} \\ 2 & 1 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \end{pmatrix} \\ &= \begin{pmatrix} \frac{\sqrt{6}}{6} \\ -\frac{\sqrt{3}}{2} \end{pmatrix}. \end{aligned}$$

Therefore, $\phi'(\sqrt{3}) = \frac{\sqrt{6}}{6}$ and $\theta'(\sqrt{3}) = -\frac{\sqrt{3}}{2}$. □

(c) Give a geometric description of the situation, and explain why the results are reasonable.



Note, the point P is after our change in x. So, $\phi'(\sqrt{3}) = \frac{\sqrt{6}}{6} > 0$ and $\theta'(\sqrt{3}) = -\frac{\sqrt{3}}{2} < 0$, which makes sense because when ρ increases, then θ decreases and ϕ increases, which is what the math tells us.

Problem 24. Consider the equation $xe^y + ye^x = 0$.

(a) Prove that this equation defines y as a C^∞ function of x in a neighborhood of $(0,0)$.

Proof. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = xe^y + ye^x$. Then, $D_1 f(x,y) = e^y + ye^x$, $D_{1,1}f(x,y) = ye^x$, $D_2 f(x,y) = xe^y + e^x$, $D_{2,2}f(x,y) = xe^y$, and $D_{1,2}f(x,y) = e^y + e^x$.

Let $P(n)$ be the statement "$\frac{\partial^n f}{\partial y^n}(x,y) = xe^y$".

Base Case: $\frac{\partial^2 f}{\partial y^2} = xe^y$. Thus, the base case is true.

Inductive Step: Suppose $P(k)$ is true for some $k \geq 2$. Then, $\frac{\partial^k f}{\partial y^k}(x,y) = xe^y$, which implies $\frac{\partial^{k+1} f}{\partial y^{k+1}}(x,y) = xe^y$. So, $P(k+1)$ is true.

Now, let $P(n)$ be the statement "$\frac{\partial^n f}{\partial x^n}(x,y) = ye^x$".

Base Case: $\frac{\partial^2 f}{\partial x^2}(x,y) = ye^x$. Thus, the base case is true.

Inductive Step: Suppose $P(k)$ is true for some $k \geq 2$. Then, $\frac{\partial^k f}{\partial x^k}(x,y) = ye^x$, which implies $\frac{\partial^{k+1} f}{\partial x^{k+1}}(x,y) = ye^x$. So, $P(k+1)$ is true. Let $m \geq 2$ be arbitrary but fixed. Then, $\frac{\partial^m f}{\partial y^m}(x,y) = xe^y$ implies $\frac{\partial^{m+1} f}{\partial y^m x}(x,y) = e^y$. Also, $\frac{\partial^m f}{\partial x^m}(x,y) = ye^x$ implies $\frac{\partial^{m+1} f}{\partial x^m y}(x,y) = e^x$.

Lastly, let $P(n)$ now be the statement "$\frac{\partial^{m+n} f}{\partial x^m y^n}(x,y) = 0$.

Base Case: $\frac{\partial^{m+1} f}{\partial x^m y} = e^x$, which implies $\frac{\partial^{m+2} f}{\partial x^m y^2}(x,y) = 0$. Thus, the base case is true.

Inductive Step: Suppose $P(k)$ is true for some $k \geq 2$. Then, $\frac{\partial^{m+k} f}{\partial x^m y^k}(x,y) = 0$, which implies $\frac{\partial^{m+k+1} f}{\partial x^m y^{k+1}} = 0$. So, $P(k+1)$ is true.

In total, we have $\frac{\partial f}{\partial x}(x,y) = e^y + ye^x$, $\frac{\partial f}{\partial y}(x,y) = xe^y + e^x$, $\frac{\partial^n f}{\partial x^n}(x,y) = ye^x$, $\frac{\partial^{n+1} f}{\partial x^n}(x,y) = e^x$, $\frac{\partial^n f}{\partial y^n}(x,y) = xe^y$, $\frac{\partial^{n+1} f}{\partial y^n}(x,y) = e^y$, and $\frac{\partial^{m+n} f}{\partial x^m y^n}(x,y) = 0$ for all $n, m \geq 2$. Clearly, they are all continuous. Therefore, f is C^∞. Note, $f(0,0) = 0$ and $D_2 f(0,0) = 1 \neq 0$, so by the implicit function theorem, there exists $r, s > 0$ and $g : B_s(0) \to B_r(0)$ defined by $f(x, g(x)) = 0$. Therefore, f and g are C^∞ and $f(x,y) = 0$ defines y as a C^∞ function of x in a neighborhood of $(0,0)$. $\square$

(b) Let $y = g(x)$ be this implicitly defined function. Find $g'(0)$ and $g''(0)$.

Proof. Note,

$$g'(x) = -\frac{D_1 f}{D_2 f} = -\frac{e^y + ye^x}{xe^y + e^x}$$

and

$$g''(x) = \frac{-(\frac{\partial f}{\partial y})^2 \frac{\partial^2 f}{\partial x^2} + 2\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}\frac{\partial^2 f}{\partial xy} - (\frac{\partial f}{\partial x})^2 \frac{\partial^2 f}{\partial y^2}}{(\frac{\partial f}{\partial y})^3}$$
$$= \frac{-(xe^y + e^x)^2(ye^x) + 2(e^y + ye^x)(xe^y + e^x)(e^y + e^x) - (e^y + ye^x)^2(xe^y)}{(xe^y + e^x)^3}.$$

So, from evaluating, we get $g'(0) = -1$ and $g''(0) = -4$. □

(c) Use this information to explain the appearance of the curve $xe^y + ye^x = 0$ near $(0, 0)$. As (x, y) approaches $(0, 0)$, the slope is directed downward at a decreasing rate.