

# MAT 473: Intermediate Real Analysis II

Trey Manuszak  
Arizona State University

April 3, 2020

**Problem 33.** Recall the Borel  $\sigma$ -algebra  $\mathcal{B}_{\mathbb{R}}$  from the course notes. Prove that  $\mathcal{B}_{\mathbb{R}}$  is generated as a  $\sigma$ -algebra by the collection of closed intervals  $\{[a, \infty) : a \in \mathbb{R}\}$ .

*Proof.* Let  $\mathcal{O}$  denote the collection of all open intervals in  $\mathbb{R}$ . Since every open set in  $\mathbb{R}$  is at most a countable union of open intervals, Then  $\mathcal{M}(\mathcal{O}) = \mathcal{B}_{\mathbb{R}}$ . Let  $\mathcal{E}$  denote the collection of intervals of the form  $[a, \infty)$  for all  $a \in \mathbb{R}$ . Let  $(a, b) \in \mathcal{O}$  for some  $a, b \in \mathbb{R}$  such that  $b > a$ . Let  $a_n = a + \frac{1}{n}$  and  $b_n = b - \frac{1}{n}$ . Then,

$$(a, b) = \bigcup_{n=1}^{\infty} [a_n, b_n) = \bigcup_{n=1}^{\infty} \{[a_n, \infty) \cap [b_n, \infty)^c\},$$

which implies that  $(a, b) \in \mathcal{M}(\mathcal{E})$ . This means that  $\mathcal{O} \subseteq \mathcal{M}(\mathcal{E})$ , which implies  $\mathcal{M}(\mathcal{O}) \subseteq \mathcal{M}(\mathcal{E})$ . But, since every element of  $\mathcal{E}$  is closed, then  $\mathcal{M}(\mathcal{E}) \subseteq \mathcal{B}_{\mathbb{R}}$ . This gives us

$$\mathcal{B}_{\mathbb{R}} = \mathcal{M}(\mathcal{O}) \subseteq \mathcal{M}(\mathcal{E}) \subseteq \mathcal{B}_{\mathbb{R}}.$$

Therefore,  $\mathcal{M}(\mathcal{E}) = \mathcal{B}_{\mathbb{R}}$ . □

**Problem 34.** Prove that for every subset  $E \subseteq \mathbb{R}$  there is a  $G_{\delta}$ -set  $A$  with  $E \subseteq A$  and  $m^*(E) = m^*(A)$ .

*Proof.* Let  $E \subseteq \mathbb{R}$  be arbitrary. By outer-measure, there exists a collection of open intervals  $I_n \subseteq \mathbb{R}$  such that  $E \subseteq \bigcup_{n=1}^{\infty} I_n$ . From the result of Problem 31, this gives us

$$m^*(E) \leq \sum_{n=1}^{\infty} m(I_n) < m^*(E) + \frac{1}{n}.$$

Also, by countable subadditivity,  $m^*(\bigcup_{n=1}^{\infty} I_n) \leq \sum_{n=1}^{\infty} m(I_n)$ . Let  $A = \bigcap_{n=1}^{\infty} \bigcup_{n=1}^{\infty} I_n$ . Then,  $A$  is a  $G_{\delta}$ -set. This gives us  $E \subseteq A$ . Then, for each  $n$ , we get

$$m^*(E) \leq m^*(A) \leq m^*(\bigcup_{n=1}^{\infty} I_n) \leq \sum_{n=1}^{\infty} m(I_n) < m^*(E) + \frac{1}{n}.$$

Therefore, by squeeze theorem, as  $n$  approaches  $\infty$ , then  $m^*(E) = m^*(A)$ . □

**Problem 35.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a function. Let  $A = \{x \in \mathbb{R} : f \text{ is continuous at } x\}$ . Prove that  $A$  is a  $G_{\delta}$ -set. (Hint: use the oscillation of  $f$  from homework 8.)

*Proof.* Let  $A = \{x \in \mathbb{R} : f \text{ is continuous at } x\}$ . Note,  $f$  is continuous at  $c$  if and only if  $\text{osc}(f, c) = 0$ . That is,  $f$  is continuous at  $c$  if and only if

$$\limsup_{x \rightarrow c} f(x) = \liminf_{x \rightarrow c} f(x). \quad (\text{I like this definition provided by [?]})$$

So, by looking at the complement of  $A$ , we get

$$\begin{aligned} A^c &= \left\{ x \in \mathbb{R} : \liminf_{x \rightarrow c} f(x) < \limsup_{x \rightarrow c} f(x) \right\} \\ &= \left\{ x \in \mathbb{R} : \exists a, b \in \mathbb{Q} \text{ s.t. } \liminf_{x \rightarrow c} f(x) \leq a < b \leq \limsup_{x \rightarrow c} f(x) \right\} \\ &= \bigcup_{a, b} \left( \left\{ x \in \mathbb{R} : \liminf_{x \rightarrow c} f(x) \leq a \right\} \cap \left\{ x \in \mathbb{R} : b \leq \limsup_{x \rightarrow c} f(x) \right\} \right). \quad (\text{with } a < b) \end{aligned}$$

Note, if  $\liminf_{x \rightarrow c} f(x) > a$ , then there must exist  $\epsilon > 0$  arbitrary but fixed such that  $\inf_{|x-c|<\epsilon} f(x) > a$ . Now, for each  $x \in B_\epsilon(c)$ , there is  $\epsilon_0 > 0$  arbitrary but fixed such that  $B_{\epsilon_0}(x) \subset B_\epsilon(c)$ . So, we get

$$\inf_{|x-c'|<\epsilon_0} f(x) \geq \inf_{|x-c|<\epsilon} f(x) > a.$$

This means that  $\left\{ x \in \mathbb{R} : \liminf_{x \rightarrow c} f(x) > a \right\}$  is open, which implies  $\left\{ x \in \mathbb{R} : \liminf_{x \rightarrow c} f(x) \leq a \right\}$  is closed. We can similarly show that  $\left\{ x \in \mathbb{R} : b \leq \limsup_{x \rightarrow c} f(x) \right\}$  is closed since  $\limsup_{x \rightarrow c} f(x) = -\left(\liminf_{x \rightarrow c} (-f(x))\right)$ .

Finally, since each set in the pair of a countable union defined above is closed, then  $A^c$  is a  $F_\sigma$ -set, which implies  $A$  is a  $G_\delta$ -set.  $\square$

**Problem 36.** For subsets  $A, B \subseteq \mathbb{R}$  recall that the *distance* between  $A$  and  $B$  is defined to be  $\text{dist}(A, B) = \inf\{|x - y| : x \in A, y \in B\}$ . Let  $A$  and  $B$  be subsets of  $\mathbb{R}$  such that  $\text{dist}(A, B) > 0$ . Prove that  $m^*(A \cup B) = m^*(A) + m^*(B)$ .

*Proof.* Let  $A, B \subseteq \mathbb{R}$ . Let  $\epsilon > 0$  be arbitrary but fixed such that  $\text{dist}(A, B) \geq \epsilon$ . Let  $E = \bigcup_{x \in A} B_{\frac{\epsilon}{2}}(x)$ . Then,  $A \subset E$ . Also, since  $E$  used the ball of radius  $\frac{\epsilon}{2}$ , then  $E \cap B = \emptyset$ . Also, since  $E$  used a countable union of open balls, then  $E$  is measurable. This gives us  $m^*(A \cup B) = m^*((A \cup B) \cap E) + m^*((A \cup B) \cap E^c)$  from Definition 17.1 in the notes. But, note that  $(A \cup B) \cap E^c = B$  and  $(A \cup B) \cap E = A$ . Therefore, we have  $m^*(A \cup B) = m^*(A) + m^*(B)$ .  $\square$

## References

@articleroyden2010real, title=Real analysis (4th Edition), author=Royden, HL and Fitzpatrick, PM, journal=New Jersey: Printice-Hall Inc, year=2010