MAT 473 **Final Exam** **Spring, 2020**

This is a take-home exam. The period of the exam is from Tuesday, May 5 at 9:50 am, to Friday, May 8 at 5:00 pm. Send to me by email a copy of your exam, together with this exam page, by 6:00 pm, Friday, May 8.

Solve four of the following five problems. If you choose to submit solutions to all five problems, only the four best solutions will be counted. Give complete proofs, and in particular, show all steps of your calculations. **Your proofs must be written in complete sentences and must include sufficient detail to allow evaluation of your understanding of the methods you use.** Be sure to verify all hypotheses of any theorem that you use, and also give the name of the theorem if it has one. You may ignore any hints, but you must follow all explicit instructions and show all of your reasoning.

You may write your exam longhand, or type it, or typset it with TeX. In all cases you must print your name and the date, and sign your name below, and submit a scan or photograph of this page with your solutions.

Honor Statement: By signing below you confirm that you have obeyed the following rules for the conduct of this exam.

During the exam period, Tuesday, May 5 at 9:50 am through Friday, May 8 at 5:00 pm,

1. you have used only your own course notes and papers, and the materials on the course webpage;
2. you have not looked in any books, notes, or other printed or written materials about the subject matter of the course other than those in 1;
3. you have not consulted in any way with any other person about the problems on this exam, the material represented on this exam, or the material covered in the course;
4. you have not accessed any electronic resources or webpages other than the materials allowed above in 1 to help you solve the problems.

PRINT NAME: Trey Manuszak

SIGN NAME: Trey Manuszak

DATE: 5/5/20

MAT 473: Intermediate Real Analysis II

Trey Manuszak
Arizona State University

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Problem 1. Let $U \subseteq \mathbb{R}^3$ be an open set, let $f : U \to \mathbb{R}$ be continuously differentiable, and let $p = (x_0, y_0, z_0) \in U$ and $c = f(p)$. We let (x, y, z) denote the coordinates in $\mathbb{R}^3$. Using the implicit function theorem, and the equation $f(x, y, z) = c$, we can give a precise meaning to the classical expression $\frac{\partial y}{\partial x}$.

(a) What is this meaning? Explain carefully.

In physical terms, we would be able to find a directional derivative, or quantify a change in direction y for a certain change in direction x. Given an example of a surface where this is not always possible, if you consider a function which goes through the cartesian point $(1, 0, 0) \in S$ with S being a unit sphere. Then, there does not exist a function that can represent y in terms of x and z on the function f, since $\frac{\partial y}{\partial x}$ would be undefined for any function going through that point on the surface of the unit sphere.

(b) What condition guarantees that $\frac{\partial y}{\partial x}$ makes sense?

Since we can supposedly find $\frac{\partial y}{\partial x}$, then we must've found a function, which approximates y defined as $y : B_s(x_0, z_0) \to B_r(y_0)$, which approximates the level set $\{(x, y, z) \in U : f(x, y, z) = c\}$ such that $f(x, y(x, z), z) = c$ for some $s, r > 0$. This function y is continuously differentiable and if $f(x, y(x, z), z) = c$ is differentiated with respect to x, will yield

$$f_x + f_y \cdot \frac{\partial y}{\partial x} + f_z \cdot 0 = 0.$$

We can then solve $\frac{\partial y}{\partial x} = -\frac{f_x}{f_y}$. So, since $\frac{\partial y}{\partial x}$ exists, f_y must be invertible, or not equal to zero. Also, since the function y represented in terms of x and z exist, we must have used implicit function theorem, which relies on $p \in U$ open with f being continuously differentiable and $f(p) = c$.

(c) Prove the "paradox of classical notation": $\frac{\partial y}{\partial x} \cdot \frac{\partial z}{\partial y} \cdot \frac{\partial x}{\partial z} = -1$. State the result as a precise theorem (in non-classical notation), and be sure to give the necessary hypothesis.

Theorem 1. Let $U \subseteq \mathbb{R}^3$ be an open set, let $f : U \to \mathbb{R}$ be continuously differentiable, and let $p = (x_0, y_0, z_0) \in U$ and $c = f(p)$. Suppose there is a continuously differentiable function $y : B_s(x_0, z_0) \to B_r(y_0)$ such that $f(x, y(x, z), z) = c$ and $f_y \neq 0$. Suppose there is a continuously differentiable function $z : B_u(x_0, y_0) \to B_t(z_0)$ such that $f(x, y, z(x, y)) = c$ and $f_z \neq 0$. Suppose there is a continuously differentiable function $x : B_w(y_0, z_0) \to B_v(x_0)$ such that $f(x(y, z), y, z) = c$ and $f_x \neq 0$. Then $\frac{\partial y}{\partial x} \cdot \frac{\partial z}{\partial y} \cdot \frac{\partial x}{\partial z} = -1$.

Proof. By applying the implicit function theorem in three different instances, we can show that $\frac{\partial y}{\partial x} = -\frac{f_x}{f_y}$, $\frac{\partial z}{\partial y} = -\frac{f_y}{f_z}$, and $\frac{\partial x}{\partial z} = -\frac{f_z}{f_x}$. Thus,

$$\begin{aligned}\frac{\partial y}{\partial x} \cdot \frac{\partial z}{\partial y} \cdot \frac{\partial x}{\partial z} &= \left(-\frac{f_x}{f_y}\right)\left(-\frac{f_y}{f_z}\right)\left(-\frac{f_z}{f_x}\right) \\ &= -1.\end{aligned}$$

□

The real problem that I see with the classical notation is that $\frac{\partial y}{\partial x}$ appears to be a fraction (which it is!) but it is hard to see domain restrictions as ∂x alone does not represent anything, hence why it looks like everything should "paradoxically" cancel to be 1, but doesn't.

Problem 2. Let $U \subseteq \mathbb{R}^2$ be an open connected set and let $f : U \to \mathbb{R}$ be continuously differentiable. Suppose that $\frac{\partial f}{\partial y} = 0$ throughout U.

(a) Suppose that U has the following property: for every vertical line $L \subseteq \mathbb{R}^2$, the intersection $U \cap L$ is a subinterval of L. Prove that for all x, y_1, y_2 such that $(x, y_1) \in U$ and $(x, y_2) \in U$, we have $f(x, y_1) = f(x, y_2)$ (we say that f is *independent of* y).

Proof. Let $U \subseteq \mathbb{R}^2$ be an open connected set and let $f : U \to \mathbb{R}$ be continuously differentiable. Suppose that $\frac{\partial f}{\partial y} = 0$ throughout U. Let $L = \{(x, y) \in \mathbb{R}^2 : x \text{ is fixed}\} \subseteq U$. Then, since f is differentiable and L is convex, by the mean value theorem, for $\overline{ab} \subseteq L$, there exists $c \in \overline{ab}$ such that $f(b) - f(a) = f'(c)(b - a)$. But, $f'(c) = 0$ by hypothesis. Thus, $f(a) = f(b)$, which implies for $(x, y_1), (x, y_2) \in U$, $f(x, y_1) = f(x, y_2)$. □

(b) Let $U = \mathbb{R}^2 \setminus ((-\infty, 0] \times \{0\})$. (Thus U equals the plane with the nonpositive x-axis removed.) Find a counterexample to the result in part (a) for U, and prove that your counterexample does the job. (In other words find a continuously differentiable function $f : U \to \mathbb{R}$ such that $\frac{\partial f}{\partial y} = 0$ throughout U, but such that there exist x, y_1, y_2 such that $(x, y_1), (x, y_2) \in U$ and $f(x, y_1) \neq f(x, y_2)$ - and prove that your function f does have all of these properties.)

Proof. Let $U = \mathbb{R}^2 \setminus ((-\infty, 0] \times \{0\})$ and $f : U \to \mathbb{R}$ be defined by

$$f(x, y) = \begin{cases} x^2 & \text{if } x, y < 0 \\ 0 & \text{else} \end{cases}$$

Then, we have that $f_y = 0$ throughout U, and $f_x = 2x$, which is continuous over U. However, $f(1, 1) = 1$, but $f(1, -1) = 0$. Thus, there exists a continuously differentiable function $f : U \to \mathbb{R}$ such that $\frac{\partial f}{\partial y} = 0$ throughout U, and there exists $(x, y_1), (x, y_2) \in U$ with $f(x, y_1) \neq f(x, y_2)$. □

Problem 3. Let $f : \mathbb{R} \to \mathbb{R}$ be Lipschitz, and let $E \subseteq \mathbb{R}$ be a measurable set with $m(E) = 0$. Prove that $m(f(E)) = 0$. (Hint: write open intervals in the form $(a - r, a + r)$ rather than in the form (a, b).) (Recall that f is *Lipschitz* if there is a constant $C > 0$ such that for all $x, y \in \mathbb{R}$, $|f(x) - f(y)| \leq C\,|x - y|$.)

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be C-Lipschitz, with $C > 0$, and let $E \subseteq \mathbb{R}$ be a measurable set with $m(E) = 0$. Since E has measure zero, for $\epsilon > 0$ arbitrary but fixed, there exists open intervals $A_n = (a - r_n, a + r_n)$, such that it creates a countable open cover, $E \subseteq \cup_{n\in\mathbb{N}} A_n$, and $m(E) \leq \sum_{n\in\mathbb{N}} m(A_n) < \epsilon$. Looking at each n, we have the following

$$\begin{aligned} m(f(A_n)) &= |f(a - r_n) - f(a + r_n)| && \text{(By Lebesgue measure on } \mathbb{R}) \\ &\leq C \cdot |a - r_n - (a + r_n)| && \text{(Since } f \text{ is } C\text{-Lipschitz)} \\ &= C \cdot |A_n|\,. \end{aligned}$$

Now, we get

$$\begin{aligned} m(f(E)) &\leq m\left(f\left(\bigcup_{n\in\mathbb{N}} A_n\right)\right) && \text{(By monotonicity)} \\ &= m\left(\bigcup_{n\in\mathbb{N}} f(A_n)\right) && \text{(By continuity of } f) \\ &\leq \sum_{n\in\mathbb{N}} m(f(A_n)) && \text{(By subadditivity and } C\text{-Lipschitz property of } f) \\ &< C \cdot \epsilon. \end{aligned}$$

Therefore, since epsilon was arbitrary, we can conclude $m(f(E)) = 0$. ☐

Problem 4. Let $f : \mathbb{R} \to \mathbb{R}$ be a nonnegative measurable function. Suppose that $\int f = 0$. Prove that $f = 0$ almost everywhere. (Hints: let $E_n = \{f \geq \frac{1}{n}\}$ and let $p_n = \frac{1}{n}\chi_{E_n}$.)

Proof. Suppose for some $X \subseteq \mathrm{dom}(f)$ that is measurable, $\int_X f = 0$. Let $E_n = \{x \in X : f(x) > \frac{1}{n}\}$. Then, $\{x \in X : f(x) > 0\} = \cup_{n\in\mathbb{N}} E_n$. Then,

$$\begin{aligned} m(E_n) &= \int_{E_n} 1 \cdot \chi_{E_n} \\ &= n \int_{E_n} \frac{1}{n} \cdot \chi_{E_n} \\ &\leq n \int_{E_n} f \\ &\leq n \int_X f \\ &= 0. \end{aligned}$$

Therefore, $m(E_n) = 0$, which implies $f = 0$ a.e., since X was an arbitrary measurable set in the domain of f and $\{x \in X : f(x) > 0\} = \cup_{n\in\mathbb{N}} E_n$. $\square$

Problem 5. For $n = 1, 2, 3, \ldots$ let $f_n : \mathbb{R} \to \mathbb{R}$ be a nonnegative measurable function. Suppose that the sequence (f_n) is decreasing, in the sense that for each $x \in \mathbb{R}$, $f_1(x) \geq f_2(x) \geq f_3(x) \geq \ldots$. Suppose further that $\lim_{n\to\infty} \int f_n = 0$. Prove that $(f_n) \to 0$ almost everywhere.

Proof. For $n = 1, 2, 3, \ldots$ let $f_n : \mathbb{R} \to \mathbb{R}$ be a nonnegative measurable function. Suppose that the sequence (f_n) is decreasing, in the sense that for each $x \in \mathbb{R}$, $f_1(x) \geq f_2(x) \geq f_3(x) \geq \ldots$. Suppose further that $\lim_{n\to\infty} \int f_n = 0$. Since each f_n is nonnegative measurable and pointwise decreasing as $n \to \infty$, then $(f_n) \to f$ pointwise for some f. By letting $g = f_1$, by Lebesgue's dominated convergence theorem, we know that $\int f = \lim_{n\to\infty} \int f_n$. Since $\lim_{n\to\infty} \int f_n = 0$ by assumption, then $\int f = 0$. By Exercise 21.6 (2), we know that since $\int f = 0$, then $f = 0$ almost everywhere. Therefore, since $\lim_{n\to\infty} f_n = f$, then $\lim_{n\to\infty} f_n \to 0$ almost everywhere, transitively. $\square$