MAT 473: Intermediate Real Analysis II

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Problem 37. Let $A_1, A_2, \ldots$ be measurable sets, and suppose that $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots$. Prove that $m(\cup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} m(A_n)$. (This is called *continuity from below* of Lebesgue measure.) (Hints: use Proposition 16.4 of the notes. It is useful also to remember that $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \sum_{i=1}^{n} a_i$.)

Proof. Let $A_1, A_2, \ldots$ be measurable sets, and suppose that $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots$. Suppose $\lim_{N\to\infty} \bigcup_{n=1}^{N} A_n = A$. From Proposition 16.4, we know

$$A = A_1 \cup \bigcup_{n=1}^{\infty} (A_{n+1} \setminus A_n).$$

Note, A_1 and $\bigcup_{n=1}^{\infty}(A_{n+1} \setminus A_n)$ are disjoint and A is a σ-algebra. Since they are disjoint and measurable, then we have

$$\begin{aligned} m(A) &= \sum_{n=1}^{\infty} m(A_n \setminus A_{n-1}) \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} m(A_i \setminus A_{i-1}) \\ &= \lim_{n\to\infty} m(A_n). \end{aligned}$$

□

Problem 38. Let $A_1, A_2, \ldots$ be measurable sets, and suppose that $A_1 \supseteq A_2 \supseteq A_3 \supseteq \ldots$. Suppose further that $m(A_1) < \infty$. Prove that $m(\cap_{n=1}^{\infty} A_n) = \lim_{n\to\infty} m(A_n)$. Be sure to indicate where the finiteness hypothesis is used. (This is called *continuity from above* of Lebesgue measure.) (Hints: as in the previous problem. Also, you will need to consider $B_\infty := \cap_{n=1}^{\infty} A_n$.) Give an example of a decreasing sequence of measurable sets of infinite measure for which the above conclusion is false.

Proof. Let $A_1, A_2, \ldots$ be measurable sets, and suppose that $A_1 \supseteq A_2 \supseteq A_3 \supseteq \ldots$. Suppose further that $m(A_1) < \infty$ and $B_\infty := \lim_{N\to\infty} \bigcap_{n=1}^{N} A_n$. Note, $m(A_1 \setminus B_\infty) = \lim_{n\to\infty} m(A_1 \setminus A_n)$. Also, $m(B_\infty) \leq m(A_n) \leq m(A_1) < \infty$. So, by Problem 37,

$$\begin{aligned} m(A_1) - m(B_\infty) &= m(A_1 \setminus B_\infty) \\ &= \lim_{n\to\infty} m(A_1 \setminus A_n) \\ &= m(A_1) - \lim_{n\to\infty} m(A_n). \end{aligned}$$

So, by subtracting the $m(A_1)$ terms from both sides, we get

$$m(B_\infty) = \lim_{n\to\infty} m(A_n).$$

□

Now, it is important that $A_k < \infty$ for some $k \in \mathbb{Z}$. For example, if not, suppose $A_n = (n, \infty)$. Then, $A_n \supseteq A_{n+1} \supseteq A_{n+2} \supseteq \dots$ and $m(A_n) = \infty$ for each n, but $\bigcap_{n=1}^{\infty} A_n = \emptyset$. So, we get

$$\infty = \lim_{n\to\infty} m(A_n) \neq m(\bigcap_{n=1}^{\infty} A_n) = 0.$$

Problem 39. Let E be a measurable set, and let $\epsilon > 0$. Prove that there are an open $U \supseteq E$ and a closed set $F \subseteq E$ such that $m(U \setminus F) < \epsilon$. Here is an outline.

(a) Suppose that $E \subseteq [a, b]$. Use the definition of outer measure to find an open set $U \supseteq E$ with $m(U \setminus E) < \epsilon$.

(b) Suppose that $E \subseteq [a, b]$. Apply the previous part to $[a, b] \setminus E$ to prove that there is a closed set $F \subseteq E$ with $m(E \setminus F) < \epsilon$.

(c) For the general case let $E_n = E \cap [n, n+1]$ for $n \in \mathbb{Z}$, and apply the previous two parts with $\epsilon 4^{-(|n|+1)}$. Use the fact that if $S_n \subseteq T_n$ then $(\cup_n T_n) \setminus (\cup_n S_n) \subseteq \cup_n (T_n \setminus S_n)$.

Proof. Let E be a measurable set with $m(E) < \infty$, and let $\epsilon > 0$ be arbitrary but fixed. Then, there are intervals (a_n, b_n) with $E \subseteq \bigcup_{n=1}^{\infty}(a_n, b_n)$ and $m(E) \leq \sum_{n=1}^{\infty} m((a_n, b_n)) + \epsilon$. Define $U = \bigcup_{n=1}^{\infty}(a_n, b_n)$. Then, $U \supseteq E$ is open and $m(U) \leq m(E) + \epsilon$. Now, since $m(U) < \infty$ and $m(E) < \infty$, then $m(U \setminus E) = m(U) - m(E) < \epsilon$.

Then, since E is measurable, then E^c is measurable. So, there exists $O \supseteq E^c$ open such that $m(O \setminus E^c) \leq \epsilon$. Let $F = O^c$, which implies F is closed and $F \subseteq E$. Then, $E \setminus F = O \setminus E^c$, which implies $m(E \setminus F) \leq \epsilon$.

Now, let $U \supseteq E$ open and $F \subseteq E$ be closed such that $m(U \setminus E) < \frac{\epsilon}{2}$ and $m(E \setminus F) < \frac{\epsilon}{2}$. Note,

$$\begin{aligned}
(U \setminus E) \cup (E \setminus F) &= (U \cap E^c) \cup (E \cap F^c) \\
&= [(U \cap E^c) \cup E] \cap [(U \cap E^c) \cup F^c] \\
&= [(U \cup E) \cap (E \cup E^c)] \cap [(U \cup F^c) \cap (E^c \cup F^c)] \\
&= (U \cup E) \cap [(F \setminus U)^c \cap F^c] \\
&= U \cap (\emptyset^c \cap F^c) \\
&= U \cap F^c \\
&= U \setminus F.
\end{aligned}$$

This gives us

$$m(U \setminus F) = m\left((U \setminus E) + m\left((E \setminus F)\right) < \epsilon.\right.$$

□

Problem 40. The *Cantor set*, C, is a subset of $[0,1]$ defined as follows. Let $F_0 = [0,1], F_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$, and in general, F_{n+1} is obtained from F_n by deleting the middle open third of each subinterval of F_n. (Thus $F_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$.) Then $C := \cap_{n=1}^{\infty} F_n$. Prove the following:

(a) F_n is the union of 2^n pairwise disjoint closed intervals each of length 3^{-n}.

Proof. Clearly, they are disjoint as you are removing the open middle third of each interval, essentialy doubling the amount of intervals each iteration. Thus, there are 2^n disjoint intervals for F_n. As for the length, F_0 has length $3^{-0} = 1$, and through induction, one can clearly see, without loss of generality, by taking the first of the 2^n intervals in F_n, call it A, that $\sup\{x - 0 : x \in A\} = 3^{-n}$. □

(b) $m(C) = 0$.

Proof. By continuity from above of Lebesgue measure, we know that

$$\begin{aligned} m(C) &= m\left(\bigcap_{n=1}^{\infty} F_n\right) \\ &= \lim_{n \to \infty} m(F_n) \\ &= 0. && \text{(Since } m(F_n = \left(\tfrac{2}{3}\right)^n)) \end{aligned}$$

□

(c) C is a closed set, C has no isolated points, and the interior of C is empty.

Proof. Since C is a countable union of closed intervals, then C is closed. □