MAT 473: Intermediate Real Analysis II

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Problem 25. Let $p, q \geq 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

(a) Use the method of Lagrange multipliers to find the minimum of $\frac{1}{p}x^p + \frac{1}{q}y^q$ subject to the constraints $xy = 1$ and $x > 0$.

Proof. Taking each partial derivative, we must find a Lagrange multiplier λ such that

$$x^{p-1} - \lambda y = 0 \tag{1}$$
$$y^{q-1} - \lambda x = 0. \tag{2}$$

Note, $x = \frac{1}{y}$ and $y = \frac{1}{x}$. So, adding λy and multiplying by x, or $\frac{1}{y}$, to equation (1), we get $x^p = \lambda$. Similarly with equation (2), we get $y^q = \lambda$. Since $xy = 1$, then $x = y = 1$. Therefore, the minimum of $\frac{1}{p}x^p + \frac{1}{q}y^q$ subject to the constraints $xy = 1$ and $x > 0$ with $p, q \geq 1$ and $\frac{1}{p} + \frac{1}{q} = 1$ is $\frac{1}{p} + \frac{1}{q} = 1$. ☐

(b) Prove that $\frac{1}{p}x^p + \frac{1}{q}y^q \geq xy$ for all $x, y \geq 0$.

Proof. Clearly, it is true if either x or y are zero, so suppose x and y are greater than zero. Note, if (x, y) satisfy the inequality, then all numbers of the form $xr^{\frac{1}{p}}$ and $yr^{\frac{1}{q}}$ are true for any $r \in \mathbb{R}^+$. Thus, we may restrict ourselves such that $xy = 1$, which implies we want to show that for all $x, y \in \mathbb{R}^+$ with $xy = 1$, we get

$$\frac{1}{p}x^p + \frac{1}{q}y^q \geq 1.$$

So, we must see if the minimum of $\frac{1}{p}x^p + \frac{1}{q}y^q$ subject to the constraints exists. This is exactly part (a). Thus, $\frac{1}{p}x^p + \frac{1}{q}y^q \geq xy$ for all $x, y \geq 0$. ☐

(c) Prove Hölder's inequality: if $u_i, v_i \geq 0$ for $i = 1, \ldots, n$, then

$$\sum_{i=1}^{n} u_i v_i \leq \left(\sum_{i=1}^{n} u_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} v_i^q\right)^{\frac{1}{q}}.$$

(Hints: for $u = (u_1, \ldots, u_n)$ let $\|u\|_p = (\sum_{i=1}^{n} u_i^p)^{\frac{1}{p}}$. If $\|u\|_p, \|v\|_q \neq 0$, let $x = \frac{u_i}{\|u\|_p}$ and $\frac{v_i}{\|v\|_q}$ in part (b).)

Proof. Let

$$u = \frac{u_i}{(\sum_{i=1}^{n} u_i^p)^{\frac{1}{p}}} \qquad \text{and} \qquad v = \frac{v_i}{(\sum_{i=1}^{n} v_i^q)^{\frac{1}{q}}},$$

such that $u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n \in \mathbb{R}^+$ and each component of (u, v) nonzero. Then, by Young's inequality, which is part (b), we get

$$\sum_{i=1}^{n} |u_i v_i| \leq \sum_{i=1}^{n} \left(\frac{u_i^p}{p} + \frac{v_i^q}{q}\right).$$

Using the fact that $\frac{1}{p}+\frac{1}{q}=1$, we get $\sum_{i=1}^n |u_i v_i| \leq 1$. One can also show that $\sum_{i=1}^n u_i^p = 1$ and $\sum_{i=1}^n v_i^p = 1$. Therefore, we get Hölder's inequality

$$\sum_{i=1}^n u_i v_i \leq \left(\sum_{i=1}^n u_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^n v_i^q\right)^{\frac{1}{q}}.$$

□

Problem 26. Let $f:\mathbb{R}^2 \to \mathbb{R}^2$ be given by $f(x) = (e^{x_1}\cos x_2, e^{x_1}\sin x_2)$.

(a) Find (with proof) the range of f.

Proof. Consider the following sets: $Z_1 = \mathbb{R}_* \times \mathbb{R}$ and $Z_2 = \mathbb{R}\times\mathbb{R}_*$. For $a \in Z_1$ or $a \in Z_2$, let $g_1 : Z_1 \to \mathbb{R}^2_*$ be defined by $g_1(a) = f(b_1, b_2)$ and let $g_2 : Z_2 \to \mathbb{R}^2_*$ be defined by $g_2(a) = f(b_1, b_2)$, respectively such that $b_1 = \ln\left(\sqrt{a_1^2+a_2^2}\right)$ and

$$b_2 = \begin{cases} \tan^{-1}\left(\frac{a_2}{a_1}\right), & \text{if } a_1 \neq 0 \\ \cot^{-1}\left(\frac{a_1}{a_2}\right), & \text{if } a_2 \neq 0. \end{cases}$$

Without loss of generality, since it can be shown similarly in both cases, let $a \in Z_1$. Then,

$$\begin{aligned} g_1(a) &= f(b_1,b_2) \\ &= \left(e^{\ln\left(\sqrt{a_1^2+a_2^2}\right)}\cos\left(\tan^{-1}\left(\frac{a_2}{a_1}\right)\right), e^{\ln\left(\sqrt{a_1^2+a_2^2}\right)}\sin\left(\tan^{-1}\left(\frac{a_2}{a_1}\right)\right)\right) \\ &= \left(\sqrt{a_1^2+a_2^2}\left(\frac{a_1}{\sqrt{a_1^2+a_2^2}}\right), \sqrt{a_1^2+a_2^2}\left(\frac{a_2}{\sqrt{a_1^2+a_2^2}}\right)\right) \\ &= (a_1, a_2). \end{aligned}$$

Thus, f is surjective if we consider the codomain to be $\mathbb{R}^2_*$. Also, clearly, we cannot have $0 \in \operatorname{ran} f$ since $e^{x_1} > 0$ for all $x_1 \in \mathbb{R}$ and if $\sin x_2 = 0$, then $\cos x_2 \neq 0$. Therefore, exhausting all possibilities, we have $\operatorname{ran} f = \mathbb{R}^2_*$. □

(b) Prove that $f'(x)$ is non-singular for every $x \in \mathbb{R}^2$, but that f is not one-to-one.

Proof. Let $f:\mathbb{R}^2\to\mathbb{R}^2$ be defined by $f(x) = (e^{x_1}\cos x_2, e^{x_1}\sin x_2)$. Then,

$$f'(x) = \begin{pmatrix} e^{x_1}\cos x_2 & -e^{x_1}\sin x_2 \\ e^{x_1}\sin x_2 & e^{x_1}\cos x_2 \end{pmatrix}.$$

Now,

$$\left|\begin{pmatrix} e^{x_1}\cos x_2 & -e^{x_1}\sin x_2 \\ e^{x_1}\sin x_2 & e^{x_1}\cos x_2 \end{pmatrix}\right| = e^{2x_1}\cos^2 x_2 + e^{2x_1}\sin^2 x_2$$
$$= e^{2x_1}.$$

Since $e^{2x_1} > 0$ for all $x_1 \in \mathbb{R}$, then $f'(x)$ is always non-singular. Also, since neither cos nor sin are injective on $\mathbb{R}$, then f is not injective. □

(c) Let $U = \{x \in \mathbb{R}^2 : |x_2| < \pi\}$. Prove that $f\big|_U$ is one-to-one, and find $f(U)$. Also prove that for any open set V properly containing U, $f\big|_V$ is not one-to-one.

Proof. Let $U = \{x \in \mathbb{R}^2 : |x_2| < \pi\} = \{x \in \mathbb{R}^2 : x_2 \in (-\pi, \pi)\}$. Let $x, y \in U$ such that $f(x) = f(y)$. Then,

$$e^{x_1}\cos x_2 = e^{y_1}\cos y_2 \qquad e^{x_1}\sin x_2 = e^{y_1}\sin y_2$$
$$e^{2x_1}\cos^2 x_2 = e^{2y_1}\cos^2 y_2 \qquad e^{2x_1}\sin^2 x_2 = e^{2y_1}\sin^2 y_2. \quad \text{(By squaring both sides)}$$

By adding the two equations together, we get

$$e^{2x_1}\cos^2 x_2 + e^{2x_1}\sin^2 x_2 = e^{2y_1}\cos^2 y_2 + e^{2y_1}\sin^2 y_2,$$

which simplifies to $e^{2x_1} = e^{2y_1}$. These are injective functions, which implies $x_1 = y_1$. By substitution, we get $e^{x_1}\cos x_2 = e^{x_1}\cos y_2$ and $e^{x_1}\sin x_2 = e^{x_1}\sin y_2$. By division, we have $\cos x_2 = \cos y_2$ and $\sin x_2 = \sin y_2$. This is broken down in the following cases:

<u>Case 1:</u> $x_2 \in (-\pi, 0)$:

<u>Case 1a:</u> Let $y_2 \in (-\pi, 0)$. Note, cos is injective on this interval, so $\cos x_2 = \cos y_2$ implies $x_2 = y_2$.

<u>Case 1b:</u> Let $y_2 \in [0, \pi)$. Since $\sin x_2 < 0$ and $\sin y_2 \geq 0$, then $\sin x_2 \neq \sin y_2$, which is impossible.

<u>Case 2:</u> $x_2 \in [0, \pi)$:

<u>Case 1a:</u> Let $y_2 \in (-\pi, 0)$. Since $\sin x_2 \geq 0$ and $\sin y_2 < 0$, then $\sin x_2 \neq \sin y_2$, which is impossible.

<u>Case 1b:</u> Let $y_2 \in [0, \pi)$. Note, cos is injective on this interval, which implies $\cos x_2 = \cos y_2$, which implies $x_2 = y_2$.

So, in all possible cases, we have $x_2 = y_2$. Therefore, $x = y$. Thus, $f\big|_U$ is injective.

Now, let $a \in \mathbb{R}^2_*$. Also, let $b_1 = \ln\left(\sqrt{a_1^2 + a_2^2}\right)$ and

$$b_2 = \begin{cases} \tan^{-1}\left(\frac{a_2}{a_1}\right), & \text{if } a_1 \neq 0 \\ \cot^{-1}\left(\frac{a_1}{a_2}\right), & \text{if } a_2 \neq 0. \end{cases}$$

Note, $b \in U$ since the range of $\tan^{-1}$ and $\cot^{-1}$ is $(\frac{-\pi}{2}, \frac{\pi}{2})$. From (a), we know that $f(b) = a$. So, $\operatorname{ran}\left(f\big|_U\right) \supseteq \mathbb{R}^2_*$.However, the cardinality of the range cannot be larger when the domain is restricted. Thus,

$$\operatorname{ran}\left(f\big|_U\right) = \mathbb{R}^2_*.$$

Now, suppose $V \subseteq \mathbb{R}^2$ open such that $U \subset V$. Then, there exists $q \in V \setminus U$. Let

$$E = \begin{cases} \{q_2 + 2\pi k : k \in \mathbb{Z}, q_2 \geq q_2 + 2\pi k > -\pi\}, & \text{if } q_2 \geq \pi \\ \{q_2 + 2\pi k : k \in \mathbb{Z}, \pi > q_2 + 2\pi k \geq q_2\}, & \text{if } q_2 \leq -\pi. \end{cases}$$

Suppose, without loss of generality, that $q_2 \geq \pi$. Now, E is finite and $E \neq \emptyset$, since $q_2 \in E$. Let $m = \min E$. Then, if $m \notin U$, then $m = \pi$, else $m - 2\pi \in E$, which is a contradiction. If $m \in U$, let $u_2 = m$ with $u \in U$. Else, there exists $r > 0$ such that $B_r(q_1, q_2) \subseteq V$ since V is open.

Let $p = \min\{q_2 + \frac{r}{2}, q_2 + \frac{3\pi}{2}\}$. So, $(q_1, p) \in V$. Also, there exists $k \in \mathbb{Z}$ such that $q_2 + 2\pi k = m$ since $m \in E$, which implies

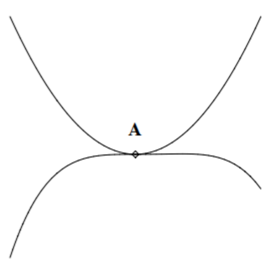
$$\pi = m = q_2 + 2\pi k < p + 2\pi k \leq q_2 + \frac{3\pi}{2} + 2\pi k = m + \frac{3\pi}{2} = \frac{5\pi}{2}.$$

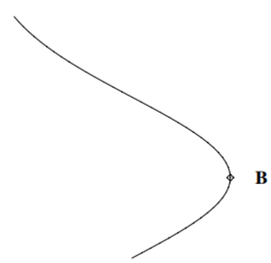
By subtracting 2π, we get $-\pi < p + 2\pi(k-1) \leq \frac{3\pi}{2}$, which implies $(q_1, p + 2\pi(k-1)) \in U$. If this is the case, let $u_2 = p + 2\pi(k-1)$. Either way, we have $(q_1, u_2) \in V$ and $(q_1, u_2 + 2\pi\alpha) \in U \subset V$ for some $\alpha \in \mathbb{Z}$. So,

$$\begin{aligned} f(q_1, u_2) &= (e^{q_1} \cos u_2, e^{q_1} \sin u_2) \\ &= (e^{q_1} \cos(u_2 + 2\pi\alpha), e^{q_1} \sin(u_2 + 2\pi\alpha)) \\ &= f(q_1, u_2 + 2\pi\alpha). \end{aligned}$$

Therefore, $f\big|_V$ is not injective. $\square$

Problem 27. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be continuously differentiable. In each part, the portion of the level set $\{x \in \mathbb{R}^2 : f(x) = 0\}$ is sketched near a point on the level set. What can you say about the derivatives $f'(A)$ and $f'(B)$? Justify your answers precisely.





Proof. Note, the important thing to consider with the two diagrams is whether, for $x = (x_1, x_2) \in \mathbb{R}^2$, x_1 can be represented as a function of x_2 or if x_2 can be represented as a function of x_1. Considering the diagram with point A, there clearly does not exist a function that can approximate one variable with the other. Thus, $D_1 f(A) = D_2 f(A) = 0$. As for the diagram with point B, x_1 can be represented as a function of x_2, but x_2 can not be represented as a function of x_1. Therefore, $D_2 f(B) = 0$. $\square$

Problem 28. Let $f : U \subseteq \mathbb{R}^3 \to \mathbb{R}^2$ be continuously differentiable, let $a \in U$, and suppose that $\frac{\partial f}{\partial(x_2,x_3)}(a)$ is non-singular (as a 2×2 matrix). Prove that there are open subsets V and W of $\mathbb{R}^3$ with $a \in W$, and a C^1-diffeomorphism $h : V \to W$, such that $f \circ h(x) = (x_2, x_3)$ for all $x \in V$. (Hint: let $F(x) = (x_1, f(x))$ and use the inverse function theorem.)

Proof. Let $f : U \subseteq \mathbb{R}^3 \to \mathbb{R}^2$ be continuously differentiable, let $a \in U$, and suppose that $\frac{\partial f}{\partial(x_2,x_3)}(a) \in M_2$ is non-singular. Define $F : \mathbb{R}^3 \to \mathbb{R}^3$ by $F(x) = (x_1, f(x))$. Since $\frac{\delta f}{\delta(a_2,a_3)}$ is invertible, then by the implicit function theorem, there exists $r, s > 0$ such that $B_r(a_2, a_3) \subseteq U$, $\frac{\delta f}{\delta(x_2,x_3)}$ is invertible for all $(x_2, x_3) \in B_r(a_2, a_3)$, and for each $x_1 \in B_s(a_1)$, there exists a unique $h(x) \in B_r(a)$. $\square$