# MAT 473: Intermediate Real Analysis II

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**Problem 13.** Define the function $r : \mathbb{R}^p \to \mathbb{R}$ by $r(x) = \sqrt{x_1^2 + \cdots + x_p^2}$.

(a) Prove that $\frac{\partial r}{\partial x_i} = \frac{x_i}{r}$.

*Proof.* Define the function $r : \mathbb{R}^p \to \mathbb{R}$ by $r(x) = \sqrt{x_1^2 + \cdots + x_p^2}$. Let $U \subseteq \mathbb{R}^p$ be open and $r$ be differentiable at some $x \in U$ and fix $i \in \{1, \ldots, p\}$. Then,

$$\begin{aligned}
\frac{\partial r}{\partial x_i}(x) &= \frac{\partial}{\partial x_i}\sqrt{x_1^2 + \cdots + x_p^2} \\
&= \frac{1}{2}\sqrt{x_1^2 + \cdots + x_p^2}^{\,-1}(2x_i) && \text{(By chain rule)} \\
&= \frac{x_i}{\sqrt{x_1^2 + \cdots + x_p^2}} \\
&= \frac{x_i}{r(x)}.
\end{aligned}$$

□

(b) Prove that $\sum_{i=1}^{p} \frac{\partial^2 r}{\partial x_i^2} = \frac{p-1}{r}$.

*Proof.* Define the function $r : \mathbb{R}^p \to \mathbb{R}$ by $r(x) = \sqrt{x_1^2 + \cdots + x_p^2}$. Let $U \subseteq \mathbb{R}^p$ be open and $r$ be twice differentiable at some $x \in U$ and fix $i \in \{1, \ldots, p\}$. Then,

$$\begin{aligned}
\frac{\partial^2 r}{\partial x_i^2}(x) &= \frac{\partial}{\partial x_i}\frac{x_i}{r(x)} \\
&= \frac{r(x) - (x_i)\frac{x_i}{r(x)}}{r(x)^2} && \text{(By quotient rule)} \\
&= \frac{r(x)^2 - x_i^2}{r(x)^3}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\sum_{i=1}^{p} \frac{\partial^2 r}{\partial x_i^2}(x) &= \sum_{i=1}^{p} \frac{r(x)^2 - x_i^2}{r(x)^3} \\
&= \left(\frac{x_1^2 + \cdots + x_p^2 - x_1^2}{r(x)^3}\right) + \left(\frac{x_1^2 + \cdots + x_p^2 - x_2^2}{r(x)^3}\right) + \cdots + \left(\frac{x_1^2 + \cdots + x_p^2 - x_p^2}{r(x)^3}\right) \\
&= \frac{(p-1)x_1^2 + \cdots + (p-1)x_p^2}{r(x)^3} \\
&= \frac{(p-1)r(x)^2}{r(x)^3} \\
&= \frac{p-1}{r(x)}.
\end{aligned}$$

$\square$

(c) Prove that $\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} \frac{1}{r^{p-2}} = 0$.

*Proof.* Let $\frac{1}{r^{p-2}} : \mathbb{R}^p \setminus \{0\} \to \mathbb{R}$ be given by $\left(\frac{1}{r^{p-2}}\right)(x) = r(x)^{-p+2}$. Fix $i \in \{1, \dots p\}$ and $x \in \mathbb{R}^p \setminus \{0\}$. Then,

$$\begin{aligned} \left(\frac{\partial}{\partial x_i} \frac{1}{r^{p-2}}\right)(x) &= (-p+2) \cdot r(x)^{-p+1} \cdot \frac{\partial r}{\partial x_i}(x) && \text{(By chain rule)} \\ &= \frac{(-p+2)x_i}{r(x)^p}. \end{aligned}$$

Then,

$$\begin{aligned} \left(\frac{\partial^2}{\partial x_i^2} \frac{1}{r^{p-2}}\right)(x) &= \frac{r(x)^p(-p+2) - (-p+2) \cdot x_i(p \cdot r(x)^{p-1} \cdot \frac{\partial r}{\partial x_i}(x))}{r(x)^{2p}} \\ & && \text{(By quotient rule)} \\ &= (-p+2)\left(\frac{r(x)^p - x_i^2 p r(x)^{p-2}}{r(x)^{2p}}\right) \\ &= \frac{(-p+2)(r(x)^{p-2})}{r(x)^{2p}}(r(x)^2 - x_i^2 p). \end{aligned}$$

Then, we have

$$\begin{aligned} \sum_{i=1}^{p} \left(\frac{\partial^2}{\partial x_i^2} \frac{1}{r^{p-2}}\right)(x) &= \sum_{i=1}^{p} \frac{(-p+2)(r(x)^{p-2})}{r(x)^{2p}}(r(x)^2 - x_i^2 p) \\ &= \frac{(-p+2)(r(x)^{p-2})}{r(x)^{2p}} \sum_{i=1}^{p}(r(x)^2 - x_i^2 p) && \text{(Summation property)} \\ &= \frac{(-p+2)(r(x)^{p-2})}{r(x)^{2p}}(r(x)^2 - x_1^2 p + r(x)^2 - x_2^2 p + \cdots + r(x)^2 - x_p^2 p) \\ &= \frac{(-p+2)(r(x)^{p-2})}{r(x)^{2p}}(r(x)^2 p - p(x_1^2 + \cdots + x_p^2)) \\ &= \frac{(-p+2)(r(x)^{p-2})}{r(x)^{2p}}(r(x)^2 p - p r(x)^2) \\ &= \frac{(-p+2)(r(x)^{p-2})}{r(x)^{2p}} \cdot 0 \\ &= 0 \end{aligned}$$

Therefore, $\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} \frac{1}{r^{p-2}} = 0$. $\square$

**Problem 14.** Let $U \subseteq \mathbb{R}^n$ be an open set, and let $f : U \to \mathbb{R}$ be a differentiable function. Suppose that $f'(x) = 0$ for all $x \in U$.

(a) Prove that for each $a \in U$ there exists $r > 0$ such that $f$ is constant in $B_r(a)$. (Here $B_r(a) = \{x \in \mathbb{R}^n : \|x - a\| < r\}$ is the open ball in $\mathbb{R}^n$ with a center $a$ and radius $r$.)

*Proof.* Let $U \subseteq \mathbb{R}^n$ be an open set, and let $f : U \to \mathbb{R}$ be a differentiable function. Suppose that $f'(x) = 0$ for all $x \in U$. Since $f$ is continuous on an open set, then there exists $r > 0$ arbitrary but fixed such that $B_r(x) \subset U$ for all $x \in U$. Let $y \in B_r(x)$ be arbitrary. Then, $[x, y] \subset U$. Since $f$ is differentiable, then by the mean value theorem, there exists $c \in [x, y]$ such that $\|f(y) - f(x)\| \leq \|f'(c)(y - x)\|$. Since $f'(x) = 0$ for all $x \in U$, then we have $\|f(y) - f(x)\| \leq \|0 \cdot (y - x)\|$. Hence, $\|f(y) - f(x)\| = 0 \implies f(y) - f(x) = 0$. Thus, $f(x) = f(y)$. Therefore, since $y \in B_r(x)$ was arbitrary, $f$ is constant in $B_r(x)$. □

(b) Suppose that $U$ is connected. Prove that $f$ is constant in $U$.

*Proof.* Let $U \subseteq \mathbb{R}^n$ be a connected open set, and let $f : U \to \mathbb{R}$ be a differentiable function. Suppose that $f'(x) = 0$ for all $x \in U$. Suppose $f$ is not constant for contradiction. Let $y \in f(U)$. Let $S = f^{-1}(\{y\})$ and $T = f^{-1}(f(U) \setminus \{y\})$. We will show the four following properties of $S$ and $T$.

(i) $S \cup T = U$:
Note, $\{y\} \cup f(U) \setminus \{y\} = f(U)$ and $f^{-1}(f(U)) = U$. Thus, $f^{-1}(\{y\} \cup f(U) \setminus \{y\}) = U$. Since $\{y\}$ and $f(U) \setminus \{y\}$ are disjoint, then $f^{-1}(\{y\}) \cup f^{-1}(f(U) \setminus \{y\}) = U$, which implies $S \cup T = U$.

(ii) $S \cap T = \emptyset$:
Suppose $S \cap T \neq \emptyset$. Then, there exists $x \in U$ such that $f(x) = y$ and $f(x) = z$ for some $z \in f(U) \setminus \{y\}$. Then, $f$ is not a function. Therefore, $S \cap T = \emptyset$.

(iii) $S \neq \emptyset$ and $T \neq \emptyset$:
Note, $y \in f(U)$. That implies there exists some $z_1 \in U$ such that $z_1 \mapsto f(U) \setminus \{y\}$. Thus, $S \neq \emptyset$. Now, there must exist some $z_2 \in U$ such that $z_2 \mapsto f(U) \setminus \{y\}$. Therefore, $T \neq \emptyset$.

(iv) $S$ and $T$ open:
Let $x \in S$ be arbitrary. That is, $f(x) = y$. Since $S \subset U$, then from part (a), there exists some $r > 0$ such that $f$ is constant in $B_r(x)$. So, $f(B_r(x)) = \{y\}$, which implies $B_r(x) \subset f^{-1}(\{y\})$. Therefore, for all $x \in S$, $B_r(x) \subset S$, which implies $S$ is open. By a similar argument, $T$ is open.

Therefore, there exist $S$ and $T$ nonempty open sets that are disjoint and whose union is $U$. Thus, $U$ is not disconnected, contradiction. Therefore, $f$ is constant. □

**Problem 15.** Recall that $GL := \{T \in M_n : T \text{ is invertible }\}$ is an open subset of $M_n$. Let $\text{inv} : GL_n \to GL_n \subseteq M_n$ be the inversion map: $\text{inv}(T) = T^{-1}$. Prove that inv is continuous on $GL_n$. (Hint: let $A \in GL_n$ and use the following outline to show that inv is continuous at $A$. Note that $T^{-1} - A^{-1} = T^{-1}(A - T)A^{-1}$. Apply the operator norm to both sides, then use the reverse triangle inequality to the left, and the operator norm inequality on the right. From the result you should be able to show that $\|T^{-1}\|$ is bounded in some ball centered at $A$. Then the righthand portion of the inequality work from before can be used to prove the continuity of inv at $A$.)

*Proof.* Let $\text{inv} : GL_n \to GL_n \subseteq M_n$ be the inversion map: $\text{inv}(T) = T^{-1}$. Let $A \in GL_n$. Define $\det : M_n \to \mathbb{R}$ to be

$$\det(a_{ij}) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) a_{1,\sigma(1)} \dots a_{n,\sigma(n)}$$

from Linear Algebra and Group Theory. Then, clearly det is continuous since it is a polynomial, which is always continuous. Now, for $A \in GL_n$, define $b_{ij}(A) = \det(A_{mk})_{m \neq j, k \neq i}$. Note, $b_{ij}$ is also continuous. Then, by Cramer's rule,

$$(A^{-1})_{ij} = \frac{(-1)^{i+j}}{\det(A)} b_{ij}(A).$$

Therefore, since $\det(A)$ is a continuous polynomial and $b_{ij}(A)$ is a continuous polynomial, then $(A^{-1})_{ij}$ is continuous. Therefore, since each index of $A^{-1}$ is continuous, then $A^{-1}$ is continuous, which implies inv is continuous since $A \in GL_n$ was arbitrary. □

**Problem 16.** Continuing from the previous problem, prove that inv is differentiable on $GL_n$, and that $\text{inv}'(A)(H) = -A^{-1}HA^{-1}$. (Hint: investigate the difference $(A + H)^{-1}$ as a geometric series (for $\|H\|$ small enough).)

*Proof.* Let $\text{inv} : GL_n \to GL_n \subseteq M_n$ be the inversion map: $\text{inv}(T) = T^{-1}$. Let $A$ in $GL_n$ be arbitrary but fixed. Note, for $r = \frac{1}{\|A^{-1}\|}$ and $H \in B_r(o)$

$$\begin{aligned}
\|-A^{-1}H\| &\leq \|A^{-1}\| \cdot \|H\| && \text{(By triangle inequality)} \\
&< \frac{\|A^{-1}\|}{\|A^{-1}\|} && \text{(Since } H \text{ in } B_r(0)) \\
&= 1
\end{aligned}$$

Then, let $H \in M_n$ arbitrary.

$$\begin{aligned}
\lim_{H\to 0} \frac{\|(A+H)^{-1} - A^{-1} - (A^{-1}HA^{-1}\|}{\|H\|} &= \lim_{H\to 0} \frac{\|(I + A^{-1}H)^{-1}A^{-1} - A^{-1} + A^{-1}HA^{-1}\|}{\|H\|} \\
&\qquad \text{(Multiplying the identity matrix in } (A+H)^{-1}\text{)} \\
&= \lim_{H\to 0} \frac{\|(I - (-A^{-1}H))^{-1}A^{-1} - A^{-1} + A^{-1}HA^{-1}\|}{\|H\|} \\
&= \lim_{H\to 0} \frac{\|\sum_{j=0}^{\infty}((-A^{-1}H)^j \cdot A^{-1}) - A^{-1} + A^{-1}HA^{-1}\|}{\|H\|} \\
&\qquad \text{(By geometric series)} \\
&= \lim_{H\to 0} \frac{\|\sum_{j=2}^{\infty}(((-A^{-1}H)^j)A^{-1}\|}{\|H\|} \qquad \text{(Simplification)} \\
&\le \lim_{H\to 0} \frac{\sum_{j=2}^{\infty}\|((-A^{-1}H)^j)\| \cdot \|A^{-1}\|}{\|H\|} \\
&\qquad \text{(By triangle inequality)} \\
&\le \lim_{H\to 0} \frac{\sum_{j=2}^{\infty}\|(-A^{-1}H)\|^j \cdot \|A^{-1}\|}{\|H\|} \\
&= \lim_{H\to 0} \frac{\left(\frac{1}{1-\|-A^{-1}H\|} - 1 - \|-A^{-1}H\|\right) \cdot \|A^{-1}\|}{\|H\|} \\
&\qquad \text{(From above, when } H \text{ is sufficiently small)} \\
&= \lim_{H\to 0} \frac{\left(\frac{1-1+\|-A^{-1}H\|-\|-A^{-1}H\|+\|-A^{-1}H\|^2}{1-\|-A^{-1}H\|}\right) \cdot \|A^{-1}\|}{\|H\|} \\
&\le \lim_{H\to 0} \frac{\left(\frac{\|-A^{-1}H\|^2}{1-\|-A^{-1}H\|}\right) \cdot \|A^{-1}\|}{\|H\|} \qquad \text{(Simplification)} \\
&= \lim_{H\to 0} \frac{\|-A^{-1}H\|^2 \cdot \|A^{-1}\|}{\|H\|(1 - \|A^{-1}H\|)} \\
&= \lim_{H\to 0} \frac{\|-A^{-1}\|^3 \cdot \|H\|}{1 - \|A^{-1}H\|} \qquad \text{(Simplification)} \\
&= \frac{0}{1} \\
&= 0.
\end{aligned}$$

Therefore, $0 \le \lim_{H\to 0} \frac{\|(A+H)^{-1}-A^{-1}-(A^{-1}HA^{-1}\|}{\|H\|} \le 0$, which implies $\lim_{H\to 0} \frac{\|(A+H)^{-1}-A^{-1}-(A^{-1}HA^{-1}\|}{\|H\|} = 0$ by squeeze theorem. Thus, inv is differentiable at $A$. Since $A \in GL_n$ was arbitrary, then inv is differentiable on $GL_n$ with $\mathrm{inv}'(A)(H) = -A^{-1}HA^{-1}$.

$\square$