MAT 473: Intermediate Real Analysis II

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Problem 5. Let $f : M_{m\times n} \to M_n$ be given by $f(A) = A^tA$. Prove that f is differentiable, and find a formula for $f'(A)$. (Hint: use facts about the operator norm and the transpose of a matrix.)

Proof. Let $f : M_{m\times n} \to M_n$ be given by $f(A) = A^tA$. Let $A \in M_{m\times n}$. Define $T \in B(M_{m\times n}, M_n)$ by $T(h) = A^th + h^tA$. Also, define $\|\cdot\|_E$ as the Euclidean norm and $\|\cdot\|_O$ as the operator norm. Then

$$\begin{aligned}
\lim_{h\to 0} \frac{\|f(A+h) - f(A) - T(h)\|_O}{\|h\|_O} &= \lim_{h\to 0} \frac{\|(A+h)^t(A+h) - A^tA - A^th - h^tA\|_O}{\|h\|_O} && \text{(Definition of } f \text{ and } T)\\
&= \lim_{h\to 0} \frac{\|(A+h)^tA + (A+h)^th - A^tA - A^th - h^tA\|_O}{\|h\|_O} && \text{(By distribution)}\\
&= \lim_{h\to 0} \frac{\|(A^t+h^t)A + (A^t+h^t)h - A^tA - A^th - h^tA\|_O}{\|h\|_O} && \text{(Property of transpose)}\\
&= \lim_{h\to 0} \frac{\|A^tA + h^tA + A^th + h^th - A^tA - A^th - h^tA\|_O}{\|h\|_O} && \text{(By distribution)}\\
&= \lim_{h\to 0} \frac{\|h^th\|_O}{\|h\|_O} && \text{(By subtraction)}\\
&\le \frac{\|h^t\|_O \cdot \|h\|_O}{\|h\|_O} && \text{(Property of operator norm)}\\
&= \lim_{h\to 0} \frac{\|h\|_O^2}{\|h\|_O} && \text{(Property of operator norm)}\\
&= \lim_{h\to 0} \|h\|_O\\
&= 0.
\end{aligned}$$

Since $0 \le \lim_{h\to 0} \frac{\|f(A+h)-f(A)-T(h)\|_O}{\|h\|_O}$, then by squeeze theorem, $\lim_{h\to 0} \frac{\|f(A+h)-f(A)-T(h)\|_O}{\|h\|_O} = 0$. Also, we know that, $\lim_{h\to 0} \frac{f(A+h)-f(A)-T(h)}{\|h\|_E} = 0 \iff \lim_{h\to 0} \frac{\|f(A+h)-f(A)-T(h)\|_E}{\|h\|_E} = 0$. Thus, since $A \in M_{m\times n} = \mathbb{R}^{mn}$, then any two norms are comparable by Corollary 2.11, which implies there exists $k_1, k_2 \in \mathbb{R}$ such that for all $x \in \mathbb{R}$, $\|x\|_0 \le k_1 \cdot \|x\|_E$ and $\|x\|_E \le k_2 \cdot \|x\|_O$.

Then, we get

$$
\begin{aligned}
\lim_{h\to 0}\frac{\|f(A+h)-f(A)-T(h)\|_E}{\|h\|_E} &\le \lim_{h\to 0}\frac{k_2\|f(A+h)-f(A)-T(h)\|_O}{\frac{1}{k_1}\|h\|_O} \\
&\qquad\qquad \text{(By comparability of } \|\cdot\|_E \text{ and } \|\cdot\|_O\text{)} \\
&= k_1k_2\lim_{h\to 0}\frac{\|f(A+h)-f(A)-T(h)\|_O}{\|h\|_O} \\
&\qquad\qquad \text{(Property of limits)} \\
&= k_1k_2\cdot 0 \\
&= 0.
\end{aligned}
$$

Since, $0 \le \lim_{h\to 0}\frac{\|f(A+h)-f(A)-T(h)\|_E}{\|h\|_E}$, then by squeeze theorem, $\lim_{h\to 0}\frac{\|f(A+h)-f(A)-T(h)\|_E}{\|h\|_E} = 0$. This implies, $\lim_{h\to 0}\frac{f(A+h)-f(A)-T(h)}{\|h\|_E} = 0$. Thus, f is differentiable at A. However, since A was arbitrary, then f is differentiable for all $A \in M_{m\times n}$. Therefore, $f'(A)(h) = A^th + h^tA$. $\square$

Problem 6. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$
f(x) = \begin{cases} \frac{x_1^2x_2}{\|x\|^2}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}
$$

Prove that all directional derivatives of f exist at 0, and that $D_vf(0)$ is not a linear function of v.

Proof. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$
f(x) = \begin{cases} \frac{x_1^2x_2}{\|x\|^2}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}
$$

Let $v \in \mathbb{R}^2 \setminus (0,0)$. Then,

$$\begin{aligned}
D_v f(0) &= \lim_{t\to 0} \frac{f(0+tv) - f(0)}{t} && \text{(Definition of } D_v f(x) \text{ in } \mathbb{R}^2) \\
&= \lim_{t\to 0} \frac{\frac{(tv_1)^2(tv_2)}{\|tv\|^2} - 0}{t} && \text{(Definition of } f \\
&= \lim_{t\to 0} \frac{t^2 v_1^2 t v_2}{t\sqrt{(tv_1)^2 + (tv_2)^2}^2} && \text{(Definition of Euclidean norm)} \\
&= \lim_{t\to 0} \frac{t^2 v_1^2 v_2}{t^2 v_1^2 + t^2 v_2^2} \\
&= \lim_{t\to 0} \frac{t^2 v_1^2 v_2}{t^2 (v_1^2 + v_2^2)} && \text{(by factoring } t^2) \\
&= \lim_{t\to 0} \frac{v_1^2 v_2}{v_1^2 + v_2^2}.
\end{aligned}$$

Also, we have

$$\begin{aligned}
D_{(0,0)} f(0) &= \lim_{t\to 0} \frac{f(0 + t\cdot 0) - f(0)}{t} \\
&= \lim_{t\to 0} \frac{0}{t} \\
&= 0.
\end{aligned}$$

So, we have $D_v f(0) : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$D_v f(0) = \begin{cases} \frac{v_1^2 v_2}{v_1^2 + v_2^2}, & \text{if } v \neq 0 \\ 0, & \text{if } v = 0. \end{cases}$$

Consider $p = (1,0)$ and $q = (0,1)$. Then, $D_p f(0) = \frac{1^2 \cdot 0}{1^2 + 0^2} = 0$ and $D_q f(0) = \frac{0^2 \cdot 1}{0^2 + 1^2} = 0$, and $D_{p+q} f(0) = \frac{1^2 \cdot 1}{1^2 + 1^2} = \frac{1}{2}$. Therefore, $D_v f(0)$ is not linear since $D_{p+q} f(0) = \frac{1}{2} \neq 0 = 0 + 0 = D_p f(0) + D_q f(0)$. $\square$

Problem 7. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} \frac{x_1 x_2^3}{x_1^2 + x_2^4}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

Prove that all directional derivatives of f exist at 0, that $D_v f(0)$ is a linear function of v, and that f is not differentiable at 0.

Proof. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} \frac{x_1 x_2^3}{x_1^2 + x_2^4}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

Let $v \in \mathbb{R}^2 \setminus (0,0)$. Then, we have

$$\begin{aligned} D_v f(0) &= \lim_{t \to 0} \frac{f(0 + tv) - f(0)}{t} \\ &= \lim_{t \to 0} \frac{\frac{tv_1 (tv_2)^3 - 0}{(tv_1)^2 + (tv_2)^4}}{t} && \text{(Definition of } f\text{)} \\ &= \lim_{t \to 0} \frac{t^3 v_1 v_2}{t^2 v_1^2 + t^4 v_2^4} \\ &= \lim_{t \to 0} \frac{t v_1 v_2}{v_1^2 + t^2 v_2^4} \\ &= \frac{0}{v_1^2} \\ &= 0. \end{aligned}$$

Also note that

$$D_0 f(0) = \lim_{t \to 0} \frac{f(0 + t \cdot 0) - f(0)}{t} = \lim_{t \to 0} \frac{0}{t} = 0.$$

Thus, $D_v f(0) : \mathbb{R}^2 \to \mathbb{R}$ is defined by $D_v f(0) = 0$, which is linear since it is the zero map. Also, the Jacobian matrix of f evaluated at 0 is $(0,0)$. To see if f is differentiable at 0, we have that

$$\lim_{h \to 0} \frac{f(0 + h) - f(0) - (0,0) \cdot h}{\|h\|} = \lim_{h \to 0} \frac{\frac{h_1 h_2^3}{h_1^2 + h_2^4}}{\sqrt{h_1^2 + h_2^2}}.$$

Consider $Z_1 = \{(t^2, t) : t \in \mathbb{R}^+\}$. Then,

$$\begin{aligned} \lim_{h \to 0} \frac{h_1 h_2^3}{(h_1^2 + h_2^4)\sqrt{h_1^2 + h_2^2}} \Bigg|_{Z_1} &= \lim_{t \to 0^+} \frac{t^2 t^3}{(t^4 + t^4)\sqrt{t^4 + t^2}} \\ &= \lim_{t \to 0^+} \frac{t}{2\sqrt{t^4 + t^2}} \\ &= \lim_{t \to 0} \sqrt{\frac{t^2}{4t^4 t^2}} \\ &= \sqrt{\frac{1}{4t^2 + 4}} \\ &= \frac{1}{2}. \end{aligned}$$

This means that the limit is $\frac{1}{2}$ or does not exist. However, since it is not equal to 0 either way, the derivative does not exist at 0. $\square$

Problem 8. Let $E = \{x \in \mathbb{R}^2 : x_1 > 0 \text{ and } 0 < x_2 < x_1^2\}$. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1, & \text{if } x \in E \\ 0, & \text{if } x \notin E. \end{cases}$$

(f is called the *characteristic function* of the set E.) Prove that all directional derivatives of f exist at 0, and equal 0, but that f is not differentiable at 0.

Proof. Let $E = \{x \in \mathbb{R}^2 : x_1 > 0 \text{ and } 0 < x_2 < x_1^2\}$. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1, & \text{if } x \in E \\ 0, & \text{if } x \notin E. \end{cases}$$

Define $A = \{x \in \mathbb{R}^2 : x_1 > 0 \text{ and } x_2 > 0 \text{ or } x_1 < 0 \text{ and } x_2 < 0\}$. Let $v \in A$ be arbitrary.

Case 1: Suppose $v_1 > 0$ and $v_2 > 0$.

Since $(tv_1) < 0$ for all $t < 0$, which implies $tv \notin E$, we have,

$$\lim_{t \to 0^-} \frac{f(0 + tv) - f(0)}{t} = \lim_{t \to 0^-} \frac{f(tv)}{t} = \lim_{t \to 0^-} \frac{0}{t} = 0,$$

which implies $f(tv) = 0$. Let $\delta \in \mathbb{R}$ such that $0 < \delta < \frac{v_2}{v_1^2}$. By multiplication of $\delta v_1^2 > 0$, $0 < (\delta v_1)^2 < \delta v_2$. This means, $\delta v \notin E$. Thus, for all $0 < t \leq \frac{v_2}{v_1^2}$,

$$\lim_{t \to 0^+} \frac{f(0 + tv) - f(0)}{t} = \lim_{t \to 0^+} \frac{f(tv)}{t} = \lim_{t \to 0^+} \frac{0}{t} = 0,$$

which implies $f(tv) = 0$.

Case 2: Suppose $v_1 < 0$ and $v_2 < 0$.

Since $(tv_1) < 0$ for all $t > 0$, which implies $tv \notin E$, we have,

$$\lim_{t \to 0^+} \frac{f(0 + tv) - f(0)}{t} = \lim_{t \to 0^+} \frac{f(tv)}{t} = \lim_{t \to 0^+} \frac{0}{t} = 0,$$

which implies $f(tv) = 0$. Let $\phi \in \mathbb{R}$ such that $\frac{v_2}{v_1^2} < \phi < 0$. By multiplication of $\phi v_1^2 < 0$, $\phi v_2 > (\phi v_1)^2 > 0$, which implies $\phi v \notin E$. Then, for all $\frac{v_2}{v_1^2} \leq t < 0$, we have

$$\lim_{t \to 0^-} \frac{f(0 + tv) - f(0)}{t} = \lim_{t \to 0^-} \frac{f(tv)}{t} = \lim_{t \to 0^-} \frac{0}{t} = 0,$$

which implies $f(tv) = 0$.

Therefore, since in all cases we have $\lim_{t\to 0^-} \frac{f(0+tv)-f(0)}{t} = \lim_{t\to 0^+} \frac{f(0+tv)-f(0)}{t}$, that implies $D_v f(0) = \lim_{t\to 0} \frac{f(0+tv)-f(0)}{t} = 0$.

Now, let $u \notin A$. Then, we have that $u_1 \geq 0$ and $u_2 \leq 0$ or $u_1 \leq 0$ and $u_2 \geq 0$. This implies that $u \notin E$. Moreover, for all $t \in \mathbb{R}$, we have $tu \notin E$ since we would still have the property mentioned. Thus, $f(tu) = 0$ for all $t \in \mathbb{R}$. This means that we have

$$\begin{aligned} D_u f(0) &= \lim_{t\to 0} \frac{f(0+tu)-f(0)}{t} \\ &= \lim_{t\to 0} \frac{f(tu)}{t} \\ &= \lim_{t\to 0} \frac{0}{t} \\ &= 0. \end{aligned}$$

Thus, for all $w \in \mathbb{R}^2$, $D_w f(0) = 0$. So, the Jacobian matrix of f evaluated at 0 is $(0,0)$. Consider $Z_1 = \{(t,t^3) : 0 < t < 1\}$. Then for all $x \in Z_1$, x is of the form (k,k^3) and $x_1 = k > 0$ and $x_1^2 = k^2 > k^3 = x_2 > 0$. This implies $x \in E$. Hence, for all $x \in Z_1$, $f(x) = 1$. Thus,

$$\lim_{h\to 0} \frac{f(0+h)-f(0)-(0,0)h}{\|h\|}\bigg|_{Z_1} = \lim_{t\to 0^+} \frac{1}{\|(t,t^3)\|} = \infty.$$

Therefore, the derivative of f at 0 does not exist. □