MAT 473: Intermediate Real Analysis II

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February 20, 2020

Problem 17. Prove that the following function $f : \mathbb{R}^2 \to \mathbb{R}$ is (once) continuously differentiable on $\mathbb{R}^2$, that all second-order partial derivatives of f exist at the origin, but that $D_1D_2f(0) \neq D_2D_1f(0)$:

$$f(x) = \begin{cases} \frac{x_1^3 x_2}{x_1^2+x_2^2}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

Proof. Let $f : U \subseteq \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \frac{x_1^3 x_2}{x_1^2+x_2^2}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

Then,

$$\begin{aligned} D_1 f(0,0) &= \lim_{h\to 0} \frac{1}{h} \frac{h^3 \cdot 0}{h^2 + 0^2} \\ &\overset{\text{L'H}}{=} \lim_{h\to 0} \frac{0}{3h^2} \\ &\overset{\text{L'H}}{=} \lim_{h\to 0} \frac{0}{6h} \\ &\overset{\text{L'H}}{=} \lim_{h\to 0} \frac{0}{6} \\ &= 0. \end{aligned}$$

Also,

$$\begin{aligned} D_2 f(0,0) &= \lim_{h\to 0} \frac{1}{h} \frac{0^3 \cdot h}{0^2 + h^2} \\ &\overset{\text{L'H}}{=} \lim_{h\to 0} \frac{0}{3h^2} \\ &\overset{\text{L'H}}{=} \lim_{h\to 0} \frac{0}{6h} \\ &\overset{\text{L'H}}{=} \lim_{h\to 0} \frac{0}{6} \\ &= 0. \end{aligned}$$

Thus, since all first-order partial derivatives of f exist and are continuous, then $f(x)$ is at least C^1. Note,

$$D_1 f(x) = \begin{cases} \frac{(x_1^2+x_2^2)(3x_1^2x_2)-(x_1^3x_2)(2x_1)}{(x_1^2+x_2^2)^2}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

and

$$D_2 f(x) = \begin{cases} \frac{(x_1^2+x_2^2)(x_1^3)-(x_1^3x_2)(2x_2)}{(x_1^2+x_2^2)^2}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

Now,

$$
\begin{aligned}
D_2D_1f(0,0) &= \lim_{h\to 0}\frac{1}{h}\frac{(h^2+0)(h^3)-(h^3\cdot 0)(0)}{(h^2+0)^2} \\
&= \lim_{h\to 0}\frac{h^5}{h^5} \\
&= 1, \\
D_2D_2f(0,0) &= \lim_{h\to 0}\frac{1}{h}\frac{(0+h^2)(0^3)-(0\cdot h)(2h)}{(0+h^2)^2} \\
&\lim_{h\to 0}\frac{0}{h^5} \\
&\overset{\text{L'H}}{=} \lim_{h\to 0}\frac{0}{120} \\
&= 0, \\
D_1D_2f(0,0) &= \lim_{h\to 0}\frac{1}{h}\frac{(0+h^2)(0\cdot h)-(0\cdot h)(0)}{(0+h^2)^2} \\
&= \lim_{h\to 0}\frac{0}{h^5} \\
&\overset{\text{L'H}}{=} \frac{0}{120} \\
&= 0, \\
D_1D_1f(0,0) &= \lim_{h\to 0}\frac{1}{h}\frac{(h^2+0)(3h^2\cdot 0)-(h^3\cdot 0)(2h)}{(h^2+0)^2} \\
&= \lim_{h\to 0}\frac{0}{h^5} \\
&\overset{\text{L'H}}{=} \frac{0}{120} \\
&= 0.
\end{aligned}
$$

Therefore, all second-order partial derivatives of f exist at the origin. But, $D_1D_2f(0,0) = 1 \neq 0 = D_2D_1f(0,0)$, which implies f is not C^2.

$\square$

Problem 18.

(a) Let (X,d) be a metric space, let $T : X \to M_n$ be a continuous function, and let $x_0 \in X$. Suppose that $T(x_0)$ has a positive eigenvalue and a negative eigenvalue. Prove that there are unit vectors v_+ and $v_- \in \mathbb{R}^n$ such that

$$\langle T(x)v_+, v_+\rangle > 0, \qquad \langle T(x)v_-, v_-\rangle < 0$$

for all x in a neighborhood of x_0.

Proof. Let (X,d) be a metric space, let $T : X \to M_n$ be a continuous function, and let $x_0 \in X$. Suppose that $T(x_0)$ has a positive eigenvalue and a negative eigenvalue. Then there exists $u, v \in \mathbb{R}^n$, $\lambda_+, \lambda_- \in \mathbb{R}$ such that $\lambda_+ > 0$ and $\lambda_- < 0$ such that $T(x_0)u = \lambda_+ u$ and $T(x_0)v = \lambda_- v$ Let $u_= = \frac{u}{\|u\|}$ and $v_- = \frac{v_2}{\|v\|}$, which are unit vectors. Then, one can show that

$$\begin{aligned}
\langle T(x_0)u_+, u_+\rangle &= \langle \frac{T(x_0)u}{\|u\|}, \frac{u}{\|u\|}\rangle \\
&= \frac{\langle \lambda_+ u, u\rangle}{\|u\|^2} \\
&= \frac{\lambda_+}{\|u\|^2} \cdot \langle u, u\rangle \\
&> 0.
\end{aligned}$$

Similarly,

$$\langle T(x_0)v, v\rangle < 0.$$

Now, there exists $r_+, r_- \in \mathbb{R}$ such that for all $x_+ \in B_{r_+}(x_0)$ and for all $x_- \in B_{r_-}(x_0)$, then

$$\|T(x_+) - T(x_0)\| < \frac{\lambda_+}{\|u\|^2} \cdot \sum_{j=1}^{n} u_j^2, \quad \text{and} \quad \|T(x_-) - T(x_0)\| \left| < \frac{\lambda_v}{\|v\|^2} \cdot \sum_{j=1}^{n} v_j^2 \right|.$$

So, for all $x_+ \in B_{r_+}(x_0)$,

$$\begin{aligned}
\langle T(x_+)u_+, u_+\rangle &= \langle T(x_+)u_+, u_+\rangle + \langle (T(x_+) - T(x_0))u_+, u_+\rangle \\
&\geq \langle T(x_+)u_+, u_+\rangle - \left|\langle (T(x_+) - T(x_0))u_+, u_+\rangle\right|.
\end{aligned}$$

So, we get

$$\begin{aligned}
\left|\langle (T(x_+) - T(x_0))u_+, u_+\rangle\right| &\leq \|T(x_+) - T(x_0)\| \cdot \|u_+\| \cdot \|u_+\| \quad \text{(By triangle inequality)} \\
&= \|T(x_+) - T(x_0)\| \\
&< \frac{\lambda_+}{\|u\|^2} \cdot \sum_{j=1}^{n} u_u^2.
\end{aligned}$$

Thus, $\langle (T(x_+) - T(x_0))u_+, u_+\rangle > 0$ for all $x_+ \in B_{r_+}(x_0)$. Also, $\langle T(x_0)u, u\rangle = \frac{\lambda_+}{\|u\|^2} \cdot sum_{j=1}^{n} u_j^2$. Thus, $\langle (T(x_+) - T(x_0))u_+, u_+\rangle > 0$ for all $x_+ \in B_{r_+}(x_0)$. In a similar argument, one can show that $\langle T(x_-)v, v\rangle < 0$ for all $x_- \in B_{r_-}(x_0)$. Now, let $r = \min\{r_+, r_-\}$. Therefore, $\langle (T(x)u_+, u_+\rangle > 0$ and $\langle T(x_-)v, v\rangle < 0$ for all $x \in B_r(x_o)$ $\square$

(b) Let $U \subseteq \mathbb{R}^n$ be open, let $a \in U$, let $f : U \to \mathbb{R}$ be a C^2 function, and suppose that $f'(a) = 0$. Suppose further that $f''(a)$ is neither positive nor negative semidefinite. Prove that f does not have a local extremum at a.

Proof. Let $U \subseteq \mathbb{R}^n$ be open, let $a \in U$, let $f : U \to \mathbb{R}$ be a C^2 function, and suppose that $f'(a) = 0$. Suppose further that $f''(a)$ is neither positive nor negative semidefinite. So, $f''(a)$ has a positive and negative eigenvalue. Hence, by the previous part, there exists $u_+, v_- \in \mathbb{R}^n$ and $r > 0$ such that for all $x \in B_r(a)$, then $\langle f''(a), u_+, u_+\rangle > 0$ and $\langle f''(a)v_-, v_-\rangle$. Let $r_1 > 0$ be arbitrary but fixed. Then let $s = \min\{r, r_1\}$. So, $(a + \frac{v_-}{2s}), (a + \frac{v_-}{2s}) \in B_r(a)$ and $(a + \frac{v_-}{2s}), (a + \frac{v_-}{2s}) \in B_{r_1}(a)$. Then,

$$\begin{aligned} f(a + \frac{u_+}{2s} &= f(a) + f'(a)\frac{u_+}{2s} + \frac{1}{2}f''(a + \theta_+\frac{u_+}{2s})(\frac{u_+}{2s}, \frac{u_+}{2s}) && (0 < \theta_+ < 1) \\ &= f(a) + \frac{1}{2}f''(a + \theta_+\frac{u_+}{2s})(\frac{u_+}{2s}, \frac{u_+}{2s}) && \text{(Since we know } f'(a) = 0) \\ &= f(a) + \frac{1}{2}\langle f''(a + \theta_+\frac{u_+}{2s})\frac{u_+}{2s}, \frac{u_+}{2s}\rangle && (1) \\ &= f(a) + \frac{1}{8s^2}\langle f''(a + \theta_+\frac{u_+}{2s})u_+, u_+\rangle. && (2) \end{aligned}$$

Thus, $\langle f''(a + \theta_+\frac{u_+}{2s})u_+, u_+\rangle > 0$ since $(a + \theta_+\frac{u_+}{2s}) \in B_r(a)$. In a similar argument, one can show that $\langle f''(a+\theta_-\frac{v_-}{2s})v_-, v_-\rangle < 0$. So, $f(a+\frac{u_+}{2s}) = f(a)+\frac{1}{8s^2}\langle f''(a+\theta_+\frac{u_+}{2s})u_+, u_+\rangle$ and $f(a + \frac{v_-}{2s}) = f(a) + \frac{1}{8s^2}\langle f''(a + \theta_+\frac{v_-}{2s})v_-, v_-\rangle$, which implies $f(a + \frac{u_+}{2s}) > f(a)$ and $f(a + \frac{v_-}{2s}) < f(a)$. Therefore, there exists $x, y \in B_{r_1}(a)$ such that $f(x) > a$ and $f(y) < f(a)$. Therefore, f has no local extrema at a. □

Problem 19. Let $f(x, y) = \frac{1}{1-x-2y}$ for (x,y) in a neighborhood of 0 in $\mathbb{R}^2$.

(a) Find $D_if(0, 0)$ and $D_{ij}f(0, 0)$ for $i, j = 1, 2$. Calculate $P_2(x, y)$ using the formula for the second order Taylor polynomial.

$$D_1f(0, 0) = D_2f(0, 0) = \frac{1}{9y^2 - 6y + 1}, \qquad D_1D_2f(0, 0) = \frac{1}{27y^2 - 27y + 9y - 1}$$

So, $P_2(x, y) = -\frac{1}{3y-1} + \frac{x}{9y^2-6y+1} + \frac{1}{9y^2-6y+1} + \frac{x^2}{27y^2-27y+9y-1} + \frac{y^2}{27y^2-27y+9y-1} + \frac{2xy}{27y^2-27y+9y-1}$

(b) Use the formula for a geometric series to calculate $P_2(x, y)$.

Problem 20. Let $0 < r < R$ and define $f : \mathbb{R}^2 \to \mathbb{R}^3$ by

$$f(\theta, \alpha) = ((R + r\cos\alpha)\cos\theta, (R + r\cos\alpha)\sin\theta, r\sin\alpha)\,.$$

(The range, T, of f is a *torus*.)

(a) Find all points of the form $f(\theta, \alpha) \in T$ such that $Df_1(\theta, \alpha) = 0$. (Hint: your answer will be a finite subset of $\mathbb{R}^3$.)

Proof. Let $0 < r < R$ and define $f : \mathbb{R}^2 \to \mathbb{R}^3$ by

$$f(\theta, \alpha) = ((R + r\cos\alpha)\cos\theta, (R + r\cos\alpha)\sin\theta, r\sin\alpha).$$

Then,

$$\begin{aligned} D_\theta f_1 &= -r\cos\alpha\sin\theta = 0 \Longrightarrow \alpha = 0, \pi, \quad \theta = \frac{\pi}{2}, \frac{3\pi}{2} \\ D_\alpha f_1 &= -r\cos\theta\sin\alpha = 0 \Longrightarrow \theta = 0, \pi, \quad \alpha = \frac{\pi}{2}, \frac{3\pi}{2}. \end{aligned}$$

So, in $\mathbb{R}^3$, we have the set of critical values $\{(0, R+r, 0), (0, R-r, 0), (0, -R-r, 0), (0, r-R, 0), (R+r, 0, r), (R-r, 0, -r), (-R-r, 0, r), (r-R, 0, -r)\}$. □

(b) Show that one of the points in part (a) corresponds to a local maximum of f_1, one corresponds to a local minimum of f_1 and the others are neither local maxima nor local minima of f_1.