MAT 473: Intermediate Real Analysis II

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Problem 29. Let $f : [a, b] \to \mathbb{R}$ and let $c \in [a, b]$. Recall that the *oscillation of f at c* is the quantity

$$\operatorname{osc}(f, c) = \lim_{r \to 0^+} \left(\sup_{x,y \in B_r(c) \cap [a,b]} |f(x) - f(y)| \right).$$

Prove that f is continuous at c if and only if $\operatorname{osc}(f, c) = 0$.

Proof. Let $f : [a, b] \to \mathbb{R}$ and $c \in (a, b)$.

($\Longrightarrow$) : Let f be continuous at c. Let $\epsilon > 0$ be arbitrary but fixed. Then, there exists $r > 0$ such that for $x, y \in B_r(c)$, we have $|f(x) - f(c)| < \frac{\epsilon}{2}$ and $|f(y) - f(c)| < \frac{\epsilon}{2}$, which from the triangle inequality implies

$$|f(x) - f(y)| \leq |f(x) - f(c)| + |f(y) - f(c)| < \epsilon.$$

Thus,

$$\lim_{r \to 0^+} \left(\sup_{x,y \in B_h(c) \cap [a,b]} |f(x) - f(y)| \right) \leq \epsilon \quad \text{if} \quad h < r.$$

This implies $\operatorname{osc}(f, c) = 0$.

($\Longleftarrow$) : Let $\operatorname{osc}(f, c) = 0$ and $\epsilon > 0$ be arbitrary but fixed. Then,

$$\lim_{r \to 0^+} \left(\sup_{x,y \in B_r(c) \cap [a,b]} |f(x) - f(y)| \right) < \epsilon$$

for some $r > 0$. So, $|f(x) - f(y)| < \epsilon$ if $x, y \in B_r(c)$. If $c = y$, then $|f(x) - f(c)| < \epsilon$. Therefore, f is continuous at c.

This argument is similar if $c = a$ or $c = d$. □

Problem 30. Let f be as in the previous problem, and let $L > 0$. Prove that the set $\{z \in [a, b] : \operatorname{osc}(f, z) \geq L\}$ is a closed set.

Proof. Let $f : [a, b] \to \mathbb{R}$ and $L > 0$. Define $A = \{z \in [a, b] : \operatorname{osc}(f, z) \geq L\}$. Notice that $A^c = (-\infty, a) \cup \{z \in [a, b] : \operatorname{osc}(f, z) < L\} \cup (b, \infty)$. Then we have the following cases.

Case 1: Let $c \in (-\infty, a) \cup (b, \infty)$. Then, since $(-\infty, a) \cup (b, \infty)$ is open, then there exists $r_0 > 0$ such that $B_{r_0}(c) \subseteq A^c$.

Case 2: Let $c \in \{z \in [a, b] : \operatorname{osc}(f, z) < L\}$. Then, there must exist $\delta > 0$ such that for all $r \in (0, \delta)$, then

$$\sup_{x,y \in B_r(c) \cap [a,b]} |f(x) - f(y)| < L.$$

Let $d \in [a,b]$. Fix $r \in (0,\delta)$. Let $d \in B_{\frac{r}{2}}(c)$. Clearly, if $d \notin [a,b]$, then we have $d \in (-\infty,a) \cup (b,\infty) \subseteq A^c$. Suppose $d \in [a,b]$. Let $x \in B_{\frac{r}{2}}(d) \cap [a,b]$. This gives us

$$\begin{aligned}
|x-c| &\leq |x-d| + |c-d| && \text{(By triangle inequality)} \\
&< \frac{r}{2} + \frac{r}{2} \\
&= r.
\end{aligned}$$

So, $x \in B_r(c)$, which implies $B_{\frac{r}{2}}(d) \subseteq B_r(c)$. Thus,

$$\sup_{x,y \in B_{\frac{r}{2}}(d) \cap [a,d]} |f(x) - f(y)| \leq \sup_{x,y \in B_r(c) \cap [a,b]} |f(x) - f(y)|. \tag{1}$$

We now have,

$$\begin{aligned}
\operatorname{osc}(f,d) &= \lim_{r \to \infty} \left(\sup_{x,y \in B_{\frac{r}{2}}(d) \cap [a,d]} |f(x) - f(y)| \right) && \text{(Definition of oscillation of } f \text{ at } d) \\
&\leq \lim_{r \to \infty} \left(\sup_{x,y \in B_r(c) \cap [a,b]} |f(x) - f(y)| \right) && \text{(By (1))} \\
&= \operatorname{osc}(f,c) && \text{(Definition of oscillation of } f \text{ at } c) \\
&< L.
\end{aligned}$$

Thus, $d \in A^c$, which implies $B_{\frac{r}{2}}(c) \subseteq A^c$.

Therefore, in both cases, we have that A^c is open, which implies that A is closed. $\square$

Problem 31. Let $[c,d]$ be a closed bounded interval, and let $(a_1,b_1),\ldots,(a_n,b_n)$ be open intervals such that $[c,d] \subseteq \cup_{i=1}^n (a_i,b_i)$. Prove that $d-c < \sum_{i=1}^n (b_i - a_i)$. (Hints: choose i_1 so that $c \in (a_{i_1}, b_{i_1})$. If $b_{i_1} \leq d$ choose i_2 so that $b_{i_1} \in (a_{i_2}, b_{i_2})$. Explain why in the continuation of this process there must be $k \leq n$ such that $d \in (a_{i_k}, b_{i_k})$.)

Proof. Let $C = [c,d]$ be a closed bounded interval and $A_1 = (a_1,b_1),\ldots,A_n = (a_n,b_n)$ be open intervals such that $[c,d] \subseteq \cup_{i=1}^n A_i$. Let $\epsilon > 0$ be arbitrary but fixed. Choose j such that $Q_j = (q_j,p_j) \supseteq A_j$ with $p_j - q_j \leq (1+\epsilon)|b_j - a_j|$. Since $\cup_{j=1}^\infty Q_j$ is an open cover of the compact set $[c,d]$, there exists a finite subcover $[c,d] \subseteq \cup_{i=1}^N Q_j$. By taking the closure of each Q_j, we have $d - c \leq \sum_{j=1}^N (p_j - q_j)$. As a result, we have

$$d - c \leq (1+\epsilon) \sum_{j=1}^N (b_j - a_j).$$

Since $\epsilon > 0$, then we have the strict inequality $d - c < \sum_{j=1}^N (b_j - a_j)$. $\square$

Problem 32. For this exercise you must recall the definition and properties of *Lebesgue outer measure* from the notes. Let $A, B \subseteq \mathbb{R}$, and suppose that $m^*(A) = 0$. Prove that $m^*(A \cup B) = m^*(A) + m^*(B)$.

Proof. Let $A, B \subseteq \mathbb{R}$ and $m^*(A) = 0$. By Theorem 15.3 (2), we have $m^*(A \cup B) \leq m^*(A) + m^*(B)$. Also,

$$\begin{aligned} m^*(A) + m^*(B) &= 0 + m^*(B) && \text{(By monotonicity of the outer measure)} \\ &= m^*(B) \\ &\leq m^*(A \cup B). \end{aligned}$$

So, since $m^*(A) + m^*(B) \leq m^*(A \cup B)$ and $m^*(A \cup B) \leq m^*(A) + m^*(B)$, then we must have $m^*(A \cup B) = m^*(A) + m^*(B)$. $\square$