Propagation Characteristics of Single-Mode Optical Fibers with Arbitrary Complex Index Profiles: A Direct Numerical Approach

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***Abstract*—We present here a rapidly converging numerical procedure for the evaluation of the propagation characteristics of a single-mode optical fiber with an arbitrary complex refractive index profile. To illustrate the procedure, we have first applied it to a fiber with a complex step index profile. As an application, we have used it to evaluate the gain in a typical 980-nm pumped erbium-doped fiber.**

***Index Terms*—EDFA, gain, single-mode fiber.**

I. Introduction

With the advent of fiber lasers and doped fiber amplifiers, attention has been drawn to the analysis of optical fibers whose refractive index profile can be described in terms of a complex function. Some approximate or numerically cumbersome methods have been reported in the literature [1]–[3] for evaluation of the propagation characteristics of such fibers. The approximate method [1] attempts to separate the real and imaginary parts of the complex scalar wave equation to obtain two real scalar wave equations, under the assumption that the imaginary part of the field is much smaller than the real part, and hence, neglects the imaginary part of the field. Thus, this method is accurate only for profiles in which the imaginary part of the refractive index is small. Another method for the analysis of such profiles has been the standard method of perturbation [2], in which the restraining assumption is that the loss or gain exhibited by the fiber does not significantly alter the field and is also valid for profiles with low values of the imaginary part of the refractive index. In order to overcome this restraint, a numerical procedure [3] was developed, in which the Rayleigh–Ritz procedure, involving expansion of the field in terms of appropriate basis functions, has been extended to complex profiles by choosing the expansion coefficients as complex. This converts the problem to a complex matrix eigenvalue equation. Although this procedure does not neglect the imaginary part of the field, the number of basis functions required in the expansion, for sufficient accuracy, can often be as large as 50. Hence, the procedure requires evaluation of the complex eigenvalues of a 50×50 matrix with complex elements. This requires a cumbersome algorithm and even by use of an optimized NAG library routines takes about 8 min on a Pentium II processor to obtain one eigenvalue.

In this paper, we present a simple direct numerical procedure to evaluate accurately the complex propagation constants in single-mode fibers with an arbitrary complex refractive index profile. It is based on the use of the Newton–Raphson method for solving a complex eigenvalue equation and the use of the fourth-order Runge–Kutta method in the complex domain for solution of differential equations to obtain the eigenvalue equation. The procedure does not make any approximation and converges to the desired accuracy rapidly, taking only a few seconds for one eigenvalue. We first establish the rapid convergence of our procedure, and then compare the results with those reported earlier on a step complex refractive index profile. As an illustration of the application of this procedure to an actual profile with an arbitrary complex refractive index profile, we have used it to calculate the gain in a typical 980-nm pumped erbium-doped fiber.

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II. Procedure

The complex refractive index profile of the fiber can, in general, be written as

$$\begin{aligned} n^2(R) &= n_{co}^2(R) = n'^2_{co}(R) + in''^2_{co}(R), \qquad R<1 \\ &= n_{cl}^2 = n'^2_{cl} + in''^2_{cl}, \qquad R>1 \end{aligned} \tag{1}$$

where

R $= r/a$;

a core radius;

n_{co}, n_{cl} refractive indices of the core and cladding, respectively, and can both, in general, be complex.

In the weakly guiding approximation, propagation in the fiber can be described in terms of the linearly polarized LP modes. For the LP modes, the transverse component of the electric field (E_x or E_y) can be written as

$$E_{x(y)} = \psi(R,\theta)e^{i(\omega t-\beta z)} \tag{2}$$

and $\psi(R,\theta)$ satisfies the scalar equation

$$\frac{\partial^2\psi}{\partial R^2} + \frac{1}{R}\frac{\partial\psi}{\partial R} + \frac{1}{R^2}\frac{\partial^2\psi}{\partial\theta^2} + a^2k_o^2[n^2(R)-n_e^2]\psi = 0 \tag{3}$$

where $k_0 = 2\pi/\lambda$ is the free space wave vector, $n_e = \beta/k_0$ is the effective index, and β is the propagation constant along the fiber. For the fundamental LP_{01} mode, $\psi(R, \theta)$ is only a function of R, and the scalar equation (3) reduces to

$$\frac{d^2\psi}{dR^2} + \frac{1}{R}\frac{d\psi}{dR} + a^2k_o^2[n_{co}^2(R) - n_e^2]\psi = 0, \qquad R < 1 \tag{4}$$

$$\frac{d^2\psi}{dR^2} + \frac{1}{R}\frac{d\psi}{dR} + a^2k_o^2[n_{cl}^2 - n_e^2]\psi = 0, \qquad R > 1. \tag{5}$$

In the region $R > 1$, the solution of (5) can be written as (see e.g., [4])

$$\psi(R) = A\frac{K_0(W^*R)}{K_0(W^*)} \tag{6}$$

where $W^* = k_0 a\sqrt{n_e^2 - n_{cl}^2}$ and $K_l(z)$ are the modified Bessel functions of the second kind of order l. The effective index n_e, the wave function $\psi = \psi_r + i\psi_i$ and W^* are all complex. The condition of continuity of $\psi(R)$ and $d\psi/dR$ at $R = 1$ leads to

$$\left(\frac{1}{\psi}\frac{d\psi}{dR}\right)_{R=1} = -\frac{W^*K_1(W^*)}{K_0(W^*)}. \tag{7}$$

For the region $R < 1$, we first transform (4) to the following first-order differential equation by the Ricatti transformation as in [5]

$$\frac{dG}{dR} = -G^2 - \frac{G}{R} - a^2k_o^2[n_{co}^2(R) - n_e^2] \tag{8}$$

where

$$G(R) = \frac{1}{\psi}\frac{d\psi}{dR}. \tag{9}$$

At $R = 0$, $\psi \neq 0$ while $d\psi/dR = 0$ (as shown in the Appendix I), leading to the initial condition

$$G(R = 0) = 0. \tag{10}$$

The boundary condition at $R = 1$ is obtained from (7) as

$$G(R = 1) = -\frac{W^*K_1(W^*)}{K_0(W^*)}. \tag{11}$$

Hence, the problem now reduces to the solution of the complex eigenvalue equation

$$F(n_e) = G_1(n_e) + \frac{W^*K_1(W^*)}{K_0(W^*)} = 0 \tag{12}$$

where $G_1(n_e)$ is the value of $G(R)$ at $R = 1$ obtained as a solution of the first-order differential equation (8) with the initial condition $G(R = 0) = 0$. In earlier analysis of planar waveguides with complex refractive index profiles [6], we found the Newton–Raphson method as a feasible and useful tool. However, this method, in addition to evaluation of the function $F(n_e)$, requires evaluation of its derivative $F'(n_e)$ (prime denotes differentiation with respect to n_e), given by

$$F'(n_e) = G_1'(n_e) - a^2k_o^2 n_e\left(1 - \frac{K_1^2(W^*)}{K_0^2(W^*)}\right) \tag{13}$$

where $G_1'(n_e)$ is the value of G' at $R = 1$. In order to evaluate $F'(n_e)$ analytically, we differentiate (8) with respect to n_e to obtain the following equation for $G'(R)$

$$\frac{dG'}{dR} = -2GG' - \frac{G'}{R} + 2a^2k_o^2n_e. \tag{14}$$

$G_1'(n_e)$ is obtained as the solution of the differential equation (14) with the initial boundary condition $G'(R = 0) = 0$. The two differential equations (8) and (14) can be solved as coupled differential equations by the fourth-order Runge–Kutta method implemented directly as in [7], with the dependent variable now being complex. The second terms on the right-hand side of both (8) and (14) are indeterminate at $R = 0$. Hence, in writing (8) at $R = 0$, the limiting form $\lim_{R\to 0}(G/R) = (dG/dR)_{R=0}$ is used, leading to

$$\left(\frac{dG}{dR}\right)_{R=0} = -\frac{1}{2}a^2k_o^2[n_{co}^2(R) - n_e^2]. \tag{15}$$

Similarly, substituting the limiting form $\lim_{R\to 0}(G'/R) = (dG'/dR)_{R=0}$ on the right-hand side of (14) at $R = 0$ leads to

$$\left(\frac{dG'}{dR}\right)_{R=0} = a^2k_o^2n_e. \tag{16}$$

In order to solve the complex eigenvalue problem, we carry out the following steps.

1) First, the real index problem is solved with the imaginary part of the core and cladding indices n''^2_{co} and n''^2_{cl} put equal to zero. The real value of n_e so obtained is used as the zeroth-order solution $n_e^{(0)}$.
2) The above value of n_e is used in (8) and (14) to calculate the values of $G_1(n_e^{(0)})$ and $G'(n_e^{(0)})$ simultaneously and, hence, $F(n_e^{(0)})$, $F'(n_e^{(0)})$.
3) The first corrected solution is obtained as
$$n_e^{(1)} = n_e^{(0)} - \frac{F\left(n_e^{(0)}\right)}{F'\left(n_e^{(0)}\right)}.$$
4) This new value of $n_e = n_e^{(1)}$ is now used to obtain $F(n_e^{(1)})$, $F'(n_e^{(1)})$ which are used to obtain the next improved value
$$n_e^{(2)} = n_e^{(1)} - \frac{F\left(n_e^{(1)}\right)}{F'\left(n_e^{(1)}\right)}.$$
5) The process is continued until convergence occurs using
$$n_e^{(p+1)} = n_e^{(p)} - \frac{F\left(n_e^{(p)}\right)}{F'\left(n_e^{(p)}\right)}.$$

In most calculations, less than six iterations are adequate for the desired accuracy. In the evaluation of $F(n_e)$ or $F'(n_e)$, the evaluation of the modified Bessel functions $K_0(W^*)$ and $K_1(W^*)$ is required for a complex argument W^*. We have used the rational approximation given by Luke *et al.* [8] for the evaluation of the modified Bessel functions $K_l(z)$. This is given in Appendix II for completeness.

III. Approximate Procedure

As mentioned, in order to simplify the analysis, Sader [1] used the approximation that the imaginary part of the field is much smaller than the real part, i.e., $\psi_i \ll \psi_r$. We also introduced a similar approximation in our numerical procedure to separate the real and imaginary parts of (8). The real part of (8) can be written as

$$\frac{dG_r}{dR} = -G_r^2 + G_i^2 - \frac{G_r}{R} - a^2k_o^2[n'^2_{co}(R) - (n_e^2)_r] \quad (17)$$

and using the approximation that $G_i^2 \ll G_r^2$, the above equation transforms to a real equation for G_r, similar to an equation obtained for a real index profile [5]

$$\frac{dG_r}{dR} = -G_r^2 - \frac{G_r}{R} - a^2k_o^2[n'^2_{co}(R) - (n_e^2)_r], \qquad R < 1 \quad (18)$$

with boundary conditions $G_r(R=0) = 0$ and $G_r(R=1) = -\mathrm{real}(W^*K_1(W^*)/K_0(W^*))$, which can be solved to obtain $(n_e^2)_r$ and $G_r(R)$.

The imaginary part of (8) is

$$\frac{dG_i}{dR} = -2G_r(R)G_i - \frac{G_i}{R} - a^2k_o^2[n''^2_{co}(R) - (n_e^2)_i] \quad (19)$$

where $G_r(R)$ is the known solution of (18). This real differential equation can be solved by the simple Runge–Kutta method [7] to obtain $G_{1i}[(n_e^2)_i]$, where G_{1i} is the value of $G_i(R)$ at $R = 1$ starting with the initial boundary condition $G_i(0) = 0$. The second term on the right-hand side of (19) is also indeterminate at $R = 0$ and, hence, the following limiting form has to be used:

$$\left(\frac{dG_i}{dR}\right)_{R=0} = -\frac{1}{2}\, a^2k_o^2[n''^2_{co}(R) - (n_e^2)_i]. \quad (20)$$

The solution for $(n_e^2)_i$ is obtained by solving the following real eigenvalue equation:

$$G_{1i}[(n_e^2)_i] = -\mathrm{imag}\left(\frac{W^*K_1(W^*)}{K_0(W^*)}\right). \quad (21)$$

The underlying assumption here is that $((1/\psi)(d\psi/dR))_i \ll ((1/\psi)(d\psi/dR))_r$ or that the imaginary part of the fractional change in ψ in a small interval $dR \to 0$ is smaller than the corresponding real part while the restraint in [1] is $\psi_i \ll \psi_r$. Like [1], the present approximation uses purely real methods in the analysis of complex refractive index profiles, but better accuracy is expected in the above approximation, which is shown clearly in our results in Table III.

IV. Results and Discussion

To illustrate the use of the procedure, test its convergence and compare results, we first chose a simple refractive index profile where both $n_r(R)$ and $n_i(R)$ follow a step profile [1]. Next, we use the procedure to calculate modal gain in a 980-nm pumped erbium-doped fiber which can be described by a radially dependent complex refractive index profile.

TABLE I
Convergence of Values of $G(R = 1)$ and $G'(R = 1)$ with Step-Size h Used in the Runge–Kutta Procedure for $n_{\rm coi} = 10^{-3}$ at Typical Values of n_e

n_e	h	$G(R=1)$	$G'(R=1)$
1.465071, 7.4508E-04	.20	-1.7138, -6.9096E-02	263.879, 8.146
	.10	-1.7138, -6.9099E-02	263.900, 8.150
	.05	-1.7138, -6.9099E-02	263.900, 8.150
	.025	-1.7138, -6.9099E-02	263.900, 8.150
1.465043, 7.4361E-04	.20	-1.7213, -6.9721E-02	264.753, 8.2378
	.10	-1.7214, -6.9722E-02	264.774, 8.2416
	.05	-1.7214, -6.9722E-02	264.773, 8.2415
	.025	-1.7214, -6.9722E-02	264.773, 8.2415

A. Step-Profile Fiber

The index profile for the step profile fiber is given by

$$\begin{aligned} n(R) &= n_{\rm cor} + in_{\rm coi} \qquad R < 1 \\ &= n_{cl} \qquad R > 1. \end{aligned} \quad (22)$$

This corresponds to $n'^2_{co}(R) = n^2_{\rm cor} - n^2_{\rm coi}$ and $n''^2_{co}(R) = 2n_{\rm cor}n_{\rm coi}$ in (1). The fiber parameters used in the calculations are $a = 2.2$ μm, $n_{\rm cor} = 1.475$, $n_{cl} = 1.458$ at $\lambda = 1.55$ μm [1]. The accuracy of the calculations of $G(R=1)$ and $G'(R=1)$ depends on the step-size in the Runge–Kutta procedure used to solve (8) and (14). Hence, we first tested the convergence of the procedure with a step-size at typical values of n_e. The results tabulated in Table I show that a step size of 0.10 or 0.05, which corresponds to 10–20 steps in the Runge–Kutta procedure, gives sufficient accuracy. Table II shows the convergence of the values of n_e with the number of iterations of the Newton–Raphson method using the converged value of step-size. Even at relatively large values of $n_{\rm coi}$, the Newton–Raphson method converges in less than six iterations.

Next, in order to compare our results with the approximate method of [1], we calculated the parameter $U^* = k_0a\sqrt{(n_{\rm cor} + in_{\rm coi})^2 - n_e^2}$ and $Gain (= 8.686k_0n_{ei})$ (dB/m) in the fiber. We also calculated the corresponding values using the approximation $G_i^2 \ll G_r^2$ as given in Section III. The results are tabulated in Table III along with the exact results taken from [1]. It is clear that the results obtained by the approximate methods become inaccurate as the imaginary part of the refractive index increases. The approximation of Section III, although similar to [1], gives better accuracy, while the present complete procedure converges to exact values in less than six iterations of the Newton–Raphson procedure.

B. Erbium-Doped Fiber

After establishing the rapid convergence of our procedure we applied it to obtain the gain characteristics of a typical erbium-doped fiber. Signal gain is obtained in erbium-doped fibers by

TABLE II
CONVERGENCE OF VALUES OF n_e WITH ITERATION NUMBER OF THE NEWTON–RAPHSON METHOD AT $h = 0.05$

Iteration Number	$n_{coi} = 10^{-5}$		$n_{coi} = 10^{-3}$		$n_{coi} = 10^{-2}$	
	n_{er}	$n_{ei} \times 10^6$	n_{er}	$n_{ei} \times 10^4$	n_{er}	$n_{ei} \times 10^3$
0	1.465051	0.0000	1.465051	0.0000	1.465051	0.0000
1	1.465051	7.4331	1.465071	7.4508	1.466449	9.1587
2	1.465051	7.4331	1.465042	7.4362	1.464257	7.3961
3	—	—	1.465043	7.4362	1.464318	7.6982
4	—	—	1.465043	7.4362	1.464314	7.6987
5	—	—	—	—	1.464314	7.6987

TABLE III
COMPARISON OF VALUES OF U^* AND GAIN (dB/m)

	$n_{coi} = 10^{-5}$			$n_{coi} = 10^{-3}$			$n_{coi} = 10^{-2}$		
	U_r	$U_i \times 10^4$	Gain	U_r	$U_i \times 10^2$	Gain	U_r	$U_i \times 10$	Gain
Present	1.5252	2.0128	261.7	1.5260	2.0095	26183	1.5891	1.7400	271073
Sec.III	1.5252	2.0128	261.7	1.5253	2.0104	26182	1.5356	1.8392	269143
Sader	1.5252	2.0128	261.7	1.5247	2.0159	26161	—	—	—
Exact	1.5252	2.0128	261.7	1.5260	2.0095	26183	—	—	—

creating a population inversion, using a pump at 980 nm. Assumption of a three-level laser system for the erbium-doped fibers provides a means to define a complex refractive index profile for the 980-nm pumped erbium-doped fiber [3]. The complex refractive index profile of the doped step index fiber can, in general, be written as

$$\begin{aligned} n(R) &= n_1 + in_i(R), \qquad R < 1 \\ &= n_2, \qquad R > 1 \end{aligned} \tag{23}$$

where n_1, n_2 are the refractive indices of the core and cladding in the absence of pump and $n_i(R)$ is the imaginary part of refractive index of the core of the pumped fiber. It has been shown in [3] that, for the signal wavelength, $n_i(R)$ is given by

$$n_i(signal) = \frac{\rho(R)\sigma_a(\lambda_s)\lambda_s(\eta_s q\psi_p^2(R) - 1)}{4\pi[1 + q\psi_p^2(R) + p\psi_s^2(R)]} \tag{24}$$

where $\rho(R)$ is the erbium-doping profile, $\psi_{p,\,s}^2(R)$ are the fiber mode profiles at the pump and signal wavelengths, given by

$$\psi_{p,\,s}^2(R < 1) = J_0^2(U_{p,\,s}R) \tag{25}$$

$$\psi_{p,\,s}^2(R > 1) = \frac{J_0^2(U_{p,\,s})}{K_0^2(W_{p,\,s})}\, K_0^2(W_{p,\,s}R) \tag{26}$$

where $U_{p,\,s}$, $V_{p,\,s}$ and $W_{p,\,s}$ are the conventional fiber parameters [4] of the undoped fiber, and $J_0(z)$ is the Bessel function of first kind of order zero. $\sigma_a(\lambda_s)$ and $\sigma_e(\lambda_s)$ are the absorption and emission cross-sections, respectively, at signal wavelength λ_s and $\eta(\lambda_s) = \sigma_e(\lambda_s)/\sigma_a(\lambda_s)$. p and q indicate the signal and pump powers guided in the fundamental mode normalized to their respective saturation powers $p = P_s(z)/P_{\text{sat}}(\lambda_s)$, $q = P_p(z)/P_{\text{sat}}(\lambda_p)$, with $P_{\text{sat}}(\lambda_s)$ and $P_{\text{sat}}(\lambda_p)$ defined as

$$P_{\text{sat}}(\lambda_s) = \frac{hc\pi w_s^2}{\lambda_s\sigma_a(\lambda_s)[1 + \eta(\lambda_s)]\tau} \tag{27}$$

$$P_{\text{sat}}(\lambda_p) = \frac{hc\pi w_p^2}{\lambda_p\sigma_a(\lambda_p)\tau} \tag{28}$$

where $\sigma_a(\lambda_p)$ is the absorption cross-section at the pump wavelength λ_p and $w_{p,\,s}$ is the mode power radius given by

$$w_{p,\,s} = a\frac{V_{p,\,s}K_1(W_{p,\,s})}{U_{p,\,s}K_0(W_{p,\,s})}\, J_0(U_{p,\,s}) \tag{29}$$

and τ is the life time of the upper laser level. We considered a step index fiber with step erbium doping in the core, $\rho = \rho_0$ for $R < 1$, and $\rho = 0$ for $R > 1$ with $a = 1.5$ μm, $n_1 = 1.46$, NA $= 0.24$, $\rho_0 = 1.6 \times 10^{25}$ m^{-3} and $\lambda_p = 0.98$ μm. The $\sigma_e(\lambda_s)$ and $\sigma_a(\lambda_s)$ profiles correspond to [9, Fig. 4.22], with

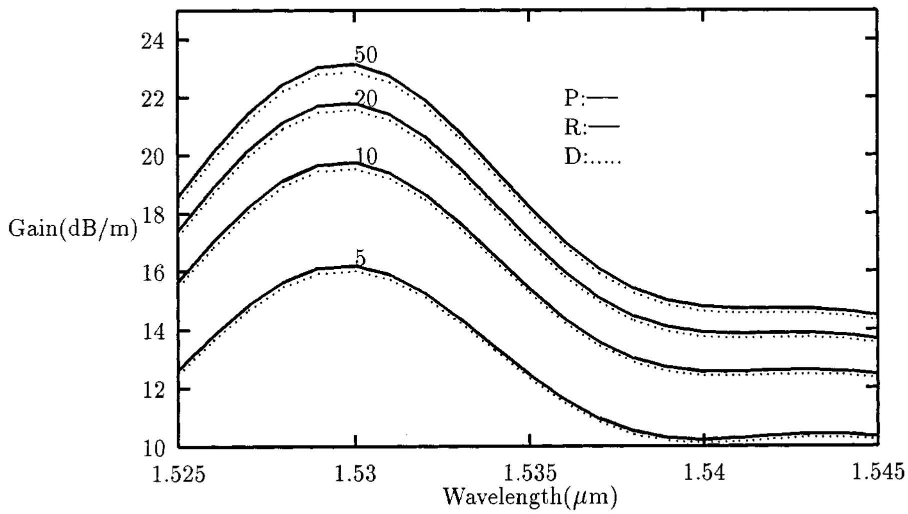

Fig. 1. Signal gain with wavelength at different input pump power levels (marked in mW) by the present method (P), Rayleigh-Ritz method (R) of [3], and perturbation calculations (D) of [9].

peak values $\sigma_a(\lambda_s) = 7 \times 10^{-25}$ m^2, $\sigma_e(\lambda_s) = 0.92\sigma_a(\lambda_s)$ and $\sigma_a(\lambda_p) = 2 \times 10^{-25}$ m^2. Using the above complex refractive index profile, we obtained values of complex n_e and corresponding values of gain at different wavelengths in the 1.53 μm range. A plot of signal gain for an input signal of 100 nW, with wavelength as obtained from our method for different values of input pump power, is shown in Fig. 1 and is coincident with the results of the more cumbersome Rayleigh–Ritz calculations (corresponding to an expansion in 30 terms) [3]. Calculations from [9, eq. 1.54], which has been shown to be equivalent to a first-order perturbation calculation in [3], are also plotted in the figure. The perturbation calculation gives slightly lower values of gain at the parameters used.

V. Conclusion

In conclusion, we have presented a simple direct numerical procedure to evaluate the exact propagation characteristics of an optical fiber with an arbitrary complex refractive index profile. The direct method is especially useful for calculations in high gain (or loss) fibers where the earlier reported approximate methods are inaccurate. Further, we have shown its practical application to evaluate gain in a typical 980-nm pumped erbium-doped fiber and compared our results with those obtained earlier by the Rayleigh–Ritz method.

Appendix I

The boundary conditions on the modal field $\psi(R)$ and its derivative at $R = 0$ can be obtained by the following procedure summarized from [10].

For the scalar wave equation in (4)

$$\frac{d^2\psi}{dR^2} + \frac{1}{R}\frac{d\psi}{dR} + \alpha^2\psi = 0 \tag{30}$$

where $\alpha^2 = a^2k_0^2[n_{co}^2 - n_e^2]$, the modal field solution ψ can be expressed in terms of a Frobenius series given by

$$\psi(R) = \sum_{n=0}^{\infty} a_n R^{(n+s)}. \tag{31}$$

By substitution of $\psi(R)$ and its derivatives, (30) transforms to

$$a_0s^2 + (1+s)^2a_1R + \sum_{n=2}^{\infty}[a_n(n+s)^2 + \alpha^2a_{n-2}]R^n = 0 \tag{32}$$

leading to the Indicial equations

$$s^2 = 0 \tag{33}$$

$$a_1(1+s)^2 = 0 \tag{34}$$

and recurrence relation

$$a_n = -\frac{\alpha^2 a_{n-2}}{n^2}. \tag{35}$$

Equation (33) gives $s = 0$ and from (34) and (35), one obtains $a_1 = a_3 = a_5 = a_7 = ----0$; a_2, a_4—can be expressed in terms of a_0 leading to the following series for $\psi(R)$ and its derivative $\psi'(R)$

$$\psi(R) = \sum_{n=0}^{\infty}(a_nR^n) = a_0\left[1 - \frac{\alpha^2R^2}{4} + \frac{\alpha^4R^4}{64} - \frac{\alpha^6R^6}{2304} + ---\right] \tag{36}$$

and

$$\psi'(R) = a_0\left[-\frac{\alpha^2R}{2} + \frac{\alpha^4R^3}{16} ----\right]. \tag{37}$$

Hence, at $R = 0$, one obtains the conditions

$$\psi(R=0) = a_0 \tag{38}$$

and

$$\psi'(R=0) = 0 \tag{39}$$

giving

$$G(R=0) = \frac{\psi'}{\psi}(R=0) = 0. \tag{40}$$

APPENDIX II

The modified Bessel function $K_l(z)$ can be expressed in terms of confluent hypergeometric function $U(a, c, z)$ as in [11].

$$K_l(z) = \sqrt{\frac{\pi}{2z}}\, e^{-z}(2z)^{(1/2+l)}U(1/2+l, 2l+1, 2z). \quad (41)$$

Hence

$$K_0(z) = \sqrt{\frac{\pi}{2z}}\, e^{-z}(2z)^{(1/2)}U(1/2, 1, 2z) \quad (42)$$

$$K_1(z) = \sqrt{\frac{\pi}{2z}}\, e^{-z}(2z)^{(3/2)}U(3/2, 3, 2z) \quad (43)$$

where $U(a, c, z)$ for complex z can be expressed in terms of a rational approximation as given in [8]

$$z^a U(a, 1+a-b, z) = \frac{A_n(z)}{B_n(z)} + R_n(z) \quad (44)$$

where $R_n(z)$ is the error term, and

$$A_0(z) = 1, \qquad B_0(z) = 1 \quad (45)$$

$$b = 1 + a - c \quad (46)$$

$$B_1(z) = 1 + zp \quad (47)$$

$$A_1(z) = B_1(z) - abp \quad (48)$$

$$B_2(z) = 1 + 3zp(1+zq) \quad (49)$$

$$A_2(z) = B_2(z) - 3ab[p(1+zq) - q] \quad (50)$$

where

$$p = \frac{2}{(a+1)(b+1)} \quad (51)$$

$$q = \frac{2}{(a+2)(b+2)}. \quad (52)$$

For $n \geq 3$, $B_n(z)$ and $A_n(z)$ are obtained by the recurrence relations

$$B_n(z) = [(zD_{2n}-C_{2n})B_{n-2}+(zD_{2n}-C_{2n})B_{n-2}-C_{3n}B_{n-3}] \quad (53)$$

$$A_n(z) = [(zD_{1n}-C_{1n})A_{n-1}+(zD_{2n}-C_{2n})A_{n-2}-C_{3n}A_{n-3}] \quad (54)$$

with

$$D_{1n} = D_{2n} = \frac{2(2n-1)}{(n+a)(n+b)} \quad (55)$$

$$C_{1n} = -\frac{(2n-1)[3n^2+(a+b-6)n+2-ab-2(a+b)]}{(2n-3)(n+a)(n+b)} \quad (56)$$

$$C_{2n} = \frac{[3n^2-(a+b+6)n+2-ab]}{(n+a)(n+b)} \quad (57)$$

and

$$C_{3n} = -\frac{(2n-1)(n-a-2)(n-b-2)}{(2n-3)(n+a)(n+b)} \quad (58)$$

$$U(a, c, z) = \frac{A_n(z)}{B_n(z)}\, z^{-a} \quad (59)$$

where n is chosen such that both real and imaginary part of $U(a, c, z)$ converge to desired accuracies.

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