FE Stokes equations

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Stokes can be written as

$$-\nabla \cdot (\nabla \mathbf{u} + pI) = \mathbf{f} \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0. \tag{2}$$

The naïve finite element approach to solving the Stokes equations is as follows. Find $(u, p) \in W$ such that

$$a((\mathbf{u}, p), (\mathbf{v}, q)) = L((\mathbf{v}, q)) \tag{3}$$

for all $(v, q) \in W$, where

$$a((\mathbf{u}, p), (\mathbf{v}, q)) = \int_\Omega \nabla \mathbf{u} \cdot \nabla \mathbf{v} - \nabla \cdot \mathbf{v} p + \nabla \cdot \mathbf{u} q \, \mathrm{d}x, \tag{4}$$

$$L((\mathbf{v}, q)) = \int_\Omega \mathbf{f} \cdot \mathbf{v} \, \mathrm{d}x. \tag{5}$$

Using first order elements in both velocity and pressure leads to stability problems for reasons that are unclear to me but clear to some in the FEA community. Replace equations 4 and 5 with

$$a((\mathbf{u}, p), (\mathbf{v}, q)) = \int_\Omega \nabla \mathbf{u} \cdot \nabla \mathbf{v} - \nabla \cdot \mathbf{v} p + \nabla \cdot \mathbf{u} q + \delta \nabla q \cdot \nabla p \, \mathrm{d}x, \tag{6}$$

$$L((\mathbf{v}, q)) = \int_\Omega \mathbf{f} \cdot \mathbf{v} \, \mathrm{d}x + \int_\Omega \nabla q \cdot \mathbf{f} \, \mathrm{d}x. \tag{7}$$

We then implement this on FEniCS with a mixed function space method.