THE GREAT COURSES

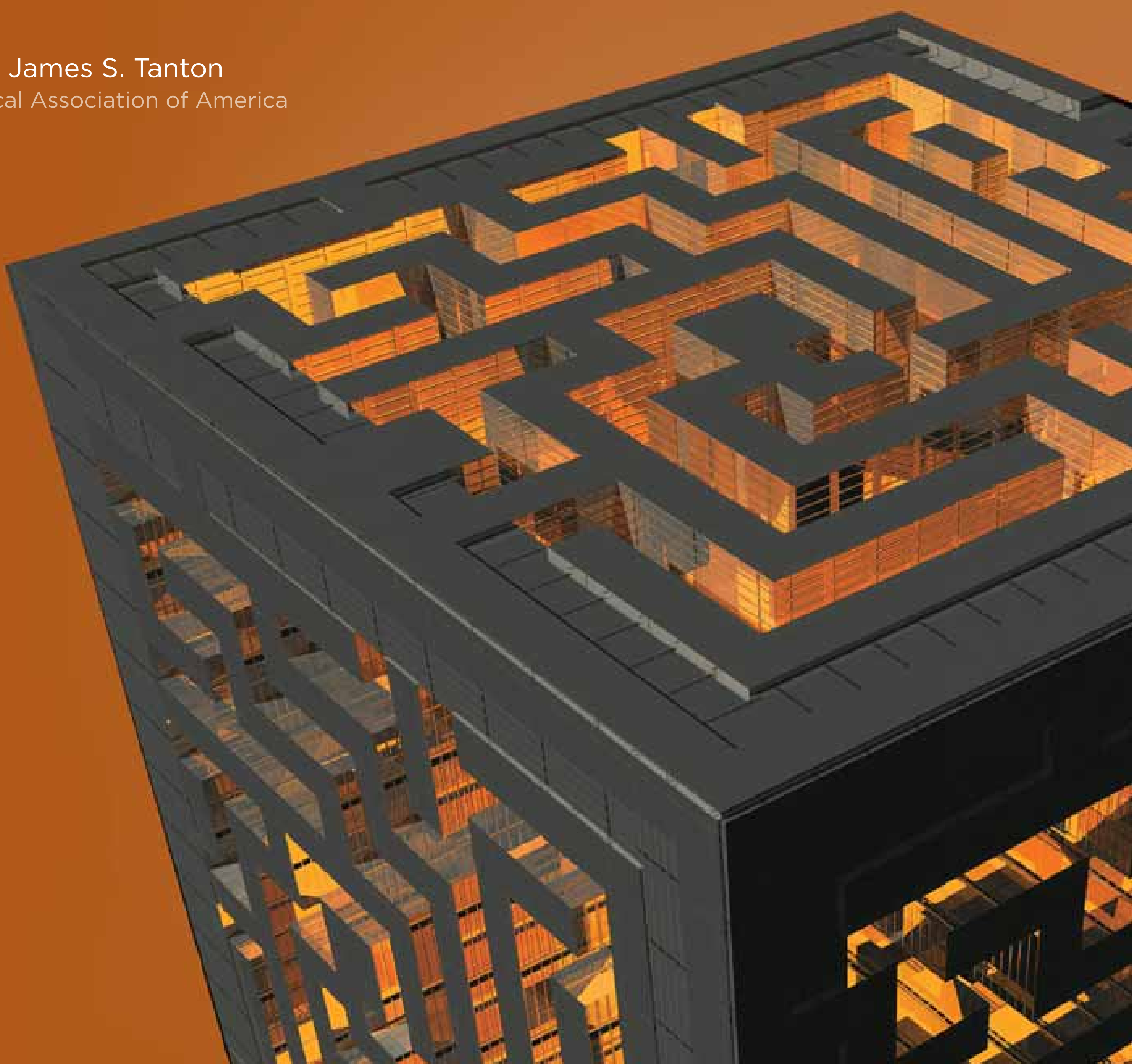
Topic
High School

Subtopic
Mathematics

Geometry: An Interactive Journey to Mastery

Course Workbook

Professor James S. Tanton
Mathematical Association of America



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Professor James S. Tanton is the Mathematician in Residence at the Mathematical Association of America (MAA) in Washington DC. He received his Ph.D. in Mathematics from Princeton University in 1994. From 2004 to 2012, he worked as a full-time high school teacher at St. Mark's School in Southborough, Massachusetts. In 2004, Professor Tanton founded the St. Mark's Institute of Mathematics, an outreach program promoting joyful and effective mathematics education for both students and educators.

Believing that mathematics really is accessible to all, Professor Tanton is committed to sharing the delight and beauty of the subject. He is actively engaged in professional development for educators in the United States, in Canada, and overseas. He also conducts the professional development program for Math for America in Washington DC.

Professor Tanton is the author of *Solve This: Math Activities for Students and Clubs*; *The Encyclopedia of Mathematics*; *Mathematics Galore!*; and 12 self-published texts. He received the 2004 Beckenbach Book Prize from the MAA, the 2006 George Howell Kidder Faculty Prize from St. Mark's School, and a 2010 Raytheon Math Hero Award for excellence in math teaching. He also publishes research and expository articles, and his extracurricular classes have helped high school students pursue research projects and publish their results. ■

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Geometry: An Interactive Journey to Mastery

Scope:

Geometry—in fact, mathematics as a whole—is an intensely human enterprise. As its name suggests, derived from the Greek *ge* for "earth" and *metria* for "measure," the subject evolved from very practical concerns: the art of accurately measuring tracts of land, performing feats of navigation, and accomplishing success in construction and architectural design, for example. But the study of geometry reveals a deep and rich layer of intellectual delight. Our human minds are drawn to elegance and beauty, both in the results of the hand and in the products of the mind. Geometry is the dance between these two joyful pursuits, and this course is the journey of that dance for you to enjoy.

Part I of the course identifies the key beliefs that make geometry work the way we think it should. (Note the sense of human choice in that statement.) These beliefs stand as the foundation of the entire subject, and the full remaining content of the course follows from them solely through deduction, stunning acts of logic, and clever application.

Part II runs through all the traditional topics expected in a typical geometry course. These topics are age-old favorites, and they are full of twists, turns, and surprises.

In Part III of the course, we present a selection of fun and unusual topics that might or might not appear in a traditional program. After you have completed parts I and II, the topics in part III essentially stand alone and can be viewed in any order that takes your fancy.

Take your time with this course. Have fun with it. Savor the ideas presented here and mull on the queries and conundrums that arise in the video lessons and in this guidebook. All will become clear as the entire story unfolds, and your understanding of geometry will be rich and deep. Give yourself permission to really enjoy the story, to wonder, to discover, and to engage in meaningful learning. ■

Geometry—Ancient Ropes and Modern Phones

Lesson 1

Topics

- What is geometry?
- The geometry of the Earth.
- Early applications of geometry.
- *The Elements*.
- Course overview.

Definitions

- **geometry**: The branch of mathematics concerned with the properties of space and of figures, lines, curves, points, and shapes drawn in space.

  If the nature of the ambient space is specified, the type of geometry might be specified. For example:

- **planar geometry**: The study of figures, lines, curves, points, and shapes drawn in a plane.
- **spherical geometry**: The study of figures, lines, curves, points, and shapes drawn on the surface of a sphere.
- **three-dimensional geometry**: The study of figures, lines, curves, points, and shapes drawn in three-dimensional space.

Summary

Welcome to geometry, one of the most fascinating, beautiful, and playful branches of mathematics!

In this opening lesson, we present an overview of what geometry is. Although the subject evolved from very practical concerns, geometry can lead to deep philosophical questions, too. What shape is the Earth? What shape is the universe? Do infinitely long lines exist?

We present some early practical and intellectual applications of geometry from history—the Egyptian use of knotted ropes, a clever use of right angles in circles, for example—but the principal idea behind this lesson is that geometry must begin by identifying beliefs about the mathematical world that we feel should be accepted as true. The great Greek geometer Euclid (c. 300 B.C.E.) identified 10 beginning assumptions to all of geometry and showed in his 13-volume text, *The Elements*, that the entire subject logically unfolds from them. His text is hailed as one of the great intellectual achievements of mankind in the entire history of mankind. The process of logical deduction Euclid demonstrated defined the very nature of modern scientific thought.

Example 1: Egyptian Ropes

A rope of length 12 divided into straight sections of lengths 3, 4, and 5 makes a triangle that sits snugly in the right angle between a wall and the floor. (See **Figure 1.1.**)

Experiment with a rope of length 30 inches. Can you divide it into three lengths that also make a triangle that sits snugly between the wall and the floor? What are the three lengths you find?

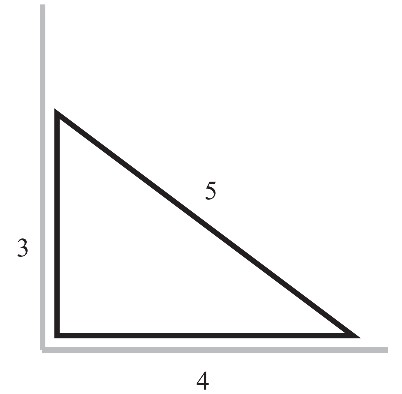


Figure 1.1

Solution

Through trial and error, you can indeed use the rope to make such a triangle. The three lengths you create this way will likely have measures that are awkward fractional numbers. If you have the patience, you can verify that the three whole-number lengths 5, 12, and 13 (adding to 30) work nicely to make a snug-fitting triangle.

Example 2: Right Angles in Circles

Trace the perimeter of a drinking glass on a piece of paper. Using a second piece of paper and a ruler, how could you find the diameter of your glass?

Solution

Lay the corner of the second piece of paper on the circle you drew as shown. (See **Figure 1.2.**) Mark where the edges of this paper intersect the circle, and measure the distance between those two points. This is the diameter of your drinking glass.

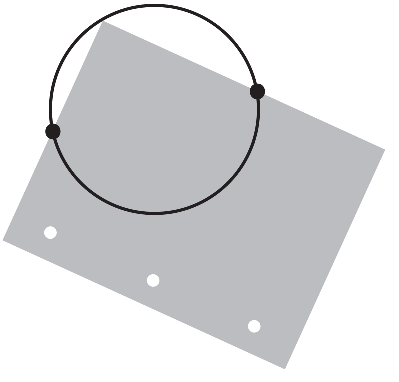


Figure 1.2

Study Tip

- The study of geometry requires a frank willingness to question ideas, assumptions, and beliefs. As we will see throughout this course, intuition can lead us astray. Don't be misled by results that only *seem* true. Success comes from taking the time to reflect on ideas and to question assumptions. There is no need to hurry in the study of geometry!

Problems

1. Is the following statement true or false? A circle drawn on the surface of a three-dimensional solid is sure to divide the surface of that solid into two parts: an “inside” and an “outside.”

2. Look around the room you are currently in. Do you see anywhere in it an example of a straight line? A *perfectly straight* line?

3. Telegraph poles, edges of buildings, and signposts are usually straight lines. Are the shadows cast from straight objects by the Sun always sure to be straight? What do you think?

Beginnings—Jargon and Undefined Terms

Lesson 2

Topics

- Notation: points, lines, line segments, distance, congruent line segments, midpoints.
- Collinear points and coplanar points.
- Definitions and circular reasoning.
- Angles.
- Vertical angle theorem.

Definitions

- **acute angle**: An angle with measure strictly between 0° and 90° is acute.
- **angle bisector**: A line that divides a given angle into two congruent angles is an angle bisector for that angle.
- **bisects**: A point M sitting on a line segment $\overline{AB}$ is said to bisect the line segment if $AM = MB$.
- **congruent angles**: Two angles of the same measure are congruent.
- **congruent line segments**: Two line segments of the same length are congruent.
- **collinear**: Two or more points are collinear if they lie on a common line.
- **coplanar**: Two or more points are coplanar if they lie on a common plane.
- **midpoint**: A point M is a midpoint of line segment $\overline{AB}$ if M lies on the segment and $AM = MB$.
- **obtuse angle**: An angle with measure strictly between 90° and 180° is obtuse.
- **perpendicular**: Two lines or line segments that intersect at an angle of 90° are perpendicular.
- **reflex angle**: An angle with measure strictly between 180° and 360° is a reflex angle.
- **right angle**: An angle with measure 90° is a right angle.

- **straight angle**: An angle with measure 180° is a straight angle.

- **vertical angles**: Two angles on opposite sides of the point of intersection of two intersecting lines are vertical angles.

Notation

- Points are labeled with capital letters.

- A line through points A and B is denoted $\overleftrightarrow{AB}$.

- The line segment connecting points A and B is denoted $\overline{AB}$.

- The length of the line segment $\overline{AB}$ is denoted AB.

- Congruent line segments are denoted $\overline{AB} \cong \overline{CD}$. On diagrams, congruent line segments are given the same markings.

- The angle given by two line segments $\overline{PQ}$ and $\overline{QR}$ is denoted $\angle PQR$ or $\angle RQP$. Unless told otherwise, one is to assume that this refers to the angle of smaller measure.

- Congruent angles are denoted $\angle PQR \cong \angle ABC$. On diagrams, congruent angles are given matching labels.

- Perpendicular lines or line segments are denoted $\overline{AB} \perp \overline{CD}$.

Result

- vertical angle theorem: Vertical angles are congruent.

Summary

Geometry is the study of points, lines (curved and straight), and surfaces and the figures one can construct with them. As we see in this lesson, it is not possible to define exactly what we mean by "point," "line," or "straight," for example, even though we feel that we have a good intuitive understanding of what these concepts mean.

We begin this lesson relying on our intuition to establish the notation and language that is now standard in the study of geometry. The philosophical concerns of pinning down what these beginning terms mean lead us to the problem of circular reasoning and the need for setting undefined terms coupled with postulates.

With philosophical difficulties in check, we end the lesson with the language and notation of angles.

Example 1

T is the midpoint of $\overline{WX}$. If $WT = 16$, what is WX?

Solution

The diagram in **Figure 2.1** shows T as the midpoint of the line segment $\overline{WX}$.

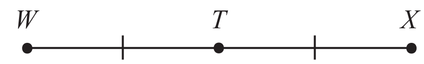


Figure 2.1

The distance WT is 16 units. It is clear from the picture that the distance WX must be 32 units.

Example 2

In the diagram in **Figure 2.2**, angles a and b are congruent.

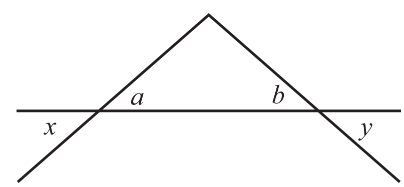


Figure 2.2

Suppose that we are asked to prove that angles x and y are also congruent. That is,

Given: $\angle a \cong \angle b$.

Prove: $\angle x \cong \angle y$.

Fill in the blanks of the following line of reasoning.

i. $\angle x \cong \angle a$ because ______________________________ .

ii. $\angle y \cong \angle b$ because ______________________________ .

iii. Because $\angle a \cong \angle b$, it follows that ________________________ .

Solution

i. Vertical angle theorem.

ii. Vertical angle theorem.

iii. $\angle x \cong \angle y$.

Example 3

Two angles are complementary if their measures sum to 90° and supplementary if their measures sum to 180°. If the complement of an angle is 15°, what is its supplement?

Solution

The angle in question has measure 75° (because $15 + 75 = 90$). Its supplement is thus $180° - 75° = 105°$.

Study Tip

- Over the last two millennia, a great deal of jargon has been developed and used in the study and practice of geometry, many more words and names than have been presented in this lesson. Be sure to understand the definitions of any new words you encounter.

Pitfall

- Diagrams in geometry need not be drawn accurately. Do not assume, for example, that an angle in a diagram is a right angle even if it appears to be one. Rely only on the information explicitly stated in a question or explicitly marked on a diagram.

Problems

1. In the diagram in **Figure 2.3**, $\overline{PQ}$ and $\overleftrightarrow{SR}$ intersect at O, which is also the midpoint of $\overline{PQ}$.

Is each of the following statements true or false?

a) $\overline{PO} \cong \overline{OQ}$.

b) P, S, and R are collinear.

c) P, R, and Q are coplanar.

d) $PO = QO$.

e) $\overline{PO}$ is congruent to $\overline{QO}$.

f) $PO + OQ = PQ$.

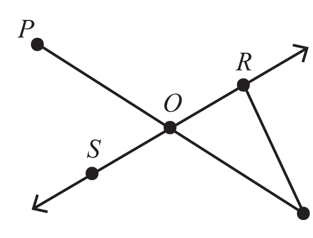


Figure 2.3

2. Which of the following statements are correct in their use of notation?

a) $AB = CD$.

b) $\overline{AB} = \overline{CD}$.

c) $AB \cong CD$.

d) $\overline{AB} \cong \overline{CD}$.

3. Draw two line segments $\overline{AB}$ and $\overline{CD}$ that

a) intersect, with neither segment bisecting the other.

b) intersect, with each bisecting the other.

c) intersect, with $\overline{AB}$ bisecting $\overline{CD}$, but not vice versa.

4. Consider the diagram in **Figure 2.4**.

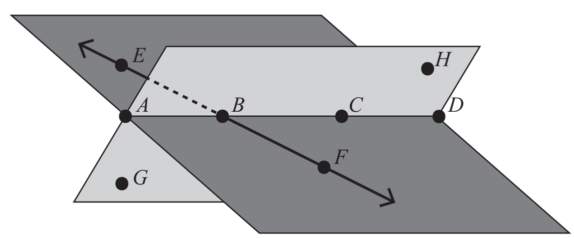


Figure 2.4

a) Are the points A, B, D, and F coplanar?

b) Are the points G, B, C, and H coplanar?

c) Are the points F, C, and H coplanar?

d) Are the points E, B, and F collinear?

e) A third plane contains the points A and D. Must it also contain the point B?

5. Points A, B, and C are collinear with $AB = 5$ and $AC = 9$. Give two possible values for BC.

6. If A, B, and M are points with $AM = MB$, does M need to be the midpoint of $\overline{AB}$?

7. In the 1790s, French scholars proposed a metric measure for angles: A right angle (a fundamental angle in architecture and engineering) was set to be assigned a measure of 100 "gradian."

a) How many degrees is 1 gradian?

b) How many gradian is 1°?

8. In the diagram in **Figure 2.5**, $\angle MNR$ has measure 98°. What is the value of x?

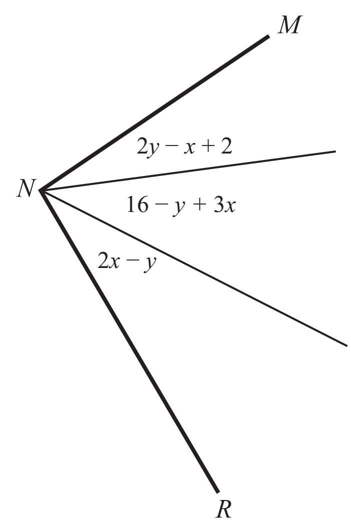


Figure 2.5

9. In the diagram in **Figure 2.6**, $m\angle ASC = 42°$, $m\angle FSO = 100°$, and $m\angle EST = 40°$.

Find the following.

a) $m\angle FSI$.

b) $m\angle FSU$.

c) $m\angle CSE$.

d) $m\angle TSI$.

e) $m\angle USC$.

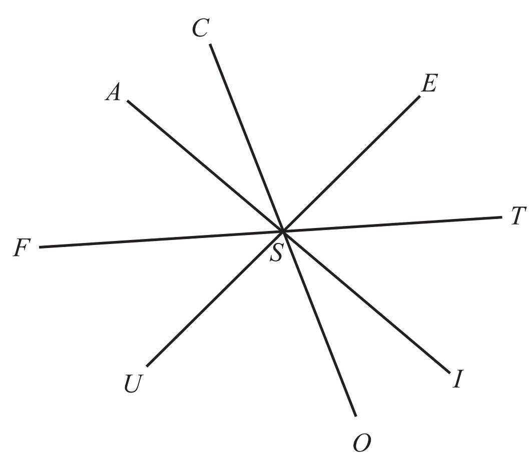


Figure 2.6

10. In the diagram in **Figure 2.7**, $\angle EUS$ is a right angle.

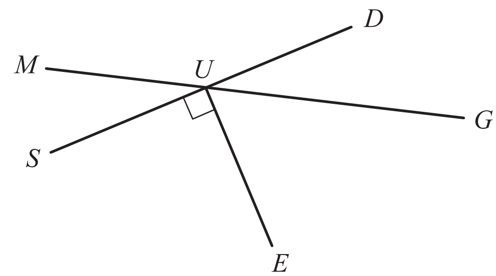


Figure 2.7

a) Name another right angle in the diagram.

b) Name an angle that must be acute.

c) Name an angle that must be obtuse.

d) Name a reflex angle.

e) Name two congruent acute angles.

f) Name two congruent obtuse angles.

g) Name two complementary angles.

h) Name two noncongruent supplementary angles.

i) Name two congruent supplementary angles.

Angles and Pencil-Turning Mysteries

Lesson 3

Topics

- The sum of the interior angles of a triangle.
- The sum of the interior angles and the sum of the exterior angles of N-sided figures.

Summary

We like to believe that the measures of the three interior angles of any given triangle sum to 180°. This is our first fundamental assumption of geometry. We also note that an N-sided figure subdivides into $N-2$ triangles with interior angles matching the interior angles of the original figure.

Example 1

Suppose that you want to compute the sum of the interior angles of a 12-sided figure.

a) Into how many triangles would you subdivide it?

b) What is the sum of the interior angles of a 12-sided polygon?

Solution

a) A 12-sided figure subdivides into 10 triangles with interior angles matching the interior angles of the figure.

b) The sum of the interior angles of the figure is thus $10 \times 180° = 1800°$.

Example 2

a) The three angles of an equilateral triangle have the same measure. Explain why each must have measure 60°.

b) Each interior angle in a square has the same measure. Explain why each must have measure 90°.

A figure composed of straight edges is regular if each edge has the same length and each interior angle of the figure has the same measure.

c) What is the measure of each angle in a regular pentagon (5-sided figure)?

d) What is the measure of each angle in a regular hexagon (6-sided figure)?

e) What is the measure of each angle in a regular heptagon (7-sided figure)?

Solution

a) If each interior angle has measure x, then $x + x + x = 3x = 180°$, forcing $x = 60°$.

b) A square subdivides into two triangles and therefore has interior angles summing to $2 \times 180° = 360°$. If each interior angle of the square has measure y, then $4y = 360°$, giving $y = 90°$.

c) A pentagon subdivides into three triangles and therefore has interior angles summing to $3 \times 180° = 540°$. If each interior angle of a regular pentagon has measure z, then $5z = 540°$, giving $z = 108°$.

d) A hexagon subdivides into four triangles and therefore has interior angles summing to $4 \times 180° = 720°$. If each interior angle of a regular hexagon has measure w, then $6w = 720°$, giving $w = 120°$.

e) In the same manner, each interior angle of a regular heptagon has measure $\frac{5\times180°}{7} \approx 128.57°$.

Study Tip

- There is very little need to memorize formulas. Clear understanding of ideas will allow you to quickly reconstruct formulas if they are needed.

Pitfall

- When dividing a figure into triangles in order to compute the sum of its interior angles, be sure to avoid erroneous interior angles. (See **Figure 3.1**.)

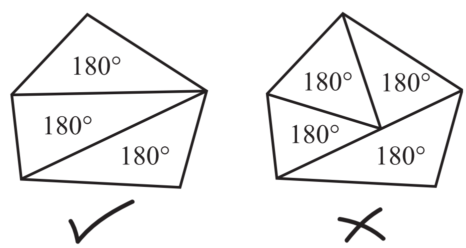


Figure 3.1

Problems

1. Find the measures of angles x and y in **Figure 3.2**.

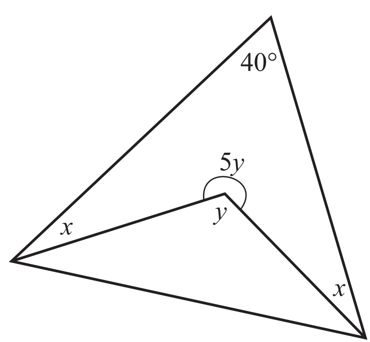


Figure 3.2

2. One interior angle of a triangle has measure 60° and another has measure less than 60°. What can you say about the measure of the third interior angle of the triangle?

3. Find the measure of *x* and *y* in each of the following.

a)

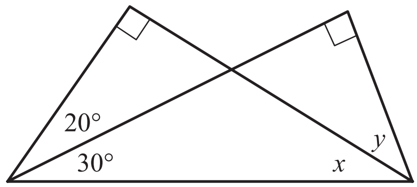


Figure 3.3

b)

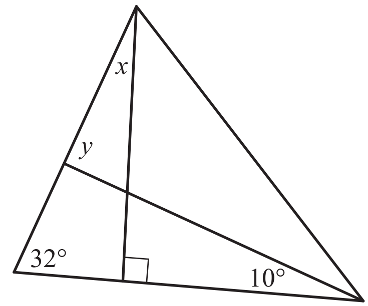


Figure 3.4

c)

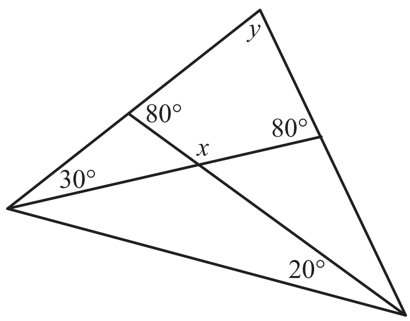


Figure 3.5

4. a) Is it possible, in flat geometry, for a triangle to have an obtuse angle? If so, sketch such a triangle. If not, why not?

b) Is it possible, in flat geometry, for a triangle to have two obtuse angles? If so, sketch such a triangle. If not, why not?

c) Is it possible, in flat geometry, for a triangle to have three obtuse angles? If so, sketch such a triangle. If not, why not?

5. Fill in the blanks of the following proof. (See **Figure 3.6**.)

Given: $m\angle IRY = m\angle IMY$.

Prove: $\angle G \cong \angle L$.

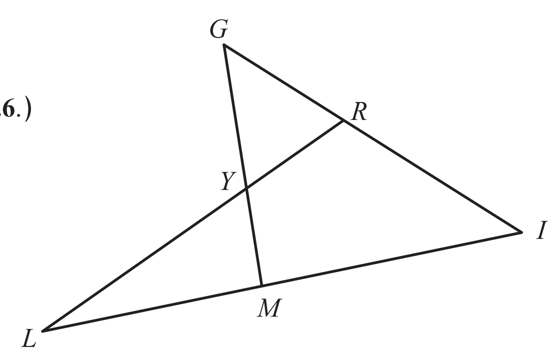


Figure 3.6

Proof

i. Label the given congruent angles x, and label the angle w as shown in **Figure 3.7**.

ii. $m\angle G + x + w = 180°$ because the angles in the triangle with vertices ____________ sum to 180°.

iii. $m\angle L + x + w = 180°$ because the angles in the triangle with vertices ____________ sum to 180°.

iv. Thus, $m\angle G =$ ____________.
$m\angle L =$ ____________.

v. So, $\angle G \cong \angle L$.

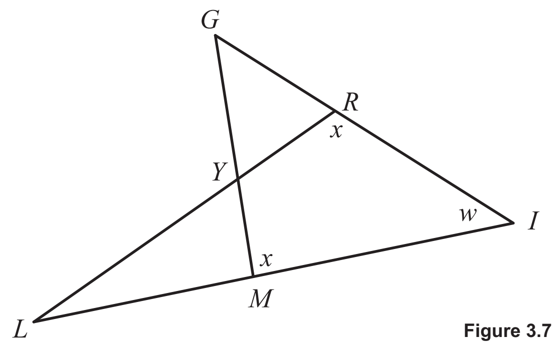


Figure 3.7

6. Consider the diagram in **Figure 3.8**.

Suppose that $m\angle W = 15°$.

Find the measure of each and every interior angle in the diagram.

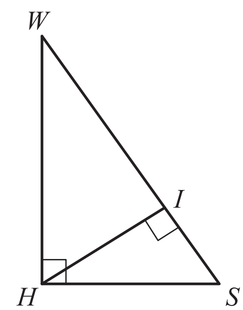


Figure 3.8

7. $\overline{LM}$ bisects $\angle GLU$. (Can you deduce what it means for an angle to be bisected by a line segment?)

If $m\angle G = 10°$ and $m\angle U = 30°$, what is $m\angle MLU$?
(See **Figure 3.9**.)

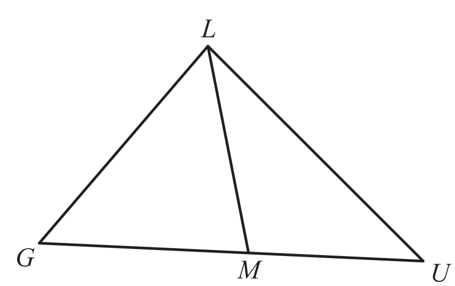


Figure 3.9

8. Fill in the blanks of the following proof. (See **Figure 3.10**.)

Given: x and y are complementary.
$\overline{OR} \perp \overline{OS}$.

Prove: $\angle x \cong \angle z$.

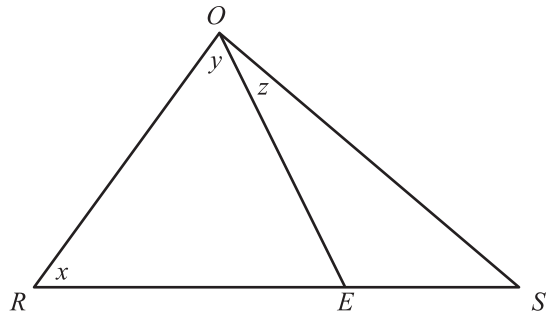


Figure 3.10

Proof

i. $x + y =$ _____ because of the definition of "complementary."

ii. $y + z =$ _____ because _________________________ .

iii. So, ____ .

9. Find the measure of x. (See **Figure 3.11**.)

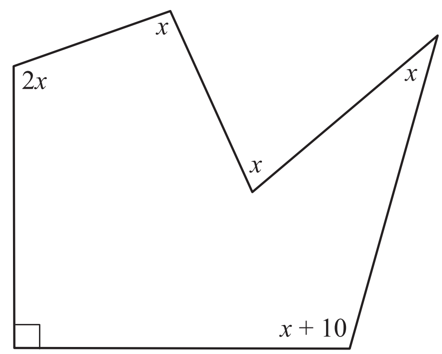


Figure 3.11

10. Despite being drawn on the page, establish that the following figure does not actually exist (in flat geometry, at least). (See **Figure 3.12**.)

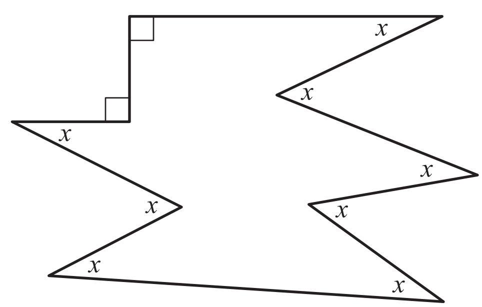


Figure 3.12

Lesson 3: Angles and Pencil-Turning Mysteries

Understanding Polygons

Lesson 4

Topics

- Definition and names of polygons.
- Regular polygons.
- Angles in regular polygons.
- Tiling.

Definitions

- **concave polygon**: A polygon that is not convex.
- **convex polygon**: A polygon with the property that for any two points A and B in the interior of the polygon, the line segment $\overline{AB}$ also wholly lies in the interior of the polygon.
- **equiangular polygon**: A polygon with interior angles all of the same measure.
- **equilateral polygon**: A polygon with all edges the same length.
- **exterior angle of a polygon**: In extending one edge of a polygon, the angle formed by that extension and the next side of the polygon.
- ***N*-gon**: A polygon with N sides.
- **polygon**: A planar figure composed of straight line segments (called sides or edges) so that
    - sides intersect only at their endpoints (the endpoints are called the vertices of the polygon);
    - precisely two sides meet at each endpoint; and
    - two sides meeting at a vertex make an angle different from 180°.
- **regular polygon**: A polygon that is both equilateral and equiangular.
- **rhombus**: An quadrilateral with four congruent sides.

Formulas

- The measure of one exterior angle of a regular N-gon is $\frac{360°}{N}$.
- The measures of an exterior angle and an interior angle of a regular polygon sum to 180°.
- A polygon with N sides subdivides into $N - 2$ triangles with angles matching the interior angles of the polygon. (Thus, the sum of the interior angles of an N-gon is $(N - 2) \times 180°$.

Summary

In this lesson, we prove, in flat geometry, that the exterior angles of a convex polygon are sure to sum to 360°, irrespective of the number of sides that the polygon has. This provides a simple means to compute the measures of the exterior and interior angles of regular polygons. This has applications to tiling the plane with regular polygons.

Example 1

Find the measure of the interior angle of a regular pentagon.

Solution

The five exterior angles of the regular pentagon sum to 360° and have the same measure.

Thus, each exterior angle has measure $\frac{360°}{5} = 72°$. Consequently, each interior angle has measure 108°.

Example 2

A regular polygon has interior angle 359°. How many sides does the polygon have?

Solution

Each exterior angle of the regular polygon has measure 1°.

If N is the number of sides of the polygon, then $\frac{360°}{N} = 1°,$ so $N = 360.$

Example 3

ABCDEFGHIJKL is a regular dodecagon. Segments $\overline{AB}$ and $\overline{CD}$ are extended to meet at a point *M*. Find the measure of $\angle BMC$.

Solution

Each exterior angle is $\frac{360}{12} = 30°$.

Using the fact that there are 180° in a triangle, we see that $\angle BMC = 120°$. (See **Figure 4.1**.)

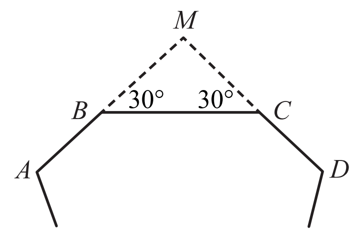


Figure 4.1

Study Tips

- Remember that the interior angle and the exterior angle of a polygon at each vertex lie on a straight line and therefore have measures that sum to 180°. Once you know the measure of an exterior angle, the measure of the interior angle follows.

- Try not to focus on the formula: The sum of the interior angles of an N-gon is $(N-2) \times 180°$. If in a problem it is possible to focus on the exterior angles of the polygon first, then do so!

Pitfalls

- Polygons are labeled by listing the names of the vertices of the polygon in a consistent clockwise or counterclockwise order. When drawing a diagram of a labeled polygon, be sure to label your diagram in a consistent direction, too.

- Remember that the exterior angle of a polygon is the angle between one side of the polygon, extended, and the next side of the polygon, so it is sure to have measure less than 180°.

Problems

1. Find the exterior angle and the interior angle of the following.

a) A regular hexagon.

b) A regular octagon.

c) A regular heptagon.

d) A regular triangle. (What is this shape normally called?)

2. The exterior angle of a regular polygon is $\frac{1}{5}$ the measure of an interior angle. How many sides does this polygon have?

3. A polygon has 1800 sides. What is the sum of its exterior angles?

4. Complete the following table for regular polygons.

Number of sides	6	15	20			
Measure of an exterior angle				10		
Measure of an interior angle					179	90

5. Is it possible for a regular polygon to have an interior angle of measure 153°? Explain.

6. a) Draw an example of an octagon that is equilateral but not equiangular.

b) Draw an example of an octagon that is equiangular but not equilateral.

7. A pentagon contains three right angles, and the remaining two interior angles are congruent. What are the measures of those angles?

8. a) Does there exist a regular heptagon with one interior angle of measure 115°?

b) Does there exist a heptagon with one interior angle of measure 115°?

9. The measure of one interior angle of a regular polygon and the measure of one exterior angle come in a 7:2 ratio. How many sides does the polygon have?

Hint: Here, "7:2 ratio" means that the measure of the interior angle is $7x$, and the measure of the exterior angle is $2x$ for some unknown number x. Start by finding what x must be.

10. The sum of the interior angles of a polygon is known to be between 3400° and 3500°. How many sides does the polygon have?

The Pythagorean Theorem
Lesson 5

Topics

- The Pythagorean theorem.
- Shortest distances.
- The triangular inequality.

Definition

- **hypotenuse**: The side opposite the right angle in a right triangle.

Results

- Pythagorean theorem: If squares are drawn in the sides of a right triangle, then the areas of the two smaller squares sum to the area of the large square. (As a statement of algebra: For a right triangle with sides of length p, q, and r, where r is the length of the side opposite the right angle, $p^2 + q^2 = r^2$.)
- triangular inequality: In a triangle, the sum of any two side lengths of the triangle is strictly greater than the length of the remaining side.

Summary

In this lesson, we introduce the famous Pythagorean theorem and develop a possible proof of it. Uncertainty about the validity of the proof leads us to accept the Pythagorean theorem as a second fundamental assumption of geometry. As logical consequences of the theorem, we demonstrate that straight paths represent the shortest routes between two points and prove the triangular inequality.

Example 1

Find the length of the line segment $\overline{AB}$, given as the interior diagonal of the rectangular box shown in **Figure 5.1**.

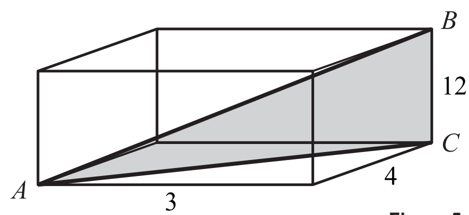


Figure 5.1

Solution

The length AC is the hypotenuse of a right triangle with legs 3 and 4. Thus, $AC^2 = 3^2 + 4^2 = 25$, giving $AC = 5$. The length AB, the length we seek, is the hypotenuse of a right triangle with legs $AC = 5$ and 12. Thus, $AB^2 = 5^2 + 12^2 = 169$, giving $AB = 13$.

Example 2

What is the missing length in the triangle that is 2 inches high?

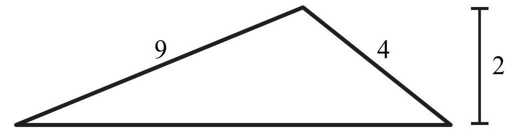


Figure 5.2

(Assume that the height of the triangle is measured from the base of the triangle to the apex of the triangle at an angle of 90°. See **Figure 5.2**.)

Warning: This triangle is *not* a right triangle!

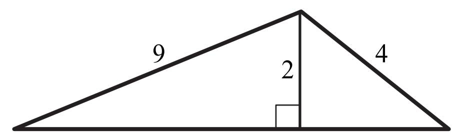


Figure 5.3

Solution

Using Pythagoras's theorem, twice, we see that the missing side length is $\sqrt{81-4}+\sqrt{16-2}=\sqrt{77}+\sqrt{12}$. (This does not equal $\sqrt{89}$. See **Figure 5.3**.)

Example 3

Which location for the point P inside a square gives the smallest value for the sum of the distances $a + b + c + d$ shown? Why? (See **Figure 5.4**.)

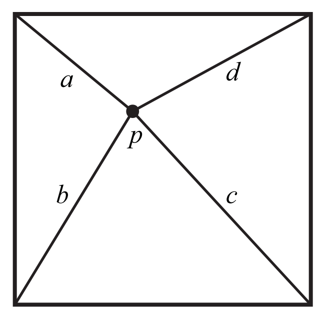


Figure 5.4

Solution

Placing the point P at the center of the square gives a sum $a + b + c + d$ that is smallest. (See **Figure 5.5**.) Here, the sum equals twice the length of the diagonal of the square.

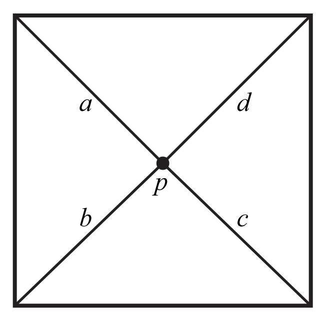


Figure 5.5

Placing P at any other location gives a sum that is greater than this.

To see why, suppose that P is placed off-center. (See **Figure 5.6**.)

The triangular inequality shows that the sum $b + d$ is longer than the diagonal of the square.

In the same way, the sum $a + c$ is longer than the diagonal of the square. Consequently, $a + b + c + d$ is longer than the two diagonals of the square.

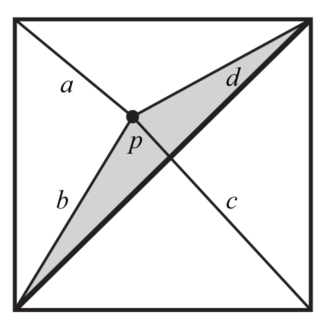


Figure 5.6

Study Tip

- Don't fixate on the formula $a^2 + b^2 = c^2$ with the specific letters a, b, and c. Remember that the Pythagorean theorem is a statement about the areas of the squares. And with the picture in mind, how the sides of those squares are named becomes immaterial.

Pitfalls

- Many students are tempted to believe that $\sqrt{a}+\sqrt{b}$ equals $\sqrt{a+b}$. But this is usually never the case. For example, $\sqrt{9}+\sqrt{16}$ (which equals 7) does not equal $\sqrt{25}$ (which equals 5).

- When a quantity such as $5x$ is squared, the result is $25x^2$. Watch out for this if $5x$ happens to be the side length of a right triangle in an application of the Pythagorean theorem.

Problems

1. Find the missing side lengths in the following right triangles.

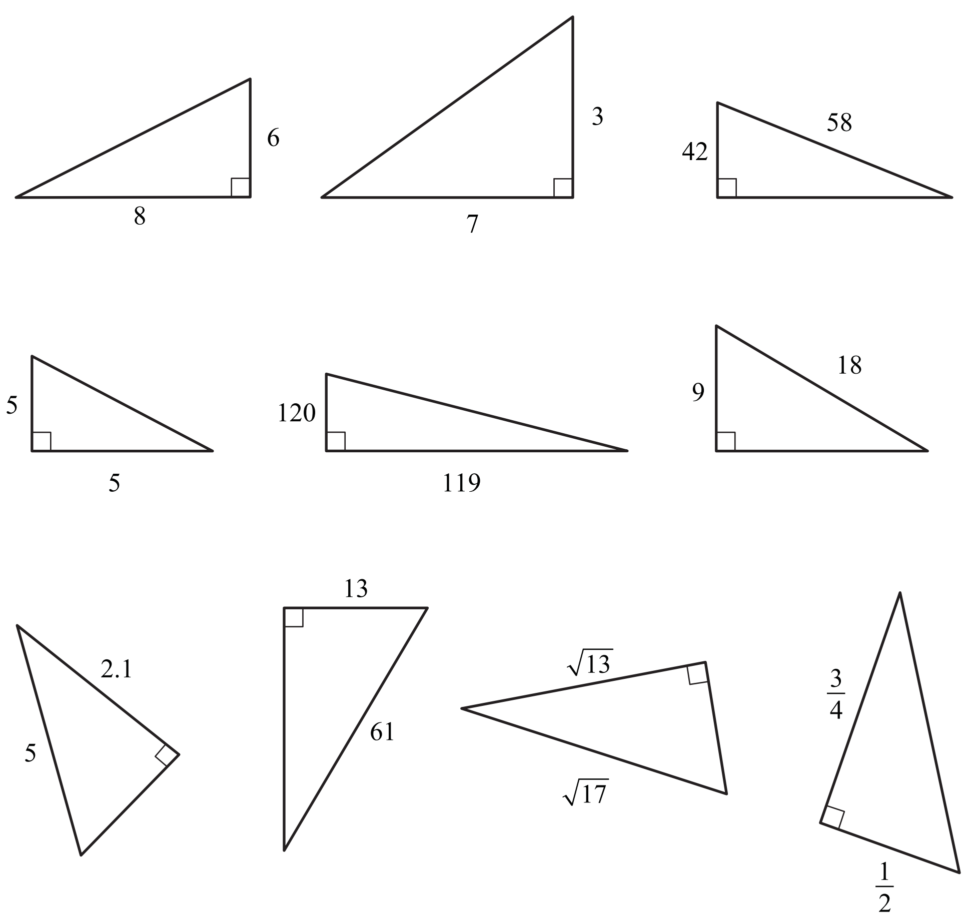


Figure 5.7

2. A rectangle has a diagonal of length 2 and a side of length $\frac{6}{5}$. What is its remaining side length?

3. A TV screen is a rectangle with sides that come in a 4:3 ratio. If the diagonal of the screen is 21 inches long, what are the length and the width of the screen?

4. Consider the picture in **Figure 5.8**.

a) Use the Pythagorean theorem to explain why we must have $a > 3$.

b) Use the Pythagorean theorem to explain why we must have $b > 5$.

c) Use the Pythagorean theorem to explain why we must have $b > a$.

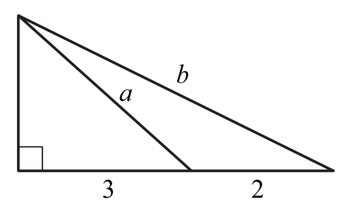


Figure 5.8

5. What is the length of the diagonal from the top-front-right corner P of a cube with side length 1 to the back-bottom-left corner Q? (See **Figure 5.9**.)

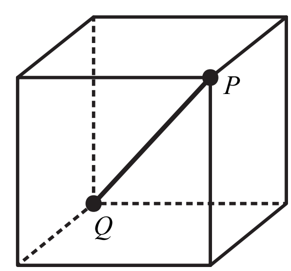


Figure 5.9

6. Find the value of h (See **Figure 5.10**.).

Hint: Set $AP = x$ and $PC = 21 - x$. Write two equations, and use them to solve for x and then for h.

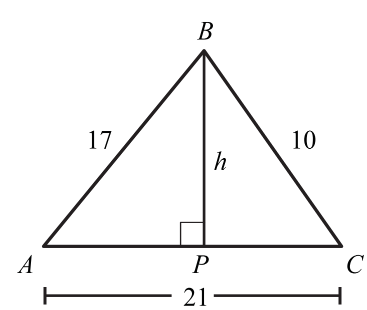


Figure 5.10

7. a) Draw a triangle with sides 6 inches, 3 inches, and 2 inches.

b) Draw a triangle with sides 67 inches, 23 inches, and 95 inches.

c) Is it possible to draw a triangle with one side 400 times longer than another?

8. The perimeter of a triangle is the sum of its three side lengths. Is the following statement true or false? It is possible to draw a triangle with one side longer than half the perimeter of the triangle.

9. Suppose that three points A, B, and C satisfy $AB + BC = AC$. Explain why the three points must be collinear.

10. A rectangular box has length a inches, width b inches, and height c inches.

Show that the length of the longest diagonal in the box, $\overline{PQ}$, is given by $PQ = \sqrt{a^2 + b^2 + c^2}$. (See **Figure 5.11**.)

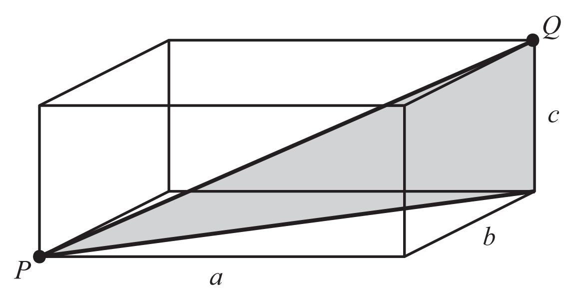


Figure 5.11

Distance, Midpoints, and Folding Ties

Lesson 6

Topics

- The coordinate plane.
- The distance formula.
- The midpoint formula.
- Applications of the midpoint formula.

Definitions

- **Cartesian coordinates**: When the plane is endowed with a pair of vertical and horizontal axes, the location of a point in the plane is specified by a pair of numbers called its Cartesian coordinates.
- **Cartesian plane/coordinate plane**: A plane for which a pair of vertical and horizontal axes have been assigned.
- **origin**: The point in the Cartesian plane with coordinates $(0, 0)$.

Formulas

- distance formula: The distance between two points in the plane that have been assigned coordinates is

$$\text{distance} = \sqrt{(\text{difference in } x\text{-values})^2 + (\text{difference in } y\text{-values})^2}.$$

- midpoint formula: The midpoint of the line segment connecting $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ is

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

Summary

Allegedly after watching a fly move across his bedroom ceiling, 17th-century French philosopher and mathematician René Descartes invented the concept of a coordinate system. By assigning coordinates to points in the plane, we can bring the power of algebra into geometry. In this lesson, we apply the Pythagorean theorem to find formulas for the distance between two points and the coordinates of the midpoint of the line segment that connects two points.

Example 1

Let $A = (3, 4)$, $B = (5, 10)$, and $R = (4, 7)$.

a) Show that $AR = RB$.

b) Suppose that $S = (1, k)$ is another point equidistant from A and B (that is, satisfies $AS = SB$). What is the value of k?

Solution

a) We have $AR = \sqrt{1^2 + 3^2}$ and $BR = \sqrt{1^2 + 3^2}$. These are the same value.

b) We need $\sqrt{2^2 + (k-4)^2} = \sqrt{4^2 + (k-10)^2}$.

Squaring each side and following through with the algebra gives the following.

$$\begin{aligned} 4 + (k-4)^2 &= 16 + (k-10)^2 \\ 4 + k^2 - 8k + 16 &= 16 + k^2 - 20k + 100 \\ 4 - 8k &= 100 - 20k \\ 12k &= 96 \\ k &= 8. \end{aligned}$$

Example 2

Find the midpoint M of the line segment connecting $A = (-1, 3)$ to $B = (2, 8)$.

Show that AM equals MB to verify that M is indeed equidistant from A and B.

Also, verify that $AM + MB$ equals AB to show that M does indeed lie on $\overline{AB}$.

Solution

The midpoint is $M = \left(\frac{-1+2}{2}, \frac{3+8}{2}\right) = \left(\frac{1}{2}, \frac{11}{2}\right)$.

The distance between $\left(\frac{1}{2}, \frac{11}{2}\right)$ and $(-1, 3)$ is

$$AM = \sqrt{\left(-1-\frac{1}{2}\right)^2 + \left(3-\frac{11}{2}\right)^2} = \sqrt{\left(\frac{-3}{2}\right)^2 + \left(\frac{-5}{2}\right)^2} = \sqrt{\frac{9}{4}+\frac{25}{4}} = \sqrt{\frac{34}{4}} = \frac{\sqrt{34}}{2}.$$

The distance between $\left(\frac{1}{2},\frac{11}{2}\right)$ and $(2, 8)$ is

$$MB = \sqrt{\left(2-\frac{1}{2}\right)^2 + \left(8-\frac{11}{2}\right)^2} = \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{5}{2}\right)^2} = \sqrt{\frac{34}{4}} = \frac{\sqrt{34}}{2}.$$

These distances are indeed the same.

Also, $AB = \sqrt{3^2+5^2} = \sqrt{34}$, so $AM + MB$ does equal AB.

Example 3

Suppose that $P = (2k, 3m)$ and $R = (5k, -m)$. Find the coordinates of a point S so that R is the midpoint of $\overline{PS}$.

Solution

We need a point $S = (a, b)$ with

$5k$ the middle value between $2k$ and a

and

$-m$ the middle value between $3m$ and b.

Can you see that $a = 8k$ does the trick for a? (It's another "$3k$ up" from $5k$.)

Can you see that $b = -5m$ works? (It's another "$4m$ down" from $-m$.)

We have $S = (8k, -5m)$.

Study Tips

- Remember that the distance formula is just the Pythagorean theorem. Work to see the picture of a right triangle in the coordinate plane in your mind (or on paper!) rather than memorize a formula.
- Keep in mind that the midpoint of a line segment $\overline{AB}$ has x-coordinate the middle value of the x-coordinates of A and B and y-coordinate the middle value of the y-coordinates of A and B. (This feels natural and right and, moreover, is mathematically correct!)

Pitfall

- Students who attempt to memorize the distance formula for the sake of memorizing it often jumble the placement of the + and − signs in the formula. Really do draw the picture of a right triangle and think of the formula as nothing more than an application of the Pythagorean theorem.

Problems

1. Find the distances between the following pairs of points. (Leave your answers in terms of square roots.)

a) $(11, 0)$ and $(5, 9)$.

b) $(4, 7)$ and $(7, 4)$.

c) $(0, 0)$ and $(-2, -5)$.

d) $(4, 400)$ and $(-3, 424)$.

e) $(-2, 11)$ and $(6, 15)$.

2. Find the coordinates of the midpoint of $\overline{PQ}$ if

a) $P = (5, 6)$ and $Q = (11, 4)$.

b) $P = (-3, 17)$ and $Q = (15, 37)$.

c) $P = (-12, 6)$ and $Q = (0, 13)$.

d) $P = (a, 0)$ and $Q = (-a, b)$.

3. a) $A = (a, 7)$ and $M = (2a, 2)$, and M is the midpoint of $\overline{AB}$. What are the coordinates of B?

b) Find a value for x so that $M = (2x + 2, 9)$ is the midpoint of $\overline{PQ}$ with $P = (x, x + 3)$ and $Q = (3x + 4, 2x)$.

4. Triangle ABC has vertices $A = (-1, 2)$, $B = (1, 6)$, and $C = (3, 0)$. Find the coordinates of the midpoint M of $\overline{BC}$ and the length of the line segment $\overline{MA}$.

Comment: A line connecting one vertex of a triangle to the midpoint of the opposite side is called a median of the triangle.

5. A point P is equidistant from two points A and B if $AP = PB$.

Let $A = (-2, 3)$ and $B = (2, -1)$.

a) Show that the point $P = (4, 5)$ is equidistant from A and B.

b) Is the point $Q = (-4, -5)$ equidistant from A and B?

c) Is the point $R = (-10, -9)$ equidistant from A and B?

d) Show that every point of the form $(k, k + 1)$ is equidistant from A and B.

6. Find a point with y-coordinate 7 that is 5 units away from $(2, 3)$.

Comment: There are two possible answers. Can you find them both?

7. a) Find an equation that must be true for the numbers x and y for the point $A = (x, y)$ to a distance of 6 units from $B = (2, 9)$.

b) When Jinny answered this question, she came up with the formula $(x - 2)^2 + (y - 9)^2 = 36$. Is she correct?

8. Let $F = (3, 4)$ and $G = (13, -6)$.

a) Find the coordinates of the point halfway along $\overline{FG}$. (That is, find the midpoint of $\overline{FG}$.)

b) Find the coordinates of the points that are $\frac{1}{4}$ and $\frac{3}{4}$ along $\overline{FG}$.

9. A triangle is isosceles if at least two of its sides are the same length.

Let $A = (0, 2)$, $B = (12, 10)$, and $C = (4, -2)$.

a) Use the distance formula to show that the triangle formed by the points A, B, and C is isosceles.

b) Compute the midpoints of the three sides of this triangle.

c) Is the triangle formed by the midpoints also isosceles?

10. Is there a point with the same x- and y-coordinates that is a distance 10 from the point $(3, 5)$? If so, what is it? (And is there more than one point with this property?)

The Nature of Parallelism
Lesson 7

Topics

- Parallel lines.
- Euclid's parallel postulate.
- Eratosthenes' measurement of the circumference of the Earth.

Definitions

- **alternate interior angles**: $\overline{PQ}$ a pair of lines and a transversal, a pair of nonadjacent angles sitting between the pair of lines and positioned on opposite sides of a transversal are alternate interior angles.
- **parallel**: Two lines, each infinite in extent, are parallel if they never meet.
- **same-side interior angles**: Given a pair of lines and a transversal, a pair of nonadjacent angles sitting between the pair of lines and positioned on the same side of a transversal are same-side interior angles.
- **transversal**: A line that crosses a pair of lines.

Results

In the diagrams in **Figure 7.1**,

- If the two lines are parallel, then $x + y = 180°$ (we have same-side interior angles summing to 180°), and conversely, if $x + y = 180°$, then the two lines are parallel.

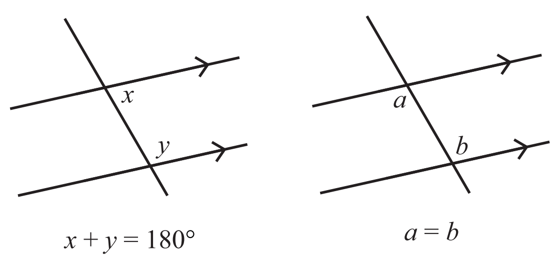


Figure 7.1

- If the two lines are parallel, then $a = b$ (we have congruent alternate interior angles), and conversely, if $a = b$, then the lines are parallel.

Notation

- The notation $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ denotes that lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are parallel.
- In diagrams, parallel lines are marked with arrows.

Summary

The concept of lines being parallel is “beyond human.” It is physically impossible in a human lifespan to check whether or not two lines, each infinite in extent, intersect. In this lesson, we describe Euclid’s clever approach to turn parallelism into a humanly accessible topic. We describe his alternative fundamental assumption of geometry—the parallel postulate—and discuss its equivalence to our fundamental assumption on angles in triangles. We also explore some applications of parallelism, including Eratosthenes’s measurement of the circumference of the Earth.

Example 1

For each quadrilateral, a pair of sides of the quadrilateral must be parallel.

Identify this pair of sides in each of the following.

a)

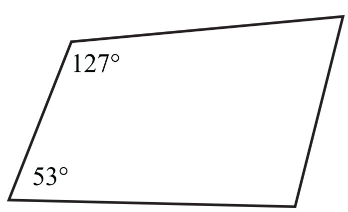


Figure 7.2

b)

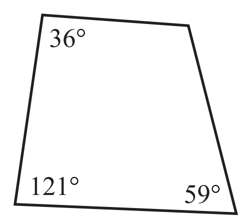


Figure 7.3

Hint: Look for a transversal across a pair of lines. (It is permissible to extend lines in a diagram if desired.)

Solution

a) The top and bottom edges are parallel because 53 + 127 = 180. (Think of the left side of the figure as a transversal for the top and bottom sides of the figure.)

b) The left and right sides are parallel because 121 + 59 = 180. (Think of the bottom edge as a transversal for the left and right sides.)

Example 2

The diagram in **Figure 7.4** is a line segment and a point not on the segment.

Using only a ruler, a protractor, and a pencil, construct or explain how you could construct a line through the point parallel to the given line segment.

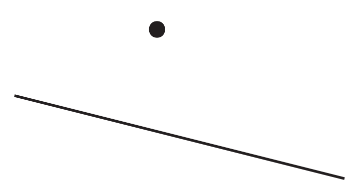


Figure 7.4

Solution

Draw any line through the point intersecting the line segment, and measure the angle a (shown in **Figure 7.5**) it happens to make.

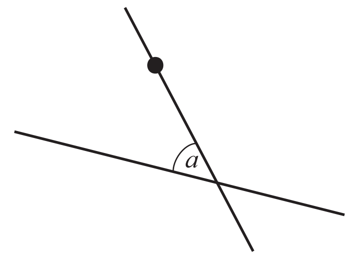


Figure 7.5

With a protractor, draw another line through the point, making the same angle a. (See **Figure 7.6**.)

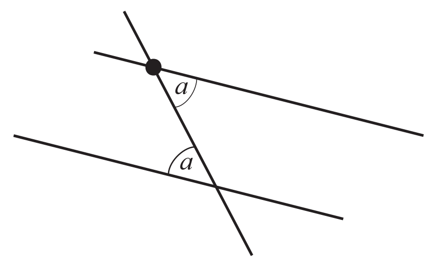


Figure 7.6

Because we have congruent alternate interior angles, we have parallel lines.

Study Tip

- It never hurts to draw over parts of complicated diagrams with colored pencils. If you highlight a pair of lines and a transversal for them, it can be much easier to identify alternate interior angles and same-side interior angles.

Pitfall

- Remember that in a pair of alternate interior angles, each angle "touches" the same transversal, but one angle touches one parallel line, and the other angle touches the other parallel line. For example, in the diagram in **Figure 7.7**, the alternate interior angle to d is $x + w$.

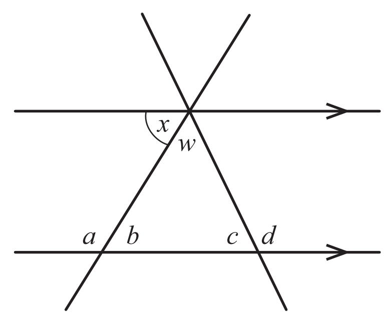


Figure 7.7

Problems

1. Angle AIB has measure 32°. Angle DJK has measure 92°.

Find the measures of angles AIK, GKF, and GKI.

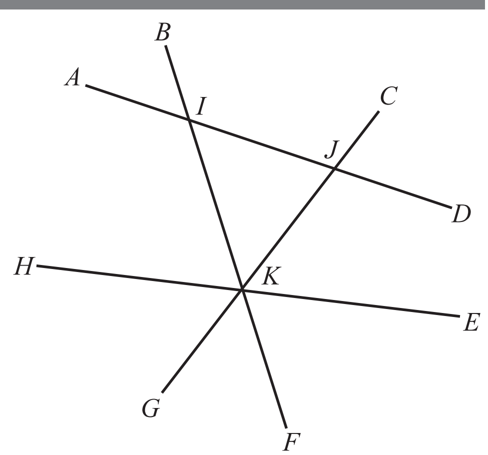


Figure 7.8

2. With a colored pencil, in each of the following diagrams, outline the one pair of parallel lines and the one transversal that contain the angle marked *x*.

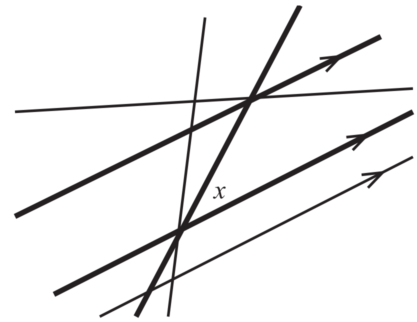


Figure 7.9

Also mark the congruent alternate interior angle to *x*.

(Example shown in **Figure 7.9**.)

Exercises:

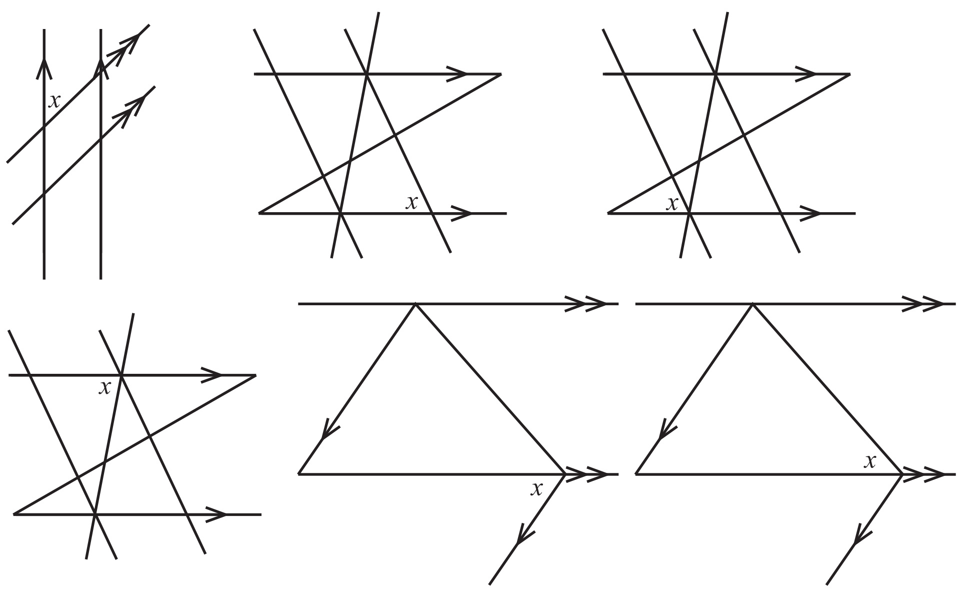


Figure 7.10

3. Find the value of *x* in each of the following.

a)

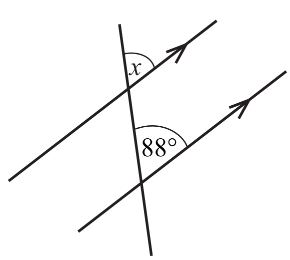


Figure 7.11

b)

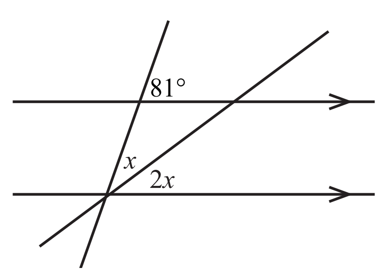


Figure 7.12

c)

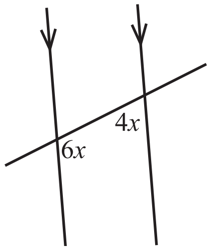


Figure 7.13

d)

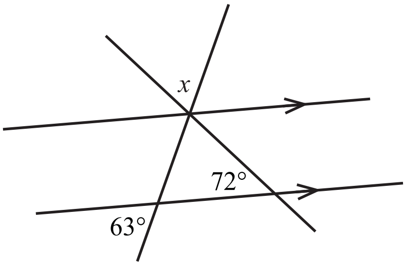


Figure 7.14

4. Find the value of w in each of the following diagrams.

a)

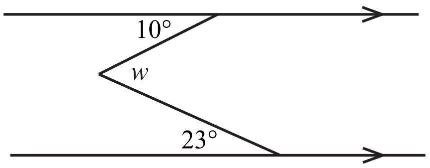


Figure 7.15

b)

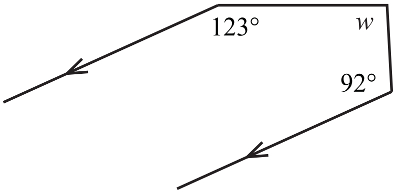


Figure 7.16

Hint: Draw a third parallel line in each diagram, and look for congruent alternate interior angles or supplementary same-side interior angles.

5. Find the values of x and y in each of the following.

a)

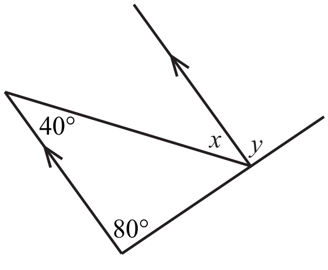


Figure 7.17

b)

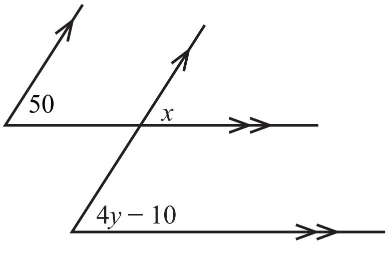


Figure 7.18

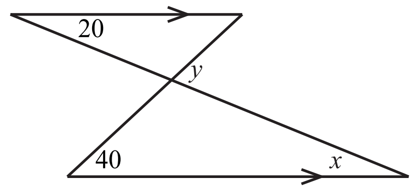
c) 

Figure 7.19

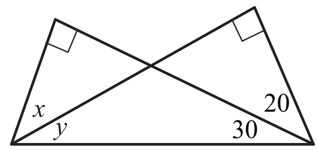
d) 

Figure 7.20

6. $PQRST$ is a pentagon. $m\angle P = 160°$, $m\angle T = 140°$, $\angle Q \cong \angle R$.

The measure of $\angle S$ is double the measure of $\angle R$.

Which two sides of $PQRST$ must be parallel?

7. Consider the diagram in **Figure 7.21**.

a) Explain why $\overline{GB} \parallel \overline{LO}$.

b) Is there another pair of parallel lines in the diagram?

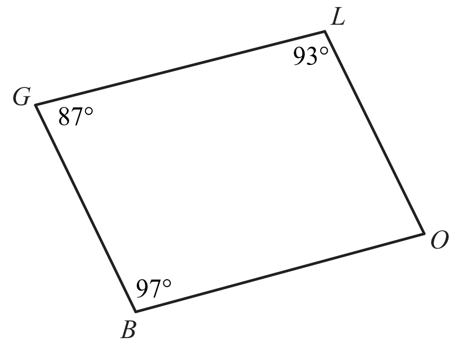


Figure 7.21

8. Consider the diagram in **Figure 7.22**.

Which lines, if any, would have to be parallel if

a) $\angle z \cong \angle r$.

b) $\angle y \cong \angle q$.

c) $m\angle y + m\angle x + m\angle r = 180°$.

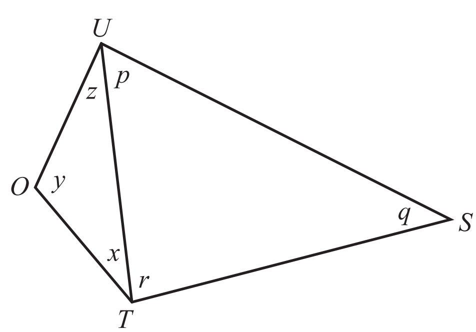


Figure 7.22

9. Solve for x, y, and z in **Figure 7.23**.

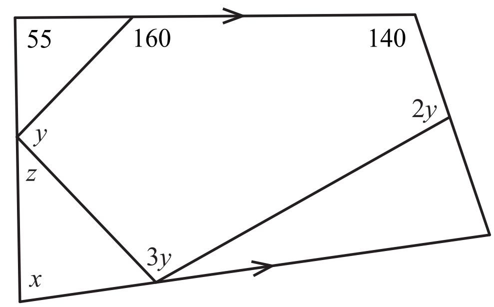


Figure 7.23

10. Fill in the missing details of the following proof. (See **Figure 7.24**.)

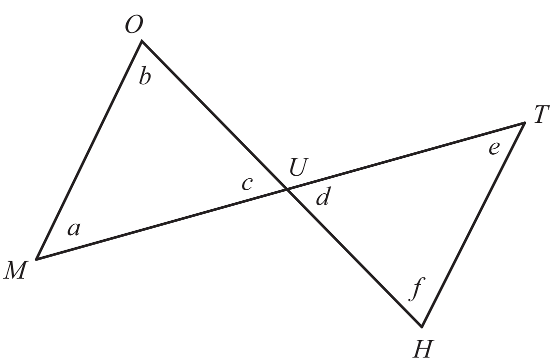


Figure 7.24

Given: $\angle a \cong \angle c$.

$\angle e \cong \angle d$.

Prove: $\overline{MO} \parallel \overline{TH}$.

Proof

i. $\angle c \cong \angle d$ because they are

____ .

ii. So, with this and the givens, we have

$\angle a \cong \angle c, \angle c \cong \angle d, \angle d \cong \angle e,$

from which we deduce $\angle a \cong \angle e$.

iii. Thus, $\overline{MO} \parallel \overline{TH}$ because ____ .

Proofs and Proof Writing

Lesson 8

Topic

- Proof writing in geometry.

Definitions

- **deductive reasoning**: The process of establishing the validity of a result by logical reasoning.

- **inductive reasoning**: The process of finding patterns and making conjectures based on those patterns.

Summary

The task of writing a mathematical proof is akin to the challenge of presenting a convincing case to a jury. One must use pristine knowledge, nothing more, based on immutable facts, nothing more, to present an argument that cannot be doubted. In this lesson, we learn the art of writing proofs in geometry.

Example 1

Given: $\angle AEC$ is a right angle.
$\overline{BD} \perp \overline{EC}$.

Prove: $\angle x \cong \angle y$.

(See **Figure 8.1**.)

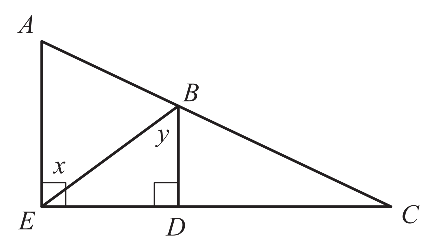


Figure 8.1

Solution

i. $\angle AEC$ is a right angle (given).

ii. $\angle BDE$ is a right angle because $\overline{BD} \perp \overline{EC}$.

iii. $\overline{AE} \mid BD$ because $\angle AEC$ and $\angle BDE$ are two same-side interior angles adding to 180°.

iv. $\angle x \cong \angle y$ because they are alternate interior angles for parallel lines.

Example 2

Given: $\overline{BD}$ bisects $\angle EBC$.

$\angle x \cong \angle z$.

Prove: $\overline{AE} \parallel \overline{BD}$.

(See **Figure 8.2**.)

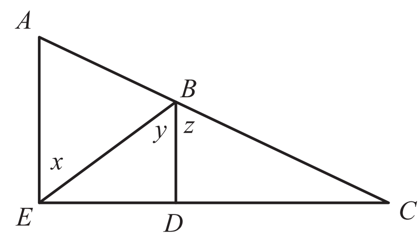


Figure 8.2

Solution

i. $\angle y \cong \angle z$ because $\overline{BD}$ bisects $\angle EBC$.

ii. $\angle x \cong \angle y$ because we have $\angle x \cong \angle z$ (given) and $\angle y \cong \angle z$.

iii. $\overline{AE} \parallel \overline{BD}$ because we have congruent alternate interior angles.

Study Tip

- Read out loud any proof you write. That way, you will be able to assess two things:
    - Does what you write "flow" well?
    - Does what you put down on paper actually say what you have in your mind?

Pitfall

- In learning to write proofs, students often make statements without justification. Use the word "because" (or some equivalent version of it) in almost every sentence you write, because this will force you to explain why each claim you make must be true.

Problems

1. Given: $\angle a \cong \angle b$.
$\angle d \cong \angle c$.

Prove: $\overline{AB} \parallel \overline{DE}$.

(See **Figure 8.3**.)

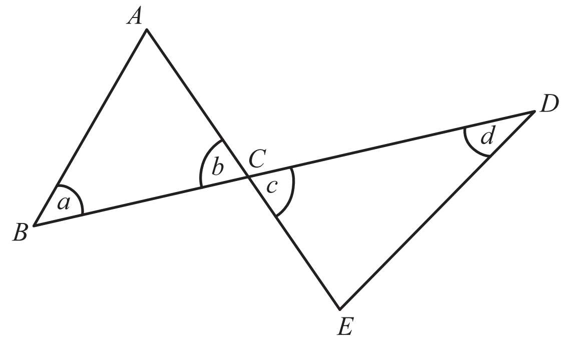


Figure 8.3

2. Given: $\overline{PS} \perp \overline{SR}$.
$\angle a$ and $\angle c$ are complementary.

Prove: $\angle b \cong \angle c$.

(See **Figure 8.4**.)

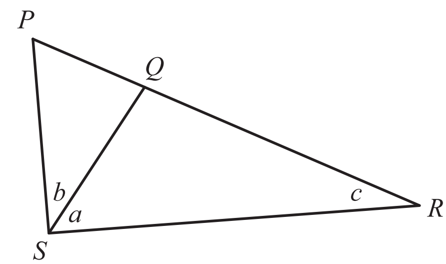


Figure 8.4

3. Given: $\overline{AB} \parallel \overline{CD}$.
$\overline{AD} \parallel \overline{BC}$.

Prove: $\angle A \cong \angle C$.

(See **Figure 8.5**.)

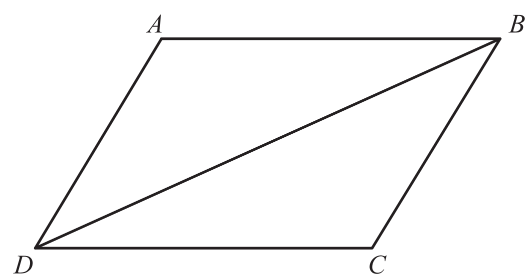


Figure 8.5

4. Given: $\overline{UF} \perp \overline{UM}$.

Prove: $FM > LM$.

(See **Figure 8.6**.)

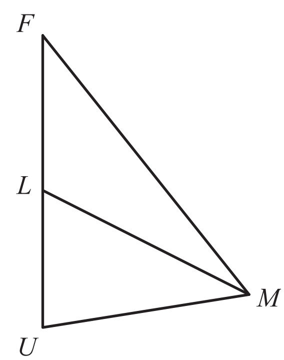


Figure 8.6

5. Given: $\angle R \cong \angle RUY$.
$\overline{RU} \parallel \overline{YB}$.
$\overline{UB} \perp \overline{YB}$.

Prove: $\angle YUB \cong \angle RBU$.

(See **Figure 8.7**.)

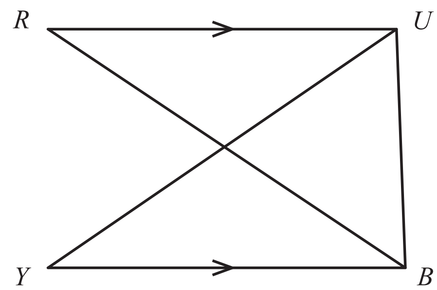


Figure 8.7

6. Prove: $z = x + y$.

(See **Figure 8.8**.)

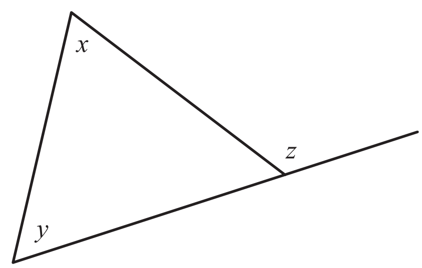


Figure 8.8

7. Given: $GT = GO$.
$\overline{GA} \perp \overline{TO}$.

Prove: A bisects $\overline{TO}$.

(See **Figure 8.9**.)

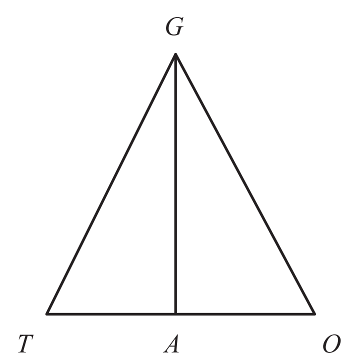


Figure 8.9

8. Given: $\overline{BE} \parallel \overline{CD}$.
$\overline{AC} \parallel \overline{ED}$.

Prove: $\angle a \cong \angle b$.

(See **Figure 8.10**.)

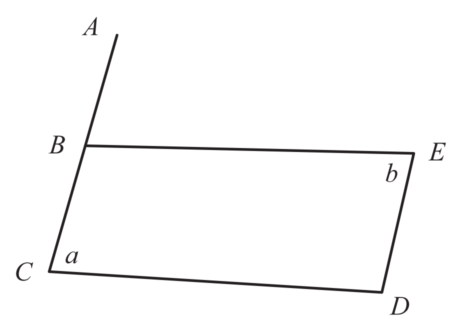


Figure 8.10

9. Given: $\overline{EI} \parallel \overline{NJ}$.

$\overline{KG}$ bisects $\angle EBH$.

$\overline{FL}$ bisects $\angle NCH$.

Prove: $\overline{FL} \parallel \overline{GK}$.

(See **Figure 8.11**.)

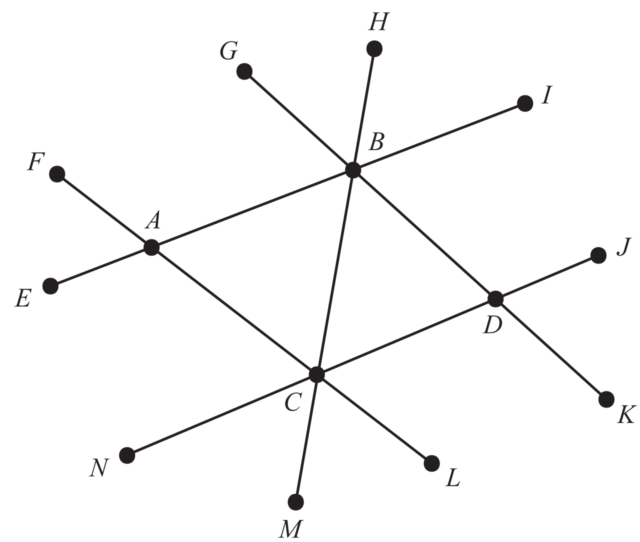


Figure 8.11

10. Prove, again, the vertical angle theorem.

Similarity and Congruence

Lesson 9

Topics

- Similar and congruent polygons.
- The SAS and AA similarity principles for triangles.
- Properties of parallel lines.
- Base angles of isosceles triangles.

Definitions

- **congruent polygons**: Two polygons that are similar with scale factor $k = 1$ are congruent.
- **isosceles triangle**: A triangle with at least two sides congruent is isosceles.
- **similar polygons**: Two polygons are similar if, in moving one direction about one polygon, it is possible to move in some direction about the second polygon so that
    - all side lengths encountered in turn match in the same ratio k.
    - all angles encountered in turn match exactly.

  The common ratio k of the side lengths is called the scale factor.

Notation

- The symbol $\sim$ is used to denote that two polygons are similar.
- The symbol $\cong$ is used to denote that two polygons are congruent.

Results

- SAS principle: If two triangles have a common angle and the sides about that angle in each triangle are in the same ratio, then the two triangles are similar. (See **Figure 9.1**.)

  This means that

  - all three angles match.
  - all three sides come in the same ratio.

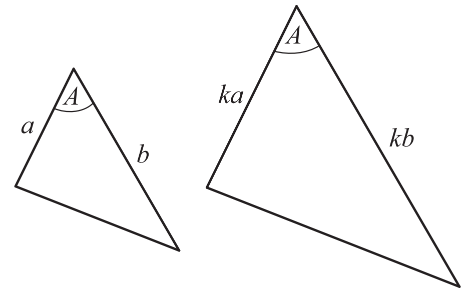


Figure 9.1

- AA principle: If two triangles have two angles that match (and, hence, all three angles match because there are 180° in a triangle), then they are similar. (See **Figure 9.2**.)

  This means that all three sides also match in the same ratio.

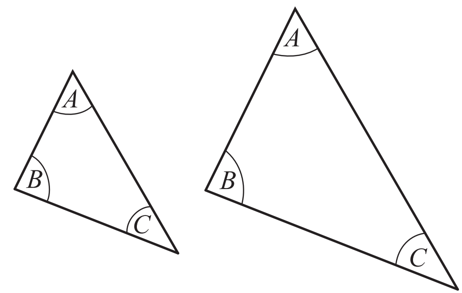


Figure 9.2

- isosceles triangles: The two base angles of an isosceles triangle are congruent, and conversely, if a triangle possesses two congruent interior angles, it is isosceles.

Summary

Attempts to capture the mathematics of what it means for a line to be "straight" leads us to two similarity beliefs about triangles. In this lesson, we introduce and discuss similarity and congruence of polygons and the SAS and AA similarity principles of triangles, and we explore logical consequences of these results.

Example 1

The left half of **Figure 9.3** is congruent to the right half.

Fill in the blanks.

a) $ABCDEF \cong$ __________ .

b) $\angle C \cong$ __________ .

c) $\overline{HG} \cong$ __________ .

d) $CD =$ __________ .

e) $\overline{AF} \cong$ __________ .

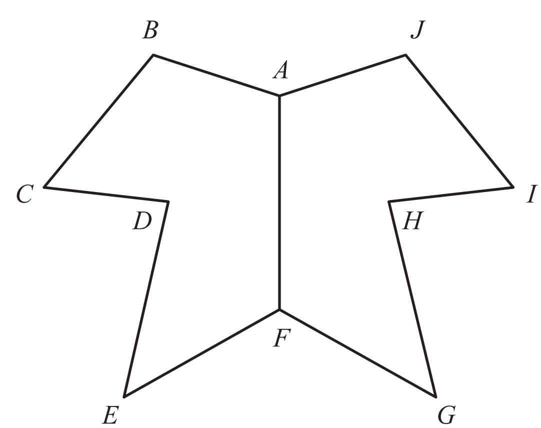


Figure 9.3

Solution

a) $AJIHGF$. (Comment: It is convention to list the names of the vertices in matching order.)

b) $\angle I$.

c) $\overline{DE}$.

d) IH.

e) $\overline{AF}$.

Example 2

Explain why the two triangles in **Figure 9.4** are similar, and find the values of x and y.

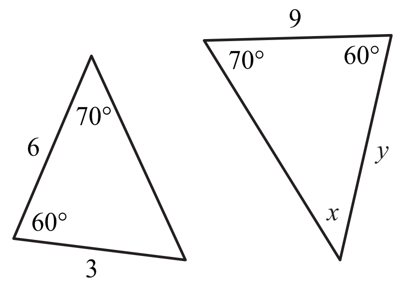


Figure 9.4

Solution

Both triangles possess an angle of measure 60° and an angle of measure 70°, so by the AA principle, they are similar.

The side of length 6 (between angles 60° and 70°) matches with the side of length 9, so $k = 1.5$.

The side of length 3 (between 60° and the unlabeled angle) matches with side y, so $y = 3k = 4.5$.

Because there are 180° in a triangle, $x = 50°$.

Example 3

a) Explain in detail why ΔABD is similar to ΔDEF.

b) Show that $xy = ab$.

(See **Figure 9.5**.)

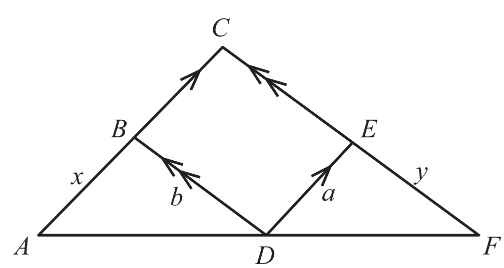


Figure 9.5

Solution

a) Extend lines and consider the angles shown. (See **Figure 9.6**.)

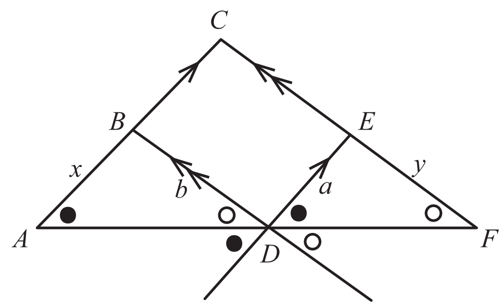


Figure 9.6

The angles marked with a solid dot are congruent because we have congruent alternate interior angles (look at the parallel lines marked with a single arrow and transversal $\overline{AF}$) and vertical angles.

The angles marked with a hollow dot are congruent because we have congruent alternate interior angles (look at the parallel lines marked with a double arrow and transversal $\overline{AF}$) and vertical angles.

Thus, ΔABD and ΔDEF share two angles, so $\Delta ABD \sim \Delta DEF$ by AA.

b) Notice:

Side x in the left triangle matches a in the right triangle.
Side b in the left triangle matches y in the right triangle.

Because in similar triangles sides come in the same ratio, we have $\frac{x}{a} = \frac{b}{y}$.

Cross multiplying yields $xy = ab$.

Study Tip

- In trying to identify similar triangles in complicated diagrams, it can be helpful to draw isolated copies of those triangles in the margin of the page. This also helps make clear the matching sides and matching angles between the two triangles.

Pitfall

- Don't forget that it is permissible for two similar figures to have opposite orientations—that is, that the scaling of one figure to produce the other may involve a reflection.

Problems

1. Fill in the following blanks. (See **Figure 9.7**.)

a) $\Delta HYB \sim$ __________.

b) Scale factor $k =$ __________.

c) $m\angle x =$ __________.

d) $a =$ __________.

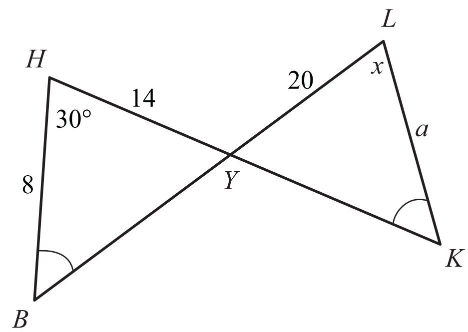


Figure 9.7

2. Fill in the blanks. (See **Figure 9.8**.)

a) This picture contains two similar triangles because of the __________ principle.

Side of length x in the small triangle matches the side of length $y + b$ in the large triangle.

b) Side of length y in the small triangle matches the side of length __________ in the large triangle.

c) Side of length w in the small triangle matches the side of length __________ in the large triangle.

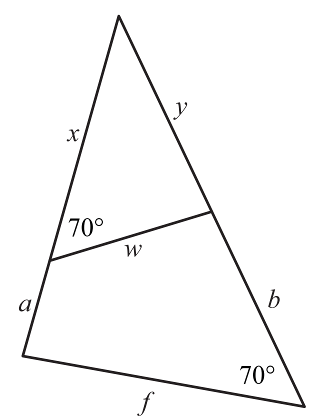


Figure 9.8

d) Because the triangles are similar, sides come in the same ratio. Consequently, we have

$$\frac{x}{y+b} = \frac{y}{} = \frac{}{f}.$$

3. Explain why the two triangles in **Figure 9.9** are similar. What is the scale factor? What is the value of length w?

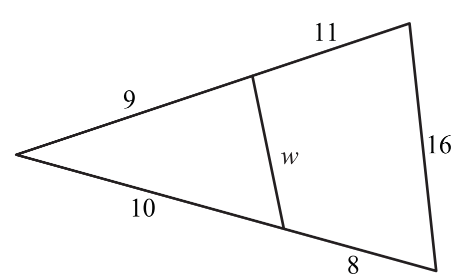


Figure 9.9

4. The two triangles in **Figure 9.10** are similar because of the AA principle.

a) Side of length a in the left triangle matches side __________ in the right triangle.

b) Side of length b in the left triangle matches side __________ in the right triangle.

c) Side of length d in the left triangle matches side __________ in the right triangle.

d) Because both triangles possess a side of length d, does this mean that the scale factor between the triangles is 1?

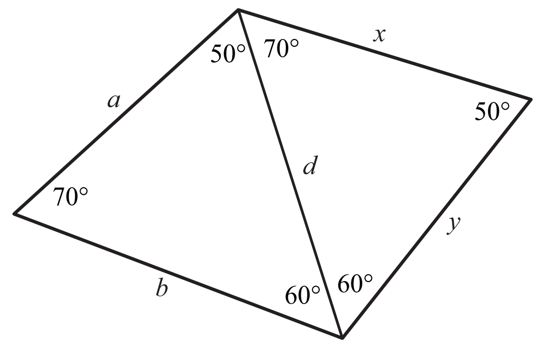


Figure 9.10

5. How tall is the tree in **Figure 9.11**?

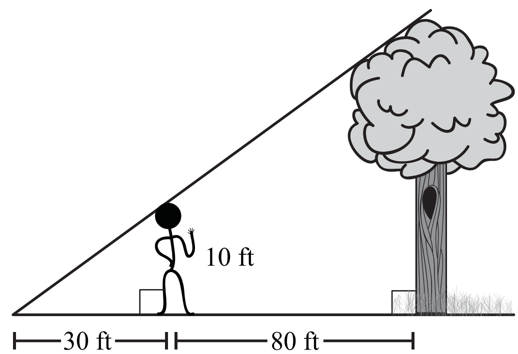


Figure 9.11

6. Three telegraph posts of different lengths stand in a row, each perpendicular to the ground. The tops of the posts are collinear. With the lengths indicated, what is the height of the middle post? (See **Figure 9.12**.)

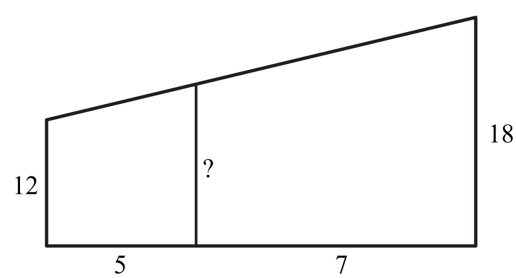


Figure 9.12

7. a) Explain why the two triangles in **Figure 9.13** are similar.

b) Explain why there is not enough information to say that they are congruent.

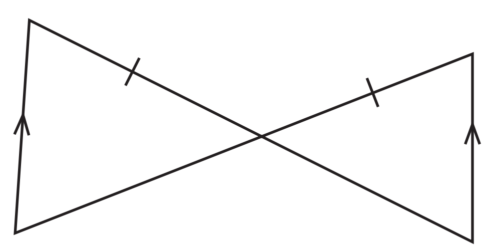


Figure 9.13

8. In ΔABC, the point P is $\frac{1}{3}$ down side $\overline{BA}$, and the point Q is $\frac{1}{3}$ down side $\overline{BC}$. (See **Figure 9.14**.)

a) Prove that $\overline{PQ}$ is parallel to $\overline{AC}$.

b) Prove that $\overline{AC}$ is 3 times as long as $\overline{PQ}$.

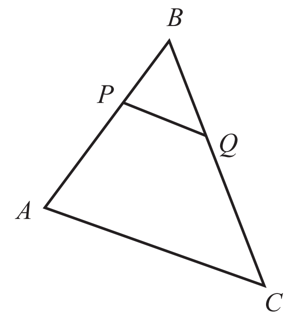


Figure 9.14

9. Consider the diagram in **Figure 9.15**.

a) Explain why $\Delta ABC \sim \Delta BDC$.

b) Establish that $x^2 = y(y + z)$.

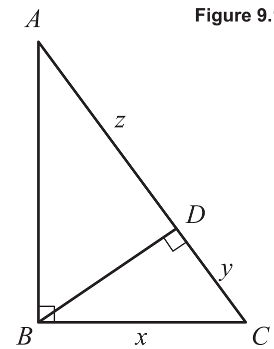


Figure 9.15

10. Use similar triangles to prove that opposite sides of a parallelogram are congruent.

That is, in the picture in **Figure 9.16** , prove that $x = y$.

Hint: Draw a diagonal line.

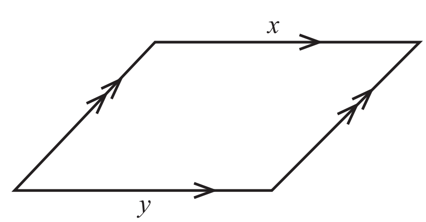


Figure 9.16

Practical Applications of Similarity

Lesson 10

Topics

- The SSS similarity principle for triangles.
- The converse of the Pythagorean theorem.
- Applications of similarity and congruence principles.

Results

- SSS principle: If two triangles have three sides that match in the same ratio, then the two triangles are similar. (See **Figure 10.1**.)

  That is, the three angles of the triangles match as well.

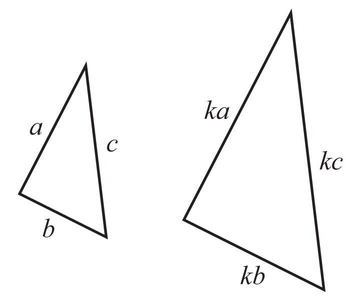


Figure 10.1

- Pythagorean converse: If a triangle has three sides of lengths p, q, and r satisfying $p^2 + q^2 = r^2$, then the triangle is a right triangle with the side of length r the hypotenuse.

Summary

In this lesson, we explore one more similarity principle for triangles, the SSS principle, and show how it follows as a logical consequence of the SAS and AA similarity principles. This allows us to establish the converse of the Pythagorean theorem. We also explore a number of practical applications of the three similarity principles.

Example 1

How many essentially different triangles can you make with 3 sticks, one of length 11 inches, one of length 15 inches, and a third of length 20 inches?

Solution

Just one. Any two triangles with these side lengths must be congruent by the SSS principle.

Example 2

Each of the following pictures shows a pair of triangles. Which pairs of triangles are certain to be similar? Which are certain to be congruent? Of those that are similar/congruent, explain which principle gives you this.

a)

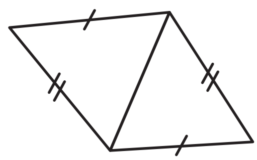


Figure 10.2

b)

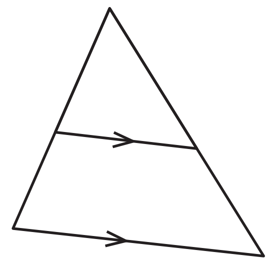


Figure 10.3

c)

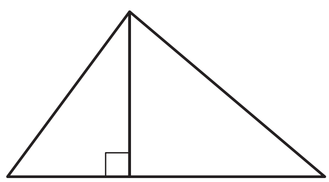


Figure 10.4

d)

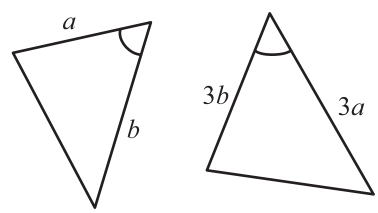


Figure 10.5

Solution

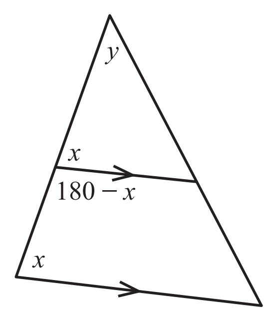


Figure 10.6

a) The two triangles are similar because of SSS. The scale factor is clearly 1, so they are, in fact, congruent.

b) Label the same-side interior angles for the parallel lines x and $180 - x$, as shown in **Figure 10.6**. We have a second angle of measure x. The two triangles in the diagram are similar by AA. (They share a common angle y, and each possess an angle of measure x.)

c) Actually, there are three triangles in this diagram! There is no reason to believe that any two of them are similar.

d) These two triangles are similar by SAS.

Study Tip

- Triangles are the "atoms" of geometry, and identifying similar triangles in complicated diagrams is often the key to unlocking results about that diagram. Become familiar with all three similarity principles for triangles: SAS, AA, and SSS.

Pitfall

- For the SAS principle, the angle in consideration must be sandwiched between the two sides in consideration. For example, in the diagram in **Figure 10.7**, triangles *PQS* and *PQR* both share the angle *A* and have two matching sides of lengths 10 and 3, yet they are clearly not similar triangles. (The angle *A* is not sandwiched between the sides of lengths 10 and 3.)

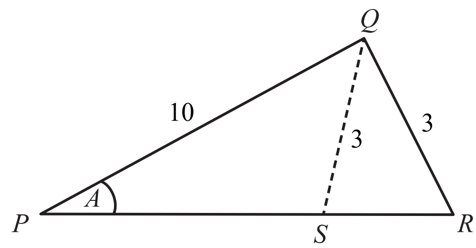


Figure 10.7

Problems

1. Each of the following pictures shows a pair of triangles. Which pairs of triangles are certain to be similar? Which are certain to be congruent? Of those that are similar/congruent, explain which principle gives you this.

a)

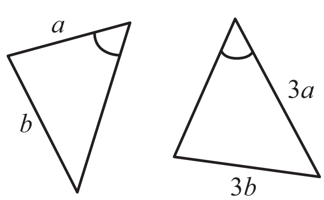


Figure 10.8

b)

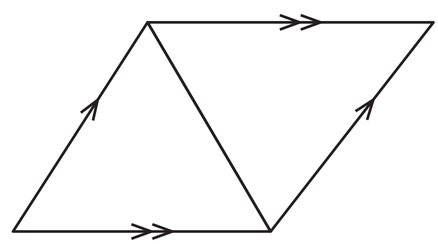


Figure 10.9

c)

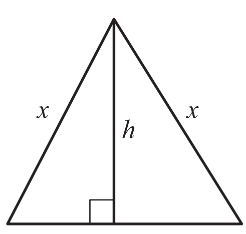


Figure 10.10

d)

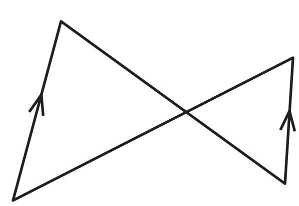


Figure 10.11

2. Are the two triangles in **Figure 10.12** similar? If so, what is the scale factor?

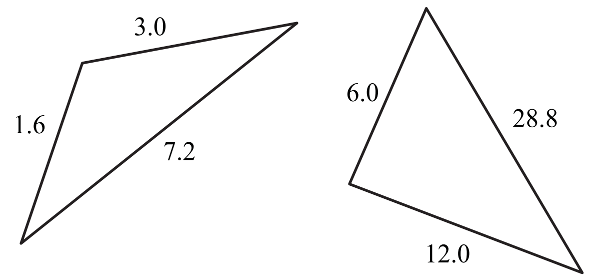


Figure 10.12

3. a) Show that there is no AAAA principle for quadrilaterals.

   That is, find examples of two quadrilaterals that have the same angles but are clearly not similar.

   b) Show that there is also no SSSS principle for quadrilaterals.

4. Explain why $\angle SNA \cong \angle A$ in **Figure 10.13**.

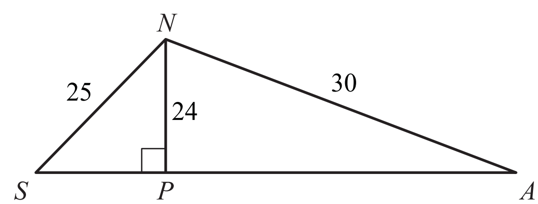


Figure 10.13

5. Explain why each interior angle of an equilateral triangle must have measure 60°.

6. An angle bisector is drawn from the apex of an isosceles triangle. Suppose that it intersects the base at point M. (See **Figure 10.14**.)

   Prove:

   a) M is the midpoint of $\overline{AC}$.

   b) $\overline{BM} \perp \overline{AC}$.

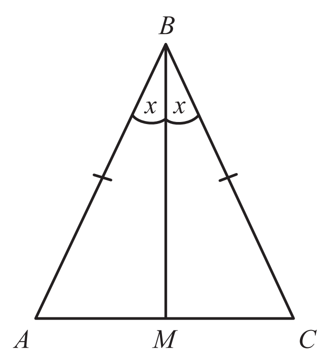


Figure 10.14

7. A line is drawn from the midpoint of the base of an isosceles triangle to its apex. (See **Figure 10.15**.)

Prove:

a) This line bisects the angle at B.

b) $\overline{BM} \perp \overline{AC}$.

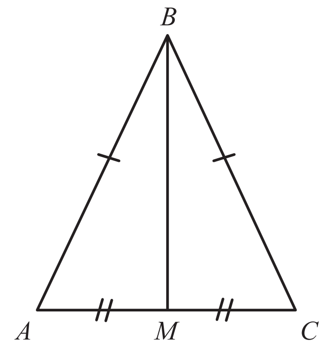


Figure 10.15

8. A line is drawn from the apex of an isosceles triangle to meet the base at 90°. (See **Figure 10.16**.)

Prove:

a) M is the midpoint of $\overline{AC}$.

b) This line bisects the angle at B.

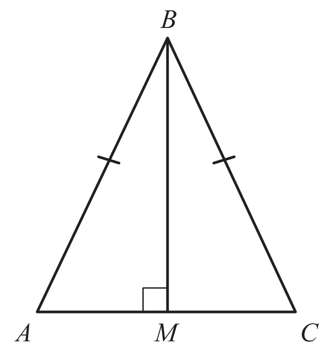


Figure 10.16

9. A line is drawn from the apex of a triangle to the midpoint M of the opposite side. Suppose that this line turns out to be perpendicular to that side. (See **Figure 10.17**.)

Prove that the triangle is isosceles.

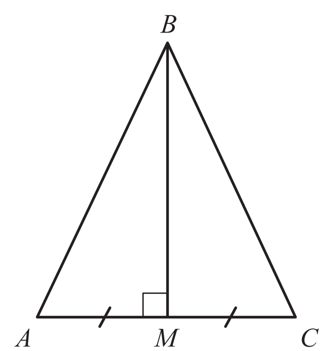


Figure 10.17

10. For **Figure 10.18**, prove that $\angle ACE \cong \angle BCD$.

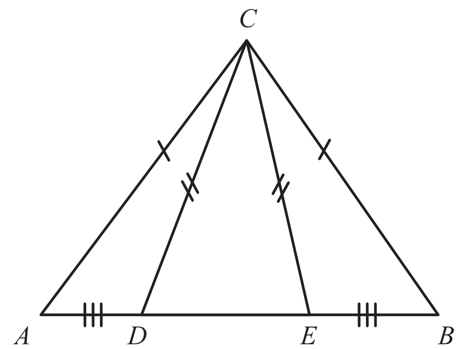


Figure 10.18

Making Use of Linear Equations

Lesson 11

Topics

- The equation of a line.
- The slopes of parallel and perpendicular lines.
- The perpendicular bisector of a line segment.
- Applications.

Definition

- **perpendicular bisector**: The perpendicular bisector of a line segment is a line through the midpoint of the segment and perpendicular to it.

Formulas

- A line of slope m through the point (a, b) has equation $y - b = m(x - a)$. (Of course, there are many algebraically equivalent versions of this equation.)
- Parallel lines have the same slope.
- If one line has slope m, then a line perpendicular to it has slope $-\frac{1}{m}$ (assuming that neither line is vertical).

Summary

The algebraic equation of a line is nothing more than a restatement of our belief that the slope of straight lines is constant. In this lesson, we develop the equation of a line, discuss the slopes of parallel and perpendicular lines, and practice our algebraic skills in a variety of geometric settings about lines.

Example 1

Sketch graphs of the following lines.

a) $y - 3 = -2(x + 1)$.

b) $y = 20x + 1$.

c) $x = 3$.

Solution

a) $y - 3 = -2(x - (-1))$ is the equation of a line of slope -2 through the point $(-1, 3)$.

(See **Figure 11.1**.)

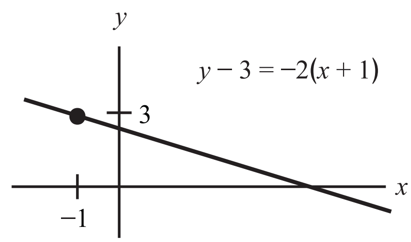


Figure 11.1

b) $y - 1 = 20(x - 0)$ is a line of slope 20 through the point $(0, 1)$.

(See **Figure 11.2**.)

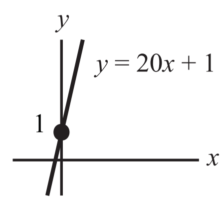


Figure 11.2

c) All points in the plane with x-coordinate 3 satisfy $x = 3$, so this is a vertical line.

(See **Figure 11.3**.)

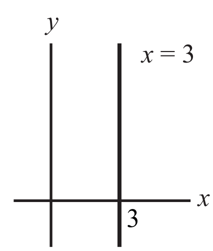


Figure 11.3

Example 2

Find the x- and y-intercepts of a line that passes through the points $(4, 6)$ and $(-2, 5)$.

Solution

Such a line has slope $\frac{5-6}{(-2)-4}=\frac{1}{6}$ and, therefore, has equation $y-6=\frac{1}{6}(x-4)$.

The x-intercept of the line is the point at which the line crosses the x-axis. We have $y = 0$ at this point. Substituting and solving gives

$$\begin{aligned} 0-6 &= \frac{1}{6}(x-4) \\ x &= -32. \end{aligned}$$

The x-intercept is $(-32, 0)$.

The y-intercept is the point at which the line crosses the y-axis—that is, the location at which $x = 0$. Substituting and solving gives

$$\begin{aligned} y-6 &= \frac{1}{6}(0-4) \\ y &= 5\frac{1}{3}. \end{aligned}$$

The y-intercept is $\left(0, 5\frac{1}{3}\right)$.

Example 3

What is the slope of the line shown in **Figure 11.4**? Name a point this line passes through. Quickly write the equation of the line.

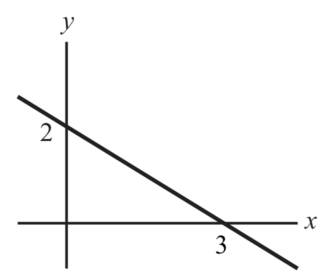


Figure 11.4

Solution

The line has slope $-\frac{2}{3}$ and passes through the point $(0, 2)$. It has equation $y-2=-\frac{2}{3}x$.

(Alternatively, noting that the line passes through $(3, 0)$, we could also write the equation of the line as $y=-\frac{2}{3}(x-3)$.)

Example 4

Write an equation for the perpendicular bisector of $\overline{PQ}$, where $P = (1, 7)$ and $Q = (4, 0)$.

Solution

The segment $\overline{PQ}$ has slope $\frac{0-7}{4-1}=-\frac{7}{3}$, so the perpendicular bisector to $\overline{PQ}$ has slope $\frac{3}{7}$.

It also passes through the midpoint of $\overline{PQ}$, which is $\left(\frac{5}{2},\frac{7}{2}\right)$.

The equation of the perpendicular bisector is thus $y-\frac{7}{2}=\frac{3}{7}\left(x-\frac{5}{2}\right)$.

Study Tip

- As Example 4 shows, there is no need to be perturbed by awkward numbers. Work to simply understand, and hold on to the basic theory behind the equations of lines.

Pitfall

- Many students are locked into the $y = mx + b$ formula for straight lines. It is rare that this is the easiest formula to use in a given situation. Be flexible in your thinking about what constitutes the equation of a line.

Problems

1. Find the slopes of the lines passing through each of the following pairs of points.

a) $(0, 0)$ and $(2, 10)$.

b) $(3, -1)$ and $(-1, -3)$.

c) $(0, 0)$ and (a, b).

d) (a, a) and $(a+b, a+b)$.

e) $(a, 0)$ and (b, c).

f) (a, b) and $(-b, a)$.

g) $(a, 3)$ and $(a, 5)$.

2. Find the slopes of each of the following.

a) A line parallel to the line through the origin and the point $(10, 11)$.

b) A line perpendicular to the line that passes through $(3, 8)$ and $(-1, 9)$.

c) A line parallel to the x-axis.

3. A line intersecting the x-axis at $(5, 0)$ has slope 2. Find its y-intercept.

4. A line of slope $-\frac{4}{3}$ contains $P = (2, 5)$ and $Q = (6, k)$. What is k?

5. A line intersects the y-axis at $y = 8$ and the x-axis at $x = -4$. Write an equation for the line.

6. $P = (3, 2)$; $Q = (4, -6)$; $R = (b, b)$. Find a value for b that makes $\overleftrightarrow{PQ} \perp \overleftrightarrow{QR}$.

7. $A = (0, 0)$; $B = (5, 3)$; $C = (2, -2)$; $D = (a, -10)$. Find a value for a that makes $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$.

8. Find an equation for each of the following lines.

a) A line through $(1, 1)$ and $(3, 20)$.

b) A line through $(-1, 2)$ and parallel to $y = 5x - 7$.

c) A line through the midpoint of $\overline{AB}$ and perpendicular to $\overline{AB}$, where $A = (-3, -7)$ and $B = (11, 5)$.

d) A line through $(15, -2)$ and parallel to the x-axis.

e) A line through $(8, 0)$ and perpendicular to $2x - 3y = 0$.

f) A line through $(2, 3)$ and perpendicular to the x-axis.

9. Very quickly write an equation for each of the following lines.

a)

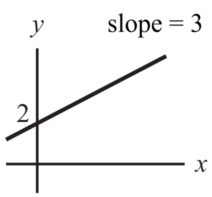


Figure 11.5

b)

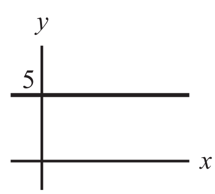


Figure 11.6

c)

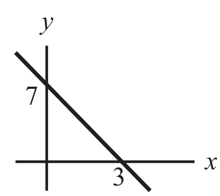


Figure 11.7

d)

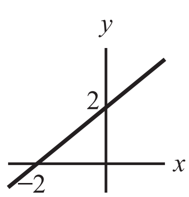


Figure 11.8

e)

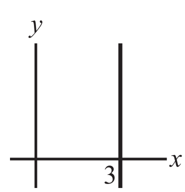


Figure 11.9

f)

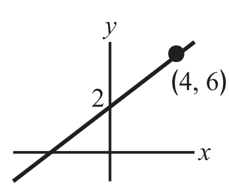


Figure 11.10

g)

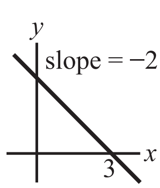


Figure 11.11

10. Warning: This is a lengthy exercise!

Let $A = (1, 2)$; $B = (3, 6)$; $C = (-3, 4)$.

Part I

a) Find the coordinates of the midpoint M of $\overline{AB}$ and the equation of the line through M and C.

b) Find the coordinates of the midpoint N of $\overline{BC}$ and the equation of the line through N and A.

c) Find the coordinates of the midpoint R of $\overline{AC}$ and the equation of the line through R and B.

d) Show that all three lines pass through the same point $\left(\frac{1}{3}, 4\right)$.

Part II

a) Find the equation of the line through M and perpendicular to $\overline{AB}$.

b) Find the equation of the line through N and perpendicular to $\overline{BC}$.

c) Find the equation of the line through R and perpendicular to $\overline{AC}$.

d) Show that all three lines pass through the same point $(0, 5)$.

Part III

a) Find the equation of the line through C and perpendicular to $\overline{AB}$.

b) Find the equation of the line through A and perpendicular to $\overline{BC}$.

c) Find the equation of the line through B and perpendicular to $\overline{AC}$.

d) Show that all three lines pass through the same point $(1, 2)$.

Comment: Part I of this question constructs the three *medians* of a triangle, part II constructs the three *perpendicular bisectors* of a triangle, and part III constructs the three *altitudes* of a triangle. You can use algebraic methods to prove that each set of three lines do always pass through a common point for any given triangle—but algebra is far from being the easiest way to establish these claims.

Equidistance—A Focus on Distance
Lesson 12

Topics

- Equidistance between points.
- Equidistance between lines.
- Circumcircles of triangles.
- Incircles of triangles.
- The Euler line.

Definitions

- **circumcenter of a triangle**: The center of the circumcircle of a triangle is its circumcenter. It is the location where the three perpendicular bisectors of the triangle coincide.
- **circumcircle of a triangle**: For each triangle, there is a unique circle that passes through the vertices of the triangle. This circle is the circumcircle of the triangle.
- **equidistant**: A point is said to be equidistant from two or more objects if its distance from each of those objects is the same.
- **incenter of a triangle**: The center of the incircle of a triangle is its incenter. It is the location where the three angle bisectors of the triangle coincide.
- **incircle of a triangle**: For each triangle, there is a unique circle sitting inside the triangle tangent to each of its three sides. This circle is the incircle of the triangle.

Results

- The set of points equidistant from a pair of points A and B is precisely the set of points on the perpendicular bisector of $\overline{AB}$.
- The three perpendicular bisectors of a triangle coincide at a single point. This point is equidistant from all three vertices of the triangle and, therefore, lies at the center of a circle that passes through all three vertices of the triangle.

- The set of points equidistant from a pair of intersecting lines is precisely the set of points on an angle bisector of those lines.

- The three angle bisectors of a triangle coincide at a single point. This point is equidistant from all three sides of the triangle and, therefore, lies at the center of a circle inside the triangle tangent to all three of its sides.

Summary

Identifying the set of all points of equal distances from a given set of objects provides a powerful tool for proving results in geometry. In this lesson, we examine equidistance between points and between lines and prove the existence of special circles for triangles. We also solve some standard schoolbook geometry problems in sophisticated ways.

Example 1

Let $A = (3, 6)$ and $B = (7, 2)$.

a) Find the equation of the perpendicular bisector of $\overline{AB}$.

b) Find a point P on this line with x-coordinate equal to 10.

c) Use the distance formula to verify that P is indeed equidistant from A and B.

Solution

a) The perpendicular bisector passes through the midpoint $(5, 2)$, and its slope is the negative reciprocal of slope $\overline{AB} = \frac{8}{-4} = -2$.

Its equation is thus $\frac{y-2}{x-5} = \frac{1}{2}$, or $y-2 = \frac{1}{2}(x-5)$.

b) Put $x = 10$ to see that $y-5 = \frac{1}{2}(5)$, giving $y = \frac{5}{2}+2 = \frac{9}{2}$.

We have $P = \left(10, \frac{9}{2}\right)$.

c) $PA = \sqrt{7^2 + \left(\frac{3}{2}\right)^2} = \sqrt{49+\frac{9}{4}} = \sqrt{\frac{205}{4}}$ and $PB = \sqrt{3^2 + \left(\frac{13}{2}\right)^2} = \sqrt{9+\frac{169}{4}} = \sqrt{\frac{205}{4}}$.

These are indeed equal.

Example 2

Figure 12.1 is a map of two intersecting paths in a park. At point A stands a statue, and at point B is a fountain.

Mark on this page the exact location Jenny should stand so that she is the same distance from each of the two paths and, at the same time, equidistant from points A and B.

In fact, show that there are *two* different places Jenny could stand to meet the requirements of this challenge.

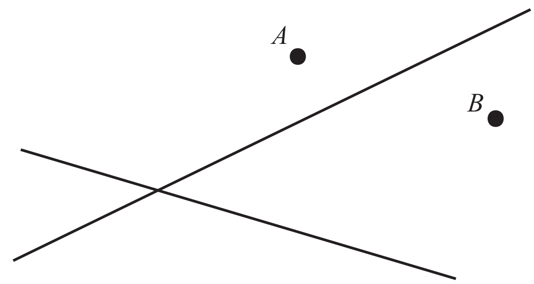


Figure 12.1

Solution

Jenny should stand at the location where the angle bisector of the two lines and the perpendicular bisector of $\overline{AB}$ intersect. (See **Figure 12.2**.)

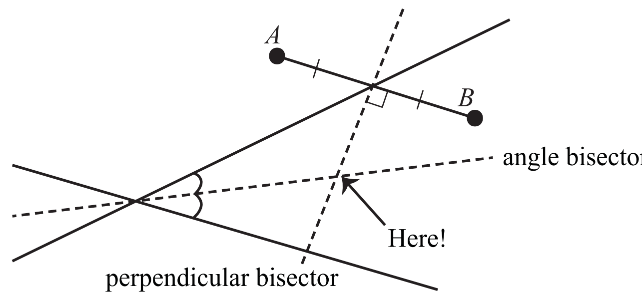


Figure 12.2

Because there are *two* angle bisectors for the pair of lines, there is a second location Jenny could stand. (See **Figure 12.3**.)

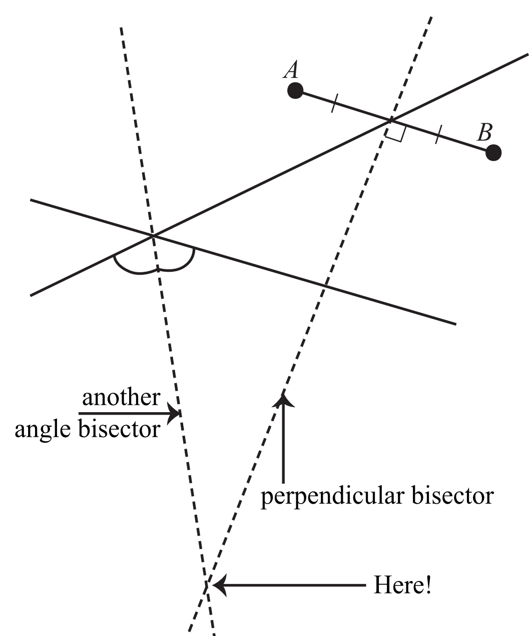


Figure 12.3

Study Tip

- If you are ever asked to prove that two given line segments are perpendicular, with at least one segment bisecting the other, consider looking for equidistant objects in the problem.

Pitfall

- If points P and Q are each equidistant from points A and B, then $\overline{PQ}$ is the perpendicular bisector of $\overline{AB}$. *Only* the segment $\overline{AB}$ is guaranteed to be bisected.

Problems

1. Which of the following points are equidistant from $A = (3, 4)$ and $B = (7, 6)$?

a) $P = (3,6)$

b) $Q = (2,7)$

c) $R = (6, 3)$

d) $S = (12, -9)$

e) $T = (2,3)$

f) $U = (10, -4)$

(What is an efficient way to handle this question?)

2. a) Without plotting the points, show that $A = (0, 0)$, $B = (4, -2)$, and $C = (1, -3)$ form an isosceles triangle.

b) Compute the midpoints of its three sides.

c) Show that the midpoints also form an isosceles triangle.

3. Given: P is on the perpendicular bisector of $\overline{AB}$.
P is on the perpendicular bisector of $\overline{BC}$.

Prove: $PA = PC$.

(See **Figure 12.4**.)

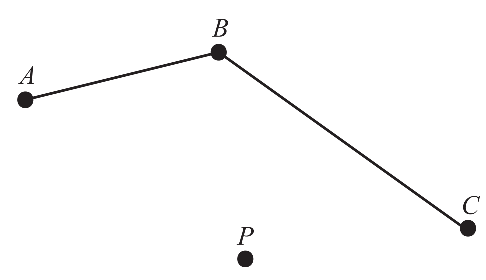


Figure 12.4

4. Fill in the following blanks. (See **Figure 12.5**.)

a) If P is on the angle bisector of $\angle RMN$, then P is equidistant from lines ______ and ______.

b) If P is on the angle bisector of $\angle RNM$, then P is equidistant from ______ and ______.

c) If P is equidistant from $\overline{MR}$ and $\overline{NR}$ then P lies on the ____________ of ______.

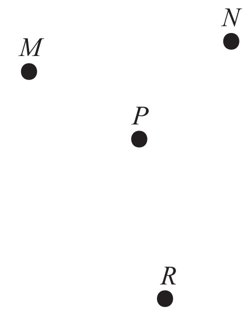


Figure 12.5

d) If R lies on the perpendicular bisector of $\overline{MN}$, then R is equidistant from ______ and ______.

e) If P is equidistant from M and R, then P lies on the ____________ of ______.

5. A kite is a convex four-sided figure with two pairs of adjacent sides equal in length. (See **Figure 12.6**.)

a) Explain why P and R each lie on the perpendicular bisector of the diagonal $\overline{AB}$.

b) Explain why the diagonals of a kite are perpendicular.

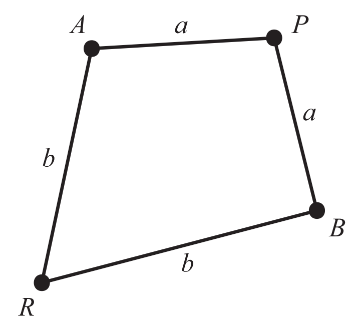


Figure 12.6

6. A deltoid is a concave four-sided figure with two pairs of adjacent sides equal in length. Must the diagonals of a deltoid be perpendicular?

7. a) A convex four-sided figure has the property that its diagonals are perpendicular. Show, with a drawing, that the figure does not need to be a kite.

b) A convex four-sided figure has the property that its diagonals are perpendicular and that one diagonal bisects the other. Prove that the figure must be a kite.

8. A treasure is buried at a location equidistant from points R and S on the ground precisely 10 meters away from point T. (See **Figure 12.7**.)

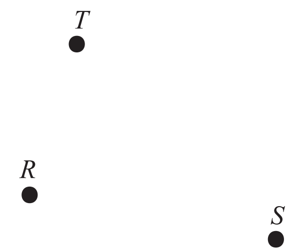


Figure 12.7

This may or may not be enough information to determine the exact location of the treasure.

a) Draw a diagram to show that the information in the clue might narrow the location of the treasure to one of two possible locations.

b) Draw a diagram to show that, if the distances between points T, R, and S are just right, the information in the clue could narrow the location of the treasure to a single location.

9. Given: $m\angle SJL = 90°$.

$\overline{TK}$ is the perpendicular bisector of $\overline{JL}$.

Prove: T bisects $\overline{SL}$.

(See **Figure 12.8**.)

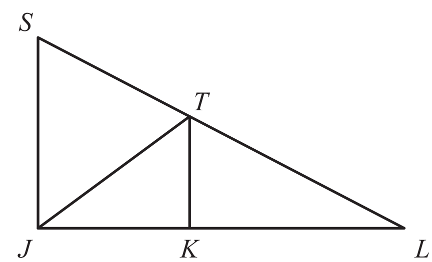


Figure 12.8

10. For four given points in the plane, is there sure to be a single location P that is equidistant from all four?

A Return to Parallelism

Lesson 13

Topics

- Midpoint segments in triangles.
- Midpoint quadrilaterals within quadrilaterals.
- Transversals and triples of parallel lines.
- The median of a trapezoid.

Definitions

- **corresponding angles**: For a transversal crossing a pair of lines, two angles on the same side of the transversal, with one between the pair of lines and one not, are corresponding angles. (Corresponding angles are congruent precisely when the two lines are parallel.)
- **median of a trapezoid**: If a trapezoid has just one pair of parallel sides, then the line segment connecting the midpoints of the two remaining sides is the median of the trapezoid.
- **parallelogram**: A quadrilateral with two pairs of parallel sides.
- **trapezoid**: A quadrilateral with at least one pair of parallel sides.

Results

- The line segment connecting midpoints of two sides of a triangle is parallel to the third side of the triangle and is half its length.
- If, for a triple of parallel lines, one transversal segment is bisected by them, then so are all transversal segments. (See **Figure 13.1**.)
- The line that connects the midpoints of two transversal segments to a pair of parallel lines is also parallel to those lines. (See **Figure 13.2**.)

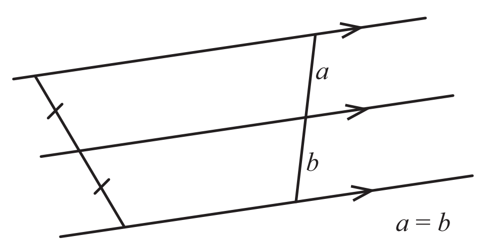


Figure 13.1

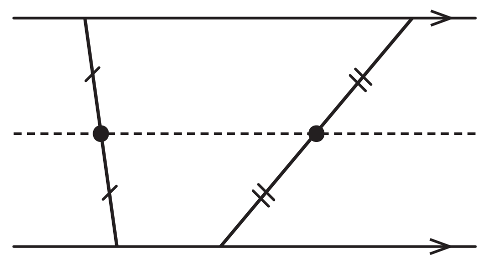


Figure 13.2

- The median of a trapezoid is parallel to the two parallel sides of the trapezoid and has length the average of the base lengths of the trapezoid. (See **Figure 13.3**.)

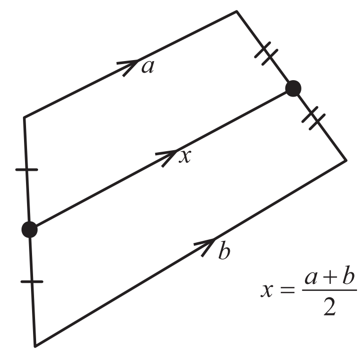


Figure 13.3

Summary

Connecting the midpoints of sides of geometric figures brings us back to parallelism. In this lesson, we explore the consequences of connecting midpoints in sides of triangles and quadrilaterals and in segments of transversals for parallel lines.

Example 1

Provide the full details that explain why, in **Figure 13.4**, length a matches length b.

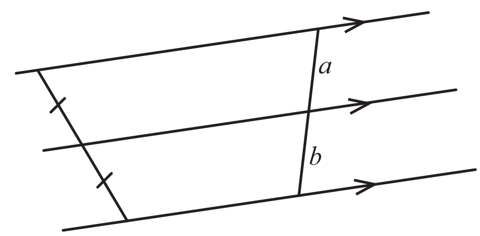


Figure 13.4

Solution

Draw an additional line, as shown in **Figure 13.5**, and mark the lengths p and q.

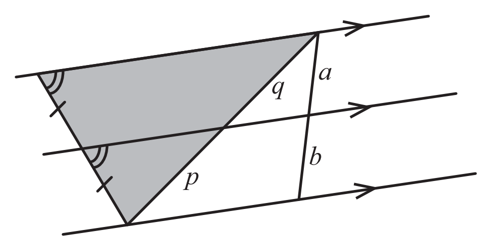


Figure 13.5

The two shaded triangles are similar by AA (corresponding angles and a shared angle), clearly with scale factor 2. Because the scale factor is 2, $p + q = 2p$, giving $p = q$.

The non-shaded triangles are also similar by AA (corresponding angles and a shared angle), with scale factor 2, because $p = q$. Consequently, $a + b = 2b$, giving $a = b$.

Example 2

Provide the full details that explain why, in **Figure 13.6**, the central line is parallel to the two given parallel lines.

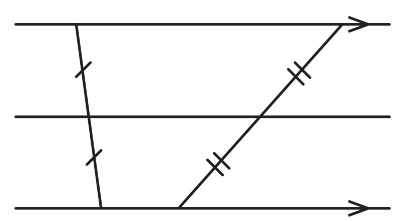


Figure 13.6

Solution

(This reasoning is sneaky!)

Label the midpoints of the two segments P and Q, as shown in **Figure 13.7**.

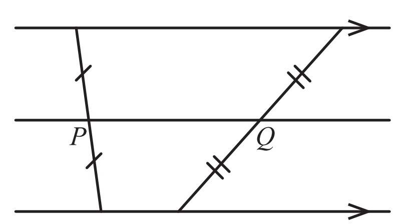


Figure 13.7

We do not know if $\overline{PQ}$ is parallel to the pair of lines.

Imagine drawing a line through P that is in fact parallel to the pair of lines. By Example 1, this line meets the second transversal segment at its midpoint—namely, Q. Thus, $\overline{PQ}$ is the parallel line through P, and therefore, $\overline{PQ}$ is indeed parallel.

Example 3

Find the values of x and y in **Figure 13.8**.

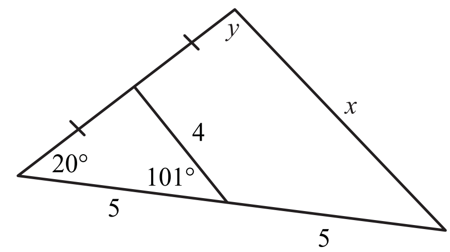


Figure 13.8

Solution

The unlabeled angle in the small triangle has measure 59°.

Because the line connecting midpoints in a triangle is parallel to the base of the triangle, this angle and y are congruent corresponding angles. Consequently, $y = 59°$.

Also, $x = 8$, because the line connecting midpoints in a triangle is half the length of the base of the triangle.

Study Tip

- Looking for lines that connect midpoints of segments can prove to be handy in establishing that given lines are parallel.

Pitfall

- Read definitions like a lawyer. Our definition of a trapezoid regards parallelograms to be trapezoids. Other textbook authors might insist that a trapezoid has precisely one pair of parallel sides (in which case, parallelograms do not sit in the class of trapezoids). The fine details of definitions like these are a matter of author's taste and, therefore, vary from author to author.

Problems

1. Given: M is midpoint of $\overline{AB}$.
$\overline{MQ} \parallel \overline{BC}$.

Prove: Q is midpoint of $\overline{AC}$.

(See **Figure 13.9**.)

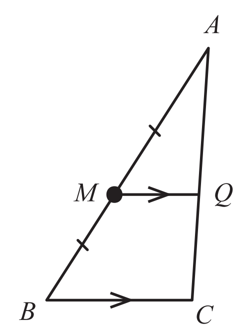


Figure 13.9

2. M, N, and O are midpoints of the sides of $\triangle ABC$. (See **Figure 13.10**.)

a) If $BC = 20$, then $MO = ?$

b) If $MN = 13$, then $AC = ?$

c) Is $\overline{NO} \parallel \overline{AB}$?

d) Explain why $\angle B$ is congruent to $\angle MON$.

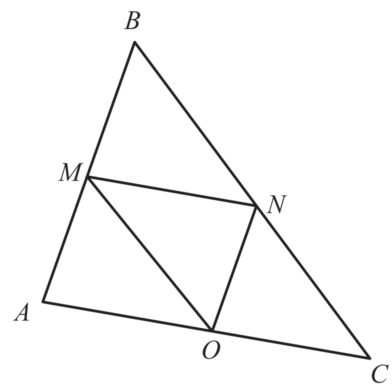


Figure 13.10

3. Consider the triangle with vertices $A = (1, 2)$, $B = (-1, 8)$, and $C = (7, 10)$.

a) Find the midpoint M of $\overline{AB}$.

b) Find the midpoint N of $\overline{AC}$.

c) Verify that $\overline{MN}$ is parallel to $\overline{BC}$.

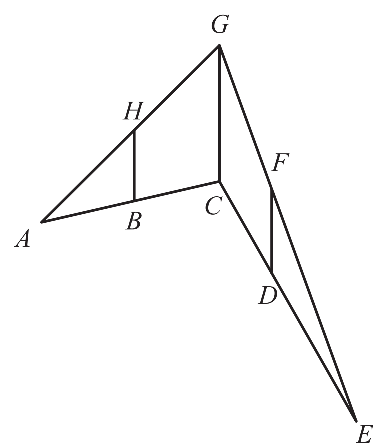


Figure 13.11

4. In the diagram in **Figure 13.11**, B, D, F, and H bisect the line segments on which they lie.

a) If $CG = 20$, what are the values of HB and FD?

b) Is $\overline{HB} \parallel \overline{FD}$?

5. The line segment inside the triangle in **Figure 13.12** connects midpoints. Find the values of x, y, and z.

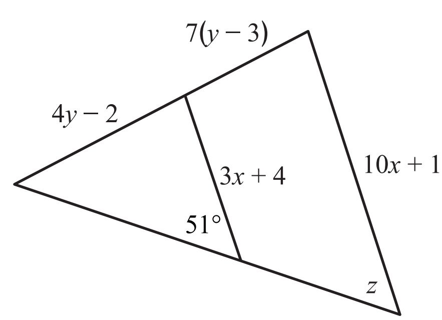


Figure 13.12

6. Points X, Y, and Z are midpoints of the line segments on which they lie. Explain why $\triangle XYZ \sim \triangle ABC$. What is the scale factor? (See **Figure 13.13**.)

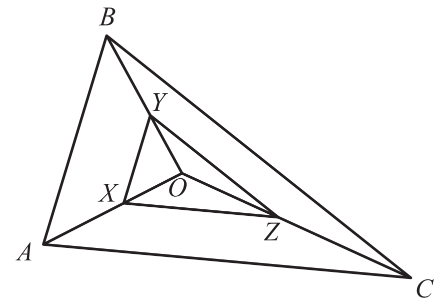


Figure 13.13

Midpoints have been the focus of this lesson. But there is actually nothing special about them. Analogous results apply for other types of points as well. The next three problems demonstrate this.

7. In $\triangle ABC$, the points M and N lie $\frac{5}{8}$ of the way down the sides on which they lie. (See **Figure 13.14**.)

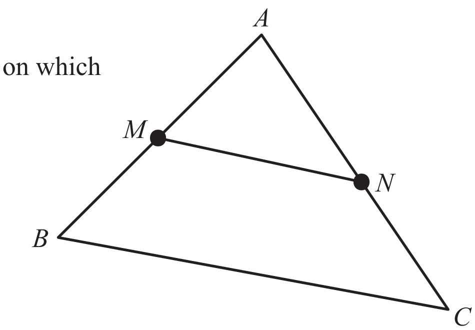


Figure 13.14

Prove:

a) $\overline{MN}$ is parallel to the base $\overline{BC}$.

b) Length MN is $\frac{5}{8}$ the length of BC.

8. Consider a set of triangles sharing a common line segment as its base. (See **Figure 13.15**.) What can you say about all the line segments connecting the points $\frac{5}{8}$ of the way down the sides of these triangles?

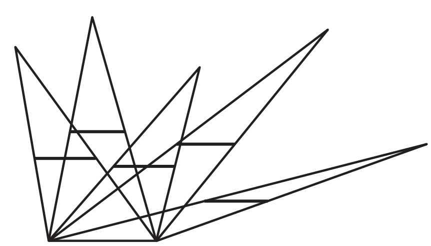


Figure 13.15

9. Explain why segments connecting the points $\frac{5}{8}$ of the way down two pairs of sides of a quadrilateral, as shown in **Figure 13.16**, must be parallel.

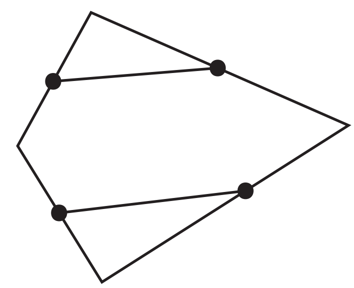
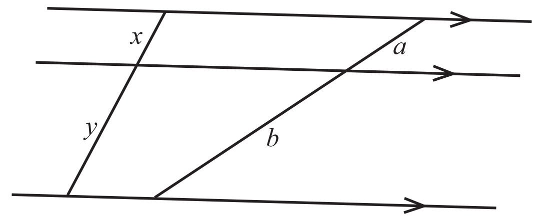


Figure 13.16

10. Consider the following general triple of parallel lines theorem.

A triple of parallel lines cuts all transversal line segments in the same ratio. (See **Figure 13.17**.)



We have: $\frac{x}{y} = \frac{a}{b}$

Figure 13.17

Prove this general theorem by filling in the details of the following steps.

First, draw a third transversal and label the lengths s and t, as shown in **Figure 13.18**.

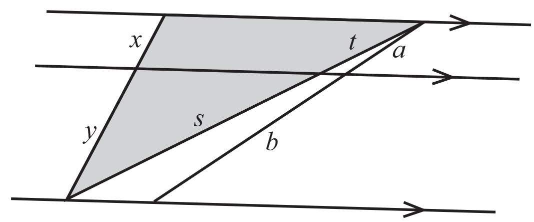


Figure 13.18

i. Use similar triangles to show that $\frac{y}{y+x} = \frac{s}{s+t}$.

ii. Perform algebra (cross multiply) to show that this means that $\frac{x}{y} = \frac{t}{s}$.

iii. Use similar triangles to show that $\frac{t}{s+t} = \frac{a}{a+b}$.

iv. Perform algebra (cross multiply) to show that this means that $\frac{t}{s} = \frac{a}{b}$.

v. Steps ii and iv then show that $\frac{x}{y} = \frac{a}{b}$.

Exploring Special Quadrilaterals

Lesson 14

Topics

- Properties of parallelograms.
- The classification of parallelograms, rectangles, squares, and rhombuses via their diagonals.

Results

- In a parallelogram,
  - opposite sides of the figure are congruent.
  - opposite angles of the figure are congruent.
  - the diagonals of the figure bisect one another.

  And the converse of each of these holds true. (For example, if a quadrilateral has opposite sides that are congruent, then that figure is a parallelogram.)

- The diagonals of a rectangle bisect one another and are congruent. Conversely, if a quadrilateral has diagonals that bisect one another and are congruent, then that figure is a rectangle.
- The diagonals of a rhombus bisect one another and are perpendicular. Conversely, if a quadrilateral has diagonals that bisect one another and are perpendicular, then that figure is a rhombus.
- The diagonals of a square bisect one another, are congruent, and are perpendicular. Conversely, if a quadrilateral has diagonals that bisect one another, are congruent, and are perpendicular, then that figure is a square.

Summary

Four-sided figures with diagonals having special properties make practical appearances in the real world. In this lesson, we explore the implications of imposing conditions on the diagonals of a quadrilateral.

Example 1

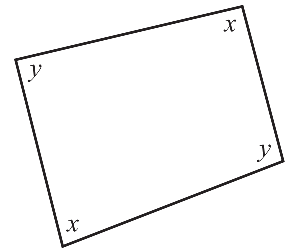


Figure 14.1

Opposite angles in a quadrilateral are congruent. Prove that the quadrilateral must be a parallelogram. (See **Figure 14.1**.)

Solution

Following the labeling given in the diagram, because the angles in a quadrilateral sum to 360°, we have $2x + 2y = 360°$, giving $x + y = 180°$. This means that each pair of angles x and y in the diagram constitute a pair of same-side interior angles summing to 180°, forcing each pair of opposite sides of the figure to be parallel. We have a parallelogram.

Example 2

Suppose that $A = (-3, 7)$, $B = (1, 9)$, $C = (0, 11)$, and $D = (-4, 9)$. What type of quadrilateral is $ABCD$?

Solution

Look at the midpoints of the diagonals.

$$\text{Midpoint } \overline{AC} = \left(-\frac{3}{2}, 9\right).$$

$$\text{Midpoint } \overline{BD} = \left(-\frac{3}{2}, 9\right).$$

The diagonals bisect each other, so we have a parallelogram, at the very least.

Look at the lengths of the diagonals.

$$AC = \sqrt{3^2 + 4^2} = 5.$$

$$BD = \sqrt{5^2 + 0^2} = 5.$$

These are the same. The parallelogram is a rectangle, at the very least.

Look at the slopes of the diagonal segments.

$$\text{Slope } \overline{AC} = \frac{4}{3}.$$

$$\text{Slope } \overline{BD} = \frac{0}{-5} = 0.$$

These are not negative reciprocals, so the diagonals are *not* perpendicular. The rectangle is not a square. $ABCD$ is a non-square rectangle.

Example 3

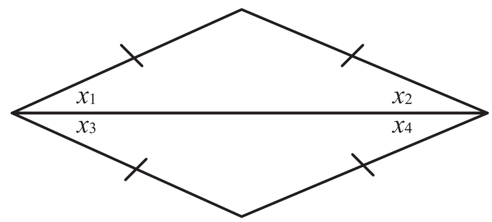


Figure 14.2

Prove that each diagonal in a rhombus bisects interior angles of the rhombus. That is, in the diagram in **Figure 14.2**, prove that $x_1 = x_3$ and $x_2 = x_4$.

Solution

The two triangles in the diagram are congruent by SSS. Thus, $x_1 = x_3$ and $x_2 = x_4$ because matching angles in similar triangles are congruent. (Do you see, in fact, that $x_1 = x_3 = x_2 = x_4$?)

Study Tip

- Don't attempt to memorize a table of facts about properties of diagonals of special quadrilaterals. Work, instead, at being adept at reconstructing the basic proofs behind the facts and deducing what the properties must be.

Pitfall

- Memorizing facts is joyless. Don't let mathematics become joyless.

Problems

1. The opposite sides of a quadrilateral are congruent. Prove that the quadrilateral must be a parallelogram. (See **Figure 14.3**.)

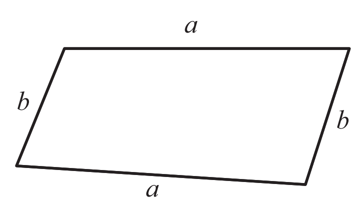


Figure 14.3

2. For one pair of sides of a quadrilateral, edges are *both* congruent and parallel. Prove that the quadrilateral must be a parallelogram. (See **Figure 14.4**.)

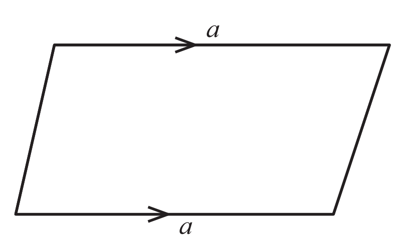


Figure 14.4

Comment: Feel free to use the results of Problems 1 and 2, and Example 1, throughout the remainder of this problem set. (And throughout the remainder of the course, too!)

3. Given: $ABCD$ is a parallelogram.
E is the midpoint of $\overline{AB}$.
F is the midpoint of $\overline{DC}$.

Prove: $EBFD$ is a parallelogram.

(See **Figure 14.5**.)

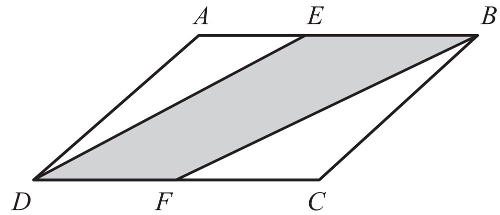


Figure 14.5

4. a) A parallelogram contains one right angle. (See **Figure 14.6**.) Explain why that parallelogram must be a rectangle.

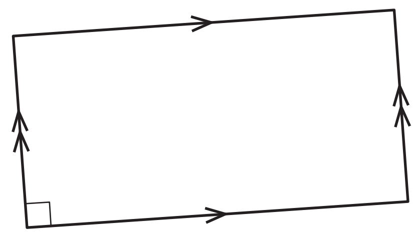


Figure 14.6

b) A parallelogram has two congruent adjacent sides. (See **Figure 14.7**.) Explain why that parallelogram must be a rhombus.

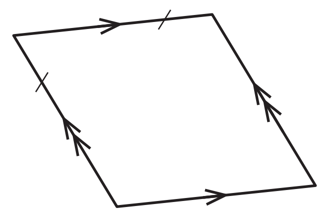


Figure 14.7

5. Two interior angles of a parallelogram come in a ratio of 11:7. What are the measures of all four interior angles of the parallelogram?

6. Suppose that $A = (7, 19)$, $B = (-4, 12)$, $C = (1, 1)$, and $D = (p, q)$. Find the values for p and q that make $ABCD$ a parallelogram.

7. The perimeter of parallelogram $FRED$ is 22 cm. The longest side of $FRED$ is 2 cm longer than the shortest side. What are the four side lengths of $FRED$?

8. Explain, in detail, why the figure formed by the midpoints of the sides of a rhombus is sure to be a rectangle.

9. Quadrilateral *DUCK* is a rhombus. Two of its vertices are $D = (4, 6)$ and $C = (8, -2)$.

a) Find the slope of diagonal $\overline{UK}$.

b) Find the equation of the line $\overleftrightarrow{UK}$.

10. Given: $TWXY$ is a parallelogram.
$\overline{YP} \perp \overline{TW}$.
$\overline{ZW} \perp \overline{TY}$.
$TP = TZ$.

Prove: $TWXY$ is a rhombus.

(See **Figure 14.8**.)

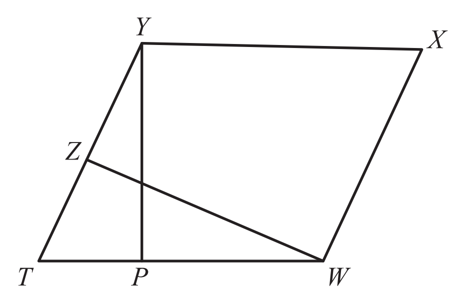


Figure 14.8

The Classification of Triangles

Lesson 15

Topics

- Classifying triangles via side lengths.
- Classifying triangles via angles.

Definitions

- **acute triangle**: A triangle with each of its three interior angles acute.
- **obtuse triangle**: A triangle with one interior angle that is an obtuse angle.
- **right triangle**: A triangle with one of its interior angles a right angle.
- **scalene triangle**: A triangle with three different side lengths.

Results

- Suppose that a triangle has three side lengths a, b, and c.
  - If $a^2 + b^2 > c^2$, then the angle between the sides of lengths a and b is acute.
  - If $a^2 + b^2 = c^2$, then the angle between the sides of lengths a and b is right.
  - If $a^2 + b^2 < c^2$, then the angle between the sides of lengths a and b is obtuse.
- The interior angle of largest measure in a triangle lies opposite the longest side of the triangle.

Summary

Having classified certain quadrilaterals in the previous lessons, it seems appropriate to also classify triangles—after all, they are the building blocks of polygons. In this lesson, we discuss two classification schemes: via side lengths and via measures of interior angles.

Example 1

Prove that the interior angle of a triangle of largest measure lies opposite the longest side of the triangle. Do this as follows.

Suppose that a triangle has interior angles of measures x and y and sides of lengths a and b, as shown in **Figure 15.1**. Assume that $b > a$.

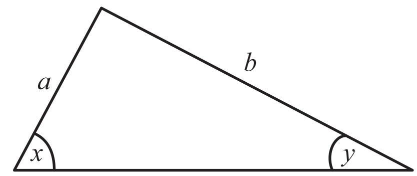


Figure 15.1

Draw a segment of length a on the side of length b to create an isosceles triangle. Mark the congruent base angles of the isosceles triangle as x_1, and mark the angle x_2, as shown in **Figure 15.2**. Clearly, $x > x_1$.

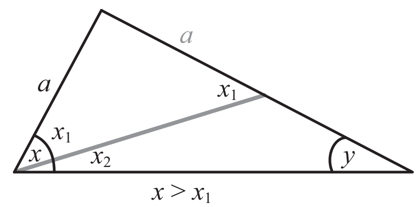


Figure 15.2

a) Prove that $x_1 > y$.

b) Explain why $x > y$.

c) Explain why the interior angle of largest measure in a triangle lies opposite the longest side of the triangle.

Solution

a) We see a triangle in the diagram with interior angles x_2, y, and $180 - x_1$. Because the angles in a triangle sum to 180°, we have $x_2 + y + 180 - x_1 = 180$. This gives $x_1 = y + x_2$, which shows that $x_1 > y$.

b) We have $x > x_1$ and $x_1 > y$. It follows that $x > y$.

c) We have just proved that if $b > a$, then the interior angle opposite b is larger than the interior angle opposite a. If b is the largest side length of the triangle, then the angle opposite b is also larger than the third interior angle (by the same reasoning). Thus, the angle opposite b is the largest interior angle.

Example 2

a) Show that a 5-7-7 triangle is an acute isosceles triangle.

b) Give an example of an obtuse isosceles triangle, and prove that your triangle really is obtuse. Also give an example of a right isosceles triangle.

c) Classify a 20-19-42 triangle.

Solution

a) The triangle is clearly isosceles. And because $5^2 + 7^2 > 7^2$, the largest angle in the triangle (one opposite a side length of 7) is acute. Thus, all three interior angles are acute, and it is an acute isosceles triangle.

b) A 2-2-3 triangle, for example, is isosceles and obtuse (because $2^2 + 2^2 > 3^2$). A 1-1-$\sqrt{2}$ triangle is isosceles and right.

c) A 20-19-42 triangle does not exist. (This is because 42 is not larger than 20 + 19.)

Example 3

What is the value of *y*, and why? (See **Figure 15.3**.)

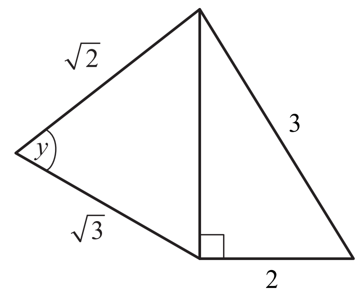


Figure 15.3

Solution

Angle *y* is part of a $\sqrt{2}$-$\sqrt{3}$-$\sqrt{5}$ triangle, and because $\left(\sqrt{2}\right)^2 + \left(\sqrt{3}\right)^2 = \left(\sqrt{5}\right)^2$, it is a right angle.

Example 4

What are the values of *z* and *w*? Explain. (See **Figure 15.4**.)

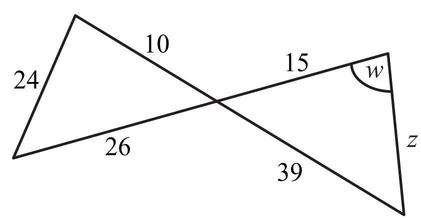


Figure 15.4

Solution

The two triangles are similar by SAS (vertical angles and the sides 10 and 15 and 26 and 39 come in a 2:3 ratio). Thus, $z = 36$. Because $15^2 + 36^2 = 39^2$, we have $w = 90°$.

Study Tip

- It is difficult to memorize the following: "If $a^2 + b^2 > c^2$, then the angle is … ." Instead, hold on to the image of the three squares on a right triangle and imagine how the area of the largest square changes as the right angle is decreased to an acute angle or increased to an obtuse angle.

Pitfall

- Save some work and apply the "$a^2 + b^2$ versus c^2" analysis to only one angle in a given triangle—namely, the largest one, opposite the largest side of the triangle.

Problems

1. Classify the following triangles as scalene, isosceles, or equilateral and as acute, right, or obtuse. Also describe where the largest angle of the triangle lies.

a) A triangle with sides 5, 8, 7.

b) A triangle with sides 10, 11, 2.

c) A triangle with sides 20, 21, 29.

d) A triangle with sides 13, 25, 39.

e) A triangle with sides 10, 10, 16.

2. Classify the following triangles with the following vertices.

a) $A = (0, 0)$, $B = (2, 0)$, $C = (0, 5)$.

b) $A = (10, 11)$, $B = (18, 5)$, $C = (19, -1)$.

c) $A = (-5, 1)$, $B = (1, 3)$, $C = (7, 1)$.

d) $A = (-2, 4)$, $B = (1, 8)$, $C = (7, 16)$.

3. a) Find an example of a right triangle with a hypotenuse that is double the length of one of its legs.

b) Find *all* isosceles right triangles with hypotenuse length 1.

4. Which is the longest side of a right triangle, and why?

5. a) Is it possible to construct a triangle with sides of lengths $\sqrt{5}$, $\frac{10}{3}$, and 5.97?

   b) A triangle has two sides of lengths 6 and 7. Give the range of possible values for the length of the third side.

   c) One side of a rectangle is 50 inches long. What can you say about the length of the diagonal of the rectangle?

   d) Two sides of a triangle have lengths *a* and *b* (with *a* smaller than *b*). Then, the third side must be greater than ______ in length and shorter than ______ .

6. Determine whether each of the following statements is *always*, *sometimes*, or *never* true.

   a) The base angles of an isosceles triangle are acute.

   b) The largest angle of a pentagon is opposite its longest side.

   c) A right triangle is scalene.

7. Arrange the angles x, y, and z in increasing order of measure. (See **Figure 15.5**.)

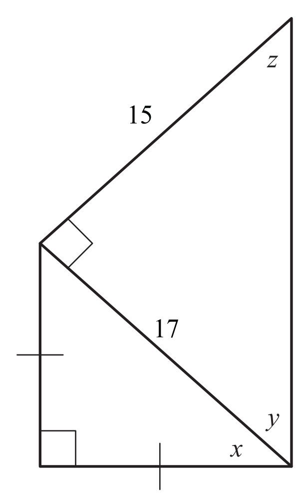


Figure 15.5

8. Prove: $AB + BC + CD + DA > 2(AC)$.

(See **Figure 15.6**.)

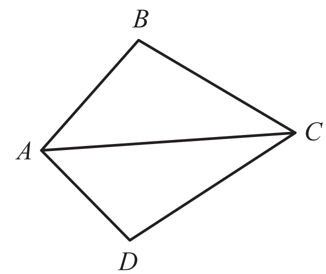


Figure 15.6

9. Given: $m\angle T > \mathrm{m}\angle \mathrm{TUS}$.

Prove: $UH + HS > TS$.

(See **Figure 15.7**.)

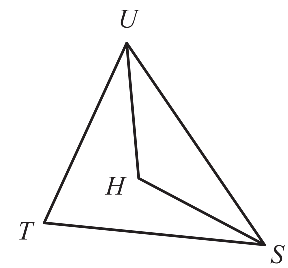


Figure 15.7

10. A triangle has sides of length x, $x + 2$, and 10.

Find the range of values of x for which *all* three of the following statements are simultaneously true.

a) This triangle exists.

b) 10 is the longest side of the triangle.

c) The triangle is acute.

"Circle-ometry"—On Circular Motion

Lesson 16

Topic

- The sine and cosine of angles computed via points on a circle.

Definitions

- **cosine of an angle (in circle-ometry)**: A point moves in a counterclockwise direction along a circle of radius 1. If the angle of elevation of the point above the positive horizontal axis is x, then the length of the horizontal displacement, left or right, of the point is denoted $\cos(x)$ and is called the cosine of the angle. (The cosine of an angle is deemed negative if the point is displaced to the left.)

- **sine of an angle (in circle-ometry)**: A point moves in a counterclockwise direction along a circle of radius 1. If the angle of elevation of the point above the positive horizontal axis is x, then the height of the point above the axis is denoted $\sin(x)$ and is called the sine of the angle. (The sine of an angle is deemed negative if the point lies below the horizontal axis.)

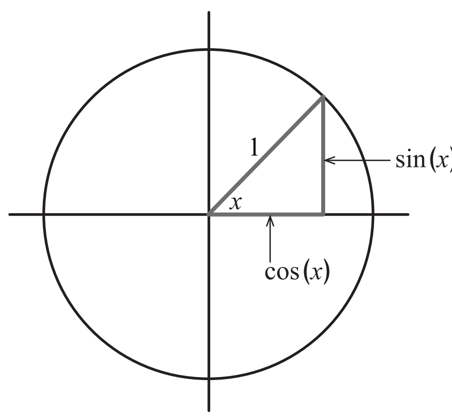


Figure 16.1

Formulas

- $\sin(x+360°)=\sin(x)$.
- $\cos(x+360°)=\cos(x)$.
- $\sin(-x)=-\sin(x)$.
- $\cos(-x)=\cos(x)$.
- $(\cos(x))^2+(\sin(x))^2=1$.

Summary

The topic of trigonometry first began as a theory of "circle-ometry," and the study of the motion of points on a circle is indeed the natural and appropriate introduction to the subject. This is a preliminary lesson that sets the stage for trigonometry. In it, we introduce the basic concepts of the sine and cosine of an angle.

Example 1

Find $\sin(120°)$ and $\cos(120°)$.

Solution

Draw a sketch. (See **Figure 16.2**.)

We see half of an equilateral triangle.

Thus, $\cos(120°) = -\frac{1}{2}$ (the displacement is in the negative direction), and an application of the Pythagorean theorem gives $\sin(120°) = \frac{\sqrt{3}}{2}$.

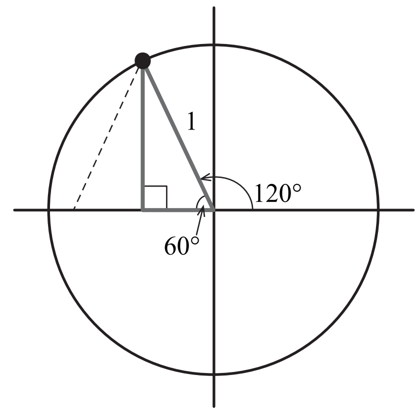


Figure 16.2

Example 2

Find $\sin(-135°)$ and $\cos(-135°)$.

Solution

Draw a sketch. (See **Figure 16.3**.)

We have a right isosceles triangle. An application of the Pythagorean theorem gives each side length of that triangle to be $\frac{1}{\sqrt{2}}$.

Thus, $\sin(-135°) = -\frac{1}{\sqrt{2}}$ and $\cos(-135°) = -\frac{1}{\sqrt{2}}$.

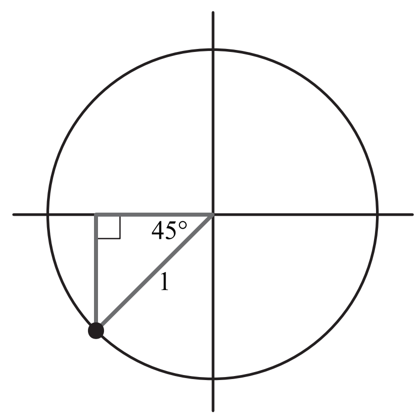


Figure 16.3

Example 3

Find *all* values for *x* satisfying $\sin(x) = \frac{1}{2}$.

Solution

A sketch shows that there are essentially two locations for which $\sin(x) = \frac{1}{2}$. (See **Figure 16.4**.)

We see half equilateral triangles, so these two locations are based on an angle of 30°.

Thus, $x = 30°$ or $150°$ or the addition of multiples of $360°$ to these two angles.

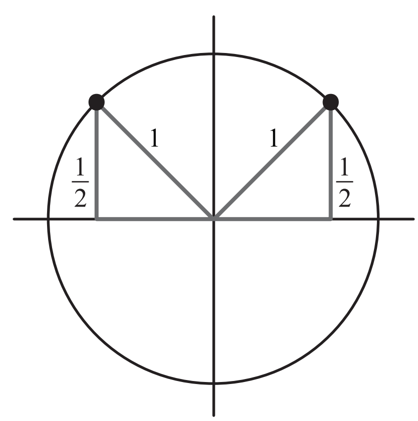


Figure 16.4

Study Tip

- To find the sine and cosine of a given angle, always sketch the location of the point in question on the circle of radius 1. (It never hurts to take the extra 15 seconds to draw a picture!) Look for right isosceles triangles and half equilateral triangles.

Pitfall

- Memorizing the values of sine and cosine for different angles is difficult and joyless. It is true that the values $\frac{1}{2}, \frac{1}{\sqrt{2}}$, and $\frac{\sqrt{3}}{2}$ will eventually stick in your mind, but don't force this. It never hurts to draw quick sketches to be sure that what you are half remembering is correct.

Problems

1. Find $\sin(-30°)$ and $\cos(-30°)$.

2. Find $\sin(150°)$ and $\cos(150°)$.

3. Find $\sin(-150°)$ and $\cos(-150°)$.

4. Find all values of x for which $\sin(x) = 0$.

5. Find all values of x for which $\cos(x) = 0$.

6. If $\sin(x) = 0.74$, what are the two possible values of $\cos(x)$?

7. Sketch a graph of $y = \cos(x)$.

8. The sine and cosine of an angle *x* are both negative. What can you say about the possible values of *x*?

9. Which is larger: $\sin(65°)$ or $\sin(130°)$?

10. Which is larger: $\sin(80°)$ or $\cos(100°)$?

Trigonometry through Right Triangles

Lesson 17

Topics

- The sine, cosine, and tangent of angles via ratios of side lengths of right triangles.
- Applications.

Definitions

- **cosine of an angle (in trigonometry)**: If x is a non–right angle in a right triangle, then the ratio of the length of the side of the triangle adjacent to the angle x (different from the hypotenuse) to the length of the hypotenuse of the right triangle is called the cosine of the angle and is denoted $\cos(x)$. (This value matches the "over-ness" of a point on a unit circle with angle of elevation x.)

- **sine of an angle (in trigonometry)**: If x is a non–right angle in a right triangle, then the ratio of the length of the side of the triangle opposite angle x to the length of the hypotenuse of the right triangle is called the sine of the angle and is denoted $\sin(x)$. (This value matches the height of a point on a unit circle with angle of elevation x.)

- **tangent of an angle (in trigonometry)**: If x is a non–right angle in a right triangle, then the ratio of the length of the side of the triangle opposite angle x to the length of the side adjacent to the angle (different from the hypotenuse of the right triangle) is called the tangent of the angle and is denoted $\tan(x)$.

Formulas

- $\sin(x) = \dfrac{\text{opp}}{\text{hyp}}$.
- $\cos(x) = \dfrac{\text{adj}}{\text{hyp}}$.
- $\tan(x) = \dfrac{\text{opp}}{\text{adj}} = \dfrac{\sin(x)}{\cos(x)}$.

Summary

By shifting the focus from circles to right triangles, the theory of "circle-ometry" is transformed into one with natural applications to architecture, surveying, and engineering. In this lesson, we define the sine and cosine (and tangent) in the setting of right triangles and exhibit some practical applications of these ratios.

Example 1

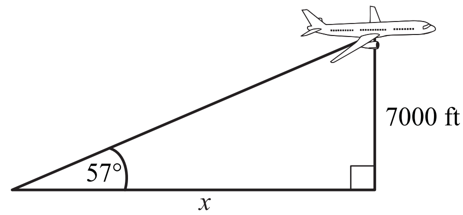


Figure 17.1

A plane is flying at an altitude of 7000 feet. From the ground, you spy the plane at an angle of elevation of 57°. What is x, your horizontal distance to the plane? (See **Figure 17.1**.)

Solution

In this problem, we have a right triangle for which we are given information about the side opposite to the angle of 57°, and we seek information about the side adjacent to it.

This suggests that we look at the ratio $\frac{\text{opp}}{\text{adj}} = \tan(57°)$.

A calculator gives $\tan(57°) \approx 1.539$, so $\frac{7000}{x} \approx 1.539$, giving $x \approx 4545$ feet.

Example 2

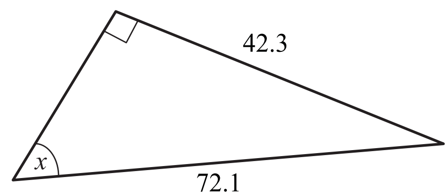


Figure 17.2

According to **Figure 17.2**, what are $\sin(x)$, $\cos(x)$, and $\tan(x)$, each to two decimal places?

Solution

$$\sin(x) = \frac{42.3}{72.1} \approx 0.59; \cos(x) = \frac{\sqrt{72.1^2 - 42.3^2}}{72.1} \approx 0.81; \tan(x) = \frac{\sin(x)}{\cos(x)} \approx \frac{0.59}{0.81} \approx 0.73.$$

Study Tip

- When solving right-triangle trigonometry problems, always ask the following questions: Which side lengths do we know? Which side length do we seek? Then, it becomes clear which of the three ratios—sine, cosine, or tangent—would be most useful for solving the problem.

Pitfall

- In trigonometry, only angles *different from the right angle* in the right triangle can be considered. Although $\sin(90°)$, for example, has meaning in the theory of circle-ometry (a star with an angle of elevation of 90° is directly overhead and, therefore, has height 1), in the context of right triangles, $\sin(90°)$ does not make sense: One cannot have a right triangle with a second right angle. When working with right triangles, one can only focus on the angles with measures strictly between 0° and 90°.

Problems

1. Calculate x and y in each diagram, each to two decimal places.

a)

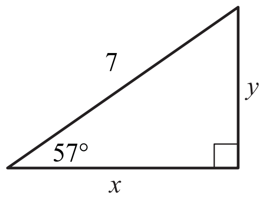


Figure 17.3

b)

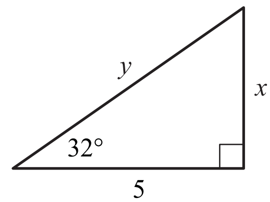


Figure 17.4

c)

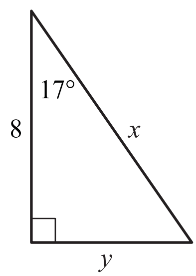


Figure 17.5

d)

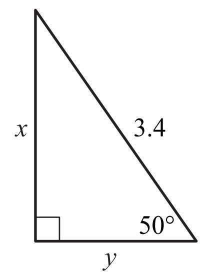


Figure 17.6

e)

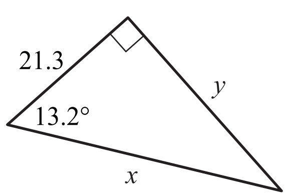


Figure 17.7

2. Find the values of a and b in each diagram, each to two decimal places.

a)

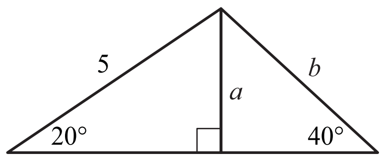


Figure 17.8

b)

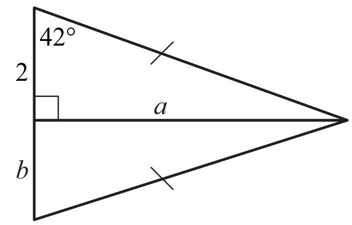


Figure 17.9

c)

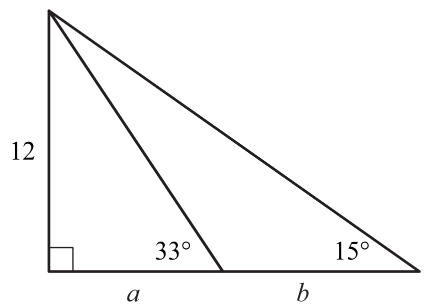


Figure 17.10

d)

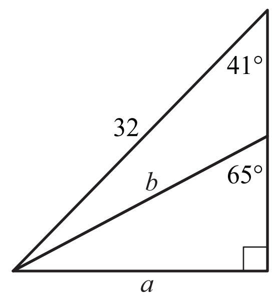


Figure 17.11

3. At 6 p.m., the Sun has an angle of elevation of 10.6°. At this time, when you stand tall, your shadow is 32 feet long. To the nearest inch, how tall are you?

4. A kite flies at an angle of elevation of 40°. If the string holding the kite is 100 m long, how high is the kite? (Assume that the string is pulled taut.)

5. A plane flying at an altitude of 850 feet sees the airport in the distance at an angle of depression of 33°. How far away (along the ground) is the airport?

6. Use **Figure 17.12** to explain why, if two angles x and y are complementary, then the sine of one angle equals the cosine of the other.

7. A jet climbs at an angle of 15°. How much altitude does it gain after it moves through the air 100 feet?

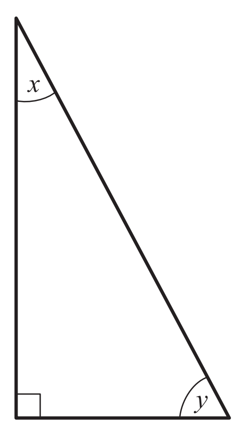


Figure 17.12

8. Compute the value of the length w. (See **Figure 17.13**.)

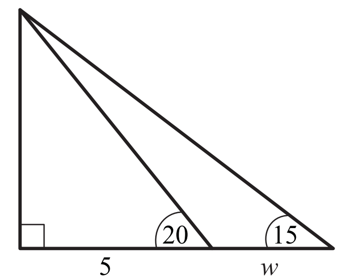


Figure 17.13

9. The angle of depression from the top of one building to the first-floor window of a neighboring building 100 feet away is 33° and to a second-floor window is 17°. To two decimal places, what is the vertical distance between the two windows?

10. Jake stands in a field and observes a bird to his north at an angle of elevation of 44°. After 3 minutes, the bird is to his south at an angle of elevation of 40°. Assuming that the bird flew in a straight path directly over Jake's head at a constant height of 20 feet and at constant speed, find the speed of the bird.

What Is the Sine of 1°?

Lesson 18

Topics

- The sine and cosine addition formulas.
- The "error term" in the Pythagorean theorem: the law of cosines.

Formulas

- addition formulas:
  - $\sin(x+y)=\sin(x)\cos(y)+\cos(x)\sin(y)$.
  - $\cos(x+y)=\cos(x)\cos(y)-\sin(x)\sin(y)$.

  Changing y to $-y$, these read (using $\sin(-y)=-\sin(y)$ and $\cos(-y)=\cos(y)$) as follows.

  - $\sin(x-y)=\sin(x)\cos(y)-\cos(x)\sin(y)$.
  - $\cos(x-y)=\cos(x)\cos(y)+\sin(x)\sin(y)$.

  Putting $y=x$ into the first equations gives the following.

  - $\sin(2x)=2\sin(x)\cos(x)$.
  - $\cos(2x)=(\cos(x))^2-(\sin(x))^2$.

  (This final equation is often abbreviated as $\cos(2x)=\cos^2(x)-\sin^2(x)$.)

- law of cosines: If a triangle has sides a, b, and c with angle x between sides a and b, then

  $$c^2=a^2+b^2-2ab\cos(x).$$

Summary

It is not at all obvious how a calculator works out the sine and cosine values for angles different from the familiar 0°, 30°, 45°, 60°, and 90°. In this lesson, we look at the sine and cosine addition formulas that allow us to compute trigonometric values for a wider range of values. However, as we see, the sine of 1° remains out of our reach—at least for this level of work in geometry. We need calculus!

Example 1

A calculator gives $\sin(1°) \approx 0.017$. What, then, is $\sin(2°)$ and $\sin(3°)$?

Solution

The obvious answer is to use a calculator again to compute these values! However, without it, we can do the following.

$$\sin(2°) = 2\sin(1°)\cos(1°).$$

Next, $\cos^2(1) + \sin^2(1) = 1$ shows that $\cos(1) = \sqrt{1-\sin^2(1)} \approx \sqrt{1-0.017^2}$, so

$$\sin(2°) = 2\sin(1°)\cos(1°) \approx 2 \times 0.017 \times \sqrt{1-0.017^2} \approx 0.034.$$

Also,

$$\begin{aligned}\sin(3°) &= \sin(1° + 2°)\\ &= \sin(1)\cos(2) + \cos(1)\sin(2)\\ &\approx 0.017 \times \sqrt{1-0.034^2} + \sqrt{1-0.017^2} \times 0.034\\ &\approx 0.051.\end{aligned}$$

Example 2

Find all three angles of the triangle in **Figure 18.1**.

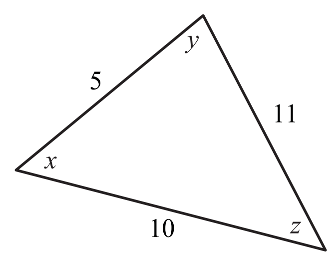


Figure 18.1

Solution

We have $5^2 = 10^2 + 11^2 - 220\cos z$, giving $\cos(z) = \dfrac{196}{220}$.

Using the "$\cos^{-1}$" button on a calculator to convert the answer to a cosine value and return the original angle, we see the following.

$$z = \cos^{-1}\left(\frac{196}{220}\right) \approx 27.0°.$$

Also, $11^2 = 5^2 + 10^2 - 100\cos x$, giving $x = \cos^{-1}\dfrac{4}{100} \approx 87.7°$, and $10^2 = 5^2 + 11^2 - 110\cos y$, giving $y = \cos^{-1}\dfrac{46}{110} \approx 65.3°$.

Study Tip

- There is no need to memorize the formulas in this lesson. Just look them up if ever you need them.

Pitfall

- Please don't memorize these formulas. (When you study calculus, you will see the true reason why these formulas look the way they do, and you will then be able to reconstruct them in your mind as you need them. Our understanding of trigonometry is still too young to see the big picture of it all.)

Problems

1. Find exact values of $\sin(75)$, $\cos(75)$, and $\tan(75)$. (Recall that $\tan(x) = \frac{\sin(x)}{\cos(x)}$.)

2. Consider **Figure 18.2**.

a) Use trigonometry to find the value of *d*.

b) Find *d* using the Pythagorean theorem.

c) Are the two answers the same?

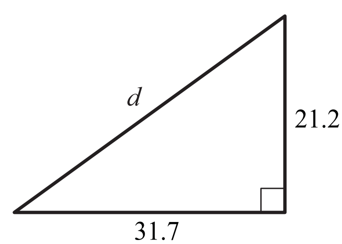


Figure 18.2

3. Consider the rectangular box shown in **Figure 18.3**.

a) What is the length of *AB*?

b) To one decimal place, what is the measure of angle *CAB*?

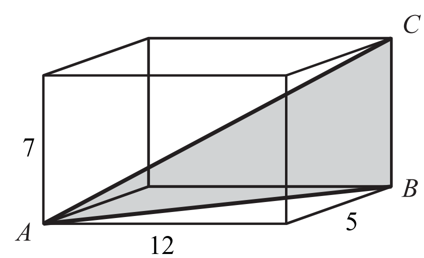


Figure 18.3

4. What is the measure of the largest angle in a 10-6-6 isosceles triangle?

5. What is the measure of the smallest angle in a 3-4-5 right triangle?

6. Show that $\tan(2x) = \dfrac{2\tan x}{1-\tan^2 x}$.

7. Show that $\tan(x+y) = \dfrac{\tan x + \tan y}{1-\tan x \tan y}$.

8. Consider a triangle with side lengths a, b, and c, as shown in **Figure 18.4**. Let m be the length of the median of the triangle to the side c. (This line connects one vertex of the triangle to the midpoint of the side of length c.)

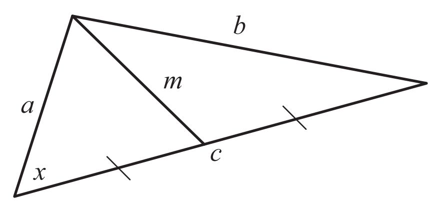


Figure 18.4

Use the law of cosines (twice—each time to the angle marked x) to establishes Apollonius's equation:

$$a^2 + b^2 = \frac{c^2}{2} + 2m^2.$$

9. The diagram in **Figure 18.5** shows that $\tan\left(\dfrac{x}{2}\right) = \dfrac{\sin x}{1+\cos x}$. How?

It also shows that $\tan\left(\dfrac{x}{2}\right) = \dfrac{1-\cos x}{\sin x}$. How?

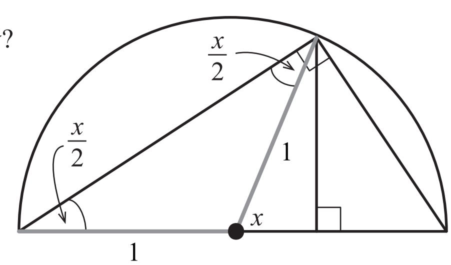


Figure 18.5

10. Prove that $\sin x + \sin y = 2\sin\left(\dfrac{x+y}{2}\right)\cos\left(\dfrac{x-y}{2}\right)$.

The Geometry of a Circle

Lesson 19

Topics

- Notation and jargon.
- The converse of Thales's theorem.
- The tangent/radius theorem.
- The two tangents theorem.
- The inscribed/central angle theorem.

Definitions

- **arc**: For two given points on a circle, a section of the circumference of the circle between them is called an arc of the circle. The measure of an arc is the measure of the angle between the two radii connecting those two given points. (There are two choices of angle between the two radii. The region between the two radii that contains the given arc defines which angle to measure.)
- **central angle**: An angle formed by two radii of a circle. (This also defines the measure of the arc contained in the region specified by the angle.)
- **chord**: A line segment connecting two points on a circle.
- **diameter**: A chord of a circle that passes through the center of the circle is called a diameter of the circle. The length of any such diameter is called the diameter of the circle. (And as lengths, the diameter of a circle is twice the radius of the circle.)
- **inscribed angle**: If P and Q are two endpoints of an arc of a circle and A is a point on the circle not on the arc, then $\angle PAQ$ is an inscribed angle.
- **radius**: A line segment connecting the center of a circle to a point on the circle is called a radius of the circle. The length of any such line segment is called the radius of the circle.
- **secant to a circle**: A line that intercepts a circle at two distinct points.
- **tangent to a circle**: A line that meets just one point of a circle.

Results

- radius/tangent theorem:

  The angle between a radius and a tangent to a circle (with the radius meeting the point of contact of the tangent line) is a right angle. (See **Figure 19.1**.)

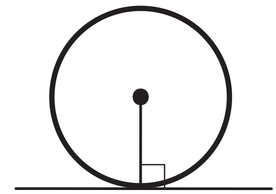


Figure 19.1

- two tangents theorem:

  Two tangent segments from a given point to a circle are congruent. (See **Figure 19.2**.)

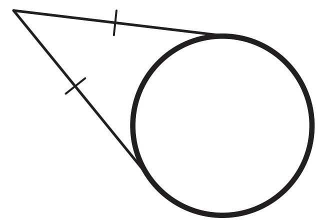


Figure 19.2

- inscribed/central angle theorem:

  All inscribed angles from a common arc are congruent and have measure half the measure of the arc. (See **Figure 19.3**.)

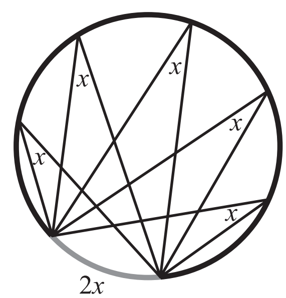


Figure 19.3

Summary

Having skirted around the topic of circles for a number of lessons, it is time to study circles as objects of interest in their own right and prove theorems about them. In this lesson, we establish three classic results: the radius/tangent theorem, the two tangents theorem, and the inscribed/central angle theorem. Our work begins with a puzzle that happens to establish the converse of Thales's theorem from Lesson 1.

Example 1

Find the values of x and y in the **Figure 19.4**.

Solution

By the two tangents theorem, $y = 2$ and $10 = 3 + x$, giving $x = 7$.

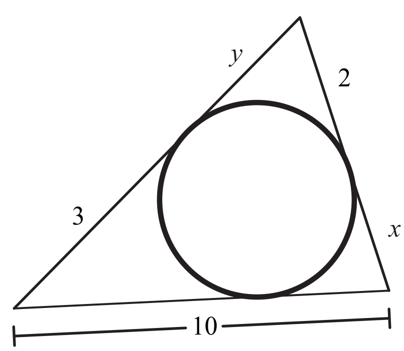


Figure 19.4

Example 2

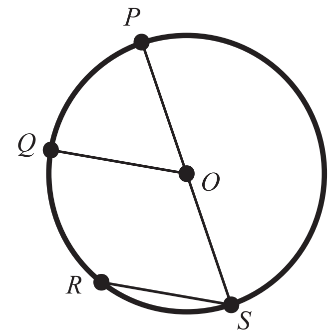


Figure 19.5

In **Figure 19.5**, O is the center of the circle.

Given: $\overline{PS}$ is the diameter.
$\overline{OQ} \parallel \overline{SR}$.

Prove: Arcs PQ and QR are congruent.

Solution

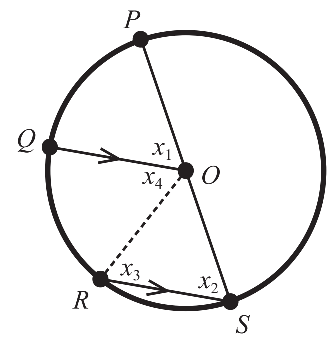


Figure 19.6

i. Draw an additional radius and label angles x_1, x_2, x_3, and x_4, as shown in **Figure 19.6**.

ii. $x_1 \cong x_2$ because they are corresponding angles for parallel lines $\overline{OQ} \parallel \overline{SR}$.

iii. $x_2 \cong x_3$ because they are base angles of an isosceles triangle ($\triangle ORS$).

iv. $x_3 \cong x_4$ because they are alternate interior angles for parallel lines.

v. Thus, $x_1 \cong x_4$.

vi. The two arcs in consideration are congruent because x_1 is the measure of arc PQ and x_4 is the measure of arc QR.

Example 3

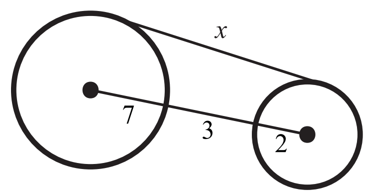


Figure 19.7

Two circles of radii 7 and 2 have centers 12 units apart.
Find the length x of the common tangent shown in **Figure 19.7**.

Solution

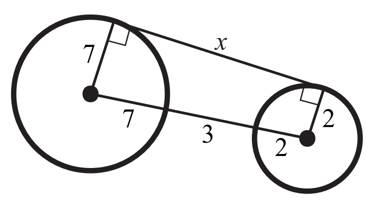


Figure 19.8

Draw in two radii to make a quadrilateral containing two right angles (noting the radius/tangent theorem) as shown in **Figure 19.8**.

Draw a dotted line to make a rectangle within this quadrilateral. (See **Figure 19.9**.)

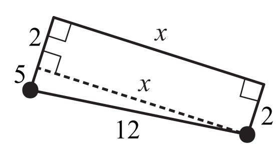


Figure 19.9

The Pythagorean theorem gives $x^2 + 5^2 = 12^2$, yielding $x = \sqrt{119}$.

Example 4

Sydney, Australia, has latitude 34° south. (See **Figure 19.10**.)

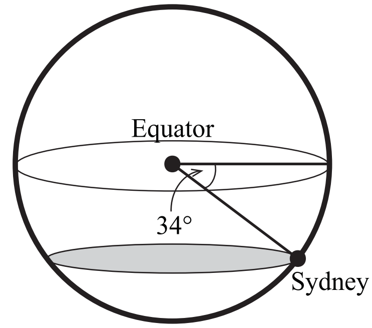


Figure 19.10

The radius of the Earth is approximately 6400 km.

Find the radius r of the circle of constant latitude 34° south.

Solution

Note that, like a circle in two dimensions, a sphere is a set of points equidistant from a given point in three-dimensional space.

The key to answering this question is to note that the line from the center of the Earth to Sydney is also a radius. (See **Figure 19.11**.)

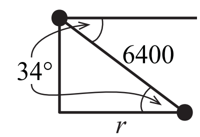


Figure 19.11

We see $\cos 34 = \dfrac{r}{6400}$, giving $r = 6400 \cos 34 \approx 5306$ km.

Example 5

Find the value of the length a in **Figure 19.12**.

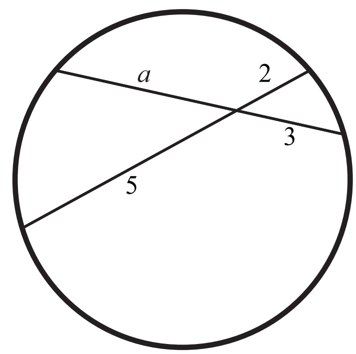


Figure 19.12

Solution

Draw two chords, as shown in **Figure 19.13**, to create two triangles.

The triangles have matching vertical angles and matching inscribed angles (subtended from the same arc) and, therefore, by AA, are similar.

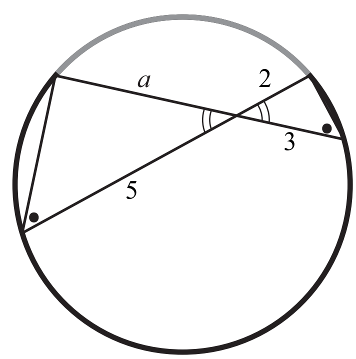


Figure 19.13

Consequently, matching sides come in the same ratio: $\dfrac{2}{a} = \dfrac{3}{5}$.
This gives $a = 3\dfrac{1}{3}$.

Study Tip

- Remember that the measure of an arc is a measure of an amount of turning (angle), not a physical length. It is the angle formed by the two radii to the endpoints of the arc.

Pitfall

- Drawing in the radii to the endpoints of an arc marked with an angle measure can make a picture overly complicated. Do this only if it seems truly helpful to make the central angle explicit.

Problems

1. Complete the proof of the inscribed/central angle theorem by showing that $y = 2x$ in this picture of a central angle and an inscribed angle on a common arc of a circle. (See **Figure 19.14**.)

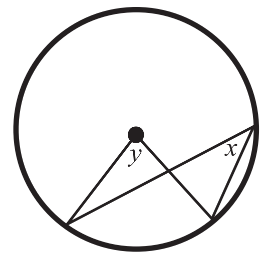


Figure 19.14

2. Find x, y, and z for the following circle with center O. (See **Figure 19.15**.)

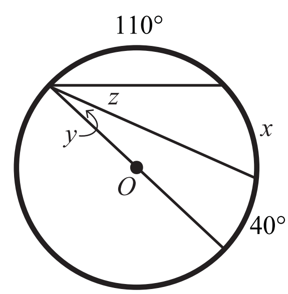
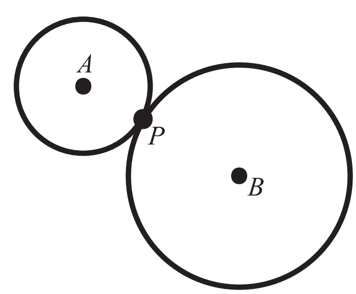


Figure 19.15

3. Two congruent circles with centers A and B each have radii of 7 inches. If $AB = 8$ inches, find the length of their common chord.

Hint: Draw a picture. Must the two circles intersect? What must "common chord" mean?



4. Circles with centers A and B are externally tangent with point of contact P. (See **Figure 19.16**.)

Explain why A, P, and B are sure to be collinear.

Figure 19.16

5. Tangent circles with centers A and B have radii of 8 and 6, respectively.

   Find the length of the common tangent segment shown in **Figure 19.17**.

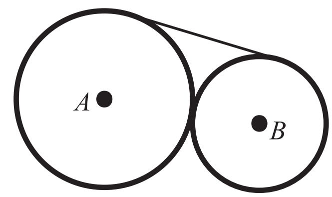


Figure 19.17

6. Explain why the two chords shown in **Figure 19.18** must be parallel.

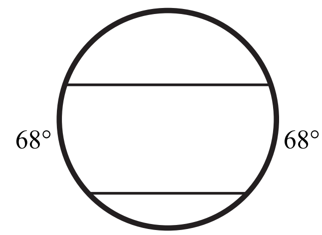


Figure 19.18

7. Find the measure of angle a in the diagram in **Figure 19.19**.

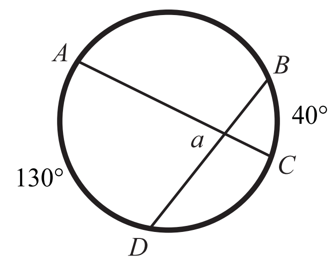


Figure 19.19

8. Prove that $ab = cd$. (See **Figure 19.20**.)

9. The four corners of a parallelogram happen to lie on a circle. Explain why that parallelogram must be a rectangle.

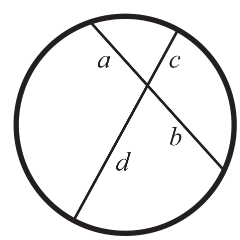


Figure 19.20

10. The circle with center P of radius 25 and the circle with center Q of radius 29 intersect at A and B. (See **Figure 19.21**.)

    If $PQ = 36$, what's AB?

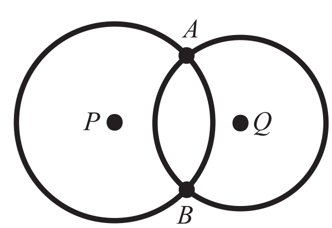


Figure 19.21

The Equation of a Circle
Lesson 20

Topics

- The equation of a circle.
- Pythagorean triples.

Formula

- The equation of a circle with center $C = (a, b)$ and radius r is $(x - a)^2 + (y - b)^2 = r^2$. (This is just an application of the Pythagorean theorem.)

Summary

In this lesson, we find an equation that must be true for a point (x, y) to lie on a circle with a given center and a given radius. As a bonus, we explore an interesting connection to Pythagorean triples.

Example 1

What is the equation of a circle with center $(0, 3)$ and radius 29? Is the point $(20, 24)$ on the circle? Is the point $(-10, 10)$ on the circle?

Solution

The equation is $x^2 + (y - 3)^2 = 29^2 = 841$. Noting that $20^2 + (24 - 1)^2 = 20^2 + 21^2 = 400 + 441 = 841$, we see that $(20, 24)$ does indeed fit the equation, so this point lies on the circle. Computing $(-10)^2 + 7^2 = 149 < 841$ shows that the point $(-10, 10)$ is not on the circle. (In fact, it is inside the circle.)

Example 2

Find the equation of the circle with center $(\pi, -\pi)$ that passes through the point $(2\pi, 0)$.

Solution

The radius of the circle is the distance between the two given points—namely, $\sqrt{\pi^2 + \pi^2} = \sqrt{2\pi^2}$. Thus, the equation of the circle is $(x - \pi)^2 + (y - (-\pi))^2 = \left(\sqrt{2\pi^2}\right)^2$. That is,

$$(x - \pi)^2 + (y + \pi)^2 = 2\pi^2.$$

Example 3

Explain why the circle $(x+13)^2 + \left(y - 2\frac{1}{2}\right)^2 = \frac{25}{4}$ must be tangent to the x-axis.

Solution

The center of the circle is $\left(-13, 2\frac{1}{2}\right)$, and its radius r satisfies $r^2 = \frac{25}{4}$, giving $r = \frac{5}{2} = 2\frac{1}{2}$.

The center of the circle is the same distance above the x-axis as the radius of the circle. The circle must therefore be tangent to the axis.

Study Tip

- A quick sketch of a right triangle shows that for (x, y) to be a distance r from a center point (a, b), we have $(x - a)^2 + (y - b)^2 = r^2$. There is no need to memorize the equation of a circle. Just see it as yet another application of the Pythagorean theorem.

Pitfall

- The equation $(x + 5)^2 + (y + 62)^2 = 28$, for example, represents a circle with center $(-5, -62)$ and radius $r = \sqrt{28}$. Watch out for two things: We need the *difference* of coordinates on the left part of the equation and the radius *squared* on the right part of the equation. (These observations make sense with a sketch and the Pythagorean theorem.)

Problems

1. Write the centers and radii of the following circles.

a) $(x-2)^2 + (y-3)^2 = 16.$

b) $(x-7)^2 + (y-5)^2 = 5.$

c) $(x+7)^2 + (y-1)^2 = 10.$

d) $(x+25)^2 + y^2 = 1.$

e) $x^2 + y^2 = 19.$

f) $(x-17)^2 + (y+17)^2 = 17.$

g) $x^2+2x+y^2-8y=8.$

h) $x^2-20x+y^2-10y+25=0.$

2. Write the equation of a circle with the following information.

a) Center $(2, 2)$ and radius 9.

b) Center $(-3, 0)$ and radius 4.

c) Center $\left(-\frac{1}{2}, -\frac{1}{3}\right)$ and radius $\frac{1}{4}$.

d) Center $(0.67, 9.21)$ and radius 0.1.

3. Consider the circle with equation $(x-2)^2+(y+1)^2=169$.

a) What is its center? What is its radius?

b) Is the point $(7, 11)$ on the circle?

c) Is the point $(7, -11)$ on the circle?

d) Find a point on the circle with x-coordinate 2. Find a second point on the circle with x-coordinate 2.

e) How many points on the circle have y-coordinate 12?

f) Find two points on the circle that are 26 units apart.

4. Find the equation of the circle with center $(-1, 2)$ that passes through $(2, 9)$.

5. Find the equation of the circle with center $(-4, 6)$ tangent to the y-axis.

6. Consider the circles with the following equations.

$$x^2 + y^2 = 25.$$
$$(x-9)^2 + (y-12)^2 = 100.$$

a) What are the radii of the circles?

b) What is the distance between the centers of the circles?

c) Make a rough sketch of the two circles to explain why the circles must be tangent to one another.

7. Consider the circles with the following equations.

$$x^2 + y^2 = 2.$$
$$(x-3)^2 + (y-3)^2 = 32.$$

a) What are the radii of the circles?

b) What is the distance between the centers of the circles?

c) Make a rough sketch of the two circles to explain why the circles must be tangent to one another.

8. Let $A = (3, 7)$ and $B = (-18, 27)$. Write the equation of the circle that has $\overline{AB}$ as its diameter.

9. Sheila claims that the two circles given by $(x + 2)^2 + (y - 4)^2 = 49$ and $x^2 + y^2 - 6x + 16y + 37 = 0$ are externally tangent. She is right. Show that she is.

10. (OPTIONAL)

a) Let $A = (2, 7)$ and $B = (4, 11)$. Find the equation of the circle with $\overline{AB}$ as its diameter.

b) Show that your answer to part a) can be rewritten as $(x - 2)(x - 4) + (y - 7)(y - 11) = 0$.

c) Show that if $A = (a_1, a_2)$ and $B = (b_1, b_2)$, then $(x - a_1)(x - b_1) + (y - a_2)(y - b_2) = 0$ is indeed the equation of the circle with $\overline{AB}$ as the diameter (albeit written in a different guise).

Understanding Area
Lesson 21

Topics

- The area congruence and area addition postulates.
- The areas of rectangles, triangles, polygons and regular polygons.

Formulas

- area of a rectangle: length × width.
- area of a triangle: $\frac{1}{2}$ base × height. (This formula applies no matter which side is considered the base.)
- area of a polygon: Subdivide into triangles.
- area of a regular N-gon: Subdivide N triangles, each with its apex at the center of the polygon.

Summary

It is very difficult to pin down exactly what we mean by the "area" enclosed by a geometric figure. We can only rely on identifying key beliefs we hold about what area means and how we operate with it. In this lesson, we establish two fundamental area postulates, make a single declaration about the areas of rectangles, and explore the concept of area for polygons.

Example 1

A parallelogram has height h and "base" of length b, as shown in **Figure 21.1**. Find a formula for its area.

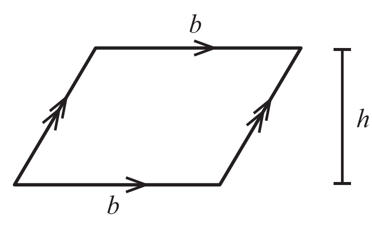


Figure 21.1

Solution

Draw a diagonal, as shown in **Figure 21.2**, to divide the figure into two triangles.

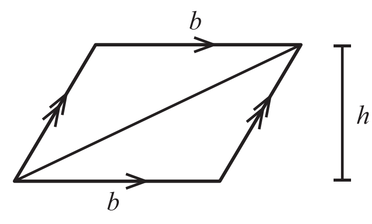


Figure 21.2

Each triangle has base b and height h and, thus, area $\frac{1}{2}bh$.

By the area addition postulate, the area of the parallelogram is $\frac{1}{2}bh + \frac{1}{2}bh = bh$.

Example 2

A trapezoid has bases of length a and b and height h, as shown in **Figure 21.3**. Find a formula for its area.

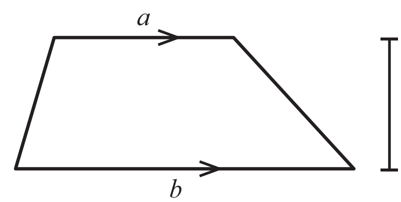


Figure 21.3

Solution

Draw a diagonal, as shown in **Figure 21.4**.

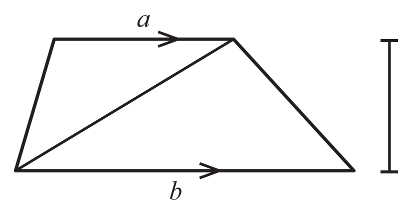


Figure 21.4

We see that $\text{area} = \frac{1}{2}ah + \frac{1}{2}bh = \frac{1}{2}(a+b)h.$

Comment: A triangle can be regarded as a trapezoid with base a of length 0.

Example 3

A regular hexagon has area 100 cm². What is its side length?

Solution

Call the side length of the regular hexagon s, as shown in **Figure 21.5**.

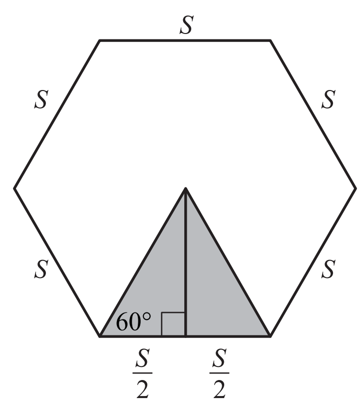


Figure 21.5

The interior angle of a regular hexagon is $180° - \frac{360^\circ}{6}° = 120°$.

Thus, the shaded triangle shown has interior angles 60° and is an equilateral triangle.

The Pythagorean theorem gives the height of the triangle as

$$\sqrt{s^2 - \left(\frac{s}{2}\right)^2} = \sqrt{\frac{3}{4}s^2} = \frac{\sqrt{3}}{2}s.$$

Thus, the area of the hexagon is $6 \times \frac{1}{2} \cdot s \cdot \frac{\sqrt{3}}{2}s = \frac{3\sqrt{3}}{2}s^2$.

Setting this equal to 100 gives $s^2 = \frac{2}{3\sqrt{3}}100 = \frac{200}{3\sqrt{3}}$, so $s = \sqrt{\frac{200}{3\sqrt{3}}}$ cm.

Study Tip

- Only keep the formulas for the area of a rectangle and the area of a triangle in mind. (In fact, they are probably already there from early school days!) All other formulas readily follow from these if you keep your wits about you.

Pitfall

- You can memorize that the area of a parallelogram is “base times height” or that the area of a rhombus is “half the product of its diagonals,” and so on, if you wish. Be selective about which formulas you feel are helpful to keep in your mind because they are easy to muddle. (Realize that all polygons can be decomposed into triangles.)

Problems

1. Show that the area of a rhombus equals half the product of the lengths of its diagonals.

2. Five students were asked to write a formula for the area A between two squares with dimensions as shown in **Figure 21.6**.

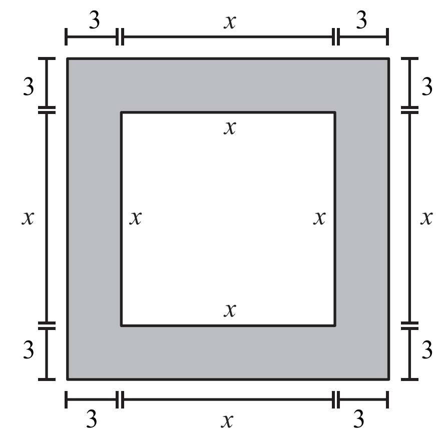


Figure 21.6

Albert thought of this shaded region as the union of four 3×3 squares and four $3 \times x$ rectangles. (See **Figure 21.7**.)

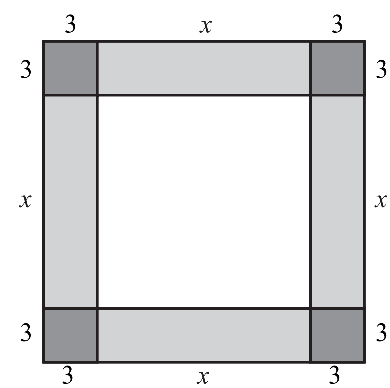


Figure 21.7

Thus, he was compelled to write $A = 12x + 36$.

a) Bilbert wrote $A = (x + 6)^2 - x^2$. How was he viewing the figure to be led to this formula?

b) Cuthbert wrote $A = 4 \times 3(x + 3)$. What was he seeing that led him to write this formula?

c) Dilbert wrote $A = 4 \cdot 3(x+6) - 4 \cdot 9$. What did he visualize to see this formula as the natural answer to the problem?

d) Egbert wrote $A = 2 \times 3(x + 6) + 2 \times 3x$. What did Egbert see to lead him to this expression?

e) Show that all five expressions are algebraically equivalent.

3. a) Lenny made a 4×9 rectangular cake. John made a square cake of the same area. What is the side length of John's cake?

b) What square has the same area as an 8×12 rectangle?

c) In general, what is the side length of a square with the same area as an $a \times b$ rectangle? (This quantity is called the geometric mean of a and b.)

4. A parallelogram has sides 8 cm and 10 cm and one interior angle of 45°. What is the area of the parallelogram?

5. Use trigonometry to find the area of an isosceles triangle with base length of 20 cm and with base angles of 72°.

6. Use trigonometry to find the area of a regular decagon of side length 2 inches.

7. Find the areas of the following figures.

a)

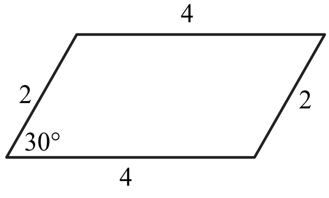


Figure 21.8

b)

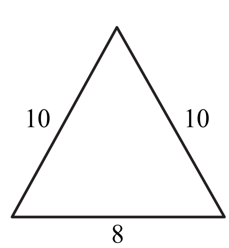


Figure 21.9

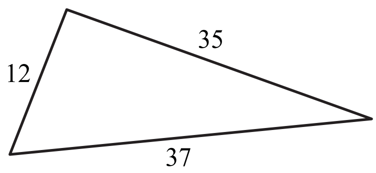
c) 

Figure 21.10

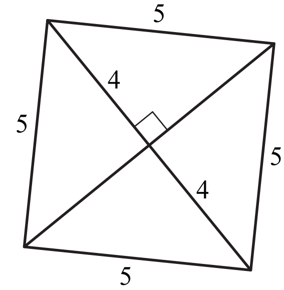
d) 

Figure 21.11

8. Find the area of each of the following.

a) An isosceles right triangle with hypotenuse 7 inches.

b) An equilateral triangle with perimeter 30 inches.

c) A right triangle containing a 30° angle and with hypotenuse 20 inches.

d) A square inscribed in a circle of radius r.

e) A rectangle with length 8 inscribed in a circle of radius 5.

9. Here in **Figure 21.12**, M is the midpoint of $\overline{AB}$.

a) What can you say about the areas of triangles I and II?

b) $\triangle ABC$ has area 50 square units, and $BM = 10$. What is the height of $\triangle AMC$ (with $\overline{AM}$ considered its base)?

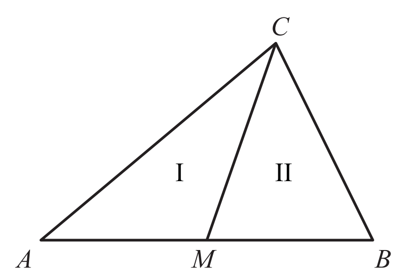


Figure 21.12

Lesson 21: Understanding Area

10. In the diagram in **Figure 21.13**, *ABCDEF* is a regular hexagon, and the area of the shaded region shown is 20 square units. What is the area of the entire hexagon? What is an elegant way to reach an answer to this question?

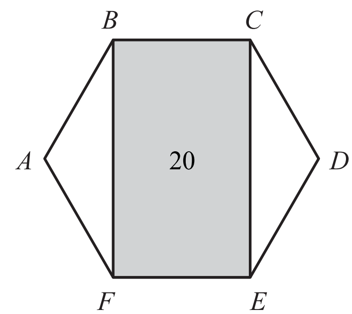


Figure 21.13

Explorations with Pi

Lesson 22

Topics

- Pi.
- Circumference, arc length, and area formulas.

Definitions

- **arc length**: The length of an arc of a circle.
- **pi**: In flat geometry, the ratio of the circumference of a circle to its diameter is the same for all circles. The common value of this ratio is called pi and is denoted π.
- **sector of a circle**: The figure formed by two radii of a circle and an arc of the circle between them.

Formulas

- If C is the circumference of a circle, D is its diameter, r is its radius $(\text{thus, } D = 2r)$, and A is its area, then

$$\pi = \frac{C}{D}.$$
$$C = 2\pi r.$$
$$A = \pi r^2.$$

- If an arc of the circle of radius r has measure $x°$, then the length of the arc is $\frac{x}{360} \cdot 2\pi r$.
- The area of the sector defined by that arc is $\frac{x}{360} \cdot \pi r^2$.

Summary

We like to believe that the ratio of the circumference of a circle to its diameter is the same value for all circles. In this lesson, we examine the basis of that belief and examine the arc length and area formulas that follow from it.

Example 1

a) What is the radius of a circle with the same area as a square of side length 4 units?

b) What is the area of the circle with equation $(x-4)^2 + y^2 = 17$? What is the circumference of this circle?

Solution

a) We need $\pi r^2 = 16$. This gives $r = \dfrac{4}{\sqrt{\pi}}$.

b) This is the equation of a circle of radius $\sqrt{17}$.

The circumference of this circle is $2\pi\sqrt{17}$, and its area is 17π.

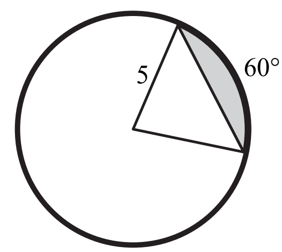


Figure 22.1

Example 2

Find the area of the shaded region shown in **Figure 22.1**.

Solution

We can compute the area of the shaded region by computing the area of the entire sector with central angle 60° and subtracting from it the area of the isosceles triangle (in fact, equilateral triangle because all angles must have measure 60°) with side length 5. (See **Figure 22.2**.)

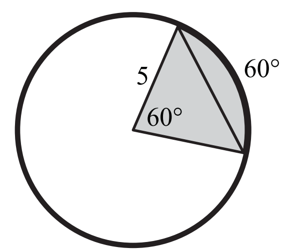


Figure 22.2

area of sector = $\dfrac{60}{360}\pi 5^2 = \dfrac{25}{6}\pi$.

area of equilateral triangle = $\dfrac{1}{2}\cdot 5\cdot\sqrt{5^2-\left(\dfrac{5}{2}\right)^2} = \dfrac{25\sqrt{3}}{4}$.
(See **Figure 22.3**.)

Thus, the area of the shaded region is $\dfrac{25}{6}\pi - \dfrac{25\sqrt{3}}{4}$ square units.

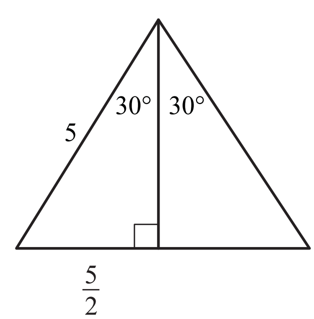


Figure 22.3

Example 3

A small circle is inscribed in an equilateral triangle, which is inscribed in a larger circle of radius 2.

Without a calculator, find the area of the shaded region shown. (See **Figure 22.4**.)

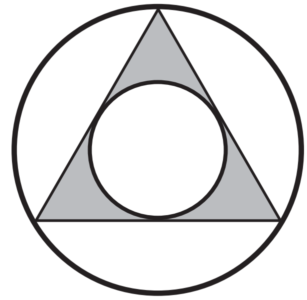


Figure 22.4

Solution

The small triangle shown in **Figure 22.5** is half an equilateral triangle. Thus, $a = 1$ and $b = \sqrt{2^2 - 1^2} = \sqrt{3}.$

The area of the small circle in the original figure is $\pi 1^2 = \pi$.

The area of the equilateral triangle in the original figure is $\frac{1}{2} \cdot 2\sqrt{3} \cdot 3 = 3\sqrt{3}.$

The area of the shaded region we seek is $3\sqrt{3} - \pi.$

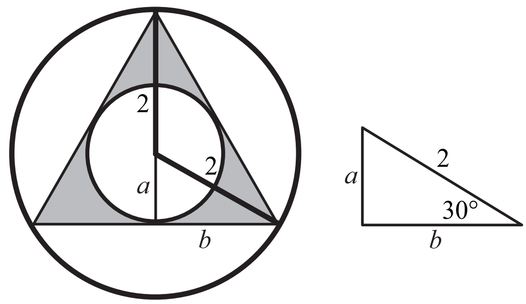


Figure 22.5

Study Tip

- Many geometry curricula have students memorize the side lengths of right triangles containing angles 30° and 60°. (Such triangles, of course, are half equilateral triangles.)

  If the short side of the triangle has length q, then the hypotenuse is double this, $2q$, and the Pythagorean theorem then gives the length of the third side as $\sqrt{3}q$. Because many textbook problems are designed to make use of these triangles, it might be handy (but not necessary) to have these side length values in your mind. (See **Figure 22.6**.)

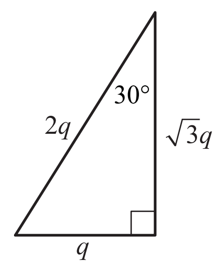


Figure 22.6

Pitfall

- Memorizing values is joyless! It is always fine to take the extra few seconds to draw half equilateral triangles in a margin and compute their side lengths.

Problems

1. An arc of a circle has length $\frac{7}{2}\pi$ and measure 315°. What is the radius of the circle?

2. A circle has radius 2. What do you notice about the area of a sector of that circle with central angle $x°$ and the length of the arc on that sector?

3. Find the areas of the shaded regions shown.

a)

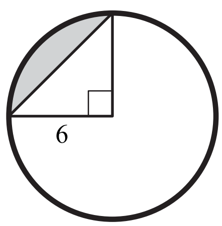


Figure 22.7

b)

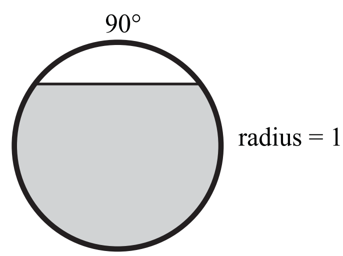


Figure 22.8

c)

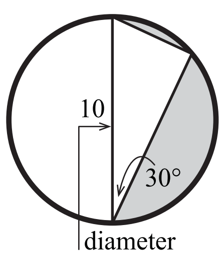


Figure 22.9

d)

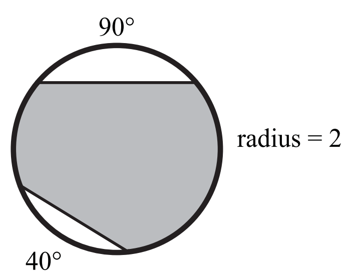


Figure 22.10

4. A dog is tied to the corner of two buildings with a leash that is 13 yards long, as shown in **Figure 22.11**. (All measurements are in yards.)

The dog is convinced that a bone is buried somewhere in the yard, but its movement, of course, is restricted by the length of the leash. Find the total area of ground the dog can reach from its leash.

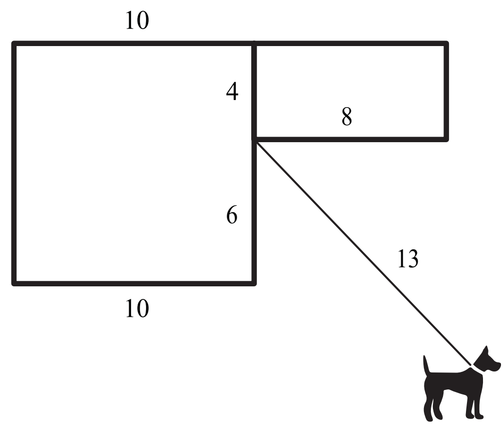


Figure 22.11

5. Two rectangles and two sectors are arranged as shown in **Figure 22.12**.

Each rectangle is 2 inches by 8 inches. What is the sum of the areas of the two sectors?

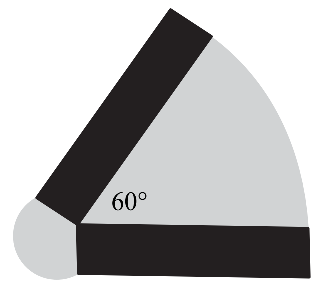


Figure 22.12

6. **Figure 22.13** shows two concentric circles and a line segment of length 7 tangent to the small circle.

If the radius of the small circle is 6, what is

a) the radius of the large circle?

b) the area between the two circles?

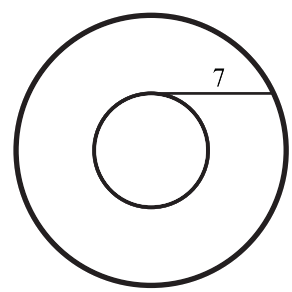


Figure 22.13

7. A sector is inscribed in a regular heptagon. The sides of the sector match the sides of the heptagon. What, to one decimal place, is the area of the shaded region shown if the side length of the figure is 2 inches? (See **Figure 22.14**.)

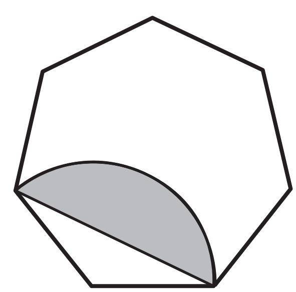


Figure 22.14

8. You have a piece of string that is 40 inches long.

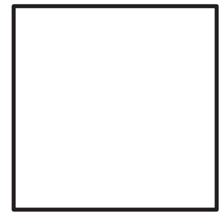


Figure 22.15

a) You use this string to make the perimeter of a square.

What is the area of the square?

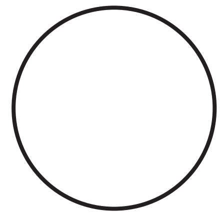


Figure 22.16

b) You use the string to make the circumference of a circle.

What is the area of the circle?

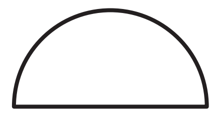


Figure 22.17

c) You use the string to make the perimeter of a semicircle (including its diameter).

What is the area of this figure?

9. a) What curious property do the following figures share?

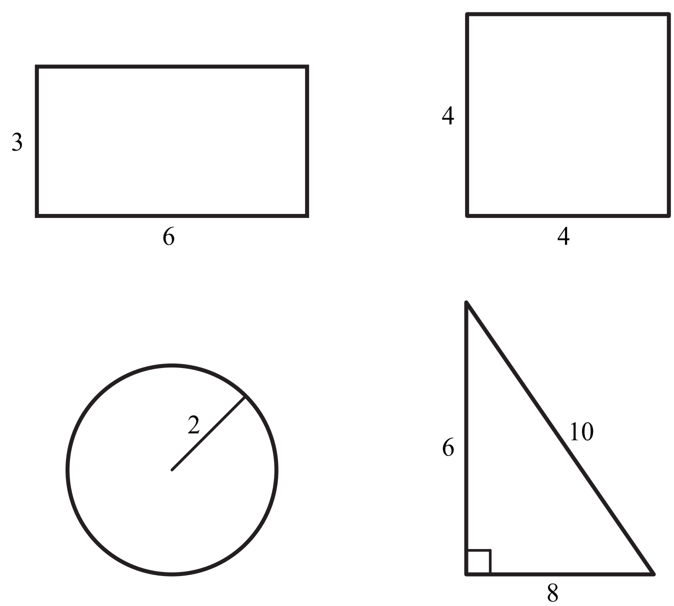


Figure 22.18

b) There is one more right triangle with integer side lengths with this property. What is it?

c) Are there any other rectangles with integer sides that have this property?

10. Proving that pi is the same for all circles via similar triangles.

The fact that π has the same value for all circles is a logical consequence of the SAS principle in flat geometry. To see why, imagine that we have two circles, one of radius r and the other of radius kr for some number k.

First, approximate each circle as a union of 12 congruent triangles, as shown in **Figure 22.19**.

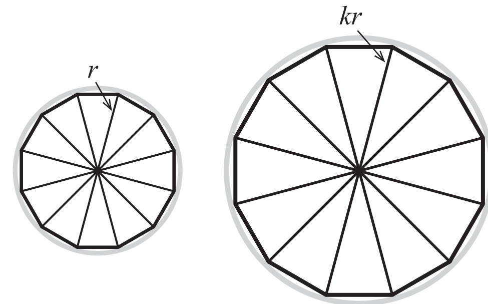


Figure 22.19

a) Explain why the angle at the center of the circle for each triangle is 30°.

b) Explain why each triangle in the figure on the left is similar to each triangle in the figure on the right with scale factor k.

c) Explain why the perimeter of the right figure—composed of the bases of 12 triangles—is k times the perimeter of the left figure.

d) Explain why the perimeter-to-diameter ratio is the same for each figure. (Hint: Think of diameter as twice the radius.)

e) Instead of using 12 triangles, suppose that we approximated the two circles using 100 triangles each. Would the perimeter-to-diameter ratio again be the same for each figure? What if, instead, we used 1000 triangles, or 10,000,000,000 triangles? Would the perimeter-to-diameter ratio for each figure agree every time?

f) In the limit of using more and more triangles, does it seem reasonable to conclude that the perimeter-to-diameter ratio for the two original circles would still be the same for each?

Three-Dimensional Geometry—Solids
Lesson 23

Topics

- Names of three-dimensional shapes.
- Cavalieri's principle and the volume cylinders.
- The volumes and surface areas of cylinders, cones, and spheres.

Definitions

- **cone**: A figure in three-dimensional space formed by
  - drawing a region in a plane. (This will be called the base of the cone.)
  - selecting a point *P* anywhere above or below the plane. (This will be called the cone point.)
  - drawing a line segment from each and every point on the boundary of the planar region to the chosen point *P*. (See **Figure 23.1.**)

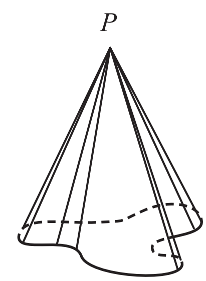


Figure 23.1

- **cylinder**: A figure in three-dimensional space formed by connecting, with straight line segments, matching boundary points of two congruent two-dimensional figures lying in parallel planes. The two planar figures (called the bases of the cylinder) are oriented so that any two connecting line segments are parallel.

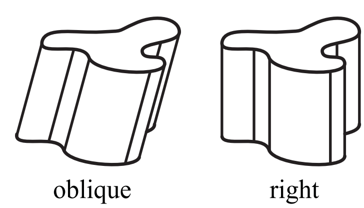


Figure 23.2

  If the line segments connecting matching boundary points are perpendicular to the planes containing the bases, then the cylinder is a right cylinder. Otherwise, the cylinder is oblique. (See **Figure 23.2**.)

- **lateral**: Refers to any feature of a cone or cylinder that is not part of a base of the figure. (For example, a lateral edge is any edge of the figure that is not an edge of a base, or a lateral face is any face of the figure that is not a base.)
- **prism**: A cylinder with a polygon for its base.
- **pyramid**: A cone with a polygon for its base.

- **regular cone** or **regular cylinder**: A cone or cylinder is said to be regular if its base is a regular polygon and all of its lateral faces are congruent.

- **slant height**: The slant height of a circular cone with cone point above the center of its base is the distance of the cone point from any point on the perimeter of the base. The slant height of a regular pyramid is the height of any one of its triangular faces (with a base edge considered the base of the triangle).

- **solid**: A figure in three-dimensional space.

Formulas

- volume of a cylinder, right or oblique

  V = area base × height.

  (See **Figure 23.3**.)

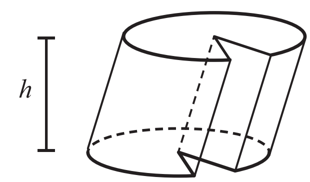


Figure 23.3

- volume of a cone

  $V = \frac{1}{3}$ area base × height.

  (See **Figure 23.4**.)

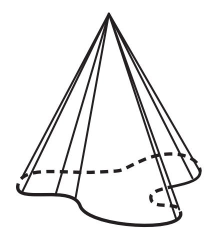


Figure 23.4

- volume and surface area of right circular cylinder

  $V = \pi r^2 h$.
  lateral area = $2\pi rh$.
  total surface area = $2\pi rh + 2\pi r^2$.

  (See **Figure 23.5**.)

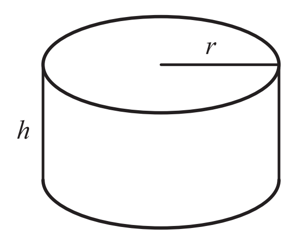


Figure 23.5

- volume and surface area of circular cone (with cone point above center of base)

  $V = \frac{1}{3}\pi r^2 h$.
  lateral area = πrs.
  total surface area = $\pi rs + \pi r^2$.

  (See **Figure 23.6**.)

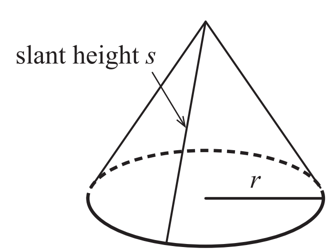


Figure 23.6

- volume and surface area of a sphere

  $V = \dfrac{4}{3}\pi r^3$.

  total surface area = $4\pi r^2$.

  (See **Figure 23.7**.)

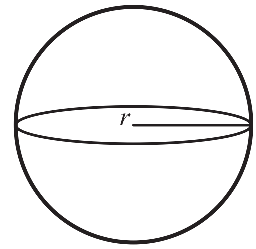


Figure 23.7

Summary

In three-dimensional space, figures are called solids. In this lesson, we review the names of classic three-dimensional shapes and develop formulas for their surface areas and volumes.

Example 1

Derive a formula for the surface area of a right circular cylinder with height h and base radius r. (See **Figure 23.8**.)

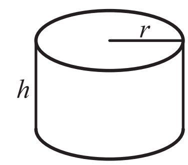


Figure 23.8

Solution

We can image a cylinder as constructed by rolling a rectangular piece of paper into the shape of a tube (and then attaching circular discs at each end). This rectangle has length 2πr and height h. (See **Figure 23.9**.)

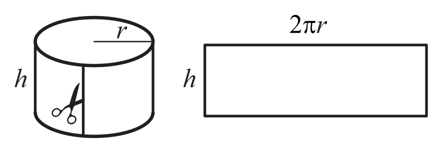


Figure 23.9

The lateral area of the cylinder is thus $2\pi rh$, and its total surface area is

$$2\pi rh + \pi r^2 + \pi r^2 = 2\pi rh + 2\pi r^2.$$

Example 2

A cone has a regular hexagon with side length 6 inches for its base. Each lateral edge of the cone has length 10 inches.

a) What is the slant height of each face of the cone?

b) What is the lateral area of the cone?

c) What is the total surface area of the cone?

d) What is the height of the cone?

e) What is the volume of the cone?

Solution

Call the slant height of the cone s and the height h. (See **Figure 23.10**.)

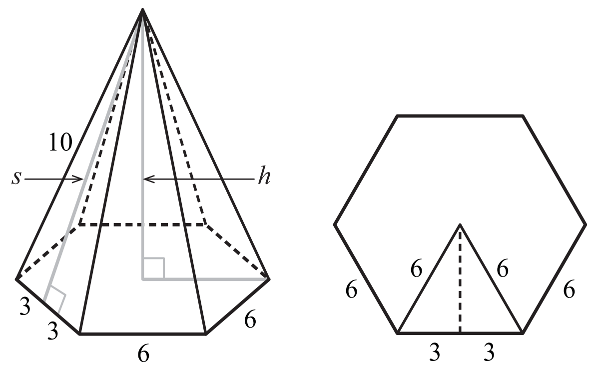


Figure 23.10

a) Each face is an isosceles 6-10-10 triangle. The slant height is the height of this triangle. By the Pythagorean theorem, we see that

$$s = \sqrt{10^2 - 3^2} = \sqrt{91}.$$

b) The lateral area is the area of the 6 triangular faces. We have the following.

$$\text{lateral area} = 6 \times \frac{1}{2} \cdot 6 \cdot \sqrt{91} = 18\sqrt{91}.$$

c) The total surface area is the lateral area plus the area of the base. The base is a regular hexagon composed of 6 equilateral triangles. We have the following.

$$\text{total surface area} = 18\sqrt{91} + 6 \times \frac{1}{2} \cdot 6 \cdot \sqrt{27} = 18\sqrt{91} + 9\sqrt{3}.$$

d) The Pythagorean theorem gives $h = \sqrt{10^2 - 6^2} = 8$.

e) $\text{Volume} = \frac{1}{3} \cdot (\text{area base}) \cdot h = \frac{1}{3} \cdot 9\sqrt{3} \cdot 8 = 24\sqrt{3}.$

Example 3

One cylinder is twice as wide as the other, but only half as tall. Which has the largest volume? (See **Figure 23.11**.)

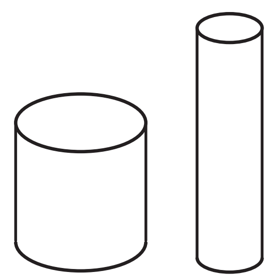


Figure 23.11

Solution

Label the dimensions of the figures as shown in **Figure 23.12**.

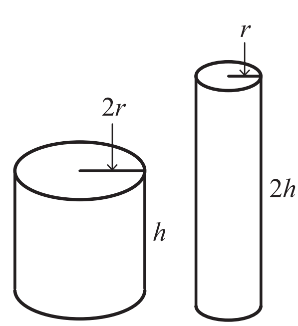


Figure 23.12

$$V_{\text{left}} = \pi (2r)^2 h = 4\pi r^2 h.$$

$$V_{\text{right}} = \pi r^2 (2h) = 2\pi r^2 h.$$

The left figure is double the volume of the right figure.

Study Tip

- The Pythagorean theorem can help compute either the slant height, height, or a lateral edge of a given pyramid or cone. Always look for right triangles on the surface or in the interior of the solid.

Pitfall

- Don't confuse the slant height of a pyramid or a circular cone with its height. (See **Figure 23.13**.)

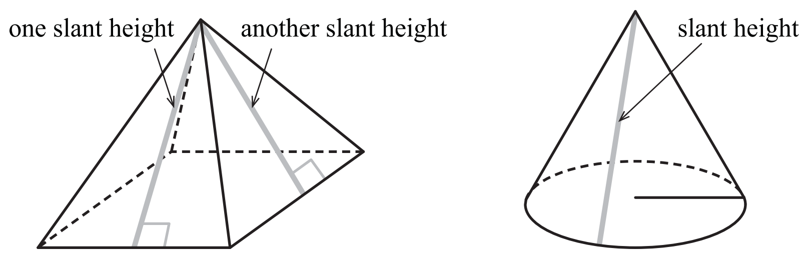


Figure 23.13

Problems

1. A right triangular prism has lateral area 100 square centimeters. If its base edges have lengths 3, 3, and 4 cm, what is the height of the prism?

2. A right triangular prism has height 6 inches, and each base is an equilateral triangle of side length 2 inches. What are the lateral area, surface area, and the volume of the prism?

3. A regular octagonal pyramid has base edge 4 meters and lateral area 40 meters squared. What is its slant height?

4. a) A circular cone has its cone point above the center of the circular base of radius 3 mm. Its slant height is 7 mm. What are the lateral area, surface area, and volume of the cone?

b) How does the volume of the cone change if the cone point is not above the center of the circular base (but the height of the cone and the radius of the base remain the same)?

5. The longest line segment that can fit in a particular cube is 3 meters. What is the volume of the cube?

6. Find the volume, lateral area, and surface area of a right triangular prism of height 100 meters and base edges 7, 24, and 25 meters.

7. a) A regular square pyramid has height 8 cm and slant height 10 cm. What is the base edge of the pyramid, its volume, and its surface area?

b) A regular square pyramid has height 20 cm and base edge 42 cm. What is the slant height of the pyramid, its volume, and its surface area?

c) A regular square pyramid has slant height 6 cm and base edge 4 cm. What is the height of the pyramid, its volume, and its surface area?

d) A regular square pyramid has volume 10 cubic centimeters and height 6 cm. What is its base edge, its slant height, and its surface area?

8. Find, to one decimal place, the volume of the regular hexagonal pyramid shown in **Figure 23.14**.

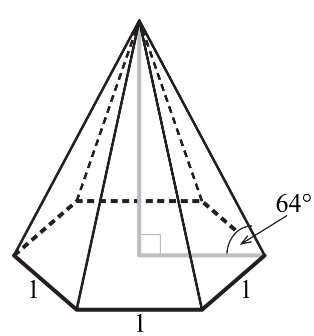


Figure 23.14

9. a) The radius of the Earth is approximately 6380 km. Approximately 70% of the surface of the Earth is covered in water. What area does this represent?

b) Ten gallons of paint are needed to paint the floor of a hemispherical room. How much paint is needed to paint the remainder of the interior of the room? (See **Figure 23.15**.)

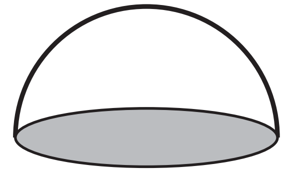


Figure 23.15

c) A capsule has the shape of a circular cylinder capped by two hemispheres, as shown in **Figure 23.16**.

Five units of paint are needed to paint one hemisphere of the capsule. How much of the paint is needed to paint the remainder of the capsule?

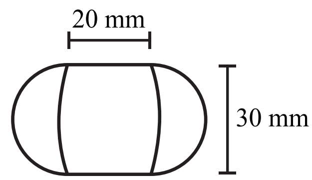


Figure 23.16

d) Four balls fit snugly in a cylindrical tube, as shown in **Figure 23.17**.

What is the volume of air around the balls compared to the volume of one ball?

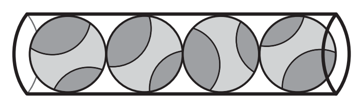


Figure 23.17

10. Consider a regular pyramid with a regular N-gon for its base. Suppose that each side of the base has length x inches and each face of the cone has slant height s. (See **Figure 23.18**.)

a) Write a formula for the lateral area of the cone.

b) Show that your formula can be rewritten and read as follows.

$$\text{lateral area} = \frac{1}{2} \times \text{perimeter of base} \times \text{slant height}.$$

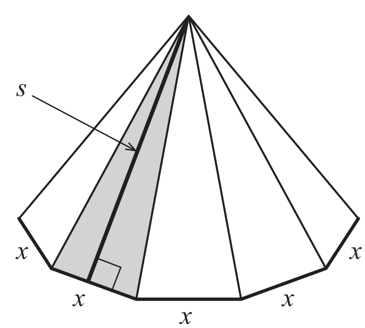


Figure 23.18

Imagine a regular pyramid with a 100-sided regular polygon as its base. Would the shape of the base be very close to being a circle? Consider instead a pyramid with a 1000-sided or 100,000,000,000,000-sided regular polygon as its base. Is the base closer still to being a circle? The following formula holds in each case.

$$\text{lateral area} = \frac{1}{2} \times \text{perimeter of base} \times \text{slant height}$$

As a leap of faith, you might be willing to say that this formula still holds for the case with the base of the cone actually being a circle. (See **Figure 23.19**.)

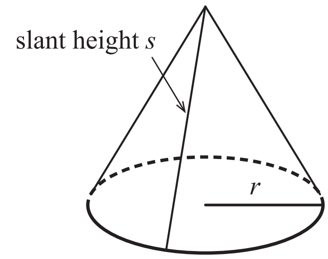


Figure 23.19

c) Show how the following formula

lateral area = $\frac{1}{2}$ × perimeter of base × slant height

translates to the following formula

lateral area = πrs

for the case of a circular cone.

The Volume and Surface Area of a Sphere

Appendix to Lesson 23

Greek scholar Archimedes of Syracuse (ca 287–212 BCE) proved that the volume of a hemisphere of radius r is given by the formula $\frac{2}{3}\pi r^3$ (and, thus, the volume of a full sphere is $\frac{4}{3}\pi r^3$).

He did this by enclosing the hemisphere in a circular cylinder of radius r and height r and comparing the space between the two with the volume of a circular cone enclosed in the same cylinder. (See **Figure 23.20**.)

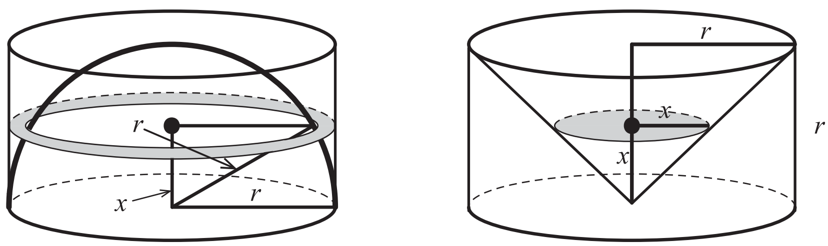


Figure 23.20

More precisely, Archimedes compared cross-sectional slices of this space with cross-sectional slices at matching heights x.

On the right, the cross section is a circle. By noting that we have isosceles right triangles, we see that the radius of this circle is x, so this cross section has area πx^2.

On the left, the cross section is a ring made of a large circle of radius r and a small circle of radius $\sqrt{r^2 - x^2}$. Its area is $\pi r^2 - \pi\left(\sqrt{r^2 - x^2}\right)^2 = \pi x^2$.

Because these areas are the same, Archimedes argued, like Cavalieri, that the volume outside the sphere equals the volume inside the cone. Now,

$$V(\text{cylinder}) = \pi r^2 \cdot r = \pi r^3.$$

$$V(\text{outside hemisphere}) = V(\text{cone}) = \frac{1}{3}\pi r^2 \cdot r = \frac{1}{3}\pi r^3.$$

Thus,

$$\begin{aligned} V(\text{hemisphere}) &= V(\text{cylinder}) - V(\text{outside hemisphere}) \\ &= \pi r^3 - \frac{1}{3}\pi r^3 \\ &= \frac{2}{3}\pi r^3. \end{aligned}$$

The volume of the entire sphere is indeed $\frac{4}{3}\pi r^3$.

To obtain a formula for the surface area of a sphere, Archimedes imagined the surface of the sphere divided into polygonal regions. Drawing radii as suggested by the diagram in **Figure 23.21** divides the volume of the sphere into cones. (The picture shows just one such cone, but imagine the entire sphere dissected this way.)

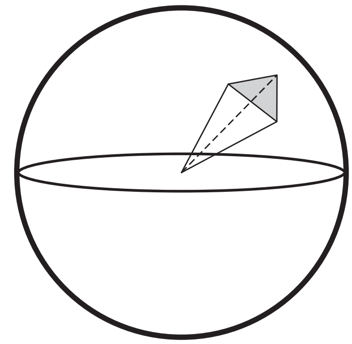


Figure 23.21

Actually, each section is not quite a cone: The base of the figure is not flat. But if we draw sufficiently small polygons, each is well approximated as a cone with a flat base. We can also assume that the height of the cone is very close to r, the radius of the sphere.

The volume of each cone is

$$\frac{1}{3}\cdot \text{area of base}\cdot r.$$

If $B_1, B_2, \ldots, B_n$ are the areas of all the individual bases, then the approximate volume of the cone is

$$\frac{1}{3}B_1r+\frac{1}{3}B_2r+\ldots+\frac{1}{3}B_nr=\frac{1}{3}\left(B_1+B_2+\ldots+B_n\right)r.$$

The true volume of the sphere is $\frac{4}{3}\pi r^3$, and this approximate formula approaches the true value if we use finer and finer polygons and cones. (The errors we introduce by assuming flatness of bases and heights of r become less and less significant.)

Now, $B_1 + B_2 + \ldots + B_n$ is the surface area of the sphere. So, we can say that

$$\frac{1}{3}\left(B_1+B_2+\ldots+B_n\right)r=\frac{1}{3}\left(\text{surface area}\right)r\approx\frac{4}{3}\pi r^3.$$

Algebra gives that surface area $\approx 4\pi r^2$, with this formula becoming more and more exact with finer and finer polygons used. To many, it seems reasonable to believe that the true surface area of a sphere can thus only be $4\pi r^2$.

Comment: This idea of taking finer and finer approximations and making the leap to believe that the formula remains true in an "ultimate" sense is the basis of calculus.

Historical comment: Archimedes was so proud of these results that the diagram of a sphere inscribed in a cylinder was placed on his tombstone at his request.

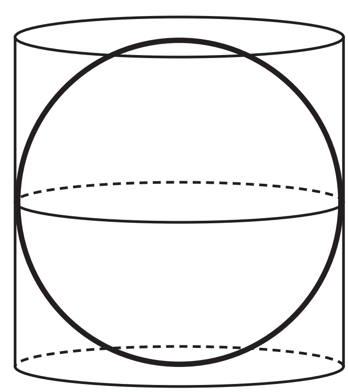


Figure 23.22

Introduction to Scale
Lesson 24

Topics

- The effect of scale on lengths, areas, and volumes.
- Changing units.
- The role of scale in the natural world.

Formula

In scaling by a factor *k*,

- all lengths change by a factor *k*.
- all areas change by a factor k^2.
- all volumes change by a factor k^3.
- all angles remain unchanged.

Summary

The mathematical effects of scaling are manifest in profound ways in the natural world. In this lesson, we examine scaling in two and three dimensions, explore scaling through the viewpoint of a change of units, and look at effects on natural phenomena.

Example 1

The triangle on the left side of **Figure 24.1** has area 18 square units. What is the area of the triangle on the right?

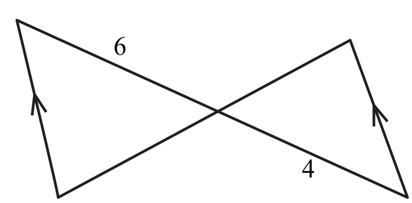


Figure 24.1

Solution

The two triangles are similar by AA (use vertical angles and congruent alternate interior angles for parallel lines) with scale factor $k = \frac{4}{6} = \frac{2}{3}$ (from the larger to the smaller triangle).

Because area scales by k^2, the area of the right triangle is $20k^2 = 20 \times \frac{4}{9} = 8\frac{8}{9}$ square units.

Example 2

Find the ratios $\dfrac{\text{area I}}{\text{area I} + \text{area II}}$ and $\dfrac{\text{area II}}{\text{area I}}$ for **Figure 24.2**.

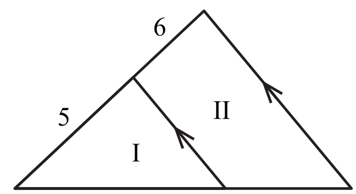


Figure 24.2

Solution

The triangles given by region I and region I + II are similar (via AA, noting congruent corresponding angles and a shared angle) with scale factor $k = \dfrac{5}{11}$ or $\dfrac{11}{5}$, depending on one's perspective. Thus,

$$\frac{\text{area I}}{\text{area I} + \text{area II}} = k^2 = \frac{25}{121}.$$

Algebra then gives the following.

$$\begin{aligned}
121 \cdot \text{area I} &= 25(\text{area I} + \text{area II}) \\
121 \cdot \text{area I} &= 25 \cdot \text{area I} + 25 \cdot \text{area II} \\
96 \cdot \text{area I} &= 25 \cdot \text{area II}.
\end{aligned}$$

So,

$$\frac{\text{area II}}{\text{area I}} = \frac{96}{25}.$$

Example 3

A model ship is built in the scale of 1:100.

a) If two gallons of paint are needed to paint the model, how many gallons are needed to paint the real ship?

b) Weight is proportional to volume. If the weight of the model is 50 pounds, what is the weight of the real ship?

c) If the length of a mast on the real ship is 50 feet, what is the length of the mast on the model?

d) If a window on the real ship has area 60 square feet, what is the area of the matching window on the model?

Solution

This question, of course, is contrived: You can't match the materials of a model ship with those of a real ship, for example, so weights don't scale with volume in this setting as claimed. Nonetheless, this is a useful exercise for practicing ideas and making first approximations.

We are told that the scale factor between the two objects is $k = 100$ (from model to real ship) or $k = \frac{1}{100}$ (from real ship to model).

a) Area scales as $k^2 = 10{,}000$. Thus, $2 \times 10{,}000 = 20{,}000$ gallons of paint are needed for the real ship.

b) Volume (and hence weight) scales as $k^3 = 1{,}000{,}000$. The weight of the real ship is 50 million pounds.

c) Length scales as $k = \frac{1}{100}$, so the length of the model's mast is $50\left(\frac{1}{100}\right) = 0.5$ feet.

d) Area scales as $k^2 = \left(\frac{1}{100}\right)^2$, so the area of the matching window on the model is $60\left(\frac{1}{100}\right)^2 = 0.006$ square feet.

Example 4

The altitude of a right triangle from the vertex of the right angle to the hypotenuse of the right triangle divides the hypotenuse into two sections of lengths x and y, as shown in **Figure 24.3**. Show that the length h of the altitude is given by $h = \sqrt{xy}$.

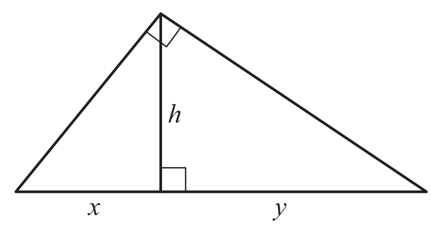


Figure 24.3

Solution

Label the vertices of the triangle A, B, and C, as shown in **Figure 24.4**, and label the point at the base of the altitude P.

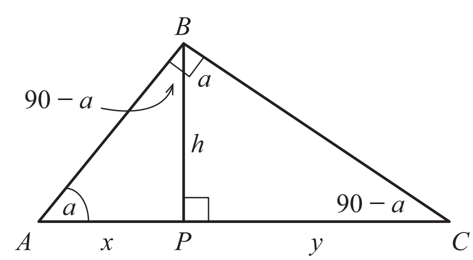


Figure 24.4

If $\angle A$ has measure $a°$, then $\angle ABP = 90 - a$, $\angle PBC = 90 - (90 - a) = a$, and $\angle BCA = 90 - a$.

By the AA principle, it follows that $\triangle ABP$ is similar to $\triangle BCP$.

Thus, matching sides in these two triangles come in the same ratio. In particular, $\frac{x}{h} = \frac{h}{y}$, from which it follows that $h = \sqrt{xy}$.

Study Tip

- Look for scaling when computing areas and volumes of figures. If one shape is just a scaled copy (scale factor k) of another shape whose size is already known, then its area or volume can be deduced with ease by multiplying values by k^2 or k^3, respectively.

Pitfall

- If you slice a cone at half its height, then, despite intuition, the volume of the top portion of the cone is just $\frac{1}{8}$ the volume of the original cone.

  (This smaller cone is a scaled copy of the original with half the height. Thus, you have a scale factor of $k = \frac{1}{2}$, and the volume changes by a factor of $k^3 = \frac{1}{8}$.)

Problems

1. A rectangle has area 6 inches squared. What scale factor on a photocopy machine would yield a scaled copy of that rectangle with area

a) 24 square inches?

b) 12 square inches?

c) 3 square inches?

2. A model airplane has scale 1:200.

a) If half a gallon of paint is needed to paint the model, how many gallons of paint are needed to paint the plane?

b) The volume of the model is 2 cubic feet. What is the volume of the plane in cubic yards?

3. Two similar cones have volumes 3 and 375. Find the ratio of their

a) base areas.

b) surface areas.

c) heights.

4. The radius of the Earth is approximately 6400 km. The radius of the Moon is approximately 1600 km. What is the ratio of the lengths of the equators of these two planetary bodies? What is the ratio of their surface areas? What is the ratio of their volumes?

5. a) 1 square foot = ____________ square inches.

b) 67 square feet = ____________ square inches.

c) 4089 square inches = ____________ square feet.

d) 1 cubic yard = ____________ cubic feet.

e) 78,713 cubic inches = ____________ cubic feet.

f) A terrace, in the shape of a rectangle, is 7.4 feet long and 5.2 feet wide. Paving stones come in the shape of 5×5 inch squares and cost \$6.39 each. How much will it cost to pave the terrace with these paving stones?

g) 1 ganz equals 173 cm. Then, 2003 square ganzes = ____________ square centimeters.

h) The volume of a sphere is 800 cubic feet. The volume of a second sphere is 800 cubic inches. What is the scale factor between the two spheres?

6. A regular hexagonal cone is divided into two parts by a plane parallel to the base of the cone halfway along the height of the cone, as shown in **Figure 24.5**.

The volume of the top portion of the cone is 20 cubic feet.

a) Explain why the volume of the entire cone must be 160 cubic feet.

b) What is the volume of just the bottom portion of the cone?

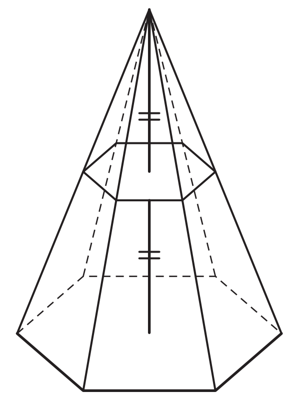


Figure 24.5

7. A cross section of a circular cone is parallel to the base of the cone at a distance of 4 cm from the base and 16 cm from the cone point, as shown in **Figure 24.6**.

This creates two similar cones.

a) What is the scale factor between the top cone and the large cone?

b) What is the ratio of the areas of the two shaded circles shown?

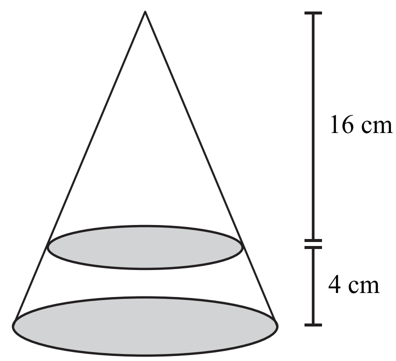


Figure 24.6

c) What is the ratio of the lateral area of the top cone to the entire cone?

d) What is the ratio of the lateral area of the top cone to the lateral area of the bottom section of height 4 cm?

e) What is the ratio of the volume of the top cone to the volume of the bottom section of height 4 cm?

8. Through their skin, earthworms absorb oxygen, which is needed for the function of the cells within their bodies.

a) Does the amount of oxygen needed by a worm depend on its surface area or on the volume of its cell material? Does the amount of oxygen it absorbs in a given period of time depend on its surface area or on its volume?

b) A small worm has length 3 cm, surface area 4 square cm, and volume 2 cubic cm. A big worm, of basically the same shape as the small worm, has length 15 cm. What is its surface area and its volume?

c) Which worm has the greatest difficulty receiving enough oxygen for its cells: the big worm or the small worm? Why?

d) Do you think that it is likely that a worm of length 100 feet could survive?

9. Little John is 6 feet tall. His friend, Big John, a giant, is 72 feet tall.

a) Little John weighs 200 pounds. Approximately how much does Big John weigh?

b) Little John's femur (leg bone) has diameter 1 inch. What is the diameter of Big John's femur?

c) Assuming that the cross section of a femur is a circle, what is the cross-sectional area of Little John's femur? Big John's?

d) When Little John stands, 200 pounds of weight are resting on his leg bones. What's the weight per unit cross-sectional area on Little John's femurs? Big John's?

e) Do you think that 72-foot-tall people could stand?

10. Brittney weighs 120 pounds and is 5 feet tall. Her surface area is 10 square feet.

a) If she were shrunk to a height of 0.05 feet, what would she weigh? What would her surface area be?

It's raining, and Brittney gets wet. A film of water 0.001 feet thick covers her body. One cubic foot of water weighs 70 pounds.

b) At her normal size, what is the weight of water covering her body? What percentage of her body weight is this?

c) When shrunk, what is the weight of water covering her body? What percentage of her body weight is this? Would Brittney likely be able to move?

Playing with Geometric Probability

Lesson 25

Topics

- The theory of geometric probability.
- Examples, and a word of caution.

Result

- geometric probability: This principle states that if a point of a region B is chosen at random, then the probability that it lands in a subregion A is as follows. (See **Figure 25.1**.)

  $$\text{probability} = \frac{\text{area } A}{\text{area } b}.$$

  This seems to be an intuitively valid notion.

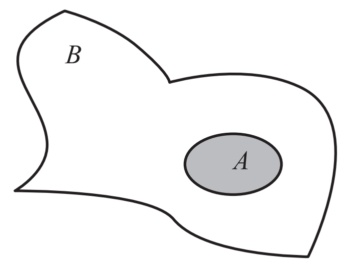


Figure 25.1

Summary

The theory of geometric probability allows us to use geometric intuition to solve problems of chance even if, at first glance, some of these problems don't seem amenable to a geometric approach. In this lesson, we state the principle of geometric probability and apply it to a variety of situations.

Example 1

Q is the midpoint of $\overline{RT}$, and S is the midpoint of $\overline{QT}$. A point is picked at random on $\overline{RT}$. What are the chances that it lands in $\overline{RS}$?

Solution

A picture helps. (See **Figure 25.2**.)

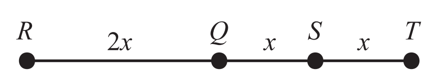


Figure 25.2

$$\text{Probability} = \frac{3x}{4x} = \frac{3}{4}.$$

Example 2

At a bus stop, a bus arrives every 8 minutes, waits exactly 1 minute, and then leaves.

a) You arrive at the stop at a random time. What are the chances that you see a waiting bus?

b) Suppose that you arrive at the stop and see no bus. What are the chances that you wait no more than 2 minutes for a bus to arrive?

Solution

Figure 25.3 is a schematic time line.

Figure 25.3

a) For every 8-minute interval, there is a 1-minute interval with a waiting bus.

The probability of seeing a bus is thus $\frac{1}{8}$.

b) We are told that we have arrived within one of the 7-minute periods with no bus. The probability that we are in the last 2 minutes of that period is $\frac{2}{7}$.

Example 3

A dart is thrown at random at the following diagram of a square inscribed in a circle, which is inscribed in a square. What are the chances that it lands in any of the shaded regions shown in **Figure 25.4**?

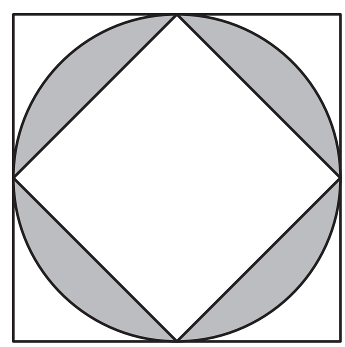


Figure 25.4

Solution

Suppose that the radius of the circle is r. Then, the side length of the large square is $2r$, and the side length of the small square is $\sqrt{r^2 + r^2} = \sqrt{2}r$. The probability we seek is as follows.

$$\text{Probability} = \frac{\pi r^2 - \left(r\sqrt{2}\right)^2}{\left(2r\right)^2} = \frac{\pi r^2 - 2r^2}{4r^2} = \frac{\pi - 2}{4}.$$

(Notice that the final result is a number that does not depend on r: $\frac{\pi - 2}{4} \approx 28.5\%$.)

Study Tip

- Drawing a picture to accompany a word problem often reveals the steps needed to solve the problem.

Pitfall

- Be confident to follow your wits! A probability value is usually computed as a ratio of areas or lengths. Even if no explicit values are given in a problem, determining an explicit value for such a ratio might still be possible.

Problems

1. You break a stick into two pieces at some point chosen at random along its length. What is the probability that the left piece is longer than the right piece?

2. A bus arrives at a particular station every 15 minutes. Each bus waits precisely 4 minutes before leaving.

a) You arrive at the stop and see a bus. What is the probability that you need to wait no more than 1 minute for the bus to leave?

b) On a different day, you arrive and don't see a bus. What is the probability that you need wait no more than 3 minutes to see one arrive?

3. A sphere sits snugly in a cube. A point inside the cube is chosen at random. What is the probability that this point also lies inside the sphere?

4. A circular pond sits in Mr. Foiglesplot's back yard. The pond has radius 5 feet, and the yard is rectangular, 30 × 60 feet. Some neighborhood kids kick a ball into his yard. What are the chances that the ball lands in the pond?

5. An equilateral triangle is inscribed in a circle. A dart is thrown at random and lands in the circle. What is the probability that the dart lands outside of the triangle?

6. It is very difficult to measure the area of an oil spill. However, someone can take an aerial photograph of the spill, digitize it, and then have a computer select 1000 points at random.

If the photograph represents a total area of 42 square km and 875 of those 1000 points chosen at random landed in a shaded region of the photograph, what is the approximate area of the spill?

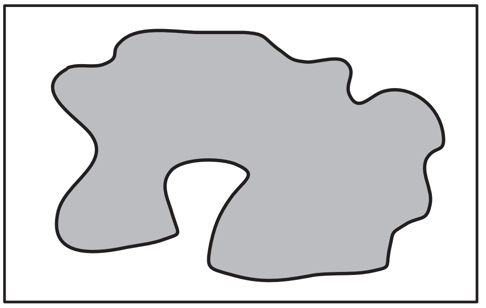


Figure 25.5

7. A rectangular room is 12 feet wide and 14 feet long, and the distance between floor and ceiling is 9 feet. Of 100,000 points that were selected at random in the room, 7143 of them landed inside the closet. What is the approximate volume of the closet?

8. A point is chosen at random somewhere in the room where you are currently sitting. Estimate the probability that the chosen point is inside your head.

9. A point is chosen at random in a square of side length 6 inches. What is the probability that that point is within a distance of 3 inches from some corner of the square?

10. A point is chosen at random in a triangle ABC. What is the probability that this point lies inside the triangle MNO formed by the midpoints of its sides? (See **Figure 25.6**.)

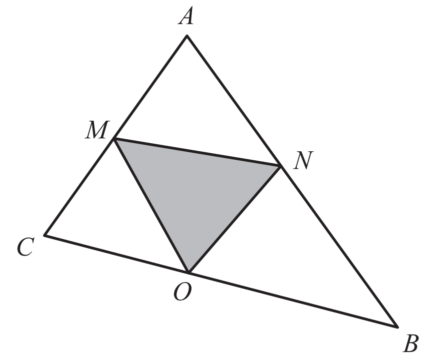


Figure 25.6

Exploring Geometric Constructions

Lesson 26

Topics

- Straightedge and compass constructions.
- Constructible numbers.
- Constructible regular polygons.

Definition

- **constructible number**: A real number a is said to be constructible if, using only a straightedge (a ruler with no markings) and a compass as tools, it is possible to draw a line segment of length a if given only a line segment of length 1 already drawn on the page.

Summary

Which figures in geometry can be constructed using only the most primitive of tools—a straightedge and a compass? This classic question from the ancient Greek scholars has proved to be surprisingly rich and complex in its mathematics. In this lesson, we explore the basic constructions we can perform with these primitive tools and survey the mathematics of constructible numbers and constructible regular polygons.

Example 1

Given a line segment $\overline{AB}$ drawn on a page, show how to construct an equilateral triangle with $\overline{AB}$ as its base.

Solution

Set the compass so that its length is AB, and draw two circles of this radius, one with endpoint A as center and one with endpoint B as center.

Let C be one of the points of intersection of the two circles.
Then, $\overline{AC}$ is a radius of the circle with center A, so $AC = AB$.
By similar reasoning, $BC = AB$. Thus, triangle ABC is equilateral.
(See **Figure 26.1**.)

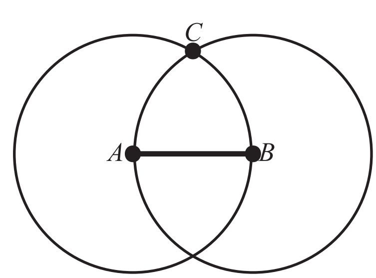


Figure 26.1

We can complete the picture by drawing the line segments $\overline{AC}$ and $\overline{BC}$.

Example 2

Given that we know how to construct segments of any whole-number length and perpendicular segments, show how to construct a line segment whose length is $\frac{13}{11}$.

Solution

Construct a segment of length 11 and a perpendicular segment of length 13, as shown in **Figure 26.2**, and draw the hypotenuse of the right triangle.

Draw a segment of length 1 along the base, and construct a perpendicular segment from its endpoint, as shown. Using matching ratios in similar triangles, we see that $\frac{x}{13} = \frac{1}{11}$, showing that this perpendicular segment has length $\frac{13}{11}$.

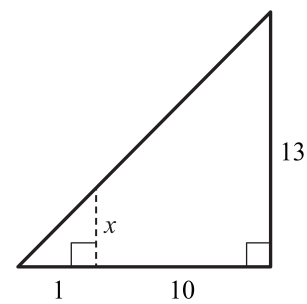


Figure 26.2

Study Tip

- If two circles have the same radius, then any line segments that represent radii have the same length. Keep this in mind as you try to construct figures with edges that are the same length.

Pitfall

- Don't be too hard on yourself! Constructing figures with only these primitive tools usually takes much ingenuity and innovation. It is often not clear how to proceed, or whether the task hoped for can even be done. Be patient, and be kind to yourself as you try out many different possible approaches.

Problems

1. Construct each of the following shapes on separate blank sheets of paper using only a straightedge and compass. Accompany each diagram—on the reverse side of your paper—with a detailed set of instructions that another person could follow to construct her or his own shape of the indicated type.

a) A perfect rectangle.

b) A perfect square.

c) A perfect hexagon.

2. Construct an angle bisector for the angle in **Figure 26.3**.

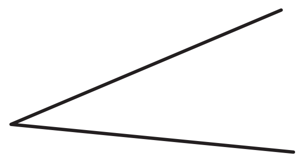


Figure 26.3

3. For the triangle in **Figure 26.4**, construct a circle that passes through each of its three vertices.

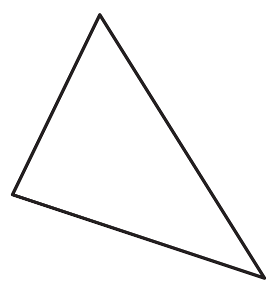


Figure 26.4

4. Using only a straightedge and compass, pinpoint the exact center of the circle from which the arc in **Figure 26.5** came.

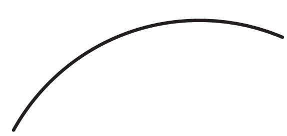


Figure 26.5

5. Is the number $\sqrt{1+\sqrt{2+\sqrt{3}}}$ constructible?

The Reflection Principle

Lesson 27

Topics

- Shortest paths and “trick shots.”
- The reflection principle.
- Fagnano’s problem.

Result

- reflection principle: The shortest path from a point A to a point B via a location along a given straight line is the path that produces congruent angles, as shown in **Figure 27.1**.

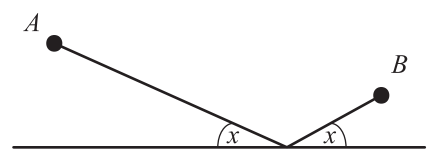


Figure 27.1

Summary

A ray of light passing between two points by first bouncing off a reflective surface follows the shortest path possible. In this lesson, we prove that this is true and make use of the general reflection principle to solve a famous problem in geometry—Fagnano’s problem—as well as explore a number of additional applications.

Example 1

Adit stands at point A at the end of a long corridor with three perpendicular walls, as shown in **Figure 27.2**.

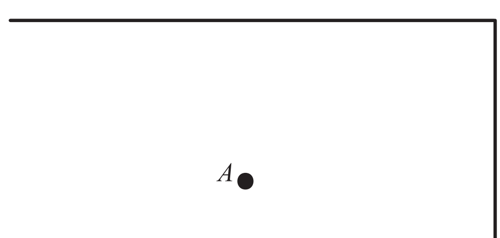


Figure 27.2

He wishes to throw a ball in some direction so that it bounces off all three walls and returns directly to him. In which direction should he throw the ball? Draw the direction on this page.

Solution

Adit should aim for the reflection of the reflection of the reflection of the point A, as shown in **Figure 27.3**.

(This, of course, is assuming ideal conditions—that the ball does not lose energy as it bounces, that there is no drag due to air resistance, and so on.)

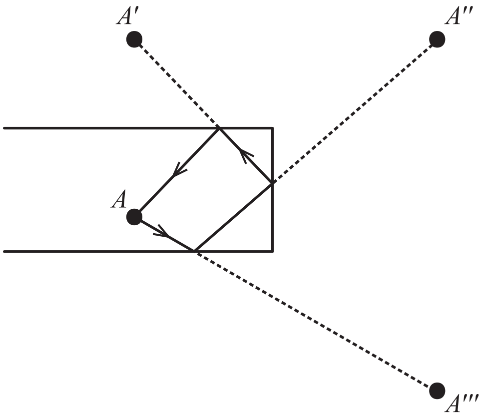


Figure 27.3

Example 2

Of all rectangles of perimeter 100 centimeters, which has the largest area?

Solution

We can write the dimensions of the rectangle as $(25 + x) \times (25 - x)$ for some value x. (See **Figure 27.4**.)

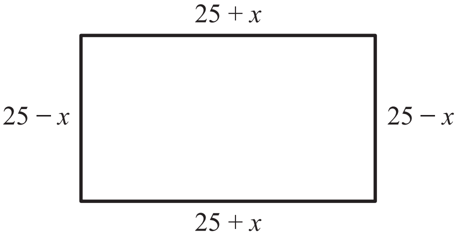


Figure 27.4

Then, the area of the rectangle is $(25 + x)(25 - x) = 625 - x^2$, which is clearly as large as it can be when $x = 0$—in which case the rectangle is a 25×25 square.

Comment: Think about the dual problem: Of all rectangles of area 100 square centimeters, which has the least perimeter? The answer turns out to be a square, but proving this is not straightforward.

Study Tip

- This lesson is purely optional, and this topic does not appear in typical geometry courses. There are no recommended study tips for this lesson other than to enjoy the lesson and let the thinking about it strengthen your understanding of geometry as a whole.

Pitfall

- Don't forget to have fun in your thinking of mathematics. This is a fun topic.

Problems

In this series of problems, a ball is shot at a 45° angle from the top-left corner of an $n \times m$ billiards table. The ball traverses the diagonals of the individual squares marked on the table, bouncing off the sides of the tables at 45° angles. There are no pockets on the sides of the table, but there is a pocket at each corner that the ball would fall into if it entered that corner.

Figure 27.5 shows the beginning of the path of a ball on a 5×9 table.

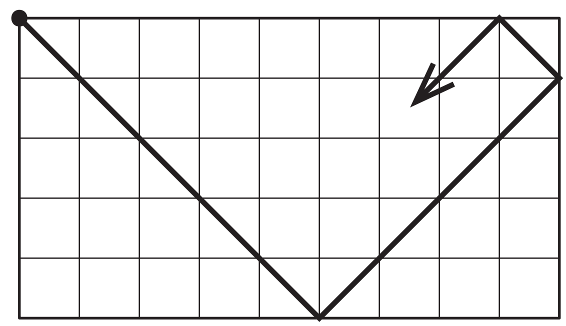


Figure 27.5

1. Trace the motion of the ball in the diagram, and verify that the ball eventually falls into the bottom-right corner.

2. Draw examples of 3×5, 5×9, and 7×3 tables. Does a ball shot from the top-left corner land in the bottom-right corner in each case?

3. Must a ball on a 103×5001 table also land in the bottom-right corner? Explain. Is this the case for all odd × odd tables?

4. Verify that a ball shot from the top-left corner of a 4×5 table lands in the bottom-left corner. Explain why this is the case for all even × odd tables.

5. Verify that a ball shot from the top-left corner of a 3×8 table lands in the top-right corner. Explain why this is the case for all odd × even tables. (Try to develop a general theory about the motion of balls on even × even tables.)

Tilings, Platonic Solids, and Theorems
Lesson 28

Topics

- Platonic solids.
- The Pythagorean theorem revisited, Napoleon's theorem, midpoint areas, and variations.

Summary

The visual beauty of regular tilings provides intellectual beauty, too: We can prove mathematical theorems via tiling patterns. In this lesson, we pick up on the tiling theme of Lesson 4 and establish mathematical results.

Example 1

Why is it not possible to make a convex three-dimensional polyhedron with faces that are congruent regular octagons?

Solution

If such a solid existed, three or more octagonal faces would meet at each vertex of the figure. For the figure to be convex, the sum of the interior angles of the octagons meeting at each vertex must be less than 360°.

However, each interior angle of a regular octagon is $180° - \frac{360°}{8} = 135°$.

Three or more of these angles sum to greater than 360°. Such a figure cannot be constructed.

Example 2

A square is inscribed in a circle, and a rectangle, with one side twice as long as the other, is inscribed in the same circle. (See **Figure 28.1**.)

Establish that the area of the rectangle is $\frac{4}{5}$ the area of the square.

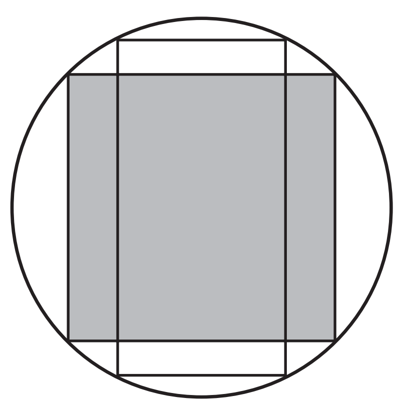


Figure 28.1

Solution

Imagine that the circle is drawn on a tiling of the plane with squares, as shown in **Figure 28.2**.

Here, the rectangle has width 2, height 4, and, therefore, area 8 units.

By the Pythagorean theorem, the side length of the square is $\sqrt{10}$, so the square has area 10 units.

The area of the rectangle is indeed $\frac{4}{5}$ the area of the square.

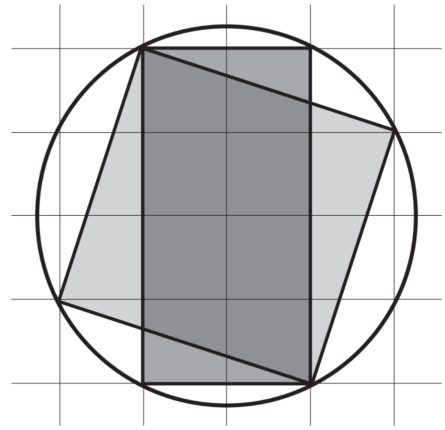


Figure 28.2

Study Tip

- This lesson is purely optional, and this topic does not appear in typical geometry courses. There are no recommended study tips for this lesson other than to enjoy the lesson and let the thinking about it strengthen your understanding of geometry as a whole.

Pitfall

- Don't forget to have fun in your thinking of mathematics. This is a fun topic.

Problems

1. A square is inscribed in a circle, as is a rectangle with one side three times as long as the other. What can you say about the areas of these two quadrilaterals?

2. Twelve congruent equilateral triangles are used to construct a tetrahedron and an octahedron. The two figures are then glued together, with a face of one glued onto a face of the other. How many faces does the resulting solid have? (Warning: The answer is not 10, even though there are now 10 equilateral triangles on the outer surface of this shape.)

3. Twelve congruent equilateral triangles are used to construct a tetrahedron and an octahedron. What is the volume of the tetrahedron compared to the volume of the octahedron?

4. Lines are drawn in a unit square, as shown in **Figure 28.3**. Find a formula for the area of the shaded inner square in terms of the value x.

(Check that your formula gives the correct values for $x = \frac{1}{2}$ and $x = \frac{1}{3}$.)

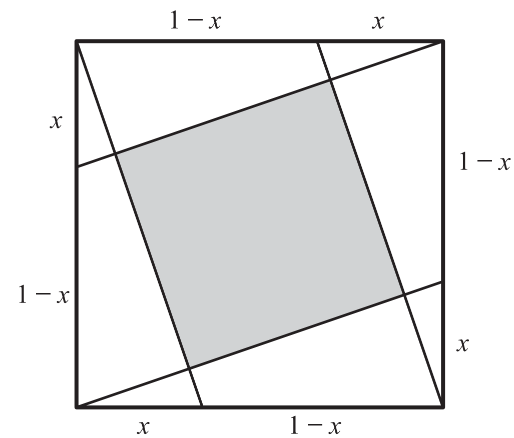


Figure 28.3

Folding and Conics

Lesson 29

Topics

- The parabola, ellipse, and hyperbola via folding.
- The parabola, ellipse, and hyperbola via conic sections.

Definitions

- **ellipse**: Given two points F and G in the plane, the set of all points P for which the sum of distances $FP + PG$ has the same constant value traces a curve called an ellipse. The points F and G are called the foci of the ellipse. (See **Figure 29.1**.)

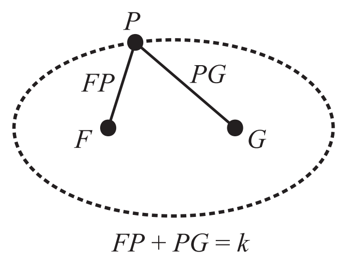


Figure 29.1

- **hyperbola**: Given two points F and G in the plane, the set of all points P for which the differences of distances $FP - PG$ and $GP - PF$ have the same constant value traces a curve called a hyperbola. The points F and G are called the foci of the hyperbola. (See **Figure 29.2**.)

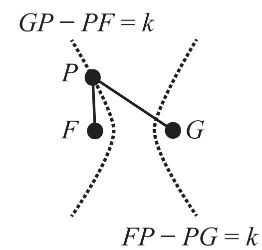


Figure 29.2

- **parabola**: Given a line m and a point F not on that line, the set of all points P equidistant from m and P form a curve called a parabola. F is called the focus of the parabola, and m is its directrix.

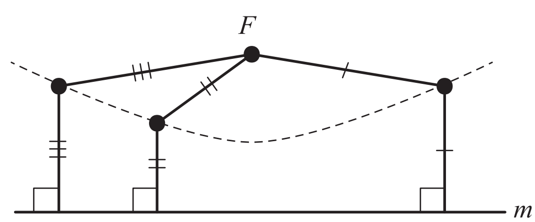


Figure 29.3

Summary

The ancient mathematical topic of conic sections makes appearances in many realms, including astronomy, physics, and medicine. In this supplemental lesson, we construct the three classic curves—the parabola, the ellipse, and the hyperbola—first by a modern approach of folding paper and then by the classic approach of slicing a cone. We show that these two methods of construction do indeed yield precisely the same set of curves.

Example 1

Pointing a flashlight directly toward a wall produces a circle of light against the wall. (See **Figure 29.4**.)

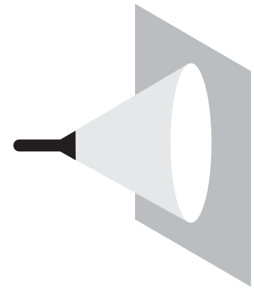


Figure 29.4

a) What shape of light appears on the wall if you angle the flashlight slightly? (See **Figure 29.5**.)

b) What shape of light appears on the wall if you place the flashlight vertically against the wall? (See **Figure 29.6**.)

Figure 29.5

Solution

A flashlight produces a cone of light, and each of the shapes of light produced on a wall represents a slice of this cone.

In a), the conic section produced is an ellipse.

In b), the conic section produced is (half of) a hyperbola. (How precisely would you need to angle the flashlight in order to see a parabola?)

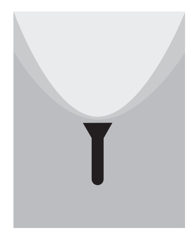


Figure 29.6

Example 2

Show that the graph of $y = x^2$, called a parabola in algebra class, really is a parabola. (Show that it is a parabola with focus $F = \left(0, \frac{1}{4}\right)$ and directrix the horizontal line $y = -\frac{1}{4}$.

Solution

Let (x, y) be a point on the parabola with focus $F = \left(0, \frac{1}{4}\right)$ and directrix $y = -\frac{1}{4}$.

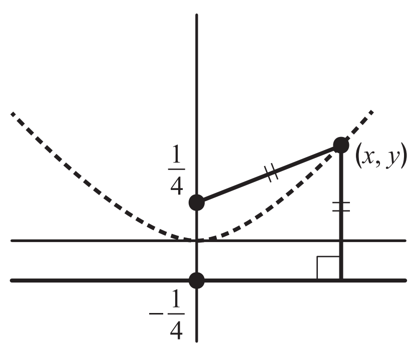


Figure 29.7

Its distance from F is $\sqrt{x^2 + \left(y - \frac{1}{4}\right)^2}$, and its distance from the directrix is $y + \frac{1}{4}$.

These need to match, giving the equation $\sqrt{x^2 + \left(y - \frac{1}{4}\right)^2} = y + \frac{1}{4}$.

Squaring and simplifying produces the following.

$$x^2+\left(y-\frac{1}{4}\right)^2=\left(y+\frac{1}{4}\right)^2$$
$$x^2+y^2-\frac{y}{2}+\frac{1}{16}=y^2+\frac{y}{2}+\frac{1}{16}$$
$$x^2-\frac{y}{2}=\frac{y}{2}$$
$$x^2=y.$$

This parabola is indeed the graph of $y=x^2$.

Study Tip

- This lesson is optional, and this topic does not usually appear in geometry courses. (Because of the hefty algebra involved, this topic often first appears in algebra courses.) There are no recommended study tips for this lesson other than to enjoy the lesson.

Pitfall

- Don't forget to keep the geometry of conics in mind if you later study conics in an algebra course.

Problems

1. A stick of butter is the shape of a rectangular prism with a square cross section.

If we cut the stick with a vertical cut parallel to a square face, the cross section obtained is a square, as shown in **Figure 29.8**.

If we tilt the plane, we obtain a rectangle, as shown in **Figure 29.9**.

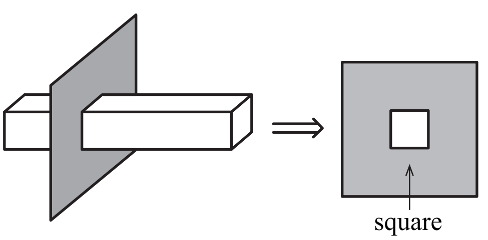


Figure 29.8

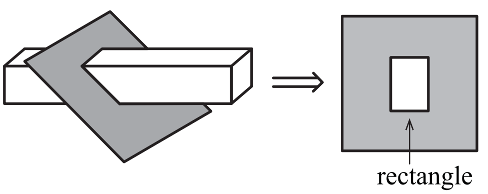


Figure 29.9

a) Is it possible to produce a non-rectangular parallelogram as a cross section from some planar cut? (See **Figure 29.10**.)

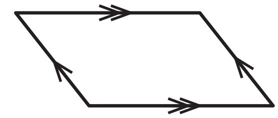


Figure 29.10

b) Is it possible to produce a kite as a cross section from some planar cut? (See **Figure 29.11**.)

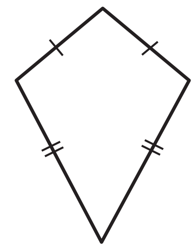


Figure 29.11

c) Is it possible to produce a non-square rhombus as a cross section? (See **Figure 29.12**.)

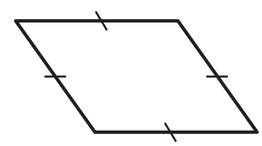


Figure 29.12

d) Is it possible to produce an irregular quadrilateral with no sides parallel and no sides congruent?

2. A circular cylinder is sliced at an angle. (See **Figure 29.13**.) Is the oval-shaped curve produced an ellipse?

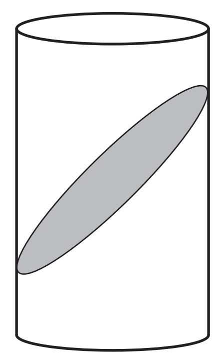


Figure 29.13

3. In Lesson 20, we saw that the equation of a circle can be written as $x^2 + y^2 = r^2$. (This is assuming that we've placed the circle so that its center lies at the origin.) This equation can be rewritten as follows.

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1.$$

We also hinted at the end of that lesson that the graph of a more general equation of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is an oval shape that looks like an ellipse.

Prove that this is indeed the case. Do this by showing that if an ellipse is centered about the origin and has foci on the x-axis at $(-c, 0)$ and $(c, 0)$, for example, then any point (x, y) on the ellipse satisfies an equation of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Warning: Answering this question requires hefty algebraic work!

4. Show that the centers of circles each tangent to a given line and a given circle lie on a parabola. (See **Figure 29.14**.)

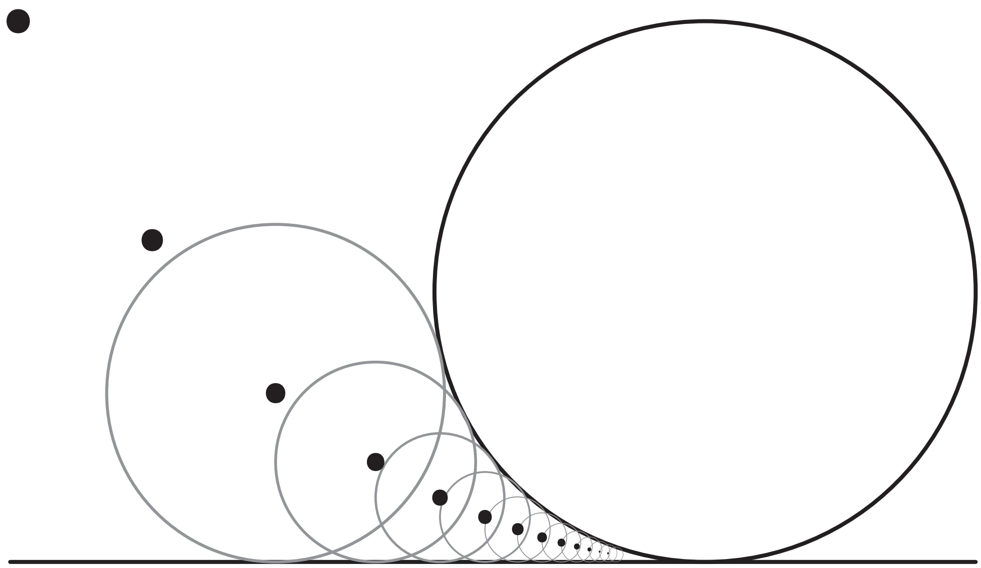


Figure 29.14

The Mathematics of Symmetry

Lesson 30

Topics

- Mappings of the plane: translations, reflections, glide reflections, rotations, and dilations.
- The symmetry of figures.
- Frieze patterns.

Definitions

- **dilation**: A dilation in the plane from a point O with scale factor k is the mapping that takes each point P different from O to a point P' on the ray $\overrightarrow{OP}$ such that $OP' = kOP$. The dilation keeps the point O itself fixed in place.
- **glide reflection**: A glide reflection along a line L in a plane is the mapping that results from performing a translation in a direction parallel to L followed by a reflection about L.
- **isometry**: A mapping that preserves distances between points.
- **mapping**: A mapping of the plane is a rule that shifts some or all points of the plane to new locations in the plane.
- **reflection**: A reflection in a plane about a line L is a mapping that takes each point P in the plane not on L to a point P' so that L is the perpendicular bisector of $\overline{PP'}$. It keeps each point on L fixed in place.
- **rotation**: A rotation in the plane about a point O through a counterclockwise angle of $x°$ is the mapping that takes each P in the plane different from O to a point P' such that the line segments $\overline{OP}$ and $\overline{OP'}$ are congruent and the angle from $\overline{OP}$ to $\overline{OP'}$, measured in a counterclockwise direction, is $x°$. The rotation keeps the point O itself fixed in place.
- **symmetry**: A figure is said to have symmetry if there is a mapping of the plane that points the figure back onto the figure itself.
- **translation**: A translation in a plane is a mapping that shifts each point of the plane a fixed distance in a fixed direction.

Summary

Even though, at first, we might not be able to pinpoint exactly what we mean by "symmetry," we nonetheless recognize it as something pleasing to our human sensibilities. In this lesson, we work to define exactly what it means for a figure to be symmetrical and explore some mathematically rich applications of symmetry.

Example 1

Explain why a translation is an isometry.

Solution

Suppose that a translation, mapping points a fixed distance d in parallel directions, takes points A and B to A' and B', respectively. (See **Figure 30.1**.)

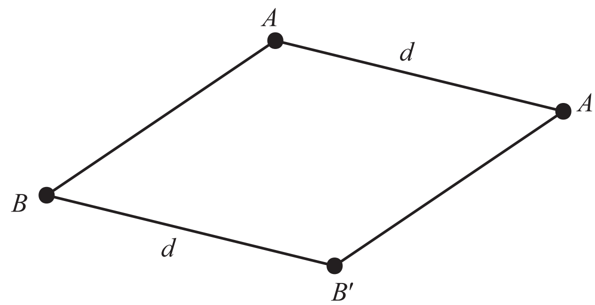


Figure 30.1

Quadrilateral $AA'B'B$ has one pair of sides that are both congruent and parallel. By Problem 2 of Lesson 14, the figure must be a parallelogram. Because both pairs of opposite sides of a parallelogram are congruent, we have that $AB = A'B'$. Thus, the translation has preserved the distance between points.

Example 2

Identify all the symmetries of a regular pentagon. How many line symmetries does it have? How many rotational symmetries does it have?

Solution

A regular pentagon has five line symmetries (reflection symmetries), one about each line through the center of the figure and a vertex. It has four rotational symmetries, rotations of 72°, 144°, 216°, and 288° about the center. Some might say that it has five rotational symmetries if they choose to include the (noneffective) rotation of 0° about the center. This is just a matter of preference. (See **Figure 30.2**.)

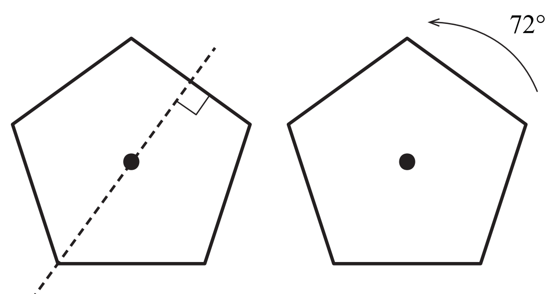


Figure 30.2

Example 3

Prove that a dilation maps three collinear points to three collinear positions.

Solution

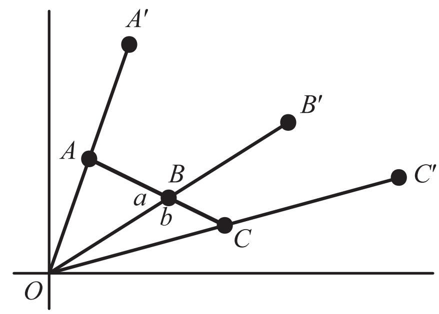


Figure 30.3

Assume that the dilation is centered about the origin O and has scale factor k. Suppose that the three collinear points A, B, and C are mapped to positions A', B', and C', respectively. Our goal is to prove that A', B', and C' are collinear. Mark the angles a and b as shown. We have that $a + b = 180°$. (See **Figure 30.3**.)

We have that $OA' = k \cdot OA$, $OB' = k \cdot OB$, and $OC' = k \cdot OC$

It follows that $\triangle A'OB' \sim \triangle AOB$ by the SAS principle. (Sides come in the same ratio k, and the triangles share a common angle at O.)

Thus, $\angle A'B'O = a$, because they are matching angles in similar triangles.

Similarly, $\triangle B'OC' \sim \triangle BOC$ and $\angle OB'C' = b$.

Thus, $\angle A'B'C' = \angle A'B'O + \angle OB'C' = a + b = 180°$, which shows that A', B', and C' are indeed collinear.

Comment: You can show, in general, that a dilation maps any set of collinear points to collinear positions. In particular, dilations map straight line segments to straight line segments.

Study Tip

- This lesson is purely optional. There are no recommended study tips for this lesson other than to enjoy the lesson and let the thinking about it strengthen your understanding of geometry as a whole.

Pitfall

- Don't forget to have fun in your thinking of mathematics. This is a fun topic.

Problems

1. Explain why a reflection is an isometry.

2. Which regular polygons have 180° rotational symmetry?

3. Identify all the symmetries that an infinite tiling of equilateral triangles across the entire plane has. (See **Figure 30.4**.)

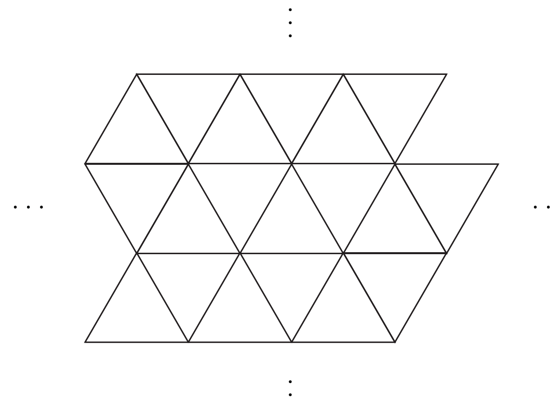


Figure 30.4

4. If possible, draw an example of each of the following.

a) A quadrilateral with exactly four lines of reflection symmetry.

b) A quadrilateral with exactly two lines of reflection symmetry.

c) A quadrilateral with exactly one line of reflection symmetry.

5. a) List all the (capital) letters of the alphabet that have horizontal line symmetry.

b) List all the (capital) letters of the alphabet that have vertical line symmetry.

c) DICED is a word with horizontal symmetry. Can you think of another word (when written with capital letters) with this property? Can you think of a word with vertical symmetry?

The Mathematics of Fractals

Lesson 31

Topics

- The definition of a fractal and fractal dimension.
- Sierpiński's triangle, mat, and sponge.
- The Koch snowflake.
- Applications in the natural world.

Definitions

- **fractal**: A geometric figure with the property that it can be divided into a finite number of congruent parts, each a scaled copy of the original figure.
- **fractal dimension**: If a fractal is composed of N parts, each a scaled copy of the original fractal with scale factor k, then its fractal dimension is the number d so that $k^d = \frac{1}{N}$.

Formula

- geometric series formula:

$$\frac{1}{N}+\frac{1}{N^2}+\frac{1}{N^3}+\cdots=\frac{1}{N-1} \text{ for whole numbers } N \geq 2.$$

Summary

Fractals are geometric objects that lie in between being a length (dimension 1) or an area (dimension 2), or in between being an area (dimension 2) or a volume (dimension 3). In this lesson, we construct examples of fractals, explore what the term "fractional dimension" means, and survey some occurrences of fractal-like objects in the natural world.

Example 1

In what way does the **Figure 31.1** illustrate the geometric series formula for $N = 3$?

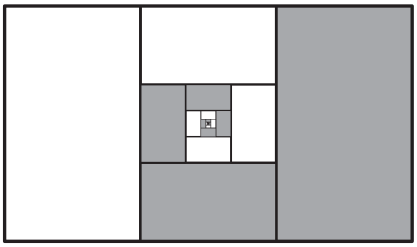


Figure 31.1

Solution

If the area of the entire rectangle is 1 square unit, then the areas of the light-gray regions sum to $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\cdots$.

The "spiral" they produce clearly occupies half the original rectangle.

Thus, $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\cdots$ must equal $\frac{1}{2}$.

(Challenge: Try to find a picture that shows $\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\frac{1}{256}+\cdots=\frac{1}{3}$.)

Example 2

What is the fractal dimension of the Koch curve? (The Koch curve is $\frac{1}{3}$ of the boundary of the Koch snowflake, as shown in **Figure 31.2**.)

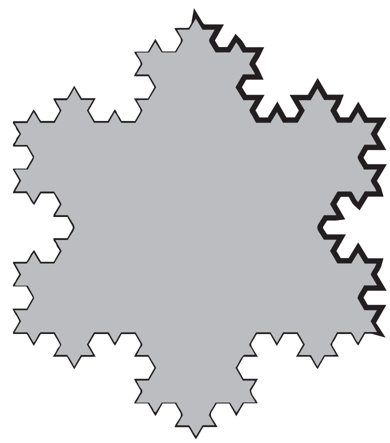


Figure 31.2

Solution

We see that the Koch curve is composed of four sections, each a $\frac{1}{3}$-scaled copy of the original curve.

Thus, its "size" satisfies the following.

$$\left(\frac{1}{3}\right)^d \cdot \text{original size} = \frac{1}{4}\cdot \text{original size}.$$

This gives $3^d = 4$. Clearly, $d = 1$ is too small a value for d, and $d = 2$ is too large. The Koch curve is somewhere between being a length and an area. Experimentation on a calculator gives $d \approx 1.26$.

Study Tip

- This lesson is purely optional, and this topic does not appear in typical geometry courses. There are no recommended study tips for this lesson other than to enjoy the lesson and let the thinking about it strengthen your understanding of geometry as a whole.

Pitfall

- Don't forget to have fun in your thinking of mathematics. This is a fun topic.

Problems

1. What is the fractal dimension of the fractal in **Figure 31.3**?

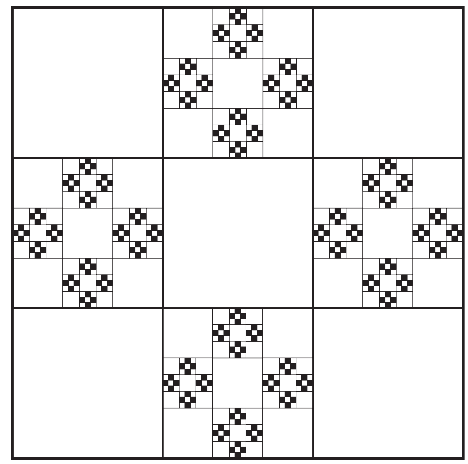


Figure 31.3

2. The Sierpiński sponge is constructed from a cube by dividing it into 27 small cubes and removing the very center cube and the center cube of each face. The action is then repeated on the 20 remaining cubes, and then repeated over and over again in the cubes that remain after each iteration.

   What is the fractal dimension of the Sierpiński sponge?

3. Use the geometric formula to show that the infinitely long decimal 0.1111111… equals the fraction $\frac{1}{9}$.

4. (Tricky!) What is the total area of the Koch snowflake? (Assume that the area of the starting equilateral triangle is 1 square unit.)

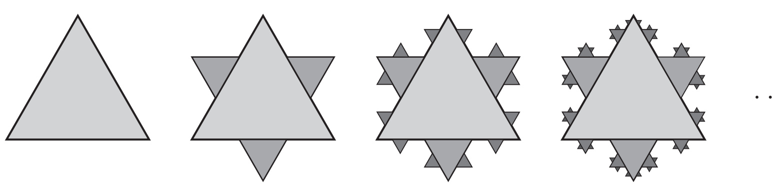


Figure 31.4

5. Two walls of identical size are to be painted. One has a smooth surface, and the other has a textured surface. Which wall will require the greater amount of paint to cover?

Dido's Problem

Lesson 32

Topics

- The legend of Dido.
- The isoperimetric problem and its solution.

Definition

- **isoperimetric problem**: The challenge of determining which figure in the plane has the greatest area given a fixed length for its perimeter.

Result

- Of all shapes with a given perimeter, the circle encloses the maximal area.

Summary

In this lesson, we use the legend of Dido to motivate a famous problem in geometry: the isoperimetric problem. We prove what the answer to the problem must be—if you believe that the problem has an answer in the first place!

Example 1

Of all the triangles with a given fixed base and a given fixed perimeter, which has the largest area?

Solution

Suppose that the base of the triangles under consideration is the horizontal line segment $\overline{FG}$. Forming a triangle with perimeter of length L requires locating a point P such that $FP + PG$ equals the fixed value $L - FG$. As we saw in Lesson 29, the locus of all possible such points P is an ellipse. (See **Figure 32.1**.)

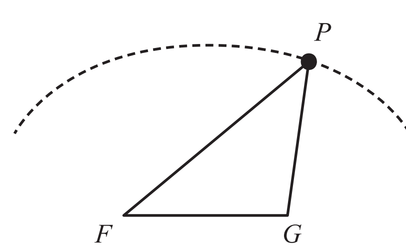


Figure 32.1

Because the area of the triangle is $\frac{1}{2} \times FG \times$ height and FG is a fixed value, the triangle of maximal area occurs when P is at the position of maximal height. This occurs when P is directly above the midpoint of $\overline{FG}$ and the triangle FPG is isosceles. (This claim is intuitively clear. Can you prove it? Which value of x gives the largest possible value for y in the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ of an ellipse?)

Thus, of all triangles with $\overline{FG}$ as its base, the isosceles triangle has the largest area.

Example 2

Of all triangles with perimeter 15 inches, which encloses the largest area?

Solution

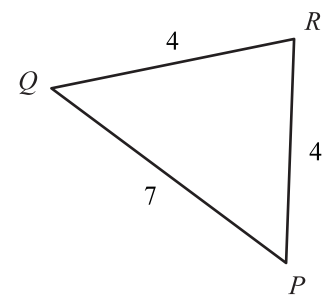


Figure 32.2

Any triangle that is not equilateral cannot be the answer. For example, we can increase the area of the triangle shown in **Figure 32.2** by moving P along the path of an ellipse (regarding $\overline{QR}$ as a fixed base akin to Example 1). This triangle fails to have maximal area, and the same argument shows that the area of any triangle that is not equilateral can be increased and, therefore, fails to have maximal area, too.

So, this leaves the 5-5-5 equilateral triangle as the only possible candidate to answer this question.

(However, are we sure this question has an answer in the first place?)

Study Tip

- This lesson is purely optional. There are no recommended study tips for this lesson other than to enjoy the lesson and let the thinking about it strengthen your understanding of geometry as a whole.

Pitfall

- Don’t forget to have fun in your thinking of mathematics. This is a fun topic.

Problems

1. A regular hexagonal hole is cut from a pan of brownies that were baked in a regular decagonal pan. (The centers of the hexagon and the decagon do not coincide.) Is there a straight-line cut that is sure to divide what remains of the brownies exactly in half? If so, give a simple description of how to find that line.

2. A regular hexagonal hole is cut from a pan of brownies that were baked in a regular decagonal pan. (The centers of the hexagon and the decagon do not coincide.) Is there a straight-line cut that is sure to divide what remains of the brownies exactly in half in area *and* divide the perimeter of the hole exactly in half *and* divide the perimeter of the decagon exactly in half?

3. Given 40 inches of string, what shape for it encloses the largest area of floor in a corner of a room between two perpendicular walls? (See **Figure 32.3**.)

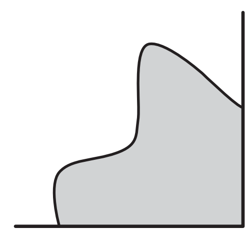


Figure 32.3

4. A point is chosen at random inside a square. Line segments from it to each of the four corners of the square divide the square into four triangles. (See **Figure 32.4**.)

If two opposite triangles are shaded gray and the remaining two opposite triangles are shaded white, prove that the total portion of the square colored gray has the same area as the total portion colored white.

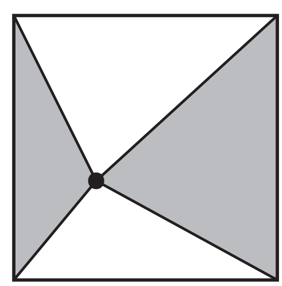


Figure 32.4

5. Does every line through the center of an equilateral triangle cut the area of the triangle in half?

The Geometry of Braids—Curious Applications

Lesson 33

Topics

- Braids with no free ends.
- Dirac's string trick.
- The waiter's trick.
- The national mathematics salute.

Result

- If strings that are attached to an object and that object is rotated two full turns, then the tangle of strings that results is physically equivalent to the beginning state of untangled strings. This is not the case if the object were given only one full rotation.

Summary

This highly unusual lesson demonstrates that the physical effect of rotating an object through one full turn is fundamentally different from rotating that object through two full turns. This topic usually does not appear in any school or undergraduate curriculum, but it is richly engaging, completely accessible, and shows a side of mathematical thinking that will astound.

Example 1

In the lesson, we proved that it is impossible to untangle three strings attached to a teacup if that teacup has undergone one full turn of rotation. Could it be possible, though, to untangle four strings attached to a cup that underwent one full turn?

Solution

If it were possible to untangle four strings, then it would be possible to untangle three: Simply imagine a fourth string accompanying the three, and untangle the four strings (one of which is imaginary). The result would be three real untangled strings. Because we cannot untangle three, we cannot, then, untangle four (or five, or more).

Study Tip

- This lesson is purely optional. There are no recommended study tips for this lesson other than to enjoy the lesson and let the thinking about it strengthen your understanding of geometry as a whole.

Pitfall

- Don't forget to have fun in your thinking of mathematics. This is a fun topic.

Problems

1. Tie three strings to the back of a chair, and braid them in any haphazard manner you choose. (Don't alternately cross the left two strands, then the right two strands. Try crossing the left two a few times, then the right two a few times, and so on, back and forth.) Make sure that the middle strand ends back in the middle position. Next, tie the three loose ends to a wooden spoon, again with the middle string tied in the middle position.

Is it now possible to maneuver the wooden spoon through the tangle of strings and completely undo the braid? Try it.

2. Show that, with three strands in a braid, a single overcrossing is physically equivalent to a single undercrossing. (See **Figure 33.1**.)

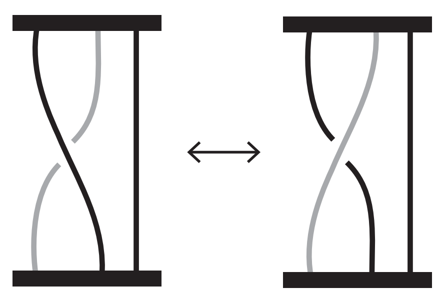


Figure 33.1

3. Show that, with three strands in a braid, two crossings in a row between the same pair of strings is physically equivalent to no crossings. (See **Figure 33.2**.)

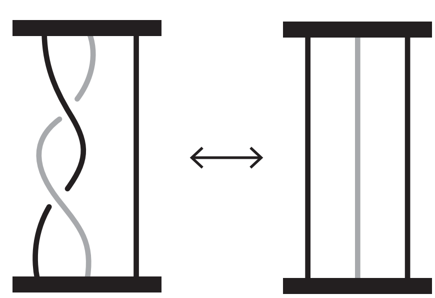


Figure 33.2

4. Show that, with three strands in a braid, a left crossing (of any kind) followed by a right crossing (of any kind) followed by a left crossing (of any kind) is physically equivalent to no crossings. (See **Figure 33.3**.)

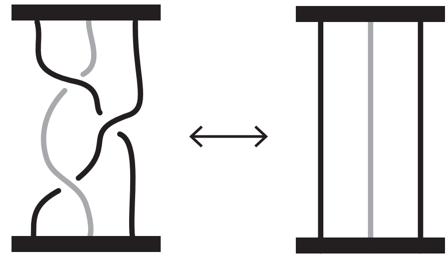


Figure 33.3

5. Explain Problem 1: Why is any braid on three strings with the middle strand ending in the middle position physically equivalent to no braid at all?

The Geometry of Figurate Numbers

Lesson 34

Topics

- Clever sums.
- Square, triangular, and squangular numbers.

Definitions

- **squangular number**: A number that is both square and triangular.
- **square number**: A number is a square number if a count of that many pebbles can be arranged in a square array.
- **triangular number**: A number is a triangular number if a count of that many pebbles can be arranged in a triangular array (with the number of pebbles in each row of the triangle one greater than the previous row).

Formulas

- $1+2+3+\cdots+(N-1)+N+(N-1)+\cdots+3+2+1=N^2$.
- The sum of the first N odd numbers is N^2.
- The sum of the first N even numbers is $N(N+1)$.
- The sum of the first N counting numbers is $1+2+3+\cdots+N=\dfrac{N(N+1)}{2}$.
- The N^{th} square number is $S(N)=N^2$.
- The N^{th} triangular number is $T(N)=\dfrac{N(N+1)}{2}$.

Summary

The Greeks of antiquity saw all of mathematics though the lens of geometry—even number theory. In this lesson, we explore the wonderful world of figurate numbers and prove, solely through the natural use of geometric arrangements, sophisticated summation formulas.

Example 1

Albert buys one piece of candy on January 1, two pieces on January 2, etc., all the way up to 365 pieces on December 31. How many total pieces of candy did Albert buy that year?

Solution

Albert bought $1+2+3+\cdots+365=\frac{365\times 366}{2}=66,795$ pieces of candy.

Example 2

If $T(N)$ denotes the N^{th} triangle number, show that $T(N+M)=T(N)+T(M)+N\times M$.

Solution

Figure 34.1 represents stacking two triangles (one with a base row of N dots and one with a base row of M dots) and a rectangle to create a triangular array of dots with a base row of $N+M$ dots.

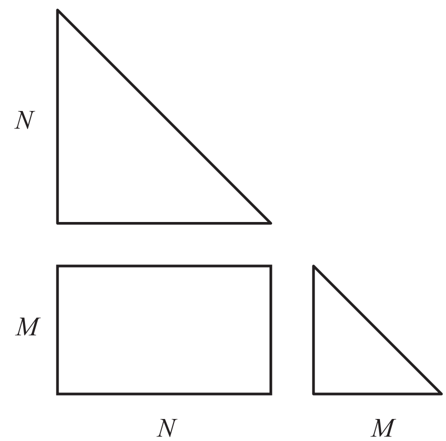


Figure 34.1

Figure 34.2 is the diagram for $N=6$ and $M=4$ with the dots drawn in.

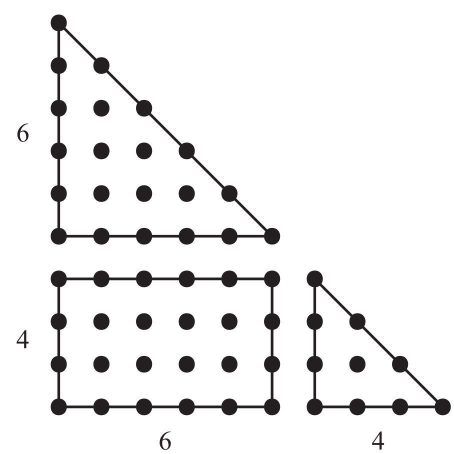


Figure 34.2

Algebraically, we have the following.

$$\begin{aligned}T(N+M)&=\frac{(N+M)(N+M+1)}{2}\\&=\frac{N^2+2MN+M^2+N+M}{2}\\&=\frac{N^2+N}{2}+\frac{2MN}{2}+\frac{M^2+M}{2}\\&=\frac{N(N+1)}{2}+MN+\frac{M(M+1)}{2}\\&=T(N)+MN+T(M).\end{aligned}$$

(This algebraic approach is less enlightening.)

Study Tip

- This lesson is purely optional. There are no recommended study tips for this lesson other than to enjoy the lesson and let the thinking about it strengthen your understanding of geometry as a whole.

Pitfall

- Don't forget to have fun in your thinking of mathematics. This is a fun topic.

Problems

1. If every year you were given a birthday cake with as many candles as your new age, how many birthday candles have you blown out in your life?

2. The number 1225 is both square and triangular. It is the 35^{th} square number (we have $S(35) = 35^2 = 1225$). Which triangular number is it?

3. Consider three consecutive triangular numbers. Use geometry—or perhaps algebra—to explain why six copies of the middle number combined with the other two is sure to give a square number. (As an example, the three triangular numbers 10, 15, and 21 give $10 + 6 \times 15 + 21 = 121$, which is square.)

4. If $S(N)$ denotes the N^{th} square number, show that $S(N + M) = S(N) + S(M) + 2MN$.

5. We have proved that

$$1+2+3+\cdots+N=\frac{N(N+1)}{2}$$

and that

$$1+2+\cdots+(N-1)+N+(N-1)+\cdots+2+1=N^2.$$

Consider the following array of multiplication problems.

$$\begin{array}{cccc}
1\times1 & 1\times2 & 1\times3 & 1\times4\\
2\times1 & 2\times2 & 2\times3 & 2\times4\\
3\times1 & 3\times2 & 3\times3 & 3\times4\\
4\times1 & 4\times2 & 4\times3 & 4\times4
\end{array}$$

The sum of the entries in the first row is $1 \times (1 + 2 + 3 + 4)$, and the sum of the entries in the second row is $2 \times (1 + 2 + 3 + 4)$, and so on.

a) Explain why the sum of all the entries in the entire table equals $(1 + 2 + 3 + 4)^2$.

b) The table can be divided into gnomons (L shapes).

$$\begin{array}{c|c|c|c}
1\times1 & 1\times2 & 1\times3 & 1\times4\\
\cline{1-1}
2\times1 & 2\times2 & 2\times3 & 2\times4\\
\cline{1-2}
3\times1 & 3\times2 & 3\times3 & 3\times4\\
\cline{1-3}
4\times1 & 4\times2 & 4\times3 & 4\times4
\end{array}$$

Explain why the sum of the entries in the entire table is also

$$1 \times 1 + 2 \times (1 + 2 + 1) + 3 \times (1 + 2 + 3 + 2 + 1) + 4 \times (1 + 2 + 3 + 4 + 3 + 2 + 1).$$

c) Explain why we have just proved that $1^3 + 2^3 + 3^3 + 4^3 = (1 + 2 + 3 + 4)^2$.

d) Explain the formula $1^3+2^3+3^3+\cdots N^3=\dfrac{N^2(N+1)^2}{4}$.

Complex Numbers in Geometry

Lesson 35

Topics

- Introduction to complex numbers.
- The geometric effect of multiplication by i.
- Applications to geometric problems.

Definition

- **complex number**: A number of the form $a + ib$, with a and b each a real number and i an alleged quantity with the mathematical property that $i^2 = -1$.
- **median of a triangle**: A line segment that connects one vertex of a triangle to the midpoint of its opposite side. (Each triangle possesses three medians, and they all pass through a common point called the centroid of the triangle.)

Result

- If a point $P = (a, b)$ is represented as a complex number $a + ib$, then "iP" $= i(a + ib) = -b + ia$ represents the point P rotated about the origin counterclockwise through an angle of 90°.

Summary

The introduction of complex numbers to the field of geometry allows us to view points as numbers and thereby be able to perform arithmetic on them. This provides a powerful new approach to proving geometric results. In this lesson, we introduce complex numbers, discuss their arithmetic, and explore some applications of their might.

Example 1

The median of a triangle is a line from one vertex of a triangle to the midpoint of its opposite side. Each triangle has three medians.

Use complex numbers to prove that the medians of a triangle are sure to be concurrent (that is, are sure to pass through a common point). Also show that the common point of intersection is $\frac{1}{3}$ along each median. (See **Figure 35.1**.)

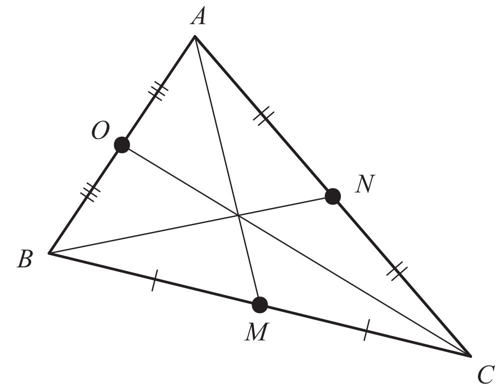


Figure 35.1

Solution

Consider a triangle *ABC* with *A*, *B*, and *C* regarded as complex numbers. Then, the midpoints of its sides are

$$M = \frac{B+C}{2},\ N = \frac{C+A}{2},\ O = \frac{A+B}{2}.$$

Let *P* be the point $\frac{1}{3}$ along the median $\overline{MA}$. We have

$$P = M + \frac{1}{3}\overline{MA} = M + \frac{1}{3}(A - M) = \frac{2}{3}M + \frac{1}{3}A = \frac{2}{3}\left(\frac{B+C}{2}\right) + \frac{1}{3}A = \frac{A+B+C}{3}.$$

Let *Q* be the point $\frac{1}{3}$ along the median $\overline{NB}$. We have

$$Q = N + \frac{1}{3}\overline{NB} = N + \frac{1}{3}(B - N) = \frac{2}{3}N + \frac{1}{3}B = \frac{2}{3}\left(\frac{A+C}{2}\right) + \frac{1}{3}B = \frac{A+B+C}{3}.$$

Let *R* be the point $\frac{1}{3}$ along the median $\overline{OC}$. We have

$$R = O + \frac{1}{3}\overline{OC} = O + \frac{1}{3}(C - O) = \frac{2}{3}O + \frac{1}{3}C = \frac{2}{3}\left(\frac{A+B}{2}\right) + \frac{1}{3}C = \frac{A+B+C}{3}.$$

These are all the same point!

Study Tip

- This lesson is purely optional. There are no recommended study tips for this lesson other than to enjoy the lesson and let the thinking about it strengthen your understanding of geometry as a whole.

Pitfall

- The mathematics in this lesson is visually overwhelming—but it is not conceptually overwhelming. Just be steady in your resolve as you work your way through the algebra.

Problems

1. If $A = (-1, 10)$ and $B = (5, 6)$, find the coordinates of the point P on $\overline{AB}$ such that $AP{:}PB = 3{:}5$.

2. Compute the following.

a) $(3 + 7i) + (2 - i)$.

b) $(3 + 7i) - (2 - i)$.

c) $(3 + 7i) \times (2 - i)$.

3. We've seen how to add, subtract, and multiply complex numbers. Is it possible to also divide (nonzero) complex numbers?

a) Find a complex number that deserves to be called "$\frac{1}{4+3i}$". (That is, find a complex number $a + ib$ such that $(a + ib)(4 + 3i) = 1$.)

b) What complex number is $\frac{1}{i}$?

c) Find a general formula for a complex number that represents $\frac{1}{p+iq}$, for p and q real numbers, not both zero.

4. What is i^{403}?

5. Four squares are drawn on the sides of a quadrilateral. Let P_1, P_2, P_3, and P_4 be the centers of those squares labeled in a clockwise order. Draw the segments $\overline{P_1P_3}$ and $\overline{P_2P_4}$ connecting opposite midpoints. (See **Figure 35.2**.)

Prove that $\overline{P_1P_3}$ and $\overline{P_2P_4}$ are congruent (that is, are the same length) and perpendicular (that is, lie at an angle of 90° to one another).

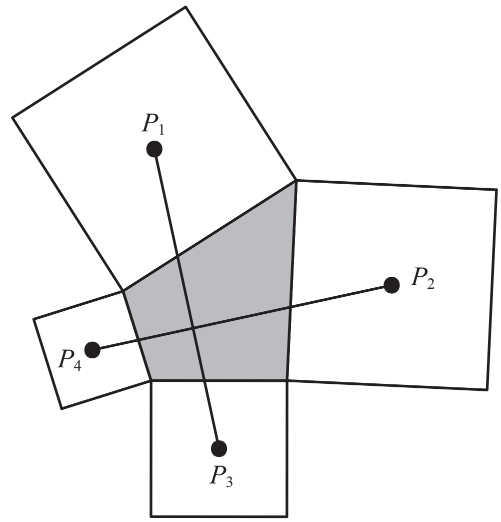


Figure 35.2

Bending the Axioms—New Geometries

Lesson 36

Topics

- Spherical geometry.
- Taxicab geometry.

Definition

- **taxicab geometry**: If distances between two points $A = (a_1, a_2)$ and $B = (b_1, b_2)$ are measured by the sum of the horizontal and vertical displacements, $d(A,B) = |b_1 - a_1| + |b_2 - a_2|$, rather than via the Pythagorean theorem, and all other structures of planar geometry are left the same, then the geometry that results is called taxicab geometry.

Formula

- The area of a spherical triangle with interior angles of measures $x°$, $y°$, and $z°$ is

$$\frac{(x+y+z-180)}{720} \times \text{total surface area of the sphere.}$$

Summary

It can be fruitful—and fun—to think of alternative ways to interpret the words "line," "point," "distance," and the like, while still obeying all but one of the postulates of geometry. That way, you can develop a feel for the degree to which a particular postulate in geometry is important and see how things drastically change if that postulate does not hold. In this lesson, we briefly return to the wonders of spherical geometry and introduce a brand new geometry called taxicab geometry.

Example 1

In taxicab geometry, what are the distances between the following pairs of points?

a) $P = (2, 5)$ and $Q = (7, 6)$.

b) $P = (-4, 8)$ and $Q = (3, -5)$.

c) $P = \left(3\frac{1}{5}, -6\right)$ and Q = $(-2, 600.76)$.

Solution

a) Distance = 5 + 1 = 6.

b) Distance = 7 + 13 = 20.

c) Distance $= 5\frac{1}{5} + 606.76 = 611.96$.

Example 2

a) Plot the points $A = (1, 1)$ and $B = (5, 3)$.

b) Find a point P whose taxicab distance from A is the same as its taxicab distance from B. In fact, find 10 such points. Actually, graph the set of *all* points that are equidistant from A and B in taxicab geometry. (This is the taxicab version of the perpendicular bisector of $\overline{AB}$.)

Solution

This graph is surprising!

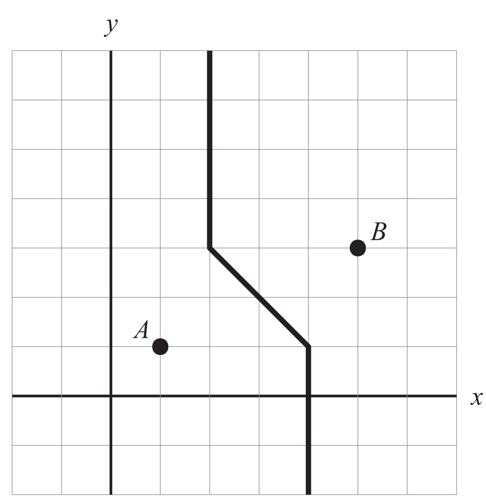


Figure 36.1

Study Tip

- This lesson is purely optional. There are no recommended study tips for this lesson other than to enjoy the lesson and let the thinking about it strengthen your understanding of geometry as a whole.

Pitfall

- Don't forget to have fun in your thinking of mathematics. This is a fun topic.

Problems

1. a) Draw a taxicab "circle" of radius 3, center $(1, 4)$.

b) Draw another taxicab circle that intersects this circle at just one point.

c) Draw another taxicab circle that intersects the first circle in exactly two points.

d) Draw yet another taxicab circle that interests the original circle at infinitely many different points.

2. Plot the points $A = (1, 2)$ and $B = (5, 6)$, and graph *all* the points that are equidistant from A and B in taxicab geometry. (This answer is exceptionally surprising!)

3. a) On the following grid, find a point P that is equidistant from each of A, B, and C.

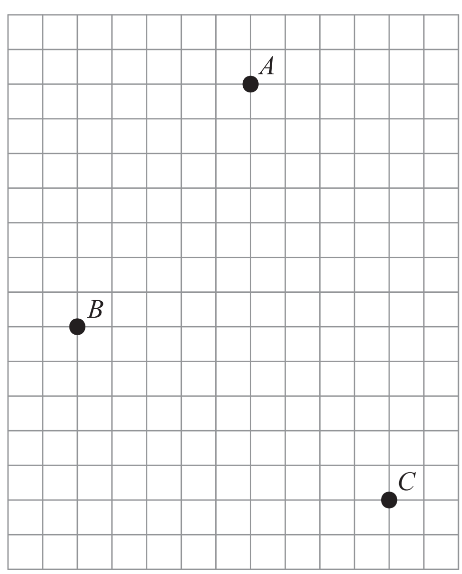


Figure 36.2

b) What is the radius of the taxicab circle that passes through each of these three points?

c) Draw the taxicab circle that passes through each of these three points.

4. Is it possible to draw an equilateral triangle on the surface of a sphere with three 60° angles?

5. The radius of the Earth is approximately 6400 km. At what latitude $x°$ can you walk directly east for 1 mile and return to start? (See **Figure 36.3**.)

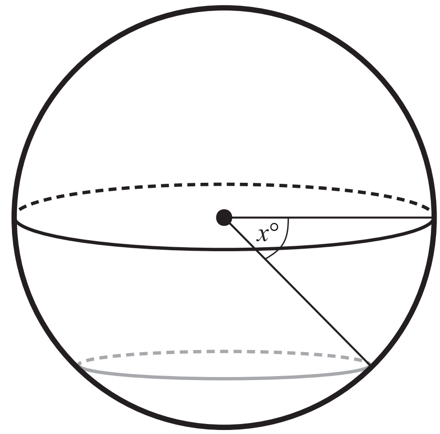


Figure 36.3

Solutions

Lesson 1

1. False. A circle drawn on the surface of a donut, for example, need not divide the surface into two parts. (See **Figure S.1.1**.)

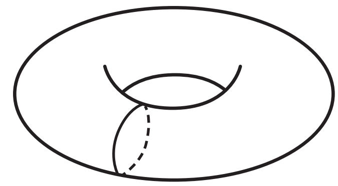


Figure S.1.1

2. There is no example of a perfectly straight line in the room you currently sit in. At the microscopic level, there are sure to be bumps and niches in anything you point out. (Isn't it curious that we all have programmed in our minds the concept of a perfectly straight line, even though we have never seen one?)

3. Like the scaling feature on smartphone maps, it seems that all shadows of straight objects are again straight (in a broad sense, putting issues of Problem 2 aside). Is it at all obvious that this should be the case? (We'll explore this in Lesson 10.)

Lesson 2

1. a) True.

b) False.

c) True (any three points are coplanar).

d) True.

e) True.

f) True.

2. a) Correct.

b) Not correct: It should read $\overline{AB} \cong \overline{CD}$.

c) Not correct: It should read $AB = CD$.

d) Correct.

3. a)

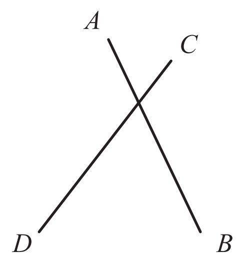


Figure S.2.1

b)

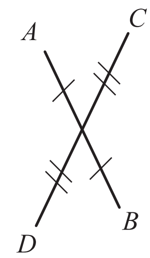


Figure S.2.2

c)

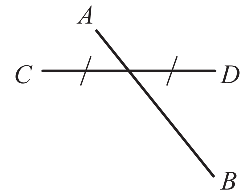


Figure S.2.3

4. a) Yes. They all lie on the shaded plane.

b) Yes. They all lie on the white plane.

c) Yes. Any three points are coplanar. (The plane doesn't happen to be shown in the picture.)

d) Yes. They all lie on a common line.

e) Intuitively, it seems that the answer is yes.

5. $BC = 4$ if we have the following.

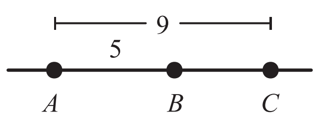


Figure S.2.4

$BC = 14$ if instead we have the following.

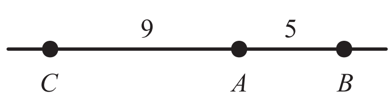


Figure S.2.5

6. No, as the following diagram shows.

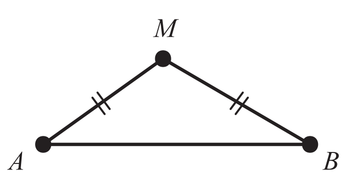


Figure S.2.6

7. $90° = 100$ gradian.

a) Divide by 100 to see that 1 gradian $= \frac{90}{100} = 0.9°$.

b) Divide by 90 to see that $1° = \frac{100}{90} = \frac{10}{9} = 1.111\ldots$ gradian.

8. We have $2x - y + 16 - y + 3x + 2y - x + 2 = 98$. Algebra then gives the following.

$$\begin{aligned} 4x + 18 &= 98 \\ 4x &= 80 \\ x &= 20. \end{aligned}$$

(Thank goodness the question didn't ask for the value of y as well!)

9. a) 142°.

b) 40°.

c) 60°.

d) 38°.

e) 120°.

10. a) $\angle EUD$.

b) Any one of $\angle DUG$ or $\angle EUG$ or $\angle SUM$ will do.

c) $\angle SUG$, for example.

d) The large version of $\angle SUM$, for example.

e) $\angle DUG \cong \angle SUM$ (vertical angle theorem).

f) $\angle DUM \cong \angle GUS$.

g) $\angle DUG$ and $\angle GUE$. (Recall the definition of "complementary" from Example 3.)

h) $\angle SUG$ and $\angle GUD$, for example. (Recall the definition of "supplementary" from Example 3.)

i) $\angle SUE$ and $\angle EUD$. (We must have two 90° angles.)

Lesson 3

1. We have $5y + y = 360°$, so $6y = 360°$, giving $y = 60$.

Because the interior angles of a four-sided shape sum to 360°, we have the following.

$$\begin{aligned} x + 40 + x + 5y &= 360 \\ x + 40 + x + 300 &= 360 \\ 2x &= 20 \\ x &= 10. \end{aligned}$$

2. Call the measures of the two remaining angles in the triangle x and y. We have $x + y + 60 = 180$, so $x + y = 120$. If $x < 60$, then we must have $y > 60$.

The third angle is greater than 60° in measure, and it is also less than 120°.

3. Look for sums of 180° in different triangles within each picture.

a) $50 + 90 + x = 180$, so $x = 40$.

$30 + 90 + (x + y) = 180$, so $x + y = 60$, and thus $y = 20$.

b) $x + 90 + 32 = 180$, so $x = 58$.

$32 + (180 - y) + 10 = 180$, giving $y = 42$.

c) $30 + 80 + y = 180$, so $y = 70$.

Look at the four-sided figure $x + 80 + 80 + y = 360$. This gives $x = 130$.

4. a) Possible. (See **Figure S.3.1**.)

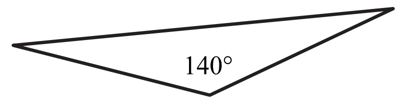


Figure S.3.1

b) No. Two obtuse angles add up to more than 180° in a triangle.

c) No. If two are not possible, then three certainly are not.

5. ii. $\triangle GMI$.

iii. $\triangle LRI$.

iv. $m\angle G = 180 - x - y$.
$m\angle L = 180 - x - y$.

6.

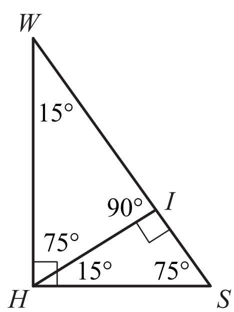


Figure S.3.2

7. $\angle L$ is divided into two congruent angles by $\overline{LM}$. Label each x.

$2x + 10 + 30 = 180$ gives $2x = 140$, so $x = 70$.

Thus, $m\angle MLU = 70°$.

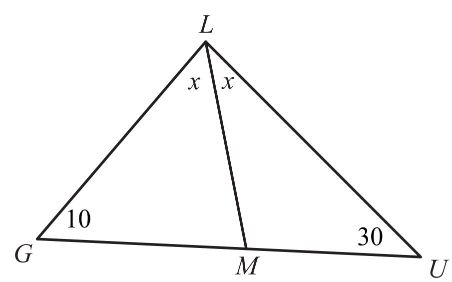


Figure S.3.3

8. i. $x + y = 90$.

ii. $y + z = 90$ because $\overline{OR} \perp \overline{OS}$.

iii. So, $x + y = y + z$, giving $x = z$.

9. Angles in a six-sided shape sum to 4 × 180 = 720°. Thus,

$$\begin{aligned} 90+2x+x+(360-x)+x+(x+10) &= 720 \\ 4x+460 &= 720 \\ 4x &= 260 \\ x &= 65. \end{aligned}$$

10. Angles in a 10-sided shape add to 8 × 180 = 1440°.

$$\begin{aligned} 270+90+x+(360+x)+x+(360-x)+x+x+(360-x)+x &= 1440 \\ 1440+2x &= 1440 \\ x &= 0. \end{aligned}$$

This means that for this figure to exist, x is an angle of 0°. This is absurd. The figure cannot exist.

Lesson 4

1. a) Exterior angle = 60°; interior angle = 120°.

b) Exterior angle = 45°; interior angle = 135°.

c) Exterior angle = 31.4°; interior angle = 128.57°.

d) Exterior angle = 120°; interior angle = 60°.

A regular triangle is usually called just an equilateral triangle. (We will later prove in Lesson 9 that an equilateral triangle is necessarily equiangular as well.)

2. We have the following.

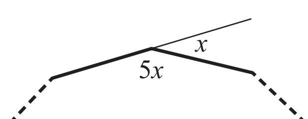


Figure S.4.1

$$\begin{aligned} x+5x &= 180 \\ 6x &= 180 \\ x &= 30. \end{aligned}$$

So, the exterior angle is 30°. Thus,

$$\frac{360}{N} = 30$$
$$360 = 30N$$
$$N = \frac{360}{30}$$
$$N = 12.$$

The polygon has 12 sides.

3. 360°, as always!

4.

Number of sides	6	15	20	36	360	4
Measure of an exterior angle	60	24	18	10	1	90
Measure of an interior angle	120	156	162	170	179	90

5. The exterior angle would be 27°. So,

$$\frac{360}{N} = 27$$
$$360 = 27N$$
$$N = \frac{360}{27}$$
$$N = 13.3.$$

That is, the polygon has 13.3 sides. This is absurd. There can be no regular polygon with interior angle 153°.

6. a)

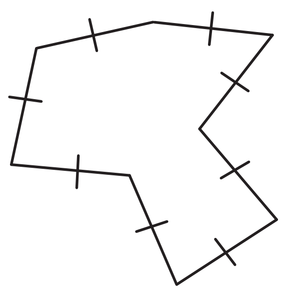
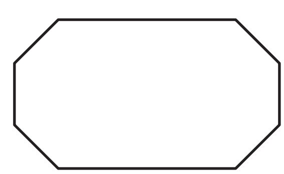


Figure S.4.2

b)



An elongated stop sign.
All angles are 135°.

Figure S.4.3

7. If the measure of each congruent angle is x, then $90 + 90 + 90 + x + x = 540$ gives $x = 135°$.

8. a) Each interior angle of a regular heptagon has measure $180 - \frac{360}{7} \approx 128.37°$. None equal 115°, so the answer is no.

b) Yes. Draw one angle of 115° and include it as part of some seven-sided shape.

9. Let N be the number of sides of the polygon.

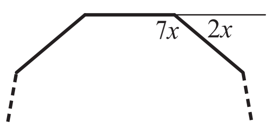


Figure S.4.4

We have $9x = 180$, so $x = 20$. The exterior angle is $2x = 40$.

$\frac{360}{N} = 40$ shows that the number of sides is $N = 9$.

10. Let N be the number of sides of the polygon.

$$3400 \le (N-2)180 \le 3500$$

$$\frac{3400}{180} \le N - 2 \le \frac{3500}{180}$$

$$18.9 \le N - 2 \le 19.4$$

$$20.9 \le N \le 21.4.$$

We must have $N = 21$.

Lesson 5

1.

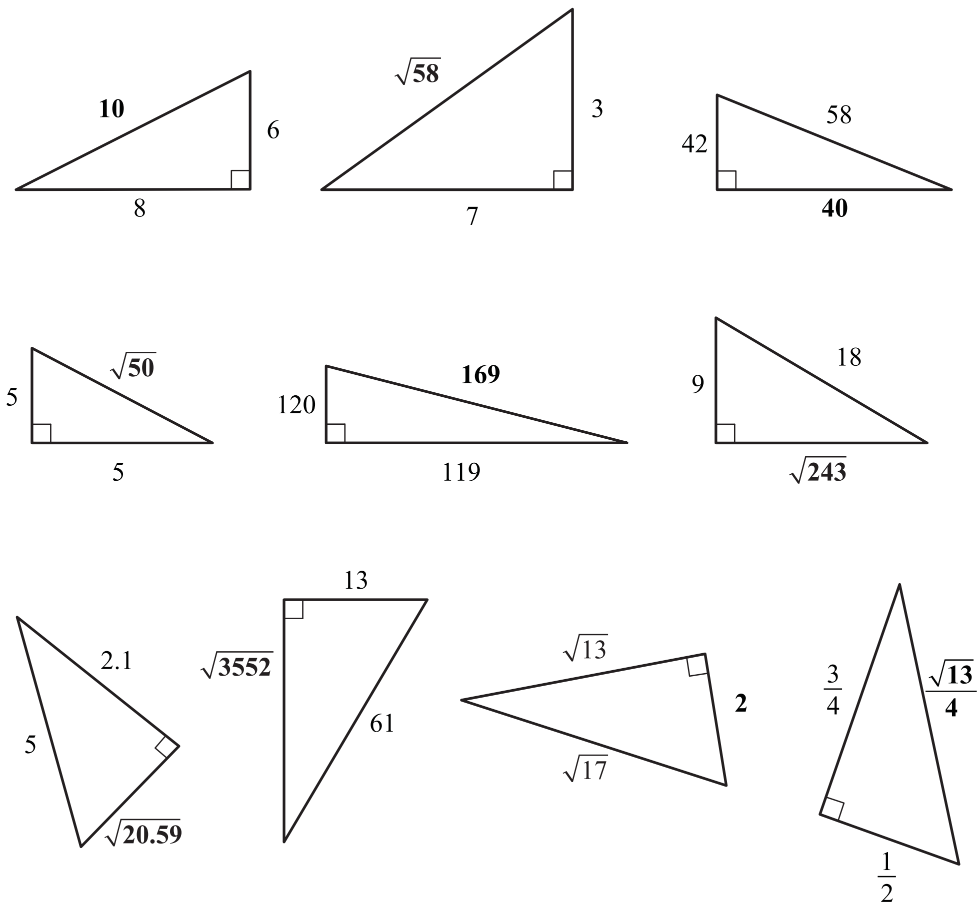


Figure S.5.1

2. Label the remaining side length x, as shown in **Figure S.5.2**.

$$\frac{36}{25} + x^2 = 4$$

$$x^2 = \frac{100}{25} - \frac{36}{25} = \frac{64}{25}$$

$$x = \frac{8}{5}.$$

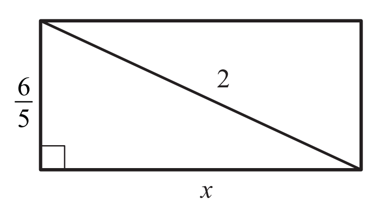


Figure S.5.2

3. If the sides of the rectangle come in a 4:3 ratio, then the side lengths are $4x$ and $3x$ for some value x, as shown in **Figure S.5.3**.

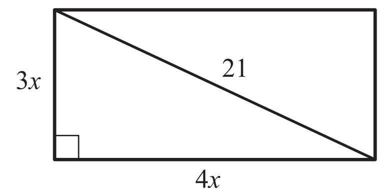


Figure S.5.3

We have the following.

$$(3x)^2 + (4x)^2 = 21^2$$
$$9x^2 + 16x^2 = 21^2$$
$$25x^2 = 21^2$$
$$x^2 = \frac{21^2}{25}$$
$$x = \frac{21}{5}.$$

So, the length and width of the rectangle are $\frac{63}{5}$ and $\frac{84}{5}$.

4. Label the remaining side h, as shown in **Figure S.5.4**.

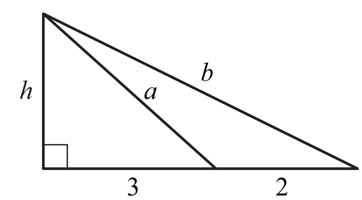


Figure S.5.4

a) By the Pythagorean theorem,

$$a^2 = 9 + h^2$$
$$a^2 = 9 + \text{more}$$
$$a^2 > 9$$
$$a > 3.$$

b) By the Pythagorean theorem,

$$b^2 = 5^2 + h^2$$
$$b^2 > 25$$
$$b > 5.$$

c) $b^2 = 25 + h^2$, which is obviously larger than $9 + h^2 = a^2$ (because 25 is larger than 9).

Thus, $b^2 > a^2$. This gives $b > a$.

Solutions

5. Label x, the length of the diagonal at the base of the cube, as shown in **Figure S.5.5**.

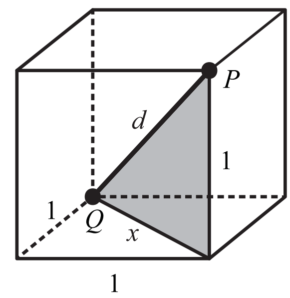


Figure S.5.5

By the Pythagorean theorem for the triangle on the bottom face, $1^2 + 1^2 = x^2$, giving $x = \sqrt{2}$.

By the Pythagorean theorem for the shaded triangle, $x^2 + 1^2 = d^2$. Thus, $2 + 1 = d^2$, giving $d = \sqrt{3}$.

6. Label as suggested by the hint. (See **Figure S.5.6**.)

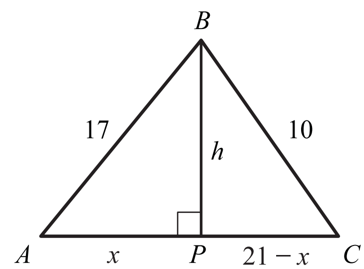


Figure S.5.6

We have the following.

$$x^2 + h^2 = 17^2 = 289$$
$$(21 - x)^2 + h^2 = 100.$$

Expand the second equation.

$$441x - 42x + x^2 + h^2 = 100.$$

Make use of the first equation.

$$\begin{aligned} 441x - 42x + 289 &= 100 \\ 42x &= 630 \\ x &= 15. \end{aligned}$$

Back to the first equation.

$$\begin{aligned} 15^2 + h^2 &= 289 \\ 225 + h^2 &= 289 \\ h^2 &= 64 \\ h &= 8. \end{aligned}$$

7. a) Impossible: Two sides cannot add to less than the third side.

b) Impossible: Two sides cannot add to less than the third side.

c) Possible. For example, we have the right triangle shown in **Figure S.5.7**.

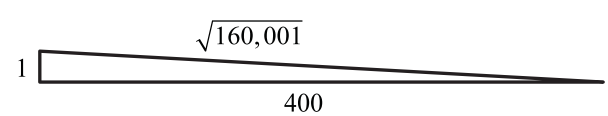


Figure S.5.7

8. Consider a triangle with side lengths a, b, and c, as shown in **Figure S.5.8**.

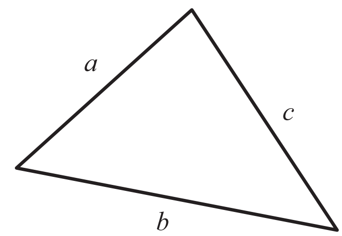


Figure S.5.8

$a + b + c =$ perimeter

If c is longer than half the perimeter, then $a + b$ is less than half the perimeter. This means that $a + b$ is smaller than c. But this is impossible. For a triangle, we need $a + b > c$.

9. If the three points A, B, and C are not collinear, then they form a triangle. (See **Figure S.5.9**.)

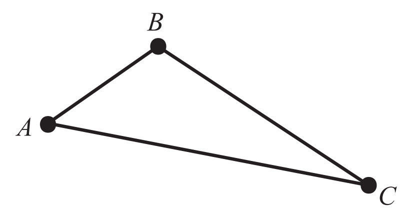


Figure S.5.9

By the triangular inequality, the three distances satisfy $AB + BC > AC$.

But we are told that $AB + BC = AC$. This means that the points *cannot* form a triangle and, therefore, must be collinear.

10. Label the lengths x and d, as shown in **Figure S.5.10**.

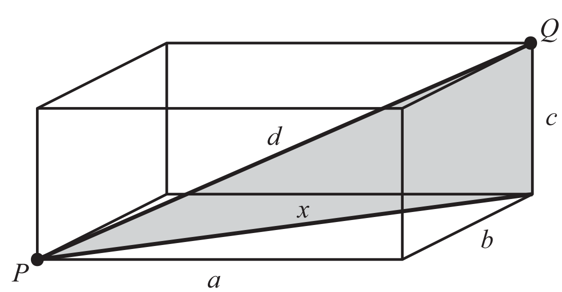


Figure S.5.10

By the Pythagorean theorem, $x^2 = a^2 + b^2$.

By the Pythagorean theorem for the shaded triangle, $d^2 = x^2 + c^2 = a^2 + b^2 + c^2$.

So, $PQ = d = \sqrt{a^2 + b^2 + c^2}$.

Lesson 6

1. a) $\sqrt{117}$.

b) $\sqrt{18}$.

c) $\sqrt{29}$.

d) 25.

e) $\sqrt{80}$.

2. a) $(8, 5)$.

b) $(6, 27)$.

c) $\left(-6, \dfrac{19}{2}\right)$.

d) $\left(0, \dfrac{b}{2}\right)$.

3. a) From a to $2a$, we have an increase of a. Another increase gives $3a$.

From 7 to 2, we have a decrease of 5. Another decrease of 5 gives -3.

This shows that $B = (3a, -3)$.

b) We need $2x+2=\dfrac{x+3x+4}{2}$, which happens to be automatically true.

We also need $9=\dfrac{x+3+2x}{2}$, which says that $18 = 3x + 3$, yielding $x = 5$.

4. $M = (2, 3)$ and $AM=\sqrt{3^2+1^2}=\sqrt{10}$.

5. a) $AP=\sqrt{6^2+2^2}$ and $BP=\sqrt{2^2+6^2}$.

These are the same.

b) $AQ=\sqrt{2^2+8^2}=\sqrt{68}$ and $QB=\sqrt{6^2+4^2}=\sqrt{52}$.

These are not the same. The point Q is not equidistant from A and B.

c) $AR=\sqrt{8^2+12^2}$ and $BR=\sqrt{12^2+8^2}$.

These are the same, and the point R is equidistant.

d) If $M = (k, k + 1)$, then $AM=\sqrt{(k+2)^2+(k+1-3)^2}=\sqrt{(k+2)^2+(k-2)^2}$ and $BM=\sqrt{(k-2)^2+(k+1+1)^2}=\sqrt{(k-2)^2+(k+2)^2}$.

These are the same, so $AM = MB$.

6. Set $A = (2, 3)$ and $B = (x, 7)$. We want $AB = 5$.

$$\begin{aligned}\sqrt{(x-2)^2+4^2} &= 5\\ (x-2)^2+16 &= 25\\ (x-2)^2 &= 9\\ x-2 &= 3 \text{ or } -3\\ x &= 5 \text{ or } -1.\end{aligned}$$

So, $B = (5, 7)$ and $B = (-1, 7)$ both work.

7. a) We need $\sqrt{(x-2)^2+(y-9)^2} = 6$.

b) Yes. She simply squared both sides of this equation.

8. a) $M = \left(\frac{16}{2}, \frac{-2}{2}\right) = (8, -1)$. (See **Figure S.6.1**.)

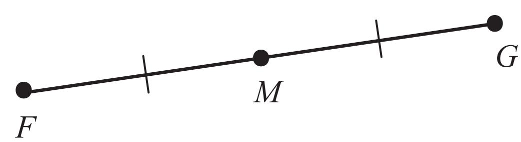


Figure S.6.1

b) The midpoint of $\overline{FM}$ is $\frac{1}{4}$ the way along $\overline{FG}$.

This is $\left(\frac{11}{2}, \frac{3}{2}\right)$.

The midpoint of $\overline{MG}$ is $\frac{3}{4}$ the way along $\overline{FG}$.

This is $\left(\frac{21}{2}, -\frac{7}{2}\right)$.

9. a) $AB = \sqrt{12^2+8^2}$.
$BC = \sqrt{8^2+12^2}$.
$AC = \sqrt{4^2+4^2}$.

We see that $AB = BC$. (See **Figure S.6.2**.)

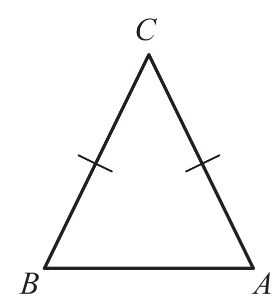


Figure S.6.2

b) M = midpoint of $\overline{AB} = (6, 6)$.
N = midpoint of $\overline{AC} = (2, 0)$.
O = midpoint of $\overline{BC} = (8, 4)$.

c) $MN = \sqrt{4^2+6^2}$ and $ON = \sqrt{6^2+4^2}$ (and $MO = \sqrt{2^2+2^2}$).

We see that $MN = ON$, so we have another isosceles triangle.

10. Let $A = (3, 5)$ and $B = (x, x)$. We need $AB = 10$.

$$\sqrt{(x-3)^2 + (x-5)^2} = 10$$
$$(x-3)^2 + (x-5)^2 = 100$$
$$x^2 - 6x + 9 + x^2 - 10x + 25 = 100$$
$$2x^2 - 16x = 66$$
$$x^2 - 8x = 33$$
$$x^2 - 8x + 16 = 49$$
$$(x-4)^2 = 49$$
$$x - 4 = 7 \text{ or } -7$$
$$x = 11 \text{ or } -3.$$

There are two points that work: $(11, 11)$ and $(-3, -3)$.

Lesson 7

1. See **Figure S.7.1**.

$$m\angle AIK = 180 - 32 = 148.$$
$$m\angle GKF = 60.$$
$$m\angle GKI = 180 - 60 = 120.$$

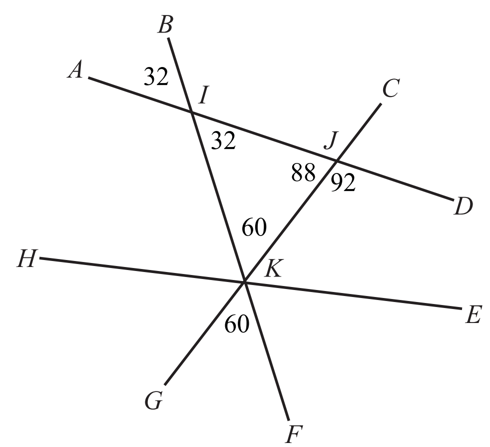


Figure S.7.1

2. See **Figure S.7.2**.

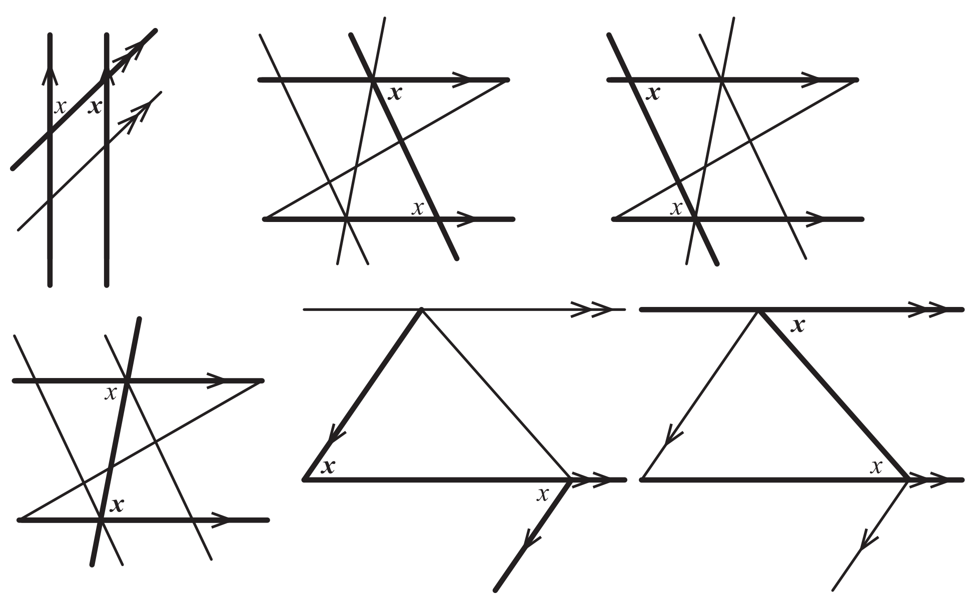


Figure S.7.2

3. a) $x = 88°$. (Use alternate interior angles and then vertical angles.)

b) $3x = 81°$ (by alternate interior angles and vertical angles), giving $x = 27°$.

c) $4x + 6x = 180°$, so $x = 18°$.

d) Use vertical angles and 180° in a triangle to see that $x + 63° + 72° = 180°$. This gives $x = 45°$.

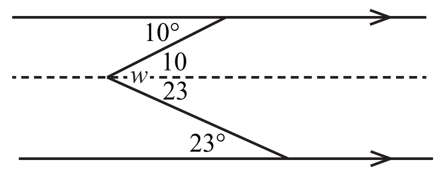


Figure S.7.3

4. a) Label congruent alternate interior angles to see that $w = 10° + 23° = 33°$. (See **Figure S.7.3**.)

b) Label supplementary same-side interior angles to see that $w = 57° + 88° = 145°$. (See **Figure S.7.4**.)

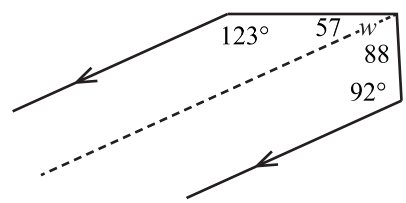


Figure S.7.4

5. a) $x = 40°$, $y = 80°$. (Look at the vertical angle to y, if you extend lines.)

b) $x = 50°$, $4y - 10 = x$, giving $y = 15°$. (Look at the vertical angle to x.)

c) $x = 20°$, $x + 40 + (180 - y) = 180$ gives $y = 60°$.

d) $y + 90 + 50 = 180°$ because there are 180° in a triangle, so $y = 40°$. $x + y + 90 + 30 = 180°$ because there are 180° in a triangle, so $x = 20°$.

6. See **Figure S.7.5**.

$$\begin{aligned} 4x + 140 + 160 &= 540 \\ 4x &= 240 \\ x &= 60°. \end{aligned}$$

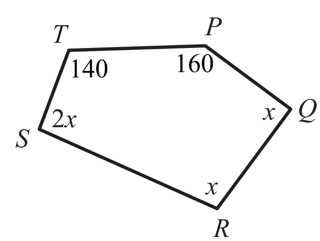


Figure S.7.5

We have the following.

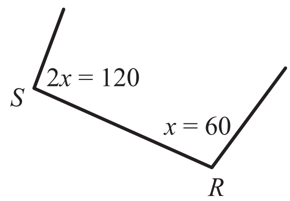


Figure S.7.6

Same-side interior angles summing to 180° gives $\overline{ST} \parallel \overline{RQ}$.

7. a) Because 87 + 93 = 180, we have same-side interior angles adding to 180°. This means that $\overline{GB} \parallel \overline{LO}$.

b) No. $87 + 97 \neq 180$.

8. a) $z \cong r$ gives congruent alternate interior angles, making $\overline{OU} \parallel \overline{TS}$.

b) $y \cong q$ doesn't imply anything.

c) $y + (x + r) = 180$ gives same-side interior angles adding to 180°, yielding $\overline{OU} \parallel \overline{TS}$.

9. Because there are 540° in a pentagon,

$$\begin{aligned} 160 + 140 + 2y + 3y + y &= 540 \\ 6y &= 240 \\ y &= 40°. \end{aligned}$$

Looking at same-side interior angles, $x + 55 = 180$, giving $x = 125°$.

Finally, $z + y + 85 = 180$ gives $z = 35°$.

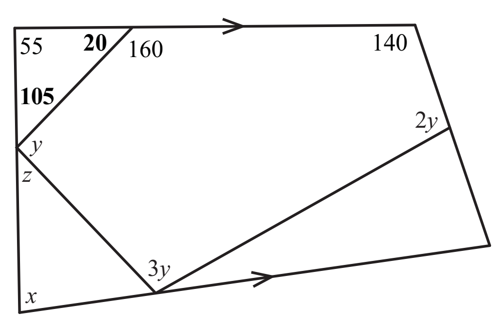


Figure S.7.7

10. i. … because they are vertical angles.

iii. … because a and e are congruent alternate interior angles.

Lesson 8

1. i. $b = c$ because vertical angles.

ii. $a = d$ because $a = b$, $b = c$, $c = d$ from the givens and the previous step.

iii. $\overline{AB} \parallel \overline{DE}$ because a and d are congruent alternate interior angles.

2. i. $a + b = 90$ because $\overline{PS} \perp \overline{SR}$.

ii. $a + c = 90$ because a and c are complementary. (Recall the definition of "complementary" from Lesson 2.)

iii. $b = 90 - a$ and $c = 90 - a$ because of algebra.

iv. So, $b = c$.

3. i. Label the angles x_1, x_2, y_1, y_2, z, w as shown in **Figure S.8.1**.

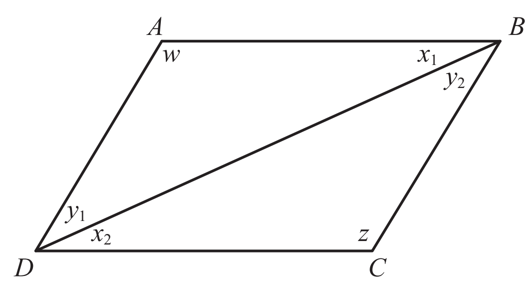


Figure S.8.1

ii. $x_1 \cong x_2$ because they are alternate interior angles for $\overline{AB} \parallel \overline{CD}$.

iii. $y_1 \cong y_2$ because they are alternate interior angles for $\overline{AD} \parallel \overline{BC}$.

iv. $w = 180 - x_1 - y_1$ and $z = 180 - x_2 - y_2$ because there are 180° in a triangle.

v. $w = z$ because $x_1 = x_2$ and $y_1 = y_2$.

That is, $\angle A \cong \angle C$.

Comment: Notice the first line. It is quite a nice technique to label angles in the diagram with letters and then refer to those letter names. If you do this, the first line of your proof should be "i. Label the diagram as shown."

4. i. Label the lengths a, b, x, y, z as shown in **Figure S.8.2**.

ii. Triangles FUM and LUM are right triangles (with right angle at U) because $\overline{UF} \perp \overline{UM}$.

iii. $a = \sqrt{(x+y)^2 + z^2}$ and $b = \sqrt{y^2 + z^2}$ because of the Pythagorean theorem.

iv. $a > b$ because $x + y > y$.

That is, $FM > LM$.

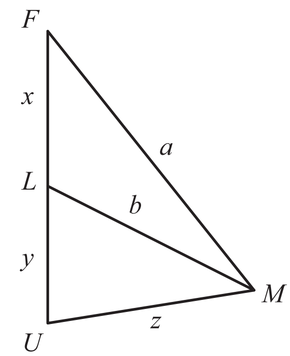


Figure S.8.2

5. i. Label angles a_1, a_2, b, c, d, e as shown in **Figure S.8.3**.

ii. $a_1 = a_2$ because given.

iii. $e = a_2$ and $d = a_1$ because alternate interior angles for parallel lines.

iv. $d = a_2$ because of the previous two steps.

v. $d + c = 90$ because $\overline{UB} \perp \overline{YB}$.

vi. $a_2 + b = 90$ because same-side interior angles add to 180°.

vii. $c = 90 - d$ and $b = 90 - a_2$ because of algebra.

viii. $b = c$ because $a_2 = d$ from step 4.

That is, $\angle YUB \cong \angle RBU$.

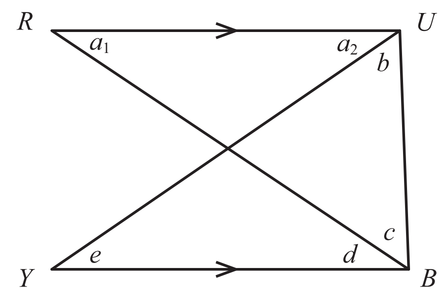


Figure S.8.3

6. i. Label angle w as shown in **Figure S.8.4**.

ii. $z + w = 180$ because on a straight line.

iii. $x + y + w = 180$ because 180° in a triangle.

iv. $z = 180 - w$ and $x + y = 180 - w$ because of algebra.

v. So, $z = x + y$.

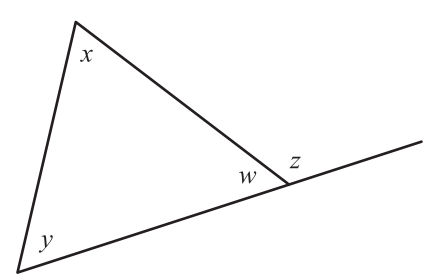


Figure S.8.4

7. i. Label lengths as shown in **Figure S.8.5**. Label the 90° angle given.

ii. $a_1 = a_2$ because $GT = GO$.

iii. $x = \sqrt{a_1^2 - h^2}$ and $y = \sqrt{a_2^2 - h^2}$ because of the Pythagorean theorem.

iv. $x = y$ because $a_1 = a_2$.

v. A bisects $\overline{TO}$ because $x = y$.

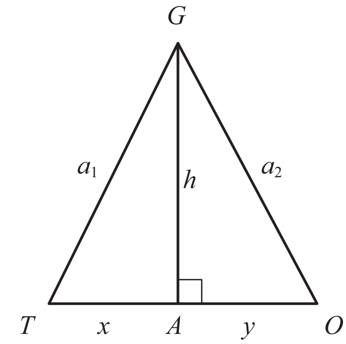


Figure S.8.5

8. i. Label angles w and z as shown in **Figure S.8.6**.

ii. $a = z$ because alternate interior angles for $\overline{BE} \parallel \overline{CD}$.

iii. $z = w$ because vertical angles.

iv. $w = b$ because alternate interior angles for $\overline{AC} \parallel \overline{ED}$.

v. $a = b$ because of algebra.

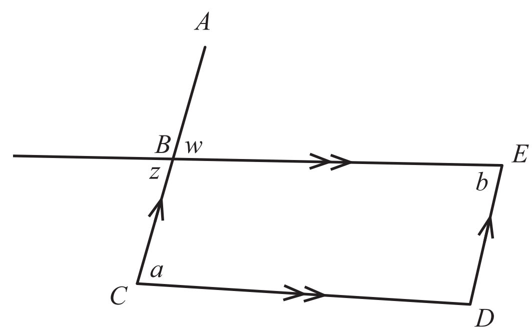


Figure S.8.6

9. This one is tricky!

i. Label angles x_1, x_2, x_3, y_1, y_2 as shown in **Figure S.8.7**.

ii. $x_1 = x_2$ because $\overline{KG}$ bisects $\angle EBH$.

iii. $y_1 = y_2$ because $\overline{FL}$ bisects $\angle NCH$.

iv. $x_1 + x_2 = m\angle CBI$ because they are vertical angles.

v. $m\angle CBI = y_1 + y_2$ because they are alternate interior angles for $\overline{EI} \parallel \overline{NJ}$.

vi. $x_1 + x_2 = y_1 + y_2$ because of the previous two steps.

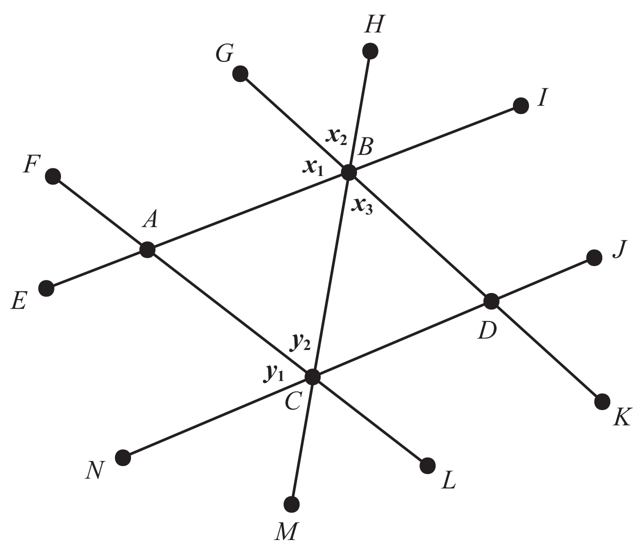


Figure S.8.7

vii. So, $2x_2 = 2y_2$ because $x_1 = x_2$ and $y_1 = y_2$.

viii. $x_2 = y_2$ because of algebra.

ix. $x_3 = x_2$ because of vertical angles.

x. $y_2 = x_3$ because of the previous two steps.

xi. $\overline{FL} \parallel \overline{GK}$ because x_3 and y_2 are congruent alternate interior angles.

10. Prove: Angles x and y have the same measure. (See **Figure S.8.8**.)

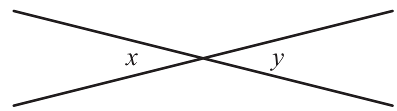


Figure S.8.8

Proof

i. Label another angle z, as shown in **Figure S.8.9**.

ii. $x + z = 180°$ and $y + z = 180°$ because each form a straight angle.

iii. So, $x + z = y + z$, giving $x = y$.

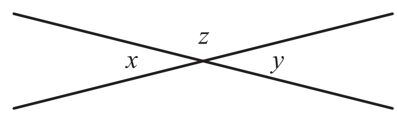


Figure S.8.9

Lesson 9

1. a) $\triangle HYB \sim \triangle LYK$ because of AA (share vertical angles and $\angle B \cong \angle K$).

b) $k = \frac{20}{14}$.

c) $x = 30°$.

d) $\frac{a}{8} = \frac{20}{14}$ gives $a = \frac{80}{7}$.

2. a) The triangles are ~ because of the AA principle. (They share the angle at the top and have angles of 70° in common.)

b) and c)

small		big
x	$\leftrightarrow$	$y+b$
y	$\leftrightarrow$	$x+a$
w	$\leftrightarrow$	f

d) We have $\frac{x}{y+b} = \frac{y}{x+a} = \frac{w}{f}$.

3. Notice that 9 $\leftrightarrow$ 10 + 8 = 18 and 10 $\leftrightarrow$ 9 + 11 = 20 with scale factor 2. Because the triangles share the angle between these sides, we have SAS in place with $k = 2$. Thus, $w = 8$.

4. a) $a \leftrightarrow x$ (the sides between 50° and 70° for their triangles).

b) $b \leftrightarrow d$ (the sides between 70° and 60° for their triangles).

c) $d \leftrightarrow y$ (the sides between 50° and 60° for their triangles).

d) No. The side d is not matching with itself between the two triangles. We cannot be sure that the scale factor is 1.

5. The two triangles in **Figure S.9.1** share an angle, and both have a 90° angle. By AA, they are similar.

We have $\frac{H}{110} = \frac{10}{30}$, giving $H = \frac{1100}{30} = 36\frac{2}{3}$ feet.

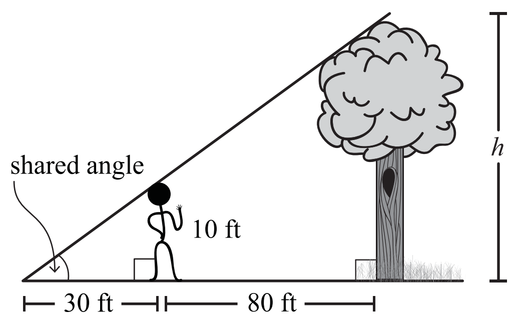


Figure S.9.1

6. See **Figure S.9.2**.

The top two triangles are similar by AA. (Both have 90°, and they share an angle.)

We have $\frac{6}{12} = \frac{x}{5}$, giving $x = 2\frac{1}{2}$.

The height of the middle pole is $12 + 2\frac{1}{2} = 14\frac{1}{2}$ feet.

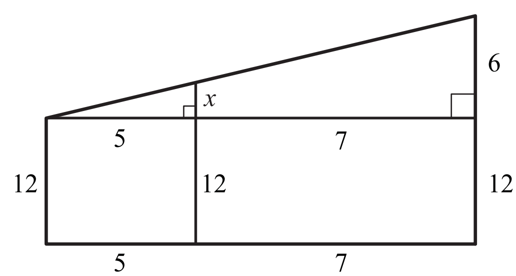


Figure S.9.2

7. a) We have congruent vertical angles and congruent alternate interior angles. So, by AA, they are similar. (See **Figure S.9.3**.)

b) The sides marked of equal length are not matching sides. We do not know the scale factor.

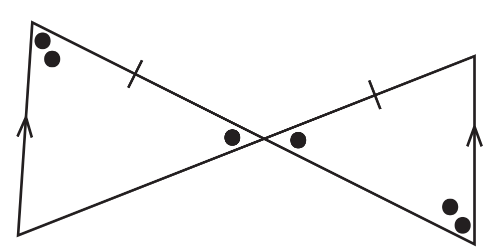


Figure S.9.3

8. a) i. Label the diagram as shown in **Figure S.9.4**.

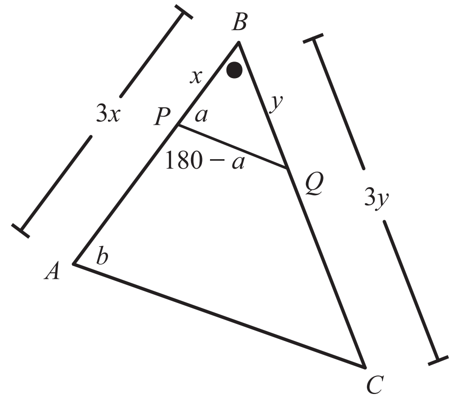


Figure S.9.4

ii. $\triangle PBQ \sim \triangle ABC$ because of SAS (x and $3x$, y and $3y$, and share angle B).

iii. $a = b$ because they are matching angles in ~ triangles.

iv. $\overline{PQ} \parallel \overline{AC}$ because the same-side interior angles marked b and $180 - a$ sum to 180°.

b) The scale factor between the two triangles is $k = 3$, so $AC = 3 \cdot PQ$.

9. a) Both triangles have a 90° angle and share angle C. By AA, they are similar. (See **Figure S.9.5**.)

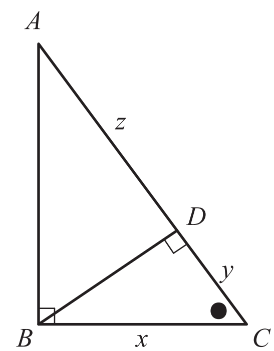


Figure S.9.5

b) Side x in the big triangle matches side y in the small triangle.

Side $y + z$ in the big triangle matches side x in the small triangle.

Because matching sides in ~ triangles come in the same ratio, $\dfrac{x}{y} = \dfrac{y+z}{x}$.

Algebra gives $x^2 = y(y + z)$.

10. i. Draw a diagonal and label the angles as shown in **Figure S.9.6**.

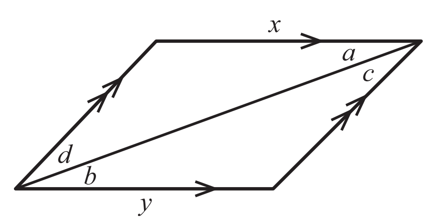


Figure S.9.6

ii. $a = b$ because they are alternate interior angles for top and bottom parallel lines.

iii. $c = d$ because they are alternate interior angles for left and right parallel lines.

iv. The two triangles are ~ because AA.

v. $k = 1$ because they share the diagonal.

vi. $x = y$ because $k = 1$.

Lesson 10

1. a) There is no reason to believe that these triangles are similar.

b) Mark congruent alternate interior angles for parallel lines, as shown in **Figure S.10.1**.

The two triangles are similar by AA. Because they share the diagonal (which does correspond to matching sides in the two triangles), the scale factor is 1. The triangles are, in fact, congruent.

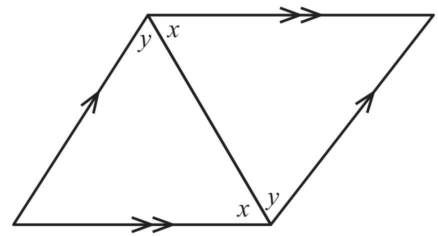


Figure S.10.1

c) Label the sides a_1 and a_2 as shown in **Figure S.10.2**.

By the Pythagorean theorem, $a_1 = \sqrt{x^2 - h^2}$ and $a_2 = \sqrt{x^2 - h^2}$; thus, $a_1 = a_2$. The two triangles are similar by SSS, with scale factor 1. They are, in fact, congruent.

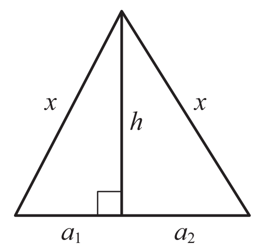


Figure S.10.2

d) See **Figure S.10.3**. Angles labeled x are congruent because they are vertical angles.

Angles labeled y are congruent because they are alternate interior angles for parallel lines.

The two triangles are similar by AA.

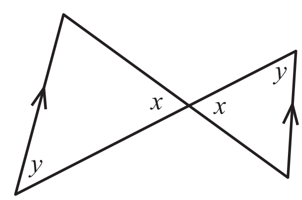


Figure S.10.3

2. Look at the ratio of the longest sides: $k = \dfrac{28.8}{7.2} = 4.$

Look at the ratio of the shortest sides: $k = \dfrac{6.0}{1.6} = 3.75.$ There is no consistent scale factor for the sides. These triangles are not similar.

3. a) A square and a non-square rectangle certainly have AAAA but are not similar. (Their side lengths do not match in the same ratio. See **Figure S.10.4**.)

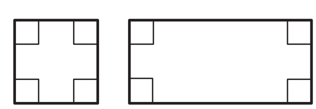


Figure S.10.4

b) A square and a non-square rhombus have SSSS but are not similar. (Angles do not match. See **Figure S.10.5**.)

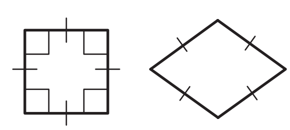


Figure S.10.5

4. By the Pythagorean theorem,

$$SP = \sqrt{25^2 - 24^2} = \sqrt{49} = 7$$
$$PA = \sqrt{30^2 - 24^2} = 18.$$

Thus, $SA = 25$, and the triangle is isosceles. Consequently, $\angle SNA \cong \angle A$.

5. Label the three interior angles of an equilateral triangle x, y, and z, as shown in **Figure S.10.6**.

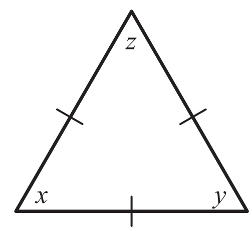


Figure S.10.6

We can view x and y as the base angles of an isosceles triangle. Thus, $x = y$.

We can view y and z as the base angles of an isosceles triangle. Thus, $y = z$.

Consequently, all three angles are congruent.

Also, their sum is 180°. It must be the case, then, that each angle has measure 60°.

6. i. Label sides a and b and angles x, y, and z as shown in **Figure S.10.7**.

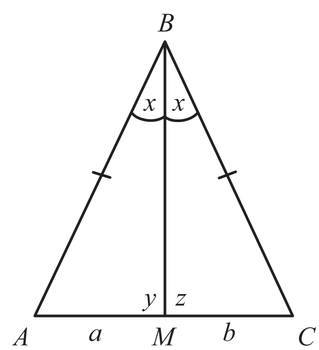


Figure S.10.7

ii. $\triangle ABM \sim \triangle CBM$ because SAS ($AB = BC$, angles x, side BM).

iii. $k = 1$ because they share side BM.

iv. $a = b$ because $k = 1$.

v. M is the midpoint of $\overline{AC}$ because $a = b$.

vi. $y = z$ because they are matching angles in ~ triangles.

vii. $y + z = 180°$ because they are on a straight line.

viii. $y = 90°$ and $z = 90°$ because the previous two steps.

iv. $\overline{BM} \perp \overline{AC}$ because $y = 90°$.

7. i. Label angles x, y, z, w as shown in **Figure S.10.8**.

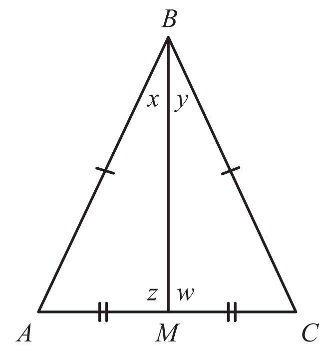


Figure S.10.8

ii. $\triangle ABM \sim \triangle CBM$ because SSS ($AB = BC$, $AM = MC$, and share BM).

iii. $x = y$ because they are matching angles in ~ triangles.

iv. $\overline{MB}$ bisects $\angle B$ because $x = y$.

v. $z = w$ because they are matching angles in ~ triangles.

vi. $z + w = 180°$ because they are on a straight line.

vii. $z = 90°$ and $w = 90°$ because the previous two steps.

viii. $\overline{BM} \perp \overline{AC}$ because $y = 90°$.

8. i. Label angles x and y and lengths a and b as shown in **Figure S.10.9**.

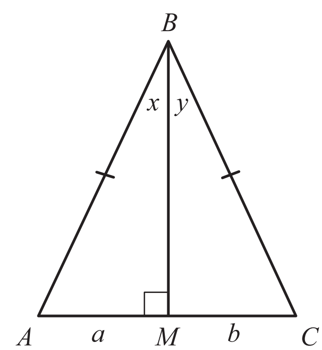


Figure S.10.9

ii. $a = \sqrt{AB^2 - BM^2}$ and $b = \sqrt{BC^2 - BM^2}$ by the Pythagorean theorem.

iii. $a = b$ because $AB = BC$.

iv. Everything now follows as if we are in the situation of the previous problem.

9. See **Figure S.10.10**.

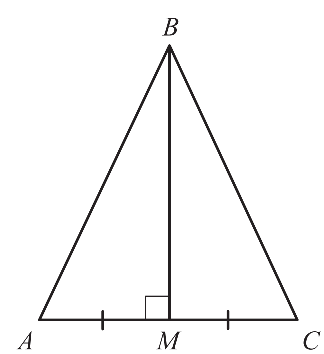


Figure S.10.10

i. $AB = \sqrt{AM^2 + BM^2}$ and $BC = \sqrt{MC^2 + BM^2}$ by the Pythagorean theorem.

ii. $AB = BC$ because $AM = MC$.

iii. Everything now follows as if we are in the situation of the previous problem.

10. i. Label angles x_1, x_2, a as shown in **Figure S.10.11**.

ii. $\triangle ACD \sim \triangle BCE$ because SSS.

iii. $x_1 = x_2$ because they are matching angles in ~ triangles.

iv. $\angle ACE = x_1 + a$ and $\angle BCD = x_2 + a$, clearly.

v. $\angle ACE \cong \angle BCD$ because $x_1 = x_2$.

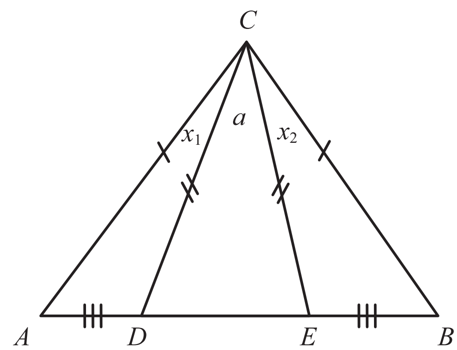


Figure S.10.11

Lesson 11

1. a) $\frac{10}{2} = 5$.

b) $\frac{-2}{-4} = \frac{1}{2}$.

c) $\frac{b}{a}$.

d) $\frac{b}{b} = 1$.

e) $\frac{c}{b-a}$.

f) $\frac{b-a}{a+b}$.

g) "$\frac{2}{0}$" = undefined.

2. a) This line connects $(0, 0)$ and $(10, 11)$ and, therefore, has slope $\frac{11}{10}$. Our parallel line has the same slope.

b) The line connecting the given points has slope $\frac{1}{-4}$. The perpendicular line we seek has slope 4.

c) This line is horizontal and, therefore, has slope 0.

3. The line has equation $\frac{y-0}{x-5} = 2$. That is, $y = 2x - 10$. Put $x = 0$ to see that the y-intercept is $y = -10$.

4. We need $\frac{k-5}{4} = -\frac{4}{3}$, which gives $k = -\frac{16}{3} + 5 = -\frac{1}{3}$.

5. The line we seek passes through $(0, 8)$ and $(-4, 0)$ and, therefore, has slope $\frac{8}{4} = 2$.

The equation is $\frac{y-8}{x} = 2$. That is, $y - 8 = 2x$.

6. Slope of $\overline{PQ} = \frac{8}{-1} = -8$; slope of $\overline{QR} = \frac{b+6}{b-4}$. We need these to be opposite reciprocals.

$$\begin{aligned} \frac{b+6}{b-4} &= \frac{1}{8} \\ 8b + 48 &= b - 4 \\ 7b &= -52 \\ b &= -\frac{52}{7}. \end{aligned}$$

7. Slope of $\overline{AB} = \frac{3}{5}$; slope of $\overline{CD} = \frac{-8}{a-2}$. We need these to match.

$$\begin{aligned} \frac{-8}{a-2} &= \frac{3}{5} \\ -40 &= 3a - 6 \\ a &= -\frac{34}{3}. \end{aligned}$$

8. a) $y - 20 = \frac{19}{2}(x-3)$.

b) We need a line of slope 5. So, $y - 2 = 5(x + 1)$.

c) Slope of $\overline{AB} = \frac{12}{14} = \frac{6}{7}$. So, we want slope $-\frac{7}{6}$.

Midpoint of $\overline{AB} = (4, -1)$.

The equation we seek is $y + 1 = -\frac{7}{6}(x-4)$.

d) Slope = 0, so $y + 2 = 0(x - 15)$; that is, $y = -2$ does the trick.

e) $2x - 3y = 0$ is equivalent to $y = \frac{2}{3}x$, and this is a line of slope $\frac{2}{3}$.

We want, then, a line of slope $-\frac{3}{2}$ through $(8, 0)$.

Then, $\frac{y}{x-8} = -\frac{3}{2}$; that is, $y = -\frac{3}{2}(x-8)$ does the trick.

f) A vertical line through $(2, 3)$ has equation $x = 2$.

9. a) $\dfrac{y-2}{x}=3$. That is, $y-2=3x$.

b) $y=5$.

c) Slope $=\dfrac{-7}{3}$. So, $\dfrac{y-7}{x-0}=-\dfrac{7}{3}$. That is, $y-7=-\dfrac{7}{3}x$.

d) Slope is 1. So, $\dfrac{y-2}{x}=1$. That is, $y-2=x$.

e) $x=3$.

f) Slope $=\dfrac{6-2}{4-0}=1$. So, $\dfrac{y-2}{x}=1$. That is, $y-2=x$.

g) $\dfrac{y-0}{x-3}=-2$. That is, $y=-2(x-3)$.

10. Part I

a) $M=(2,4)$. The line $\overleftrightarrow{MC}$ has equation $y=4$.

b) $N=(0,5)$. The line $\overleftrightarrow{NA}$ has equation $y-5=-3x$.

c) $R=(-1,3)$. The line $\overleftrightarrow{RB}$ has equation $y-3=\frac{3}{4}(x+1)$.

d) By substituting $x=\dfrac{1}{4}$ and $y=3$ into each of these equations, we see that $\left(\dfrac{1}{3},4\right)$ fits all three equations.

Part II

a) $y-4=-\frac{1}{2}(x-2)$.

b) $y-5=-3x$.

c) $y-3=2(x+1)$.

d) The point $(0,5)$ does indeed fit all three equations.

Part III

a) $y-4=-\frac{1}{2}(x+3)$, which is the same as $x+2y=5$.

b) $y-2=-3(x-1)$, which is the same as $3x+y=5$.

c) $y - 6 = 2(x - 3)$, which is the same as $y = 2x$.

d) The point $(1, 2)$ does indeed fit all three equations.

Comment: In the mid-1700s, Swiss mathematician Leonhard Euler proved that three points arising this way from the coincidence of each of these three sets of special lines in a triangle are sure to be collinear.

Can you verify that $\left(\frac{1}{3}, 4\right)$, $(0, 5)$, and $(1, 2)$ do indeed lie on a common line?

Lesson 12

1. There are two ways to approach this question: Either use the distance formula twice for each point and check to see if distances match, or find the equation of the perpendicular bisector of $\overline{AB}$ and check to see if the given points lie on this line. In this solution, we adopt the second, more efficient, approach.

Slope of $\overline{AB} = \frac{2}{4} = \frac{1}{2}$.

Slope of perpendicular bisector $= -2$.

Midpoint of $\overline{AB} = (5, 5)$.

Equation of perpendicular bisector: $y - 5 = -2(x - 5)$.

a) $P = (3, 6)$ does not fit this equation. It is not equidistant.

b) $Q = (2, 7)$ does not fit this equation. It is not equidistant.

c) $R = (6, 3)$ does fit the equation. It is equidistant.

d) $S = (2, -9)$ is equidistant.

e) $T = (2, 3)$ is not equidistant.

f) $U = (10, -4)$ is not equidistant.

2. a) $AB = \sqrt{4^2 + 2^2} = \sqrt{20}$.
$BC = \sqrt{3^2 + 1^2} = \sqrt{10}$.
$AC = \sqrt{1^2 + 3^2} = \sqrt{10}$.

We have $AC = BC$, so the triangle is isosceles.

b) $M = \text{midpoint of } \overline{AB} = (2, -1)$.

$N = \text{midpoint of } \overline{BC} = \left(\frac{5}{2}, -\frac{5}{2}\right)$.

$O = \text{midpoint of } \overline{AC} = \left(\frac{1}{2}, -\frac{3}{2}\right)$.

c) $MO = \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2}$.

$NM = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{3}{2}\right)^2}$.

These are the same. We have another isosceles triangle.

3. i. $PA = PB$ because P is on the perpendicular bisector of $\overline{AB}$.

ii. $PB = PC$ because P is on the perpendicular bisector of $\overline{BC}$.

iii. $PA = PC$ because of algebra.

4. a) $\overline{MR}$ and $\overline{MN}$.

b) $\overline{MN}$ and $\overline{NR}$.

c) angle bisector of $\angle MRN$.

d) M and N.

e) perpendicular bisector of $\overline{MR}$.

5. a) P and R are each equidistant from A and B. Therefore, they each lie on the perpendicular bisector of $\overline{AB}$.

b) The line connecting P and R, by part a), must be the perpendicular bisector of $\overline{AB}$. That is, at the very least, $\overline{PQ}$ and $\overline{AB}$ are perpendicular.

6. See **Figure S.12.1**.

Precisely the same reasoning of Problem 5 applies to deltoids. Their diagonals are indeed perpendicular (though you must extend one of the diagonals of the figure before they intersect).

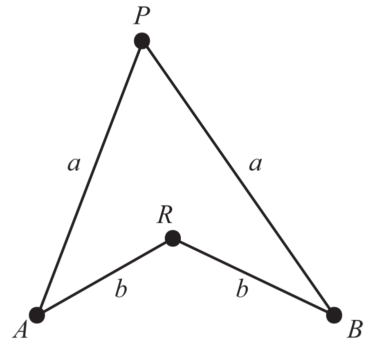


Figure S.12.1

7. a) Something like the following won't form the diagonals of a kite.

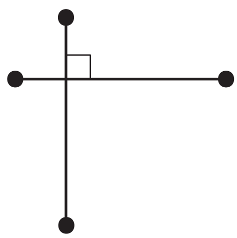


Figure S.12.2

b) Suppose that $ABCD$ is a convex figure with perpendicular diagonals $\overline{BD}$ bisecting $\overline{AC}$.

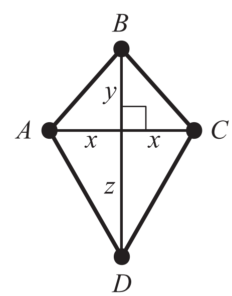


Figure S.12.3

i. Label lengths x, y, z as shown in **Figure S.12.3**, noting that $\overline{AC}$ is bisected.

ii. $AB = \sqrt{x^2 + y^2}$ and $CB = \sqrt{x^2 + y^2}$ because of the Pythagorean theorem.

iii. $AD = \sqrt{x^2 + z^2}$ and $CD = \sqrt{x^2 + z^2}$ because of the Pythagorean theorem.

iv. We have a kite because neighboring edges are congruent.

8. a) See **Figure S.12.4**.

b) If the radius of the circle were such that the circle just touches the equidistant line at one point, then the location of the treasure would be clear.

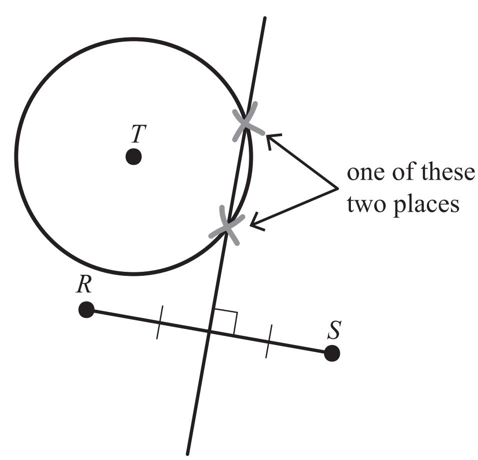


Figure S.12.4

Solutions

9. i. $JL = 2 \cdot KL$ because $\overline{TK}$ bisects $\overline{JL}$.

ii. $\triangle LKT \sim \triangle LJS$ because AA (90° and share angle L).

iii. $k = 2$ because step 1.

iv. $SL = 2 \cdot TL$ because $k = 2$.

v. T bisects $\overline{SL}$ because step 4.

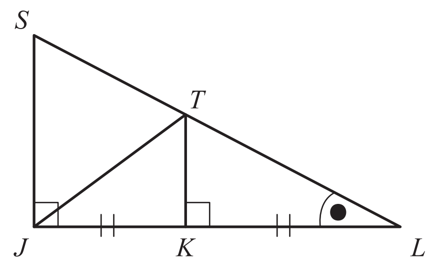


Figure S.12.5

10. Suppose that three points A, B, and C lie on a circle with center O and D does not. For P to be equidistant from all four points, it must be, in particular, equidistant from A, B, and C. Thus, P must coincide with the center O. But then P would not be the same matching distance from D.

Lesson 13

1. Draw a third parallel line as shown in **Figure S.13.1**.

By the triple parallel lines result, Q is the midpoint of the transversal segment $\overline{AC}$.

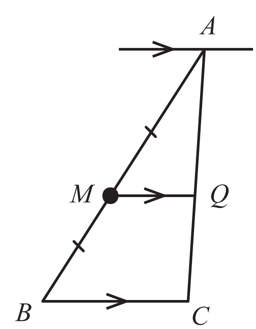


Figure S.13.1

2. a) $MO = \frac{1}{2} BC = 10.$

b) $AC = 2 \cdot MN = 26.$

c) Yes. A line connecting midpoints in a triangle is parallel to the base.

d) $\angle B \cong \angle AMO$ because corresponding angles for $\overline{BC} \parallel \overline{MO}$. $\angle AMO \cong \angle MON$ because alternate interior angles for $\overline{AB} \parallel \overline{NO}$. Thus, $\angle B \cong \angle MON$.

3. a) $M = (0, 5)$.

b) $N = (4, 6)$.

c) The slope of $\overline{MN} = \frac{1}{4}$, and the slope of $\overline{BC} = \frac{2}{8} = \frac{1}{4}$.

These are the same—as one expects for a line connecting midpoints in a triangle.

4. A line connecting midpoints in a triangle is half the length of the base of the triangle (and parallel to it).

a) $HB = \frac{1}{2}CG = 10$ and $FD = \frac{1}{2}CG = 10.$

b) Yes. They are both parallel to $\overline{CG}$.

5. Via congruent corresponding angles, we have $z = 51°$. Also,

$$\begin{aligned} 10x+1 &= 2(3x+4) \\ 10x+1 &= 6x+8 \\ x &= \frac{7}{4} \end{aligned}$$

and

$$\begin{aligned} 4y-2 &= 7(y-3) \\ 4y-2 &= 7y-21 \\ 19 &= 3y \\ y &= \frac{19}{3}. \end{aligned}$$

6. Each side of the small triangle is half the length of its matching side in the big triangle (because they are lines connecting midpoints in a triangle). Thus, by SSS, the two triangles are similar with $k = 2$.

7. i. $\triangle MAN \sim \triangle BAC$ because SAS (share $\angle A$, and sides about it are in same ratio).

ii. $k = \frac{5}{8}$ because $AM = \frac{5}{8}AB.$

iii. $MN = \frac{5}{8}BC$ because $k = \frac{5}{8}.$

iv. $\angle AMN \cong \angle ABC$ because matching angles in similar triangles.

v. $\overline{MN} \parallel \overline{BC}$ because $\angle AMN \cong \angle ABC$ give congruent corresponding angles.

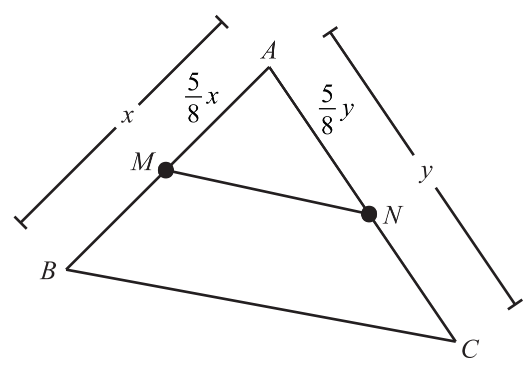


Figure S.13.2

8. These segments are all the same lengths ($\frac{5}{8}$ the length of the base segment) and parallel (because each is parallel to the base segment).

9. By Problem 7, each is parallel to the same diagonal. (See **Figure S.13.3**.)

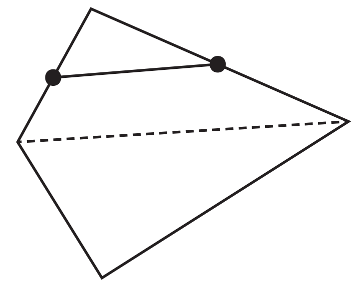


Figure S.13.3

10. i. The shaded triangles are similar by AA (we have congruent corresponding angles, and they share an angle).

Thus, $\frac{y}{y+x} = \frac{s}{s+t}$ because matching sides in similar triangles come in the same ratio.

ii. $$\frac{y}{y+x} = \frac{s}{s+t}$$
$$ys + yt = ys + xs$$
$$yt = xs$$
$$\frac{t}{s} = \frac{x}{y}.$$

iii. The unshaded triangles are similar by AA (we have congruent corresponding angles, and they share an angle). Thus, $\frac{t}{s+t} = \frac{a}{a+b}$ because matching sides in similar triangles come in the same ratio.

iv. Follow the same algebra as Step 2.

v. $\frac{x}{y} = \frac{a}{b}$ because they both equal $\frac{t}{s}$.

Lesson 14

1. Draw a diagonal and label the angles as shown in **Figure S.14.1**.

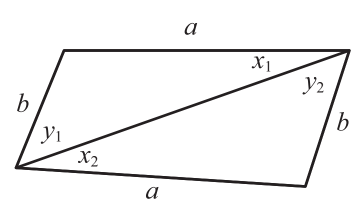


Figure S.14.1

i. The two triangles we see are similar by SSS.

ii. $x_1 = x_2$ and $y_1 = y_2$ because they are matching angles in similar triangles.

iii. The sides labeled *a* are parallel because $x_1 = x_2$ are congruent alternate interior angles.

iv. The sides labeled *b* are parallel because $y_1 = y_2$ are congruent alternate interior angles.

v. We have a parallelogram.

2. Draw a diagonal and label the sides b_1 and b_2 as shown in **Figure S.14.2**.

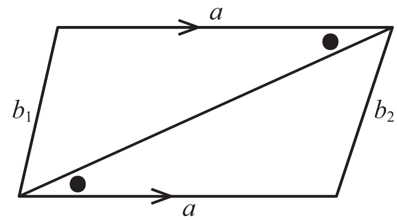


Figure S.14.2

i. The angles marked with a dot are congruent because they are alternate interior angles for parallel lines.

ii. The two triangles we see are similar by SAS (the side labeled a, the angle labeled with a dot, and the shared diagonal).

iii. The triangles are congruent because the scale factor is clearly 1.

iv. Thus, $b_1 = b_2$ because they are matching sides in congruent triangles.

v. The figure is a parallelogram because of the previous problem. (We just showed that both pairs of opposite sides are congruent.)

3. i. $\overline{EB} \parallel \overline{DF}$ because opposite edges of a parallelogram are parallel.

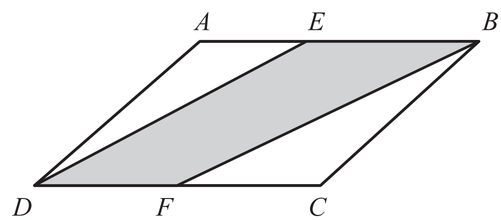


Figure S.14.3

ii. $AB = CD$ because opposite edges of a parallelogram are congruent.

iii. $EB = \frac{1}{2}AB$ and $DF = \frac{1}{2}DC$ because E and F are midpoints.

iv. $EB = DF$ because of steps 2 and 3.

v. $EBFD$ is a parallelogram because it has one pair of opposite edges that are both congruent and parallel.

4. a) The fact that same-side interior angles must add to 180° in the presence of parallel lines forces two more interior angles to be right angles. The fact that all angles in a quadrilateral sum to 360° forces the fourth angle to be right as well. We thus have a figure with four right angles. It is a rectangle.

b) Opposite sides in a parallelogram are congruent. This forces all four sides to be congruent. We have a rhombus.

5. We have $11x + 7x = 180°$, giving $x = 10$. The four interior angles are thus 70°, 110°, 70°, and 110° (using the fact that opposite angles are congruent).

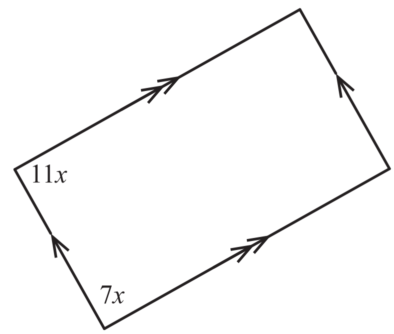


Figure S.14.4

6. We need the diagonals to bisect and, therefore, have the same midpoint.

Midpoint $\overline{AC} = (4, 10)$.

Midpoint $\overline{BD} = \left(\frac{p-4}{2}, \frac{q+12}{2}\right)$.

We want $\frac{p-4}{2} = 4$, giving $p = 12$, and we want $\frac{q+12}{2} = 10$, giving $q = 8$.

7. Call the length of the shorter side x. Then, the four sides of the parallelogram are x, $x + 2$, x, and $x + 2$. Thus,

$$x + x + 2 + x + x + 2 = 22$$
$$4x = 18$$
$$x = 4\frac{1}{2}.$$

The four side lengths are $4\frac{1}{2}, 6\frac{1}{2}, 4\frac{1}{2}, 6\frac{1}{2}$.

8. Consider a rhombus $ABCD$ with its midpoints marked.

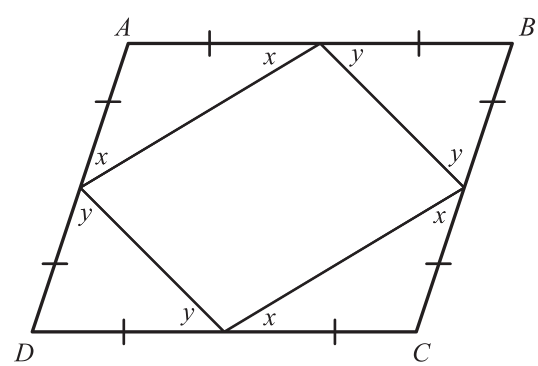


Figure S.14.5

Because $AB = BC = CD = DA$ and we are considering midpoints, all eight lengths shown as congruent in **Figure S.14.5** are indeed congruent. We thus have four isosceles triangles.

Because opposite angles are congruent, $\angle A \cong \angle C$, and the two isosceles triangles with these angles are similar by SAS. This means that all angles match, so all four angles labeled x are indeed congruent—and similarly for all four angles labeled y.

Each interior angle of the inside figure has measure $180 - x - y$, the same.

Thus, the inside figure is a four-sided shape with interior angles equal in measure and summing to 360°. Each of those angles must be 90°. The interior figure is a rectangle.

9. a) The slope of $\overline{DC} = -2,$ so the slope of $\overline{UK}$ must be $\frac{1}{2}$ because the diagonals of a rhombus are perpendicular.

b) This is a line of slope $\frac{1}{2}$ that goes through the midpoint of $\overline{DC}$ (because the diagonals bisect), which is $(6, 2)$. So, the line is $y-2=\frac{1}{2}(x-6)$.

10. See **Figure S.14.6**.

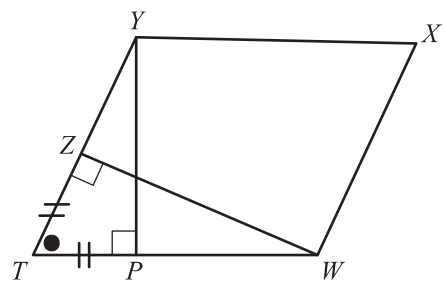


Figure S.14.6

i. $\triangle TPY \sim \triangle TZW$ because AA (90° and share angle T).

ii. $k = 1$ because $TP = TZ$.

iii. $TY = TW$ because $k = 1$.

iv. $TW = YX$ and $TY = WX$ because opposite sides of a parallelogram are $\cong$.

v. $WX = TY = TW = YX$ because algebra.

vi. $TWXY$ is a rhombus because all four sides are congruent.

Lesson 15

1. a) Acute scalene with largest angle between sides 5 and 7.

b) Obtuse scalene with largest angle between sides 2 and 10.

c) Right scalene with right angle opposite side 29.

d) This triangle does not exist.

e) Obtuse isosceles with obtuse angle opposite side 16.

Solutions

2. Use the distance formula to find the side lengths.

a) Here, $AB = 2$, $AC = 5$, $BC = \sqrt{29}$. This is a 2-5-$\sqrt{29}$ triangle. It is right scalene because $2^2 + 5^2 = \left(\sqrt{29}\right)^2$.

b) This is a 10-15-$\sqrt{37}$ triangle. It is obtuse scalene.

c) This is a $\sqrt{40}$-$\sqrt{40}$-12 triangle. It is obtuse isosceles.

d) This is a 5-10-15 "triangle." It does not exist. The points are collinear.

3. a) It is possible. Consider, for example, **Figure S.15.1**.

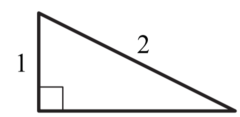


Figure S.15.1

The third side must be $\sqrt{3}$.

b) See **Figure S.15.2**.

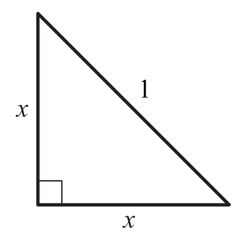


Figure S.15.2

We must have the following.

$$\begin{aligned} x^2 + x^2 &= 1 \\ 2x^2 &= 1 \\ x^2 &= \frac{1}{2} \\ x &= \frac{1}{\sqrt{2}}. \end{aligned}$$

There is only one such triangle: the $\frac{1}{\sqrt{2}}$-$\frac{1}{\sqrt{2}}$-1 right triangle.

4. We have already answered this question in Lesson 5. But in the context of this lesson, we can argue that the side opposite the right angle, the hypotenuse, is the longest side of the triangle because it is opposite the largest angle.

5. a) No. $\sqrt{5}+\frac{10}{3}$ is not greater than 5.97.

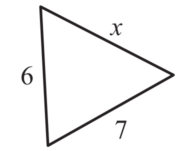


Figure S.15.3

b) We need $x + 6 > 7$ (that is, $x > 1$), and $x + 7 > 6$ (already true), and $6 + 7 > x$ (that is, $x < 13$).

Thus, $1 < x < 13$. (See **Figure S.15.3**.)

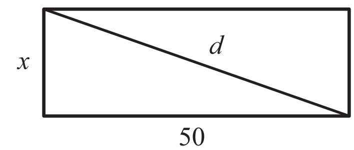


Figure S.15.4

c) We have $d^2 = x^2 + 50^2$. So, $d^2 > 50^2$, giving $d > 50$. That is about all we can say. (See **Figure S.15.4**.)

d) We need $a + b > x$ and $x + a > b$ (giving $x > b - a$) and $x + b > a$ (automatically true because b is larger than a).

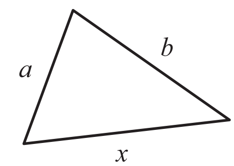


Figure S.15.5

Putting these together gives $b - a < x < b + a$. (See **Figure S.15.5**.)

6. a) Always. Because the angles in a triangle sum to 180°, the two congruent base angles must each have measure less than 90°.

b) Sometimes. For example, consider **Figure S.15.6**.

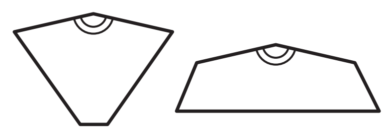


Figure S.15.6

c) Sometimes. Consider **Figure S.15.7**.

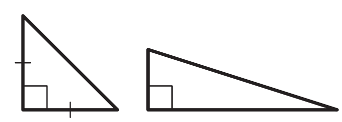


Figure S.15.7

7. Angle $x = 45°$. Now, $y + z = 90°$, with z larger than y because it is opposite a longer side than y. (We proved this general property in the lesson.) This means that z is larger than 45° and y is smaller than 45°. We have $y < x < z$.

8. By the triangular inequality, $AB + BC > AC$ and $AD + DC > AC$.

Now add to get $AB + BC + AD + DC > 2(AC)$.

9. i. $UH + HS > US$ because of the triangle inequality.

ii. $US > TS$ because $\angle T > \angle TUS$.

iii. $UH + HS > TS$ because of algebra.

10. We want the triangle to exist, so three things must be true.

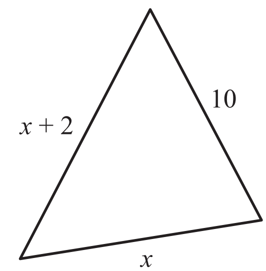


Figure S.15.8

$x + (x + 2) > 10$, giving $2x > 8$; that is, $x > 4$.

$x + 10 > x + 2$ (already true).

$x + 2 + 10 > x$ (already true).

We want 10 to be the longest side, so $x + 2 < 10$; that is, $x < 8$.

We want all three angles to be acute. Thus, the largest angle, the one opposite 10, must be acute. This means that

$$x^2 + (x+2)^2 > 10^2$$
$$x^2 + x^2 + 4x + 4 > 100$$
$$2x^2 + 4x + 4 > 100$$
$$x^2 + 2x + 2 > 50$$
$$x^2 + 2x + 1 > 49$$
$$(x+1)^2 > 49$$
$$x + 1 > 7 \; (\text{or } x + 1 < -7)$$
$$x > 6.$$

Putting it all together, we have $6 < x < 8$. That is, x can be any number between 6 and 8 (for example, 7 or $6\frac{1}{4}$ or 7.99999993878).

Lesson 16

1. $\sin(-30°) = -\frac{1}{2}$, $\cos(-30°) = \frac{\sqrt{3}}{2}$.

2. $\sin(150°) = \frac{1}{2}$, $\cos(150°) = -\frac{\sqrt{3}}{2}$.

3. $\sin(-150°) = -\frac{1}{2}$, $\cos(-150°) = -\frac{\sqrt{3}}{2}$.

4. If $\sin(x) = 0$, then the point on the unit circle has no height. Thus, $x = 0°$ or $180°$ or $360°$ or $540°$, and so on; or $-180°$ or $-360°$ or $-540°$, and so on.

5. If $\cos(x) = 0$, then the point on the unit circle must be directly overhead or directly below. We have $x = 90°$ or $-90°$ or $270°$ or $-270°$ or $450°$ or $-450°$, and so on.

6. By the Pythagorean theorem, $(\sin(x))^2 + (\cos(x))^2 = 1$. So, $(\cos(x))^2 = 1 - 0.74^2 = 0.4524$, so $\cos(x) \approx 0.67$ or -0.67.

7. See **Figure S.16.1**.

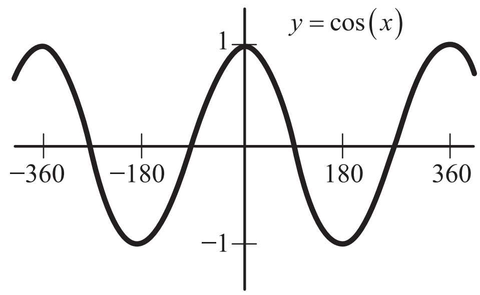


Figure S.16.1

8. The point on the unit circle must be in the third quadrant, so $180° < x < 270°$ (or $540° < x < 630°$ or $-180° < x < -90°$, and so on).

9. From the sketch in **Figure S.16.2**, it is clear that a point at angle of elevation 65° is higher than a point at angle of elevation 130°.

Thus, $\sin(65°) > \sin(130°)$.

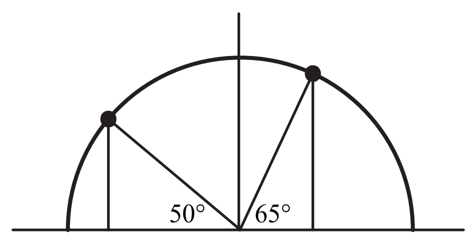


Figure S.16.2

10. From the triangle on the left in **Figure S.16.3**, it is clear that $\cos(100°) < \sin(100°)$.

And it is also clear from the sketch that $\sin(80°) = \sin(100°)$.

Thus, $\sin(80°)$ is larger than $\cos(100°)$.

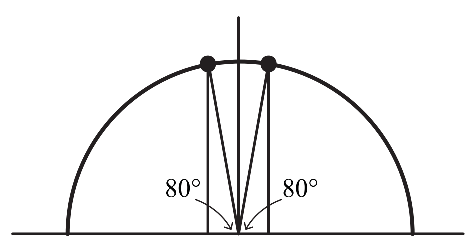


Figure S.16.3

Solutions

Lesson 17

1. a) $x = 7\cos(57) \approx 3.81$; $y = 7\sin(57) \approx 5.87$.

b) $x = 5\tan(32) \approx 3.12$; $y = \dfrac{5}{\cos(32)} \approx 5.90$.

c) $x = \dfrac{8}{\cos(17)} \approx 8.37$; $y = 8\tan(17) \approx 2.44$.

d) $x = 3.4\sin(50) \approx 2.60$; $y = 3.4\cos(50) \approx 2.19$.

e) $x = \dfrac{21.3}{\cos(13.2)} \approx 21.88$; $y = 21.3\tan(13.2) \approx 4.99$.

2. a) $a = 5\sin(20) \approx 1.71$ and $\sin(40) = \dfrac{a}{b}$, so $b = \dfrac{a}{\sin(40)} \approx \dfrac{1.71}{\sin(40)} \approx 2.66$.

b) $b = 2$ (because we have congruent triangles) and $\tan(42) = \dfrac{a}{2}$, so $a = 2\tan(42) \approx 1.80$.

c) $\tan(33) = \dfrac{12}{a}$, so $a = \dfrac{12}{\tan(33)} \approx 18.5$. $\tan(15) = \dfrac{12}{a+b}$, so $a+b = \dfrac{12}{\tan(15)} \approx 44.8$.

Thus, $b \approx 44.8 - a \approx 44.8 - 18.5 = 26.3$.

d) $\sin(41) = \dfrac{a}{32}$, so $a = 32\sin(41) \approx 21.0$.

$\sin(65) = \dfrac{a}{b}$, so $b = \dfrac{a}{\sin(65)} \approx \dfrac{21.0}{\sin(65)} \approx 23.2$.

3. Let H be your height. (See **Figure S.17.1**.)

We have $\tan(10.6) = \dfrac{H}{32}$, so $H = 32 \cdot \tan(10.6) \approx 5.99$ feet, which is close to 6 feet and 0 inches.

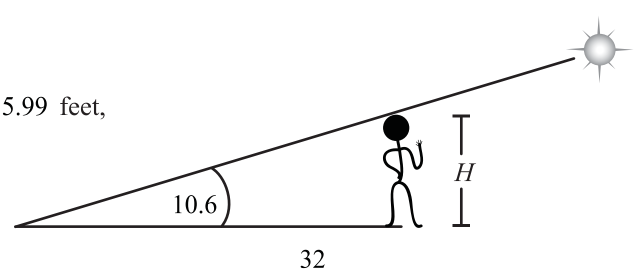


Figure S.17.1

4. See **Figure S.17.2**.

$\sin(40) = \dfrac{H}{100}$, so $H = 100\sin(40) \approx 64.3$ feet.

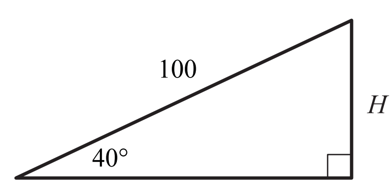


Figure S.17.2

5. See **Figure S.17.3**.

$$\tan(33) = \frac{850}{x}, \text{ so } x = \frac{850}{\tan(33)} \approx 1308 \text{ feet.}$$

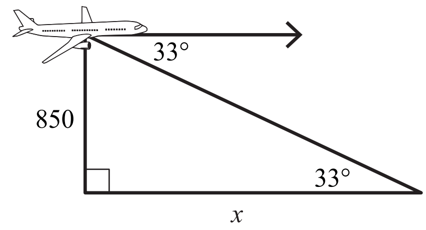


Figure S.17.3

6. See **Figure S.17.4**.

$$\sin(x) = \frac{b}{c}.$$

$$\cos(y) = \frac{b}{c}.$$

These are the same!

$$\sin(y) = \frac{a}{c}.$$

$$\cos(x) = \frac{a}{c}.$$

These are the same!

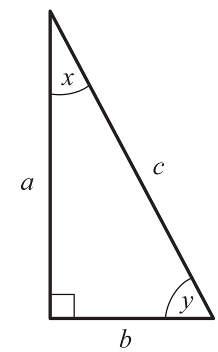


Figure S.17.4

7. See **Figure S.17.5**.

$$\text{rise} = 100\sin(15) \approx 25.9 \text{ feet.}$$

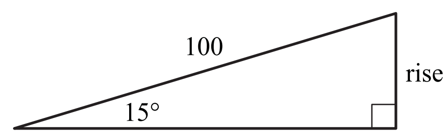


Figure S.17.5

8. See **Figure S.17.6**.

$$\tan(20) = \frac{h}{5}, \text{ so } h = 5\tan(20) = 1.82.$$

$$\tan(15) = \frac{h}{5+w} = \frac{1.82}{5+w}, \text{ so } 5 + w = \frac{1.82}{\tan(15)} = 6.79.$$

Thus, $w \approx 1.79$.

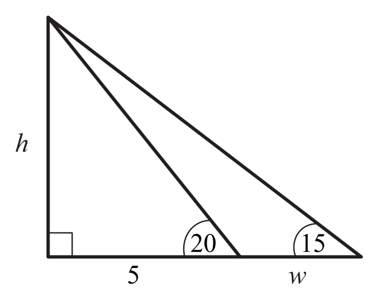


Figure S.17.6

9. See **Figure S.17.7**.

$$\tan(17) = \frac{x}{100}, \text{ so } x = 100\tan(17) \approx 30.6.$$

$$\tan(33) = \frac{x+d}{100}, \text{ so } x+d = 100\tan(33) \approx 64.9.$$

Thus, $d = 64.9 - 30.6 = 34.3$ feet.

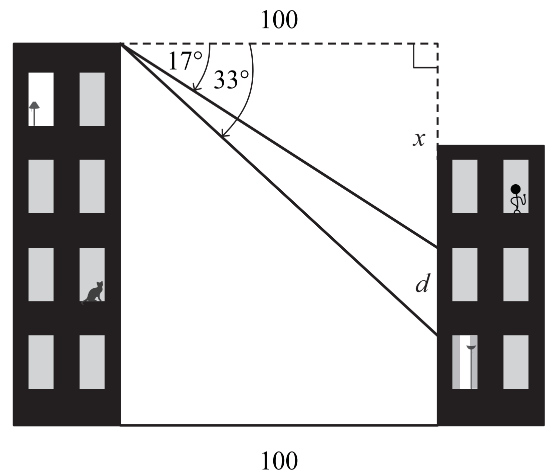


Figure S.17.7

10. See **Figure S.17.8**.

$$x = \frac{20}{\tan(40)} \approx 23.8 \text{ and } y = \frac{20}{\tan(44)} \approx 20.7.$$

So, the bird traveled 23.8 + 20.7 = 44.5 feet in 3 minutes. That's a speed of 44.5 × 20 = 890 feet per hour. (That's a *very* slow bird!)

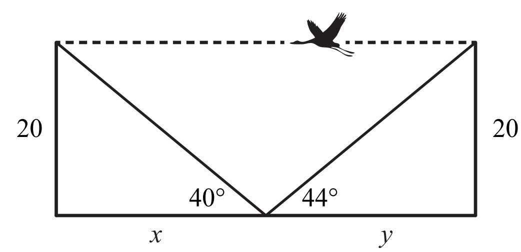


Figure S.17.8

Lesson 18

1.
$$\begin{aligned}\sin(75°) &= \sin(30+45)\\ &= \sin(30)\cos(45)+\cos(30)\sin(45)\\ &= \frac{1}{2}\cdot\frac{1}{\sqrt{2}}+\frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}}\\ &= \frac{1+\sqrt{3}}{2\sqrt{2}}\end{aligned}$$

$$\cos(75°) = \cos(30)\cos(45)-\sin(30)\sin(45) = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

and

$$\tan(75°) = \frac{\sin(75)}{\cos(75)} = \frac{\sqrt{3}+1}{\sqrt{3}-1}.$$

2. See **Figure S.18.1**.

a) $\tan(x) = \frac{21.2}{31.7}$, so $x = \tan^{-1}\left(\frac{21.2}{31.7}\right) \approx 33.8°$.

And $\sin(x) = \frac{21.2}{d}$, so $d = \frac{21.2}{\sin(x)} \approx \frac{21.2}{\sin(33.8°)} \approx 38.11$.

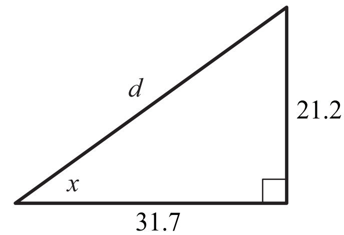


Figure S.18.1

b) $d = \sqrt{31.7^2 + 21.2^2} \approx 38.14$.

c) There were rounding errors. Within those range of errors, these answers match.

3. a) $AB = \sqrt{12^2 + 5^2} = 13$.

b) $m\angle CAB = \tan^{-1}\left(\frac{7}{13}\right) \approx 28.3°$.

4. Let x be the angle between the two sides of length 6. (This is the largest.) We have

$$10^2 = 6^2 + 6^2 - 2 \cdot 6 \cdot 6 \cdot \cos(x),$$

giving $\cos(x) = -\frac{28}{72}$. Thus, $x = \cos^{-1}\left(-\frac{28}{72}\right) \approx 112.9°$.

5. Let x be the angle opposite the side of length 3. (This is the smallest angle.)

Then, $\sin(x) = \frac{3}{5}$, giving $x = \sin^{-1}\left(\frac{3}{5}\right) \approx 36.9°$.

6. Now, $\tan(2x) = \frac{\sin(2x)}{\cos(2x)} = \frac{2\sin(x)\cos(x)}{\cos^2(x) - \sin^2(x)}$. Divide the numerator and denominator each by $\cos^2(x)$ to obtain

$$\tan(2x) = \frac{\frac{2\sin(x)\cos(x)}{\cos^2(x)}}{\frac{\cos^2(x) - \sin^2(x)}{\cos^2(x)}} = \frac{2\tan(x)}{1 - \tan^2(x)}.$$

Recall: "$\cos^2(x)$" is shorthand for $(\cos(x))^2$, and "$\tan^2(x)$" is shorthand for $(\tan(x))^2$.

7. Now, $\tan(x+y) = \dfrac{\sin(x)\cos(y)+\cos(x)\sin(y)}{\cos(x)\cos(y)-\sin(x)\sin(y)}$.

Divide the numerator and denominator each by $\cos(x)\cos(y)$ to obtain

$$\tan(x+y) = \tan(x+y) = \frac{\dfrac{\sin(x)\cos(y)}{\cos(x)\cos(y)} + \dfrac{\cos(x)\sin(y)}{\cos(x)\cos(y)}}{\dfrac{\cos(x)\cos(y)}{\cos(x)\cos(y)} - \dfrac{\sin(x)\sin(y)}{\cos(x)\cos(y)}} = \frac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)}.$$

8. We have the following.

$$m^2 = a^2 + \frac{c^2}{4} - 2a\cdot\frac{c}{2}\cos(x).$$

$$b^2 = a^2 + c^2 - 2ac\cos(x).$$

According to the second equation, $2ac\cos(x) = a^2 + c^2 - b^2$.

Substituting this into the first gives

$$m^2 = a^2 + \frac{c^2}{4} - \frac{a^2+c^2-b^2}{2}.$$

Multiplying through by 4 yields

$$4m^2 = 4a^2 + c^2 - 2a^2 - 2c^2 + 2b^2.$$

Thus, $4m^2 = 2a^2 + 2b^2 - c^2$, from which we obtain $a^2 + b^2 = 2m^2 + \dfrac{c^2}{2}$.

9. See **Figure S.18.2**. First notice that the leftmost triangle is isosceles. Thus, it has two angles equal in measure, here marked as $\frac{x}{2}$.

The remaining angle in that triangle is $180 - x$, and the angle marked of measure x in the diagram is correct.

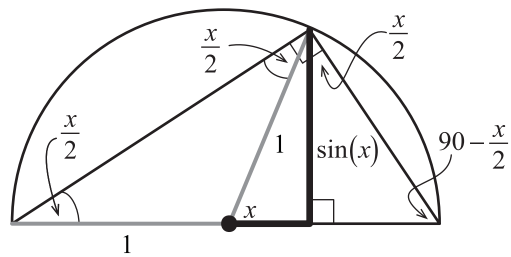


Figure S.18.2

Now, look at the bottom-left angle $\frac{x}{2}$. We see

$$\tan\left(\frac{x}{2}\right) = \frac{\text{opp}}{\text{adj}} = \frac{\sin(x)}{1+\cos(x)}.$$

The bottom right angle is $90 - \frac{x}{2}$ because there are 180° in the large triangle.

Thus, there is a third angle of measure $\frac{x}{2}$, as shown. It is part of a right triangle with

$$\tan\left(\frac{x}{2}\right) = \frac{\text{opp}}{\text{adj}} = \frac{1-\cos(x)}{\sin(x)}.$$

10. (This isn't pleasant!)

$$2\sin\left(\frac{x}{2}+\frac{y}{2}\right)\cos\left(\frac{x}{2}-\frac{y}{2}\right)$$

$$= 2\left(\sin\frac{x}{2}\cos\frac{y}{2}+\cos\frac{x}{2}\sin\frac{y}{2}\right)\left(\cos\frac{x}{2}\cos\frac{y}{2}+\sin\frac{x}{2}\sin\frac{y}{2}\right)$$

$$= 2\left(\sin\frac{x}{2}\cos\frac{x}{2}\cos^2\frac{y}{2}+\sin\frac{x}{2}\cos\frac{x}{2}\sin^2\frac{y}{2}+\sin\frac{y}{2}\cos\frac{y}{2}\cos^2\frac{x}{2}+\sin\frac{y}{2}\cos\frac{y}{2}\sin^2\frac{x}{2}\right)$$

$$= 2\sin\frac{x}{2}\cos\frac{x}{2}\left(\cos^2\frac{y}{2}+\sin^2\frac{y}{2}\right)+2\sin\frac{y}{2}\cos\frac{y}{2}\left(\cos^2\frac{x}{2}+\sin^2\frac{x}{2}\right)$$

$$= 2\sin\frac{x}{2}\cos\frac{x}{2}\cdot 1+2\sin\frac{y}{2}\cos\frac{y}{2}\cdot 1$$

$$= 2\sin\frac{x}{2}\cos\frac{x}{2}+2\sin\frac{y}{2}\cos\frac{y}{2}$$

$$= \sin(x)+\sin(y).$$

Lesson 19

1. Draw an additional radius and label points as shown in **Figure S.19.1**.

$\triangle OAC$ is isosceles, so the two angles labeled a are congruent.

$\triangle OBC$ is isosceles, so $\angle OBC = x + a$.

The two vertical angles at M are congruent, and these have measures $180 - y - a$ and $180 - x - (x + a)$.

Consequently, $180 - y - a = 180 - x - x - a$, from which $y = 2x$ follows.

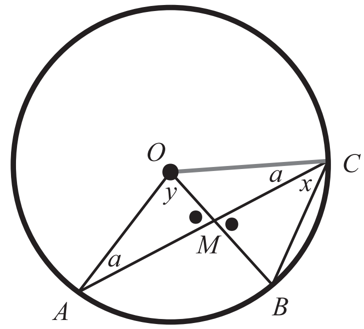


Figure S.19.1

2. $y = 20°$; $x = 180 - 110 - 40 = 30°$; $z = \frac{1}{2}x = 15°$.

3. See **Figure S.19.2**.

$4^2 + h^2 = 7^2$, so $h = \sqrt{33}$, giving $AB = 2\sqrt{33}$.

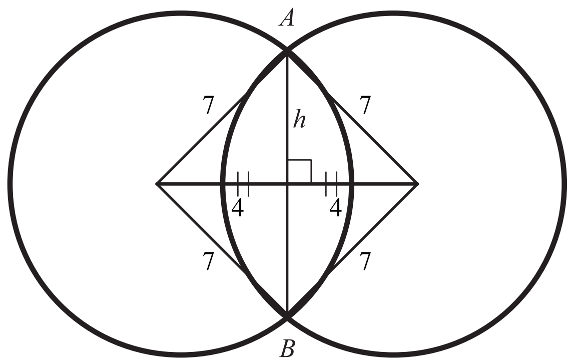


Figure S.19.2

4. Draw a common tangent through P as shown in **Figure S.19.3**.

By the tangent/radius theorem, twice, we see two right angles.

Thus, $\angle APB = 90 + 90 = 180°$, and A, P, and B are indeed collinear.

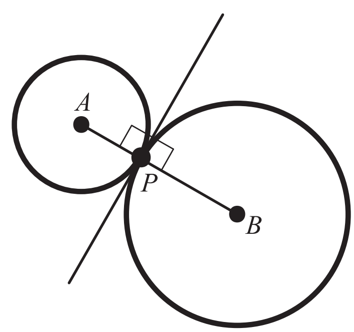


Figure S.19.3

5. See **Figure S.19.4**.

$2^2 + x^2 = 14^2$ gives $x = \sqrt{192}$.

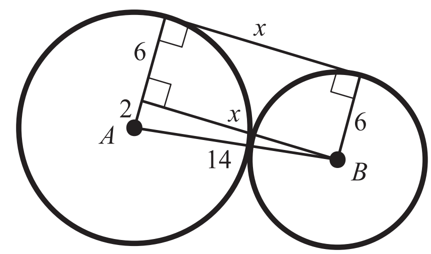


Figure S.19.4

6. Draw one line as shown in **Figure S.19.5**.

The inscribed/central angle theorem shows that we have congruent alternate interior angles, so the chords are parallel.

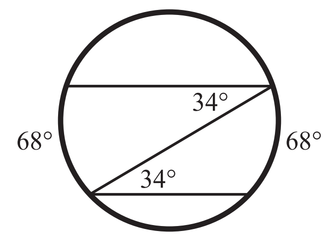


Figure S.19.5

7. Draw a chord as shown in **Figure S.19.6**, and use the inscribed/central angle theorem to identify an angle of 20° and an angle of 65°.

Because the angles in a triangle sum to 180°, we have $(180 - a) + 20 + 65 = 180$, giving $a = 85°$.

Comment: It is not a coincidence that 85 happens to be the average of 130 and 40, the two original angles mentioned. Can you see why?

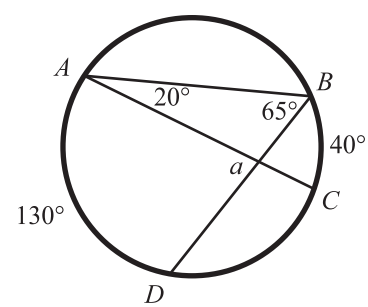


Figure S.19.6

8. Draw two chords as shown in **Figure S.19.7**.

Using vertical angles and the fact that two inscribed angles from the same arc have the same measure, we see that the two triangles formed are similar by AA.

Consequently, matching sides of those triangles come in the same ratio.

We have $\frac{a}{d} = \frac{c}{b}$. The result follows by cross multiplying.

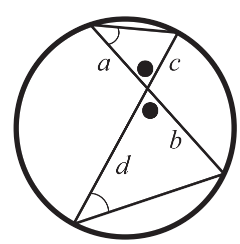


Figure S.19.7

9. Recall that opposite angles of a parallelogram are congruent. Suppose that a parallelogram with two angles of measure x and two angles of measure y sits in a circle, as shown in **Figure S.19.8**.

Each angle x is half the measure of the arc AB: One arc AB has measure $2x$, and the other arc AB also has measure $2x$. Because the two arcs cover the entire circle, $2x + 2x = 360°$, giving $x = 90°$. By the same reasoning, $y = 90°$. The figure has four right angles and is therefore a rectangle.

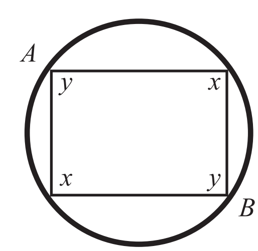


Figure S.19.8

Solutions

10. *PAQB* is a kite.

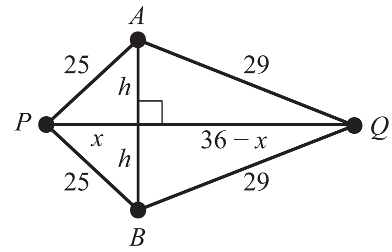


Figure S.19.9

Recall that the diagonals of a kite are perpendicular, and one diagonal bisects the other. Label the lengths h, x, and $36 - x$ as shown in **Figure S.19.9**.

We have $x^2 + h^2 = 25^2$. We also have $(36 - x)^2 + h^2 = 29^2$.

That is, $36^2 - 72x + x^2 + h^2 = 29^2$. That is, $36^2 - 72x + 25^2 = 29^2$, yielding $1296 - 72x + 625 = 841$.

This gives $x = 15$.

So, $h = \sqrt{25^2 - x^2} = \sqrt{25^2 - 15^2} = 20$ and $AB = 2h = 40$.

Lesson 20

1. a) $C = (2, 3);\ r = 4.$

b) $C = (7, 5);\ r = \sqrt{5}.$

c) $C = (-7, 1);\ r = \sqrt{10}.$

d) $C = (-25, 0);\ r = 1.$

e) $C = (0, 0);\ r = \sqrt{19}.$

f) $C = (17, -17);\ r = \sqrt{17}.$

For g) and h), we need to conduct some algebraic work, as follows.

g) $x^2 + 2x + 1 + y^2 - 8y + 16 = 8 + 1 + 16$

$$(x+1)^2 + (y-4)^2 = 25$$

$C = (-1, 4);\ r = 5.$

h) $x^2 - 20x + 100 + y^2 - 10y + 25 = 0 + 100$

$$(x-10)^2 + (y-5)^2 = 100$$

$C = (10, 5);\ r = 10.$

2. a) $(x-2)^2+(y-2)^2=81.$

b) $(x+3)^2+y^2=16.$

c) $\left(x+\frac{1}{2}\right)^2+\left(y+\frac{1}{3}\right)^2=\frac{1}{16}.$

d) $(x-0.67)^2+(y-9.21)^2=0.01.$

3. a) $C=(2,-1);\ r=13.$

b) $(7-2)^2+(11+1)^2=169.$ Yes.

c) $(7-2)^2+(-11+1)^2\neq 169.$ No.

d)
$$\begin{aligned}(2-2)^2+(y+1)^2&=169\\ 0+(y+1)^2&=169\\ y+1&=13 \text{ or } -13\\ y&=12 \text{ or } -14.\end{aligned}$$

There are two points: (2, 12) and (2, −14).

e)
$$\begin{aligned}(x-2)^2+(12+1)^2&=169\\ (x-2)^2+169&=169\\ (x-2)^2&=0\\ x-2&=0\\ x&=2.\end{aligned}$$

There is only one point: (2, 12).

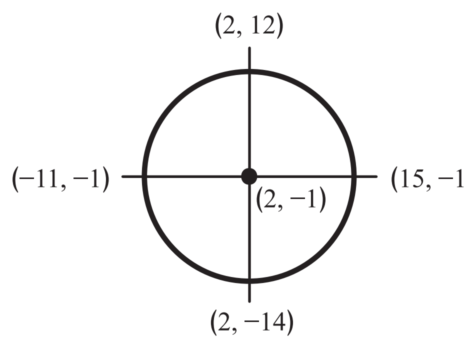


Figure S.20.1

f) r = 13, so the diameter = 26. (See **Figure S.20.1**.)
The horizontal pair or the vertical pair is the easiest to find.

4. $r=\sqrt{3^2+7^2}=\sqrt{58}$, so $(x+1)^2+(y-2)^2=58$.

5. The radius is 4. So, $(x+4)^2+(y-6)^2=16$. (See **Figure S.20.2**.)

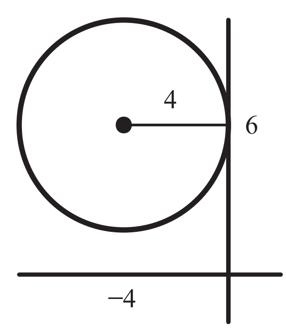


Figure S.20.2

Solutions

6. a) $r_1 = 5$ and $r_2 = 10$.

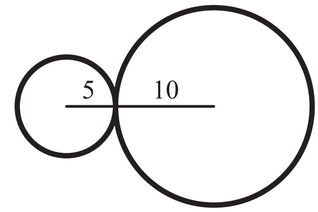


Figure S.20.3

b) $C_1 = (0, 0)$ and $C_2 = (9, 12)$. Distance $= \sqrt{9^2 + 12^2} = 15$.

c) Notice that the distance between the centers is the sum of the two radii. The two circles must be tangent. (See **Figure S.20.3**.)

7. a) $r_1 = \sqrt{2}$ and $r_2 = \sqrt{32} = 4\sqrt{2}$.

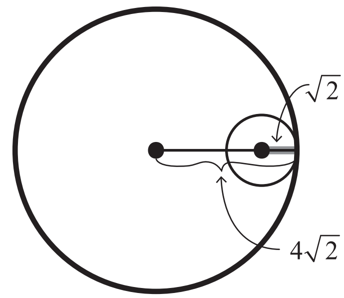


Figure S.20.4

b) $C_1 = (0, 0)$ and $C_2 = (3, 3)$. Distance $= \sqrt{3^2 + 3^2} = \sqrt{18} = 3\sqrt{2}$.

c) Notice that the distance between the centers, $3\sqrt{2}$, is the difference of the two radii: $4\sqrt{2} - \sqrt{2}$. The two circles must be tangent as illustrated in **Figure S.20.4**.

8. The center of the circle is the midpoint: $\left(-\frac{15}{2}, 17\right)$.

The radius is half of AB, which equals $\frac{1}{2}\sqrt{21^2 + 20} = \frac{29}{2}$.

The equation of the circle is $\left(x + \frac{15}{2}\right)^2 + (y - 17)^2 = \frac{841}{4}$.

9. For $(x + 2)^2 + (y - 4)^2 = 49$, we have $C_1 = (-2, 4)$ and $r_1 = 7$.

For

$$
\begin{aligned}
x^2 + y^2 - 6x + 16y + 37 &= 0 \\
x^2 - 6x + 9 + y^2 + 16y + 64 &= 9 + 64 - 37 \\
(x-3)^2 + (y+8)^2 &= 36
\end{aligned}
$$

we have $C_2 = (3, -8)$ and $r_2 = 6$.

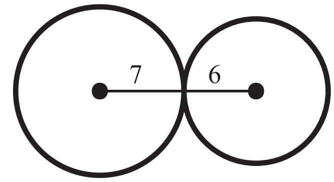


Figure S.20.5

The distance between the centers is $\sqrt{5^2 + 12^2} = 13 = r_1 + r_2$, the sum of the radii.

The two circles must be tangent as shown in **Figure S.20.5**.

10. a) Center = midpoint = $(3, 9)$.

Radius = $\frac{1}{2}AB = \frac{1}{2}\sqrt{2^2+4^2} = \frac{1}{2}\sqrt{20} = \sqrt{5}$.

Equation: $(x-3)^2 + (y-9)^2 = 5$.

b) $$\begin{aligned}(x-2)(x-4)+(y-7)(y-11)&=0\\ x^2-6x+8+y^2-18y+77&=0\\ x^2-6x+9+y^2-18y+81&=5\\ (x-3)^2+(y-9)^2&=5.\end{aligned}$$

It's the same equation.

c) $$(x-a_1)(x-b_1)+(y-a_2)(y-b_2)=0$$
$$x^2-(a_1+b_1)x+a_1b_1+y^2-(a_2+b_2)y+a_2b_2=0$$
$$\left(x-\frac{a_1+b_1}{2}\right)^2+\left(y-\frac{a_2+b_2}{2}\right)^2=\left(\frac{a_1+b_1}{2}\right)^2+\left(\frac{a_2+b_2}{2}\right)^2-a_1b_1-a_2b_2$$
$$\left(x-\frac{a_1+b_1}{2}\right)^2+\left(y-\frac{a_2+b_2}{2}\right)^2=\frac{a_1^2+b_1^2-2a_1b_1+a_2^2+b_2^2-2a_2b_2}{4}$$
$$\left(x-\frac{a_1+b_1}{2}\right)^2+\left(y-\frac{a_2+b_2}{2}\right)^2=\frac{(a_1-b_1)^2+(a_2-b_2)^2}{4}.$$

This is a circle with center $\left(\frac{a_1+b_1}{2}, \frac{a_2+b_2}{2}\right)$, the midpoint, and radius $\frac{\sqrt{(a_1-b_1)^2+(a_2-b_2)^2}}{2}$, which is half the distance of AB, as hoped.

Lesson 21

1. Recall that the diagonals of a rhombus bisect one another and are perpendicular.

Suppose that their lengths are $2a$ and $2b$, as shown in **Figure S.21.1**.

We see four right triangles, each with base a and height b, so the area of the rhombus is $4\times\frac{1}{2}ab = 2ab = \frac{1}{2}(2a)(2b)$.

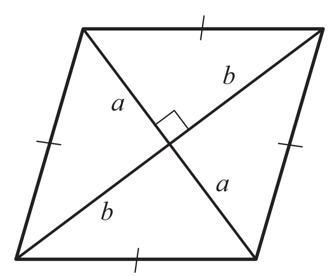


Figure S.21.1

Solutions

2. a) Area = big square − small square = $(x+6)^2 - x^2$.

b) Four rectangles as shown in **Figure S.21.2**.

c) Four overlapping rectangles and subtracting the area of four 3×3 squares to compensate, as shown in **Figure S.21.3**.

d) Two rectangles of one size plus two rectangles of a smaller size, as shown in **Figure S.21.4**.

e) Complete algebra on each of them to see that they all equal $12x + 36$.

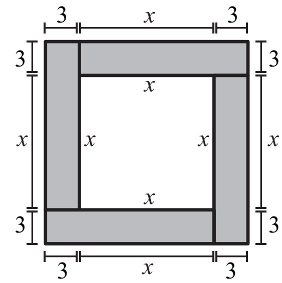


Figure S.21.2

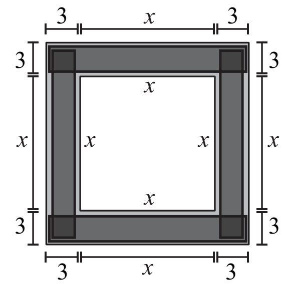


Figure S.21.3

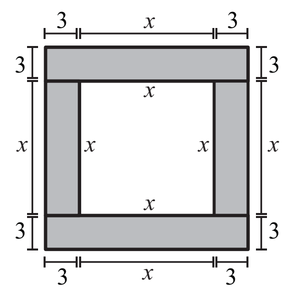


Figure S.21.4

3. a) $x^2 = 4 \times 9 = 36$, so the side length is $x = 6$.

b) $x^2 = 96$, so $x = \sqrt{96}$.

c) $x = \sqrt{ab}$.

4. The triangle on the left is a right isosceles triangle. (See **Figure S.21.5**.)

By the Pythagorean theorem, we have $h^2 + h^2 = 8^2$, giving $h = \dfrac{8}{\sqrt{2}}$.

By Example 1 in the lesson, the area of the parallelogram is $10h = \dfrac{80}{\sqrt{2}}$.

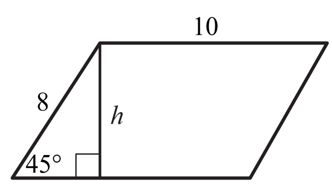


Figure S.21.5

5. Area = $\dfrac{1}{2} \cdot 20 \cdot h = 10h = 10 \times 10 \tan(72°) = 100 \tan(72°) \approx 307.8$. (See **Figure S.21.6**.)

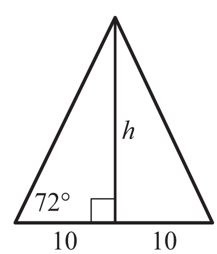


Figure S.21.6

6. A decagon has 10 sides. On each side of the figure, draw a triangle with apex at the center of the decagon, as shown in **Figure S.21.7**.

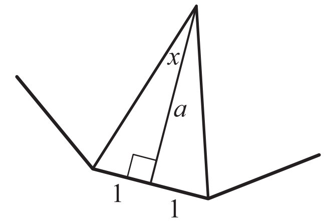


Figure S.21.7

With the labels given, we see that $\tan(x) = \frac{1}{a}$, giving $a = \frac{1}{\tan(x)}$.

Now, $2x$ is $\frac{1}{10}$ of a full turn, so $2x = \frac{360}{10} = 36$.

Thus, $x = 18°$, $a = \frac{1}{\tan(18°)}$, and

$$\text{area} = 10 \times \frac{1}{2} \cdot 2 \cdot a = 10a = 10\frac{1}{\tan(18)} \approx 30.8.$$

7. a) See **Figure S.21.8**.

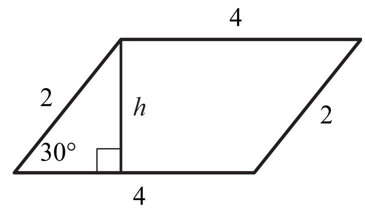


Figure S.21.8

The triangle on the left is half an equilateral triangle (the mirror half is below it).

Thus, $h = 1$. Using Example 1 from the lesson for the area of a parallelogram, we have area = 4 × 1 = 4.

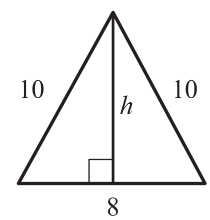


Figure S.21.9

b) See **Figure S.21.9**.

$h = \sqrt{10^2 - 4^2} = \sqrt{84}$ and area $= \frac{1}{2} \cdot 8h = 4\sqrt{84}$.

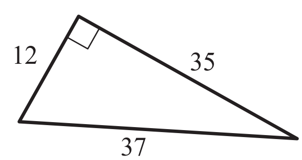


Figure S.21.10

c) $12^2 + 35^2 = 37^2$. We have a right triangle. (See **Figure S.21.10**.)

Area $= \frac{1}{2} \cdot 12 \times 35 = 210$.

d) See **Figure S.21.11**.

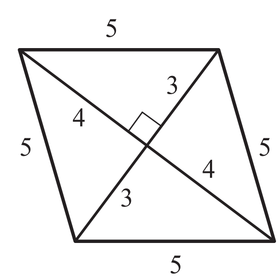


Figure S.21.11

Area $= 4 \times \frac{1}{2} \cdot 3 \cdot 4 = 24$.

8. a) See **Figure S.21.12**.

$x^2 + x^2 = 7^2$, so $x = \frac{7}{\sqrt{2}}$.

Thus, area $= \frac{1}{2} \cdot \frac{7}{\sqrt{2}} \cdot \frac{7}{\sqrt{2}} = \frac{49}{4}$.

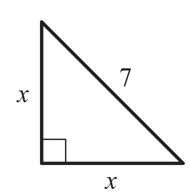


Figure S.21.12

b) See **Figure S.21.13**.

$h = \sqrt{75} = 5\sqrt{3}$ and area $= 25\sqrt{3}$.

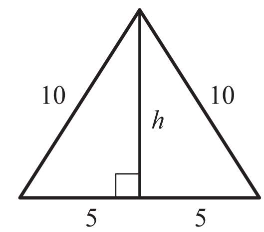


Figure S.21.13

c) We have half an equilateral triangle. (See **Figure S.21.14**.)

Thus, the side opposite the angle of 30° is 10, and the Pythagorean theorem gives the remaining side as $10\sqrt{3}$.

We have area = $50\sqrt{3}$.

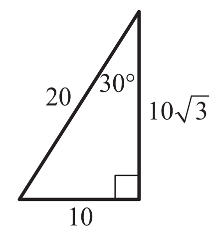


Figure S.21.14

d) See **Figure S.21.15**.

Area = $4 \times \frac{1}{2} r \cdot r = 2r^2$.

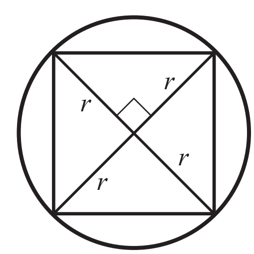


Figure S.21.15

e) The diagonal of the rectangle does pass through the center of the circle. Because of the inscribed/central angle theorem, we have an arc of 180°. (See **Figure S.21.16**.)

$x = \sqrt{100 - 64} = 6$, so area $= 8 \times 6 = 48$.

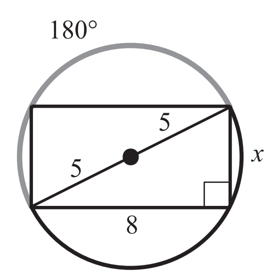


Figure S.21.16

9. a) Both triangles have the same height.

Because M is the midpoint of $\overline{AB}$, they have the same base lengths, too. Thus, they have equal areas.

b) Each triangle has base 10 and area 25.

So, $\frac{1}{2} 10 \cdot H = 25$ gives height $h = 5$.

10. We have 12 congruent triangles. (See **Figure S.21.17**.)

The shaded 8 of them have area 20. Thus, the full 12 of them (half again more) makes area 30.

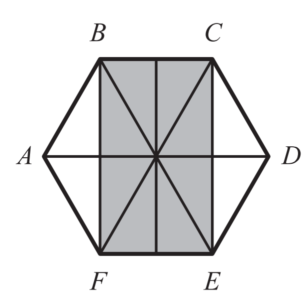


Figure S.21.17

Lesson 22

1. Arc length $= \frac{315}{360} \cdot 2\pi r = \frac{7}{2}\pi$. This gives $\frac{7}{8} \cdot 2r = \frac{7}{2}$, so $r = 2$.

2. We have area $= \frac{x}{360}\pi 2^2 = \frac{4\pi x}{360}$, and arc length $= \frac{x}{360} 2\pi \cdot 2 = \frac{4\pi x}{360}$.

They are the same numerical value.

3. a) Area = area sector − area triangle $= \frac{1}{4}\pi 6^2 - \frac{1}{2}6 \times 6 = 9\pi - 18$.

b) Area of shaded region $= \frac{3}{4}$ area circle $+ \frac{1}{2}1 \times 1 = \frac{3}{4}\pi 1^2 + \frac{1}{2} = \frac{3}{4}\pi + \frac{1}{2}$. (See **Figure S.22.1**.)

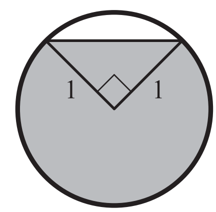


Figure S.22.1

c) By Thales's theorem, the triangle has a right angle. It has a 30° angle and, therefore, half an equilateral triangle.

Its sides are 5, 10, and $5\sqrt{3}$, and its area is $\frac{1}{2} \cdot 5 \times 5\sqrt{3} = \frac{25\sqrt{3}}{2}$.

The shaded region is half the circle minus this triangle and, therefore, has area $\frac{1}{2}\pi 5^2 - \frac{25\sqrt{3}}{2} = \frac{25\pi}{2} - \frac{25\sqrt{3}}{2}$.

d) First notice that for a circle of radius r, the area of the shaded wedge shown in **Figure S.22.2** is given as follows.

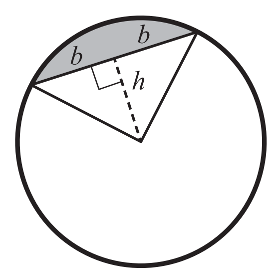


Figure S.22.2

Area shaded = area sector − area triangle

$$= \frac{x}{360}\pi r^2 - \frac{1}{2} \cdot 2b \cdot h$$

$$= \frac{x}{360}\pi r^2 - bh$$

$$= \frac{x}{360}\pi r^2 - r\sin\left(\frac{x}{2}\right) \cdot r\cos\left(\frac{x}{2}\right)$$

$$= \frac{x}{360}\pi r^2 - r^2 \sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right).$$

So, for our question, we do the following.

Area = area circle − area of each wedge

$$= \pi 2^2 - \left(\frac{40}{360}\pi 2^2 - 2^2\sin(20)\cos(20)\right) - \left(\frac{90}{360}\pi 2^2 - 2^2\sin(45)\cos(45)\right)$$

$$= 4\pi - \frac{4}{9}\pi - \pi + 4\sin(20)\cos(20) + 4\sin(45)\cos(45)$$

$$\approx 11.31.$$

4. See **Figure S.22.3**.

Area $= \frac{1}{4}\pi 7^2 + \frac{1}{4}\pi 13^2 + \frac{1}{4}\pi 5^2 + \frac{1}{4}\pi 1^2 = \frac{244\pi}{4}.$

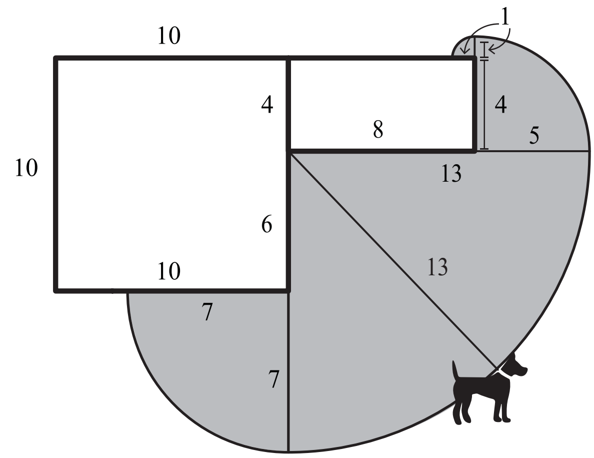


Figure S.22.3

5. $\frac{60}{360}\pi 8^2 + \frac{120}{360}\pi 2^2 = \frac{1}{6}64\pi + \frac{1}{3}4\pi = 12\pi.$

6. By the tangent/radius theorem, we have a right triangle, as shown in **Figure S.22.4**.

The radius of the large circle is $\sqrt{6^2+7^2} = \sqrt{85}$, and the area between the circles is $\pi\left(\sqrt{6^2+7^2}\right)^2 - \pi 6^2 = \pi 7^2 = 49\pi.$

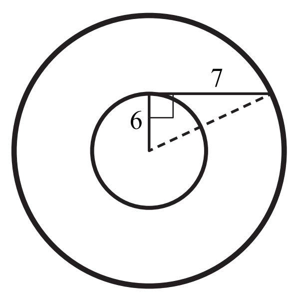


Figure S.22.4

7. The shaded region in **Figure S.22.5** is a "wedge" in a circle of radius 2. We are working with an interior angle $x = 180 - \frac{360}{7} \approx 128.6°$.

By the work of Problem 3, d), we have the following.

$$\text{Area} = \frac{126.6}{360}\pi 2^2 - 2^2\sin(63.4)\cos(63.4)$$

$$= \frac{126.6}{360}4\pi - 4\sin(63.4)\cos(63.4) \approx 2.9.$$

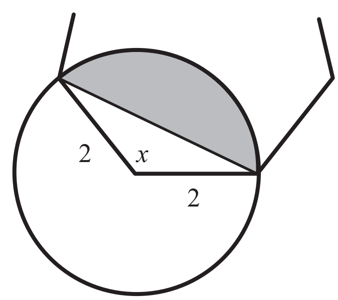


Figure S.22.5

8. a) Area = 10 × 10 = 100.

b) $2\pi r = 40$ gives $r = \dfrac{20}{\pi}$, so area $= \pi r^2 = \pi \dfrac{400}{\pi^2} = \dfrac{400}{\pi}$.

c) $2r + \dfrac{1}{2} 2\pi r = 40$. That is, $2r + \pi r = 40$, which gives $r = \dfrac{40}{2+\pi}$.

So, area $= \dfrac{1}{2}\pi r^2 = \dfrac{\pi}{2}\left(\dfrac{40}{2+\pi}\right)^2 = \dfrac{800\pi}{(2+\pi)^2}$.

9. a) Their areas have the same numerical value as their perimeters.

b) The 5-12-13 triangle has perimeter 30 and area 30.

c) No. If $a \times b = 2a + 2b$, then $b = \dfrac{2a}{a-2} = \dfrac{2a-4+4}{a-2} = 2 + \dfrac{4}{a-2}$.

For this to be an integer, we need $a - 2$ to be a factor of 4.

If $a - 2 = 1$, then $a = 3$ and $b = 6$, and we have the 3 × 6 rectangle.

If $a - 2 = 2$, then $a = 4$ and $b = 4$, and we have the 4 × 4 square.

If $a - 2 = 4$, then $a = 6$ and $b = 3$, and we have the 3 × 6 rectangle again.

These are the only possibilities.

10. a) We have a full turn of 360° divided into 12 congruent parts.

b) SAS holds: sides r and r with 30° between them and sides kr and kr with 30° between them.

c) Because the scale factor is k, the base of each triangle in the large figure is k times the length of each base in the small figure. Summing them shows that the perimeters also differ by a factor of k.

d) If in the small figure each base is x, then this ratio is $\dfrac{12x}{2r}$.

For the large figure, the matching ratio is $\dfrac{12(kx)}{2(kr)} = \dfrac{12x}{2r}$, which is the same.

e) There is nothing special about the number 12 here. The perimeter-to-twice-radius ratio would always match in the small and large figures.

f) Yes. For 1,000,000 triangles, the small and large figures each closely resemble a circle—so closely that we humans could probably not detect the difference. Many would then say that it seems reasonable to conclude that the perimeter-to-twice-radius ratio is the same for two circles.

Lesson 23

1. $3h + 4h + 3h = 100$ gives $h = 10$. (See **Figure S.23.1**.)

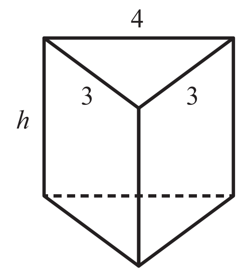


Figure S.23.1

2. The area of the base is $\frac{1}{2} \cdot 2 \cdot \sqrt{3} = \sqrt{3}$.

$$LA = 12 + 12 + 12 = 36.$$
$$SA = 36 + 2 \times \sqrt{3} = 36 + 2\sqrt{3}.$$
$$V = \sqrt{3} \times 6 = 6\sqrt{3}.$$

(See **Figure S.23.2**.)

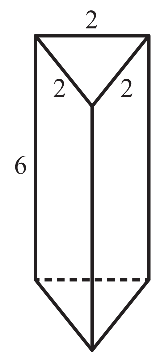


Figure S.23.2

3. The area of one face is $\frac{40}{8} = 5$. So, $\frac{1}{2} \cdot 4 \cdot s = 5$, giving $s = \frac{5}{2}$.

4. a) See **Figure S.23.3**.

$$h = \sqrt{49 - 9} = \sqrt{40}.$$
$$LA = \pi rs = 21\pi.$$
$$SA = 21\pi + 9\pi = 30\pi.$$
$$V = \frac{1}{3} 9\pi\sqrt{40} = 3\sqrt{40}\pi.$$

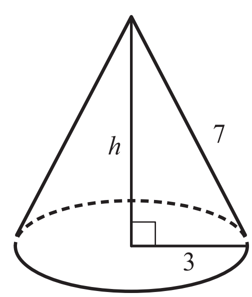


Figure S.23.3

b) By Cavalieri's principle, the volume does not change.

5. $\sqrt{3}x = 3$ gives $x = \frac{3}{\sqrt{3}} = \sqrt{3}$, so $V = \left(\sqrt{3}\right)^3 = 3\sqrt{3}$. (See **Figure S.23.4**.)

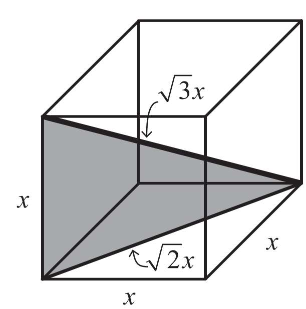


Figure S.23.4

6. The base with sides 7, 24, 25 is a right triangle $(7^2 + 24^2 = 25^2)$ and, therefore, has area $\frac{1}{2} \cdot 7 \cdot 24 = 84$.

$$LA = 7 \times 100 + 24 \times 100 + 25 \times 100 = 5600.$$

$$SA = 5600 + 84 + 84 = 5768.$$

$$V = 84 \cdot 100 = 8400.$$

7. a) Base edge = 12; $V = \frac{1}{3} \cdot 12^2 \cdot 8 = 384$; $SA = 12^2 + 4 \cdot \frac{1}{2} \cdot 12 \cdot 10 = 384$. (See **Figure S.23.5**.)

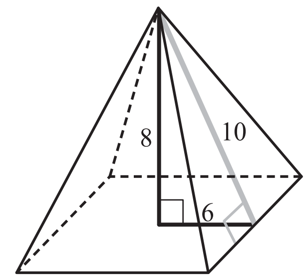


Figure S.23.5

b) Slant height = 29; V = 11,760; LA = 4200. (See **Figure S.23.6**.)

c) Height $= \sqrt{6^2 - 2^2} = \sqrt{32}$; $V = \frac{1}{3} 16\sqrt{32}$; $SA = 16 + 8\sqrt{32}$.

d) Let e be the base edge. Then, $10 = \frac{1}{3} e^2 6$ gives $e = \sqrt{5}$.

Slant height: $s = \sqrt{6^2 + \left(\frac{e}{2}\right)^2} = \sqrt{36 + \frac{5}{4}} = \frac{\sqrt{149}}{2}$.

$$SA = e^2 + 4 \times \frac{1}{2} e \cdot s = 5 + \sqrt{5} \cdot \sqrt{149} = 5 + \sqrt{745}.$$

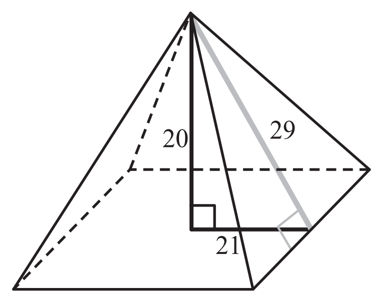


Figure S.23.6

8. Area base $= 6 \times \frac{1}{2} \cdot 1 \cdot \frac{1}{2}\sqrt{3} = \frac{3\sqrt{3}}{2}$.

$$\tan(64) = \frac{h}{1}, \text{ so } h = \tan(64) \approx 2.05.$$

Thus,

$$V \approx \frac{1}{3} \cdot \frac{3\sqrt{3}}{2} \cdot 2.05 \approx 1.8.$$

9. a) 70% of $4\pi r^2 = 0.70 \times 4\pi(6380)^2 = 358054603.2 \text{ km}^2$.

b) Area of floor = πr^2. Area of hemisphere = $\frac{1}{2} 4\pi r^2 = 2\pi r^2 =$ double the floor. So, we need 20 gallons.

Solutions

c) $$SA = \frac{1}{2} \cdot 4\pi r^2 + \frac{1}{2} \cdot 4\pi r^2 + 2\pi rh$$

$$= \frac{1}{2} \cdot 4\pi (15)^2 + \frac{1}{2} \cdot 4\pi (15)^2 + 2\pi 15 \cdot 20 = 1500\pi \text{ mm}^2.$$

One hemisphere $= \frac{1}{2} 4\pi 15^2 = 450\pi \text{ mm}^2$.

$450\pi \leftrightarrow 5$ gallons.

$150\pi \leftrightarrow \frac{5}{3}$ gallons.

$1050\pi \leftrightarrow \frac{35}{3}$ gallons.

We need $11\frac{2}{3}$ more units of paint.

d) $$V_{\text{space}} = V_{\text{cylinder}} - 4 \times V_{\text{sphere}}$$

$$= \pi r^2 (8r) - 4 \times \frac{4}{3} \pi r^3$$

$$= 8\pi r^3 - \frac{16}{3} \pi r^3 = \left(8 - \frac{16}{3}\right) \pi r^3$$

$$= \frac{8}{3} \pi r^3 = 2 \times \frac{4}{3} \pi r^3$$

$$= 2 \times V_{\text{sphere}}.$$

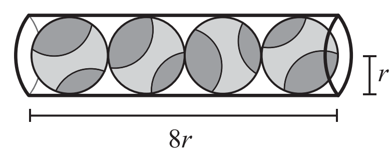


Figure S.23.7

The volume of the space equals the volume of the two balls. (See **Figure S.23.7**.)

10. a) $LA = n \times \frac{1}{2} xs$.

b) Rewrite as $LA = \frac{1}{2} \times nx \times s$. Because nx = perimeter of the base, this reads

lateral area = $\frac{1}{2}$ × perimeter of base × slant height.

c) If the base is a circle of radius r, then the perimeter = $2\pi r$. The formula becomes

lateral area = $\frac{1}{2} \times 2\pi r \times s = \pi rs$.

Lesson 24

1. a) $6k^2 = 24$ gives $k = 2$.

b) $6k^2 = 12$ gives $k = \sqrt{2}$.

c) $6k^2 = 3$ gives $k = \dfrac{1}{\sqrt{2}}$.

2. $k = 200$.

a) $\dfrac{1}{2}k^2 = \dfrac{1}{2}(200)^2 = 20{,}000$ gallons.

b) $2k^3 = 2(200)^3 = 16{,}000{,}000$ cubic feet

$= 16{,}000{,}000\left(\dfrac{1}{3}\right)^3 \approx 592592.6$ cubic yards.

3. $3k^3 = 375$ gives $k = 5$.

a) $k^2 = 25$.

b) $k^2 = 25$.

c) $k = 5$.

4. $k = 4$.

Equators are lengths and have ratio $k = 4$.

Surface areas have ratio $k^2 = 16$.

Volumes have ratio $k^3 = 64$.

5. a) 144.

b) $67k^2 = 67(12)^2 = 9648$.

c) $4089\left(\dfrac{1}{12}\right)^2 \approx 28.4$.

d) $1 \times (3)^3 = 27$.

e) $78{,}713\left(\frac{1}{12}\right)^3 \approx 45.6$

f) Area = $7.4 \times 5.2 = 38.48$ ft^2 = $38.48 \times 144 = 5541.12$ in^2.

Number of tiles $= \frac{5541.12}{25} = 221.6448 \approx 222$ tiles.

Cost = $222 \times 6.39 = \$1418.58$.

g) $2003(173)^2 = 59{,}947{,}787$ cm^2.

h) $V_1 = 800(12)^3$ in^3 and $V_2 = 800$ in^3. So, $k^3 = 12^3$, and $k = 12$.

6. $k = 2$.

a) $V = 20k^3 = 20 \cdot 8 = 160$ ft^3.

b) $V_{\text{bottom}} = 160 - 20 = 140$ ft^3.

7. a) $k = \frac{16}{20} = \frac{4}{5}$ (or $k = \frac{5}{4}$).

b) $k^2 = \frac{16}{25}$.

c) $k^2 = \frac{16}{25}$.

d) Suppose that the lateral area of the top cone is A.

Then, the lateral area of the entire cone is $Ak^2 = \frac{25}{16}A$, so the lateral area of the bottom portion is $\frac{25}{16}A - A = \frac{9}{16}A$.

The ratio of the two lateral areas, top over bottom, is $\frac{A}{\frac{9}{16}A} = \frac{16}{9}$.

e) Suppose that the volume of the top cone is V.

Then, the volume of the entire cone is $Vk^3 = \frac{125}{64}V$, so the volume of the bottom portion is $\frac{125}{64}V - V = \frac{61}{64}V$.

The ratio of the two volumes, top over bottom, is $\frac{V}{\frac{61}{64}V} = \frac{64}{61}$.

8. a) Amount of oxygen needed depends on volume. Amount of oxygen absorbed depends on surface area.

b) $k = 5$, so $SA = 4k^2 = 100$ cm^2 and $V = 2k^3 = 250$ cm^3.

c) The ratio $\frac{SA}{V}$ is $\frac{4}{2} = 2$ for the small worm and $\frac{100}{250} = 0.4$ for the big worm. This is much worse for the big worm.

d) The ratio $\frac{SA}{V}$ would be considerably worse for a 100-foot worm. Unlikely to survive.

9. a) $200k^3 = 345{,}600$ pounds.

b) $1 \cdot k = 12$ inches.

c) $A_{\text{little}} = \pi\left(\frac{1}{2}\right)^2 \approx 0.786 \text{ in}^2$, and $A_{\text{big}} \approx 0.786k^2 = 113.097 \text{ in}^2$.

d) Little $\approx \frac{200}{2\times.0786} \approx 127 \text{ lbs/in}^2$.

Big $\approx \frac{345{,}600}{2\times113.097} \approx 6112 \text{ lbs/in}^2$.

e) Most likely not.

10. a) $k = \frac{1}{100}$, so weight $= 120k^3 = 0.00012$ pounds, and $SA = 10k^2 = 0.001$ ft^2.

b) Volume of water $= 10 \times 0.001 = 0.01$ ft^3. This weighs 0.7 pounds, which is $\frac{0.7}{120} = 0.6\%$ of her weight.

c) Volume of water $= 0.001 \times 0.001 = 0.000001$ ft^3. This weighs 0.00007 pounds, which is $\frac{0.00007}{0.00012} \approx 60\%$ of her body weight. She would not be able to move.

Lesson 25

1. Probability $= \frac{1}{2}$.

2. a) $\frac{1}{4}$.

b) $\frac{3}{11}$.

3. If the radius of the sphere is r, then the side of the cube is $2r$. (Draw a picture to see this.)

$$\text{Probability} = \frac{\frac{4}{3}\pi r^3}{(2r)^2} = \frac{\frac{4}{3}\pi r^3}{8r^3} = \frac{4\pi}{24} = \frac{\pi}{6}.$$

4. Probability $= \dfrac{\pi 5^2}{30\times 60} = \dfrac{25\pi}{1800} = \dfrac{\pi}{72}.$

5. Let r be the radius of the circle.

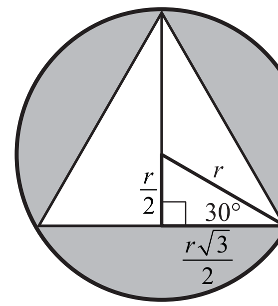


Figure S.25.1

Note the half equilateral triangle in **Figure S.25.1** with hypotenuse r (and, hence, with one leg $\frac{r}{2}$ and the other $\sqrt{r^2 - \frac{r^2}{4}} = \frac{r\sqrt{3}}{2}$).

Area of the circle = πr^2.

$$\text{Area of the triangle } = \frac{1}{2}\times \text{base} \times \text{height} = \frac{1}{2}\left(2\frac{r\sqrt{3}}{2}\right)\left(\frac{r}{2}+r\right)$$

$$= \frac{1}{2}r\sqrt{3}\,\frac{3r}{2} = \frac{3\sqrt{3}}{4}r^2.$$

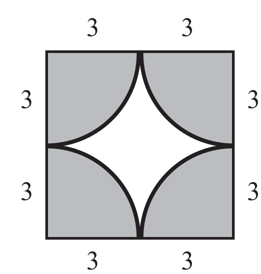
$$\text{Probability } \frac{\pi r^2 - \frac{3\sqrt{3}}{4}r^2}{\pi r^2} = \frac{\pi - \frac{3\sqrt{3}}{4}}{\pi} = \frac{4\pi - 3\sqrt{3}}{4\pi}.$$

6. We have that approximately $\frac{875}{1000}$; that is, 0.875 of the photograph represents oil.

Because the total area represented by the photograph is 42 square kilometers, the area of the spill is approximately $0.875 \times 42 = 36.75$ km^2.

7. $\dfrac{7143}{100{,}000}\cdot 12\times 14\times 9 = 108 \text{ ft}^3.$

8. You need to estimate the volume of the room and the volume of your head to compute this probability.



9. The probability we seek is $\frac{\pi 3^2}{36} = \frac{\pi}{4}$. (See **Figure S.25.2**.)

Figure S.25.2

10. $\triangle ANM$ is similar to $\triangle ABC$ by SAS with scale factor $k = 2$. The area of $\triangle ANM$ is $\frac{1}{4}$ the area of $\triangle ABC$. The same is true for $\triangle CMO$ and $\triangle BON$.

Thus, the non-shaded regions represent $\frac{3}{4}$ of the area of the triangle, and the shaded region represents just $\frac{1}{4}$.

The probability that the chosen point lies in this shaded region is thus $\frac{1}{4}$.

Lesson 26

1. a) and b) (Constructing a square constructs a rectangle.)

Figure S.26.1

Draw a line and a line segment $\overline{AB}$ along it.

Using a circle of radius AB and center A, mark a point B' on the line so that $AB' = AB$, as shown in **Figure S.26.1**.

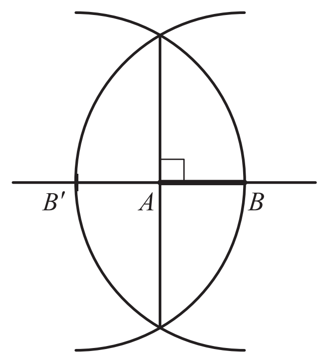


Figure S.26.2

Drawing two circles, each of radius BB', one with center B and the other with center B', gives two intersecting points that define a perpendicular line through A, as shown in **Figure S.26.2**.

Using A as the center and a circle of radius AB, mark a point C along the perpendicular segment, as shown in **Figure S.26.3**. We now have two sides of a square.

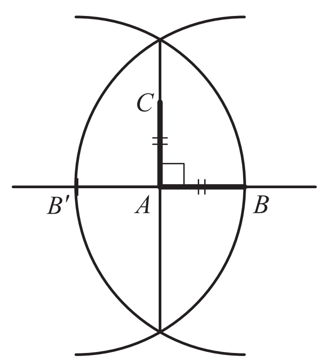


Figure S.26.3

Turn the page 90° and repeat this work starting with line segment $\overline{CA}$ to construct a perpendicular segment through C to make the third side of the square. Connect the ends of the first and third sides to draw the fourth edge of the square.

c) **Figure S.26.4** shows how to draw an equilateral triangle with a given segment as one of its sides.

To draw a hexagon, draw an equilateral triangle. And then draw a second equilateral triangle on one of its sides.

Do this six times to construct a hexagon. (See **Figure S.26.5**.)

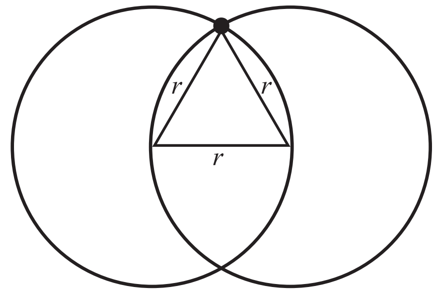


Figure S.26.4

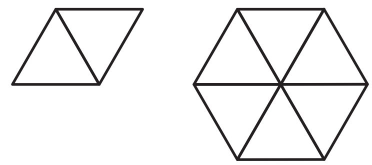


Figure S.26.5

2. Draw a circle with the vertex of the angle its center. This circle intersects the rays at two points, A and B. (See **Figure S.26.6**.)

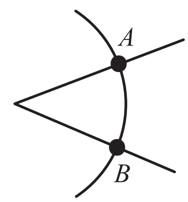


Figure S.26.6

Draw circles each of radius AB with centers at A and B.

Let P be one of their points of intersection. Also, call the vertex of the angle O.

Then, the line $\overleftrightarrow{OP}$ is the angle bisector of $\angle AOB$. (See **Figure S.26.7**.)

To see why, notice that $OA = OB$ by our first step and $AP = BP$ by our second step.

So, triangles OAP and OBP are similar by SSS and, therefore, $\angle AOP \cong \angle BOP$.

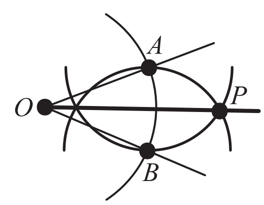


Figure S.26.7

3. Draw the perpendicular bisector of each side of the triangle—by the method shown in the lesson. We know from Lesson 12 that these three line segments meet at a common point P that is the center of the circle that passes through each vertex of the triangle. Set one point of the compass at P and the other at one corner of the triangle, and draw the circle of this radius with center P.

(Was it necessary to construct all *three* perpendicular bisectors?)

4. Draw a chord across part of the arc, and construct a perpendicular bisector of the chord. Do the same for a second chord. The intersection point of these two perpendicular bisectors is the center of the circle. (Can you explain why this is the case?)

Comment: If you rip the page from the book and hold it up to the light, you can fold a section of the arc onto itself. The crease is a perpendicular bisector of a chord. Do this again. Where the two creases intersect is the center of the circle.

5. Yes. We have seen that if a and b have been constructed, then it is possible to construct $a + b$ and also $\sqrt{a}$. We have seen that is it possible to construct a segment of any whole-number length.

So, use the following steps to construct $\sqrt{1+\sqrt{2+\sqrt{3}}}$.

First, construct segments of lengths 2 and 3 (from a given segment of length 1).

Next, construct a segment of length $\sqrt{3}$.

Next, construct a segment of length $2+\sqrt{3}$.

Next, construct a segment of length $\sqrt{2+\sqrt{3}}$.

Next, construct a segment of length $1+\sqrt{2+\sqrt{3}}$.

Finally, construct a segment of length $\sqrt{1+\sqrt{2+\sqrt{3}}}$.

Lesson 27

1. Verify that it does.

2. In each case, they do.

3. Color the corners of the squares of the table black and white according to a checkerboard pattern as shown in **Figure S.27.1**, with the top-left corner black.

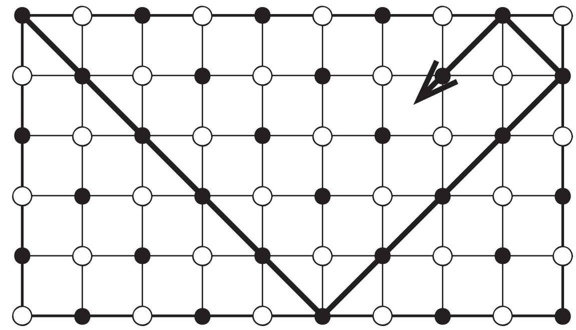


Figure S.27.1

It is clear that the ball will only ever pass from black corner to black corner. So, if the ball is to land in a corner, it must land in the bottom-right corner of the table, the only black corner. (All odd × odd boards have just the bottom-right corner black.)

To prove that the ball must land in a corner, we need to establish two things.

a) The ball cannot, after some point in time, fall into a repeating path.

b) The ball cannot return to start and fall back into the starting corner.

The motion of the ball is time-reversible: If we see it traverse one square of the table, we know precisely from which square it just came. So, if the ball traverses some square more than one time, the previous square in the ball's path was also repeated more than once, and so on.

Consequently, there can be no "first" square that is traversed more than once, and we conclude that the ball will never pass through the same square of the table twice. This means that the ball cannot bounce about the table indefinitely (there are only a finite number of squares), and it cannot return to start (this requires traversing the top-left square more than once). The ball, therefore, must fall into one of the remaining three corners.

Comment: This conclusion that the ball falls into one of the remaining three corners applies to tables of all dimensions.

4. With the coloring scheme implied by Problem 3, only the bottom-left corner is black in even × odd tables. This is the corner into which the ball must fall.

5. With the coloring scheme implied by Problem 3, only the top-right corner is black in odd × even tables. This is the corner into which the ball must fall.

Lesson 28

1. On a square grid, mark a point O, and mark a point P 1 unit to the east and 3 units to the north. Use OP as a radius to draw a circle with center O. In this picture, we see a 2×6 rectangle and a square of side length $\sqrt{20}$. The ratio of their areas is $\frac{12}{20} = \frac{3}{5}$.

2. This is a very difficult question!

The following is the picture of a tetrahedron with the midpoints of its sides marked and line segments connecting drawn to create an inner "midpoint figure." (See **Figure S.28.1**.)

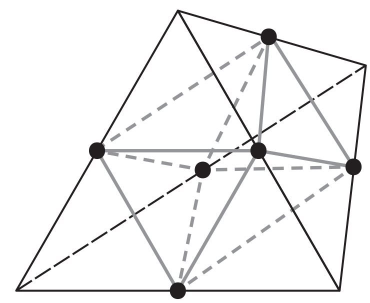


Figure S.28.1

We see that the inner figure is a regular octahedron.

This shows that one octahedron and one tetrahedron stack together to make a figure with unexpected flat faces. There are 7 flat faces in all.

3. Suppose the volume of the large tetrahedron in the solution to Problem 2 is V.

Each small tetrahedron in the corner of this large one is a scaled copy with scale factor $k = \frac{1}{2}$.

Thus, its volume is $k^3V = \frac{V}{8}$. This means that the volume of the interior octahedron is $V - 4 \times \frac{V}{8} = \frac{V}{2}$.

The ratio of volumes of one small tetrahedron to the octahedron is $\dfrac{\frac{V}{8}}{\frac{V}{2}} = \frac{1}{4}$.

4. Label angle w and sides a and b as shown in **Figure S.28.2**. Notice that $PQ = \sqrt{1+x^2}$.

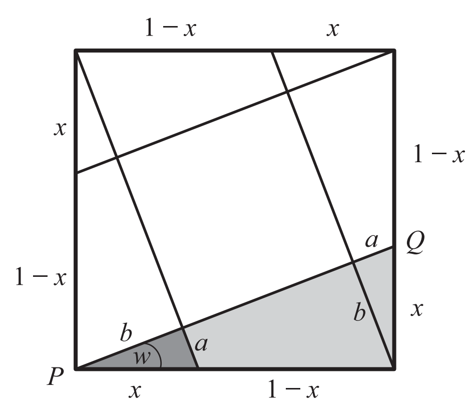


Figure S.28.2

Looking at the small shaded triangle and the large shaded triangle, we see

$$\sin(w) = \frac{a}{x} = \frac{x}{\sqrt{1+x^2}}$$

$$\cos(w) = \frac{b}{x} = \frac{1}{\sqrt{1+x^2}}.$$

So, $a = \frac{x^2}{\sqrt{1+x^2}}$ and $b = \frac{x}{\sqrt{1+x^2}}$.

Solutions

The area of the inner square is

$$\left(\sqrt{1+x^2}-a-b\right)^2 = \left(\frac{1+x^2-x(1+x)}{\sqrt{1+x^2}}\right)^2$$

$$= \left(\frac{1-x}{\sqrt{1+x^2}}\right)^2$$

$$= \frac{(1-x)^2}{1+x^2}.$$

Lesson 29

1. Because opposite faces of a stick of butter are parallel, any cross section that intersects a pair of opposite faces has a pair of parallel sides. Thus, all cross sections are parallelograms—or trapezoids if the slicing plane intercepts a square end of the butter, as shown in **Figure S.29.1**. (Actually, cross sections could be triangular, too. Do you see how?)

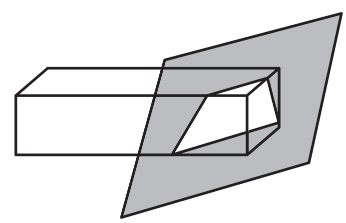


Figure S.29.1

2. It is. Follow precisely the same argument discussed in the lesson given by inserting two spheres in the cylinder that each just touch the plane of the cross section.

3. Suppose that a point $P = (x, y)$ is on an ellipse with foci $F = (-c, 0)$ and $G = (c, 0)$, satisfying $FP + PG = k$, for some number k. Notice by the triangular inequality that $FP + PG > FG$. That is, $k > 2c$. See **Figure S.29.2**.

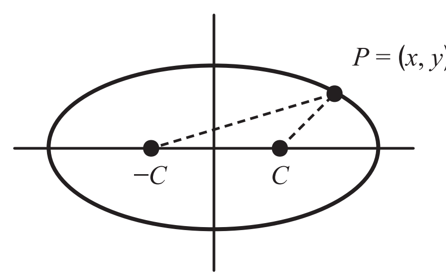


Figure S.29.2

The equation $FP + PG = k$ reads

$$\sqrt{(x+c)^2+y^2}+\sqrt{(x-c)^2+y^2}=k.$$

Rewrite this as

$$\sqrt{(x+c)^2+y^2}=k-\sqrt{(x-c)^2+y^2},$$

and square each side to obtain

$$(x+c)^2+y^2=k^2+(x-c)^2+y^2-2k\sqrt{(x-c)^2+y^2}.$$

This gives

$$x^2+2cx+c^2+y^2=k^2+x^2-2cx+c^2+y^2-2k\sqrt{(x-c)^2+y^2},$$

yielding

$$2k\sqrt{(x-c)^2+y^2}=k^2-4cx$$

or

$$\sqrt{(x-c)^2+y^2}=\frac{k}{2}-\frac{2cx}{k}.$$

Squaring again gives

$$(x-c)^2+y^2=\frac{k^2}{4}-2cx+\frac{4c^2x^2}{k^2},$$

and expanding gives

$$x^2-2cx+c^2+y^2=\frac{k^2}{4}-2cx+\frac{4c^2x^2}{k^2},$$

yielding

$$x^2\left(1-\frac{4c^2}{k^2}\right)+y^2=\frac{k^2}{4}-c^2.$$

Because $k>2c$, the quantities $1-\frac{4c^2}{k^2}$ and $\frac{k^2}{4}-c^2$ are both positive numbers. Dividing through by $\frac{k^2}{4}-c^2$ gives

$$\frac{x^2}{\left(\dfrac{\frac{k^2}{4}-c^2}{1-\frac{4c^2}{k^2}}\right)}+\frac{y^2}{\left(\frac{k^2}{4}-c^2\right)}=1,$$

which is indeed an equation of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $\left(\text{with } a=\sqrt{\dfrac{\frac{k^2}{4}-c^2}{1-\frac{4c^2}{k^2}}} \text{ and } b=\sqrt{\frac{k^2}{4}-c^2}\right)$.

Solutions

4. Suppose that the given circle has radius r. Call its center F.

Draw a line parallel to the given line r units below it. Call this line m.

The diagram in **Figure S.29.3** shows that the center of any circle tangent to the given circle and the given line is equidistant from F and m. They thus lie on a parabola with focus F and directrix m.

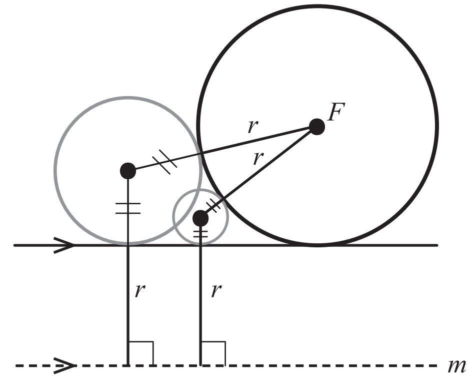


Figure S.29.3

Lesson 30

1. Suppose that points A and B are reflected about a line L to the points A' and B', respectively. First assume that A and B lie on the same side of L.

Also, let P be the point of intersection of $\overline{AA'}$ and line L and Q be the intersection of $\overline{BB'}$ and L. (See **Figure S.30.1**.)

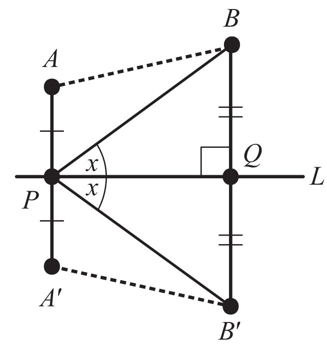


Figure S.30.1

Now, $\triangle BQP$ and $\triangle B'QP$ are congruent by SAS ($BQ = B'Q$, both have a right angle, and both share $\overline{PQ}$). Thus, the two angles labeled x are congruent (because they are matching angles in congruent triangles), and $PB = PB'$, (because they are matching sides in congruent triangles).

Thus, $\angle APB \cong \angle A'PB'$, because they both have measure $90 - x$.

So, $\triangle APB$ is congruent to $\triangle A'PB'$ by SAS ($AP = A'P$, $\angle APB \cong \angle A'PB'$, and $PB = PB'$).

Thus, $AB = A'B'$ (because they are matching sides in congruent triangles).

So, the reflection has indeed preserved this distance between A and B.

In the same way, you can show that this distance is preserved in the case where A and B lie on opposite sides of the line, in the case where one of the points A or B lies on the line, and in the case where they both do. (Check this!)

Thus, a reflection preserves the distances between all pairs of points in the plane and, therefore, is an isometry.

2. Only the regular polygons with an even number of sides have 180° rotational symmetry.

3. This question is difficult to answer because there are many symmetries.

There are line-reflection symmetries about the lines that follow an edge of any triangle and about lines that go through a vertex of a triangle and bisect its opposite side.

There are rotational symmetries of multiples of 60° about each vertex in the figure and multiples of 120° about the center of each triangle and 180° about the center of each edge of each triangle.

There are translation symmetries in many different directions (not just the three obvious directions), and there are plenty of glide symmetries.

4. i) Draw a square.

ii) Draw a non-square rectangle.

iii) Draw a kite that is not a rhombus.

5. This question, of course, depends on the font you use to write these letters. (For example, in one font, "W" does have vertical line symmetry, but not in another.)

a) B, C, D, E, H, I, O, X.

b) A, H, I, M, O, T, V, W, X, Y.

c) CHIDE and CODE, for example, have horizontal line symmetry. MOM and TOOT, for example, have vertical line symmetry.

Lesson 31

1. This fractal is obtained from dividing a square into 9 sub-squares, removing 5 of them, and iterating. We see that the fractal is composed of 4 scaled copies of itself, each at $\frac{1}{3}$ the scale. Thus,

$$\frac{1}{4} \cdot \text{original size} = \left(\frac{1}{3}\right)^d \cdot \text{ original size},$$

giving $3^d = 4$. Thus, $d \approx 1.26$.

2. We have 20 copies of the entire figure, each at $\frac{1}{3}$ the scale. Thus,

$$\frac{1}{20}\cdot \text{original size} = \left(\frac{1}{3}\right)^d \cdot \text{ original size},$$

giving $3^d = 20$. This yields $d \approx 2.72$.

3. $0.1111\ldots = 0.1+0.01+0.001+0.0001+\cdots = \frac{1}{10}+\frac{1}{10^2}+\frac{1}{10^3}+\frac{1}{10^4}+\cdots = \frac{1}{9}.$

4. We start with 1 equilateral triangle and add to it 3 triangles at $\frac{1}{3}$ the scale, then 12 triangles at $\frac{1}{9}$ the scale, then 48 at $\frac{1}{27}$ the scale, and so on.

With each iteration, the number of triangles added is 4 times the number of triangles added in the previous iteration, and these triangles are each $\frac{1}{3}$ the previous scale. (See **Figure S.31.1**.)

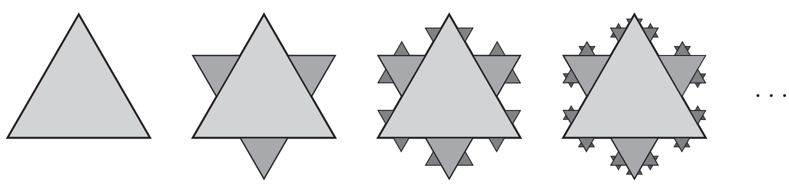


Figure S.31.1

Because area scales by k^2, the total area of Koch's snowflake is

$$\begin{aligned}
&1+3\cdot\left(\frac{1}{3}\right)^2\cdot 1+12\cdot\left(\frac{1}{9}\right)^2\cdot 1+48\cdot\left(\frac{1}{27}\right)^2\cdot 1+\cdots\\
&=1+3\cdot\left(\frac{1}{3}\right)^2\cdot 1+3\cdot 4\cdot\left(\frac{1}{3}\right)^4\cdot 1+3\cdot 4^2\cdot\left(\frac{1}{3}\right)^6\cdot 1+\cdots\\
&=1+3\cdot\left(\frac{1}{9}\right)^1\cdot 1+3\cdot 4\cdot\left(\frac{1}{9}\right)^2\cdot 1+3\cdot 4^2\cdot\left(\frac{1}{9}\right)^3\cdot 1+\cdots\\
&=1+\frac{1}{3}+3\cdot\frac{1}{9}\left(4\cdot\left(\frac{1}{9}\right)+4^2\cdot\left(\frac{1}{9}\right)^2+\cdots\right)\\
&=1+\frac{1}{3}+\frac{1}{3}\left(\frac{4}{9}+\left(\frac{4}{9}\right)^2+\cdots\right).
\end{aligned}$$

If we believe that the geometric formula works for fractions as well as whole numbers (read on!), then

$$\begin{aligned}\frac{4}{9}+\left(\frac{4}{9}\right)^2+\left(\frac{4}{9}\right)^3+\cdots &= \frac{1}{\frac{9}{4}}+\frac{1}{\left(\frac{9}{4}\right)^2}+\frac{1}{\left(\frac{9}{4}\right)^3}+\cdots \\ &= \frac{1}{\frac{9}{4}-1} \\ &= \frac{4}{9-4} \\ &= \frac{4}{5},\end{aligned}$$

giving the total area of the snowflake to be $1+\frac{1}{3}+\frac{1}{3}\cdot\frac{4}{5}=\frac{8}{5}$.

The question that remains is as follows. Should $\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}+\cdots$ equal $\frac{1}{x-1}$, even if x is not a whole number?

Notice that $\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}+\cdots=\frac{1}{x}+\frac{1}{x}\left(\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}+\cdots\right)$.

If $\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}+\cdots$ equals A, then this equation reads $A=\frac{1}{x}+\frac{1}{x}\cdot A$.

Multiplying through by x and solving for A gives

$$\begin{aligned} xA &= 1+A \\ (x-1)A &= 1 \\ A &= \frac{1}{x-1}.\end{aligned}$$

So, it seems that we do indeed have $\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}+\cdots=\frac{1}{x-1}$.

Challenge: Devise a paper-tearing exercise that shows that $\frac{1}{\frac{9}{4}}+\frac{1}{\left(\frac{9}{4}\right)^2}+\frac{1}{\left(\frac{9}{4}\right)^3}+\cdots=\frac{1}{\frac{9}{4}-1}$.

5. The textured surface has the greatest surface area and, therefore, requires more paint to cover.

Lesson 32

1. Any line through the center of a regular polygon with an even number of sides divides the area of that polygon exactly in half. (Draw a picture to see why.) Thus, the straight line that passes through the center of the hexagon and the center of the decagon divides the areas of the hexagonal hole in half and the area of the entire decagon in half. It now follows that it cuts the area of the brownies with the hole missing in half, too.

2. Any line through the center of a regular polygon with an even number of sides divides the perimeter of that polygon exactly in half. (Draw a picture to see why.) Thus, the line described in Problem 1 accomplished all of the feats listed in this problem.

3. By imagining a mirror reflection of the string and the region it encloses, we see that we are trying to enclose a region along a straight line using a perimeter of 80 inches. From Dido's problem, we know that a semicircle gives the maximum possible area. Thus, using the 40 inches of string to create a quarter circle gives the maximal area in this problem. (See **Figure S.32.1**.)

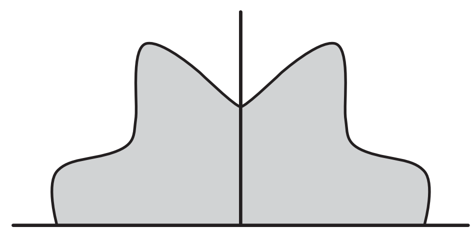


Figure S.32.1

4. Suppose that the square has side length x.

Draw the heights of the two shaded triangles, calling their lengths a and b, as shown in **Figure S.32.2**. Clearly, $a + b = x$.

Then, the total area of the portion of the square shaded gray is $\frac{1}{2}xa+\frac{1}{3}bx=\frac{1}{2}x\left(a+b\right)=\frac{1}{2}x^2$, which is half the area of the square.

The total area of the white portions must be the remaining half of the area of the square.

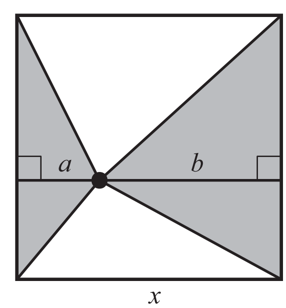


Figure S.32.2

5. No. The line through the center parallel to the base, for example, does not. (See **Figure S.32.3**.)

(What is an elegant way to see that the ratio of the areas of the two regions of the triangle cut by this line is 8:10?)

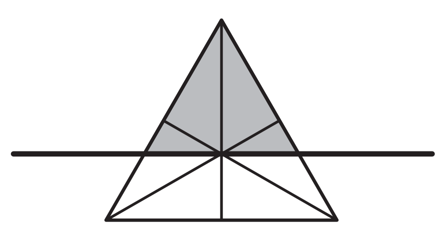


Figure S.32.3

Lesson 33

1. It can always be done! The solutions to the next four problems explain why.

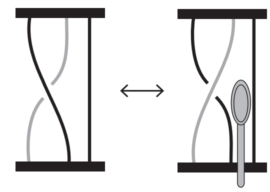


Figure S.33.1

2. Push the wooden spoon between the gap on the other side of the crossing, as shown in **Figure S.33.1**.

This result shows that we can always convert undercrossings to overcrossings, and vice versa, as convenient.

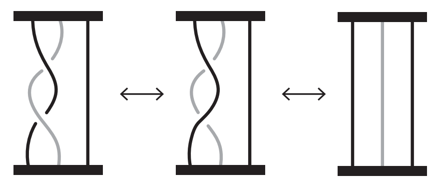


Figure S.33.2

3. Use Problem 2 to convert one of the crossings from an undercrossing to an overcrossing, or vice versa. Then, it is clear that the strands can be untangled. (See **Figure S.33.2**.)

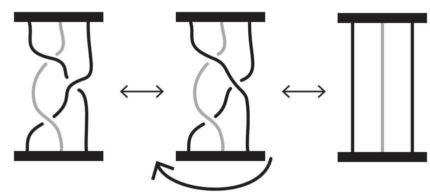


Figure S.33.3

4. Using the result of Problem 2, we can transform any undercrossings or overcrossings so that the braid is equivalent to just a 180° rotation of the base. (See **Figure S.33.3**.)

5. For three strands, let "L" denote the action of crossing the left two strands and "R" denote the action of crossing the right two strands. (By Problem 2, it does not matter if these are undercrossings or overcrossings.) Then, any braid you create can be encoded as a list of letters:

LLLRRRLRLLRRRLLLRRLLLLLRRRRRRLLRLRLRLLLRRRL,

for example.

By Problem 3, any two consecutive Ls or two consecutive Rs can be deleted. Thus, any braid is physically equivalent to one of the form LRLRLRLRLRL… or RLRLRLRLR… .

By Problem 4, any three consecutive terms of the form LRL or RLR can be deleted. A little thought shows that this means that the braid is equivalent to one of the following: L, R, LR, RL, or the braid with no crossings whatsoever (the untangled state).

You can check that none of L, R, LR, or RL have the middle strand in the middle position. Because the question demanded this, the only option that remains is that the original braid is physically equivalent to the untangled state.

Lesson 34

1. If you are N years old, then you have blown out $1+2+3+\cdots+N=\dfrac{N(N+1)}{2}$ candles.

2. $T(49)=\dfrac{49\times 50}{2}=1225.$

3. **Figure S.34.1** generalizes for any size of triangle.

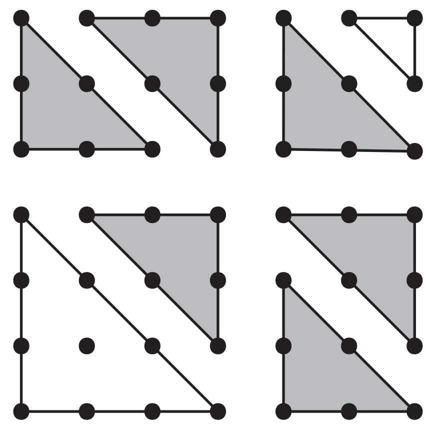


Figure S.34.1

Via algebra,

$$
\begin{aligned}
&T(N-1)+6T(N)+T(N+1)\\
&=\frac{(N-1)N}{2}+\frac{6N(N+1)}{2}+\frac{(N+1)(N+2)}{2}\\
&=\frac{8N^2+8N+2}{2}\\
&=4N^2+4N+1\\
&=(2N+1)^2.
\end{aligned}
$$

4. $S(N+M)=(N+M)^2=N^2+M^2+2MN=S(N)+S(M)+2MN.$

Geometrically:

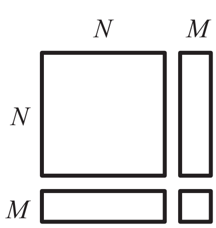


Figure S.34.2

5. a) Adding the rows gives

$$
\begin{aligned}
&1\times(1+2+3+4)+2\times(1+2+3+4)+3\times(1+2+3+4)+4\times(1+2+3+4)\\
&=(1+2+3+4)(1+2+3+4)\\
&=(1+2+3+4)^2.
\end{aligned}
$$

b) The first gnomon has a common factor of 1, the second a common factor of 2, the third a common factor of 3, and the fourth a common factor of 4.

$$
\begin{aligned}
&1\times 1\\
&\quad +1\times 2+2\times 2+2\times 1\\
&\qquad +1\times 3+2\times 3+3\times 3+3\times 2+3\times 1\\
&\qquad\quad +1\times 4+2\times 4+3\times 4+4\times 4+4\times 3+4\times 2+4\times 1\\
&=1\times 1+2\times\left(1+2+1\right)+3\times\left(1+2+3+2+1\right)+4\times\left(1+2+3+4+3+2+1\right).
\end{aligned}
$$

c) The two sums come from the same diagram, so they are the same. Because

$$
\begin{aligned}
&1\times 1+2\times\left(1+2+1\right)+3\times\left(1+2+3+2+1\right)+4\times\left(1+2+3+4+3+2+1\right)\\
&=1\times 1+2\times 2^2+3\times 3^2+4\times 4^2\\
&=1^3+2^3+3^3+4^3,
\end{aligned}
$$

we have established the stated formula in the question.

d) The same argument works for an $n\times n$ table. Summing all the entries by rows gives $\left(1+2+\cdots+n\right)^2$. Summing by gnomons gives $1^3+2^3+\cdots+n^3$. These must be the same.

Lesson 35

1. $A=-1+10i$ and $B=5+6i$. The point we seek is

$$
\begin{aligned}
P&=A+\frac{3}{5}\overline{AB}\\
&=A+\frac{3}{5}\left(B-A\right)\\
&=\frac{2}{5}A+\frac{3}{5}B\\
&=\frac{2}{5}\left(-1+10i\right)+\frac{3}{5}\left(5+6i\right)\\
&=\frac{13}{5}+\frac{38}{5}i.
\end{aligned}
$$

Thus, $P=\left(\frac{13}{5},\frac{38}{5}\right)$.

Solutions

2. a) $(3+7i)+(2-i)=5+6i.$

b) $(3+7i)-(2-i)=1+8i.$

c) $(3+7i)\times(2-i)=13+11i.$

3. a) From $(a+ib)(4+3i)=1$, we see that we need

$$(4a-3b)+i(3a+4b)=1.$$

This suggests solving

$$4a-3b=1$$
$$3a+4b=0.$$

The second equation gives $b=-\frac{3}{4}a.$

Substituting into the first yields $4a+\frac{9}{4}a=1,$ giving $a=\frac{4}{25}$, so $b=-\frac{3}{25}.$

It seems that $\frac{1}{4+3i}=\frac{4}{25}-\frac{3}{25}i.$

b) From $(-i)i=1$, it seems that $\frac{1}{i}=-i.$

c) Following the same line of reasoning as part a), it seems that $\frac{1}{p+iq}=\frac{p}{p^2+q^2}-\frac{q}{p^2+q^2}i.$

4. From $i^2=-1$, we see that $i^4=(i^2)^2=(-1)^2=1$. Thus, $i^{400}=(i^4)^{100}=1^{100}=1$. Consequently,

$$i^{403}=i^{400}\cdot i^2\cdot i=1\cdot(-1)\cdot i=-i.$$

5. Label the four vertices of the quadrilateral A, B, C, and D as shown in **Figure S.35.1**, and regard these as complex numbers.

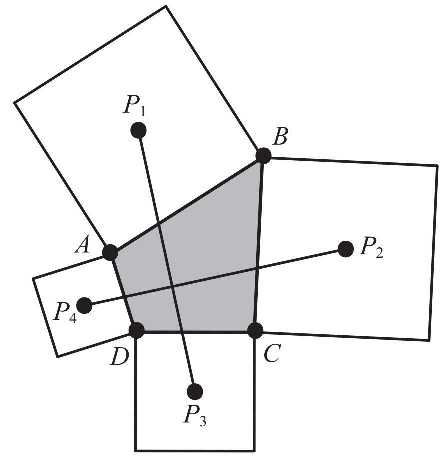


Figure S.35.1

Let's express each of P_1, P_2, P_3, and P_4 in terms of A, B, C, and D.

Starting at A, to reach P_1, we must first walk halfway along $\overrightarrow{AB}$ and then turn counterclockwise 90° and walk the same distance again.

Thus,

$$P_1 = A + \frac{1}{2}\overrightarrow{AB} + i\frac{1}{2}\overrightarrow{AB} = A + \frac{1}{2}(B - A) + \frac{i}{2}(B - A) = \frac{1-i}{2}A + \frac{1+i}{2}B.$$

In the same way,

$$P_3 = C + \frac{1}{2}\overrightarrow{CD} + \frac{i}{2}\overrightarrow{CD} = \frac{1-i}{2}C + \frac{1+i}{2}D.$$

Thus, $\overrightarrow{P_1P_3}$ is given by

$$\overrightarrow{P_1P_3} = P_3 - P_1 = -\frac{1-i}{2}A - \frac{1+i}{2}B + \frac{1-i}{2}C + \frac{1+i}{2}D.$$

In the same way,

$$\begin{aligned}\overrightarrow{P_2P_4} &= P_4 - P_2 \\ &= \left(D + \frac{1}{2}(A - D) + \frac{i}{2}(A - D)\right) - \left(B + \frac{1}{2}(C - B) + \frac{i}{2}(C - B)\right) \\ &= \frac{1+i}{2}A - \frac{1-i}{2}B - \frac{1+i}{2}C + \frac{1-i}{2}D.\end{aligned}$$

Now, notice that

$$i\overrightarrow{P_2P_4} = i\left(\frac{1+i}{2}A - \frac{1-i}{2}B - \frac{1+i}{2}C + \frac{1-i}{2}D\right) = -\frac{1-i}{2}A - \frac{1+i}{2}B + \frac{1-i}{2}C + \frac{1+i}{2}D = \overrightarrow{P_1P_3}.$$

Thus, $\overrightarrow{P_1P_3}$ is $\overrightarrow{P_2P_4}$ rotated counterclockwise 90°. Consequently, the line segments are congruent and perpendicular.

Lesson 36

1.

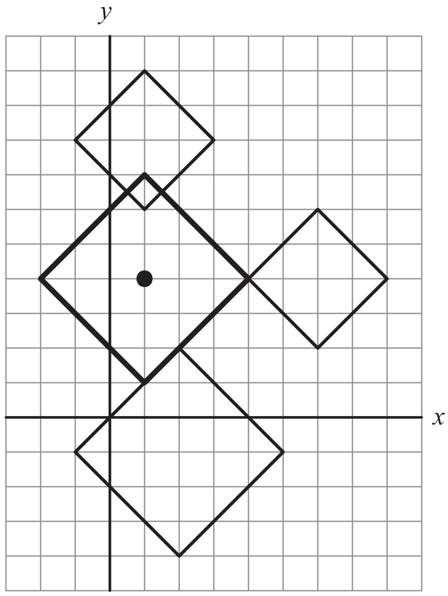


Figure S.36.1

2.

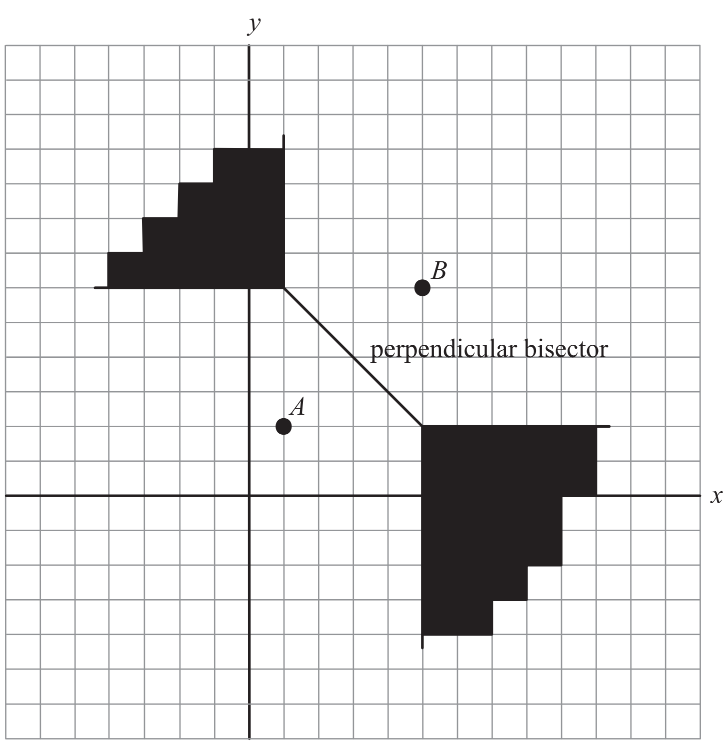


Figure S.36.2

3.

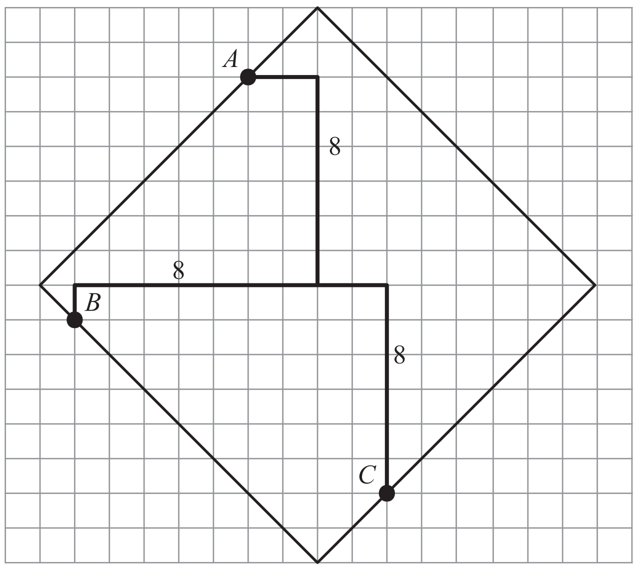


Figure S.36.3

4. No. The area of any such triangle would be $\frac{(60+60+60-180)}{720}\times$ total surface area = 0. The angles of any spherical triangle sum to more than 180°.

5. Let R = 6400 km be the radius of the Earth and r the radius of the circle of circumference 1 mile we seek. Label these lengths as shown in **Figure S.36.4**. (Notice the congruent alternate interior angles.)

We need 2πr = 1 mile = 1.609 km, so $r=\frac{1.609}{2\pi}\approx 0.256$ km.

We see that

$$\cos(x)=\frac{r}{R}\approx\frac{0.256}{6400}\text{, so } x\approx\cos^{-1}\left(\frac{0.256}{6400}\right)\approx 89.998°.$$

This circle just scoots right near the pole!

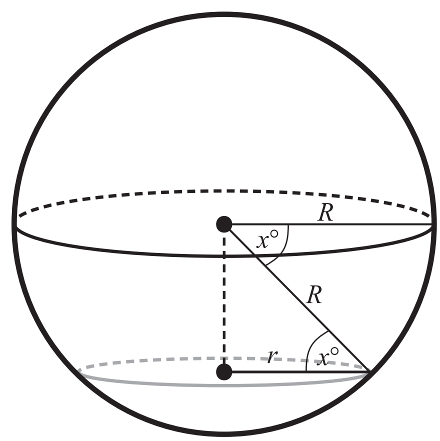


Figure S.36.4

Solutions

Formulas

Polygons

A polygon with N sides subdivides into $N - 2$ triangles with angles matching the interior angles of the polygon. (Thus, the sum of the interior angles of an N-gon is $(N - 2) \times 180°$.

Regular Polygons

The measure of one exterior angle of a regular N-gon is $\frac{360°}{N}$.

The measures of an exterior angle and an interior angle of a regular polygon sum to 180°.

Distance and Midpoints

The distance between two points in the plane that have been assigned coordinates is

$$\text{distance} = \sqrt{(\text{difference in } x\text{-values})^2 + (\text{difference in } y\text{-values})^2}.$$

The midpoint of the line segment connecting $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ is $M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

Lines

A line of slope m through the point (a, b) has equation $y - b = m(x - a)$. (Of course, there are many algebraically equivalent versions of this equation.)

Parallel lines have the same slope.

If one line has slope m, then a line perpendicular to it has slope $-\frac{1}{m}$. (assuming that neither line is vertical).

Area

area of a rectangle: length × width.

area of a triangle: $\frac{1}{2}$ base × height. (This formula applies no matter which side is considered the base.)

area of a polygon: Subdivide into triangles.

area of a regular N-gon: Subdivide N triangles, each with its apex at the center of the polygon.

Circles

The equation of a circle with center $C = (a, b)$ and radius r is $(x - a)^2 + (y - b)^2 = r^2$. (This is just an application of the Pythagorean theorem.)

If C is the circumference of a circle, D is its diameter, r is its radius (thus, $D = 2r$), and A is its area, then

$$\pi = \frac{C}{D}.$$

$$C = 2\pi r.$$

$$A = \pi r^2.$$

If an arc of the circle of radius r has measure $x°$, then the length of the arc is $\frac{x}{360} \cdot 2\pi r$.

The area of the sector defined by that arc is $\frac{x}{360} \cdot \pi r^2$.

Volumes and Surface Areas

volume of a cylinder, right or oblique

V = area base × height.

(See **Figure F.1**.)

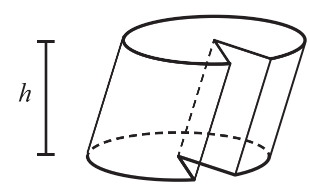


Figure F.1

volume of a cone

$V = \frac{1}{3}$ area base × height.

(See **Figure F.2**.)

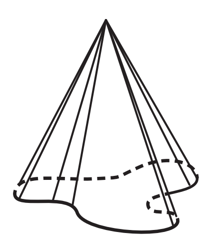


Figure F.2

Formulas

volume and surface area of right circular cylinder

$V = \pi r^2 h$.

lateral area = $2\pi rh$.

total surface area = $2\pi rh + 2\pi r^2$.

(See **Figure F.3**.)

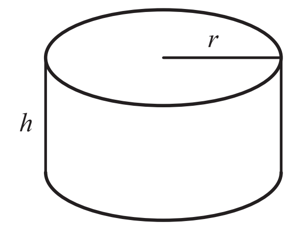


Figure F.3

volume and surface area of circular cone (with cone point above center of base)

$V = \dfrac{1}{3}\pi r^2 h$.

lateral area = πrs.

total surface area = $\pi rs + \pi r^2$.

(See **Figure F.4**.)

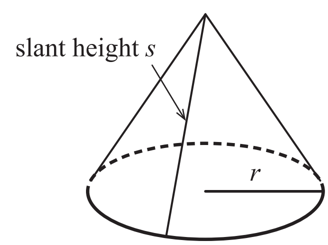


Figure F.4

volume and surface area of a sphere

$V = \dfrac{4}{3}\pi r^3$.

total surface area = $4\pi r^2$.

(See **Figure F.5**.)

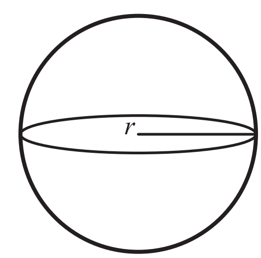


Figure F.5

Trigonometry: Through "Circle-ometry"

$\sin(x + 360°) = \sin(x)$.

$\cos(x + 360°) = \cos(x)$.

$\sin(-x) = -\sin(x)$.

$\cos(-x) = \cos(x)$.

$(\cos(x))^2 + (\sin(x))^2 = 1$.

Trigonometry: Through Right Triangles

$$\sin(x) = \frac{\text{opp}}{\text{hyp}}.$$

$$\cos(x) = \frac{\text{adj}}{\text{hyp}}.$$

$$\tan(x) = \frac{\text{opp}}{\text{adj}} = \frac{\sin(x)}{\cos(x)}.$$

Addition Formulas

$$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y).$$

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y).$$

Changing y to $-y$, these read (using $\sin(-y) = -\sin(y)$ and $\cos(-y) = \cos(y)$) as follows.

$$\sin(x-y) = \sin(x)\cos(y) - \cos(x)\sin(y).$$

$$\cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y).$$

Putting $y = x$ into the first equations gives the following.

$$\sin(2x) = 2\sin(x)\cos(x).$$

$$\cos(2x) = (\cos(x))^2 - (\sin(x))^2.$$

(This final equation is often abbreviated as $\cos(2x) = \cos^2(x) - \sin^2(x).$)

law of cosines: If a triangle has sides a, b, and c with angle x between sides a and b, then

$$c^2 = a^2 + b^2 - 2ab\cos(x).$$

Scale

In scaling by a factor k,

- all lengths change by a factor k.
- all areas change by a factor k^2.

- all volumes change by a factor k^3.

- all angles remain unchanged.

Fractal Dimension

If a fractal is composed of N parts, each a scaled copy of the original fractal with scale factor k, then its fractal dimension is the number d so that $k^d = \frac{1}{N}$.

Geometric Series Formula

$\frac{1}{N} + \frac{1}{N^2} + \frac{1}{N^3} + \cdots = \frac{1}{N-1}$ for whole numbers N ≥ 2.

Figurate Number Formulas

$$1+2+3+\cdots+(N-1)+N+(N-1)+\cdots+3+2+1 = N^2.$$

The sum of the first N odd numbers is N^2.

The sum of the first N even numbers is $N(N+1)$.

The sum of the first N counting numbers is $1+2+3+\cdots+N = \frac{N(N+1)}{2}$.

The N^{th} square number is $S(N) = N^2$.

The N^{th} triangular number is $T(N) = \frac{N(N+1)}{2}$.

Spherical Triangles

The area of a spherical triangle with interior angles of measures $x°$, $y°$, and $z°$ is

$\frac{(x+y+z-180)}{720} \times$ total surface area of the sphere.

Glossary

Note: The number in parentheses indicates the lesson in which the concept or term is first introduced.

acute angle (2): An angle with measure strictly between 0° and 90° is acute.

acute triangle (15): A triangle with each of its three interior angles acute. *See also* **acute angle**.

alternate interior angles (7): Given a pair of lines and a transversal, a pair of nonadjacent angles sitting between the pair of lines and positioned on opposite sides of a transversal are alternate interior angles. (Alternate interior angles are congruent precisely when the two lines are parallel.)

angle bisector (2): A line that divides a given angle into two congruent angles is an angle bisector for that angle.

arc (19): For two given points on a circle, a section of the circumference of the circle between them is called an arc of the circle. The measure of an arc is the measure of the angle between the two radii connecting those two given points. (There are two choices of angle between the two radii. The region between the two radii that contains the given arc defines which angle to measure.)

arc length (22): The length of an arc of a circle. *See also* **arc**.

bisects (2): A point M sitting on a line segment $\overline{AB}$ is said to bisect the line segment if $AM = MB$.

Cartesian coordinates (6): When the plane is endowed with a pair of vertical and horizontal axes, the location of a point in the plane is specified by a pair of numbers called its Cartesian coordinates.

Cartesian plane/coordinate plane (6): A plane for which a pair of vertical and horizontal axes have been assigned.

central angle (19): An angle formed by two radii of a circle. (This also defines the measure of the arc contained in the region specified by the angle.) *See also* **arc**.

chord (19): A line segment connecting two points on a circle.

circumcenter of a triangle (12): The center of the circumcircle of a triangle is its circumcenter. It is the location where the three perpendicular bisectors of the triangle coincide. *See also* **circumcircle of a triangle**.

circumcircle of a triangle (12): For each triangle, there is a unique circle that passes through the vertices of the triangle. This circle is the circumcircle of the triangle.

collinear (2): Two or more points are collinear if they lie on a common line.

complex number (35): A number of the form $a + ib$, with a and b each a real number and i an alleged quantity with the mathematical property that $i^2 = -1$.

concave polygon (4): A polygon that is not convex. *See also* **convex polygon**.

cone (23): A figure in three-dimensional space formed by

- drawing a region in a plane. (This will be called the base of the cone.)
- selecting a point P anywhere above or below the plane. (This will be called the cone point.)
- drawing a line segment from each and every point on the boundary of the planar region to the chosen point P.

congruent angles (2): Two angles of the same measure are congruent.

congruent line segments (2): Two line segments of the same length are congruent.

congruent polygons (9): Two polygons that are similar with scale factor $k = 1$ are congruent. *See also* **similar polygons**.

constructible number (26): A real number a is said to be constructible if, using only a straightedge (a ruler with no markings) and a compass as tools, it is possible to draw a line segment of length a if given only a line segment of length 1 already drawn on the page.

convex polygon (4): A polygon with the property that for any two points A and B in the interior of the polygon, the line segment $\overline{AB}$ also wholly lies in the interior of the polygon. See also **concave polygon**.

coordinate plane (6): *See* Cartesian plane/coordinate plane.

coplanar (2): Two or more points are coplanar if they lie on a common plane.

cosine of an angle (in circle-ometry) (16): A point moves in a counterclockwise direction along a circle of radius 1. If the angle of elevation of the point above the positive horizontal axis is x, then the length of the horizontal displacement, left or right, of the point is denoted $\cos(x)$ and is called the cosine of the angle. (The cosine of an angle is deemed negative if the point is displaced to the left.) *See also* **cosine of an angle (in trigonometry)**.

cosine of an angle (in trigonometry) (17): If x is a non–right angle in a right triangle, then the ratio of the length of the side of the triangle adjacent to the angle x (different from the hypotenuse) to the length of the hypotenuse of the right triangle is called the cosine of the angle and is denoted $\cos(x)$. (This value matches the "over-ness" of a point on a unit circle with angle of elevation x.) *See also* **cosine of an angle (in circle-ometry)**.

corresponding angles (13): For a transversal crossing a pair of lines, two angles on the same side of the transversal, with one between the pair of lines and one not, are corresponding angles. (Corresponding angles are congruent precisely when the two lines are parallel.)

cylinder (23): A figure in three-dimensional space formed by connecting, with straight line segments, matching boundary points of two congruent two-dimensional figures lying in parallel planes. The two planar figures (called the bases of the cylinder) are oriented so that any two connecting line segments are parallel. If the line segments connecting matching boundary points are perpendicular to the planes containing the bases, then the cylinder is a right cylinder. Otherwise, the cylinder is oblique.

deductive reasoning (8): The process of establishing the validity of a result by logical reasoning.

diameter (19): A chord of a circle that passes through the center of the circle is called a diameter of the circle. The length of any such diameter is called the diameter of the circle. (And as lengths, the diameter of a circle is twice the radius of the circle.) *See also* **chord**.

dilation (30): A dilation in the plane from a point O with scale factor k is the mapping that takes each point P different from O to a point P' on the ray $\overrightarrow{OP}$ such that $OP' = kOP$. The dilation keeps the point O itself fixed in place. *See also* **mapping**.

ellipse (29): Given two points F and G in the plane, the set of all points P for which the sum of distances $FP + PG$ has the same constant value traces a curve called an ellipse. The points F and G are called the foci of the ellipse.

equiangular polygon (4): A polygon with interior angles all of the same measure.

equidistant (12): A point is said to be equidistant from two or more objects if its distance from each of those objects is the same.

equilateral polygon (4): A polygon with all edges the same length.

exterior angle of a polygon (4): In extending one edge of a polygon, the angle formed by that extension and the next side of the polygon.

fractal (31): A geometric figure with the property that it can be divided into a finite number of congruent parts, each a scaled copy of the original figure.

fractal dimension (31): If a fractal is composed of N parts, each a scaled copy of the original fractal with scale factor k, then its fractal dimension is the number d so that $k^d = \frac{1}{N}$. *See also* **fractal**.

geometry (1): The branch of mathematics concerned with the properties of space and of figures, lines, curves, points, and shapes drawn in space. *See also* **planar geometry**, **three-dimensional geometry**, **spherical geometry**.

glide reflection (30): A glide reflection along a line L in a plane is the mapping that results from performing a translation in a direction parallel to L followed by a reflection about L. *See also* **mapping**.

hyperbola (29): Given two points F and G in the plane, the set of all points P for which the differences of distances $FP - PG$ and $GP - PF$ have the same constant value traces a curve called a hyperbola. The points F and G are called the foci of the hyperbola.

hypotenuse (4): The side opposite the right angle in a right triangle.

incenter of a triangle (12): The center of the incircle of a triangle is its incenter. It is the location where the three angle bisectors of the triangle coincide. *See also* **incircle of a triangle**.

incircle of a triangle (12): For each triangle, there is a unique circle sitting inside the triangle tangent to each of its three sides. This circle is the incircle of the triangle.

inductive reasoning (8): The process of finding patterns and making conjectures based on those patterns.

inscribed angle (19): If P and Q are two endpoints of an arc of a circle and A is a point on the circle not on the arc, then $\angle PAQ$ is an inscribed angle.

isometry (30): A mapping that preserves distances between points. *See also* **mapping**.

isoperimetric problem (32): The challenge of determining which figure in the plane has the greatest area given a fixed length for its perimeter.

isosceles triangle (9): A triangle with at least two sides congruent is isosceles.

lateral (23): Refers to any feature of a cone or cylinder that is not part of a base of the figure. (For example, a lateral edge is any edge of the figure that is not an edge of a base, or a lateral face is any face of the figure that is not a base.) *See also* **cone**, **cylinder**.

mapping (30): A mapping of the plane is a rule that shifts some or all points of the plane to new locations in the plane.

median of a trapezoid (13): If a trapezoid has just one pair of parallel sides, then the line segment connecting the midpoints of the two remaining sides is the median of the trapezoid. *See also* **trapezoid**.

median of a triangle (35): A line segment that connects one vertex of a triangle to the midpoint of its opposite side. (Each triangle possesses three medians, and they all pass through a common point called the centroid of the triangle.)

midpoint (2): A point M is a midpoint of line segment $\overline{AB}$ if M lies on the segment and $AM = MB$.

***N*-gon (4)**: A polygon with N sides. *See also* **polygon**.

obtuse angle (2): An angle with measure strictly between 90° and 180° is obtuse.

obtuse triangle (15): A triangle with one interior angle that is an obtuse angle. *See also* **obtuse angle**.

origin (6): The point in the Cartesian plane with coordinates (0, 0).

parabola (29): Given a line m and a point F not on that line, the set of all points P equidistant from m and P form a curve called a parabola. F is called the focus of the parabola, and m is its directrix.

parallel (7): Two lines, each infinite in extent, are parallel if they never meet.

parallelogram (13): A quadrilateral with two pairs of parallel sides.

perpendicular (2): Two lines or line segments that intersect at an angle of 90° are perpendicular.

perpendicular bisector (11): The perpendicular bisector of a line segment is a line through the midpoint of the segment and perpendicular to it. *See also* **midpoint**.

pi (22): In flat geometry, the ratio of the circumference of a circle to its diameter is the same for all circles. The common value of this ratio is called pi and is denoted π.

planar geometry (1): The study of figures, lines, curves, points, and shapes drawn in a plane.

polygon (4): A planar figure composed of straight line segments (called sides or edges) so that

- sides intersect only at their endpoints (the endpoints are called the vertices of the polygon);
- precisely two sides meet at each endpoint; and
- two sides meeting at a vertex make an angle different from 180°.

prism (23): A cylinder with a polygon for its base. *See also* **cylinder**.

pyramid (23): A cone with a polygon for its base. *See also* **cone**.

radius (19): A line segment connecting the center of a circle to a point on the circle is called a radius of the circle. The length of any such line segment is called the radius of the circle.

reflection (30): A reflection in a plane about a line L is a mapping that takes each point P in the plane not on L to a point P' so that L is the perpendicular bisector of $\overline{PP'}$. It keeps each point on L fixed in place. *See also* **mapping**.

reflex angle (2): An angle with measure strictly between 180° and 360° is a reflex angle.

regular polygon (4): A polygon that is both equilateral and equiangular. *See also* **equiangular**, **equilateral**.

regular cone or **regular cylinder (23)**: A cone or cylinder is said to be regular if its base is a regular polygon and all of its lateral faces are congruent. *See also* **cone**, **cylinder**.

rhombus (4): An quadrilateral with four congruent sides.

right angle (2): An angle with measure 90° is a right angle.

right triangle (15): A triangle with one of its interior angles a right angle. *See also* **right angle**.

rotation (30): A rotation in the plane about a point O through a counterclockwise angle of $x°$ is the mapping that takes each P in the plane different from O to a point P' such that the line segments $\overline{OP}$ and $\overline{OP'}$ are congruent and the angle from $\overline{OP}$ to $\overline{OP'}$, measured in a counterclockwise direction, is $x°$. The rotation keeps the point O itself fixed in place. *See also* **mapping**.

same-side interior angles (7): Given a pair of lines and a transversal, a pair of nonadjacent angles sitting between the pair of lines and positioned on the same side of a transversal are same-side interior angles. (The measures of same-side interior angles sum to 180° precisely when the two lines are parallel.)

scalene triangle (15): A triangle with three different side lengths.

secant to a circle (19): A line that intercepts a circle at two distinct points.

sector of a circle (22): The figure formed by two radii of a circle and an arc of the circle between them.

similar polygons (9): Two polygons are similar if, in moving one direction about one polygon, it is possible to move in some direction about the second polygon so that

- all side lengths encountered in turn match in the same ratio k.
- all angles encountered in turn match exactly.

The common ratio k of the side lengths is called the scale factor.

sine of an angle (in circle-ometry) (16): A point moves in a counterclockwise direction along a circle of radius 1. If the angle of elevation of the point above the positive horizontal axis is x, then the height of the point above the axis is denoted $\sin(x)$ and is called the sine of the angle. (The sine of an angle is deemed negative if the point lies below the horizontal axis.) *See also* **sine of an angle (in trigonometry)**.

sine of an angle (in trigonometry) (17): If x is a non–right angle in a right triangle, then the ratio of the length of the side of the triangle opposite angle x to the length of the hypotenuse of the right triangle is called the sine of the angle and is denoted $\sin(x)$. (This value matches the height of a point on a unit circle with angle of elevation x.) *See also* **sine of an angle (in circle-ometry)**.

slant height (23): The slant height of a circular cone with cone point above the center of its base is the distance of the cone point from any point on the perimeter of the base. The slant height of a regular pyramid is the height of any one of its triangular faces (with a base edge considered the base of the triangle). *See also* **cone**, **pyramid**.

solid (23): A figure in three-dimensional space.

spherical geometry (1): The study of figures, lines, curves, points, and shapes drawn on the surface of a sphere.

squangular number (34): A number that is both square and triangular. *See also* **square number**, **triangular number**.

square number (34): A number is a square number if a count of that many pebbles can be arranged in a square array.

straight angle (2): An angle with measure 180° is a straight angle.

symmetry (30): A figure is said to have symmetry if there is a mapping of the plane that points the figure back onto the figure itself. *See also* **mapping**.

tangent of an angle (in trigonometry) (17): If x is a non–right angle in a right triangle, then the ratio of the length of the side of the triangle opposite angle x to the length of the side adjacent to the angle (different from the hypotenuse of the right triangle) is called the tangent of the angle and is denoted $\tan(x)$.

tangent to a circle (19): A line that meets just one point of a circle.

translation (30): A translation in a plane is a mapping that shifts each point of the plane a fixed distance in a fixed direction. *See also* **mapping**.

taxicab geometry (36): If distances between two points $A = (a_1, a_2)$ and $B = (b_1, b_2)$ are measured by the sum of the horizontal and vertical displacements, $d(A,B) = |b_1 - a_1| + |b_2 - a_2|$, rather than via the Pythagorean theorem, and all other structures of planar geometry are left the same, then the geometry that results is called taxicab geometry.

three-dimensional geometry (1): The study of figures, lines, curves, points, and shapes drawn in three-dimensional space.

transversal (7): A line that crosses a pair of lines.

trapezoid (13): A quadrilateral with at least one pair of parallel sides.

triangular number (34): A number is a triangular number if a count of that many pebbles can be arranged in a triangular array (with the number of pebbles in each row of the triangle one greater than the previous row).

vertical angles (2): Two angles on opposite sides of the point of intersection of two intersecting lines are vertical angles.

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———. *Thinking Mathematics! Vol. 2: Advanced Counting and Advanced Algebra Systems.* Raleigh, NC: Lulu, 2009. This resource uses complex numbers (Lesson 35) to prove the geometry of Euler's line (Lesson 12).

———. *Thinking Mathematics! Vol. 5: e, i, pi and All That.* Raleigh, NC: Lulu, 2009. This text provides discussion on trigonometry (Lessons 16 and 17) and the work on "squine" and "cosquine," extending "circle-ometry" to squares.

———. *Weird Ways to Work with Pi.* Raleigh, NC: Lulu, 2011. This text explains, among many curious things, why setting the value of π for a square as 4 (Lesson 22) is mathematically appropriate.

Online Resources

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