Astr 513: Homework 2

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1 Parts A and B

a) We can use three kinematic equations to find u_x and u_y in terms of R, g, and h:

(1) $0 = u_y^2 - 2gh \Rightarrow \boxed{u_y = \sqrt{2gh}}$

(2) $0 = u_y - \frac{gt}{2} = \sqrt{2gh} - \frac{gt}{2} \Rightarrow t = \sqrt{\frac{8h}{g}}$

(3) $R = u_x t = u_x\sqrt{\frac{8h}{6}} \Rightarrow \boxed{u_x = R\sqrt{\dfrac{g}{8h}}}$

In the error propagation method, the expected values for u_x and u_y are just the values u_x^0 and u_y^0 we obtain when we insert the mean values R_0, g_0, and h_0 into the expressions above:

$$\begin{aligned} u_x^0 &= R_0\sqrt{\frac{g}{8h}} \\ &= 10.0\mathrm{m}\sqrt{\frac{9.81\mathrm{m}\cdot\mathrm{s}^{-2}}{8\cdot 1.0\mathrm{m}}} \end{aligned}$$

$$\boxed{u_x^0 \approx 11.1\mathrm{m}\cdot\mathrm{s}^{-1}} \tag{1a}$$

$$u_y^0 = \sqrt{2g_0h_0}$$
$$= \sqrt{2 \cdot 9.81\text{m} \cdot \text{s}^{-2} \cdot 1.0\text{m}}$$
$$\boxed{u_y^0 \approx 4.43\text{m} \cdot \text{s}^{-1}} \tag{2a}$$

When using propagation of errors, the fractional error on some function $f(\vec{x})$ with mean $\vec{x}$ values $\vec{x}^0$ and errors σ_i can be found by $\frac{\sigma_f}{f} = \sqrt{\sum_{i=1}^{n} \left(\frac{\partial \ln f}{\partial x_i}\right)_{\vec{x}^0} \sigma_i^2}$. So we first take the natural log of u_x and u_y:

$$\ln u_x = \ln R + \frac{1}{2}\ln g - \frac{1}{2}\ln h - \frac{3}{2}\ln 2$$
$$\ln u_y = \ln 2 + \ln g + \ln h$$

Then we find that

$$\frac{\sigma_{u_x}}{u_x} = \sqrt{\left(\frac{\sigma_R}{R_0}\right)^2 + \frac{1}{4}\left(\frac{\sigma_g}{g_0}\right)^2 + \frac{1}{4}\left(\frac{\sigma_h}{h_0}\right)^2}$$
$$= \sqrt{\left(\frac{0.2\text{m}}{10.0\text{m}}\right)^2 + \frac{1}{4}\left(\frac{0.05\text{m} \cdot \text{s}^{-2}}{9.81\text{m} \cdot \text{s}^{-2}}\right)^2 + \frac{1}{4}\left(\frac{0.2\text{m}}{1.0\text{m}}\right)^2}$$
$$\boxed{\frac{\sigma_{u_x}}{u_x} \approx 0.10} \tag{3a}$$

for σ_{u_x}, and for σ_{u_y},

$$\frac{\sigma_{u_y}}{u_y} = \sqrt{\frac{1}{4}\left(\frac{\sigma_g}{g_0}\right)^2 + \frac{1}{4}\left(\frac{\sigma_h}{h_0}\right)^2}$$
$$= \frac{1}{2}\sqrt{\left(\frac{0.05\text{m} \cdot \text{s}^{-2}}{9.81\text{m} \cdot \text{s}^{-2}}\right)^2 + \left(\frac{0.2\text{m}}{1.0\text{m}}\right)^2}$$
$$\boxed{\frac{\sigma_{u_y}}{u_y} \approx 0.10} \tag{4a}$$

b) Now we use Monte Carlo realizations of R, g, and h assuming Gaussian uncertainties to find the distributions of u_x and u_y. Using this method, we find that $u_x \approx 11.1^{+1.32}_{-0.99}$ and $u_y \approx 4.43^{+0.42}_{-0.46}$. So the means for u_x and u_y seem to agree with the error propagation method, but not the errors. The marginalized distributions for u_x and u_y, as well as the distribution of u_y vs u_x, can be seen in the corner plot in Figure 1.

2 Part C

The analytic approach to this problem involves a Jacobian transformation from (R, h, g) to (u_x, u_y, g). To do so, we invert

$$h = \frac{u_y^2}{2g} \quad \text{and} \quad R = \frac{2u_x u_y}{g}$$

to give

$$u_x = \sqrt{2gh} \quad \text{and} \quad u_y = \sqrt{\frac{g}{8h}}.$$

Of course, $g = g$. The probability distribution is

$$P(R, h, g)dRdhdg = P(u_x, u_y, g)du_x du_y dg. \tag{5}$$

It can be written in terms of three Gaussian functions

$$P(R, h, g) = G(R, R_0, \sigma_R)G(h, h_0, \sigma_h)G(g, g_0, \sigma_g). \tag{6}$$

The Jacobian transformation of the probability distribution is

$$P(u_x, u_y, g) = G(R, R_0, \sigma_R)G(h, h_0, \sigma_h)G(g, g_0, \sigma_g)J\left(\frac{R, h, g}{u_x, u_y, g}\right) \tag{7}$$

where J is the Jacobian. In this transformation from 3 variables to 3 inferences,

$$J = \begin{vmatrix} \frac{\partial R}{\partial u_x} & \frac{\partial R}{\partial u_y} & \frac{\partial R}{\partial g} \\ \frac{\partial h}{\partial u_x} & \frac{\partial h}{\partial u_y} & \frac{\partial h}{\partial g} \\ \frac{\partial g}{\partial u_x} & \frac{\partial g}{\partial u_y} & \frac{\partial g}{\partial g} \end{vmatrix} = \frac{2u_y^2}{g^2}. \tag{8}$$

With the Jacobian,

$$\begin{aligned} P(u_x, u_y, g) \quad = \quad & \frac{u_y^2}{\sqrt{2}\pi^{3/2}\sigma_R\sigma_h\sigma_g g^2}\exp\left[\frac{-(\frac{2u_y u_x}{g} - R_0)^2}{2{\sigma_R}^2}\right] \\ & \exp\left[\frac{-(\frac{u_y^2}{2g} - h_0)^2}{2{\sigma_h}^2}\right]\exp\left[\frac{-(g - g_0)^2}{2{\sigma_g}^2}\right]. \end{aligned} \tag{9}$$

In order to obtain the probability distribution of u_y, we need to marginalize Equation 9 over u_x and g. Thus,

$$\begin{aligned} P(u_y) \quad = \quad & \frac{u_y^2}{\sqrt{2}\pi^{3/2}\sigma_R\sigma_h\sigma_g}\exp\left[\frac{-(\frac{u_y^2}{2g} - h_0)^2}{2{\sigma_h}^2}\right] \\ & \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left[\frac{-(\frac{2u_y u_x}{g} - R_0)^2}{2{\sigma_R}^2}\right]\exp\left[\frac{-(g - g_0)^2}{2{\sigma_g}^2}\right]\frac{1}{g^2}du_x dg. \end{aligned} \tag{10}$$

In order to obtain the probability distribution of u_x, we need to marginalize Equation 9 over u_y and g. Thus,

$$\begin{aligned} P(u_x) &= \frac{1}{\sqrt{2}\pi^{3/2}\sigma_R\sigma_h\sigma_g}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left[\frac{-(\frac{u_y^2}{2g}-h_0)^2}{2\sigma_h{}^2}\right] \\ &\quad \exp\left[\frac{-(\frac{2u_yu_x}{g}-R_0)^2}{2\sigma_R{}^2}\right]\exp\left[\frac{-(g-g_0)^2}{2\sigma_g{}^2}\right]\frac{u_y^2}{g^2}du_ydg. \end{aligned} \tag{11}$$

As seen in Figure 2 and Figure 3, the shapes of the analytic probability distributions match the shapes of the probability distributions from Monte Carlo simulations.

3 Part D

Bayes' Theorem says

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)} \tag{12}$$

where $P(X|Y)$ is a conditional probability of X given Y, and $P(Z)$ is referred to as a *prior* belief on what the value of Z should be. Frequentists equate the two conditional probabilities, but this is incorrect in the Bayesian approach precisely because you wish to be informed by the priors. In this part, we assume *uniform* priors on our posterior parameters u_x, u_y and g, and in addition to this there is no Jacobian transformation. Thus we can write for the conditional probability

$$P_B(u_x,u_y,g|R,h,g) = \frac{1}{(2\pi)^{3/2}\sigma_R\sigma_h\sigma_g}\exp\left(-\frac{(2u_xu_y/g-R_0)^2}{2\sigma_R^2}\right)\times \tag{13}$$

$$\exp\left(-\frac{(u_y^2/2g-h_0)^2}{2\sigma_h^2}\right)\exp\left(-\frac{(g-g_0)^2}{2\sigma_g^2}\right) \tag{14}$$

where we have written R and h as functions of u_x, u_y and g. Just as in class, this expression is missing the Jacobian term, $2u_y^2/g^2$, as calculated in section 2. Thus, if we assume flat priors

$$P_{pr}(u_x) = P_{pr}(u_y) = 1 \tag{15}$$

then the ratio of the two probabilities is simply this Jacobian

$$\frac{P_f}{P_b} = \frac{2u_y^2}{g^2}. \tag{16}$$

Interestingly, this ratio is not unitless, but this is not an issue since in the Bayesian framework we neglect the normalization terms anyway, which carry units with them.

We can marginalize over any two parameters to acquire the third. See Figure 4Figure 5 for figures showing these distributions with no priors.

4 Part E

We can also introduce logarithmic priors

$$P_{pr}(X) \propto x^{-1}$$

on any of our parameters. The effect is to shift (sometimes slightly) the distributions down slightly. In the following three figures we plot the marginalized probability for each parameter with and without a logarithmic prior. These probabilities were generated by performing a numerical integral over each of the marginalized parameters.

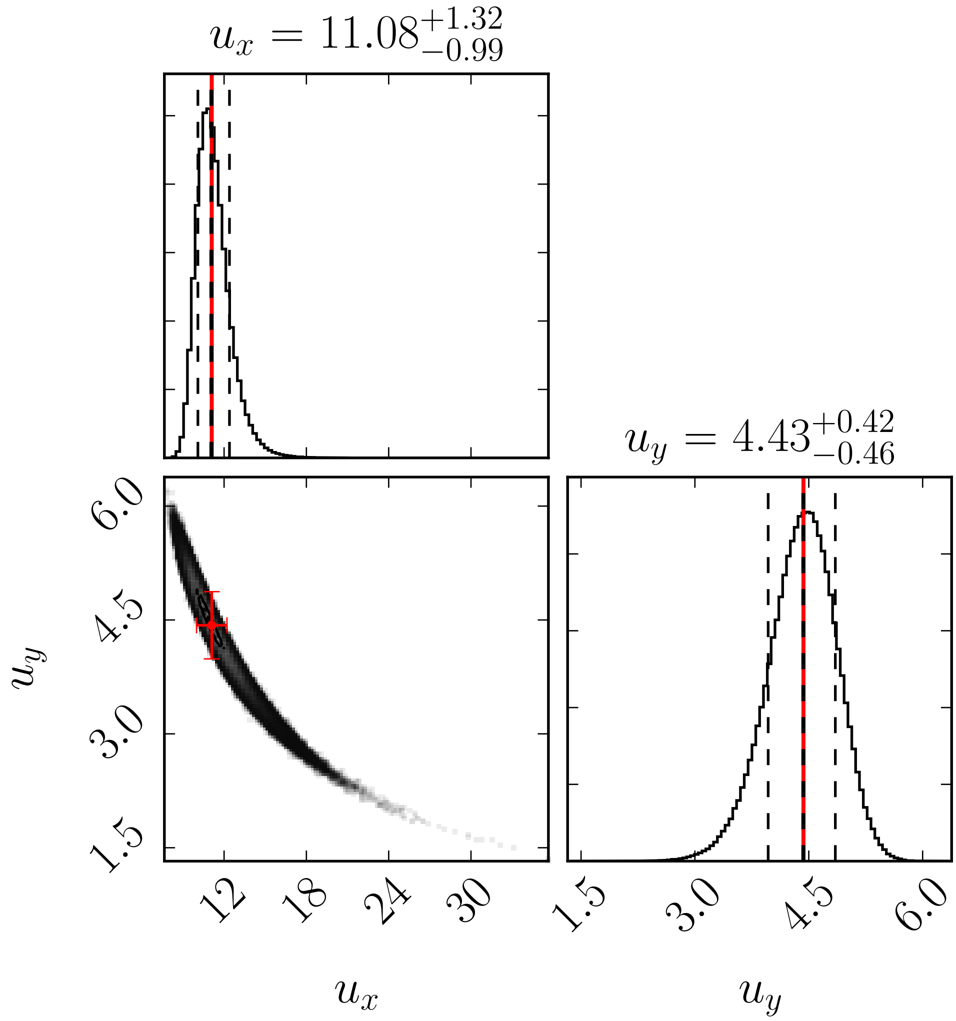

Figure 1: The distribution of u_y vs u_x as well as the marginalized distributions for u_x and u_y. The red line in the marginalized distributions shows the answer from the error propagation method, and the red point with error bars is the expected value and error from the error propagation method.

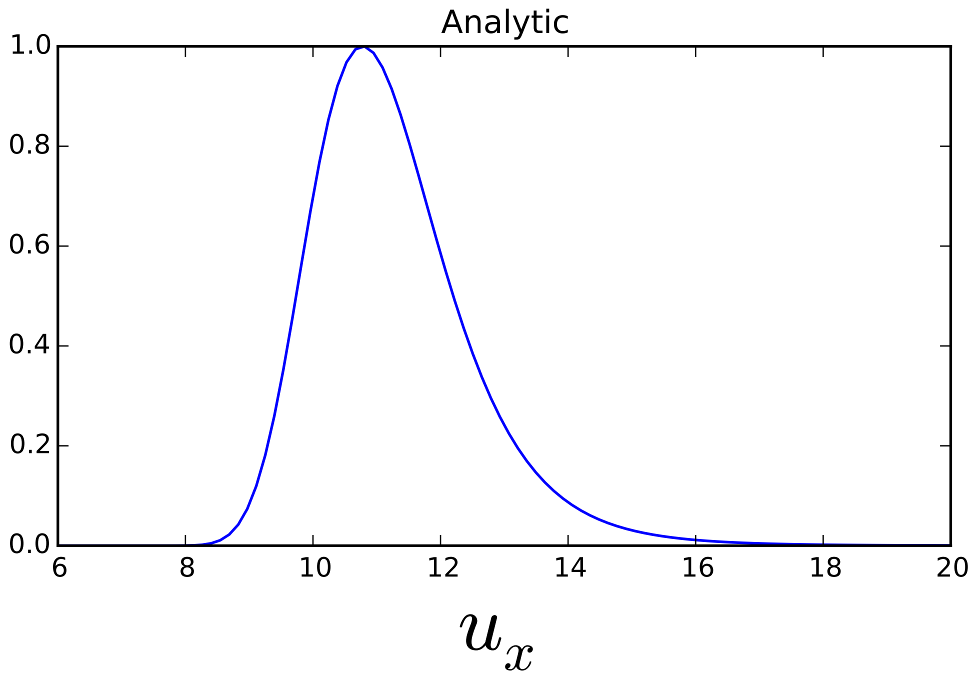

Figure 2: The analytic, marginalized probability of u_x. The marginalization over the other two parameters was done numerically.

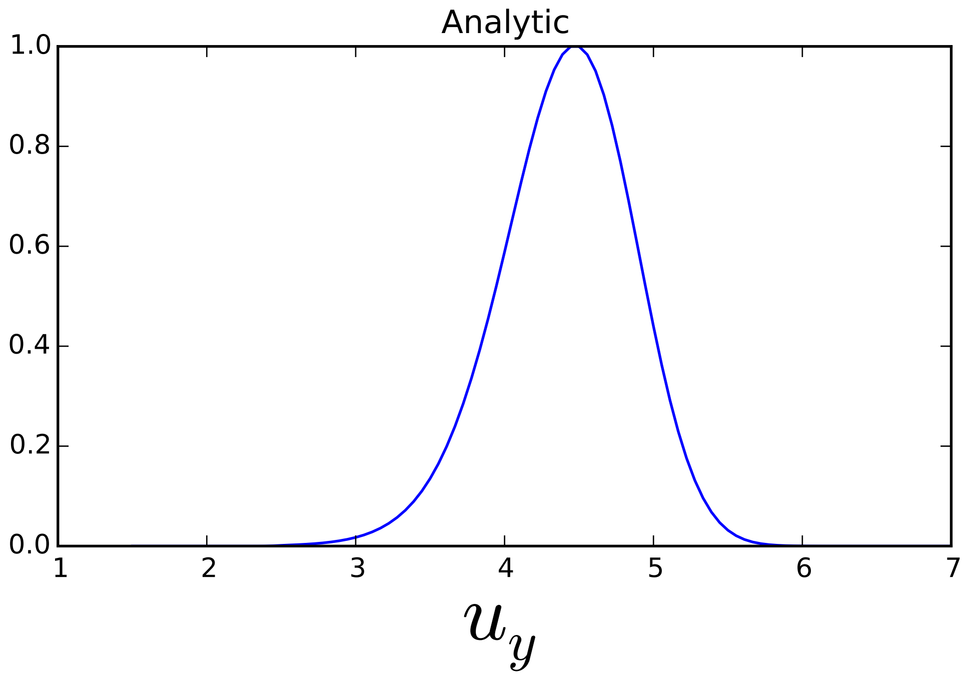

Figure 3: The analytic, marginalized probability of u_y. The marginalization over the other two parameters was done numerically.

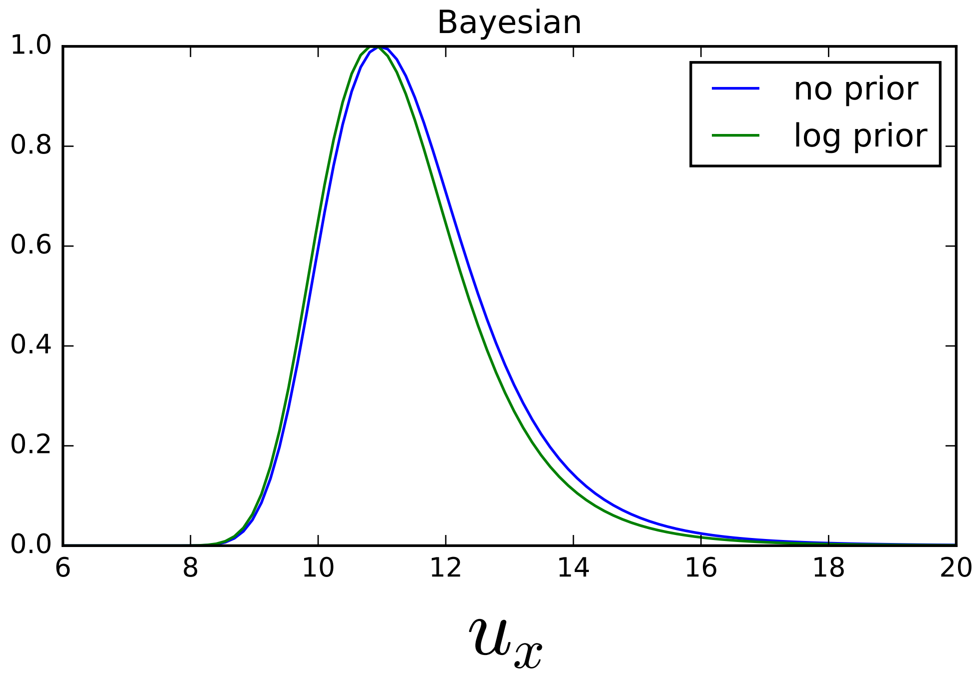

Figure 4: The marginalized probability of u_X. The marginalization over the other two parameters was done numerically.

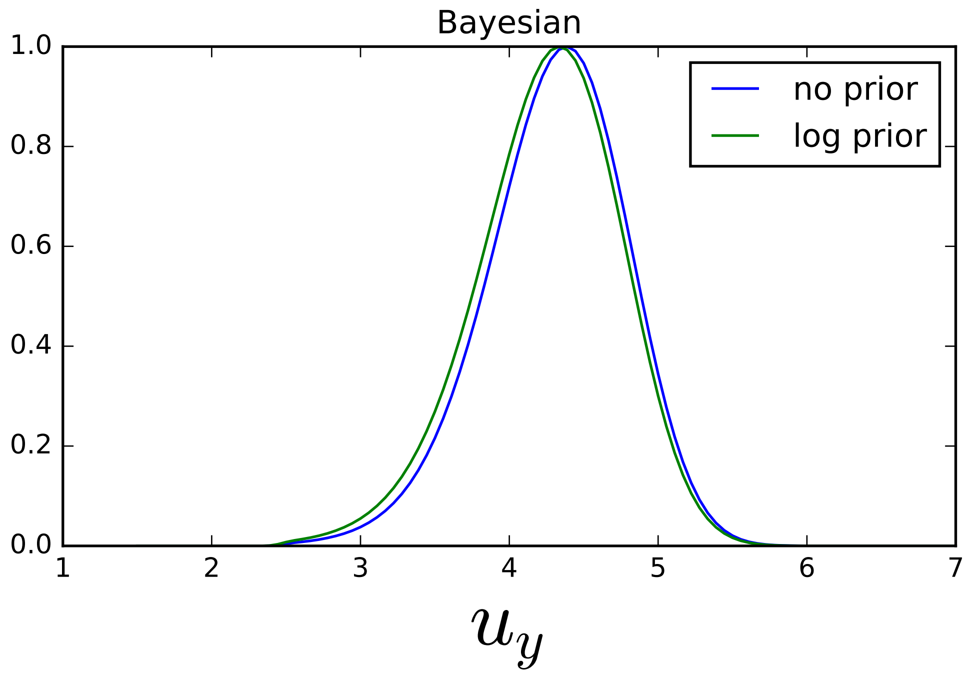

Figure 5: The marginalized probability of u_y. The marginalization over the other two parameters was done numerically, however one may perform the marginalization over u_x analytically in this exercise.

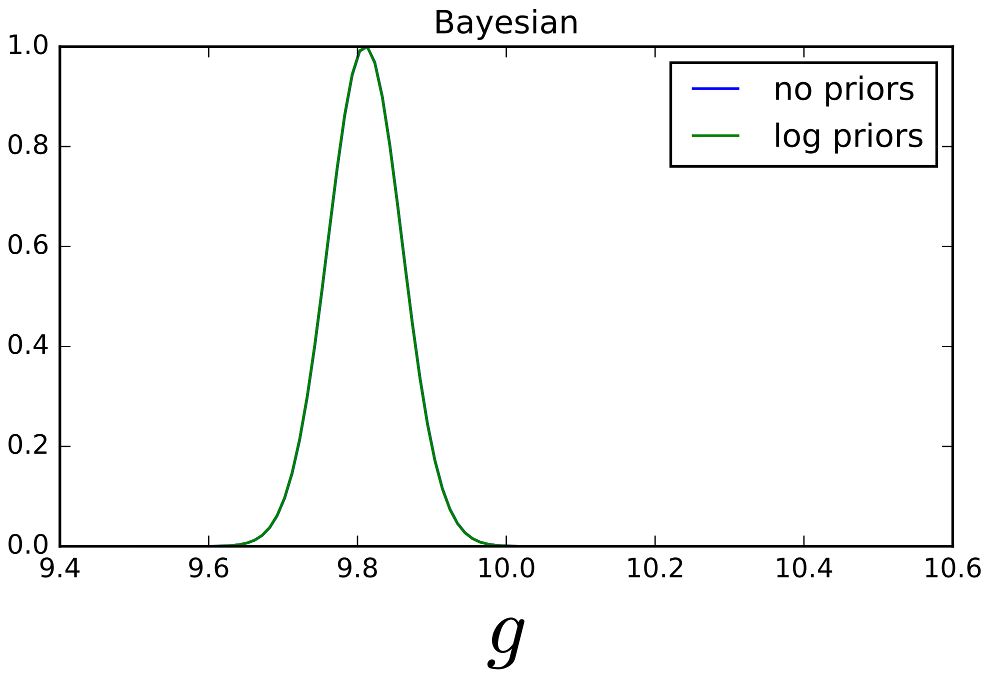

Figure 6: The marginalized probability of g. The marginalization over the other two parameters was done numerically, however one may perform the marginalization over u_x analytically in this exercise.