Astr 513: Homework 3

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1 Linear models of data with correlated uncertainties

The definition for Δ_i will not change when $\rho_i \neq 0$, as it does not depend on the uncertainties. So plugging in for $\vec{u}^\top$, $\vec{Z}_i$, and $\cos\theta$, we have

$$\Delta_i = \frac{1}{\sqrt{1+b^2}} \left(\begin{bmatrix} -b & 1 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \end{bmatrix} - a \right), \tag{1}$$

and simplifying and squaring gives

$$\Delta_i^2 = \frac{(y_i - a - bx_i)^2}{1+b^2}. \tag{2}$$

However, Σ_i^2 is defined with the covariance matrix, which will change when the uncertainties are correlated. We have that $\Sigma_i^2 = \vec{u}^\top \mathbf{S_i} \vec{u}$, and the covariance matrix for correlated uncertainties with $\sigma_{xyi} = \rho_i \sigma_{xi} \sigma_{yi} \neq 0$ is given as, for the data point (x_i, y_i),

$$\mathbf{S_i} = \begin{bmatrix} \sigma_{xi}^2 & \rho_i \sigma_{xi} \sigma_{yi} \\ \rho_i \sigma_{xi} \sigma_{yi} & \sigma_{yi}^2 \end{bmatrix}. \tag{3}$$

Thus, we find

$$\begin{aligned} \Sigma_i^2 &= \frac{1}{1+b^2} \begin{bmatrix} -b & 1 \end{bmatrix} \begin{bmatrix} \sigma_{xi}^2 & \rho_i \sigma_{xi} \sigma_{yi} \\ \rho_i \sigma_{xi} \sigma_{yi} & \sigma_{yi}^2 \end{bmatrix} \begin{bmatrix} -b \\ 1 \end{bmatrix} \\ &= \frac{1}{1+b^2} \begin{bmatrix} \sigma_{xi} \left(-b\sigma_{xi} + \rho_i \sigma_{yi} \right) & \sigma_{yi} \left(-b\rho_i \sigma_{xi} + \sigma_{yi} \right) \end{bmatrix} \begin{bmatrix} -b \\ 1 \end{bmatrix} \\ &= \frac{1}{1+b^2} \left(b^2 \sigma_{xi}^2 - b\rho_i \sigma_{xi} \sigma_{yi} - b\rho_i \sigma_{xi} \sigma_{yi} + \sigma_{yi}^2 \right) \\ \Rightarrow \Sigma_i^2 &= \frac{\sigma_{yi}^2 - 2b\rho_i \sigma_{xi} \sigma_{yi} + b^2 \sigma_{xi}^2}{1+b^2}. \end{aligned} \tag{4}$$

When we then take $\Delta_i^2 / 2\Sigma_i^2$, we will see the factor of $1+b^2$ cancel from both terms, and are left with

$$\boxed{\frac{\Delta_i^2}{2\Sigma_i^2} = \frac{(y_i - a - bx_i)^2}{2\left(\sigma_{yi}^2 - 2b\rho_i \sigma_{xi} \sigma_{yi} + b^2 \sigma_{xi}^2 \right)}} \tag{5}$$

2 The Tully-Fisher relation