…if it was so, it might be; and if it were so, it would be: but as it isn't, it ain't. That's logic…

--Tweedledee in *Through the Looking-Glass* by Lewis Carroll (1832-1898)

CSCE 222
Discrete Structures

Logic – Part 2

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Mathematical Reasoning

Rosen, 8^{th} Edition, section 1.7

Mathematical Reasoning

We need **mathematical reasoning** to

- determine whether a mathematical argument is correct or incorrect and
- construct mathematical arguments.

Mathematical reasoning is not only important for conducting **proofs** and **program verification**, but also for **artificial intelligence** systems (drawing logical inferences from knowledge and facts).

We focus on **deductive** proofs

Terminology (1.7.2)

An **axiom** is a basic assumption about mathematical structure that needs no proof.

- Things known to be true (facts or proven theorems)
- Things believed to be true but cannot be proved

We can use a **proof** to demonstrate that a particular statement is true. A proof consists of a sequence of statements that form an argument.

The steps that connect the statements in such a sequence are the **rules of inference**.

Cases of incorrect reasoning are called **fallacies**.

Terminology

A **theorem** is a statement that can be shown to be true.

A **lemma** is a simple theorem used as an intermediate result in the proof of another theorem.

A **corollary** is a proposition that follows directly from a theorem that has been proved.

A **conjecture** is a statement whose truth value is unknown. Once it is proven, it becomes a theorem.

Proofs

A **theorem** often has two parts

- Conditions (premises, hypotheses)
- conclusion

A **correct (deductive) proof** is to establish that

- If the conditions are true then the conclusion is true
- I.e., Conditions $\rightarrow$ conclusion is a tautology

Often there are missing pieces between conditions and conclusion.
Fill it by an argument

- Using conditions and axioms
- Statements in the argument connected by proper rules of inference

Valid Arguments
(reminder)

Recall:

An argument is a sequence of propositions. The final proposition is called the conclusion of the argument while the other propositions are called the premises or hypotheses of the argument.

An argument is valid whenever the truth of all its premises implies the truth of its conclusion.

How to show that **q** logically follows from the hypotheses $(p_1 \wedge p_2 \wedge \ldots \wedge p_n)$?

Show that

$$(p_1 \wedge p_2 \wedge \ldots \wedge p_n) \rightarrow q \quad \text{is a tautology}$$

One can use the rules of inference to show the validity of an argument.

Vacuous proof - if one of the premises is false then $(p_1 \wedge p_2 \wedge \ldots \wedge p_n) \rightarrow q$ is vacuously True, since False implies anything.

Arguments involving universally quantified variables

Note: Many theorems involve statements for universally quantified variables:

e.g., the following statements are equivalent:

- "If $x>y$, where x and y are positive real numbers, then $x^2 > y^2$"
- "$\forall x \forall y$ (if $x > y > 0$ then $x^2 > y^2$)"

Quite often, when it is clear from the context, theorems are proved without explicitly using the laws of universal instantiation and universal generalization.

Methods of Proof (1.7.4)

- Direct Proof
- Proof by Contraposition
- Proof by Contradiction
- Proof of Equivalences
- Proof by Cases
- Exhaustive Proof
- Existence Proofs
- Uniqueness Proofs
- Counterexamples

Direct Proof (1.7.5)

Proof of a statement $p \to q$

Assume p

From p derive q.

Direct Proof

Direct proof:

An implication $p \rightarrow q$ can be proved by showing that if p is true, then q is also true.

Example: Give a direct proof of the theorem
"If n is odd, then n^2 is odd."

Idea: Assume that the hypothesis of this implication is true (n is odd). Then use rules of inference and known theorems of math to show that q must also be true (n^2 is odd).

Example - direct proof

Here's what you know:

Premises:

Mary is a Math major or a CS major.

If Mary does not like discrete math, she is not a CS major.

If Mary likes discrete math, she is smart.

Mary is not a math major.

Can you conclude Mary is smart?

Let

M - Mary is a Math major

C – Mary is a CS major

D – Mary likes discrete math

S – Mary is smart

Informally, what's the chain of reasoning?

$$((M \vee C) \wedge (\neg D \rightarrow \neg C) \wedge (D \rightarrow S) \wedge (\neg M)) \rightarrow S$$

?

Example - direct proof

In general, to prove $p \rightarrow q$, assume p is true and show that q must also be true

$$((M \vee C) \wedge (\neg D \rightarrow \neg C) \wedge (D \rightarrow S) \wedge (\neg M)) \rightarrow S$$

?

- Since, p is a conjunction of all the premises, we instead make the equivalent assumption that all of the following premises are true
  - $M \vee C$
  - $\neg D \rightarrow \neg C$
  - $D \rightarrow S$
  - $\neg M$
- Then the truth of these premises are used to prove S is true

Example - direct proof

1. $M \vee C$	Given
2. $\neg D \rightarrow \neg C$	Given
3. $D \rightarrow S$	Given
4. $\neg M$	Given
5. C	Disjunctive Syllogism (1,4)
6. D	Modus Tollens (2,5)
7. S	Modus Ponens (3,6)

Mary is smart! QED

QED or Q.E.D. --- quod erat demonstrandum

"which was to be demonstrated"

or "I rest my case" ☺

Direct Proof

(Reminder) Direct proof:

An implication $p \rightarrow q$ can be proved by showing that if p is true, then q is also true.

Example: Give a direct proof of the theorem
"If n is odd, then n^2 is odd."

Idea: Assume that the hypothesis of this implication is true (n is odd). Then use rules of inference and known theorems of math to show that q must also be true (n^2 is odd).

Proving Theorems

n is odd.

Then $n = 2k + 1$, where k is an integer.

Consequently, $n^2 = (2k + 1)^2.$

$$\begin{aligned} &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1 \end{aligned}$$

Since n^2 can be written in this form, it is odd.

Proving Theorems (1.7.6)

Indirect proof:

An implication $p \rightarrow q$ is equivalent to its **contra-positive** $\neg q \rightarrow \neg p$. Therefore, we can prove $p \rightarrow q$ by showing that whenever q is false, then p is also false.

Example: Give an indirect proof of the theorem "If $3n + 2$ is odd, then n is odd."

Idea: Assume that the conclusion of this implication is false (n is even). Then use rules of inference and known theorems to show that p must also be false ($3n + 2$ is even).

Proving Theorems

n is even.

Then $n = 2k$, where k is an integer.

It follows that
$$\begin{aligned} 3n + 2 &= 3(2k) + 2 \\ &= 6k + 2 \\ &= 2(3k + 1) \end{aligned}$$

Therefore, $3n + 2$ is even.

We have shown that the contrapositive of the implication is true, so the implication itself is also true (If $3n + 2$ is odd, then n is odd).

Proving Theorems (1.7.7)

Indirect Proof is a special case of proof by contradiction

Suppose n is even (negation of the conclusion).

Then $n = 2k$, where k is an integer.

It follows that
$$\begin{aligned} 3n + 2 &= 3(2k) + 2 \\ &= 6k + 2 \\ &= 2(3k + 1) \end{aligned}$$

Therefore, $3n + 2$ is even.

However, this is a contradiction since $3n + 2$ is given to be odd, so the conclusion (n is odd) holds.

Proof by Contradiction

A – We want to prove p.

We show that:

(1) $\neg p \rightarrow \mathbf{F}$ (i.e., a **False** statement , say $r \wedge \neg r$)

(2) We conclude that $\neg p$ is false since (1) is **True** and therefore p is **True**.

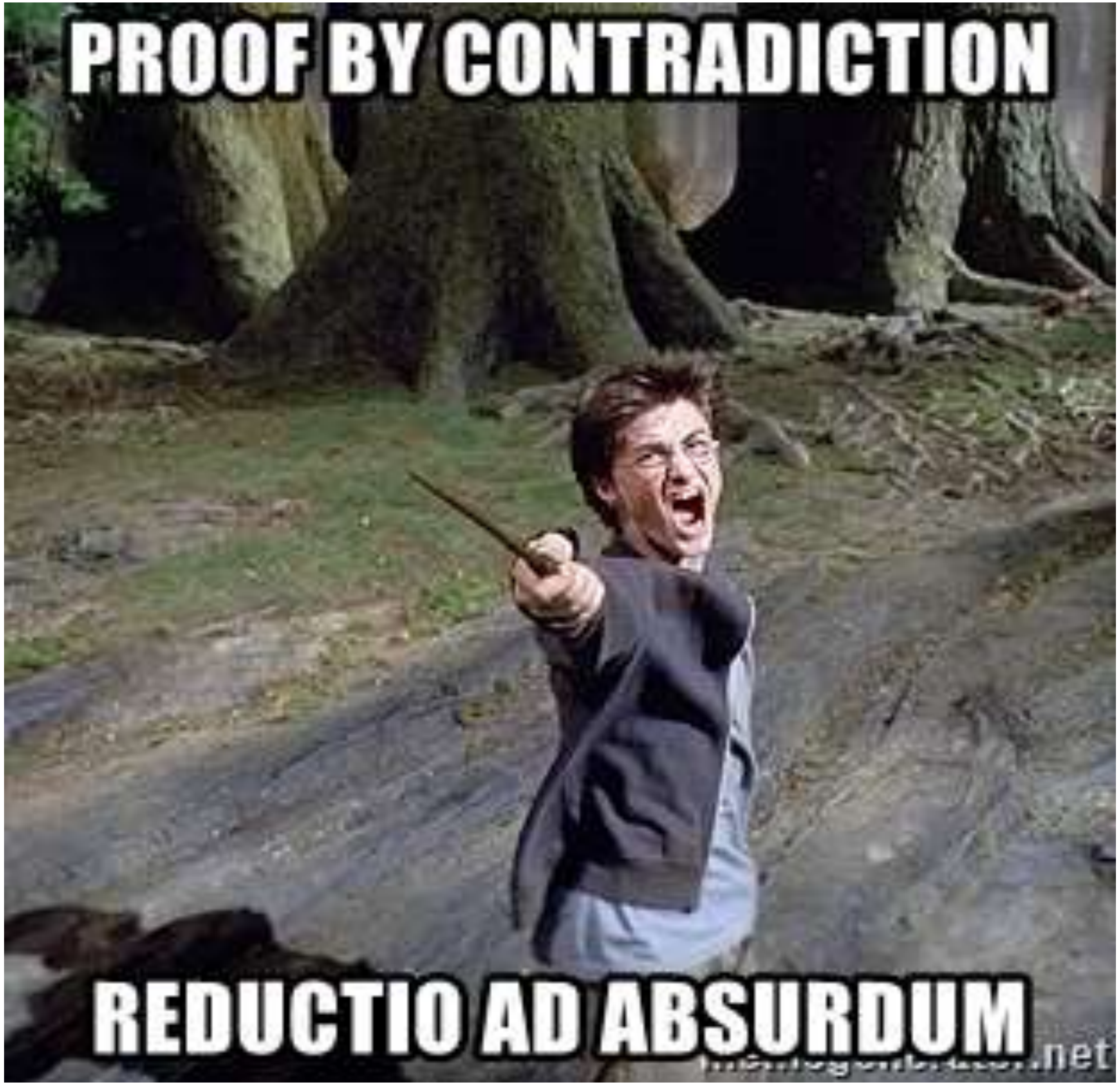
B – We want to show $p \rightarrow q$

(1) Assume the negation of the conclusion, i.e., $\neg q$

(2) Show that $(p \wedge \neg q) \rightarrow \mathbf{F}$

(3) Since $((p \wedge \neg q) \rightarrow \mathbf{F}) \Leftrightarrow (p \rightarrow q)$ (why?) we are done

$$
\begin{aligned}
((p \wedge \neg q) \rightarrow \mathbf{F}) &\Leftrightarrow \neg(p \wedge \neg q) \\
&\Leftrightarrow p \rightarrow q
\end{aligned}
$$



Section 1.7.7

Example: Proof by Contradiction

Classic proof that $\sqrt{2}$ is irrational.

It's quite clever!!

Suppose $\sqrt{2}$ is rational. Then $\sqrt{2} = a/b$ for some integers a and b (relatively prime, no factor in common).

Note: Here we again first go to the definition of concepts ("rational"). Makes sense! Definitions provide information about important concepts. In a sense, math is all about "What follows from the definitions and premises!

$\sqrt{2} = a/b$ implies

$2 = a^2/b^2$

$2b^2 = a^2$

a^2 is even, and so a is even ($a = 2k$ for some k)

$2b^2 = (2k)^2 = 4k^2$

$b^2 = 2k^2$

b^2 is even, and so b is even ($b = 2k$ for some k)

But if a and b are both even, then they are not relatively prime!

Q.E.D.

Example2: Proof by Contradiction

You're going to let me get away with that? ☺

Lemma: a^2 is even implies that a is even (i.e., $a = 2k$ for some k)??

Suppose to the contrary that a is not even.

Then $a = 2k + 1$ for some integer k

Then $a^2 = (2k + 1)(2k + 1) = 4k^2 + 4k + 1$

and a^2 is odd.

Then, as discussed earlier, a^2 is not even

contradiction

So, a really is even.

Corollary: An integer n is even if and only if n^2 is even

Why does the above statement follow immediately from previous work???

Example 3: Proof by Contradiction

Theorem:

"There are infinitely many prime numbers"

(Euclid's proof, c 300 BC)
One of the most famous early proofs. An early intellectual "tour the force".

Proof by contradiction

Let P – "There are infinitely many primes"

- Assume ¬P, i.e., "there is a finite number of primes" , call largest p_r.
- Let's define R the product of all the primes, i.e, R = p_1 × p_2 × … × p_r.
- Consider R + 1. *(Clever "trick". The key to the proof.)*
- Now, R+1 is either prime or not:
  - If it's prime, we have prime larger than p_r.
  - If it's not prime, let p* be a prime dividing (R+1). But p* cannot be any of p_1, p_2, … p_r (remainder 1 after division); so, p* not among initial list and thus p* is larger than p_r.
- This contradicts our assumption that there is a finite set of primes, and therefore such an assumption has to be false which means that there are infinitely many primes.

See e.g. http://primes.utm.edu/notes/proofs/infinite/euclids.html

Example 4: Proof by Contradiction

Theorem “If 3n+2 is odd, then n is odd”

Let p = “3n+2 is odd” and q = “n is odd”

1 – assume p and ¬q i.e., 3n+2 is odd and n is not odd

2 – because n is not odd, it is even

3 – if n is even, n = 2k for some k, and therefore 3n+2 = 3 (2k) + 2 = 2 (3k + 1), which is even

4 – So, we have a contradiction, 3n+2 is odd and 3n+2 is even.

Therefore, we conclude $p \rightarrow q$, i.e., “If 3n+2 is odd, then n is odd”

Q.E.D.

Proof of Equivalences

To prove $p \leftrightarrow q$

show that $p \rightarrow q$

and $q \rightarrow p$.

The validity of this proof results from the fact that

$$(p \leftrightarrow q) \leftrightarrow [(p \rightarrow q) \wedge (q \rightarrow p)] \text{ is a tautology}$$

Counterexamples

Show that $\forall(x)\ P(x)$ is false

We need only to find a counterexample.

Counterexample

Show that the following statement is false:

"Every day of the week is a weekday"

Proof:

Saturday and Sunday are weekend days. ☺

Proof by Cases

To show

$$(p_1 \vee p_2 \vee \ldots \vee p_n) \rightarrow q$$

We use the tautology

$$[(p_1 \vee p_2 \vee \ldots \vee p_n) \rightarrow q] \leftrightarrow [(p_1 \rightarrow q) \wedge (p_2 \rightarrow q) \wedge \ldots \wedge (p_n \rightarrow q)]$$

A particular case of a proof by cases is an exhaustive proof in which all the cases are considered

Theorem

"If n is an integer, then $n^2 \geq n$ "

Proof by cases

Case 1 $n=0$ $0^2 = 0$

Case 2 $n > 0$, i.e., $n \geq 1$. We get $n^2 \geq n$ since we can multiply both sides of the inequality by n, which is positive.

Case 3 $n < 0$. Then $n \times n > 0 \times n$ since n is negative and multiplying both sides of inequality by n changes the direction of the inequality). So, we have $n^2 > 0$ in this case.

In conclusion, $n^2 \geq n$ since this is true in all cases.

Existence Proofs

Existence Proofs:

- Constructive existence proofs
  - Example: "there is a positive integer that is the sum of cubes of positive integers in two different ways"
  - Proof: Show by brute force using a computer $1729 = 10^3 + 9^3 = 12^3 + 1^3$
- Non-constructive existence proofs
  - Example: "$\forall n$ (integers), $\exists p$ so that p is prime, and $p > n$."
  - Proof: Recall proof used to show there were infinitely many primes.
  - Very subtle – does not give an example of such a number, but shows one exists. (Let P = product of all primes $< n$ and consider P+1.)

Uniqueness proofs involve

- Existence proof
- Uniqueness proof

Example - Existence Proof

$\forall n$ (integers), $\exists p$ so that p is prime, and $p > n$.

Proof: Let n be an arbitrary integer, and consider $n! + 1$. If $(n! + 1)$ is prime, we are done since $(n! + 1) > n$. But what if $(n! + 1)$ is composite?

If $(n! + 1)$ is composite then it has a prime factorization, $p_1 p_2 \ldots p_n = (n! + 1)$

Consider the smallest p_i, and call it p. How small can it be?

Can it be 2?

Can it be 3?

Can it be 4?

Can it be n?

So, $p > n$, and we are done. BUT WE DON'T KNOW WHAT p IS!!!

Fallacies

Fallacies are incorrect inferences.
Some common fallacies:

1. The Fallacy of Affirming the Consequent
2. The Fallacy of Denying the Antecedent
3. Begging the question or circular reasoning

Logic: another thing that penguins aren't very good at.

The Fallacy of Affirming the Consequent

If the butler did it he has blood on his hands.
The butler had blood on his hands.
Therefore, the butler did it.

This argument has the form

$$\begin{array}{l} P \to Q \\ \underline{Q \qquad\quad} \\ \therefore\ P \end{array}$$

or $((P \to Q) \wedge Q) \to P$ which is not a tautology and therefore not a valid rule of inference

The Fallacy of Denying the Antecedent

If the butler is nervous, he did it.

The butler is really mellow.

Therefore, the butler didn't do it.

This argument has the form

$$\begin{array}{l} P \to Q \\ \neg P \\ \hline \therefore\ \neg Q \end{array}$$

or $((P \to Q) \wedge \neg P) \to \neg Q$ which is not a tautology and therefore not a valid rule of inference


PLAYER IS BEGGING THE QUESTION
TRIED TO ASSUME THEIR
PREMISE WAS ALREADY CORRECT
imgflip.com

Begging the question or circular reasoning

This occurs when we use the truth of the statement being proved (or something equivalent) in the proof itself.

Example:

Conjecture: *if n^2 is even then n is even.*

Proof: If n^2 is even then $n^2 = 2k$ for some k. Let $n = 2m$ for some m. Hence, x must be even.

Note that the statement $n = 2m$ is introduced without any argument showing it.

Additional Proof Methods Covered in CSCE 222

- Induction Proofs

- Combinatorial proofs

But first we have to cover some basic notions on sets, functions, and counting.