



CSCE 222

Discrete Structures

Recurrence Relations

Dr. Tim McGuire

Grateful acknowledgement to Professor Bart Selman, Cornell University, and Prof. Johnnie Baker, Kent State, for some of the material upon which these notes are adapted.

Learning Objectives

- Learn about recurrence relations
- Learn the relationship between sequences and recurrence relations
- Explore how to solve recurrence relations by iteration
- Learn about linear homogeneous recurrence relations and how to solve them
- Become familiar with linear nonhomogeneous recurrence relations

Recurrence Relations

A **recurrence relation** for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, a_0, a_1, \dots, a_{n-1} , for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer.

A sequence is called a **solution** of a recurrence relation if its terms satisfy the recurrence relation.

Recurrence Relations

In other words, a recurrence relation is like a recursively defined sequence, but **without specifying any initial values (initial conditions)**.

Therefore, the same recurrence relation can have (and usually has) **multiple solutions**.

If **both** the initial conditions and the recurrence relation are specified, then the sequence is **uniquely** determined.

Recurrence Relations

Example:

Consider the recurrence relation

$$a_n = 2a_{n-1} - a_{n-2} \text{ for } n = 2, 3, 4, \dots$$

Is the sequence $\{a_n\}$ with $a_n = 3n$ a solution of this recurrence relation?

For $n \geq 2$ we see that

$$2a_{n-1} - a_{n-2} = 2(3(n-1)) - 3(n-2) = 3n = a_n.$$

Therefore, $\{a_n\}$ with $a_n = 3n$ is a solution of the recurrence relation.

Recurrence Relations

Is the sequence $\{a_n\}$ with $a_n=5$ a solution of the same recurrence relation?

For $n \geq 2$ we see that

$$2a_{n-1} - a_{n-2} = 2 \cdot 5 - 5 = 5 = a_n.$$

Therefore, $\{a_n\}$ with $a_n=5$ is also a solution of the recurrence relation.

Modeling with Recurrence Relations

Example:

Someone deposits \$10,000 in a savings account at a bank yielding 5% per year with interest compounded annually. How much money will be in the account after 30 years?

Solution:

Let P_n denote the amount in the account after n years. How can we determine P_n on the basis of P_{n-1} ?

Modeling with Recurrence Relations

We can derive the following **recurrence relation**:

$$P_n = P_{n-1} + 0.05P_{n-1} = 1.05P_{n-1}.$$

The initial condition is $P_0 = 10,000$.

Then we have:

$$P_1 = 1.05P_0$$

$$P_2 = 1.05P_1 = (1.05)^2P_0$$

$$P_3 = 1.05P_2 = (1.05)^3P_0$$

...

$$P_n = 1.05P_{n-1} = (1.05)^nP_0$$

We now have a **formula** to calculate P_n for any natural number n and can avoid the iteration.

Modeling with Recurrence Relations

Let us use this formula to find P_{30} under the initial condition $P_0 = 10,000$:

$$P_{30} = (1.05)^{30} \cdot 10,000 = 43,219.42$$

After 30 years, the account contains \$43,219.42.

Modeling with Recurrence Relations

Another example:

Let a_n denote the number of bit strings of length n that do not have two consecutive 0s (“valid strings”). Find a recurrence relation and give initial conditions for the sequence $\{a_n\}$.

Solution:

Idea: The number of valid strings equals the number of valid strings ending with a 0 plus the number of valid strings ending with a 1.

Modeling with Recurrence Relations

Let us assume that $n \geq 3$, so that the string contains at least 3 bits.

Let us further assume that we know the number a_{n-1} of valid strings of length $(n - 1)$.

Then how many valid strings of length n are there, if the string ends with a 1?

There are a_{n-1} such strings, namely the set of valid strings of length $(n - 1)$ with a 1 appended to them.

Note: Whenever we append a 1 to a valid string, that string remains valid.

Modeling with Recurrence Relations

Now we need to know: How many valid strings of length n are there, if the string ends with a **0**?

Valid strings of length n ending with a 0 **must have a 1 as their $(n - 1)$ st bit** (otherwise they would end with 00 and would not be valid).

And what is the number of valid strings of length $(n - 1)$ that end with a 1?

We already know that there are a_{n-1} strings of length n that end with a 1.

Therefore, there are a_{n-2} strings of length $(n - 1)$ that end with a 1.

Modeling with Recurrence Relations

So there are a_{n-2} valid strings of length n that end with a 0 (all valid strings of length $(n - 2)$ with 10 appended to them).

As we said before, the number of valid strings is the number of valid strings ending with a 0 plus the number of valid strings ending with a 1.

That gives us the following **recurrence relation**:

$$a_n = a_{n-1} + a_{n-2}$$

Modeling with Recurrence Relations

What are the **initial conditions**?

$$a_1 = 2 \text{ (0 and 1)}$$

$$a_2 = 3 \text{ (01, 10, and 11)}$$

$$a_3 = a_2 + a_1 = 3 + 2 = 5$$

$$a_4 = a_3 + a_2 = 5 + 3 = 8$$

$$a_5 = a_4 + a_3 = 8 + 5 = 13$$

...

This sequence satisfies the same recurrence relation as the **Fibonacci sequence**.

Since $a_1 = f_3$ and $a_2 = f_4$, we have $a_n = f_{n+2}$.

Solving Recurrence Relations

In general, we would prefer to have an **explicit formula** to compute the value of a_n rather than conducting n iterations.

For one class of recurrence relations, we can obtain such formulas in a systematic way.

Those are the recurrence relations that express the terms of a sequence as **linear combinations** of previous terms.

Solving Recurrence Relations

Definition: A **linear homogeneous recurrence relation of degree k with constant coefficients** is a recurrence relation of the form:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k},$$

Where c_1, c_2, \dots, c_k are real numbers, and $c_k \neq 0$.

A sequence satisfying such a recurrence relation is uniquely determined by the recurrence relation and the k initial conditions

$$a_0 = C_0, a_1 = C_1, a_2 = C_2, \dots, a_{k-1} = C_{k-1}.$$

Solving Recurrence Relations

Examples:

The recurrence relation $P_n = (1.05)P_{n-1}$ is a linear homogeneous recurrence relation of **degree one**.

The recurrence relation $f_n = f_{n-1} + f_{n-2}$ is a linear homogeneous recurrence relation of **degree two**.

The recurrence relation $a_n = a_{n-5}$ is a linear homogeneous recurrence relation of **degree five**.

Solving Recurrence Relations

Basically, when solving such recurrence relations, we try to find solutions of the form $a_n = r^n$, where r is a constant.

$a_n = r^n$ is a solution of the recurrence relation

$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$ if and only if

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \dots + c_k r^{n-k}.$$

Divide this equation by r^{n-k} and subtract the right-hand side from the left:

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \dots - c_{k-1} r - c_k = 0$$

This is called the **characteristic equation** of the recurrence relation.

Solving Recurrence Relations

The solutions of this equation are called the **characteristic roots** of the recurrence relation. (They yield an explicit formula for the sequence.)

Let us consider linear homogeneous recurrence relations of **degree two**.

Theorem 1: Let c_1 and c_2 be real numbers. Suppose that $r^2 - c_1r - c_2 = 0$ has two distinct roots r_1 and r_2 .

Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1a_{n-1} + c_2a_{n-2}$ if and only if $a_n = \alpha_1r_1^n + \alpha_2r_2^n$ for $n = 0, 1, 2, \dots$, where α_1 and α_2 are constants.

See section 2 of chapter 8 in the text for the proof

Solving Recurrence Relations

Example: What is the solution of the recurrence relation

$$a_n = a_{n-1} + 2a_{n-2} \text{ with } a_0 = 2 \text{ and } a_1 = 7 ?$$

Solution: The characteristic equation of the recurrence relation is $r^2 - r - 2 = 0$, which is equivalent to

$$(r - 2)(r + 1) = 0$$

Its roots are $r = 2$ and $r = -1$.

Hence, the sequence $\{a_n\}$ is a solution to the recurrence relation if and only if:

$$a_n = \alpha_1 2^n + \alpha_2 (-1)^n \text{ for some constants } \alpha_1 \text{ and } \alpha_2.$$

Solving Recurrence Relations

Given the equation $a_n = \alpha_1 2^n + \alpha_2 (-1)^n$ and the initial conditions $a_0 = 2$ and $a_1 = 7$, it follows that

$$a_0 = 2 = \alpha_1 + \alpha_2$$

$$a_1 = 7 = \alpha_1 \cdot 2 + \alpha_2 \cdot (-1)$$

Solving these two equations gives us

$$\alpha_1 = 3 \text{ and } \alpha_2 = -1.$$

Therefore, the solution to the recurrence relation and initial conditions is the sequence $\{a_n\}$ with

$$a_n = 3 \cdot 2^n - (-1)^n.$$

Solving Recurrence Relations

Example: Give an explicit formula for the Fibonacci numbers.

Solution: The Fibonacci numbers satisfy the recurrence relation

$f_n = f_{n-1} + f_{n-2}$ with initial conditions $f_0 = 0$ and $f_1 = 1$.

The characteristic equation is $r^2 - r - 1 = 0$.

Its roots are

$$r_1 = \frac{1 + \sqrt{5}}{2}, \quad r_2 = \frac{1 - \sqrt{5}}{2}$$

Solving Recurrence Relations

Therefore, the Fibonacci numbers are given by

$$f_n = \alpha_1 \left(\frac{1 + \sqrt{5}}{2} \right)^n + \alpha_2 \left(\frac{1 - \sqrt{5}}{2} \right)^n$$

for some constants α_1 and α_2 .

We can determine values for these constants so that the sequence meets the conditions $f_0 = 0$ and $f_1 = 1$:

$$\begin{aligned} f_0 &= \alpha_1 + \alpha_2 = 0 \\ f_1 &= \alpha_1 \left(\frac{1 + \sqrt{5}}{2} \right) + \alpha_2 \left(\frac{1 - \sqrt{5}}{2} \right) = 1 \end{aligned}$$

Solving Recurrence Relations

The unique solution to this system of two equations and two variables is

$$\alpha_1 = \frac{1}{\sqrt{5}}, \quad \alpha_2 = -\frac{1}{\sqrt{5}}$$

So finally we obtained an explicit formula for the Fibonacci numbers:

$$f_n = \frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1 - \sqrt{5}}{2} \right)^n$$

Solving Recurrence Relations

But what happens if the characteristic equation has only one root?

How can we then match our equation with the initial conditions a_0 and a_1 ?

Theorem: Let c_1 and c_2 be real numbers with $c_2 \neq 0$. Suppose that $r^2 - c_1r - c_2 = 0$ has only one root r_0 .

A sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1a_{n-1} + c_2a_{n-2}$ if and only if $a_n = \alpha_1r_0^n + \alpha_2nr_0^n$, for $n = 0, 1, 2, \dots$, where α_1 and α_2 are constants.

Solving Recurrence Relations

Example: What is the solution of the recurrence relation $a_n = 6a_{n-1} - 9a_{n-2}$ with $a_0 = 1$ and $a_1 = 6$?

Solution: The only root of $r^2 - 6r + 9 = 0$ is $r_0 = 3$. Hence, the solution to the recurrence relation is

$a_n = \alpha_1 3^n + \alpha_2 n 3^n$ for some constants α_1 and α_2 .

To match the initial condition, we need

$$a_0 = 1 = \alpha_1$$

$$a_1 = 6 = \alpha_1 \cdot 3 + \alpha_2 \cdot 3$$

Solving these equations yields $\alpha_1 = 1$ and $\alpha_2 = 1$.

Consequently, the overall solution is given by

$$a_n = 3^n + n 3^n.$$

Solving Recurrence Relations

Example: What is the solution of the recurrence relation
 $a_n = 2a_{n-1}$ with $a_0 = 5$?

Solution: This is a linear homogeneous recurrence relation of order 1. The characteristic equation is $r - 2 = 0$ with a root of 2. Hence, the solution to the recurrence relation is

$$a_n = \alpha 2^n \text{ for some constant } \alpha.$$

To match the initial condition, we need

$$a_0 = 5 = \alpha$$

Consequently, the overall solution is given by

$$a_n = 5 \cdot 2^n$$

What about nonlinear recurrences?

- These require different techniques, and the general solution is beyond the scope of this course. But we can solve some using forward substitution. For example:

$$a_n = na_{n-1} \text{ with } a_0 = 2$$

$$a_1 = 1 \cdot 2 = 2$$

$$a_2 = 2 \cdot 2 = 4 = 2 \cdot 2!$$

$$a_3 = 3 \cdot 4 = 12 = 2 \cdot 3!$$

$$a_4 = 4 \cdot 12 = 48 = 2 \cdot 4!$$

So the general solution to $a_n = na_{n-1}$ is $a_n = a_0 \cdot n!$