CSCE 222
Discrete Structures

Functions

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Grateful acknowledgement to Professor Bart Selman, Cornell University, and Prof. Johnnie Baker, Kent State, for some of the material upon which these notes are adapted.

Based on Chapter 2 of Rosen

Discrete Mathematics and its Applications

Functions

A function f from a set A to a set B is an assignment of exactly one element of B to each element of A.

We write

$$f(a) = b$$

if b is the unique element of B assigned by the function f to the element a of A.

If f is a function from A to B, we write

$$f: A \to B$$

(note: Here, "$\to$" has nothing to do with if… then)

Note: Functions are also called **mappings** or **transformations.**

Functions

If $f:A \to B$, we say that A is the domain of f and B is the codomain of f.

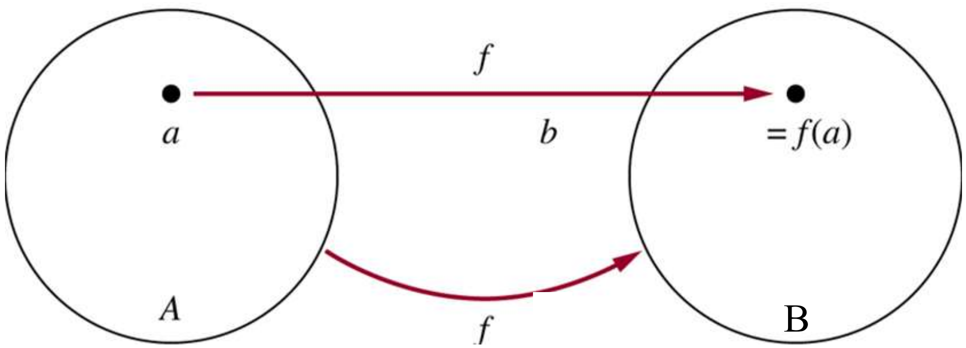
If $f(a) = b$, we say that b is the image of a and a is the pre-image of b.

The range of $f:A \to B$ is the set of all images of all elements of A.

We say that $f:A \to B$ maps A to B.

Functions

More generally:



A - Domain of f

B- Co-Domain of f

For example: f: R→R, f(x) = -(1/2)x – 1/2

domain

co-domain

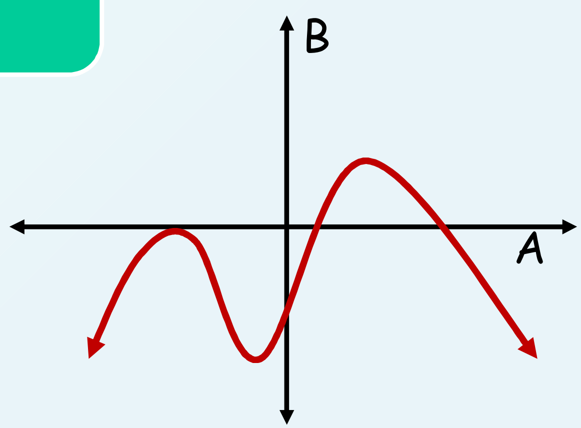
Functions

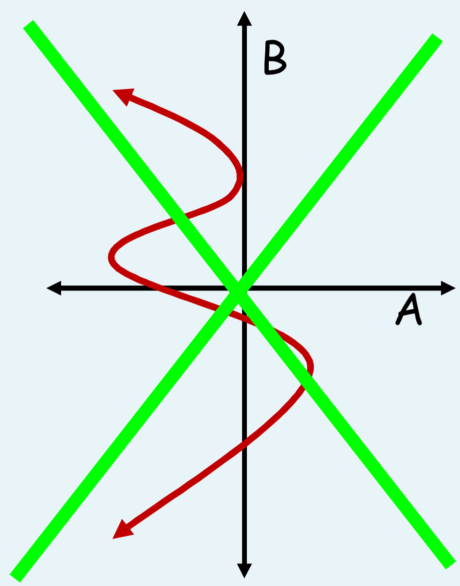
a collection of points!

More formally: a function $f : A \to B$ is a subset of $A \times B$ where
$\forall a \in A, \exists! b \in B$ and $\langle a,b \rangle \in f$.

a point!

Remember, this means "there exists a unique …"



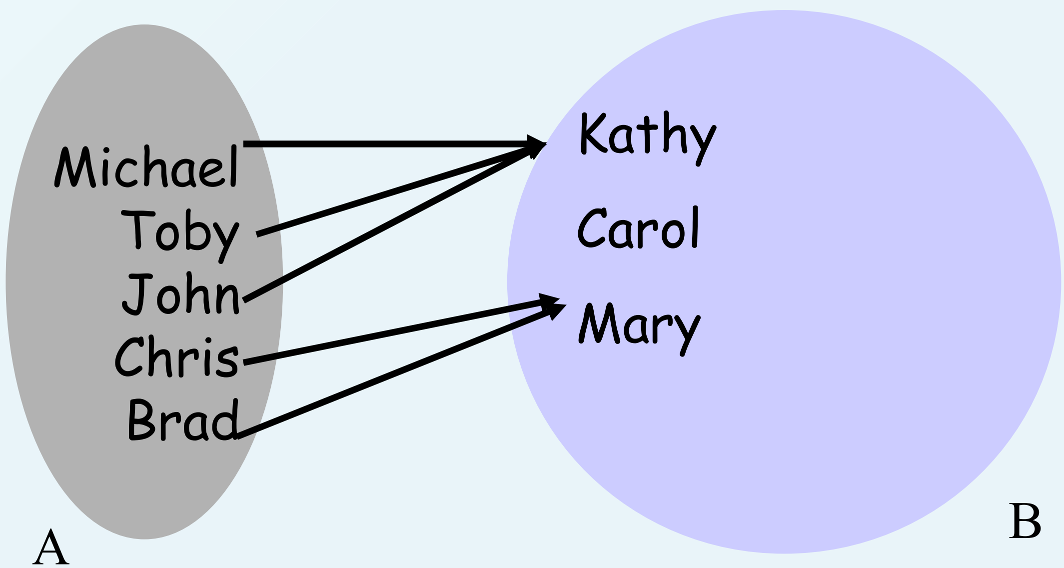


Why not?

Functions

A = {Michael, Toby , John , Chris , Brad }
B = { Kathy, Carla, Mary}

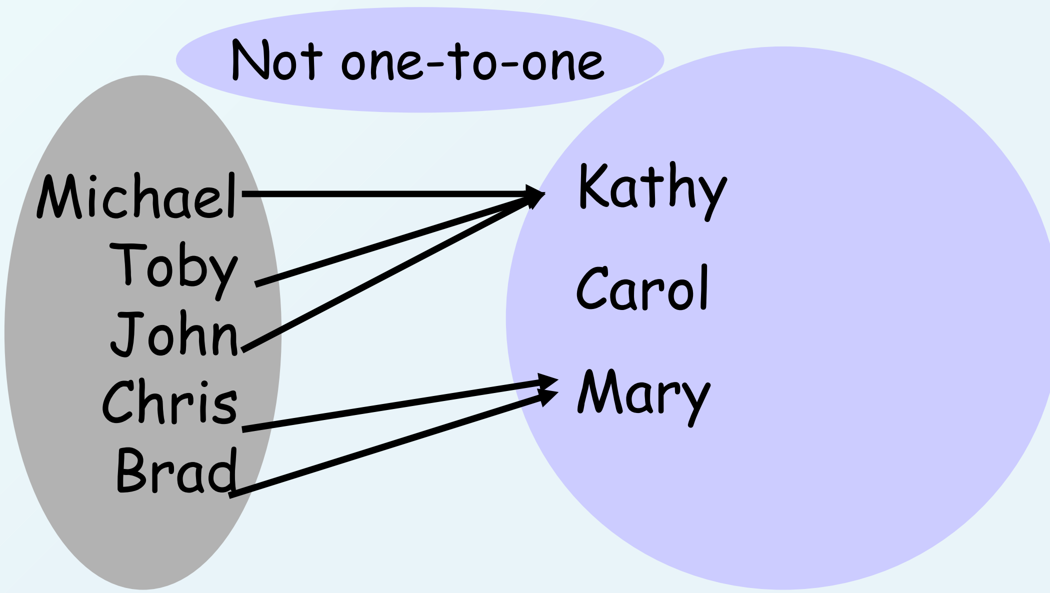
Let f: $A \rightarrow B$ be defined as f(a) = mother(a).



Functions - injection

Every $b \in B$ has at most 1 preimage.

A function $f: A \rightarrow B$ is one-to-one (injective, an injection) if
$\forall a,b,c, (f(a) = b \wedge f(c) = b) \rightarrow a = c$



In other words: f is **one-to-one** if and only if it does not map two distinct elements of A onto the same element of B.

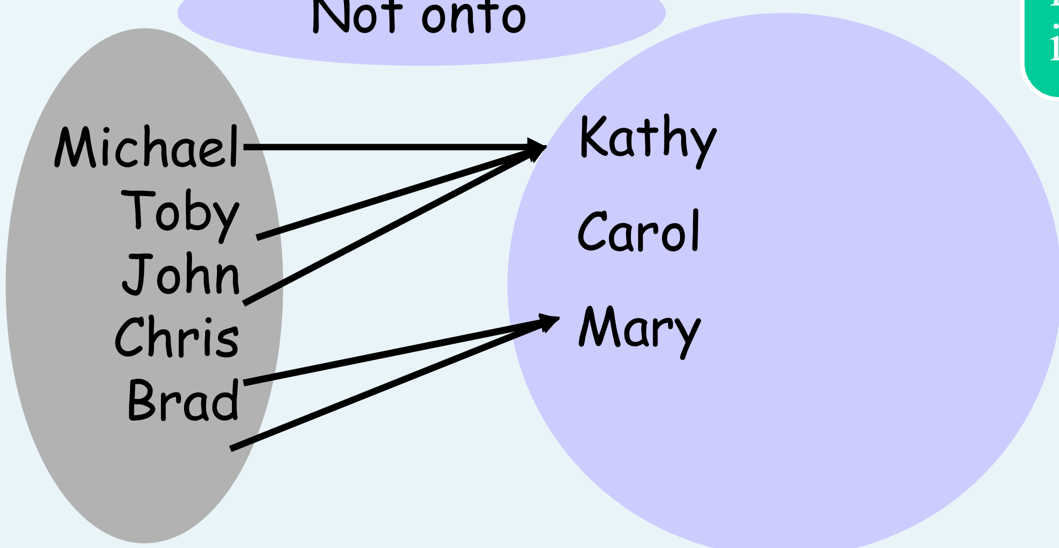
Functions - surjection

Every b $\in$ B has at least 1 preimage.

A function f: $A \rightarrow B$ is onto (surjective, a surjection) if $\forall b \in B, \exists a \in A\ f(a) = b$

In other words, f is **onto** if and only if its range is its entire codomain.

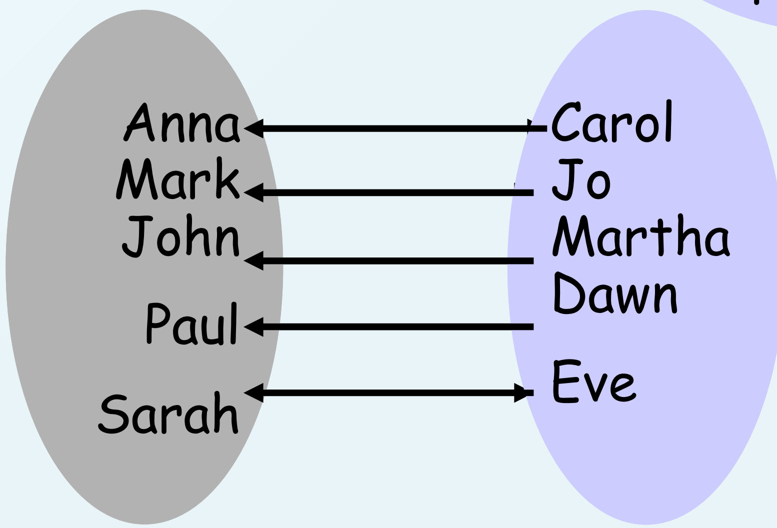
Not onto



Functions – one-to-one-correspondence or bijection

A function $f: A \to B$ is bijective if it is one-to-one and onto.

Every $b \in B$ has exactly 1 preimage.

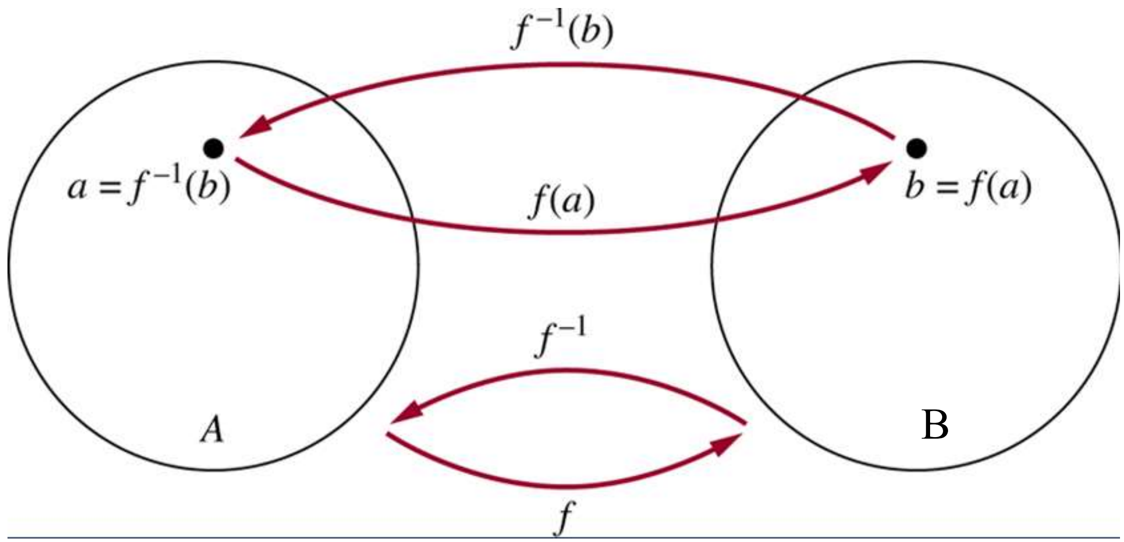


An important implication of this characteristic: The preimage (f^{-1}) is a function! They are **invertible**.

Functions: inverse function

Definition:

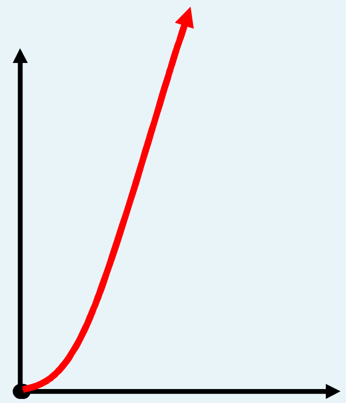
Given f, a one-to-one correspondence from set A to set B, the **inverse function of f** is the function that assigns to an element b belonging to B the unique element a in A such that f(a)=b. The inverse function is denoted f^{-1} . f^{-1} (b)=a, when f(a)=b.



Functions - examples

Suppose $f: R^+ \rightarrow R^+$, $f(x) = x^2$.

Is f one-to-one? yes
Is f onto? yes
Is f bijective? yes



This function is invertible.

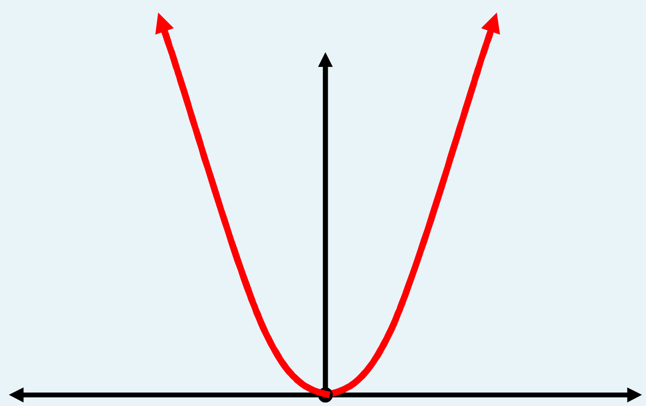
Functions - examples

Suppose $f: R \to R^+$, $f(x) = x^2$.

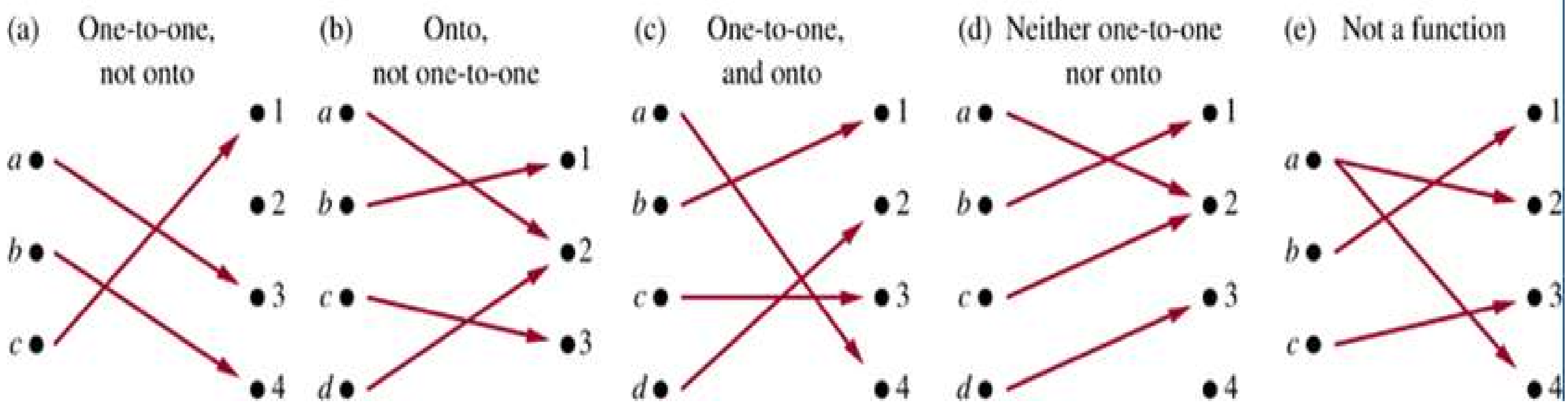
Is f one-to-one? no

Is f onto? yes

Is f bijective? no



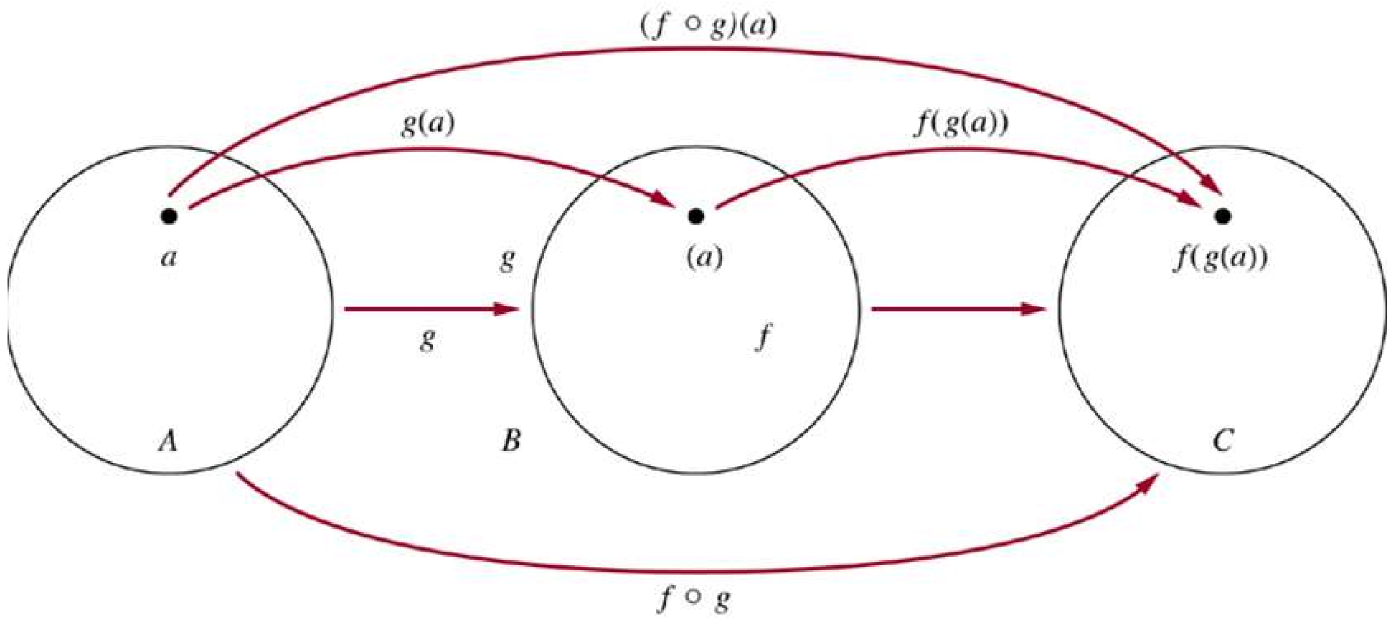
This function is not invertible.


(a) One-to-one, not onto
(b) Onto, not one-to-one
(c) One-to-one, and onto
(d) Neither one-to-one nor onto
(e) Not a function
a
b
c
d
1
2
3
4

Functions - composition

Let f: A→B, and g: B→C be functions. Then the ***composition*** of f and g is:

$(f \circ g)(x) = f(g(x))$



Note: $(f \circ g)$ cannot be defined unless the range of g is a subset of the domain of f.

Example:

Let $f(x) = 2x + 3$; $g(x) = 3x + 2$;

$(f \circ g)(x) = f(3x + 2) = 2(3x + 2) + 3 = 6x + 7.$

$(g \circ f)(x) = g(2x + 3) = 3(2x + 3) + 2 = 6x + 11.$

As this example shows, $(f \circ g)$ and $(g \circ f)$ are not necessarily equal – i.e, the composition of functions is not commutative.

Composition

Composition of a function and its inverse:

$(f^{-1} \circ f)(a) = f^{-1}(f(a)) = f^{-1}(b) = a.$

$(f \circ f^{-1})(b) = f(f^{-1}(b)) = f(a) = b$

$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = x$

The composition of a function and its inverse is the **identity function** $I(x) = x$.

Graphs

The **graph** of a function $f:A\rightarrow B$ is the set of ordered pairs $\{(a, b) \mid a \in A \text{ and } f(a) = b\}$.

The graph is a subset of $A \times B$ that can be used to visualize f in a two-dimensional coordinate system.

Some important functions

Absolute value:

Domain R; Co-Domain = $\{0\} \cup R^+$

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Ex: $|-3| = 3$; $|3| = 3$

Floor function (or greatest integer function):

Domain = R; Co-Domain = Z

$\lfloor x \rfloor$ = largest integer not greater than x

Ex: $\lfloor 3.2 \rfloor = 3$; $\lfloor -2.5 \rfloor = -3$

Some important functions

Ceiling function:

Domain = R;

Co-Domain = Z

$\lceil x \rceil$ = smallest integer greater than or equal to x

Ex: $\lceil 3.2 \rceil = 4$; $\lceil -2.5 \rceil = -2$

Table 1, Section 2-3

TABLE 1 Useful Properties of the Floor and Ceiling Functions. (*n* is an integer)
(1a) $\lfloor x \rfloor = n$ if and only if $n \leq x < n + 1$
(1b) $\lceil x \rceil = n$ if and only if $n - 1 < x \leq n$
(1c) $\lfloor x \rfloor = n$ if and only if $x - 1 < n \leq x$
(1d) $\lceil x \rceil = n$ if and only if $x \leq n < x + 1$
(2) $x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x + 1$
(3a) $\lfloor -x \rfloor = -\lceil x \rceil$
(3b) $\lceil -x \rceil = -\lfloor x \rfloor$
(4a) $\lfloor x + n \rfloor = \lfloor x \rfloor + n$
(4b) $\lceil x + n \rceil = \lceil x \rceil + n$

Some important functions

Factorial function: Domain = **N**, Range = $\mathbf{N}^+$

$n! = n\,(n-1)(n-2)\ \ldots,\ 3 \times 2 \times 1$

Ex: $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

Note: $0! = 1$ by convention.

Some important functions

Mod (or remainder):

Domain = $\mathbf{N} \times \mathbf{N}^+ = \{(m,n) \mid m \in \mathbf{N}, n \in \mathbf{N}^+\}$

Co-domain Range = $\mathbf{N}$

$m \; mod \; n = m - \lfloor m/n \rfloor n$

Ex: $8 \bmod 3 = 8 - \lfloor 8/3 \rfloor 3 = 2$

$57 \bmod 12 = 9$;

In C++ this is computed using the **%** operator. That is, to compute m *mod* n, we write `m % n`

Note: This function computes the remainder when m is divided by n.
The name of this function is an abbreviation of m modulo n, where modulus means with respect to a modulus (size) of n, which is defined to be the remainder when m is divided by n. Note also that this function is an example in which the domain of the function is a 2-tuple.

Some important functions: Exponential Function

Exponential function:

Domain = $R^+ \times R$ = $\{(a,x) | a \in R+, x \in R\}$

Co-domain Range = R^+

$f(x) = a^x$

Note: *a* is a **positive** constant; *x* varies.

Ex: $f(n) = a^n = a \times a \times \ldots \times a$ (n times)

How do we define f(x) if x is not a positive integer?

Some important functions: Exponential function

Exponential function:
How do we define f(x) if x is not a positive integer?
Important properties of exponential functions:

(1) $a^{(x+y)} = a^x a^y$; (2) $a^1 = a$; (3) $a^0 = 1$

See:

$$a^2 = a^{1+1} = a^1 a^1 = a \times a;$$
$$a^3 = a^{2+1} = a^2 a^1 = a \times a \times a;$$
$$\ldots$$
$$a^n = a \times \cdots \times a \quad (n\ times)$$

We get: $a = a^1 = a^{1+0} = a \times a^0 \quad therefore \quad a^0 = 1$

$$1 = a^0 = a^{b+(-b)} = a^b \times a^{-b} \quad therefore \quad a^{-b} = 1/a^b$$

$$a = a^1 = a^{\frac{1}{2}+\frac{1}{2}} = a^{\frac{1}{2}} \times a^{\frac{1}{2}} = (a^{\frac{1}{2}})^2 therefore \quad a^{\frac{1}{2}} = \sqrt{a}$$

By similar arguments:

$$a^{\frac{1}{k}} = \sqrt[k]{a}$$

$$a^{mx} = a^x \times \cdots a^x \ (m \quad times) = (a^x)^m, \quad therefore \quad a^{\frac{m}{n}} = (a^{\frac{1}{n}})^m = (\sqrt[n]{a})^m$$

Note: This determines a^x for all x rational. x is irrational by continuity (we'll skip "details").

Some important functions:
Logarithm Function

$\log_a x = y$ **means** $a^y = x$

A one-word definition for *logarithm* is ***exponent***

Logarithm base a:

Domain = $R^+ \times R$ = $\{(a,x) | a \in R+, a>1, x \in R\}$

Co-domain Range = R

$y = \log_a (x) \Leftrightarrow a^y = x$

Ex: $\log_2 (8) = 3$; $\log_2 (16) = 4$; $3 < \log_2 (15) < 4$.

Some important functions: Logarithm Function

$\log_a x = y$ **means** $a^y = x$

A one-word definition for *logarithm* is ***exponent***

Key properties of the log function (they follow from those for exponential):

1. $\log_a (1)=0$ (because $a^0 =1$)
2. $\log_a (a)=1$ (because $a^1 =a$)
3. $\log_a (xy) = \log_a (x) + \log_a (x)$ (similar arguments)
4. $\log_a (x^r) = r \log_a (x)$
5. $\log_a (1/x) = - \log_a (x)$ (note $1/x = x^{-1}$)
6. $\log_b (x) = \log_a (x) / \log_a (b)$

Logarithm Functions

Examples:

$\log_2 (1/4) = -\log_2 (4) = -2.$

$\log_2 (-4)$ undefined

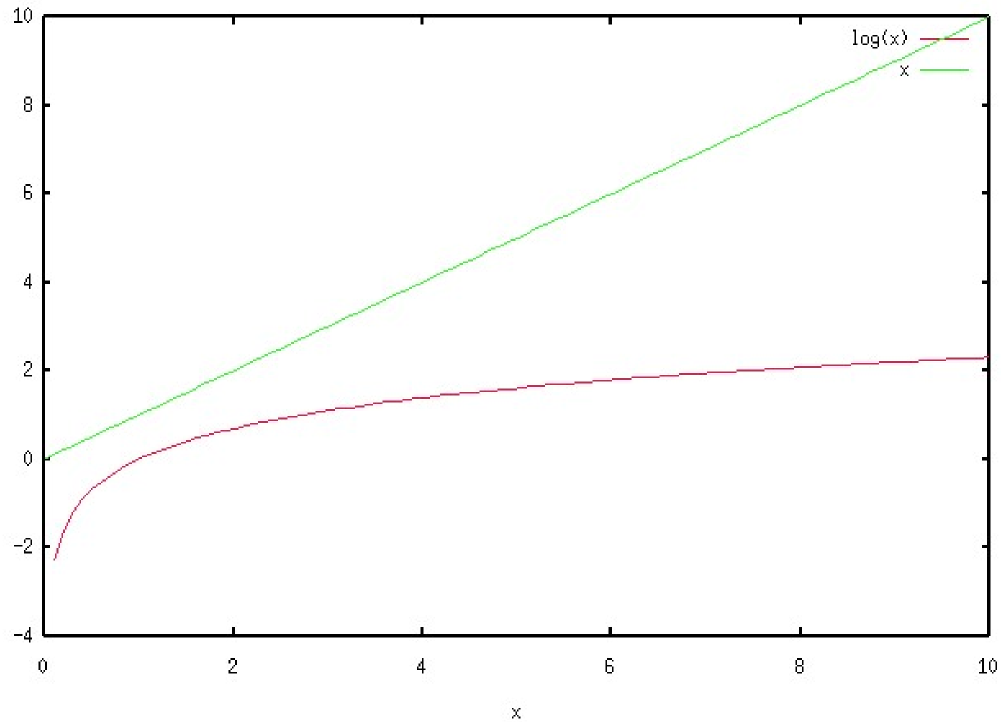
$\log_2 (2^{10}\, 3^5) = \log_2 (2^{10}) + \log_2 (3^5) = 10 \log_2 (2) + 5\log_2 (3)$

$= 10 + 5 \log_2 (3)$

Limit Properties of Log Function

$$\lim_{x \to \infty} \log(x) = \infty$$

$$\lim_{x \to \infty} \frac{\log(x)}{x} = 0$$



As x gets large, *log(x)* grows without bound.
But *x* grows **MUCH** faster than *log(x)*…more soon on growth rates.

Some important functions: Polynomials

Polynomial function:

Domain = usually R

Co-domain Range = usually R

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x^1 + a_0$$

n, a nonnegative integer is the degree of the polynomial; $a_n \neq 0$ (so that the term $a_n x^n$ actually appears)

$(a_n, a_{n-1}, \ldots, a_1, a_0)$ are the coefficients of the polynomial.

Ex:

$y = P_1(x) = a_1 x^1 + a_0$ linear function

$y = P_2(x) = a_2 x^2 + a_1 x^1 + a_0$ quadratic polynomial or function

Exponentials grow MUCH faster than polynomials:

$$\lim_{x \to \infty} \frac{a_0 + \cdots + a_k x^k}{b^x} = 0 \; if \;\; b > 1$$

We’ll talk more about growth in Chapter 3