CSCE 222
Discrete Structures
Logic – Part 1

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Goals of CSCE 222

Introduce students to a range of mathematical tools from discrete mathematics that are key in computer science

Mathematical Sophistication

How to write statements rigorously

How to read and write theorems, lemmas, etc.

How to write rigorous proofs

Practice works!

Actually, *only* practice works!

Areas we will cover:

- Logic and proofs
- Set Theory
- Induction and Recursion
- Counting and combinatorics
- Probability theory
- Number Theory (if time permits)

Note: Learning to do proofs from watching the lecture is like trying to learn to play tennis from watching it on TV! So, do the exercises!

Aside: We're not after the shortest or most elegant proofs; verbose but rigorous is just fine! ☺

Topics CSCE 222

Logic and Methods of Proof

- Propositional Logic --- SAT as an encoding language!
- Predicates and Quantifiers
- Methods of Proofs

Sets

- Sets and Set operations
- Functions

Counting

- Basics of counting
- Pigeonhole principle
- Permutations and Combinations

Number Theory (if time permits)

- Modular arithmetic
- RSA cryptosystems

Topics CSCE 222

Probability

- Probability Axioms, events, random variable
- Independence, expectation, example distributions
- Birthday paradox
- Monte Carlo method

Graphs and Trees

(light coverage if time permits. Covered in Data Structures and Algorithms)

- Graph terminology
- Example of graph problems and algorithms:
  - graph coloring
  - TSP
  - shortest path

Based on Chapter 1 of Rosen

Discrete Mathematics and its Applications

Introduction to Logic

- We will approach logic in a fairly intuitive fashion.
- In this module we will prescribe certain rules of logic which we will follow through the rest of this course.
- They should be self-evident. We won't attempt to prove these rules.
- The rules we present here are universally accepted in mathematics and in most of science and analytic thought.

Introduction to Logic

- We begin with sentential logic and elementary connectives.
- This is called ***propositional logic*** (to distinguish it from *predicate logic*, which will be treated later.)
- In other words, we will be discussiong propositions which are built up from atomic statements and connectives.
- The elementary connectives include “and”, “or”, “not”, “if-then”, and “if-and-only-if”
- Each of these have a precise meaning and will have exact relationships with the other connectives.

Introduction to Logic

- An atomic statement is a sentence with a subject and a verb (and sometimes an object) but no connectives (and, or, etc.)
- For example, these are all atomic statements.
  - Juan is good
  - Mary has bread
  - Lavinia reads books

Introduction to Logic

- Later, we will look at the quantifiers “for all” and “there exists” and their relationships with the connectives.
- These give rise to predicate logic.
- Connectives and quantifiers will prove to be the building blocks for all of of study in this course.

1.1 Propositional Logic

Syntax: Elements of the language

A proposition is a declarative statement that is either true or false (but not both).

Primitive propositions --- statements like*:*

Bob loves Alice $\longrightarrow$ P

Alice loves Bob $\longrightarrow$ Q

Propositional Symbols (atomic propositions)

Compound propositions

Bob loves Alice *and* Alice loves Bob

$\longrightarrow$ $P \wedge Q$ ($\wedge$ - stands for **and**)

Connectives

- $\neg$ - not
- $\wedge$ - and
- $\vee$ - or
- $\rightarrow$ - implies
- $\leftrightarrow$ - equivalent (if and only if)

Syntax

- **Syntax of Well Formed Formulas (wffs) or sentences**

  - Atomic sentences are wffs:
    - Examples: P, Q, R, BlockIsRed; SeasonIsWinter;

  - Complex or compound wffs examples, assuming that w1 and w2 are wwfs:
    - $\neg$ w1 (negation)
    - (w1 $\wedge$ w2) (conjunction)
    - (w1 $\vee$ w2) (disjunction)
    - (w1 $\rightarrow$ w2) (implication; w1 is the antecedent; w2 is the consequent)
    - (w1 $\leftrightarrow$ w2) (biconditional)

Propositional logic: Examples

Additional Examples of wffs

- $P \land Q$
- $(P \lor Q) \to R$
- $P \lor Q \to P$
- $(P \to Q) \to (\neg Q \to \neg P)$
- $\neg \neg P$

Comments:

- Atoms or negated atoms are called <u>*literals*</u>;
  - Examples: p and $\neg$p are literals.
- $P \land Q$ is a compound statement or compound proposition.
- Parentheses are important to ensure that the syntax is unambiguous. Quite often parentheses are omitted;
- The order of precedence in propositional logic is (from highest to lowest): $\neg$,$\land$, $\lor$, $\to$, $\leftrightarrow$

TABLE 8
Precedence of Logical Operators.

Operator	*Precedence*
$\neg$	1
$\wedge$ $\vee$	2 3
$\rightarrow$ $\leftrightarrow$	4 5

Propositional Logic: Syntax vs. Semantics

- Syntax involves whether notation is correctly formed
- Semantics has to do with "meaning":
  - it associates the elements of a logical language with the elements of a domain of discourse.

- Propositional Logic – involves associating *atomic sentences* with *propositions* or *assertions* about the world (therefore called "propositional logic").

Truth Assignment to Propositions

- **Interpretation** or **Truth Assignment**:
- In an application, a truth assignment (True or False) must be made to each proposition.
- So if for **n** atomic propositions, there are $\mathbf{2^n}$ truth assignments or interpretations.
- This makes the representation powerful: the propositions implicitly capture $\mathbf{2^n}$ possible states of the world.

Sematics Example

- We might associate the atom (just a symbol!) **BlockIsRed** with the proposition: “The block is Red”,
- However, we could also associate it with the proposition “The block is Black” even though this would be quite confusing…
- **BlockIsRed** has value **True** just in the case the block is red; otherwise BlockIsRed is False.
  - Computers manipulate symbols. The string “BlockIsRed” does not “mean” anything to the computer.
  - Meaning has to come from how to come from relations to other symbols and the “external world”. Hmm

Sematics Example (cont.)

- How can a computer / robot obtain the meaning ``The block is Red''?
- The fact that computers only "push around symbols" led to quite a bit of confusion in the early days or Artificial Intelligence, Robotics, and natural language understanding.

The Statement/Proposition Game

- "Elephants are bigger than mice."

Is this a statement? yes

Is this a proposition? yes

What is the truth value of the proposition? true

The Statement/Proposition Game

- “520 < 111”

Is this a statement? yes

Is this a proposition? yes

What is the truth value of the proposition? false

The Statement/Proposition Game

- “$y > 5$”

Is this a statement? yes

Is this a proposition? no

Its truth value depends on the value of y, but this value is not specified.

We call this type of statement a propositional function or open sentence.

The Statement/Proposition Game

- “The year is 2019 and 99 < 5.”

Is this a statement? yes

Is this a proposition? yes

What is the truth value of the proposition? false

Propositions Review

- Which ones are propositions?
  - Texas A&M University is in College Station, Texas
  - 1 + 1 = 2
  - what time is it?
  - 2 + 3 = 10
  - watch your step!

Propositions Review

- What is the negation of the proposition “At least ten inches of rain fell today in Miami”?

Propositions Review

- What is the negation of the proposition “At least 10 inches of rain fell today in Miami”?
  - It is not the case that at least 10 inches of rain fell today in Miami
  - (Simpler) Less than 10 inches of rain fell today in Miami.

Logical Operators (Connectives)

- We will examine the following logical operators:
  - Negation (NOT, $\neg$)
  - Conjunction (AND, $\wedge$)
  - Disjunction (OR, $\vee$)
  - Exclusive-or (XOR, $\oplus$)
  - Implication (if – then, $\rightarrow$)
  - Biconditional (if and only if, $\leftrightarrow$)
- Truth tables can be used to show how these operators can combine propositions to compound propositions.

Propositional Logic: Semantics

Truth table for connectives

- Given the values of atoms under some interpretation, we can use a truth table to compute the value for any wff under that same interpretation; the truth table establishes the semantics (meaning) of the propositional connectives.

P	Q	$\neg P$	$P \wedge Q$	$P \vee Q$	$P \rightarrow Q$	$P \leftrightarrow Q$
$false$	$false$	$true$	$false$	$false$	$true$	$true$
$false$	$true$	$true$	$false$	$true$	$true$	$false$
$true$	$false$	$false$	$false$	$true$	$false$	$false$
$true$	$true$	$false$	$true$	$true$	$true$	$true$

We can use the truth table to compute the value of any wff given the values of the constituent atom in the wff.
Note: In table, P and Q can be compound propositions themselves.
<u>Note</u>: Implication is not necessarily aligned with English usage.

Implication ($p \to q$)

- This is only False (violated) when q is False and p is True.
- Related implications:
  - Converse: $q \to p$;
  - Contra-positive: $\neg q \to \neg p$;
  - Inverse $\neg p \to \neg q$;

.

Important: only the contra-positive of $p \to q$ is equivalent to $p \to q$ (i.e., has the same truth values in all models); the converse and the inverse are equivalent;

Implication ($p \rightarrow q$)

- Implication plays an important role in reasoning. A variety of terminologies are used to refer to implication:

- conditional statement
- if p then q
- if p, q
- p is sufficient for q
- q if p
- q when p
- a necessary condition for p is q (*)
- p implies q
- p only if q (*)
- a sufficient condition for q is p
- q whenever p
- q is necessary for p (*)
- q follows from p

Note: the mathematical concept of implication is independent of a cause and effect relationship between the hypothesis (p) and the conclusion (q), that is normally present when we use implication in English.
Note: Focus on the case, when is the statement False. That is, p is True and q is False, should be the only case that makes the statement false.

(*) assuming the statement true, for p to be true, q has to be true

Implication Questions

- Let P be the statement "Rock the Good Ag learns discrete mathematics" and Q the statement "Rock the Good Ag will find a good job". Express $P \rightarrow Q$ as a statement in English.
- You can access the internet from campus only if you are a computer science major or you are not a freshman

Implication Question (cont.)

- Question:
- Let P be the statement “Rock the Good Ag learns discrete mathematics” and Q the statement “Rock the Good Ag will find a good job”. Express $P \rightarrow Q$ as a statement in English.
- Solution: Any of the following.
- If Rock learns discrete mathematics, then he will find a good job.
- Rock will find a good job when he learns discrete mathematics
- For Rock to get a good job, it is sufficient for him to learn discrete mathematics.

Second Conditional Question

- You can access the internet from campus only if you are a computer science major or you are not a freshman.
- Solution:
- Let a, c and f represent “you can access the Internet from campus” , “you are a computer science major”, and “you are a freshman”.
- Then above statement can be stated more simply as “You can access the internet implies that you are a computer science major major or you are not a freshman
- $a \rightarrow (c \vee \neg f)$

Bi-Conditionals $(p \leftrightarrow q)$

- Variety of terminology :
  - p is necessary and sufficient for q
  - if p then q, and conversely
  - p if and only if q
  - p iff q

$$p \leftrightarrow q \text{ is equivalent to } (p \rightarrow q) \wedge (q \rightarrow p)$$

<u>Note:</u> the if and only if construction used in biconditionals is rarely used in common language;
<u>Example</u>: “if you finish your meal, then you can play;” really means: “If you finish your meal, then you can play” and ”You can play, only if you finish your meal”.

TABLE 6 The Truth Table for the Biconditional $p \leftrightarrow q$.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Exclusive Or

- Truth Table

P	Q	$P \oplus Q$
T	T	F
T	F	T
F	T	T
F	F	F

$P \oplus Q$ is equivalent to $(P \wedge \neg Q) \vee (\neg P \wedge Q)$
and also equivalent to $\neg (P \leftrightarrow Q)$

Use a truth table to check these equivalences.


PENGUINS ARE BLACK AND WHITE.
SOME OLD TV SHOWS ARE BLACK AND WHITE.
THEREFORE, SOME PENGUINS ARE OLD TV SHOWS.
GLASBERGEN
Logic: another thing that penguins aren't very good at.

Propositional Logic: Satisfiability and Models

Satisfiability and Models

- **An interpretation or truth assignment *satisfies* a *wff*, if the *wff* is assigned the value *True, under that interpretation*.**

- **An interpretation that satisfies a *wff* is called a *model* of that *wff*.**

Given an interpretation (i.e., the truth values for the n atoms) then one can use the truth table to find the value of any ***wff***.

Tautologies and Contradictions

- A *tautology* is a statement that is always true.
- Examples:
  - $R \vee (\neg R)$
  - $\neg(P \wedge Q) \leftrightarrow (\neg P) \vee (\neg\, Q)$
- A *contradiction* is a statement that is always false.
- Examples:
  - $R \wedge (\neg R)$
  - $\neg(\neg(P \wedge Q) \leftrightarrow\ (\neg P) \vee (\neg Q))$
- The negation of any tautology is a contradiction, and the negation of any contradiction is a tautology.

The first rule of Tautology Club is the first rule of Tautology Club

1.2 Propositional Equivalences: Inconsistency (Unsatisfiability) and Validity

- **Inconsistent or Unsatisfiable set of WFFs**
  - **It is possible that no interpretation satisfies a set of wffs**
  - **In that case we say that the set of wffs is *inconsistent* or *unsatisfiable* or a *contradiction***
  - **Examples:**

    1 – $\{P \land \neg P\}$

    2 – $\{ P \lor Q,\ P \lor \neg Q,\ \neg P \lor Q,\ \neg P \lor \neg Q\}$

    (use the truth table to confirm that this set of wffs is inconsistent)

- **Validity (Tautology) of a set of WFFs**
  - **If a wff is True under all the interpretations of its constituents atoms, we say that the wff is *valid or it is a tautology*.**

Showing a Set of wwfs are Inconsistent

- **Consider { $P \vee Q$, $P \vee \neg Q$, $\neg P \vee Q$, $\neg P \vee \neg Q$}**
- Must show that the following wwf is unsatisfiable

$$(P \vee Q) \wedge (P \vee \neg Q) \wedge (\neg P \vee Q) \wedge (\neg P \vee \neg Q)$$

- List the following 11 terms in your truth table in following order:

$\underline{P}$ $\underline{Q}$ $\underline{\neg P}$ $\underline{\neg Q}$ $\underline{(P \vee Q)}$ $\underline{(P \vee \neg Q)}$ $\underline{(P \vee Q) \wedge (P \vee \neg Q)}$

$\underline{(\neg P \vee Q)}$ $\underline{(\neg P \vee \neg Q)}$ $\underline{(\neg P \vee Q) \wedge (\neg P \vee \neg Q)}$

$\underline{(P \vee Q) \wedge (P \vee \neg Q) \wedge (\neg P \vee Q) \wedge (\neg P \vee \neg Q)}$

Logical equivalence

Two sentences **p** an **q** are logically equivalent ($\equiv$ or $\Leftrightarrow$) iff p $\leftrightarrow$ q is a tautology

(and therefore p and q have the same truth value for all truth assignments)

$$
\begin{array}{rcll}
(\alpha \wedge \beta) & \equiv & (\beta \wedge \alpha) & \text{commutativity of } \wedge \\
(\alpha \vee \beta) & \equiv & (\beta \vee \alpha) & \text{commutativity of } \vee \\
((\alpha \wedge \beta) \wedge \gamma) & \equiv & (\alpha \wedge (\beta \wedge \gamma)) & \text{associativity of } \wedge \\
((\alpha \vee \beta) \vee \gamma) & \equiv & (\alpha \vee (\beta \vee \gamma)) & \text{associativity of } \vee \\
\neg(\neg\alpha) & \equiv & \alpha & \text{double-negation elimination} \\
(\alpha \rightarrow \beta) & \equiv & (\neg\beta \rightarrow \neg\alpha) & \text{contraposition} \\
(\alpha \rightarrow \beta) & \equiv & (\neg\alpha \vee \beta) & \text{implication elimination} \\
(\alpha \leftrightarrow \beta) & \equiv & ((\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha)) & \text{biconditional elimination} \\
\neg(\alpha \wedge \beta) & \equiv & (\neg\alpha \vee \neg\beta) & \text{de Morgan} \\
\neg(\alpha \vee \beta) & \equiv & (\neg\alpha \wedge \neg\beta) & \text{de Morgan} \\
(\alpha \wedge (\beta \vee \gamma)) & \equiv & ((\alpha \wedge \beta) \vee (\alpha \wedge \gamma)) & \text{distributivity of } \wedge \text{ over } \vee \\
(\alpha \vee (\beta \wedge \gamma)) & \equiv & ((\alpha \vee \beta) \wedge (\alpha \vee \gamma)) & \text{distributivity of } \vee \text{ over } \wedge
\end{array}
$$

Note: logical equivalence (or iff) allows us to make statements about propositional logic, pretty much like we use = in in ordinary mathematics.

The truth table method

The distributive law of OR over AND

TABLE 5 A Demonstration That $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ Are Logically Equivalent.

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

(Propositional) logic has a "truth compositional semantics":
Meaning is built up from the meaning of its primitive parts (just like English text).

Truth Tables

Truth table for connectives

We can use the truth table to compute the value of any wff given the values of the constituent atom in the wff.

Example:

Suppose P and Q are False and R has value True.
Given this interpretation, what is the truth value of $[(P \rightarrow Q) \rightarrow R] \rightarrow P$? False

If a system is described using *n features* (corresponding to *propositions*), and these features are represented by a corresponding set of *n atoms*, then there are 2^n different ways the system can be. Why? Each of the ways the system can be corresponds to an interpretation. Therefore there are 2^n interpretations.

Logic and Bit Operations

- Computers represent information using bits.
- A bit has only two possible values, namely 0 and 1.
- A 1 represents T (true) and 0 represents F (false)
- A variable is called a boolean variable if its value is either true or false.
- By replacing true by 1 and false by 0, a computer can perform logical operations.
- These replacements provides the following table for bit operators.

x	y	$x \vee y$	$x \wedge y$	$x \oplus y$
0	0	0	0	0
0	1	1	0	1
1	0	1	0	1
1	1	1	1	0

Example: Binary valued featured descriptions

- Consider the following description:
  - The router can send packets to the edge system only if it supports the new address space. For the router to support the new address space it is necessary that the latest software release be installed. The router can send packets to the edge system if the latest software release is installed. The router does not support the new address space.

  - Features:
    - Router
      - P - router can send packets to the edge of system
      - Q - router supports the new address space
    - Latest software release
      - R – latest software release is installed

Formal:

- The router can send packets to the edge system only if it supports the new address space. (constraint between feature 1 and feature 2)
  - If <u>Feature 1</u> (P) (router can send packets to the edge of system) then <u>Feature 2</u> (Q) (router supports the new address space) **$P \rightarrow Q$**

- For the router to support the new address space it is necessary that the latest software release be installed. (constraint between feature 2 and feature 3);
  - If <u>Feature 2</u> (Q) (router supports the new address space) then <u>Feature 3</u> (R) (latest software release is installed) **$Q \rightarrow R$**

- The router can send packets to the edge system if the latest software release is installed. (constraint between feature 1 and feature 3);
  If <u>Feature 3</u> (R) (latest software release is installed) then <u>Feature 1</u> (P) (router can send packets to the edge of system) **$R \rightarrow P$**

- The router does not support the new address space. **$\neg Q$**

Section 1.5 Rules of Inference

1.5 Propositional logic: Rules of Inference or Methods of Proof

How to produce additional wffs (sentences) from other ones? What steps can we perform to show that a conclusion follows logically from a set of hypotheses?

Example

Modus Ponens (the law of detachment)

$$\begin{array}{l} P \\ \underline{P \rightarrow Q} \\ \\ \therefore\ Q \end{array}$$

The hypotheses (premises) are written in a column and the conclusions below the bar

The symbol $\therefore$ denotes "therefore". Given the hypotheses, the conclusion follows.

The basis for this rule of inference is the **tautology** $\mathbf{(P \land (P \rightarrow Q)) \rightarrow Q)}$

[aside: check tautology with truth table to make sure]

In words: when P and $P \rightarrow Q$ are True, then Q must be True also. (meaning of second implication)

Propositional logic: Rules of Inference or Methods of Proof

- Example: Modus Ponens

If you study the CSCE 222 material $\rightarrow$ You will pass
<u>You study the CSCE 222 material</u>
$\therefore$ you will pass

Nothing "deep", but again remember the formal reason is that

$((P \wedge (P \rightarrow Q)) \rightarrow Q$ is a tautology.

Propositional logic: Rules of Inference

See Table 1, sec. 1.5, Rosen.

Rule of Inference	Tautology (Deduction Theorem)	Name
P $\therefore\ P \vee Q$	$P \rightarrow (P \vee Q)$	**Addition**
$P \wedge Q$ $\therefore\ P$	$(P \wedge Q) \rightarrow P$	**Simplification**
P Q $\therefore\ P \wedge Q$	$[(P) \wedge (Q)] \rightarrow (P \wedge Q)$	**Conjunction**
P $P \rightarrow Q$ $\therefore\ Q$	$[(P) \wedge (P \rightarrow Q)] \rightarrow P$	**Modus Ponens**
$\neg Q$ $P \rightarrow Q$ $\therefore\ \neg P$	$[(\neg Q) \wedge (P \rightarrow Q)] \rightarrow \neg P$	**Modus Tollens**
$P \rightarrow Q$ $Q \rightarrow R$ $\therefore\ P \rightarrow R$	$[(P \rightarrow Q) \wedge (Q \rightarrow R)] \rightarrow (P \rightarrow R)$	**Hypothetical Syllogism** **(“chaining”)**
$P \vee Q$ $\neg P$ $\therefore\ Q$	$[(P \vee Q) \wedge (\neg P)] \rightarrow Q$	**Disjunctive syllogism**
$P \vee Q$ $\neg P \vee R$ $\therefore\ Q \vee R$	$[(P \vee Q) \wedge (\neg P \vee R)] \rightarrow (Q \vee R)$	**Resolution**

Valid Arguments

An argument is a sequence of propositions. The final proposition is called the conclusion of the argument while the other proposition are called the premises or hypotheses of the argument.

An argument is valid whenever the truth of all its premises implies the truth of its conclusion.

How to show that **q** logically follows from the hypotheses $(p_1 \wedge p_2 \wedge \ldots \wedge p_n)$?

Show that

$$(p1 \wedge p2 \wedge \ldots \wedge p_n) \rightarrow q \quad \text{is a tautology}$$

One can use the rules of inference to show the validity of an argument.

Arguments

- Just like a rule of inference, an **argument** consists of one or more hypotheses (or premises) and a conclusion.
- We say that an argument is **valid**, if whenever all its hypotheses are true, its conclusion is also true.
- However, if any hypothesis is false, even a valid argument can lead to an incorrect conclusion.
- Proof: show that hypotheses $\rightarrow$ conclusion is true using rules of inference

Arguments

- **Example:**

- "If 101 is divisible by 3, then 101^2 is divisible by 9. 101 is divisible by 3. Consequently, 101^2 is divisible by 9."

- Although the argument is **valid**, its conclusion is **incorrect**, because one of the hypotheses is false ("101 is divisible by 3.").

- If in the above argument we replace 101 with 102, we could correctly conclude that 102^2 is divisible by 9.

Arguments

- Which rule of inference was used in the last argument?

- p: “101 is divisible by 3.”
- q: “101^2 is divisible by 9.”

$$\begin{array}{l} p \\ p \rightarrow q \\ \hline \therefore q \end{array}$$

Modus ponens

Unfortunately, one of the hypotheses (p) is false.
Therefore, the conclusion q is incorrect.


GOOD AFTERNOON Y'ALL
I AM HERE TO POINT OUT LOGICAL FALLACIES
memegenerator.net

Review: Modus ponens

- *modus ponens* (Latin for "mode that affirms")
- The modus ponens argument is built in the following way:
  A) If x [is true], then y [is true].
  B) x [is true]
  C) Therefore, y [is true].
- Let's flesh this out in an example:
  A) If Fido is a dog, then Fido is an animal.
  B) Fido is a dog.
  C) Therefore, Fido is an animal.

In AI, modus ponens is often referred to as "forward chaining"

Arguments

- **Another example:**
- “If it rains today, then we will not have a bonfire today. If we do not have a bonfire today, then we will have a bonfire tomorrow.
  Therefore, if it rains today, then we will have a bonfire tomorrow.”
- This is a **valid** argument: If its hypotheses are true, then its conclusion is also true.

Arguments

- Let us formalize the previous argument:
- p: "It is raining today."
- q: "We will not have a bonfire today."
- r: "We will have a bonfire tomorrow."
- So the argument is of the following form:

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore\ p \rightarrow r \end{array}$$

Hypothetical syllogism

Hypothetical syllogism is the transitive application of *modus ponens*

Review: Disjunctive Syllogism

- Remember that “disjunction” is just a big word meaning “OR”
- This argument is built in the following way:
  A) Either x [is true] or y [is true]
  B) $\neg$ x [x is not true]
  C) Therefore, y [is true].
- Let’s flesh this out in an example:
  A) Libby is a dog or Libby is a cat.
  B) Libby is not a cat.
  C) Therefore, Libby is a dog.

Review: Modus tollens

- *modus tollens* (Latin for "mode that denies”)
- The modus tollens argument is built in the following way:
  A) If x [is true], then y [is true].
  B) $\neg$ y [y is not true]
  C) Therefore, $\neg$ x [x is not true].
- Let’s flesh this out in an example:
  A) If Fido detects an intruder, then Fido will bark.
  B) Fido did not bark.
  C) Therefore, Fido did not detect an intruder.

Arguments

- **Another example:**

- Rock the Good Ag is either intelligent or a good actor.
- If Rock the Good Ag is intelligent, then he can count from 1 to 10.
- Rock the Good Ag can only count from 1 to 3.
- Therefore, Rock the Good Ag is a good actor.

- i: “Rock the Good Ag is intelligent.”
- a: “Rock the Good Ag is a good actor.”
- c: “Rock the Good Ag can count from 1 to 10.”

Arguments

i: "Rock the Good Ag is intelligent."
a: "Rock the Good Ag is a good actor."
c: "Rock the Good Ag can count from 1 to 10."

- Step 1: $\neg$ **c** — Hypothesis — (Rock can only count to 3)
- Step 2: **i** $\rightarrow$ **c** — Hypothesis
- Step 3: $\neg$ **i** — Modus tollens Steps 1 & 2
- Step 4: **a** $\vee$ **i** — Hypothesis
- Step 5: **a** — Disjunctive Syllogism Steps 3 & 4

- Conclusion: **a** ("Rock the Good Ag is a good actor.")


IF, BY SOME PARADOX IN THE SPACE/TIME CONTINUUM,
CHUCK NORRIS WERE EVER TO FIGHT HIMSELF,
HE'D WIN.
quickmeme.com

Arguments

Yet another example:

If you listen to me, you will pass CSCE 222.
You passed CSCE 222.
Therefore, you have listened to me.

Is this argument valid?

No, it assumes $((p \rightarrow q) \wedge q) \rightarrow p$.

This statement is not a tautology. It is false if p is false and q is true.

1.3-1.4 Beyond Propositional Logic: Predicates and Quantifiers

Predicates

- Propositional logic assumes the world contains facts that are true or false.
- But let’s consider a statement containing a variable:
- $x > 3$ since we don’t know the value of x we cannot say whether the expression is true or false
- $x > 3$ which corresponds to “x is $\underbrace{\text{greater than 3}}_{\text{Predicate, i.e. a property of } x}$”

"x is greater than 3" can be represented as P(x), where P denotes "greater than 3"

In general a statement involving n variables $x_1, x_2, \dots x_n$ can be denoted by

$$P(x_1, x_2, \dots x_n)$$

P is called a ***predicate*** or the ***propositional function*** P at the n-tuple ($x_1, x_2, \dots x_n$).

Propositional Functions & Predicates

- Propositional function (open sentence):
- statement involving one or more variables,
- e.g.: $x-3 > 5$.
- Let us call this propositional function $P(x)$, where P is the predicate and x is the variable.

What is the truth value of $P(2)$? false

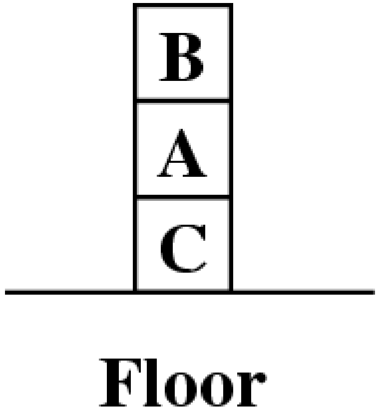
What is the truth value of $P(8)$? false

What is the truth value of $P(9)$? true

When a variable is given a value, it is said to be instantiated

Truth value depends on value of variable

A predicate (aka propositional function) becomes a proposition when all its variables are instantiated.



Predicate: On(x,y)

Propositions:

ON(A,B) is False (in figure)

ON(B,A) is True

Clear(B) is True

Variables and Quantification

- How would we say that every block in the world has a property – say "clear"? We would have to say:
- Clear(A); Clear(B); … for all the blocks… (it may be long or, worse, we may have an infinite number of blocks…)

(Note: $\forall x$ P(x) is either true or false, so it is a proposition, not a propositional function.)

What we need is: Quantifiers

$\forall$ ***Universal quantifier***

$\forall$***x*** **P(*x*)** - P(*x*) is true for all the values *x* in the universe of discourse

$\exists$ ***Existential quantifier***

$\exists$***x*** **P(*x*)** - there exists an element *x* in the universe of discourse such that P(*x*) is true

Universal quantification

Everyone in Aggieland is smart:

$$\forall x \;\; In(x,Aggieland) \rightarrow Smart(x)$$

Implicitly equivalent to the conjunction of instantiations of Predicate “In"

$$
\begin{aligned}
& In(Demelza,Aggieland) \rightarrow Smart(Demelza) \\
\wedge \quad & In(Ross,Aggieland) \rightarrow Smart(Ross) \\
\wedge \quad & In(José,Aggieland) \rightarrow Smart(José) \\
\wedge \; & \ldots
\end{aligned}
$$

A common mistake to avoid

- Typically, $\rightarrow$ is the main connective with $\forall$
- Common mistake: Using $\wedge$ as the main connective with $\forall$:

$\forall$x In(x,Aggieland) $\wedge$ Smart(x)
means:

"Everyone is in Aggieland and everyone is smart."

Existential quantification

Someone in Aggieland is smart:

$\exists x$ (In(x,Aggieland) $\wedge$ Smart(x))

$\exists x$ P(x) " There exists an element x in the universe of discourse such that P(x) is true"

Equivalent to the disjunction of instantiations of *P*

(In(Demelza,Aggieland) $\wedge$ Smart(Demelza))

$\vee$ (In(Ross,Aggieland) $\wedge$ Smart(Ross))

$\vee$ (In(José,Aggieland) $\wedge$ Smart(José))

$\vee$ ...

Quantification

- Another example:
- Let the universe of discourse be the real numbers.

- What does $\forall x \exists y\ (x + y = 320)$ mean ?

- "For every x there exists a y so that x + y = 320."

Is it true? yes

Is it true for the natural numbers? no

Another common mistake to avoid

Typically, $\wedge$ is the main connective with $\exists$

Common mistake: using $\rightarrow$ as the main connective with $\exists$:

$$\exists x \; \text{In}(x,\text{Austin}) \rightarrow \text{Smart}(x)$$

when is this true?

Is true if there is either (anyone who is not in Austin) or (there is anyone who is smart)

Above is equivalent to

$$\exists x \; [\neg \, \text{In}(x,\text{Austin}) \vee \text{Smart}(x)]$$

Properties of quantifiers

$\forall x\ \forall y$ is the same as $\forall y\ \forall x$

$\exists x\ \exists y$ is the same as $\exists y\ \exists x$

$\exists x\ \forall y$ is not the same as $\forall y\ \exists x$

$\forall x\ \exists y\ \ Loves(x,y)$

"Everybody loves somebody sometime" – Dean Martin (1964)

"Everyone in the world is loves at least one person"

Properties of quantifiers

- $\exists y\ \forall x\ \ Loves(x,y)$
  - "There is a person who is loved by everyone in the world"

- Quantifier duality: each can be expressed using the other
- $\forall x\ Likes(x,IceCream) \qquad \neg\exists x\ \neg Likes(x,IceCream)$
- $\exists x\ Likes(x,Broccoli) \qquad \neg\forall x\ \neg Likes(x,Broccoli)$

Love Affairs
Loves(x,y) x loves y

Everybody loves Raymond

$\forall x$ Loves (x, Raymond)

Everybody loves somebody

$\forall x \, \exists y$ Loves (x, y)

There is somebody whom somebody loves

$\exists y \, \exists x$ Loves (x, y)

Nobody loves everybody

$\neg \exists x \, \forall y \text{ Loves } (x, y) \equiv \forall x \, \exists y \, \neg \text{Loves } (x, y)$

There is somebody whom Lydia doesn't love

$\exists y \, \neg$Loves (Lydia, y)

Note: flipping quantifiers when $\neg$ moves in.

Love Affairs continued…

There is somebody whom no one loves

$$\exists y\ \forall x\ \neg \text{Loves}(x, y)$$

There is exactly one person whom everybody loves (**uniqueness**)

$$\exists y\ (\forall x\ \text{Loves}(x,y) \wedge \forall z((\forall w\ \text{Loves}(w,z) \rightarrow z=y))$$

There are exactly two people whom Lynn Loves

$$\exists x\ \exists y\ ((x\neq y) \wedge \text{Loves}(\text{Lynn},x)\ \text{Loves}(\text{Lynn},y) \wedge$$
$$\forall z(\ \text{Loves}(\text{Lynn},z) \rightarrow (z=x \vee z=y)))$$

Everybody loves himself or herself

$$\forall x\ \text{Loves}(x,x)$$

There is someone who loves no one besides herself or himself

$$\exists x\ \forall y\ \ \text{Loves}(x,y) \leftrightarrow (x=y)$$ (note biconditional – why?)

Let $Q(x,y)$ denote "$x+y = 0$"; consider the domain of discourse the real

numbers

What is the truth value of

a) $\exists y\ \forall x\ \ Q(x,y)$? **False**

b) $\forall x\ \exists y\ \ Q(x,y)$? **True** **(additive inverse)**

Statement	When True	When False
$\forall x\ \forall y\ P(x,y)$ $\forall y\ \forall x\ P(x,y)$	P(x,y) is true for every pair	There is a pair for which P(x.y) is false
$\forall x\ \exists y\ P(x,y)$	For every x there is a y for which P(x,y) is true	There is an x such that P(x,y) is false for every y.
$\exists x\ \forall y\ P(x,y)$	There is an x such that P(x,y) is true for every y.	For every x there is a y for which P(x,y) is false
$\exists x\ \exists\ y\ P(x,y)$ $\exists y\ \exists\ x\ P(x,y)$	There is a pair x, y for which P(x,y) is true	P(x,y) is false for every pair x,y.

Negation

Negation	Equivalent Statement	When is the negation True	When is it False
$\neg \exists x\, P(x)$	$\forall x\, \neg P(x)$	For every x, P(x) is false	There is an x for which P(x) is true.
$\neg\, \forall x\, P(x)$	$\exists x\, \neg P(x)$	There is an x for which P(x) is false.	For every x, P(x) is true.

The kinship domain:

Brothers are siblings

$\forall$x,y *Brother(x,y)* $\rightarrow$ *Sibling(x,y)*

One's mother is one's female parent

$\forall$m,c *Mother(c)* = m $\leftrightarrow$ *(Female(m)* $\wedge$ *Parent(m,c))*

“Sibling” is symmetric

$\forall$x,y *Sibling(x,y)* $\leftrightarrow$ *Sibling(y,x)*

Rules of Inference for Quantified Statements

$(\forall x)\ P(x)$ $\therefore P(c)$	Universal Instantiation
P(c) for an arbitrary c $\therefore (\forall x)\ P(x)$	Universal Generalization
$\exists(x)\ P(x)$ $\therefore$ P(c) for some element c	Existential Instantiation
P(c) for some element c $\therefore \exists(x)\ P(x)$	Existential Generalization

- Example:

- Let CSCE222(x) denote: x is taking the CSCE222 class
- Let CSE(x) denote: x is taking a course in CSE

- Consider the premises $\forall x\ (\mathrm{CSCE222}(x) \rightarrow \mathrm{CSE}(x))$
- CSCE222(Rock)

- We can conclude CSE(Rock)

Arguments

- Argument (formal):

Step	Reason
1 $\forall x$ (CSCE222(x) $\rightarrow$ CSE(x))	premise
2 CSCE222(Rock) $\rightarrow$ CSE(Rock)	Universal Instantiation
3 CSCE222(Rock)	Premise
4 CSE(Rock)	Modus Ponens (2 and 3)

Example

Show that the premises:

1- A student in this class has not read the textbook;

2- Everyone in this class passed the first homework

Imply

Someone who has passed the first homework has not read the textbook

Example

Solution:

Let $C(x)$ denote that x is in this class;

$T(x)$ denote that x has read the textbook;

$P(x)$ denote that x has passed the first homework

Premises:

$\exists x\ (C(x) \wedge \neg T(x))$

$\forall x\ (C(x) \rightarrow P(x))$

Conclusion: we want to show $\exists x\ (P(x) \wedge \neg T(x))$

Step	Reason
1 $\exists x\,(Cx \wedge \neg T(x))$	Premise
2 $C(a) \wedge \neg T(a)$	Existential Instantiation from 1
3 $C(a)$	Simplification 2
4 $\forall x\,(C(x) \to P(x))$	Premise
5 $C(a) \to P(a)$	Universal Instantiation from 4
6 $P(a)$	Modus ponens from 3 and 5
7 $\neg T(a)$	Simplification from 2
8 $P(a) \wedge \neg T(a)$	Conjunction from 6 and 7
9 $\exists x\, P(x) \wedge \neg T(x)$	Existential generalization from 8

Next: methods for proving theorems.