CSCE 222
Discrete Structures
Counting

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Based on Chapter 6 of Rosen
Discrete Mathematics and its Applications

Combinatorics: the study of arrangements of objects

Enumeration: the counting of objects with certain properties

Basic Counting Principles

- **Counting problems** are of the following kind:
  - "**How many** different 8-letter passwords are there?"
  - "**How many** possible ways are there to pick 11 soccer players out of a 20-player team?"
- Most importantly, counting is the basis for computing **probabilities of discrete events**.
  - ("What is the probability of winning the lottery?")

Basic Counting Principles

- **The sum rule:** If a task can be done either in one of n_1 ways or in one of n_2 ways, where none of the set of n_1 ways is the same as any of the set of n_2 ways, then there are n_1+n_2 ways to do the task
- **Example:**
  The department will award a free computer to either a CSE student or a CSE professor.
  How many different choices are there, if there are 1229 students and 50 professors?

  There are 1229 + 50 = 1279 choices.

Basic Counting Principles

- **Generalized sum rule:**
- If a task can be done in $n_1, n_2, \ldots, n_m$ ways, where none of the set of n_i ways is the same as any of the set of n_j ways ($i \neq j$), then there are $n_1 + n_2 + \ldots + n_m$ ways to do the task.

Basic Counting Principles

- **The product rule:**
- Suppose that a procedure can be broken down into two successive tasks. If there are n_1 ways to do the first task and n_2 ways to do the second task after the first task has been done, then there are $n_1 n_2$ ways to do the procedure.

Basic Counting Principles

- **Example:**

  How many different license plates are there that containing exactly three English letters ?

- **Solution:**

  There are 26 possibilities to pick the first letter, then 26 possibilities for the second one, and 26 for the last one.

  So there are $26 \cdot 26 \cdot 26 = 17576$ different license plates.

Basic Counting Principles

- **Generalized product rule:**

  If we have a procedure consisting of sequential tasks T_1, T_2, …, T_m that can be done in n_1, n_2, …, n_m ways, respectively, then there are $n_1 \cdot n_2 \cdot \ldots \cdot n_m$ ways to carry out the procedure.

Basic Counting Principles

- The sum and product rules can also be phrased in terms of **set theory**.
- **Sum rule:** Let A_1, A_2, …, A_m be disjoint sets. Then the number of ways to choose any element from one of these sets is $|A_1 \cup A_2 \cup \ldots \cup A_m| =$
  $|A_1| + |A_2| + \ldots + |A_m|$.
- **Product rule:** Let A_1, A_2, …, A_m be finite sets. Then the number of ways to choose one element from each set in the order A_1, A_2, …, A_m is
  $|A_1 \times A_2 \times \ldots \times A_m| = |A_1| \cdot |A_2| \cdot \ldots \cdot |A_m|$.

How many strings of length 3 can be formed from the letters {A, B, C, D, E} if repetition is not allowed?

- Task 1: choosing the first letter: 5 ways
- Task 2: choosing the remaining letters: ? Ways
- Task 2.1: choosing the second letter: 4 ways
- Task 2.2 choosing the third letter: 3 ways
- Task 2: 4*3=12 ways
- 5*12 = 60 strings of length 3

How many strings of length 3 can be formed from the letters {A, B, C, D, E} if repetition is allowed?

- Task 1: choosing the first letter: 5 ways
- Task 2: choosing the remaining letters: ? Ways
- Task 2.1: choosing the second letter: 5 ways
- Task 2.2 choosing the third letter: 5 ways
- Task 2: 5*5=25 ways
- 5*25 = 125 strings of length 3

How many bitstrings of length 7 are there?

$$\underline{\ \ }\ \underline{\ \ }\ \underline{\ \ }\ \underline{\ \ }\ \underline{\ \ }\ \underline{\ \ }\ \underline{\ \ }$$
$$2\ 2\ 2\ 2\ 2\ 2\ 2 = 2^\wedge 7 = 128$$

More complex counting problems

- In a version of the BASIC programming language, the name of a variable is a string of 1 or 2 alphanumeric characters, where uppercase and lowercase letters are not distinguished.
- Moreover, a variable name must begin with a letter and must be different from the five strings of two characters that are reserved for programming use
- How many different variables names are there?
- Let V_1 be the number of these variables of 1 character, and likewise V_2 for variables of 2 characters
- So, $V_1=26$, and $V_2=26\cdot 36-5=931$
- In total, there are 26+931=957 different variables

Example

- Each user on a computer system has a password, which is 6 to 8 characters long, where each character is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?
- Let P be the number of all possible passwords and $P=P_6+P_7+P_8$ where P_i is a password of i characters
- $P_6=36^6-26^6=1,867,866,560$
- $P_7=36^7-26^7=70,332,353,920$
- $P_8=36^8-26^8=2,612,282,842,880$
- $P=P_6+P_7+P_8=2,684,483,063,360$

Example: Internet address

Bit Number	0	1	2	3	4	8	16	24	31
Class A	0	netid				hostid			
Class B	1	0	netid				hostid		
Class C	1	1	0	netid				hostid	
Class D	1	1	1	0	Multicast Address				
Class E	1	1	1	1	0	Address			

- Internet protocol (IPv4)
  - Class A: largest network
  - Class B: medium-sized networks
  - Class C : smallest networks
  - Class D: multicast (not assigned for IP address)
  - Class E: future use
  - Some are reserved: netid 1111111, hostid all 1's and 0's
- Neither class D or E addresses are assigned as the IPv4 addresses
- How may different IPv4 addresses are available?

Example: Internet address

Bit Number	0	1	2	3	4	8	16	24	31
Class A	0	netid					hostid		
Class B	1	0	netid					hostid	
Class C	1	1	0	netid					hostid
Class D	1	1	1	0	Multicast Address				
Class E	1	1	1	1	0	Address			

- Let the total number of address be x, and $x=x_A+x_B+x_C$
- Class A: there are 2^7-1=127 netids (1111111 is reserved). For each netid, there are 2^{24}-2=16,777,214 hostids (as hostids of all 0s and 1s are reserved), so there are x_A=127·16,777,214=2,130,706,178 addresses
- Class B, C: 2^{14}=16,384 Class B netids and 2^{21}=2,097,152 Class C netids. 2^{16}-2=65,534 Class B hostids, and 2^8-2=254 Class C hostids. So, x_B=1,073,709,056, and x_C=532,676,608
- So, $x=x_A+x_B+x_C$=3,737,091,842

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Inclusion-exclusion principle

- Suppose that a task can be done in n_1 or in n_2 ways, but some of the set of n_1 ways to do the task are the same as some of the n_2 ways to do the task
- Cannot simply add n_1 and n_2, but need to subtract the number of ways to the task that is common in both sets
- This technique is called **principle of inclusion-exclusion** or **subtraction principle**

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Inclusion-Exclusion

- How many bit strings of length 8 either start with a 1 or end with 00?
- **Task 1:** Construct a string of length 8 that starts with a 1.

  There is one way to pick the first bit (1),
  two ways to pick the second bit (0 or 1),
  two ways to pick the third bit (0 or 1),
  .
  .
  .
  two ways to pick the eighth bit (0 or 1).
- **Product rule:** Task 1 can be done in $1 \cdot 2^7 = 128$ ways.

Inclusion-Exclusion

- **Task 2:** Construct a string of length 8 that ends with 00.

  There are two ways to pick the first bit (0 or 1),
  two ways to pick the second bit (0 or 1),
  .
  .
  .
  two ways to pick the sixth bit (0 or 1),
  one way to pick the seventh bit (0), and
  one way to pick the eighth bit (0).
- **Product rule:** Task 2 can be done in $2^6 = 64$ ways.

Inclusion-Exclusion

- **Since there are 128 ways to do Task 1 and 64 ways to do Task 2, does this mean that there are 192 bit strings either starting with 1 or ending with 00 ?**

  No, because here Task 1 and Task 2 can be done **at the same time**.

  When we carry out Task 1 and create strings starting with 1, some of these strings end with 00.

- Therefore, we sometimes do Tasks 1 and 2 at the same time, so **the sum rule does not apply**.

Inclusion-Exclusion

- If we want to use the sum rule in such a case, we have to subtract the cases when Tasks 1 and 2 are done at the same time.

  How many cases are there, that is, how many strings start with 1 **and** end with 00?

  There is one way to pick the first bit (1),
  two ways for the second, …, sixth bit (0 or 1),
  one way for the seventh, eighth bit (0).

- **Product rule:** In $2^5 = 32$ cases, Tasks 1 and 2 are carried out at the same time.

Inclusion-Exclusion

- Since there are 128 ways to complete Task 1 and 64 ways to complete Task 2, and in 32 of these cases Tasks 1 and 2 are completed at the same time, there are

  128 + 64 – 32 = 160 ways to do either task.

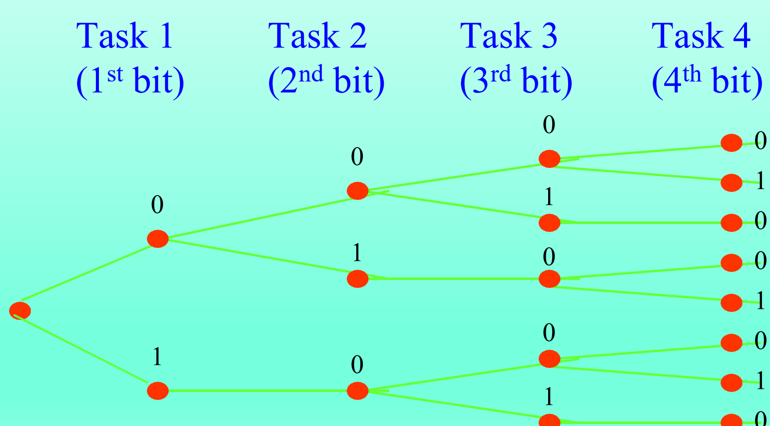
- In set theory, this corresponds to sets A_1 and A_2 that are **not** disjoint. Then we have:

  $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$

- This is called the **principle of inclusion-exclusion.**

Tree Diagrams

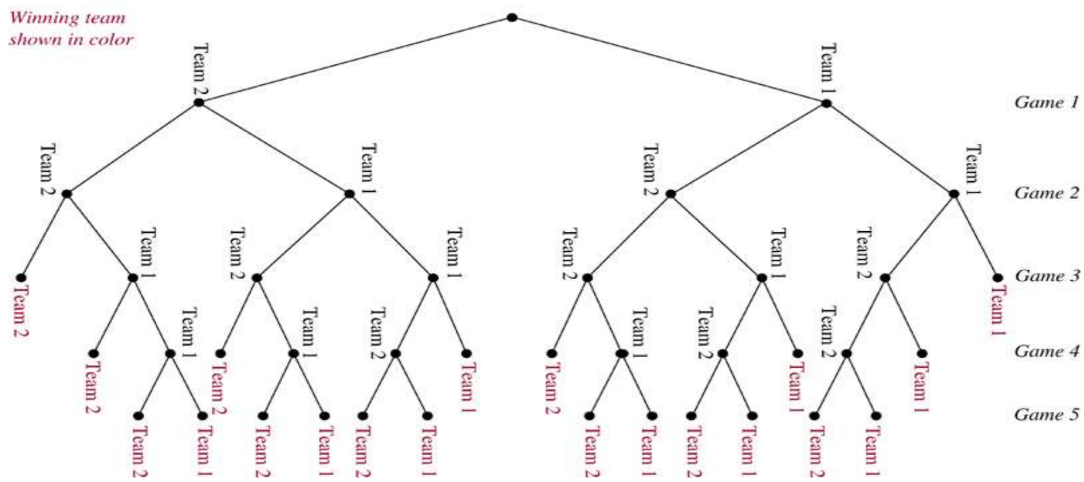
- How many bit strings of length four do not have two consecutive 1s?



There are 8 strings.

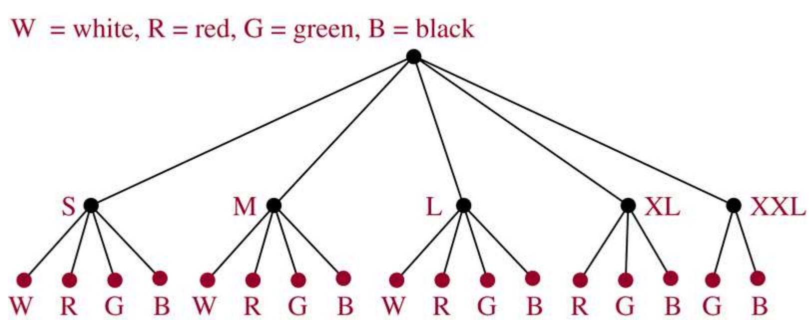
Example

- A playoff between 2 teams consists of at most 5 games. The 1st team that wins 3 games wins the playoff. How many different ways are there?



Example

- Suppose a T-shirt comes in 5 different sizes: S, M, L, XL, and XXL. Further suppose that each size comes in 4 colors, white, green, red, and black except for XL which comes only in red, green and black, and XXL which comes only in green and black. How many possible size and color of the T-shirt?



Pigeonhole Principle

If $k + 1$ or more objects are placed into $k > 0$ boxes, then at least 1 box has more than 1 object.

$k = 9$ boxes
10 pigeons

Proof

By **contraposition**.

Show "if no box contains more than 1 object, then there are at most k objects in the k boxes".

Assume no box contains more than 1 object.

There are k boxes.

Thus, there are at most k objects.

QED

This is a contradiction as there are at least k+1 objects

The Pigeonhole Principle

- **The pigeonhole principle:** If (k + 1) or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.
- **Example 1:** If there are 11 players in a soccer team that wins 12-0, there must be at least one player in the team who scored at least twice.
- **Example 2:** If you have 6 classes from Monday to Friday, there must be at least one day on which you have at least two classes.

More Examples

- Among any group of 367 people, there must be at least 2 with the same birthday
- How many students must be in a class to guarantee that at least 2 students receive the same score on the final exam, if the exam is graded on a scale from 0 to 100 points?

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The Pigeonhole Principle

- **The generalized pigeonhole principle:** If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ of the objects.
- **Example 1:** In our 50-student class, at least 10 students will get the same letter grade (A, B, C, D, or F).

The Pigeonhole Principle

- **Example 2:** Assume you have a drawer containing a random distribution of a dozen brown socks and a dozen black socks. It is dark, so how many socks do you have to pick to be sure that among them there is a matching pair?
- There are two types of socks, so if you pick at least 3 socks, there must be either at least two brown socks or at least two black socks.
- Generalized pigeonhole principle: $\lceil 3/2 \rceil = 2$.

How many cards must be drawn from a standard deck to guarantee that at least 5 of the same suit are drawn?

- Ceil(N/4) = 5
- N = 21
- 21 cards will suffice
- Deal 21 and show that at least 5 of some suit were drawn.

Permutations and Combinations

- How many ways are there to pick a set of 3 people from a group of 6?

  There are 6 choices for the first person, 5 for the second one, and 4 for the third one, so there are
  $6 \cdot 5 \cdot 4 = 120$ ways to do this.

  This is not the correct result!

- For example, picking person C, then person A, and then person E leads to the **same group** as first picking E, then C, and then A.

  However, these cases are counted **separately** in the above equation.

Permutations and Combinations

So how can we compute how many different subsets of people can be picked (that is, we want to disregard the order of picking) ?

To find out about this, we need to look at **permutations**.

A **permutation** of a set of distinct objects is an ordered arrangement of these objects.

An ordered arrangement of r elements of a set is called an **r-permutation**.

A permutation is an <u>ordered</u> <u>arrangement</u> of *distinct* objects.

How many ways can a deck of cards be ordered?

- Use product rule.
- 52 ways for card 1
- 51 ways for card 2
- …
- 1 way for card 52
- = 52!

80,658,175,170,943,878,571,660,
636,856,403,766,975,289,505,440,
883,277,824,000,000,000,000

A number so big that 1 trillion planets, each with 1 trillion humanoids, each shuffling a deck 1 trillion times per second (and always uniquely), non-stop since the beginning of time in the universe (approx. 14 billion years ago), would need 200 trillion more lifetimes of the universe to get through all possible orderings.

Permutations and Combinations

Example: Let S = {1, 2, 3}.
The arrangement 3, 1, 2 is a permutation of S.
The arrangement 3, 2 is a 2-permutation of S.

The number of r-permutations of a set with n distinct elements is denoted by **P(n, r).** **(Read "n Pick r")**

We can calculate P(n, r) with the product rule:

$P(n, r) = n \cdot (n - 1) \cdot (n - 2) \cdot \ldots \cdot (n - r + 1)$.

(n choices for the first element, (n – 1) for the second one, (n – 2) for the third one…)

Permutations and Combinations

- **Example:**

  $P(8, 3) = 8 \cdot 7 \cdot 6 = 336$
  $= (8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)/(5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)$

- **General formula:**

  $P(n, r) = n!/(n - r)!$

- Knowing this, we can return to our initial question:
- **How many ways are there to pick a set of 3 people from a group of 6 (disregarding the order of picking)?**

Permutations and Combinations

- An **r-combination** of elements of a set is an unordered selection of r elements from the set.
- Thus, an r-combination is simply a subset of the set with r elements.
- **Example:** Let $S = \{1, 2, 3, 4\}$.
- Then $\{1, 3, 4\}$ is a 3-combination from S.
- The number of r-combinations of a set with n distinct elements is denoted by C(n, r). (Read "n Choose r")
- Note that C(n,r) is also denoted by $\binom{n}{r}$ and is called a ***binomial coefficient***
- **Example:** C(4, 2) = 6, since, for example, the 2-combinations of a set $\{1, 2, 3, 4\}$ are $\{1, 2\}$, $\{1, 3\}$, $\{1, 4\}$, $\{2, 3\}$, $\{2, 4\}$, $\{3, 4\}$.

Permutations and Combinations

- **How can we calculate C(n, r)?**
- Consider that we can obtain the r-permutation of a set in the following way:
- **First,** we form all the r-combinations of the set (there are C(n, r) such r-combinations).
- **Then,** we generate all possible orderings in each of these r-combinations (there are P(r, r) such orderings in each case).
- Therefore, we have:
- $P(n, r) = C(n, r) \cdot P(r, r)$

Permutations and Combinations

$$
\begin{aligned}
C(n, r) &= P(n, r)/P(r, r) \\
&= n!/(n - r)!/(r!/(r - r)!) \\
&= n!/(r!(n - r)!)
\end{aligned}
$$

Now we can answer our initial question:

How many ways are there to pick a set of 3 people from a group of 6 (disregarding the order of picking)?

$C(6, 3) = 6!/(3! \cdot 3!) = 720/(6 \cdot 6) = 720/36 = 20$

- **There are 20 different ways, that is, 20 different groups to be picked.**

Combinations

- We also have the following:

$$C(n, n-r) = \frac{n!}{(n-r)!\,[n-(n-r)]!} = \frac{n!}{(n-r)!\,r!}.$$

Hence, $C(n, r) = C(n, n-r)$.

This symmetry is intuitively plausible. For example, let us consider a set containing six elements (n = 6).

Picking two elements and **leaving four** is essentially the same as **picking four** elements and **leaving two**.

In either case, our number of choices is the number of possibilities to **divide** the set into one set containing two elements and another set containing four elements.

Permutations and Combinations

- **Example:**
- A youth soccer club 8 female and 7 male members. For today's match, the coach wants to have 6 female and 5 male players on the grass. How many possible configurations are there?

$$\begin{aligned} C(8, 6) \cdot C(7, 5) &= 8!/(6!\cdot 2!) \cdot 7!/(5!\cdot 2!) \\ &= 28\cdot 21 \\ &= 588 \end{aligned}$$

Note that here we are using both combinations *and* the product rule

Combinations

- **Pascal's Identity:**
- Let n and k be positive integers with $n \geq k$.
  Then $C(n + 1, k) = C(n, k - 1) + C(n, k)$.
- How can this be explained?
- What is it good for?

Combinations

- Imagine a set S containing n elements and a set T containing (n + 1) elements, namely all elements in S plus a new element **a**.
- Calculating $C(n + 1, k)$ is equivalent to answering the question: How many subsets of T containing k items are there?
- **Case I**: The subset contains (k – 1) elements of S plus the element **a**: $C(n, k - 1)$ choices.
- **Case II:** The subset contains k elements of S and does not contain **a**: $C(n, k)$ choices.
- **Sum Rule:** $C(n + 1, k) = C(n, k - 1) + C(n, k)$.

Pascal's Triangle

- In Pascal's triangle, each number is the sum of the numbers to its upper left and upper right:

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
\dots & & \dots & & \dots & & \dots & & \dots & & \dots
\end{array}
$$

Pascal's Triangle

- Since we have C(n + 1, k) = C(n, k – 1) + C(n, k) and C(0, 0) = 1, we can use Pascal's triangle to simplify the computation of C(n, k):

k

n

C(0, 0) = 1

C(1, 0) = 1 C(1, 1) = 1

C(2, 0) = 1 C(2, 1) = 2 C(2, 2) = 1

C(3, 0) = 1 C(3, 1) = 3 C(3, 2) = 3 C(3, 3) = 1

C(4, 0) = 1 C(4, 1) = 4 C(4, 2) = 6 C(4, 3) = 4 C(4, 4) = 1

Binomial Coefficients

- Expressions of the form C(n, k) are also called **binomial coefficients**.
- **Why?**
- A **binomial expression** is the sum of two terms, such as (a + b).
  Now consider $(a + b)^2 = (a + b)(a + b)$.
- When expanding such expressions, we have to form all possible products of a term in the first factor and a term in the second factor:
  $(a + b)^2 = a \cdot a + a \cdot b + b \cdot a + b \cdot b$
- Then we can sum identical terms:
  $(a + b)^2 = a^2 + 2ab + b^2$

Binomial Coefficients

- For $(a + b)^3 = (a + b)(a + b)(a + b)$ we have
- $(a + b)^3 = aaa + aab + aba + abb + baa + bab + bba + bbb$
- $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
- There is only one term a^3, because there is only one possibility to form it: Choose **a** from all three factors: C(3, 3) = 1.
- There is three times the term a^2b, because there are three possibilities to choose **a** from two out of the three factors: C(3, 2) = 3.
- Similarly, there is three times the term ab^2
  (C(3, 1) = 3) and once the term b^3 (C(3, 0) = 1).

Binomial Coefficients

- This leads us to the following formula:

$$(a+b)^n = \sum_{k=0}^{n} C(n,k)\, a^{n-k} b^k \qquad \text{(Binomial Theorem)}$$

With the help of Pascal's triangle, this formula can considerably simplify the process of expanding powers of binomial expressions.

For example, the fifth row of Pascal's triangle (1 – 4 – 6 – 4 – 1) helps us to compute $(a + b)^4$:

$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

Many Useful Identities!

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n$$

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} = 0$$

$$\sum_{i=0}^{n} 2^i \binom{n}{i} = 3^n$$

Pascal's Identity

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Review Exercises

- **There must be at least two people in New York city with exactly the same number of hairs on their heads. Why?**

Typical head of hair has around 100,000 hairs. So, it is reasonable to assume that no one has more than 1,000,000 hairs on their head (m = 1 million holes).

There are more than 1,000,000 people in NYC (n is bigger than 1 million objects).
If we assign a pigeonhole for each number of hairs on a head, and assign people to the pigeonhole with their number of hairs on it, there must be at least two people with the same number of hairs on their heads.

Useful stuff to know… ☺

A bowl contains 10 red and 10 yellow balls

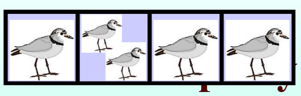
How many balls must be selected to **ensure** 3 balls of the same color?

- One solution: consider the "worst" case
  - Consider 2 balls of each color
  - You can't take another ball without hitting 3
  - Thus, the answer is 5
- Via generalized pigeonhole principle
  - How many balls are required if there are 2 colors, and one color must have 3 balls?
  - How many pigeons are required if there are 2 pigeon holes, and one must have 3 pigeons?
  - number of boxes: $k = 2$
  - We want $\lceil N/k \rceil = 3$
  - What is the minimum N?
  - $N = 5$

A bowl contains 10 red and 10 yellow balls

How many balls must be selected to **ensure** 3 yellow balls?

- Consider the “worst” case
  - Consider 10 red balls and 2 yellow balls
  - You can’t take another ball without hitting 3 yellow balls
  - Thus, the answer is 13

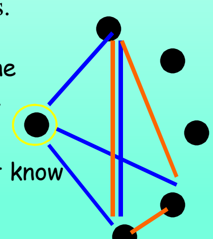


problem”

Dinner party of six: Either there is a group of 3 who all know each other, or there is a group of 3 who are all strangers.

By contradiction. Assume we have a party of six where no three people all know each other and no three people are all strangers.

Consider one person.



She either knows or doesn’t know each other person.

But there are 5 other people! So, she knows, or doesn’t know, at least 3 others.

(GPH)

Let’s say she knows 3 others.

If any of those 3 know each other, we have a blue △, which means 3 people know each other. Contradicts assumption.

So they all must be strangers. But then we have three strangers. Contradicts assumption.

The case where she *doesn’t* know 3 others is similar. Also, leads to constradiction.

So, such a party does not exist! QED

Permutations

In how many ways can 5 distinct Martians and 3 distinct Jovians stand in line, if no two Jovians stand together? Hmm…

___ M1 ___ M2 ___ M3 ___ M4 ___ M5 ___

5! X P(6,3)

Horse races

- How many ways are there for 4 horses to finish if ties are allowed?
  - Note that order does matter!
- Solution by cases
  - No ties
    - The number of permutations is P(4,4) = 4! = 24
  - Two horses tie
    - There are $C(4,2) = 6$ ways to choose the two horses that tie
    - There are $P(3,3) = 6$ ways for the "groups" to finish
      - A "group" is either a single horse or the two tying horses
    - By the product rule, there are 6*6 = 36 possibilities for this case

Horse races

- Two groups of two horses tie
  - There are $C(4,2) = 6$ ways to choose the two winning horses
  - The other two horses tie for second place
- Three horses tie with each other
  - There are $C(4,3) = 4$ ways to choose the three horses that tie
  - There are $P(2,2) = 2$ ways for the "groups" to finish
  - By the product rule, there are 4*2 = 8 possibilities for this case
- All four horses tie
  - There is only one combination for this
- By the sum rule, the total is 24+36+6+8+1 = 75

Exercises if time permits

1. List all permutations of $\{a,b,c\}$
2. How many permutations of $\{a,b,c,d,e,f,g\}$ end with a?
3. How many permutations of $\{a,b,c,d,e,f,g\}$ contain bcd?
4. How many words of length 6 over the English alphabet of 26 letters contain exactly one vowel (a,e,i,o,u)?
5. Find the coefficient of x^2y^4 in $(x+y)^6$
6. How many ways are there to get from the point (0,0) to the point (m,n) using only steps of unit up (0,+1) or one unit right (+1,0)?

TABLE 1 Combinations and Permutations With and Without Repetition.

Type	*Repetition Allowed?*	*Formula*
r-permutations	No	$\frac{n!}{(n-r)!}$
r-combinations	No	$\frac{n!}{r!\,(n-r)!}$
r-permutations	Yes	n^r
r-combinations	Yes	$\frac{(n+r-1)!}{r!\,(n-1)!}$

Combinations with Repetition (Example 5, section 6.5.3)

How many solutions does the equation

$x1 + x2 + x3 = 11$

have, where $x1$, $x2$, and $x3$ are nonnegative integers?

Solution: To count the number of solutions, we note that a solution corresponds to a way of selecting 11 items from a set with three elements so that $x1$ items of type one, $x2$ items of type two, and $x3$ items of type three are chosen.

Example, continued

Hence, the number of solutions is equal to the number of 11-combinations with repetition allowed from a set with three elements. From Theorem 2 of Section 6.5.3 it follows that there are

$C(3 + 11 - 1, 11) = C(13, 11) = C(13, 2) = (13 \cdot 12)/(1 \cdot 2) = 78$

solutions.

Example, continued

What if we have constraints, such as $x1 \geq 1$, $x2 \geq 2$, and $x3 \geq 3$?

A solution to the equation subject to these constraints corresponds to a selection of 11 items where there is at least one item of type one, two items of type two, and three items of type three. So, a solution corresponds to a choice of one item of type one, two of type two, and three of type three, together with a choice of five additional items of any type. By Theorem 2 this can be done in

$C(3 + 5 - 1, 5) = C(7, 5) = C(7, 2) = (7 \cdot 6)/(1 \cdot 2) = 21$ ways.

Thus, there are 21 solutions of the equation subject to the given constraints.