CSCE 222
Discrete Structures
Sequences & Series

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Based on Chapter 2 of Rosen
Discrete Mathematics and its Applications

Sequences (Rosen 2.4)

- **Sequences** represent **ordered lists** of elements.
- A **sequence** is defined as a function from a subset of **N** to a set S. We use the notation a_n to denote the image of the integer n. We call a_n a term of the sequence.
- **Example:**
- subset of **N**:

subset of **N**:	1	2	3	4	5	…
	↓	↓	↓	↓	↓	
S:	2	4	6	8	10	…

Sequences

- We use the notation $\{a_n\}$ to describe a sequence.
- Important: Do not confuse this with the {} used in set notation.
- It is convenient to describe a sequence with a **formula**.
- For example, the sequence on the previous slide can be specified as $\{a_n\}$, where $a_n = 2n$.

Sequences

- A more formal definition:
  A sequence $\{a_i\}$ is a function f: $A \subseteq N \rightarrow S$, where we write a_i to indicate f(i). We call a_i term I of the sequence.
- Examples:
- Sequence $\{a_i\}$, where $a_i = i$ is just $a_0 = 0, a_1 = 1, a_2 = 2, \ldots$
- Sequence $\{a_i\}$, where $a_i = i^2$ is just $a_0 = 0, a_1 = 1, a_2 = 4, \ldots$

Sequences of the form $a_0, a_1, \ldots, a_n$ are often used in computer science. (always check whether sequence starts at a_0 or a_1)
These finite sequences are also called strings. The length of a string is the number of terms in the string. The empty string, denoted by λ, is the string that has no terms.

The Formula Game

What are the formulas that describe the following sequences $a_1, a_2, a_3, \ldots$?

1, 3, 5, 7, 9, …	$a_n = 2n - 1$
-1, 1, -1, 1, -1, …	$a_n = (-1)^n$
2, 5, 10, 17, 26, …	$a_n = n^2 + 1$
0.25, 0.5, 0.75, 1, 1.25 …	$a_n = 0.25n$
3, 9, 27, 81, 243, …	$a_n = 3^n$

Strings

•Finite sequences are also called **strings**, denoted by $a_1a_2a_3…a_n$.

•The **length** of a string S is the number of terms that it consists of.

•The **empty string** contains no terms at all. It has length zero.

Geometric and Arithmetic Progressions

•Definition: A **geometric progression** is a sequence of the form

$$a, ar, ar^2, ar^3, \cdots, ar^n, \cdots$$

The **initial term *a*** and the common **ratio *r*** are real numbers

Definition: An **arithmetic progression** is a sequence of the form

$$a, a+d, a+2d, a+3d, \cdots, a+nd, \cdots$$

The **initial term *a*** and the common **difference *d*** are real numbers

Note: An arithmetic progression is a discrete analogue of the linear function f(x) = dx + a

Table 1, section 2.4

TABLE 1 Some Useful Sequences.

nth Term	*First 10 Terms*
n^2	1, 4, 9, 16, 25, 36, 49, 64, 81, 100, . . .
n^3	1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, . . .
n^4	1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, 10000, . . .
2^n	2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, . . .
3^n	3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049, . . .
$n!$	1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800, . . .

Notice differences in growth rate.

Summation

- The symbol $\sum$ (Greek letter sigma) is used to denote summation.

$$\sum_{i=1}^{k} a_i = a_1 + a_2 + \ldots + a_k$$

i is the **index of the summation**, and the choice of letter i is arbitrary;

the index of the summation runs through all integers, with its **lower limit** 1 and ending **upper limit** k.

- The limit: $\sum_{i=1}^{\infty} a_i = \lim_{n \to \infty} \sum_{i=1}^{n} a_i$

Summation

- The laws for arithmetic apply to summations

$$\sum_{i=1}^{k}(ca_i + b_i) = c\sum_{i=1}^{k} a_i + \sum_{i=1}^{k} b_i$$

Use associativity to separate the b terms from the a terms.

Use distributivity to factor the c's.

Summations you should know…

(little) Gauss in 4th grade. ☺

- What is S = 1 + 2 + 3 + … + n?

$$\begin{array}{ccccccccc} S & = & 1 & + & 2 & + & \dots & + & n \\ S & = & n & + & n-1 & + & \dots & + & 1 \\ \hline 2s & = & n+1 & + & n+1 & + & \dots & + & n+1 \end{array}$$

Write the sum.

Write it again.

Add together.

You get n copies of (n+1). But we've over added by a factor of 2. So just divide by 2.

Why is the result a whole number?

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

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Sum of first n odds.

- What is S = 1 + 3 + 5 + … + (2n - 1)?

$$\sum_{k=1}^{n}(2k-1) = 2\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1$$

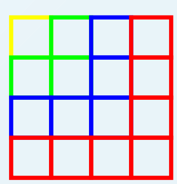
$$= 2\left(\frac{n(n+1)}{2}\right) - n$$

$$= n^2$$

Sum of first n odds.

- What is S = 1 + 3 + 5 + … + (2n - 1)?

$$= n^2$$



- What is S = 1 + r + r^2 + … + r^n

Geometric Series

$$\sum_{k=0}^{n} r^k = 1 + r + \ldots + r^n$$

Multiply by r

$$r\sum_{k=0}^{n} r^k = r + r^2 + \ldots + r^{n+1}$$

Subtract the summations

$$\sum_{k=0}^{n} r^k - r\sum_{k=0}^{n} r^k = 1 - r^{n+1}$$

factor

$$(1-r)\sum_{k=0}^{n} r^k = 1 - r^{n+1}$$

divide

$$\sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{(1-r)}$$

DONE!

- What about:

$$\sum_{k=0}^{\infty} r^k = 1 + r + \ldots + r^n + \ldots$$

If r ≥ 1 this blows up.

If r < 1 we can say something.

$$\sum_{k=0}^{\infty} r^k = \lim_{n\to\infty} \sum_{k=0}^{n} r^k$$

$$= \lim_{n\to\infty} \frac{1-r^{n+1}}{(1-r)} \qquad = \frac{1}{(1-r)}$$

Try r = ½.

Useful Summations

- Table 2, Section 2.4

Sum	Closed Form
$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\frac{ar^{n+1}-a}{r-1}, r \neq 1$
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \lvert x\rvert < 1$	$\frac{1}{1-x}$
$\sum_{k=1}^{\infty}, kx^{k-1}, \lvert x\rvert < 1$	$\frac{1}{(1-x)^2}$

Double Summations

•Corresponding to nested loops in C or Java, there is also double (or triple etc.) summation:

•Example:

$$\sum_{i=1}^{4}\sum_{j=1}^{3} ij = \sum_{i=1}^{4}(i + 2i + 3i)$$
$$= \sum_{i=1}^{4} 6i$$
$$= 6 + 12 + 18 + 24 = 60.$$

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Cardinality of Sets (Rosen 2.5)

- (Definition 4, Section 2.1)
  - Let S be a set. If there are exactly n distinct elements in S where n is a nonnegative integer, we say that S is a ***finite set*** and that n is the ***cardinality*** of S. The cardinality of S is denoted by $|S|$

Infinite Cardinality

- We previously defined the cardinality of a finite set as the number of elements in the set.
- We use the cardinalities of finite sets to tell us when they have the same size, or when one is bigger than the other.
- Now, we extend this notion to infinite sets.
- That is, we will define what it means for two infinite sets to have the same cardinality, providing us with a way to measure the relative sizes of infinite sets.

Infinite Cardinality

- How can we extend the notion of cardinality to infinite sets?
- Definition: Two sets **A and B have the same cardinality** if and only if there exists a bijection (or a one-to-one correspondence) between them, A ~ B.

We split infinite sets into two groups:

1. Sets with the **same cardinality as the set of natural numbers**
2. Sets with **different cardinality as the set of natural numbers**

Infinite Cardinality

- Definition: A set is **countable** if it is **finite** or has the same **cardinality as the set of positive integers.**
- Definition: A set is un**countable** if it is **not countable.**
- Definition: The cardinality of an infinite set S that is countable is denotes by $\aleph_0$ (where $\aleph$ is aleph, the first letter of the Hebrew alphabet). We write $|S| = \aleph_0$ and say that S has cardinality "aleph null".

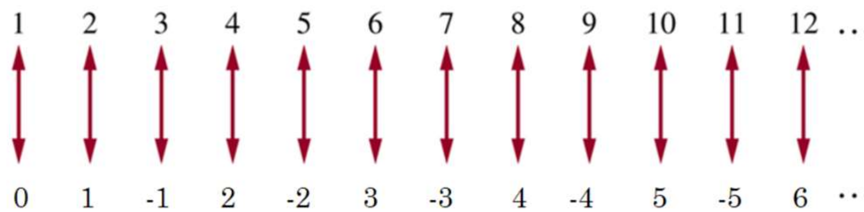
Note: Georg Cantor defined the notion of cardinality and was the first to realize that infinite sets can have different cardinalities. $\aleph_0$ is the cardinality of the natural numbers; the next larger cardinality is aleph-one $\aleph_1$, then, $\aleph_2$ and so on.

Infinite Cardinality: Integers

- Example: The set of integers is a countable set.

- Lets consider the sequence of all integers, starting with 0: 0,1,-1,2,-2,….
- We can define this sequence as a function:

$$f(n) = \begin{cases} n/2 & n \in N, even \\ \frac{-(n-1)}{2} & n \in N, odd \end{cases}$$



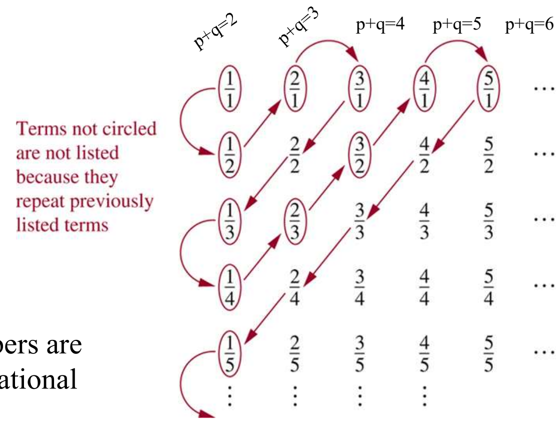
Infinite Cardinality: Rational Numbers

- Example: The set of **positive rational numbers** is a **countable** set. Hmm…

Infinite Cardinality: Rational Numbers

- Example: The set of **positive rational numbers** is a **countable** set
- Key aspect to list the rational numbers as a sequence – every positive number is the quotient p/q of two positive integers.
- Visualization of the proof.

If you think about it, all possible fractions will be in the list. For example, 145/8793 will be in the table at the intersection of the 145th row and 8793rd column, and will eventually get listed in the "waiting line."



Since all positive rational numbers are listed once, the set of positive rational numbers is countable.

Uncountable Sets

- The set of all **infinite sequences of zeros and ones** is **uncountable.**
- The set of real numbers is an uncountable set.
- The proof of this is beyond the scope of this course.
  - The text uses a diagonalization argument originated by Georg Cantor in 1891.

Matrices (Rosen 2.6)

- A **matrix** is a rectangular array of numbers.
- A matrix with m rows and n columns is called an **m×n matrix.**

- **Example: A =** $\begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & 3 \end{bmatrix}$ **is a 3 × 2 matrix**.

A matrix with the same number of rows and columns is called **square**.

Two matrices are **equal** if they have the same number of rows and columns and the corresponding entries in every position are equal.

Matrices

- A general description of an m×n matrix A = $[a_{ij}]$:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \begin{bmatrix} a_{1j} \\ a_{2j} \\ \cdot \\ \cdot \\ \cdot \\ a_{mj} \end{bmatrix}$$

j-th column of A

$$[a_{i1}, a_{i2}, \ldots, a_{in}].$$

i-th row of A

Matrix Addition (2.6.2)

Let A = [a_{ij}] and B = [b_{ij}] be m×n matrices.
The sum of A and B, denoted by A+B, is the m×n matrix that has $a_{ij} + b_{ij}$ as its (i, j)th element.
In other words, A+B = [$a_{ij} + b_{ij}$].

•Example:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 0 \\ 2 & 1 \end{bmatrix}$$

Matrix Multiplication

- Let A be an m×k matrix and B be a k×n matrix.
- The **product** of A and B, denoted by AB, is the m×n matrix with (i, j)th entry equal to the sum of the products of the corresponding elements from the i-th row of A and the j-th column of B.
- In other words, if AB = [c_{ij}], then

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ik}b_{kj}.$$

Matrix Multiplication

•A more intuitive description of calculating C = AB:

$$\begin{bmatrix} \boxed{3 \quad 0 \quad 1} \\ -2 \quad -1 \quad 4 \\ 0 \quad 0 \quad 5 \\ -1 \quad 1 \quad 0 \end{bmatrix} \qquad \begin{bmatrix} \boxed{\begin{matrix} 2 \\ 0 \\ 3 \end{matrix}} & \begin{matrix} 1 \\ -1 \\ 4 \end{matrix} \end{bmatrix}$$

- Take the first column of B
- Turn it counterclockwise by 90° and superimpose it on the first row of A
- Multiply corresponding entries in A and B and add the products:
  $3 \times 2 + 0 \times 0 + 1 \times 3 = 9$
- Enter the result in the upper-left corner of C

Matrix Multiplication

- Now superimpose the first column of B on the second, third, …, m-th row of A to obtain the entries in the first column of C (same order).
- Then repeat this procedure with the second, third, …, n-th column of B, to obtain to obtain the remaining columns in C (same order).
- After completing this algorithm, the new matrix C contains the product AB.

Matrix Multiplication

- Let us calculate the complete matrix C:

$$\begin{bmatrix} 3 & 0 & 1 \\ -2 & -1 & 4 \\ 0 & 0 & 5 \\ -1 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 2 & 1 \\ 0 & -1 \\ 3 & 4 \end{bmatrix}$$

$$C = \begin{bmatrix} 9 & 7 \\ 8 & 15 \\ 15 & 20 \\ -2 & -2 \end{bmatrix}$$

Identity Matrices (2.6.3)

- The **identity matrix of order n** is the n×n matrix $I_n = [\delta_{ij}]$, where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$:

$$\mathbf{I}_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

Multiplying an $m \times n$ matrix A by an identity matrix of appropriate size does not change this matrix:

$$AI_n = I_mA = A$$

Powers and Transposes of Matrices

- The **power function** can be defined for **square** matrices. If A is an $n \times n$ matrix, we have:

$$\mathbf{A}^0 = \mathbf{I}_n, \qquad \mathbf{A}^r = \underbrace{\mathbf{AAA}\cdots\mathbf{A}}_{r \text{ times}}.$$

- The **transpose** of an $m \times n$ matrix $A = [a_{ij}]$, denoted by A^t, is the $n \times m$ matrix obtained by interchanging the rows and columns of A.

- In other words, if $A^t = [b_{ij}]$, then $b_{ij} = a_{ji}$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$.

Powers and Transposes of Matrices

- Example: A = $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$ $\qquad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

A square matrix A is called **symmetric** if $A = A^t$.
Thus $A = [a_{ij}]$ is symmetric if $a_{ij} = a_{ji}$ for all $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$.

$$A = \begin{bmatrix} 5 & 1 & 3 \\ 1 & 2 & -9 \\ 3 & -9 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 3 & 1 \\ 1 & 3 & 1 \end{bmatrix}$$

A is symmetric, B is not.

Zero-One Matrices (2.6.4)

- A matrix with entries that are either 0 or 1 is called a **zero-one matrix**. Zero-one matrices are often used like a "table" to represent discrete structures.
- We can define Boolean operations on the entries in zero-one matrices:

a	b	$a \wedge b$
0	0	0
0	1	0
1	0	0
1	1	1

a	b	$a \vee b$
0	0	0
0	1	1
1	0	1
1	1	1

Zero-One Matrices

- Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be $m \times n$ zero-one matrices.
- Then the **join** of A and B is the zero-one matrix with (i, j)th entry $a_{ij} \vee b_{ij}$. The join of A and B is denoted by $A \vee B$.
- The **meet** of A and B is the zero-one matrix with (i, j)th entry $a_{ij} \wedge b_{ij}$. The meet of A and B is denoted by $A \wedge B$.

Zero-One Matrices

Example: $\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ $\quad \mathbf{B} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$

Join: $\mathbf{A} \vee \mathbf{B} = \begin{bmatrix} 1 \vee 0 & 0 \vee 1 & 1 \vee 0 \\ 0 \vee 1 & 1 \vee 1 & 0 \vee 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

Meet: $\mathbf{A} \wedge \mathbf{B} = \begin{bmatrix} 1 \wedge 0 & 0 \wedge 1 & 1 \wedge 0 \\ 0 \wedge 1 & 1 \wedge 1 & 0 \wedge 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

Zero-One Matrices

- Let $A = [a_{ij}]$ be an $m \times k$ zero-one matrix and $B = [b_{ij}]$ be a $k \times n$ zero-one matrix.
- Then the **Boolean product** of A and B, denoted by $A \odot B$, is the $m \times n$ matrix with (i, j)th entry $[c_{ij}]$, where
- $c_{ij} = (a_{i1} \wedge b_{1j}) \vee (a_{i2} \wedge b_{2i}) \vee \ldots \vee (a_{ik} \wedge b_{kj})$.

- Basically, Boolean multiplication works like the multiplication of matrices, but with computing $\wedge$ instead of the product and $\vee$ instead of the sum.
- In Boolean algebra, AND ≡ multiplication, and OR ≡ addition

Zero-One Matrices

Example:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\mathbf{A} \odot \mathbf{B} = \begin{bmatrix} (1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \\ (0 \wedge 1) \vee (1 \wedge 0) & (0 \wedge 1) \vee (1 \wedge 1) & (0 \wedge 0) \vee (1 \wedge 1) \\ (1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 \vee 0 & 1 \vee 0 & 0 \vee 0 \\ 0 \vee 0 & 0 \vee 1 & 0 \vee 1 \\ 1 \vee 0 & 1 \vee 0 & 0 \vee 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Zero-One Matrices

Let A be a square zero-one matrix and r be a positive integer.

The **r-th Boolean power** of A is the Boolean product of r factors of A. The r-th Boolean power of A is denoted by $A^{[r]}$.

$A^{[0]} = I_n$,
$A^{[r]} = A \odot A \odot \ldots \odot A$ (r times the letter A)