Electronically Designed Controllers using Op-Amps

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Project Outline

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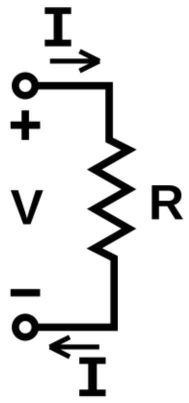
- Building Blocks of Electronically Design Controllers
- Characteristics of Op-Amp Circuits
- Different Feedback Amplifiers
- DC Motor Speed Control using Op-Amp PID Controller
- Conclusion

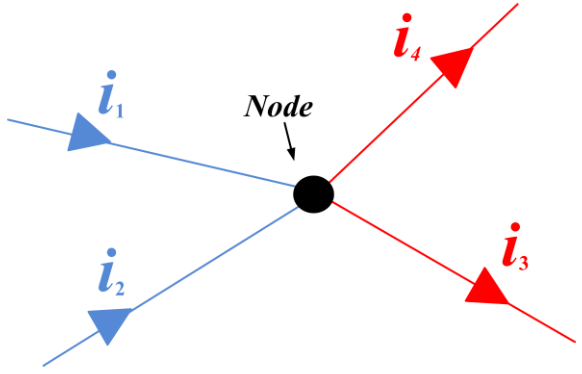
741 Op-Amp IC

Building Blocks of Electronic Controllers

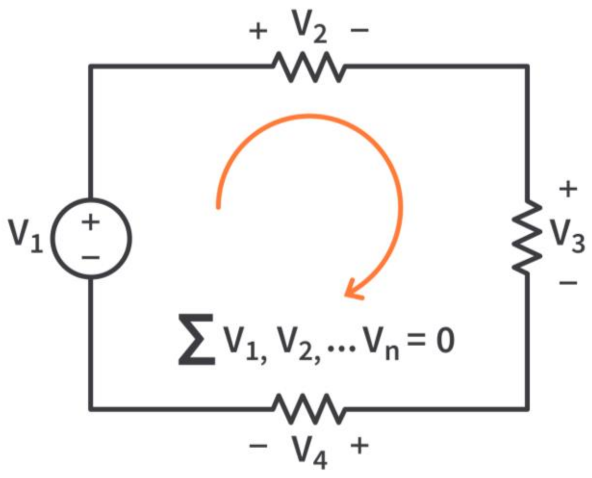
- Circuits are designed using the following concepts:
  - Current: I – the rate of change of electrical charge
  - Voltage: V – potential energy between two nodes
  - Resistance: R – resistance to the flow of current
  - Capacitance: C – the ratio of the charge on one plate of the capacitor to the voltage difference between two plates
  - Impedance: Z – the complex ratio of phasor **V** to phasor **I** where **R = Re Z** and **X = Im Z** and **Z = R ± jX**
  - We use analysis methods such as Ohm's Law, Kirchoff's Voltage and Current Laws to derive the equations for the voltage drop across various components, the current entering or leaving a node in the circuit, and the differential equations that govern these electrical circuits



Ohm's Law



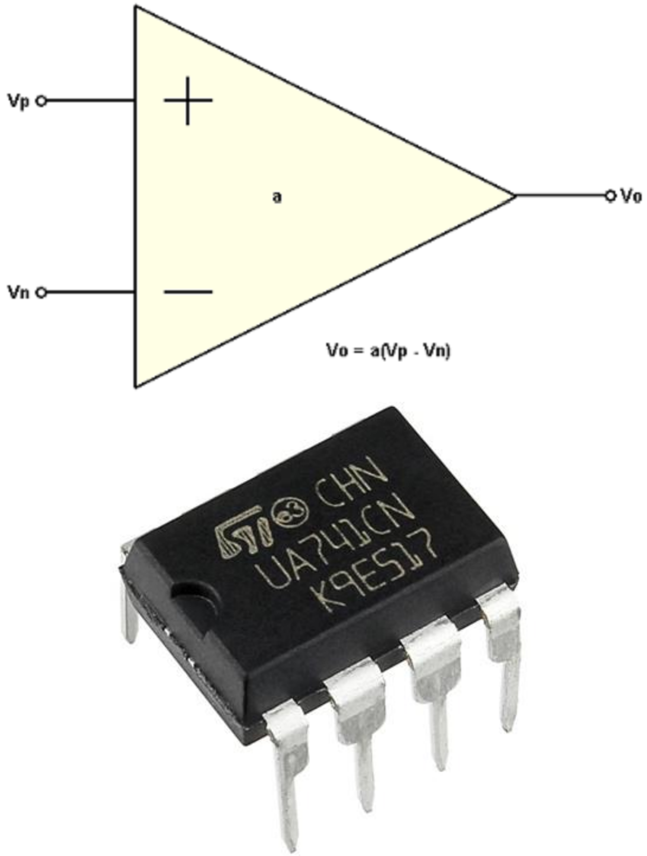
Kirchoff's Current Law



Kirchoff's Voltage Law

Operational Amplifier Description

- **Operational Amplifier**: An active circuit element designed to perform mathematical operations depending on the arrangement of passive circuit elements
  - addition/subtraction,
  - multiplication/division,
  - differentiation, and integration.
- Modern implementation of Harold Black's feedback amplifier and is a universal component that is widely used for instrumentation, control, and communication
- Very versatile as any linear system can be implemented by combining operational amplifiers with resistors and capacitors, most notably, analog proportional-integral-derivative controllers



Noninverting Op Amp Schematic Symbol and Physical Op Amp IC

Operational Amplifier Characteristics

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- **Ideal Operational Amplifiers**
  - Infinite Open Loop Gain
  - Infinite Input Impedance
  - Zero Output Impedance

The Open-Loop Transfer Function can be described as:

$$A(s) = \frac{A_0}{(1 + s/\omega_1)(1 + s/\omega_2)}$$

The value of A_0 is ideally ∞ but typically within the range of $10^6 - 10^8$ in most practical Op Amp ICs

Transfer Function has two real poles at

$\omega_1 = 10,000 \text{and} \omega_2 = 1,000,000$

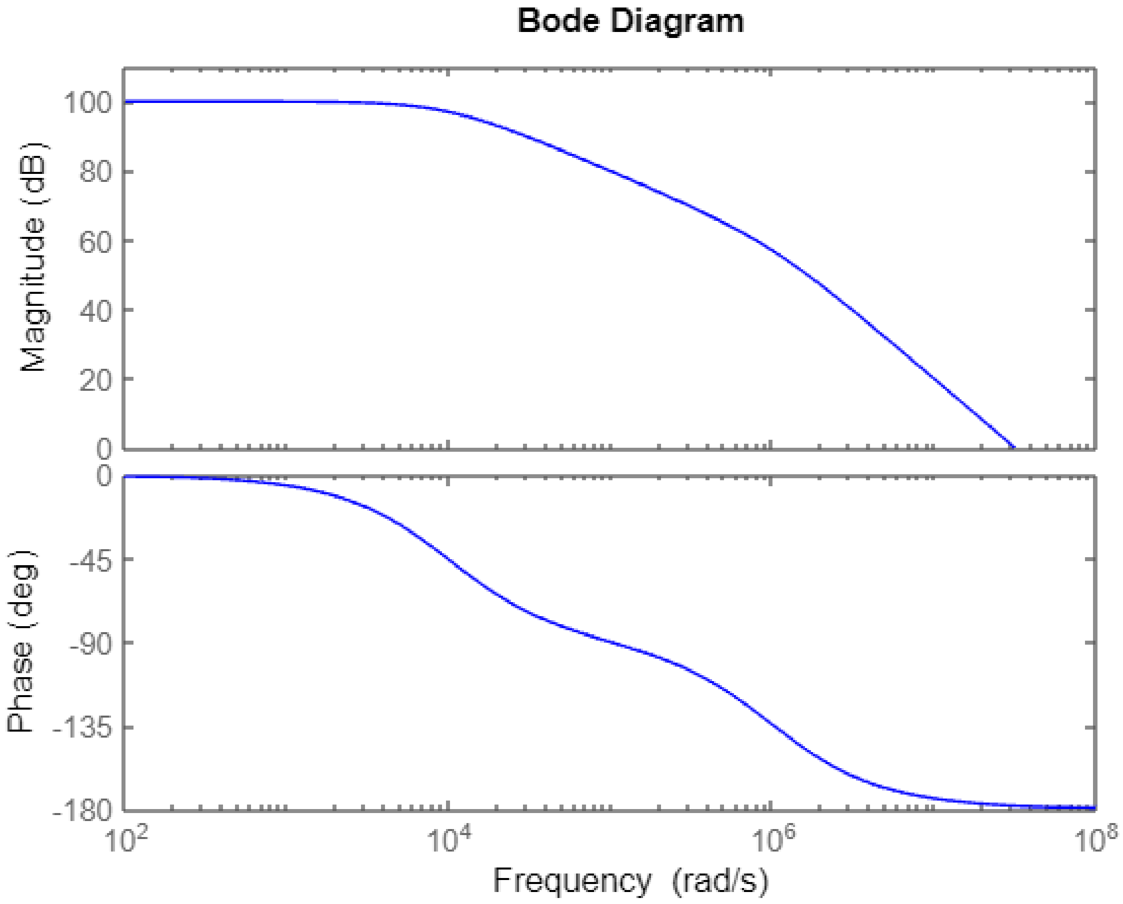
Assume Values and Create Transfer Function

```
A0 = 1e5; w1 = 1e4; w2 = 1e6;
s = tf('s');
A_open = A0/(1 + s/w1)/(1 + s/w2)
```

```
A_open =

           1e15
  ----------------------
  s^2 + 1.01e06 s + 1e10

Continuous-time transfer function.
```



Open Loop Gain of Op-Amp provides 100 dB gain for low frequencies and -180 degrees phase shift

Closing the Loop for Op Amps

- Placing two resistors attached between the output and one of the input terminals creates a feedback network, thus closing the loop
- The Feedback network B(s):
  - voltage divider between input and output.

Design Amplifier with DC Gain of 10 with $R_1 = 10k\Omega$. Solve for R_2 :

```
A0 = 10; B = 1/A0;
R1 = 1e4; R2 = R1 * (1/B - 1)
```

```
R2 = 90000
```

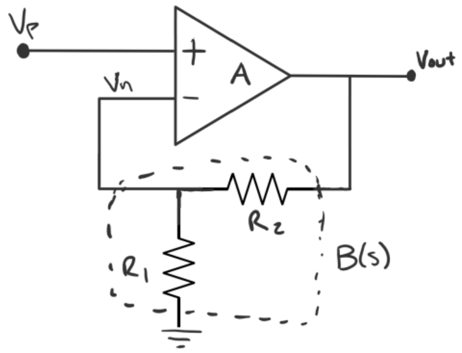
Thus, $R_2 = 90k\Omega$ would produce the desired Gain of $A = 10V/V$

Construct Closed-Loop System using Feedback() Function

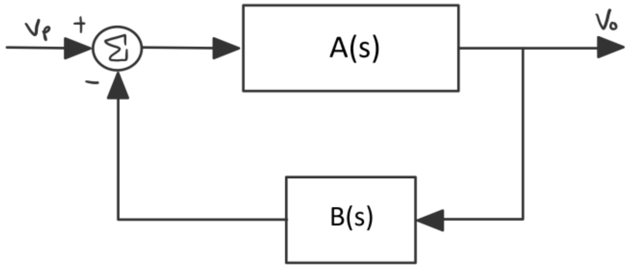
```
A_closed = feedback(A_open,B)
```

```
A_closed =

          1e15
  ----------------------
  s^2 + 1.01e06 s + 1e14
```



Noninverting Feedback Amplifier

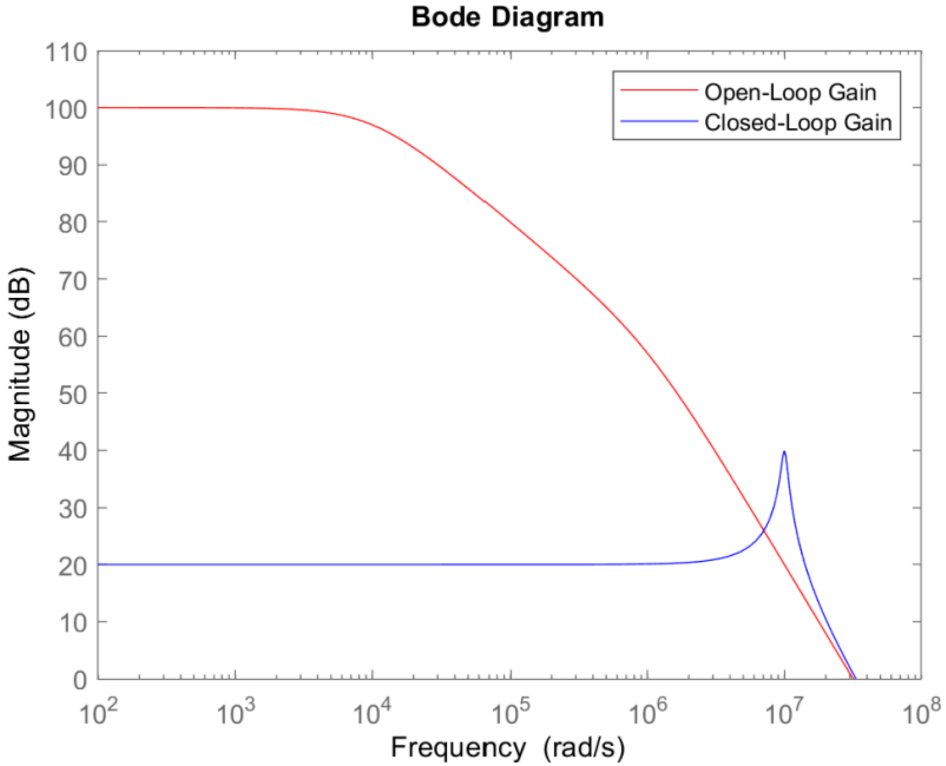


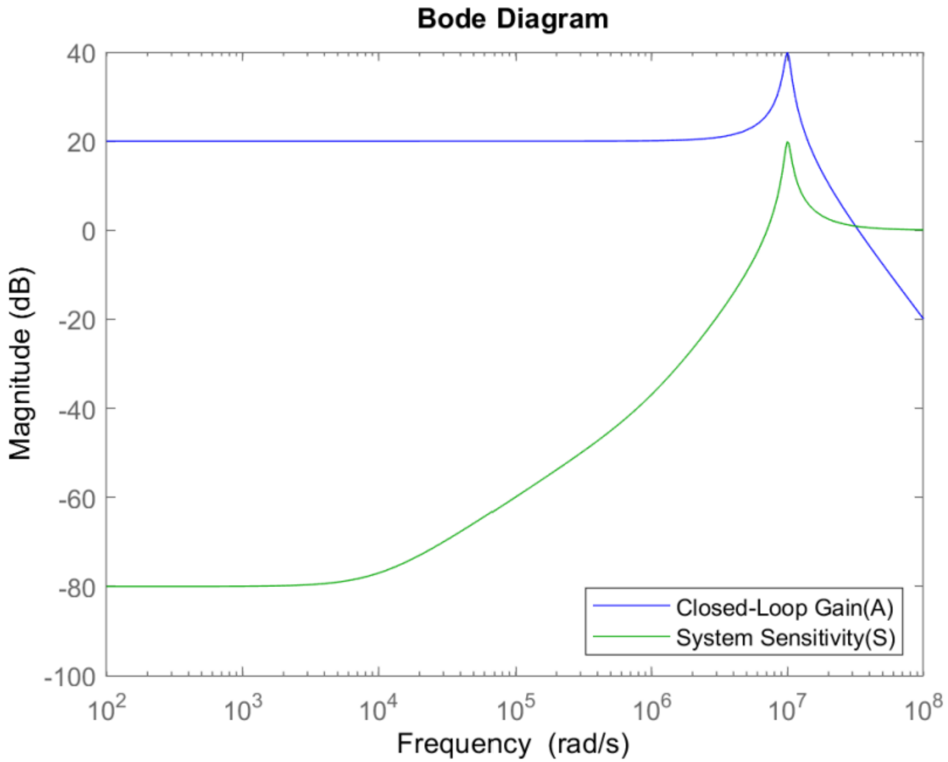
Closed-Loop Block Diagram

$$B = \frac{V_o}{V_{in}} = \frac{R_1}{R_1 + R_2}$$

$$A = \frac{V_o}{V_p} = \frac{A}{1 + AB}$$

Op-Amp Trade-Offs





- We can see that the Closed-Loop Gain is much lower for Low Frequencies, but bandwidth has increased as a trade-off
- This Gain vs Bandwidth trade-off presents a powerful tool when designing feedback circuits as it gives a drastic reduction in gain sensitivity to variations that allows making precise systems from uncertain components
- Feedback can be used to trade high gain and low robustness for low gain and high robustness

Feedback Amplifiers: Proportional Controller

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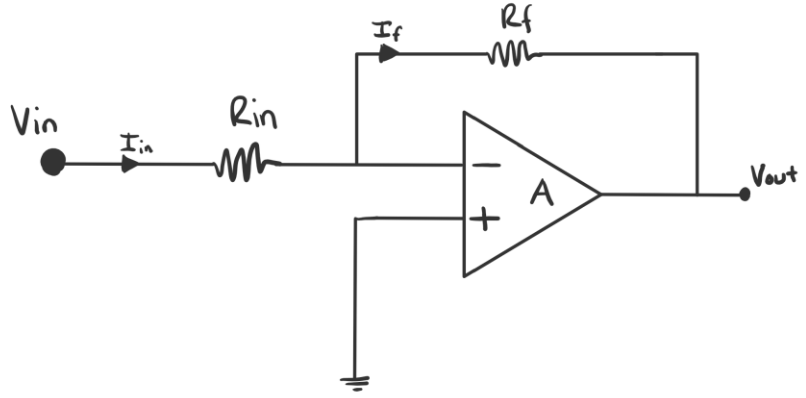
- Since the Op-Amp has ideally infinite input impedance, thus no current will flow through the + or - terminals and will result in zero voltage drop. After performing some KCL to solve for the transfer function of the system, we find:

$$V_{\text{out}} = -\frac{R_f}{R_{\text{in}}} V_{\text{in}} \leftrightarrow H(s) = -\frac{R_f}{R_{\text{in}}}$$

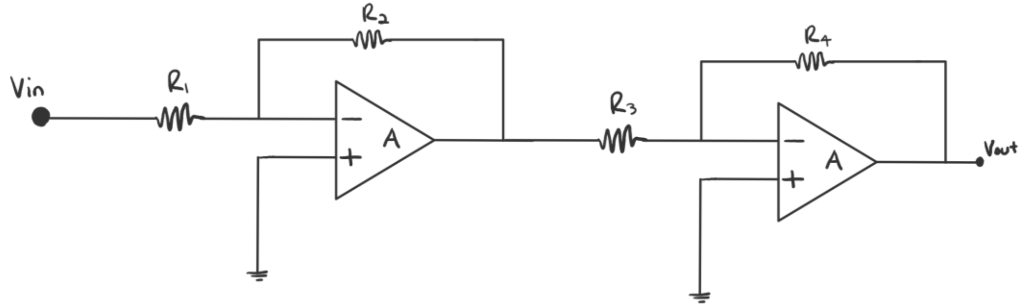
- Thus, this arrangement of components provides a negative amplitude gain to any signal that is passed through this circuit. If we cascade a second inverting amplifier with equivalent resistors, we get a Proportional Controller

$$V_o = \frac{R_2 R_4}{R_1 R_3} V_{\text{in}} \leftrightarrow H(s) = \frac{R_2 R_4}{R_1 R_3}$$

Inverting Amplifier



Proportional Controller



Feedback Amplifiers: Integrator Controller

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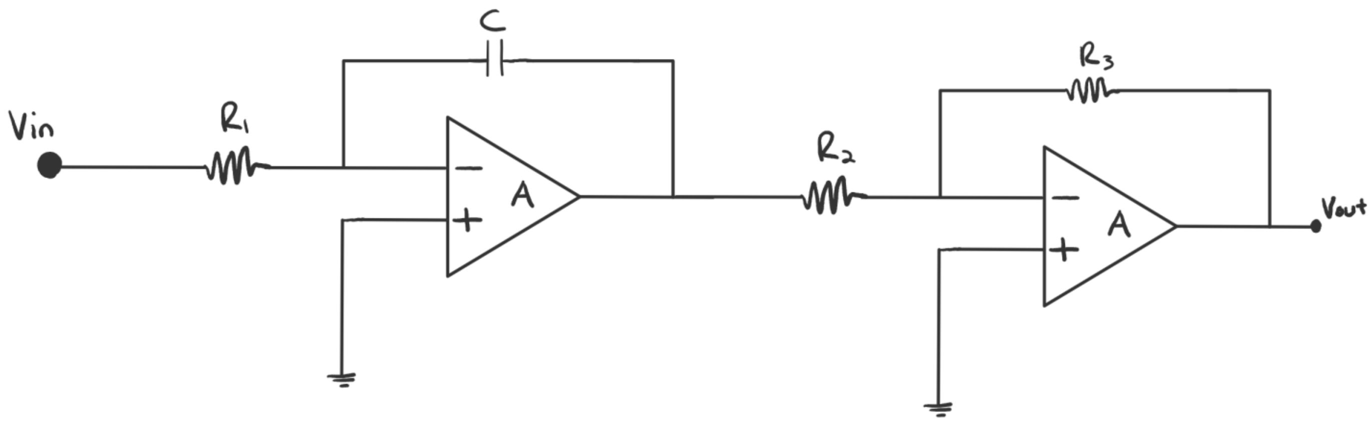
- Using two op-amps, two resistors, and two capacitors, we can form an integrator controller
- Produces an output voltage that is proportional to the area contained under the input waveform
- This controller utilizes two inverting amplifiers that serve to either prove an amplitude boost or unity gain depending on the values of resistors R2/R3
- The transfer function can be derived as:

$$V_o = \left(-\frac{1}{R_1 C}\int_0^t v_{in}(\tau)d\tau\right)\left(-\frac{R_3}{R_2}\right)V_i \leftrightarrow H(s) = \frac{R_3}{sCR_1R_2}$$

Integrator Controller

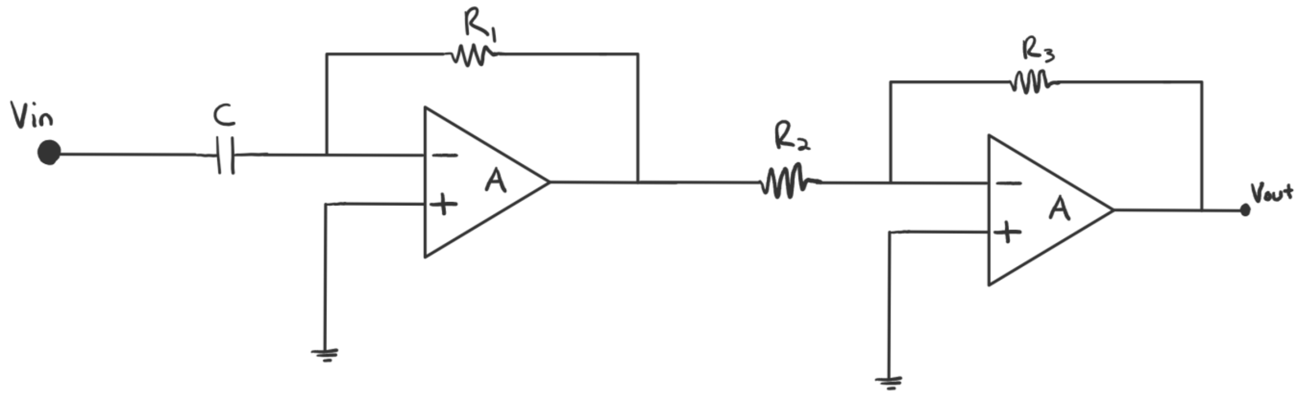


Feedback Amplifiers: Differential Controller

- Again, using two op-amps, two resistors, and two capacitors, we can form a differential controller by swopping the locations of the capacitor and first resistor in the Integrator design.
- This controller is used to provide anticipation action, as it can provide a gain to the rapid change of the control

$$V_o = \left(-R_1 C \frac{dv_{in}(t)}{dt}\right)\left(-\frac{R_3}{R_2}\right) V_i \leftrightarrow H(s) = \frac{sCR_1R_3}{R_2}$$

Differential Controller



DC Motor Speed Control

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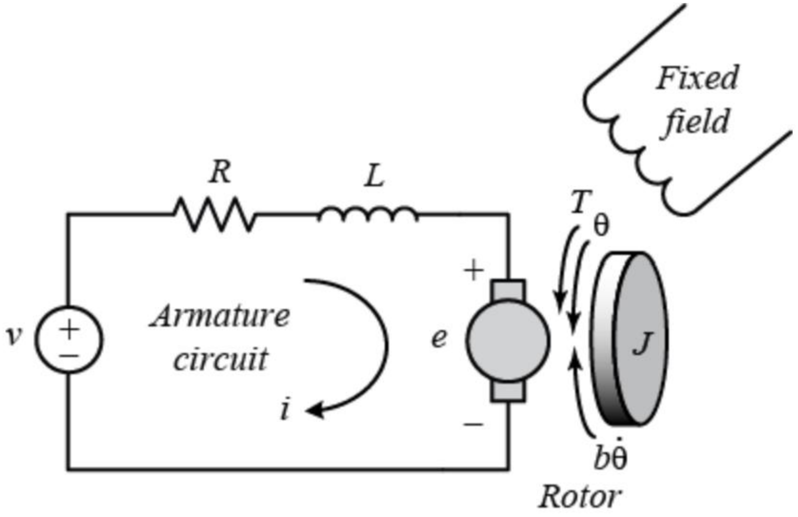
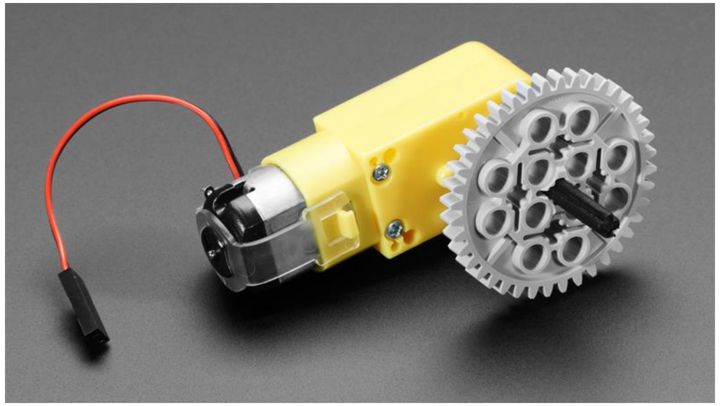
Physical Parameters and Governing Equations for Electric Motors

- Electric Motor converts between mechanical energy and electrical energy
- Inductor with a magnet that produces a current from an applied voltage across its terminals. This current then gives way to torque on the shaft of the motor which can be used to power rotation loads like wheels, propellers, etc...
- Energy can be converted from mechanical into electrical, where external torque produces a current
- Then applying some voltage can produce a desired torque
- Thus, an electrical motor's angular velocity can be controlled through the design of electrical circuits that apply this desired voltage, which confirms we can design Operational Amplifier circuits that are analogous to PID control.



Electric equivalent circuit of armature



Lego Motor

Derive Governing Eqs for Motor

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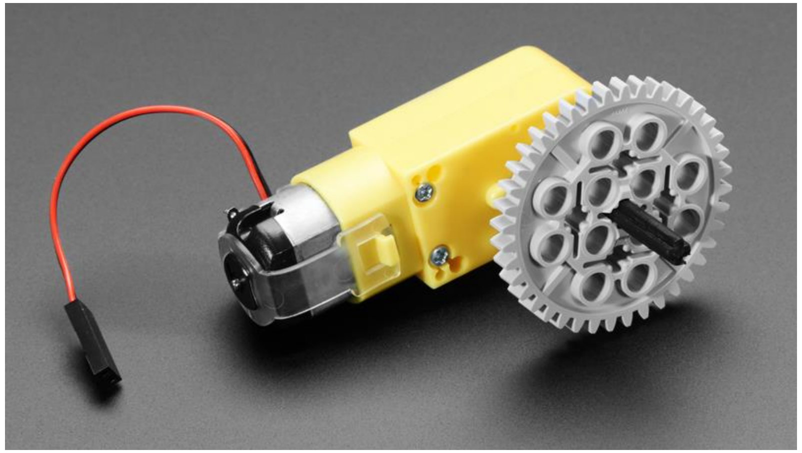
- Lego Motor
  - Used for some electronics circuits/projects
  - Operational voltage 9V DC
  - Low friction and used for high-speed design
  - Motor Constant derived from MIT Robotics Seminar
- Torque of Lego Motor
  - $I = (9V/25\Omega)$
  - Kt = 0.25 Nm/A
  - T = (Kt)(I) = (9/25)(0.25) = 0.089Nm
- EMF of Lego Motor
  - Vemf = (Kω)(ω)
  - 9V = (Kω)(35.6rad/s)
  - Kω = 0.2528, which is approximately 0.25



Lego Motor

$$T \propto K_T \times I \qquad V_{\mathrm{emf}} \propto K_\omega \times \omega$$

For all brushed motors Kt = Kw

Derive Governing Eqs for Motor

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Applying Newton's 2nd Law and KVL:

- The Current and Torque equation is:

$$J\ddot{\theta} + b\dot{\theta} = K_M \cdot I$$

where,

- J – resistance to angular acceleration
- b – resistnace to angular velocity
- I – current through the motor
- K_M – motor constant

- The EMF Voltage and Angular Velocity equations are:

$$L\frac{di(t)}{dt} + Ri(t) = V_{emf} - K_M \cdot \dot{\theta}$$

then substituting $\omega = \dot{\theta}$, we get the two equations:

$$J\dot{\omega} + b\omega = K_M \cdot i$$

$$L\frac{di(t)}{dt} + Ri(t) = V_{emf} - K_M \cdot \omega$$

Laplace Transform of Equations:

$$J\dot{\omega} + b\omega = K_M \cdot i \leftrightarrow sJ\Omega(s) + b\Omega(s) = K_M I(s)$$

$$\Omega(s)(Js + b) = K_M I(s)$$

$$L\frac{di(t)}{dt} + Ri(t) = V_{emf} - K_M \cdot \omega \leftrightarrow sLI(s) + RI(s) = V(s) - K_M\Omega(s)$$

$$I(s)(Ls + R) = V(s) - K_M\Omega(s)$$

We can substitute in equivalence of I(s) to get:

$$\frac{\Omega(s)}{V(s)} = \frac{K}{(Js + b)(Ls + R) + K^2}$$

where,

- K – motor constant
- J – resistance to angular acceleration
- b – resistance to angular velocity
- R – internal resistance
- L – inductance

MATLAB Analysis of Plant: P(s)

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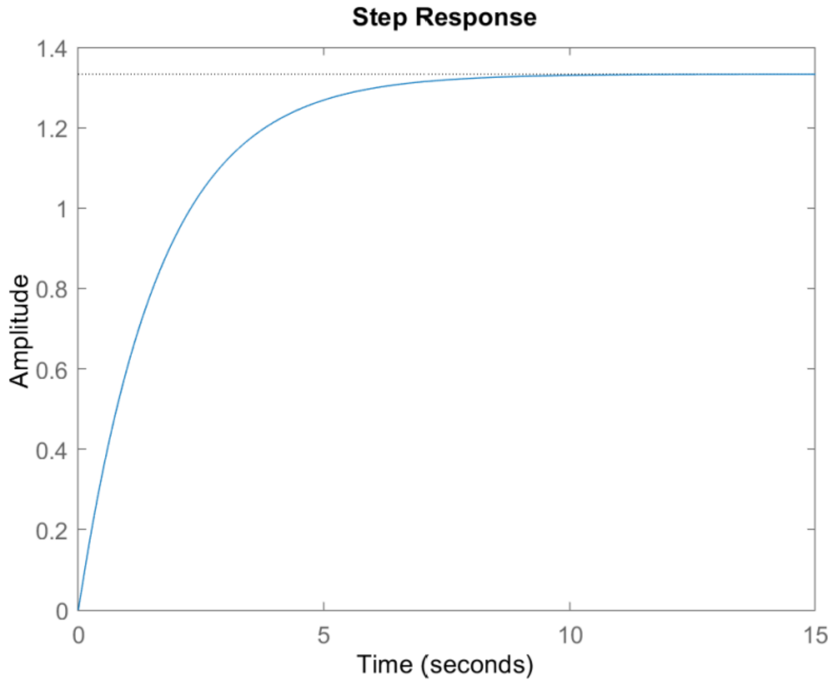
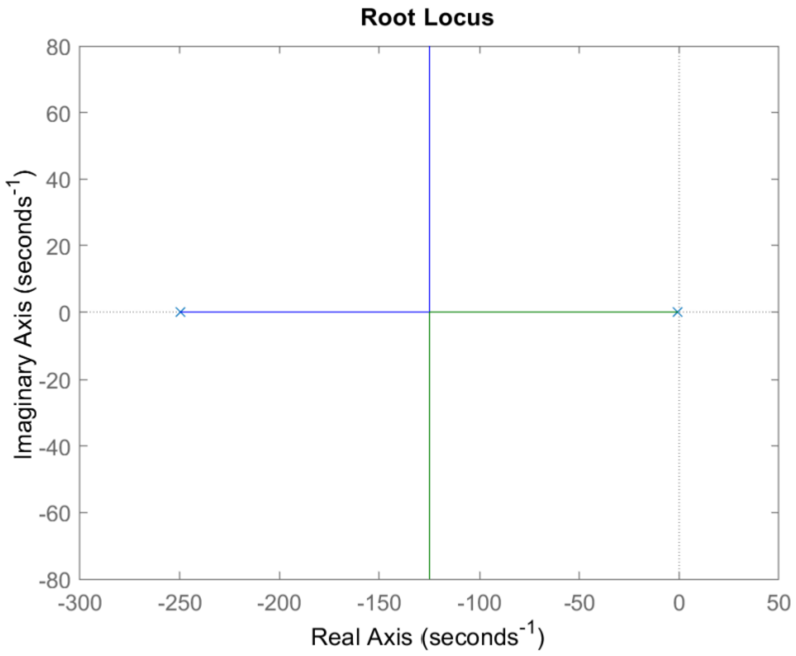
```
s = tf('s');
J = 0.0124; % kg m^2
b = 0.005; % N m s
K = 0.25; % N m / A
R = 25; % ohms
L = 0.1; % H
P = K/((J*s+b)*(L*s+R)+K^2)
```

```
P =

                0.25
  ---------------------------------
  0.00124 s^2 + 0.3105 s + 0.1875

Continuous-time transfer function.
```





- System Function contains poles at -250 and -0.61
- So, the system is open-loop stable but want to increase the step response to implement PID parameters to reach the desired velocity in a faster response
- Plan to show how different PID parameters affect performance of velocity

Effects of Proportional Control on Motor

- Proportional Control provides amplification to the error term by some proportional constant - Kp, which results in faster response by the system
- System approaches steady-state quicker with proportional control, but there is a trade-off that introduces overshoot % the faster the rise-time.
- Introduces an underdamped system the higher the proportional gain becomes

$$u(t) = K_p e(t)$$
$$U(s) = K_p E(s)$$
$$\frac{U(s)}{E(s)} = K_p$$

```
Y =

               25
  ---------------------------
  0.00124 s^2 + 0.3105 s + 25.19

Continuous-time transfer function.
```

```
t = 0:.0001:.05;
Kp = 100
```

```
Kp = 100
```

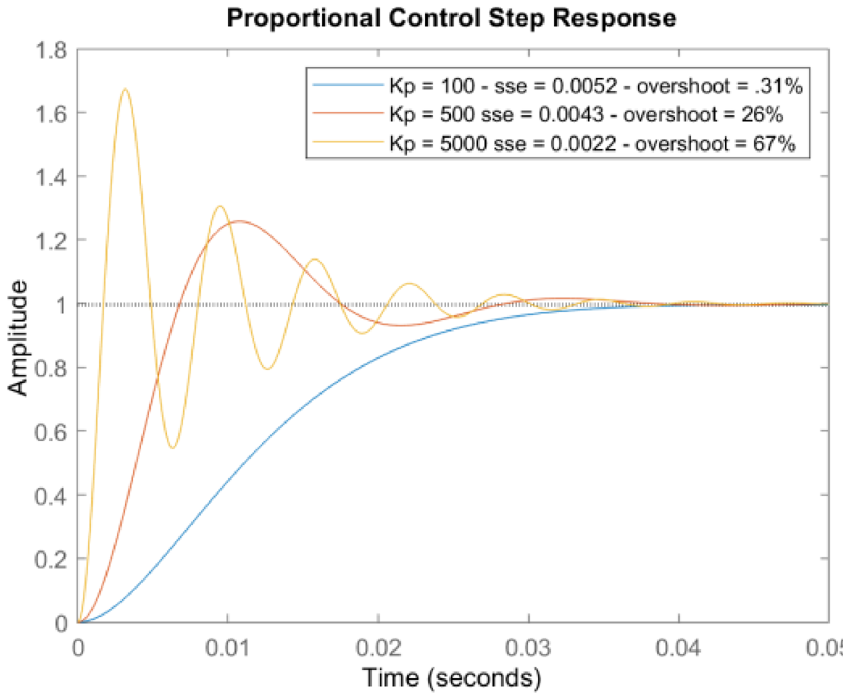
```
C = pid(Kp);
Y = feedback(C*P,1)
```

```
Y =

               25
  ---------------------------
  0.00124 s^2 + 0.3105 s + 25.19

Continuous-time transfer function.
```

```
figure
step(Y,t), hold on
y = step(Y);
sserror100=abs(1-y(end));
S1 = stepinfo(Y); overshoot1 = S1.Overshoot;
Kp = 500;
C = pid(Kp);
Y = feedback(C*P,1);
step(Y,t), hold on
y = step(Y);
sserror1000=abs(1-y(end));
S2 = stepinfo(Y); overshoot2 = S2.Overshoot;
Kp = 5000;
C = pid(Kp);
Y = feedback(C*P,1);
step(Y,t), hold off
y = step(Y);
sserror5000=abs(1-y(end));
S3 = stepinfo(Y); overshoot3 = S3.Overshoot;
title('Proportional Control Step Response')
legend('Kp = 100 - sse = 0.0052 - overshoot = .31%'
```



Effects of PI Control on Motor

- Increasing value Ki results in a decrease in steady-state error, but there is an increase in overshoot %, which solidifies that there is an inverse relationship between these system specifications.
- This phenomenon occurs because if a steady-state error exists, the integration will continue to grow larger and larger over time, which takes the control signal longer to increase enough to eliminate the error

$$u(t) = K_p e(t) + K_i \int e(t)\,dt$$

$$U(s) = K_p E(s) + K_i \frac{E(s)}{s}$$

$$U(s) = E(s)\left(K_p + \frac{K_i}{s}\right)$$

```
% Keep Kp constant and observe effects of varying Ki
t = 0:.0001:0.3;
Kp = 100, Ki = 10
```

```
Kp = 100
Ki = 10
```

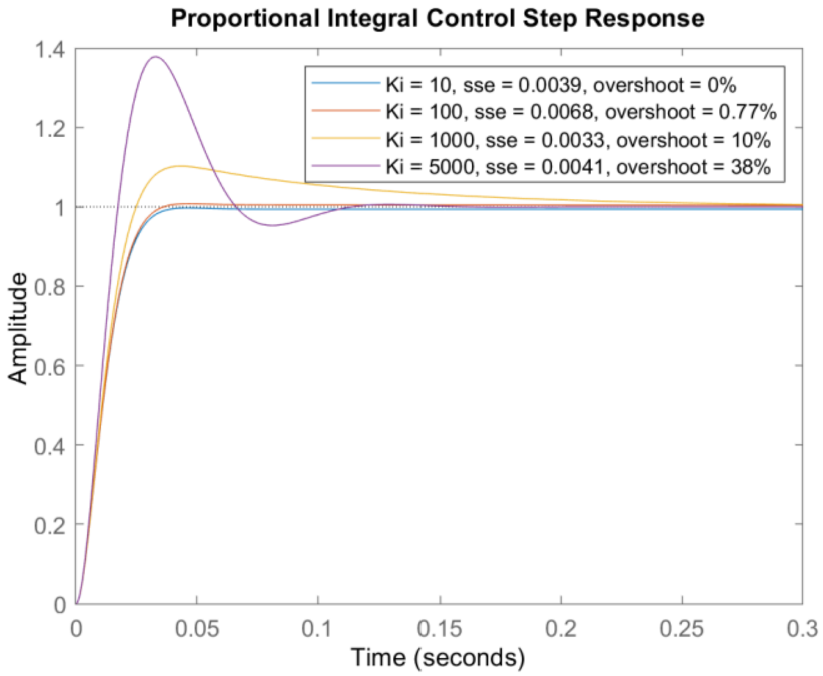
```
C = pid(Kp,Ki);
Y = feedback(C*P,1)
```

```
Y =

                25 s + 2.5
  ----------------------------------------
  0.00124 s^3 + 0.3105 s^2 + 25.19 s + 2.5

Continuous-time transfer function.
```

```
figure
step(Y,t), hold on
y = step(Y);
sserror10=abs(1-y(end));
S1 = stepinfo(Y); overshoot1 = S1.Overshoot;
Ki = 100;
C = pid(Kp,Ki);
Y = feedback(C*P,1);
step(Y,t), hold on
y = step(Y);
sserror100=abs(1-y(end));
S2 = stepinfo(Y); overshoot2 = S2.Overshoot;
Ki = 1000;
C = pid(Kp,Ki);
Y = feedback(C*P,1);
step(Y,t), hold on
y = step(Y);
sserror1000=abs(1-y(end));
S3 = stepinfo(Y); overshoot3 = S3.Overshoot;
Ki = 5000;
C = pid(Kp,Ki);
Y = feedback(C*P,1);
step(Y,t), hold off
y = step(Y);
sserror5000=abs(1-y(end));
S4 = stepinfo(Y); overshoot4 = S4.Overshoot;
title('Proportional Integral Control Step Response')
legend('Ki = 10, sse = 0.0039, overshoot = 0%','Ki = 100,
```



Effects of PD on Motor

- PD control doesn't decrease the steady-state error and the higher the Kd term the larger the steady-state error becomes.
- Once the error term stops changing, then the derivative of the error is no longer significant to affect the control signal.
- The PD control can cause overdamping to the system, though the system has a faster rise time and settling time,i.e., reaches steady-state more rapidly.

$$u(t) = K_p e(t) + K_D \frac{de(t)}{dt}$$

$$U(s) = K_p E(s) + K_D s E(s)$$

$$U(s) = E(s)\,(K_p + K_D s)$$

$$\frac{U(s)}{E(s)} = K_p + K_D s$$

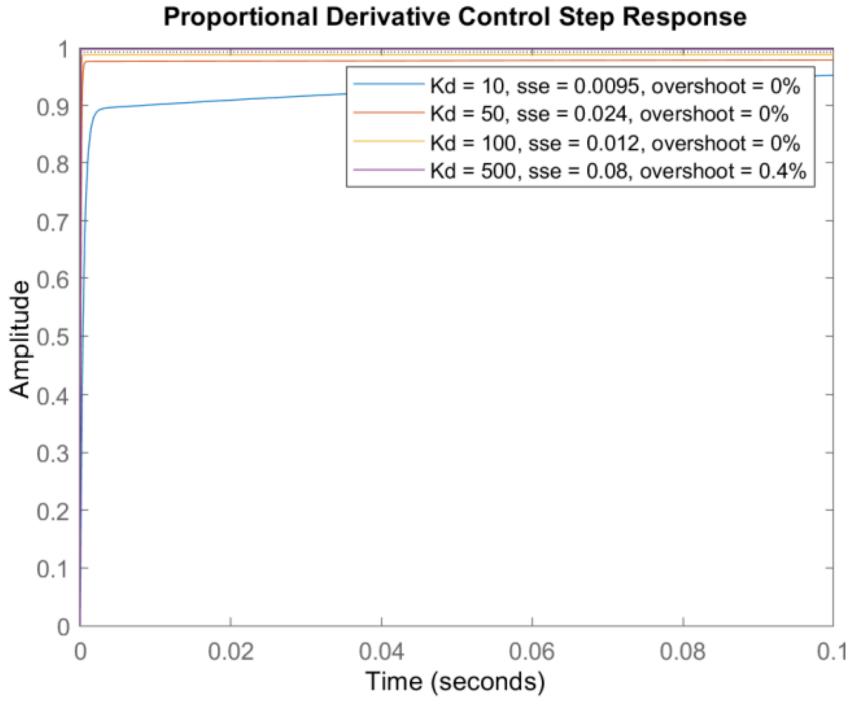
```
C = pid(Kp,Ki,Kd);
Y = feedback(C*P,1)
```

```
Y =

          2.5 s + 25
  ----------------------------
  0.00124 s^2 + 2.811 s + 25.19

Continuous-time transfer function.
```

```
figure
step(Y,t), hold on
y = step(Y);
sserror10=abs(1-y(end));
S1 = stepinfo(Y); overshoot1 = S1.Overshoot;
Kd = 50;
C = pid(Kp,Ki,Kd);
Y = feedback(C*P,1);
step(Y,t), hold on
y = step(Y);
sserror50=abs(1-y(end));
S2 = stepinfo(Y); overshoot2 = S2.Overshoot;
Kd = 100;
C = pid(Kp,Ki,Kd);
Y = feedback(C*P,1);
step(Y,t), hold on
y = step(Y);
sserror100=abs(1-y(end));
S3 = stepinfo(Y); overshoot3 = S3.Overshoot;
Kd = 500;
C = pid(Kp,Ki,Kd);
Y = feedback(C*P,1);
step(Y,t), hold off
y = step(Y);
sserror500=abs(1-y(end));
S4 = stepinfo(Y); overshoot4 = S4.Overshoot;
title('Proportional Derivative Control Step Response')
legend('Kd = 10, sse = 0.0095, overshoot = 0%','Kd = 50,
```



Effects of PID on Motor

- After experimenting:
  - Kp = 6000
  - Ki = 3600
  - Kd = 100
- This design includes all of the bene from the previously described controllers:
  - Fast reaction of proportional controller
  - Reduction in steady-state error from the PI controller
  - Dampening effect of the PD controller
- System approaches steady-state in about 25 msecs, has steady-state error of 0.009 and 0% overshoot

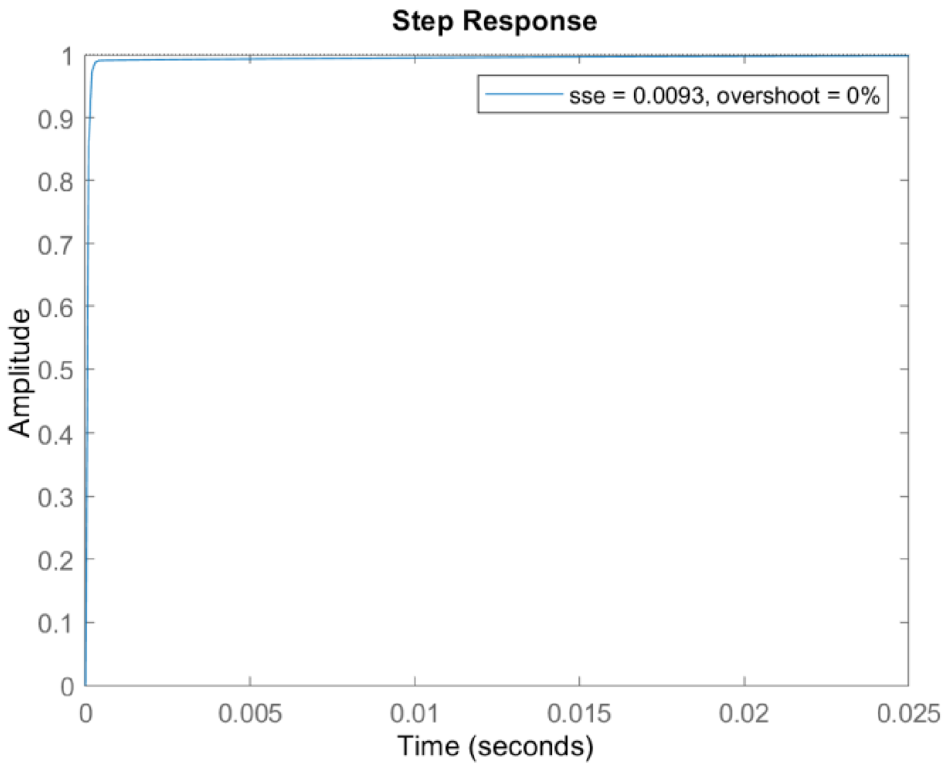
$$u(t) = K_p e(t) + K_i \int e(t)\,dt + K_D \frac{de(t)}{dt}$$

$$U(s) = K_p E(s) + K_i E(s)/s + K_D s E(s)$$

$$U(s) = E(s)(K_p + K_i/s + K_D s)$$

$$\frac{U(s)}{E(s)} = K_p + K_i/s + K_D s$$

$$\frac{Y(s)}{R(s)} = \frac{25s^2 + 1500s + 900}{0.00124s^3 + 25.31s^2 + 1500s + 900}$$



Op-Amp PID Controller Realization

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- Here, in order to realize the necessary values for the PID constants, appropriate values for the resistors and capacitors need to be chosen such that the Op-Amp circuits are analogous to the generated MATLAB transfer functions.
- The values for the PID constants were:

$$K_p = 6000 = \frac{R_2}{R_1} \Rightarrow \frac{600k\Omega}{1k\Omega}$$

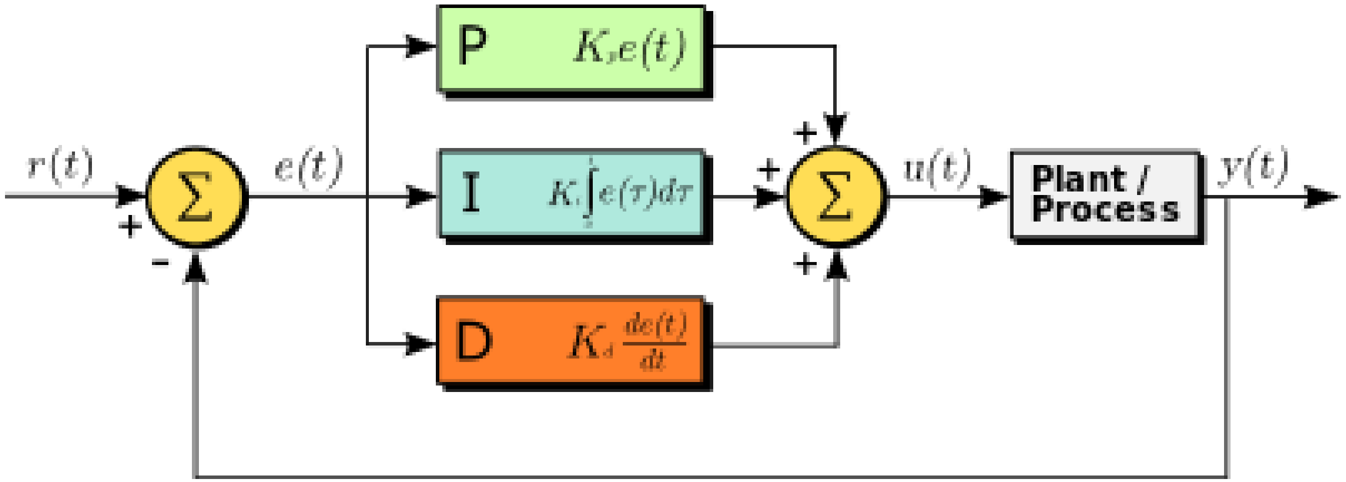
$$K_i = 3500 = -\frac{1}{R_3C_1} \Rightarrow -\frac{1}{(1k\Omega)(.278\mu F)}$$

$$K_D = 100 = -R_4C_2 \Rightarrow -(1M\Omega)(100\mu F)$$

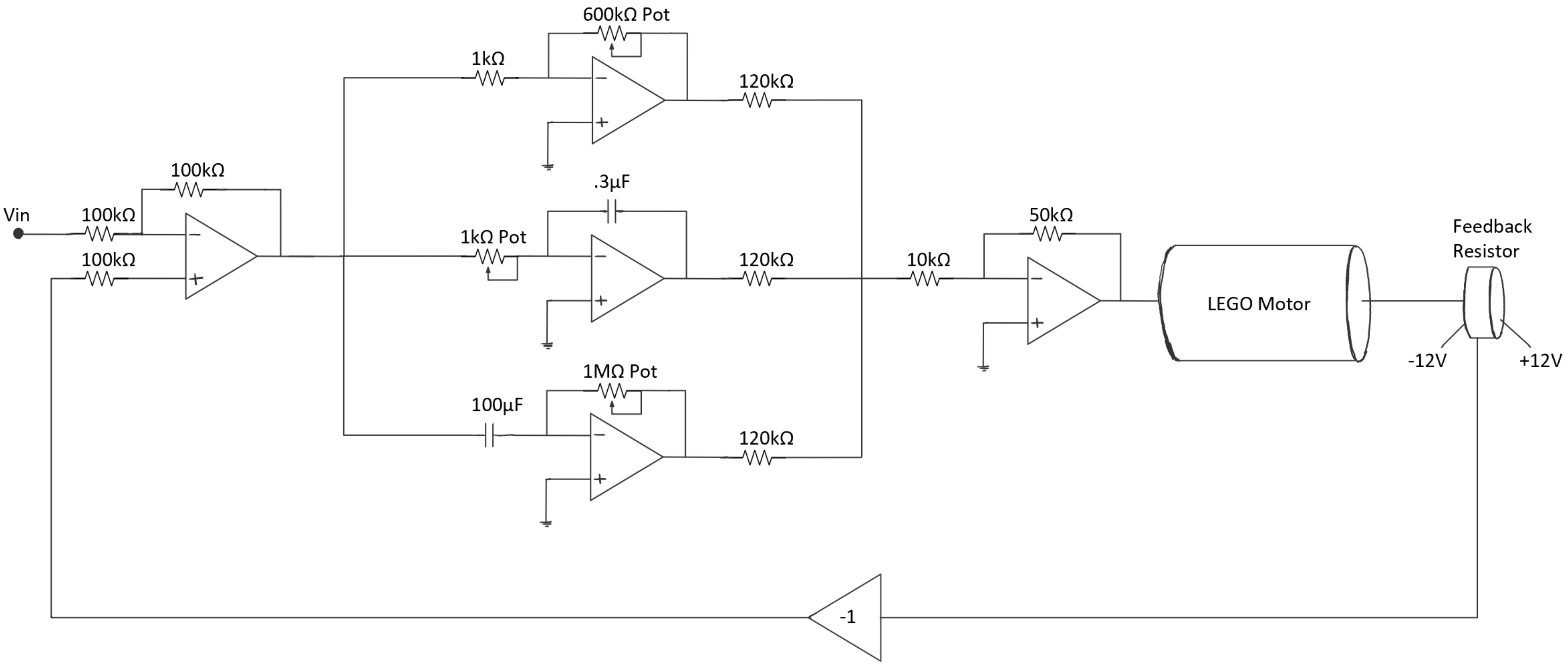
$$R_1 = 1k\Omega,\ R_2 = 600k\Omega,\ R_3 = 1k\Omega,\ R_4 = 1M\Omega$$

$$C_1 = .278\mu F \approx .3\mu F$$

$$C_2 = 100\mu F$$



Analog PID Controller using Op-Amps



References

[1] Åström Karl J., & Murray, R. M. (2021). Feedback systems: An introduction for scientists and Engineers. Princeton University Press.

[2] Alexander, C. K., & O., S. M. N. (2021). Fundamentals of Electric Circuits. McGraw-Hill.

[3] Introduction: PID Controller Design. Control Tutorials for MATLAB and Simulink - Introduction: PID Controller Design. (n.d.). Retrieved April 25, 2022, from https://ctms.engin.umich.edu/CTMS/index.php? example=Introduction§ion=ControlPID

[4] "Emf Equation of a DC Generator." Circuit Globe, November 16, 2015. https://circuitglobe.com/emf-equationof-dc-generator.html

[5] MIT . (n.d.). Random Hall Lego Robotics Seminar. Retrieved from http://web.mit.edu/sp.742/www/motor.html

[6] Feedback Amplifier Design. Feedback Amplifier Design - MATLAB & Simulink Example. (n.d.). Retrieved April 25, 2022, from https://www.mathworks.com/help/control/ug/feedback-amplifier-design.html