Digital Control Systems - Chapter 7 Notes

Stability Analysis Techniques

7.1 Introduction

In this chapter stability analysis techniques for LTI discrete-time systems are emphasized.

In general, the stability analysis techniques applicable to LTI CT systems may also be applied to analysis of LTI DT systems, if certain modifications are made. These techniques include:

- Routh-Hurwitz criterion
- Root-Locus procedures
- Frequency-Response methods
- Jury stability test is presented as well

7.2 Stability

To introduce the concepts of stability, consider the system:

$$C(z) = \frac{G(z)R(z)}{1 + \overline{GH}(z)} = \frac{K \prod^{m} (z - z_i)}{\prod^{n} (z - p_i)} R(z)$$

where z_i are the zeros and p_i are the poles of the system transfer function. Using partial-fraction expansion, we can express $C(z)$ as

$$C(z) = \frac{k_1 z}{z - p_1} + \ldots + \frac{k_n z}{z - p_n} + C_R(z)$$

where $C_R(z)$ contains the terms of $C(z)$ which originate in the poles of $R(z)$. The first n terms of the above equation are the natural-response terms of $C(z)$. If the inverse z-transform of these terms tends to zero as time increases, the system is stable, and these terms are called the $\textit{transient response}$. The inverse z-transform of the ith term is

$$z^{-1}\left[\frac{k_i z}{z - p_i}\right] = k_i (p_i)^k$$

Thus, the magnitude of p_i is less than 1, this term approaches zero as k approaches ∞. Note that the factors $(z - p_i)$ originate in the characteristic equation of the system, that is, in

$$1 + \overline{GH}(z) = 0$$

The system is stable provided that all the roots of the above characteristic equation lie inside the unit circle in the z-plane. This can also be expressed as

$$1 + \overline{GH}^*(s) = 0$$

ans since the area within the unit circle of the z-plane corresponds to the left half of the s-plane, the roots must lie in the left half plane of the s-plane for stability. The system characteristic equation may be calculated by either equation above.

For the case that a root of the characteristic equation is unity in magnitude (e.g., $p_i = 1\angle\theta$), the term would provide a constant in magnitude. Hence the natural reponse has a term that neither dies out nor becomes unbounded as k approaches infinity. If the natural response approaches a bounded non-zero steady state → system said to be *marginally stable*.

Hence, for a marginally stable system, the characteristic equation has at least one zero on the unit circle, with no zeros outside the unit circle

When having a transfer function with repeated poles, the same conditions for stability apply!

Its been demonstrated previously that for a certain discrete-time control systems, transfer functions cannot be derived. A method for finding the characteristic equation for control systems of this type will now be developed. To illustrate this method, consider the sytrem below

$$C(z) = \left[\frac{R}{2+G_2}\right](z) + \frac{\left[\dfrac{G_1G_2}{2+G_2}\right](z)}{1+\left[\dfrac{G_1G_2}{2+G_2}\right](z)}\left[\frac{(1+G_2)R}{2+G_2}\right](z)$$

Hence that part of the denominator of $C(z)$ that is independent of the input R is

$$1+\left[\frac{G_1G_2}{2+G_2}\right](z)$$

and this function set equal to zero is then the characteristic equation.

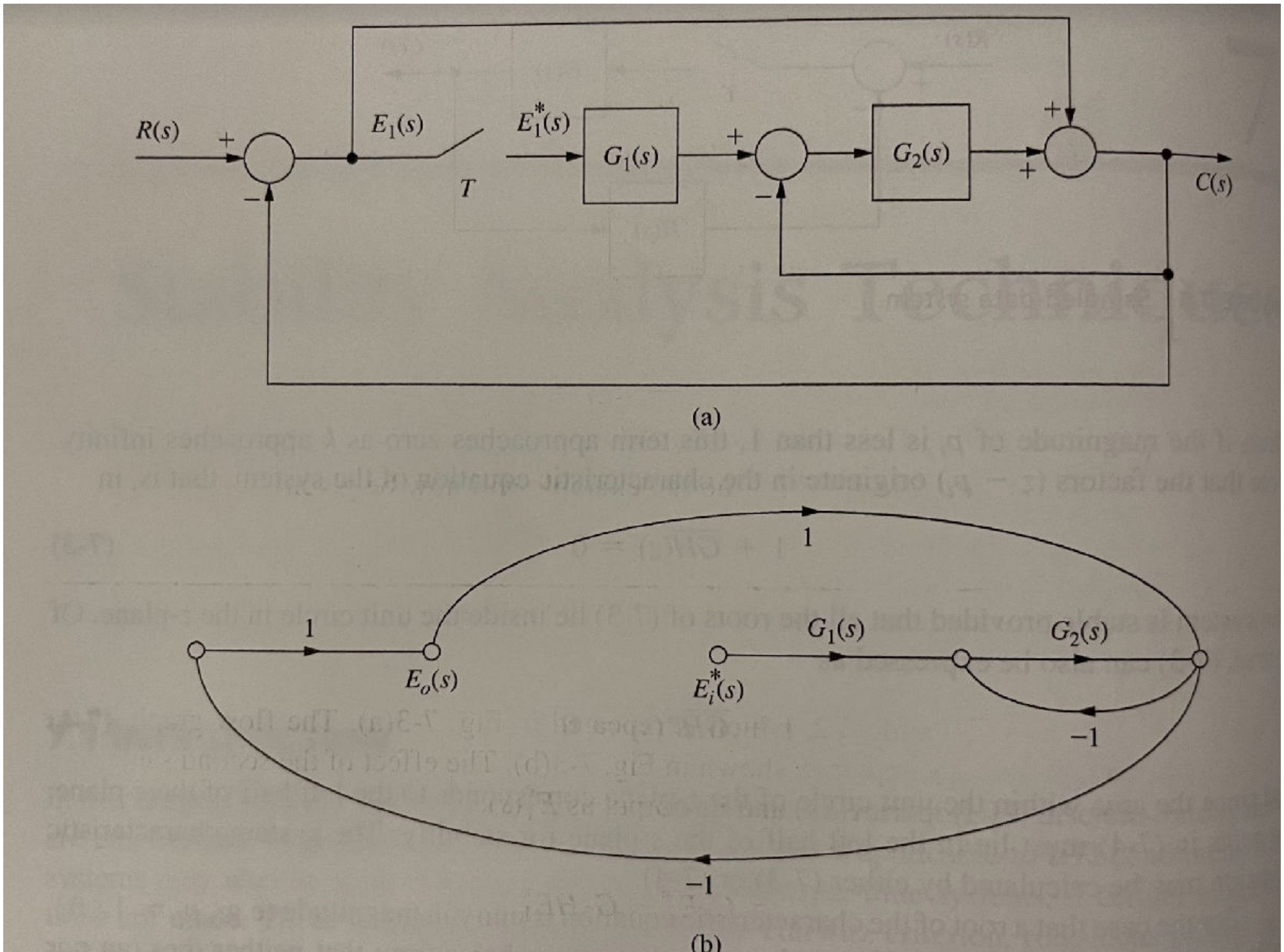


FIGURE 7-2 Discrete-time system.

This characteristic function can be developed by a different procedure. Since the stability of a linear system is independent of the input, we set $R(s) = 0$ in thet system. In addition, we open the system in front of a sampler and derive a transfer function at this open. We open at a sampler, since we can always write a transfer function if an input signal is sampled prior to being applied to a CT part of a system. If we opened the system at any other point, we would not be able to find a transfer function. We denote the input signal at this open as $E_i(s)$, and the outpu signal as $E_o(s)$. Thus, the system flow graph above with no system input and an open at the sampler appears. By Mason's gain formula, we write

$$E_o(s) = \frac{-G_1G_2}{2 + G_2} E_i^*(s)$$

Taking the z-transform of this equation, we obtain

$$E_o(z) = -\left[\frac{G_1G_2}{2 + G_2}\right](z)E_i(z)$$

We will denote this open-loop transfer function as

$$G_{op}(z) = \frac{E_0(z)}{E_i(z)} = -\left[\frac{G_1G_2}{2+G_2}\right](z)$$

For the closed-loop system, $E_i(z) = E_o(z)$, and the foregoing equations yield

$$[1 - G_{op}(z)]E_o(z) = 0$$

Since we can set initial conditions on the sytem such that $E_o(z) \neq 0$, then

$$1 - G_{op}(z) = 0$$

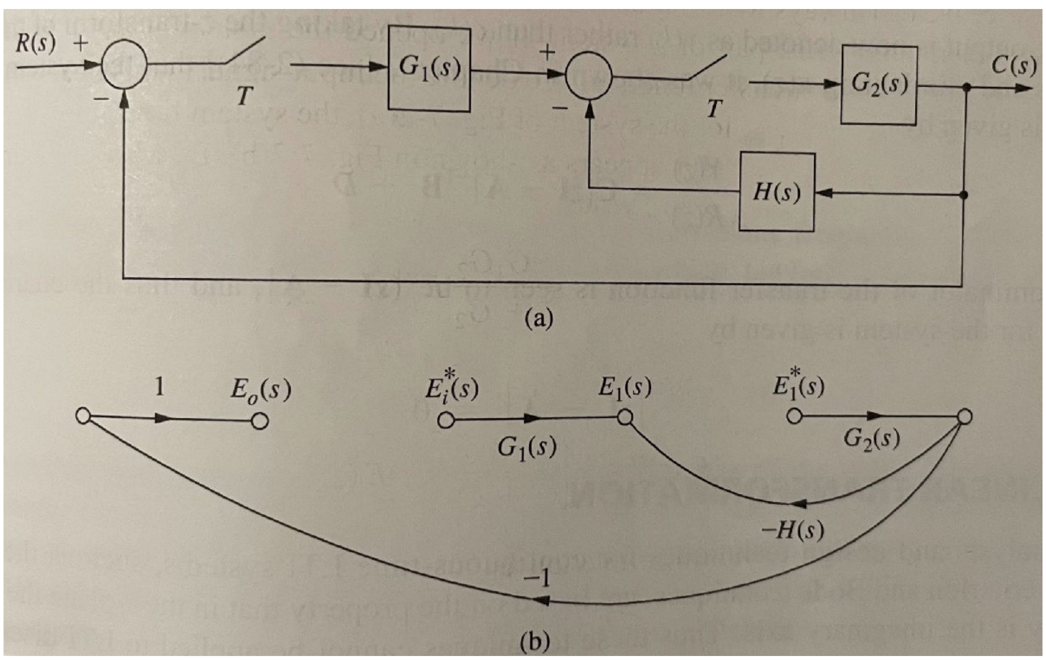
and this relationship must be the system characteristic equation. Hence the characteristic equation for this system is

$$1 + \left[\frac{G_1G_2}{2+G_2}\right](z) = 0$$

which checks the results of Example 5.3. Another example will now be given.

Example 7.1

Consider the system of Example 5.2, which is repeated in Fig. 7-3(a). The flow graph of the system is opened at the first sampler as shown in Fig. 7-3(b). The effect of the second sampler is included, by denoting its input as $E_1(s)$ and its output as $E_1^*(s)$.



From this flow graph we write

$$E_1 = G_1E_1^* - G_2HE_1^*$$

$$E_o = -G_1E_1^*$$

Starring the first equation and solving for E_1^*, we obtain

$$E_1^* = \frac{G_1^* E_i^*}{1 + \overline{G_2 H}^*}$$

Starring the second equation and substituting in the value for E_1^*, we obtain

$$E_o^* = \frac{-G_1^* G_2^*}{1 + \overline{G_2 H}^*} E_i^*$$

Since $E_i(z) = E_o(z)$ in the closed-loop system,

$$\left[1 + \frac{G_1(z) G_2(z)}{1 + \overline{G_2 H}(z)}\right] E_o(z) = 0$$

Thus, we can write the characteristic equation

$$1 + G_1(z) G_2(z) + \overline{G_2 H}(z) = 0$$

This result is verified in Example 5.2. We leave the derivation of the characteristic equation by opening the system at the second sampler as an exercise for the reader (see Problem 7.2-2)

From the discussion above, in general, the characteristic equation of a discrete system can be expressed as

$$1 + F(z) = 1 - G_{op}(z) = 0$$

where $G_{op}(z)$ is the $open\ loop\ transfer\ \ function$. The function $F(z)$ is important in analysis and design, and we will call it the $open-loop\ \ function$. For the system of Fig. 7-1, the open-loop function is $\overline{GH}(z)$ and the open-loop transfer function is $-\overline{GH}(z)$.

The characteristic equation of an LTI discrete system can also be calculated from a state-variable approach. Suppose that the state-variable model of the system of Fig. 7-1 is

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}r(k)$$

$$y(k) = \mathbf{C}\mathbf{x}(k) + Dr(k)$$

where the output is now denoted as $y(k)$ rather than $c(k)$. By taking the z-transform of these state equations and eleminating $\mathbf{x}(z)$, it was shown in Chapter 2 that the system transfer function is given by

$$\frac{Y(z)}{R(z)} = \mathbf{C}[z\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B} + D$$

The denominator of the transfer function is seen to be $|z\mathbf{I} - \mathbf{A}|$, and thus the characteristic equation for the system is given by

$$|z\mathbf{I} - \mathbf{A}| = 0$$

7.3 Bilinear Transformation

Many analysis and design techniques for CT LTI systems → Routh-Hurwitz Criterion and Bode techniques, are based on the property that in the s-plane the stability boundary is the imaginary axis. Thus these techniques cannot be applied to LTI discrete-time systems in the z-plane since the stability boundary is the unit circle. However, through the use of the transformation

$$z = \frac{1 + (T/2)w}{1 - (T/2)w}$$

or solving for w,

$$w = \frac{2}{T}\frac{z-1}{z+1}$$

the unit circle of the z-plane transforms into the imaginary axis of the w-plane. This can be seen through the following development. On the unit circle in the z-plane, $z = \varepsilon^{j\omega T}$ and

$$w = \frac{2}{T}\frac{z-1}{z+1}\Big|_{z=\varepsilon^{j\omega T}} = \frac{2}{T}\frac{\varepsilon^{j\omega T}-1}{\varepsilon^{j\omega T}+1} = \frac{2}{T}\frac{\varepsilon^{j\omega T/2}-\varepsilon^{-j\omega T/2}}{\varepsilon^{j\omega T/2}+\varepsilon^{-j\omega T/2}} = j\frac{2}{T}\tan\frac{\omega T}{2}$$

Thus it is seen that the unit circle of the z-plane transforms into the imaginary axis of the w-plane. The mappings of the primary strip of the s-plane into both the z-plane $(z = \varepsilon^{sT})$ and the w-plane are shown in Fig. 7-4. It is noted that the stable region of the w-plane is the left half-plane.

Let $j\omega_w$ be the imaginary part of w. We will refer to ω_w as the w-plane frequency. Then the above equation can be expressed as

$$\omega_w = \frac{2}{T}\tan\frac{\omega T}{2}$$

and this expression gives the relationship between frequencies in the s-plane and frequencies in the w-plane.

For small values of real frequency (s-plane frequency) such that ωT is small, the above equation becomes

$$\omega_w = \frac{2}{T}\tan\frac{\omega T}{2} \approx \frac{2}{T}\left(\frac{\omega T}{2}\right) = \omega$$

Thus, the w-plane frequency is approximately equal to the s-plane frequency for this case. The approximation is valid for those values of frequency for which $\tan(\omega T/2) \approx \omega T/2$. For

$$\frac{\omega T}{2} \le \frac{\pi}{10}, \qquad \omega \le \frac{2\pi}{10T} = \frac{\omega_s}{10}$$

the error in this approximation is less than 4%. Because of the phase lag introduced by the ZOH, we usually choose the sample period T such that the above equation is satisfied over most if not all of the band of frequencies that the system will pass (the system bandwidth). At $\omega = \omega_s/10$, the ZOH introduces a phase lag of 18°, which is an appreciable amount and, as we shall see, can greatly affect system stability.

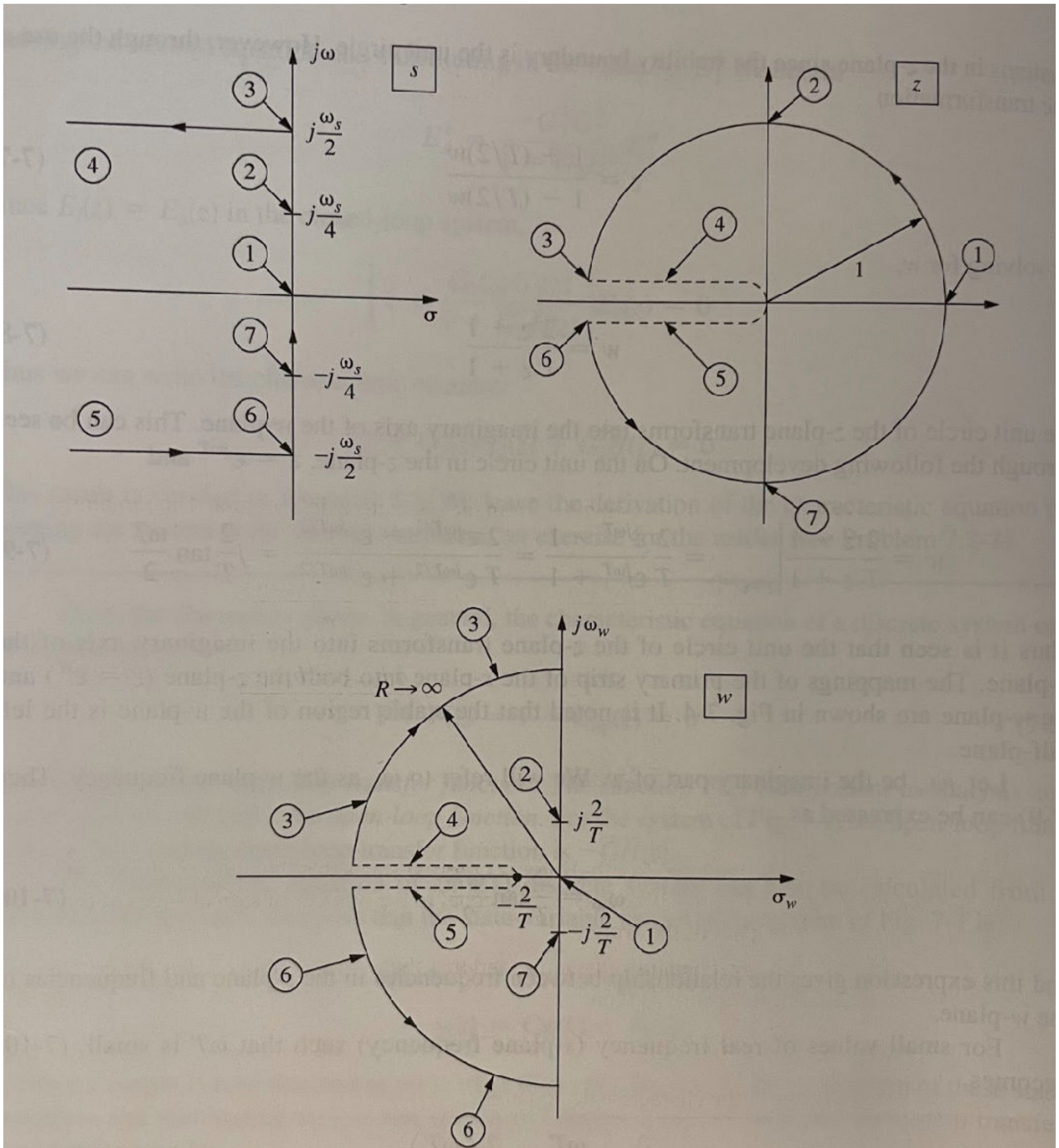


FIGURE 7-4 Mapping from s-plane to z-plane to w-plane.

7.4 The Routh-Hurwitz Criterion

The Routh-Hurwitz criterion may be used in the analysis of LTI CT systems to determine if any roots of a given equation are in the right half od the s-plane. If this criterion is applied to the characteristic of an LTI DT system when expressed as a function of z, no useful information on stability is obtained. However, if the characteristic equation is expressed as a function of the bilinear transform variable w, then the stability of the system may be determined by directly applying the Routh-Hurwitz criterion.

Table 7-1 Basic Procedure for APplying the Routh-Hurwitz Criterion

1. Given a characteristic equaiton of the form

$$F(w) = b_n w^n + b_{n-1} w^{n-1} + \dots b_1 w + b_0 = 0$$

form the Routh array as

$$\begin{bmatrix} w^n \\ w^{n-1} \\ w^{n-2} \\ \vdots \\ w^1 \\ w^0 \end{bmatrix}\begin{bmatrix} b_n & b_{n-2} & b_{n-4} & \dots \\ b_{n-1} & b_{n-3} & b_{n-5} & \dots \\ c_1 & c_2 & c_3 & \dots \\ d_1 & d_2 & d_3 & \dots \\ j_1 & & & \\ k_1 & & & \end{bmatrix}$$

2. Only the fist two rows of the array are obtained from the characteristic equation. The remaining rows are claculated as follows.

$$c_1 = \frac{b_{n-1}b_{n-2} - b_n b_{n-3}}{b_{n-1}} \qquad d_1 = \frac{c_1 b_{n-3} - b_{n-1}c_2}{c_1}$$

$$c_2 = \frac{b_{n-1}b_{n-4} - b_n b_{n-5}}{b_{n-1}} \qquad d_2 = \frac{c_1 b_{n-5} - b_{n-1}c_3}{c_1}$$

$$c_3 = \frac{b_{n-1}b_{n-6} - b_n b_{n-7}}{b_{n-1}} \qquad \vdots$$

3. Once the array has been formed, the Routh-hurwitz criterion states that the number of roots of the characteristic equation with positive real parts is equal to the number of sign changes of the coefficients in the first column of the array.

4. Suppose that the w^{i-2}th row contains only zeros, and that the w^ith row above it has the coefficients $\alpha_1, \alpha_2, \dots$ The auxiliary equation is then

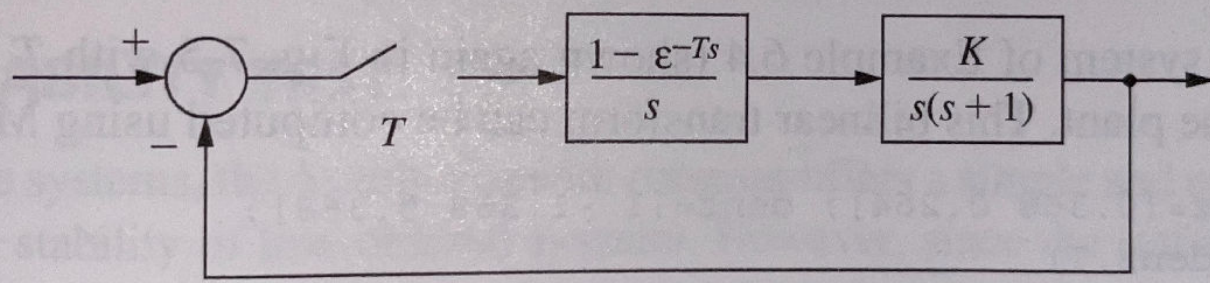
$$\alpha_1 w^i + \alpha_2 w^{i-2} + \alpha_3 w^{i-4} + \dots = 0$$

This equation is a factor of the characteristic equation.

Example 7.2

Consider the system of Example 6.7 shown in Fig. 7-5, with $T = 0.1s$. With $K = 1$, the open-loop function is

$$G(s) = \frac{1 - \varepsilon^{-Ts}}{s}\left[\frac{1}{s(s+1)}\right]$$



System for Examples 7.2 and 7.3.

Hence, from the z-transform tables we obtain

$$G(z) = \frac{0.00484z + 0.00468}{(z-1)(z-0.905)}$$

Then $G(w)$ is given by

$$G(w) = G(z)|_{z=[1+(T/2)w]/[1-(T/2)w]}$$

or

$$G(w) = \frac{-0.0000420w^2 - 0.0491w + 1}{w^2 + 0.997w}$$

The bilinear transform can also be computed using MATLAB:

```
T = 0.1; numz = [0.00484 0.00468]; denomz = [conv([1 -1],[1 -0.905])];
Gz = tf(numz, denomz,T)
```

```
Gz =

   0.00484 z + 0.00468
  ---------------------
  z^2 - 1.905 z + 0.905

Sample time: 0.1 seconds
Discrete-time transfer function.
```

```
% d2c returns a function of s, but is w in our notation here
Gw = d2c(Gz,'tustin') % Bilinear transform from discrete to continuous;
```

```
Gw =

  -4.199e-05 s^2 - 0.04913 s + 0.9995
  -----------------------------------
      s^2 + 0.9974 s + 4.429e-15

Continuous-time transfer function.
```

Then the characteristic equation for any value of K is given by

$$1 + KG(w) = (1 - 0.000042K)w^2 + (0.997 - 0.0491K)w + K = 0$$

The Routh array derived from this equation is

$$\begin{bmatrix} w^2 \\ w^1 \\ w^0 \end{bmatrix}\begin{bmatrix} 1 - 0.000042K & 3.81K \\ 0.997 - 0.491K & \\ K & \end{bmatrix} \rightarrow \begin{bmatrix} K < 23,800 \\ K < 20.3 \\ K > 0 \end{bmatrix}$$

Hence, for no sign changes to occur in the first column, it is necessary that K be in the range $0 < K < 20.3$, and this is the range of K for stability.

Example 7.3

Consider again the system of Example 6.4 but this time $T = 1s$ with gain factor of K added to the plant. This bilinear transform can be computed using MATLAB:

```
T = 1; numz = [0.368 0.264]; denomz = [1 -1.368 0.368];
Gz = tf(numz,denomz,T)
```

```
Gz =

     0.368 z + 0.264
  ---------------------
  z^2 - 1.368 z + 0.368

Sample time: 1 seconds
Discrete-time transfer function.
```

```
Gw = d2c(Gz,'tustin') % Bilinear transform; change s to w
```

```
Gw =

  -0.03801 s^2 - 0.386 s + 0.924
  ------------------------------
    s^2 + 0.924 s - 2.052e-16

Continuous-time transfer function.
```

Then the characteristic equation is given by

$$1 + KG(w) = (1 - 0.03801K)w^2 + (0.0924 - 0.386K)w + 0.924K = 0$$

The Routh array derived from this equation is

$$\begin{bmatrix} w^2 \\ w^1 \\ w^0 \end{bmatrix}\begin{bmatrix} 1 - 0.003801K & 0.924K \\ 0.924 - 0.386K & \\ 0.924K & \end{bmatrix} \rightarrow \begin{bmatrix} K < 26.3 \\ K < 2.39 \\ K > 0 \end{bmatrix}$$

Hence, the system is stable for $0 < K < 2.39$.

From our knowledge of continuous-time system, we know that the Routh-Hurwitz criterion can be used to determine the value fo K at which the root locus crosses into the right half-plane (i.e., the value of K at which the system becomes unstable). That value of K is the gain at which the system is $mariginally\ \ stable$, and thus can also be used to determine the resultant frequency of steady-state oscillation. Therefore, $K = 2.39$ in Example 7.3 is the gain for which the system is marignally stable.

In a manner similar to that employed in CT systems, the frequency of oscillation at $K = 2.39$ can be found from the w^2 row of the array. Recalling that ω_w is the imaginary part of w, we obtain the auziliary equation (see Table 7.1)

$$(1 - 0.03801K)w^2 + 0.924K|_{K=2.39} = 0.9092w^2 + 2.2084 = 0$$

or

$$w = \pm j\sqrt{\frac{2.2084}{0.9092}} = \pm j1.5585$$

Then $\omega_w = 1.5585$ and from (7-10),

$$\omega = \frac{2}{T}\tan^{-1}\frac{\omega_w T}{2} = \frac{2}{1}\tan^{-1}\left[\frac{(1.5585)(1)}{2}\right] = 1.324 \text{ rad/s}$$

and is the s-plane (real) frequency at which this system will oscillate with $K = 2.39$.

The same system was used in both examples in this section, but with different sample periods. For $T = 0.1s$, the system is stable for $0 < K < 20.3$. For $T = 1s$, the system is stable for $0 < K < 2.39$.

Hence we can see the dependency of system stability on the sample period. The degradation of stability with increasing T (decreasing sampling frequency) is due to the delay (phase lag) introduced by the sampler and data hold, which is illustrated in the frequency response of the data hold, shown in Fig. 3-13.

7.5 Jury's Stability Test

For CT systems, the Routh-Hurwitz criterion offers a simple and convenient technique for determining the stability of low-ordered systems. However, since the stability boundary in the z-plane is different from that in the s-plane, the Routh-Hurwitz criterion cannot be directly applied to discrete-time systems if the system characteristic equation is expressed as a functino of z. A stability criterion for DT systems that is similar to the Routh-Hurwitz criterion and can be applied to the characteristic equation written as a function of z is the Jury stability test.

Jury's test will now be presented. Let the characteristic equation of a DT system be expressed as

$$Q(z) = a_n z^n + a_{n-1}z^{n-1} + \ldots a_1 z + a_0 = 0, \qquad a_n > 0$$

Then form the array as shown in Table 7-2. Note that the elements of each of the even-numbered rows are the elements of the preceding row in reverse order. The elements of the odd-numbereed rows are defined as

$$b_k = \begin{bmatrix} a_0 & a_{n-k} \\ a_n & a_k \end{bmatrix}, \qquad c_k = \begin{bmatrix} b_0 & b_{n-1-k} \\ b_{n-1} & b_k \end{bmatrix}$$

$$d_k = \begin{bmatrix} c_0 & c_{n-2-k} \\ c_{n-2} & c_k \end{bmatrix} \dots$$

The necessary and sufficient conditions for the polynomial $Q(z)$ to have no roots outside or on the unit circle, with $a_n > 0$, are as follows:

$$\begin{aligned} Q(1) &> 0 \\ (-1)^n Q(-1) &> 0 \\ |a_0| &< a_n \\ |b_0| &> |b_{n-1}| \\ |c_0| &> |c_{n-2}| \\ |d_0| &> |d_{n-3}| \\ &\vdots \\ |m_0| &> |m_2| \end{aligned}$$

TABLE 7-2 Array for Jury's Stability Test

z^0	z^1	z^2	$\dots$	z^{n-k}	$\dots$	z^{n-1}	z^n
a_0	a_1	a_2	$\dots$	a_{n-k}	$\dots$	a_{n-1}	a_n
a_n	a_{n-1}	a_{n-2}	$\dots$	a_k	$\dots$	a_1	a_0
b_0	b_1	b_2	$\dots$	b_{n-k}	$\dots$	b_{n-1}	
b_{n-1}	b_{n-2}	b_{n-3}	$\dots$	b_{k-1}	$\dots$	b_0	
c_0	c_1	c_2	$\dots$	c_{n-k}	$\dots$		
c_{n-2}	c_{n-3}	c_{n-4}	$\dots$	c_{k-2}	$\dots$		
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$			
l_0	l_1	l_2	l_3				
l_3	l_2	l_1	l_0				
m_0	m_1	m_2					

Note that for a second--order system, the array contains only one row. For each additional order, two additional rows are added to the array. Note also that for an nth-order system, there are a total of $n+1$ constraints

Jury's test may be applied in the following manner:

1. Check the three conditions $Q(1) > 0, (-1)^n Q(-1) > 0,$ and$|a_0| < a_n$, which requires no calculations. Stop if any of these conditions are not satisfied.
2. Construct the array, checking the conditions above as each row is calculated. Stop if any condition is not satisfied.

Example 7.4

Consider again teh system of Example 6.4 (and Example 7.3). Suppose that a gain factor K is added to the plant, and it is desired to determine the range of K for which the system is stable. Now, for Example 6.4, the system characteristic equation is

$$1 + KG(z) = 1 + \frac{(0.368z + 0.264)K}{z^2 - 1.368z + 0.368} = 0$$

or

$$z^2 + (0.368K - 1.368)z + (0.368 + 0.264K) = 0$$

The Jury array is

$$\begin{bmatrix} z^0 & z^1 & z^2 \\ 0.368 + 0.264K & 0.368K - 1.368 & 1 \end{bmatrix}$$

The constraint $Q(1) > 0$ yields

$$1 + (0.368K - 1.368) + (0.368 + 0.264K) = 0.632K > 0 \rightarrow K > 0$$

The constraint $(-1)^2Q(-1) > 0$ yields

$$1 - 0.368K + 1.368 + 0.368 + 0.264K > 0 \rightarrow K < \frac{2.736}{0.104} = 26.3$$

The constraint $|a_0| < a_2$ yields

$$0.368 + 0.264K < 1 \rightarrow K < \frac{0.632}{0.264} = 2.39$$

Thus the system is stable for

$$0 < K < 2.39$$

The system is marignally stable for $K = 2.39$. For this value of K, the characteristic equation is

$$z^2 + (0.368K - 1.368)z + (0.368 + 0.264K)|_{K=2.39} = z^2 - 0.488z + 1 = 0$$

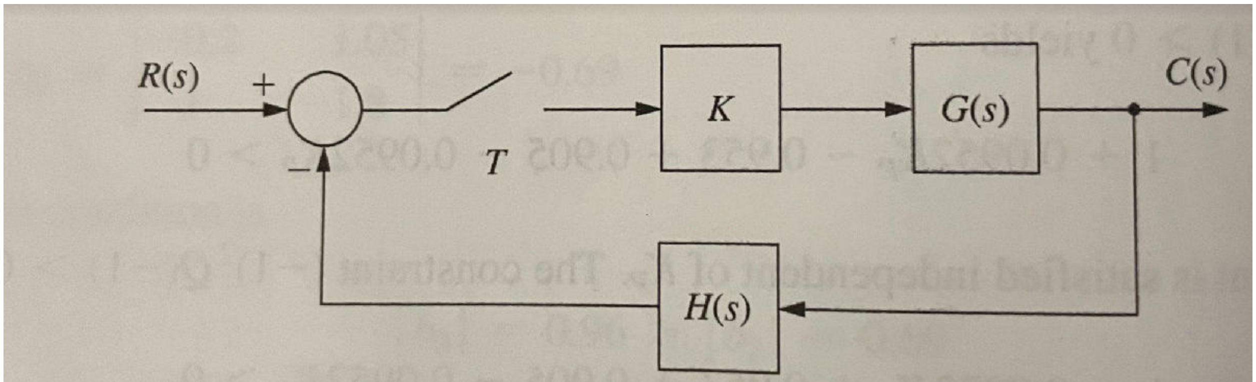
The roots of this equation are

$$z = 0.244 \pm j0.970 = 1\angle(\pm 75.9^\circ) = 1\angle(\pm 1.32\text{rad}) = 1\angle(\pm \omega T)$$

Since $T = 1$s, the system will oscillate at a frequency of 1.32 rad/s. These results verify those of Example 7.3

7.6 Root Locus

For the LTI sampled-data system of Fig. 7-6,



Sampled-data system.

$$\frac{C(z)}{R(z)} = \frac{KG(z)}{1 + K\overline{GH}(z)}$$

The system characterstic equation is, then,

$$1 + K\overline{GH}(z) = 0$$

The root locus for this system is a plot of the locus of roots from teh above equation in the z-plane as a function of K. Thus the rules of root-locus construction for DT systems are identical to those for CT systems, since the roots of any equation are dependent only on the coefficients of the equation and are independent of the designation of the variable. Since the rules for root-locus construction are numerous and appear in any standard text for CT control systems, only the most important rules will be repeated here in abbreviated form. These rules are given in Table 7-3.

Although the rules for the construction of both s-plane and z-plane root loci are the same, there are important differences in the interpretation of the root loci. For example, in the z-plane, the stable region is the interior of the unit circle. In addition, root locations in the z-plane have different meanings from those in the s-plane from the standpoint of the system time response, as seen in Fig. 6-11.

Table 7-3 Rules for Root-Locus Construction

For the characteristic equation

$1 + K\overline{GH}(z) = 0$

1. Loci originate on poles of $\overline{GH}(z)$ and terminate on the zeros of $\overline{GH}(z)$.

2. The root locus on the real axis always lies in a section of the real axis to the left of an odd number of poles and zeros on the real axis

3. The root locus is symmetrical with respect to the real axis

4. The number of aymptotes is equal to the number of poles of $\overline{GH}(z)$, n_p, minus the number of zeros of $\overline{GH}(z)$, n_z, with angles given by $(2k+1)\pi/(n_p - n_z)$.

5. The asymptotes intersect the real axis at σ, where

$$\sigma = \frac{\sum poles\ \ of\ \ \overline{GH}(z) - \sum zeros\ \ of\ \ \overline{GH}(z)}{n_p - n_z}$$

6. The breakaway points are given by the roots of

$$\frac{d[\overline{GH}(z)]}{dz} = 0$$

or equivalently,

$$den(z)\frac{d(num(z))}{dz} - num(z)\frac{d(den(z))}{dz} = 0, \quad \overline{GH}(z) = \frac{num(z)}{den(z)}$$

Example 7.7

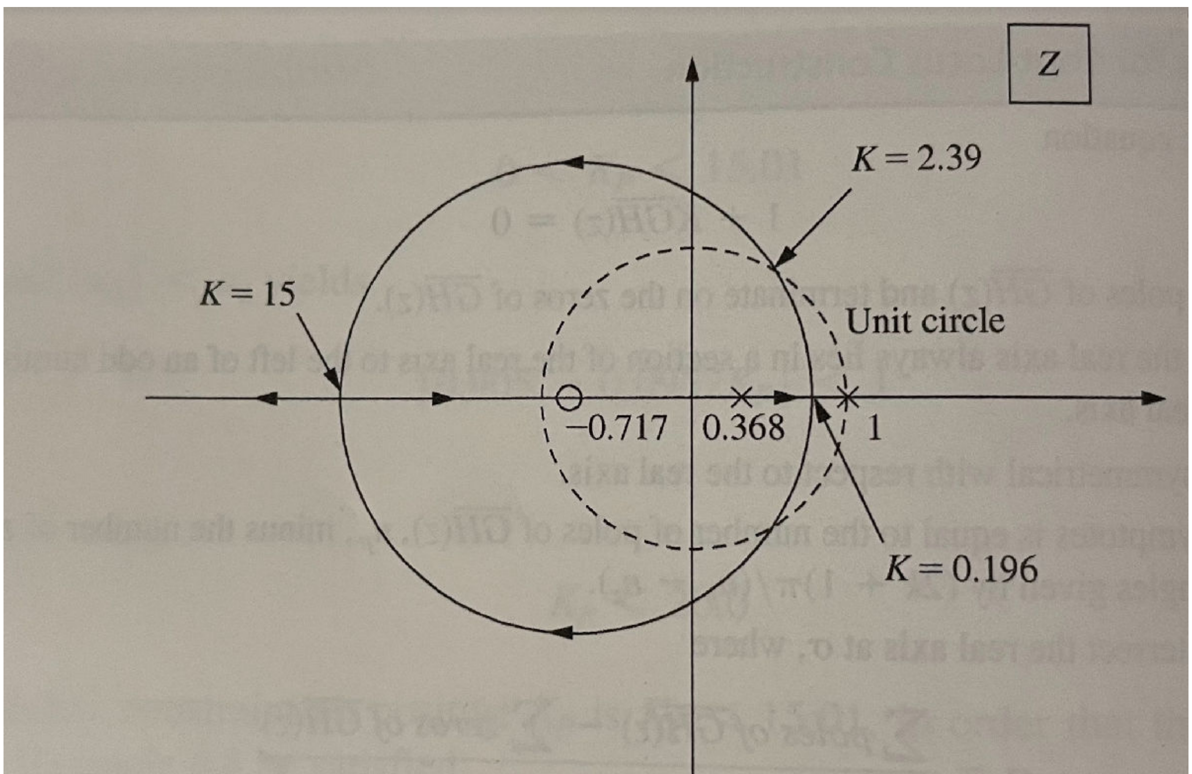
Consider again the system of Example 6.4. For this system

$$KG(z) = \frac{0.368K(z+0.717)}{(z-1)(z-0.368)}$$

Thus the loci originate at $z = 1$ and $z = 0.368$, and terminate at $z = -0.717$ and $z = \infty$. There is one asymptote, at 180°. The breakaway points, obtained from

$$\frac{d}{dz}[G(z)] = 0$$

occur at $z = 0.65$ for $K = 0.196$, and $z = -2.08$ for $K = 15$. The root locus is then shown in Fig. 7-7



Root locus for Example 7.7.

The points of intersection of the root loci with the unit circle may be found by graphical construction, the Jury stability test, or the Routh-Hurwitz criterion. To illustrate the use of Jury stability test, consider the results of Example 7.4. The value of gain for marginal stability (i.e., for the roots to appear on the unit circle) is $K = 2.39$. For this value of gain, the characteristic equation is, from Example 7.4,

$$z^2 - 0.488z + 1 = 0$$

The roots of this equation are

$$z = 0.244 \pm j0.970 = 1\angle(\pm 75.9^\circ) = 1\angle(\pm 1.32\text{rad}) = 1\angle(\pm \omega T)$$

and thus these are the points at which the root locus ccrosses the unit circle. Note that the frequency of oscillation for this case is $\omega = 1.32$ rad/s, since $T = 1$s. This was also calculated in Example 7.3 using Routh-Hurwitz criterion, and in Example 7.4 using the Jury test.

The value of the gain at points where the root locus crosses the unit circle can also be determined using the root-locus condition that at any point along the locus the magnitude of the open-loop function must be equal to 1 (i.e., $|K\overline{GH}(z)| = 1$). Using the condition and Fig. 7-8, we note that

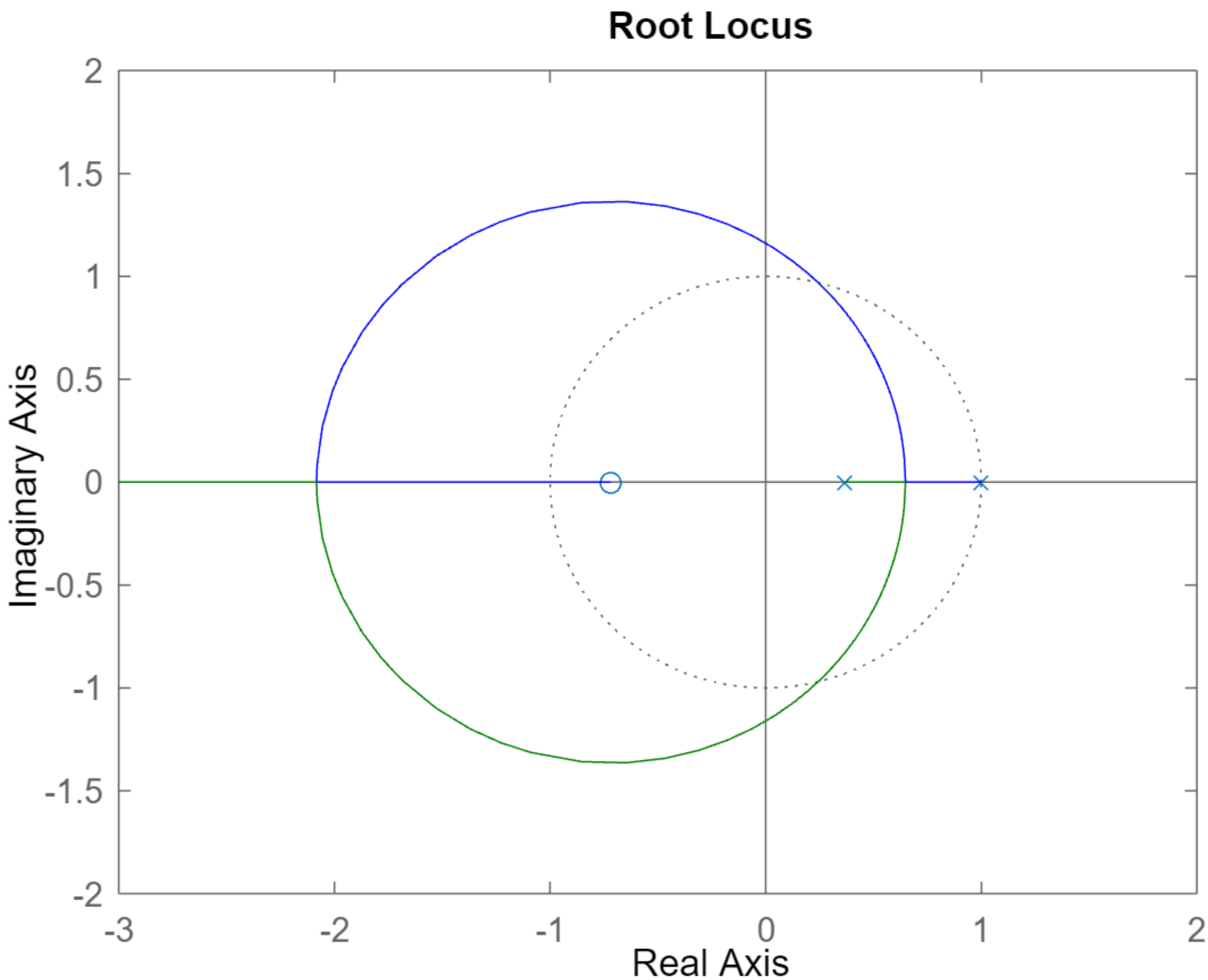
$$\frac{0.368K(Z_1)}{(P_1)(P_2)} = 1$$

From Fig. 7-8 the following values can be calculated: $Z_1 = 1.364, P_1 = 1.229$, and $P_2 = 0.978$. Using these values in the equation above yields $K = 2.39$. A MATLAB program that solves for and plots a root locus for this example is given by

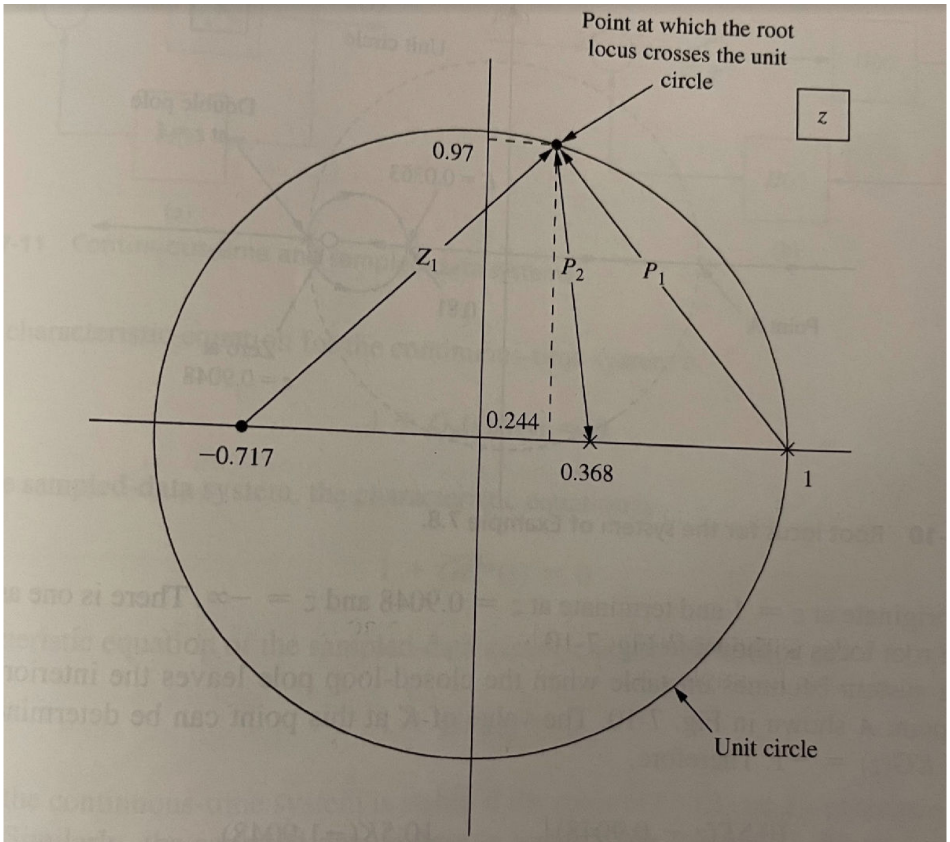
```
T = 1;
num = [0 0.368 0.264];
den = [1 -1.368 0.368];
Gz = tf(num,den,T)
```

```
Gz =

     0.368 z + 0.264
  ---------------------
  z^2 - 1.368 z + 0.368

Sample time: 1 seconds
Discrete-time transfer function.
```

```
figure()
rlocus(Gz)
axis([-3 2 -2 2]) % Set to match Figure 7-7
```



Note that the breakaway points and their respective gains K can be found directly from the MATLAB *rlocus* plot, as can the gain and locations of the roots as they cross the unit circle.



Determination of the system gain at the crossover point on the unit circle.

7.7 The Nyquist Criterion

7.8 The Bode Diagram

7.9 Interpretation of the Frequency Response

7.10 Closed-Loop Frequency Response

7.11 Summary