Digital Control Systems Homework #4

Problem 1

Problem 4.5-1 (d) and (f)

Find the modified z-transform of the following functions

(d) $E(s) = \dfrac{s+2}{s^2(s+1)}$

Start by solving the partial fraction expansion

$$E(s) = \frac{A_1}{s} + \frac{A_2}{s^2} + \frac{A_3}{s+1}$$

$$\frac{s+2}{s^2(s+1)} = \frac{A_1 s(s+1) + A_2(s+1) + A_3 s^2}{s^2(s+1)}$$

$$s + 2 = A_1 s(s+1) + A_2(s+1) + A_3 s^2$$

Substitute for $s = 0$

$$2 = A_1(0)(0+1) + A_2(0+1) + A_3(0)^2$$

$$2 = A_2$$

Thus, $A_2 = 2$

Substitute for $s = -1$

$$-1 + 2 = A_1(-1)(-1+1) + A_2(-1+1) + A_3(-1)^2$$

$$1 = A_3$$

Thus, $A_3 = 1$

Use previous values to solve for A_1 by expanding and combing like terms

$$s + 2 = A_1 s^2 + 2s + 2 + s^2$$

$$s = s^2(A_1 + 1) + s(A_1 + 2)$$

Solve for $A_1 + 2 = 1$

$$s(A_1 + 2) = s$$

$$A_1 + 2 = 1$$

$$A_1 = -1$$

Thus, the partial fraction expansion of $E(s)$ is:

$$E(s) = -\frac{1}{s} + \frac{2}{s^2} + \frac{1}{s+1}$$

Verify Partial Fraction Expansion using MATLAB residue() function

```
num = [1 2];
den = [1 1 0 0];
[R,P,K] = residue(num,den);
R', P'
```

```
ans = 1×3
     1    -1     2
ans = 1×3
    -1     0     0
```

```
syms s
Es = R(1)/(s - P(1)) + R(2)/(s - P(2)) + R(3)/(s^2 - P(3))
```

Es =

$$\frac{1}{s+1} - \frac{1}{s} + \frac{2}{s^2}$$

Now using the modified z-transform tables and using common pairs:

$$\frac{1}{s} \rightarrow \frac{1}{z-1}$$

$$\frac{1}{s^2} \rightarrow \frac{\mathrm{mT}}{z-1} + \frac{T}{(z-1)^2}$$

$$\frac{1}{s+a} \rightarrow \frac{\varepsilon^{-\mathrm{amT}}}{z - \varepsilon^{-\mathrm{aT}}}$$

We find that the modified z-transform for the following function is:

$$E(z,m) = -\frac{1}{z-1} + \frac{2\mathrm{mT}}{z-1} + \frac{2T}{(z-1)^2} + \frac{\varepsilon^{-\mathrm{mT}}}{z - \varepsilon^{-T}}$$

$$= \frac{2\mathrm{mT} - 1}{z-1} + \frac{2T}{(z-1)^2} + \frac{\varepsilon^{-\mathrm{mT}}}{z - \varepsilon^{-T}}$$

(f) $E(s) = \frac{2}{s^2 + 2s + 5}$

After examining this function we see that we can represent it as:

$$E(s) = \frac{2}{s^2 + 2s + 1 + 4} \rightarrow 2\frac{1}{(s+1)^2 + 2^2}$$

Which this function has a common modified z-transform pair:

$$\frac{1}{(s+a)^2+b^2} \to \frac{1}{b}\left[\frac{\varepsilon^{-amT}[z\sin bmT + \varepsilon^{-aT}\sin(1-m)bT]}{z^2 - 2z\varepsilon^{-aT}\cos bT + \varepsilon^{-2aT}}\right]$$

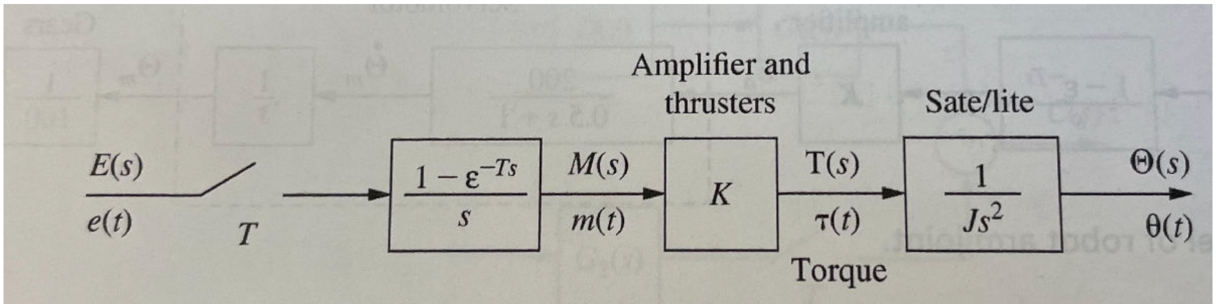
where in our case, $a = 1$ and $b = 2$. So filling in the correct values, we get

$$E(z,m) = \left[\frac{\varepsilon^{-mT}[z\sin 2mT + \varepsilon^{-T}\sin(1-m)2T]}{z^2 - 2z\varepsilon^{-T}\cos 2T + \varepsilon^{-2T}}\right]$$

Problem 2

Problem 4.3-10 (a) and (b)

Given in Fig. P4.3-10 is the block diagram of a rigid-body satellite. The control signal is the voltage $e(t)$. The zero-order hold output $m(t)$ is converted into a torque $\tau(t)$ by an amplifier and the thrusters (see Section 1.4). The system output is the attitude angle $\theta(t)$ of the satellite.



Block diagram for a satellite.

(a) Find the transfer function $\frac{\Theta(z)}{E(z)}$.

Here we define the pulse transfer function from the data hold and amplifier, and transfer function of the satellite as:

$$G_1(s) = \frac{K(1-\varepsilon^{-\mathrm{Ts}})}{s}$$

$$G_2(s) = \frac{1}{Js^2}$$

The output $\Theta(z)$ can be defined as;

$$\Theta(z) = \overline{G_1G_2}(z)E(z)$$

$$\frac{\Theta(z)}{E(z)} = \overline{G_1G_2}(z)$$

where $\overline{G_1G_2}(z) = z\{G_1(s)G_2(s)\}$

Solving for the z-transform of the product of the two transfer functions, we get:

$$\overline{G_1G_2}(z) = z\left\{(K(1-\varepsilon^{-\mathrm{Ts}}))\left(\frac{1}{\mathrm{Js}^3}\right)\right\}$$

Using the z-transform pairs for the data-hold and the satellite transfer function:

$$z\{(1-\varepsilon^{-\mathrm{Ts}})\} \to 1 - z^{-1} = \frac{z-1}{z}$$

$$z\left\{\frac{1}{s^3}\right\} = \frac{T^2 z(z+1)}{2(z-1)^3}$$

Then evaluating with these transform pairs we find that the transfer function of the system is:

$$\frac{\Theta(z)}{E(z)} = \frac{K}{J}\frac{T^2 z(z+1)}{2(z-1)^3}\frac{z-1}{z}$$

$$\frac{\Theta(z)}{E(z)} = \frac{KT^2(z+1)}{2J(z-1)^2}$$

(b) Use the results of part (a) to find the system's unit-step response, that is, the response with $e(t) = u(t)$.

Here, we make the substitution for $e(t) = u(t) \to E(z) = \frac{z}{z-1}$, and solve for $\Theta(z)$

$$\Theta(z) = \frac{KT^2(z+1)}{2J(z-1)^2}E(z)$$

$$= \frac{KT^2(z+1)}{2J(z-1)^2}\frac{z}{z-1}$$

$$= \frac{KT^2 z(z+1)}{2J(z-1)^3}$$

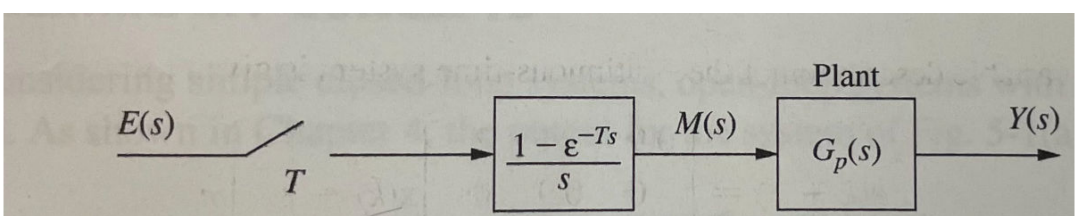
Then converting back to the time-domain using the transform pair table, we solve for $\theta(t)$:

$$\theta(t) = \frac{K}{J}\frac{t^2}{2}$$

Problem 3

Problem 4.10.1

(Hint: use MATLAB commands tf2ss, c2d, and ss2tf)



System for Problem 4.10-1.

Consider the system in Fig. P4.10-1. The plant is described by the first-order differential equation

$$\frac{\mathrm{dy}(t)}{\mathrm{dt}} + 0.05y(t) = 0.1m(t)$$

Let $T = 2\,\mathrm{secs}$.

(a) Find the system transfer function $Y(z)/E(z)$

First we solve for the plant transfer function in the s-domain

$$G_p(s) \rightarrow Y(s)(s+0.05) = M(s)(0.1)$$

$$G_p(s) = \frac{Y(s)}{M(s)} = \frac{0.1}{s+0.05}$$

Then the system's transfer function can be described as the pulse transfer function:

$$\frac{Y(z)}{E(z)} = z\left\{\frac{(1-\varepsilon^{-\mathrm{Ts}})}{s}G_p(s)\right\}$$

$$= z\left\{\frac{(1-\varepsilon^{-\mathrm{Ts}})}{s}\frac{0.1}{s+0.05}\right\}$$

$$= 1 - z^{-1}\,z\left\{\frac{0.1}{s(s+0.05)}\right\}$$

Here, we can factor out a value of 2 from the numerator of $G_p(s)$ and use the transform pair:

$$\frac{a}{s(s+a)} \leftrightarrow \frac{z(1-\varepsilon^{-\mathrm{aT}})}{(z-1)(z-\varepsilon^{-\mathrm{aT}})}$$

where $a = 0.05$ and $T = 2$ secs. Substituting and evaluating, we find:

$$\frac{Y(z)}{E(z)} = 2\left(\frac{z-1}{z}\right)\left(\frac{z(1-\varepsilon^{-0.1})}{(z-1)(z-\varepsilon^{-0.1})}\right)$$

$$= 2\frac{(1-0.905)}{(z-0.905)}$$

$$= \frac{0.1903}{z-0.905}$$

Thus, the discrete transfer function for this system is

$$\frac{Y(z)}{E(z)} = \frac{0.1903}{z-0.905}$$

```
% Verify with MATLAB
syms z
T = 2; a = 0.05;
num = 2*(1-exp(-a*T))
```

```
num = 0.1903
```

```
den = z - round(exp(-a*T), 3)
```

den =

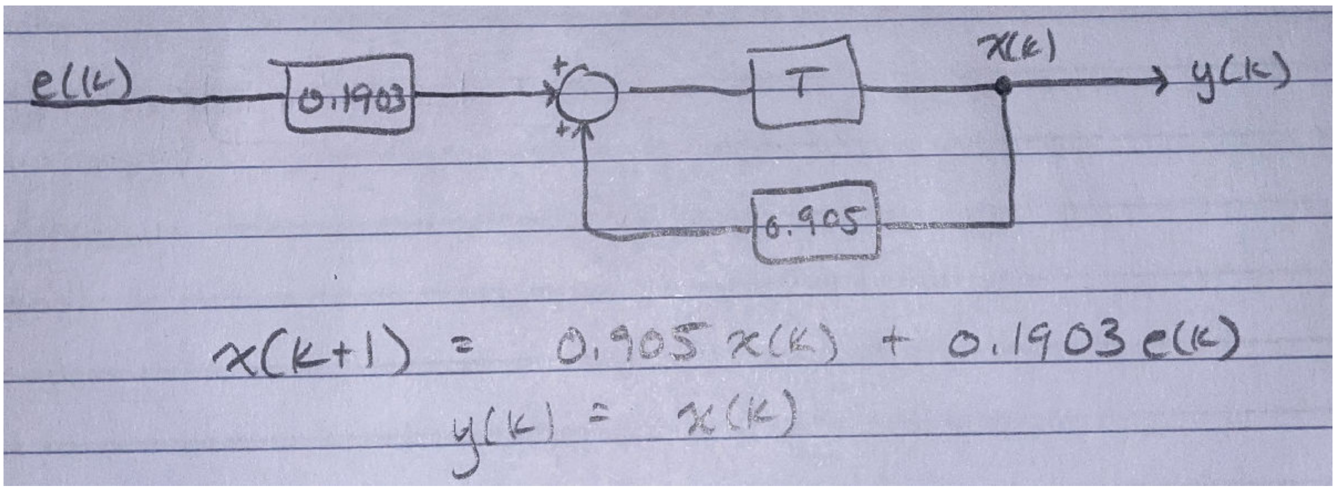
$$z - \frac{181}{200}$$

```
den = z - 0.905
```

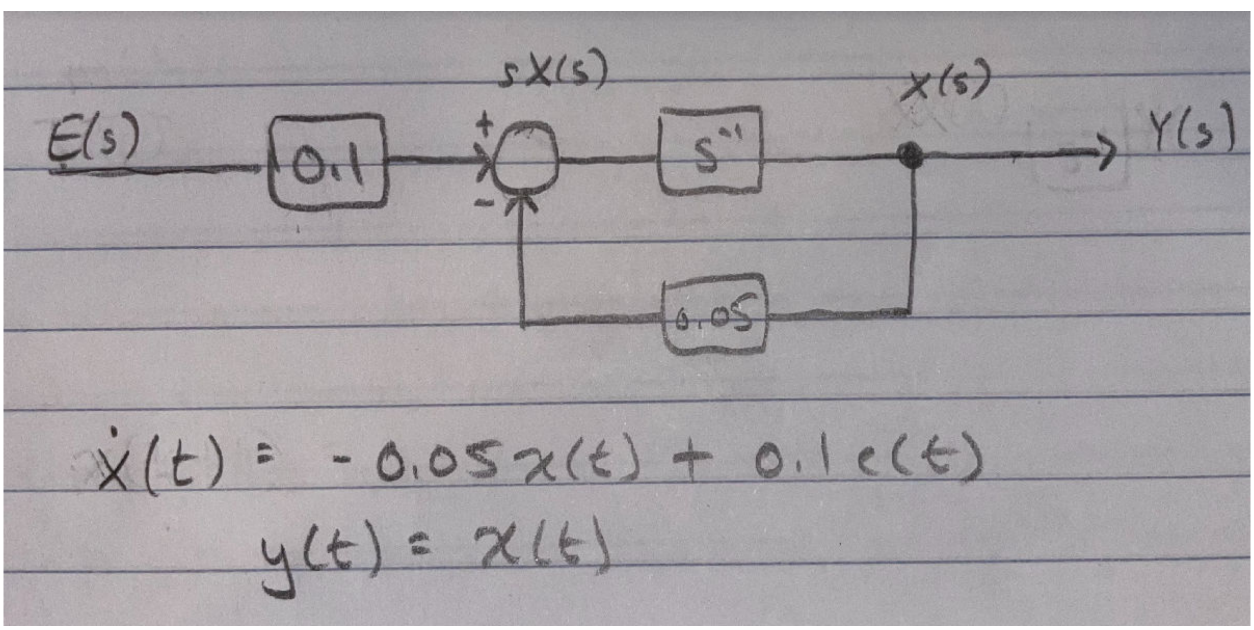
den =

$$z - \frac{181}{200}$$

(b) Draw a discrete simulation diagram, using the results of part (a), and give the state equations for this diagram.



(c) Draw a continuous-time simulation diagram for $G_p(s)$, and given the state equations for this diagram.



(d) Use the state-variable model of part (c) to find a discrete state model for the system. The state vectors of the discrete system and the continuous-time system are to be the same.

The continuous-time state model can be decribed by

$$\dot{x}(t) = \mathbf{A}_c x(t) + \mathbf{B}_c e(t)$$

$$y(t) = \mathbf{C}_c x(t) + D_c e(t)$$

Here, the state model has only one state variable $x(t)$, so the matrix notation does not apply, as we are only dealing with a scalar state variable.

We wish to find the discrete state model for the system, which will be described by the following state model:

$$x(k+1) = \mathbf{A}x(k) + \mathbf{B}e(k)$$

$$y(k) = \mathbf{C}x(k) + De(k)$$

From the continuous-time state model, we state the known variables that will be used to convert to a discrete state model:

$A_c = -0.05$

$B_c = 0.1$

$C_c = 1$

We must next solve for the state-transition matrix $\mathbf{\Phi}_c(t)$ which will gives the discrete-system $\mathbf{A}$ and $\mathbf{B}$ matrices

$$\mathbf{A} = \mathbf{\Phi}_c(t) = L^{-1}[[s\mathbf{I} - \mathbf{A}_c]^{-1}]$$

$$= L^{-1}\{(s - (-0.05))^{-1}\}$$

$$= L^{-1}\left\{\frac{1}{s + 0.05}\right\}$$

Using the transform pair to solve for $\mathbf{\Phi}_c(t)$:

$$\frac{1}{s+a} \leftrightarrow \varepsilon^{-\text{at}}$$

$$\mathbf{\Phi}_c(t) = \varepsilon^{-0.05t}$$

Then, solving for $\mathbf{\Phi}_c(T)$ at the sampling instants

$$\mathbf{A} = \mathbf{\Phi}_c(T)|_{T=2} = \varepsilon^{-0.05(2)} = 0.9048$$

Now, solving the discrete $\mathbf{B}$ matrix, we use the definition:

$$\mathbf{B} = \left[\int_0^T \mathbf{\Phi}_c(\tau)d\tau\right]\mathbf{B}_c$$

$$= \left[\int_0^2 \varepsilon^{-0.05t} d\tau\right](0.1)$$

$$= -\frac{0.1}{0.05}[\varepsilon^{-0.05(2)} - \varepsilon^{-0.05(0)}]$$

$$= -2[0.9048 - 1]$$

$\mathbf{B} = 0.1903$

Then, using all the discrete matrices solved above, we find the discrete state model to be described by:

$$x(k+1) = 0.9048x(k) + 0.1903e(k)$$

$$y(k) = x(k)$$

```
% Verify results using MATLAB
T = 2;
Ac = -0.05;
Bc = 0.1;
C = 1;
D = 0;
[A,B] = c2d(Ac,Bc,T)
```

```
A = 0.9048
B = 0.1903
```

```
[numz,denz] = ss2tf(A,B,C,D)
```

```
numz = 1×2
         0    0.1903
denz = 1×2
    1.0000   -0.9048
```

```
Gz = tf(numz,denz,T)
```

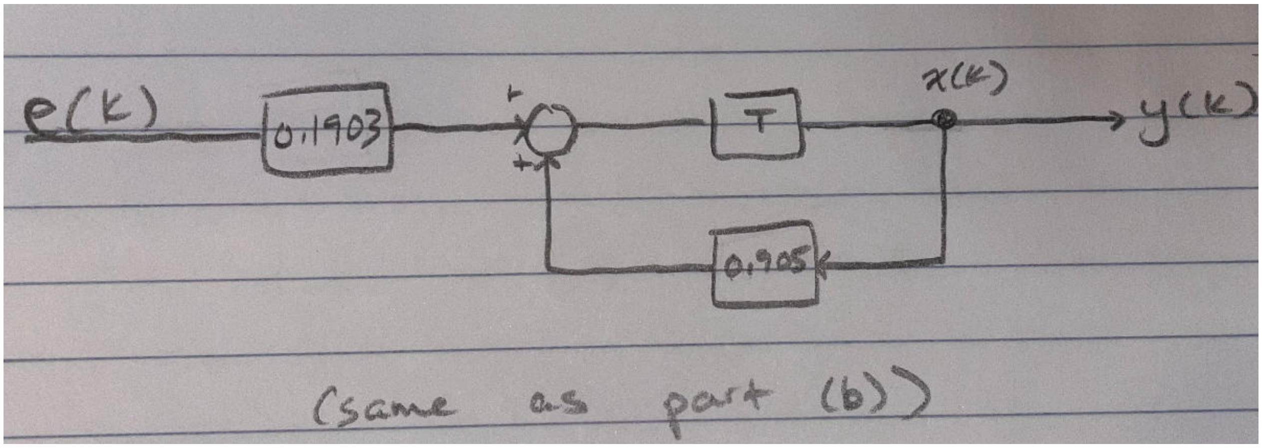
```
Gz =

    0.1903
  ----------
  z - 0.9048

Sample time: 2 seconds
Discrete-time transfer function.
```

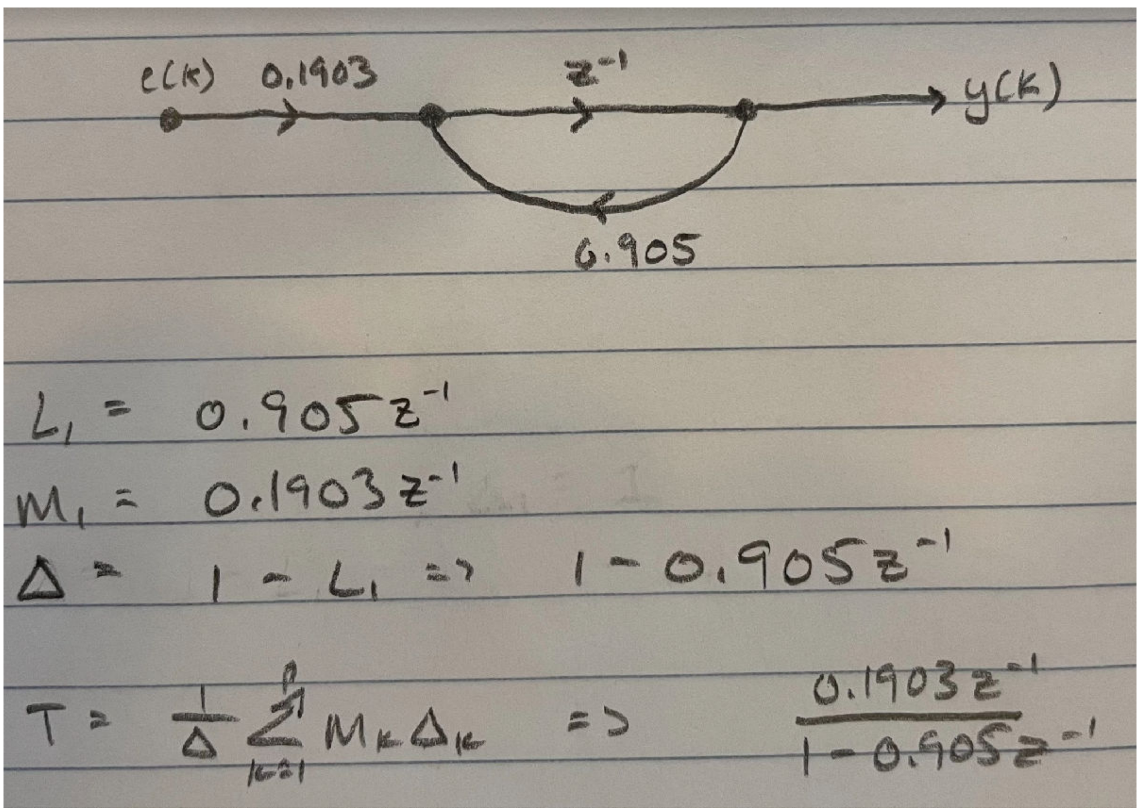
Thus, both calculations are the same.

(e) Draw a simulation diagram for the discrete state model in part (d)



(f) Use Mason's gain formula to find the transfer function in part (e), which must be the same as found in part (a).

First, we draw a signal flowgraph, and then we can evaluate the Forward Path Gains, and Loops of the discrete state model.



If we evaluate the transfer function discovered using Mason's Formula further, we find

$$T = \frac{0.1903z^{-1}}{1 - 0.905z^{-1}} \rightarrow \frac{0.1903}{z - 0.905}$$

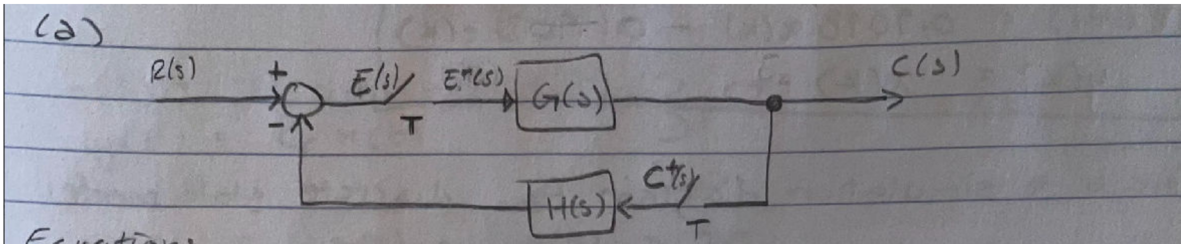
which is what we found in part (a).

Problem 4

Problem 5.3-1

For each of the systems of Fig. P5.3-1, express $C(z)$ as a function of the input and the transfer functions shown.

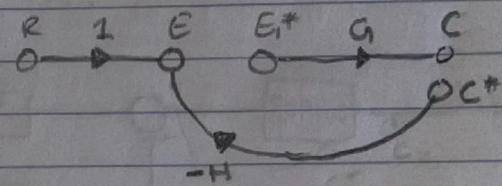
(a)



<u>Equations</u>

$E_1 = R - HC^*$

$C = GE^*$



<u>Starred Equations</u>

$E^* = R^* - H^*C^*$

$C^* = G^*(R^* - H^*C^*)$

$\quad = G^*R^* - G^*H^*C^*$

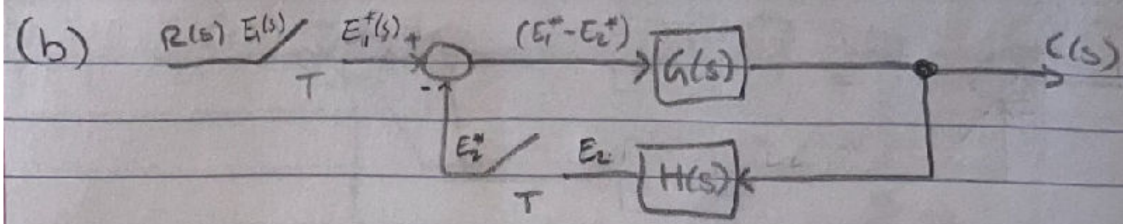
$C^* + G^*H^*C^* = G^*R^*$

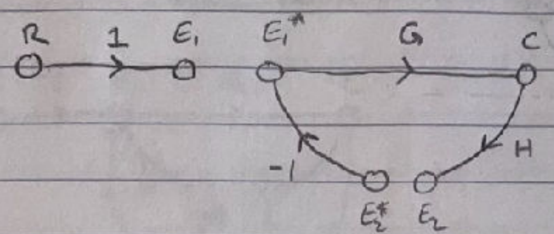
$C^*(1 + G^*H^*) = G^*R^*$

$C^* = \dfrac{G^*R^*}{1 + G^*H^*}$, $\boxed{C(z) = \dfrac{G(z)R(z)}{1 + G(z)H(z)}}$

$$\boxed{\frac{C(z)}{R(z)} = \frac{G(z)}{1 + G(z)H(z)}}$$

(b)





$E_1 = R$

$E_2 = GHE_1^* - GHE_2^*$

$C = (E_1^* - E_2^*)G$

Starred Transforms

$E_1^* = R^*$

$E_2^* = \overline{GH}^* R^* - \overline{GH}^* E_2^*$

$E_2^*(1 + \overline{GH}^*) = \overline{GH}^* R^*$

$$E_2^* = \frac{\overline{GH}^*}{1+\overline{GH}^*} R^*$$

$$C^* = G^* R^* - \frac{\overline{GH}^* R^* G^*}{1+\overline{GH}^*}$$

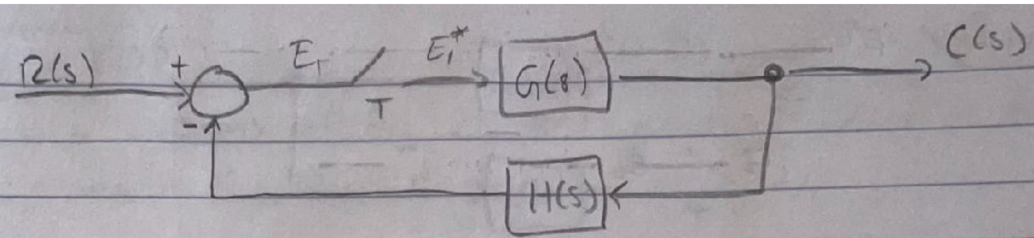
$$C^* = G^* R^* \left(1 - \frac{\overline{GH}^*}{1+\overline{GH}^*} \right)$$

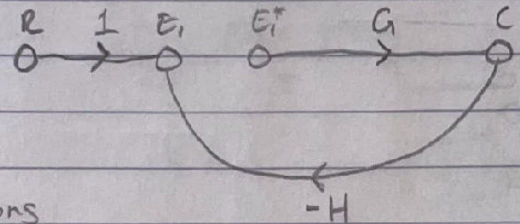
$$= G^* R^* \left(\frac{1 \cancel{+ \overline{GH}^*} - \cancel{\overline{GH}^*}}{1+\overline{GH}^*} \right)$$

$$\boxed{C^* = \frac{G^*}{1+\overline{GH}^*} R^* \quad \text{or} \quad C(z) = \frac{G(z)}{1+\overline{GH}(z)} R(z)}$$

$$\boxed{\frac{C(z)}{R(z)} = \frac{G(z)}{1+\overline{GH}(z)}}$$

(c)





<u>Equations</u>

$$E_1 = R - GHE_1^*$$

$$C = G E_1^*$$

<u>Starred Equations</u>

$$E_1^* = R^* - \overline{GH}^* E_1^*$$

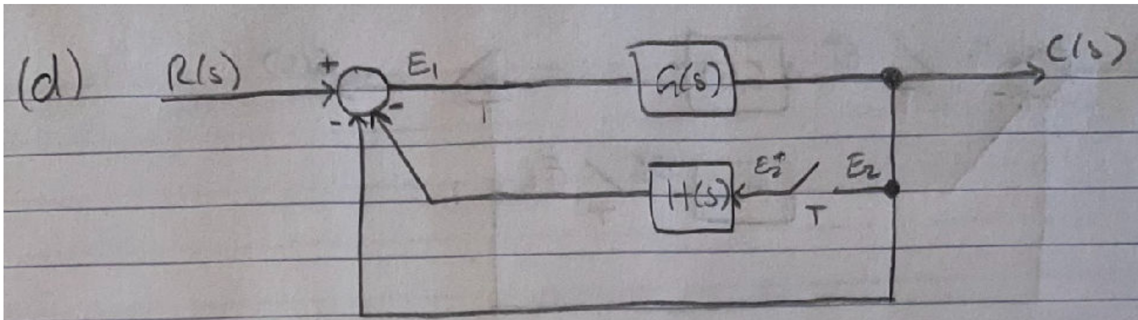
$$E_1^*(1+\overline{GH}^*) = R^* \implies E_1^* = \frac{R^*}{1+\overline{GH}^*}$$

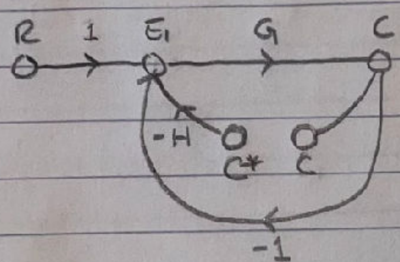
$$C^* = G^* E_1^* = \frac{G^*}{1+\overline{GH}^*} R^*$$

$$\boxed{C^* = \frac{G^*}{1+\overline{GH}^*} R^* \quad \text{or} \quad C(z) = \frac{G(z)}{1+\overline{GH}(z)} R(z)}$$

$$\boxed{\frac{C(z)}{R(z)} = \frac{G(z)}{1+\overline{GH}(z)}}$$

(d)





Eqations

$E_1 = R - HC^* + GHC^*$

$C = GE_1 = GR - GHC^* + G^2HC^*$

Starred Equations

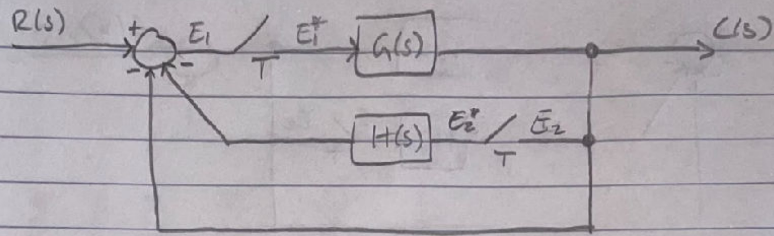
$C^* = \overline{GR}^* - \overline{GH}^*C^* + \overline{G^2H}^*C^*$

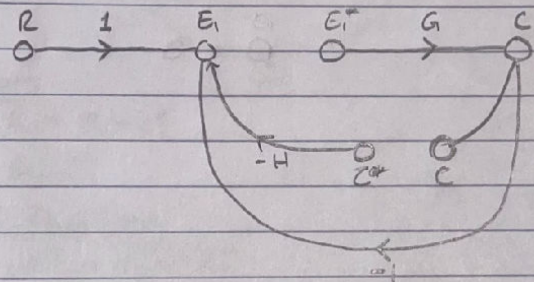
$C^*(1 + \overline{GH}^* - \overline{G^2H}^*) = \overline{GR}^*$

$$\boxed{C^* = \frac{\overline{GR}^*}{1 + \overline{GH}^* - \overline{G^2H}^*} \quad \text{or} \quad C(z) = \frac{\overline{GR}(z)}{1 + \overline{GH}(z) - \overline{G^2H}(z)}}$$

Can't express transfer function because we can't R^* cannot be factored from $\overline{GR}^*$

(c)





Equations

$E_1 = R - HC^* - C$

$C = G E_1^*$

Starred Equations

$E_1^* = R^* - H^*C^* - C^*$

$C^* = G^* E_1^* \;\to\; G^*R^* - G^*H^*C^* - G^*C^*$

$C^*(1 + G^* + G^*H^*) = G^*R^*$

$$\boxed{C^* = \frac{G^*}{1+G^*+G^*H^*} R^* \quad \text{or} \quad C(z) = \frac{G(z)}{1+G(z)+G(z)H(z)} R(z)}$$

$$\boxed{\frac{C(z)}{R(z)} = \frac{G(z)}{1+G(z)+G(z)H(z)}}$$