Routh Hurwitz Method

Given a polynomial in *s*

$$a_n \cdot s^n + a_{n-1} \cdot s^{n-1} + \ldots + a_1 \cdot s + a_0 = 0$$

Routh Hurwitz Method

Given a polynomial in *s*

$$a_n \cdot s^n + a_{n-1} \cdot s^{n-1} + \ldots + a_1 \cdot s + a_0 = 0$$

Procedure:

Step 1: Build the Routh array.

(a) For rows 1 and 2, build h columns, where h = **Largest integer** [$(n+1)/2$],

Routh Hurwitz Method

Given a polynomial in *s*

$$a_n \cdot s^n + a_{n-1} \cdot s^{n-1} + \ldots + a_1 \cdot s + a_0 = 0$$

Procedure:

Step 1: Build the Routh array.

(a) For rows 1 and 2, build h columns, where h = **Largest integer** [$(n+1)/2$],

If n is odd:

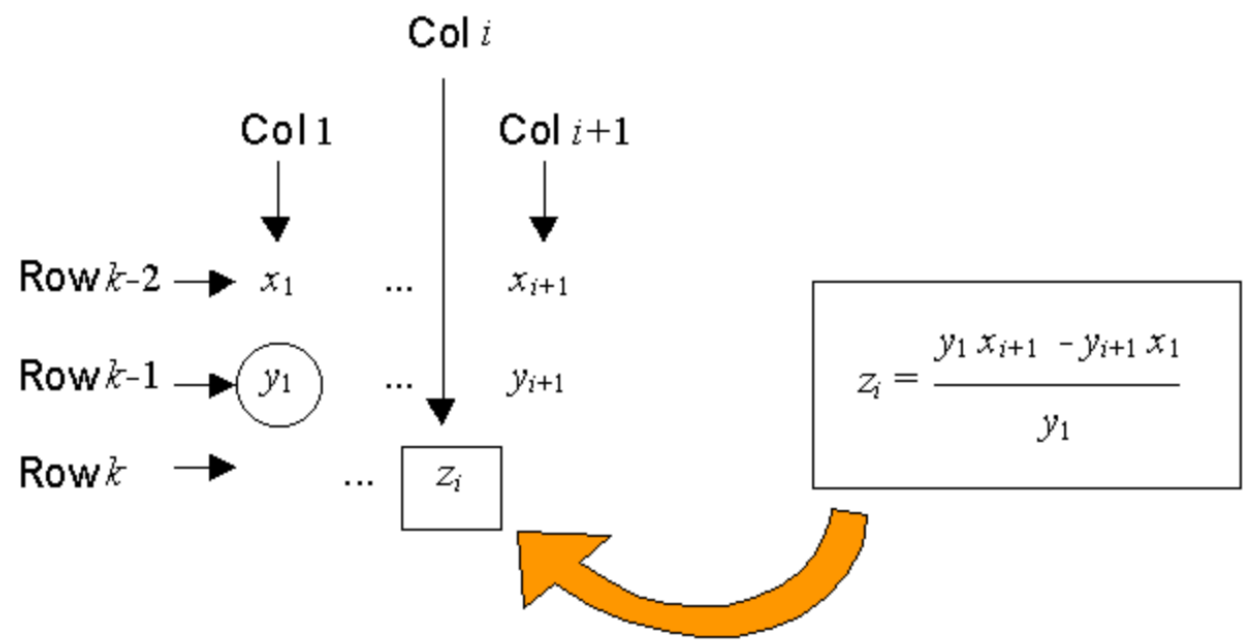
$$\begin{array}{lllll} \text{Row1:} & a_n & a_{n-2} & \ldots & a_1 \\ \text{Row2:} & a_{n-1} & a_{n-3} & \ldots & a_0 \end{array}$$

If n is even:

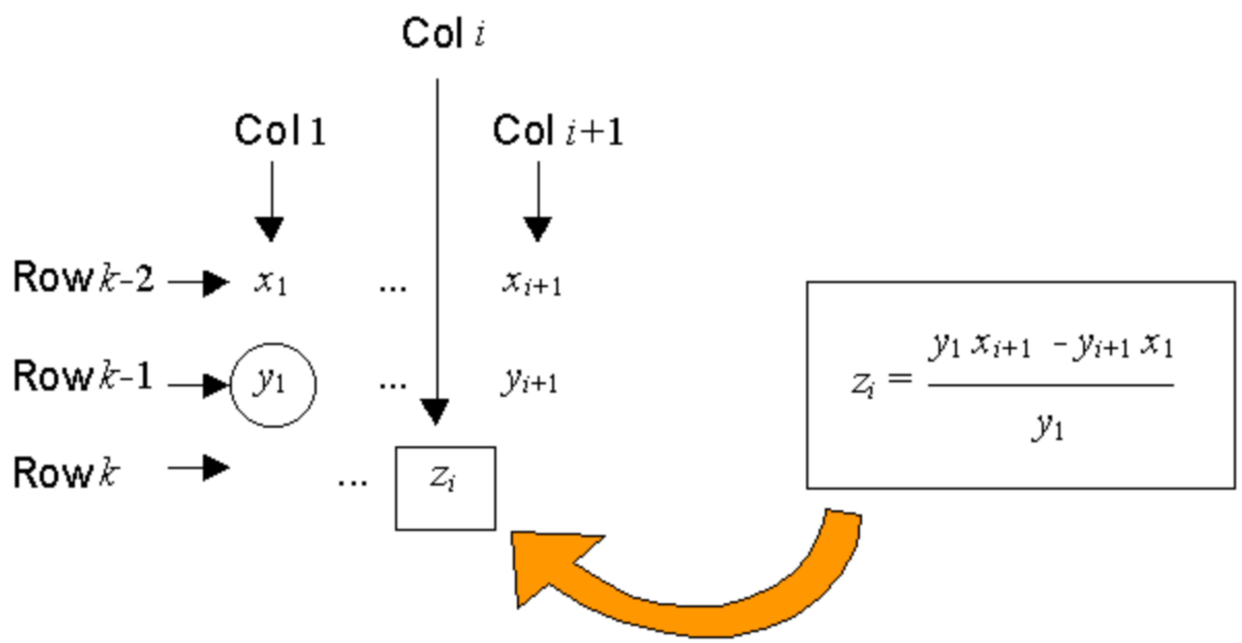
$$\begin{array}{lllll} \text{Row1:} & a_n & a_{n-2} & \ldots & a_0 \\ \text{Row2:} & a_{n-1} & a_{n-3} & \ldots & 0 \end{array}$$

(b) For row 3 to row $n+1$,

(b) For row 3 to row *n+1*.

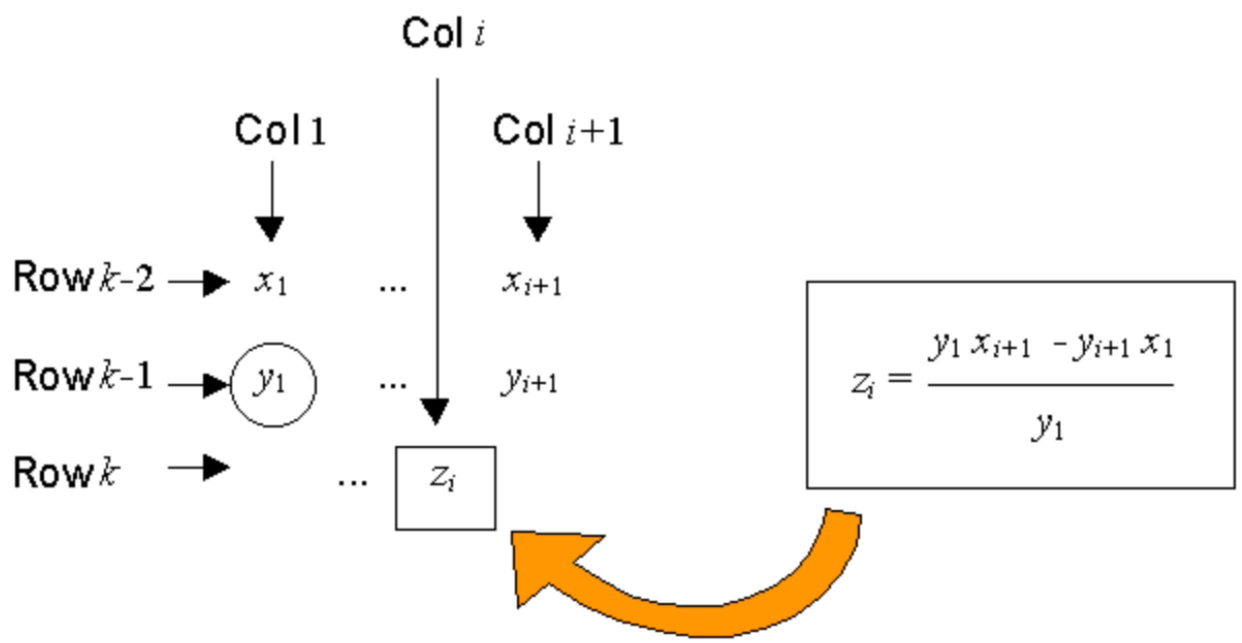


(b) For row 3 to row $n+1$.



Step 2: <u>Extract the first column of the array and count the number of sign changes.</u>

(b) For row 3 to row *n+1*.



Step 2: Extract the first column of the array and count the number of sign changes.

The number of sign changes gives the number of roots of the polynomial which have positive real parts.

Example:

$$2s^6 + 4s^5 + 2s^4 - s^3 + 2s - 2 = 0$$

Example:

$$2s^6 + 4s^5 + 2s^4 - s^3 + 2s - 2=0$$

2	2	0	-2
4	-1	2	0
2.5	-1	-2	0
0.6	5.2	0	0
-22.67	-2	0	0
5.142	0	0	0
-2	0	0	0

Example:

$$2s^6 + 4s^5 + 2s^4 - s^3 + 2s - 2=0$$

2	2	0	-2
4	-1	2	0
2.5	-1	-2	0
0.6	5.2	0	0
-22.67	-2	0	0
5.142	0	0	0
-2	0	0	0

There are 3 sign changes in the first column, thus the conclusion is that there are three roots that have positive real parts.

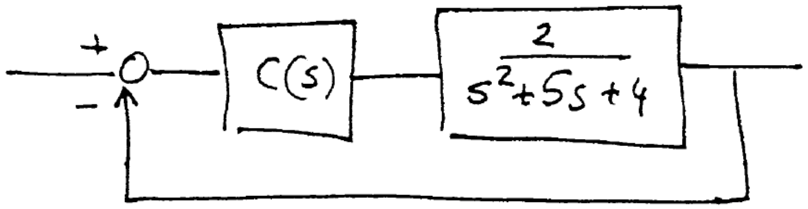
EECE 5610 Digital Control Systems

Lecture 16

Milad Siami
Assistant Professor of ECE
Email: m.siami@northeastern.edu

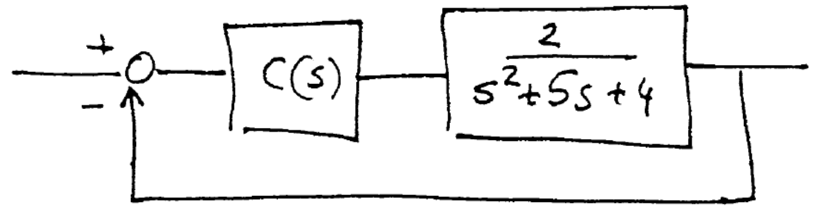
Northeastern University
College *of* Engineering

Laser example :



If $C(s) = K \Rightarrow k_p = \frac{K}{2}$, $e_{ss}^{step} = \frac{1}{1+K/2} = \frac{2}{2+K}$ <u>provided</u> that the closed loop is stable

Laser example :



If $C(s) = k \Rightarrow k_p = \frac{k}{2}$, $e_{ss}^{step} = \frac{1}{1+k/2} = \frac{2}{2+k}$ provided that the closed loop is stable

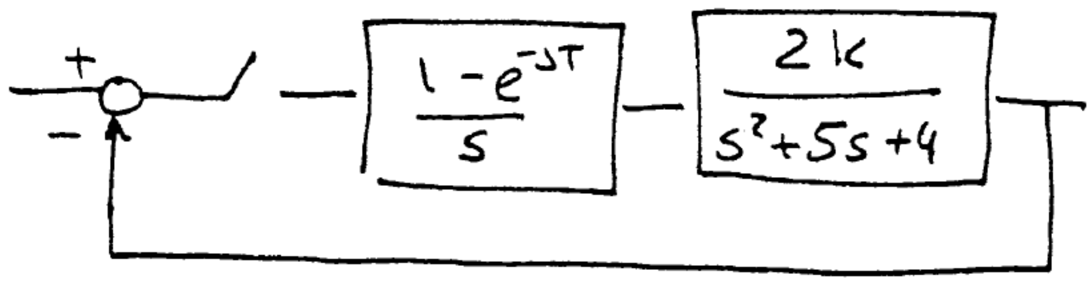
Q: How do we assess stability?

A: Use Routh Hurwitz: Char eq: $s^2 + 5s + 4 + 2k = 0 \Rightarrow$

stable for all $k > 0$

What about the sampled data case. Assume $C(z) = k$

$T = 0.1\ s$



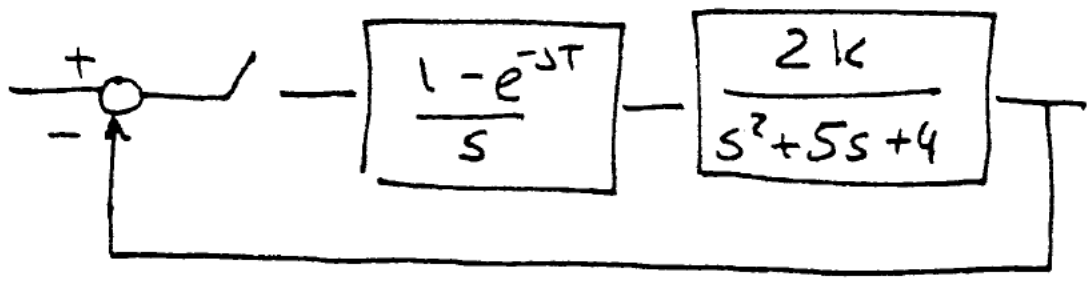
Matlab yields:

$$\mathcal{Z}\left[\frac{1-e^{-sT}}{s}\,\frac{2}{s^2+5s+4}\right] = \frac{0.0085z + 0.0072}{z^2 - 1.5752z + 0.6065}$$

K=55 —> A- Stable B- Unstable ?

What about the sampled data case. Assume $C(z) = k$

$T = 0.1\ s$



Matlab yields:

$$\mathcal{Z}\left[\frac{1-e^{-sT}}{s}\,\frac{2}{s^2+5s+4}\right] = \frac{0.0085z + 0.0072}{z^2 - 1.5752z + 0.6065}$$

$\Rightarrow$ Characteristic equation:

$$z^2 + (0.0085 \cdot k - 1.5752)z + (0.6065 + 0.0072k) = 0$$

$\Rightarrow$ 2 roots: In this case one can solve for $z_{1,2}$ as a function of k and find out for what value we become unstable (i.e $|z_{1,2}| = 1$)

(Turns out that $k \sim 55$)

However, this is fairly tedious even for a relatively simple system $\Rightarrow$
We need a <u>better way of assessing stability</u>

2 options (1) Map the z-plane to the s-plane and use continuous-time techiques (i.e. Routh Hurwitz)

(2) Derive an "equivalent" Routh Hurwitz criterion for discrete time systems

We will explore both options

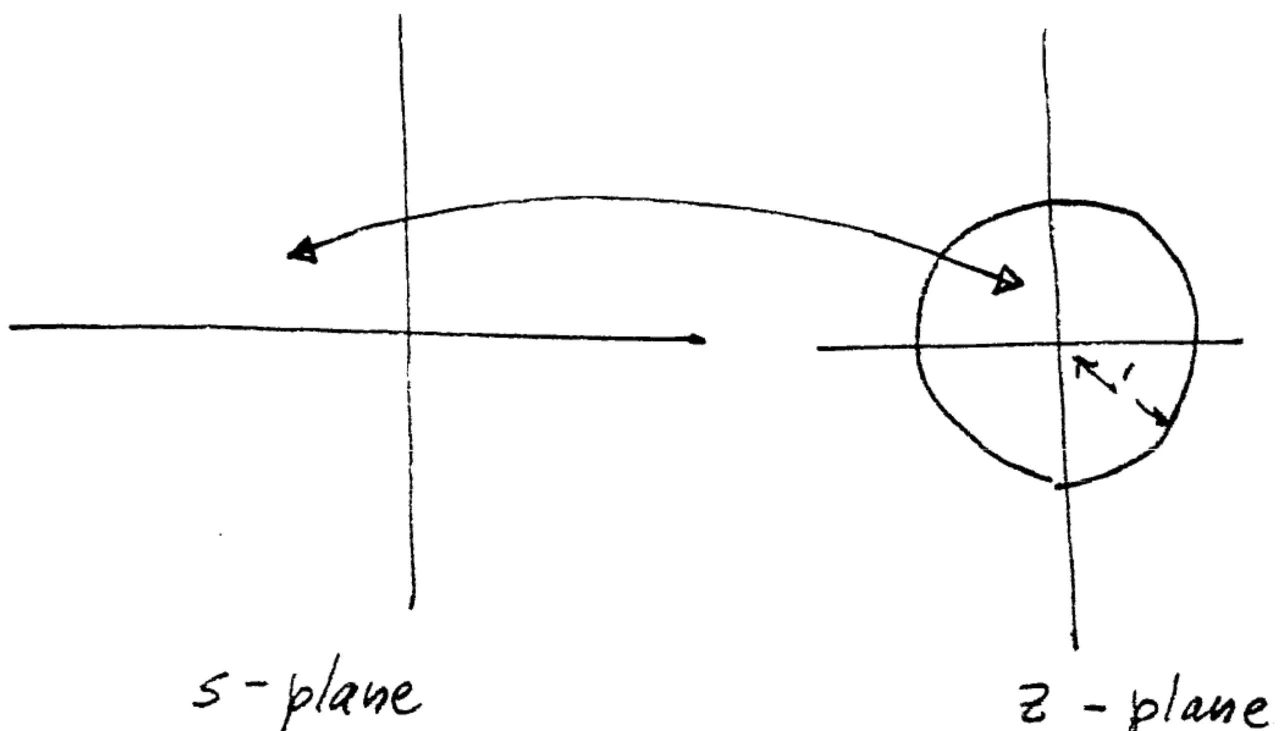
- <u>Bilinear transformation</u> (section 7.3)

One way to map the z-plane to the s-plane is via the transformation

$$z = e^{sT} \iff s = \frac{1}{T} \ln z \iff \mathrm{Re}(s) = \frac{1}{T} \ln |z|$$

$$\mathrm{Im}(s) = \frac{1}{T} \angle \theta$$

$$|z| < 1 \iff \mathrm{Re}(s) < 0$$



So in principle we could assess stability as follows

1) Use the transformation $s = \frac{1}{T} \ln(z)$ to map the discrete time char equation $D(z) = 0$ to a continuous time equivalent $D_{eq}(s)$

$$D_{eq}(s) = D(z)\Big|_{z = e^{sT}}$$

So in principle we could assess stability as follows

1) Use the transformation $s = \frac{1}{T} \ln(z)$ to map the discrete time char equation $D(z) = 0$ to a continuous time equivalent $D_{eq}(s)$

$$D_{eq}(s) = D(z)\Big|_{z = e^{sT}}$$

2) Assess stability of $D_{eq}(s)$ using EECE 5580 techniques

Let's try it in our laser example:

$$D(z) = z^2 + (0.0085\,k - 1.5752)z + 0.6065 + 0.0072\,k = 0$$

$$\Downarrow \; z = e^{sT}$$

$$D_{eq}(s) = e^{2sT} + (0.0085\,k - 1.5752)e^{sT} + 0.6065 + 0.0072\,k = 0$$

But trouble!! We got a char equation that is <u>not</u> a polynomial in s (we have e^{sT} dependence) $\Rightarrow$ can't use Routh Hurwitz

$\Rightarrow$ we are stuck!

But trouble!! We got a char equation that is <u>not</u> a polynomial in s (we have e^{sT} dependence) $\Rightarrow$ can't use Routh Hurwitz

$\Rightarrow$ we are stuck!

<u>Solution</u>: look for a different transformation.

Desirable properties (1) unit disk in z domain $\Longleftrightarrow$ Left half plane in s domain

(2) char eq polynomial in z $\Longleftrightarrow$ char eq polynomial in s.

One transformation that has these properties is the <u>bilinear</u> (or Tustin) transformation

$$z = \frac{1 + Ts/2}{1 - Ts/2} \Longleftrightarrow s = \frac{2}{T}\frac{z-1}{z+1}$$

But trouble!! We got a char equation that is <u>not</u> a polynomial in s (we have e^{sT} dependence) $\Rightarrow$ can't use Routh Hurwitz

$\Rightarrow$ we are stuck!

<u>Solution</u>: look for a different transformation.

Desirable properties (1) unit disk in z domain $\Leftrightarrow$ Left half plane in s domain

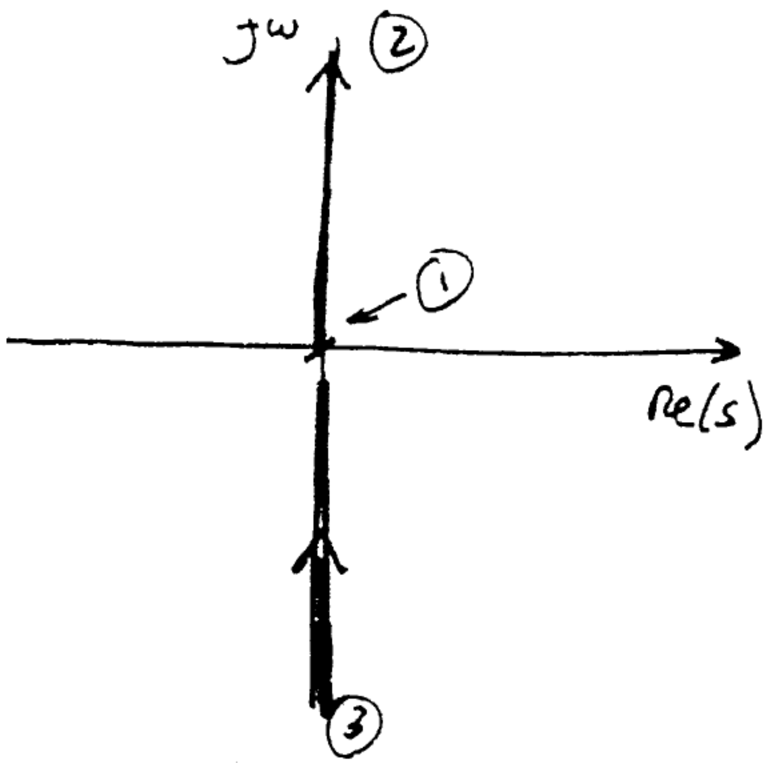
(2) char eq polynomial in z $\Leftrightarrow$ char eq polynomial in s.

One transformation that has these properties is the <u>bilinear</u> (or Tustin) transformation

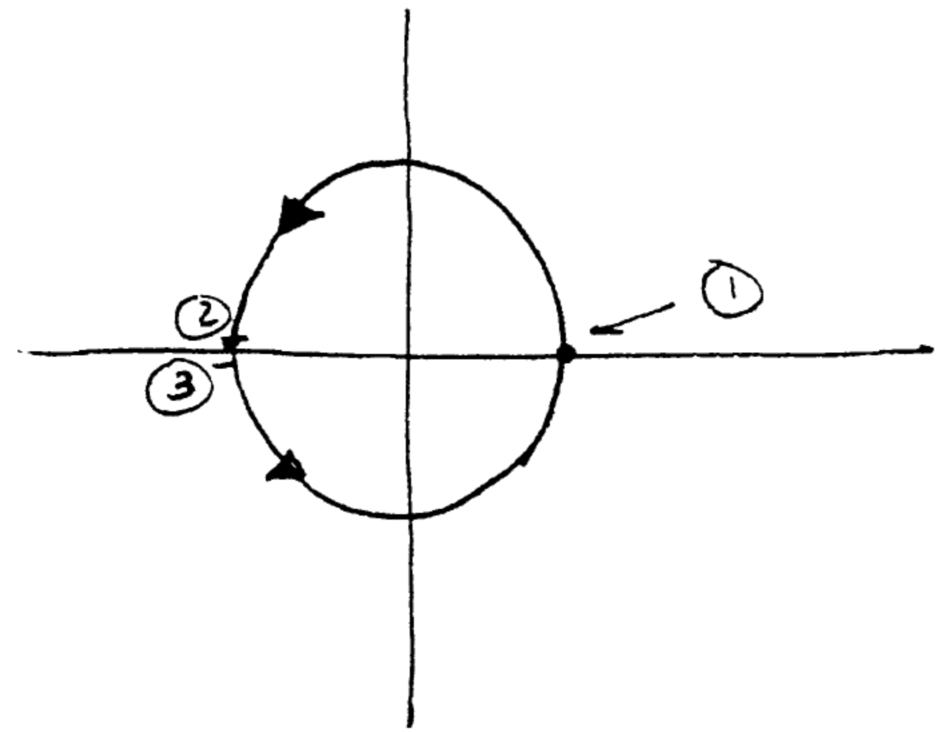
$$z = \frac{1 + Ts/2}{1 - Ts/2} \Leftrightarrow s = \frac{2}{T}\,\frac{z-1}{z+1}$$

(this is a special case of a conformal mapping: a mapping analytic in the LHP with inverse analytic in the open unit disk)

Let's look at the image of the $j\omega$ axis (in the s plane)

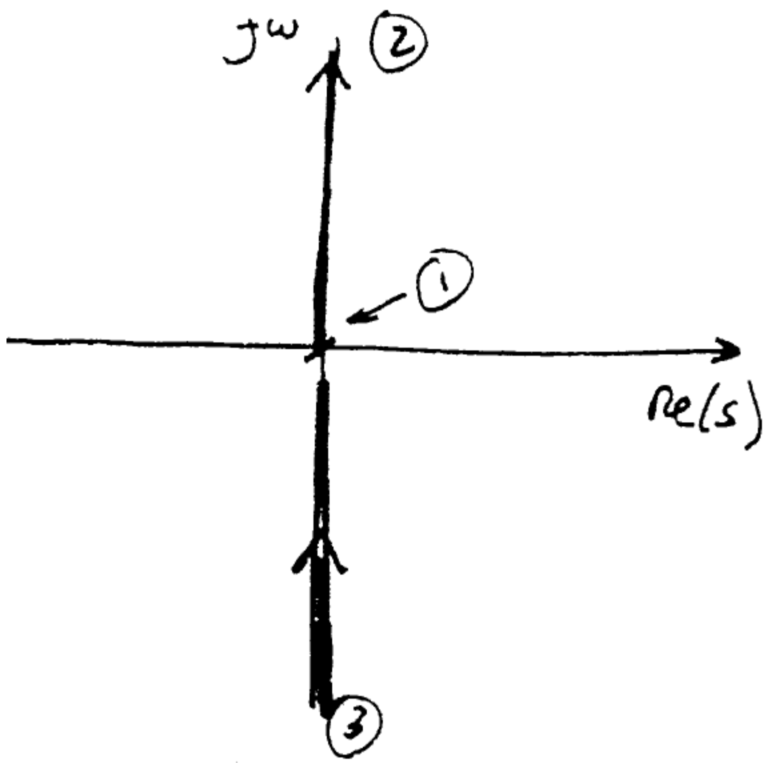


s - plane

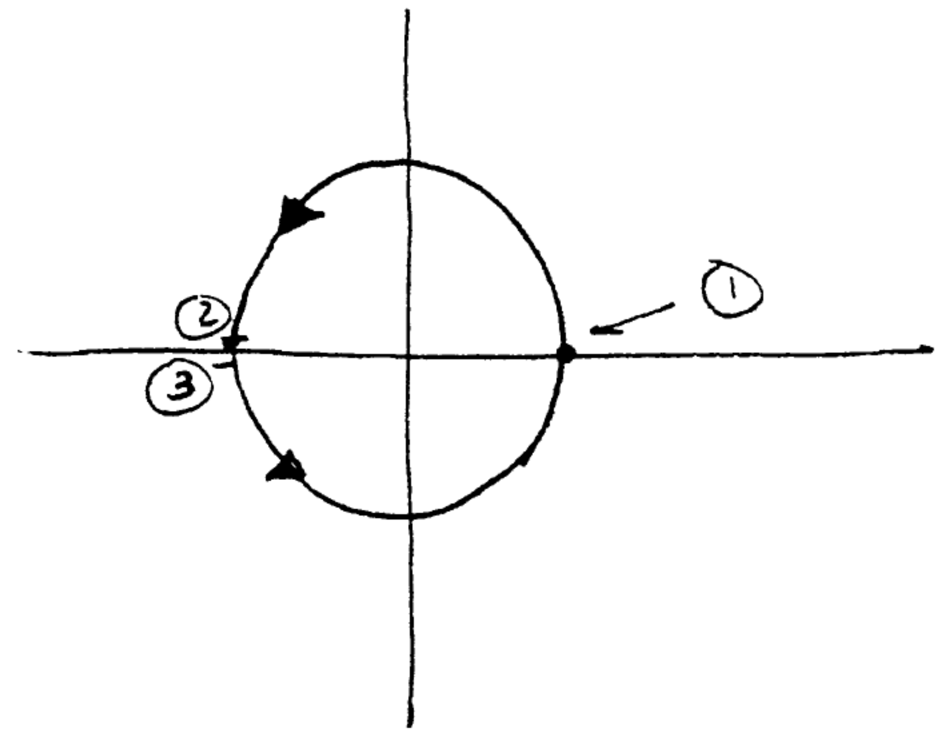


z - plane

Let's look at the image of the $j\omega$ axis (in the s plane)



s - plane



z - plane

If $s = j\omega \Rightarrow z = \dfrac{1 + j\frac{\omega T}{2}}{1 - j\frac{\omega T}{2}} \Rightarrow |z| = \sqrt{\dfrac{1 + \left(\frac{\omega T}{2}\right)^2}{1 + \left(\frac{\omega T}{2}\right)^2}} = 1 \Rightarrow z = 1 \angle 2\tan^{-1}\left(\frac{\omega T}{2}\right)$

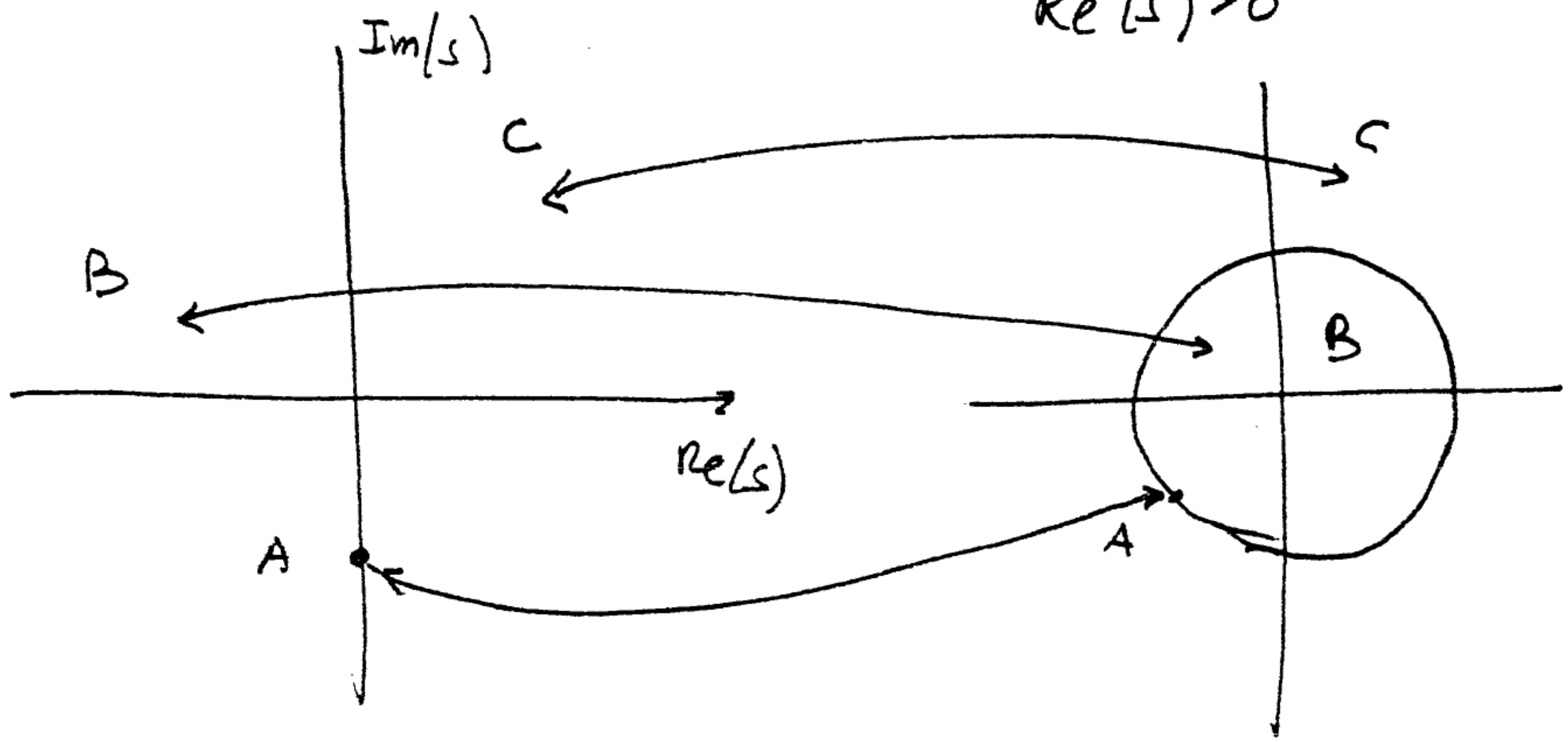
$\angle z = 2 \cdot \tan^{-1}\left(\frac{\omega T}{2}\right)$

If we have a generic point $s = \sigma + j\omega \Rightarrow z = \dfrac{1 + \frac{\sigma T}{2} + j\frac{\omega T}{2}}{1 - \frac{\sigma T}{2} - j\frac{\omega T}{2}}$

$$\Rightarrow |z|^2 = \frac{\left(1 + \frac{\sigma T}{2}\right)^2 + \left(\frac{\omega T}{2}\right)^2}{\left(1 - \frac{\sigma T}{2}\right)^2 + \left(\frac{\omega T}{2}\right)^2} \Rightarrow \begin{array}{l} |z| < 1 \iff \sigma < 0 \\ > 1 \iff \sigma > 0 \end{array}$$

In other words, the region $Re(s) < 0 \xleftrightarrow{\text{mapped to}} |z| < 1$

$Re(s) > 0 \qquad |z| > 1$

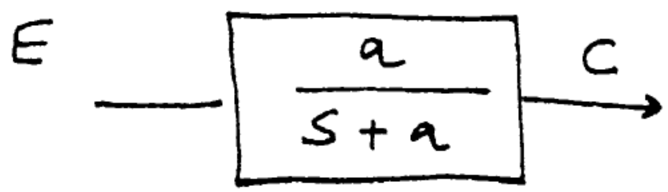


s-plane $\qquad$ z plane

s-domain		z-domain
$j\omega$ axis	$\longleftrightarrow$	unit circle
LHP	$\longleftrightarrow$	int of unit disk
RHP	$\longleftrightarrow$	exterior of unit disk.

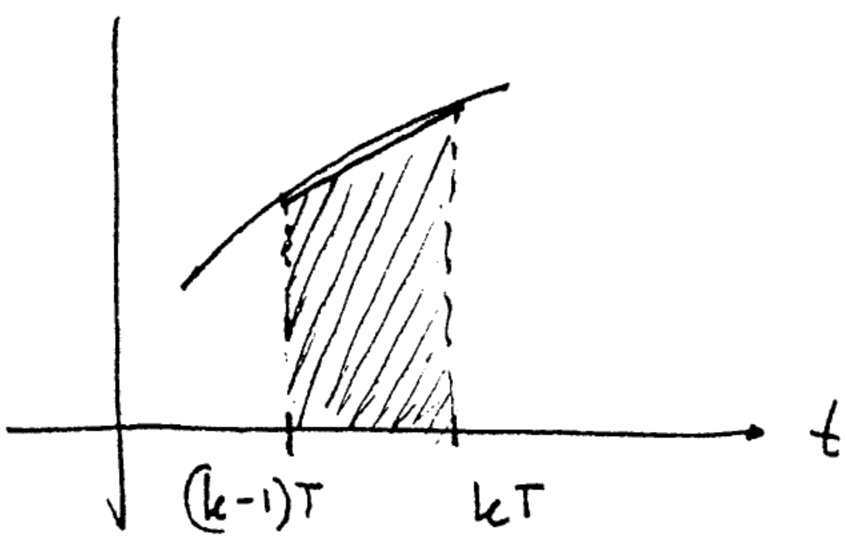
- ## Physical motivation for Tustin's method

Suppose that we have a continous time system and we decide to find a discrete time equivalent by numerical integration

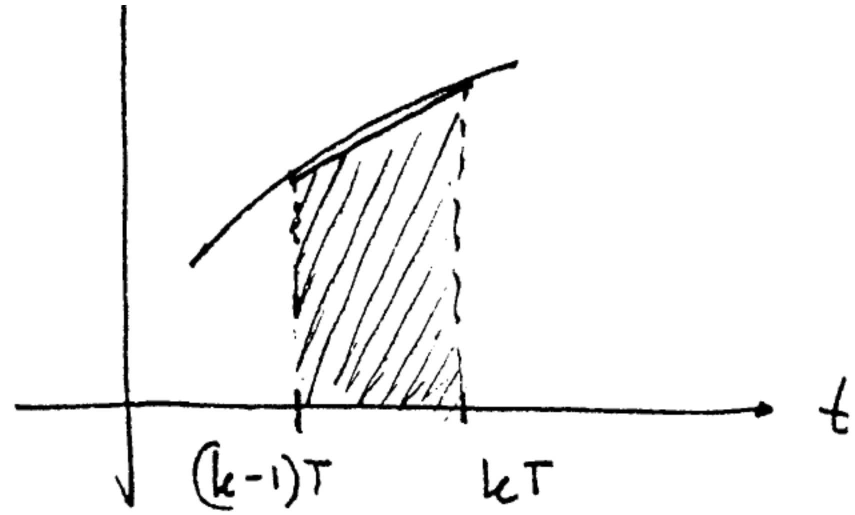


$$C(s) = \frac{a}{s+a} E(s) \quad // \quad \dot{c} + a\,c(t) = a\,e(t)$$

$$c(kT) = \int_{(k-1)T}^{kT} \left[a\,e(\lambda) - a\,c(\lambda)\right] d\lambda + c\left[(k-1)T\right]$$



$$= c\left[(k-1)T\right] + \int_{0}^{T} f(\lambda)\, d\lambda$$

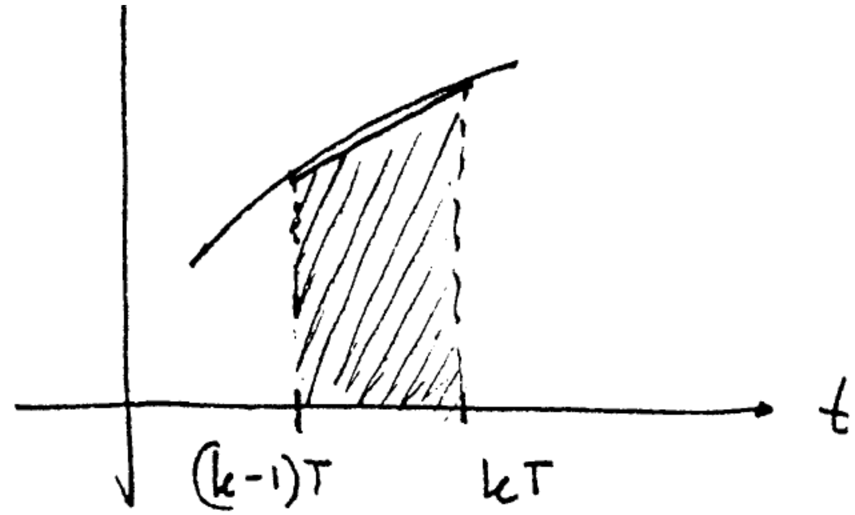


If we approximate the integral using the trapezoidal rule, we get:

$$\int_{(k-1)T}^{kT} \sim \left[\frac{f[(k-1)T] + f(kT)}{2} \right] T$$

$$c(kT) = c[(k-1)T] + \frac{aT}{2}\Big[e[(k-1)T] + e(kT) - c[(k-1)T] - c(kT) \Big]$$

$$\left(1 + \frac{aT}{2}\right) c(kT) - \left(1 - \frac{aT}{2}\right) c[(k-1)T] = \frac{aT}{2}\Big[e(kT) + e[(k-1)T] \Big]$$



Thus the corresponding <u>discrete</u> transfer function is:

$$\left[\left(1+\frac{aT}{2}\right)-\left(1-\frac{aT}{2}\right)\frac{1}{z}\right]C(z)=\frac{aT}{2}\left[1+\frac{1}{z}\right]E(z)$$

$$\frac{C(z)}{E(z)}=\left(\frac{z+1}{\cancel{z}}\right)\frac{\cancel{z}}{z\left(1+\frac{aT}{2}\right)+\frac{aT}{2}-1}=\frac{aT(z+1)}{(2+aT)z+aT-2}$$

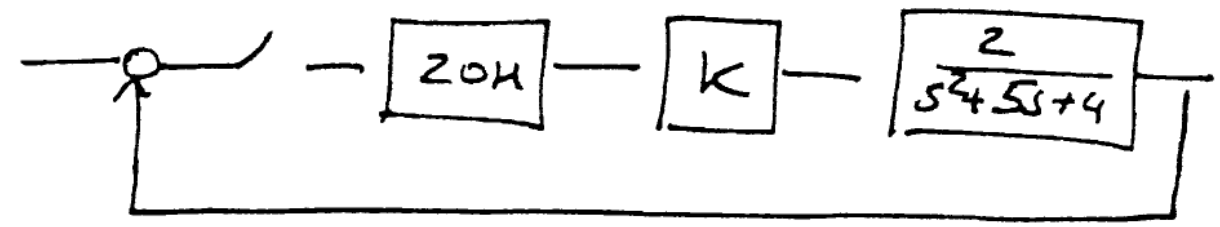
$$=\frac{a}{\frac{2}{T}\left(\frac{z-1}{z+1}\right)+a}\quad\#$$

Compared with the original TF in the s-domain $H(s) = \frac{a}{s+a}$

we see that the trapezoidal rule amounts to the substitution:

$$s \leftarrow \frac{2}{T} \frac{z-1}{z+1} \quad \#$$

- **Back to the laser example:**

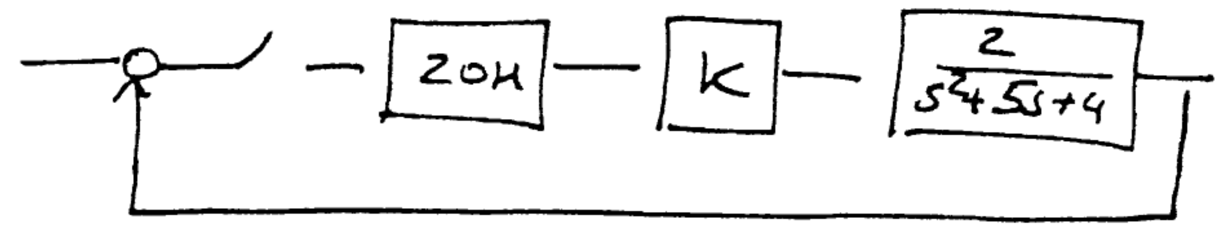


Recall that for $T=0.1$ we found experimentally that for $k \cong 55$ it goes unstable. Let's see if we can show this analytically

$$G(z) = \mathcal{Z}\left[\frac{1-e^{-sT}}{s}\,\frac{2}{s^2+5s+4}\right] = 2\left(\frac{z-1}{z}\right)\mathcal{Z}\left[\frac{1}{s(s^2+5s+4)}\right]$$

$$= \frac{0.0085z + 0.0072}{z^2 - 1.5752z + 0.6065}$$

- Back to the laser example:



Recall that for $T=0.1$ we found experimentally that for $k \gtrsim 55$ it goes unstable. Let's see if we can show this analytically

$$G(z) = \mathcal{Z}\left[\frac{1-e^{-sT}}{s}\,\frac{2}{s^2+5s+4}\right] = 2\left(\frac{z-1}{z}\right)\mathcal{Z}\left[\frac{1}{s(s^2+5s+4)}\right]$$

$$= \frac{0.0085\,z + 0.0072}{z^2 - 1.5752z + 0.6065}$$

The discrete time characteristic equation is given by:

$$1 + K\,G(z) = 0$$

Using the bilinear transformation $\quad z = \dfrac{1 + \frac{Ts}{2}}{1 - \frac{Ts}{2}} = \dfrac{1 + 0.05s}{1 - 0.05s}$

yields:

$$G(s) = \frac{-0.0004s^2 - 0.0904s + 1.9721}{s^2 + 4.9467s + 3.9442}$$

and the corresponding "continuous time equivalent" char equation is:

$$1 + K\,G(s) = 0 \iff \left(1 - 4\cdot 10^{-3} K\right) s^2 + \left(4.9467 - 9.04\cdot 10^{-2} K\right) s + \left(3.9442 + 1.9721\, K\right) = 0$$

Routh Hurwitz array:

$$\begin{array}{lll} s^2 & 1 - 4\cdot 10^{-3} K & 3.9442 + 1.9721\, K \\ s^1 & 4.9467 - 9.04\cdot 10^{-2} K & \\ s^0 & 3.9442 + 1.9721\, K & \end{array}$$

$-2 < K < 54.713$

(A)

$K < 54.7128$

(B)

?

and the corresponding "continuous time equivalent" char equation is:

$$1 + K\,G(s) = 0 \iff (1 - 4\cdot10^{-3}K)s^2 + (4.9467 - 9.04\cdot10^{-2}K)s + (3.9442 + 1.9721K) = 0$$

Routh Hurwitz array:

$$\begin{array}{lll} s^2 & 1 - 4\cdot10^{-3}K & 3.9442 + 1.9721K \\ s^1 & 4.9467 - 9.04\cdot10^{-2}K & \\ s^0 & 3.9442 + 1.9721K & \end{array}$$

$\Rightarrow$ stable iff:

$-2 < K < 54.713$

(A)

$K < 54.7128$

(B)

?

Suppose that we want to find out the point where the system becomes marginally stable and the frequency of oscillation. From the Routh Hurwitz array we have that the system is marginally stable for $k = 54.713$

The corresponding auxiliary equation is: $(1 - 4.10^{-3}k)s^2 + 3.9442 + 1.9721k = 0$

$$\Rightarrow 0.9776 s^2 + 111.844 = 0$$

$$\boxed{s = \pm j\, 10.7}$$

We should expect an oscillation with frequency $\omega = 10.7$ rad/sec <u>provided</u> that $\omega << \omega_s = \frac{2\pi}{T}$ (say $\omega \sim \frac{\omega_s}{10}$)

In our case $\omega_s = 62.83$ (so $\omega \sim \frac{\omega_s}{6}$ and the approx should be ok)

We should expect an oscillation with frequency $\omega = 10.7$ rad/sec <u>provided</u> that $\omega << \omega_s = \frac{2\pi}{T}$ (say $\omega \sim \frac{\omega_s}{10}$)

In our case $\omega_s = 62.83$ (so $\omega \sim \frac{\omega_s}{6}$ and the approx should be ok)

- Q: What happens if we increase the sampling interval?

A: Intuitively the system should become <u>less</u> stable (due to increased time delay)

Consider the same system as before, but let $T=1$, rather than $T=0.1$

$$G(z) = 2\left(\frac{z-1}{z}\right) \mathcal{Z}\left[\frac{1}{s(s^2+5s+4)}\right] = \frac{0.2578z + 0.0525}{z^2 - 0.3862z + 0.0067}$$

$$\Big\Downarrow \quad z = \frac{1+0.5s}{1-0.5s}$$

$$G(s) = \frac{-0.1474s^2 - 0.1507s + 0.8910}{s^2 + 2.8523s + 1.7820}$$

Char. eq: $(1-0.1474\,k)s^2 + (2.8523 - 0.1507k)s + 1.7820 + 0.8910\,k = 0$

stable iff:

$$\left.\begin{array}{lll} 1-0.1474\,k>0 & \Rightarrow & k<6.784 \\ 2.8523-0.1507\,k>0 & \Rightarrow & k<18.93 \\ 1.782+0.891\,k>0 & \Rightarrow & k>-2 \end{array}\right\} \boxed{-2<k<6.784}$$

<u>Comparison</u>:

$T = 0.1$	$T = 1$	cont. time
$-2 < K < 54.713$	$-2 < K < 6.784$	$-2 < K < \infty$

Note: recall that from the bilinear transf we have

$$s = j\omega_c \Longleftrightarrow z = 1 \angle 2\tan^{-1}\left(\frac{\omega_c T}{2}\right)$$

A cont time frequency of oscillation ω_c corresponds to a discrete time oscillation with frequency $\omega_d = \frac{2}{T}\tan^{-1}\left(\frac{\omega_c T}{2}\right)$

$\Rightarrow \quad \omega_d \sim \omega_c \iff T\omega_c \ll 1$

(in this case $\tan^{-1} \frac{\omega_c T}{2} \cong \frac{\omega_c T}{2}$ and $\frac{2}{T} \tan^{-1}\left(\frac{\omega_c T}{2}\right) \sim \omega_c$)

$\omega_c T \ll 1 \iff \omega_c \ll \omega_{Sampling}$

- **Jury's stability test** (section 7.5 book)

Jury's test is similar to Routh Hurwitz in the sense that it counts the number of unstable roots of the (discrete time) char. equation.

You form an array using the coefficients of the polynomial, starting with two rows of length n. From these you compute a succesor row of length n-1, then another one of length n-2 and so on, until we get a row of length 1. Stability is related to the entries of the first column, as follows:

Assume that the characteristic polynomial is given by:

$$\varphi(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots \qquad a_1 z + a_0, \qquad a_n > 0$$

- <u>Step 1</u>: Form Jury's Array:

$$\begin{array}{cccccc}
z^0 & z^1 & z^2 & \cdots & z^{n-1} & z^n \\
 & & & & & \\
a_0 & a_1 & a_2 & & a_{n-1} & a_n \\
a_n & a_{n-1} & a_{n-2} & & a_1 & a_0 \\
 & & & & & \\
b_0 & b_1 & b_2 & & b_{n-1} & \\
b_{n-1} & b_{n-1} & & & b_0 & \\
 & & & & & \\
c_0 & c_1 & - - - - - & & c_{n-2} & \\
c_{n-2} & c_{n-3} & & & c_0 & \\
\vdots & & & & & \\
l_0 & l_1 & l_2 & l_3 & & \\
l_3 & l_2 & l_1 & l_0 & & \\
 & & & & & \\
m_0 & m_1 & m_2 & & &
\end{array}$$

Remark: the elements of the <u>even numbered</u> rows are the elements of the preceeding row in reverse order

$$b_k = \begin{vmatrix} a_0 & a_{n-k} \\ a_n & a_k \end{vmatrix}$$

$$c_k = \begin{vmatrix} b_0 & b_{n-1-k} \\ b_{n-1} & b_k \end{vmatrix}; \qquad d_k = \begin{vmatrix} c_0 & c_{n-2-k} \\ c_{n-2} & c_k \end{vmatrix}, \ldots$$

$$\boxed{Q(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots \quad a_1 z + a_0, \qquad a_n > 0}$$

Step 2 : Check the following conditions (necessary and sufficient for having all roots in $|z|<1$)

(a) $Q(1) > 0$

(b) $(-1)^n Q(-1) > 0$

(c) $|a_0| < a_n,$

$|b_0| > |b_{n-1}|$

$|c_0| > |c_{n-2}|$

$|d_0| > |d_{n-3}|$

$\vdots$

$|m_0| > |m_2|$

Remark : Check first $\varphi(1) > 0$, $(-1)^n \varphi(-1) > 0$, $a_n > |a_0|$

If any of these conditions fails the system is unstable and there is no need to proceed any further

Example 1: (Laser example with $T=0.1$)

$$\varphi(z) = 1 + K\,G(z) = z^2 + (0.0085K - 1.5752)\,z + (0.0072K + 0.6065)$$

Conditions: $\varphi(1) = 1 + (0.0085K - 1.5752) + (0.0072K + 0.6065) = 0.0314 + 0.0157K > 0$

$$(-1)^2\varphi(-1) = 3.1817 - 0.0013K > 0$$

$$a_2 > |a_0| \Rightarrow |0.0072K + 0.6065| < 1 \Leftrightarrow -1 < 0.0072K + 0.6065 < 1$$

From these conditions we have:

$$K > -2$$
$$K < 2.4475 \cdot 10^3$$
$$K < 54.713$$
$$K > -223.39$$

$$\Rightarrow \boxed{-2 < K < 54.713}$$

same conditions as before

__Example 2__ (a robust stability example)

Consider the following second order system:

$$P(z) = z^2 + \alpha z + \beta \qquad \text{where } \alpha \,\&\, \beta \text{ are parameters}$$

The Jury array is:

$$\begin{array}{ccc} z^0 & z^1 & z^2 \\ \beta & \alpha & 1 \\ 1 & \alpha & \beta \\ \beta^2-1 & \alpha(\beta-1) & \end{array}$$

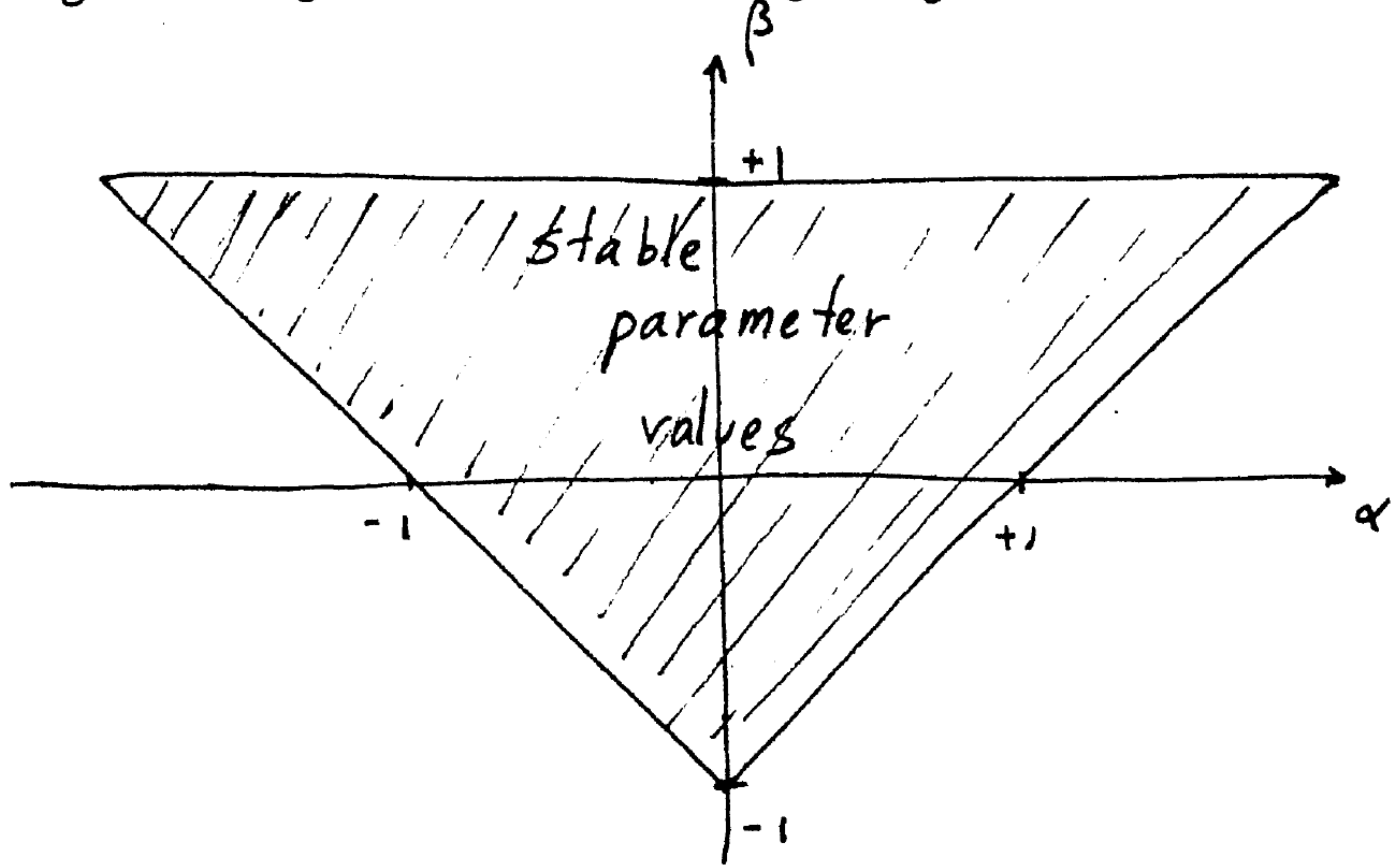
conditions:

$$P(1) > 0 \Rightarrow \alpha + \beta + 1 > 0$$
$$(-1)^2 P(-1) > 0 \qquad \beta + 1 - \alpha > 0$$

$$|a_0| < a_n \Rightarrow |\beta| < 1 \Leftrightarrow -1 < \beta < 1$$
$$|b_0| > |b_{n-1}| \Rightarrow |\beta^2 - 1| > |\alpha(\beta-1)|$$
$$|\cancel{(\beta-1)}(\beta+1)| > |\alpha| \cdot |\cancel{\beta-1}| \quad // \quad |(\beta+1)| > |\alpha|$$

$$\text{or} \quad 1+\beta > \alpha$$
$$1+\beta > -\alpha$$

Graphically we get the following region:



Example 3, a third order system: $\varphi(z) = z^3 - 1.8z^2 + 1.05z - 0.20$

$\varphi(1) = 1 - 1.8 + 1.05 - 0.20 = 0.05 > 0$, ✓ $(-1)^3\varphi(-1) = -1\cdot(-1 - 1.8 - 1.05 - 0.2) = 4.05 > 0$ ✓

$|a_0| = 0.2 < a_3 = 1$ ✓

Jury array:

z^0	z^1	z^2	z^3
-0.2	1.05	-1.8	1
1	-1.8	1.05	-0.2
-0.96	1.59	-0.69	

$\Rightarrow$ $|b_0| = 0.96 > |b_2| = 0.69$ ✓

All conditions hold:
<u>System is stable</u>

$b_0 = \begin{vmatrix} -0.2 & 1 \\ 1 & -0.2 \end{vmatrix} = -0.96$; $b_1 = \begin{vmatrix} -0.2 & -1.8 \\ 1 & 1.05 \end{vmatrix} = 1.59$, $b_2 = \begin{vmatrix} -0.2 & 1.05 \\ 1 & -1.8 \end{vmatrix} = -0.69$