

# EECE 5610 Digital Control Systems

## *Lecture 12*

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## Chapter 6: System Time-Response Characteristics

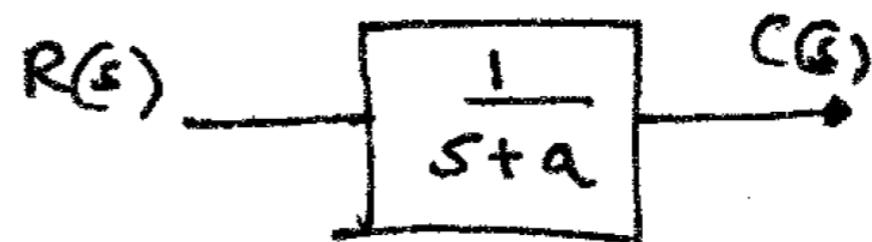
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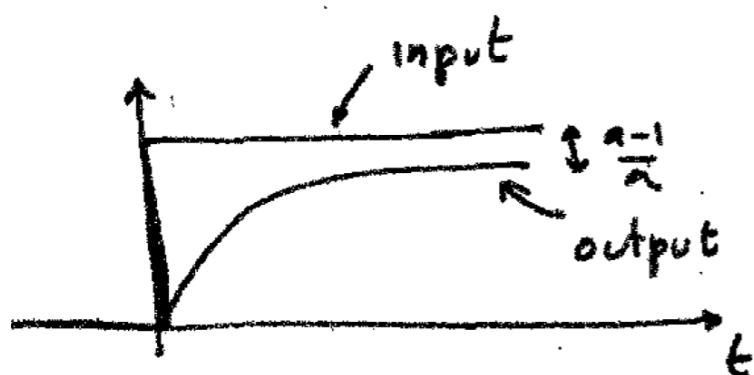
## Chapter 6: System Time-Response Characteristics

- Review of time-responses of continuous time systems:

- First order plant:



$$\text{if } R(s) = \frac{1}{s} \Rightarrow C(s) = \frac{1}{s(s+a)} \Leftrightarrow c(t) = \frac{1}{a} \mathcal{L}^{-1} \left[ \frac{1}{s} - \frac{1}{s+a} \right] = \frac{1}{a} (1 - e^{-at})$$



$\Rightarrow$  has non-zero steady state error

$$\text{Time constant } T = \frac{1}{a}$$

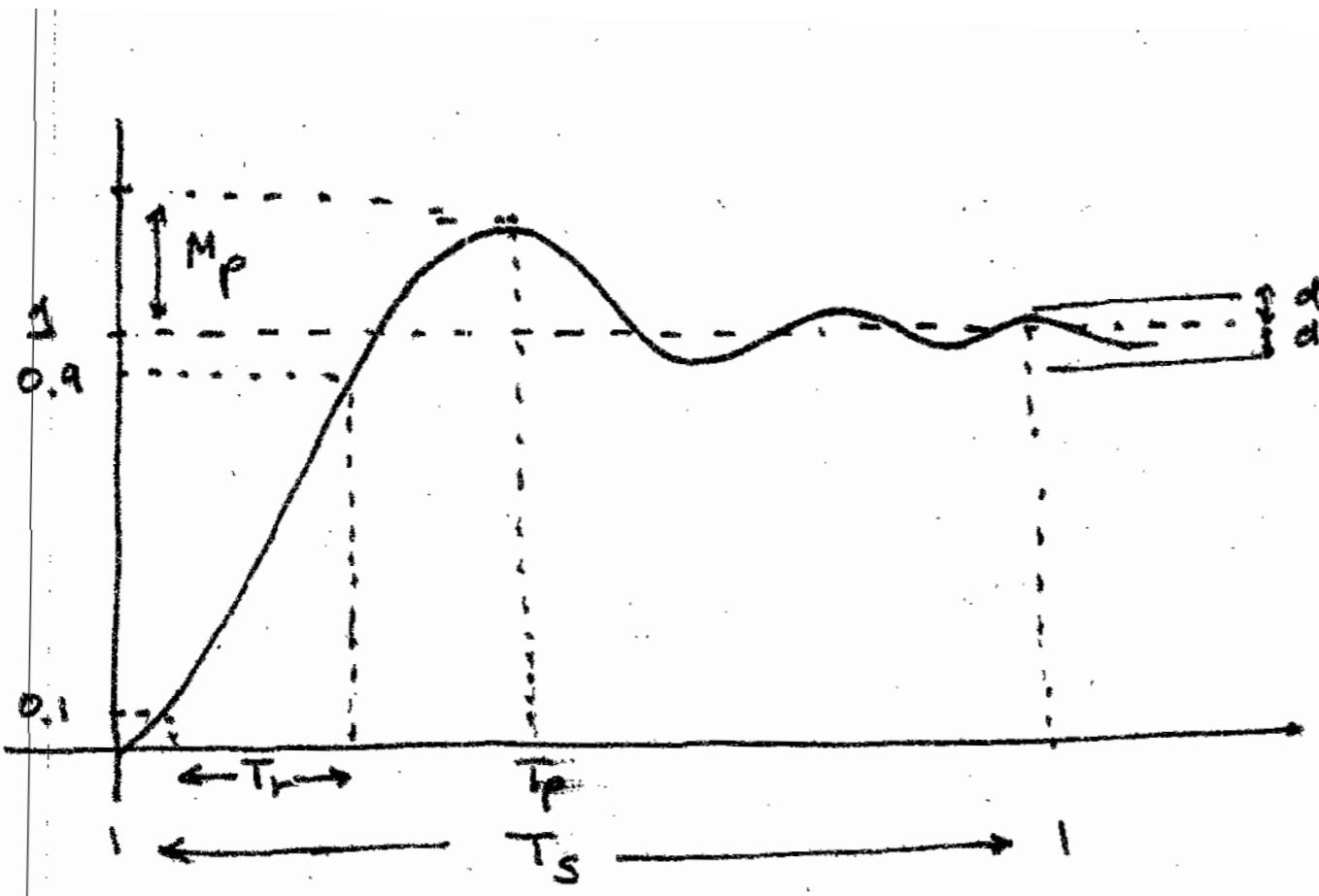
Steady state is reached in about  $4T$

• 2<sup>nd</sup> order plant :

$$R(s) \rightarrow \boxed{\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}} \rightarrow C(s)$$

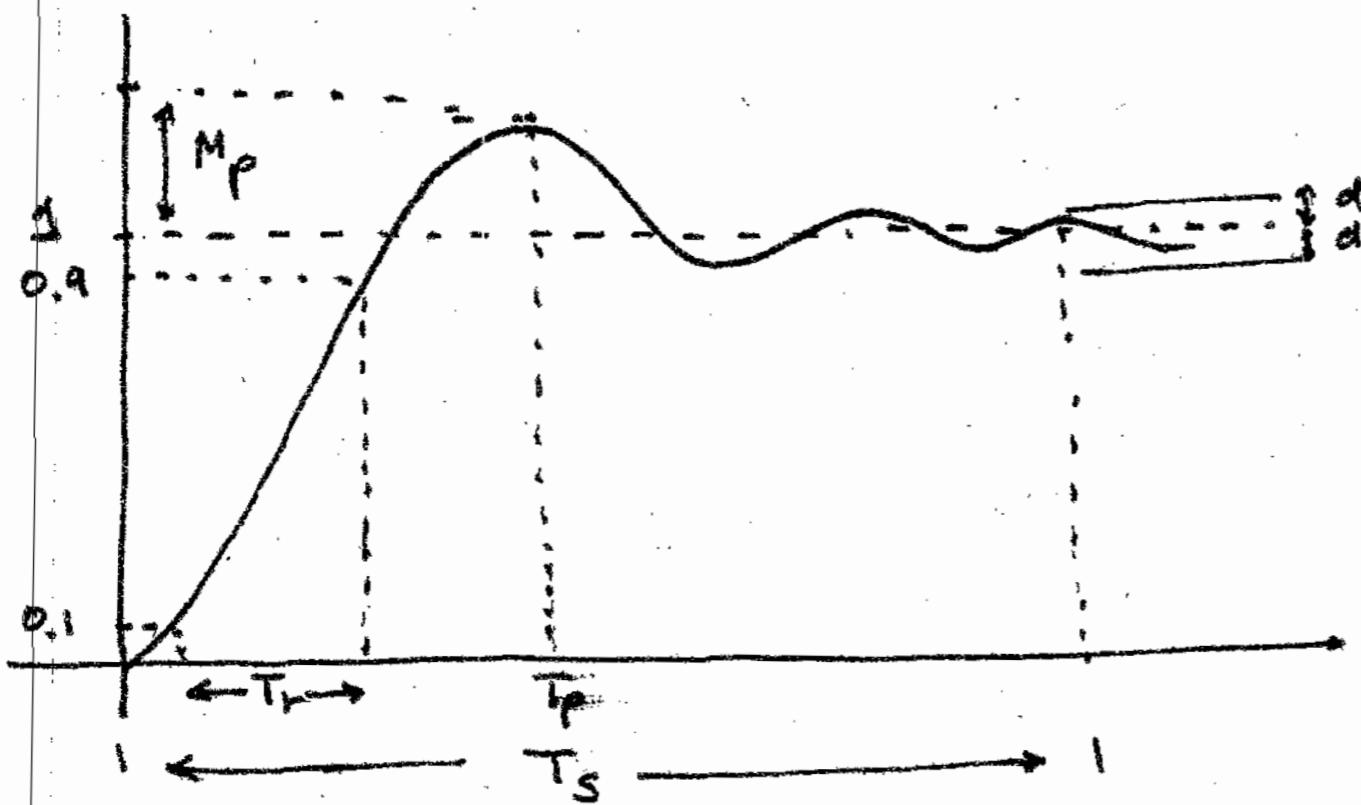
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$\zeta$ : damping ratio  
 $0 < \zeta < 1$ : underdamped

$\zeta = 1$  critically damped

$\zeta > 1$  overdamped

$\omega_n$ : natural frequency

~ 1%, 5%

$T_r$ :	rise time	(function of $\zeta, \omega_n$ )	$T_r \approx \frac{4}{\omega_n}$ for $\zeta = 0.5$
$T_p$ :	peak time	(function of $\zeta, \omega_n$ )	$T_p = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}}$
$T_s$ :	settling time	(function of $\zeta, \omega_n$ )	$T_s = 4/\zeta \omega_n$
$M_p$ :	% overshoot	(function of $\zeta$ <u>only</u> )	$M_p = e^{-\frac{\pi \zeta}{\sqrt{1-\zeta^2}}}$

$T_r$ : rise time

(function of  $\zeta, \omega_n$ )

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(function of  $\zeta, \omega_n$ )

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$T_s$ : settling time

(function of  $\zeta, \omega_n$ )

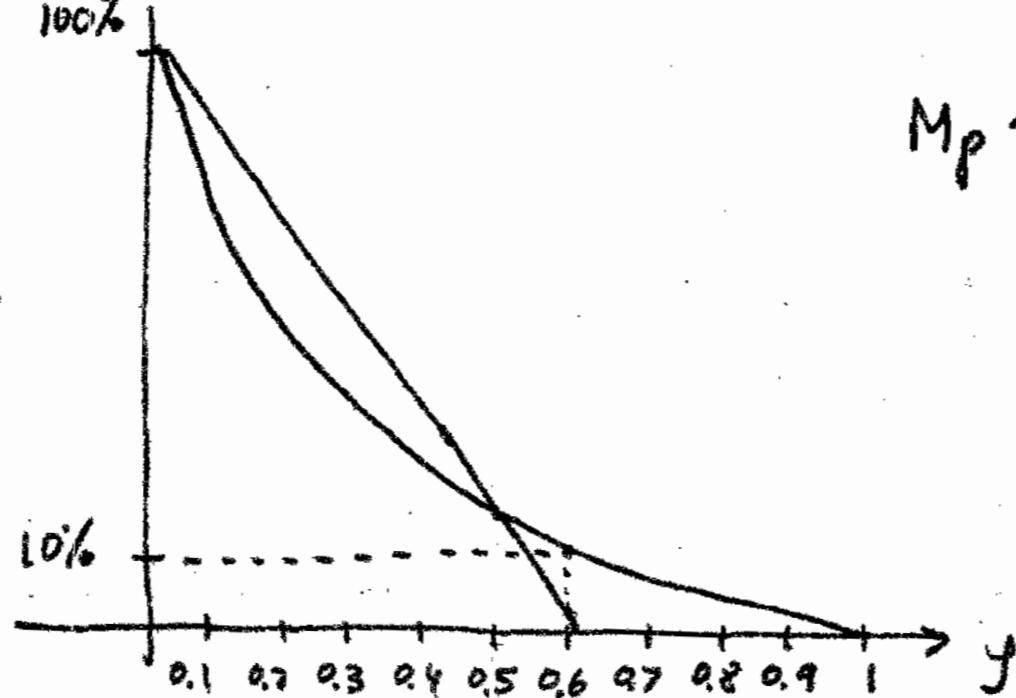
$$T_s = 4/\zeta \omega_n$$

$M_p$ : % overshoot

(function of  $\zeta$  only)

$$M_p = e^{-\frac{\pi \zeta}{\sqrt{1-\zeta^2}}}$$

Approximation: 100%



$$M_p \approx \left(1 - \frac{\zeta}{0.6}\right)$$

$$0 \leq \zeta \leq 0.6$$

(10% max error)  
e  $\zeta = 0.6$

$T_r$ : rise time

(function of  $\zeta, \omega_n$ )

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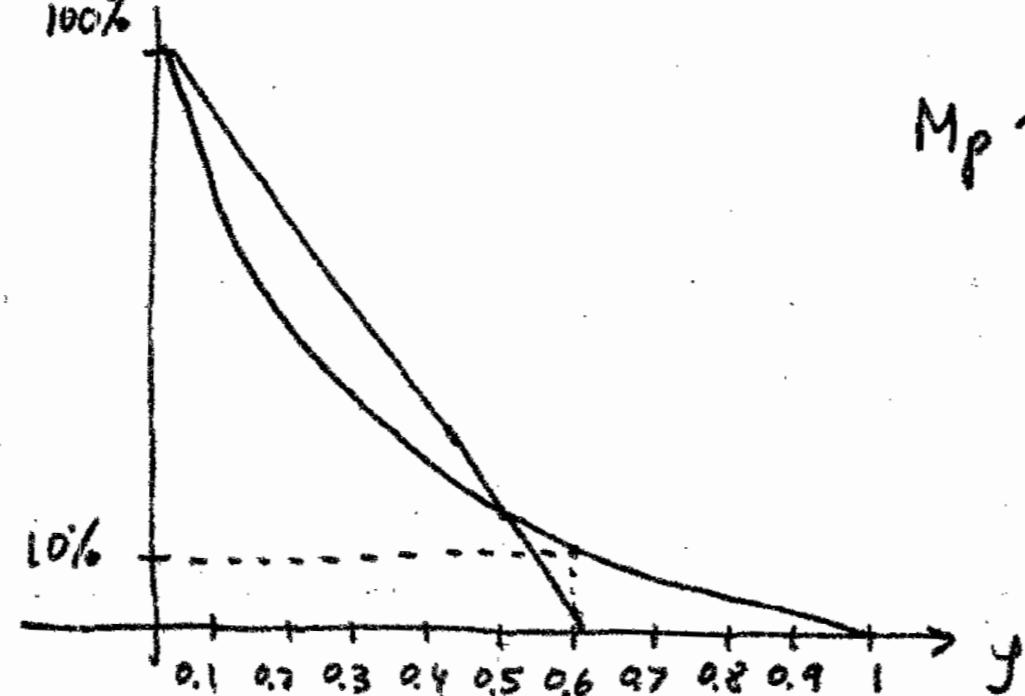
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e  $\zeta = 0.6$

$\zeta$  is also related to the phase margin:  $PM \approx 100 \cdot \zeta$

- Now we need to find the equivalent responses for discrete time systems.
  - 2 alternatives — a) derive everything again as in 580 Tedious!!
  - b) Find a relationship between locations in the s & z planes and use directly the results obtained in 580

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We will pursue approach (b):

Start by recalling the relationship between the poles of  $E(s)$ ,  $E^*(\bar{s})$ ,  $E(z)$

$$\frac{e(t)}{E(s)} \xrightarrow{\quad} \frac{e^*(t)}{E^*(\bar{s})}$$

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Remember:

$$E(z) = \sum_{\text{at poles } E(\lambda)} \left[ \text{residues } E(\lambda) \frac{z}{z - e^{\lambda T}} \right]$$

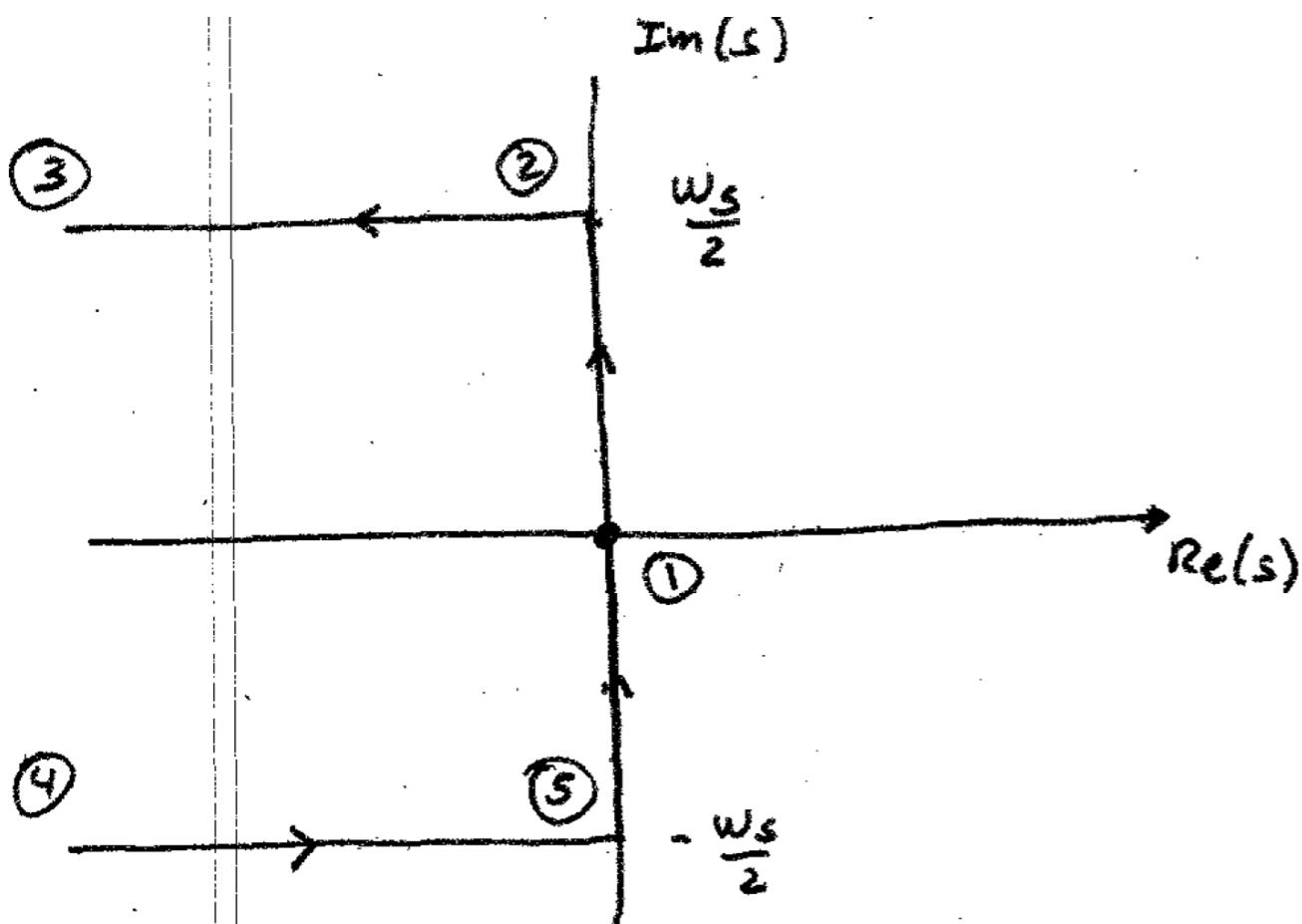
$$\Rightarrow E(s) \text{ pole at } s=p \Leftrightarrow E(z) \text{ pole at } z=e^{pT}$$

For instance if  $E(s) = \frac{1}{s+a}$   $\Rightarrow E^* = \frac{e^{sT}}{e^{sT} - e^{-aT}}$

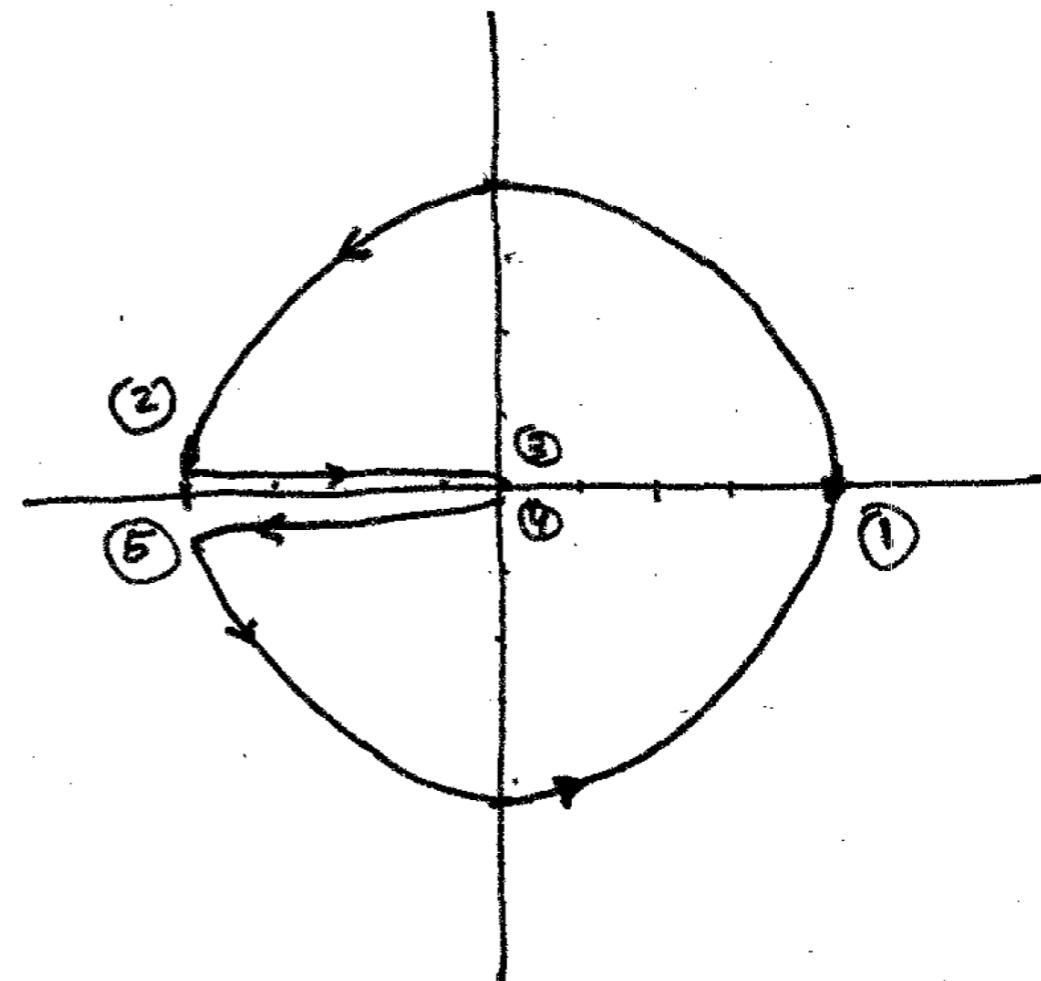
$$E(z) = \frac{z}{z - e^{-aT}}$$

= A pole at  $z=z_1$  results in the transient response characteristics (at the sampling instants) of the continuous time function that has a pole at  $s=s_1$  where  $z_1 = e^{s_1 T}$ . Using this relationship we have:

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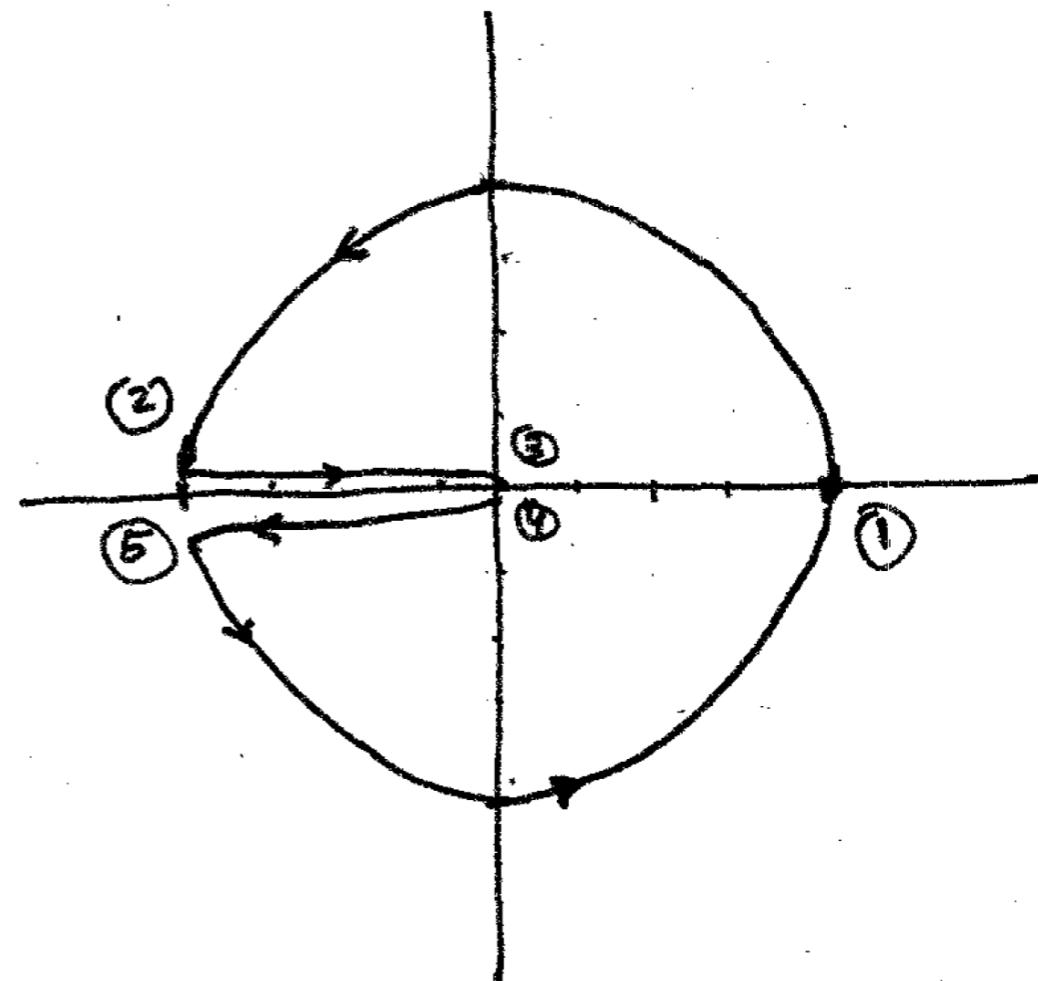
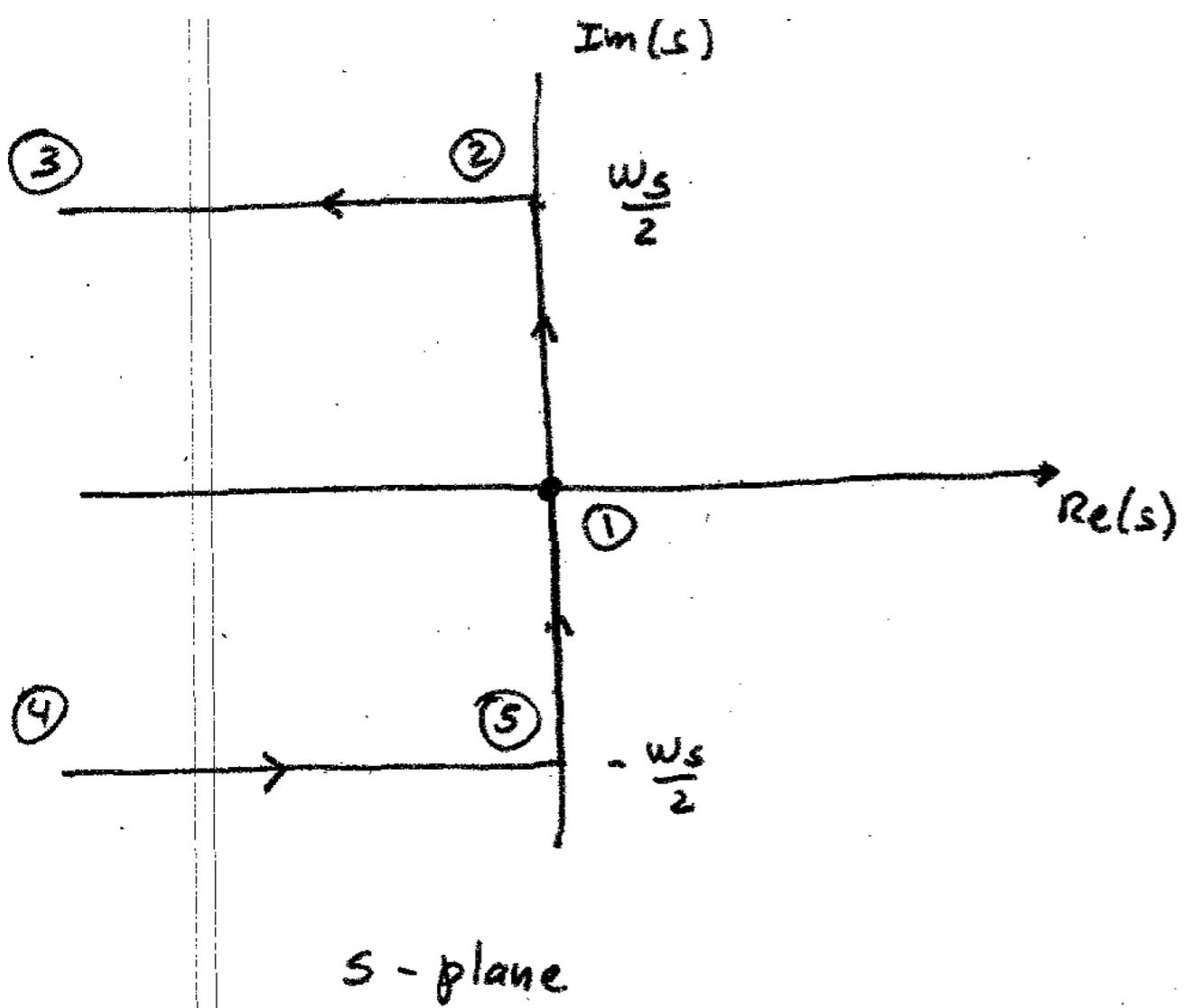


$s$ -plane



$z$ -plane (after sampling)

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$s$ -plane

$z$ -plane (after sampling)

(Note: since  $e^{sT} = e^{\frac{s2\pi}{\omega}}$  is periodic with period  $j\omega_s$ , we need to consider only one such period, for instance  $-\frac{\omega_s}{2} < \omega \leq \frac{\omega_s}{2}$ )

- The  $j\omega$  axis gets mapped to the unit circle:

i.e. if  $s = j\omega \Rightarrow z = e^{j\omega T} = 1 \angle \omega T$

Therefore, a pole at  $z_1 = 1 \angle \omega_1 T$  generates oscillations with freq  $\omega_1$

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- The line  $s = \sigma + j\frac{\omega_s}{2}$  gets mapped to:  $z = e^{\sigma T + j\frac{\omega_s}{2}} = e^{\sigma T} \cdot e^{j\frac{\omega_s}{2}} = -e^{\sigma T}$

so if  $\sigma < 0$ , we get the portion of the real axis  $-1 \leq z < 0$

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so if  $\sigma < 0$ , we get the portion of the real axis  $-1 \leq z < 0$

- Similarly, the line  $s = \sigma - j\frac{\omega_s}{2}$  gets mapped to the same place

- A pole at  $z = z_1 = r \angle \theta \leftrightarrow s = s_1 = \sigma + j\omega$  with  $r = e^{\sigma T}$   
 $\angle \theta = \omega T$

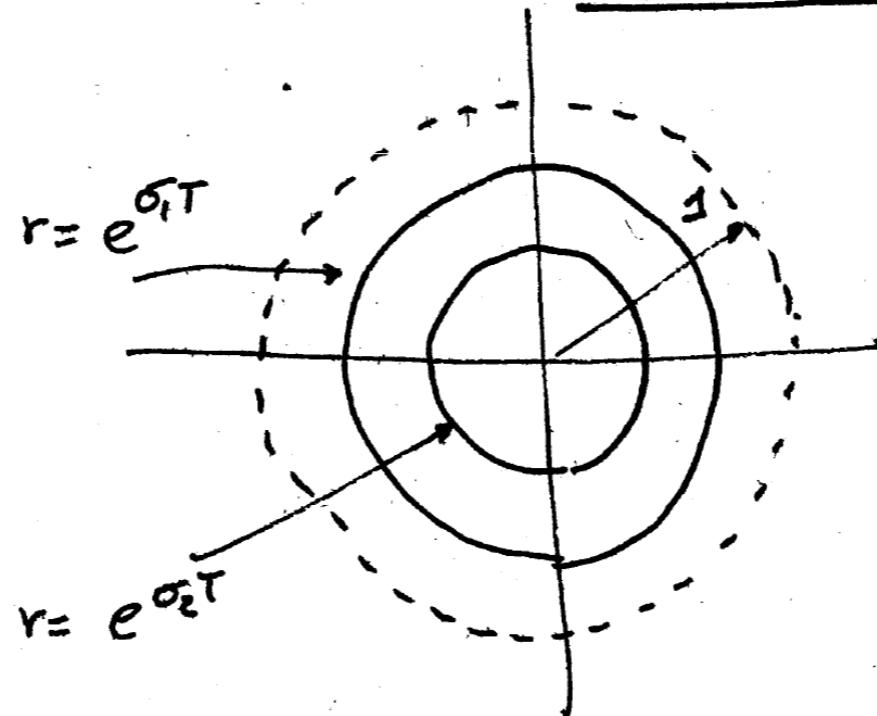
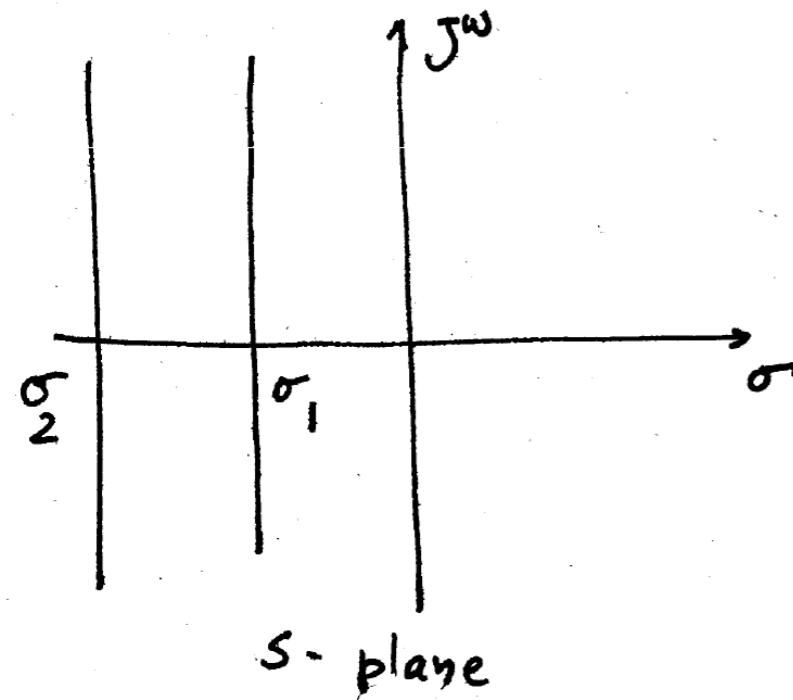
$$c_k = e^{k\sigma T} (\cos(k\omega T + \phi))$$

$$\leftarrow c(t) = e^{\sigma t} \cos(\omega t + \phi)$$

$$c_k = (r)^k \cos[k\theta + \phi]$$

(consistent with what we knew, if  $r < 1 \Rightarrow c_k \rightarrow 0$  as  $k \rightarrow \infty$ )  
 $r > 1 \Rightarrow c_k \rightarrow \infty$ , unstable)

- What happens to the loci of constant time constant? (ie. constant real part,



$$s = σ_1 + jω \Rightarrow z = e^{sT} = e^{\sigma_1 T} e^{j\omega T} = e^{\sigma_1 T} \text{ circle of radius } e^{\sigma_1 T}$$

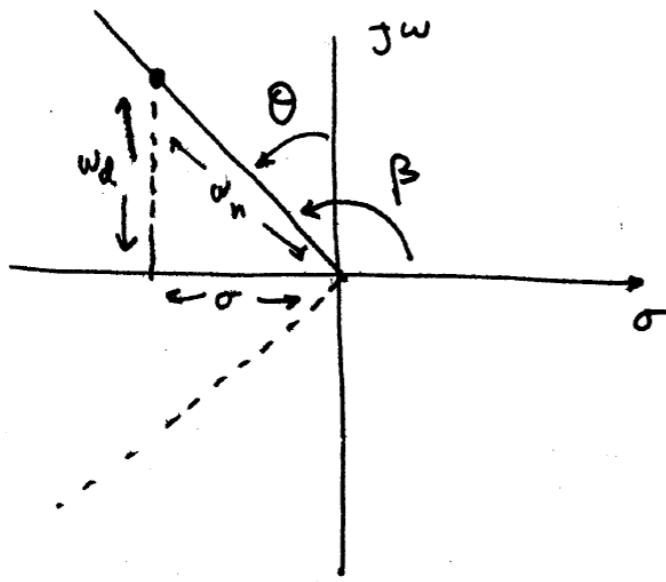
In the discrete time case:

time constant related to  $|z|$

- What about constant damping ratio?

- What about constant damping ratio?

Recall that  $y = \sin \theta \Rightarrow$  in the continuous time case  
the locus of constant  $y$  is a ray through the origin



$$y = \sin \theta$$

$$\frac{\omega_d}{\sigma} = \tan \beta$$

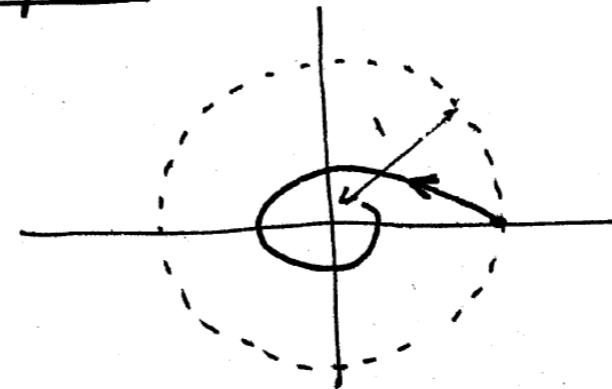
$$z = e^{sT} = e^{\sigma T} e^{j\omega T} = e^{\sigma T} e^{j(\sigma T + \tan \beta)}$$

$$\Rightarrow \beta \text{ constant: } z = e^{\sigma T} e^{j(\sigma T + \tan \beta)}$$

s-plane

this is the equation of a  
logarithmic spiral

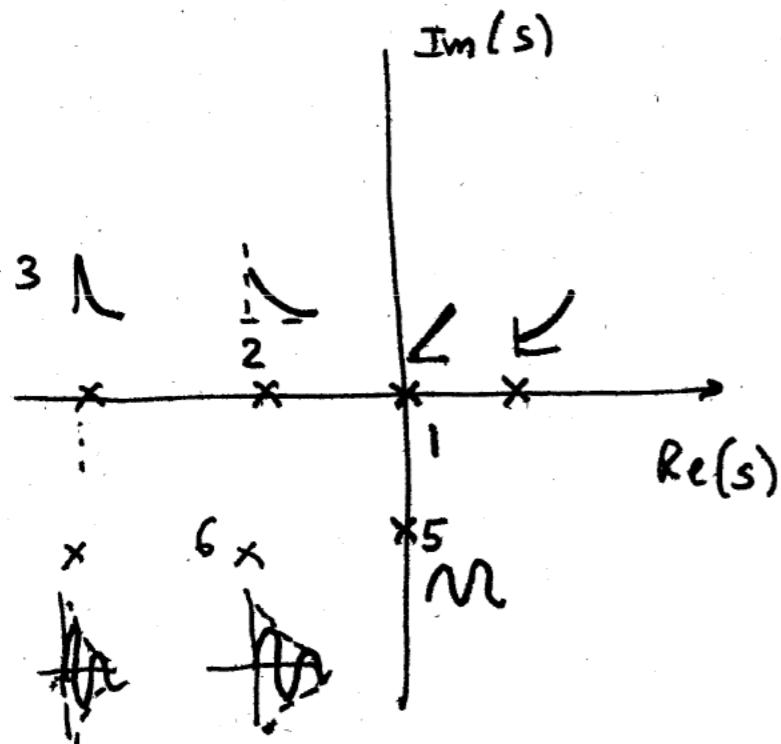
All poles on the straight line in s-domain  
or spiral in z-domain will have the same  
damping ratio



z-plane

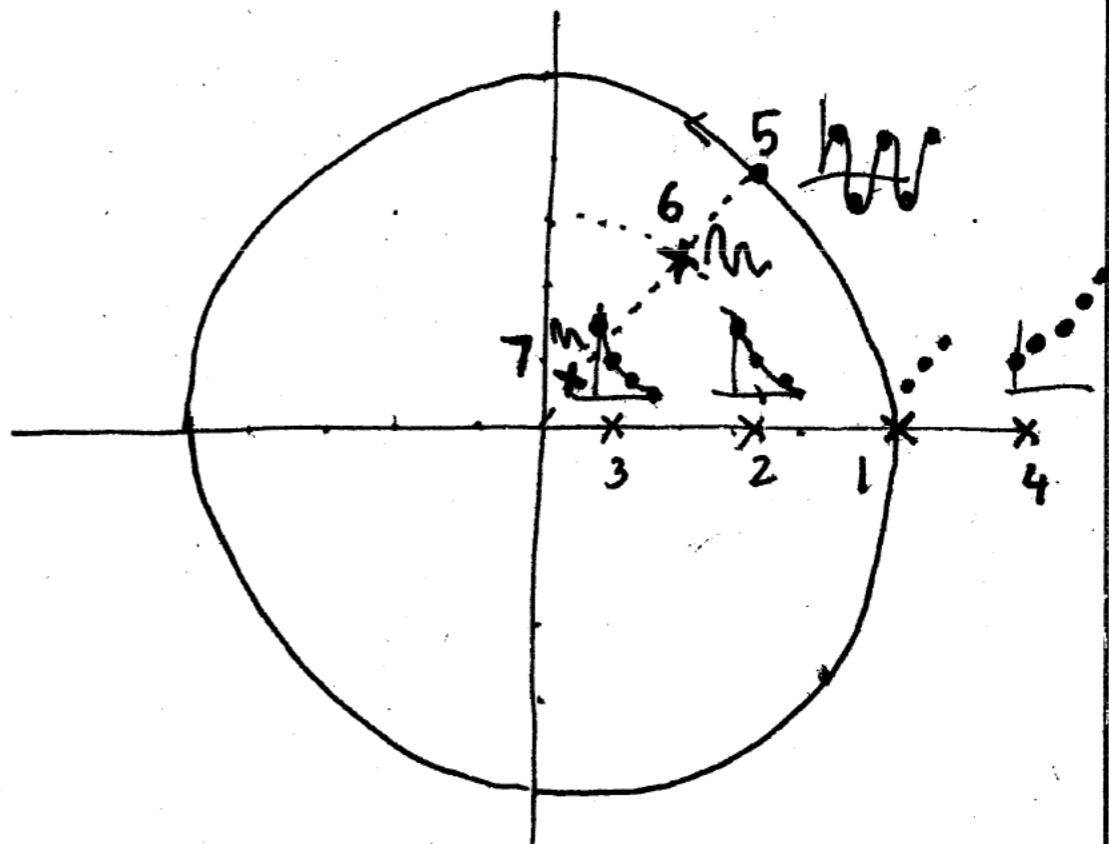
Recap:  $z = e^{sT}$

Time responses in the s-plane



$s$ -plane

Time responses in the z-plane



Note 2 & 6 are at the same distance from  $z=0$   
3 & 7 are at the same distance from  $z=0$

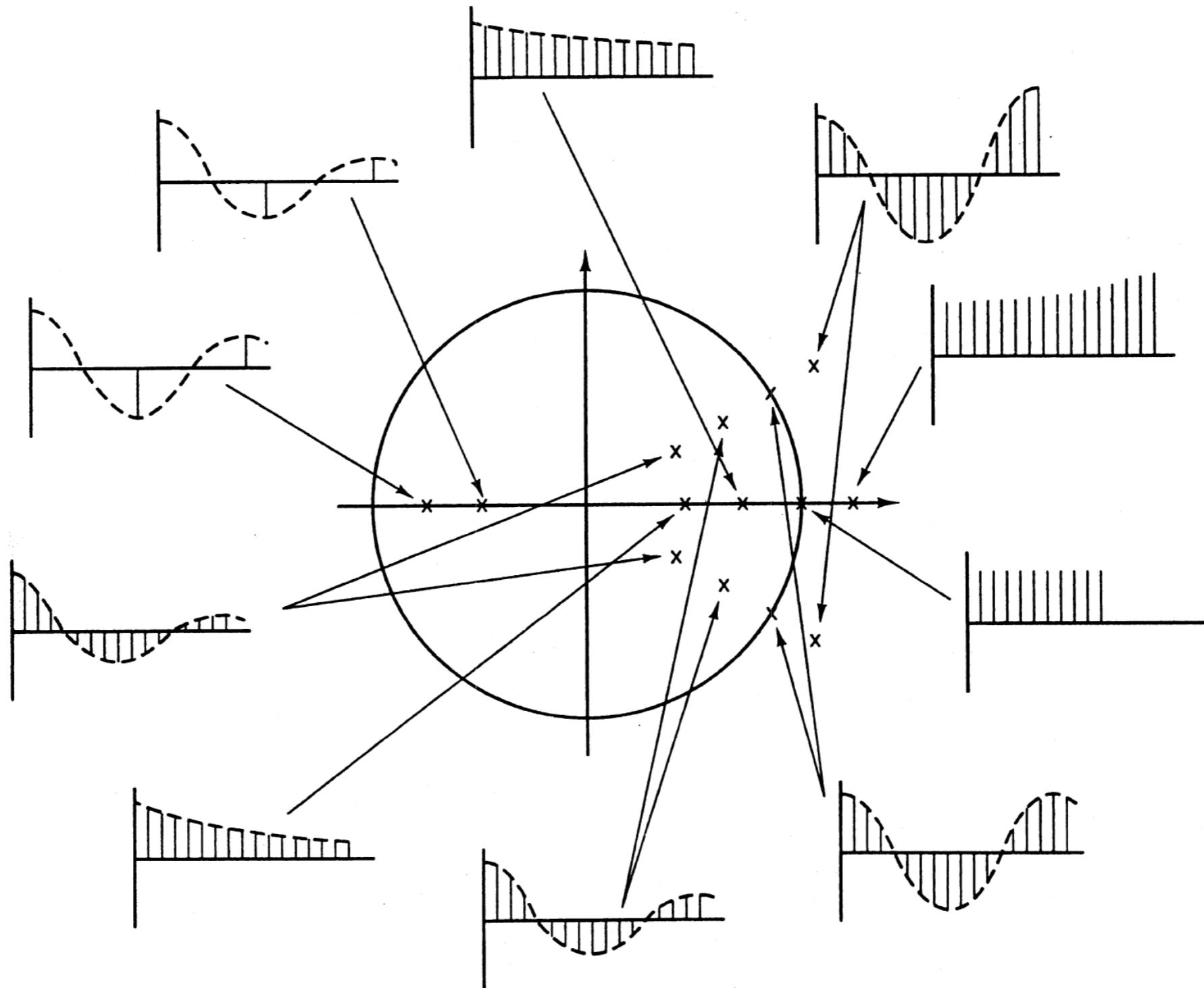
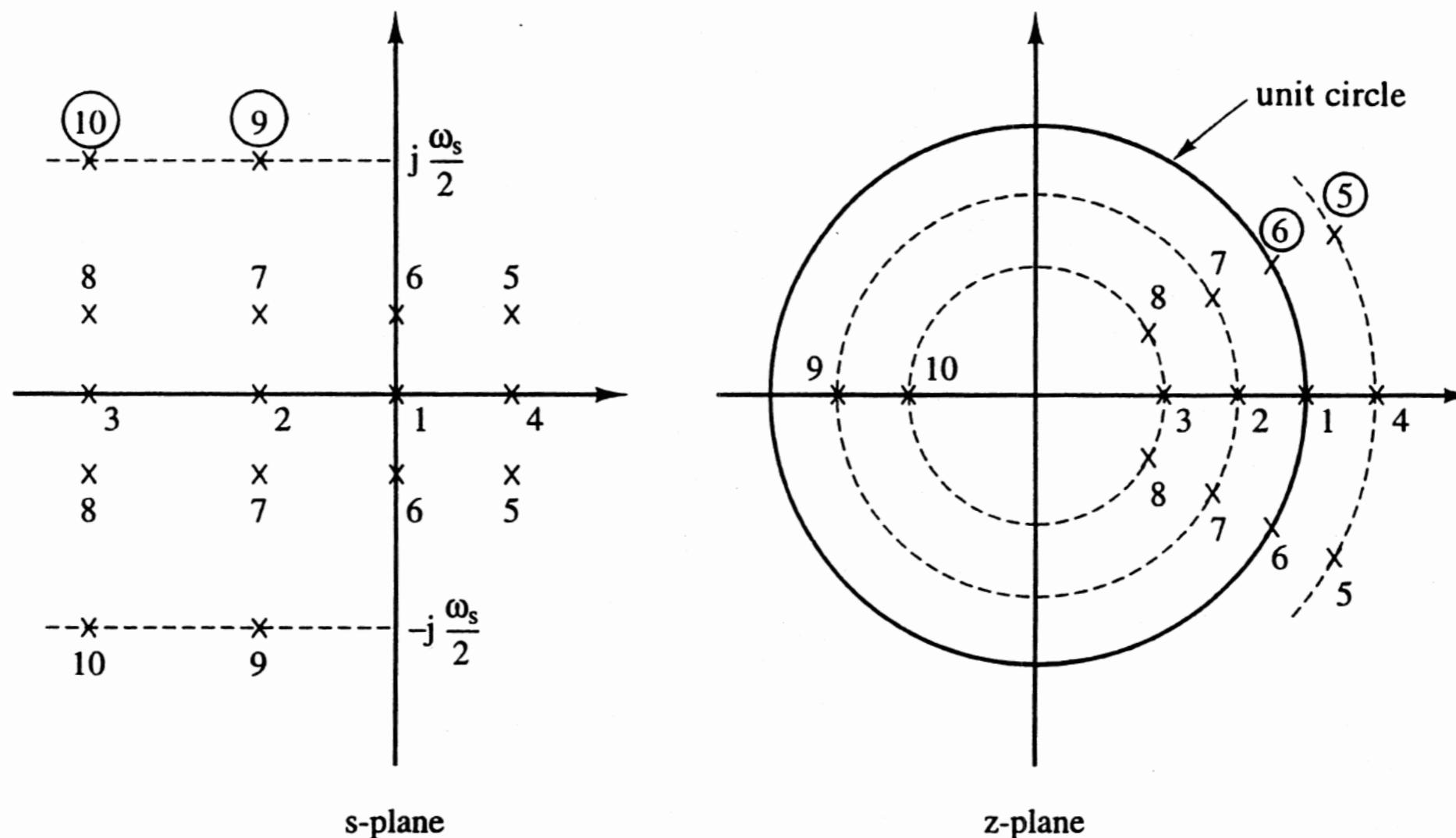


Figure 6-11 Transient response characteristics of the z-plane pole locations.



**Figure 6-10** Corresponding pole locations between the *s*-plane and the *z*-plane.

- If we have a second order system (in the s-plane):

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \Rightarrow s_{1,2} = -\zeta\omega_n \pm j\omega_n \sqrt{1 - \zeta^2}$$

The equivalent z-plane poles are located at:  $z = e^{\frac{sT}{\zeta\omega_n}} = e^{-\frac{\zeta\omega_n T}{\omega_n \sqrt{1-\zeta^2}}} = r e^{j\theta}$

where (1)  $r = e^{-\zeta\omega_n T} \Rightarrow \zeta\omega_n T = -L_n r$

(2)  $\theta = \omega_n T \sqrt{1 - \zeta^2} \Rightarrow \omega_n T = \frac{\theta}{\sqrt{1 - \zeta^2}}$

From (1), (2):  $\frac{-L_n r}{\theta} = \frac{\zeta}{\sqrt{1 - \zeta^2}} \Rightarrow$

$$\zeta = -\frac{L_n r}{\sqrt{\theta^2 + L_n^2 r^2}}$$

$$\omega_n = \frac{1}{T} \sqrt{\theta^2 + L_n^2 r^2}$$

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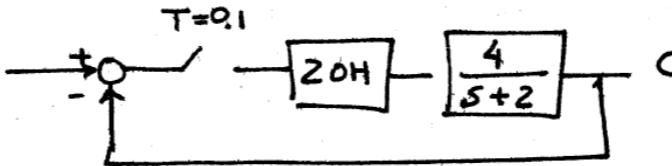
$$\boxed{\begin{aligned}\zeta &= -\frac{L_n r}{\sqrt{\theta^2 + L_n^2 r^2}} \\ \omega_n &= \frac{1}{T} \sqrt{\theta^2 + L_n^2 r^2}\end{aligned}}$$

Finally, the time constant is given by:

$$\boxed{Z = \frac{1}{\zeta\omega_n} = \frac{T}{L_n r}}$$

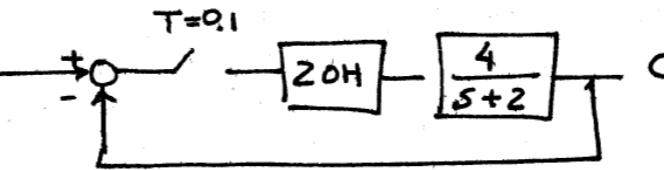
$\Rightarrow$  From  $r, \theta$  we can compute  $\zeta, \omega_n, Z$  and from there  $T_s, M_p, T_r, \dots$

- Example 1 (A first order plant)



$$C(z) = \frac{G(z)}{1+G(z)} R(z) \quad \text{where} \quad G(z) = \mathcal{Z} \left[ \frac{1-e^{-sT}}{s} \cdot \frac{4}{s+2} \right] = \left( \frac{z-1}{z} \right) \mathcal{Z} \left[ \frac{4}{s(s+2)} \right] = \frac{0.3625}{z - 0.8187}$$

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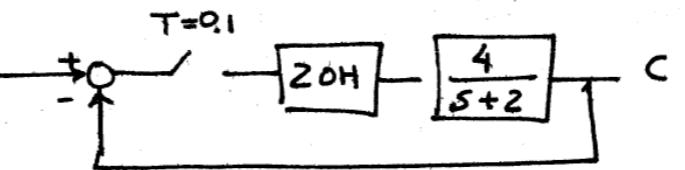


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$$E(z) = \sum_{\text{at poles of } E(\lambda)} \left[ \text{residues of } E(\lambda) \frac{1}{1 - z^{-1} e^{T\lambda}} \right]$$

$$= \frac{z-1}{z} \frac{2(1 - e^{-2T})z}{(z-1)(z - e^{-2T})} = \frac{0.3625}{z - 0.8187}, \quad T = 0.1 \text{ s}$$

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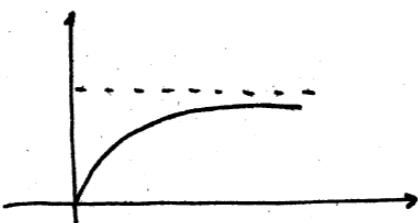
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$$C(z) = \frac{0.3625}{z - 0.4562} R(z)$$

Since we have a pole at  $z = 0.4562 = e^{-0.7848} = e^{-7.848 \cdot (0.1)} = e^{-7.848 T}$

we should expect a time response of the form



with a time constant  $\tau \sim 0.127 \text{ sec.}$

Specifically,

$$R(z) = \frac{z}{z-1}, \quad C(z) = \frac{0.3625}{z - 0.4562} \cdot \frac{z}{z-1} = \frac{2}{3} \left[ \frac{z}{z-1} - \frac{z}{0.4562} \right]^{\frac{z-1}{\tau}} \Rightarrow C(nT) = \frac{2}{3} \left[ 1 - (0.4562)^k \right]$$

$$c(nT) = \frac{2}{3} \left[ 1 - (0.4562)^k \right] = \frac{2}{3} \left[ 1 - e^{-7.848 k T} \right]$$

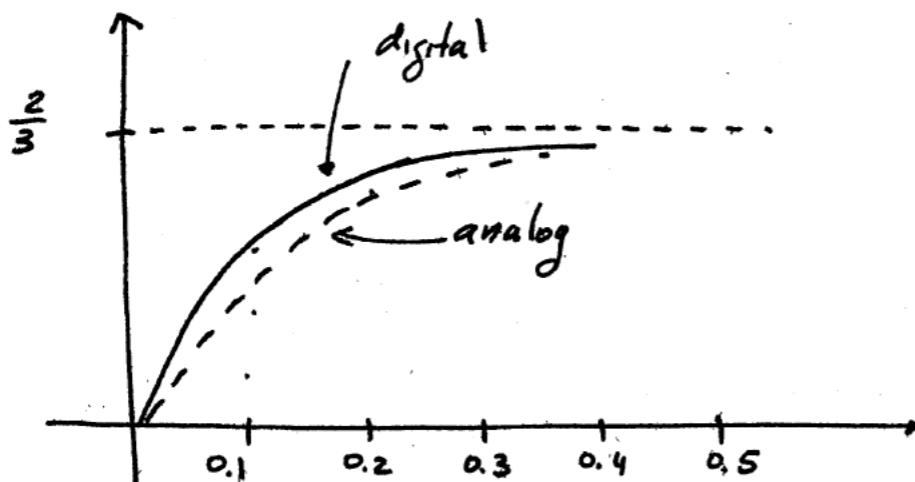
If we remove the sample & hold, we get:

$$T_c(s) = \frac{4}{s+6} \Rightarrow C(s) = \frac{4}{s(s+6)} = \frac{2}{3} \left[ \frac{1}{s} - \frac{1}{s+6} \right] \Rightarrow c(t) = \frac{2}{3} \left[ 1 - e^{-6t} \right] u(t)$$

$$c(nT) = \frac{2}{3} \left[ 1 - (0.4562)^k \right] = \frac{2}{3} \left[ 1 - e^{-7.848 k T} \right]$$

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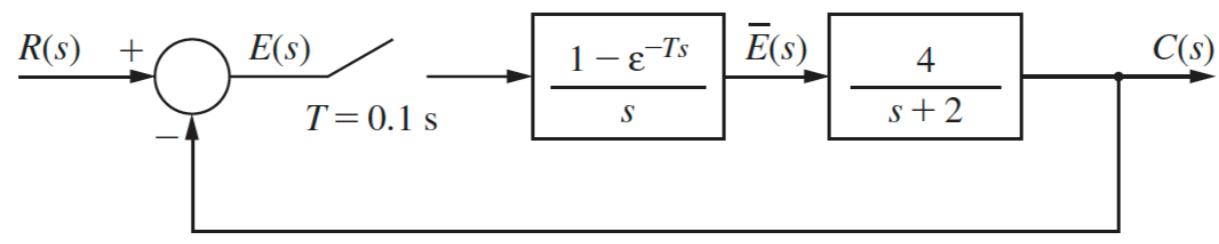
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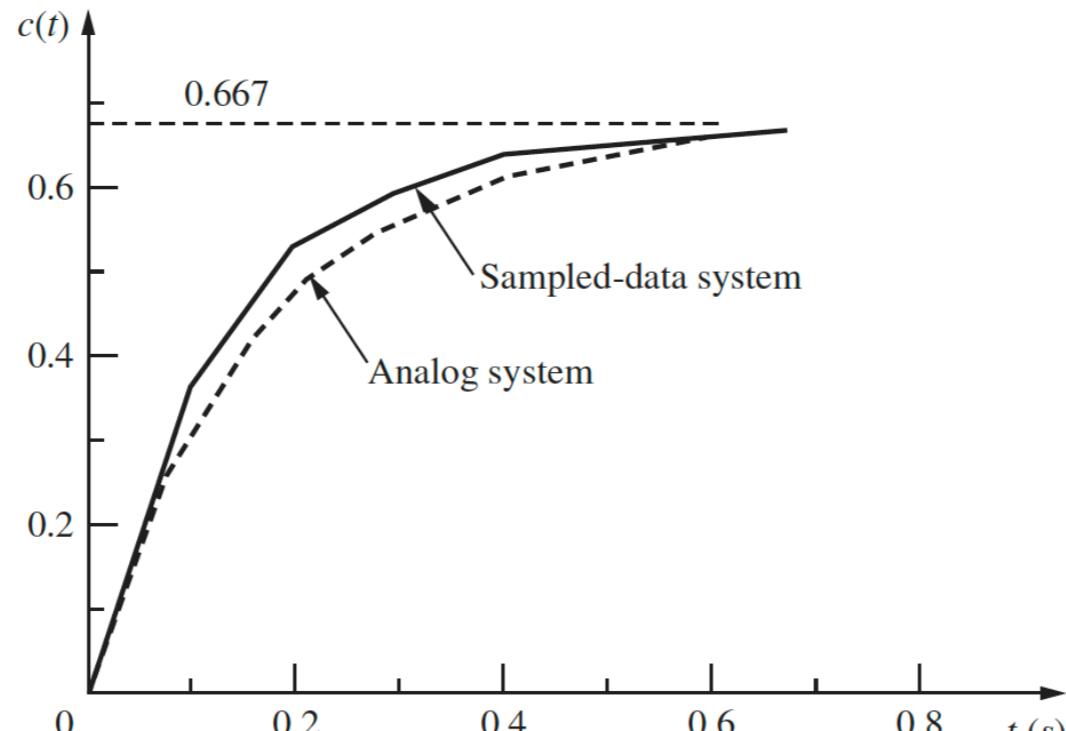
Note that the final value is the same for both systems.  
This is a general result:

Since the ZOH does not have any effect in steady-state, the output of both systems should be the same (provided that both are stable)

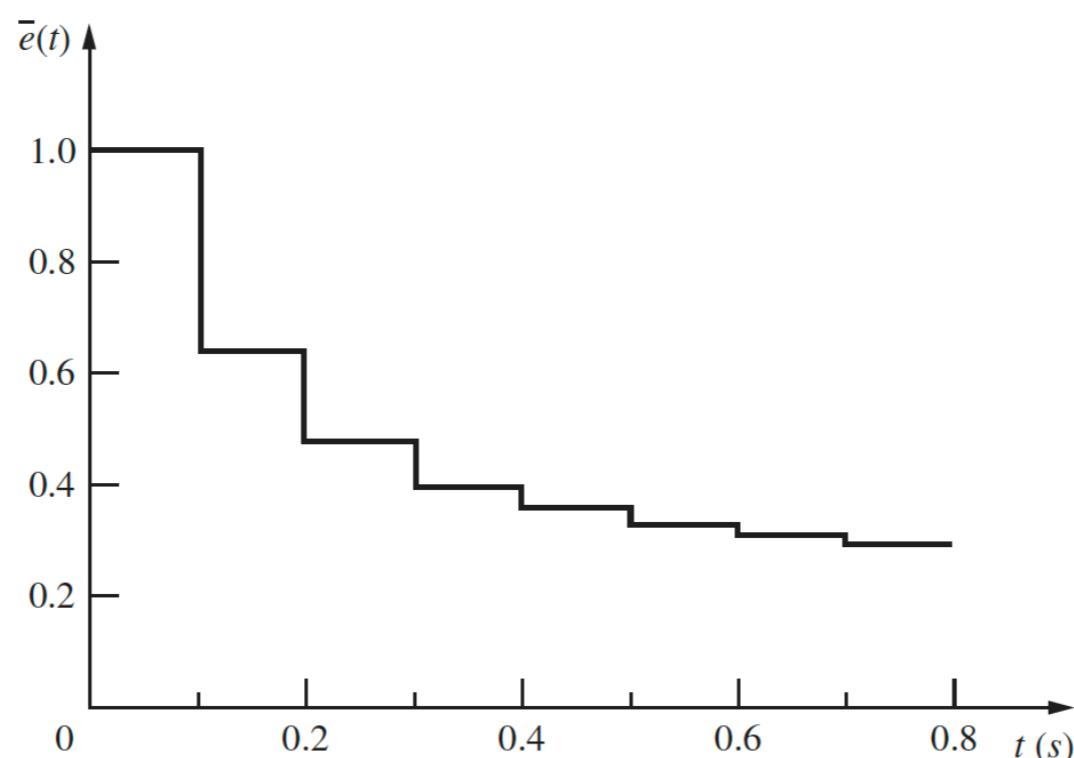
$$\Rightarrow \text{Same DC gain, i.e.: } \lim_{s \rightarrow 0} \frac{G(s)}{1+G(s)} = \lim_{z \rightarrow 1} \frac{G(z)}{1+G(z)} = \frac{2}{3} \#$$



(a)



(b)



(c)

The computations may be done in MATLAB:

```
>> T = 0.1; Gs = tf([4], [1 2]); Gz = c2d(Gs, T);
Tz = feedback(Gz, 1); Rz = tf([1 0], [1 -1], T);
Cz = zpk(Rz*Tz)

Cz =
0.3625 z
-----
(z-1) (z-0.4562)

>> % Partial fraction expansion of C(z)/z
[r,p,k]=residue([0.3625], [1 -1.4562 0.4562])

r =
0.6666
-0.6666

p =
1.0000
0.4562

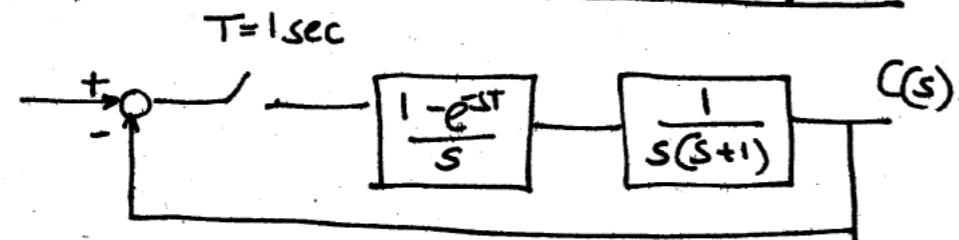
k =
[]
```

If you want to find out what happens between sampling instants you can solve for

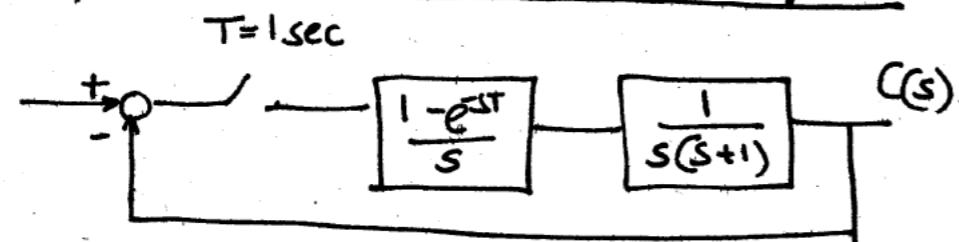
$$C(s) = G(s) E^* = G(s) \left[ \frac{R}{1+G} \right]^* \text{ and use the inverse Laplace transf}$$

or use the modified z-transform.

- Example 2: second order plant



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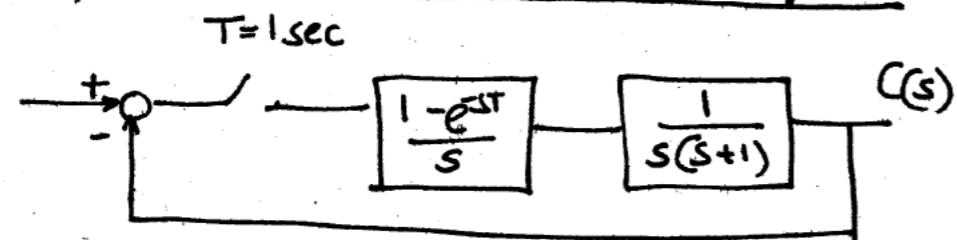


$$G_{op} = Z \left[ \frac{1 - e^{-ST}}{s^2(s+1)} \right] = \frac{0.368Z + 0.264}{Z^2 - 1.368Z + 0.368}$$

$$G_d = \frac{G_{op}}{1 + G_{op}} = \frac{0.368Z + 0.264}{Z^2 - Z + 0.632}$$

Char equation:  $Z^2 - Z + 0.632 = 0 \Rightarrow \text{poles at } z = 0.7950 \angle \pm 0.8906$

• Example 2: second order plant



$$G(z) = \left( \frac{z-1}{z} \right) \tilde{\mathcal{Z}} \left[ \frac{1}{s^2(s+1)} \right]_{T=1} = \frac{z-1}{z} \left[ \frac{z[(1-1+\varepsilon^{-1})z + (1-\varepsilon^{-1}-\varepsilon^{-1})]}{(z-1)^2(z-\varepsilon^{-1})} \right]$$

$$= \frac{0.368z + 0.264}{z^2 - 1.368z + 0.368}$$

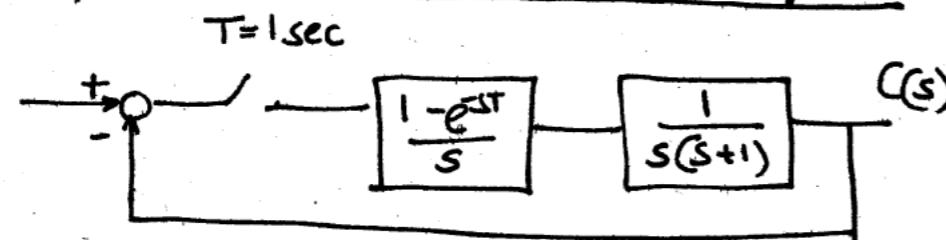
$$G_{op} = z \left[ \frac{1 - e^{-ST}}{s^2(s+1)} \right] = \frac{0.368z + 0.264}{z^2 - 1.368z + 0.368}$$

?

$$G_{cl} = \frac{G_{op}}{1+G_{op}} = \frac{0.368z + 0.264}{z^2 - z + 0.632}$$

Char equation:  $z^2 - z + 0.632 = 0 \Rightarrow$  poles at  $\varepsilon = 0.7950 \angle \pm 0.8906$

• Example 2: second order plant



$$G(z) = \left( \frac{z-1}{z} \right) \mathcal{Z} \left[ \frac{1}{s^2(s+1)} \right]_{T=1} = \frac{z-1}{z} \left[ \frac{z[(1-1+\varepsilon^{-1})z + (1-\varepsilon^{-1}-\varepsilon^{-1})]}{(z-1)^2(z-\varepsilon^{-1})} \right]$$

$$= \frac{0.368z + 0.264}{z^2 - 1.368z + 0.368}$$

$$G_{ol} = z \left[ \frac{1 - e^{-ST}}{s^2(s+1)} \right] = \frac{0.368z + 0.264}{z^2 - 1.368z + 0.368}$$

?

$$G_{cl} = \frac{G_{ol}}{1+G_{ol}} = \frac{0.368z + 0.264}{z^2 - z + 0.632}$$

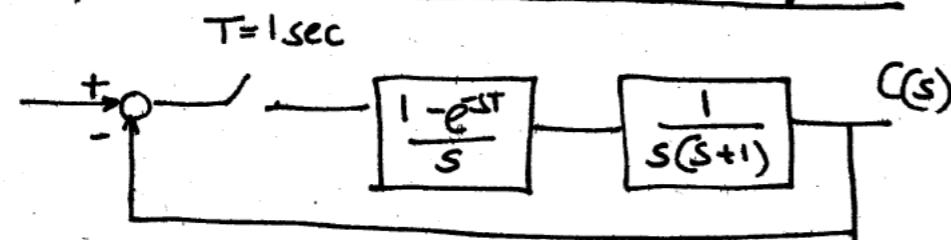
Char equation:  $z^2 - z + 0.632 = 0 \Rightarrow \text{poles at } z = 0.7950 \angle \pm 0.8906$

$$\zeta = \frac{-\ln(0.7950)}{\sqrt{[0.8906]^2 + (\ln 0.795)^2}} \approx 0.25$$

$$\omega_n = \frac{1}{\zeta} \sqrt{\zeta^2 + (\ln r)^2} \approx 0.92$$

$$T = \frac{1}{\zeta \ln(r)} \approx 4.36 \text{ sec}$$

• Example 2: second order plant



$$G(z) = \left( \frac{z-1}{z} \right) \tilde{\mathcal{Z}} \left[ \frac{1}{s^2(s+1)} \right]_{T=1} = \frac{z-1}{z} \left[ \frac{z[(1-1+\varepsilon^{-1})z + (1-\varepsilon^{-1}-\varepsilon^{-1})]}{(z-1)^2(z-\varepsilon^{-1})} \right]$$

$$= \frac{0.368z + 0.264}{z^2 - 1.368z + 0.368}$$

$$G_{ol} = z \left[ \frac{1 - e^{-ST}}{s^2(s+1)} \right] = \frac{0.368z + 0.264}{z^2 - 1.368z + 0.368}$$

$$G_{cl} = \frac{G_{ol}}{1+G_{ol}} = \frac{0.368z + 0.264}{z^2 - z + 0.632}$$

Char equation:  $z^2 - z + 0.632 = 0 \Rightarrow$  poles at  $\varepsilon = 0.7950 \angle \pm 0.8906$

$$\varphi = \frac{-\ln(0.7950)}{\sqrt{[0.8906]^2 + (\ln[0.795])^2}} \approx 0.25$$

$$\omega_n = \frac{1}{T} \sqrt{\Omega^2 + (\ln r)^2} \approx 0.92$$

$$T = \frac{1}{\ln(r)} \approx 4.36 \text{ sec}$$

$$\varphi = -\frac{\ln r}{\sqrt{\Omega^2 + \ln^2 r}}$$

$$\omega_n = \frac{1}{T} \sqrt{\Omega^2 + \ln^2 r}$$

$$T = \frac{1}{\varphi \omega_n} = \frac{T}{\ln r}$$

Digital system ( $T = 1 \text{ sec}$ )

$$\zeta = 0.25 \Rightarrow M_p \approx 0.58$$

$$\omega_n \approx 0.92 \text{ rad/sec}$$

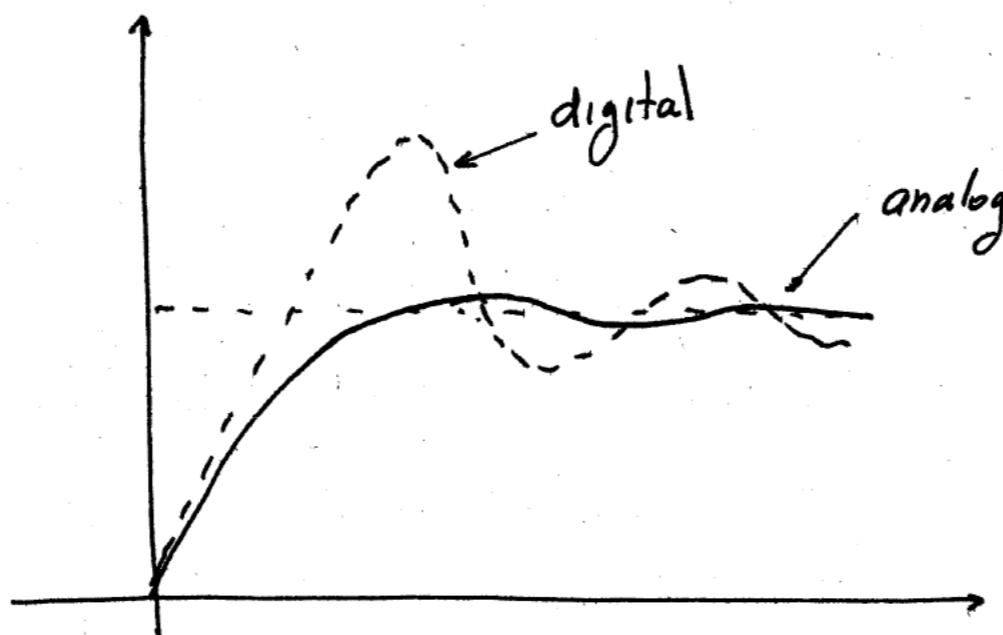
$$\tau \approx 4.36 \text{ sec}$$

Analog system ( $\frac{\text{no}}{\text{ZOH}}$ )

$$\zeta = 0.5 \Rightarrow M_p \approx 16\%$$

$$\omega_n = 1$$

$$\tau = 2 \text{ sec}$$



You can see here that the effects of sampling are destabilizing

(The problem in this specific case is that the sampling rate is too low compared with the system's time constant: rule of thumb: want  $T \ll \tau$ )