ECE 5610 0, Fall 2019, First Midterm Exam, 100 minutes

Name:______________________________ Signature: ______________________________

Instructions:

1. This is a closed–book test but **one** $8\frac{1}{2} \times 11$ ~~single~~–sided cheat–sheet is allowed.
2. Work as many problems as you can. Try not to spend too much time working on a single problem. If you get stuck, try working on a different question.
3. Show all your work, but try to be as concise as possible.
4. • **DO NOT LOOK** at the problems until told to do so.
5. • **STOP** working after the "time's up" announcement.
6. • **GOOD LUCK !**

(1) --- 15 points

A function $e(t)$ is sampled, and the resultant sequence has the z-transform

$$E(z) = \frac{z^3 - 2z}{z^4 - 0.9z^2 + 0.8}$$

Solve this problem using $E(z)$ and the properties of the z-transform.

(a) Find the z-transform of $e(t-2T)u(t-2T)$.

(b) Find the z-transform of $e(t+2)u(t)$.

(c) Find the z-transform of $e(t-T)u(t-2T)$.

Note that *T* is sampling time, and *u(t)* is a unit step function.

Solution:

(a) $\mathcal{Z}[e(t-2T)u(t-2T)] = \dfrac{(z^3-2z)z^{-2}}{z^4-0.9z^2+0.8}$

(b) $e(0)=0,\ e(1)=1$

$$\therefore \mathcal{Z}[e(t+T)u(t)] = z[E(z) - e(0) - e(1)z^{-1}]$$

$$= z\left[\frac{z^3-2z}{z^4-0.9z^2+0.8} - \frac{1}{z}\right] = \frac{-1.1z^2+0.8}{z^4-0.9z^2+0.8}$$

(c) $\mathcal{Z}[e(t-T)u(t-2T)] = e(T)z^{-2} + e(2T)z^{-3} + \cdots$

$$= z^{-1}[E(z) - e(0)] = z^{-1}E(z), \quad \text{since } e(0) = 0$$

$$= \frac{z^2 - z}{z^4 - 0.9z^2 + 0.8}$$

(2) – 10 points

.Find $E^*(s)$, with $T = 0.5$ s, for

$$E(s) = \frac{\left(1 - \varepsilon^{-0.5s}\right)^2}{0.5s^2(s+1)}$$

and

$$E(s) = \frac{1 - \varepsilon^{-Ts}}{s(s+1)}$$

Solution:

Consider $E_1(s) = \dfrac{1}{s^2(s+1)}$; Then $E^*(s) = E_1^*(s)[2(1-\varepsilon^{-Ts})^2]$

$$(\text{residue})\Big|_{\lambda=0} = \frac{d}{d\lambda}\left[\frac{1}{(\lambda+1)(1-\varepsilon^{-T(s-\lambda)})}\right]_{\lambda=0}$$

$$= \frac{-(1-\varepsilon^{-T(s-\lambda)}) - (-T\varepsilon^{-T(s-\lambda)})(\lambda+1)}{(\lambda+1)^2(1-\varepsilon^{-T(s-\lambda)})^2}\Bigg|_{\lambda=0} = \frac{-1+\varepsilon^{-Ts}+T\varepsilon^{-Ts}}{(1-\varepsilon^{-Ts})^2}$$

$$(\text{residue})_{\lambda=-1} = \left[\frac{1}{\lambda^2(1-\varepsilon^{-T(s-\lambda)})}\right]_{\lambda=-1} = \frac{1}{1-\varepsilon^{-T(s+1)}}$$

$$\therefore E_1^*(s) = \frac{1}{1-\varepsilon^{-T(s+1)}} + \frac{-1+\varepsilon^{-Ts}+T\varepsilon^{-Ts}}{(1-\varepsilon^{-Ts})^2}$$

$$\therefore E^*(s) = \frac{1-\varepsilon^{-Ts}-1+\varepsilon^{-Ts}+T\varepsilon^{-Ts}}{(1-\varepsilon^{-T(s+1)})(1-\varepsilon^{-Ts})^2}[2(1-\varepsilon^{-Ts})^2] = \frac{2T\varepsilon^{-Ts}}{1-\varepsilon^{-T(s+1)}}$$

Solution:

$$E(s) = \frac{1-\varepsilon^{-Ts}}{s(s+1)}; \text{ define } E_1(s) = \frac{1}{s(s+1)}$$

$$\therefore E_1^*(s) = \left.\frac{1}{(\lambda+1)(1-\varepsilon^{-T(s-\lambda)})}\right|_{\lambda=0} + \left.\frac{1}{\lambda(1-\varepsilon^{-T(s-\lambda)})}\right|_{\lambda=-1}$$

$$= \frac{1}{1-\varepsilon^{-Ts}} - \frac{1}{1-\varepsilon^{-T(s+1)}}$$

$$E(s) = E_1(s) - E_1(s)\varepsilon^{-Ts}; \therefore E^*(s) = E_1^*(s) - E_1^*(s)\varepsilon^{-Ts}$$

$$= \left[\frac{1}{1-\varepsilon^{-Ts}} - \frac{1}{1-\varepsilon^{-T(s+1)}}\right](1-\varepsilon^{-Ts}) = \frac{\varepsilon^{-Ts}(1-\varepsilon^{-T})}{1-\varepsilon^{-T(s+1)}}$$

(3) – 20 points

Given the system described by the state equations

$$x(k+1) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(k),$$

$$y(k) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x(k).$$

(a) Find the transfer function $Y(z)/U(z)$.
(b) Draw a simulation diagram for this system, from the state equations given.
(c) Use Mason's gain formula and the simulation diagram to verify the transfer function found in part (a).

Hint: To find $(zI - A)^{-1}$ just use the following:

$$\begin{pmatrix} a & 0 & 0 \\ b & d & 0 \\ c & e & f \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{a} & 0 & 0 \\ -\frac{b}{a\,d} & \frac{1}{d} & 0 \\ \frac{-(c\,d)+b\,e}{a\,d\,f} & -\frac{e}{d\,f} & \frac{1}{f} \end{pmatrix}$$

Solution:

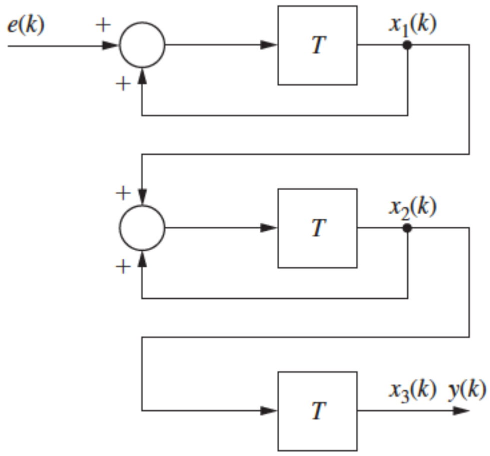
(a) $$z\mathbf{I} - \mathbf{A} = \begin{bmatrix} z-1 & 0 & 0 \\ -1 & z-1 & 0 \\ 0 & -1 & z \end{bmatrix};\ \Delta = z^3 - 2z^2 + z = z(z-1)^2$$

$$\text{Cof }(z\mathbf{I} - \mathbf{A}) = \begin{bmatrix} z(z-1) & z & 1 \\ 0 & z(z-1) & z-1 \\ 0 & 0 & (z-1)^2 \end{bmatrix},\ (z\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} \frac{1}{z-1} & 0 & 0 \\ \frac{1}{(z-1)^2} & \frac{1}{z-1} & 0 \\ \frac{1}{z(z-1)^2} & \frac{1}{z(z-1)} & \frac{1}{z} \end{bmatrix}$$

$$G(z) = \mathbf{C}[z\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B} = [0 \quad 0 \quad 1][z\mathbf{I} - \mathbf{A}]^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{z(z-1)^2} & \frac{1}{z(z-1)} & \frac{1}{z} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{z(z-1)^2} = \frac{1}{z^3 - 2z^2 + z}$$

(b)



(c) $\Delta = 1 - z^{-1} - z^{-1} + z^{-2} = 1 - 2z^{-1} + z^{-2}$

$$\therefore G(z) = \frac{z^{-3}}{\Delta} = \frac{1}{z^3 - 2z^2 + z}$$

(4)-- 20 points

Solve the given difference equation for $x(k)$ using:

$$x(k)-3x(k-1)+2x(k-2)=e(k),\ \ e(k)=\begin{cases}1,\ k=0,1\\0,\ k\geq 2\end{cases}$$

$$x(-2)=x(-1)=0$$

(a) By calculating x(0), x(1), x(2), x(3), and x(4).

(b) The z-transform.

(c) Will the final-value theorem give the correct value of $x(k)$ as $k\to\infty$?

Solution:

(a) $x(0)=e(0)=1$

$x(1)=e(1)+3x(0)=4$

$x(2)=e(2)+3x(1)-2x(0)=10$

$x(3)=0+3(10)-2(4)=22$

$x(4)=0+3(22)-2(10)=46$

(b) $[1-3z^{-1}+2z^{-2}]X(z)=E(z)=1+z^{-1}=\dfrac{z+1}{z}$

$$X(z)=\frac{z^2}{(z-1)(z-2)}\times\frac{z+1}{z}=\frac{z(z+1)}{(z-1)(z-2)}=z\left[\frac{-2}{z-1}+\frac{3}{z-2}\right]$$

$\therefore x(k)=-2+3(2)^k$

(c) No, since the final value does not exist.

(5) – 15+10 points

(a)

Find a state-variable formulation for the system described by the coupled second-order difference equations given. The system output is $y(k)$, and $e_1(k)$ and $e_2(k)$ are the system inputs. *Hint:* Draw a simulation diagram first.

$$x(k+2) + v(k+1) = 4e_1(k) + e_2(k)$$

$$v(k+2) - v(k) + x(k) = 2e_1(k)$$

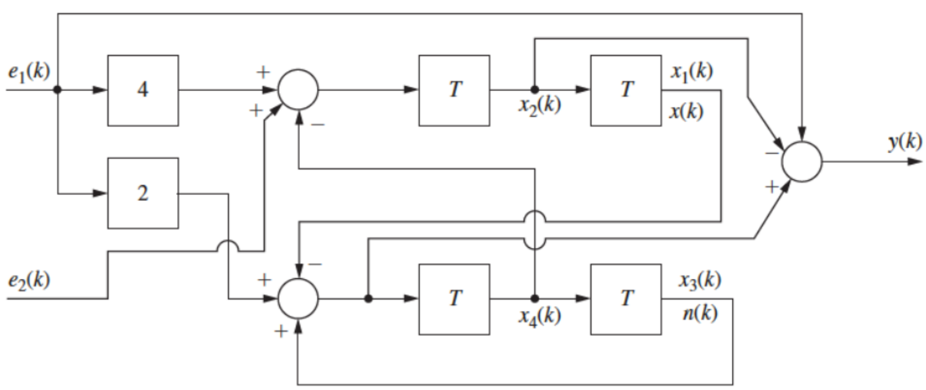
$$y(k) = v(k+2) - x(k+1) + e_1(k)$$

(b)

Assume $e_1(k) = 0$ for all k, find the transfer function from $e_2(k)$ to $y(k)$.

Hint: To find the solution of this part you do not need to find an inverse of 4 by 4 matrix. One way could be using Mason's formula and the obtained simulation diagram from part (a). But you can use any other methods that you prefer.

Solution:



$$\mathbf{x}(k+1) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 0 & 0 \\ 4 & 1 \\ 0 & 0 \\ 2 & 0 \end{bmatrix} e(k)$$

$$y(k) = x_4(k+1) - x_2(k) + e_1(k) = -x_1(k) + x_3(k) - x_2(k) + 3e_1(k)$$

$$\therefore y(k) = \begin{bmatrix} -1 & -1 & 1 & 0 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 3 & 0 \end{bmatrix} e(k)$$

(b) Just take Z transform for the difference equations and put e_1=0.

(6) – 10 points

The continuous time transfer function of first order hold is given by

$$G_{foh} \;=\; = \frac{1+Ts}{T}\left[\frac{1-e^{-Ts}}{s}\right]^2,$$

where T is sample time. Find an analytic expression (in the time domain) for its impulse response and sketch it.

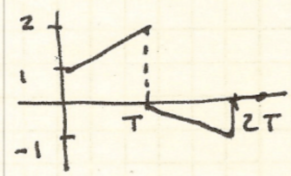
Solution

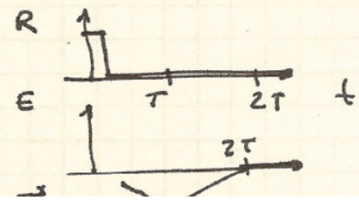
1.1) $G_{foh} = \left(\frac{1+Ts}{T}\right)\left(\frac{1-e^{-sT}}{s}\right)^2 = \left(\frac{1}{s}+\frac{1}{Ts^2}\right)\left(1+e^{-2sT}-2e^{-sT}\right)$

$$g_{foh} = \mathcal{L}^{-1}\left[G_{foh}\right] = \frac{1}{T}\left[t\,u(t)-2(t-T)u(t-T)+(t-2T)u(t-2T)\right] + \left[u(t)-2u(t-T)+u(t-2T)\right]$$

collecting terms yields:

$$\boxed{g_{fho}(t) = \left(1+\frac{t}{T}\right)\left[u(t)-u(t-T)\right] + \left(1-\frac{t}{T}\right)\left[u(t-T)-u(t-2T)\right]}$$





Laplace transform $E(s)$	**Time function** $e(t)$	**z-Transform** $E(z)$	**Modified z-transform** $E(z, m)$
$\frac{1}{s}$	$u(t)$	$\frac{z}{z-1}$	$\frac{1}{z-1}$
$\frac{1}{s^2}$	t	$\frac{Tz}{(z-1)^2}$	$\frac{mT}{z-1}+\frac{T}{(z-1)^2}$
$\frac{1}{s^3}$	$\frac{t^2}{2}$	$\frac{T^2z(z+1)}{2(z-1)^3}$	$\frac{T^2}{2}\left[\frac{m^2}{z-1}+\frac{2m+1}{(z-1)^2}+\frac{2}{(z-1)^3}\right]$
$\frac{(k-1)!}{s^k}$	t^{k-1}	$\lim_{a\to 0}(-1)^{k-1}\frac{\partial^{k-1}}{\partial a^{k-1}}\left[\frac{z}{z-\varepsilon^{-aT}}\right]$	$\lim_{a\to 0}(-1)^{k-1}\frac{\partial^{k-1}}{\partial a^{k-1}}\left[\frac{\varepsilon^{-amT}}{z-\varepsilon^{-aT}}\right]$
$\frac{1}{s+a}$	ε^{-at}	$\frac{z}{z-\varepsilon^{-aT}}$	$\frac{\varepsilon^{-amT}}{z-\varepsilon^{-aT}}$
$\frac{1}{(s+a)^2}$	$t\varepsilon^{-at}$	$\frac{Tz\varepsilon^{-aT}}{(z-\varepsilon^{-aT})^2}$	$\frac{T\varepsilon^{-amT}[\varepsilon^{-aT}+m(z-\varepsilon^{-aT})]}{(z-\varepsilon^{-aT})^2}$
$\frac{(k-1)!}{(s+a)^k}$	$t^k\varepsilon^{-at}$	$(-1)^k\frac{\partial^k}{\partial a^k}\left[\frac{z}{z-\varepsilon^{-aT}}\right]$	$(-1)^k\frac{\partial^k}{\partial a^k}\left[\frac{\varepsilon^{-amT}}{z-\varepsilon^{-aT}}\right]$
$\frac{a}{s(s+a)}$	$1-\varepsilon^{-at}$	$\frac{z(1-\varepsilon^{-aT})}{(z-1)(z-\varepsilon^{-aT})}$	$\frac{1}{z-1}-\frac{\varepsilon^{-amT}}{z-\varepsilon^{-aT}}$
$\frac{a}{s^2(s+a)}$	$t-\frac{1-\varepsilon^{-at}}{a}$	$\frac{z[(aT-1+\varepsilon^{-aT})z+(1-\varepsilon^{-aT}-aT\varepsilon^{-aT})]}{a(z-1)^2(z-\varepsilon^{-aT})}$	$\frac{T}{(z-1)^2}+\frac{amT-1}{a(z-1)}+\frac{\varepsilon^{-amT}}{a(z-\varepsilon^{-aT})}$
$\frac{a^2}{s(s+a)^2}$	$1-(1+at)\varepsilon^{-at}$	$\frac{z}{z-1}-\frac{z}{z-\varepsilon^{-aT}}-\frac{aT\varepsilon^{-aT}z}{(z-\varepsilon^{-aT})^2}$	$\frac{1}{z-1}-\left[\frac{1+amT}{z-\varepsilon^{-aT}}+\frac{aT\varepsilon^{-aT}}{(z-\varepsilon^{-aT})^2}\right]\varepsilon^{-amT}$
$\frac{b-a}{(s+a)(s+b)}$	$\varepsilon^{-at}-\varepsilon^{-bt}$	$\frac{(\varepsilon^{-aT}-\varepsilon^{-bT})z}{(z-\varepsilon^{-aT})(z-\varepsilon^{-bT})}$	$\frac{\varepsilon^{-amT}}{z-\varepsilon^{-aT}}-\frac{\varepsilon^{-bmT}}{z-\varepsilon^{-bT}}$