- If we have a second order system (in the s-plane):

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \Rightarrow s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

The equivalent z-plane poles are located at: $z = e^{sT}\Big|_{s_1, s_2} = e^{-\zeta\omega_n T} \angle \omega_n T\sqrt{1-\zeta^2} = r\angle \pm\theta$

where (1) $r = e^{-\zeta\omega_n T} \Rightarrow \zeta\omega_n T = -\ln r$

(2) $\theta = \omega_n T\sqrt{1-\zeta^2}$

From (1), (2): $\dfrac{-\ln r}{\theta} = \dfrac{\zeta}{\sqrt{1-\zeta^2}} \Rightarrow$

$$\boxed{\begin{aligned} \zeta &= \frac{-\ln r}{\sqrt{\theta^2 + \ln^2 r}} \\ \omega_n &= \frac{1}{T}\sqrt{\theta^2 + \ln^2 r} \end{aligned}}$$

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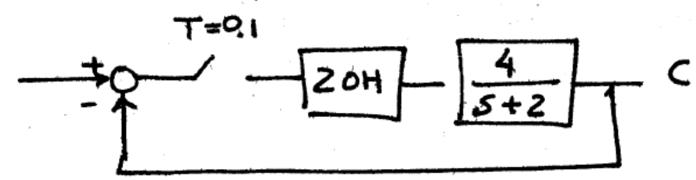
From (1), (2): $\dfrac{-\ln r}{\theta} = \dfrac{\zeta}{\sqrt{1-\zeta^2}} \Rightarrow$ $\boxed{\zeta = \dfrac{-\ln r}{\sqrt{\theta^2 + \ln^2 r}} \qquad \omega_n = \dfrac{1}{T}\sqrt{\theta^2 + \ln^2 r}}$

Finally, the time constant is given by: $\boxed{\tau = \dfrac{1}{\zeta\omega_n} = \dfrac{-T}{\ln r}}$

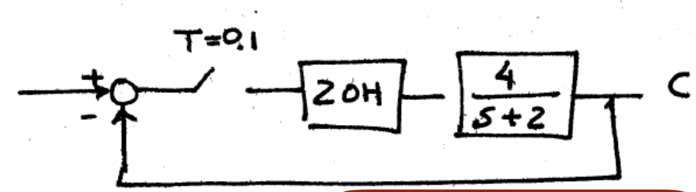
$\Rightarrow$ From $r, \theta$ we can compute $\zeta, \omega_n, \tau$ and from there $T_s, M_p, T_r, \ldots$

- Example 1 (A first order plant)




$$C(z) = \frac{G(z)}{1+G(z)} R(z) \quad \text{where} \quad G(z) = \mathcal{Z}\left[\frac{1-e^{-sT}}{s} \cdot \frac{4}{s+2}\right] = \left(\frac{z-1}{z}\right) \mathcal{Z}\left[\frac{4}{s(s+2)}\right] = \frac{0.3625}{z-0.8187}$$

• **Example 1** (A first order plant)

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$$\boxed{E(z) = \sum_{\substack{\text{at poles}\\ \text{of } E(\lambda)}} \left[\text{residues of } E(\lambda) \frac{1}{1 - z^{-1}\varepsilon^{T\lambda}}\right]}$$

$$\boxed{= \frac{z-1}{z} \frac{2(1-\varepsilon^{-2T})z}{(z-1)(z-\varepsilon^{-2T})} = \frac{0.3625}{z-0.8187}, \quad T = 0.1 \text{ s}}$$

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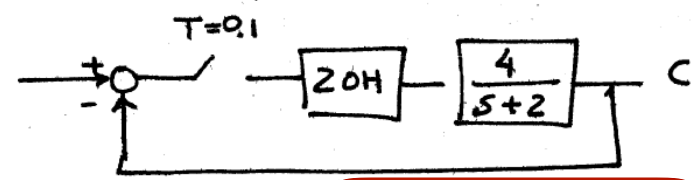


$$C(z) = \frac{G(z)}{1+G(z)}\,R(z) \quad \text{where} \quad G(z) = \mathcal{Z}\left[\frac{1-e^{-sT}}{s}\cdot\frac{4}{s+2}\right] = \boxed{\left(\frac{z-1}{z}\right)\mathcal{Z}\left[\frac{4}{s(s+2)}\right]} = \frac{0.3625}{z-0.8187}$$

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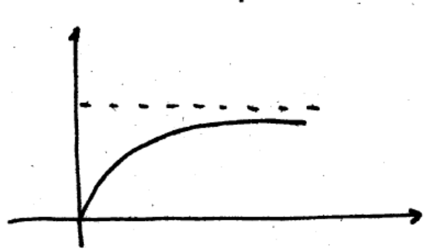
$$C(z) = \frac{0.3625}{z-0.4562}\,R(z)$$

Since we have a pole at $z = 0.4562 = e^{-0.7848} = e^{-7.848\cdot(0.1)} = e^{-7.848T}$ (with $0.1 = T$)

we should expect a time response of the form



with a time constant $\tau \sim 0.127$ sec. Specifically,

$$R(z) = \frac{z}{z-1}, \qquad C(z) = \frac{0.3625}{z-0.4562}\cdot\frac{z}{z-1} = \frac{2}{3}\left[\frac{z}{z-1} - \frac{z}{z-0.4562}\right] \overset{\mathcal{Z}^{-1}}{\Longrightarrow} c(nT) = \frac{2}{3}\left[1-(0.4562)^k\right]$$

$$c(nT) = \frac{2}{3}\left[1 - (0.4562)^k\right] = \frac{2}{3}\left[1 - e^{-1.848\,kT}\right]$$

If we remove the sample & hold, we get:

$$T_c(s) = \frac{4}{s+6} \Rightarrow C(s) = \frac{4}{s(s+6)} = \frac{2}{3}\left[\frac{1}{s} - \frac{1}{s+6}\right] \Rightarrow c(t) = \frac{2}{3}\left[1 - e^{-6t}\right]u(t)$$

$$c(nT) = \frac{2}{3}\left[1-(0.4562)^k\right] = \frac{2}{3}\left[1 - e^{-1.848\,kT}\right]$$

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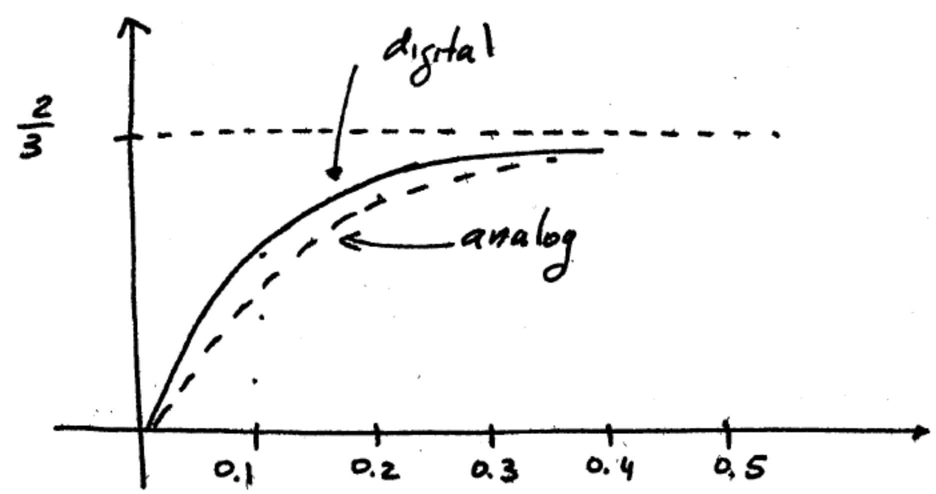
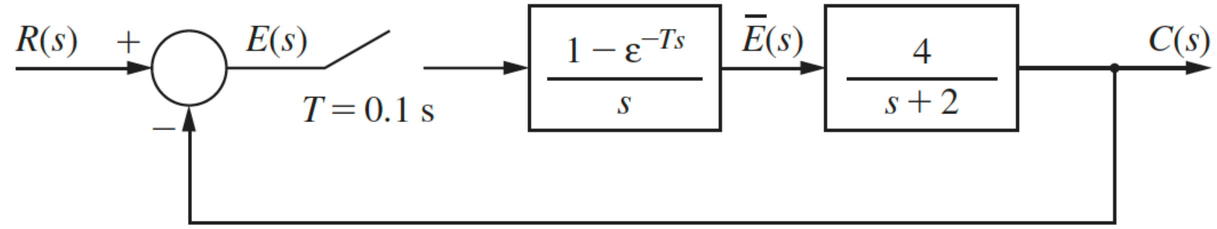


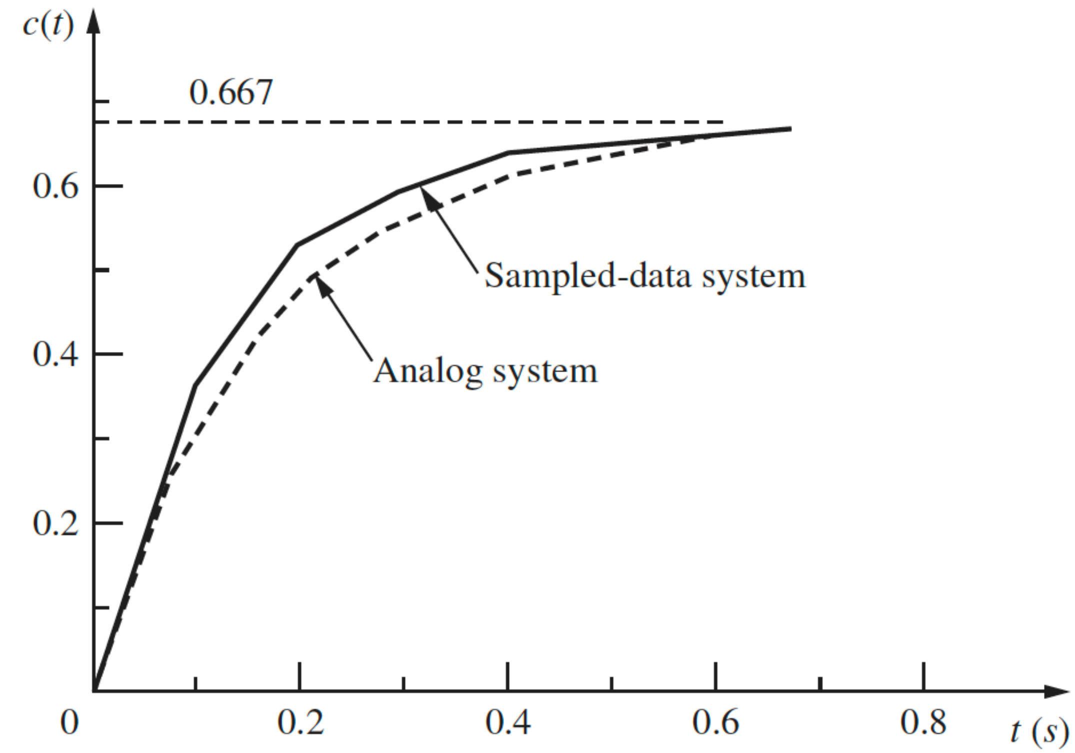

Note that the final value is the same for both systems.
This is a general result:

Since the ZOH does not have any effect in steady-state, the output of both systems should be the same (provided that both are stable)

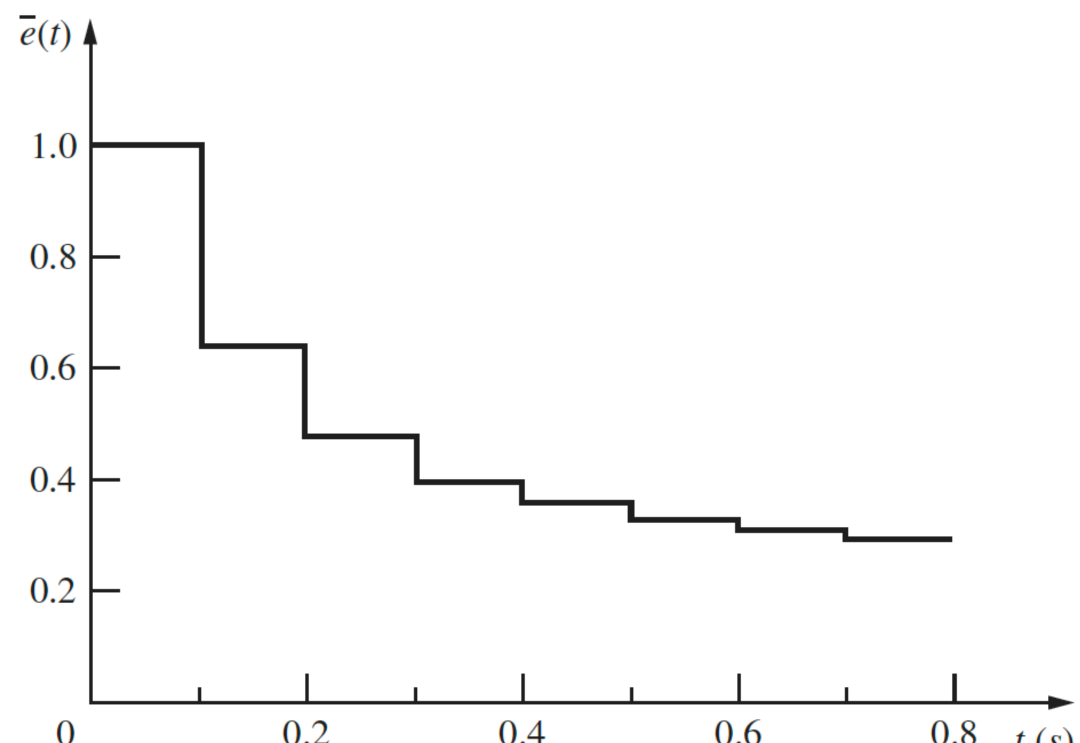
$\Rightarrow$ Same DC gain, i.e.: $\lim_{s\to 0} \frac{G(s)}{1+G(s)} = \lim_{z\to 1} \frac{G(z)}{1+G(z)} = \frac{2}{3}$ #


R(s)
+
−
E(s)
T = 0.1 s
1 − ε^{−Ts}
s
Ē(s)
4
s + 2
C(s)

(a)


c(t)
0.667
0.6
0.4
0.2
0
0.2
0.4
0.6
0.8
t (s)
Sampled-data system
Analog system

(b)


ē(t)
1.0
0.8
0.6
0.4
0.2
0
0.2
0.4
0.6
0.8
t (s)

(c)

The computations may be done in MATLAB:

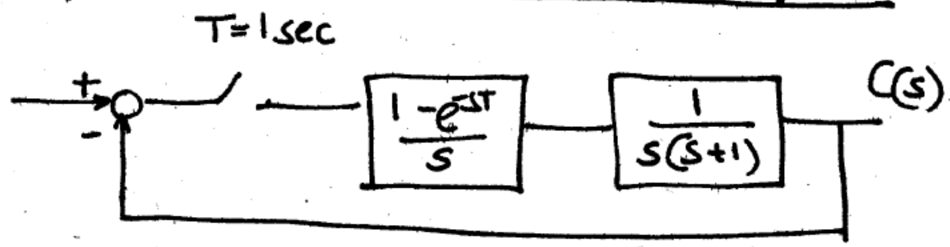
```
>> T = 0.1; Gs = tf([4],[1 2]); Gz = c2d(Gs,T);
Tz = feedback(Gz,1); Rz = tf([1 0],[1 -1],T);
Cz = zpk(Rz*Tz)

Cz =
 0.3625 z
 ----------------
 (z-1) (z-0.4562)

>> % Partial fraction expansion of C(z)/z
 [r,p,k]=residue([0.3625],[1 -1.4562 0.4562])

r =
 0.6666
 -0.6666

p =
 1.0000
 0.4562

k =
  []
```

If you want to find out what happens between sampling instants you can solve for

$C(s) = G(s)\,E^* = G(s)\left[\dfrac{R}{1+G}\right]^*$ and use the inverse Laplace transf

or use the modified z-transform.

- Example 2: <u>second order plant</u>



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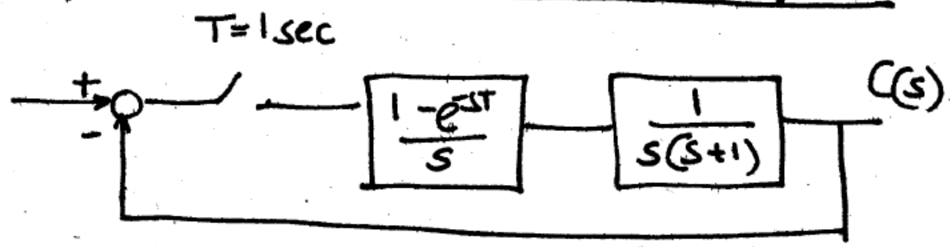
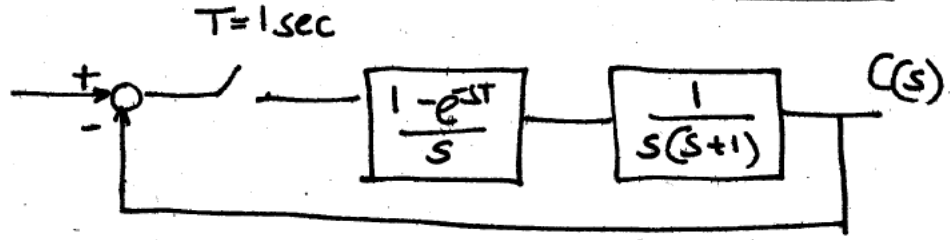



$$G_{ol} = Z\left[\frac{1-e^{-sT}}{s^2(s+1)}\right] = \frac{0.368z + 0.264}{z^2 - 1.368z + 0.368}$$

$$G_{cl} = \frac{G_{ol}}{1+G_{ol}} = \frac{0.368z + 0.264}{z^2 - z + 0.632}$$

Char equation: $z^2 - z + 0.632 = 0 \Rightarrow$ poles at $z = 0.7950 \angle \pm 0.8906$

- Example 2: <u>second order plant</u>




$$G(z) = \left(\frac{z-1}{z}\right)\mathcal{Z}\left[\frac{1}{s^2(s+1)}\right]_{T=1} = \frac{z-1}{z}\left[\frac{z[(1-1+\varepsilon^{-1})z + (1-\varepsilon^{-1}-\varepsilon^{-1})]}{(z-1)^2(z-\varepsilon^{-1})}\right]$$

$$= \frac{0.368z + 0.264}{z^2 - 1.368z + 0.368}$$

?

$$G_{o\ell} = Z\left[\frac{1-e^{-sT}}{s^2(s+1)}\right] = \frac{0.368z + 0.264}{z^2 - 1.368z + 0.368}$$

$$G_{c\ell} = \frac{G_{o\ell}}{1+G_{o\ell}} = \frac{0.368z + 0.264}{z^2 - z + 0.632}$$

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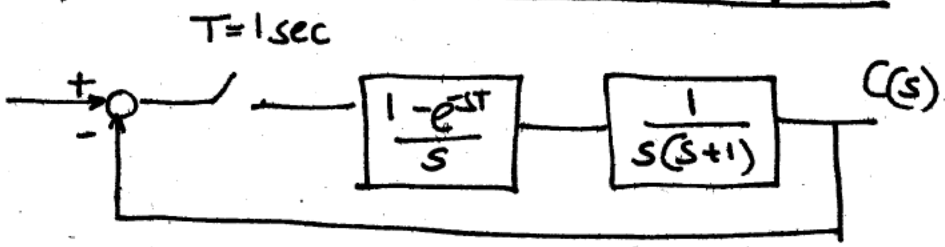


$$G(z) = \left(\frac{z-1}{z}\right)\mathfrak{z}\left[\frac{1}{s^2(s+1)}\right]_{T=1} = \frac{z-1}{z}\left[\frac{z[(1-1+\varepsilon^{-1})z + (1-\varepsilon^{-1}-\varepsilon^{-1})]}{(z-1)^2(z-\varepsilon^{-1})}\right]$$

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$$G_{oe} = Z\left[\frac{1-e^{-sT}}{s^2(s+1)}\right] = \frac{0.368z + 0.264}{z^2 - 1.368z + 0.368}$$

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Char equation: $z^2 - z + 0.632 = 0 \Rightarrow$ poles at $z = 0.7950 \angle \pm 0.8906$

$$\zeta = \frac{-\ln(0.7950)}{\sqrt{[0.8906]^2 + (\ln[0.795])^2}} \simeq 0.25$$

$$\omega_n = \frac{1}{1}\sqrt{\theta^2 + (\ln r)^2} \simeq 0.92$$

$$\tau = \frac{-1}{\ln(r)} \simeq 4.36 \text{ sec}$$

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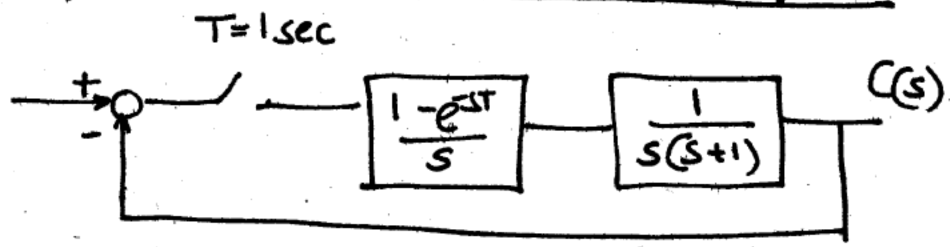


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$$\boxed{\zeta = \frac{-\ln r}{\sqrt{\theta^2 + \ln^2 r}} \qquad \omega_n = \frac{1}{T}\sqrt{\theta^2 + \ln^2 r}}$$

$$\boxed{\tau = \frac{1}{\zeta\omega_n} = \frac{-T}{\ln r}}$$

Digital system $(T = 1 \text{ sec})$

$\zeta = 0.25 \Rightarrow M_p \cong 0.58$

$\omega_n \cong 0.92$ rad/sec

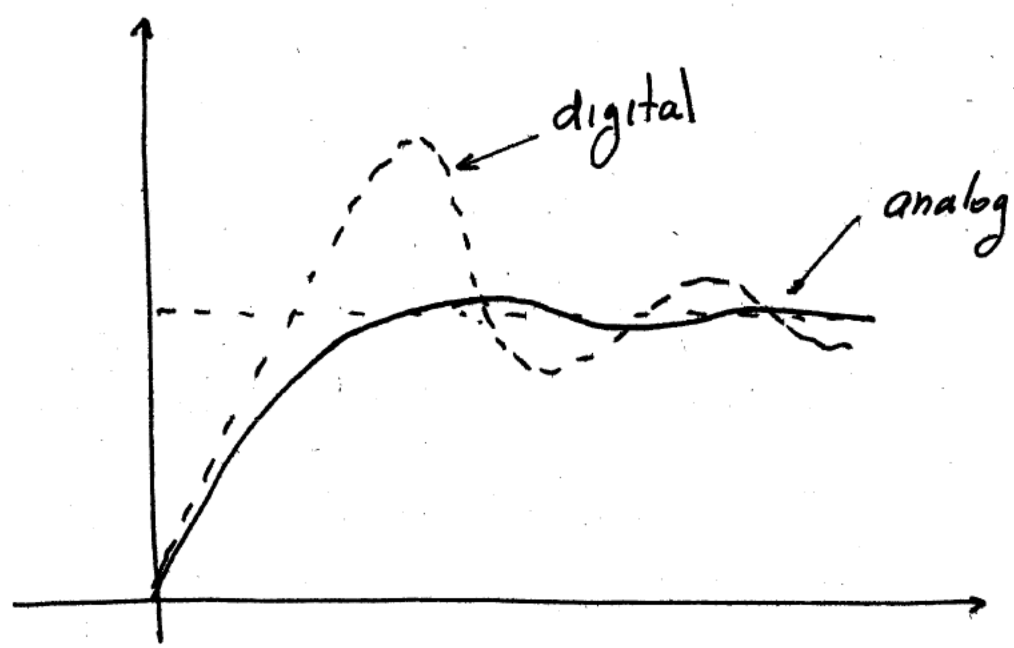
$\tau \cong 4.36$ sec

Analog system (no ZOH)

$\zeta = 0.5 \Rightarrow M_p \cong 16\%$

$\omega_n = 1$

$\tau = 2$ sec




You can see here that the effects of sampling are <u>destabilizing</u>

(The problem in this specific case is that the sampling rate is too low compared with the system's time constant: rule of thumb: want $T \ll \tau$)

# EECE 5610 Digital Control Systems

*Lecture 13*

**Milad Siami**
Assistant Professor of ECE
Email: m.siami@northeastern.edu

Northeastern University
College of Engineering

# Control Systems Specifications

# Control Systems Specifications

The typical steps in designing a control system are:

(a) model the system (chapters 2 - 5)

(b) decide performance specifications — You are here

(c) analyze the behavior of the system

(d) (if necessary) design a controller to modify the behavior so that these specs are met (or show that no such controller exists)

(e) check performance via simulations (using a detailed model) or in a prototype

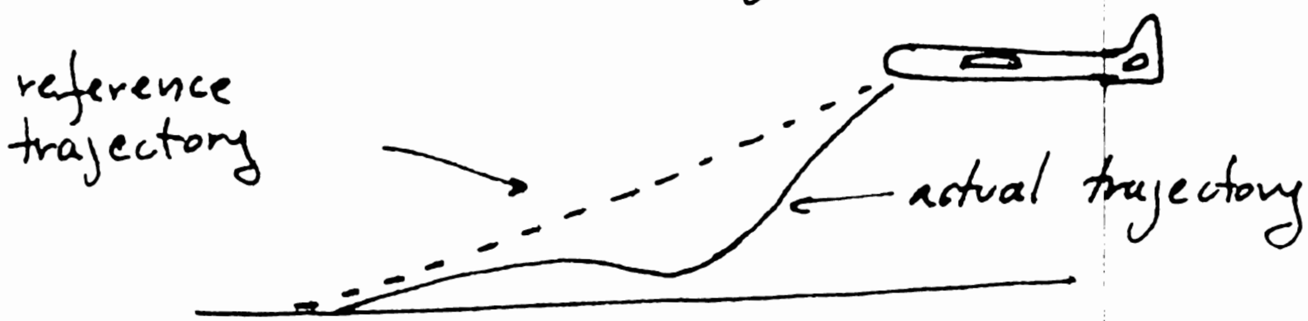
(f) check performance in the actual plant

(g) done: go to the beach

iterate

- General features that we require from a control system:

Example: automatic landing system




inputs: $\left\{ \begin{array}{l} \text{reference trajectory} \\ \text{disturbances (wind)} \end{array} \right.$

output: actual trajectory

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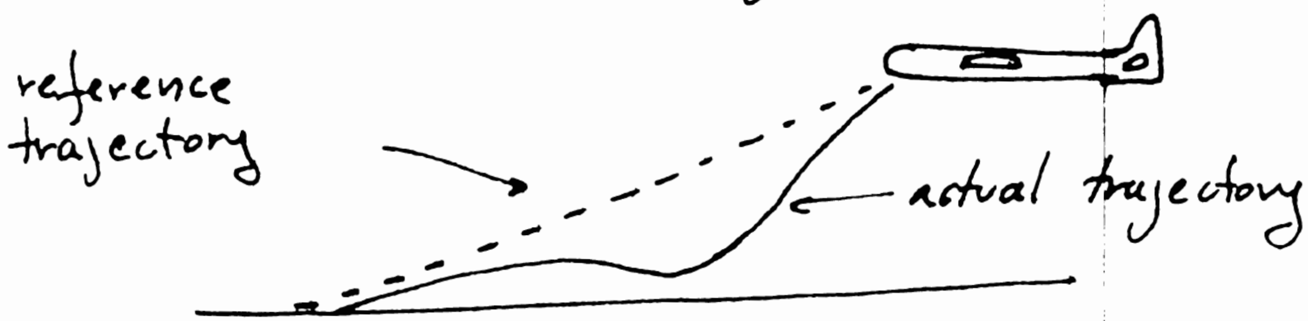


inputs: $\begin{cases} \text{reference trajectory} \\ \text{disturbances (wind)} \end{cases}$

output: actual trajectory

(a) Stability: We want the system to remain "stable", in the sense that the output corresponding to any **bounded** input should also be bounded.

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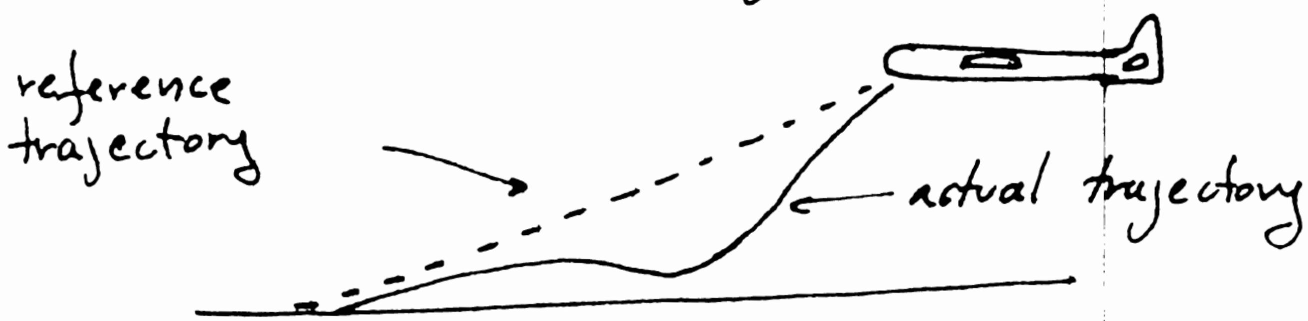
Example: automatic landing system



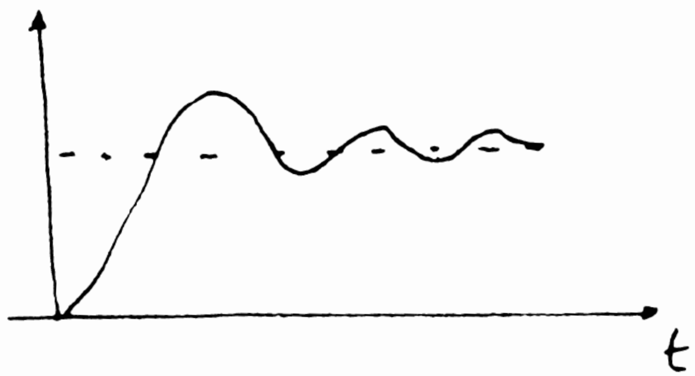

inputs: { reference trajectory, disturbances (wind) }

output: actual trajectory

(a) <u>Stability</u>: We want the system to remain "stable", in the sense that the output corresponding to any <u>bounded</u> input should also be bounded.

(b) <u>Steady state accuracy</u>: The final error (after all transients decay) must be stable.
In some cases we require <u>zero</u> steady state error $\Rightarrow$ need to impose special structure on the controller to achieve this.

(c) <u>Transient response</u> (or dynamic response): Impose specifications upon overshoot, rise-time, settling-time.



These specs are related to both how the steady state is achieved <u>and</u> the ability of the system to track a time varying input.

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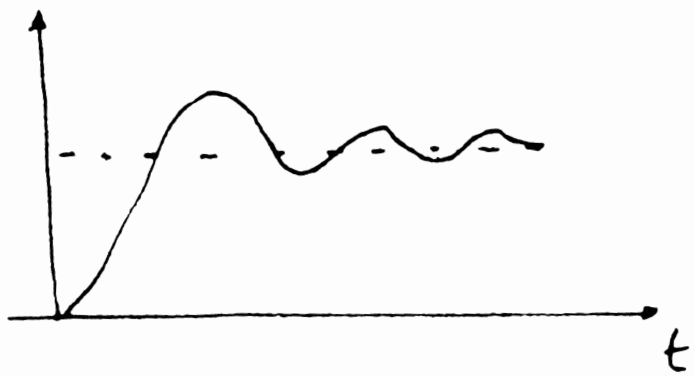
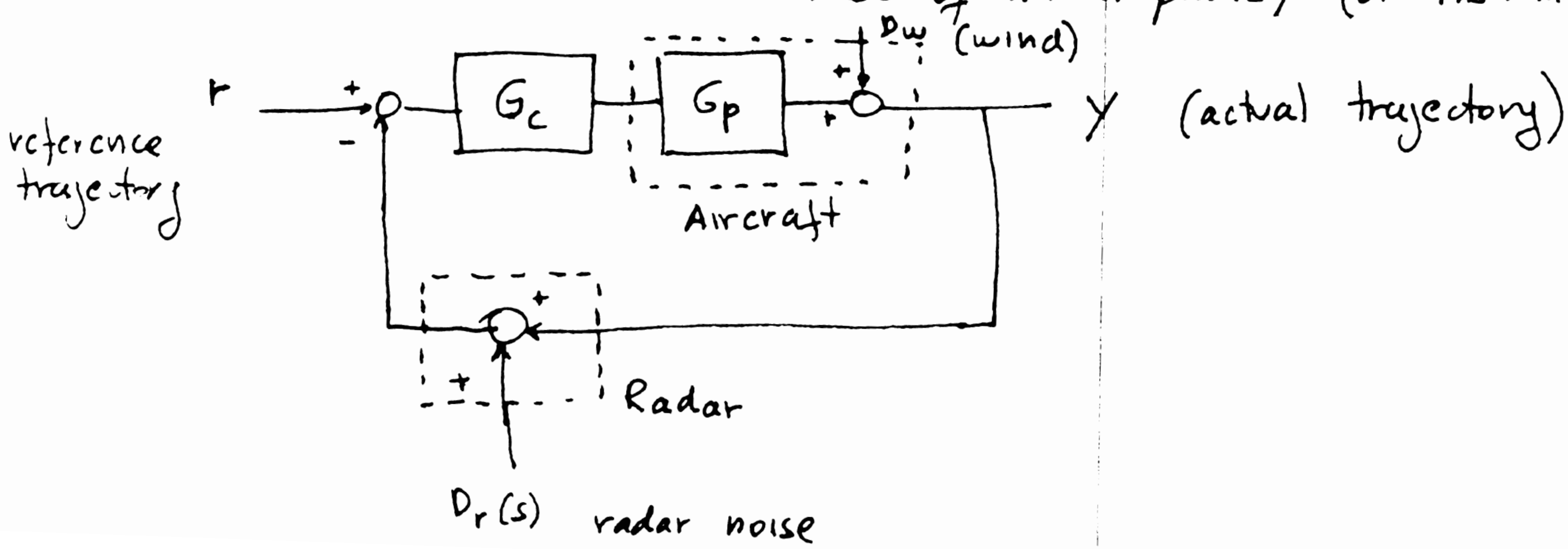



These specs are related to both how the steady state is achieved <u>and</u> the ability of the system to track a time varying input.

(d) <u>Sensitivity to parameter changes</u>: We want the overall behavior of the system (or at least some key properties) to remain invariant even if some of the parameters of the system change (related to "robust" control)

New aspects of the problem <u>specific</u> to digital control: effects of the sampling rate, round-off errors, etc.

(e) <u>Disturbance rejection</u>: Ability of the system to reject unwanted disturbances (e.g. wind gusts in the case of an airplane) (or instrument noise)



(f) Minimization of some "cost": peak control effort, energy, time to intercept, ...

These specs lead to problems of the form:

$$\min \left\{ \max_t |u(t)| \right\}$$ (an "$L_\infty$" type problem) (solved in early 90's)

$$\min \int_0^\infty |u|\, dt$$ ("$L_1$" optimal control) (solved in mid 80's)

$$\min \int_0^\infty [y^2(t) + u^2(t)]\, dt$$ (LQR control) (solved in mid 60's)

Solving these problems requires tools beyond the scope of 5610. Some are covered in ECEG 7214 (Optimal control problems)

## (Preliminary) Stability Analysis:

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- Def: Bounded Input Bounded Output (BIBO) Stability

  A system is BIBO stable if and only if the output is bounded for every possible bounded input.

(There are many different definitions of stability: BIBO, asymptotic, Lyapunov, exponential, .... Turns out that for LTI systems all these definitions are equivalent)

# (Preliminary) Stability Analysis:

- **Def:** Bounded Input Bounded Output (BIBO) Stability

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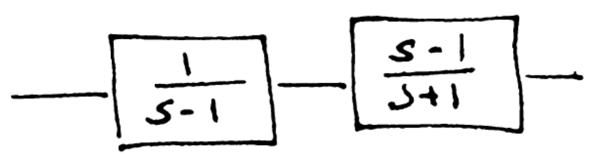
(There are many different definitions of stability: BIBO, asymptotic, Lyapunov, exponential, .... Turns out that for LTI systems all these definitions are equivalent)

Aside: "internal" versus input-output or "external" stability

"external" stability: purely input/output, does not care about internal signals that do not show up at the output "internal" stability: all internal signals <u>must</u> remain bounded, even if they do not show up at the output

Important case: unstable pole-zero cancellation: we get a system that is input-output stable but not internally stable (in practice it will not work)




These issues are related to the concepts of controllability / observability and minimal realizations. More on this latter and in ECE 7200

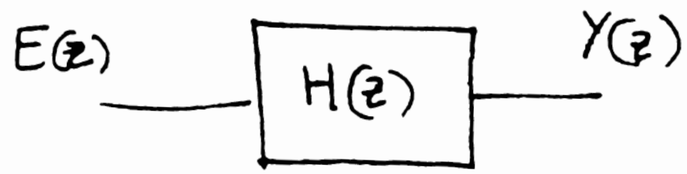
Back to stability analysis:

Our definition of BIBO stability provides a nice conceptual definition but it is useless as a practical tool.

According to that definition, in order to assess whether or not a system is BIBO stable we would need to try every possible bounded input (an infinite number!) and check if the corresponding output is bounded.

Obviously this is not feasible: we need to find an equivalent, practically implementable definition of stability

- <u>Necessary and Sufficient Conditions for Stability</u>:




Suppose that we know the pulse transfer function $H(z)$. What conditions on $H(z)$ guarantee stability?

Let's look at the output $Y(z)$ corresponding to a generic input $E(z)$:

$$Y(z) = H(z)\, E(z) \iff y(nT) = \sum_{k=0}^{n} h(nT) \cdot e\left[(k-n)T\right]$$

where $h(nT) = \mathcal{Z}^{-1}\{H(z)\}$ (the impulse response, also known as the Markov parameters)

- Assume that the following condition holds:

$$\sum_{n=0}^{\infty} |h(nT)| \leq K < \infty \qquad \text{(for some } K \text{ large enough)}$$

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Then:

$$|y(nT)| = \left| \sum_{0}^{n} h(kT)\, e[(n-k)T] \right| \leq \sum_{0}^{n} |h(kT)|\, |e[(n-k)T]|$$

$$\leq \sum_{0}^{\infty} |h(kT)| \cdot |e[(n-k)T]| \leq \sup_{k} |e(kT)| \cdot \sum_{0}^{\infty} |h(kT)|$$

if the input $E(z)$ is bounded, i.e $|e(kT)| < M$, all $k$

then: $$|y(nT)| \leq \underbrace{\sup_{k} |e(kT)|}_{\leq M} \cdot \underbrace{\sum_{0}^{\infty} |h(kT)|}_{\leq K} \leq M \cdot K \Rightarrow \text{also bounded}$$

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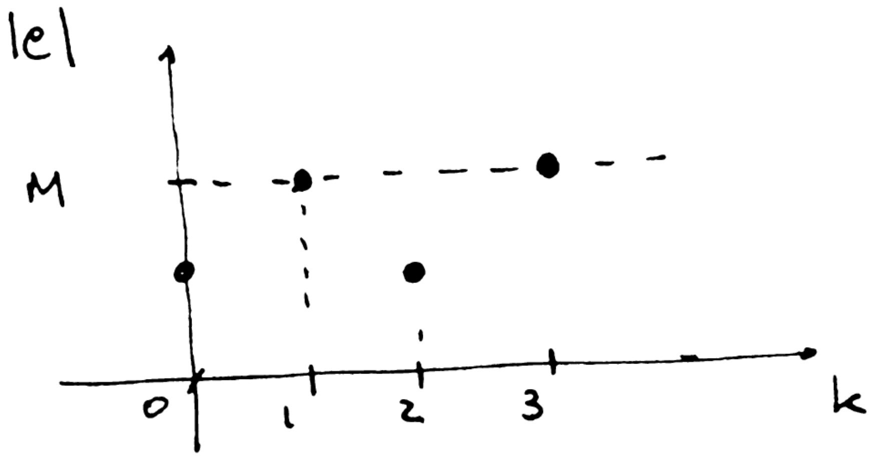


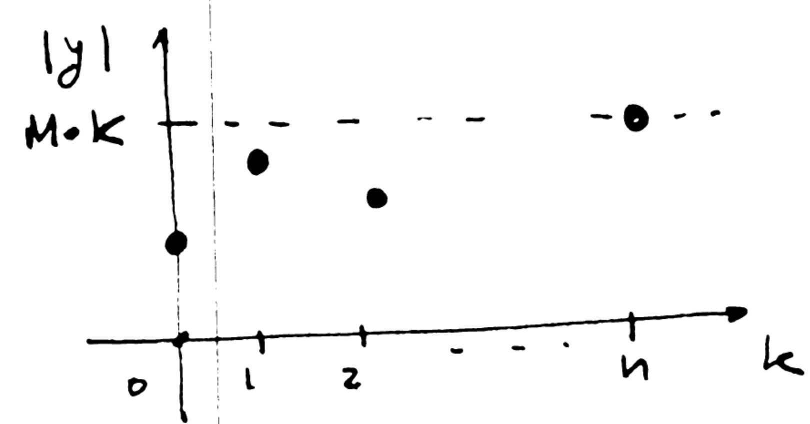

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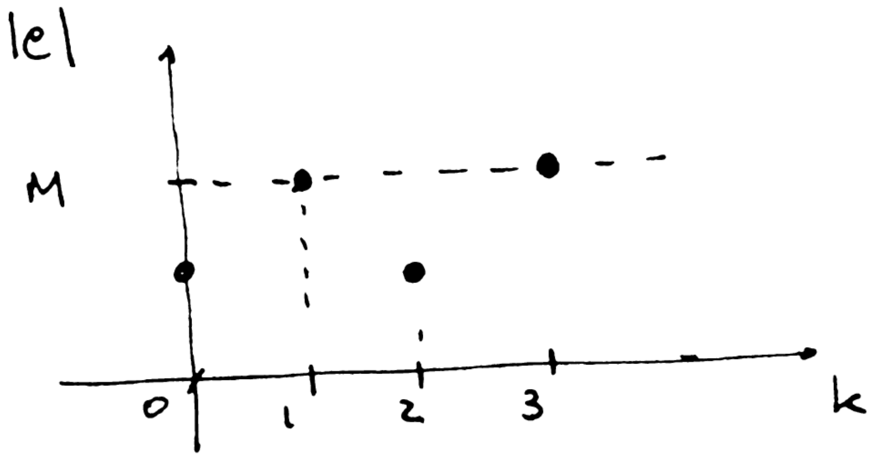


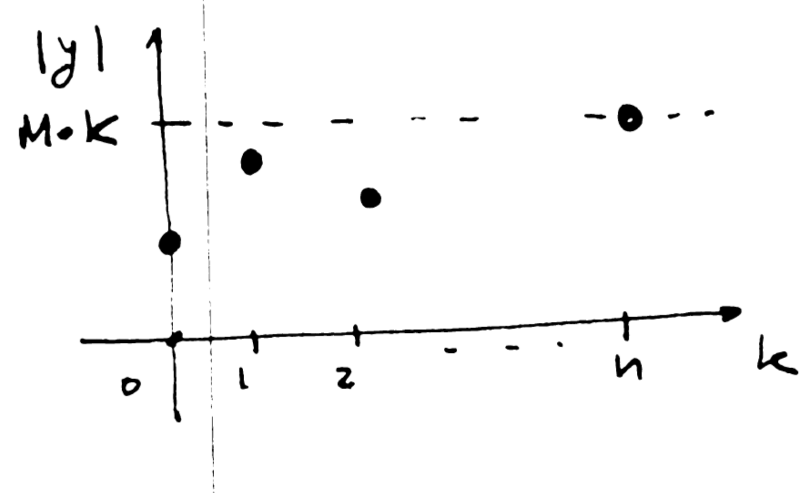


$\Rightarrow$ The system is BIBO stable!

**Recap:** $$\boxed{\text{If } \sum_0^\infty |h(kT)| < \infty \Rightarrow \text{BIBO stable}}$$

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(i.e., if it holds then the system is stable)

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<u>Proof</u>: Assume that the condition fails, i.e for any $K > 0$ we can find some $n$ such that:

$$\sum_{k=0}^{n} |h(kT)| > K$$

Take now the following input: $e[(n-k)T] = \text{signum}\{h(kT)\}$

$$\left( \text{signum}\{x\} = \begin{matrix} 1 & x>0 \\ -1 & x<0 \\ 0 & x=0 \end{matrix} \right)$$

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Obviously $e$ is bounded by 1. Let's look now at the output

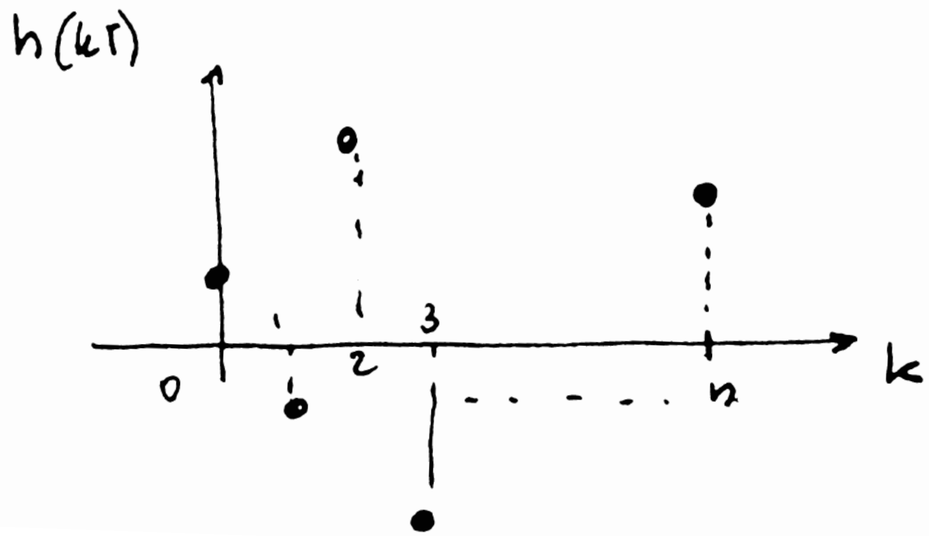
$$|y(nT)| = \left| \sum_0^n h(kT)\, e[(n-k)T] \right| = \left| \sum_0^n h(kT)\, \text{signum}\{h(kT)\} \right| = \left| \sum_0^n |h(kT)| \right|$$

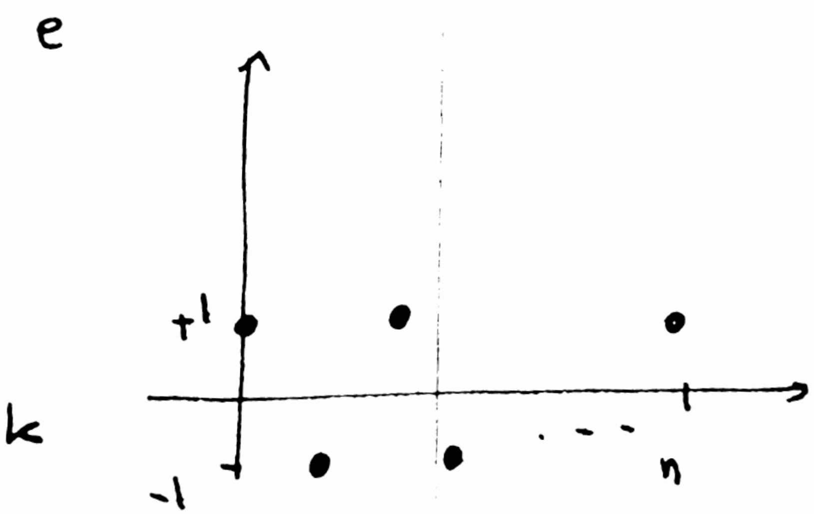
$$= \sum_0^n |h(kT)| > K \quad \text{(by assumption)}$$

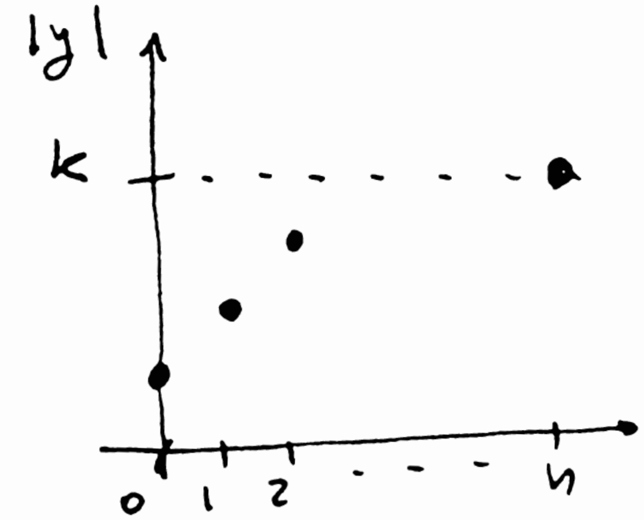
$\Rightarrow$ Given <u>any level $K$</u>, we can always find an input bounded by one and such that $|y(nT)| > K$

(i.e we can make the peak value of the output as large as desired)

$\Rightarrow$ **NOT** BIBO stable



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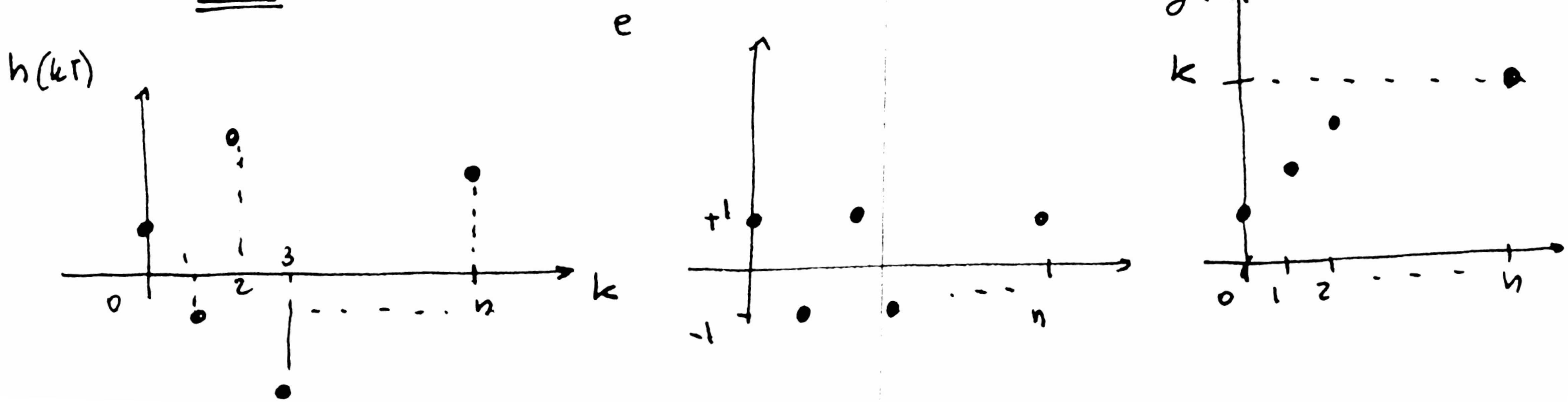
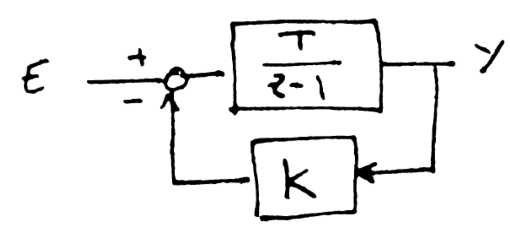



<u>Note</u>: as a byproduct we also found out the <u>worst-case</u> signal (the one that will do you in):
Is an input that it is "aligned" with the impulse response

**Recap:**

$$\boxed{\text{BIBO stable} \iff \sum_{0}^{\infty} |h(kT)| < \infty}$$

Example:




$$H(z) = \frac{\frac{T}{z-1}}{1 + \frac{KT}{z-1}} = \frac{T}{z-1+KT}$$

$$H(z) = \frac{T}{z-(1-kT)} = \frac{T}{z}\,\frac{1}{\left(1-\left(\frac{1-kT}{z}\right)\right)} \iff h[nT] = \begin{cases} 0 & n=0 \\ T\cdot(1-KT)^{n-1} & n \geq 1 \end{cases}$$

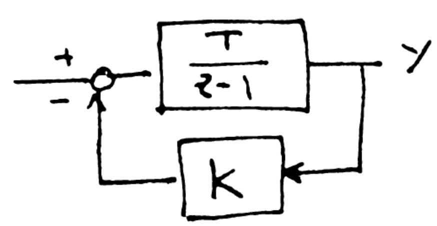
$$\Rightarrow \sum_{n=0}^{\infty} |h(nT)| = T\sum_{0}^{\infty} |(1-kT)|^{n} = \begin{cases} T\cdot\frac{1}{1-|1-kT|} < \infty & \text{if } |1-kT| < 1 \\ \infty & \text{if } |1-kT| \geq 1 \end{cases}$$

$\Rightarrow$ System is BIBO stable iff: $|1-kT| < 1$ or $kT < 2$

**Recap:**

$$\text{BIBO stable} \iff \sum_{0}^{\infty} |h(kT)| < \infty$$

So now we have a testable condition:

Example: $E$  $Y$

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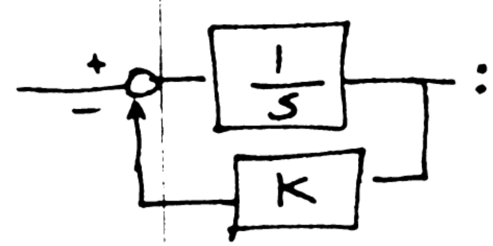
( Compare to the continuous time case: 

: $H(s) = \frac{1}{s+k}$

$$h(t) = e^{-kt}, \qquad \int_0^\infty |h(t)|\,dt = \int_0^\infty e^{-kt}\,dt = \frac{1}{k} < \infty \quad \text{all } k > 0$$

$\Rightarrow$ cont. time system is stable for all $k$ )

Now we have a testable condition for stability. However it is hard to use: You need to (1) find $\mathcal{Z}^{-1}[H]$

(2) compute $\sum_0^\infty |h_k|$

We'd like to have something simpler. Turns out that if your system is finite dimensional linear time invariant (FDLTI) (as always the case in 429) we can assess stability by looking at the <u>location of the poles</u>

- **Relationship between BIBO stability and the location of the poles:**

Suppose $G(z) = \frac{C(z)}{E(z)} = \frac{(z-z_1)\cdots(z-z_m)}{(z-p_1)\cdots(z-p_n)}$

Assume for simplicity that all roots are simple. Then:

$$G(z) = \sum_i k_i \frac{z}{z-p_i} = \sum_i k_i \frac{1}{1-\frac{p_i}{z}} \overset{Z^{-1}}{\Longleftrightarrow} g_k = \sum k_i (p_i)^k$$

Note that $|p_i|^k \to \infty$ if $|p_i| > 1$

In fact, it can be shown that $\sum |p_i|^k < \infty \Leftrightarrow |p_i| < 1$

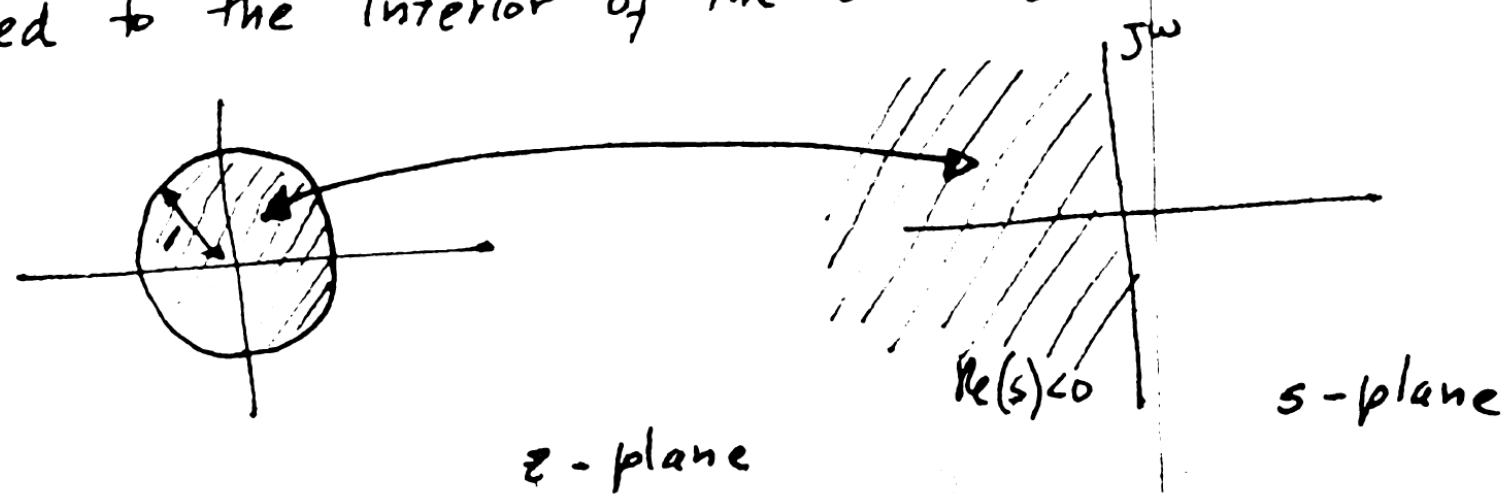
$\Rightarrow \sum_{0}^{\infty} |g(k)|$ bounded $\Leftrightarrow |p_i| < 1$, i.e. all poles <u>must</u> be inside the unit disk.

(if we have repeated poles we get terms of the form $n p_i^n$ and the conclusion still stands)

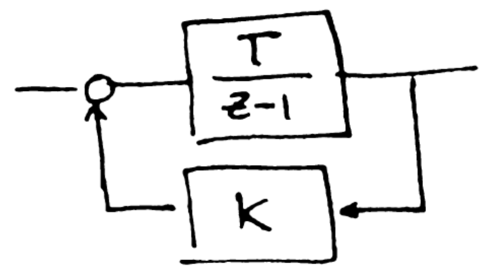
$$\boxed{\text{System BIBO stable} \Leftrightarrow \text{all poles inside the unit disk}}$$

This is not surprising. Recall from the HW that $G(z)$ has a pole at $z = z_0$ $\Longleftrightarrow$ $G(s)$ has a pole at $s_0$ where $z_0 = e^{s_0 T}$

Thus the stable region in the s plane $(Re(s) < 0)$ gets mapped to the interior of the unit disk



Example revisted :

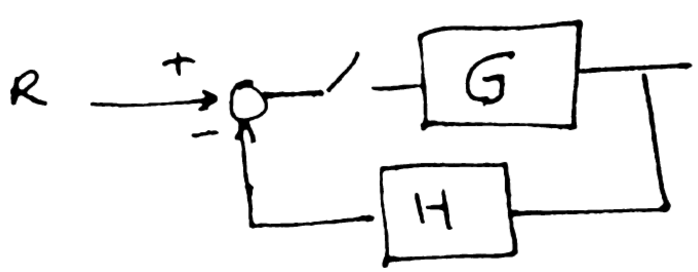



$\Rightarrow H(z) = \dfrac{T}{z-1+kT}$

has a single pole at $z = 1-kT$

$\Rightarrow$ stable $\Leftrightarrow$ $|1-kT|<1 \Leftrightarrow kT<2$

- So now we have an easy way of checking stability



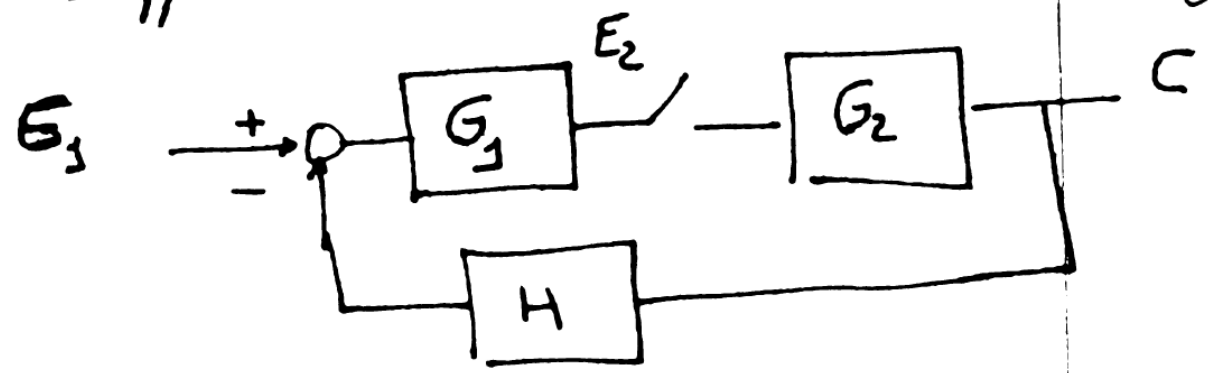

$\Leftrightarrow$ Char equation: $1+G(z)H(z)=0$

poles: roots of this equation

(a) find all the poles of the transfer function (i.e roots of the characteristic equation: Mason's $\Delta = 0$)

(b) stable $\Leftrightarrow$ all poles inside unit disk

But **trouble**: Not all sampled data systems have a Transfer Function
Suppose that we have something like




We know that in this case the pulse transfer function from $E(z)$ to $C(z)$ does not exist. So: how do we handle these cases?