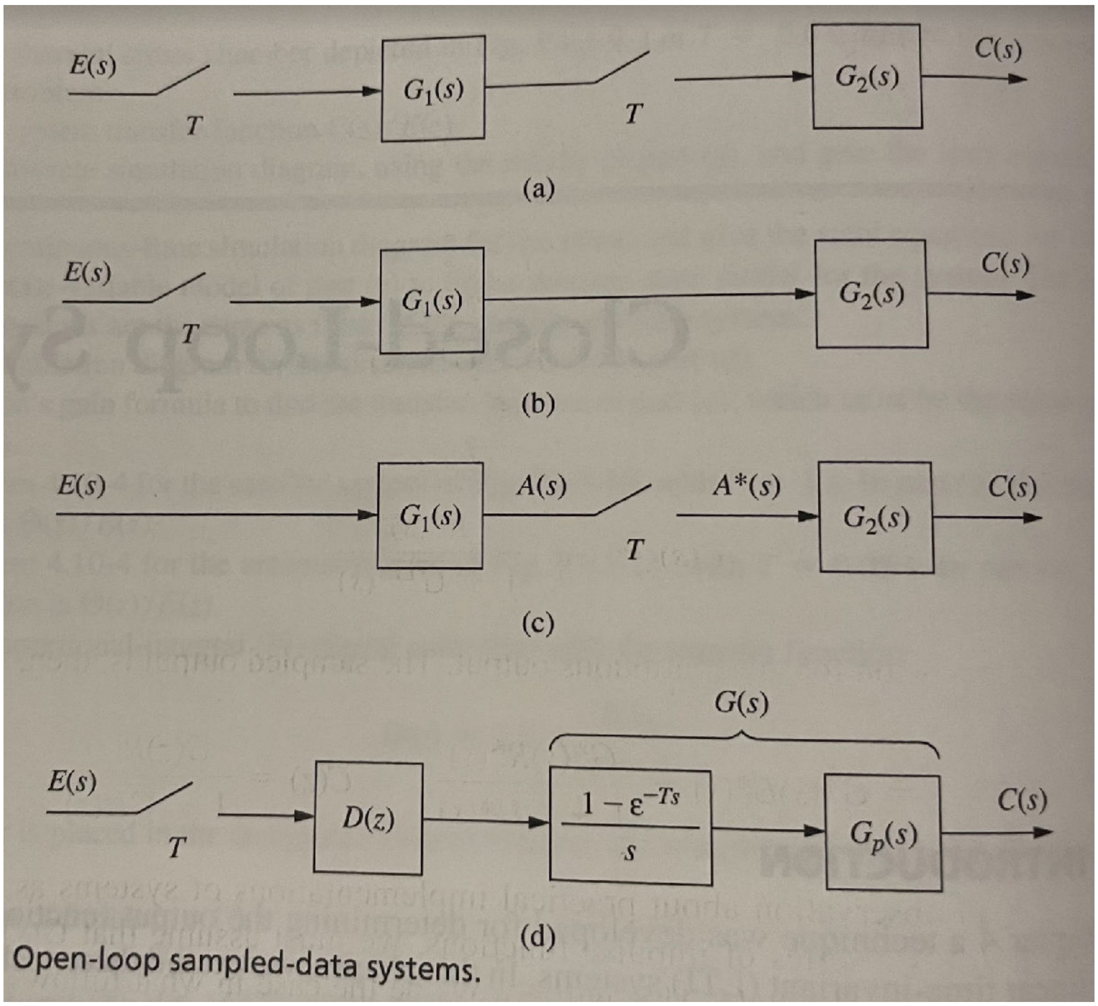
Digital Control Systems - Chapter 5 Notes

Closed-Loop Systems

5.1 Introduction

In this chapter we extend the techniques of Chapter 4 to determine the output functions of closed-loop discrete-time LTI systems. Also presented is a technique for developing state-variable models for closed-loop discrete-time systems. For convenience, we refer to discrete-time systems as simply discrete systems.

5.2 Preliminary Concepts



Open-loop sampled-data systems.

The output for the system of Fig. 5-1(a) is

$$C(z) = G_1(z)G_2(z)E(z)$$

the output for the system of Fig. 5-1(b) is

$$C(z) = \overline{G_1G_2}(z)E(z)$$

for the system of Fig. 5-1(c)

$$C(s) = G_2(s)A^*(s) = G_2(s)\overline{G_1E^*}(s)$$

and

$$C(z) = G_2(z)\overline{G_1E}(z)$$

In this last case, no transfer function can be found since $E(z)$ cannot be factored from $\overline{G_1E}(z)$

- In general, no transfer function can be written for the system in which the input is applied to an analog element before being sampled.
- However, the output can always be expressed as a function of the input, and will be shown later that this type of system presents no particular difficulties in either analysis or design

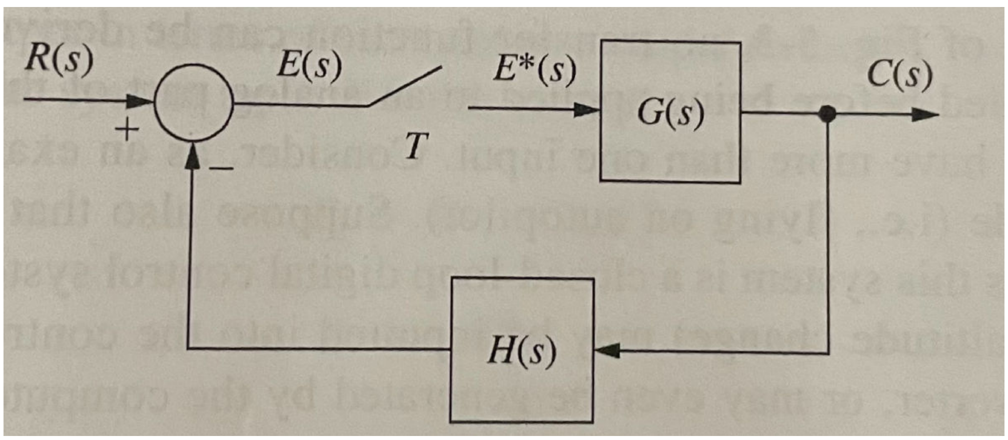
The output for Fig. 5-1(d) is

$$C(z) = D(z)G(z)E(z)$$

where

$$G(z) = z\left\{\frac{1-\varepsilon^{-Ts}}{s}G_p(s)\right\} = \frac{z-1}{z}z\left\{\frac{G_p(s)}{s}\right\}$$

We now derive the output function for the system of Fig. 5-2



$$C(s) = G(s)E^*(s)$$

and

$$E(s) = R - H(s)C(s)$$

Substituting equations we get:

$$E(s) = R - H(s)G(s)E^*(s)$$

and by taking the starred transform, we have

$$E^*(s) = R^*(s) - \overline{GH}^*(s)E^*(s)$$

Solving for $E^*(s)$, we obtain

$$E^*(s) = \frac{R^*(s)}{1 + \overline{GH}^*(s)}$$

and from the previous equation,

$$C(s) = G(s)\frac{R^*(s)}{1 + \overline{GH}^*(s)}$$

which yields an expression for the continuous output. The sampled output is, then,

$$C^*(s) = G^*(s)E^*(s) = \frac{G^*(s)R^*(s)}{1 + \overline{GH}^*(s)}, \qquad C(z) = \frac{G(z)R(z)}{1 + \overline{GH}(z)}$$

- Anytime a starred transform is applied to a Laplace transfer function, assume that the transfer function contains a *zero-order hold* to process its starred-transform input

Problems can be encountered in deriving the output function of a closed-loop system. This can be illustrated for the case above. Suppose that $E(s)$ was starred and substituted into $C(s)$

$$C(s) = G(s)R^*(s) - G(s)\overline{HC}^*(s)$$

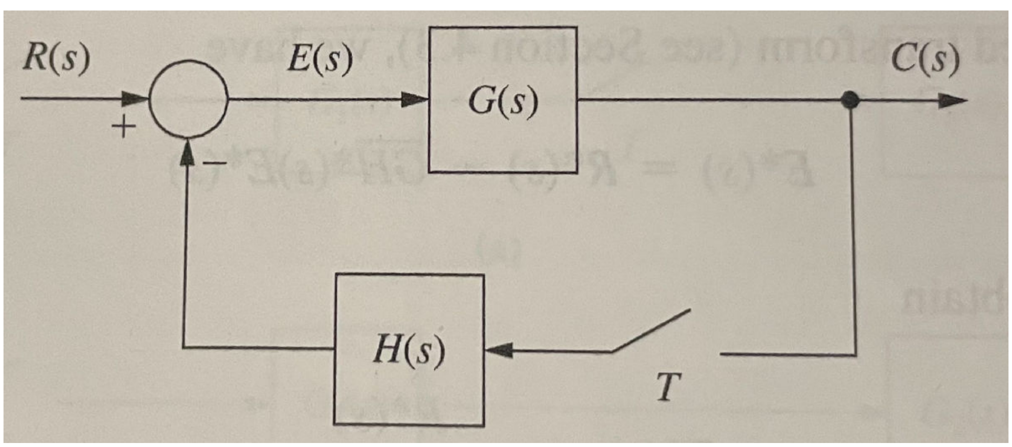
and $C^*(s)$ is

$$C^*(s) = G^*(s)R^*(s) - G^*(s)\overline{HC}^*(s)$$

In general, $C^*(s)$ cannot be factored from $\overline{HC}^*(s)$. Thus, this equation cannot be solved for $C^*(s)$.

- In general, in analyzing a system, an equation should NOT be starred if a system signal is lost as a factor, as shown above.
- However, for systems more complex than the one above, solving the system equations can become quite complex.

A simpler method of analysis will now be developed that avoids this problem.



First, we consider the system of Fig. 5-3. Since the input is not sampled before being applied to an analog element, no transfer function can be derived. Nevertheless, the system can be analyzed.

$$C(s) = G(s)E(s)$$

and

$$E(s) = R(s) - H(s)C^*(s)$$

Substituting, we obtain

$$C(s) = G(s)R(s) - G(s)H(s)C^*(s)$$

Starring the above equation yields

$$C^*(s) = \overline{GR}^*(s) - \overline{GH}^*(s)C^*(s)$$

In this system, the forcing function $R(s)$ is necessarily lost as a factor, since $r(t)$ is not sampled. Solving for $C^*(s)$, we obtain

$$C^*(s) = \frac{\overline{GR}^*(s)}{1 + \overline{GH}^*(s)}, \qquad C(z) = \frac{\overline{GR}(z)}{1 + \overline{GH}(z)}$$

The continuous output is obtained as

$$C(s) = G(s)R(s) - \frac{G(s)H(s)\overline{GR}^*(s)}{1 + \overline{GH}^*(s)}$$

For the system of Fig. 5-3, no transfer function can be derived; the problem is that the input is not sampled before being applied to an analog part of the system.

5.3 Derivation Procedure

The determination of transfer functions for samlped-data systems is difficult because a transfer function for the idela sampler does not exist. A procedure for finding transfer functions will now be developed using the system of Fig. 5-4, which has the flowgraph in Fig. 5-5.

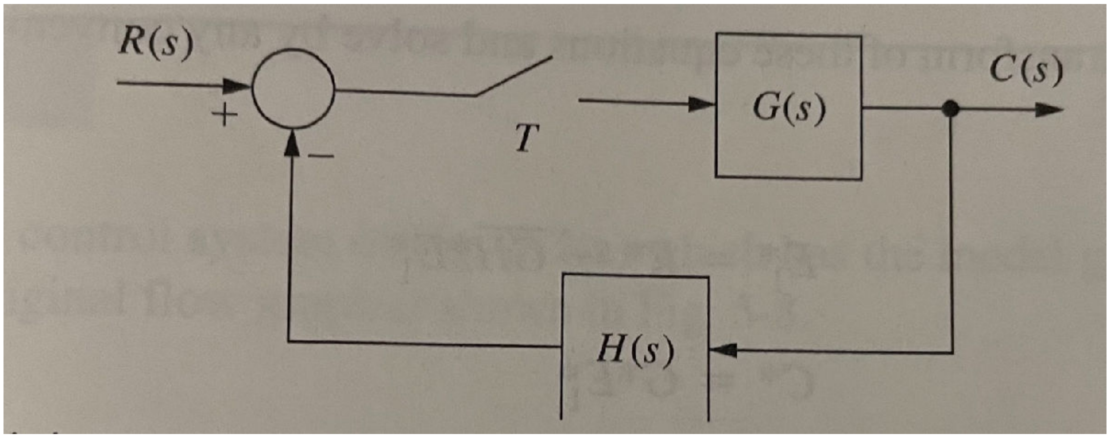


Fig. 5-4

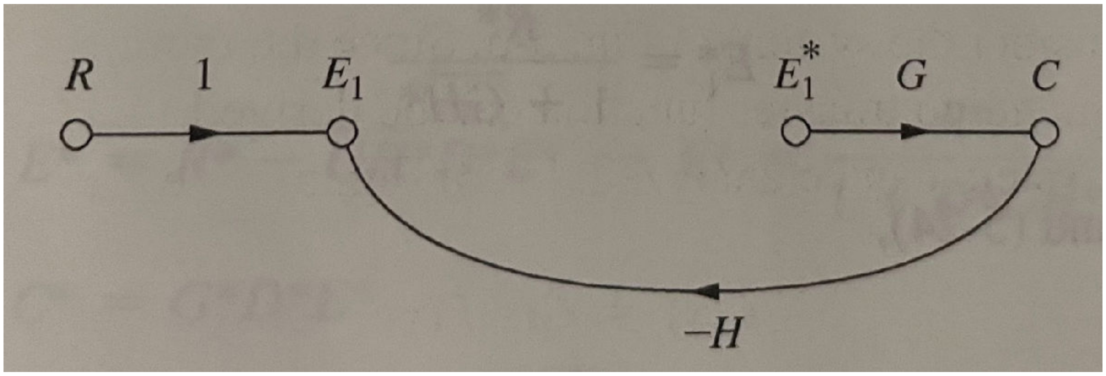


Fig. 5-5

1. Construct the *original signal flow graph.*
2. Assign a variable to each sampler input. Then the sampler output is this variable starred.
3. Considering each sampler output to be a source node (input), express the sampler inputs and the system output in terms of each sampler output (which is treated as an input in the flowgraph), and the system input.
4. Take the starred transform of these equations and solve by any convenient method.

$$E_1 = R - GHE_1^*$$

$$C = GE_1^*$$

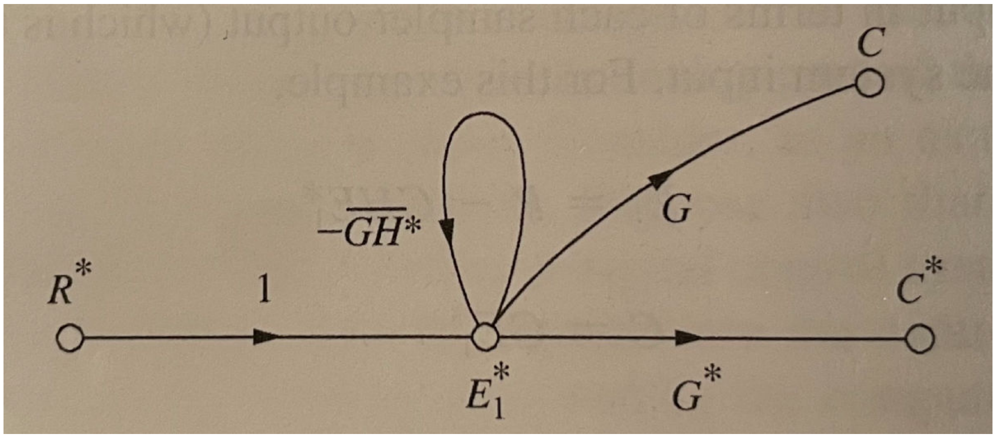
Starred transform the equations

$$E_1^* = \frac{R*}{1 + \overline{GH}^*}$$

and, then,

$$C^*(s) = \frac{G^*(s)R^*(s)}{1 + \overline{GH}^*(s)}$$

The system equations can also be solved by constructing a signal flow graph *from these equations* and applying Mason's gain formula. This flow graph is called the *sampled signal flow graph.* This method is sometimes superior to Cramer's rule, provided that the sampled signal flow graph is simple enough so that all loops are easily identified.



From the flow graph

$$C^*(s) = \frac{G^*(s)R^*(s)}{1+\overline{GH}^*(s)}$$

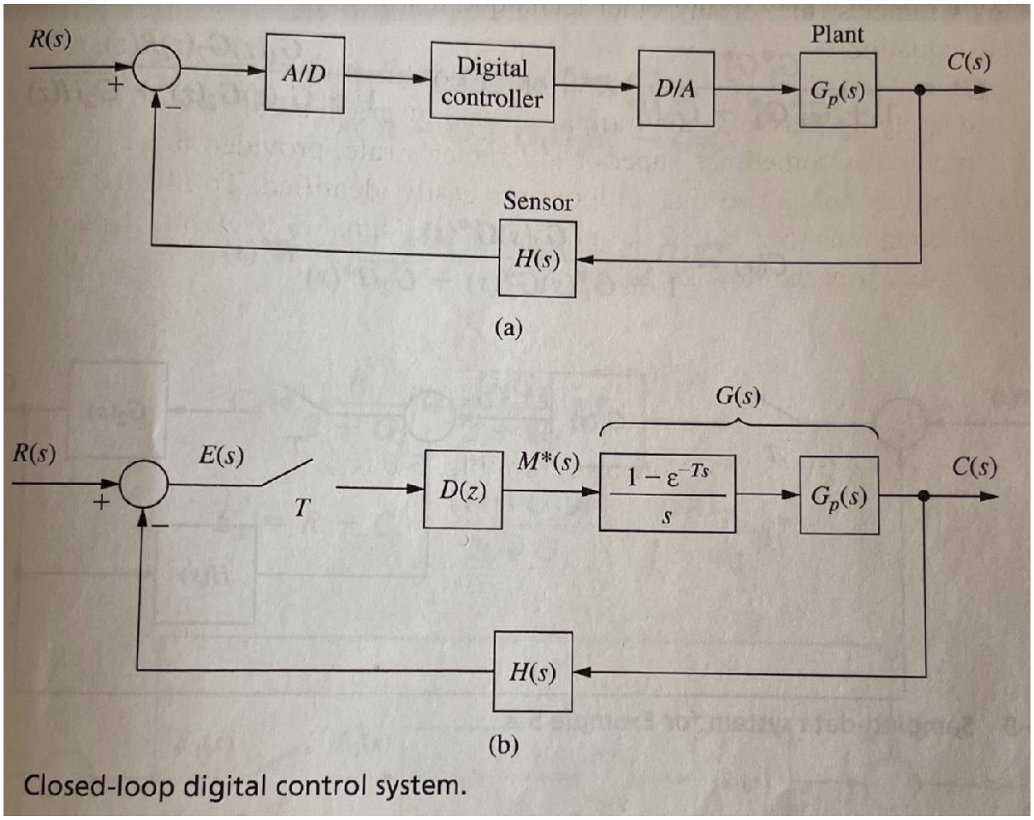
and this result agrees with previous definition of $C^*(s)$. Note that

$$C(s) = \frac{G(s)R^*(s)}{1+\overline{GH}^*(s)}$$

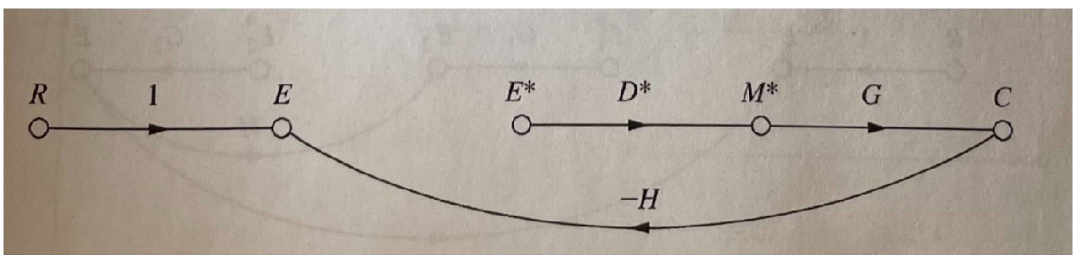
This method for finding the output function for closed-loop systems will now be illustrated via the following examples

Example 5.1

Consider the digital control system of Fig. 5-7(a) which has the model given in Fig. 5-7(b), with original flow graph shown in Fig. 5-8



Closed-loop digital control system.



From the original flow graph, we write

$$E = R - GHD^*E^*$$

$$C = GD^*E^*$$

Hence

$$E^* = R^* - \overline{GH}^*D^*E^* \rightarrow E^* = \frac{R^*}{1 + D^*\overline{GH}^*}$$

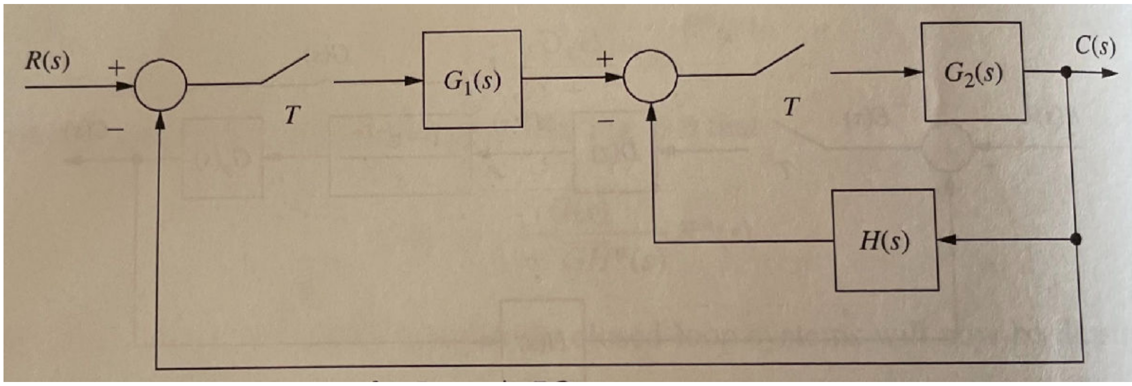
$$C^* = G^*D^*E^*$$

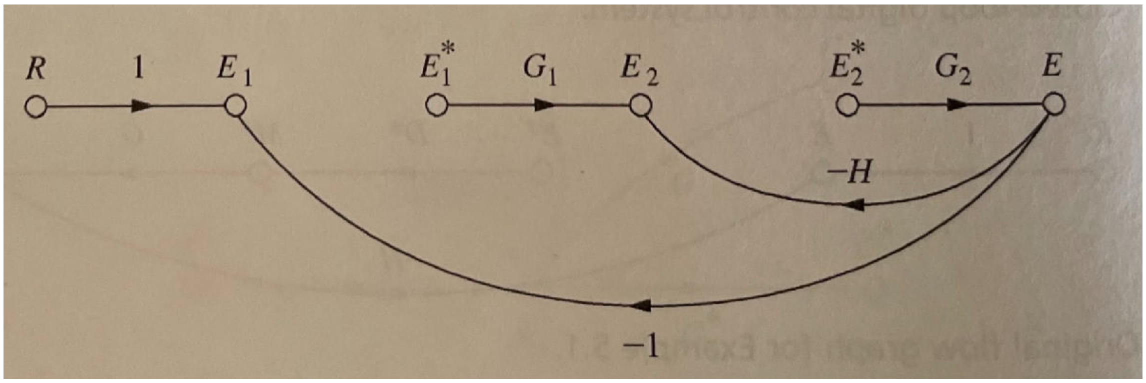
Thus,

$$C(z) = \frac{D(z)G(z)}{1 + D(z)\overline{GH}(z)} R(z)$$

Example 5.2

Consider the system in Fig. 5-9. The original flow graph is shown in Fig. 5-10.





The system equations are

$$E_1 = R - G_2 E_2^*$$

$$E_2 = G_1 E_1^* - G_2 H E_2^*$$

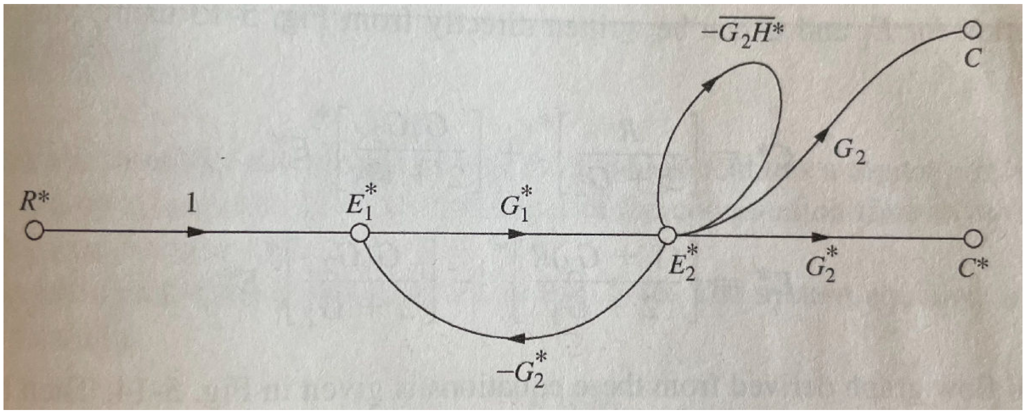
$$C = G_2 E_2^*$$

Starring these equations, we obtain

$$E_1^* = R^* - G_2^* E_2^*$$

$$E_2^* = G_1^* E_1^* - \overline{G_2 H}^* E_2^*$$

$$C^* = G_2^* E_2^*$$



The sampled flow graph can then be drawn from these equations as shown in Fig. 5-11. Then applying Mason's formula, we get

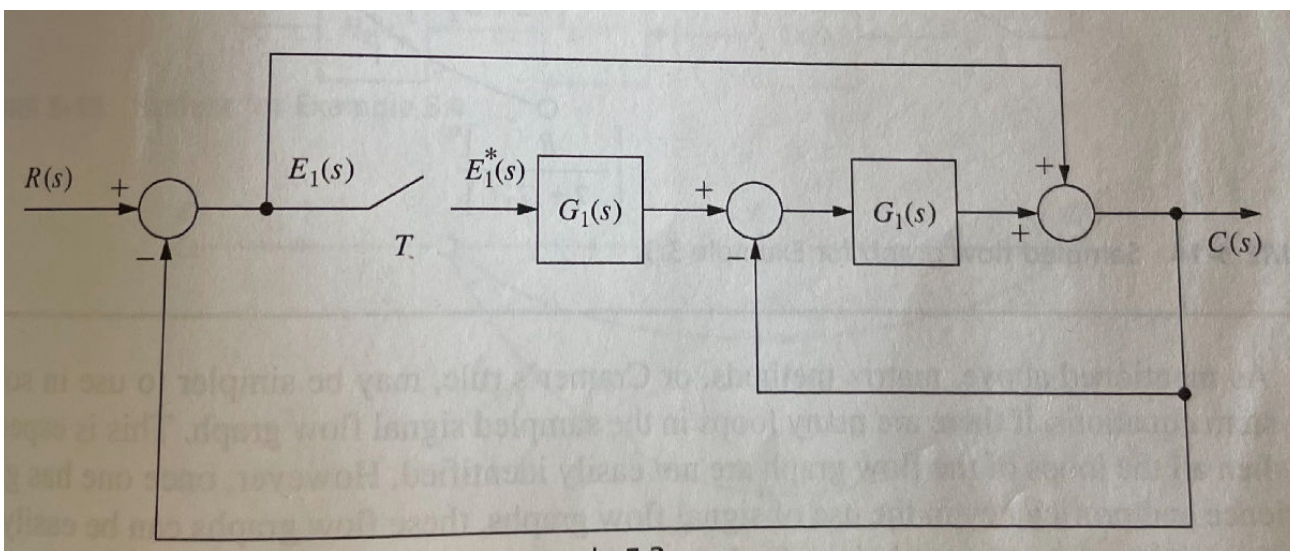
$$C^* = \frac{G_1^* G_2^*}{1 + G_1^* G_2^* + \overline{G_2 H}^*} R^* \quad \text{or} \quad C(z) = \frac{G_1(z) G_2(z)}{1 + G_1(z) G_2(z) + \overline{G_2 H}(z)} R(z)$$

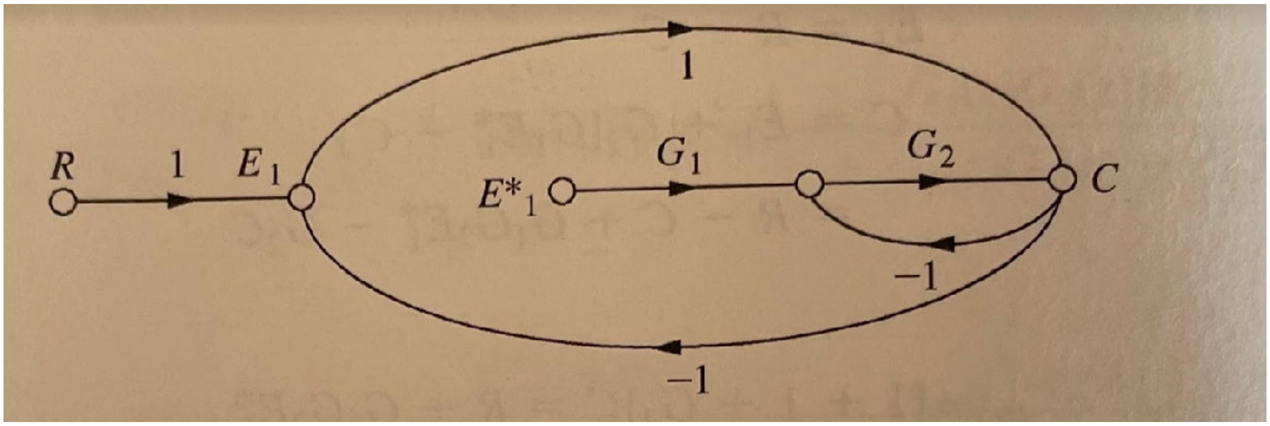
and

$$C(s) = \frac{G_2(s) G_1^*(s)}{1 + G_1^*(s) G_2^*(s) + \overline{G_2 H}^*(s)} R^*(s)$$

Example 5.3

Consider the system of Fig. 5-12. Note that no transfer function may be derived for this system, since the input $R(s)$ reaches $G_2(s)$ without being sampled. The original flow graph is given in Fig. 5-13.





5.4 State-Variable Models

5.5 Summary