EECE 5610 Digital Control Systems

Lecture 6

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<u>A</u>: Not necessarily. Note that we started out with a <u>second order</u> T.F. Hence we would expect to be able to realize it with <u>just two delays</u>. However our realization uses 4 !

Summary :

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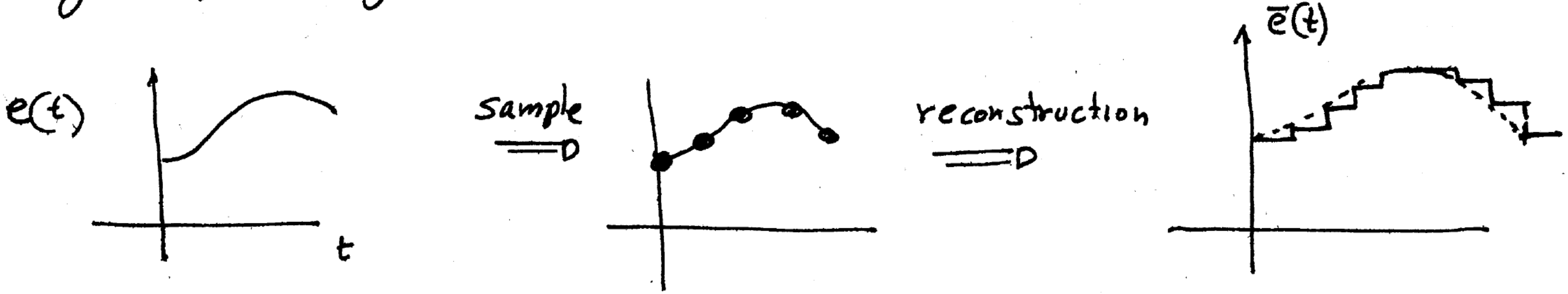
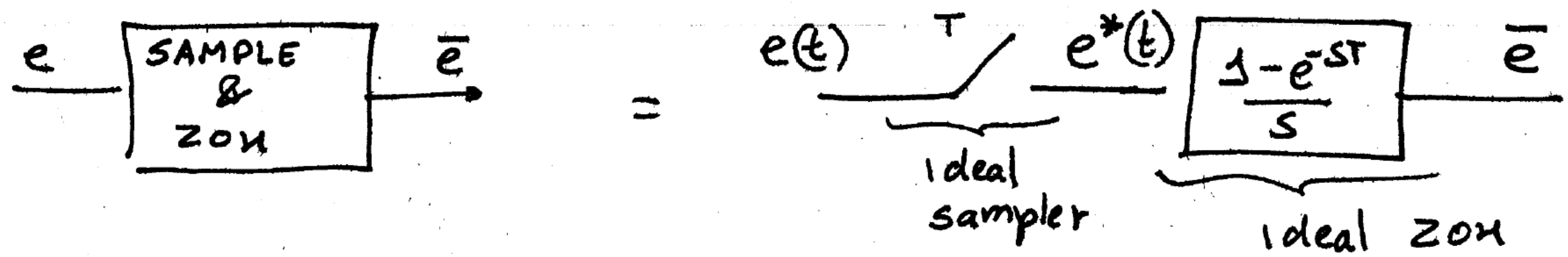
Analysis of sampling and reconstruction:





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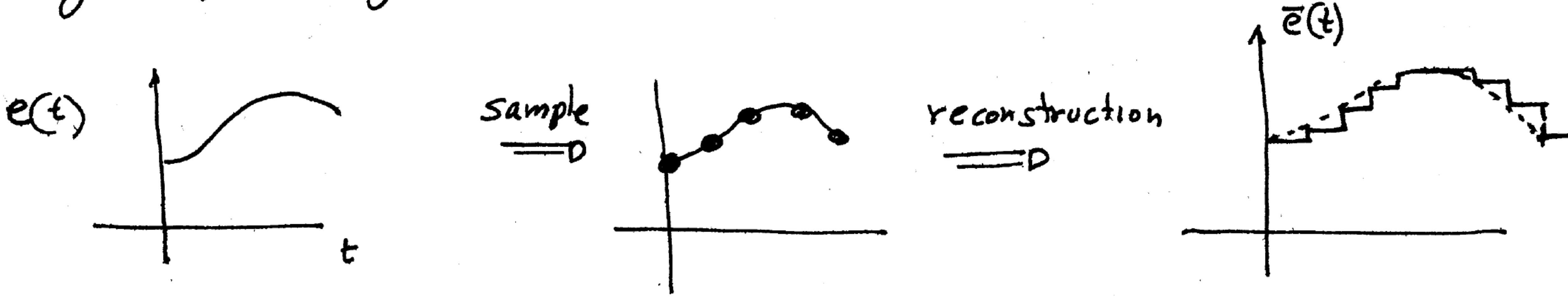
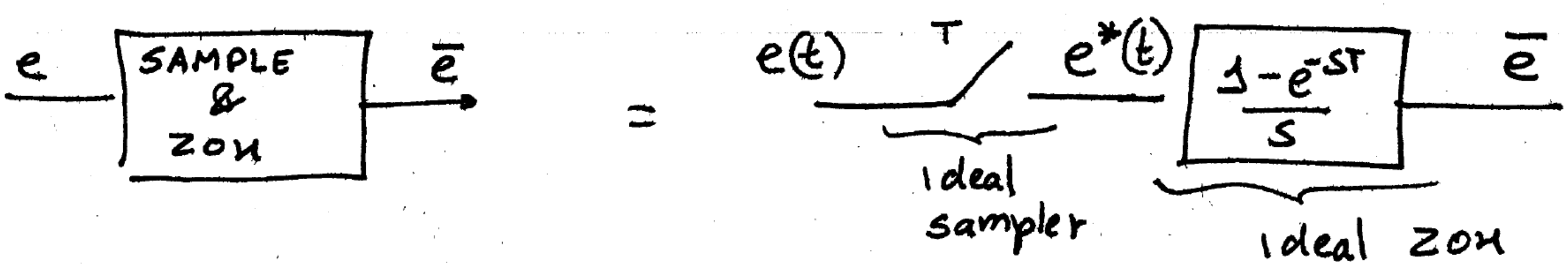
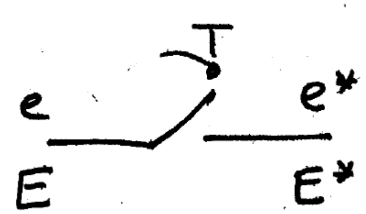




Ideal sampler

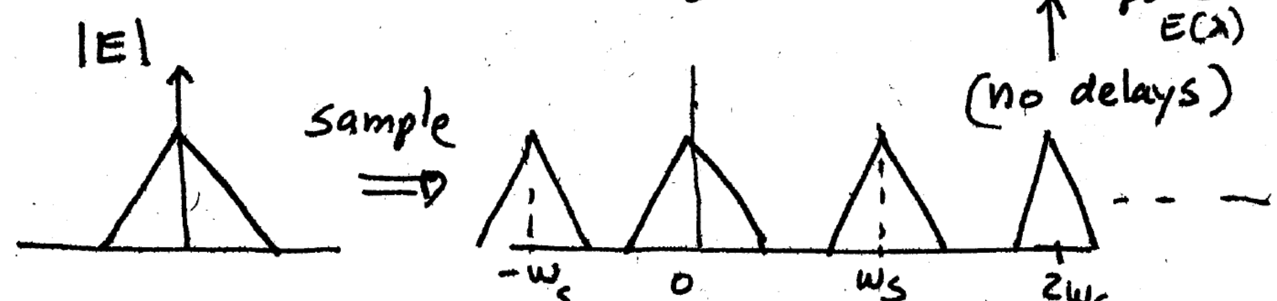


$$e^*(t) = \sum_{0}^{\infty} e(kT)\,\delta(t-kT) = e(t)\cdot\delta_T(t)$$

$$E^*(s) = \sum_{0}^{\infty} e(kT)\, e^{-kTs} = \sum_{\substack{\text{poles}\\ E(\lambda)}} \text{Res}\left\{E(\lambda)\frac{1}{(1-e^{-T(s-\lambda)})}\right\}$$

(no delays)

Spectrum of $E^*(s)$:



Shannon’s Theorem

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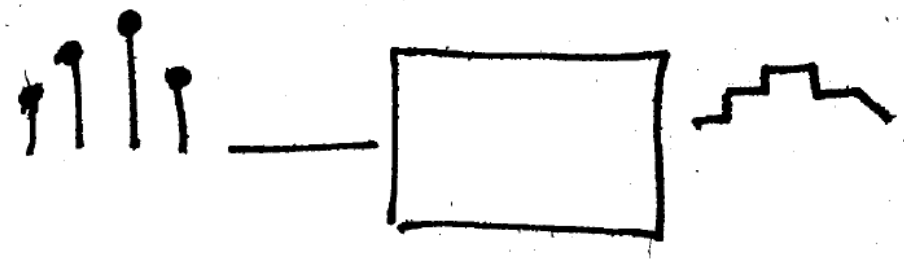
Can reconstruct E from E^* <u>iff</u> $\omega_1 \le \frac{\omega_s}{2}$

Reconstruction: 

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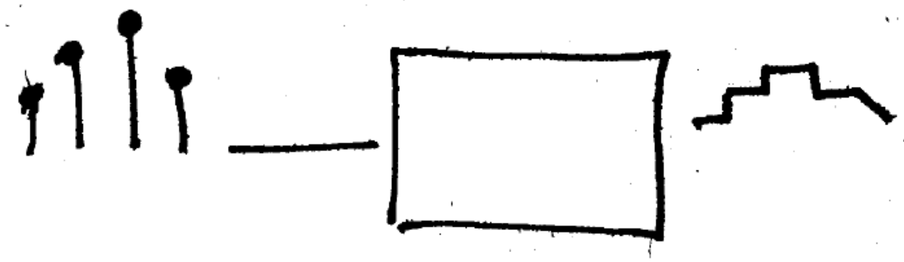
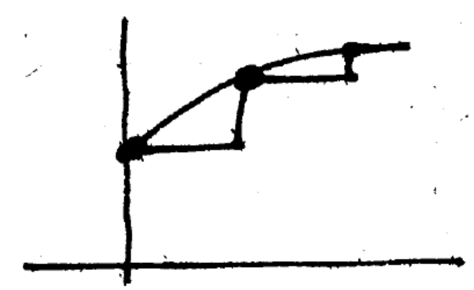
Reconstruction: 

Polynomial interpolation: $e_n(t) = e(T) + e'(T)(t-T) + \cdots$

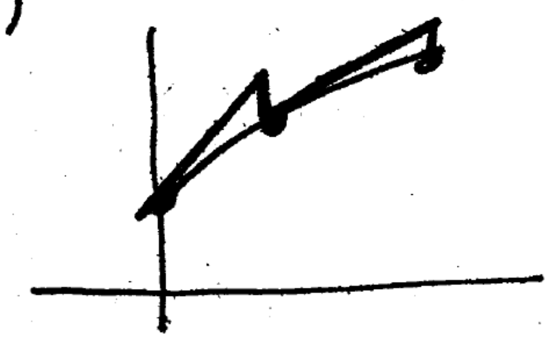
$\underline{\text{ZOH}}$: $e_n(t) = e(kT) \qquad kT \le t < (k-1)T$

$$G_{ZOH} = \frac{1-e^{-sT}}{s}$$



F.OH: $e_n(t) = e(kT) + e'(kT)(t-kT)$

$$G_{FOH} = \left(\frac{1-e^{-sT}}{s}\right)^2 \left(\frac{1+sT}{T}\right)$$



EECE 5610 Digital Control Systems

Lecture 9

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Open Loop Discrete Time Systems (Chapter 4)

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The first step is to examine:

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(this relationship allows for considering the z-transform as a special case of the Laplace transform)

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Old way:

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Alternative

1) find $E^*(s)$
2) $E(z) = E^*(s)\Big|_{z = e^{Ts}}$

Obviously the alternative works better only if we have an efficient way of computing $E^*(s)$ from $E(s)$. Here is where the residues formula becomes useful.

Recall that
$$E^*(s) = \sum_{\substack{\text{Poles} \\ E(\lambda)}} \text{Res}\left[E(\lambda)\frac{1}{1-e^{-T(s-\lambda)}}\right]$$

$$\Rightarrow \quad \boxed{E(z) = \sum_{\substack{\text{poles} \\ E(\lambda)}} \left[\text{residues of } E(\lambda)\frac{1}{1-\frac{1}{z}e^{\lambda T}}\right]}$$

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3) $$e(kT) = e^{-kT} - e^{-2kT}$$

4) $$E(z) = \sum_{0}^{\infty} e^{-kT} \cdot z^{-k} - \sum_{0}^{\infty} e^{-2kT} \cdot z^{-k}$$

$$= \frac{1}{1 - \frac{e^{-T}}{z}} - \frac{1}{1 - \frac{e^{-2T}}{z}}$$

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<u>"New" way</u>

$$E(z) = \sum_{\substack{\lambda=-1 \\ \lambda=-2}} \mathrm{Res}\left\{ \frac{1}{(\lambda+1)(\lambda+2)} \; \frac{1}{1 - e^{\lambda T} \cdot \frac{1}{z}} \right\}$$

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2 poles: $z = e^{-T}$

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$\Rightarrow$ Any analysis based on pole/zeros will be far easier to carry out in the <u>z-plane</u>. We will see more advantages latter on.

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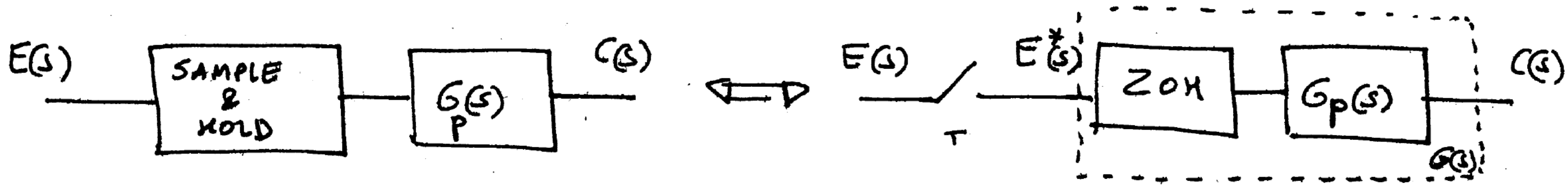

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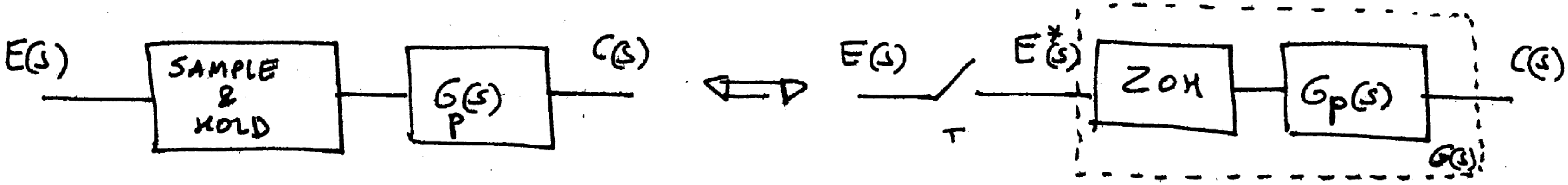

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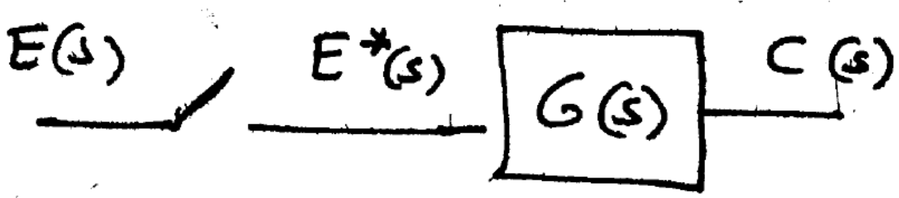
$\Rightarrow$



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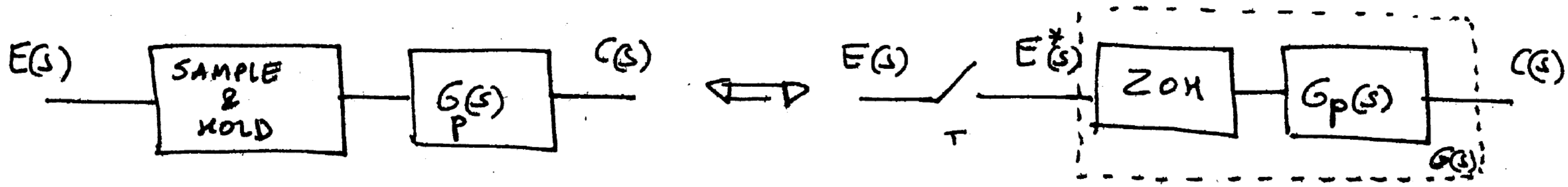

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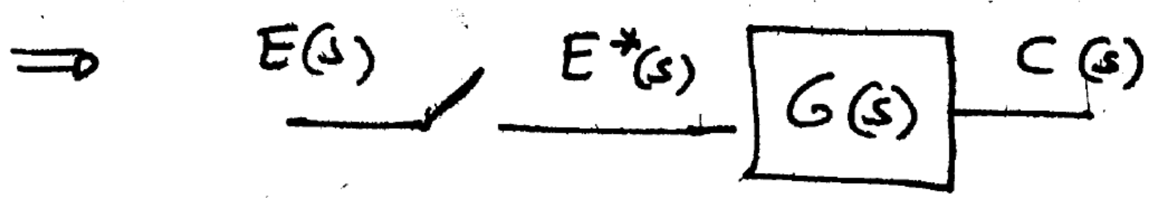

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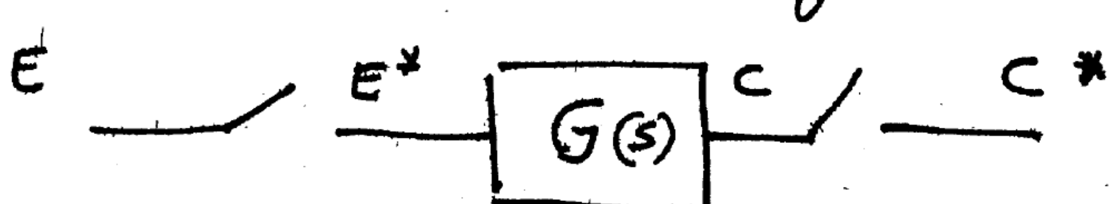
Assume now that we only look at the output at discrete time instants: kT



We want to find out and expression for $C^*(s)$ (as an intermediate step to finding $c(kT)$)

$$C^*(s) = \frac{1}{T}\sum_{-\infty}^{+\infty} C(s+jn\omega_s) = \frac{1}{T}\sum_{-\infty}^{+\infty} G(s+jn\omega_s)E^*(s+jn\omega_s)$$

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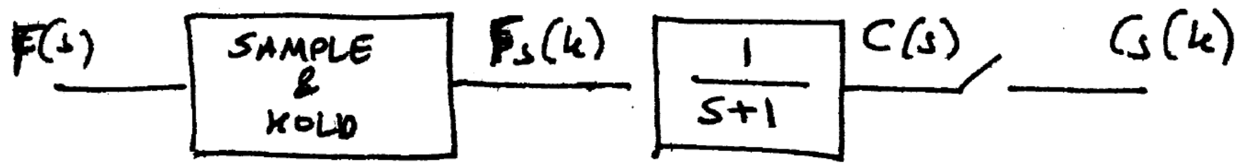
We will call $G(z)$ the <u>pulse transfer function</u>

(Roughly speaking: TF between sampled input & sampled output at the sampling instants)

Suppose that we want to find the TF of a simple plant:



Brute force approach: $C(s) = \frac{1}{(s+1)} F_s \overset{\mathcal{L}^{-1}}{\Longrightarrow} \dot{c}(t) + c(t) = f_s(t)$

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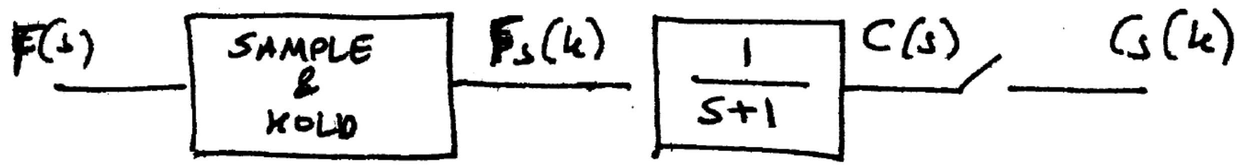

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Solving this first order differential equation yields:

$$c(t) = A e^{-(t-kT)} + f(kT) \qquad \text{(recall that } f_s(t) = f(kT) \quad kT \le t < (k+1)T\text{)}$$

Now we need to impose the initial condition $c(t)\big|_{t=kT} = c(kT)$ and solve for A:

$$A = c(kT) - f(kT) \Rightarrow$$

$$c(t) = c(kT)\, e^{-(t-kT)} + \left[1 - e^{-(t-kT)}\right] f(kT) \qquad kT \leq t < (k+1)T$$

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$$\Rightarrow c((k+1)T) = c(kT)\, e^{-T} + (1 - e^{-T}) f(kT)$$

$\Downarrow$ z transform

$$(z - e^{-T})\, C(z) = (1 - e^{-T})\, F(z) \Rightarrow \boxed{\frac{C(z)}{F(z)} = \frac{1 - e^{-T}}{z - e^{-T}}}$$

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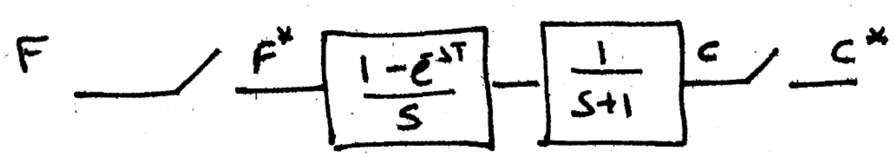
Lot's of work, even for a very simple plant! Let's try now our tools:



$$G(s) = \frac{1 - e^{-sT}}{s(s+1)} = \frac{C(s)}{F^*}$$

$$\Rightarrow \frac{C(z)}{F(z)} = Z\left\{\frac{1 - e^{-sT}}{s(s+1)}\right\} = \left[\frac{1 - e^{-sT}}{s(s+1)}\right]^* \Bigg|_{z = e^{sT}}$$

Note that we have a delay $\Rightarrow$ (3-10) <u>does not</u> apply. However since the delay is an integer number of periods we can use the alternative formula:

$$\left[e^{-mTs}G_1(s)\right]^* = e^{-mTs}\left[G_1(s)\right]^* \Longleftrightarrow Z\left[e^{-mTs}G_1(s)\right] = \frac{1}{z^m}G_1(z)$$

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In this case we have:

$$G_1(s) = \frac{1}{s(s+1)} \Longleftrightarrow G_1(z) = \sum_{\substack{\lambda=0 \\ \lambda=-1}} \mathrm{Res}\left[\frac{1}{\lambda(\lambda+1)} \cdot \frac{1}{\left(1-\frac{e^{\lambda T}}{z}\right)}\right]$$

$$= \frac{1}{1-\frac{1}{z}} - \frac{1}{1-e^{-T}/z} = \frac{z\left(1-e^{-T}\right)}{(z-1)\left(z-e^{-T}\right)}$$

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$$= \frac{1}{1-\frac{1}{z}} - \frac{1}{1-e^{-T}/z} = \frac{z\left(1-e^{-T}\right)}{(z-1)(z-e^{-T})}$$

and $G(z) = G_1(z) - \frac{1}{z} G_1(z) = \left(\frac{z-1}{z}\right) G_1(z) = \boxed{\frac{1-e^{-T}}{z-e^{-T}}}$ same as before

(but we did not have to solve a differential eq.)

Assume now that the input is a step: $f(kT) = 1, 1, \ldots$

$$F(z) = \frac{z}{z-1}$$

$$\Rightarrow C(z) = G(z)F(z) = \frac{z}{(z-1)}\frac{(1-e^{-T})}{(z-e^{-T})} = \frac{z}{z-1} - \frac{z}{z-e^{-T}}$$

Going back to the time domain we get:

$$c(kT) = \mathcal{Z}^{-1}\left[C(z)\right] = \mathcal{Z}^{-1}\left[\frac{z}{z-1} - \frac{z}{z-e^{-T}}\right] = 1 - e^{-kT} \quad \#$$

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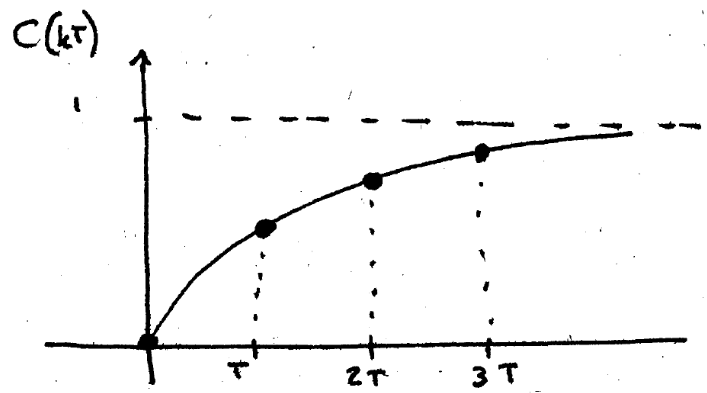
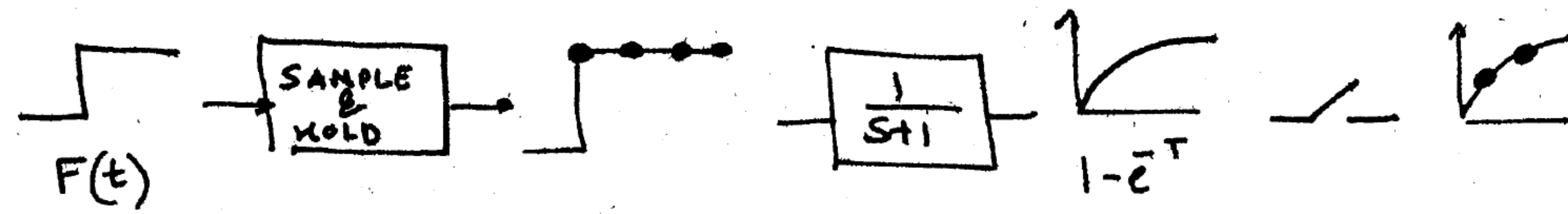
(Note: you can get the same result from residues, without having to do partial fractions:

$$c_k = \sum_{\substack{z=1 \\ z=e^{-T}}} \text{Res}\left\{ z^{k-1} \cdot \frac{z\,(1-e^{-T})}{(z-1)(z-e^{-T})} \right\} = \left(\frac{1-e^{-T}}{1-e^{-T}}\right) + e^{-kT}\left(\frac{1-e^{-T}}{e^{-T}-1}\right) = 1 - e^{-kT} \# \,)$$



Sanity check: Physically



Important: You can get $c(kT)$ from $c(t)$ but the converse <u>does not</u> work. You cannot recover $c(t)$ from $c(kT)$
(Obvious, since $c(kT)$ does not contain information about what happens in between samples)

- **Definition:** DC gain:

$$\frac{\lim_{k \to \infty} c_k}{\lim_{k \to \infty} F_k} = \frac{c_{ss}}{F_{ss}} \overset{\text{(Final value theorem)}}{=} \lim_{z \to 1} \frac{(z-1)\,C(z)}{F_{ss}}$$

$$= \lim_{z \to 1} (z-1)\,\frac{G(z)\,F(z)}{F_{ss}} = \lim_{z \to 1} \cancel{(z-1)}\,\frac{G(z)}{\cancel{F_{ss}}}\,\cancel{F_{ss}}\,\frac{z}{\cancel{(z-1)}}$$

$$= G(1)$$

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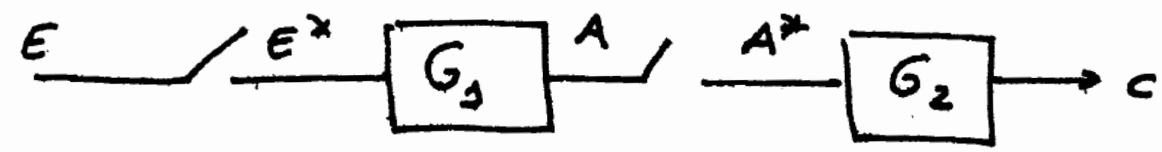
$$= G(1)$$

$$\Rightarrow \boxed{\text{DC gain: } G(1)}$$

(compare with continuous time: DC gain $= G(s)\big|_{s=0}$)

but then again: $z = e^{Ts} \Rightarrow s=0$ gets mapped to $z=1$

- <u>Other configurations:</u>

a) 

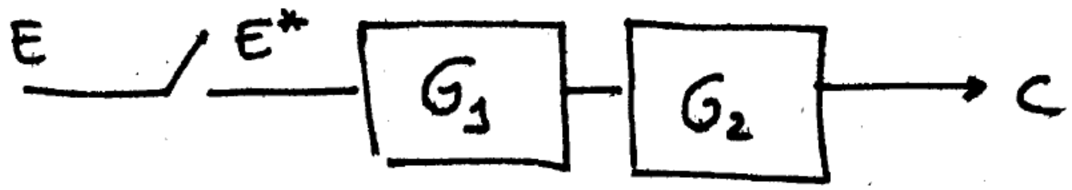
$$C(s) = G_2(s)\,A^*(s) \qquad A(s) = G_1(s)\,E^*(s) \Rightarrow A^* = (G_1 E^*)^* = G_1^* E^*$$

$$C^*(s) = G_2^* A^* = G_2^*(s)\,G_1^*(s)\,E^*(s)$$

$$\Downarrow$$

$$\boxed{C(z) = G_2(z)\,G_1(z)\,E(z)} \quad \text{or} \quad \boxed{G(z) = G_1(z) \cdot G_2(z)}$$

Key feature: we have an ideal sampler between G_1 & G_2

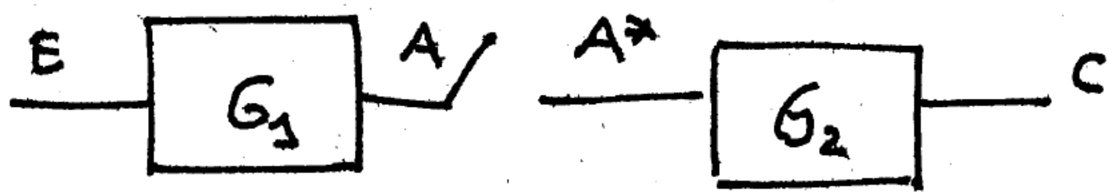
b) 

$$C(s) = G_2(s)\, G_1(s)\, E^*(s) \Rightarrow C^*(s) = [G_1 \cdot G_2]^* E^*(s) \quad \text{or}$$

$$C(z) = \mathcal{Z}\{G_1 \cdot G_2\} \cdot E(z) \Rightarrow \boxed{G(z) = \mathcal{Z}\,[G_1 \cdot G_2]}$$

Note: $\mathcal{Z}\,[G_1 G_2] \neq G_1(z)\, G_2(z)$!!

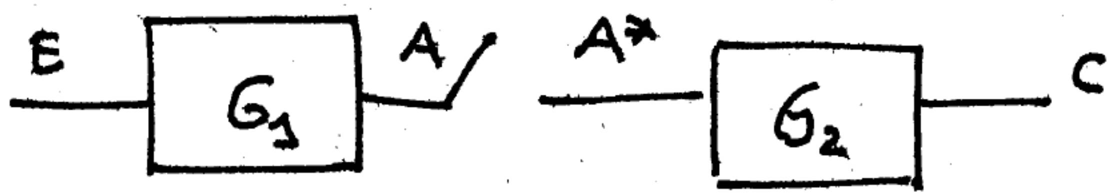
c)



$$C^* = [G_2 A^*]^*$$

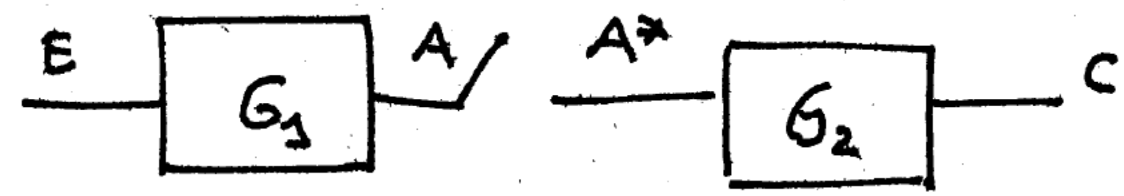
$$A = G_1 E \Rightarrow A^* = [G_1 E]^*$$

$$C^* = G_2^* [G_1 E]^*$$

c) 

$$C = G_2 A^* \Rightarrow C^* = G_2^* [G_1 E]^*$$

For this system a transfer function <u>cannot</u> be written!
The reason is that you can't factor $E(z)$ out of $[G_1 E](z)$

c) 

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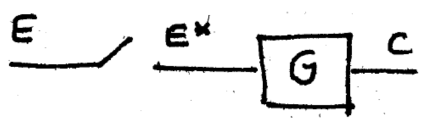
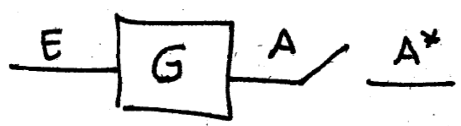
<u>Physical reason</u>:

When you have 

, E^* contains information about E <u>only</u> at the sampling instants. $\Rightarrow$ When you compute C^* you can factor E^* out.

On the other hand, if you have something like this:



$$A(s) = G(s)\,E(s)$$

$$a(t) = \int_0^t g(t-\tau)\,e(\tau)\,d\tau \Rightarrow$$

to compute A^* (and latter C^*) you need information about $e(t)$ at <u>all</u> times, not just the sampling instants.

Important consequence: If you want to have a discrete T.F you have to select as your unknowns the inputs to the sampler

This variables will always "come free" after the equation sampling process and give a set of starred variables for which we can solve.

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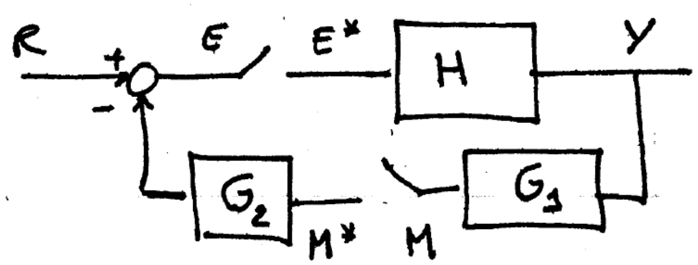
Example:



$$\left.\begin{aligned} &E(s) = R - G_2 M^* \\ &E^* = R^* - [G_2 M^*]^* = R^* - G_2^* M^* \\ &M = (G_1 H) E^* \Rightarrow M^* = (G_1 H)^* E^* \end{aligned}\right\} \quad E^* = R^* - G_2^* [G_1 H]^* E^*$$

$$\Rightarrow \boxed{E^* = \frac{R^*}{1 + (G_1 H)^* G_2^*}}$$

or: $\boxed{\dfrac{E(z)}{R(z)} = \dfrac{1}{1 + [G_1 H](z)\, G_2(z)}}$

To obtain Y we can use the equation: $Y(s) = H E^{*} \Rightarrow Y^{*} = H^{*} E^{*}$

$$\Rightarrow \frac{Y^{*}}{R^{*}} = \frac{H^{*}}{1 + (G_1 H)^{*} G_2^{*}} \Leftrightarrow \boxed{\frac{Y(z)}{R(z)} = \frac{H(z)}{1 + (HG_1)(z)\, G_2(z)}}$$

Note that in this case we have a TF. This is because R goes right through a sampler before entering other blocks.

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- <u>Open-loop Systems with Digital Filters</u>

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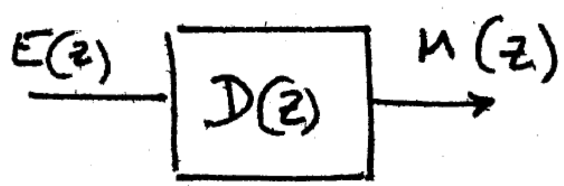
• Open-loop Systems with Digital Filters

We consider now the case where the sampled-data system contains a digital filter



$$M(z) = D(z)\, E(z)$$
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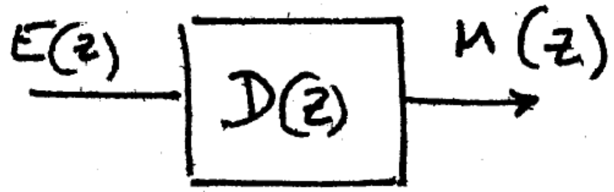

$$M(z) = D(z)\,E(z)$$
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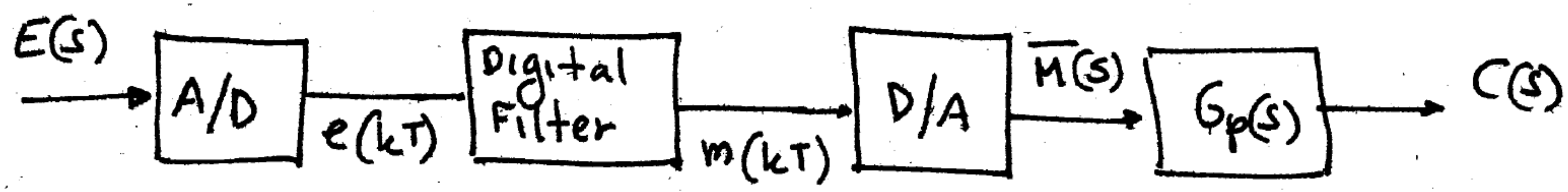
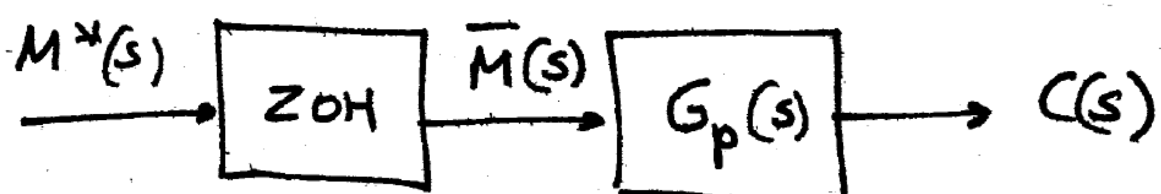
<u>Physically</u>:



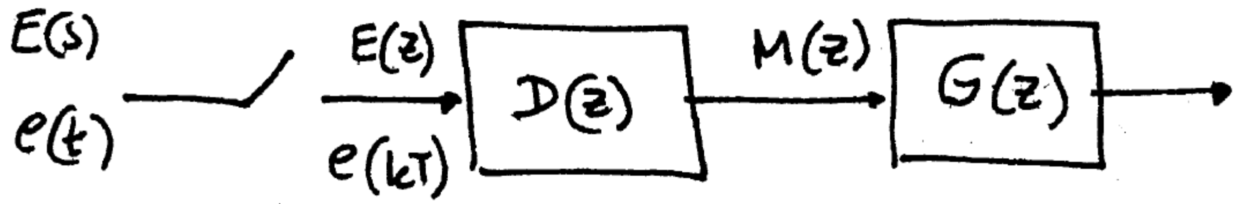
We will assume that the D/A can be represented as a zero order hold

$\Rightarrow$



(Recall that $G_{h_o} = \dfrac{1-e^{-sT}}{s}$)

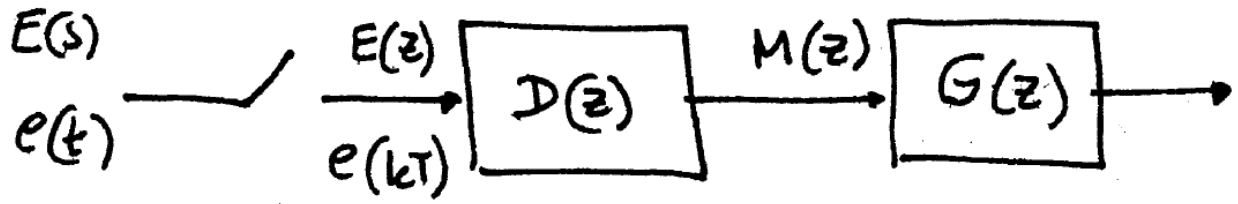
$\Rightarrow$



$$C(s) = \underbrace{G_{h_o}(s)\cdot G_p(s)}_{G(s)}\, M^*(s) \quad \Longrightarrow \quad C(z) = G(z)\,M(z) = G(z)\,D(z)\,E(z)$$

(Recall that $G_{h_0} = \frac{1-e^{-sT}}{s}$)

$\Rightarrow$



$$C(s) = \underbrace{G_{h_0}(s) \cdot G_p(s)}_{G(s)} \, M^*(s) \quad \Rightarrow \quad C(z) = G(z)M(z) = G(z)D(z)E(z)$$

<u>Example</u>: Suppose that the digital filter is given by the following diff. eq.

$$m(kT) = 2e(kT) - e[(k-1)T]$$

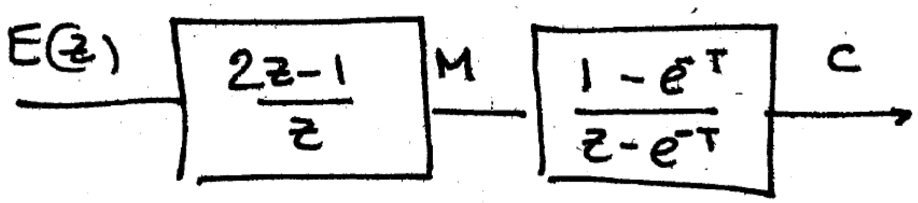
$$M(z) = 2E(z) - \frac{E(z)}{z} = \frac{2z-1}{z}E(z)$$

$$\Rightarrow D(z) = \frac{M}{E} = \boxed{\frac{2z-1}{z}}$$

$$G_p(s) = \frac{1}{(s+1)} \Rightarrow G(s) = G_{h0}\, G_p = \frac{1-e^{-sT}}{s(s+1)}$$

$$G(z) = \left(1-\frac{1}{z}\right) Z\left[\frac{1}{s(s+1)}\right] = \left(1-\frac{1}{z}\right)\left[\frac{1}{1-1/z} - \frac{1}{1-e^{-T}/z}\right]$$

$$= \frac{1-e^{-T}}{z-e^{-T}}$$



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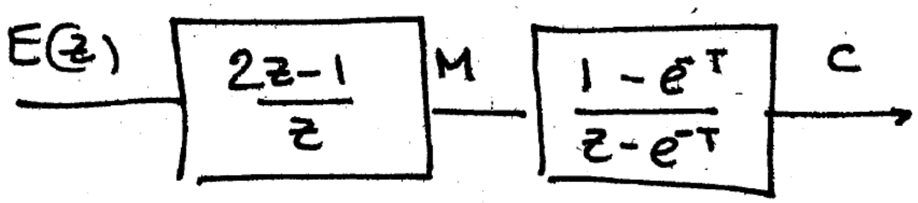

Assume that $E(z)$ is a step: $E(z) = \frac{z}{z-1} \Rightarrow$

$$C(z) = \left(\frac{1-e^{-T}}{z-e^{-T}}\right)\left(\frac{2z-1}{\cancel{z}}\right)\left(\frac{\cancel{z}}{z-1}\right)$$

From here we can get $c(kT)$ either by doing partial fraction expansion or using the residues formula.

Effect of T_s

Recall that $C^*(s)$ or $C(z)$ gives you information <u>only</u> on what happens at the sampling instants, but not in-between.

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Q: Is this a problem?

Effect of T_s

Recall they $c^*(s)$ or $C(z)$ gives you information only on what happens at the sampling instants, but not in-between.

Q: Is this a problem?

A: Depends on how fast the dynamics of your plant are, compared to the sampling rate

- Slow Plant = Large Sample Rate
- Fast Plant = Small Sample Rate

Example:

$$G_p(s) = \frac{25}{s^2 + 2s + 25} \quad \Rightarrow \quad \omega_n^2 = 25 \qquad (\omega_n = 5)$$

$$\zeta = 0.2 \qquad \text{(underdamped)}$$

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Analyze step response of original plant

It can be shown that if we sample at $T=0.1$ and $T=1$ we get the following discrete time equivalents:

$$G(z) = \mathcal{Z}\left\{\left(\frac{1-e^{-sT}}{s}\right)\cdot G_p(s)\right\} = \left(1-\frac{1}{z}\right)\mathcal{Z}\left[\frac{25}{s(s^2+2s+25)}\right]$$

$$G_{T=1} = \frac{z - 0.007}{z^2 - 0.1365z + 0.1353}$$

$$G_{T=0.1} = \frac{0.115z - 0.107}{z^2 - 1.6z + 0.82}$$

To see differences between these Plants, look at the step response to see how different sample rate affect them