EECE 5610 Digital Control Systems

State-space representation

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Learning outcomes

By the end of *this* lecture, you should be able to:

- Discretize a continuous-time system in a state-space form to its discrete counterpart

- Compute the transfer function of a discrete-time system expressed in a state-space representation.

Based on the feedback I got from you and your exam!

Introduction

- Often, physical equations that describe a system mathematically are already available in *state-space representation*, where the states are some set of physical variables (e.g., displacements, velocities, etc.), i.e.,

$$\begin{aligned} \text{System model: } \dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + B\mathbf{u}(t), \quad x(0) = x_0 \\ \text{Observation model: } \mathbf{y}(t) &= C\mathbf{x}(t) + D\mathbf{u}(t) \end{aligned}$$

$$\dim(\mathbf{x}) = n \quad \dim(\mathbf{u}) = r \quad \dim(\mathbf{y}) = p$$

- Advantages of the system description using the state-space representation in comparison to the conventional methods are:
  - Possibility to describe the state of the entire system each time; unlike the transfer function, which connects the input $u(t)$ with the output $y(t)$
  - They facilitate the solution of control problems, such as stability and optimized control
  - The simulation and scheduling in computer systems is quite easy since they are represented by a set of linear differential (later difference) equations
  - They are also able to describe some nonlinear systems, which cannot be done using the transfer function

Homogeneous set of differential equations

e.g., autonomous systems

- In several occasions the evolutions of the system is given by

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t), \quad \mathbf{x}(0) = \mathbf{x}_0$$

- By using Laplace transforms:

$$\begin{aligned} &sX(s) - \mathbf{x}_0 = AX(s) \\ &(sI - A)X(s) = \mathbf{x}_0 \\ &(sI - A)^{-1}(sI - A)X(s) = (sI - A)^{-1}\mathbf{x}_0 \\ &X(s) = (sI - A)^{-1}\mathbf{x}_0 \end{aligned}$$

- The solution is obtained by the inverse-Laplace transform:

$$\begin{aligned} \mathbf{x}(t) &= \mathcal{L}^{-1}\left((sI - A)^{-1}\mathbf{x}_0\right) \\ &= \underbrace{\mathcal{L}^{-1}\left((sI - A)^{-1}\right)}_{e^{At}}\mathbf{x}_0 = e^{At}\mathbf{x}_0 \end{aligned}$$

e^{At} : *state transition* matrix

Solution to the general state-space representation

$$\text{System model: } \dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t), \quad x(0) = x_0$$

$$\text{Observation model: } \mathbf{y}(t) = C\mathbf{x}(t) + D\mathbf{u}(t)$$

- The solution is:

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0 + \int_0^t e^{A(t-\tau)}B\mathbf{u}(\tau)d\tau$$

$$\mathbf{y}(t) = C\left(e^{At}\mathbf{x}_0 + \int_0^t e^{A(t-\tau)}B\mathbf{u}(\tau)d\tau\right) + D\mathbf{u}(t)$$

- **Note 1:** To prove the solution check the initial condition and differentiate the solution to see that the original differential equation holds

- **Note 2:** Let A be a square matrix, then

$$e^{At} = I + tA + \frac{1}{2}t^2A^2 + \frac{1}{6}t^3A^3 + \ldots = \sum_{n=0}^{\infty}\frac{1}{n!}t^nA^n$$

which is always convergent. From this definition: $\dfrac{de^{At}}{dt} = Ae^{At} = e^{At}A$

From continuous-time to discrete-time

Zero-Order Hold (ZOH) and sampling

- Solution at any time t after the sampling instant t_k

$$\mathbf{x}(t) = e^{A(t-t_k)}\mathbf{x}(t_k) + \int_{t_k}^{t} e^{A(t-\tau)}B\mathbf{u}(\tau)d\tau$$

$$= e^{A(t-t_k)}\mathbf{x}(t_k) + \int_{t_k}^{t} e^{A(t-\tau)}d\tau B\mathbf{u}(t_k)$$

$\boldsymbol{u}(t)$ is constant between sampling instants, ZOH

$$= \underbrace{e^{A(t-t_k)}}_{\Phi(t-t_k)}\mathbf{x}(t_k) + \underbrace{\left(\int_0^{t-t_k} e^{As}dsB\right)}_{\Gamma(t-t_k)}\mathbf{u}(t_k)$$

Change the integration variable: $s = t - \tau$

$$= \Phi(t-t_k)\mathbf{x}(t_k) + \Gamma(t-t_k)\mathbf{u}(t_k)$$

A state transition matrix Φ and control matrix Γ are obtained (independent of $\boldsymbol{x}$ and $\boldsymbol{u}$)

From continuous-time to discrete-time

Zero-Order Hold (ZOH) and sampling

- At the next sampling instant, i.e., $t = t_{k+1}$

$$\mathbf{x}(t_{k+1}) = \Phi(t_{k+1} - t_k)\mathbf{x}(t_k) + \Gamma(t_{k+1} - t_k)\mathbf{u}(t_k)$$
$$\mathbf{y}(t_k) = C\mathbf{x}(t_k) + D\mathbf{u}(t_k)$$

where

$$\Phi(t_{k+1} - t_k) = e^{A(t_{k+1} - t_k)}$$
$$\Gamma(t_{k+1} - t_k) = \int_0^{t_{k+1} - t_k} e^{As} ds B$$

- For periodic sampling: $t_k = kh \text{ and } t_{k+1} - t_k = h$. Therefore,

$\boxed{h = T}$

$$\mathbf{x}(kh + h) = \Phi(h)\mathbf{x}(kh) + \Gamma(h)\mathbf{u}(kh)$$
$$\mathbf{y}(kh) = C\mathbf{x}(kh) + D\mathbf{u}(kh)$$

where

$$\Phi(h) = e^{Ah}$$
$$\Gamma(h) = \int_0^h e^{As} ds B$$

Example: discretization by direct calculus

- Sampling interval: h=*0.1*

$$\begin{cases} \dot{x}(t) = 2x(t) + u(t) \\ y(t) = 3x(t) \end{cases}$$

of the form: $$\begin{cases} \dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t) \\ \mathbf{y}(t) = C\mathbf{x}(t) \end{cases}$$

- We compute Φ and Γ as follows:

$$\Phi(h) = e^{Ah} = e^{2(0.1)} = e^{0.2}$$

$$\Gamma(h) = \int_0^h e^{As} ds B = \int_0^{0.1} e^{2s} ds = \left[\frac{1}{2} e^{2s}\right]_0^{0.1} = \frac{1}{2}\left(e^{2(0.1)} - e^0\right) = \frac{1}{2}\left(e^{0.2} - 1\right)$$

- Therefore, the discrete-time system is given by

$$x(kh+h) = e^{0.2} x(kh) + \frac{1}{2}\left(e^{0.2} - 1\right) u(kh)$$

$$y(kh) = 3x(kh)$$

or

$$x[k+1] = e^{0.2} x[k] + \frac{1}{2}\left(e^{0.2} - 1\right) u[k]$$

$$y[k] = 3x[k]$$

Example: discretization by using the series expansion

State-space representation of the double integrator

$$\begin{cases} \dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(t) \end{cases}$$

$$\boxed{\Phi = e^{Ah} = I + hA + \frac{1}{2}h^2A^2 + \frac{1}{6}h^3A^3 + \ldots = \sum_{n=0}^{\infty} \frac{1}{n!} h^n A^n}$$

Example: discretization by using the series expansion

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$$\Phi = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & h \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \ldots = \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix}$$

$$\Gamma(h) = \int_0^h e^{As} ds B = \int_0^h \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} ds \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \int_0^h \begin{bmatrix} s \\ 1 \end{bmatrix} ds = \begin{bmatrix} \frac{1}{2}s^2 \\ s \end{bmatrix}_0^h = \begin{bmatrix} \frac{1}{2}h^2 \\ h \end{bmatrix}$$

Example: discretization by using the series expansion

State-space representation of the double integrator

$$\begin{cases} \dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(t) \end{cases}$$

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$$\Phi = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & h \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \ldots = \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix}$$

$$\Gamma(h) = \int_0^h e^{As} ds B = \int_0^h \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} ds \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \int_0^h \begin{bmatrix} s \\ 1 \end{bmatrix} ds = \begin{bmatrix} \frac{1}{2}s^2 \\ s \end{bmatrix}_0^h = \begin{bmatrix} \frac{1}{2}h^2 \\ h \end{bmatrix}$$

- Hence, the corresponding discrete time model becomes:

$$\begin{cases} \mathbf{x}(kh+h) = \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} \mathbf{x}(kh) + \begin{bmatrix} \frac{h^2}{2} \\ h \end{bmatrix} u(kh) \\ y(kh) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(kh) \end{cases}$$

Example: discretization by using the Laplace transform

State-space representation of the DC motor

$$\begin{cases} \dot{\mathbf{x}}(t) = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \mathbf{x}(t) \end{cases}$$

$$\boxed{\Phi = e^{Ah} = e^{At}\big|_{t=h} = \mathcal{L}^{-1}\left\{(sI - A)^{-1}\right\}\big|_{t=h}}$$

Example: discretization by using the Laplace transform

State-space representation of the DC motor

$$\begin{cases} \dot{\mathbf{x}}(t) = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \mathbf{x}(t) \end{cases}$$

$$\boxed{\Phi = e^{Ah} = e^{At}\big|_{t=h} = \mathcal{L}^{-1}\left\{(sI - A)^{-1}\right\}\big|_{t=h}}$$

$$\Phi = \mathcal{L}^{-1}\left\{\begin{bmatrix} s+1 & 0 \\ -1 & s \end{bmatrix}^{-1}\right\}\Bigg|_{t=h} = \mathcal{L}^{-1}\left\{\frac{1}{s(s+1)}\begin{bmatrix} s & 0 \\ 1 & s+1 \end{bmatrix}\right\}\Bigg|_{t=h}$$

$$= \mathcal{L}^{-1}\left\{\begin{bmatrix} \frac{1}{s+1} & 0 \\ \frac{1}{s(s+1)} & \frac{1}{s} \end{bmatrix}\right\}\Bigg|_{t=h} = \begin{bmatrix} e^{-t} & 0 \\ 1 - e^{-t} & 1 \end{bmatrix}\Bigg|_{t=h} = \begin{bmatrix} e^{-h} & 0 \\ 1 - e^{-h} & 1 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \underbrace{\frac{1}{ad-bc}}_{\text{determinant}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

(inverse of a 2x2 square matrix)

Example: discretization by using the Laplace transform

State-space representation of the DC motor

$$\Gamma(h) = \int_0^h e^{As} ds B = \int_0^h \begin{bmatrix} e^{-s} & 0 \\ 1 - e^{-s} & 1 \end{bmatrix} ds \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \int_0^h \begin{bmatrix} e^{-s} \\ 1 - e^{-s} \end{bmatrix} ds = \begin{bmatrix} 1 - e^{-h} \\ h - 1 + e^{-h} \end{bmatrix}$$

- Hence, the corresponding discrete time model becomes:

$$\begin{cases} \mathbf{x}(kh+h) = \begin{bmatrix} e^{-h} & 0 \\ 1 - e^{-h} & 1 \end{bmatrix} \mathbf{x}(kh) + \begin{bmatrix} 1 - e^{-h} \\ h - 1 + e^{-h} \end{bmatrix} u(kh) \\ y(kh) = \begin{bmatrix} 0 & 1 \end{bmatrix} \mathbf{x}(kh) \end{cases}$$

How do we solve the discrete-time state-space system?

Input/Output (I/O) models

- The state-space representation of the discrete-time system is given by:

$$
\begin{aligned}
\text{System model:} \qquad & \mathbf{x}[k+1] = \Phi \mathbf{x}[k] + \Gamma \mathbf{u}[k], \quad x[k_0] = x_0 \\
\text{Observation model:} \qquad & \mathbf{y}[k] = C\mathbf{x}[k] + D\mathbf{u}[k]
\end{aligned}
$$

- Solution by direct recursive calculus:

$$
\begin{aligned}
\mathbf{x}[k_0+1] &= \Phi \mathbf{x}[k_0] + \Gamma \mathbf{u}[k_0] \\
\mathbf{x}[k_0+2] &= \Phi \mathbf{x}[k_0+1] + \Gamma \mathbf{u}[k_0+1] \\
&= \Phi^2 \mathbf{x}[k_0] + \Phi\Gamma \mathbf{u}[k_0] + \Gamma \mathbf{u}[k_0+1] \\
&\vdots
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{x}[k] &= \Phi^{k-k_0} \mathbf{x}[k_0] + \Phi^{k-k_0-1}\Gamma \mathbf{u}[k_0] + \ldots + \Gamma \mathbf{u}[k-1] \\
&= \Phi^{k-k_0} \mathbf{x}[k_0] + \sum_{j=k_0}^{k-1} \Phi^{k-j-1}\Gamma \mathbf{u}[j]
\end{aligned}
$$

How do we solve the discrete-time state-space system?

Input/Output (I/O) models

- The state-space representation of the discrete-time system is now:

$$\mathbf{x}[k] = \Phi^{k-k_0}\mathbf{x}[k_0] + \sum_{j=k_0}^{k-1} \Phi^{k-j-1}\Gamma\mathbf{u}[j]$$

$$\mathbf{y}[k] = C\left(\Phi^{k-k_0}\mathbf{x}[k_0] + \sum_{j=k_0}^{k-1} \Phi^{k-j-1}\Gamma\mathbf{u}[j] \right) + D\mathbf{u}[k]$$

Pulse response:

- Assume $k_0 = 0$. Then,

$$\mathbf{y}[k] = C\Phi^{k}\mathbf{x}[0] + \sum_{j=0}^{k-1} C\Phi^{k-j-1}\Gamma\mathbf{u}[j] + D\mathbf{u}[k]$$

$$\mathbf{h}[k] = \begin{cases} \mathbf{0}, & k < 0 \\ D, & k = 0 \\ C\Phi^{k-1}\Gamma, & k \geq 1 \end{cases}$$

Recall that:

$$\mathbf{y}[k] = \sum_{j=0}^{k} \underbrace{\mathbf{h}[k-j]}_{\text{pulse response}}\mathbf{u}[j] + \underbrace{\mathbf{y}_p[k]}_{\text{Initial conditions}}$$

Transfer function of a state-space model

- The state-space representation of the discrete-time system is given by:

$$
\begin{aligned}
\text{System model:} \qquad & \mathbf{x}[k+1] = \Phi\mathbf{x}[k] + \Gamma\mathbf{u}[k], \quad x[k_0] = x_0 \\
\text{Observation model:} \qquad & \mathbf{y}[k] = C\mathbf{x}[k] + D\mathbf{u}[k]
\end{aligned}
$$

- We want to find the transfer function ***G(z)*** for this model.

- First, note that:

$$
\mathcal{Z}\{\mathbf{x}[k]\} = \mathcal{Z}\left\{\begin{bmatrix} x_1[k] \\ x_2[k] \\ \vdots \\ x_n[k] \end{bmatrix}\right\} = \begin{bmatrix} X_1(z) \\ X_2(z) \\ \vdots \\ X_n(z) \end{bmatrix} = \mathbf{X}(z)
$$

Transfer function of a state-space model

- Taking the z-transform of the system model:

$$z\mathbf{X}(z) - z\mathbf{x}[0] = \Phi\mathbf{X}(z) + \Gamma\mathbf{U}(z)$$

- Then, the state vector is given by:

$$\mathbf{X}(z) = (zI - \Phi)^{-1} z\mathbf{x}[0] + (zI - \Phi)^{-1}\Gamma\mathbf{U}(z)$$

- The output (observation model) is given by

$$\begin{aligned}
\mathbf{Y}(z) &= C\mathbf{X}(z) + D\mathbf{U}(z) \\
&= C\left[(zI - \Phi)^{-1} z\mathbf{x}[0] + (zI - \Phi)^{-1}\Gamma\mathbf{U}(z)\right] + D\mathbf{U}(z) \\
&= C(zI - \Phi)^{-1} z\mathbf{x}[0] + \left[C(zI - \Phi)^{-1}\Gamma + D\right]\mathbf{U}(z)
\end{aligned}$$

free response transfer function

In-class exercise

- Consider the linear system given by the following state-space representation

$$\begin{aligned} x[k+1] &= 0.5x[k] + 0.5u[k] \\ y[k] &= 2x[k] \end{aligned}$$

  Find its transfer function.

In-class exercise

- Consider the linear system given by the following state-space representation

$$x[k+1] = 0.5x[k] + 0.5u[k]$$
$$y[k] = 2x[k]$$

Find its transfer function.

Solution:

In the general case, the transfer function is given by

$$\mathbf{G}(z) = C(zI - \Phi)^{-1}\Gamma + D$$

But here we have a scalar system. Hence:

$$G(z) = C(z - \Phi)^{-1}\Gamma + D = \frac{2(0.5)}{z - 0.5} = \frac{1}{z - 0.5}$$

Transfer function of a state-space model

- The transfer function can be written as:

$$\begin{aligned}\mathbf{G}(z) &= C(zI-\Phi)^{-1}\Gamma + D \\ &= \frac{C\,\mathrm{adj}(zI-\Phi)\Gamma + \det(zI-\Phi)D}{\det(zI-\Phi)}\end{aligned}$$

where

$\mathrm{adj}(A)$: adjoint of matrix A

$\det(A)$: determinant of matrix A

- The denominator of ***G(z)*** is the determinant $\det(zI-\Phi)$

⇒ poles = roots of the determinant = eigenvalues of Φ (the system matrix)

$$\det(zI-\Phi) = |zI-\Phi| \triangleq \chi(z)$$ (Characteristic polynomial of the system)

$$A^{-1} = \frac{1}{\det(A)}\mathrm{adj}(A)$$ (inverse of a square matrix)

Example

- The state-space representation of the continuous-time system is:

$$\begin{cases} \dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 0 \\ 1 & -0.1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0.1 \\ 0 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \mathbf{x}(t) \end{cases}$$

- Find the discrete-time transfer function using MATLAB for sampling period of T_s=*0.2*

Solution:

```
>> F= [0 0;1 -0.1]; G= [0.1; 0]; C = [0 1]; D = 0; Ts = 0.2;
>> sys = ss (F,G,C,D);
>> sysd = c2d(sys,Ts,'zoh');
>> H= tf(sysd)
H =

  0.001987 z + 0.001974
  ---------------------
  z^2 - 1.98 z + 0.9802
```

Mapping of poles

Continuous-time system:

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t)$$
$$\mathbf{y}(t) = C\mathbf{x}(t) + D\mathbf{u}(t)$$

Poles: $\lambda_i(A), \quad i = 1, \ldots, n$

Discrete-time system:

$$\mathbf{x}[k+1] = \Phi\mathbf{x}[k] + \Gamma\mathbf{u}[k]$$
$$\mathbf{y}[k] = C\mathbf{x}[k] + D\mathbf{u}[k]$$

Poles: $\lambda_i(\Phi), \quad i = 1, \ldots, n$

$$\boxed{\Phi = e^{Ah} \Rightarrow \lambda_i(\Phi) = e^{\lambda_i(A)h}}$$

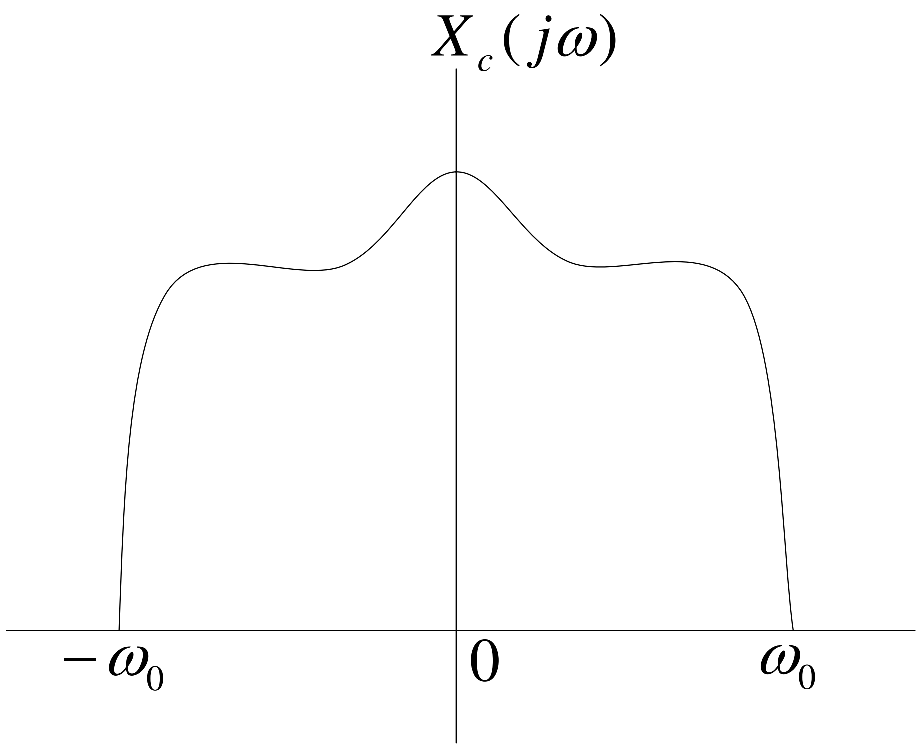
- Interpretation: Let $\lambda_i(A) = -\sigma_i + j\omega_i, \quad \sigma_i > 0$. Then,

$$\lambda_i(\Phi) = e^{(-\sigma_i + j\omega_i)h} = e^{-\sigma_i h} e^{j\omega_i h} \Rightarrow |\lambda_i(\Phi)| = e^{-\sigma_i h}|e^{j\omega_i h}| = e^{-\sigma_i h} < 1$$

- Therefore, **stability of the system is preserved**!

Recall: Sampling criterion/theorem

- Suppose $x_c(t)$ is a low-pass signal with $X_c(j\omega) = 0, \forall\ |\omega| > \omega_0$, e.g.,



- Then, $x_c(t)$ can be uniquely determined by its samples $x_c(nT_s), n = 0, \pm 1, \pm 2, \ldots$ if the sampling angular frequency is at least twice as big as ω_0, i.e.,

$$\omega_s = \frac{2\pi}{T_s} > 2\omega_0$$

- The minimum sampling angular frequency, for which the inequality holds, is called the *Nyquist angular frequency*

Mapping of poles

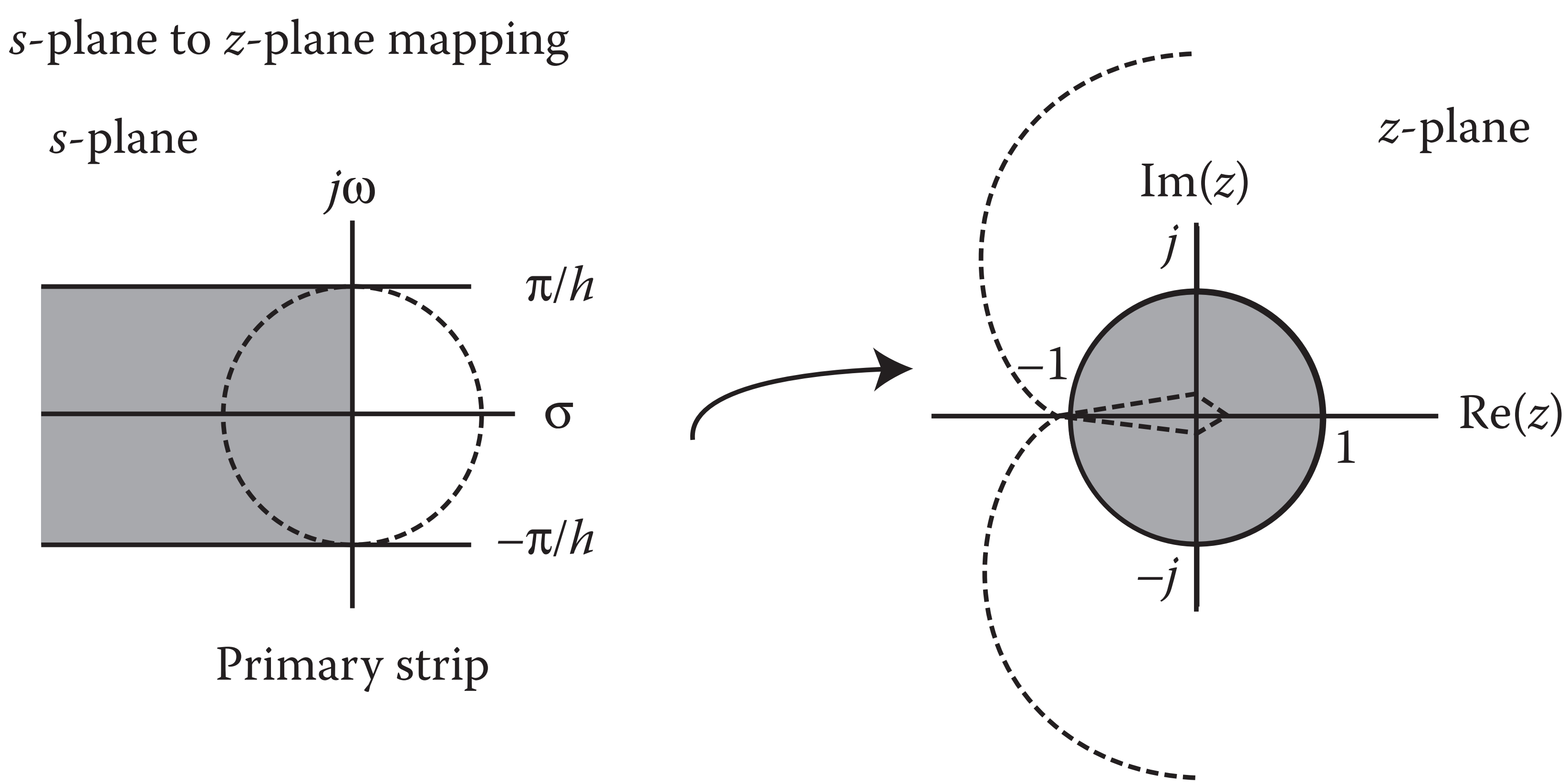
- Interpretation: Let $\lambda_i(A) = -\sigma_i + j\omega_i, \quad \sigma_i > 0$. Then,

$$\lambda_i(\Phi) = e^{(-\sigma_i + j\omega_i)h} = e^{-\sigma_i h} e^{j\omega_i h}$$

$$\boxed{h = T}$$

- To avoid aliasing (**why?**),

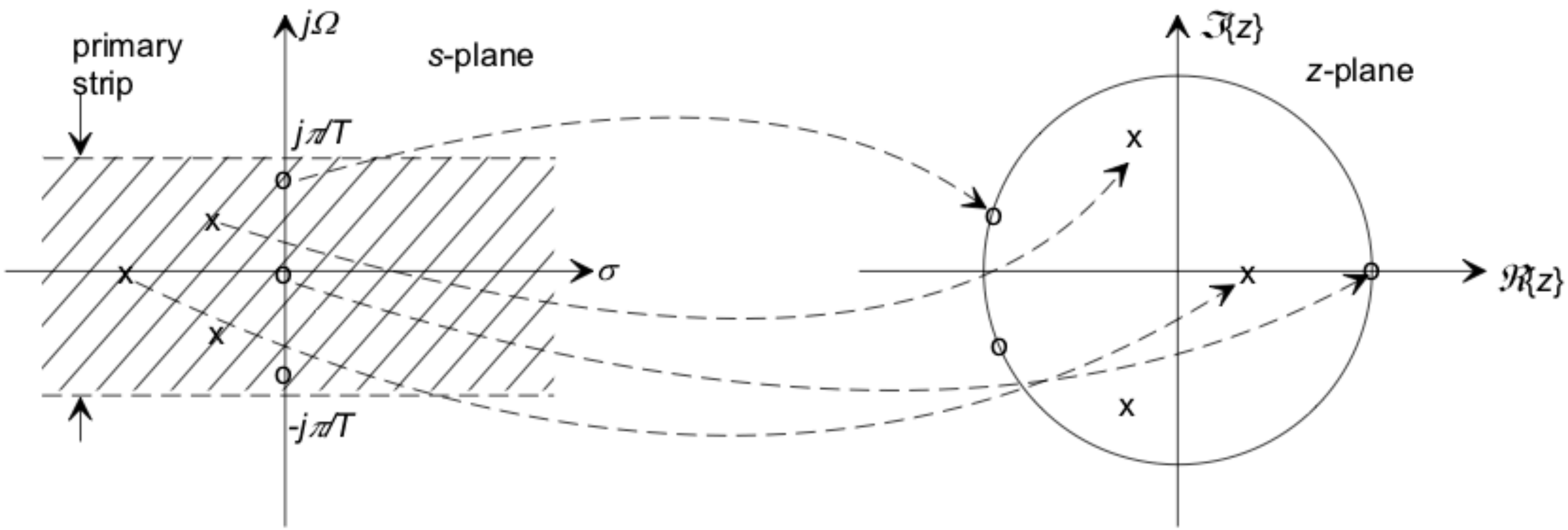
$$\frac{2\pi}{h} > 2\omega_i \Rightarrow \omega_i h < \pi$$



Mapping of poles demystified

- In discrete time systems, frequency response is repeated every *2π* steps

- From *π* to *2π*, the frequency response is the reflection of that from *0* to *π*.



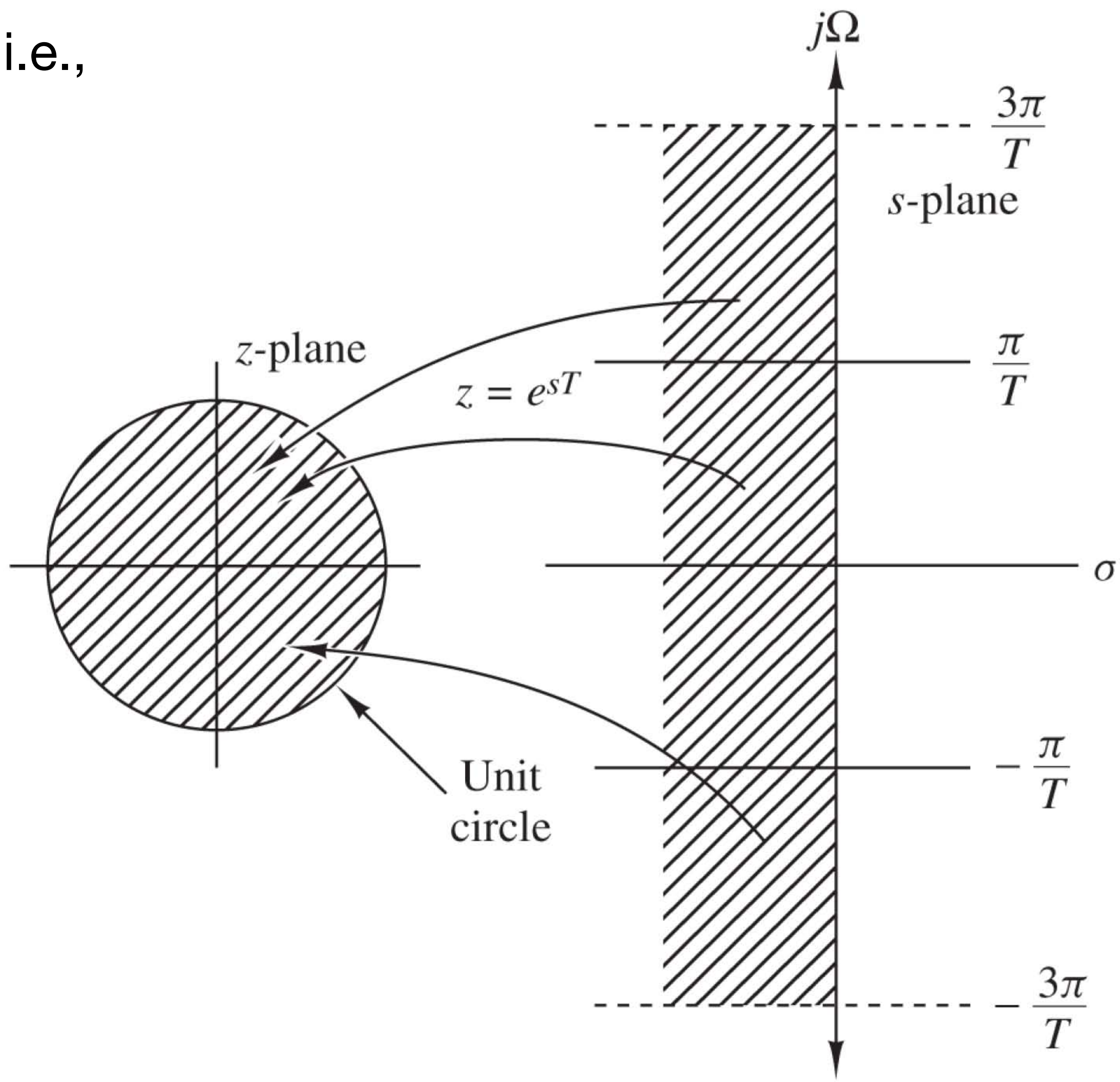
- If the imaginary part of a pole of the continuous-time system is bigger than *π/h* then the frequency response has a peak at a higher frequency than the cut-off frequency in the discrete-time domain.

Mapping of poles demystified

- Mapping introduces aliasing, i.e.,

$$s \text{ and } s + j2\pi k/T$$

  map to the same z.



Proof

$$\boxed{\Phi = e^{Ah} \Rightarrow \lambda_i(\Phi) = e^{\lambda_i(A)h}}$$

- Before proving the statement above, we need some theorems.

- **The Cayley-Hamilton Theorem:** Let

$$\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ldots + a_n = 0$$

be the characteristic polynomial of square matrix M. Then M satisfies

$$M^n + a_1 M^{n-1} + a_2 M^{n-2} + \ldots + a_n I = 0$$

- Proof: $M^k = V\Lambda^k V^{-1} \Rightarrow \chi(M) = V\chi(\Lambda)V^{-1} = V\mathrm{diag}_i(\chi(\lambda_i))V^{-1} = \mathbf{0}$

- **Eigenvalues of a matrix function:** If *f(M)* is a polynomial in M and $\boldsymbol{v}_i$ is the eigenvector of M associated with eigenvalue λ_i, then

$$f(M)\mathbf{v}_i = f(\lambda_i)\mathbf{v}_i$$

Proof

$$\boxed{\Phi = e^{Ah} \Rightarrow \lambda_i(\Phi) = e^{\lambda_i(A)h}}$$

- We already know that

$$\Phi = f(A) = e^{Ah} = I + hA + \frac{1}{2}h^2A^2 + \frac{1}{6}h^3A^3 + \ldots = \sum_{n=0}^{\infty} \frac{1}{n!}h^nA^n$$

- Hence

$$\begin{aligned}
\Phi\mathbf{v}_i &= f(A)\mathbf{v}_i = [I + hA + \frac{1}{2}h^2A^2 + \frac{1}{6}h^3A^3 + \ldots]\mathbf{v}_i \\
&= I\mathbf{v}_i + h\lambda_i(A)\mathbf{v}_i + \frac{1}{2}h^2\lambda_i^2(A)\mathbf{v}_i + \frac{1}{6}h^3\lambda_i^3(A)\mathbf{v}_i + \ldots \\
&= [1 + h\lambda_i(A) + \frac{1}{2}h^2\lambda_i^2(A) + \ldots]\mathbf{v}_i \\
&= f(\lambda_i(A))\mathbf{v}_i = \underbrace{e^{\lambda_i(A)h}}_{\lambda_i(\Phi)}\mathbf{v}_i
\end{aligned}$$

Mapping of zeros

- A simple relationship for the mapping of zeros does not hold. Even the number of zeros does not necessarily remain invariant. The mapping of zeros is a complicated issue.

- A *continuous-time* system is *non-minimum phase*[1], if it has zeros on the right half plane (RHP) or if it contains a delay.

- A *discrete-time* system has an unstable inverse[2], if it has zeros outside the unit circle.

- Zeros are not mapped in a similar way as poles, so a minimum phase continuous system may have a discrete counterpart with an unstable inverse and a non-minimum phase continuous system may have a discrete counterpart with a stable inverse.

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[1] *A LTI system is minimum-phase if the system and its inverse are causal and stable.*

[2] *A system is invertible if we can uniquely determine its input from its output.*

In-class exercise

- The considered continuous-time process has first-order dynamics and is given by

$$\tau \dot{y}(t) + y(t) = u(t)$$

  Discretize the process with a sampling time of h and assuming that the control signal $u(t)$ is piecewise constant between the sampling instants (ZOH). Use

  i) discretization of the state-space model

  ii) discretization using the step-invariance method (what we learned in the previous lectures)

  Compare the discretized models.

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Solution:

i) First, we write the system in state-space form:

$$\begin{cases} \dot{x}(t) = -\frac{1}{\tau}x(t) + \frac{1}{\tau}u(t) \\ y(t) = x(t) \end{cases}$$

In-class exercise

Next, we compute Φ and Γ:

$$\Phi(h) = e^{Ah} = e^{-\frac{h}{\tau}}$$

$$\Gamma(h) = \int_0^h e^{As} ds B = \int_0^h e^{-\frac{s}{\tau}} ds \frac{1}{\tau} = \frac{1}{\tau}\left[\frac{1}{-\frac{1}{\tau}} e^{-\frac{s}{\tau}}\right]_0^h = 1 - e^{-\frac{h}{\tau}}$$

Therefore:

$$x[k+1] = e^{-\frac{h}{\tau}} x[k] + \frac{1}{2}\left(1 - e^{-\frac{h}{\tau}}\right) u[k]$$

$$y[k] = x[k]$$

The transfer function is thus

$$G(z) = C(zI - \Phi)^{-1}\Gamma + D = \frac{1 - e^{-\frac{h}{\tau}}}{z - e^{-\frac{h}{\tau}}}$$

In-class exercise

ii) Discretization via the step-invariance method:

$$G(z) = \frac{z-1}{z}\mathcal{Z}\left(\mathcal{L}^{-1}\left(\frac{G(s)}{s}\right)_{t=kh}\right)$$

$$= \frac{z-1}{z}\mathcal{Z}\left(\mathcal{L}^{-1}\left(\frac{1}{s(\tau s+1)}\right)_{t=kh}\right)$$

$$= \frac{z-1}{z}\mathcal{Z}\left(1 - e^{-\frac{t}{\tau}}\Big|_{t=kh}\right) = \frac{z-1}{z}\mathcal{Z}\left(1 - \left(e^{-\frac{h}{\tau}}\right)^k\right)$$

$$= \frac{z-1}{z}\left(\frac{z}{z-1} - \frac{z}{z-e^{-\frac{h}{\tau}}}\right) = 1 - \frac{z-1}{z-e^{-\frac{h}{\tau}}}$$

$$= \frac{1-e^{-\frac{h}{\tau}}}{z-e^{-\frac{h}{\tau}}}$$

Both discretization methods give the same result!