Digital Control Systems - Chapter 2 Notes

Discrete-Time Systems & z-Transform

2.1 Introduction

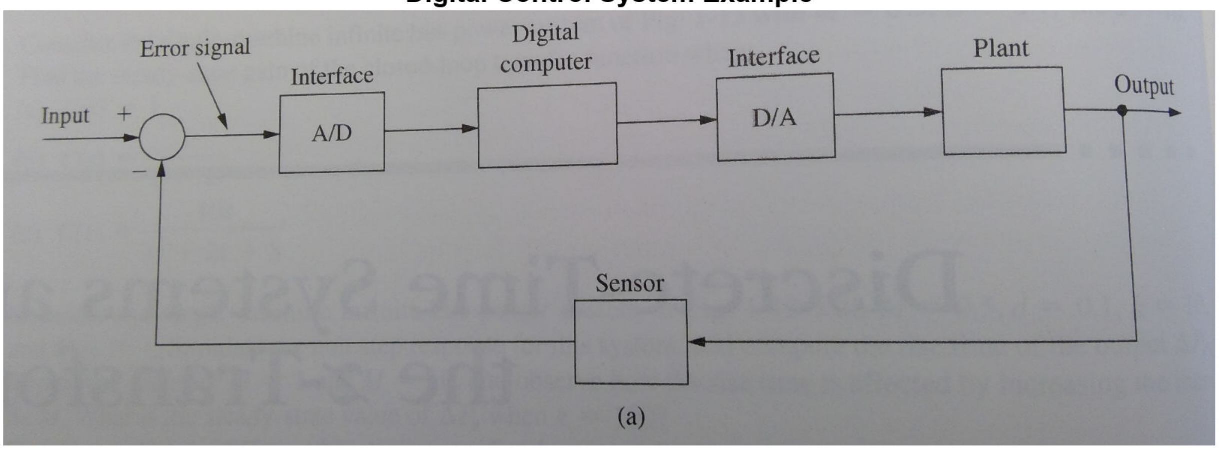
- Discrete-Time System is one whose operations is described (modeled) by a set of **difference equations**
- The transform used in the analysis of linear time-invariant discrete-time systems is the z**-transform**

2.2 Discrete-Time Systems

Examine an example of a Digital Control System

```
f = imread('DigitalControlSystem.png'); imshow(f)
title('Digital Control System Example')
```

Digital Control System Example



- Digital Computer performs the comensation function within the system
- A/D converter converts error signal from continuous-time, into a form that's processed by computer
- D/A converter at output to convert binary signals of computer into form necessary to drive the Plant

Consider example where A/D converter, digital computer, and D/A converter are replaced with CT proportional-integral controller

$$m(t) = K_p e(t) + K_I \int_0^t e(\tau) d\tau$$

where $e(t)$ is controller input, $m(t)$ is controller output signal, and K_p and K_I are constant gains ndetermined by design process

- Controller can be realized using digital computer because it can multiply, add, and integrate numerically

Can use Rectangular rule → the area under a curve is approximated by the sum of the rectangular areas underneath it

The numerical integral of $e(t)$ can be written

$$x(kT) = x[(k-1)T] + Te(kT)$$

where T is the numerical algorithm step size, in seconds. Then the digital compensator becomes

$$m(kT) = K_p e(kT) + K_I x(kT)$$

The general form for a *first-order difference equation* can be formed from the equations above omitting T for convenience

$$x(k) = b_1 e(k) + b_0 e(k-1) - a_0 x(k-1)$$

This equation is first-order since the signals from only the last sampling instant appear explicitly in the equation

The general form on an nth-order linear difference equation is

$$x(k) = b_n e(k) + b_{n-1} e(k-1) + ... + b_0 e(k-n) - a_{n-1} x(k-1) - ... - a_0 x(k-n)$$

Two approaches for design of digital compensators

1. Analog compensator may be designed and then converted by some approxiamte procedure to a digital compensator (example above)
2. Design a device that realizes a digital filter

Problems of the Control System Designer:

1. Choosing T, the sampling period
2. Choosing n, the order of the difference equation
3. Choosing $a_i \text{ and } b_i$, the filter coefficients for the digital filter

2.3 Transform Methods

The function $E(z)$ is defined as a power series in z^{-k} with coefficients equal to the values of the number sequences $\{e(k)\}$

- $E(z) = z[e(k)] = e(0) + e(1)z^{-1} + e(2)z^{-1} + \ \ldots$
- $e(k) = z^{-1}\{E(z)\} = \dfrac{1}{2\pi j} \oint E(z) z^{k-1} \mathrm{dz},\ j = \sqrt{-1}$

Compact form of single-sided z-transform

- $E(z) = z\{e(k)\} = \displaystyle\sum_{k=0}^{\infty} e(k) z^{-k}$

If the sequence $e(k)$ is generated from a time function $e(t)$ by sampling every Tseconds, $e(k)$ is understood to be $e(\mathrm{kT})$

Example 2.2

Given that $e(k) = 1$ for all k, find $E(z)$.

$$E(z) = \frac{1}{1 - z^{-1}} = \frac{z}{z - 1}, \qquad |z^{-1}| < 1$$

```
syms k
ek = 1^k;
Ez = ztrans(ek)
```

```
Ez =
```

$$\frac{z}{z - 1}$$

```
syms k T
ek = 1^(k*T);
Ez = ztrans(ek)
```

```
Ez =
```

$$\frac{z}{z - 1}$$

Example 2.3

Given that $e(k) = \epsilon^{-\mathrm{akT}}$, find $E(z)$, which can be written as a power series form as

$$E(z) = 1 + \epsilon^{-\mathrm{aT}} z^{-1} + \epsilon^{-2\mathrm{aT}} z^{-2} + \ \ldots$$

$$= 1 + (\epsilon^{-\mathrm{aT}} z^{-1}) + (\epsilon^{-\mathrm{aT}} z^{-1})^2 + \ \ldots$$

$$E(z) = \frac{1}{1 - \epsilon^{-\mathrm{aT}} z^{-1}} = \frac{z}{z - \epsilon^{-\mathrm{aT}}}, \qquad |\epsilon^{-\mathrm{aT}} z^{-1}| < 1$$

```
syms k a T
ek = exp(-a*k*T)
```

ek = $\mathrm{e}^{-T\,a\,k}$

```
Ez = ztrans(ek)
```

```
Ez =
```

$$\frac{z}{z - \mathrm{e}^{-T\,a}}$$

2.4 Properties of *z*-Transform

Several properties of the z-Transform will now be developed

Addition and Subtraction

$z[e_1(k) \pm e_2(k)] = E_1(k) \pm E_2(k)$

Multiplication by a Constant - Linearity Property

$z[\mathrm{ae}(k)] = \mathrm{az}[e(k)] = \mathrm{aE}(z)$

Real Translation

Complex Translation

Initial Value

$e(0) = \lim_{z\to\infty} E(z)$

Since $E(z) = e(0) + e(1)z^{-1} + e(2)z^{-2} + \ \ldots$

$= e(0) + 0 + 0 + 0 + \ \ldots$

Thus, $e(0) = \lim_{z\to\infty} E(z)$

Final Value

$\lim_{n\to\infty} e(n) = \lim_{z\to 1}(z-1)E(z)$

Example 2.7

To illustrate the initial-value property and the final-value property, consider the z-transform of $e(k) = 1, k = 0, 1, 2, \ldots$

$$E(z) = z\{1\} = \frac{z}{z-1}$$

Applying the initial-value property, we see that

$$e(0) = \lim_{z\to\infty} \frac{z}{z-1} = \lim_{z\to\infty} \frac{1}{1-1/z} = \frac{1}{1-1/\infty} = 1$$

Since the final value of $e(k)$ exists, we may apply the final-value property:

$$\lim_{k\to\infty} e(k) = \lim_{z\to 1}(z-1)E(z) = \lim_{z\to 1}(z-1)\left(\frac{z}{z-1}\right) = \lim_{z\to 1} z = 1$$

2.5 Finding *z*-Transforms

TABLE 2-2 Properties of the z-Transform

Sequence	Transform
$e(k)$	$E(z) = \sum_{k=0}^{\infty} e(k)z^{-k}$
$a_1e_1(k) + a_2e_2(k)$	$a_1E_1(z) + a_2E_2(z)$
$e(k-n)u(k-n);\quad n \geq 0$	$z^{-n}E(z)$
$e(k+n)u(k);\quad n \geq 1$	$z^n\left[E(z) - \sum_{k=0}^{n-1} e(k)z^{-k}\right]$
$\varepsilon^{akT}e(k)$	$E(z\varepsilon^{-aT})$
$ke(k)$	$-z\dfrac{dE(z)}{dz}$
$e_1(k) * e_2(k)$	$E_1(z)E_2(z)$
$e_1(k) = \sum_{n=0}^{k} e(n)$	$E_1(z) = \dfrac{z}{z-1}E(z)$

Initial value: $e(0) = \lim_{z\to\infty} E(z)$

Final value: $e(\infty) = \lim_{z\to 1}(z-1)E(z)$, if $e(\infty)$ exists

TABLE 2-3 z-Transforms

Sequence	Transform
$\delta(k-n)$	z^{-n}
1	$\dfrac{z}{z-1}$
k	$\dfrac{z}{(z-1)^2}$
k^2	$\dfrac{z(z+1)}{(z-1)^3}$
a^k	$\dfrac{z}{z-a}$
ka^k	$\dfrac{az}{(z-a)^2}$
$\sin ak$	$\dfrac{z\sin a}{z^2 - 2z\cos a + 1}$
$\cos ak$	$\dfrac{z(z-\cos a)}{z^2 - 2z\cos a + 1}$
$a^k \sin bk$	$\dfrac{az\sin b}{z^2 - 2az\cos b + a^2}$
$a^k \cos bk$	$\dfrac{z^2 - az\cos b}{z^2 - 2az\cos b + a^2}$

Example 2.8

```
syms a b k
ek = a^k*cos(b*k)
```

ek = $a^k \cos(b\,k)$

```
Ez = ztrans(ek)
```

Ez =

$$-\frac{z\left(\cos(b)-\frac{z}{a}\right)}{a\left(\frac{z^2}{a^2}-\frac{2\,z\cos(b)}{a}+1\right)}$$

```
pretty(Ez)
```

```
        /           z \
      z | cos(b) -  - |
        \           a /
- -------------------------
     /  2                  \
     | z    2 z cos(b)     |
   a | -- - ---------- + 1 |
     |  2        a         |
     \ a                   /
```

```
syms a b k T
Ez = ztrans(a^(k*T)*cos(b*k*T))
```

Ez =

$$-\frac{z\left(\cos(T\,b)-\frac{z}{a^T}\right)}{a^T\left(\frac{z^2}{a^{2\,T}}-\frac{2\,z\cos(T\,b)}{a^T}+1\right)}$$

```
pretty(Ez)
```

```
          /             z \
        z | cos(T b) - -- |
          |             T |
          \            a  /
- --------------------------------
   T /   1   2   2 z cos(T b)     \
  a  | ---- z  - ------------ + 1 |
     |  2 T            T          |
     \ a              a           /
```

Thus the relationship is

$$z\{a^{\mathrm{kT}}\cos(\mathrm{bkT})\}=\frac{z^2-a^Tz\cos\mathrm{bT}}{z^2-2a^Tz\cos\mathrm{bT}+a^{2T}}$$

Example 2.9

Find z-transform of $E(s)=\frac{s^2+4s+3}{s^3+6s^2+8s}$

First find partial fraction expansion

```
b = [1 4 3]; a = [1 6 8 0];
[R,P,K] = residue(b,a)
```

```
R = 3×1
    0.3750
    0.2500
    0.3750
P = 3×1
    -4
    -2
     0
K =

     []
```

$$E(s) = \frac{0.375}{s+4} + \frac{0.25}{s+2} + \frac{0.375}{s}$$

Next find the z-transform of each term in the expansion:

$$E(z) = 0.375\ \frac{z}{z-1} + 0.25\frac{z}{z-e^{-2T}} + 0.375\ \frac{z}{z-e^{-4T}}$$

For $T = 0.1,$ we cna express $E(z)$ as a ratio of two polynomials:

$$E(z) = 0.375\ \frac{z}{z-1} + 0.25\frac{z}{z-0.8187} + 0.375\ \frac{z}{z-0.6703}$$

$$= \frac{z^3 - 1.658z^2 + 0.6804z}{z^3 - 2.489z^2 + 2.038z - 0.5488}$$

```
T = 0.1;
b = [1 4 3]; a = [1 6 8 0];
n = length(a);
Es = tf(b,a)
```

```
Es =

    s^2 + 4 s + 3
  -----------------
  s^3 + 6 s^2 + 8 s

Continuous-time transfer function.
```

```
[R,P,K] = residue(b,a)
```

```
R = 3×1
    0.3750
    0.2500
    0.3750
P = 3×1
    -4
    -2
     0
```

```
K =

     []
```

```
for i=1:n-1
    pz(i) = exp(P(i)*T);
end
[numz,denomz] = residue(R,pz,K)
```

```
numz = 1×3
    1.0000   -1.6580    0.6804
denomz = 1×4
    1.0000   -2.4891    2.0379   -0.5488
```

```
numz=conv(numz,[1 0]);
Ez = tf(numz,denomz,T)
```

```
Ez =

      z^3 - 1.658 z^2 + 0.6804 z
  ----------------------------------
  z^3 - 2.489 z^2 + 2.038 z - 0.5488

Sample time: 0.1 seconds
Discrete-time transfer function.
```

Alternative Method using *ztrans* function

```
T = 0.1;
syms s t k z
Es = (s^2+4*s+3)/(s^3+6*s^2+8*s)
```

Es =

$$\frac{s^2+4\,s+3}{s^3+6\,s^2+8\,s}$$

```
et = ilaplace(Es)
```

et =

$$\frac{e^{-2\,t}}{4}+\frac{3\,e^{-4\,t}}{8}+\frac{3}{8}$$

```
ekT = subs(et,t,k*T)
```

ekT =

$$\frac{e^{-\frac{k}{5}}}{4}+\frac{3\,e^{-\frac{2\,k}{5}}}{8}+\frac{3}{8}$$

```
Ez = ztrans(ekT)
```

Ez =

$$\frac{3\,z}{8\,(z-1)}+\frac{z}{4\,\left(z-e^{-\frac{1}{5}}\right)}+\frac{3\,z}{8\,\left(z-e^{-\frac{2}{5}}\right)}$$

```
pretty(Ez)
```

```
   3 z                z                     3 z
--------- + -------------------- + --------------------
8 (z - 1)     /         /   1 \ \     /         /   2 \ \
            4 | z - exp| - - | |   8 | z - exp| - - | |
              \         \   5 / /     \         \   5 / /
```

2.6 Solution of Difference Equations

2.7 The Inverse *z*-Transform

- **Power Series Method - Divide Numerator by Denominator**
- **Partial-Fraction Expansion - Real and Complex Poles**
- **Inversion-Formula Method**

$$[\text{residue}]_{z=a} = \frac{1}{(m-1)!}\frac{d^{m-1}}{\mathrm{dz}^{m-1}}[(z-a)^m E(z) z^{k-1}]|_{z=a}$$

Example 2.15

Consider the function $E(z)$ from Examples 2.12 and 2.13

$$E(z) = \frac{z}{(z-1)(z-2)}$$

Substitute equation for residues

$$e(k) = \frac{z^k}{z-2}|_{z=1} + \frac{z^k}{z-1}|_{z=2} = -1 + 2^k$$

2.8 Simulation Diagrams and Flow Graphs

2.9 State Variables

Example 2.18

It is desired to find a state-variable model of the system described by the difference equation

$$y(k+2) = u(k) + 1.7y(k+1) - 0.72y(k)$$

Let

$$x_1(k) = y(k)$$
$$x_2(k) = x_1(k+1) = y(k+1)$$

Then

$$x_2(k+1) = y(k+2) = u(k) + 1.7x_2(k) - 0.72x_1(k)$$

or, from these equations, we write

$x_1(k+1) = x_2(k)$

$x_2(k+1) = -0.72x_1(k) + 1.7x_2(k) + u(k)$

$y(k) = x_1(k)$

We may express these equations in vector matrix form of (2-49) and (2-50) as

$$\mathbf{x}(k+1) = \begin{bmatrix} 0 & 1 \\ -0.72 & 1.7 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ -1 \end{bmatrix} u(k)$$

$y(k) = [1 \ \ 0]\, \mathbf{x}(k)$

2.10 Other State-Variable Formulations

2.11 Transfer Functions

2.12 Solutions of State Equations

2.13 Linear Time-Varying Systems