# HW #3

Solution 1:

(a) $e(k) = \sum_{\text{residues}} \dfrac{0.1z^{k-1}}{z(z-0.9)} = \sum_{\text{residues}} \dfrac{0.1z^{k-2}}{z-0.9}$

$k=0: \text{fcn} = \dfrac{0.1}{z^2(z-0.9)}, \quad \therefore \text{residue}\Big|_{z=0.9} = \dfrac{0.1}{(0.9)^2} = 0.1235$

$\text{residue}\Big|_{z=0} = \dfrac{d}{dz}\left[\dfrac{0.1}{z-0.9}\right]_{z=0} = \left.\dfrac{-0.1(1)}{(z-0.9)^2}\right|_{z=0} = \dfrac{-0.1}{(0.9)^2} = -0.1235$

$\therefore e(0) = 0$

$k=1: e(1) = \left.\dfrac{0.1}{z-0.9}\right|_{z=0} + \left.\dfrac{0.1}{z}\right|_{z=0.9} = 0$

$k=10: \; e(10) = 0.1(0.9)^8$

(b) $e(0) = \lim_{z\to\infty} E(z) = \lim_{z\to\infty} \dfrac{0.1}{z(z-0.9)} = 0$

(c) $\dfrac{E(z)}{z} = \dfrac{0.1}{z^2(z-0.9)} = \dfrac{k_1}{z^2} + \dfrac{k_2}{z} + \dfrac{k_3}{z-0.9}$

$k_1 = \dfrac{-0.1}{0.9} = -\dfrac{1}{9}; \; k_3 \dfrac{0.1}{(0.9)^2} = \dfrac{1}{8.1}$

$k_2 = \dfrac{d}{dz}\left[\dfrac{0.1}{z-0.9}\right]_{z=0} = \dfrac{-1}{8.1}$, from (a)

$$\therefore e(k) = \frac{-1}{8.1}\delta(k) - \frac{1}{9}\delta(k-1) + \frac{1}{8.1}(0.9)^k$$

$$x(0) = -\frac{1}{8.1} + 0 + \frac{1}{8.1} = 0; \;\; x(1) = -0 - \frac{1}{9} + \frac{0.9}{8.1} = 0$$

$$x(10) = -0 - 0 + \frac{0.1}{(0.9)^2}(0.9)^{10} = 0.1(0.9)^8$$

(d) $E(z) = \dfrac{1.98z}{z^5 + \cdots} = 1.98z^{-4} + (\cdot)z^{-5} + (\cdot)z^{-6} + \cdots$

$$\therefore e(0) = e(1) = e(2) = e(3) = 0; \;\; e(4) = 1.98$$

(e) $E(z) = \dfrac{2z}{z - 0.8} = \dfrac{2z}{z - \varepsilon^{-aT}} \quad \therefore \varepsilon^{-aT} = 0.8 \Rightarrow aT = 0.2231$

$$\therefore a = \frac{0.2231}{0.1} = 2.231, \quad \therefore e(t) = 2\varepsilon^{-2.231t}u(t)$$

(f) $E(z) = \dfrac{2z}{z - (-0.8)}; \; \therefore \varepsilon^{-aT}\varepsilon^{j\pi} = -0.8 \Rightarrow aT = 2.231$

$$\therefore e(t) = 2e^{-2.231t}\cos 10\pi t \;\; \text{where} \;\; \frac{\omega_s}{2} = 10\pi$$

(g) (e) $e(k) = (0.8)^k$; (f) $e(k) = (-0.8)^k$

$\therefore$ sign alternates on $e(k)$.

Solution 2:

(a) $z\mathbf{I}-\mathbf{A}=\begin{bmatrix} z & -1 \\ 0 & z-3 \end{bmatrix}$; $\Delta = |z\mathbf{I}-\mathbf{A}| = z(z-3) = \Delta$

$$\frac{Y(z)}{U(z)} = \mathbf{C}[z\mathbf{I}-\mathbf{A}]^{-1}\mathbf{B} = \frac{1}{\Delta}[-2 \quad 1]\begin{bmatrix} z-3 & 1 \\ 0 & z \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{\Delta}[-2z+6 \quad z-2]\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{-z+4}{z(z-3)}$$

(b) $\mathbf{P} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$; $\mathbf{P}^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$

$$\mathbf{A}_w = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$$

$$\mathbf{B}_w = \mathbf{P}^{-1}\mathbf{B} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\mathbf{C}_w = \mathbf{C}\mathbf{P} = [-2 \quad 1] \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = [-1 \quad 3]$$

$$\therefore \mathbf{w}(k+1) = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \mathbf{w}(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mathbf{u}(k)$$

$$\mathbf{y}(k) = [-1 \quad 3]\mathbf{w}(k)$$

(c) $z\mathbf{I} - \mathbf{A}_w = \begin{bmatrix} z-2 & -2 \\ -1 & z-1 \end{bmatrix}$; $\Delta = |z\mathbf{I} - \mathbf{A}_w| = z^2 - 3z + 2 - 2 = z(z-3)$

$$\frac{Y(z)}{U(z)} = \mathbf{C}_w[z\mathbf{I} - \mathbf{A}_w]^{-1}\mathbf{B}_w = \frac{1}{\Delta}[-1 \quad 3] \begin{bmatrix} z-1 & 2 \\ 1 & z-2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{\Delta}[-1 \quad 3] \begin{bmatrix} z-1 \\ 1 \end{bmatrix} = \frac{-z+4}{z(z-3)}$$

Note that solution of part (b) could be different depending on how you choose your coordinate transform however you should get the same result for part (c)

Solution 3:

**(a)** From the code, we can get the difference equations as follows:

$$s_2(k) = e(k) - s_1(k), \tag{1}$$

$$m(k) = \frac{1}{2}s_2(k) - s_1(k), \tag{2}$$

and

$$s_1(k+1) = s_2(k). \tag{3}$$

Using (1) and (3), we get

$$s_1(k+1) = e(k) - s_1(k) \rightarrow e(k) = s_1(k+1) + s_1(k). \tag{4}$$

Using (1) and (2), we have

$$m(k) = \frac{1}{2}(e(k) - s_1(k)) - s_1(k) = \frac{1}{2}e(k) - \frac{3}{2}s_1(k). \tag{5}$$

Let's substitute $k = k+1$ in (5):

$$m(k+1) = \frac{1}{2}e(k+1) - \frac{3}{2}s_1(k+1). \tag{6}$$

Then, adding the both sides of (6) to the both sides of (5), we get

$$\begin{aligned} m(k+1) + m(k) &= \frac{1}{2}e(k+1) - \frac{3}{2}s_1(k+1) + \frac{1}{2}e(k) - \frac{3}{2}s_1(k) \\ &= \frac{1}{2}(e(k+1) + e(k)) - \frac{3}{2}(s_1(k+1) + s_1(k)). \end{aligned}$$

Now based on this equation and (4), it follows that

$$m(k+1) + m(k) = \frac{1}{2}(e(k+1) + e(k)) - \frac{3}{2}e(k) = \frac{1}{2}e(k+1) - e(k). \tag{7}$$

Using the $z$-transform, we get

$$M(z) = \frac{1}{2}E(z) - z^{-1}E(z) - z^{-1}M(z)$$

$$\frac{M(z)}{E(z)} = \frac{0.5z - 1}{z+1}$$

**(b)**

$$e(k+1) = e(k) + 1 = e(k-1) + 1 + 1 = \cdots = e(0) + k + 1 = k + 1$$

$$e(k) = k \quad \longrightarrow \quad E(z) = \sum_{k=0}^{\infty} kz^{-k} = \frac{z}{(z-1)^2}$$

**(c)** Using partial fraction formula, we get

$$M(z) = \frac{0.5z - 1}{z+1}\frac{z}{(z-1)^2} = \frac{(0.5z-1)z}{(z+1)(z-1)^2}$$

$$\frac{M(z)}{z} = \frac{A}{z+1} + \frac{B}{z-1} + \frac{C}{(z-1)^2}$$

Solving for $A$:

$$A = \left.\frac{0.5z - 1}{(z-1)^2}\right|_{z=-1} = -\frac{3}{8}$$

Solving for $B$:

$$B = \frac{1}{(2-1)!}\frac{d}{dz}\left.\frac{0.5z - 1}{z+1}\right|_{z=1} = \frac{3}{8}$$

Solving for $C$:

$$C = \frac{0.5z - 1}{z + 1}\Big|_{z=1} = \frac{1}{4}$$

Using $A$, $B$, and $C$, we have

$$\frac{M(z)}{z} = \frac{-3}{8(z+1)} + \frac{3}{8(z-1)} + \frac{-1}{4(z-1)^2}$$
$$M(z) = \frac{-0.375z}{z+1} + \frac{0.375z}{z-1} + \frac{-0.25z}{(z-1)^2}$$

Then, applying the inverse $z$-transform it follows that

$$m(k) = -0.375(-1)^k - 0.25(k) + 0.375$$

Checking some results based on the result of the Matlab code:

$$m(10) = -0.375(-1)^1 0 - 0.25(10) + 0.375 = -2.5.$$

```
>> hw3

ans =

     0     0     0


ans =

    1.0000    0.5000    1.0000


ans =

    2.0000   -0.5000    1.0000


ans =

     3     0     2


ans =

     4    -1     2


ans =

    5.0000   -0.5000    3.0000


ans =

    6.0000   -1.5000    3.0000


ans =

     7    -1     4


ans =

     8    -2     4


ans =

    9.0000   -1.5000    5.0000


ans =

   10.0000   -2.5000    5.0000

>>
```

Solution 4:

x 2 pi are missing in all G_h0

(a) $$\frac{1}{T}G_{h0}(j2) = \frac{\sin\left(\pi \frac{2\pi(2)}{2\pi(12)}\right)}{2\pi/12}\varepsilon^{-j\pi/6} = \frac{0.50\angle -30°}{0.5236} = 0.9549\angle -30°$$

(b) $$\frac{1}{T}G_{h0}(j10) = \frac{\sin\left(\pi 10/12\right)}{10\pi/12}e^{-j10\pi/12} = \frac{0.50\angle -150°}{2.618} = 0.1910\angle -150°$$

(c) (a) $$\frac{1}{T}\left|G_{h1}(j\omega)\right| = \left[1+\left(\frac{2\pi\omega}{\omega_s}\right)^2\right]^{1/2}\left[\frac{1}{T}G_{h0}(j\omega)\right]^2$$

$$\frac{1}{T}\left|G_{h1}(j2z)\right| = \left[1+\left(\frac{2\pi}{6}\right)^2\right]^{1/2}(0.9549)^2 = 1.3203$$

$$\angle G_{h1}(j2) = \tan^{-1}\left(\frac{2\pi}{6}\right) - \left(\frac{2\pi}{6}\right) = 46.32° - 60° = -13.7°$$

(b) $$\frac{1}{T}\left|G_{h1}(j10)\right| = \left[1+\left(\frac{20\pi}{12}\right)^2\right]^{1/2}[0.1910]^2 = 0.194$$

$$\angle G_{h1}(j10) = \tan^{-1}\left(\frac{20\pi}{12}\right) - \left(\frac{20\pi}{12}\right) = 79.2° - 300° = -220.18°$$

(d) The components at $\omega = 10$ are approximately equal.

The component at $\omega =$ 2 for the 1st order hold is approximately 30% too large, which that for the zero-order-hold is approximately 10% too small. ∴ the zero-order hold is better in this case.