EECE 5610 Digital Control Systems

Lecture 7

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- Ideal sampler:

- **Ideal sampler:**

Consider $E^*(s) = \sum_0^\infty e(kT)\, e^{-kTs}$

$$\Downarrow \mathcal{L}^{-1}$$

$$e^*(t) = \sum_0^\infty e(kT)\, \mathcal{L}^{-1}\left[e^{-kTs}\right] = \sum_0^\infty e(kT)\, \delta(t-kT) = \sum_0^\infty e(t)\, \delta(t-kT)$$

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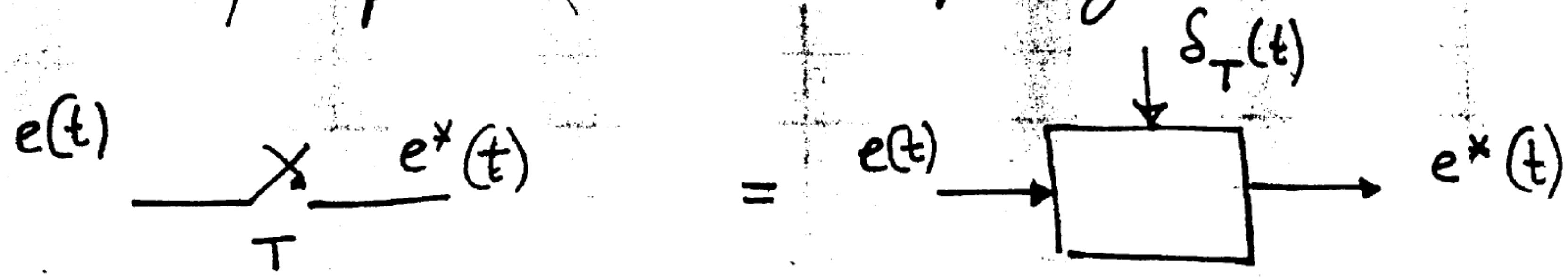

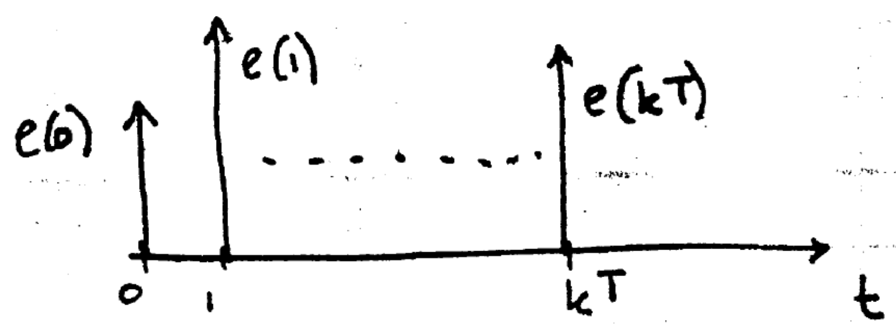

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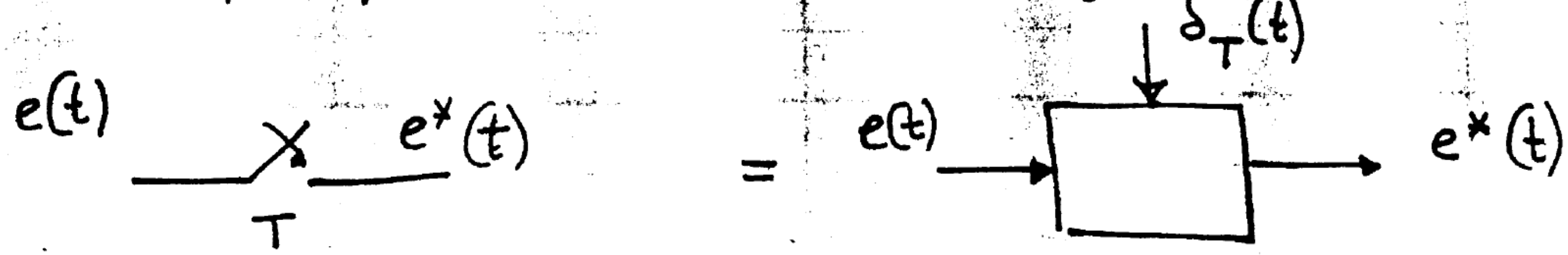

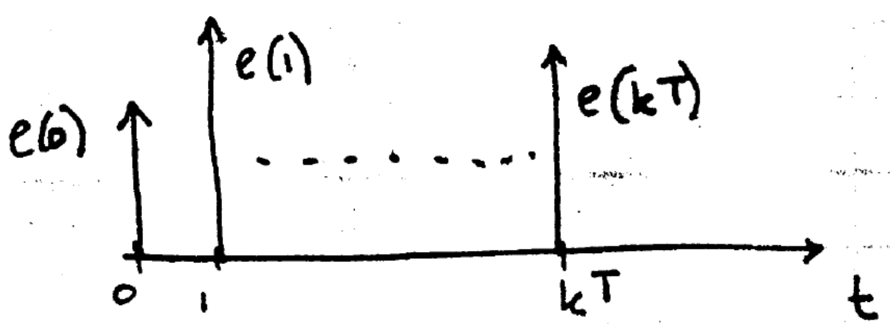

⇒ In the future we will think of $E^*(s)$ as the output of an ideal sampler.

- As mentioned before neither the ideal sampler nor the $\left[\frac{1-e^{-Ts}}{s}\right]$ block by themselves model the operation of a physical device. However, their combination does give the correct mathematical description of the sample & hold operation.

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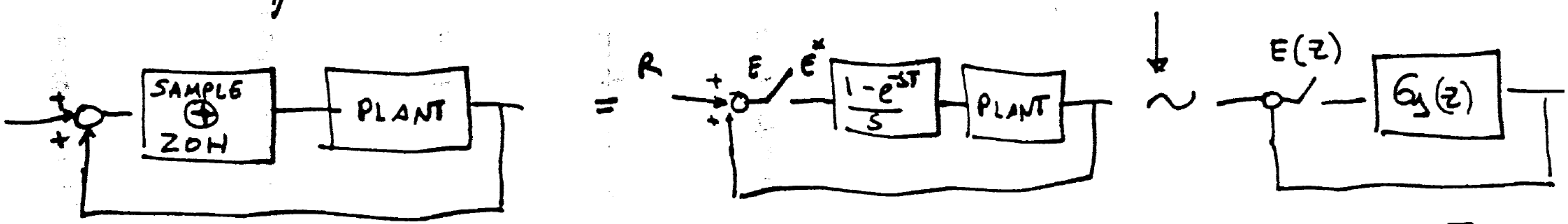
- Q: Why do we go through all this trouble?

A: We want to get a T.F model of the hold operation and an input-output model of the sample and hold suitable for using in combination with linear systems analysis tools, such as the z-transform

we'll see later



where $G_1(z) = Z\left[\left(\frac{1-e^{-sT}}{s}\right)\cdot \text{Plant}\right]$

Example: $e(t) = u(t)$

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(Note that the expression for $E^*(s)$ resembles that of $E(z)$, in fact, they are identical if we define $z = e^{sT}$. This is no accident, more on this later)

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(b) $E^*(s) = \frac{1}{T} \left[\sum_{-\infty}^{+\infty} E(s + jn\omega_s) + \frac{e(0)^+}{2} \right]$ where $\omega_s = \frac{2\pi}{T}$

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The proof uses again Cauchy's theorem. Sketch of the proof:

$$e^*(t) = e(t)\,\delta_T(t) \qquad \text{where} \qquad \delta_T(t) = \sum_{n=0}^{\infty} \delta(t - nT)$$

$$E^*(s) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} E(\lambda)\,\Delta_T(s-\lambda)\,d\lambda \qquad \Delta_T(s) = \mathcal{L}\left[\delta_T(t)\right]$$

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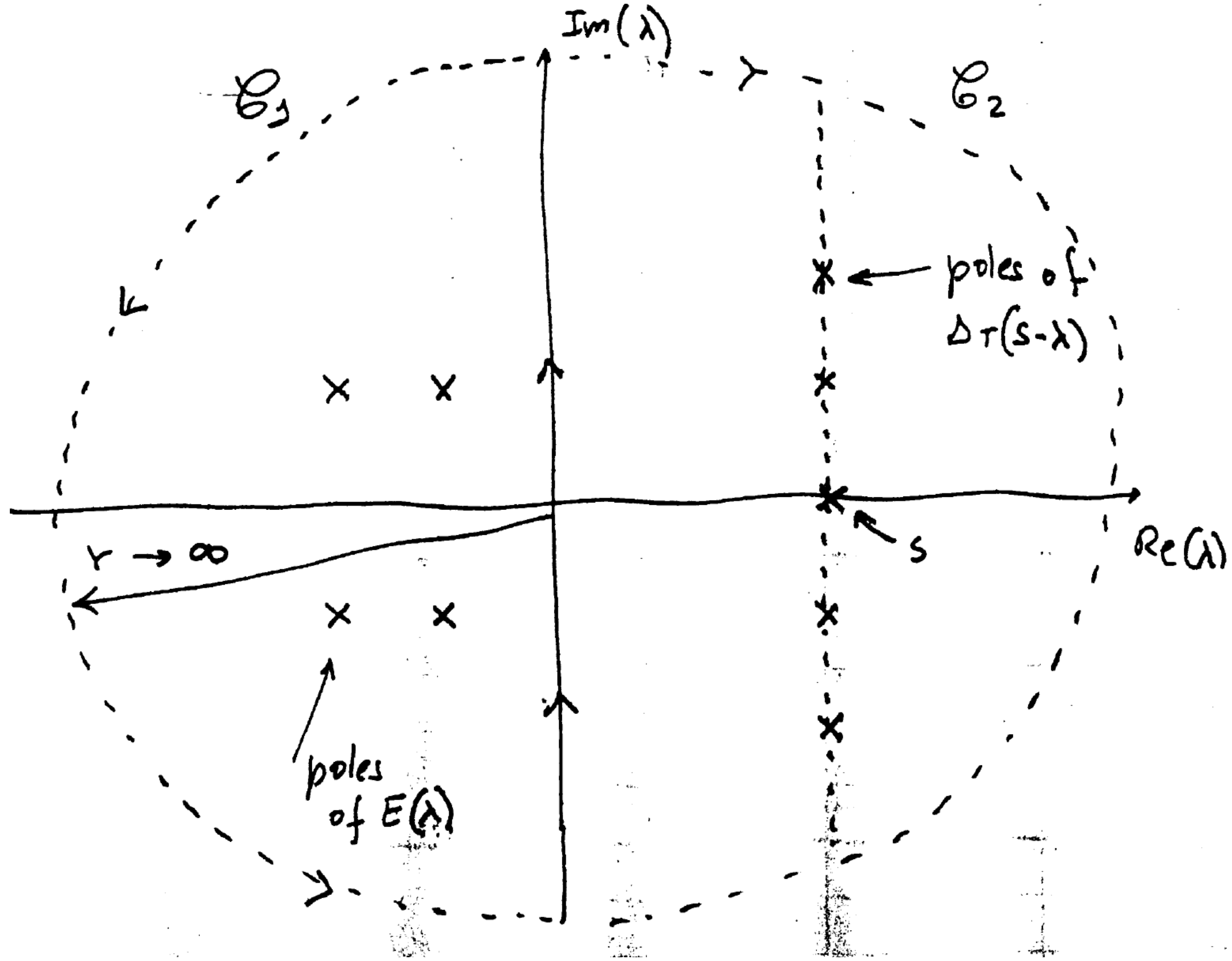
Note that $\Delta_T(s) = \mathcal{L}\left[\delta_T(t)\right] = \sum_0^{\infty} e^{-nTs} = \dfrac{1}{1-e^{-sT}} \Rightarrow$ poles at $e^{-Ts} = 1$

$$\Rightarrow s = \pm j\frac{2\pi n}{T} = \pm j\omega_s n$$

so that $\Delta_T(s)$ has an <u>infinite</u> number of poles, all on the $j\omega$ axis, spaced ω_s



$\Delta_T(s)$ poles at $s = \pm jn\omega_s$

$\Delta_T(s-\lambda)$ poles at $\lambda = s \pm jn\omega_s$

Closing the contour with $\mathcal{C}_1$ (a semicircle in the LHP with radius $r \to \infty$) yields the first equality. If, on the other hand we close the contour with $\mathcal{C}_2$ (semicircle in RHP, $r \to \infty$) we obtain the second formula

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$\text{Res}(z_i) = \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \left[(z-z_i)^n F(z) \right] \Big|_{z=z_i}$ if $f(z)$ has a singularity of order n at z_i

$\text{Res}(z_i) = (z-z_i) F(z) \Big|_{z=z_i}$ for singularities of order 1

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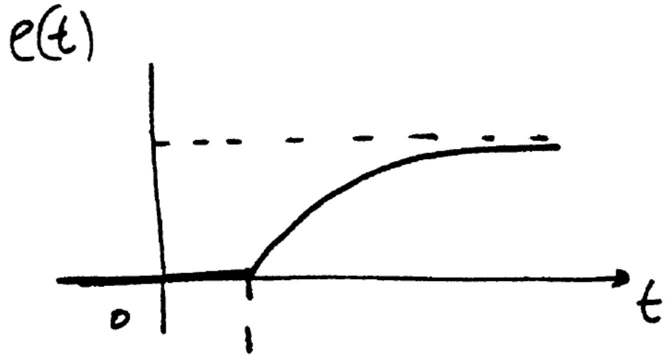

$e(k) = \left[1 - e^{-(0.5k-1)}\right] \quad k \geq 2; \qquad e(k) = 0 \quad k = 0, 1 \qquad T=1/2$

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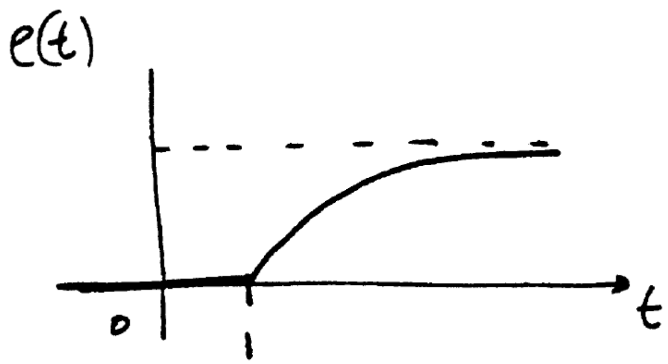

Starting Summation from k = 2 because first 2 terms are zero due to step delay

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From the definition we have:

$$E^*(s) = \sum e(k) e^{-kTs} = \sum_{k=2}^{\infty} \left(1 - e^{-(0.5k-1)}\right) e^{-kTs} = \frac{e^{-2Ts}}{1 - e^{-Ts}} - e^{1} \cdot \frac{e^{-2(0.5+Ts)}}{1 - e^{-(0.5+Ts)}}$$

$$= \frac{e^{-s}}{1 - e^{-0.5s}} - \frac{\cancel{e^{1}} \cancel{e^{-1}} \cdot e^{-s}}{1 - e^{-0.5(s+1)}} = \boxed{\frac{\left(1 - e^{-0.5}\right) e^{-1.5s}}{\left(1 - e^{-0.5s}\right)\left(1 - e^{-0.5(s+1)}\right)}} \quad \#$$

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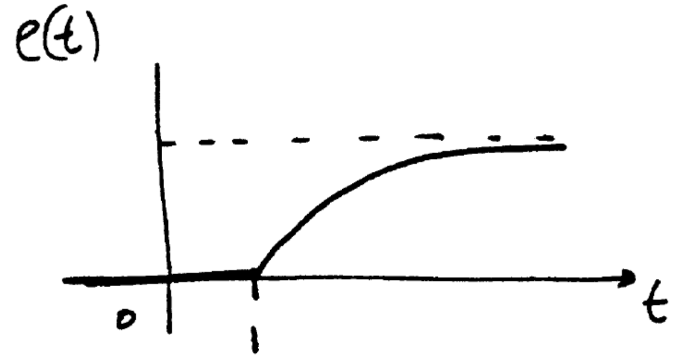

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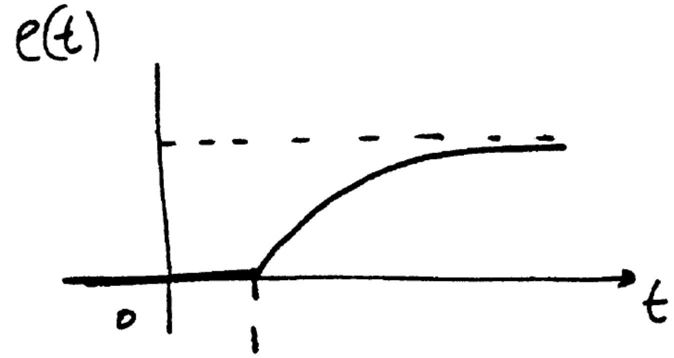

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Now let's try our residues formula:

$$E(s) = \frac{e^{-s}}{s(s+1)} \Rightarrow E^*(s) = \sum_{\substack{\lambda=0 \\ \lambda=-1}} \mathrm{Res}\left\{ \frac{e^{-\lambda}}{\lambda(\lambda+1)} \frac{1}{1-e^{-T(s-\lambda)}} \right\} =$$

$$= \frac{1}{1-e^{-Ts}} + \frac{e^{1}}{(-1)} \cdot \frac{1}{1-e^{-T(s+1)}} = \boxed{\frac{1}{1-e^{-0.5s}} + \frac{e^{1}}{1-e^{-0.5(s+1)}}}$$

Surprise! we got <u>different</u> answers

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Q: What went wrong here? → Cauchy's Integral does not work well with time-delays!!

A: A techical point: the "proof" of the residues formula is not valid for systems having time delays

The reason is that $e^{-sT} \not\to 0$ on the infinite portion of the contour B_1 and thus we can't close the contour and compute the $\int$ using residues

<u>Solution</u>

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(b) Modify the formula:

It can be shown that if the delay is an <u>integer</u> number of periods then:

$$E^*(s) = \left[e^{-kTs} E_1(s)\right]^* = e^{-kTs} \sum_{\substack{\text{at poles} \\ \text{of } E_1}} \left\{ \text{Res } E_1(\lambda) \frac{1}{1 - e^{-T(s-\lambda)}} \right\}$$

$E_1(s)$: non delayed signal

Applying this modified formula to our earlier example we get

$$E_1(s) = \frac{1}{s(s+1)} \qquad \sum_{\substack{\lambda=0 \\ \lambda=-1}} \operatorname{Res} \frac{1}{\lambda(\lambda+1)} \cdot \frac{1}{\left(1 - e^{-T(s-\lambda)}\right)}$$

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$$= \frac{1}{\left(1 - e^{-Ts}\right)} - \frac{1}{\left(1 - e^{-T(s+1)}\right)}$$

$$\Rightarrow E^*(s) = e^{-2Ts}\left[\frac{1}{1-e^{-0.5s}} - \frac{1}{1-e^{-0.5(s+1)}}\right] = e^{-s}\left[\frac{1}{1-e^{-0.5s}} - \frac{1}{1-e^{-0.5(s+1)}}\right]$$

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which coincides with our earlier result

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proof:

$$E^*(s) = \sum_0^\infty e(kT)\, e^{-skT}$$

$$E^*(s+j\omega_s) = \sum_0^\infty e(kT)\, e^{-k\left[s+j\frac{2\pi}{T}\right]T} = \sum_0^\infty e(kT)\, e^{-skT}\, \underbrace{e^{-jk\frac{2\pi}{\cancel{T}}\cdot\cancel{T}}}_{1} = E^*(s) \quad \#$$

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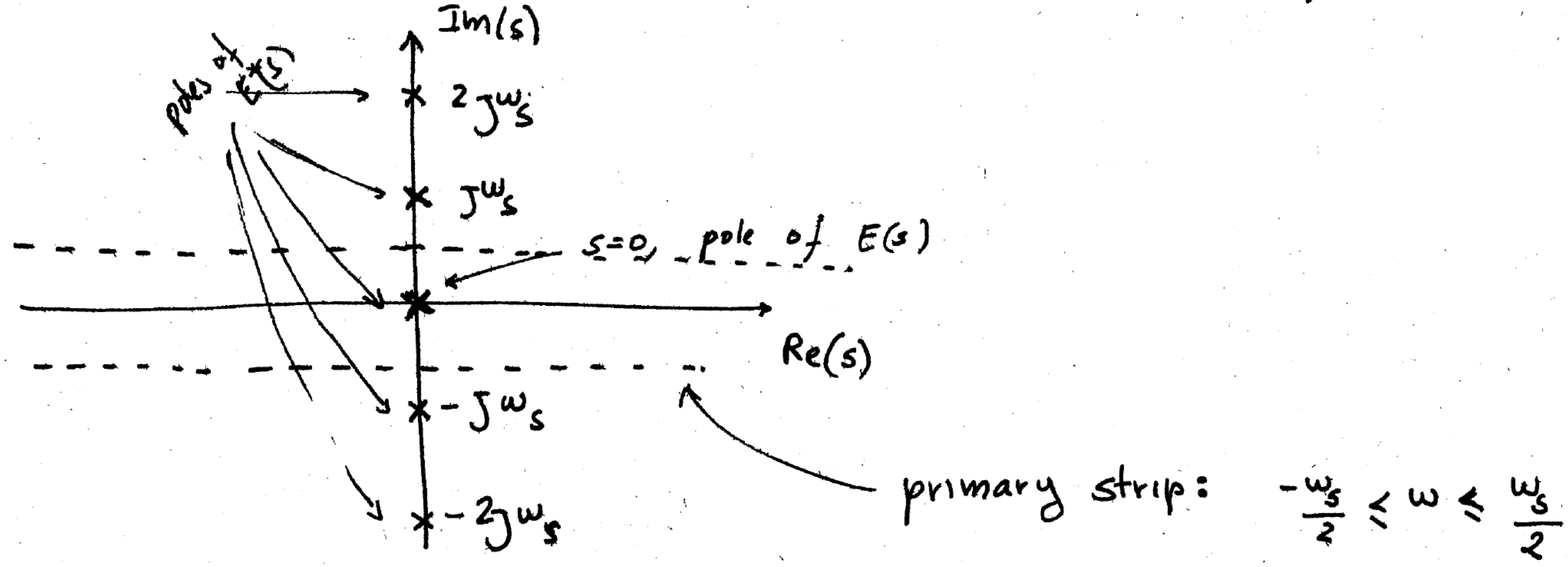
2) If $E(s)$ has a pole at $s = s_1 \Rightarrow E^*(s)$ has poles at $s = s_1 + jm\omega_s$ $\quad m = 0, \pm 1 \dots$

Note: some property does not apply to zeros of $E^*(s)$

Example: assume that $E(s)$ has a pole at $s=0 \Rightarrow E^*(s)$ poles at $s_m = \pm j n \omega_s$



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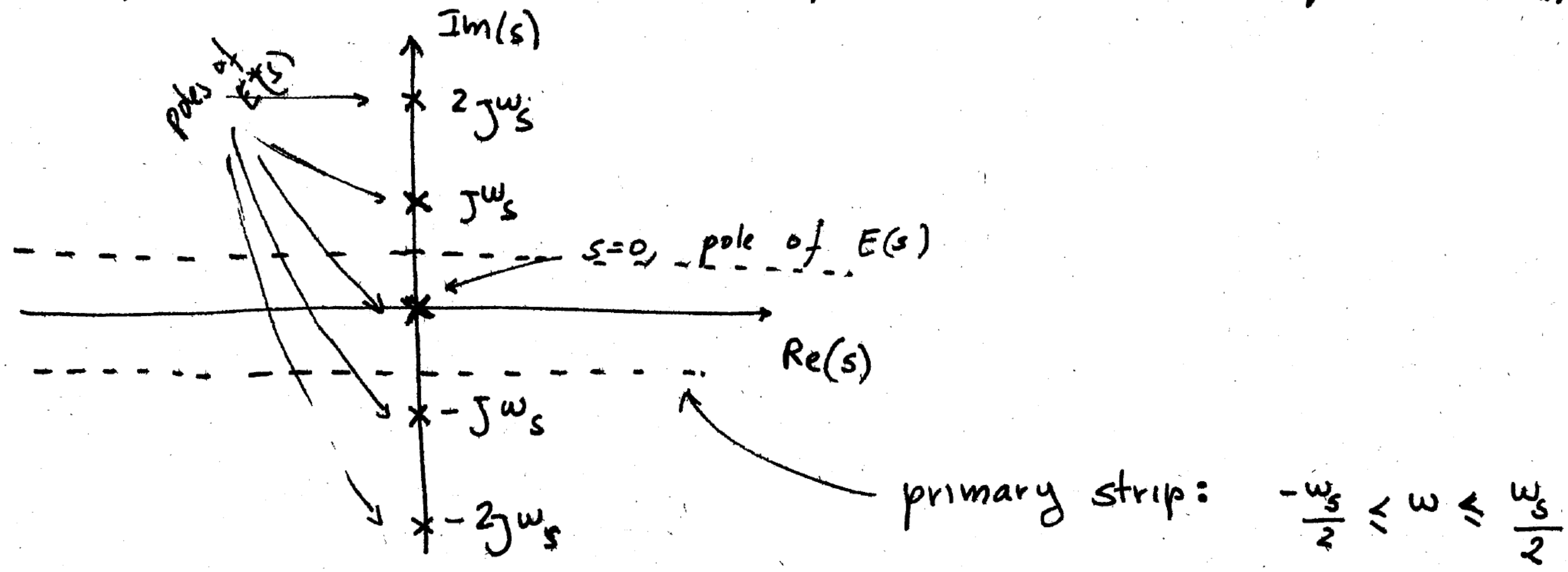
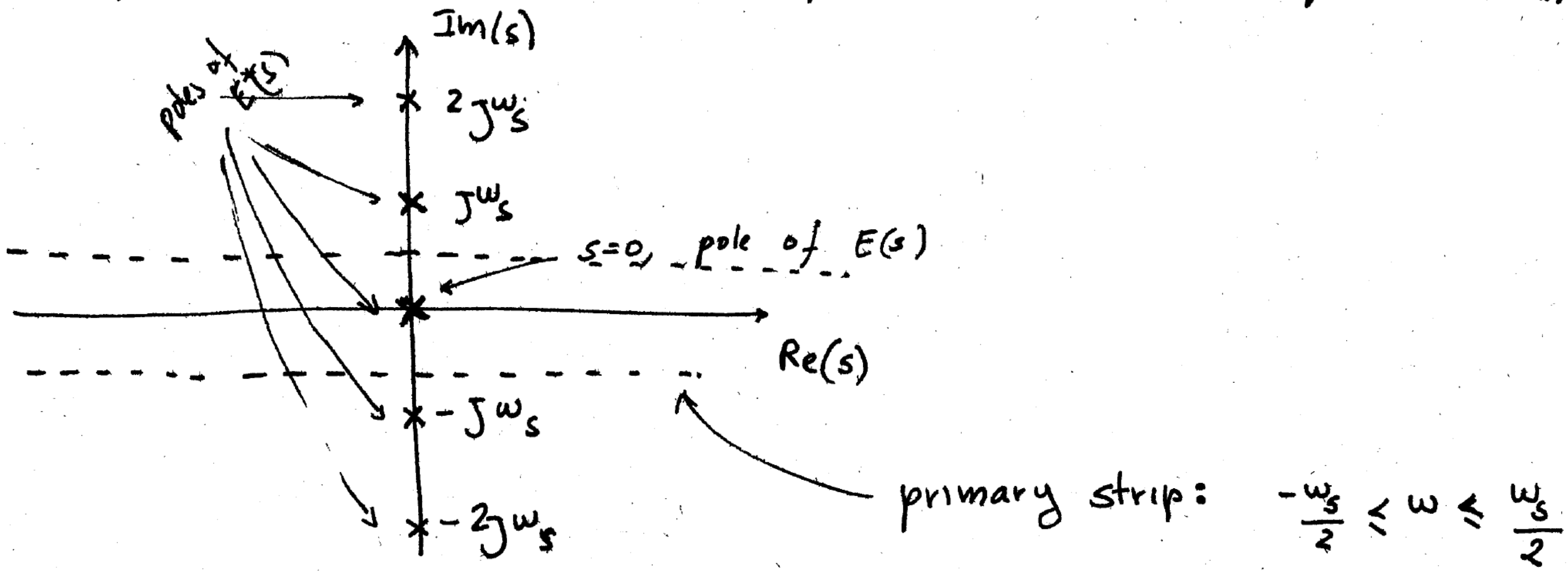


proof: $$E^*(s) = \frac{1}{T} \sum_{n=-\infty}^{n=+\infty} E(s + jn\omega_s) = \frac{1}{T}\left[E(s) + E(s + j\omega_s) + \cdots \right]$$

Example: assume that $E(s)$ has a pole at $s=0 \Rightarrow E^*(s)$ poles at $s_m = \pm j n \omega_s$



proof: $$E^*(s) = \frac{1}{T} \sum_{n=-\infty}^{n=+\infty} E(s+jn\omega_s) = \frac{1}{T}\left[E(s) + E(s+j\omega_s) + \cdots \right]$$

if $E(s)$ has a pole at $s=s_1$, the first term contributes a pole at $s=s_1$

second $s = s_1 - j\omega_s$

$\vdots$

n^{th} . . . $s = s_1 - jn\omega_s$

Example 2: Recall that we have shown that:

$$F(s) = \frac{1}{s}\cdot\frac{1}{(s+1)} \Longleftrightarrow F^*(s) = \frac{1}{1-e^{-sT}} - \frac{1}{1-e^{-T(s+1)}}$$

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

poles at $s=0$ $\qquad$ poles at $s = \pm j n \frac{2\pi}{T}$ $\quad (e^{-jTs}=1)$

$s=-1$ $\qquad$ $s = -1 \pm j n \frac{2\pi}{T}$

- **Spectrum of a Sampled Signal**

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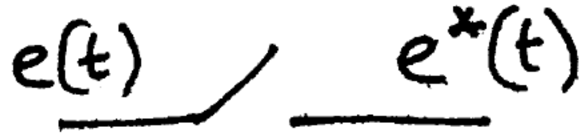
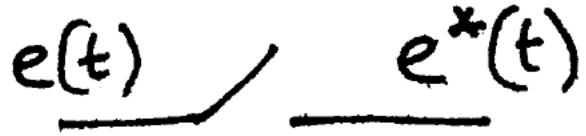


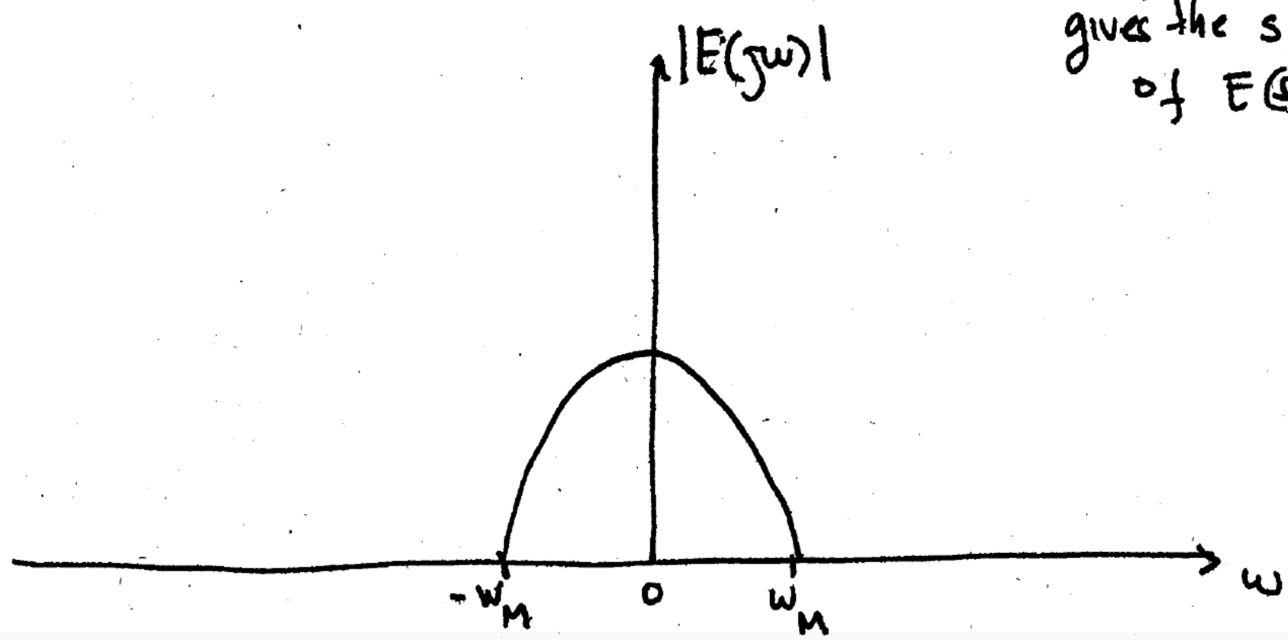
- ## Spectrum of a Sampled Signal

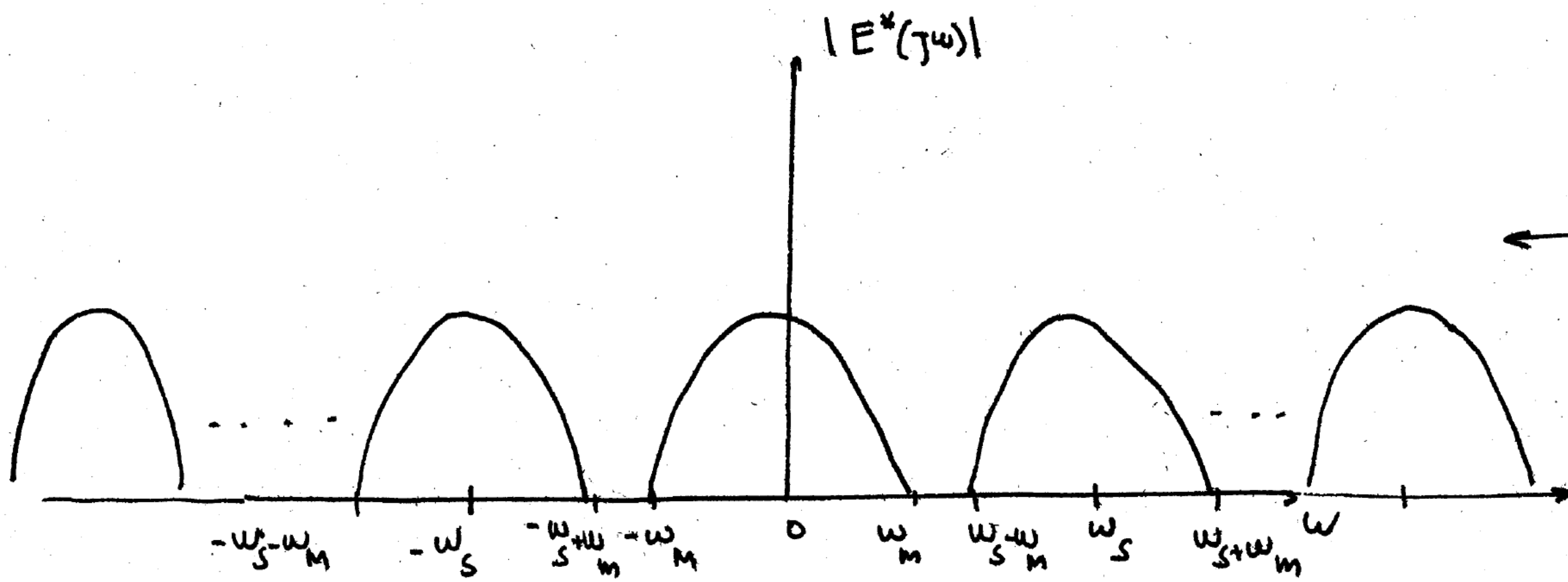


We'd like to relate the spectrum (i.e. fourier transform) of $e(t)$ and $e^*(t)$. This will become relevant when we discuss how to <u>reconstruct</u> (if possible) $e(t)$ from $e^*(t)$.

Let $$e^*(t) = \sum_{k=-\infty}^{k=+\infty} e(t)\,\delta(t-kT)$$

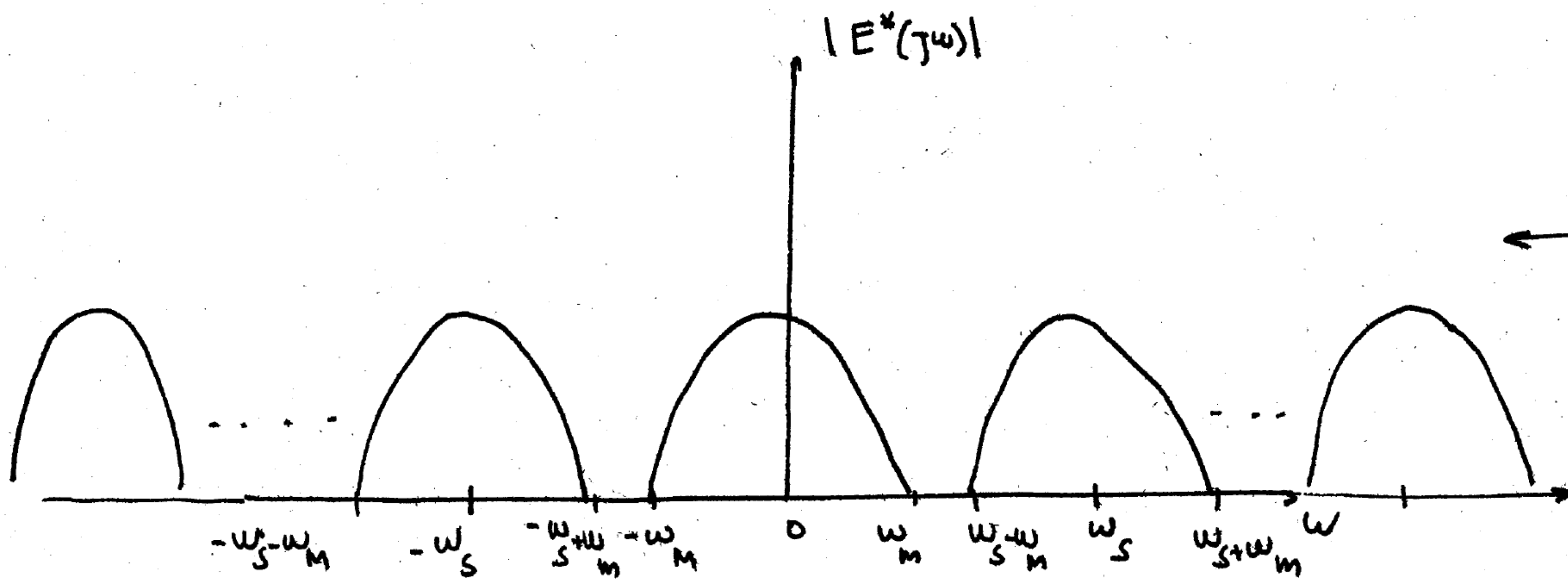
$$E^*(s) = \frac{1}{T}\sum_{-\infty}^{+\infty} E(s+jn\omega_s) = \frac{1}{T}\Big[\underbrace{E(s)}_{\text{gives the spectrum of } E(s)} + E(s+j\omega_s) + \cdots \underbrace{E(s+jn\omega_s)}_{\text{gives the spectrum of } E\text{, shifted b } n\omega_s} + \cdots\Big]$$





original spectrum replicated at

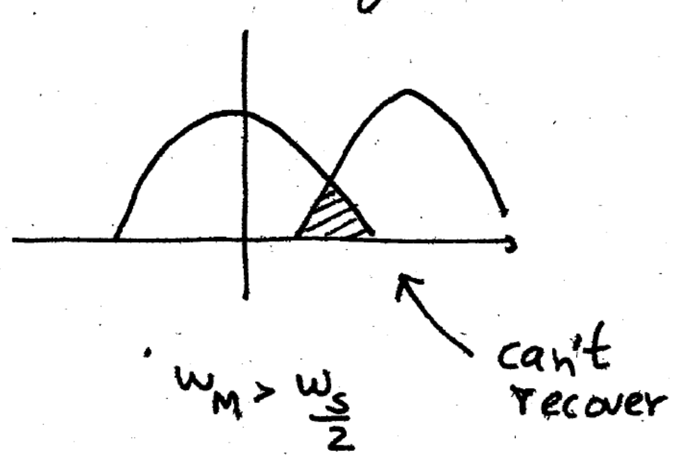
$\omega_s,\ 2\omega_s, \ldots$

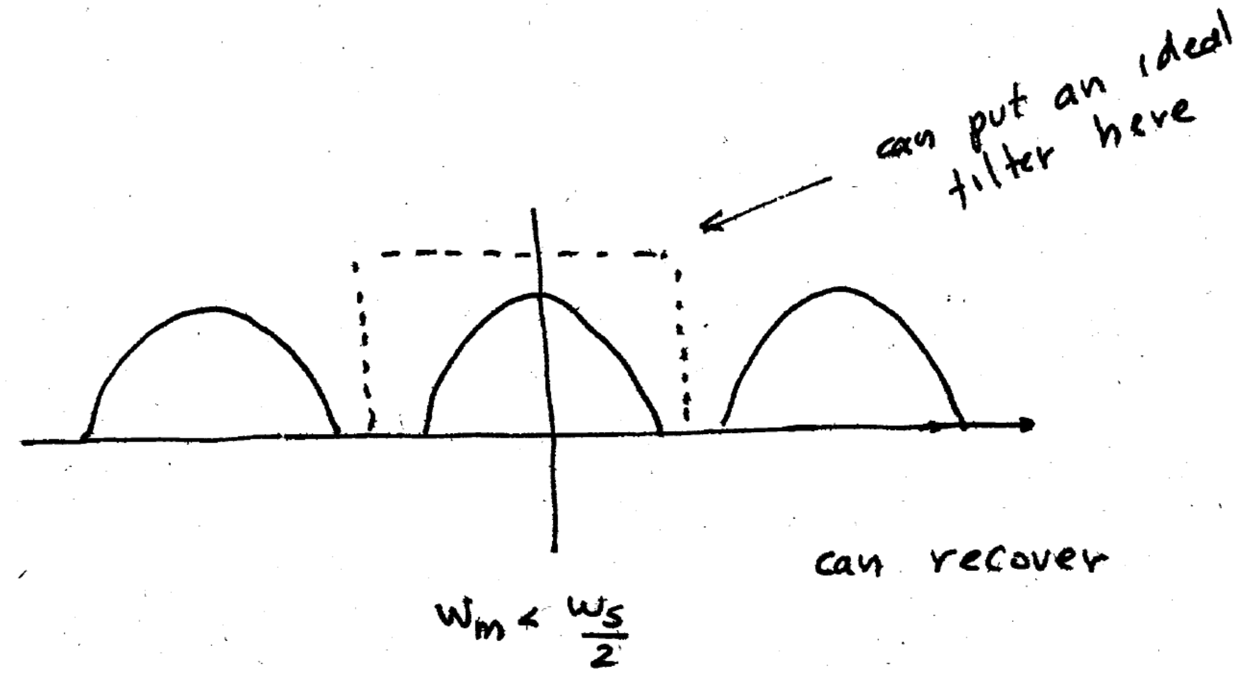


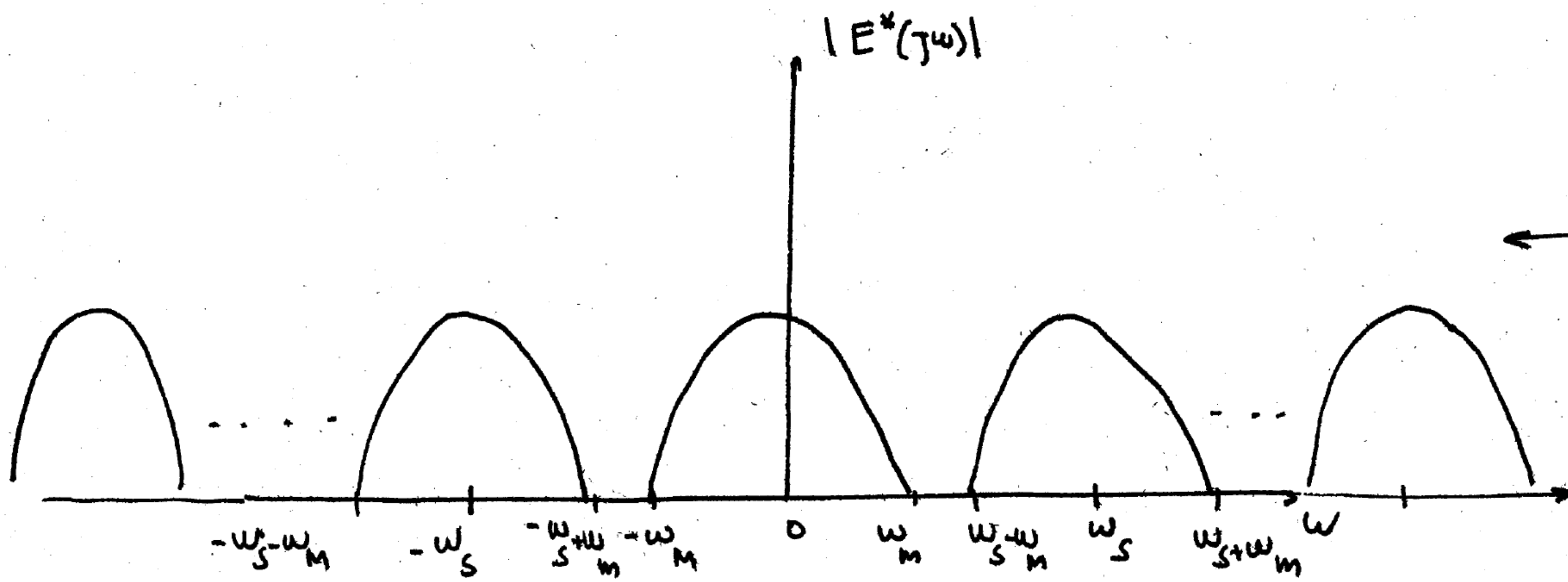
original spectrum replicated at ω_s, $2\omega_s$, . . .

From these plots it follows that we can recover $E(s)$ from $E^*(s)$ <u>only</u> if the highest frequency present in $e(t)$ is smaller than $\frac{\omega_s}{2}$ (the "Nyquist frequency")

We can't have any overlapping:







original spectrum replicated at $\omega_s, 2\omega_s, \ldots$

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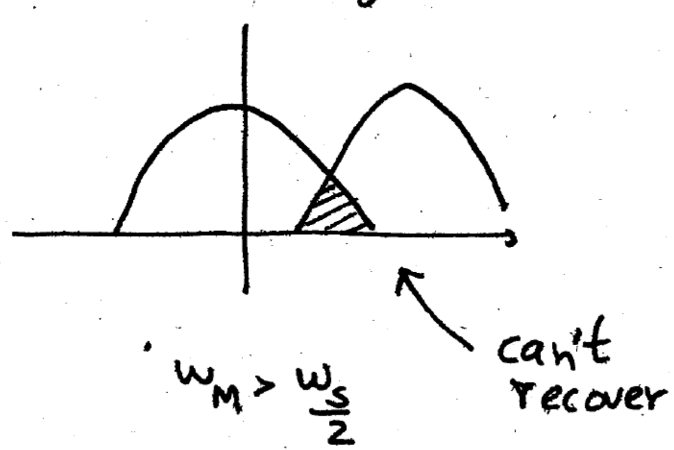

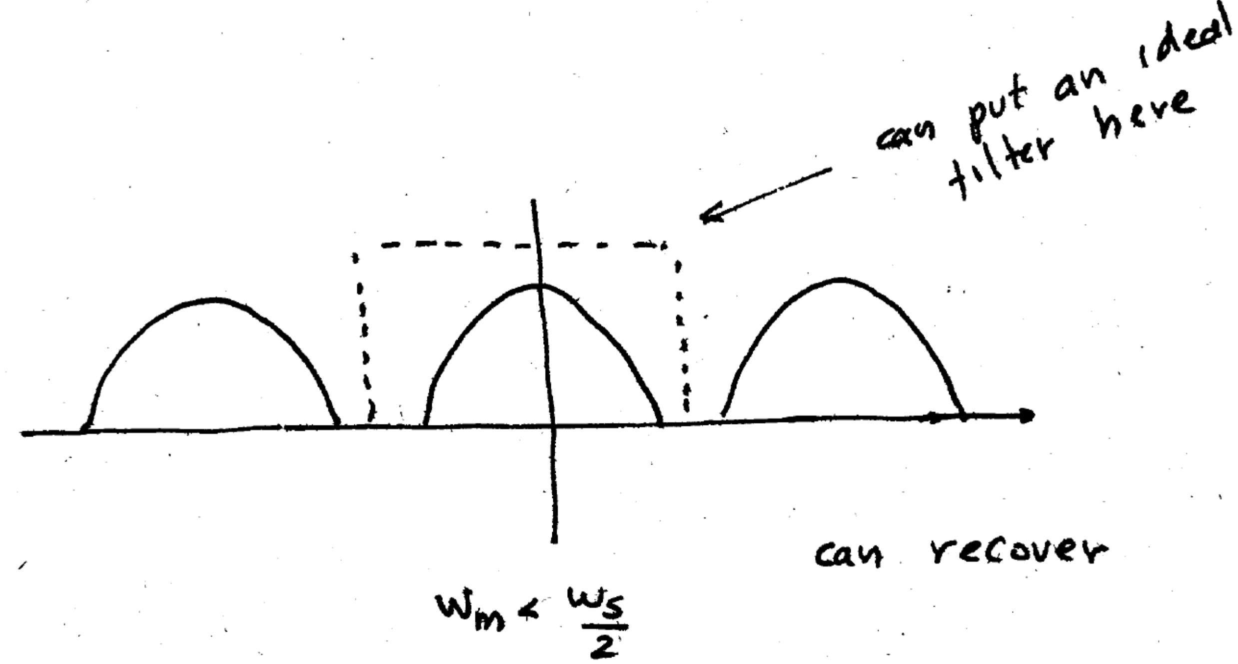

This is the celebrated <u>Shannon's Sampling Theorem</u>:

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- A function $e(t)$ which contains no frequency component higher than f_0 is <u>uniquely</u> determined by the values of $e(t)$ at any set of sampling points spaced $T = \frac{1}{2f_0}$

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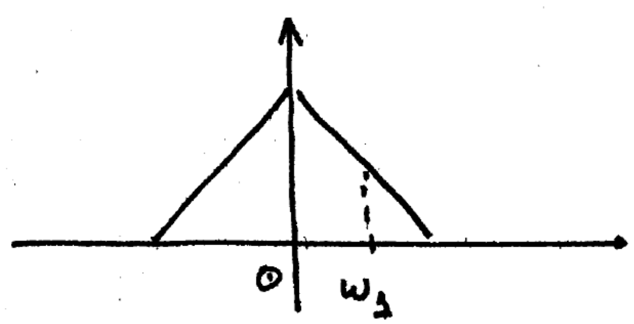
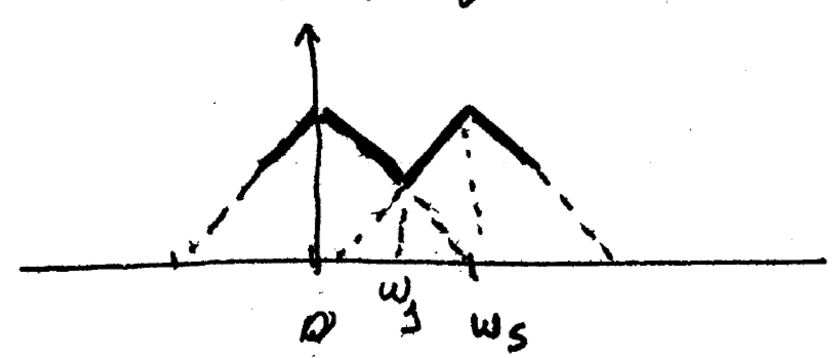
If $e(t)$ has components above the Nyquist frequency we have the following situation:



sample $\Longrightarrow$



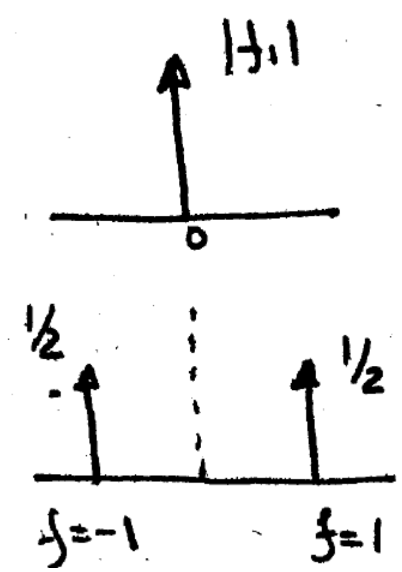
In the sampled signal the contributions from the frequencies ω_1 and $\omega_2 = \omega_1 - \omega_s$ <u>both</u> show up at ω_1. This phenomenon is called <u>aliasing</u>

Implications: 2 sinusoids of different frequencies appear at the same place when sampled $\Rightarrow$ can't tell them appart

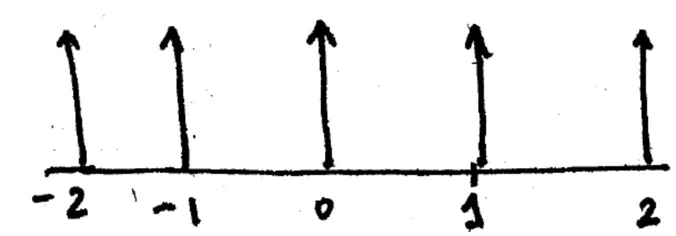
Example:

$f_1(t) = 1$

$f_2(t) = \cos 2\pi t$

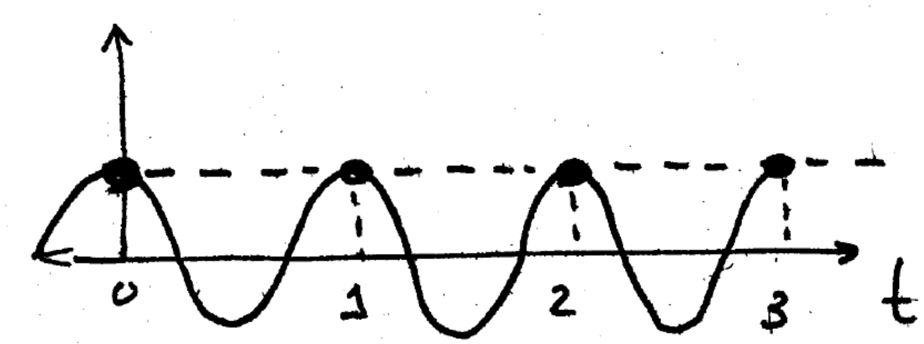


If sampled at $f_s = 1$ Hz both yield:



(the same spectrum!)

With 20/20 hindsight this is not surprising:



Related phenomenon: <u>Hidden oscillations</u>: We can have a signal that <u>does not</u> show up at all when sampled