EECE 5610 Digital Control Systems

Lecture 14

Milad Siami
Assistant Professor of ECE
Email: m.siami@northeastern.edu

Northeastern University
College *of* Engineering

FAQ Regarding Midterm#2:

- Will we be expected to do the Fourier Transform on the exam? (3.5)

I'll provide a Fourier Transform Table if needed.

- Should we have an understanding on non-synchronous sampling? (4.7)

Not more than what we discussed in our lectures.

- Will most or all of Chapter 6 (System Time-Response characteristics) be on the exam? I know it was covered between Lectures 12 and 13, but we haven't had any assignments on it.

Not more than what we discussed in our lectures. In the sample questions, there are some questions from 6 and 7.

- Will there be any questions on Stability (Chapter 7), since it was partially covered in Lecture 13?

Ditto!

Recap:

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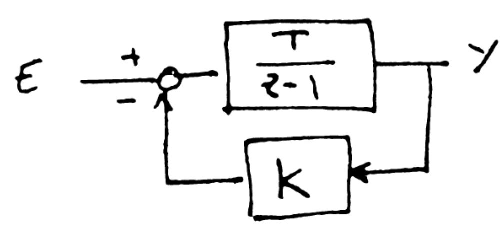
Example:

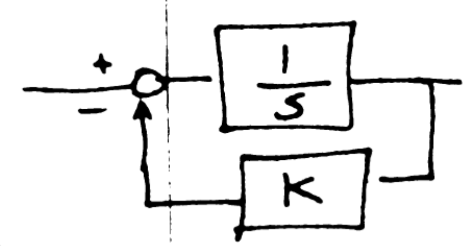


$$H(z) = \frac{\frac{T}{z-1}}{1 + \frac{KT}{z-1}} = \frac{T}{z-1+KT}$$

$$H(z) = \frac{T}{z-(1-kT)} = \frac{T}{z}\,\frac{1}{\left(1-\left(\frac{1-kT}{z}\right)\right)} \iff h[nT] = \begin{cases} 0 & n = 0 \\ T\cdot(1-KT)^{n-1} & n \geq 1 \end{cases}$$

$$\Rightarrow \sum_{n=0}^{\infty} |h(nT)| = T\sum_{0}^{\infty} |(1-kT)|^{n} = \begin{cases} T\cdot\dfrac{1}{1-|1-kT|} < \infty & \text{if } |1-kT| < 1 \\ \infty & \text{if } |1-kT| \geq 1 \end{cases}$$

$\Rightarrow$ System is BIBO stable iff: $|1-kT| < 1$ or $KT < 2$

(Compare to the continuous time case: 

: $H(s) = \frac{1}{s+k}$

$h(t) = e^{-kt}$, $\int_0^\infty |h(t)|\,dt = \int_0^\infty e^{-kt}\,dt = \frac{1}{k} < \infty$ all $k > 0$

$\Rightarrow$ cont. time system is stable for all k)

Now we have a testable condition for stability. However it is hard to use: You need to (1) find $\mathcal{Z}^{-1}[H]$

(2) compute $\sum_{0}^{\infty} |h_k|$

We'd like to have something simpler. Turns out that if your system is finite dimensional linear time invariant (FDLTI) (as always the case in 429) we can assess stability by looking at the <u>location of the poles</u>

- **Relationship between BIBO stability and the location of the poles:**

Suppose $G(z) = \dfrac{C(z)}{E(z)} = \dfrac{(z-z_1)\cdots(z-z_m)}{(z-p_1)\cdots(z-p_n)}$

- **Relationship between BIBO stability and the location of the poles:**

Suppose $G(z) = \dfrac{C(z)}{E(z)} = \dfrac{(z-z_1)\cdots(z-z_m)}{(z-p_1)\cdots(z-p_n)}$

Assume for simplicity that all roots are simple. Then:

$$G(z) = \sum_i k_i \frac{z}{z-p_i} = \sum_i k_i \frac{1}{1-\frac{p_i}{z}} \overset{Z^{-1}}{\Longleftrightarrow} g_k = \sum k_i (p_i)^k$$

Note that $|p_i|^k \to \infty$ if $|p_i| > 1$

In fact, it can be shown that $\sum |p_i|^k < \infty \Leftrightarrow |p_i| < 1$

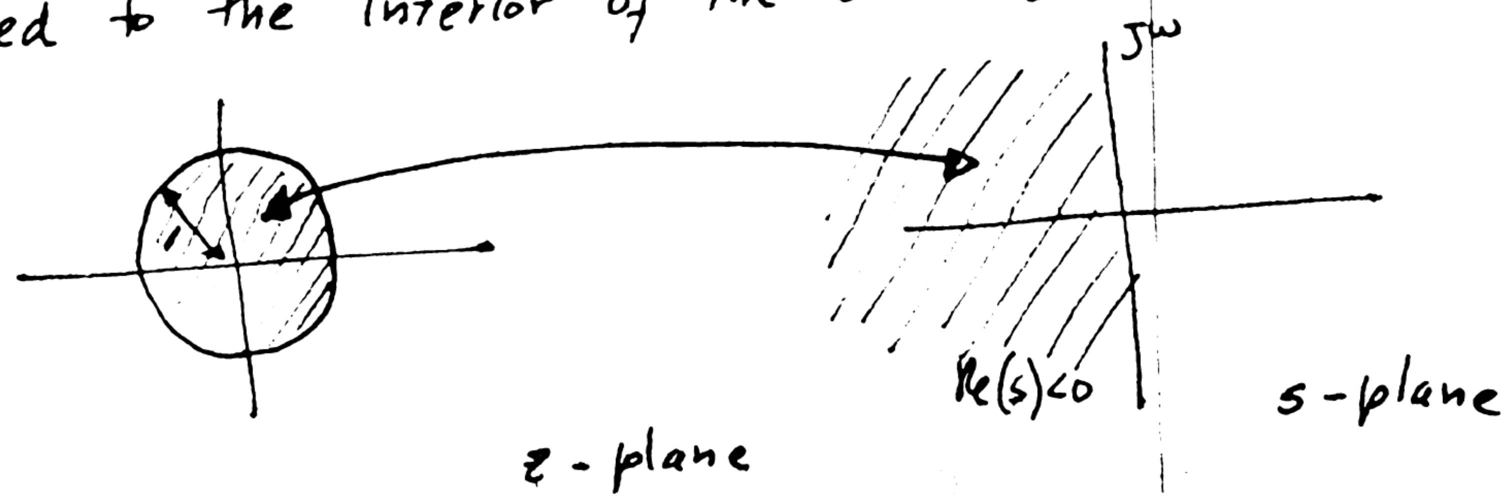
$\Rightarrow \sum_0^\infty |g(k)|$ bounded $\Leftrightarrow |p_i| < 1$, i.e. all poles <u>must</u> be inside the unit disk.

(if we have repeated poles we get terms of the form $n p_i^n$ and the conclusion still stands)

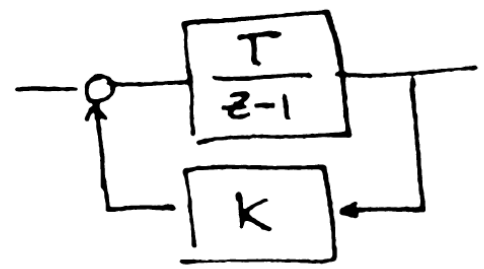
$$\boxed{\text{System BIBO stable} \Leftrightarrow \text{all poles inside the unit disk}}$$

This is not surprising. Recall from the HW that $G(z)$ has a pole at $z = z_0$ $\Leftrightarrow$ $G(s)$ has a pole at s_0 where $z_0 = e^{s_0 T}$

Thus the stable region in the s plane $(Re(s) < 0)$ gets mapped to the interior of the unit disk



__Example revisted:__



$\Rightarrow H(z) = \dfrac{T}{z-1+kT}$

has a single pole at $z = 1-kT$

$\Rightarrow$ stable $\Leftrightarrow |1-kT| < 1 \Leftrightarrow kT < 2$

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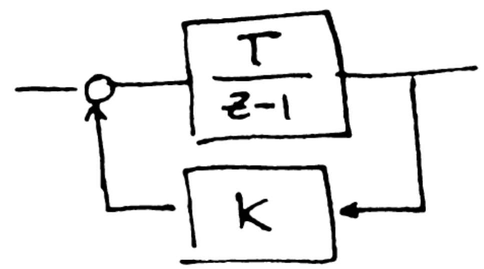

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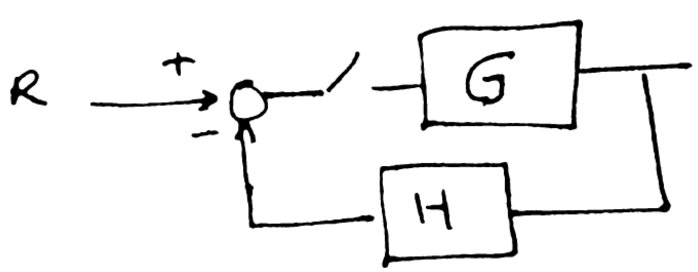
- So now we have an easy way of checking stability



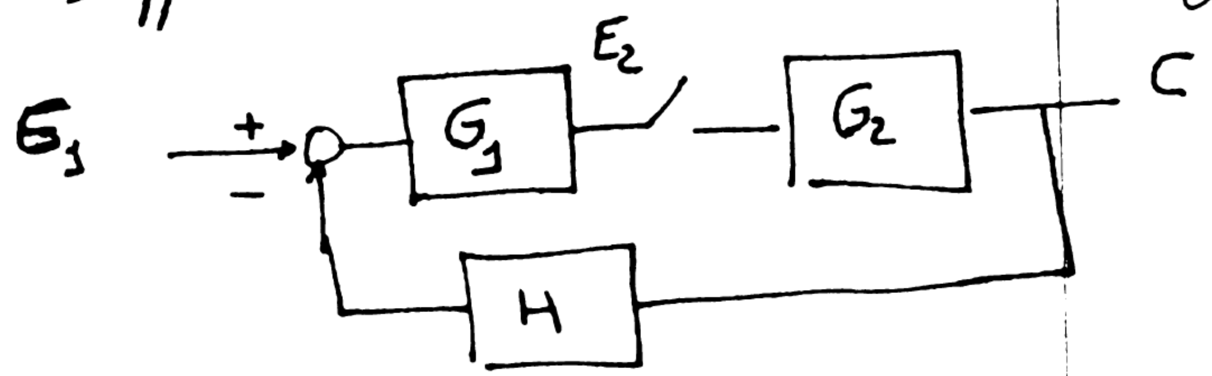
$\Leftrightarrow$ Char equation: $1+G(z)H(z)=0$

poles: roots of this equation

(a) find all the poles of the transfer function (i.e roots of the characteristic equation: Mason's $\Delta = 0$)

(b) stable $\Leftrightarrow$ all poles inside unit disk

But **trouble:** Not all sampled data systems have a Transfer Function
Suppose that we have something like



We know that in this case the pulse transfer function from $E(z)$ to $C(z)$ does not exist. So: how do we handle these cases?

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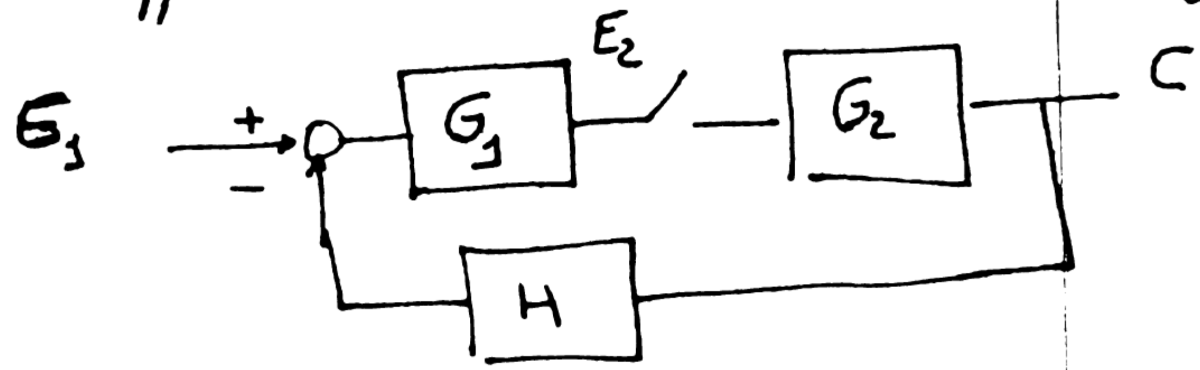

We know that in this case the pulse transfer function from $E(z)$ to $C(z)$ does not exist. So: how do we handle these cases?

A: It turns out that we can still define a characteristic equation

Recall that stability is an intrinsic property of the system, i.e. it does not depend on which signals we choose as inputs and outp

(provided that there are no pole/zero cancellations)

⇒ Instead of considering the signal E_1 in the diagram above, we could take one that is more convenient (i.e one such that the TF exists)

Specifically, we can open the system <u>before</u> the sampler (at E_2) (and set $E_1 = 0$)

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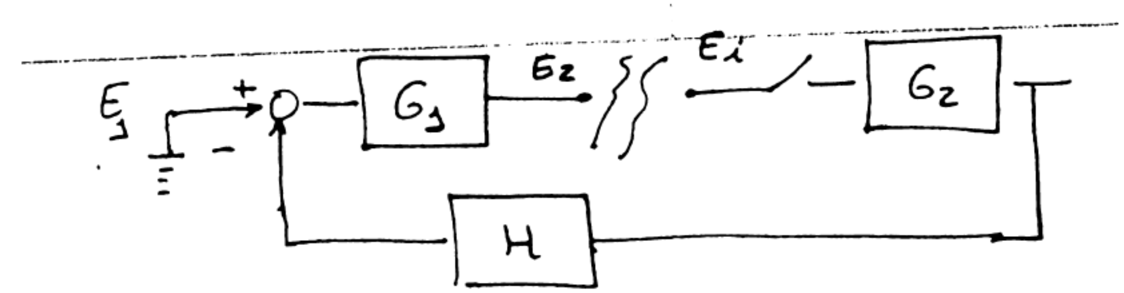

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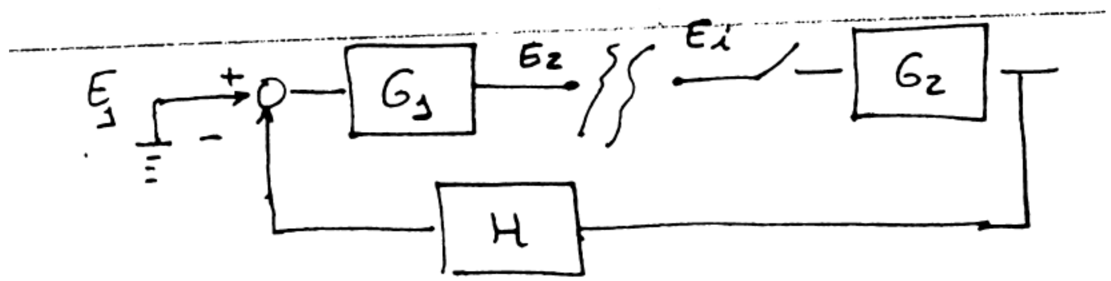
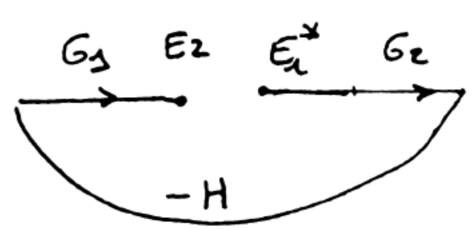


Now we can find the TF from E_i to E_2. (We will call this TF the <u>"open loop" function</u>)



$$\Rightarrow \quad G_{op} = \frac{E_2^*}{E_i^*} = -(G_1 H G_2)^*$$

When you close the loop, you get:

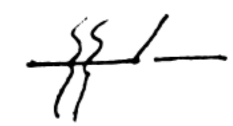
$$E_i(z) = E_2(z)$$

and since $E_2(z) = G_{op}(z) E_i(z) \Rightarrow \left[1 - G_{op}(z)\right] E_i = 0 \Rightarrow \left[1 - G_{op}(z)\right] E_o = 0$

Solutions: $\begin{cases} E_0 = 0 & \text{or} \\ 1 - G_{op} = 0 & \text{(in which case } E_0 \text{ can be arbitrarily large)} \end{cases}$

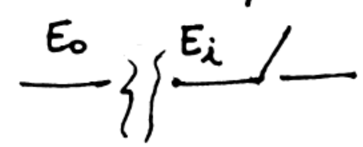
This is a general result: $\boxed{1 - G_{op}(z) = 0 \text{ is the characteristic equation}}$

Recap: If a TF does not exist, we can still find the char. equation as follows:

(1) Open the system in front of a sampler 

(2) Set all inputs to zero

(3) find the TF between the points where the loop was opened

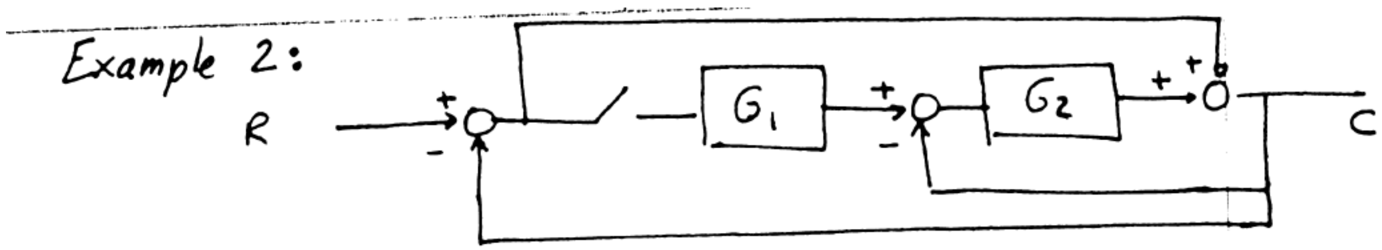


$$G_{op} = \frac{E_o(z)}{E_i(z)}$$

(4) Char equation: $1 - G_{op} = 0$

(5) "poles": roots of Char equation

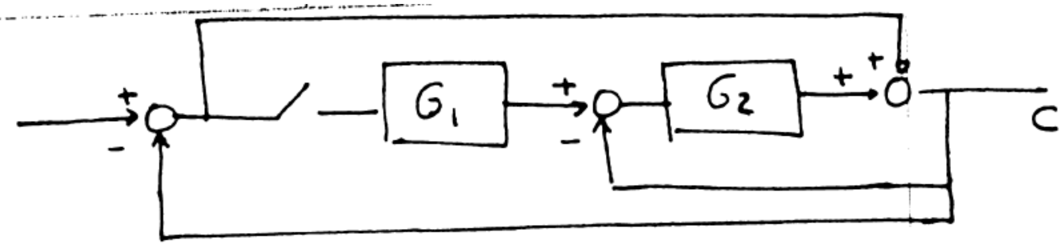
Quiz



We saw that this system does not have a TF $\Rightarrow$ can't find the characteristic equation by finding the poles of $G_{ce}(z)$

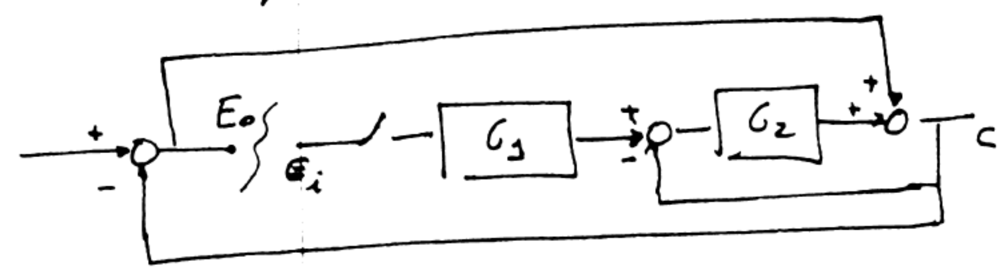
Quiz

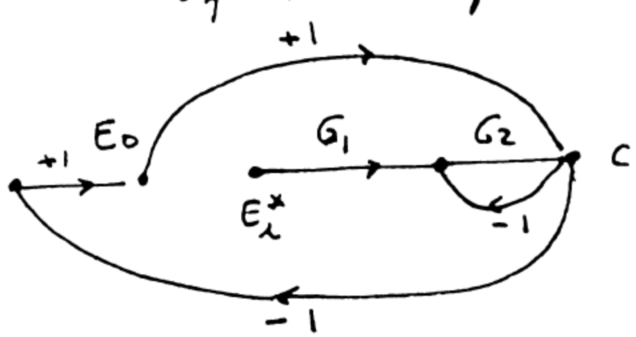
Example 2:



We saw that this system does not have a TF ⇒ can't find the characteristic equation by finding the poles of $G_{ce}(z)$

⇒ (1) set $R = 0$
(2) open the loop in front of the sampler



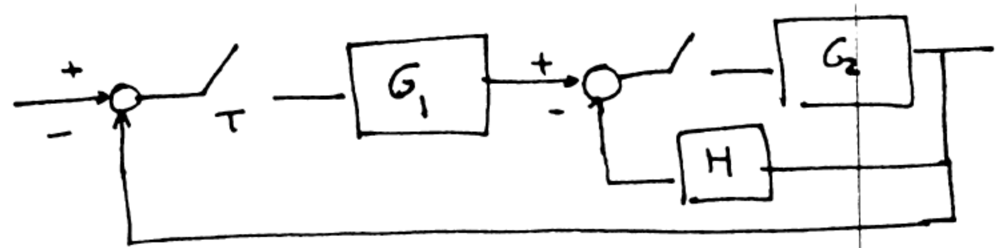


$$\Delta = 1 + G_2 + 1 = 2 + G_2$$

$$\frac{E_o}{E_i^*} = -\frac{G_1 G_2}{2 + G_2} \Rightarrow \frac{E_o^*}{E_i^*} = -\left[\frac{G_1 G_2}{2 + G_2}\right]^*$$

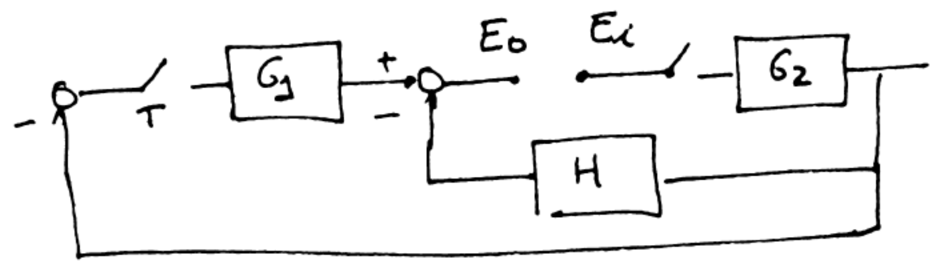
$$\Rightarrow G_{op} = -\mathcal{Z}\left[\frac{G_1 G_2}{2 + G_2}\right] \Rightarrow \text{Char equation: } 1 + \mathcal{Z}\left[\frac{G_1 G_2}{2 + G_2}\right] = 0$$

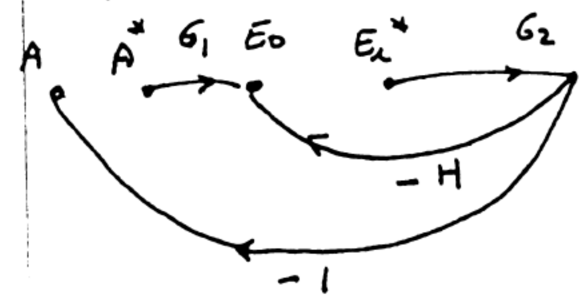
Example 3:



Here we have <u>two</u> samplers: Q: which one do we open
A: it doesn't matter (you get the same result)

The textbook opens the first one, so let's open the second:





$$\left.\begin{array}{l} E_o = G_1 A^* - HG_2 E_i^* \\ A = -G_2 E_i^* \Rightarrow A^* = -G_2^* E_i^* \end{array}\right\} \begin{array}{l} E_o = -G_1 G_2^* E_i^* - (HG_2) E_i^* \\ E_o^* = -\left(G_1^* G_2^* + (HG_2)^*\right) E_i^* \end{array}$$

$$\Rightarrow \quad G_{op} = -\left[G_1(z) G_2(z) + \mathcal{Z}[HG_2] \right]$$

$\Rightarrow$ Char equation: $\quad 1 + G_1(z)G_2(z) + \mathcal{Z}[HG_2] = 0$

(same as in the book)