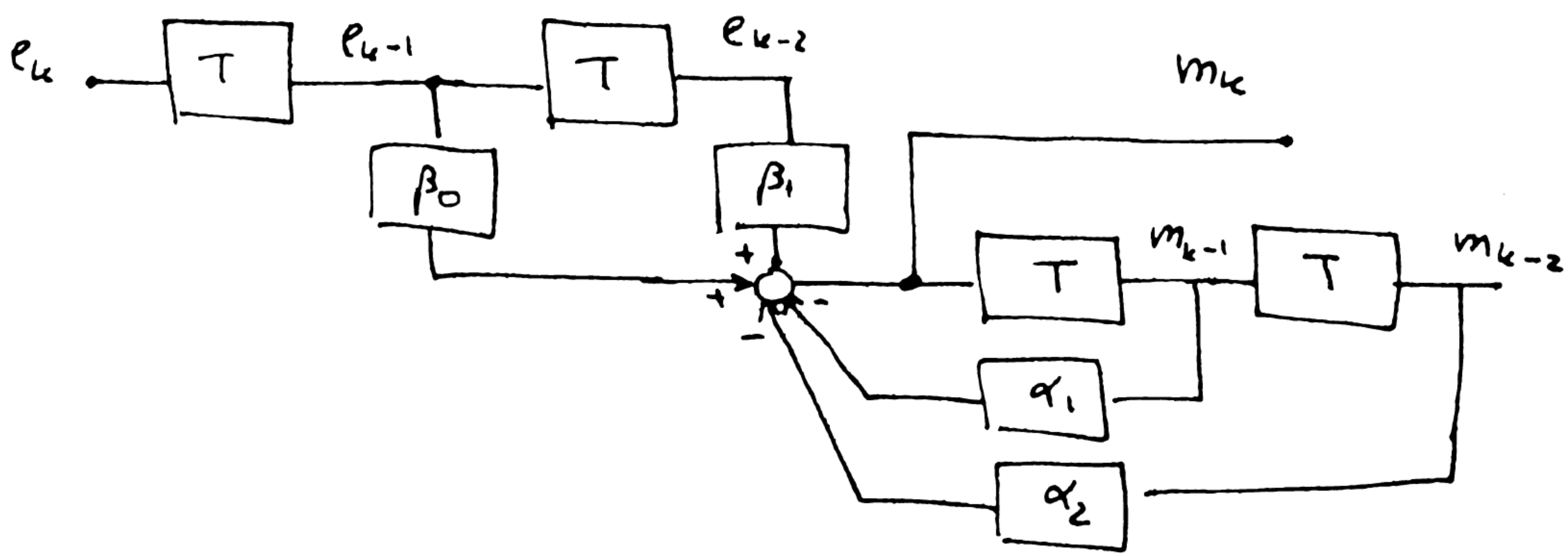
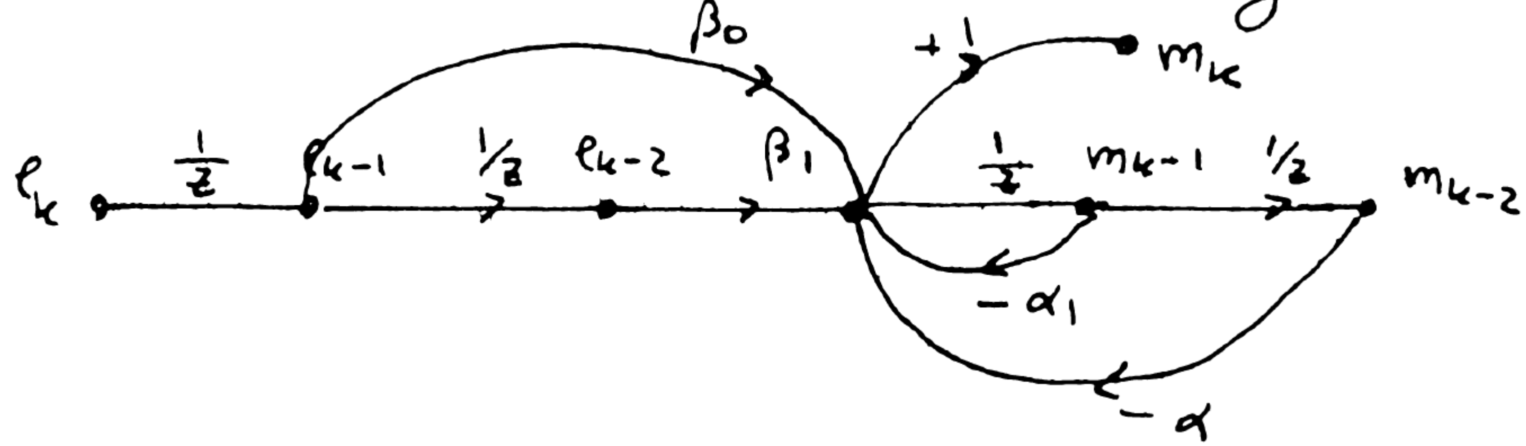
Taking inverse Z transforms on both sides yields:

$$m_k + \alpha_1 m_{k-1} + \alpha_2 m_{k-2} = \beta_0 e_{k-1} + \beta_1 e_{k-2}$$

Now we can "build" a system that is described precisely by this equation:



Sanity check: We can transform this diagram back to the z-domain $\left(\boxed{T} \Rightarrow \frac{1}{z} \right)$ and check the T.F. using Mason's formula:



Exercise: check that indeed you get the right T.F.

EECE 5610 Digital Control Systems

Lecture 6

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<u>A</u>: Not necessarily. Note that we started out with a <u>second order</u> T.F. Hence we would expect to be able to realize it with <u>just two delays</u>. However our realization uses 4 !

Let's look at the problem again and try <u>reverse engineering</u>:
first find a signal flow graph and then the simulation diagram

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$\Rightarrow$ Let's build something that has

$$\Delta = 1 + \frac{\alpha_1}{z} + \frac{\alpha_2}{z^2}$$

and $\sum M_i \Delta_i = \frac{\beta_0}{z} + \frac{\beta_1}{z^2}$

(Note: we needed only two "$\frac{1}{z}$" blocks)

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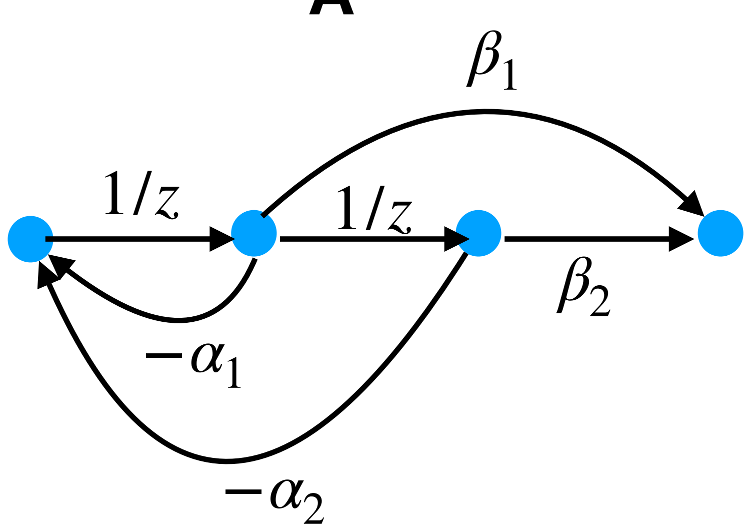
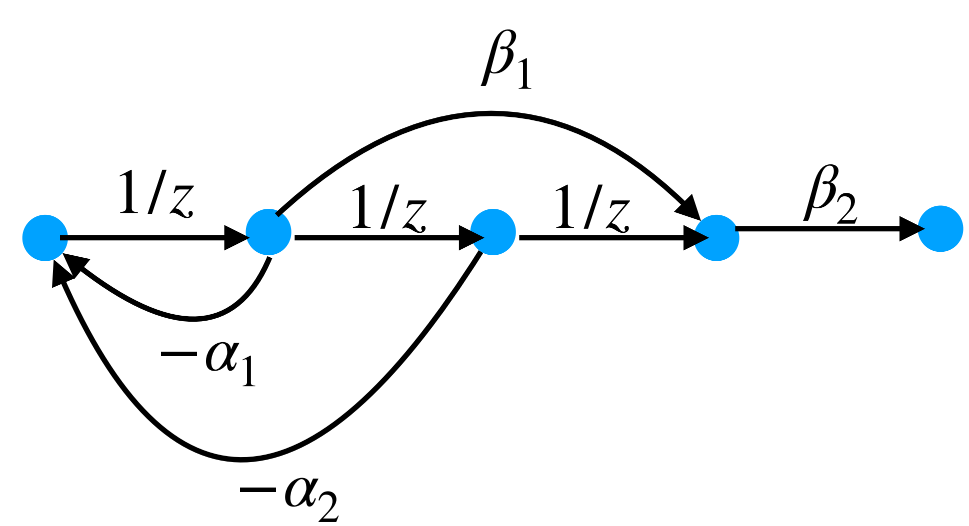
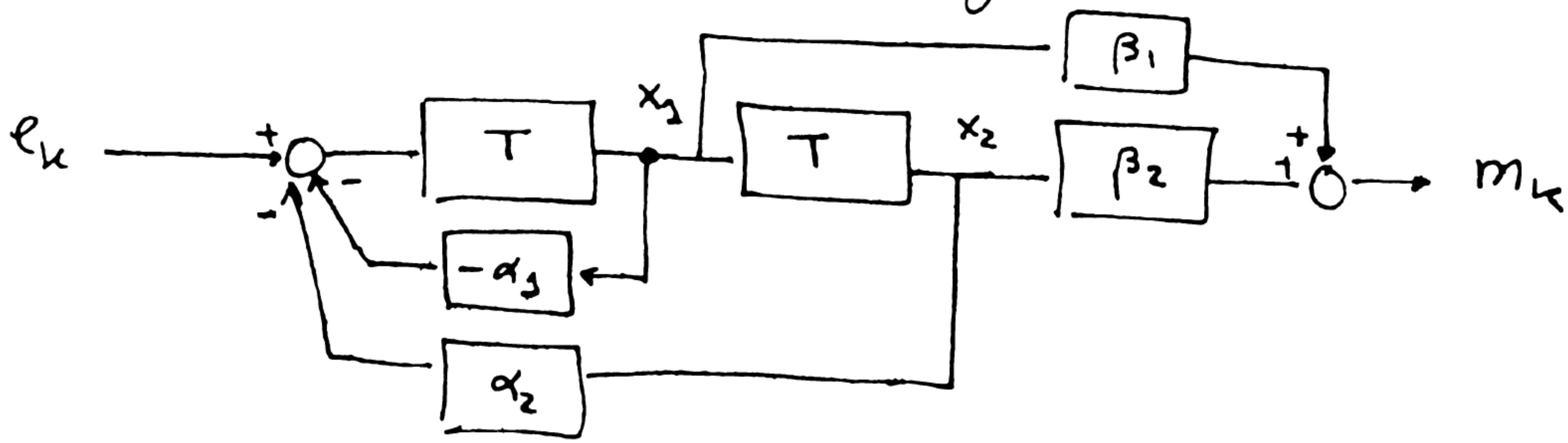


B



From here we get the following simulation diagram:



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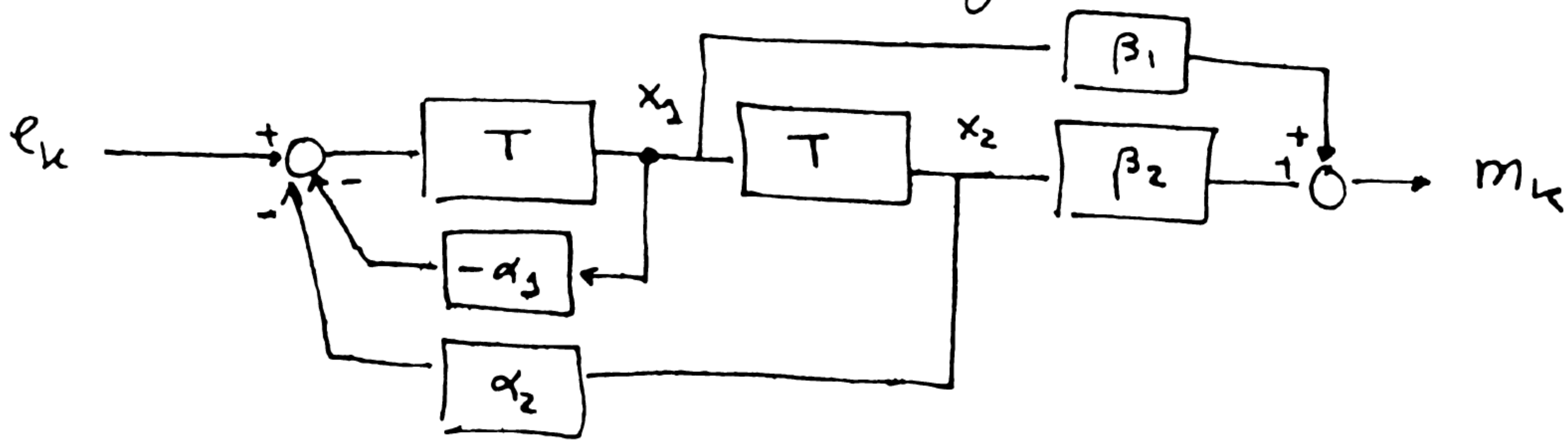

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Do the intermediate variables "x_i" have any significance?

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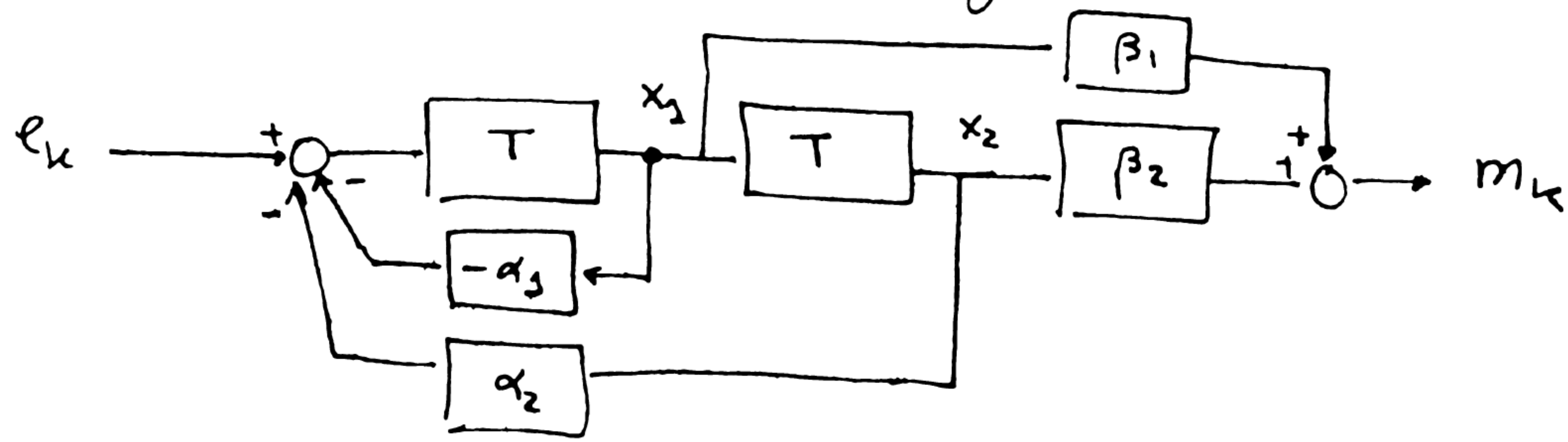

Q: Is this the "minimal" realization? Is it unique?
Do the intermediate variables "x_i" have any significance?

- Turns out that to answer these questions we need to introduce the concept of <u>state variables</u>

- State Space Models:

Consider a generic transfer function of the form

$$G(z) = \frac{M(z)}{E(z)} = \frac{b_0 + b_1 z + \quad + b_{n-1} z^{n-1}}{a_0 + a_1 z + \quad . \quad a_{n-1} z^{n-1} + z^n}$$

(note that it is strictly proper)

Dividing by z^{-n} yields:

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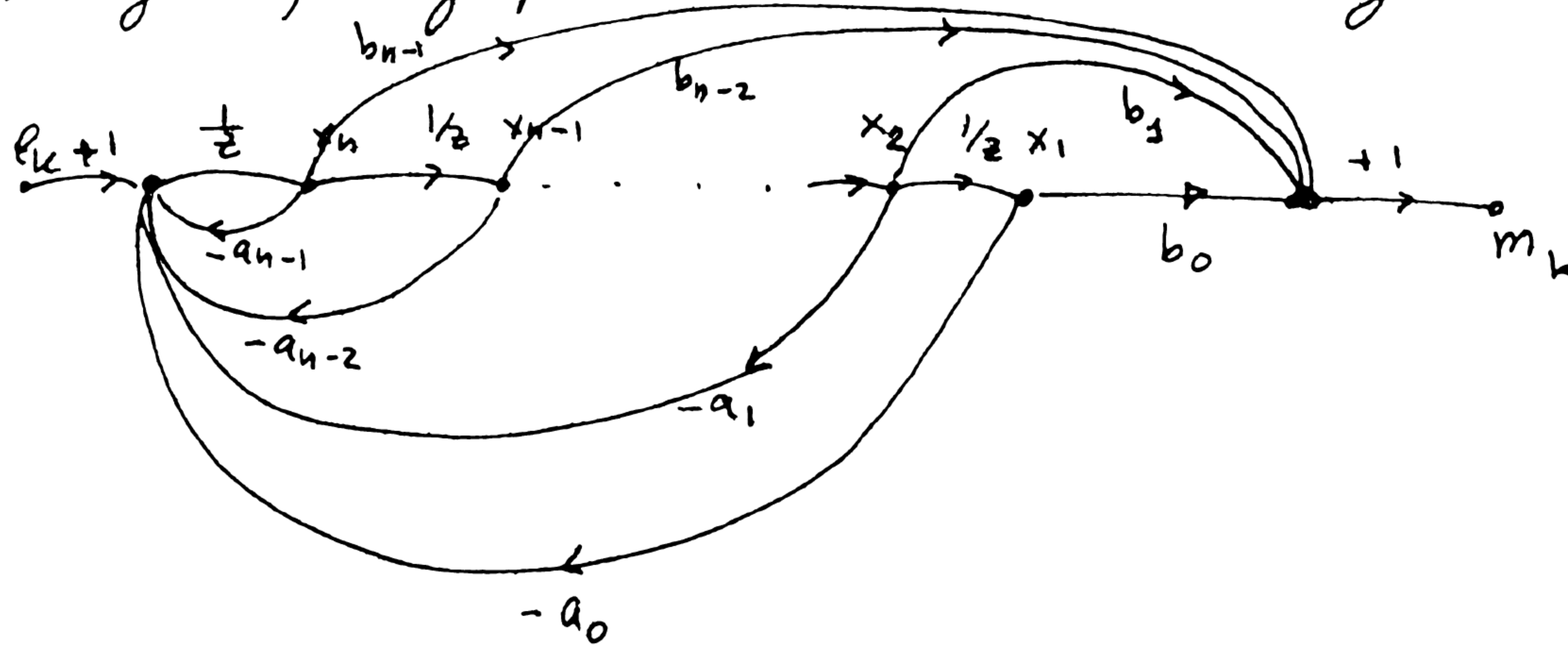
A signal flow graph that realizes this T.F is given by:



Note that we have n loops (all touching) with gains $L_i = -a_{n-i}\left(\frac{1}{z}\right)^i$ and n forward paths, each with $\Delta_i = \Delta$ and $M_i = \dfrac{b_{n-i}}{z^i}$

According to Mason's formula: $$G(z) = \frac{\Sigma M_i \Delta_i}{\Delta} = \frac{b_{n-1} z^{-1} + \cdots \; b_0 z^{-n}}{1 + a_{n-1} z^{-1} + \cdots \; a_0 z^{-n}}$$

(precisely what we wanted)

Q: Here e_k is the input and u_k is the output, but what are the (internal) variables x_k?

Q: Here e_k is the input and m_k is the output, but what are the (internal) variables x_k?

A: x_k are called the <u>states of the system</u>. These state variables represent the minimum amount of information that (together with the input) is necessary to determine the future evolution of the system. In other words, they encapsulate <u>all</u> the past history.

From the signal flow graph we get the following equations:

$$x_n(k+1) = e(k) - a_{n-1}x_n(k) - a_{n-2}x_{n-1}(k) \cdots - a_0 x_1(k)$$

$$x_{n-1}(k+1) = x_n(k)$$

$$\vdots$$

$$x_1(k+1) = x_2(k)$$

$$m(k) = b_0 x_1(k) + b_1 x_2(k) + \cdots + b_{n-1} x_n(k)$$

or, in matrix form:

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{bmatrix} 0 & 1 & \cdots & & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & & & \\ -a_0 & & & & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1(k) \\ \vdots \\ x_n(k) \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix} e(k)$$

$$m(k) = \begin{bmatrix} b_0 & b_1 & \cdots & b_{n-1} \end{bmatrix} \begin{bmatrix} x_1(k) \\ \vdots \\ x_n(k) \end{bmatrix}$$

In compact notation:

$$x(k+1) = A\,x(k) + B\,e_k$$

$$m(k) = C\,x(k)$$

where $\left\{\begin{array}{l} A \text{ is a } n \times n \\ B \text{ is a } n \times n_u \\ C \text{ is a } n_z \cdot n \end{array}\right\}$ matrix

$X = \begin{bmatrix} x_1 \\ \\ x_n \end{bmatrix}$ is an $n \times 1$ vector

- State space formulations were introduced in Control theory in the early 60's

Advantages of state space methods:

1) can handle multiple-input / multiple-output plants

2) Can answer our earlier question on what is the minimum number of delays required to realize a given transfer function (minimal realizations) and analyze why some realizations are <u>not</u> minimal (unobservable and/or uncontrollable states)

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For instance, it can be shown that the minimum number of delays required in our earlier example is indeed two, except for some special values of $\beta_0, \beta_1, \alpha_1$ and α_2 where it may reduce to 1

3) Give a systematic way of designing controllers that place all closed-loop poles at desired locations

<u>However</u>: Dealing with state-space representations requires the use of tools from linear algebra and linear vector spaces that are beyond the scope of this course (at NU these tools are covered in ECE 7200)

Additional drawback of state space methods: "traditional" state space methods are less robust than frequency domain based methods (such as the ones learned in ECE 5580/5610)

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<u>Current state of the art</u>: mix of both techniques. We use state space tools to design controllers based on a generalization of the methods covered in 5580/5610

SAMPLING AND RECONSTRUCTION (Chapter 3)

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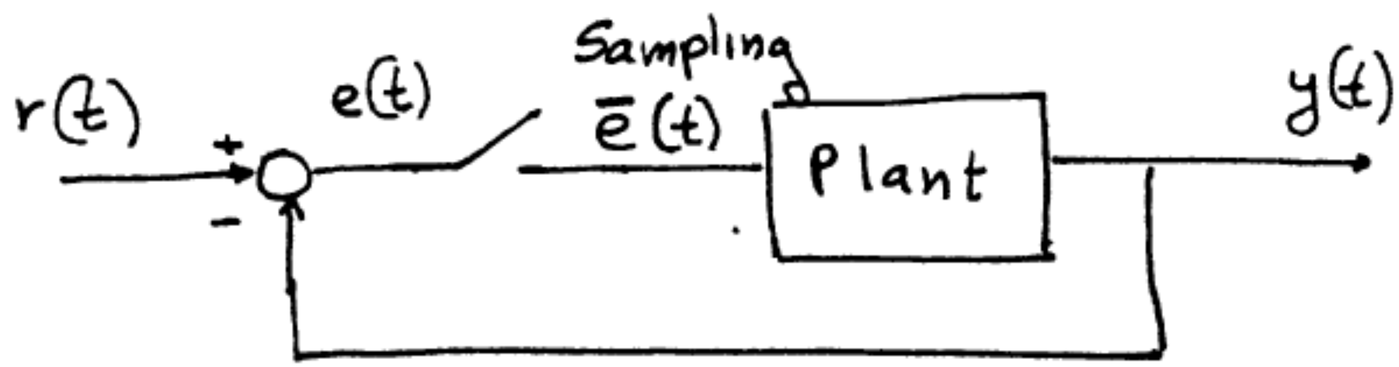
Suppose that we want to control a <u>continuous time</u> plant using a digital controller. We could have a loop of the form



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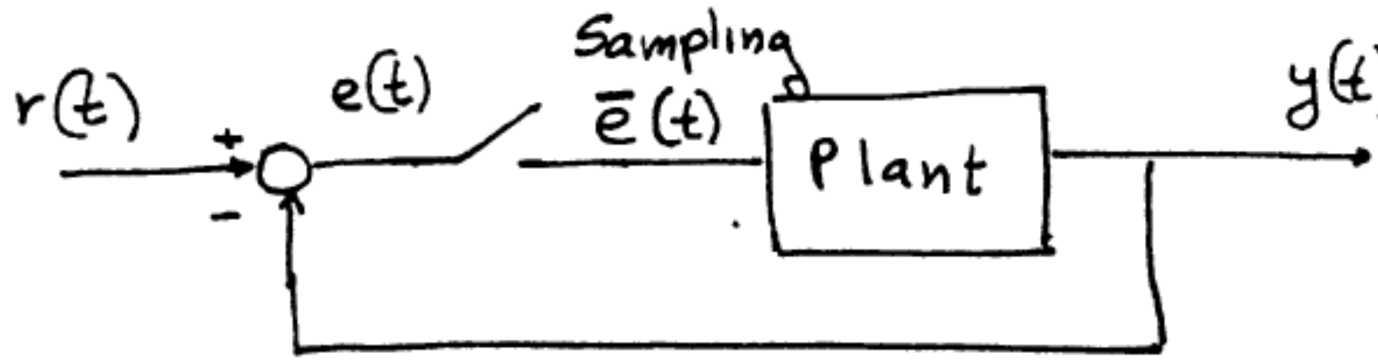
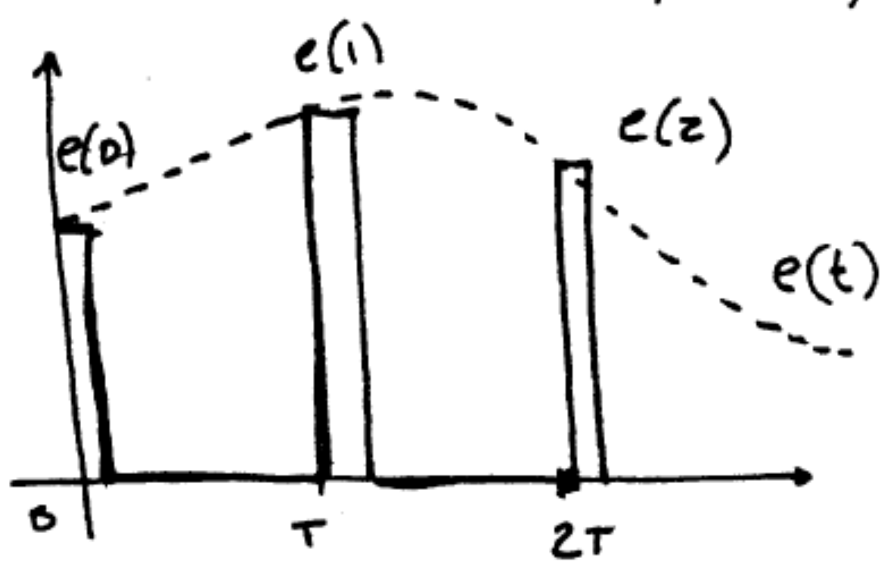
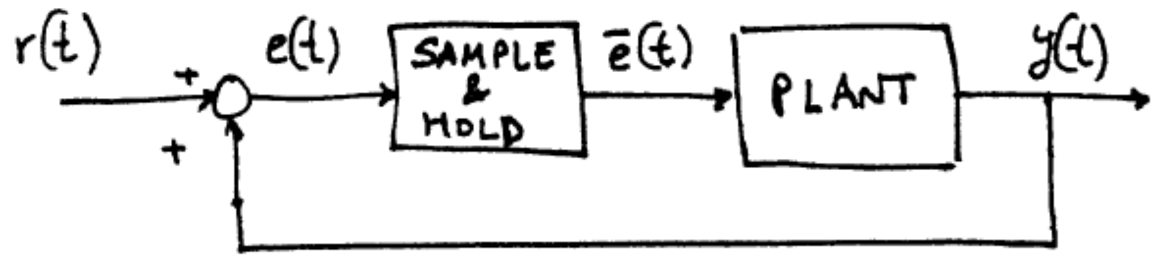


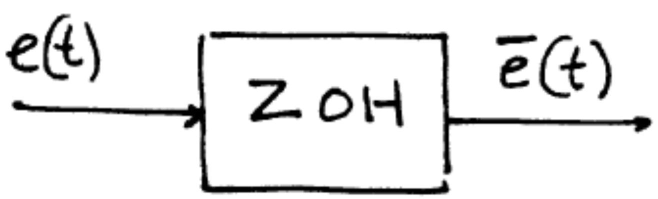
where ē(t), the output of the sampler is of the form:



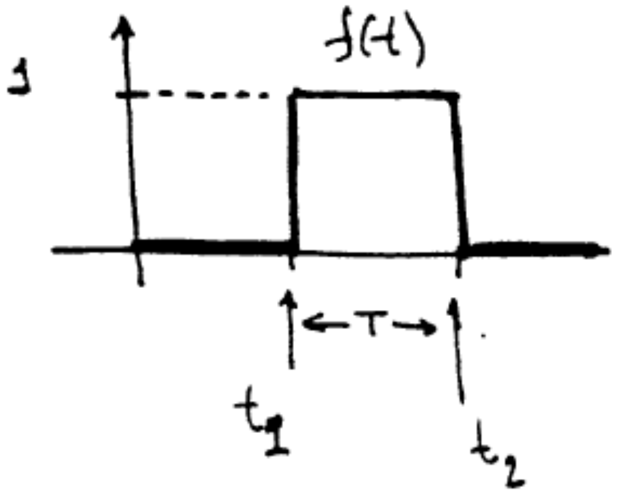
However: you <u>don't</u> want to apply a signal like this to the plant because of its high frequency components (which could excite high frequency resonant modes). <u>Solution</u>: use some sort of device to "reconstruct" the original signal:



We could use as sample & hold device a <u>zero-order hold</u> that holds the output signal constant during the sampling period



Suppose that we want to write down $\bar{e}(t)$ in terms of $e(t)$. We will use the following property:



$$f(t) = u(t-t_1) - u(t-t_1-T)$$

$$F(s) = \frac{1}{s}\left[e^{-t_1 s} - e^{-(t_1+T)s}\right] = \frac{e^{-t_1 s}}{s}\left(1 - e^{-Ts}\right)$$

$$\bar{e}(t) = e(0)\left[u(t) - u(t-T)\right] + e(T)\left[u(t-T) - u(t-2T)\right] + \cdots +$$

$\Downarrow$ (Laplace transform)

$$\bar{E}(s) = e(0)\left(\frac{1-e^{-Ts}}{s}\right) + e(T)\, e^{-Ts}\left(\frac{1-e^{-Ts}}{s}\right) + e(2T)\, e^{-2Ts}\left(\frac{1-e^{-Ts}}{s}\right) + \cdots$$

$$\bar{E}(s) = \left(\frac{1-e^{-Ts}}{s}\right) * \left[e(0) + e(T)\, e^{-Ts} + e(2T)\, e^{-2Ts} + \cdots\right]$$

$$= \underbrace{\left(\frac{1-e^{-Ts}}{s}\right)}_{} \cdot \underbrace{\sum_{0}^{\infty} e(kT)\, e^{-kTs}}_{}$$

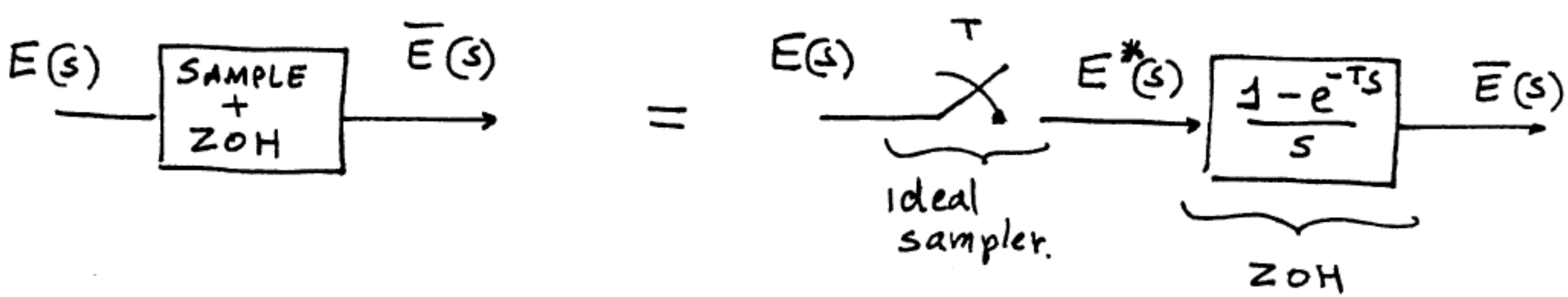
this is related only to the input $e(t)$ and T, but not the ZOH

This is independent of $e(t)$ an can be tought off as a transfer function associated with the Z.O.H

Thus it is <u>convenient</u> to represent the sample and hold operation as a <u>combination</u> of two operations

(a) an "ideal" sampler that generates $\sum_{0}^{\infty} e(kT)\, e^{-kTs}$ from $E(s)$

(b) a ZOH with transfer function $G_{ZOH}(s) = \dfrac{1-e^{-sT}}{s}$



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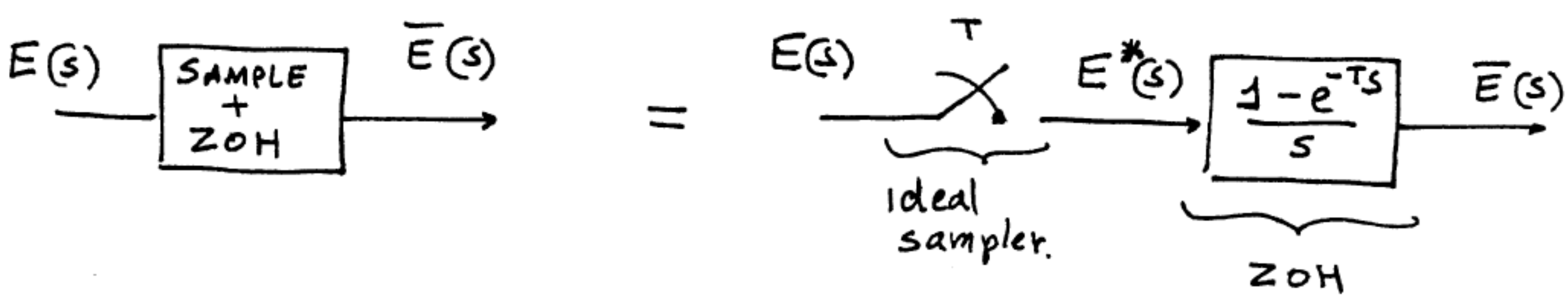

We will call the intermediate variable $E^*(s)$ the <u>"starred transform"</u>:

$$\boxed{\text{Definition:} \quad E^*(s) = \sum_{k=0}^{\infty} e(kT)\, e^{-kTs}}$$

Note: The ideal sampler $E(s) \xrightarrow{T} E^*(s)$ **does not** model a physical sampler and it does **not** have a Transfer function (because it is a many to one mapping: different $E(s)$ can yield the same $E^*(s)$)

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However: The **combination** does model the operation of the combined sample & hold. and gives the correct mathematical description.