EECE 5610 Digital Control Systems

Lecture 4

Milad Siami
Assistant Professor of ECE
Email: m.siami@northeastern.edu

Northeastern University
College *of* Engineering

- **Scaling in z-plane:**

$$Z\{(r^{-k}e_k)\} = E(r\dot{z})$$

Proof: $$Z\{(r^{-k}e_k)\} = \sum_{0}^{\infty} (e_k\, r^{-k})z^{-k} = \sum_{0}^{\infty} e_k\,(rz)^{-k} = E(rz) \quad \#$$

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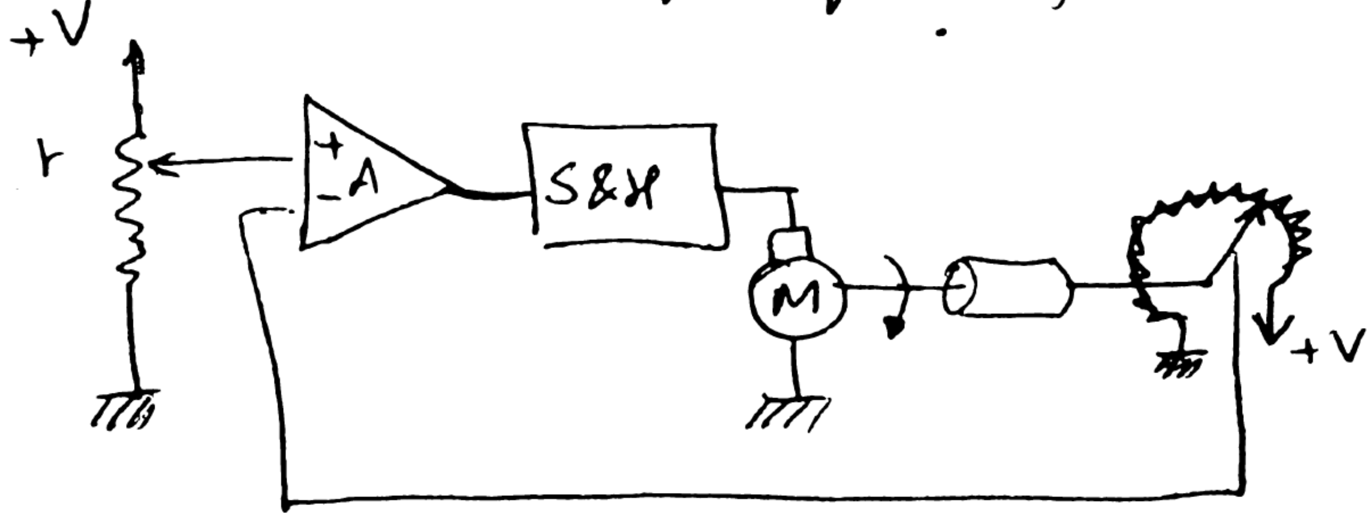
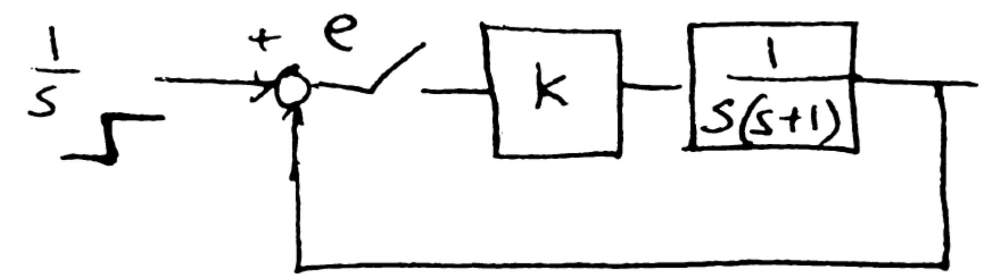
- **Two important Theorems:** (Initial and Final value theorems)

They allow for the calculation of limits and steady state values without having to actually find the transform

For instance, suppose that we want to analyze the steady-state error to a step input for a position control system:





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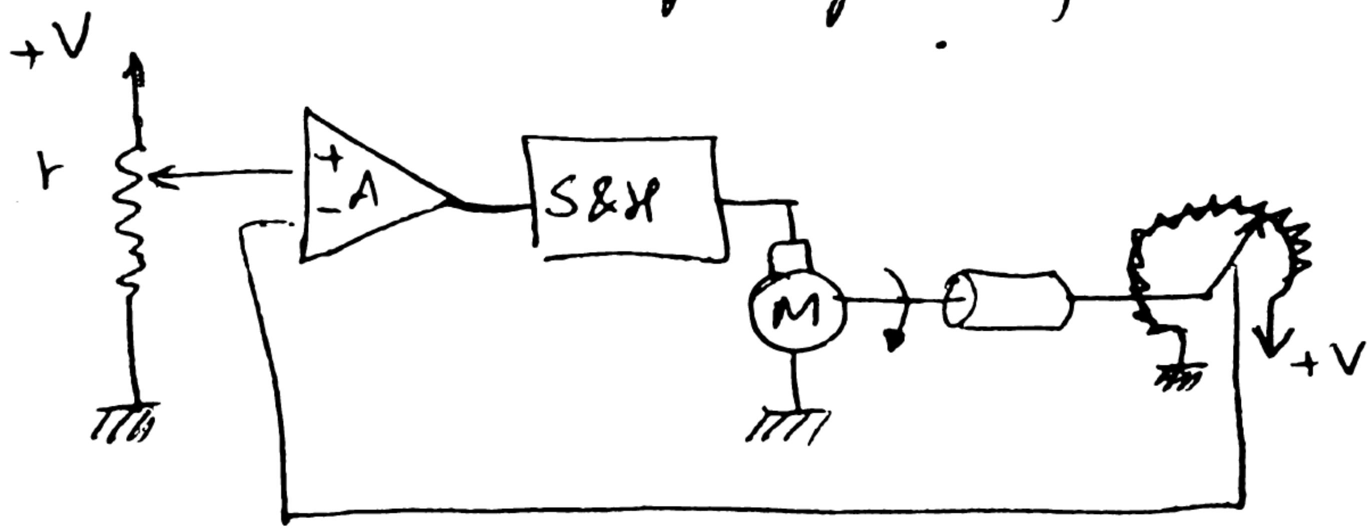
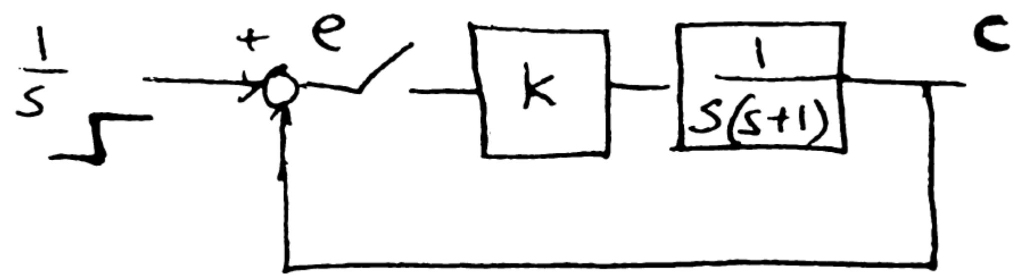
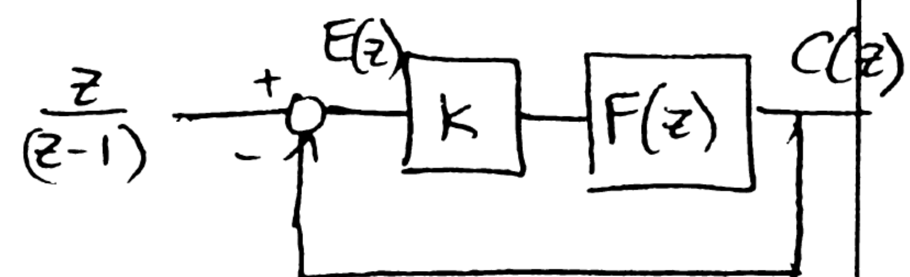




We could proceed as follows

(1) find the discrete time equivalent:



(2) find $C(z)$

(3) Take the inverse z-transform $\Rightarrow \{c_k\}$

(4) find $c_{ss} = \lim_{k\to\infty} \{c_k\}$

However, this is an overkill if we are only interested in c_{ss} (and not the transient)

- <u>Final value theorem</u>:

$$\boxed{\lim_{k\to\infty} e(k) = \lim_{z\to 1} (z-1)\,E(z)}$$

(Physical significance: the behavior of $e(k)$ as $k\to\infty$ is related to the low frequency $(s=0 \longleftrightarrow z=1)$ components of $E(z)$)

Proof: Consider the sequence $\{e(k+1)-e(k)\}$. We can compute its Z transform 2 different ways (a) directly
(b) using the time shift theorem

(a) yields:

$$Z\{e_{k+1}-e_k\} = \lim_{n\to\infty}\sum_0^n \{e_{k+1}-e_k\} z^{-k} = \lim_{n\to\infty}\left\{e_1-e_0+(e_2-e_1)\frac{1}{z}+\cdots(e_{n+1}-e_n)\frac{1}{z^n}\right\}$$

$$= \lim_{n\to\infty}\left\{-e_0+e_1\left(1-\frac{1}{z}\right)+\frac{e_2}{z}\left(1-\frac{1}{z}\right)+\cdots\frac{e_{n+1}}{z^n}\right\}$$

$$\Rightarrow \lim_{z\to 1} Z\{e_{k+1}-e_k\} = \lim_{n\to\infty}\{-e(0)+e(n+1)\} = -e(0)+\lim_{n\to\infty} e(n) \qquad (1)$$

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$$= \lim_{n\to\infty} \left\{ -e_0 + e_1\left(1 - \frac{1}{z}\right) + \frac{e_2}{z}\left(1 - \frac{1}{z}\right) + \cdots \frac{e_{n+1}}{z^n} \right\}$$

$$\Rightarrow \lim_{z\to 1} Z\{e_{k+1} - e_k\} = \lim_{n\to\infty} \{-e(0) + e(n+1)\} = -e(0) + \lim_{n\to\infty} e(n) \qquad (1)$$

(b) yields:

$$Z\{e_{k+1}\} = z\left[E(z) - e_0\right] \Rightarrow Z\{e_{k+1} - e_k\} = zE(z) - ze_0 - E(z)$$

$$= (z-1)E(z) - ze_0$$

$$\Rightarrow \lim_{z\to 1} Z\{E(z) - e_0\} = -e(0)$$

(2)

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(b) yields:

$f(k)$

$$Z\{e_{k+1}\} = z[E(z) - e_0] \Rightarrow Z\{e_{k+1} - e_k\} = zE(z) - ze_0 - E(z)$$

$$= \cancel{(z-1)E(z)} - ze_0 \quad (2)$$

$$\Rightarrow \lim_{z\to 1} Z\{E(z) - e_0\} = -e(0)$$

Comparing (1) and (2) we have

$$\lim_{n\to\infty} e(n) - \cancel{e(0)} = \lim_{z\to 1} (z-1)E(z) - \cancel{z e_0} \Rightarrow \boxed{\lim_{n\to\infty} e(n) = \lim_{z\to 1} (z-1)E(z)}$$

Examples (1) unit step: $e_k = 1$ for all k

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$$E(z) = \frac{1}{1 - 1/z} = \frac{z}{z-1}$$

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Examples

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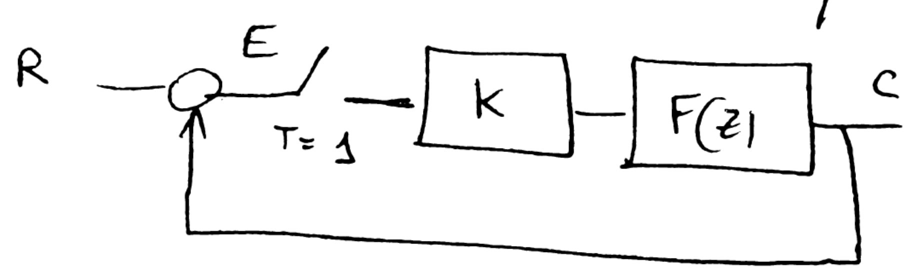
(2) Back to the position control system. (we will see it later in the semester) It can be shown that if $T=1$ then the discrete time equivalent is



with $F(z) = \dfrac{0.368z + 0.264}{z^2 - 1.368z + 0.368}$

$R = \dfrac{z}{z-1}$

The overall transfer function: $G(z) = \dfrac{C}{R} = \dfrac{kF}{1+kF}$, $H(z) = \dfrac{E}{R} = \dfrac{1}{1+kF}$

$R \times G = C \Rightarrow$

$$\lim_{z \to 1} (z-1)\,G(z) \Rightarrow C_\infty$$

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Proof: follows immediately from the definition:

$$E(z) = e_0 + \cancel{\frac{e_1}{z}} + \cancel{\frac{e_2}{z^2}} + \cdots \cancel{\frac{e_n}{z^n}} + \cdots$$

when $z \to \infty$ all the terms in the RHS except the first drop out

- **Convolution of time sequences:**

Given two time sequences $e_1(k)$ and $e_2(k)$ its convolution $f = e_1 * e_2$ is defined as:

$$f_k = \sum_{\ell=-\infty}^{\ell=+\infty} e_1(\ell)\, e_2(k-\ell)$$

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If $e_1(k)$ and $e_2(k) = 0$ for $k<0$ (as in this course) then the formula above reduces to:

$$f_k = \sum_{\ell=0}^{k} e_1(\ell) \cdot e_2(k-\ell)$$

$$\boxed{Z\{e_1 * e_2\} = E_1(z) \cdot E_2(z)}$$

convolution in time domain $\Longleftrightarrow$ product in z-domain

Very useful result: allows for analyzing the combination of dynamic systems by just using linear algebra

$$\longrightarrow \boxed{G_1(z)} \longrightarrow \boxed{G_2(z)} \longrightarrow$$

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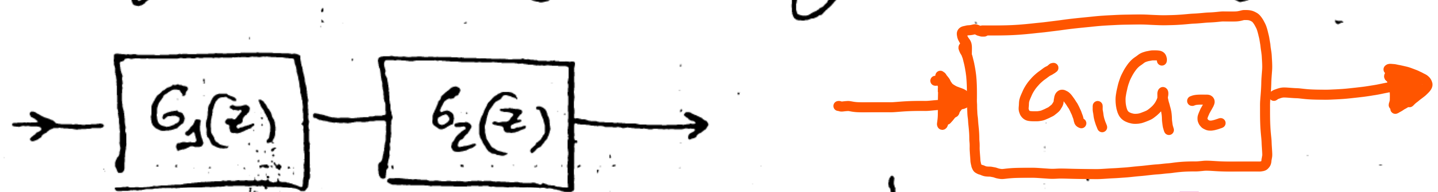

Proof: $\quad f_k z^{-k} = \sum_{\ell=0}^{k} g_1(\ell)\, g_2(k-\ell)\, z^{-k} = \sum_{\ell=0}^{k} g_1(\ell)\, z^{-\ell} \cdot g_2(k-\ell)\, z^{-(k-\ell)}$

$$= \sum_{\ell=0}^{\infty} g_1(\ell)\, z^{-\ell} \cdot g_2(k-\ell)\, z^{-(k-\ell)}$$

(where we exploited the fact that $g_2(n)=0$ for $n<0$) $\Rightarrow$

$$F(z) = \sum_{k=0}^{\infty} f_k z^{-k} = \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} g_1(\ell)\, z^{-\ell}\, g_2(k-\ell)\, z^{-(k-\ell)}$$

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<u>Proof</u>: $$f_k z^{-k} = \sum_{\ell=0}^{k} e_1(\ell)\, e_2(k-\ell)\, z^{-k} = \sum_{\ell=0}^{k} e_1(\ell) z^{-\ell} \cdot e_2(k-\ell) z^{-(k-\ell)}$$

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Now let $k-\ell=m$ $\Rightarrow$ $F(z) = \sum_{k=0}^{\infty} e_1(\ell) z^{-\ell} \sum_{m=0}^{\infty} e_2(m) z^{-m} = E_1(z) \cdot E_2(z)$ #

- <u>Solution of difference equations</u>

Recall that one of the motivations for looking into the z-transform was the expectation that it will help us solve difference equations (as the Laplace transform helped with differential equations). Next we will see how to accomplish this

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<u>Example:</u> consider the difference equation:

$$m(k) = e(k) - e(k-1) - m(k-1) \qquad k \geq 0$$

- ## Solution of difference equations

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Example: consider the difference equation:

$$m(k) = e(k) - e(k-1) - m(k-1) \qquad k \geq 0$$

Assume initial conditions $m(-1)=0$, $e(-1)=0$ and take z-transforms on both sides:

$$\begin{array}{ccccccc} m(k) & = & e(k) & - & e(k-1) & - & m(k-1) \\ \downarrow z & & \downarrow z & & \downarrow z & & \downarrow z \\ M(z) & = & E(z) & - & \frac{1}{z}E(z) & - & \frac{1}{z}M(z) \end{array}$$

$$\Rightarrow \left(1+\frac{1}{z}\right)M(z) = \left(1-\frac{1}{z}\right)E(z)$$

solving for $M(z)$ yields: $\boxed{M(z) = \left(\frac{z-1}{z+1}\right)E(z)}$

take z-inverse of $M(z)$ at end

Assume that $e_k = \begin{cases} 0 & k < 0 \\ 1 & k \geq 0 \end{cases} \Rightarrow E(z) = \dfrac{z}{z-1}$

$$\Rightarrow \quad M(z) = \frac{z}{z+1} = \frac{1}{1+1/z} \Rightarrow \boxed{m_k = (-1)^k} \quad \#$$

Sanity check:

$$m(0) = e(0) - e(-1) - m(-1) = 1$$
$$m(1) = e(1) - e(0) - m(0) = -1$$
$$\vdots$$
$$m(k) = \cancel{e(k)} - \cancel{e(k-1)} = -m(k-1)$$

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Features: (1) We solve the difference equation by solving a single algebraic equation.

(2) method incorporates initial conditions

__Caveat:__ we need to find an inverse z-transform

In general we have:

$$x(k+n) + a_1 x(k+n-1) + \cdots + a_n x(k) = b_0 e(k+n) + \cdots + b_m e(k+n-m)$$

Taking z transforms on both sides yields:

(Recall that $x(k+n) \overset{Z}{\longleftrightarrow} z^n \left[X(z) - \sum_0^{n-1} x(k) z^{-k} \right]$)

$$\left(z^n + a_1 z^{n-1} + \cdots + a_n\right) X(z) + (IC) = \left(b_0 z^n + b_1 z^{n-1} + \cdots + b_m z^{n-m}\right) E(z) + (IC)$$

where $IC = \left\{ \begin{array}{l} x(0),\ x(1) \cdots, x(n-1) \\ e(0),\ e(1) \cdots e(n-1) \end{array} \right\}$

If $IC = 0$ then $X(z) = \left[\dfrac{b_0 z^n + b_1 z^{n-1} + \cdots + b_m z^{n-m}}{z^n + a_1 z^{n-1} + \cdots + a_n} \right] E(z)$

• Concept of Transfer Function:

Transfer function: Ratio of the z-transform of the output to the z-transform of the input when all initial conditions are set to zero

From linearity it follows that we can represent the I/O relationship for an LTI system as:

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<u>Fact</u>: For finite dimensional systems $G(z)$ is rational, i.e.

$$G(z) = \frac{N(z)}{D(z)}$$ where N and D are polynomials

Moreover, if the system is causal then $G(z)$ is proper, i.e. degree $(N) \leq$ Degree (D)

<u>Notation</u>: Roots of $N(z)=0$: zeros of the system (related to performance)

Roots of $D(z)=0$: poles of the system (related to stability)

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- <u>Inverse z-transform</u>

Our method for solving difference equations works as follows

$$\text{difference equation(s)} \xrightarrow{z\ \text{transf}} \text{algebraic equation(s)} \xrightarrow{\text{solve}} X(z) \xrightarrow{\mathcal{Z}^{-1}} \{x_k\}$$

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Three methods

(1) partial fraction expansion

(2) power series

(3) inversion formula: $e_k = \frac{1}{2\pi j} \oint E(z)\, z^{k-1}\, dz$

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However, note that <u>most</u> z transforms are of the form

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Example: $E_1(z) = \dfrac{z}{(z-1)(z-2)}$; $\dfrac{E_1(z)}{z} = \dfrac{1}{(z-1)(z-2)} = \dfrac{-1}{z-1} + \dfrac{1}{z-2}$

$$\Rightarrow E(z) = \frac{-z}{z-1} + \frac{z}{z-2} \xrightarrow{Z^{-1}} e_k = -1 + 2^k$$

In general:

$$F(z) = \frac{E(z)}{z} = \frac{N(z)}{D(z)} = b_0 \frac{\prod (z - z_i)}{\prod (z - p_i)} = \frac{k_1}{z-p_1} + \frac{k_2}{z-p_2} + \cdots + \frac{k_n}{z-p_n}$$

(assuming all p_i different) where $k_i = (z - p_i) F(z) \big|_{z = p_i}$

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If we have repeated poles the situation gets more complicated. Suppose that the pole p_1 has multiplicity $r \Rightarrow$ we have r terms associated with this pole:

$$\frac{E(z)}{z} = F(z) = \frac{N(z)}{(z-p_1)^r (z-p_2) \cdots (z-p_m)} = \frac{k_{11}}{(z-p_1)} + \frac{k_{12}}{(z-p_1)^2} + \cdots \frac{k_{1r}}{(z-p_1)^r} + \frac{k_2}{(z-p_2)} + \cdots \frac{k_m}{(z-p_m)}$$

where:
$$k_{1j} = \frac{1}{(r-j)!} \frac{d^{r-j}}{dz^{(r-j)}} \left[(z-p_1)^r F(z) \right] \Big|_{z = p_1}$$

<u>Example</u>:

$$E(z) = \frac{z}{(z-1)^2(z-2)} \Rightarrow \frac{E(z)}{z} = \frac{1}{(z-1)^2(z-2)} = \frac{k_{11}}{(z-1)} + \frac{k_{12}}{(z-1)^2} + \frac{k_2}{(z-2)}$$

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$k_2 = 1;$ $$k_{11} = \frac{1}{1!}\frac{d}{dz}\left[\cancel{(z-p_1)^2}\frac{1}{\cancel{(z-1)^2}(z-2)}\right]\Bigg|_{z=1} = 1 \cdot \frac{d}{dz}\left(\frac{1}{z-2}\right)\Bigg|_{z=1} = \frac{-1}{(z-2)^2}\Bigg|_{z=1} = -1$$

$$k_{12} = \frac{1}{0!}\,\cancel{(z-p_1)^2}\frac{1}{\cancel{(z-1)^2}(z-2)}\Bigg|_{z=1} = -1$$

$$E(z) = \frac{-z}{(z-1)^2} - \frac{z}{(z-1)} + \frac{z}{(z-2)} \Rightarrow e_k = -k - 1 + 2^k \quad \#$$