Digital Control Systems Homework #3

Problem 1

(a) Find $e(0), e(1),$ and $e(10)$ for

$$E(z) = \frac{0.1}{z(z - 0.9)}$$

using the inversion formula

Inversion Formula: $\frac{1}{2\pi j}\oint E(z)z^{k-1}\mathrm{dz}$

The Inversion formula can be rewritten as the sum of the residues because of the property of complex functions

$$e(k) = \sum_{\text{at poles }\left[E(z)z^{k-1}\right]} \text{Residues of } \frac{0.1z^{k-1}}{z(z-0.9)}$$

For $e(0)$,

$$e(0) = \sum_{z=0,0.9} \text{residues of } \frac{0.1}{z(z-0.9)}$$

$$= \frac{0.1}{(z-0.9)}\Big|_{z=0} + \frac{0.1}{z}\Big|_{z=0.9} = -\frac{0.1}{0.9} + \frac{0.1}{0.9} = 0$$

Thus, $e(0) = 0$

For $e(1)$,

$$e(k) = \sum_{\text{at poles }\left[E(z)z^{k-1}\right]} \text{Residues of } \frac{0.1z^{k-1}}{z(z-0.9)}$$

$$e(1) = (z-0.9)\frac{0.1}{z(z-0.9)}z^{k-1}\Big|_{z=0.9}$$

$$= \frac{0.1}{(0.9)^2}(0.9)^k \rightarrow \frac{0.1}{(0.9)^2}(0.9)^1 = \frac{0.1 \times 0.9}{0.81} = \frac{0.09}{0.81} = \frac{1}{9} \text{ or } 0.1111$$

Thus, $e(1) = 0.1111$

For $e(10)$,

$$e(10) = \frac{0.1}{(0.9)^2}(0.9)^k \rightarrow \frac{0.1}{(0.9)^2}(0.9)^{10} \rightarrow 0.1(0.9)^8 = 0.043$$

Thus, $e(10) = 0.043$

(b) Check the value of $e(0)$ using the initial-value property

The initial-value property states:

$e(0) = \lim_{z\to\infty} E(z)$

So,

$e(0) = \lim_{z\to\infty} \frac{0.1}{z(z-0.9)} = \frac{0.1}{\infty(\infty-0.9)} = 0$

Thus, the initial-value property states that $e(0) = 0,$ which was derived in part (a) as well.

(c) Check the values calculated in part (a) using partial fractions

In order to break $E(z)$ into partial fractions, we use the formula

$\frac{E(z)}{z} = \frac{0.1}{z^2(z-0.9)}$

$\frac{E(z)}{z} = \frac{A_0}{z} + \frac{A_1}{z^2} + \frac{A_2}{z-0.9}$

$\frac{0.1}{z^2(z-0.9)} = \frac{A_0}{z} + \frac{A_1}{z^2} + \frac{A_2}{z-0.9}$

Multiplying the right-hand side by the denominator of $\frac{E(z)}{z}$, we get:

$0.1 = A_0 z(z-0.9) + A_1(z-0.9) + A_2 z^2$

Start by plugging in the value for $z = 0$

$0.1 = A_0(0)(0-0.9) + A_1(0-0.9) + A_2 0^2,$ which simplifies to

$0.1 = -0.9A_1$

Thus, $A_1 = -\frac{1}{9}$

Using this value fo A_1 and now plugging in value for $z = 0.9$

$0.1 = A_0(0.9)(0.9-0.9) - \frac{1}{9}(0.9-0.9) + A_2(0.9)^2,$ which simplifies to:

$0.1 = 0.81A_2$

Thus, $A_2 = \frac{0.1}{0.81} = \frac{10}{81}$

Now solving for A_0

$0.1 = z^2\left(A_0 + \frac{10}{81}\right) - z\left(0.9A_0 + \frac{1}{9}\right) + 0.1$

$$z^2\left(A_0+\frac{10}{81}\right)-z\left(0.9A_0+\frac{1}{9}\right)=0$$

Solve for $-0.9A_0-\frac{1}{9}=0$

Thus, $A_0=-\frac{1/9}{0.9}=-\frac{10}{81}$

Putting all the coefficients together and multiplying by z

$$E(z)=-\frac{10}{81}-\frac{1}{9}z^{-1}+\frac{10}{81}\frac{z}{z-0.9}$$

Using some common z-transform pairs we solve for $e(k)$

$$e(k)=-\frac{10}{81}\delta(k)-\frac{1}{9}\delta(k-1)+\frac{10}{81}(0.9)^k$$

Now that we have $e(k)$ we solve for $e(0), e(1), \text{and } e(10)$

$$e(0)=-\frac{10}{81}\delta(0)-\frac{1}{9}\delta(0-1)+\frac{10}{81}(0.9)^0$$

$$e(0)=-\frac{10}{81}+\frac{10}{81}=0$$

$$e(1)=-\frac{10}{81}\delta(1)-\frac{1}{9}\delta(1-1)+\frac{10}{81}(0.9)^1$$

$$e(1)=\frac{10}{81}(0.9)=\frac{9}{81}=\frac{1}{9}$$

$$e(10)=-\frac{10}{81}\delta(10)-\frac{1}{9}\delta(10-1)+\frac{10}{81}(0.9)^{10}$$

$$e(10)=\frac{10}{81}(0.9)^{10}=0.043$$

Thus, the values for $e(0), e(1), \text{and } e(10)$ match the values calculated in part (a)

(d) Find $e(k)$ for $k=0,1,2,3,4$ if $Z[e(k)]$ is given by

$$E(z)=\frac{1.98z}{(z^2-0.9z+0.9)(z-0.8)(z^2-1.2z+0.27)}$$

using MATLAB to calculate the roots of the polynomials in the denominator, we find:

```
a1 = [1 -0.9 0.9]; a2 = [1 -0.8]; a3 = [1 -1.2 0.27];
roots(a1)
```

```
ans = 2×1 complex
   0.4500 + 0.8352i
   0.4500 - 0.8352i
```

```
roots(a3)
```

```
ans = 2×1
    0.9000
    0.3000
```

Thus,

$$E(z) = \frac{1.98z}{(z-0.45+j0.835)(z-0.45-j0.835)(z-0.8)(z-0.9)(z-0.3)}$$

$$\frac{E(z)}{z} = \frac{1.98}{(z-0.45+j0.835)(z-0.45-j0.835)(z-0.8)(z-0.9)(z-0.3)}$$

Then, using MATLAB to solve for the Partial Fraction Expansion, we get

```
denom = conv(a1,a2); denom = conv(denom,a3);
num = 1.98;
[R,P,K] = residue(num,denom)
```

```
R = 5×1 complex
   1.2297 - 1.0641i
   1.2297 + 1.0641i
  36.6667 + 0.0000i
 -48.2927 + 0.0000i
   9.1667 + 0.0000i
P = 5×1 complex
   0.4500 + 0.8352i
   0.4500 - 0.8352i
   0.9000 + 0.0000i
   0.8000 + 0.0000i
   0.3000 + 0.0000i
K =

     []
```

So, from these vectors of the residues and poles, we have

$$\frac{E(z)}{z} = \frac{1.23-j1.064}{z-0.45+j0.835} + \frac{1.23+j1.064}{z-0.45-j0.835} + \frac{36.67}{z-0.9} - \frac{48.3}{z-0.8} + \frac{9.17}{z-0.3}$$

Then multiplying z back to the right-hand side we get:

$$E(z) = \frac{1.23-j1.064z}{z-0.45+j0.835} + \frac{1.23+j1.064z}{z-0.45-j0.835} + \frac{36.67z}{z-0.9} - \frac{48.3z}{z-0.8} + \frac{9.17z}{z-0.3}$$

```
R_mag = abs(R(1:2))
```

```
R_mag = 2×1
    1.6262
    1.6262
```

```
R_deg = angle(R(1:2))*180/pi
```

```
R_deg = 2×1
  -40.8720
   40.8720
```

```
P_mag = abs(P)
```

```
P_mag = 5×1
    0.9487
    0.9487
    0.9000
    0.8000
    0.3000
```

```
P_deg = angle(P)*180/pi
```

```
P_deg = 5×1
   61.6835
  -61.6835
         0
         0
         0
```

Or expressed in Polar form:

$$E(z) = \frac{1.67e^{-j40.87^\circ}z}{z - 0.95e^{j61.7^\circ}} + \frac{1.67e^{j40.87^\circ}z}{z - 0.95e^{-j61.7^\circ}} + \frac{36.67z}{z - 0.9} - \frac{48.3z}{z - 0.8} + \frac{9.17z}{z - 0.3}$$

Then using common transform pairs, we solve for $e(k)$ and get:

$$e(k) = (1.23 - j1.064)(0.45 + j0.835)^k + (1.23 + j1.064)(0.45 - j0.835)^k + 36.67(0.9)^k - 48.3(0.8)^k + 9.17(0.3)^k$$

Then, solving for $e(k)$ for $k = 0, 1, 2, 3, 4$

```
e0 = (1.23-1i*1.064)*(0.45+1i*0.835)^0 + (1.23+1i*1.064)*(0.45-1i*0.835)^0 + 36.67*(0.9)^0 - 48
```

```
e0 = 5.3291e-15
```

Thus, $e(0) \approx 0$

```
e1 = (1.23-1i*1.064)*(0.45+1i*0.835) + (1.23+1i*1.064)*(0.45-1i*0.835) + 36.67*(0.9) - 48.3*(0.
```

```
e1 = -0.0021
```

$e(1) = -2.1 \times 10^{-3}$

```
e2 = (1.23-1i*1.064)*(0.45+1i*0.835)^2 + (1.23+1i*1.064)*(0.45-1i*0.835)^2 + 36.67*(0.9)^2 - 48
```

```
e2 = -0.0018
```

$e(2) = -1.8 \times 10^{-3}$

```
e3 = (1.23-1i*1.064)*(0.45+1i*0.835)^3 + (1.23+1i*1.064)*(0.45-1i*0.835)^3 + 36.67*(0.9)^3 - 48
```

```
e3 = -3.2728e-04
```

$e(3) = -3.27 \times 10^{-4}$

```
e4 = (1.23-1i*1.064)*(0.45+1i*0.835)^4 + (1.23+1i*1.064)*(0.45-1i*0.835)^4 + 36.67*(0.9)^4 - 48
```

```
e4 = 1.9803
```

And lastly, $e(4) = 1.98$

(e) A continuous time function $e(t)$, when sampled at a rate of $10\,\text{Hz}\,(T = 0.1s)$, has the following z-transform $E(z) = \dfrac{2z}{z - 0.8}$. Find function $e(t)$.

$$E(z) = \frac{2z}{z - 0.8}$$

Using the transform pair: $\left(\dfrac{z}{z-a}\right) \leftrightarrow a^k$, we find that $e(t)$ is:

$$e(t) = 2(0.8)^k$$

Then unsing the fact that $t = \text{kT}$ and rearranging, we find that $k = \dfrac{t}{T}$ with $T = 0.1$ and substituting, we get

$$k = \frac{t}{0.1} = 10t$$

Thus, $e(t) = 2(0.8)^{10t}$

(f) Repeat part (e) for $E(z) = \dfrac{2z}{z + 0.8}$.

Repeating the steps of part (e), we find that

$$e(t) = 2(-0.8)^{10t}$$

(g) From parts (e) and (f), what is the effect on the inverse z-transform of changing the sign on a real pole?

We observe that $e_1(k) = 2(0.8)^k \text{ and } e_2(k) = 2(-0.8)^k$

Investigating further into $e_2(k)$, we find:

$$\begin{aligned} e_2(k) &= 2(-0.8)^k \\ &= 2(-1)^k(0.8)^k \\ &= (-1)^k e_1(k) \end{aligned}$$

Thus, the effect of changing the sign of the real pole translates to a factor of $(-1)^k$ being multiplied by the inverse z-transform which then oscillates between positive and negative values as the sequence goes on until infinity.

Problem 2

Consider the system described by

$$x(k+1) = \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k),$$

$$y(k) = [-2 \quad 1]x(k)$$

(a) Find the transfer function $Y(z)/U(z)$

From this system we observe that

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix}; \ \mathbf{B} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}; \ \mathbf{C} = [-2 \quad 1]; \text{ and } \mathbf{D} = 0$$

We then solve for the characteristic equation of the matrix $\mathbf{A}$ and the characteristic values of the matrix by

$$|z\mathbf{I} - \mathbf{A}| = z\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} z & -1 \\ 0 & z-3 \end{bmatrix}$$

Taking the determinant of this matrix we find:

$$|z\mathbf{I} - \mathbf{A}| = z(z-3) - (-1)(0) = z^2 - 3z$$

Then, we define the transpose of the coefficient matrix $Cof|z\mathbf{I} - \mathbf{A}|^T$ as

$$[Cof|z\mathbf{I} - \mathbf{A}|]^T = \begin{bmatrix} z-3 & 1 \\ 0 & z \end{bmatrix}$$

Then the Inverse of the characteristic matrix $|z\mathbf{I} - \mathbf{A}|^{-1}$ is

$$|z\mathbf{I} - \mathbf{A}|^{-1} = \frac{[Cof|z\mathbf{I} - \mathbf{A}|]^T}{|z\mathbf{I} - \mathbf{A}|} = \frac{\begin{bmatrix} z-3 & 1 \\ 0 & z \end{bmatrix}}{z^2 - 3z}$$

The transfer function for the system can be described by the equation:

$$G(z) = \mathbf{C}[z\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B} + \mathbf{D}$$

and now filling in the corresponding matrices and vectors, we get:

$$G(z) = [-2 \quad 1]\frac{\begin{bmatrix} z-3 & 1 \\ 0 & z \end{bmatrix}}{z^2 - 3z}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 0$$

$$= \left(\frac{1}{z^2 - 3z}\right)[-2 \quad 1]\begin{bmatrix} z-3 & 1 \\ 0 & z \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \left(\frac{1}{z^2 - 3z}\right)[-2z + 6 \quad -2 + z]\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \left(\frac{1}{z^2 - 3z}\right)(-2z + 6 - 2 + z)$$

$$= \left(\frac{-z + 4}{z^2 - 3z}\right)$$

Thus, the transfer function $G(z) = \frac{Y(z)}{U(z)} = \frac{-z+4}{z(z-3)}$

(b) Using any similarity transformation, find a different state model for this system

Choosing any arbitrary linear transformation matrix $\mathbf{P} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

The inverse of this matrix $\mathbf{P}^{-1} = \frac{[Cof[\mathbf{P}]]^T}{|\mathbf{P}|}$

where, $Cof[\mathbf{P}] = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $|\mathbf{P}| = (1)(1) - (-1)(1) = 2$

Thus, $\mathbf{P}^{-1} = \frac{\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^T}{2} = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}$

Now, that we have our values for $\mathbf{P}$ and $\mathbf{P}^{-1}$, we can alter the other matrices and vectors of the system to complete the linear transformation to the state model with a state vector $w(k+1)$ and $w(k)$

$\mathbf{A}_w = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$

$= \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix}$

$\mathbf{A}_w = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$

$\mathbf{B}_w = \mathbf{P}^{-1}\mathbf{B}$

$= \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\mathbf{B}_w = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\mathbf{C}_w = \mathbf{C}\mathbf{P}$

$= [-2 \quad 1] \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

$\mathbf{C}_w = [-1 \quad 3]$

```
% Calculate with MATLAB
A = [0 1; 0 3]; B = [1; 1]; C = [-2 1];
```

```
P = [1 -1; 1 1];
invP = inv(P);
Aw = invP*A*P
```

```
Aw = 2×2
     2     2
     1     1
```

```
Bw = invP * B
```

```
Bw = 2×1
     1
     0
```

```
Cw = C*P
```

```
Cw = 1×2
    -1     3
```

Thus, the transformed state model can be defined by the new state equations:

$$w(k+1) = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} w(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k),$$

$$y(k) = [-1 \quad 3] w(k)$$

(c) Find the transfer function of the system from the transformed state equations

To find the transfer function of the transformed system, we follow the steps as in part (a)

From this transformed system we observe that

$\mathbf{A}_w = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$; $\mathbf{B}_w = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$; $\mathbf{C}_w = [-1 \quad 3]$; and $\mathbf{D}_w = 0$

We then solve for the characteristic equation of the matrix $\mathbf{A}$ and the characteristic values of the matrix by

$$|z\mathbf{I} - \mathbf{A}_w| = z\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} z-2 & -2 \\ -1 & z-1 \end{bmatrix}$$

Taking the determinant of this matrix we find:

$$|z\mathbf{I} - \mathbf{A}_w| = (z-2)(z-1) - (-2)(-1) = z^2 - 3z + 2 - 2 = z^2 - 3z$$

Then, we define the transpose of the coefficient matrix $Cof\,|z\mathbf{I} - \mathbf{A}_w|$ and its transpose as

$$Cof\,[z\mathbf{I} - \mathbf{A}_w] = \begin{bmatrix} z-1 & 1 \\ 2 & z-2 \end{bmatrix}$$

$$[Cof\,|z\mathbf{I} - \mathbf{A}_w|]^T = \begin{bmatrix} z-1 & 2 \\ 1 & z-2 \end{bmatrix}$$

Then the Inverse of the characteristic matrix $|z\mathbf{I} - \mathbf{A}|^{-1}$ is

$$\left|z\mathbf{I}-\mathbf{A}_w\right|^{-1} = \frac{[Cof\,|z\mathbf{I}-\mathbf{A}_w|]^T}{|z\mathbf{I}-\mathbf{A}_w|} \quad = \frac{\begin{bmatrix} z-1 & 2 \\ 1 & z-2 \end{bmatrix}}{z^2-3z}$$

The transfer function for the system can be described by the equation:

$$G_w(z) = \mathbf{C}_w[z\mathbf{I}-\mathbf{A}_w]^{-1}\mathbf{B}_w + \mathbf{D}_w$$

and now filling in the corresponding matrices and vectors, we get:

$$G(z) = [-1 \quad 3]\frac{\begin{bmatrix} z-1 & 2 \\ 1 & z-2 \end{bmatrix}}{z^2-3z}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0$$

$$= \left(\frac{1}{z^2-3z}\right)[-1 \quad 3]\begin{bmatrix} z-1 & 2 \\ 1 & z-2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \left(\frac{1}{z^2-3z}\right)[-1 \quad 3]\begin{bmatrix} z-1 \\ 1 \end{bmatrix}$$

$$= \left(\frac{1}{z^2-3z}\right)(-z+1+3)$$

$$= \left(\frac{-z+4}{z^2-3z}\right)$$

Thus, the transfer function $G_w(z) = \frac{Y_w(z)}{U_w(z)} = \frac{-z+4}{z(z-3)}$

which is still the same transform function as before we conducted the linear transformation to the state model.

Problem 3

Given the MATLAB program

```
clear all;
m_old = zeros(11,1); % vector to store all values of m(k)
s1 = 0;
e = 0;  % input signal e(0)
for k = 0:10
    s2 = e - s1;
    m = 0.5*s2 - s1; % output signal m(k)
    s1 = s2;
    [k,m,s1]
    e = e + 1; % input signal e(k)
    m_old(k+1,:) = m;
end
```

```
ans = 1×3
     0     0     0
```

```
ans = 1×3
    1.0000    0.5000    1.0000
ans = 1×3
    2.0000   -0.5000    1.0000
ans = 1×3
     3     0     2
ans = 1×3
     4    -1     2
ans = 1×3
    5.0000   -0.5000    3.0000
ans = 1×3
    6.0000   -1.5000    3.0000
ans = 1×3
     7    -1     4
ans = 1×3
     8    -2     4
ans = 1×3
    9.0000   -1.5000    5.0000
ans = 1×3
   10.0000   -2.5000    5.0000
```

that solves the difference equation of a digital controller.

(a) Find the transfer function of the controller from input $e(.)$ to output $m(.)$.

From the first several lines of the code, we can see that

$$e(0) = 0,\ s_1(0) = 0, \text{and}\ m(k) = \frac{1}{2}s_2(k) - s_1(k)$$

Then, the future value of s_1 is assigned from the present value of s_2 or simply $\rightarrow s_1(k+1) = s_2(k)$

Using this information, we can make a substitution into $m(k)$:

$$m(k) = \frac{1}{2}s_1(k+1) - s_1(k)$$

Observing that the present value of $s_2(k) = e(k) - s_1(k)$ can now be updated from previous findings, such that:

$s_1(k+1) = e(k) - s_1(k)$ or rearranging, $e(k) = s_1(k+1) + s_1(k)$

Now adding together definitions of $m(k)\ \text{and}\ e(k)$ we find:

$$m(k) + e(k) = \frac{1}{2}s_1(k+1) - s_1(k) + s_1(k+1) + s_1(k)$$

$m(k) + e(k) = \frac{3}{2}s_1(k+1)$ or rearranging in terms of s_1

$$s_1(k+1) = \frac{2}{3}(m(k) + e(k))$$

then we can substitute k for $k-1$

$$s_1(k) = \frac{2}{3}(m(k-1) + e(k-1))$$

Then using the previous definitions of $s_1(k)$ and $s_1(k+1)$, we get:

$s_1(k+1) = e(k) - s_1(k) \rightarrow \frac{2}{3}(m(k) + e(k)) = e(k) - \frac{2}{3}(m(k-1) + e(k-1))$, and then lastly we arrange the equation in terms of the output $m(k)$:

$$m(k) = \frac{3}{2}e(k) - e(k) - e(k-1) - m(k-1)$$

$$m(k) = \frac{1}{2}e(k) - e(k-1) - m(k-1)$$

Now, gathering like-terms on each side and taking the z-transform we find the tranfer function to be:

$$m(k) + m(k-1) = \frac{1}{2}e(k) - e(k-1)$$

$$M(z)(1 + z^{-1}) = E(z)\left(\frac{1}{2} - z^{-1}\right)$$

$$\frac{M(z)}{E(z)} = \frac{1/2 - z^{-1}}{1 + z^{-1}} \text{ or } \frac{1/2z - 1}{z + 1}$$

(b) Find the z-transform of the controller input $\{e(k)\}_{k=0}^{\infty}$

Since the sequence $e(k)$ increased linearly with each value of k

$$e(k) = k$$

which, we can use the common transform pair $k \leftrightarrow \frac{z}{(z-1)^2}$. So,

$$E(z) = \frac{z}{(z-1)^2}$$

(c) Use the results of parts (a) and (b) to find the inverse z-transform of the controller output.

$$M(z) = E(z)\left(\frac{1/2z - 1}{z + 1}\right) \rightarrow \frac{z}{(z-1)^2}\frac{1/2z - 1}{z + 1} \rightarrow \frac{z(1/2z - 1)}{(z-1)^2(z+1)}$$

Then we can use Partial Fraction Expansion of the rational function:

$$\frac{M(z)}{z} = \frac{(1/2z - 1)}{(z-1)^2(z+1)}$$

$$\frac{M(z)}{z} = \frac{A_0}{(z-1)} + \frac{A_1}{(z-1)^2} + \frac{A_2}{(z+1)}$$

Using MATLAB to solve for the poles and corresponding residues:

```
b = [1/2 -1]; a = conv([1 -2 1],[1 1])
```

```
a = 1×4
     1    -1    -1     1
```

```
[R,P,K] = residue(b,a)
```

```
R = 3×1
    0.3750
   -0.2500
   -0.3750
P = 3×1
    1.0000
    1.0000
   -1.0000
K =

     []
```

We find that the Z-Transform of the Output is:

$$\frac{M(z)}{z} = \frac{3/8}{(z-1)} - \frac{1/4}{(z-1)^2} - \frac{3/8}{(z+1)}$$

$$M(z) = \frac{3/8z}{(z-1)} - \frac{1/4z}{(z-1)^2} - \frac{3/8z}{(z+1)}$$

Then, again using commons transform pairs: $k \leftrightarrow \frac{z}{(z-1)^2}$, $1 \leftrightarrow \frac{z}{z-1}$, $a^k \leftrightarrow \frac{z}{z-a}$

$m(k) = \frac{3}{8} - \frac{1}{4}k - \frac{3}{8}(-1)^k$ or simplifying like terms:

$$m(k) = \frac{3}{8}(1-(-1)^k) - \frac{1}{4}k$$

(d) Run the program to check the results of part (c). Attach results to report.

```
syms k0
mk = (3/8) * (1-(-1)^k0) - (1/4)*k0
```

```
mk =
```

$$\frac{3}{8} - \frac{3\ (-1)^{k_0}}{8} - \frac{k_0}{4}$$

```
m_new = zeros(11,1);
for k = 0:10
m_new(k+1,:) = subs(mk,k0,k);
end
disp(m_old)
```

```
         0
    0.5000
   -0.5000
         0
   -1.0000
   -0.5000
   -1.5000
   -1.0000
   -2.0000
```

```
   -1.5000
   -2.5000
```

```
disp(m_new)
```

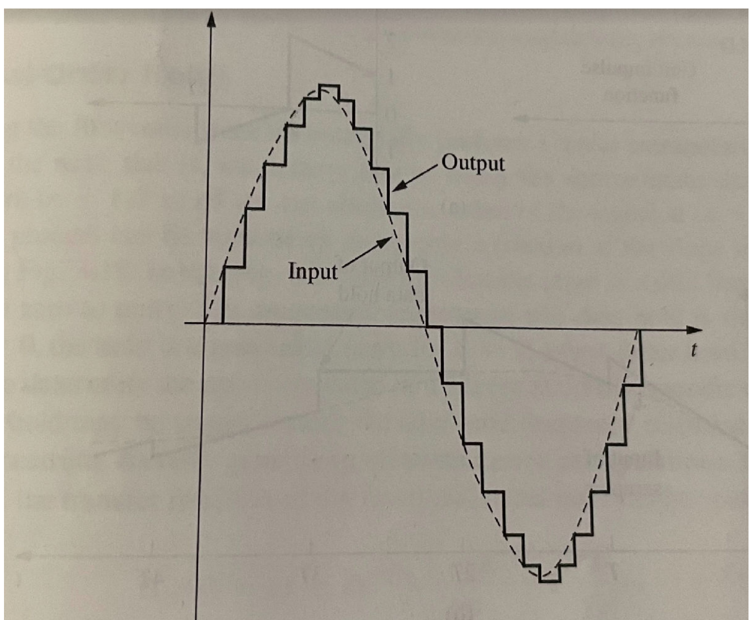
```
         0
    0.5000
   -0.5000
         0
   -1.0000
   -0.5000
   -1.5000
   -1.0000
   -2.0000
   -1.5000
   -2.5000
```

After storing the original values of the output $m(k)$ for $k = 0$ to 10, and comparing them to the calculated output, we can see that all the values match.

Thus, the results from part (c) are equivalent to all the original program values for $m(k)$

Problem 4

A sinusoid is applied to a sampler/zero-order-hold device, with a distorted sine wave appearing at the output, as shown in Fig. 3-15



Output of a sampler/zero-order hold for a sinusoidal input.

(a) With the sinusoid of unity amplitude and frequency $2\,\text{Hz}$, and with $f_s = 12\,\text{Hz}$, find the amplitude and phase of the component in the output at $f_1 = 2\,\text{Hz}$.

Magnitude and Phase Responses for Zero-Order Hold are:

$\left|G_{h0}(j\omega)\right| = T\left|\frac{\sin(\pi\omega/\omega_s)}{\pi\omega/\omega_s}\right|$ and $\angle G_{\text{h0}}(j\omega) = -\frac{\pi\omega}{\omega_s} + \theta$

Substituting values for $T = 1$ (unity), $\omega = 2\pi(2\,\text{Hz}) = 4\pi$, and $\omega_s = 2\pi(12\,\text{Hz}) = 24\pi$, we get:

$$\left|G_{h0}(j\omega)\right| = \left|\frac{\sin(\pi(4\pi)/(24\pi))}{\pi(4\pi)/(24\pi)}\right| = \left|\frac{\sin(\pi/6)}{\pi/6}\right| = 0.955$$

$$\angle G_{\text{h0}}(j\omega) = -\frac{\pi\omega}{\omega_s} + \theta$$

To evaluate the value of θ, we check that when

$$\sin\left(\frac{\pi\omega}{\omega_s}\right) > 0 \rightarrow \theta = 0$$

$$\sin\left(\frac{\pi\omega}{\omega_s}\right) < 0 \rightarrow \theta = \pi$$

Evaluating $\sin\left(\frac{\pi\omega}{\omega_s}\right)$ we get :

$\sin\left(\frac{\pi(4\pi)}{(24\pi)}\right) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2} > 0$, Thus, $\theta = 0$

Using all this information, we find the phase response of the distorted sine wave to be:

$$\begin{aligned}\angle G_{\text{h0}}(j\omega) &= -\frac{\pi\omega}{\omega_s} + \theta \\ &= -\frac{\pi(4\pi)}{(24\pi)} + 0 \\ &= -\frac{\pi}{6} \\ &= -30^\circ\end{aligned}$$

Thus, the magnitude and phase response of the distorted sine wave is: $0.955\angle -30^\circ$

```
%Prove with MATLAB
fs = 12; f1 = 2; T = 1;
ws = 2*pi*fs; w1 = 2*pi*f1;
Gh0_mag = T*abs((sin(pi*w1/ws)/(pi*w1/ws)))
```

```
Gh0_mag = 0.9549
```

```
if (sin(pi*w1/ws) > 0)
    theta = 0
else
    theta = pi
end
```

```
theta = 0
```

```
Gh0_deg = - ((pi*w1/ws) + theta)*(180/pi)
```

```
Gh0_deg = -30.0000
```

(b) Repeat part (a) for the component in the output at $(f_s - f_1) = 10\,\text{Hz}$.

$\left|G_{h0}(j\omega)\right| = T\left|\frac{\sin(\pi\omega/\omega_s)}{\pi\omega/\omega_s}\right|$ and $\angle G_{\text{h0}}(j\omega) = -\frac{\pi\omega}{\omega_s} + \theta$

Substituting values for $T = 1$ (unity), $\omega = 2\pi(12 - 2) = 20\pi$, and $\omega_s = 2\pi(12\,\text{Hz}) = 24\pi$, we get:

$$\left|G_{h0}(j\omega)\right| = \left|\frac{\sin(\pi(20\pi)/(24\pi))}{\pi(20\pi)/(24\pi)}\right| = \left|\frac{\sin(\pi(5/6)}{\pi(5/6)}\right| = 0.191$$

$$\angle G_{\text{h0}}(j\omega) = -\frac{\pi\omega}{\omega_s} + \theta$$

To evaluate the value of θ, we check that when

$$\sin\left(\frac{\pi\omega}{\omega_s}\right) > 0 \rightarrow \theta = 0$$

$$\sin\left(\frac{\pi\omega}{\omega_s}\right) < 0 \rightarrow \theta = \pi$$

Evaluating $\sin\left(\frac{\pi\omega}{\omega_s}\right)$ we get :

$\sin\left(\frac{\pi(20\pi)}{(24\pi)}\right) = \sin(\pi(5/6)) = \frac{1}{2} > 0$, Thus, $\theta = 0$

Using all this information, we find the phase response of the distorted sine wave to be:

$$\begin{aligned}\angle G_{\text{h0}}(j\omega) &= -\frac{\pi\omega}{\omega_s} + \theta \\ &= -\frac{\pi(20\pi)}{(24\pi)} + 0 \\ &= -\pi\left(\frac{5}{6}\right) \\ &= -150^\circ\end{aligned}$$

Thus, the magnitude and phase response of the component in the output at $(f_s - f_1) = 10\,\text{Hz}$ is: $0.191\angle -150^\circ$

```
%Prove with MATLAB
fs = 12; f2 = fs - f1; T = 1;
ws = 2*pi*fs; w2 = 2*pi*f2;
Gh0_mag = T*abs((sin(pi*w2/ws)/(pi*w2/ws)))
```

```
Gh0_mag = 0.1910
```

```
if (sin(pi*w2/ws) > 0)
    theta = 0
```

```
else
    theta = pi
end
```

```
theta = 0
```

```
Gh0_deg = - ((pi*w2/ws) + theta)*(180/pi)
```

```
Gh0_deg = -150.0000
```

(c) Repeat parts (a) and (b) for a sampler-first-order-hold device.

Magnitude Response of First-Order Hold:

$$|G_{\mathrm{h1}}(j\omega)| = T\sqrt{1+\frac{4\pi^2\omega^2}{\omega_s^2}}\left[\frac{\sin(\pi\omega/\omega_s)}{\pi\omega/\omega_s}\right]^2$$

Phase Response of First-Order Hold:

$$\angle G_{\mathrm{h1}}(j\omega) = \tan^{-1}\left(\frac{2\pi\omega}{\omega_s}\right) - \frac{2\pi\omega}{\omega_s}$$

```
% Calculate part (a) specifications using First-Order-Hold with MATLAB
fs = 12; f1 = 2; T = 1;
ws = 2*pi*fs; w1 = 2*pi*f1;
Gh1_mag = T*sqrt(1 + (4*pi^2*w1^2)/(ws^2))*(sin(pi*w1/ws)/(pi*w1/ws))^2
```

```
Gh1_mag = 1.3204
```

```
Gh1_deg = (atan(2*pi*w1/ws) - (2*pi*w1/ws))*(180/pi)
```

```
Gh1_deg = -13.6793
```

Thus, the magnitude and phase response of the distorted sine wave using First-Order-Hold is: $1.32\angle -13.7^{\circ}$

```
% Calculate part (b) specifications using First-Order-Hold with MATLAB
fs = 12; f2 = fs - f1; T = 1;
ws = 2*pi*fs; w2 = 2*pi*f2;
Gh1_mag = T*sqrt(1 + (4*pi^2*w2^2)/(ws^2))*(sin(pi*w2/ws)/(pi*w2/ws))^2
```

```
Gh1_mag = 0.1944
```

```
Gh1_deg = (atan(2*pi*w2/ws) - (2*pi*w2/ws))*(180/pi)
```

```
Gh1_deg = -220.8125
```

Thus, the magnitude and phase response of the component in the output at $(f_s - f_1) = 10\,\mathrm{Hz}$ using First-Order-Hold is: $0.194\angle -220.8^{\circ}$

(d) Comment on the distortion in the data-hold output for the cases considered in parts (a), (b), and (c).

For the frequency of the component at $f_s - f_1 = 10\,\mathrm{Hz}$ the magnitude response of both the Zero-Order-Hold and First-Order Hold are about equal with different phases.

However, when analyzing the frequency component at $f_1 = 2\,\mathrm{Hz}$, the Zero-Order-Hold yielded a magnitude response of 0.955 and the First-Order-Hold yielded a magnitude response of 1.32, which shows that the Zero-Order-Hold was more consistent in the case for lower frequency components.