EECE 5610 Digital Control Systems

Lecture 5

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- **Power series method:** Divide the denominator into the numerator and obtain a power series of the form

$$E(z) = e_0 + e_1 z^{-1} + e_2 z^{-2} + \cdots +$$

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Example: $E(z) = \dfrac{z}{(z-1)(z-2)} = \dfrac{z}{z^2 - 3z + 2}$

$$\begin{array}{r l}
 & \frac{1}{z} + \frac{3}{z^2} + \frac{7}{z^3} + \cdots \\ \hline
z^2 - 3z + 2 \; \big) & z \\
 & z - 3 + \frac{2}{z} \\ \hline
 & 3 - \frac{2}{z} \\
 & 3 - 9z + \frac{6}{z^2} \\ \hline
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\end{array}$$

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So the first terms of the sequence are: $\{0, 1, 3, 7 \cdots$

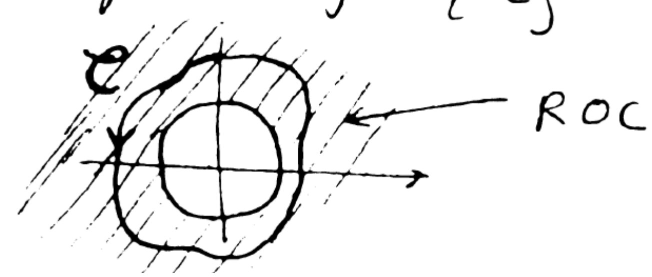
Is it consistent with the formula $-1+2^k$ that we found earlier?

Drawback of the method: We don't get a closed-form expression for $\{e_k\}$

- **Inversion Formula method:**

We will see that a closed form expression for $\{e_k\}$ is given by:

$$e_k = \frac{1}{2\pi j} \oint_{\mathcal{C}} E(z)\, z^{k-1}\, dz$$



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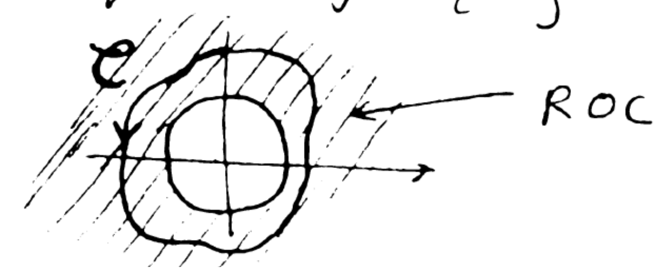
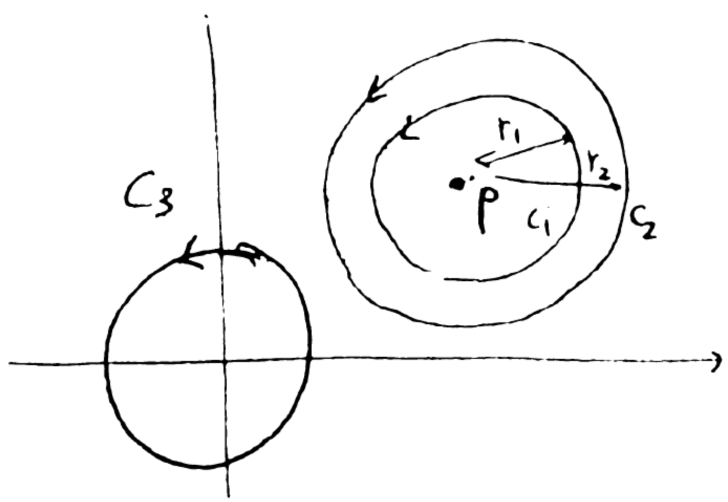


where $\mathcal{C}$ denotes a closed path encircling the origin and inside the ROC. Obviously, this method works only if there is a convenient way of computing this integral. We will see that this is the case, but first we need to introduce some concepts from complex analysis and analytic function theory

Motivation: Consider the function $\frac{1}{z-p}$ (single pole at $z=p$)

Lets compute $\frac{1}{2\pi J}\oint_{\mathcal{C}} \frac{1}{(z-p)}\,dz$ where $\mathcal{C}$ is a curve that may o may not enclose p

(For simplicity will take circles)



$(z-p)$ is a vector from a generic point on $\mathcal{C}$ to the point z

$\Rightarrow$ we can write it as: $(z-p) = r e^{J\theta}$

$\Rightarrow \frac{1}{(z-p)} = \frac{1}{r} e^{-J\theta}$

$$\frac{1}{2\pi J}\oint_{C_1} \frac{1}{(z-p)}\,dz = \frac{1}{2\pi J}\int_0^{2\pi} \frac{1}{r} e^{-J\theta}\, d\left[r e^{J\theta}\right] = \frac{1}{2\pi \cancel{J}}\int_0^{2\pi} \frac{1}{\cancel{r}} \cdot \cancel{e^{-J\theta}}\, \cancel{r}\, \cancel{J}\, \cancel{e^{J\theta}}\, d\theta$$

2π

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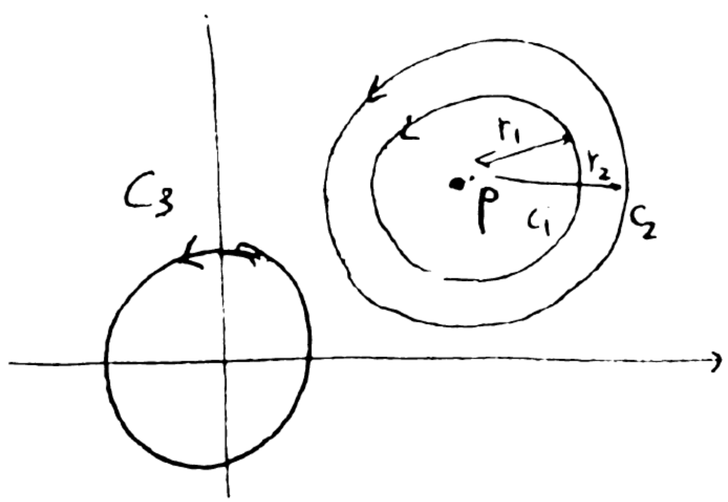

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= ?

A) 0

B) 1

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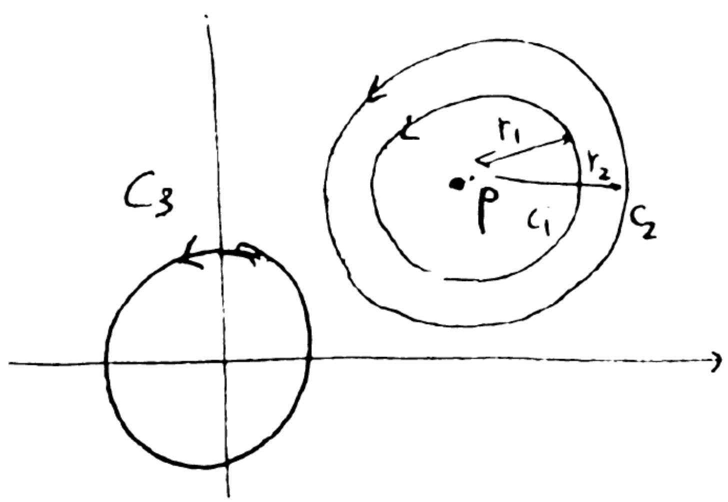

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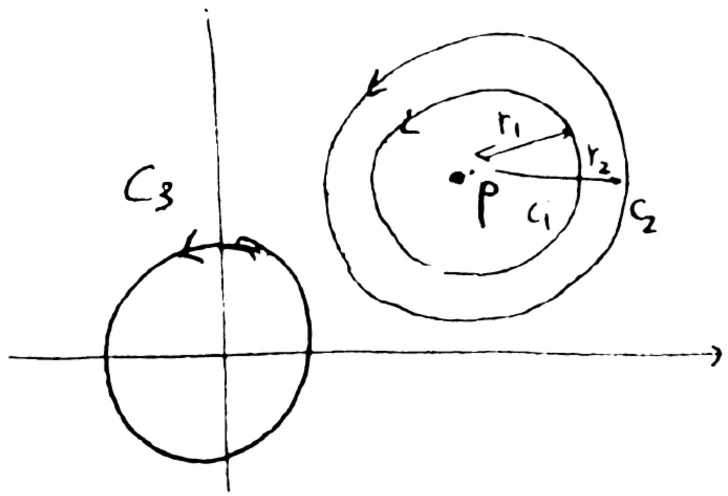

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$$= \frac{1}{2\pi}\int_0^{2\pi} d\theta = \frac{2\pi}{2\pi} = \boxed{1}$$

Note that the answer is 1, regardless of the <u>radius</u>.
(In fact it can be shown that we get this answer for <u>any</u> curve encircling p)

On the other hand, it can be shown that $\oint_{C_3} = 0$

$\Rightarrow$ If C $\begin{cases} \text{encircles } p \Rightarrow \frac{1}{2\pi}\oint = 1 \\ \text{does not encircle } p \Rightarrow \frac{1}{2\pi}\oint = 0 \end{cases}$

This is a special case of Cauchy's Theorem:

<u>Facts</u>:

1) A function $F(z)$ is analytic at a point z_0 if it is continuously differentiable at z_0 (i.e $F'(z)$ cont. at z_0)

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where $\operatorname{Res}(z_i)$ (the "residues") are given by:

$$\operatorname{Res}(z_i) = \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \left[(z - z_i)^n F(z) \right] \Big|_{z = z_i}$$ if $f(z)$ has a singularity of ord n at z_i

$$\operatorname{Res}(z_i) = (z - z_i) F(z) \Big|_{z = z_i}$$ for singularities of order 1

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(b) $F(z) = \frac{1}{z}$ ⇒ isolated singularity at $z = 0$

⇒ $\frac{1}{2\pi j} \oint_{\mathcal{C}} \frac{1}{z} \, dz = \text{Res}\,(z=0) = \cancel{z} \cdot \frac{1}{\cancel{z}} \Big|_{z=0} =$ [?] as before

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Question: Is this relevant to us at all?

Answer: Yes! we can use this both to <u>prove</u> the inversion formula <u>and</u> to compute the $\oint$ in an efficient way.

Let $E(z) = \sum_0^\infty e_k z^{-k} = e_0 + \frac{e_1}{z} + \cdots + \frac{e_k}{z^k} + \cdots$

Multiply both sides by z^{k-1} and integrate along a closed curve $\mathcal{C}$ enclosing the origin

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$$\frac{1}{2\pi j} \oint_{\mathscr{C}} E(z) z^{k-1} dz = \frac{1}{2\pi j} \oint \left(\sum_0^{\infty} e_n z^{-n} \right) z^{k-1} dz = \frac{1}{2\pi j} \oint \left(e_0 z^{k-1} + e_1 z^{k-2} + \cdots + \frac{e_k}{z} + \frac{e_{k+1}}{z^2} + \cdots \right) dz$$

$$= \frac{1}{2\pi j} \Big[e_0 \oint z^{k-1} dz + e_1 \oint z^{k-2} dz + \cdots e_{k-1} \oint dz + \Big\} = \phi$$

$$+ \; e_k \oint \frac{1}{z} dz \qquad \Big\} = e_k$$

$$+ \; e_{k+1} \int \frac{1}{z^2} dz + e_{k+2} \int \frac{1}{z^3} dz + \cdots \Big\} = 0$$

Assuming that we can interchange Σ and $\oint$

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Assuming that we can interchange Σ and $\oint$

$$\Rightarrow \boxed{\frac{1}{2\pi j}\oint_{\mathscr{C}} E(z) z^{k-1} dz = e_k}$$

provided that $\mathscr{C}$ is inside the region of convergence so that indeed we can interchange Σ and $\oint$

We have proved the inversion formula!

Q: what about the second issue (how to compute the $\oint$ on the left)?

A: Let's use the residue formula:

$$e_k = \frac{1}{2\pi j} \oint_{\mathcal{C}} E(z)\, z^{k-1}\, dz = \sum \mathrm{Res}\left\{ E(z)\, z^{k-1} \right\}$$

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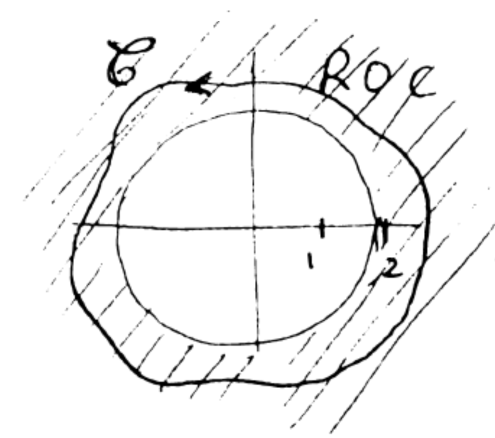

ROC: $|z| > 2$

$\Rightarrow \mathcal{C}$ in $|z|>2$

<u>Example</u>: $E(z) = \dfrac{z}{(z-1)(z-2)}$ poles at $z = 1, 2$

$$E(z)\, z^{k-1} = \frac{z^k}{(z-1)(z-2)}$$

$\text{Res}\left[E(z) z^{k-1}\right]$ at $z=1$:

$\text{Res}\left[\quad\right]$ at $z=2$:

$$\mathrm{Res}(z_i) = \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \left[(z - z_i)^n F(z) \right] \Big|_{z=z_i}$$ if $f(z)$ has a singularity of ord n at z_i

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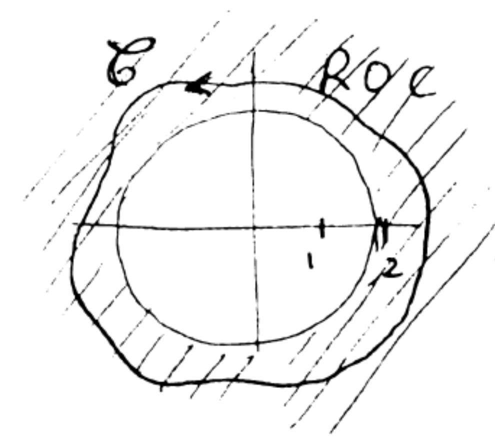

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$$\boxed{\begin{array}{ll} \operatorname{Res}(z_i) = \dfrac{1}{(n-1)!} \dfrac{d^{n-1}}{dz^{n-1}} \left[(z-z_i)^n F(z) \right] \Big|_{z=z_i} & \text{if } f(z) \text{ has a singularity of ord } n \text{ at } z_i \\ \operatorname{Res}(z_i) = (z-z_i) F(z) \Big|_{z=z_i} & \text{for singularities of order } 1 \end{array}}$$

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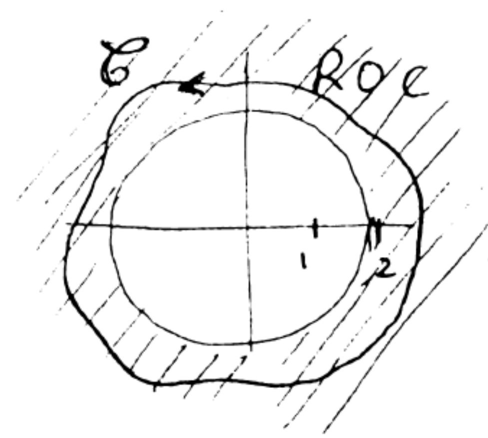

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$$E(z)\, z^{k-1} = \frac{z^k}{(z-1)(z-2)}$$

$\operatorname{Res}\left[E(z) z^{k-1}\right]$ at $z=1$: $\left.\dfrac{\cancel{(z-1)}\, z^k}{\cancel{(z-1)}(z-2)}\right|_{z=1} = \boxed{-1}$

$\operatorname{Res}\left[\quad\right]$ at $z=2$: $\left.\dfrac{\cancel{(z-2)}\, z^k}{(z-1)\cancel{(z-2)}}\right|_{z=2} = \boxed{2^k}$

$\Rightarrow \boxed{e_k = -1 + 2^k}$ <u>as before</u>!

Example 2: Assume $f_3(k) = f_1(k)\, f_2(k)$

want $F_3(z)$

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$$F_3(z) = \sum_0^\infty f_3(k) z^{-k} ;$$

$$f_3(k) z^{-k} = f_1(k) \cdot f_2(k) z^{-k}$$
$$= f_1(k) \cdot \frac{1}{2\pi j}\left(\oint_{\mathscr{C}} F_2(\lambda) \lambda^{k-1} d\lambda\right) z^{-k}$$
$$= f_1(k) \frac{1}{2\pi j} \oint_{\mathscr{C}} F_2(\lambda) \left(\frac{\lambda}{z}\right)^k \frac{d\lambda}{\lambda}$$

$$F_3(z) = \sum_0^\infty f_1(k) \frac{1}{2\pi j} \oint_{\mathscr{C}} F_2(\lambda)\left(\frac{\lambda}{z}\right)^k \frac{d\lambda}{\lambda} = \frac{1}{2\pi j}\oint_{\mathscr{C}} \Big[\underbrace{\sum_0^\infty f_1(k) \left(\frac{z}{\lambda}\right)^{-k}}_{F_1\left(\frac{z}{\lambda}\right)} F_2(\lambda) \frac{d\lambda}{\lambda}$$
$$= \frac{1}{2\pi j} \oint_{\mathscr{C}} F_1\left(\frac{z}{\lambda}\right) F_2(\lambda) \frac{d\lambda}{\lambda} \quad \#$$

If we let $z = 1$, $f_1 = f_2 = f$ we get

$$\sum_0^\infty f_3(k) = \sum_0^\infty f^2(k) = \frac{1}{2\pi j} \oint_{\mathcal{C}} F\left(\frac{1}{\lambda}\right) F(\lambda) \frac{d\lambda}{\lambda}$$

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Take now the circle $|\lambda|=1$ as the contour $\mathcal{C}$:

$$\Rightarrow \quad \sum_0^\infty f^2(k) = \frac{1}{2\pi j}\oint_{|\lambda|=1} F\left(\frac{1}{\lambda}\right)F(\lambda)\,\frac{d\lambda}{\lambda} \underset{\lambda=e^{j\theta}}{=} \frac{1}{2\pi}\int_0^{2\pi} F(e^{j\theta})\,F(e^{-j\theta})\,d\theta$$

$$\boxed{\sum_0^\infty f^2(k) = \frac{1}{2\pi}\int_0^{2\pi} F(e^{j\theta})\,F(e^{-j\theta})\,d\theta}$$

If we let $z = 1$, $f_1 = f_2 = f$ we get

$$\sum_0^\infty f_3(k) = \sum_0^\infty f^2(k) = \frac{1}{2\pi j} \oint_{\mathscr{C}} F\left(\frac{1}{\lambda}\right) F(\lambda) \frac{d\lambda}{\lambda}$$

Take now the circle $|\lambda| = 1$ as the contour $\mathscr{C}$:

$$\Rightarrow \quad \sum_0^\infty f^2(k) = \frac{1}{2\pi j} \oint_{|\lambda|=1} F\left(\frac{1}{\lambda}\right) F(\lambda) \frac{d\lambda}{\lambda} \underset{\lambda = e^{j\theta}}{=} \frac{1}{2\pi} \int_0^{2\pi} F(e^{j\theta}) F(e^{-j\theta}) \, d\theta$$

$$\boxed{\sum_0^\infty f^2(k) = \frac{1}{2\pi} \int_0^{2\pi} F(e^{j\theta}) F(e^{-j\theta}) \, d\theta}$$

This is known as Parseval's theorem: energy in the time domain
= energy in the freq. domain

Summary of the z-transform

- **Definition:** $$E(z) = \sum_{k=0}^{\infty} e_k z^{-k}$$ for some region $r_0 < |z| < R_0$

- **Properties:**

1) Linearity $$Z\left\{\alpha e_1(k) + \beta e_2(k)\right\} = \alpha Z(e_1) + \beta Z(e_2)$$

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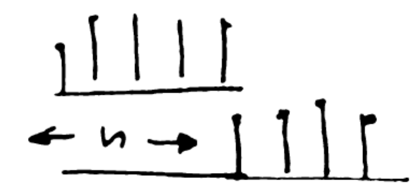
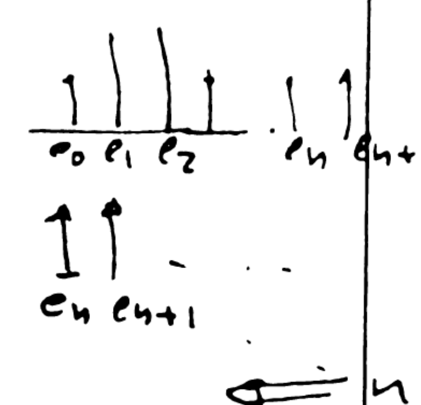
2) Time shift:



a) $$Z\{e(k-n)\} = z^{-n} E(z)$$

b) $$Z\{e(k+n)\} = z^{n}\left[E(z) - \sum_{0}^{n-1} e(k) z^{-k}\right]$$



3) scaling in z-plane:

$$Z\{(r^{-k} e_k)\} = E(rz)$$

4) Initial Value Theorem:

$$e(0) = \lim_{z \to \infty} E(z)$$

5) Final Value Theorem:

$$\lim_{k \to \infty} e(k) = \lim_{z \to 1} (z-1) E(z)$$

Provided that the left hand side exist $\Leftrightarrow$ $(z-1)F(z)$ has all poles inside the unit circle

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6) Convolution of Time Sequences:

$$f_k = e_1(k) * e_2(k) = \sum_{\ell=0}^{k} e_1(\ell) \cdot e_2(k-\ell)$$

$$Z\{e_1 * e_2\} = E_1(z)\, E_2(z)$$

7) Inversion Formula: Let $E(z) = Z(e_k)$, then:

$$e_k = \frac{1}{2\pi j} \oint_{\mathcal{C}} E(z)\, z^{k-1}\, dz$$

- <u>Representation of Linear Time Invariant Discrete Time Systems</u>

So far we have seen that a LTI system described by difference equations can also be described by a transfer function

- **Representation of Linear Time Invariant Discrete Time Systems**

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physical model + Newton equations $\Longrightarrow$ set of difference equations $\xRightarrow{z \text{ transform}}$ (rational) Transfer function $G(z)$

<u>Question</u> : Suppose that we are given a transfer function $C(z)$, can we find a system with that t.f? how?

(This is relevant because we will carry out the <u>design</u> of controllers in the z-domain, but then we will need to <u>implement</u> them)

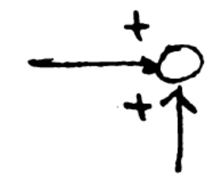
Question: Suppose that we are given a transfer function $C(z)$, can we find a system with that t.f? how?

(This is relevant because we will carry out the design of controllers in the z-domain, but then we will need to implement them)

Answer: Yes, using as an intermediate step yet another representation: simulation diagrams.

- <u>Simulation diagrams</u>: elements: (a) Time Delay (shift register) 

(b) Product by a constant 

(c) Summing junction 

Turns out that with these three elements we can built a simulation diagram that realizes <u>any</u> T. F.

Example: Suppose that we want to realize the T.F

$$G(z) = \frac{M(z)}{E(z)} = \frac{\beta_0 z + \beta_1}{z^2 + \alpha_1 z + \alpha_2}$$

Dividing numerator & denominator by $1/z^2$ yields:

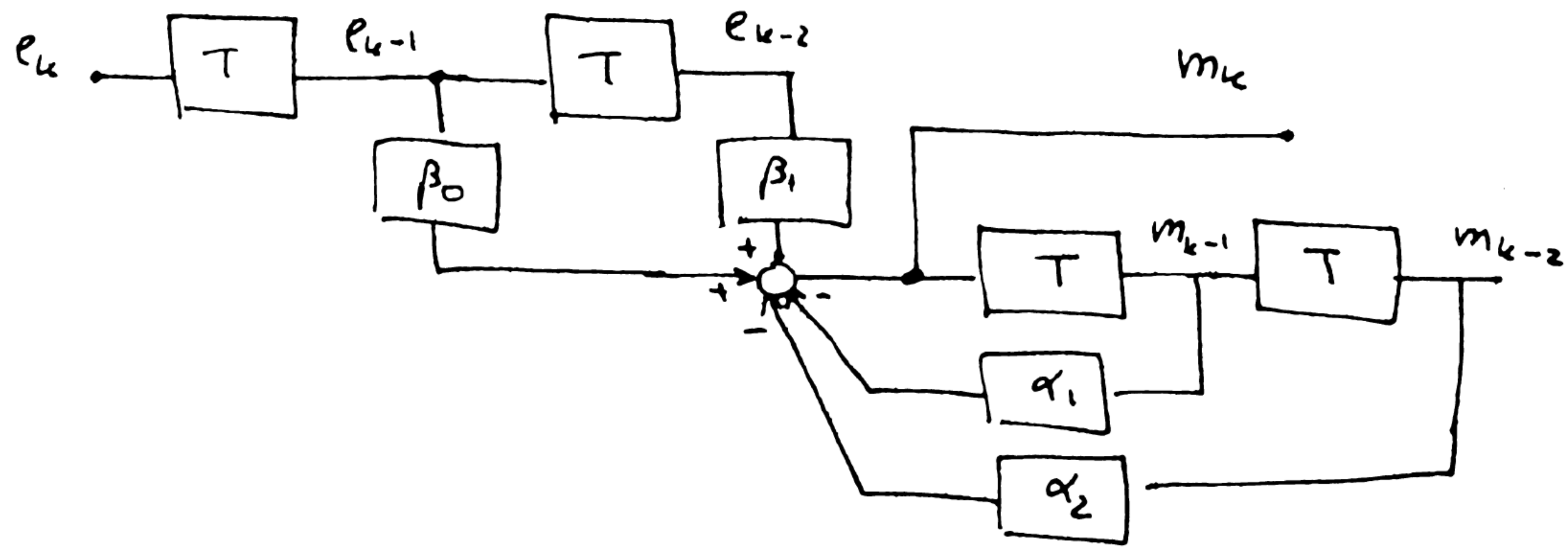
$$G(z) = \frac{M(z)}{E(z)} = \frac{\frac{\beta_0}{z} + \beta_1 \cdot \frac{1}{z^2}}{1 + \alpha_1 \cdot \frac{1}{z} + \frac{\alpha_2}{z^2}} \Rightarrow \left(1 + \alpha_1 \frac{1}{z} + \frac{1}{z^2}\alpha_2\right) M(z) = \left(\frac{\beta_0}{z} + \frac{\beta_1}{z^2}\right) E(z)$$

Now recall that $\frac{1}{z} \overset{z^{-1}}{\Longleftrightarrow}$ unit time delay

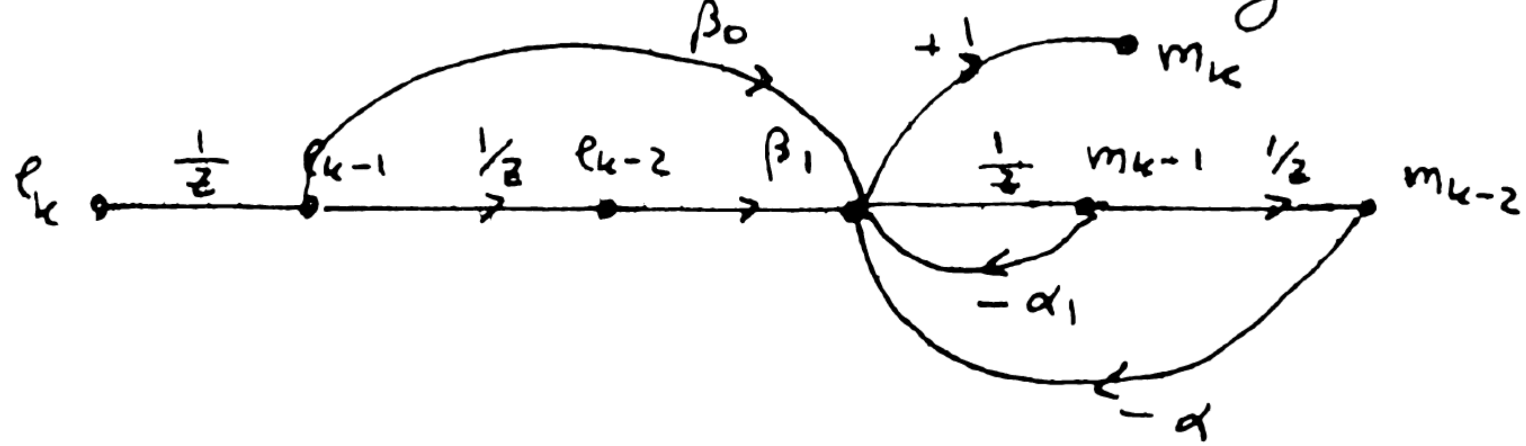
Taking inverse Z transforms on both sides yields:

$$m_k + \alpha_1 m_{k-1} + \alpha_2 m_{k-2} = \beta_0 e_{k-1} + \beta_1 e_{k-2}$$

Now we can "build" a system that is described precisely by this equation:



__Sanity check:__ We can transform this diagram back to the z-domain $\left(\boxed{T} \to \frac{1}{z}\right)$ and check the T.F. using Mason's formula:



Exercise: check that indeed you get the right T.F.

Q? Is this the "best" way to proceed? Is this the <u>only</u> way to proceed

<u>A</u>: Not necessarily. Note that we started out with a <u>second order</u> T.F. Hence we would expect to be able to realize it with <u>just two delays</u>. However our realization uses 4!

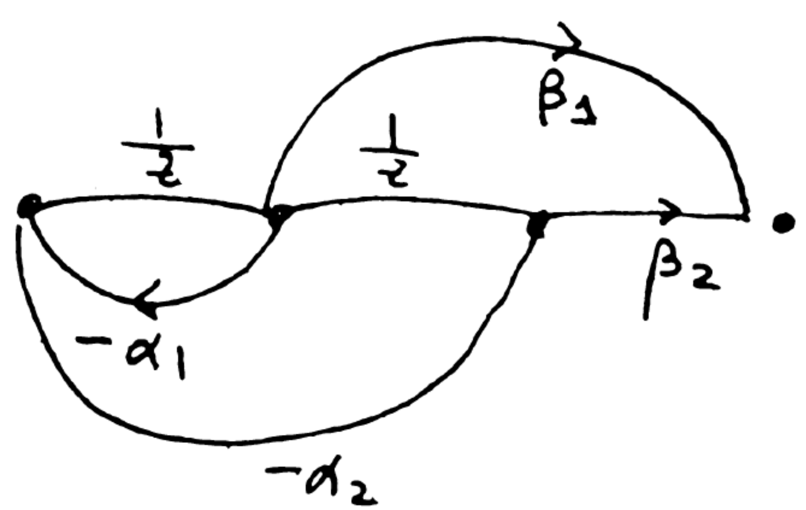
Let's look at the problem again and try <u>reverse engineering</u>: first find a signal flow graph and then the simulation diagram

$$G(z) = \frac{\frac{\beta_0}{z} + \frac{\beta_1}{z^2}}{1 + \frac{\alpha_1}{z} + \frac{\alpha_2}{z^2}}$$

$\Rightarrow$ Let's build something that has

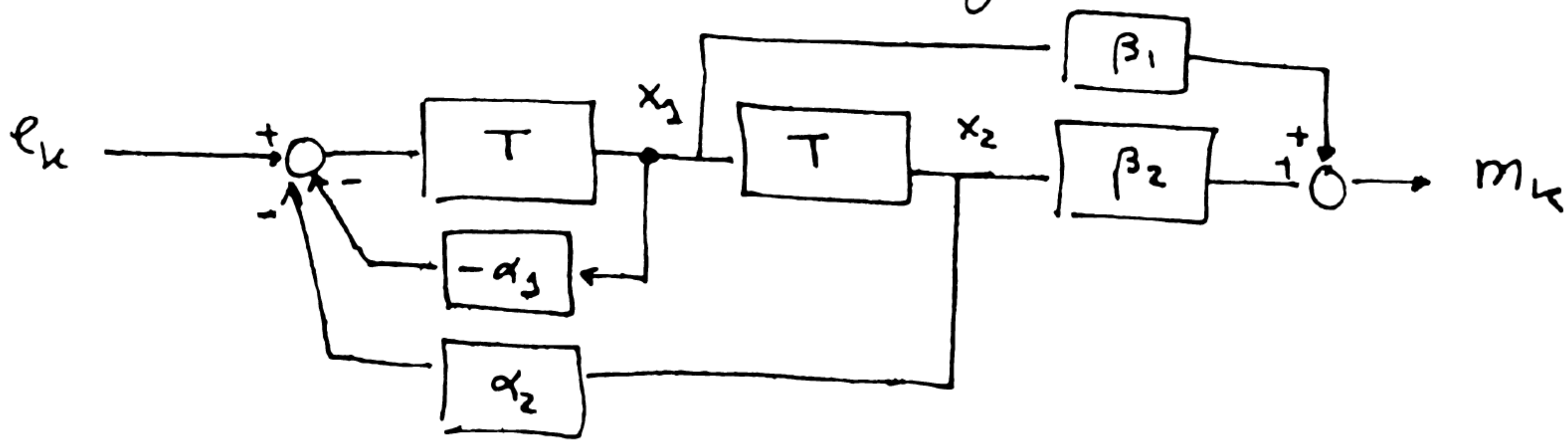
$$\Delta = 1 + \frac{\alpha_1}{z} + \frac{\alpha_2}{z^2}$$

and $\sum M_i \Delta_i = \frac{\beta_0}{z} + \frac{\beta_1}{z^2}$



(Note: we needed only two "$\frac{1}{z}$" blocks)

From here we get the following simulation diagram:



Q: Is this the "minimal" realization? Is it unique?
Do the intermediate variables "x_i" have any significance?

- Turns out that to answer these questions we need to introduce the concept of <u>state variables</u>

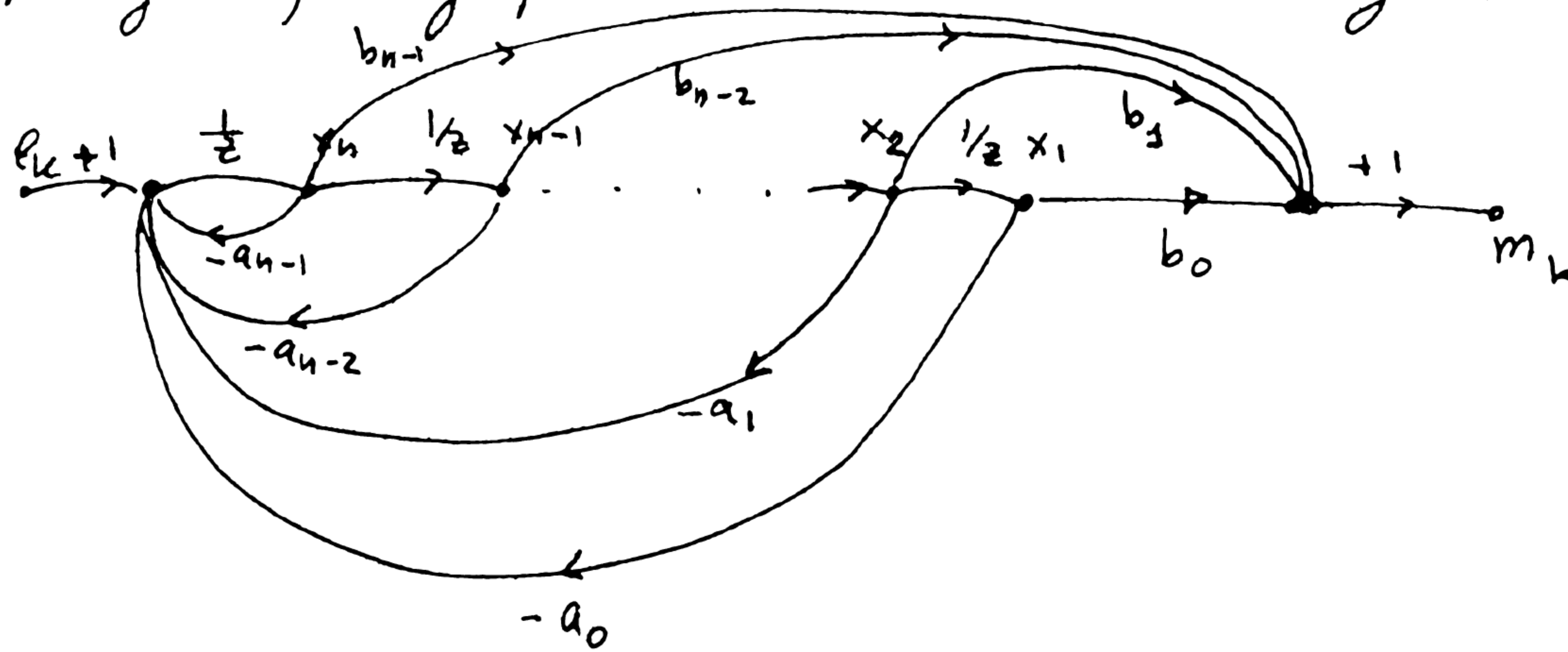
- State Space Models:

Consider a generic transfer function of the form

$$G(z) = \frac{M(z)}{E(z)} = \frac{b_0 + b_1 z + \qquad + b_{n-1} z^{n-1}}{a_0 + a_1 z + \quad . \quad a_{n-1} z^{n-1} + z^n}$$ (note that it is strictly proper)

Dividing by z^{-n} yields: $$G(z) = \frac{b_{n-1} z^{-1} + \quad . \quad . \quad . \quad + b_0 z^{-n}}{1 + a_{n-1} z^{-1} + \quad . \quad . \quad a_0 z^{-n}}$$

A signal flow graph that realizes this T.F is given by:



Note that we have n loops (all touching) with gains $L_i = -a_{n-i}\left(\frac{1}{z}\right)^i$ and n forward paths, each with $\Delta_i = \Delta$ and $M_i = \frac{b_{n-i}}{z^i}$

According to Mason's formula: $G(z) = \frac{\sum M_i \Delta_i}{\Delta} = \frac{b_{n-1} z^{-1} + \cdots b_0 z^{-n}}{1 + a_{n-1} z^{-1} + \cdots a_0 z^{-n}}$

(precisely what we wanted)