EECE 5610 Digital Control Systems

Lecture 11

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Problem 1

(a) Find $e(0)$, $e(1)$, and $e(10)$ for

$$E(z) = \frac{0.1}{z(z-0.9)}$$

using the inversion formula.

(b) Check the value of $e(0)$ using the initial-value property.

(c) Check the values calculated in part (a) using partial fractions.

(d) Find $e(k)$ for $k = 0, 1, 2, 3, 4$ if $\mathcal{Z}[e(k)]$ is given by

$$E(z) = \frac{1.98z}{(z^2 - 0.9z + 0.9)(z - 0.8)(z^2 - 1.2z + 0.27)}$$

(e) A continuous time function $e(t)$, when sampled at a rate of 10 Hz ($T = 0.1s$), has the following z-transform $E(z) = \frac{2z}{z-0.8}$. Find function $e(t)$.

(f) Repeat part (e) for $E(z) = \frac{2z}{z+0.8}$.

(g) From parts (e) and (f), what is the effect on the inverse z-transform of changing the sign on a real pole?

Problem 2

Consider the system described by

$$x(k+1) = \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k),$$

$$y(k) = \begin{bmatrix} -2 & 1 \end{bmatrix} x(k).$$

(a) Find the transfer function $Y(z)/U(z)$.
(b) Using any similarity transformation, find a different state model for this system.
(c) Find the transfer function of the system from the transformed state equations.

Problem 3

Given the MATLAB program

```
clear all;
s1 = 0;
e = 0;      %input signal e(0)
for k = 0:10
s2 = e - s1;
m = 0.5*s2 - s1;  % output signal m(k)
s1 = s2;
[k,m,s1]
e = e + 1;      %input signal e(k)
end
```

that solves the difference equation of a digital controller.

(a) Find the transfer function of the controller from input $e(.)$ to output $m(.)$.
(b) Find the z-transform of the controller input $\{e(k)\}_{k=0}^{\infty}$.
(c) Use the results of parts (a) and (b) to find the inverse z-transform of the controller output.
(d) Run the program to check the results of part (c). Please attach your MATLAB code/result (from the command window) to your report.

3.7-7. A sinusoid is applied to a sampler/zero-order-hold device, with a distorted sine wave appearing at the output, as shown in Fig. 3-15.

(a) With the sinusoid of unity amplitude and frequency 2 Hz, and with $f_s = 12$ Hz, find the amplitude and phase of the component in the output at $f_1 = 2$ Hz.

(b) Repeat part (a) for the component in the output at $(f_s - f_1) = 10$ Hz.

(c) Repeat parts (a) and (b) for a sampler-first-order-hold device.

(d) Comment on the distortion in the data-hold output for the cases considered in parts (a), (b), and (c).

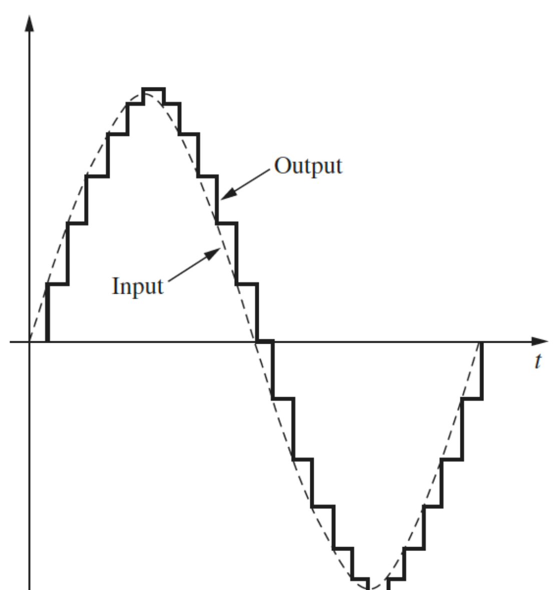
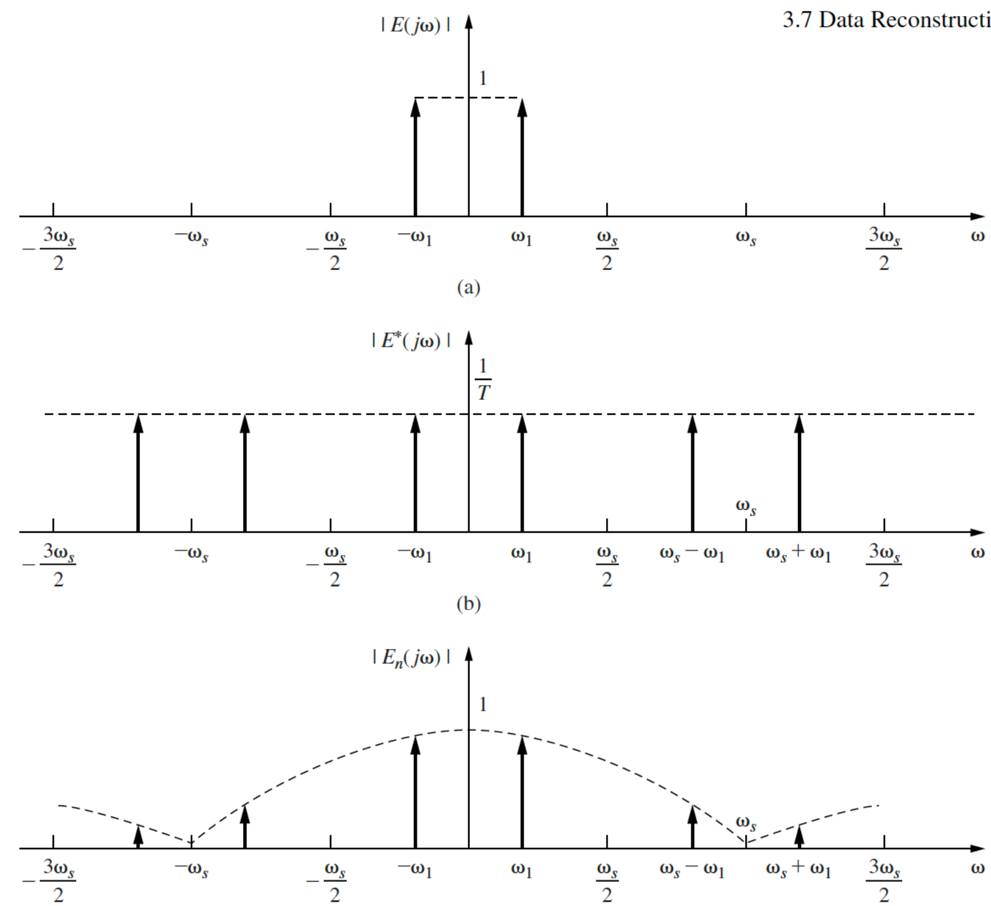


FIGURE 3-15 Output of a sampler/zero-order hold for a sinusoidal input.



$$G_{h0}(j\omega) = \frac{1 - \varepsilon^{-j\omega T}}{j\omega} e^{j(\omega T/2)} \varepsilon^{-j(\omega T/2)} = \frac{2\varepsilon^{-j(\omega T/2)}}{\omega}\left[\frac{\varepsilon^{j(\omega T/2)} - \varepsilon^{-j(\omega T/2)}}{2j}\right]$$

$$= T\frac{\sin(\omega T/2)}{\omega T/2}\varepsilon^{-j(\omega T/2)}$$

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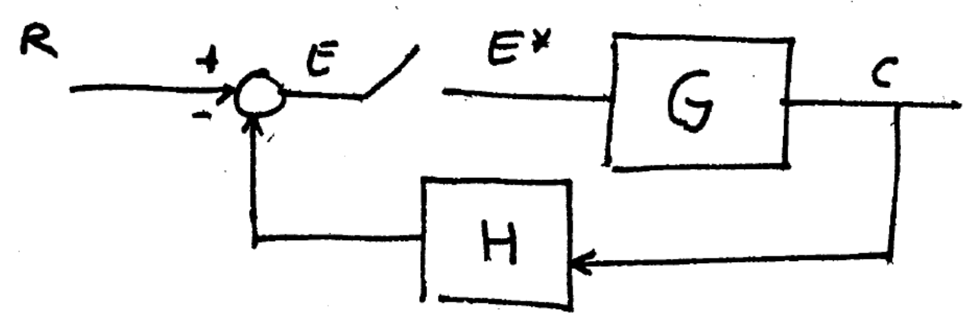

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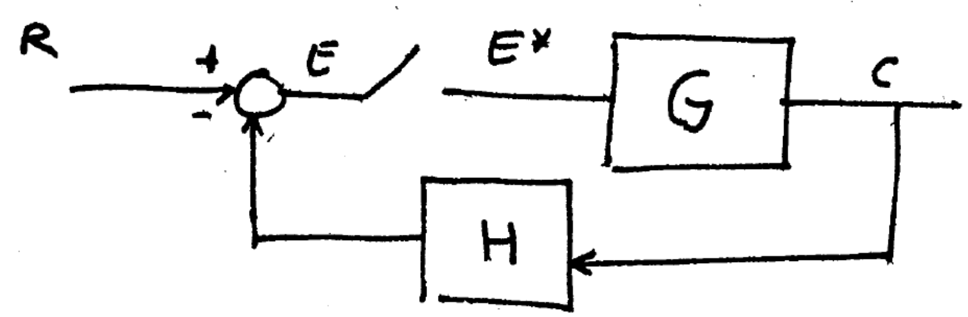

Want to find closed loop transfer function: $\Rightarrow$

$$E = R - HC = R - (HG)E^* \Rightarrow E^* = R^* - (HG)^* E^* \Rightarrow$$

$$E^* = \frac{R^*}{1+[HG]^*} \quad \text{and} \quad C = \frac{GR^*}{1+(HG)^*}$$

However, if we had selected C instead of E as variable we get

$$C = GE^*$$
$$E = R - HC \Rightarrow E^* = R^* - [HC]^* \Rightarrow C = GR^* - G[HC]^*$$
$$C^* = G^*R^* - G^*(HC)^*$$

Now we are stuck!! we can't proceed because we can't factor C^* out of $[HC]^*$.

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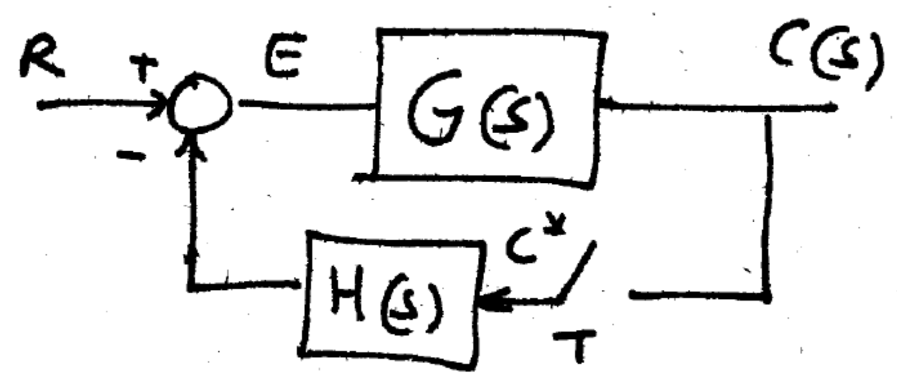

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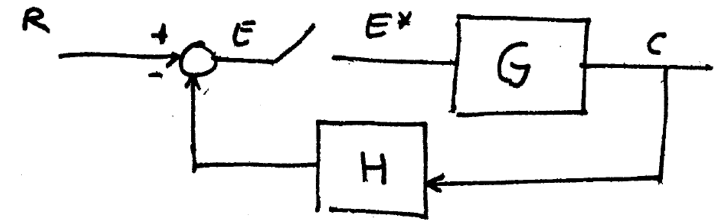

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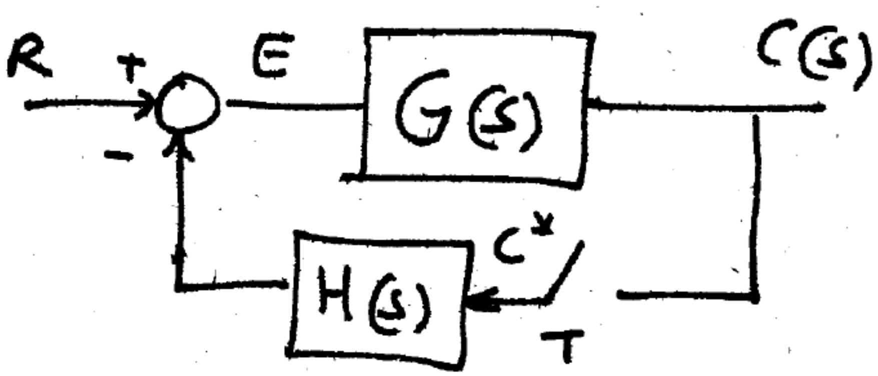

$$E = R - HC^*$$
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On the other hand, if we select E as a variable we have:

$$E = R - HC^*, \qquad C = GE$$

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$$\Rightarrow \quad E = R - H[GE]^* \qquad \text{or} \qquad E^* = R^* - H^* [GE]^*$$ and

stuck again!! (can't solve for E^*)

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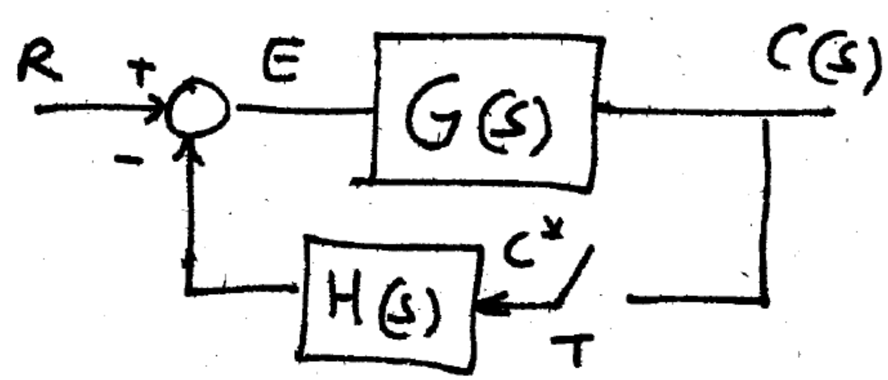
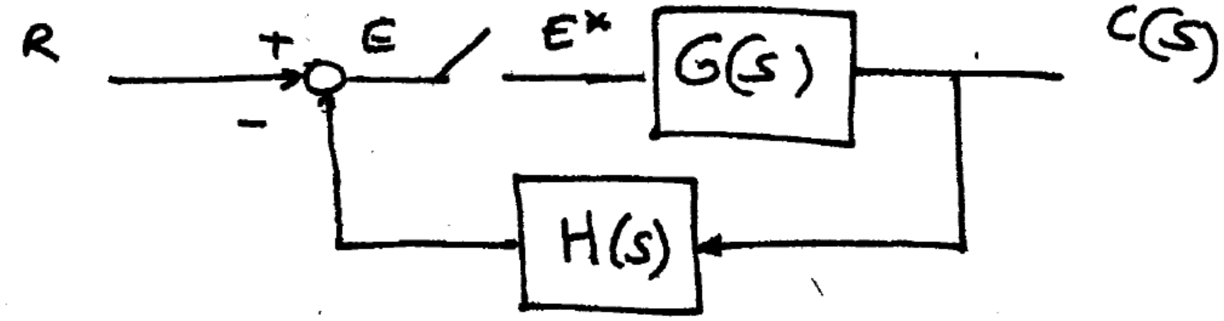


General Procedure



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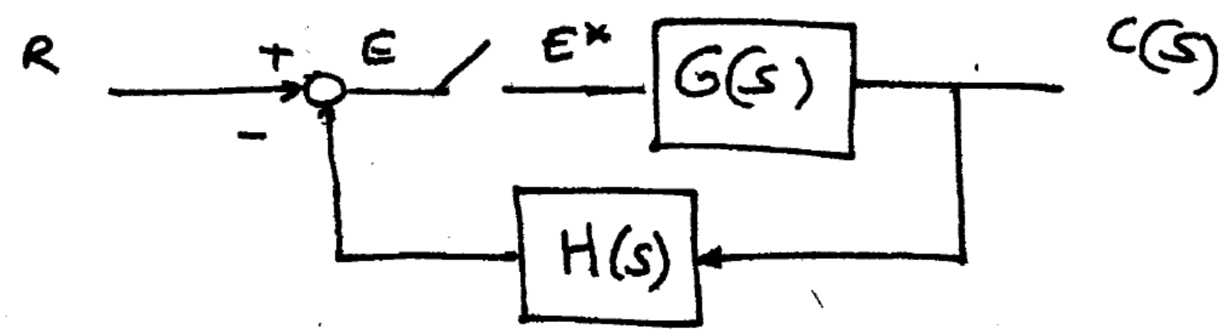

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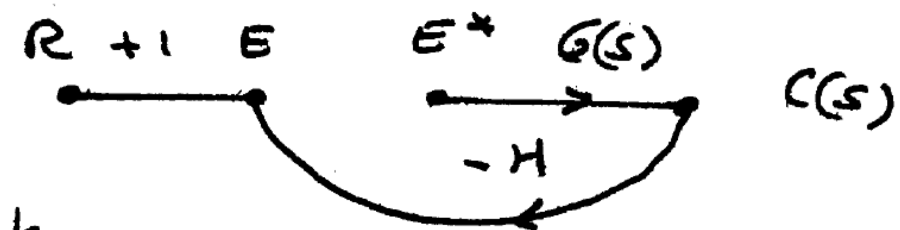

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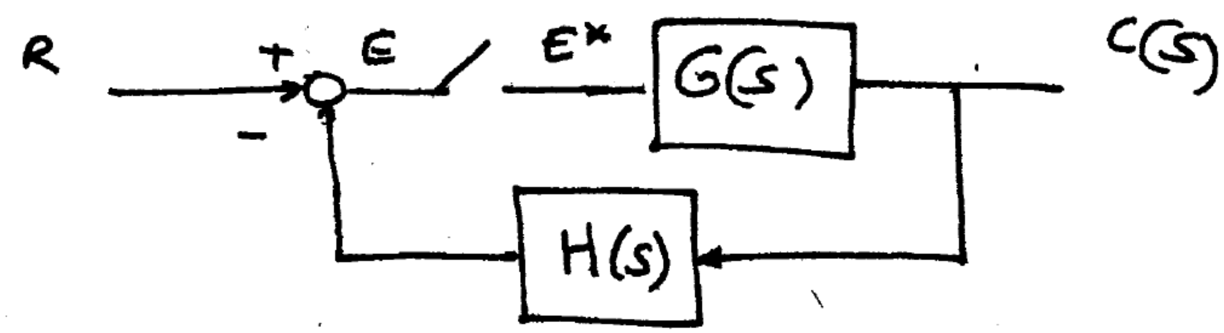

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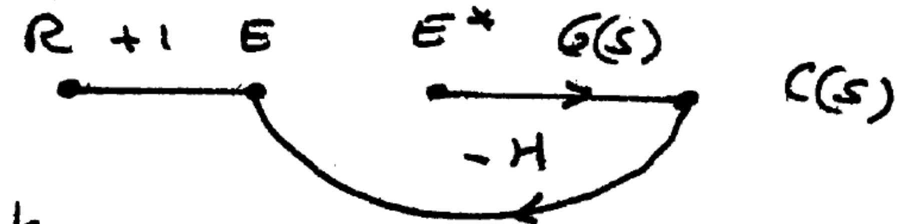

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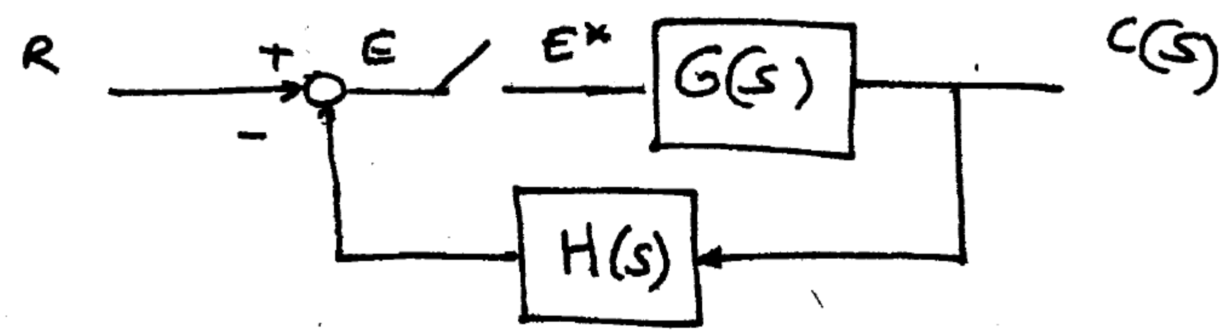

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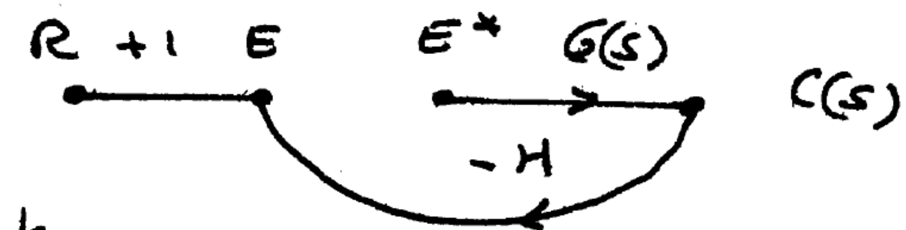

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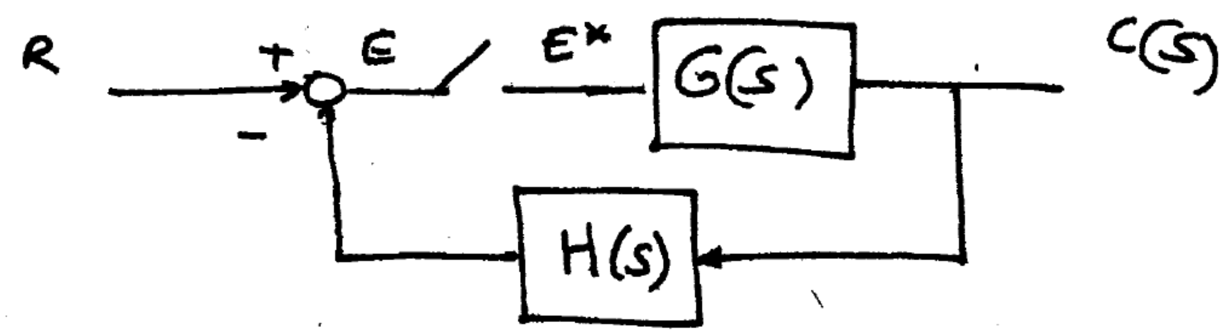

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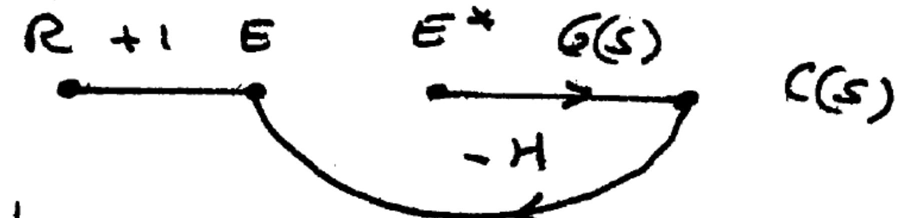

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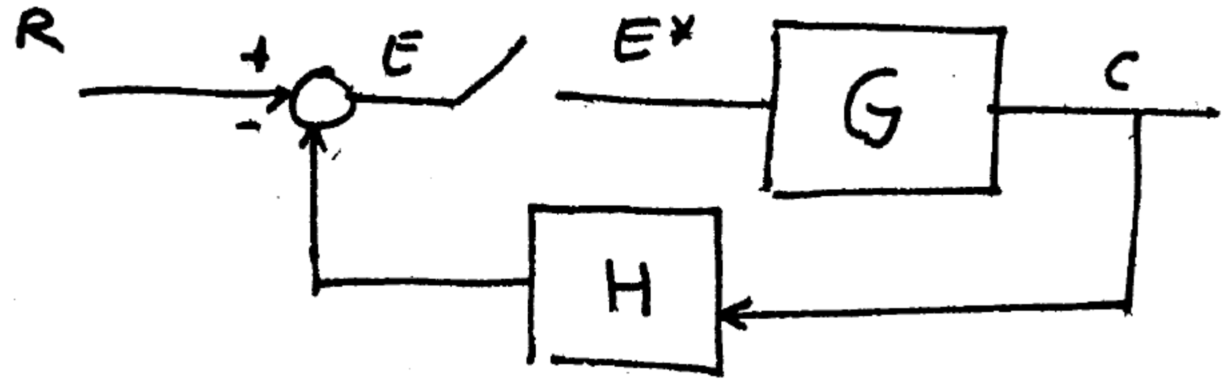
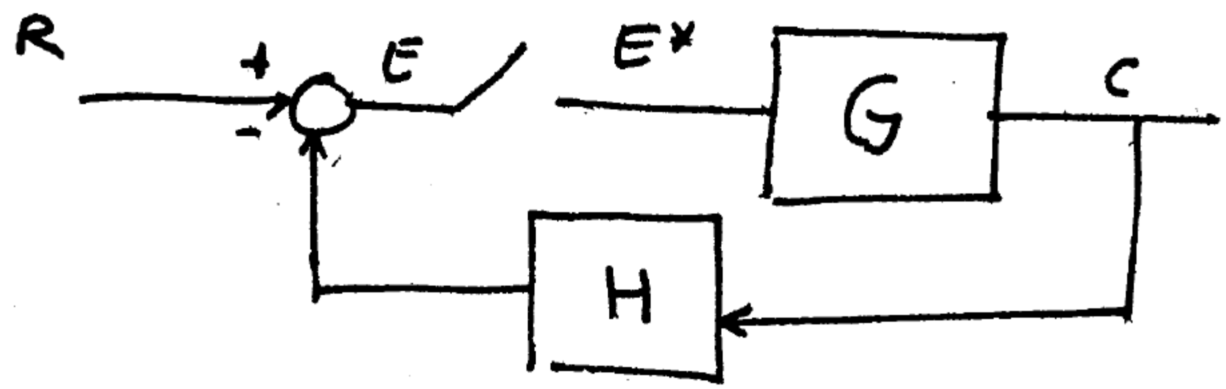

R
+
-
E
E*
G
C
H



Example:

(2) sampler output: E^*

(3) sampler input: $E = R - (HG)\, E^*$

System output: $C = G(s)\, E^*$

(4) $E^* = R^* - (HG)^* E^* \Rightarrow E^* = \dfrac{R^*}{1 + (GH)^*}$,

$C^* = G^* E^* \Rightarrow C^* = \dfrac{G^*}{1+(GH)^*} R^*$

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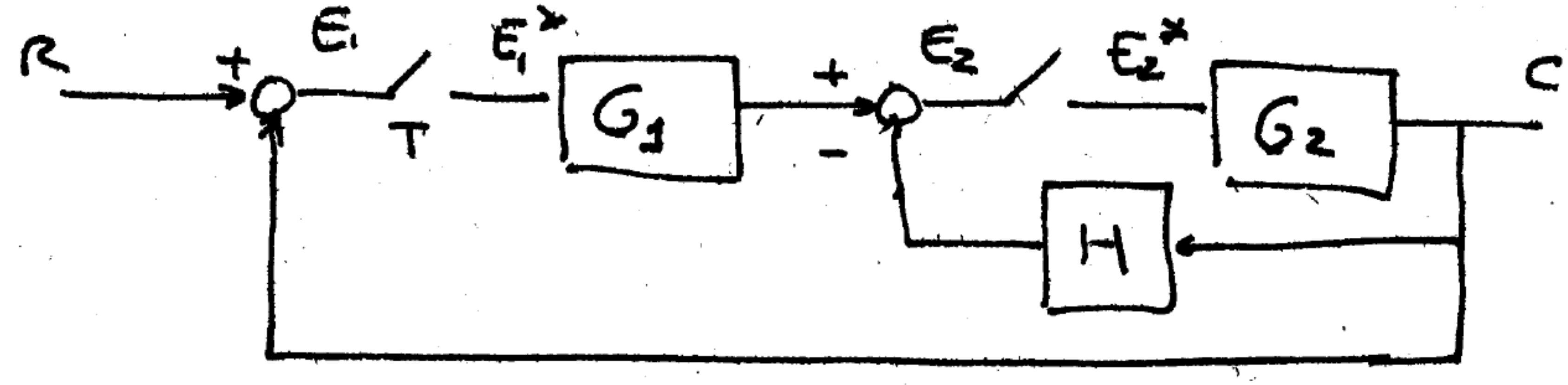

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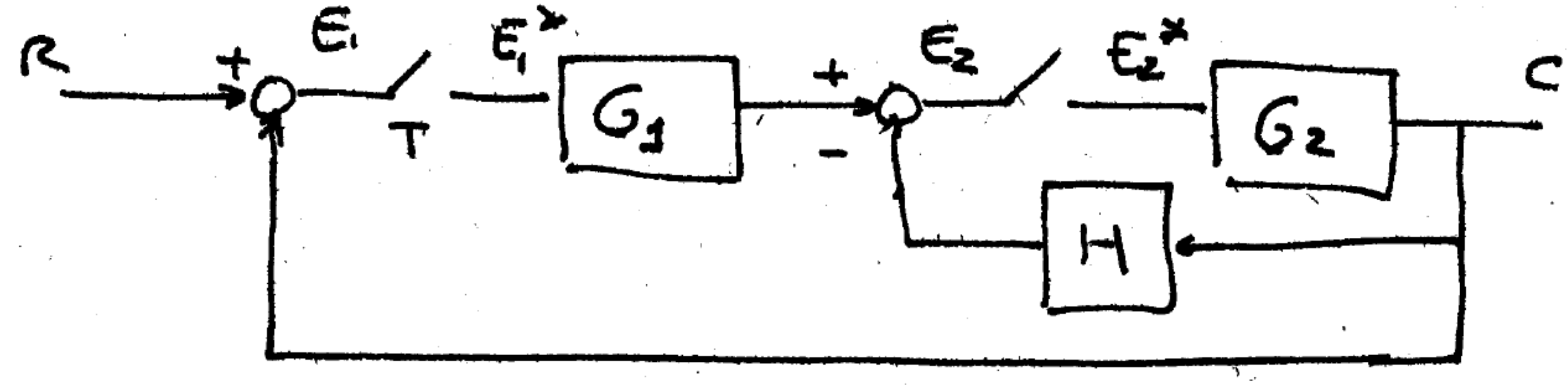

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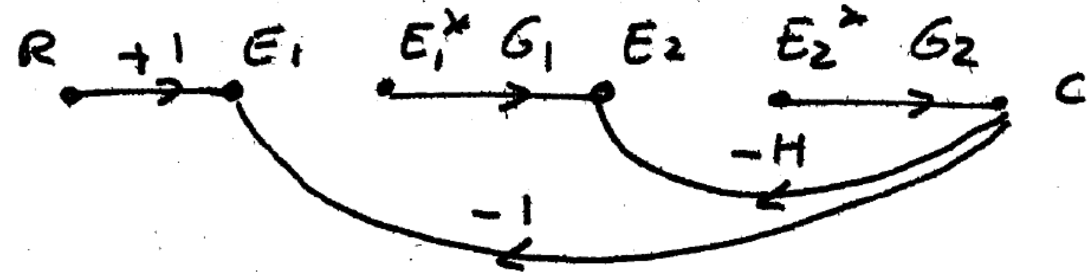

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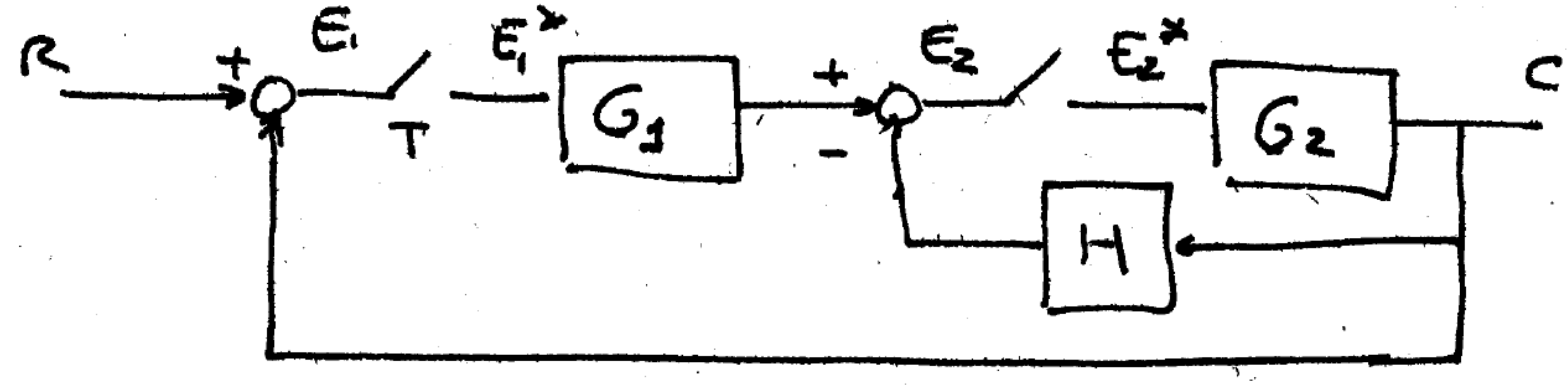

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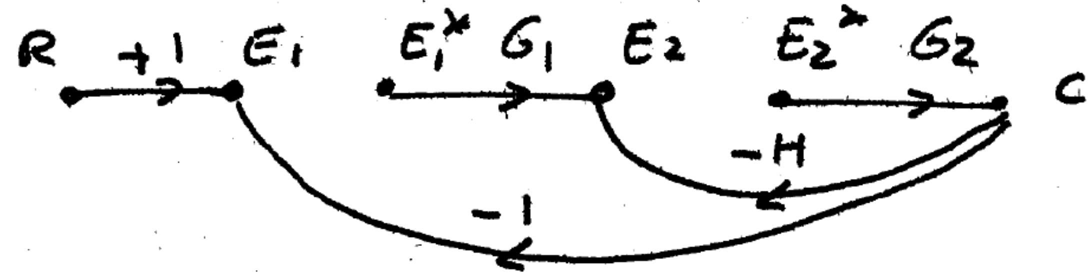

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$$\begin{aligned} E_1 &= R - G_2 E_2^* \\ E_2 &= G_1 E_1^* - H G_2 E_2^* \\ C &= G_2 E_2^* \end{aligned} \quad \Longrightarrow \quad \left.\begin{aligned} E_1^* &= R^* - G_2^* E_2^* \\ E_2^* &= G_1^* E_1^* - (H G_2)^* E_2^* \\ C^* &= G_2^* E_2^* \end{aligned}\right\} \text{need to solve these}$$

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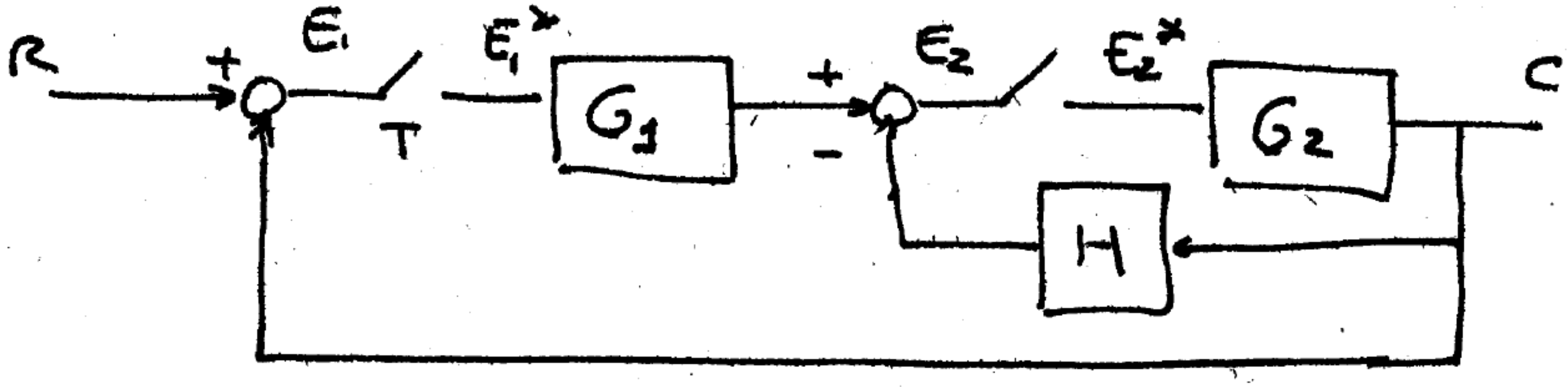

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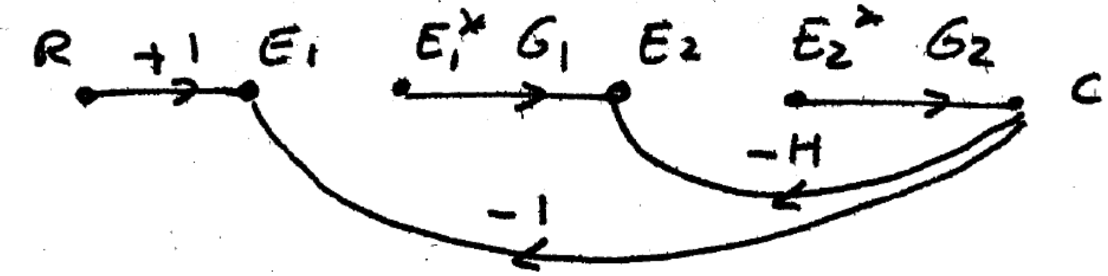

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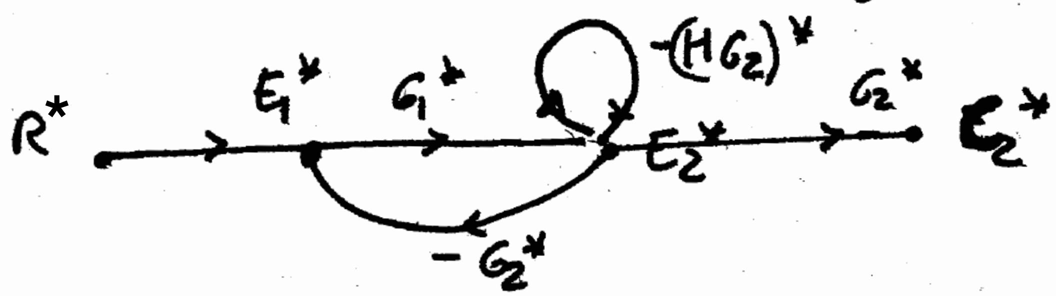
How do we solve these equations?

(1) use Cramer's rule (tedious)
(2) Use Mason's formula

To use Mason's formula we need to find the corresponding flow graph diagram: the <u>sampled</u> flow graph



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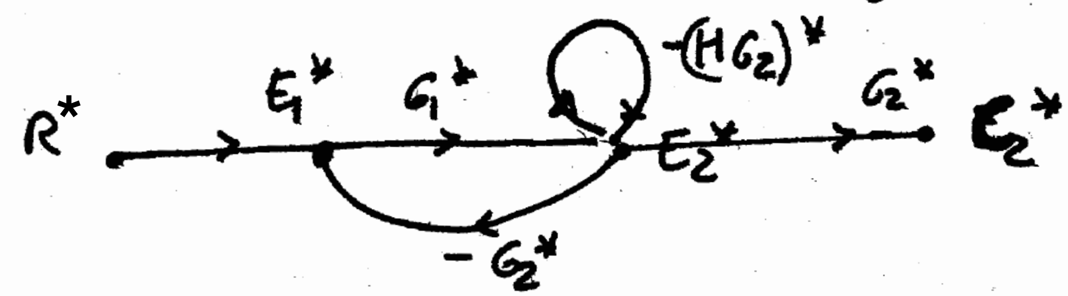

$$\Delta = 1 + G_1^* G_2^* + (HG_2)^* \Rightarrow \boxed{\frac{C^*}{R^*} = \frac{G_1^* G_2^*}{1 + G_1^* G_2^* + (G_2 H)^*}}$$

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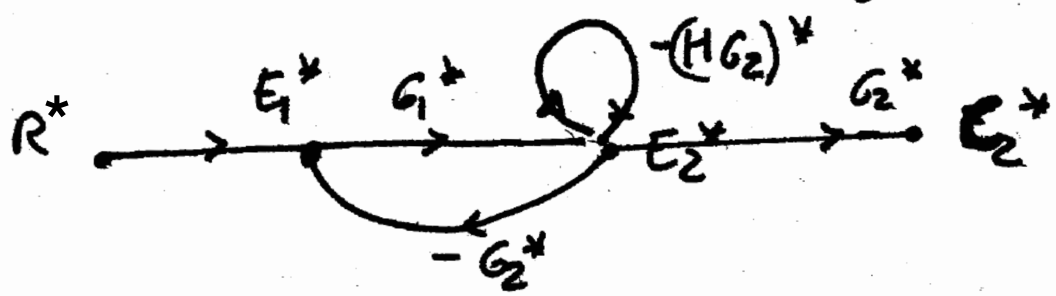

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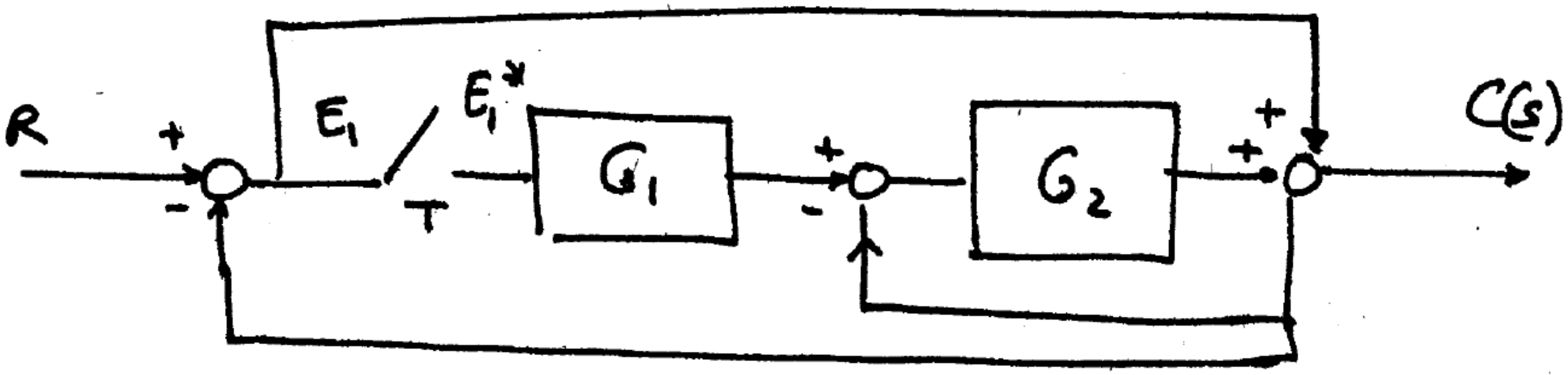

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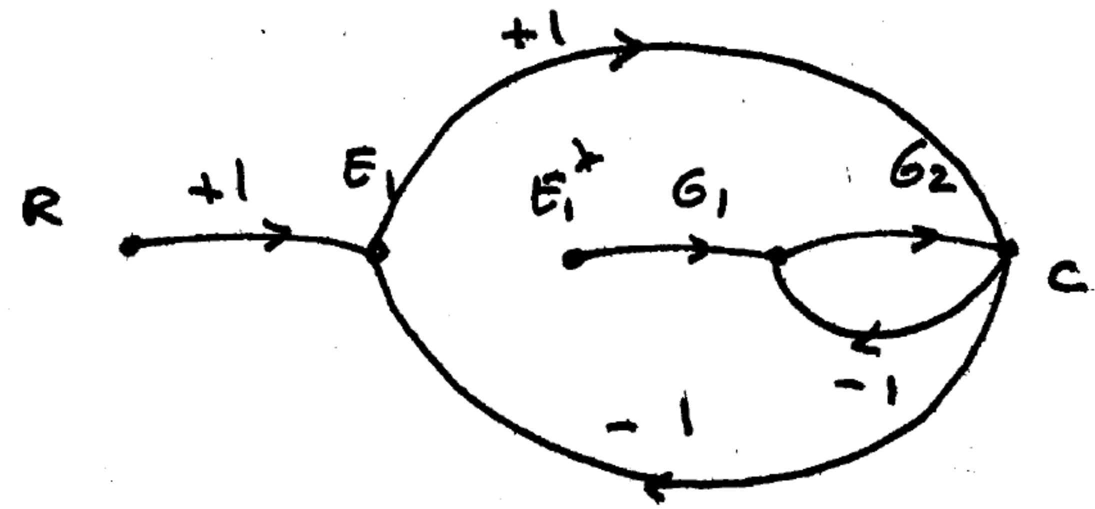

(2) variables: E_1^*

(3) need to find E_1, C as functions of R, E_1^* $\Rightarrow$ use Mason's formula on the "original" graph:

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$$\Delta = 1 + 1 + G_2 = 2 + G_2$$

$$\frac{E_1}{R} = \frac{1 \cdot (1+G_2)}{2+G_2} \, ; \quad \frac{E_1}{E_1^*} = \frac{-G_1 G_2}{2+G_2} \Rightarrow \boxed{E_1 = \frac{(1+G_2)R}{2+G_2} - \frac{G_1 G_2}{2+G_2} E_1^*} \quad (1)$$

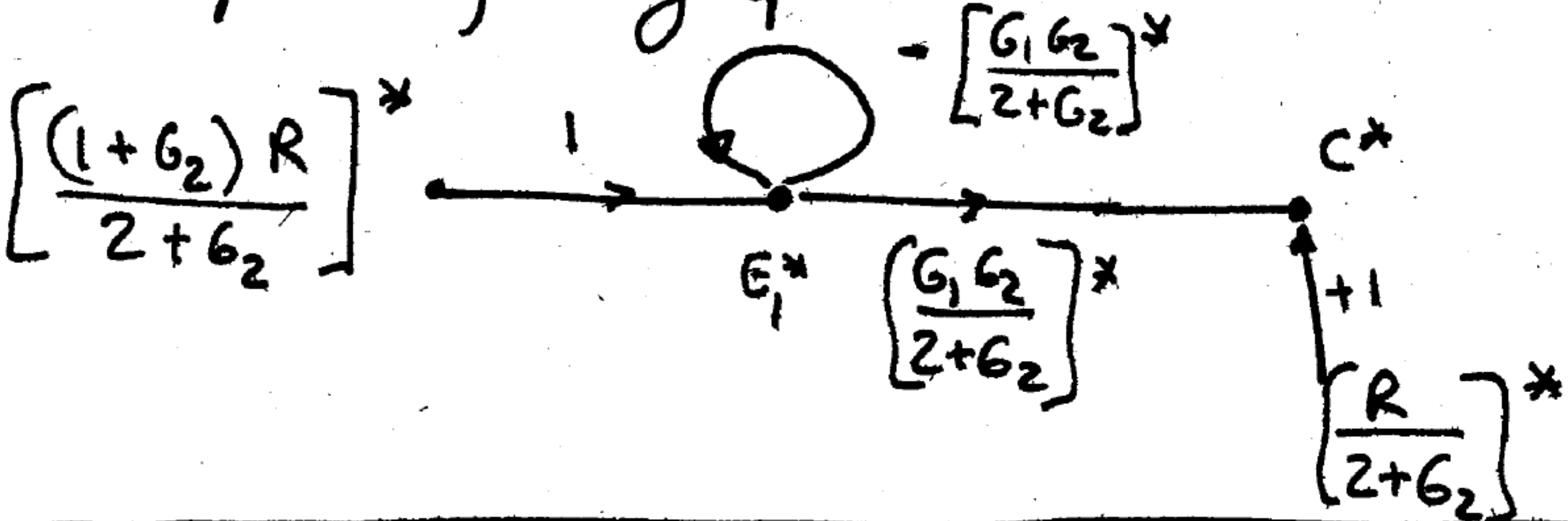
$$\frac{C}{R} = \frac{1}{2+G_2} \, ; \quad \frac{C}{E_1^*} = \frac{G_1 G_2}{2+G_2} \Rightarrow \boxed{C = \frac{1}{(2+G_2)} R + \frac{G_1 G_2}{2+G_2} E_1^*} \quad (2)$$

(4) Take starred transforms of (1) and (2):

$$E_1^* = \left[\frac{(1+G_2)R}{2+G_2}\right]^* - \left[\frac{G_1 G_2}{2+G_2}\right]^* E_1^*$$

$$C^* = \left[\frac{R}{2+G_2}\right]^* + \left[\frac{G_1 G_2}{2+G_2}\right]^* E_1^*$$

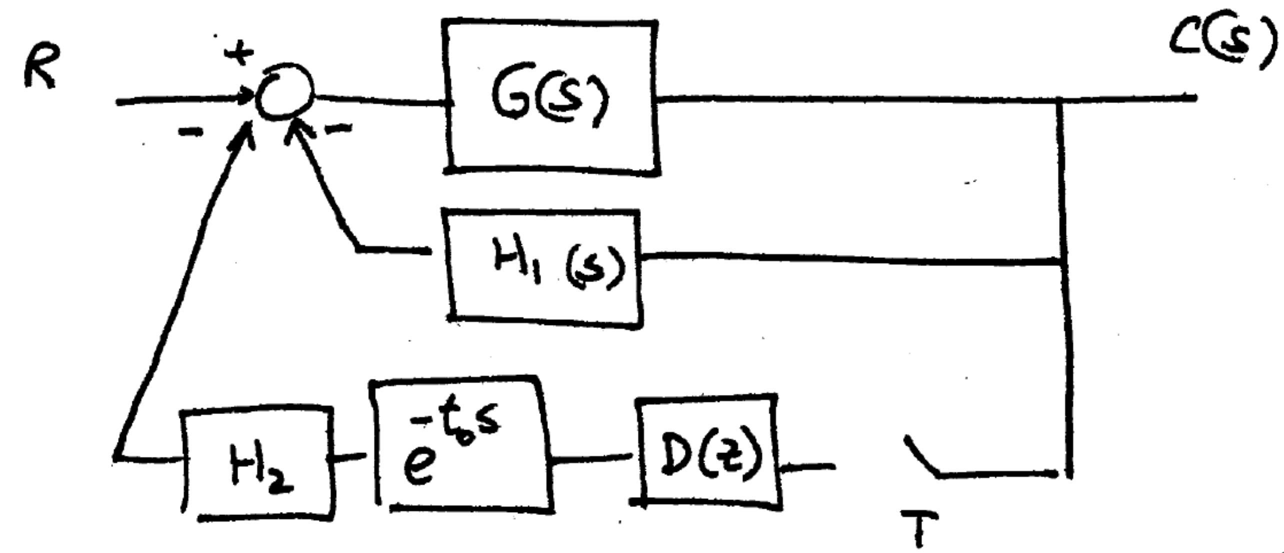
(5) sampled flow graph:



$$\Rightarrow \boxed{C^* = \left[\frac{R}{2+G_2}\right]^* + \frac{\left[\frac{G_1G_2}{2+G_2}\right]^*}{1+\left[\frac{G_1G_2}{2+G_2}\right]^*}\left[\frac{(1+G_2)R}{2+G_2}\right]^*}$$

<u>Note</u>: No transfer function exists from $R(z)$ to $C(z)$ since you can't factor R^* out. This was expected since the continuous time input $R(s)$ goes into the block $G_2(s)$ without going through a sampler

Example 3:



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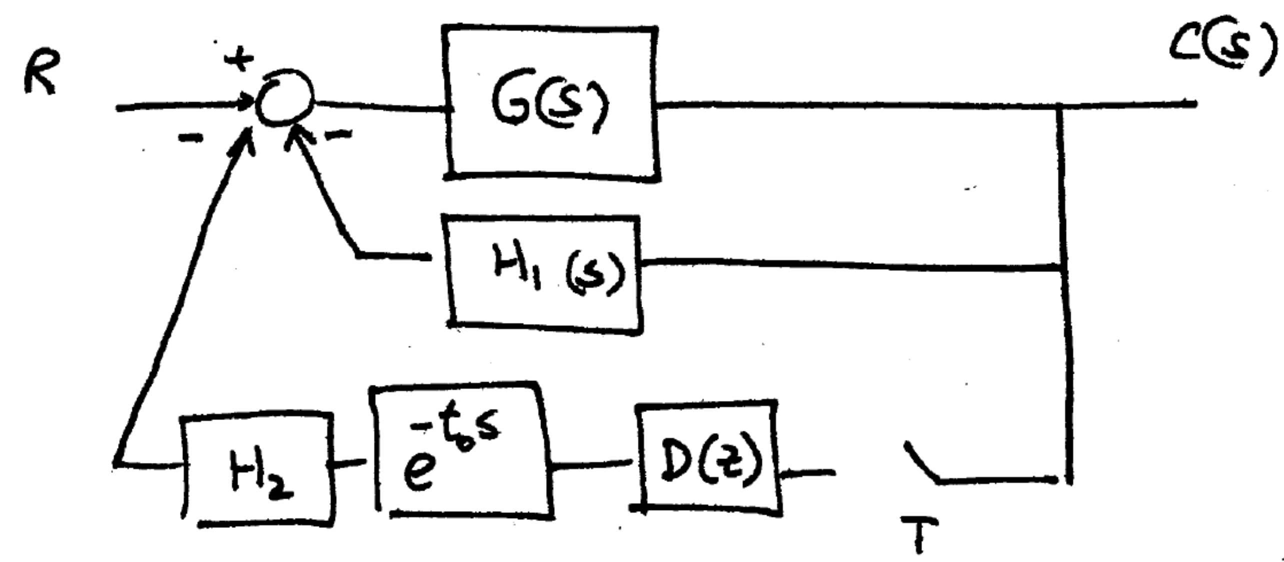
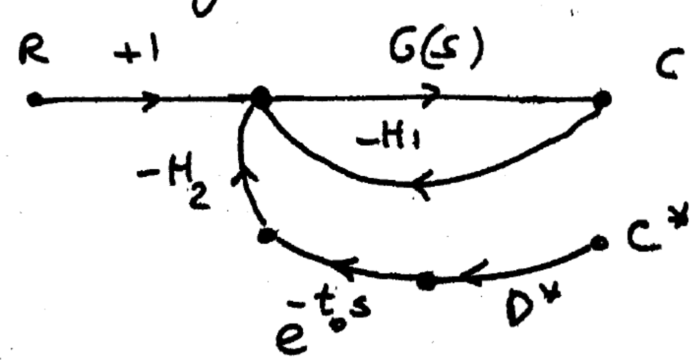


A system containing a digital controller and non-negligible computation time:

(1)



(Assume $0 \le t_0 < T$).

(2) variable: C^*

(3) $$C = \left(\frac{G}{1+GH_1}\right)R - \frac{GH_2 e^{-t_0 s} D^*}{1+GH_1} C^*$$

(4) $$C^* = \left[\frac{GR}{1+GH_1}\right]^* - \left(\frac{GH_2 e^{-t_0 s}}{1+GH_1}\right)^* D^* C^*$$

modified z transform

$$C(z) = \left[\frac{GR}{1+GH_1}\right](z) - \left[\frac{GH_2}{1+GH_1}\right](z,m) \cdot D(z)\, C(z)$$

with $t_0 = \Delta T$
$m = 1 - \Delta$
$mT = T - t_0$

Solving this equation for $C(z)$ yields:

$$C(z) = \frac{\left[\frac{GR}{1+GH_1}\right](z)}{1 + \left[\frac{GH_2}{1+GH_1}\right](z,m)\, D(z)}$$