Digital Control Systems - Chapter 3 Notes

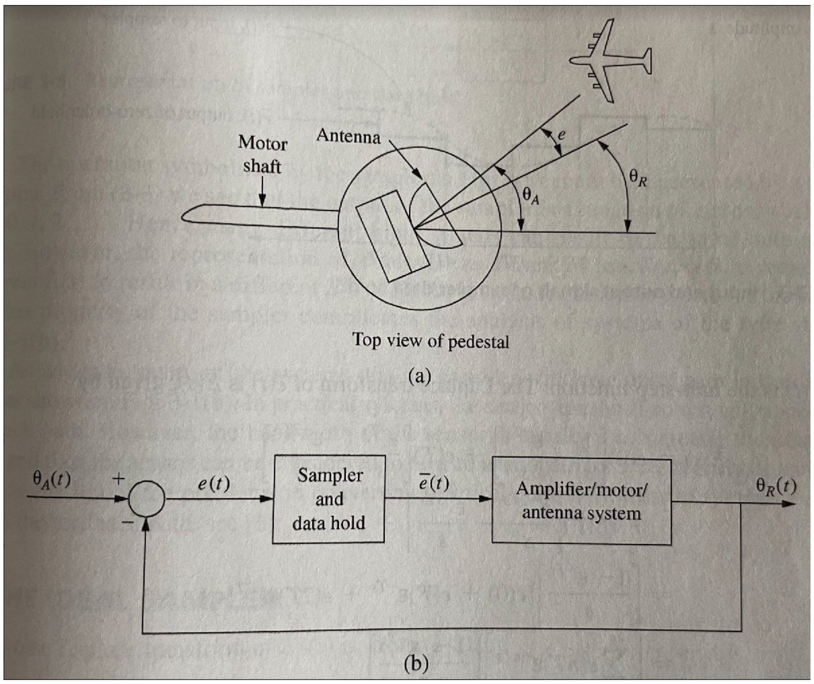
Sampling & Reconstruction

3.1 Introduction

- Necessary to determine the effects of sampling a continuous-time signal

3.2 Sampled-Data Control Systems

Introduce concept of sampled-data systems by examining the radar tracking system



Shown is the closed-loop system for tracking the aircraft automatically, with:

- $\theta_R(t) \rightarrow$ yaw angle of antenna
- $\theta_A(t) \rightarrow$ angle of the aircraft
- $e(t) = \theta_A(t) - \theta_R(t) \rightarrow$ tracking error
- $T \rightarrow$ sample period that the radar transmits
- $\bar{e}(t) = e(0)[u(t) - u(t-T)] + e(T)[u(t-T) - u(t-2T)] + e(2T)[u(t-2T) - u(t-3T)] + \ldots$

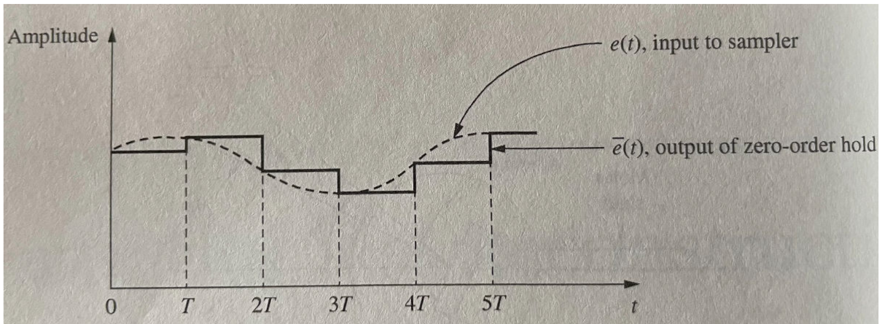
Note: undesired to apply a train of narrow rectangular pulses to a plan, because of the high-frequency components present in signal.

Data-Hold → inserted into system following the sampler with a purpose to reconstruct the sampled signal into a form that closely resembles the signal before sampling

- Simplest → **Zero-Order Hold**

Expression for sampled-ZOH signal is

$\bar{e}(t) = e(0)[u(t) - u(t-T)] + e(T)[u(t-T) - u(t-2T)] + e(2T)[u(t-2T) - u(t-3T)] + \ldots$



3-3 Input and output signals of sampler/data hold.

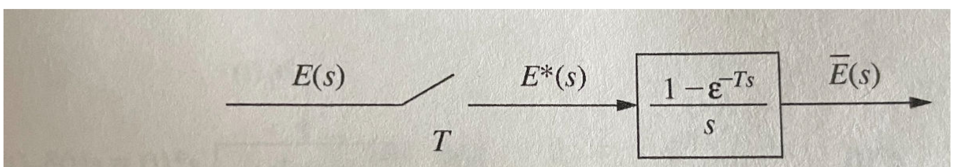
The Laplace transform of $\bar{e}(t)$ is $\bar{E}(s)$, given by

$$\bar{E}(s) = e(0)\left[\frac{1}{s} - \frac{\varepsilon^{-Ts}}{s}\right] + e(T)\left[\frac{\varepsilon^{-Ts}}{s} - \frac{\varepsilon^{-2Ts}}{s}\right] + e(2T)\left[\frac{\varepsilon^{-2Ts}}{s} - \frac{\varepsilon^{-3Ts}}{s}\right] + \ldots$$

$$= \left[\frac{1-\varepsilon^{-Ts}}{s}\right][e(0) + e(T)\varepsilon^{-Ts} + e(2T)\varepsilon^{-2Ts} + \ldots]$$

$$= \left[\sum_{n=0}^{\infty} e(\mathrm{nT})\varepsilon^{-\mathrm{nTs}}\right]\left[\frac{1-\varepsilon^{-Ts}}{s}\right]$$

The Starred Transform $\rightarrow E^*(s) = \sum_{n=0}^{\infty} e(\mathrm{nT})\varepsilon^{-\mathrm{nTs}} \rightarrow$ **Ideal Sampler**



Representation of sampler and data hold.

Switch $\rightarrow$ 2nd component of Laplace of $\bar{E}(s)$ called **Data Hold**

- Combined, these accurately model the input-output characteristics of the sampler-data hold device without being physically modelled
- Above figure **cannot** be represented by a transfer function

3.3 The Ideal Sampler

The inverse Laplace transform of $E^*(s)$ is

$$e^*(t) = L^{-1}[E^*(s)] = e(0)\delta(t) + e(T)\delta(t-T) + e(2T)\delta(t-2T) + \ldots$$

- $e^*(t)$ is a train of impulse functions whose weights are equal to the values of the signal at the instants of sampling

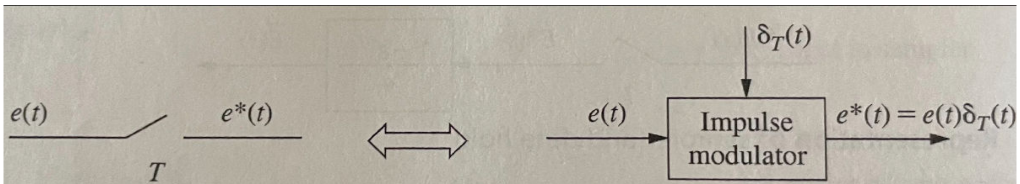
The sampler that appears in a sampler/hold → *ideal sampler or impulse modulator* → since nonphysical signals (impulse functions) appear on its output

$$\delta_T(t) = \sum_{n=0}^{\infty} \delta(t - \mathrm{nT}) = \delta(t) + \delta(t - T) + \ \dots$$

Then $e^*(t)$ can be expressed as

$$e^*(t) = e(t)\delta_T(t) = e(0)\delta(t) + e(T)\delta(t - T) + e(2T)\delta(t - 2T) + \ \dots$$

- $\delta_T(t)$ → carrier of modulation process
- $e(t)$ → the modulating signal



Representations of the ideal sampler.

Definition. The output signal of an ideal sampler is defined as the signal whose Laplace transform is

$$E^*(s) = \sum_{n=0}^{\infty} e(\mathrm{nT})\varepsilon^{-\mathrm{nTs}}$$

where $e(t)$ is the input to the sampler.

- If $e(t)$ is discontinuous at $t = \mathrm{kT}$, where k is an integer, then $e(\mathrm{kT})$ is taken to be $e(\mathrm{kT})^+$, which indicates the value of $e(t)$ as t approaches kT from the right (i.e., at $t = \mathrm{kT} + \Delta,$ where Δ is arbitrarily small)

Zero-Order Hold Transfer Function

$$G_{\mathrm{ho}}(s) = \frac{1 - \varepsilon^{-\mathrm{Ts}}}{s}$$

- If the signal to be sampled contains an impulse function at a sampling instant, the Laplace transform of the sampled signal <u>does not exist</u>; but is of no practical concern

Example 3.1

Determine $E^*(s)$ for $e(t) = u(t)$. For the unit step, $e(\mathrm{nT}) = 1, n = 0, 1, 2, \dots$

$$E^*(s) = \sum_{n=0}^{\infty} e(\mathrm{nT})\varepsilon^{-\mathrm{nTs}} = e(0) + e(\mathrm{nT})\varepsilon^{-\mathrm{Ts}} + e(\mathrm{nT})\varepsilon^{-2\mathrm{Ts}} + \ \dots$$

or

$E^*(s) = 1 + \varepsilon^{-\mathrm{Ts}} + \varepsilon^{-2\mathrm{Ts}} + \ \ldots$

$E^*(s)$ can be expressed in closed form using the following relationship. For $|x| < 1$,

$$\frac{1}{1-x} = 1 + x + x^2 + \ \ldots$$

The condition $|x| < 1$ guarantees convergence of the series. Hence the expression for $E^*(s)$ above can be written in closed form as

$$E^*(s) = \frac{1}{1-\varepsilon^{-\mathrm{Ts}}},\ |\varepsilon^{-\mathrm{Ts}}| < 1$$

Example 3.2

Determine $E^*(s)$ for $e(t) = \varepsilon^{-t}$

$$E^*(s) = \sum_{n=0}^{\infty} e(\mathrm{nT})\varepsilon^{-\mathrm{nTs}}$$

$$= 1 + \varepsilon^{-T}\varepsilon^{-\mathrm{Ts}} + \varepsilon^{-2T}\varepsilon^{-2\mathrm{Ts}} + \ \ldots$$

$$= 1 + \varepsilon^{-(1+s)T} + (\varepsilon^{-(1+s)T})^2 + \ \ldots$$

$$= \frac{1}{1-\varepsilon^{-(1+s)T}},\ |\varepsilon^{-(1+s)T}| < 1$$

3.4 Evaluation of $E^*(s)$

$E^*(s)$ has limited usefulness in analysis because it is expressed as an infinite series. However, for many useful time functions, $E^*(s)$ can be expressed in closed form.

If we then take the Laplace transform of $e^*(t)$ using the complex convolution integral, we can derive two additional expressions for $E^*(s)$

1. $$E^*(s) = \sum_{\text{at poles of } E(\lambda)} \left[\text{residues of } E(\lambda)\frac{1}{1-\varepsilon^{-T(s-\lambda)}}\right]$$
2. $$E^*(s) = \frac{1}{T}\sum_{n=-\infty}^{\infty} E(s + \mathrm{j}n\omega_s) + \frac{e(0)}{2}$$

where ω_s is the radian sampling frequency $\rightarrow \omega_s = 2\pi/T$

Expression 1.) is useful in generating tables for the starred transform $E^*(s)$

Expression 2.) will prove to be useful in analysis in next section

Example 3.3

Determine $E^*(s)$ given that

$$E(s) = \frac{1}{(s+1)(s+2)}$$

From Expression 1.)

$$E(\lambda)\frac{1}{1-\varepsilon^{-T(s-\lambda)}} = \frac{1}{(\lambda+1)(\lambda+2)(1-\varepsilon^{-T(s-\lambda)})}$$

Then

$$E^*(s) = \sum_{\text{poles of } E(\lambda)} \left[\text{residues of } E(\lambda)\frac{1}{1-\varepsilon^{-T(s-\lambda)}}\right]$$

$$= \frac{1}{(\lambda+2)(1-\varepsilon^{-T(s-\lambda)})}|_{\lambda=-1} + \frac{1}{(\lambda+1)(1-\varepsilon^{-T(s-\lambda)})}|_{\lambda=-2}$$

$$= \frac{1}{(1-\varepsilon^{-T(s+1)})} - \frac{1}{(1-\varepsilon^{-T(s+2)})}$$

Example 3.4

We wish to determine the starred transform of $e(t) = \sin\beta t$. The corresponding $E(s)$ is

$$E(s) = \frac{\beta}{s^2+\beta^2} = \frac{\beta}{(s-j\beta)(s+j\beta)}$$

$E^*(s)$ can be evaluated from the expression

$$E^*(s) = \sum_{\text{poles of } E(\lambda)} \left[\text{residues of } \frac{\beta}{(\lambda-j\beta)(\lambda+j\beta)(1-\varepsilon^{-T(s-\lambda)})}\right]$$

$$= \frac{\beta}{(\lambda+j\beta)(1-\varepsilon^{-T(s-\lambda)})}|_{\lambda=j\beta} + \frac{1}{(\lambda-j\beta)(1-\varepsilon^{-T(s-\lambda)})}|_{\lambda=-j\beta}$$

$$= \frac{1}{2j}\left[\frac{1}{1-\varepsilon^{-\text{Ts}}\varepsilon^{j\beta T}} - \frac{1}{1-\varepsilon^{-\text{Ts}}\varepsilon^{-j\beta T}}\right]$$

$$= \frac{\varepsilon^{-\text{Ts}}\sin\beta T}{1-2\varepsilon^{-\text{Ts}}\cos\beta T + \varepsilon^{-2\text{Ts}}}$$

using the equations from Euler's relation:

$$\cos\beta T = \frac{\varepsilon^{j\beta T}+\varepsilon^{-j\beta T}}{2}; \sin\beta T = \frac{\varepsilon^{j\beta T}-\varepsilon^{-j\beta T}}{2j}$$

Example 3.5

Given $e(t) = 1-\varepsilon^{-t}$, determine $E^*(s)$, using **Starred Transform method** and **Residues method**

Starred Transform Method:

$$E^*(s) = \sum_{n=0}^{\infty} e(\mathrm{nT})\varepsilon^{-\mathrm{nTs}}$$

$$= \sum_{n=0}^{\infty} (1 - \varepsilon^{-t})\varepsilon^{-\mathrm{nTs}}$$

$$= \sum_{n=0}^{\infty} \varepsilon^{-\mathrm{nTs}} - \sum_{n=0}^{\infty} \varepsilon^{-(1+s)\mathrm{nT}}$$

$$= \frac{1}{1 - \varepsilon^{-\mathrm{Ts}}} - \frac{1}{1 - \varepsilon^{-(1+s)T}}$$

Residues Method:

$$E^*(s) = \sum_{\lambda=0,\, \lambda=-1} \left[\text{residues of } \frac{1}{\lambda(\lambda+1)} \frac{1}{1 - \varepsilon^{-T(s-\lambda)}}\right]$$

$$= \frac{1}{1 - \varepsilon^{-\mathrm{Ts}}} - \frac{1}{1 - \varepsilon^{-(1+s)T}}$$

Time-Shifting Property of Laplace Transform:

$$E(s) = \varepsilon^{-t_0 s} L[e_1(t)] = \varepsilon^{-t_0 s} E_1(s)$$

Special techniques are required to find the starred transform (regular formula will not work)

Special case in which the time signal is delayed a whole number of sampling periods the Residue Method can be applied:

$$\left[\varepsilon^{-\mathrm{kTs}} E_1(s)\right]^* = \varepsilon^{-\mathrm{kTs}} \sum_{\text{at poles of } E(\lambda)} \left[\text{residues of } E_1(\lambda) \frac{1}{1 - \varepsilon^{-T(s-\lambda)}}\right]$$

where k is a positive integer

Example 3.6

The starred transform of $e(t) = [1 - \varepsilon^{-(t-1)}]u(t-1)$, with $T = 0.5s$, will now be found. First we find $E(s)$:

$$E(s) = \frac{\varepsilon^{-s}}{s} - \frac{\varepsilon^{-s}}{s+1} = \frac{\varepsilon^{-s}}{s(s+1)}$$

From the Time-Shifting Property formula, $k = 2$ and

$$E_1(s) = \frac{1}{s(s+1)}$$

Then, from Residue Method:

$$\left[\frac{\varepsilon^{-s}}{s(s+1)}\right]^* = \sum_{\lambda=0,\, -1} \left[\text{residues of } \frac{1}{\lambda(\lambda+1)} \frac{1}{1 - \varepsilon^{-0.5(s-\lambda)}}\right]$$

$$= \varepsilon^{-s}\left[\frac{1}{(\lambda+1)(1-\varepsilon^{-0.5(s-\lambda)})}|_{\lambda=0} + \frac{1}{\lambda(1-\varepsilon^{-0.5(s-\lambda)})}|_{\lambda=-1}\right]$$

$$= \varepsilon^{-s}\left[\frac{1}{(1-\varepsilon^{-0.5})} + \frac{-1}{(1-\varepsilon^{-0.5(s+1)})}\right] = \frac{(1-\varepsilon^{-0.5})\varepsilon^{-1.5s}}{(1-\varepsilon^{-0.5s})(1-\varepsilon^{-0.5s(s+1)})}$$

3.5 Results From The Fourier Transform

In this section we present some results regarding the Fourier Transform, which is helpful for understanding the effects of sampling a signal.

Fourier Transform:

$$F\{e(t)\} = E(j\omega) = \int_{-\infty}^{\infty} e(t)\varepsilon^{-j\omega t}\mathrm{dt}$$

For the unilateral Laplace Transform, where signal $e(t)$ is zero for $t < 0$, its Fourier Transform is given by:

$$F\{e(t)\} = \int_{0}^{\infty} e(t)\varepsilon^{-j\omega t}\mathrm{dt} = L\{e(t)\}$$

provided that both transforms exists. This results can be expressed as

$$F\{e(t)u(t)\} = L\{e(t)u(t)\}|_{s=j\omega}$$

⋆ Hence, for the case that $e(t)$ is zero for negative time, the Fourier Transform of $e(t)$ is equal to the Laplace Transform of $e(t)$ with s replaced with $j\omega$ ⋆

- Also applies to *causal systems* where Transfer function $G(s)$ or $g(t) = 0$ for $t < 0$

Frequency Spectrum:

$$E(j\omega) = |E(j\omega)|\varepsilon^{j\theta(j\omega)} = |E(j\omega)|\angle\theta(j\omega)$$

Frequency Response:

$$Y(j\omega) = G(j\omega)E(j\omega)$$

if $e(t) = \delta(t)$ then $E(s) = 1$ and the amplitude and phase changes in the output are determined by Transfer function $G(j\omega)$

3.6 Properties of $E^*(s)$

Two s-plane properties of $E^*(s)$ are given:

Property 1. $E^*(s)$ is periodic in s with period $j\omega_s$.

$$E^*(s + jm\omega_s) = \sum_{n=0}^{\infty} e(\mathrm{nT})\varepsilon^{-\mathrm{nT}(s+jm\omega_s)}$$

Since $\omega_s T = (2\pi/T)T = 2\pi$, and from Euler's relationship,

$$\varepsilon^{j\theta} = \cos\theta + j\sin\theta$$

then

$$\varepsilon^{-jnm\omega_s T} = \varepsilon^{-jnm2\pi} = 1, \text{ for } m \text{ an integer}$$

Thus,

$$E^*(s + jm\omega_s) = \sum_{n=0}^{\infty} e(\mathrm{nT})\varepsilon^{-\mathrm{nTs}} = E^*(s)$$

Property 2. If $E(s)$ has a pole at $s = s_1$, then $E^*(s)$ must have poles at $s = s_1 = jm\omega_s,\ m = 0, \pm 1, \pm 2, \ldots$

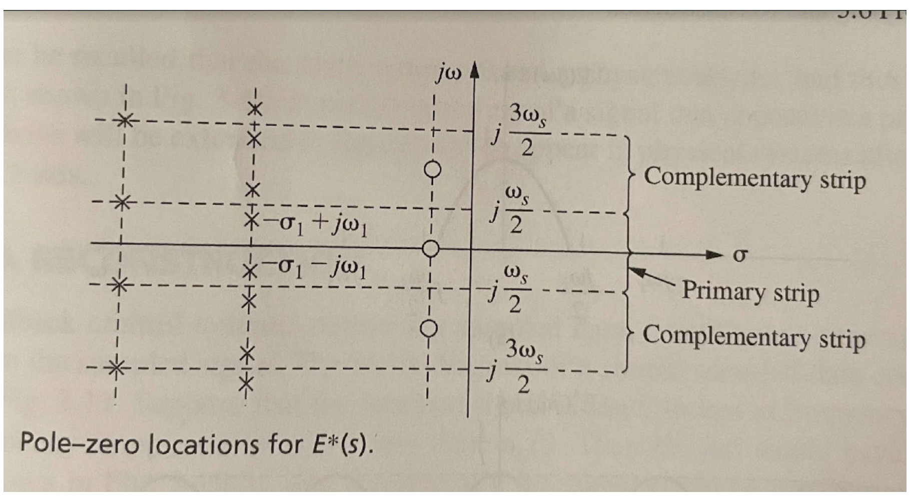
This property can be proved from **Expression 2.** Consider $e(t)$ to be continuous at all sampling instants. Then

$$E^*(s) = \frac{1}{T}\sum_{n=-\infty}^{\infty} E(s + \mathrm{j}n\omega_s) = \frac{1}{T}[E(s) + E(s + j\omega_s) + E(s + 2j\omega_s) + \ \ldots\ + E(s - j\omega_s) + E(s - 2j\omega_s) + \ \ldots]$$

Note: No equivalent statement can be made concerning the zeros of $E^*(s)$; the zero locations of $E(s)$ do not uniquely determine the zero locations of $E^*(s)$.

However, the zero locations are periodic with period $j\omega_s$, as indicated from **Property 1.** of $E^*(s)$.

Example of Pole-Zero Locations of $E^*(s)$:

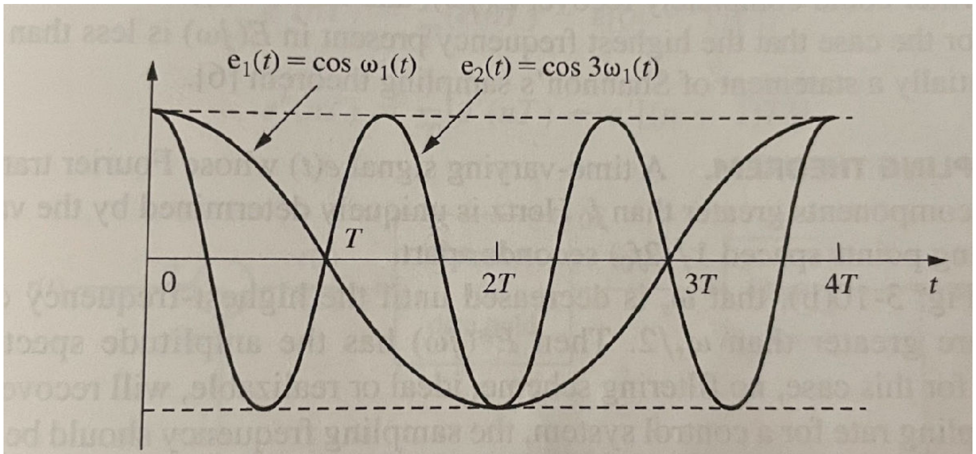


Pole–zero locations for $E^*(s)$.

- The primary strip in the s-plane is defined as the strip for which $-\frac{\omega_s}{2} \le \omega \le \frac{\omega_s}{2}$
- If the pole-zero locations are known for $E^*(s)$ in the primary strip, then the pole-zero locations in the entire s-plane are also known
- If $E(s)$ has a pole at $-\sigma_1 + j\omega_1$ → sampling operation will generate a pole in $E^*(s)$ at $-\sigma_1 + j(\omega_1 + \omega_s)$
- Conversely, if $E(s)$ has a pole at $-\sigma_1 + j(\omega_1 + \omega_s)$ → pole in $E^*(s)$ at $-\sigma_1 + j\omega_1$

- Pole Location in $E(s)$ at $-\sigma_1 + j(\omega_1 + k\omega_s)$, k an integer, will result in identical pole locations in $E^*(s)$, regardless of the integer value of k

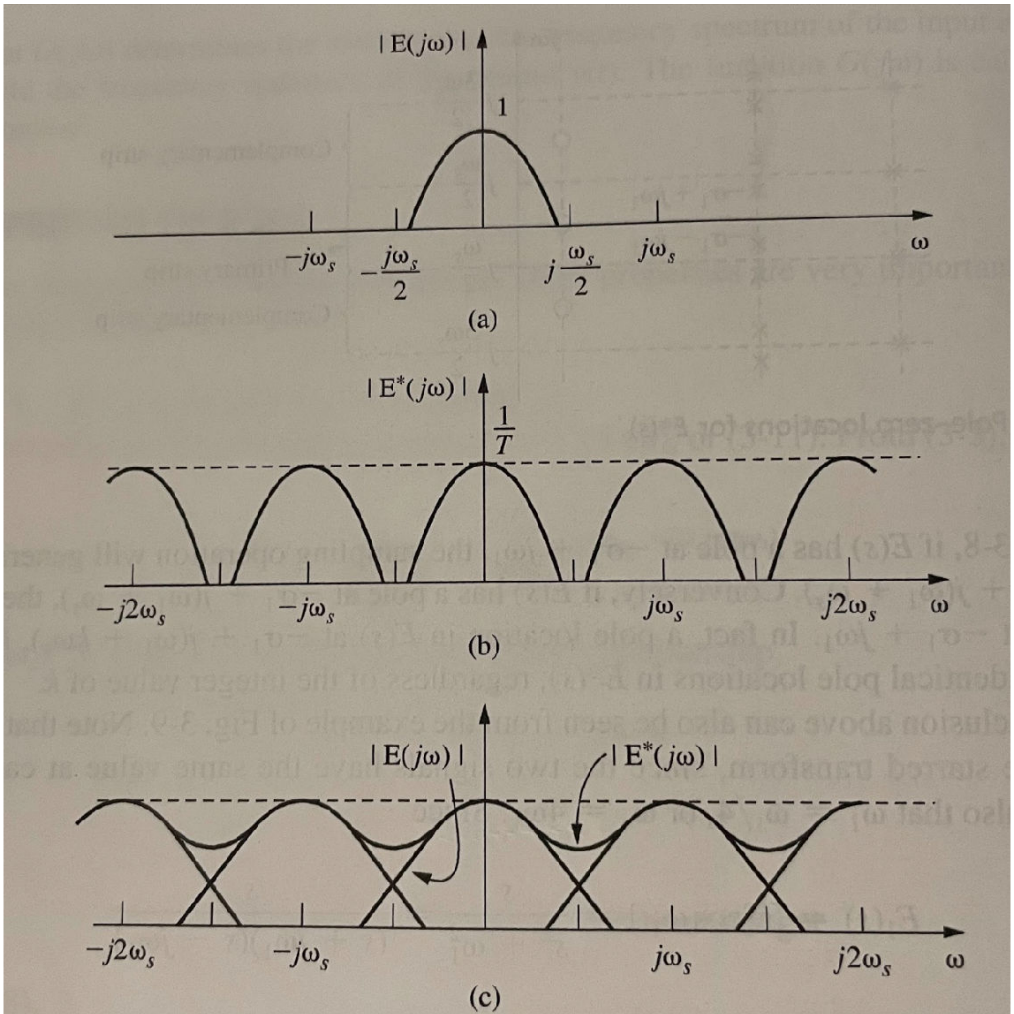
Example of Two Signals having same Starred Transform



Two signals that have the same starred transform.

- $E_1(s) = L\{\cos \omega_1 t\} = \dfrac{s}{s^2 + \omega_1^2} = \dfrac{s}{(s + j\omega_1)(s - j\omega_1)}$; poles at $s = \pm j\omega_1$
- $E_2(s) = L\{\cos 3\omega_1 t\} = \dfrac{s}{(s + 3j\omega_1)(s - 3j\omega_1)}$; poles at $s = \pm 3j\omega = \pm j(\omega_1 - \omega_s)$

Spectrum of $E(j\omega)$ and $E^*(j\omega)$



Frequency spectra for $E(j\omega)$ and $E^*(j\omega)$.

$$E^*(j\omega) == \frac{1}{T}[E(j\omega) + E(j\omega + j\omega_s) + E(j\omega + 2j\omega_s) + \ \dots \ + E(j\omega - j\omega_s) + E(j\omega - 2j\omega_s) + \ \dots \ ] + \frac{e(0)}{2}$$

Hence, the effect of ideal sampling is to replicate the origianl spectrum centered at ω_s, at $2\omega_s$, at $-\omega_s$, at $-2\omega_s$, and so on.

Shannon's Sampling Theorem

A time-varying signal $e(t)$ whose Fourier Transform contains no frequency components greater than f_0 Hertz is uniquely determined by the values of $e(t)$ at any set of sampling points spaced $\frac{1}{(2f_0)}$ seconds apart.

- Spectrum (c) above shows what happens to signal when sampling frequency f_0 is less then $1/2$ the highest frequency component of $E(j\omega) \rightarrow$ can't recover original signal

$\star$ Thus, in choosing the sampling rate for a control system, the sampling frequency should be greater than twice the highest-frequency component of *significant amplitude* of the signal being sampled $\star$

3.7 Data Reconstruction

In most feedback control systems employing sampled data, a continuous-time signal is reconstructed from the sampled signal

Since ideal filters do not exist in physically realizable systems, we must employ approximations → Practical data holds approximate ideal low-pass filters

Commonly used method of data reconstruction is polynomial extrapolation. Using a Taylor's series expansion about $t = \mathrm{nT}$, we can express $e(t)$ as

$$e(t) = e(\mathrm{nT}) + e'(\mathrm{nT})(t - \mathrm{nT}) + \frac{\mathrm{e''(nT)}}{2!}(t - \mathrm{nT})^2 + \ \ldots$$

$e_n(t)$ is defined as the reconstructed version of $e(t)$ for the nth sample period; that is,

$e_n(t) \cong e(t)$ for $\mathrm{nT} \leq t \leq (n+1)T$

Derivatives may be approximated by the backward difference:

- $e'(\mathrm{nT}) = \frac{1}{T}[e(\mathrm{nT}) - e[(n-1)T]]$
- $e''(\mathrm{nT}) = \frac{1}{T}[e'(\mathrm{nT}) - e'[(n-1)T]]$ or $e''(\mathrm{nT}) = \frac{1}{T^2}[e(\mathrm{nT}) - 2e[(n-1)T] + e[(n-2)T]]$

Three Types of Data-Holds

- Zero-Order Hold
- First-Order Hold
- Fractional-Order Hold

Zero-Order Hold

Here, assume that $e(t)$ is approximately constant within the sampling interval at a value equal to that of the function at the preceding samploing instant.

$e_n(t) = e(\mathrm{nT}), \quad \mathrm{nT} \leq t < (n+1)T$

- Simpliest to construct as it requires no memory of previous value of $e(t)$

$e_0(t) = u(t) - u(t-T)$

and

$$E_0(s) = \frac{1}{s} - \frac{\varepsilon^{-\mathrm{Ts}}}{s}$$

Since $E_i(s) = 1$, the transfer function of the zero-order hold is

$$G_{\mathrm{h0}}(s) = \frac{E_0(s)}{E_i(s)} = \frac{1 - \varepsilon^{-\mathrm{Ts}}}{s}$$

Frequency Response of ZOH:

$$G_{h0}(j\omega) = T\frac{\sin(\omega T/2)}{\omega T/2}\varepsilon^{-j(\omega T/2)}$$

Since

$$\frac{\omega T}{2} = \frac{\omega}{2}\left(\frac{2\pi}{\omega_s}\right) = \frac{\pi\omega}{\omega_s}$$

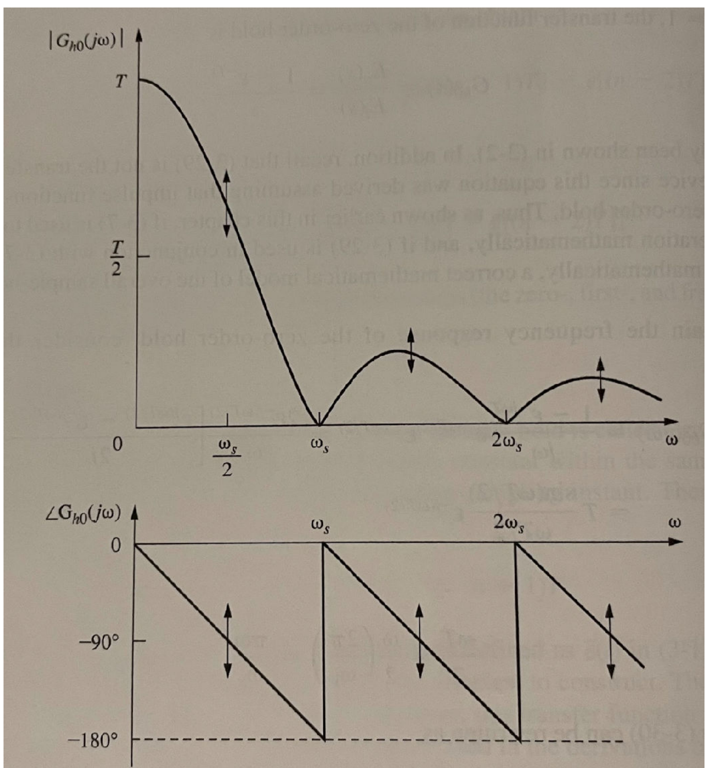
Expression can be rewritten as:

$$G_{h0}(j\omega) = T\frac{\sin(\pi\omega/\omega_s)}{\pi\omega/\omega_s}\varepsilon^{-j(\pi\omega/\omega_s)}$$

Thus

$|G_{h0}(j\omega)| = T|\frac{\sin(\pi\omega/\omega_s)}{\pi\omega/\omega_s}|$ and $\angle G_{h0}(j\omega) = -\frac{\pi\omega}{\omega_s} + \theta$,

- $\theta = 0, \text{ for } \sin(\pi\omega/\omega_s) > 0$
- $\theta = \pi, \text{ for } \sin(\pi\omega/\omega_s) < 0$



Frequency response of the zero-order hold.

Thus, the frequency response of the zero-order hold may be used to determine the amplitude spectrum of the data-hold output signal.

First-Order Hold

Using the first two terms from the expression can realize the first-order hold. Therefore,

$$e_n(t) = e(\mathrm{nT}) + e'(\mathrm{nT})(t - \mathrm{nT}), \quad \mathrm{nT} \le t < (n+1)T$$

where,

$$e'(\mathrm{nT}) = \frac{e(\mathrm{nT}) - e[(n-1)T]}{T}$$

- The extrapolated function within a given interval outputs a straight line and its slope is determined by the values of the function at the sampling instants in previous intervals
- Memory is required in realization of this data hold

Transfer Function of First-Order Hold:

$$G_{\mathrm{h1}}(s) = \frac{1 + \mathrm{Ts}}{T}\left[\frac{1 - \varepsilon^{-\mathrm{Ts}}}{s}\right]^2$$

Frequency Response of First-Order Hold:

$$G_{\mathrm{h1}}(j\omega) = \frac{1 + j\omega T}{T}\left[\frac{1 - \varepsilon^{-j\omega T}}{j\omega}\right]^2$$

Magnitude Response of First-Order Hold:

$$|G_{\mathrm{h1}}(j\omega)| = T\sqrt{1 + \frac{4\pi^2\omega^2}{\omega_s^2}}\left[\frac{\sin(\pi\omega/\omega_s)}{\pi\omega/\omega_s}\right]^2$$

Phase Response of First-Order Hold:

$$\angle G_{\mathrm{h1}}(j\omega) = \tan^{-1}\left(\frac{2\pi\omega}{\omega_s}\right) - \frac{2\pi\omega}{\omega_s}$$

- Provides a better approximation of the ideal low-pass filter in the vicinity of zero frequency than does the zero-order hold
- Yet, when ω is larger the zero-order hold is a better approximation

Fractional-Order Holds

- FOH performs a linear extrapolation from one sampling interval to the next
- The error generated in this process can be reduced b y using only a fraction of the slope in the previous interval
- FOH may be used to match the data-hold frequency response to the sampled signal's frequency spectrum, thereby generating minimum error extrapolations

Transfer Function of Fractional-Order Holds:

$$G_{\mathrm{hk}}(s) = (1 - k\varepsilon^{-\mathrm{Ts}})\frac{1 - \varepsilon^{-\mathrm{Ts}}}{s} + \frac{k}{\mathrm{Ts}^2}(1 - \varepsilon^{-\mathrm{Ts}})^2$$

3.8 Summary

Key takeaways from this chapter:

- Development of approximate rules for the choice of the sample period T for a given signal
- Later chapters will show the importance of a system's frequency response in determining the sample rate to be used in the system