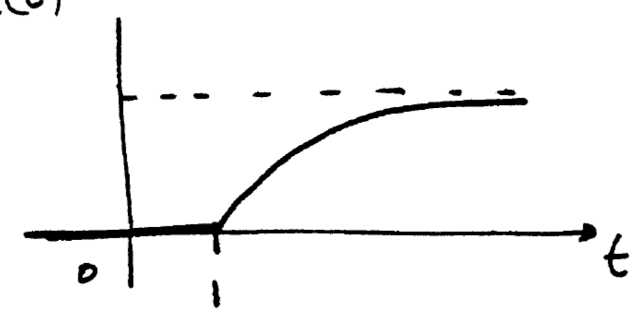
Example 3:

A function with a time-delay. This example will become relevant later on when we will look into the effects of sampling:

Let $e(t) = \left[1 - e^{-(t-1)}\right] u(t-1)$ (i.e.: $e(t) = (1-e^{-t})$ delayed by 1 second)

$e(t)$



$e(k) = \left[1 - e^{-(0.5k-1)}\right] \quad k \geq 2; \qquad e(k) = 0 \quad k = 0, 1$

Now let's try our residues formula:

$$E(s) = \frac{e^{-s}}{s(s+1)} \Rightarrow E^*(s) = \sum_{\substack{\lambda=0 \\ \lambda=-1}} \mathrm{Res}\left\{ \frac{e^{-\lambda}}{\lambda(\lambda+1)} \frac{1}{1-e^{-T(s-\lambda)}} \right\} =$$

$$= \frac{1}{1-e^{-Ts}} + \frac{e^{1}}{(-1)} \cdot \frac{1}{1-e^{-T(s+1)}} = \boxed{\frac{1}{1-e^{-0.5s}} - \frac{e^{1}}{1-e^{-0.5(s+1)}}}$$

Surprise! we got <u>different</u> answers

Q: What went wrong here?

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A: <u>A techical point</u>: the "proof" of the residues formula is <u>not</u> valid for systems having time delays

The reason is that $e^{-sT} \not\to 0$ on the infinite portion of the contour B_1 and thus we <u>can't</u> close the contour and compute the $\int$ using residues

Solution

(a) don't use the residues formula for systems with delays
Not too convenient. It defeats the whole purpose
of introducing the * transform!

<u>Solution</u>

(a) don't use the residues formula for systems with delays
Not too convenient. It defeats the whole purpose of introducing the * transform!

(b) Modify the formula:

It can be shown that if the delay is an <u>integer</u> number of periods then:

$$E^*(s) = \left[e^{-kTs} \underset{\substack{\uparrow \\ \text{non delayed} \\ \text{signal}}}{E_1(s)}\right]^* = e^{-kTs} \sum_{\substack{\text{at poles} \\ \text{of } E_1}} \left\{ \text{Res } E_1(\lambda) \frac{1}{1 - e^{-T(s-\lambda)}} \right\}.$$

Applying this modified formula to our earlier example we get

$$E_1(s) = \frac{1}{s(s+1)}$$

$$\sum_{\substack{\lambda=0 \\ \lambda=-1}} \operatorname{Res} \frac{1}{\lambda(\lambda+1)} \cdot \frac{1}{\left(1 - e^{-T(s-\lambda)}\right)}$$

$$= \frac{1}{\left(1 - e^{-Ts}\right)} - \frac{1}{\left(1 - e^{-T(s+1)}\right)}$$

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$$= \frac{1}{\left(1 - e^{-Ts}\right)} - \frac{1}{\left(1 - e^{-T(s+1)}\right)}$$

$$\Rightarrow E^*(s) = e^{-2Ts}\left[\frac{1}{1 - e^{-0.5s}} - \frac{1}{1 - e^{-0.5(s+1)}}\right] = e^{-s}\left[\frac{1}{1 - e^{-0.5s}} - \frac{1}{1 - e^{-0.5(s+1)}}\right]$$

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which coincides with our earlier result

Properties of $E^*(s)$

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proof:

$$E^*(s) = \sum_0^\infty e(kT)\, e^{-skT}$$

$$E^*(s+j\omega_s) = \sum_0^\infty e(kT)\, e^{-k\left[s+j\frac{2\pi}{T}\right]T} = \sum_0^\infty e(kT)\, e^{-skT} \underbrace{e^{-jk\frac{2\pi}{\cancel{T}}\cdot\cancel{T}}}_{1} = E^*(s) \quad \#$$

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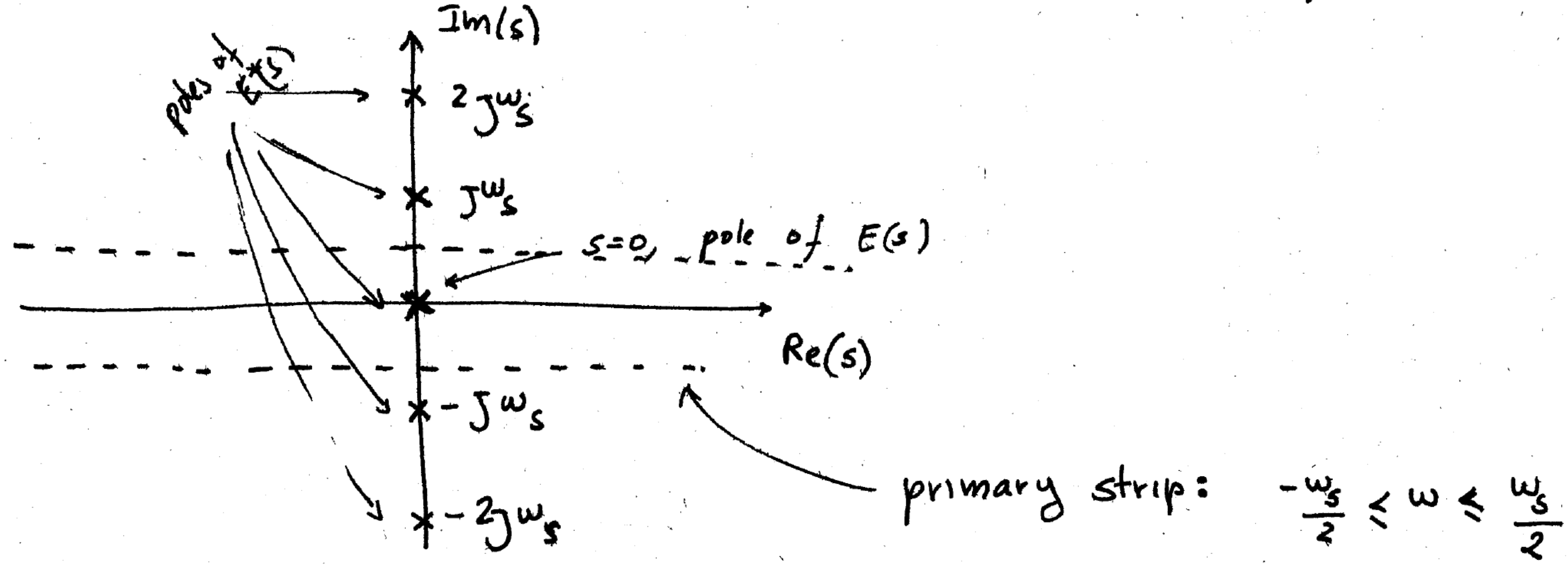
2) If $E(s)$ has a pole at $s = s_1 \Rightarrow E^*(s)$ <u>has poles at $s = s_1 + jm\omega_s$</u> $m = 0, \pm 1 \ldots$

<u>Note</u>: some property <u>does not</u> apply to zeros of $E^*(s)$

Example: assume that $E(s)$ has a pole at $s=0 \Rightarrow E^*(s)$ poles at $s_m = \pm j n \omega_s$



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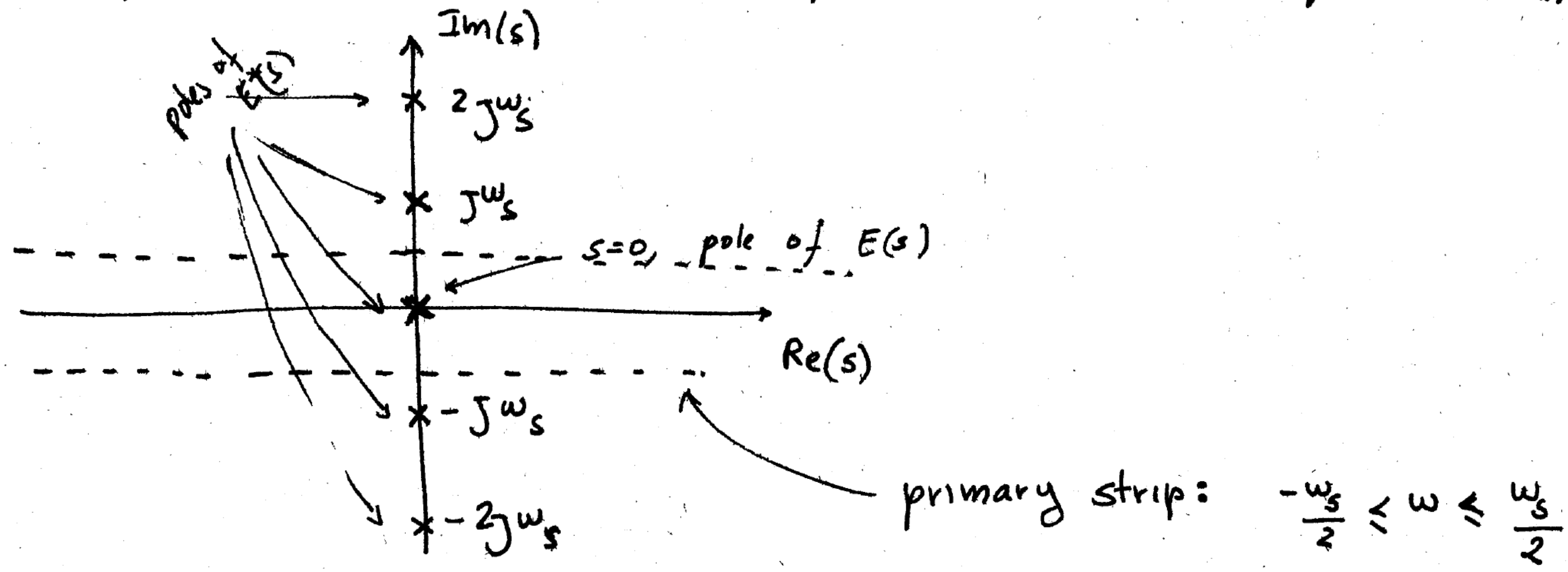

proof: $$E^*(s) = \frac{1}{T} \sum_{n=-\infty}^{n=+\infty} E(s + jn\omega_s) = \frac{1}{T}\left[E(s) + E(s+j\omega_s) + \cdots \right]$$

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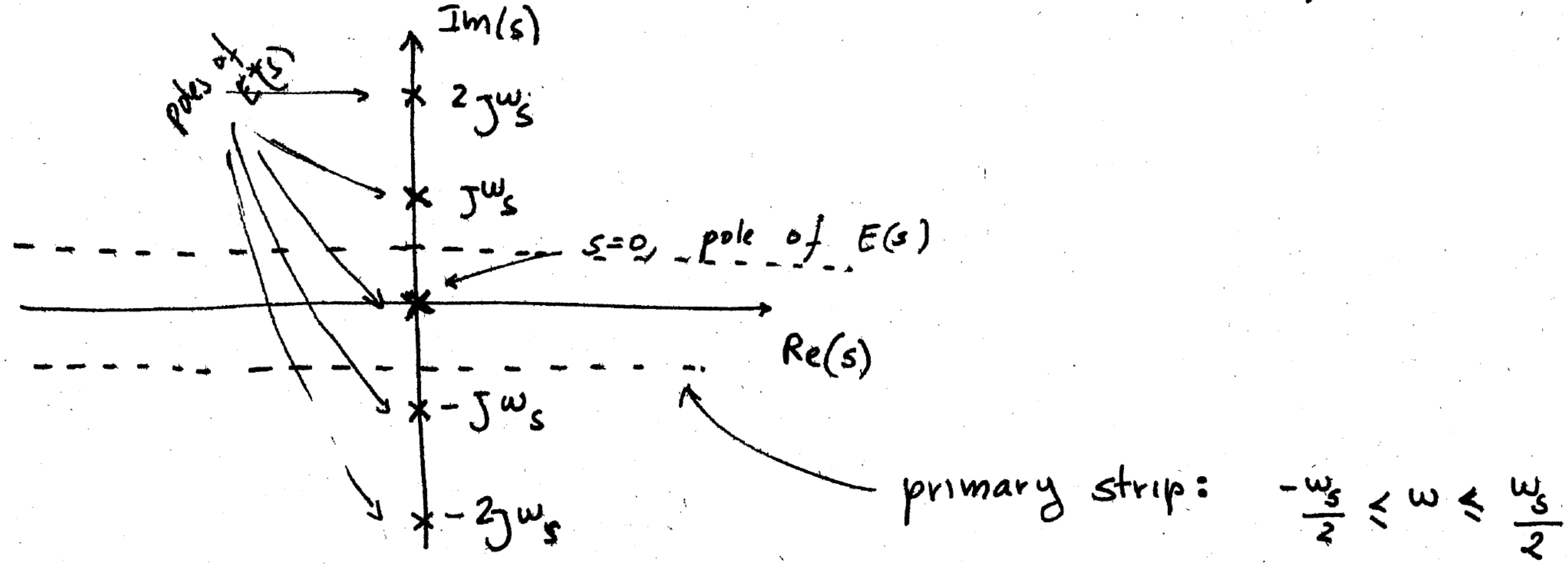

proof: $$E^*(s) = \frac{1}{T} \sum_{n=-\infty}^{n=+\infty} E(s+jn\omega_s) = \frac{1}{T}\left[E(s) + E(s+j\omega_s) + \cdots \right]$$

if $E(s)$ has a pole at $s=s_1$, the first term contributes a pole at $s=s_1$

second $\quad s=s_1-j\omega_s$

$\vdots$

n^{th} . . . $\quad s=s_1-jn\omega_s$

<u>Example</u> 2: Recall that we have shown that:

$$F(s) = \frac{1}{s} \cdot \frac{1}{(s+1)} \Leftrightarrow F^*(s) = \frac{1}{1-e^{-sT}} - \frac{1}{1-e^{-T(s+1)}}$$

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

poles at $s=0$ poles at $s = \pm j n \frac{2\pi}{T}$ $\left(e^{-jTs}=1\right)$

$s=-1$ $s = -1 \pm j n \frac{2\pi}{T}$

EECE 5610 Digital Control Systems

Lecture 8

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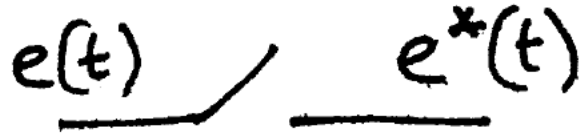
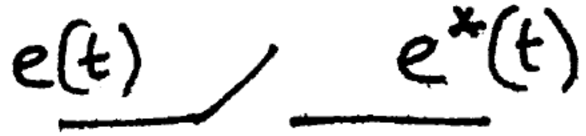
Northeastern University
College *of* Engineering

- <u>Spectrum of a Sampled Signal</u>

- <u>Spectrum of a Sampled Signal</u>



Spectrum of a Sampled Signal



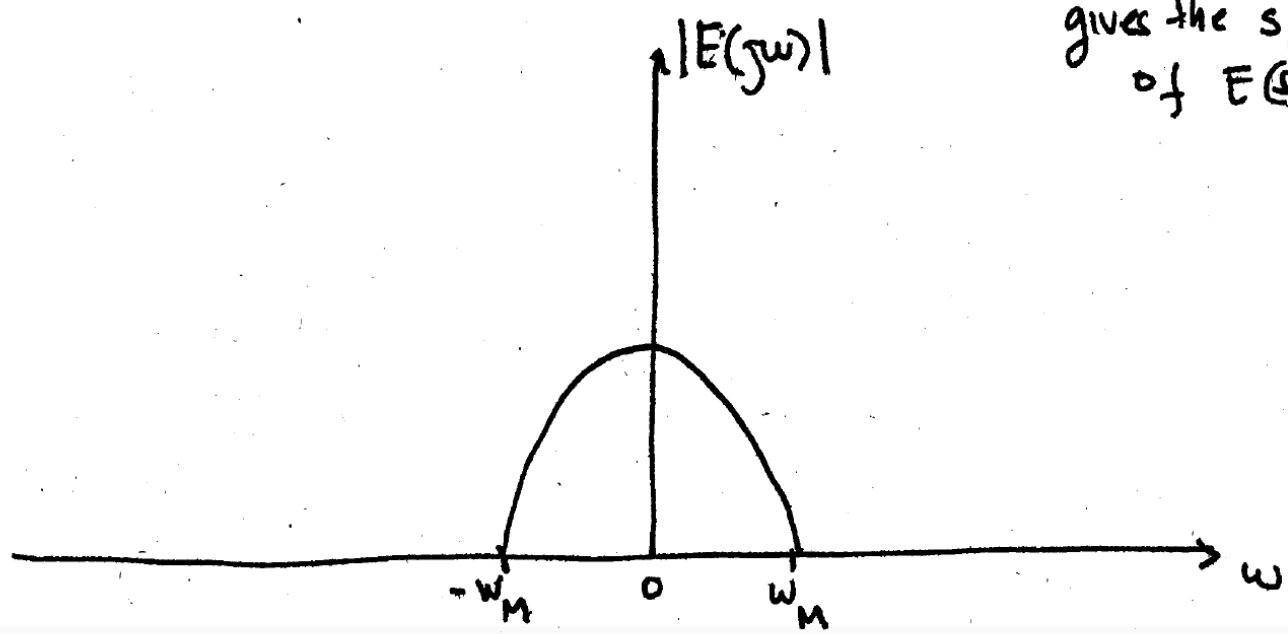
We'd like to relate the spectrum (i.e. fourier transform) of $e(t)$ and $e^*(t)$. This will become relevant when we discuss how to reconstruct (if possible) $e(t)$ from $e^*(t)$.

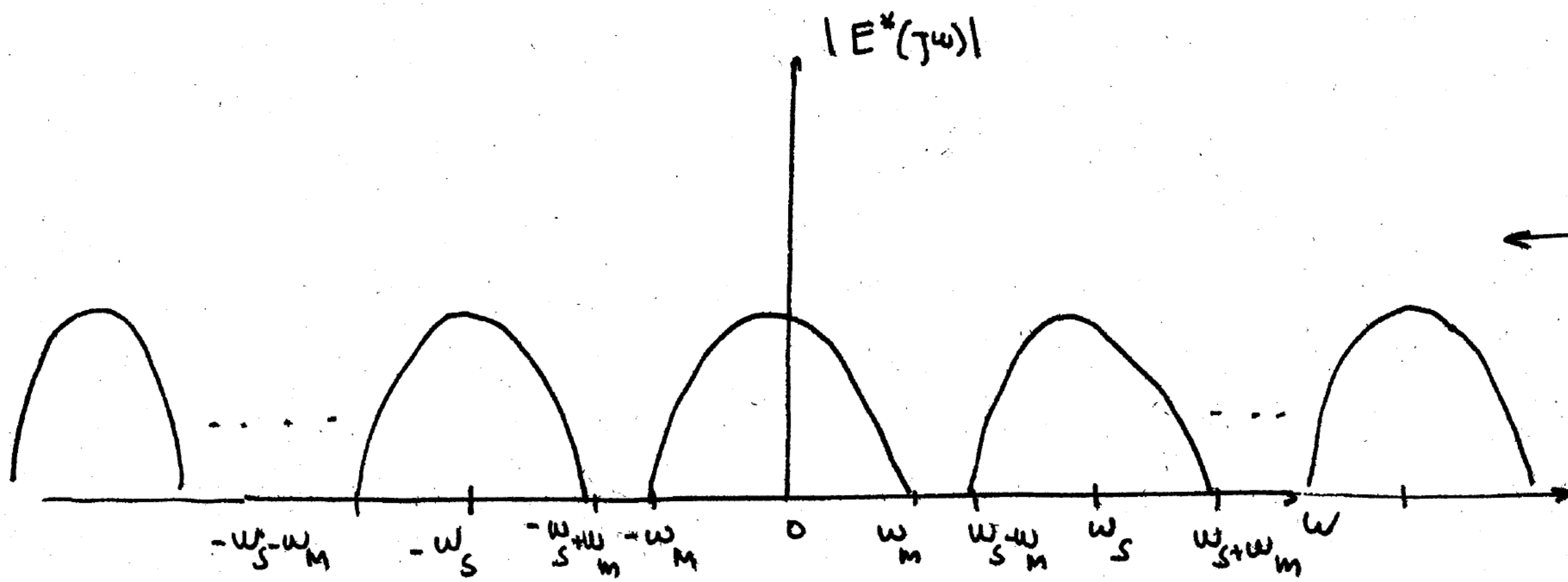
Let $$e^*(t) = \sum_{k=-\infty}^{k=+\infty} e(t)\,\delta(t-kT)$$

gives the spectrum of E, shifted by $n\omega_s$

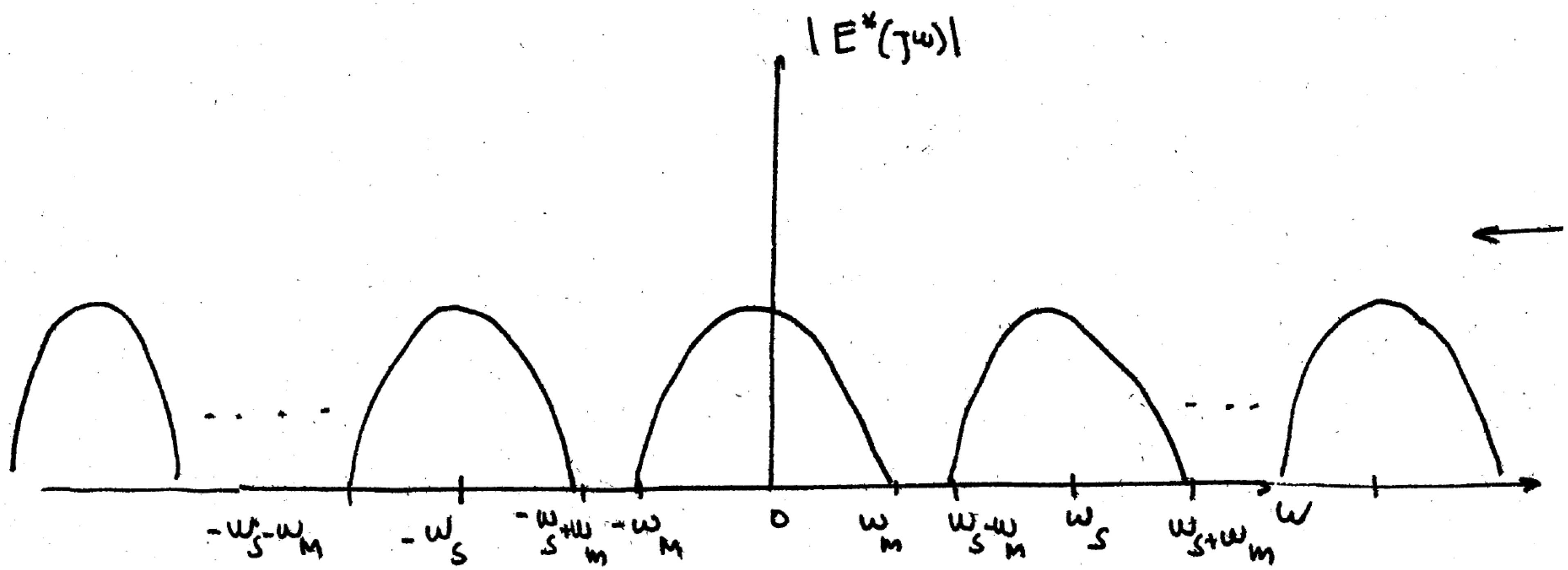
$$E^*(s) = \frac{1}{T}\sum_{-\infty}^{+\infty} E(s+jn\omega_s) = \frac{1}{T}\left[E(s) + E(s+j\omega_s) + \cdots E(s+jn\omega_s) + \cdots\right]$$

gives the spectrum of $E(s)$





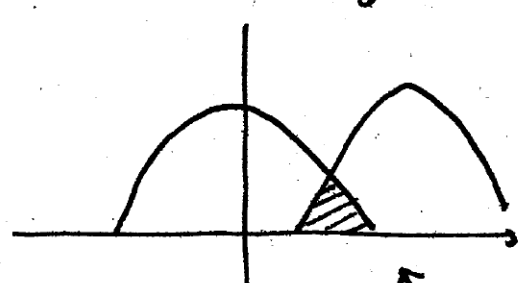
original spectrum replicated at ω_s, $2\omega_s$, . . .



original spectrum replicated at $\omega_s, 2\omega_s, \ldots$

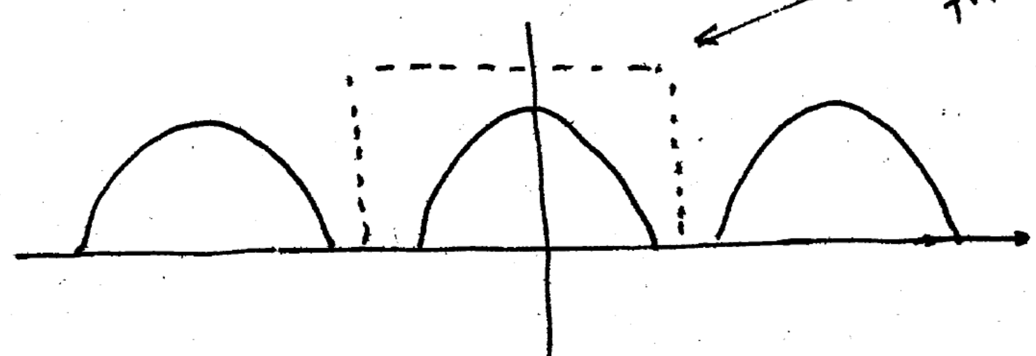
From these plots it follows that we can recover $E(s)$ from $E^*(s)$ <u>only</u> if the highest frequency present in $e(t)$ is smaller than $\frac{\omega_s}{2}$ (the "Nyquist frequency")

We can't have any overlapping:

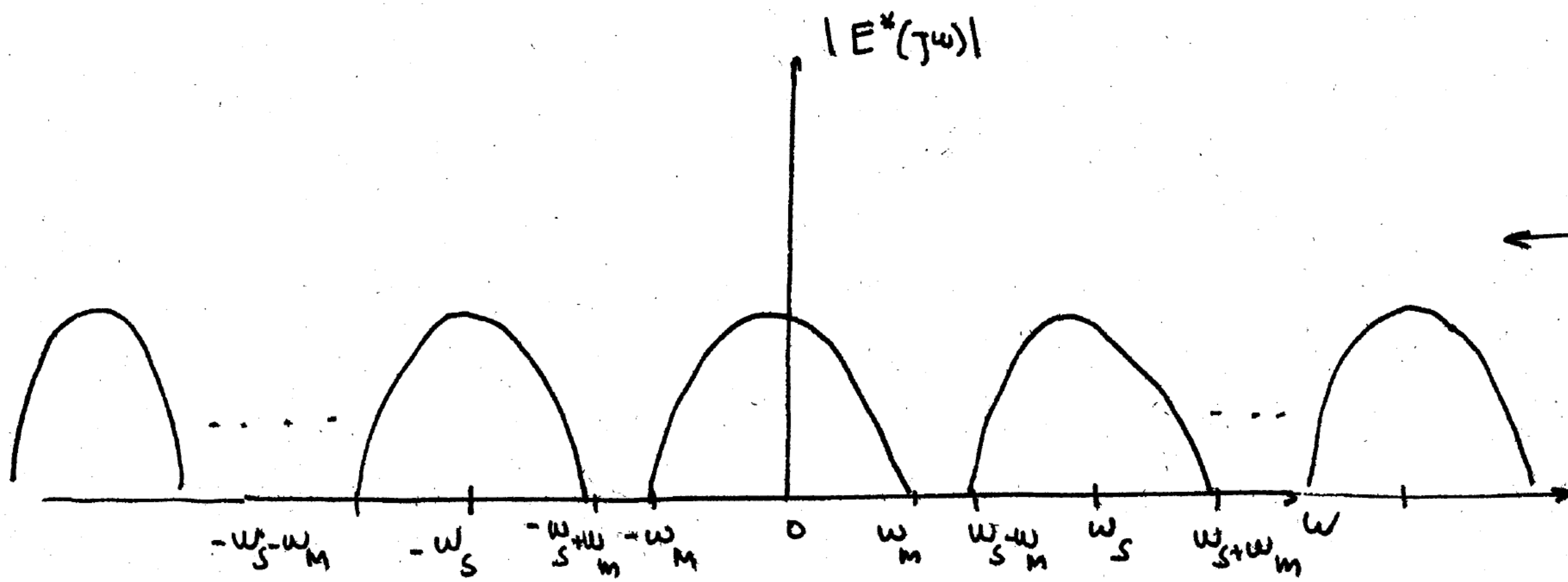


$\omega_M > \frac{\omega_s}{2}$ can't recover

can put an ideal filter here



$\omega_m < \frac{\omega_s}{2}$ can recover



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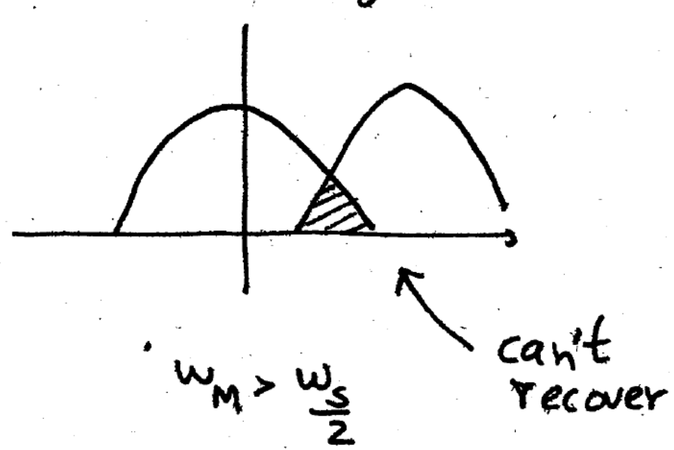

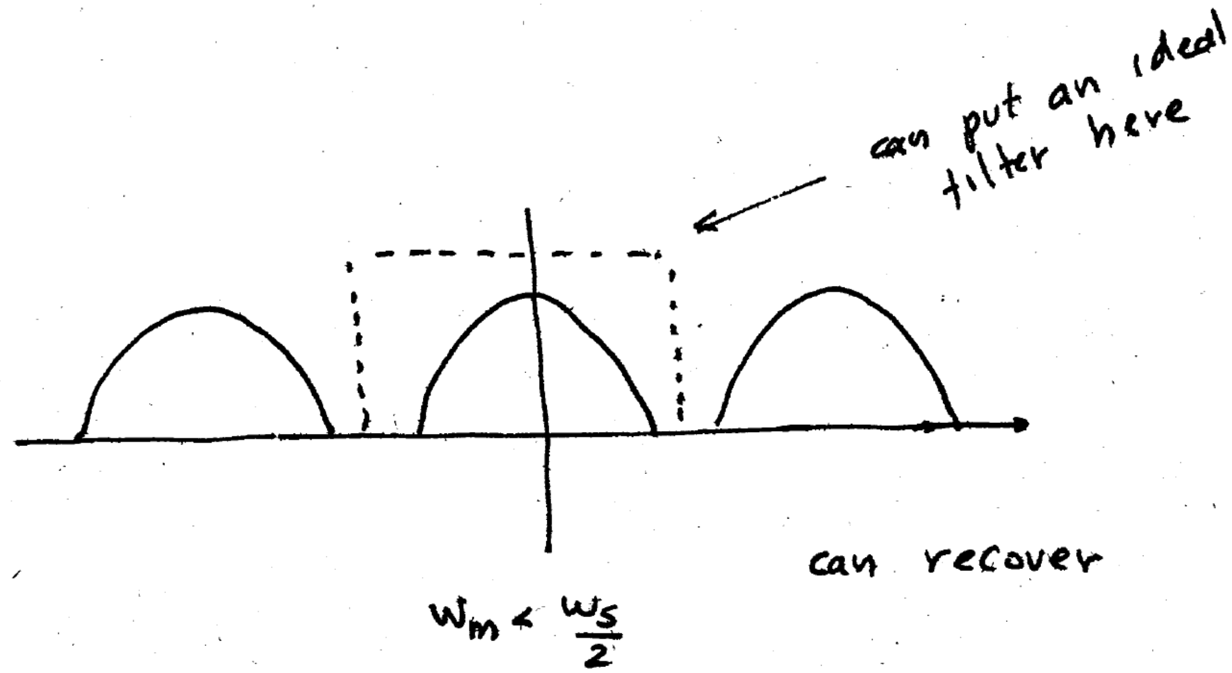

This is the celebrated <u>Shannon's Sampling Theorem</u>:

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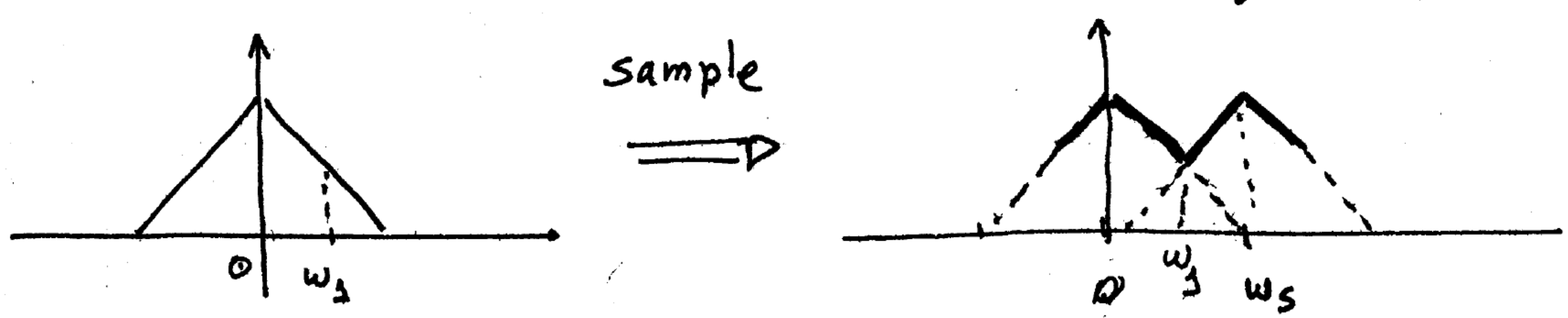
This is the celebrated <u>Shannon's Sampling Theorem</u>:

- A function $e(t)$ which contains no frequency component higher than f_0 is <u>uniquely determined</u> by the values of $e(t)$ at any set of sampling points spaced $T = \frac{1}{2f_0}$

If $e(t)$ has components above the Nyquist frequency we have the following situation:



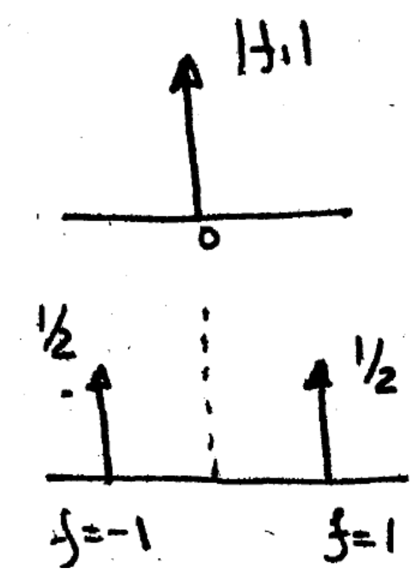
In the sampled signal the contributions from the frequencies ω_1 and $\omega_2 = \omega_1 - \omega_s$ <u>both</u> show up at ω_1. This phenomenon is called <u>aliasing</u>

Implications: 2 sinusoids of different frequencies appear at the same place when sampled $\Rightarrow$ can't tell them appart

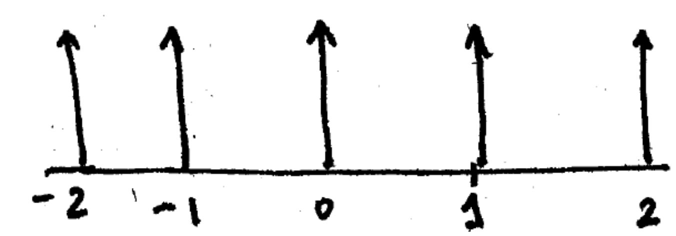
Example:

$f_1(t) = 1$

$f_2(t) = \cos 2\pi t$

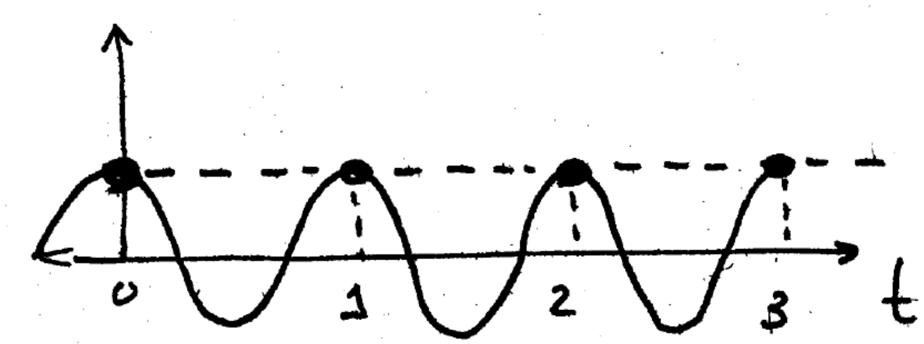


If sampled at $f_s = 1$ Hz both yield:

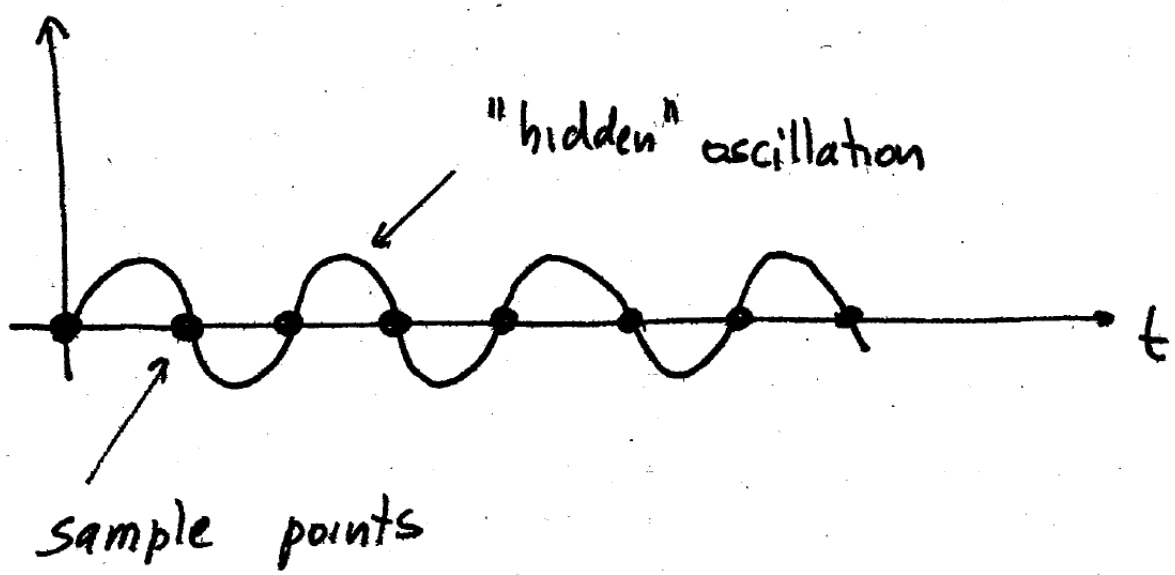


(the same spectrum!)

With 20/20 hindsight this is not surprising:

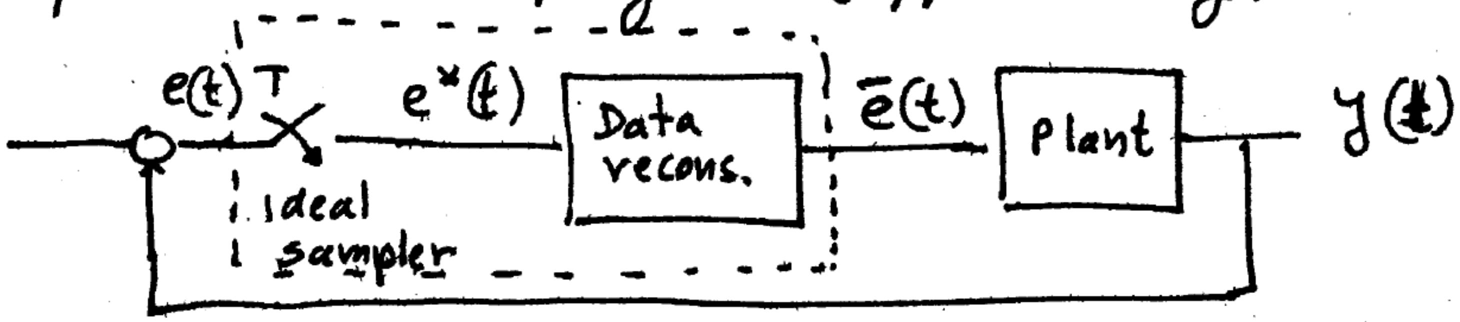


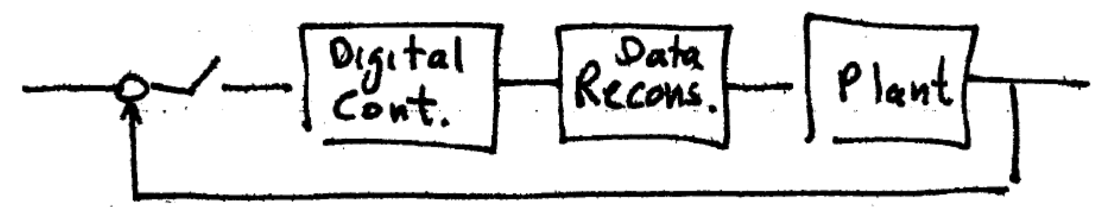
Related phenomenon: **Hidden oscillations**: We can have a signal that **does not** show up at all when sampled



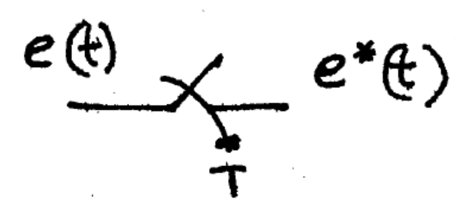
• Data Reconstruction:

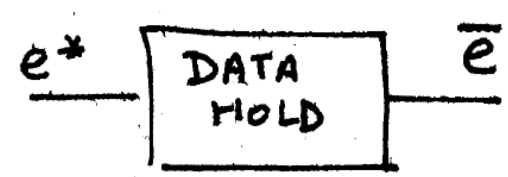
Dual operation to sampling: (approximately) reconstruct a signal from its samples



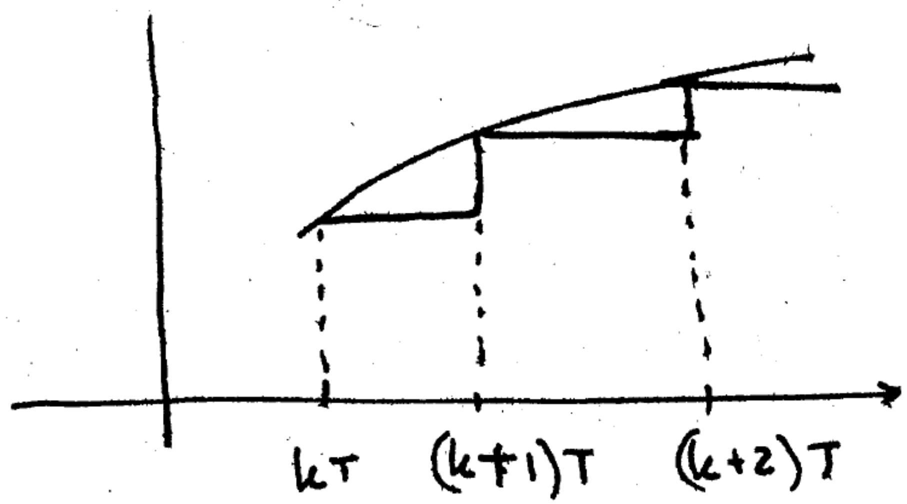
(We want to analyze this as a first step to: 

)

We already analyzed the first half: 

Now we will analyze the second half: 

Simplest approximation: zero order hold



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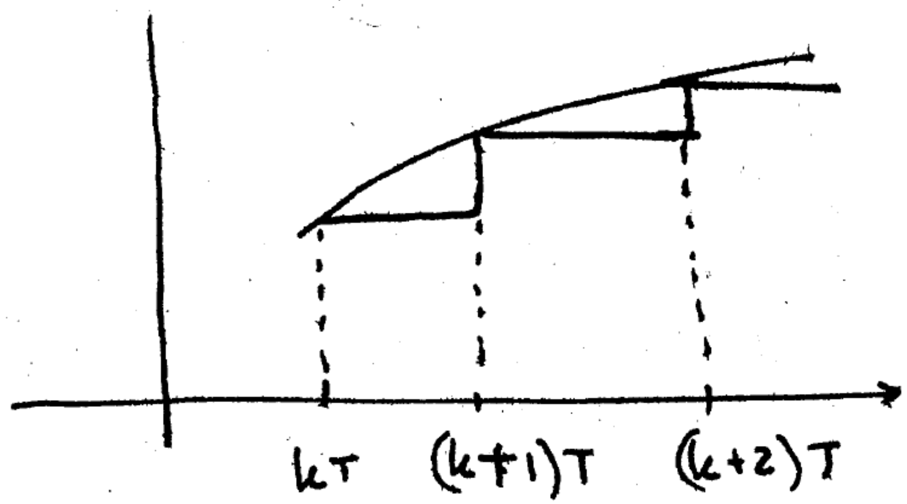
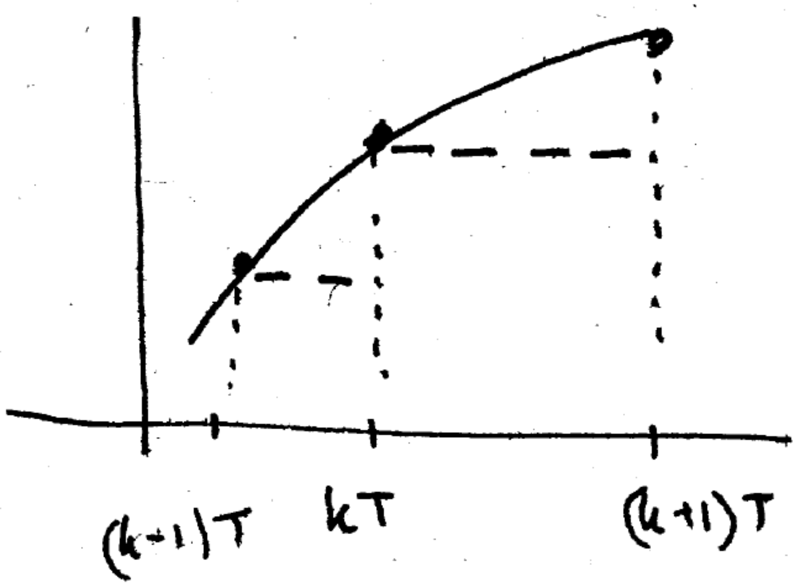


Better idea (perhaps): Try to get a tighter fit by using a polynomial approximation.

Recall from calculus that we can approximate any signal (with arbitrary accuracy) by considering enough terms of its Taylor series expansion:

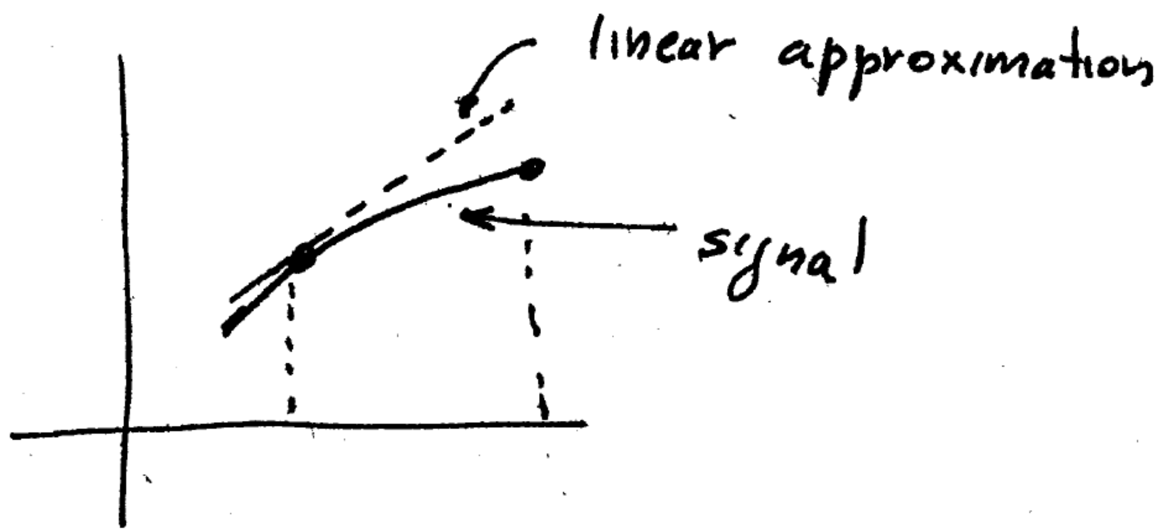
$$e(t) = e(kT) + e'(kT)(t-kT) + e''(kT)\frac{(t-kT)^2}{2} + \cdots$$

If we use only the first term we get back the zero order approximation ($e(t) = e(kT)$ $kT \le t < (k+1)T$)



If we consider the first <u>two</u> terms we get a first order hold:

$$e(t) = e(kT) + e'(kT)(t - kT)$$



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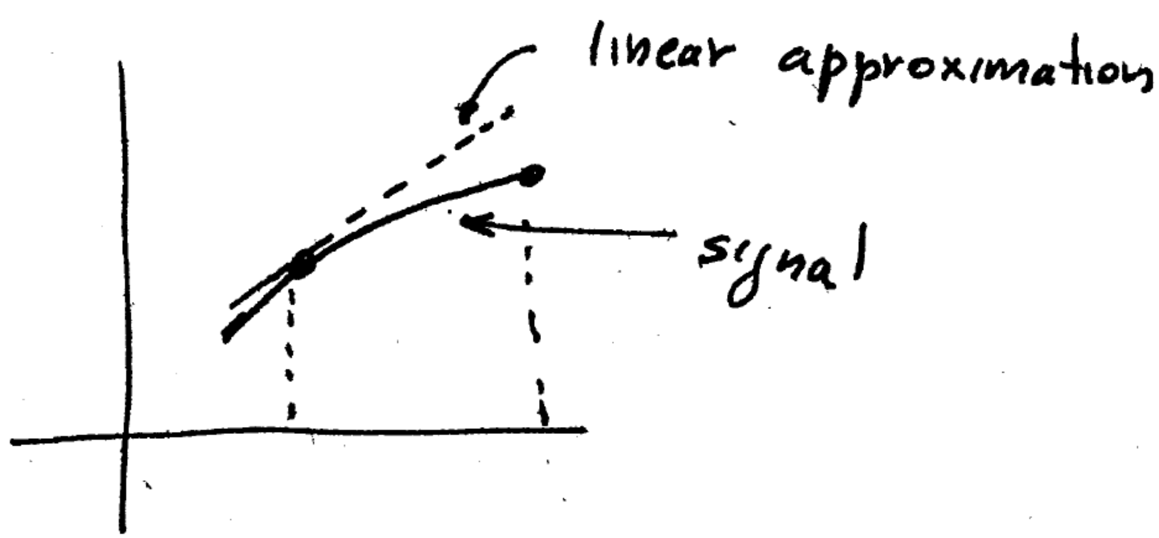
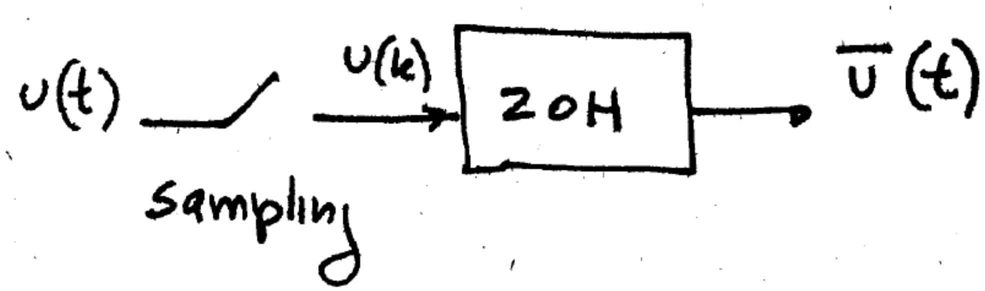


• <u>Transfer function for a zero order hold</u> (ZOH)



Recall (from EECE5580) that we can get the T.F by finding the impulse response and then taking its Laplace transform.

Since the ZOH keeps the output clamped at the input value for one period we have:

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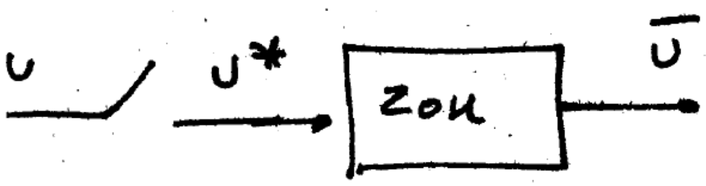
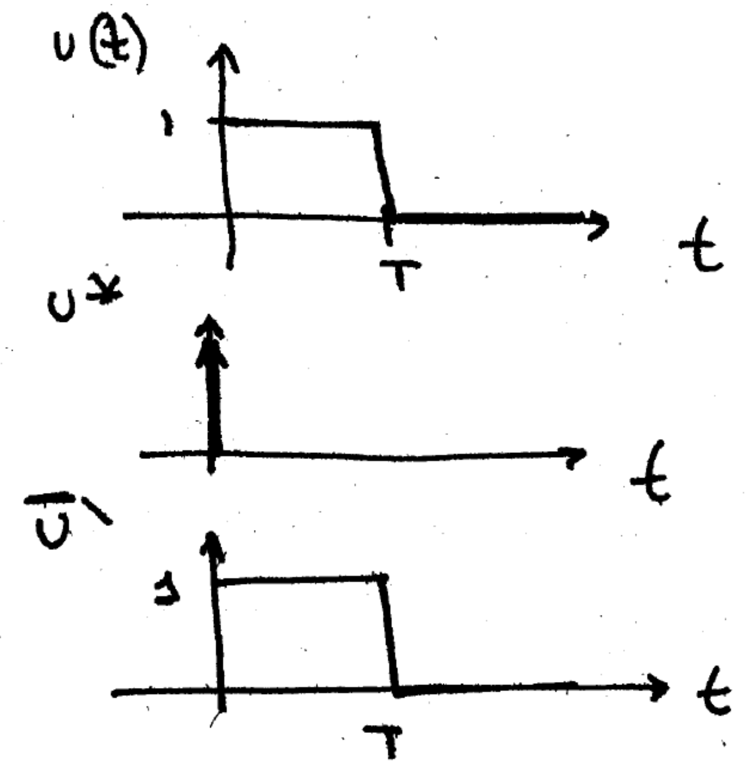




$\Leftarrow$ impulse response of ZOH

$$g(t) = u(t) - u(t-T)$$

$$\Downarrow \mathcal{L}$$

$$\boxed{G_{h_0}(s) = \frac{1-e^{-sT}}{s}}$$

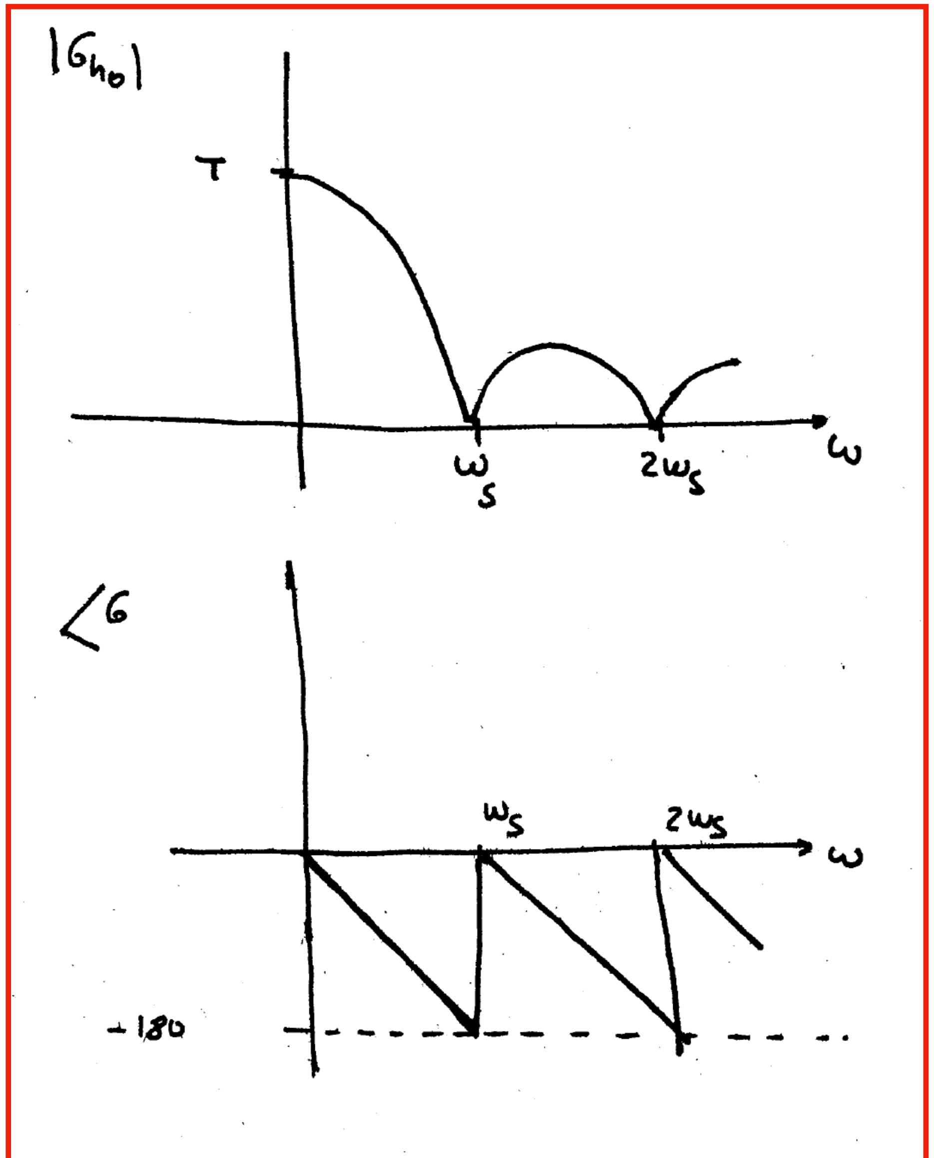
(same as before)

- **Frequency response of a ZOH:**

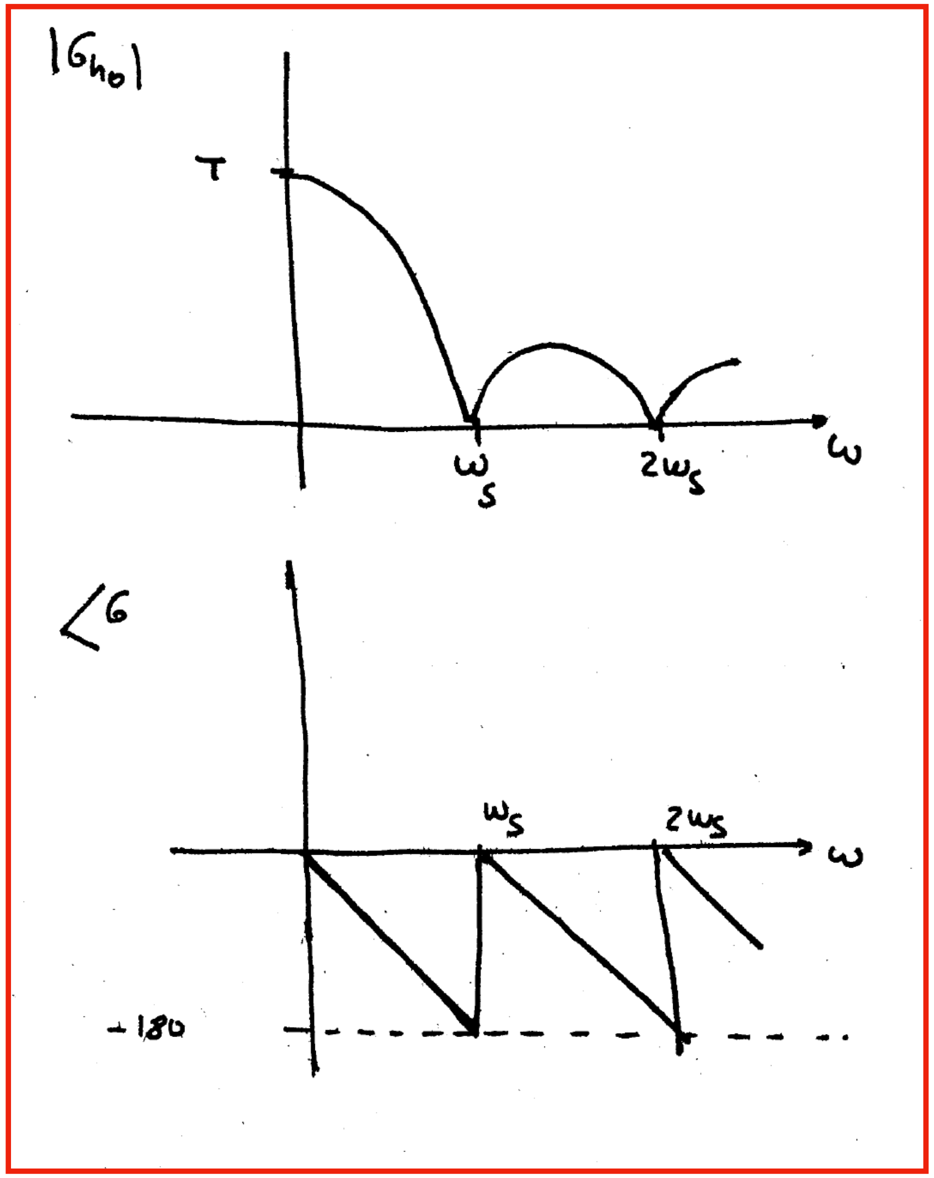
$$G_{h_o}(j\omega) = \frac{1 - e^{-j\omega T}}{j\omega} = \frac{\left(e^{j\frac{\omega T}{2}} - e^{-j\omega T/2}\right) T e^{-j\frac{\omega T}{2}}}{2 j \frac{\omega T}{2}} = T e^{-j\frac{\omega T}{2}} \left(\frac{\sin \frac{\omega T}{2}}{\frac{\omega T}{2}}\right)$$

recall that $T = \frac{2\pi}{\omega_s} \Rightarrow$

$$G_{h_o}(j\omega) = T\left[\frac{\sin \pi\left(\frac{\omega}{\omega_s}\right)}{\pi\left(\frac{\omega}{\omega_s}\right)}\right] e^{-j\left(\pi\frac{\omega}{\omega_s}\right)} \Rightarrow \begin{cases} |G_{h_o}| = T \cdot \left|\operatorname{sinc} \pi\left(\frac{\omega}{\omega_s}\right)\right| \\ \angle G_{h_o} = -\pi\frac{\omega}{\omega_s} + \theta \quad - \begin{cases} \theta = 0 & \text{if } \sin \pi\left(\frac{\omega}{\omega_s}\right) > 0 \\ \pi & \text{if } \sin \pi\left(\frac{\omega}{\omega_s}\right) < 0 \end{cases} \end{cases}$$

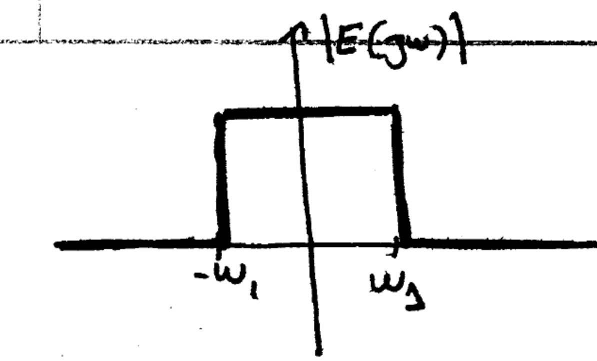


|G_ho|
T
ω
ω_s
2ω_s
∠G
ω_s
2ω_s
ω
-180

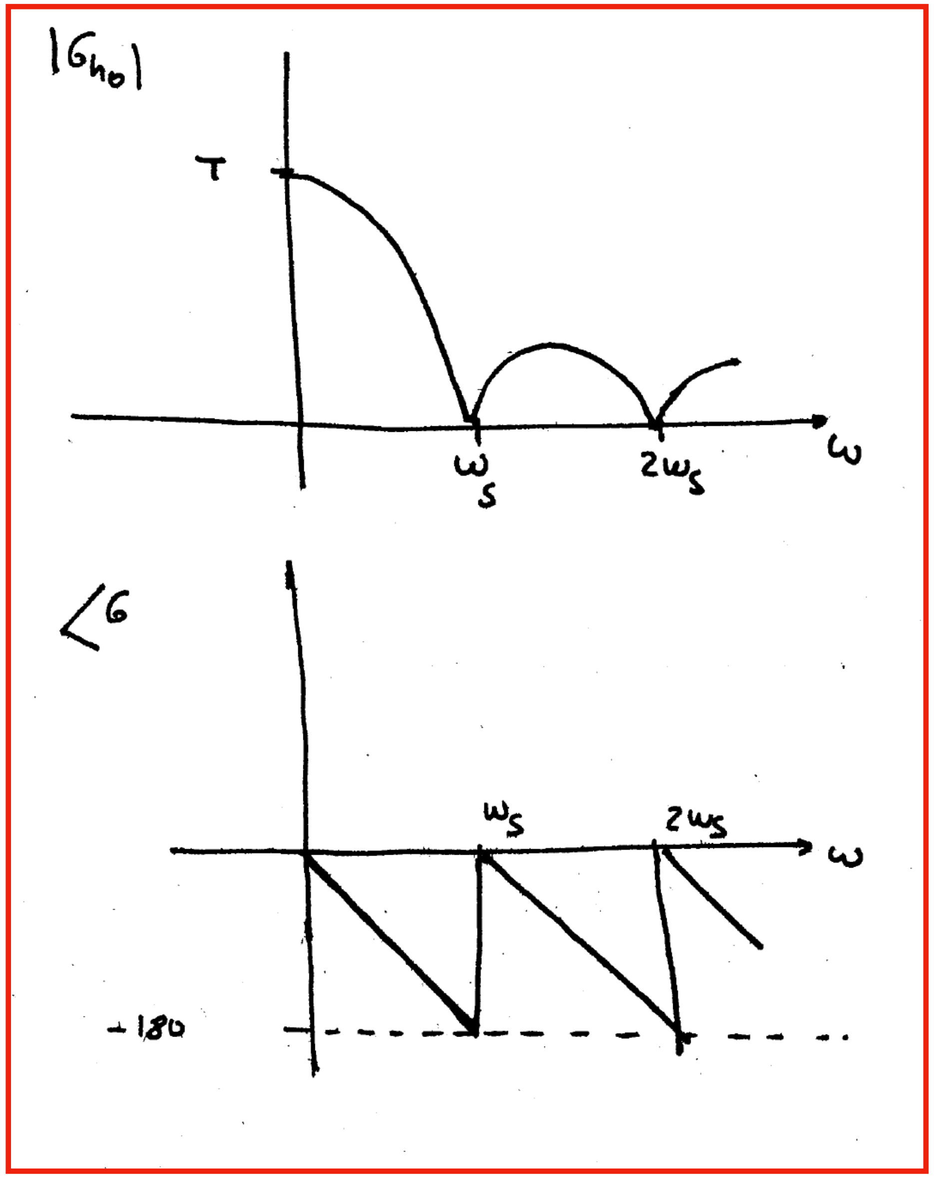


$|G_{ho}|$
T
ω_s
$2\omega_s$
ω
$\angle G$
ω_s
$2\omega_s$
ω
-180

Assume:

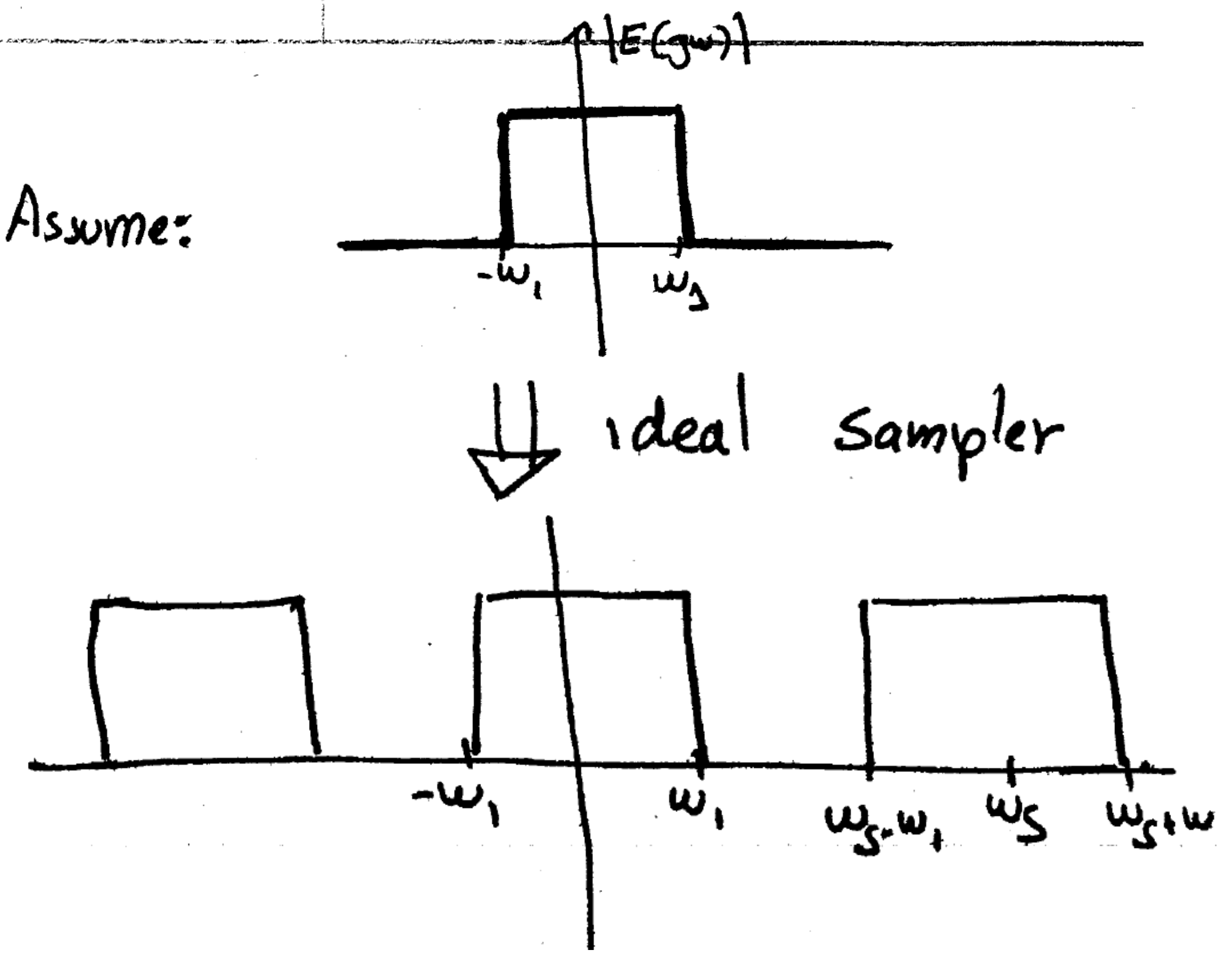


$|E(j\omega)|$
$-\omega_1$
ω_1

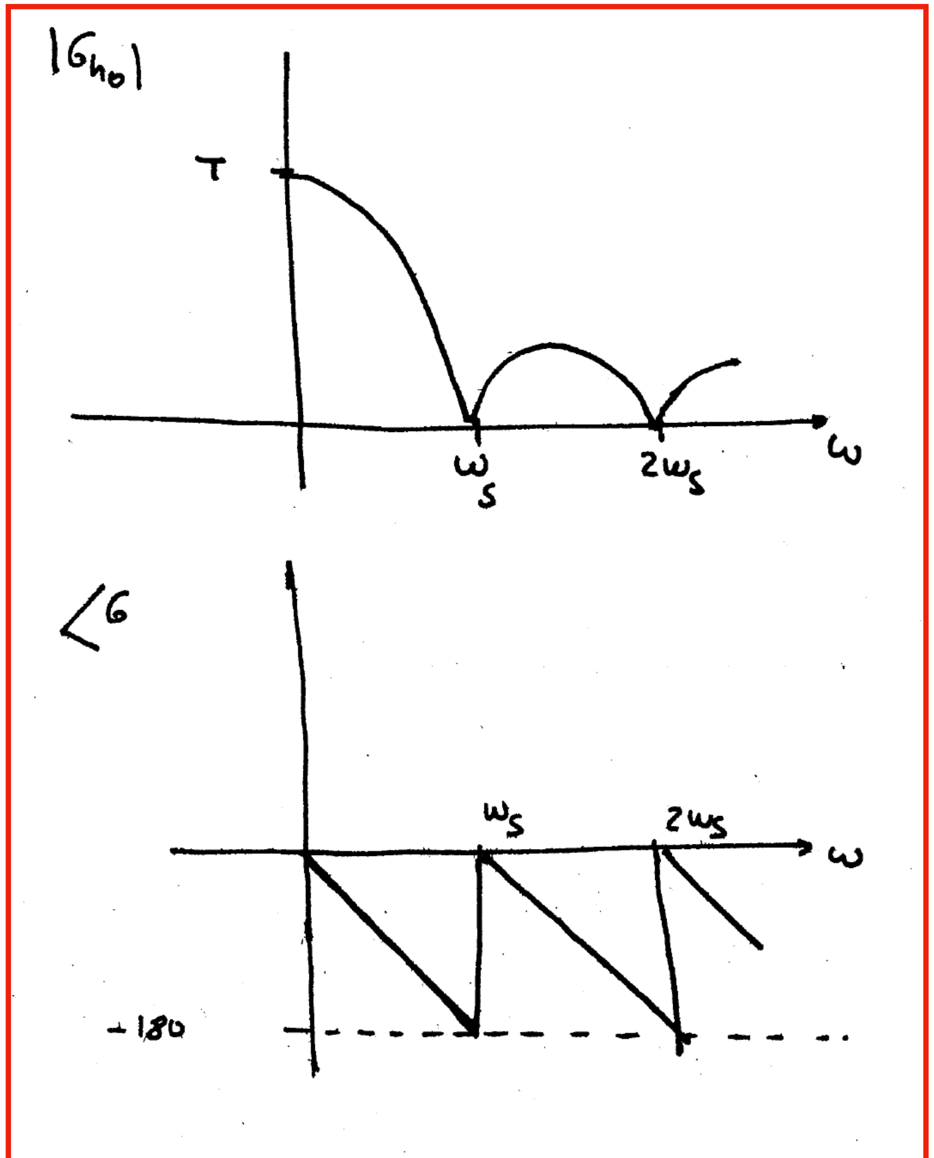


$|G_{ho}|$
T
ω_s
$2\omega_s$
ω
$\angle G$
ω_s
$2\omega_s$
ω
-180

Assume:

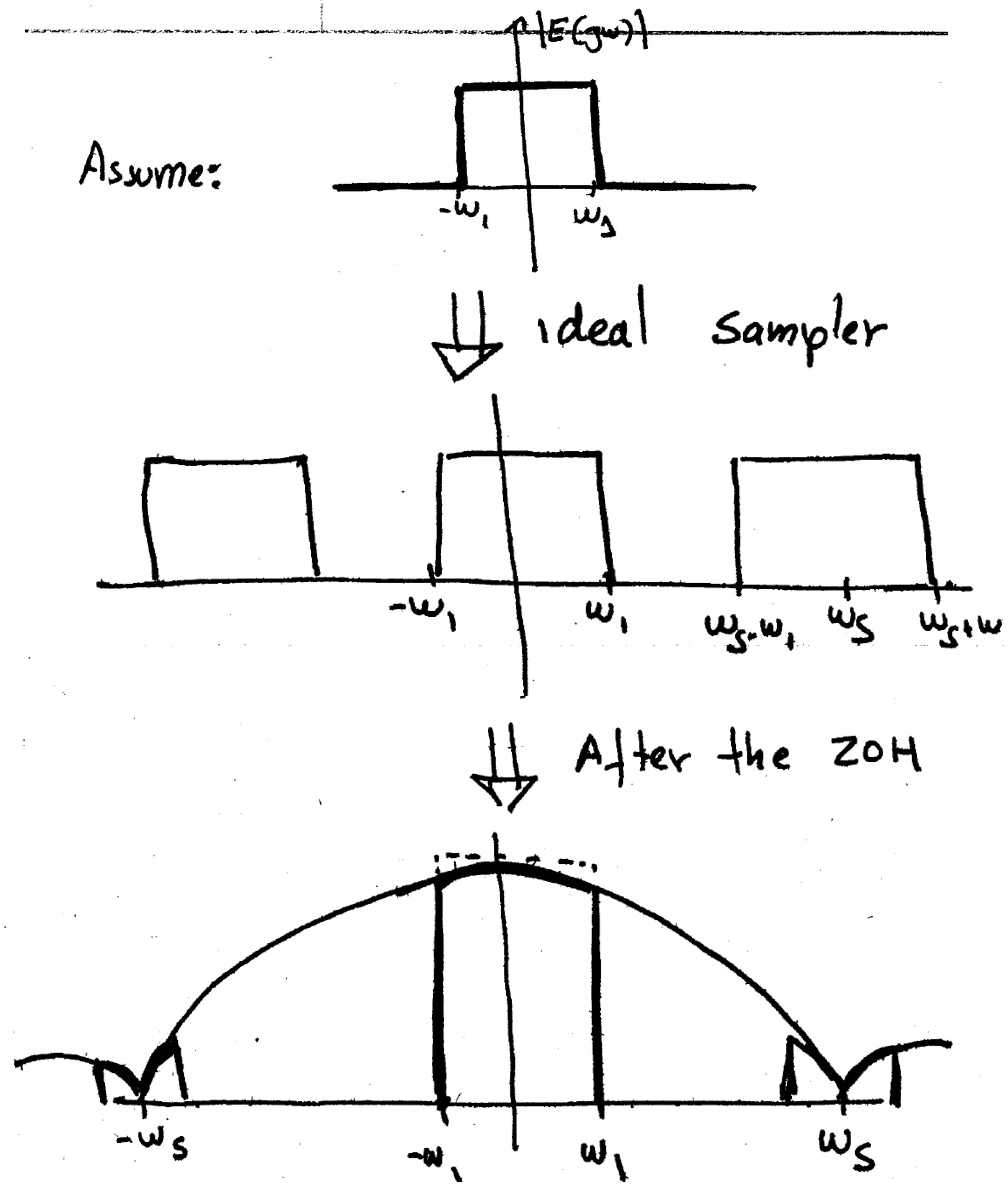


$|E(j\omega)|$
$-\omega_1$
ω_1
ideal sampler
$-\omega_1$
ω_1
$\omega_s-\omega_1$
ω_s
$\omega_s+\omega$

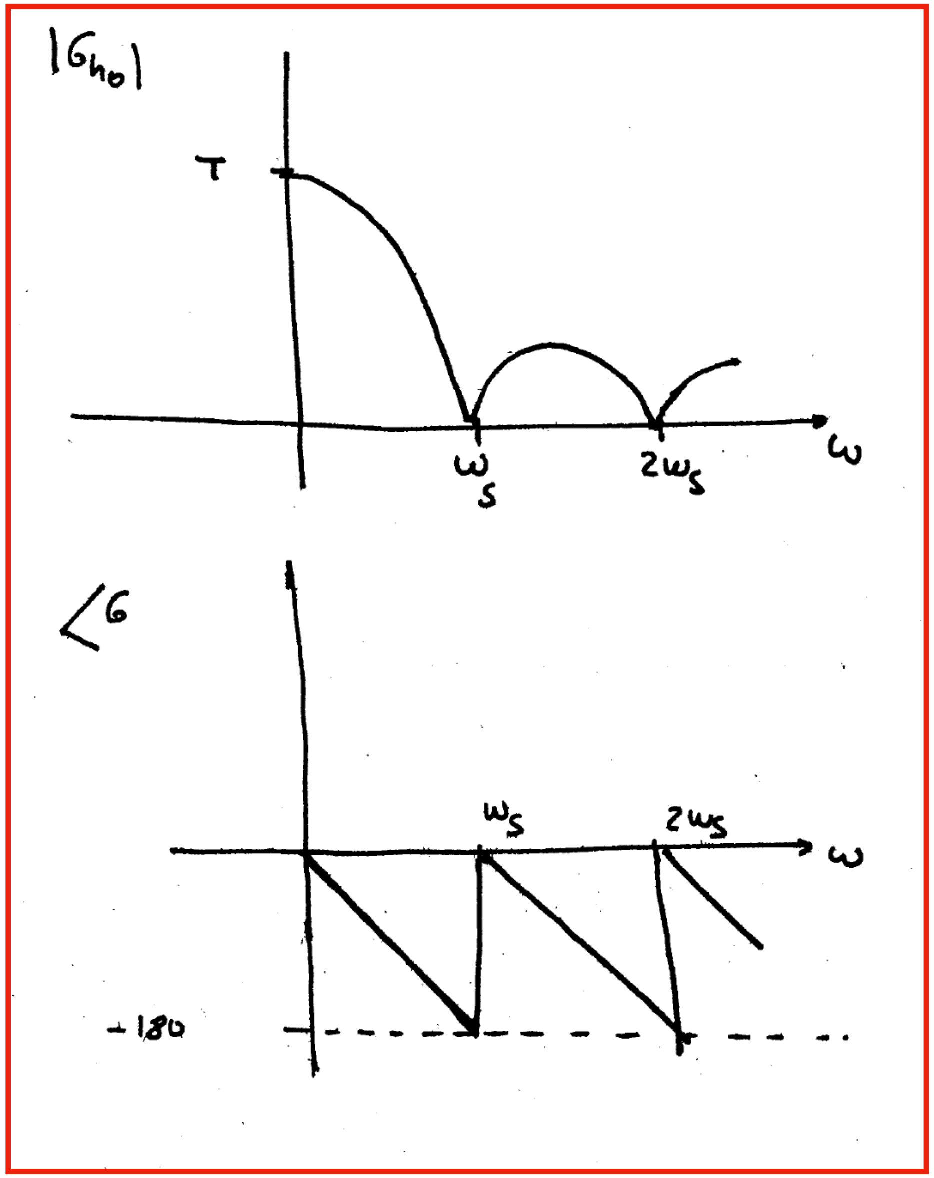


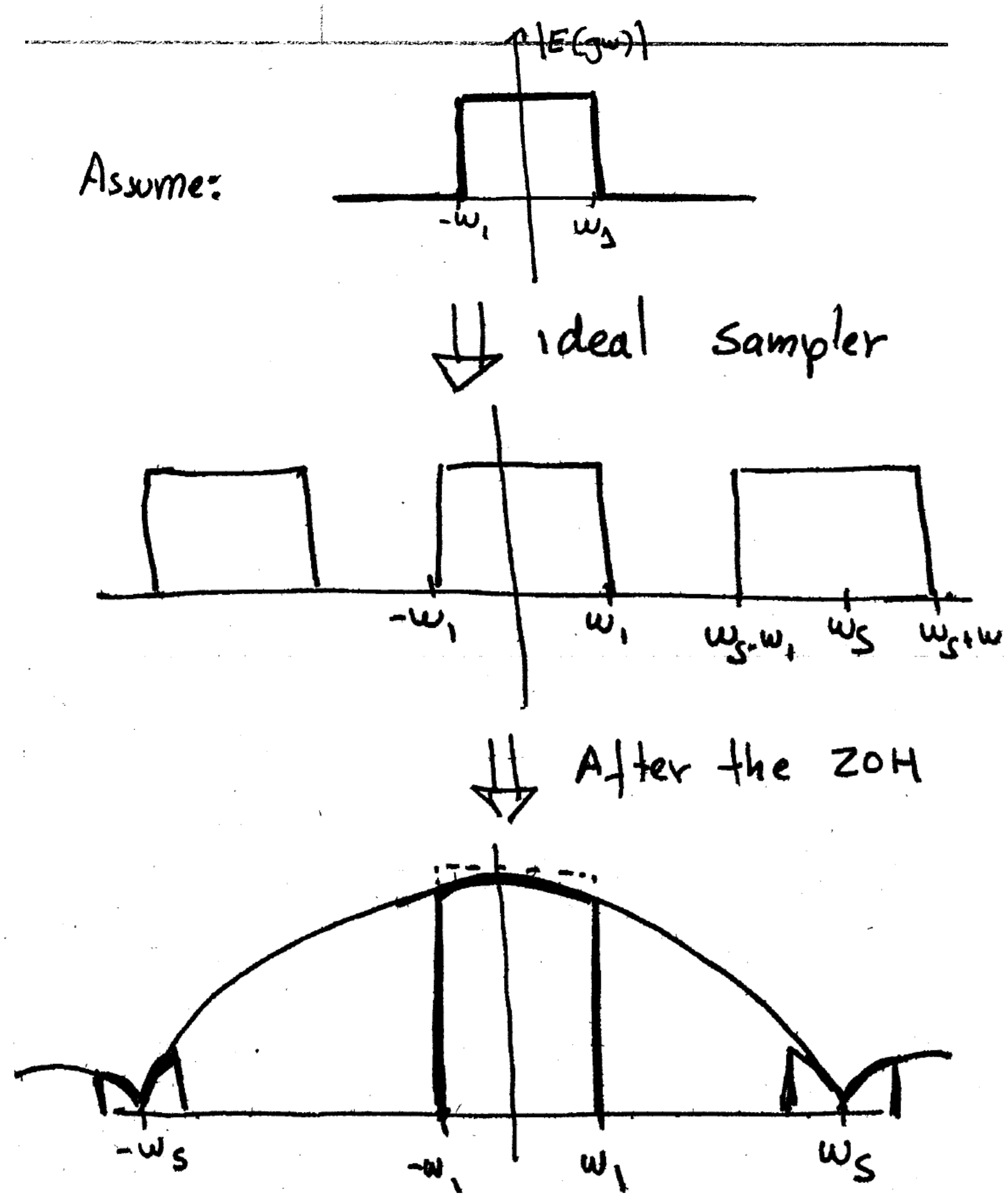
|Gho|
T
ω
ωs
2ωs
∠G
-180

Assume:



|E(jω)|
-ω1
ω1
ideal sampler
-ω1
ω1
ωs-ω1
ωs
ωs+ω
After the ZOH
-ωs
-ω1
ω1
ωs





If $\omega_1 \ll \omega_s$ the <u>high</u> frequency components of E^* fall close to zeros of G_{h_0} $\Rightarrow$ Zero order hold works like a <u>low pass</u> filter and we recover $E(s)$

- **First order hold:** Use the first 2 terms of the polynomial expansion

$$\bar{e}(t) = e(kT) + e'(kT)(t - kT) \qquad kT \leq t < (k+1)T$$

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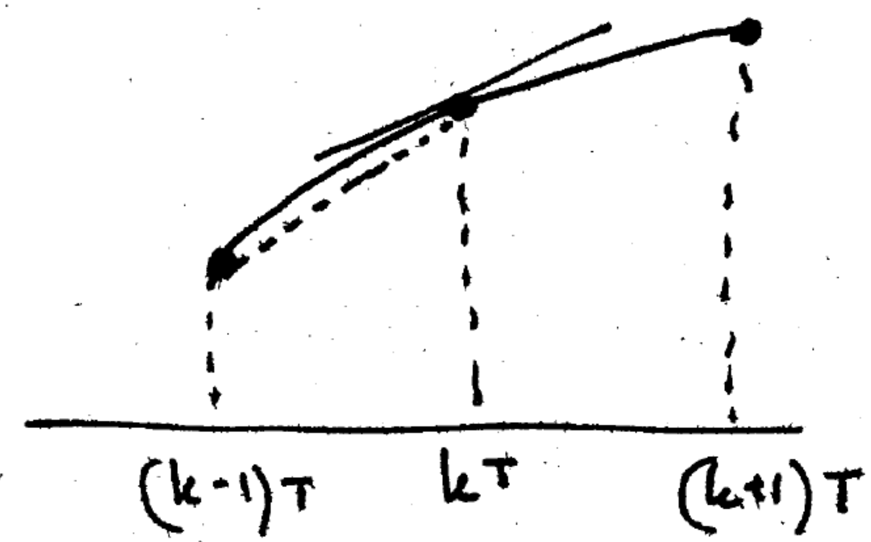
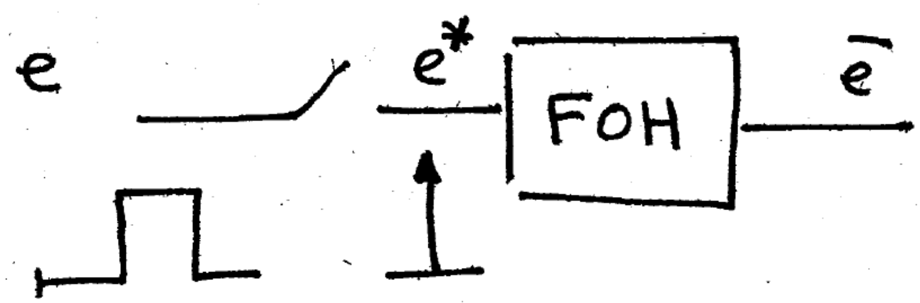


Note that you need memory to accomplish this.



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$e^*(t) = \delta(t)$

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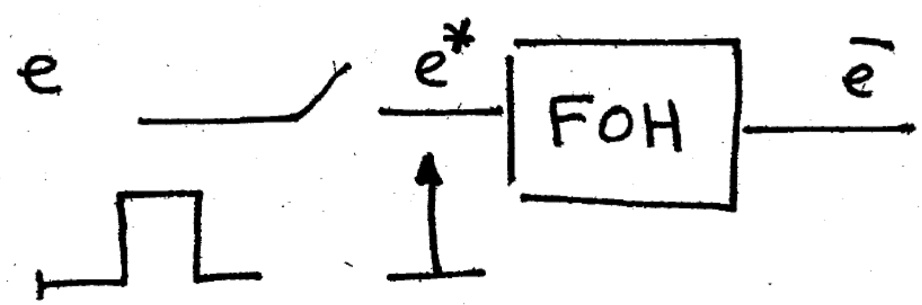

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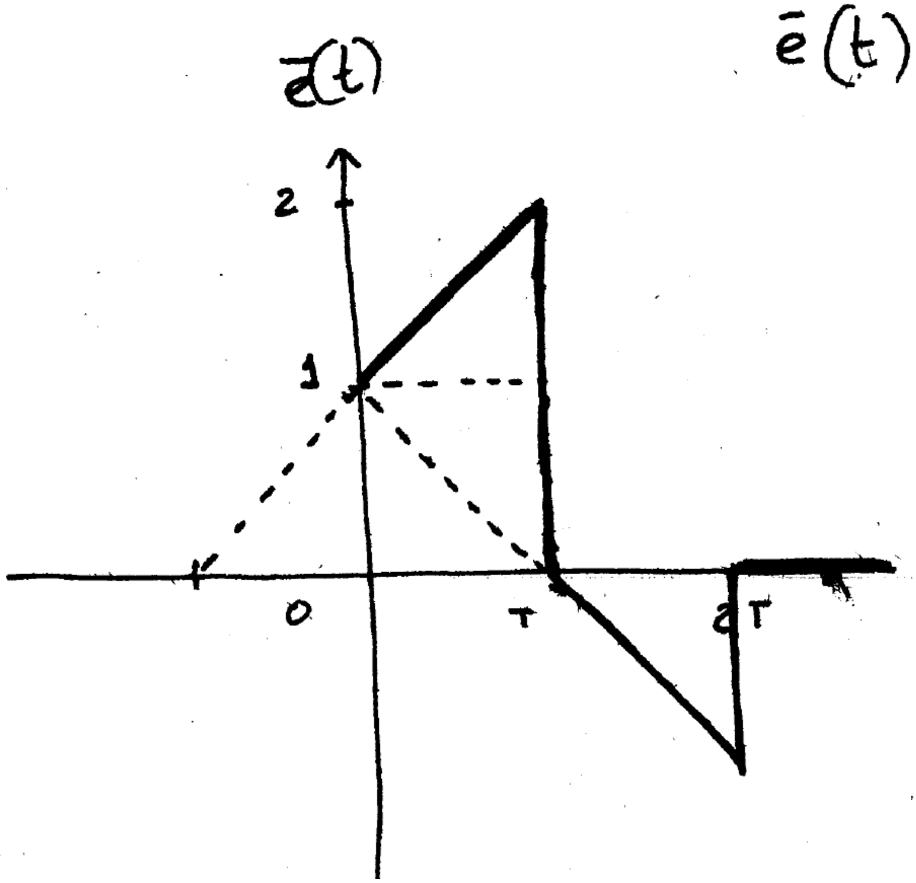
Desired output:

$$\bar{e}(t) = 1 + e'(0)\,t = 1 + \frac{t}{T} \qquad 0 \leq t < T$$

$$\bar{e}(t) = 0 + e'(T)(t-T) = 0 - \left(\frac{t-T}{T}\right) = 1 - \frac{t}{T} \qquad T \leq t < 2T$$



Frequency response:

$$G_{h_1}(j\omega) = \left(\frac{1+j\omega T}{T}\right)\left(\frac{1-e^{-j\omega T}}{j\omega}\right)^2$$

For low frequences ($\omega \ll \omega_s$) FOH has less phase lag (but more amplitude distortion)

For large ω, ZOH has less phase lag

In practice we almost always use the ZOH due to the increased hardware complexity entailed by higher order holds. If the steps from the ZOH have detrimental effects on the plant the solution is to add a low-pass filter to the output of the ZOH (with time constant of the order of the sampling period)

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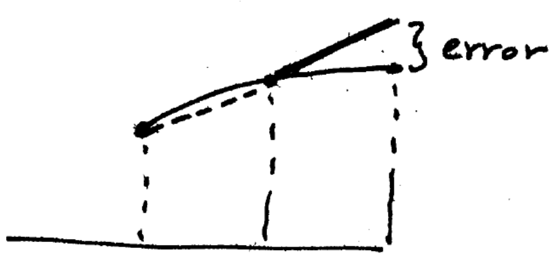

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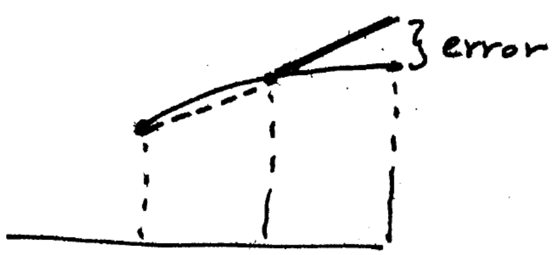
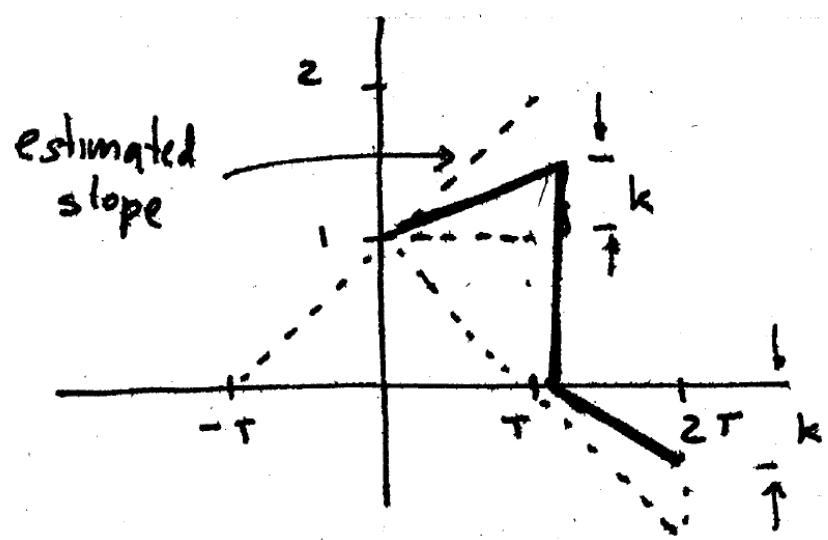


⇒ We may attempt to reduce the error by using only a fraction of the slope



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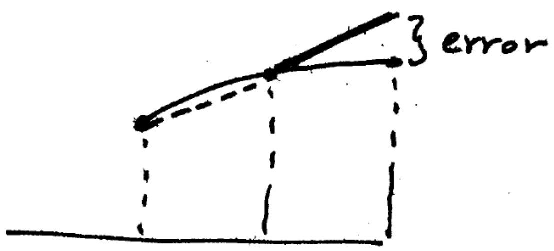

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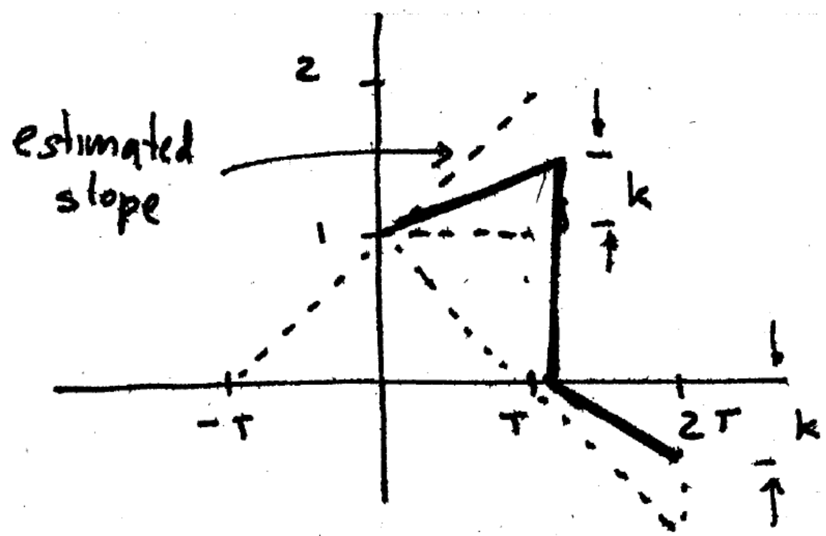

for $k=0$ you get ZOH
for $k=1$ you get FOH

$$G_{h_k}(s) = \left(1 - k e^{-Ts}\right)\left(\frac{1 - e^{-Ts}}{s}\right) + \frac{k}{Ts^2}\left(1 - e^{-Ts}\right)^2$$

Summary:

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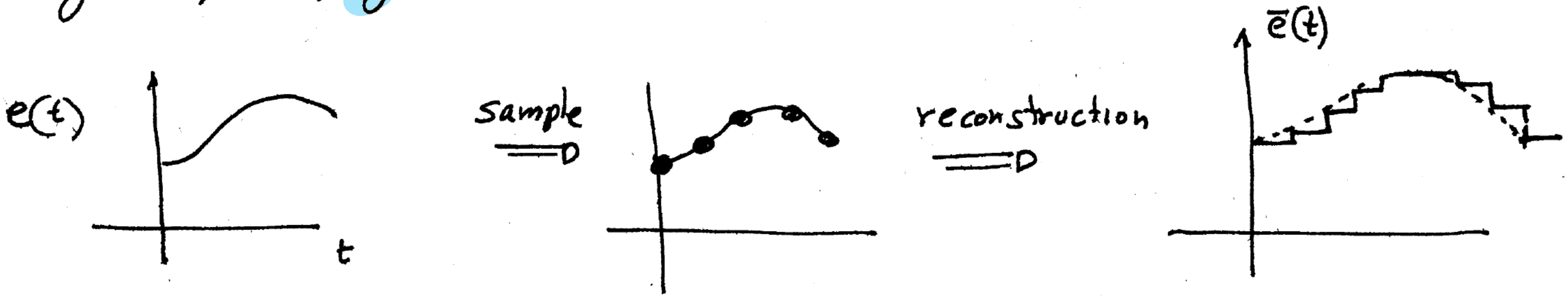
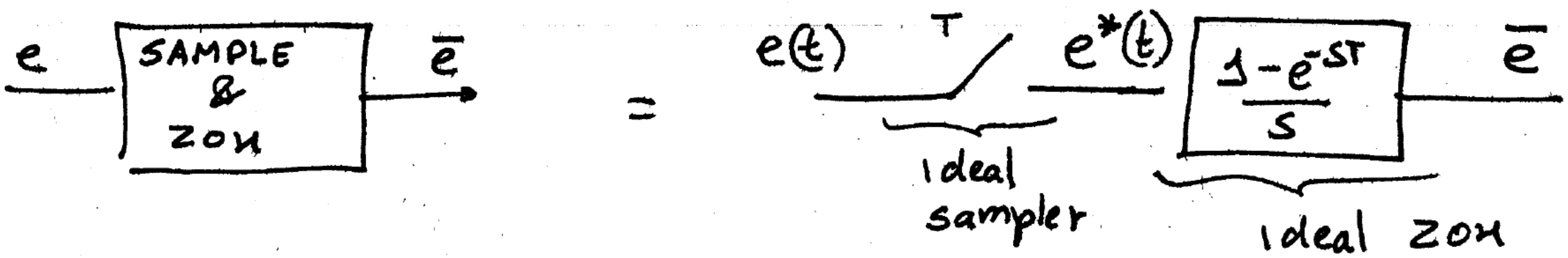
Analysis of sampling and reconstruction:





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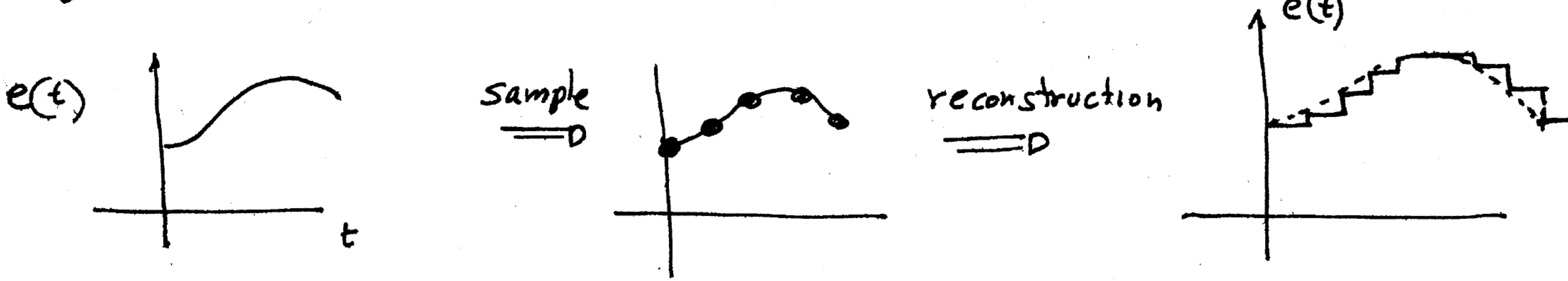
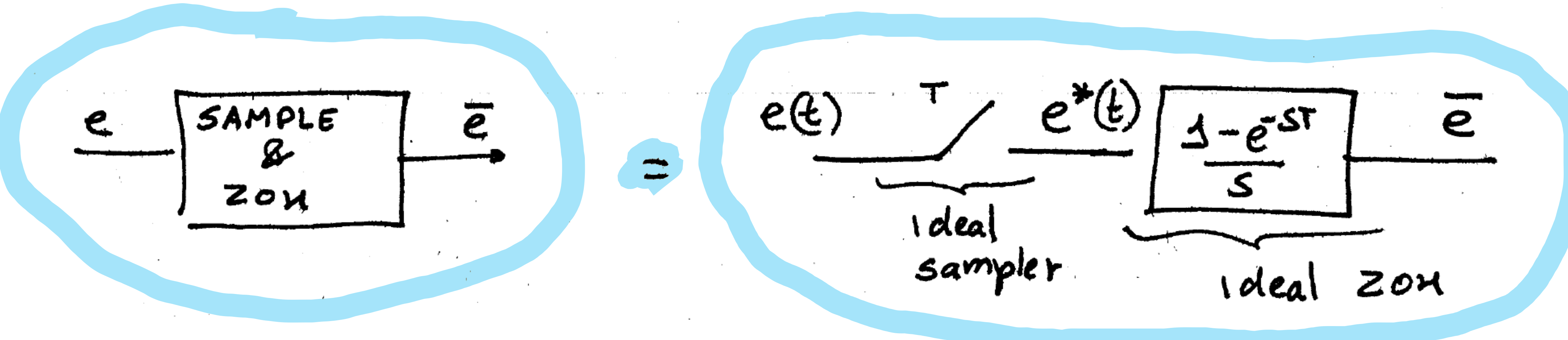
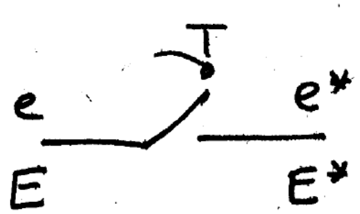




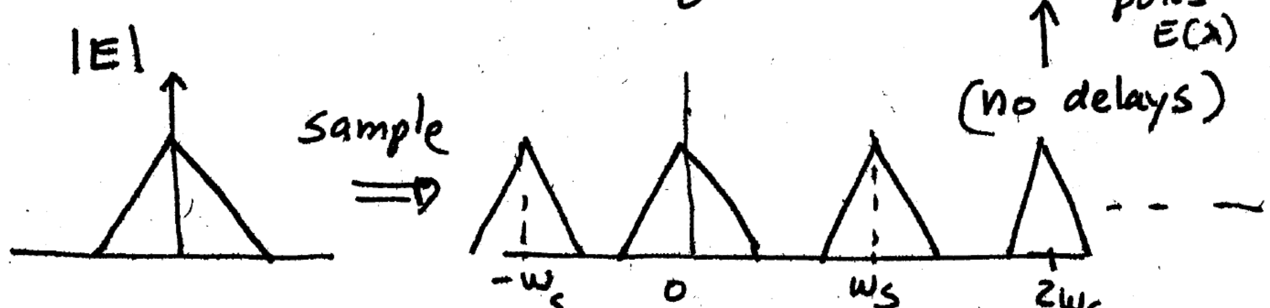
Ideal sampler



$$e^*(t) = \sum_{0}^{\infty} e(kT)\,\delta(t-kT) = e(t)\cdot\delta_T(t)$$

$$E^*(s) = \sum_{0}^{\infty} e(kT)\,e^{-kTs} \underset{\text{(no delays)}}{=} \sum_{\substack{\text{poles}\\ E(\lambda)}} \text{Res}\left\{E(\lambda)\frac{1}{(1-e^{-T(s-\lambda)})}\right\}$$

Spectrum of $E^*(s)$:



Shannon’s Theorem

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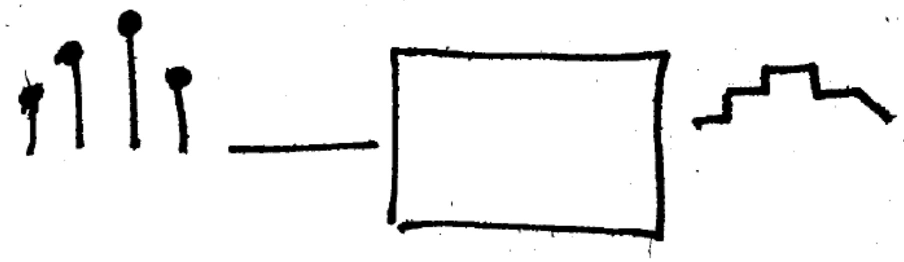
Can reconstruct E from E^* <u>iff</u> $\omega_1 \leq \frac{\omega_s}{2}$

Reconstruction: 

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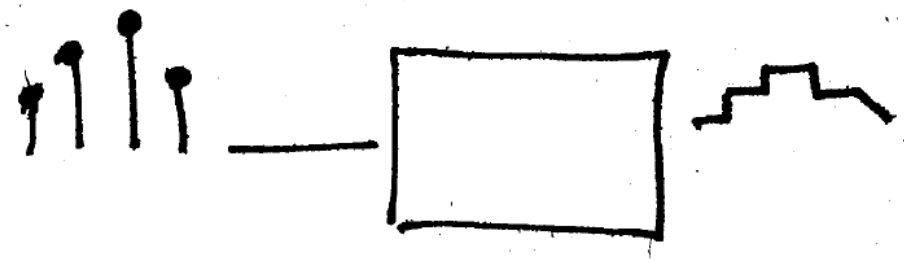
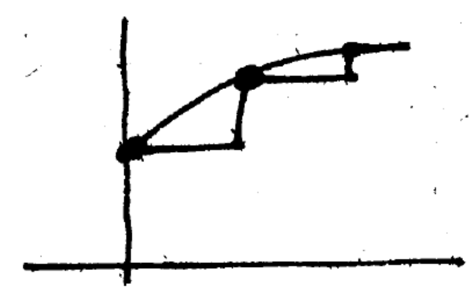
Reconstruction: 

Polynomial interpolation: $e_n(t) = e(T) + e'(\cancel{T})(t-T) + \cdots$

ZOH: $e_n(t) = e(kT) \qquad kT \le t < (k-1)T$



$$G_{ZOH} = \frac{1-e^{-sT}}{s}$$

F.O.H: $e_n(t) = e(kT) + e'(kT)(t-kT)$

$$G_{FOH} = \left(\frac{1-e^{-sT}}{s}\right)^2\left(\frac{1+sT}{T}\right)$$