EECE 5610 Digital Control Systems

Lecture 15

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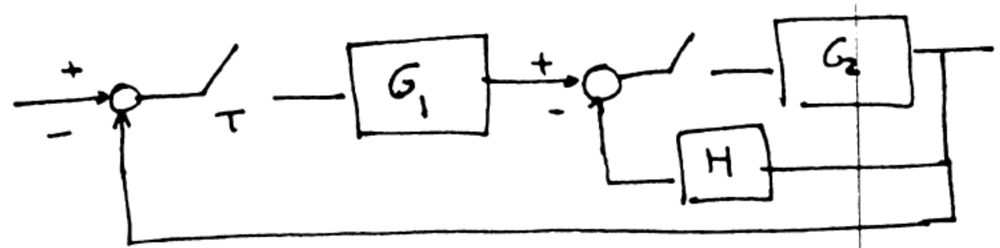
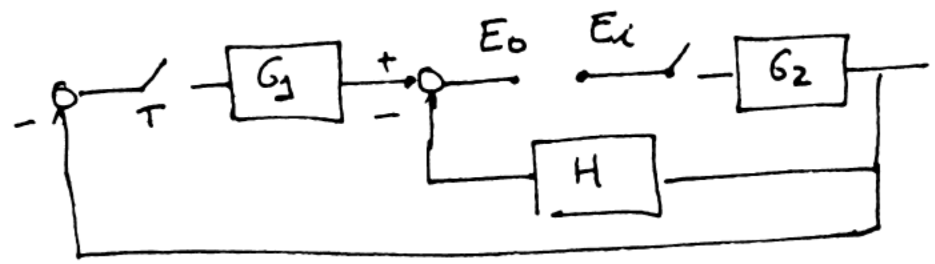
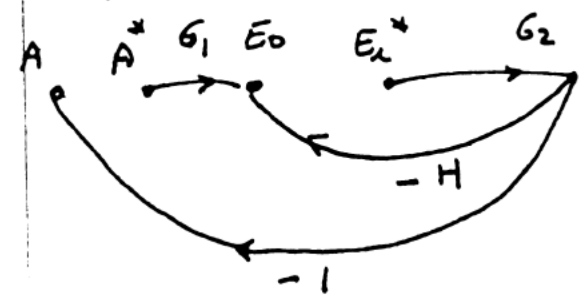
Example 3:



Here we have two samplers: Q: which one do we open
A: it doesn't matter (you get the same result)

The textbook opens the first one, so let's open the second:





$$E_o = G_1 A^* - H G_2 E_i^*$$

$$A = -G_2 E_i^* \Rightarrow A^* = -G_2^* E_i^*$$

$$\left.\begin{matrix} E_o = G_1 A^* - H G_2 E_i^* \\ A = -G_2 E_i^* \Rightarrow A^* = -G_2^* E_i^* \end{matrix}\right\} \begin{matrix} E_o = -G_1 G_2^* E_i^* - (H G_2) E_i^* \\ E_o^* = -\left(G_1^* G_2^* + (H G_2)^*\right) E_i^* \end{matrix}$$

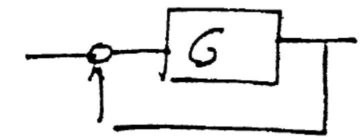
$$\Rightarrow \quad G_{op} = -\left[G_1(z) G_2(z) + \mathcal{Z}[H G_2] \right]$$

$\Rightarrow$ Char equation: $\quad 1 + G_1(z) G_2(z) + \mathcal{Z}[H G_2] = 0$

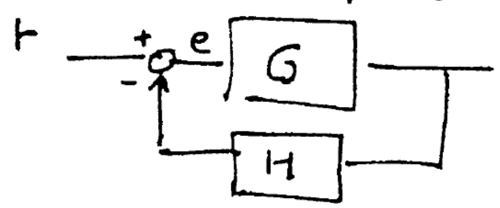
(same as in the book)

- Steady state accuracy

• Steady state accuracy

(For the time being we will assume unity feedback: 

a non-unity feedback system can always be recast into this form with a prefilter:



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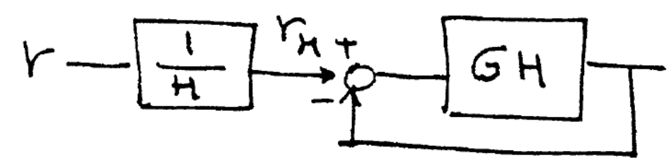

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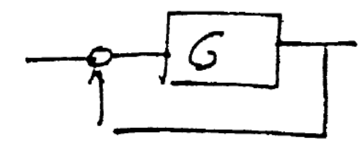
(For the time being we will assume unity feedback: 

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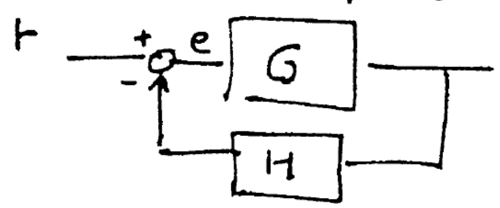

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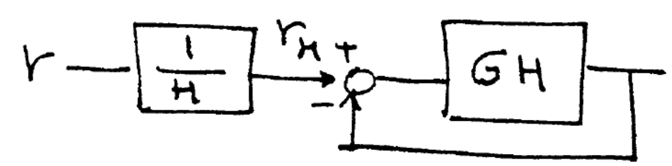
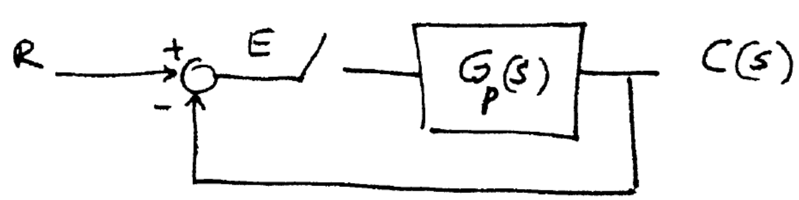


)

The goal is to follow a reference trajectory (ideally without error)



Want $\lim_{k \to \infty} e(kT) \to 0$

- Steady state accuracy

- Steady state accuracy

Assume that $G(z) = \dfrac{k\prod(z - z_i)}{(z-1)^N \prod(z - p_j)} = \dfrac{P(z)}{(z-1)^N Q_1(z)}$ where $P(1) \neq 0$, $Q_1(1) \neq 0$

- ## Steady state accuracy

Assume that
$$G(z) = \frac{k\prod(z - z_i)}{(z-1)^N \prod(z - p_j)} = \frac{P(z)}{(z-1)^N Q_1(z)} \quad \text{where} \quad \begin{array}{l} P(1) \neq 0 \\ Q_1(1) \neq 0 \end{array}$$

As in the continuous time case, N is called the <u>system type</u>
As we will see next, this determines the ability of the system to track a reference input.

We are going to consider the following types of inputs : step
ramp
parabolic

- ## Steady state accuracy

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ramp
parabolic

Reasons
1) most commonly used inputs
2) any other input can be approximated by a combination of these three (to a large extent)

$$r(t) = \underbrace{r(0)}_{\text{step}} + \underbrace{\left.\frac{dr}{dt}\right|_0 t}_{\text{ramp}} + \underbrace{\left.\frac{d^2r}{dt^2}\right|_0 \frac{t^2}{2}}_{\text{parabolic}} + \cdots$$

- Steady state accuracy

- **Steady state accuracy**

IF the system is stable (standing assumption for the next two lectures) then we can use the FVT

$$E(z) = \frac{1}{1+G(z)} R(z) \Rightarrow e_{ss} = \lim_{k \to \infty} e(kT) = \lim_{z \to 1} (z-1) E(z) =$$

$$= \lim_{z \to 1} (z-1) \frac{R(z)}{1+G(z)}$$

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$$= \lim_{z\to 1} (z-1)\frac{R(z)}{1+G(z)}$$

- Step input: $R(z) = \frac{z}{z-1}$

$$\Rightarrow e_{ss} = \lim_{z\to 1} \cancel{(z-1)} \cdot \cancel{\frac{z}{(z-1)}} \frac{1}{1+G(z)} = \frac{1}{1+\lim_{z\to 1} G(z)} = \frac{1}{1+k_p}$$

where $\boxed{k_p = \lim_{z\to 1} G(z)}$ (position error constant)

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(a) Type 0: $k_p = G(1) = \text{DC gain} \Rightarrow e_{ss} = \dfrac{1}{1+k_p}$ <u>finite</u> (non zero)

(b) Type 1 or higher: $k_p \to \infty \Rightarrow e_{ss} = 0$

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where $\boxed{K_v = \lim_{z \to 1} \dfrac{(z-1)\,G(z)}{T}}$ (velocity error constant)

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(a) Type 0: $k_v = \lim_{z \to 1} (z-1) \dfrac{G(0)}{T} = 0 \Rightarrow e_{ss_{ramp}} = \infty$

(b) Type (1) $k_v = \lim_{z \to 1} (z-1) \dfrac{G}{T} =$ finite value $\Rightarrow e_{ss_{ramp}} = \dfrac{1}{k_v}$

(c) Type (2) $k_v = \lim_{z \to 1} (z-1) \dfrac{G}{T} = \infty \Rightarrow e_{ss_{ramp}} = 0$

- <u>Parabolic input</u>: $r = \frac{1}{2}t^2 \Rightarrow R = \frac{T^2 z(z+1)}{2(z-1)^3} \Rightarrow \epsilon(z) = \lim_{z \to 1} \frac{T^2}{(z-1)^2 G(z)} = \frac{1}{k_a}$

where $k_a = \lim_{z \to 1} \frac{(z-1)^2 G(z)}{T^2}$

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(a) Type 0 or 1: $k_a = 0 \Rightarrow e_{ss} = \infty$

(b) Type 2 k_a finite $\Rightarrow e_{ss} = \frac{1}{k_a}$ (finite)

(c) Type 3 or higher $k_a = \infty \Rightarrow e_{ss} = 0$

General property: A system of type N can follow without error an input of the form: $\dfrac{A}{(z-1)^k}$ with $k \leq N$, and with finite error an input $\dfrac{1}{z^{N+1}}$

(for $k > N+1$, $e_{ss} \to \infty$)

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(for $k > N+1$, $e_{ss} \to \infty$)

Summary:

N \ R(z)	$\dfrac{z}{z-1}$	$\dfrac{Tz}{(z-1)^2}$	$\dfrac{T^2 z(z+1)}{2(z-1)^3}$
0	$\dfrac{1}{1+k_p}$	∞	∞
1	0	$\dfrac{1}{k_v}$	∞
2	0	0	$\dfrac{1}{k_a}$

where
$$k_p = \lim_{z \to 1} G(z)$$
$$K_v = \lim_{z \to 1} \frac{(z-1)\, G(z)}{T}$$
$$K_a = \lim_{z \to 1} (z-1)^2 \frac{G(z)}{T^2}$$

Trade-off: The higher the type, the more accurate the system. <u>However</u>, it is more difficult to stabilize.

Laser example :

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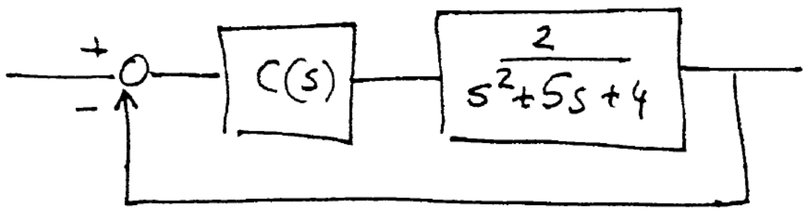

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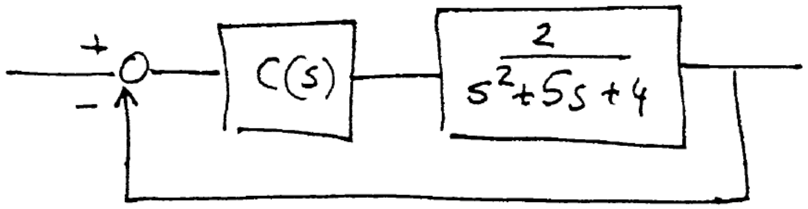

If $C(s) = k \Rightarrow k_p = \frac{k}{2}$, $e_{ss}^{step} = \frac{1}{1+k/2} = \frac{2}{2+k}$ provided that the closed loop is stable

Q: How do we assess stability?

A: Use Routh Hurwitz: Char eq: $s^2 + 5s + 4 + 2k = 0 \Rightarrow$

stable for all $k > 0$

Now let's try to make it a type 1: $C(s) = \frac{k}{s}$

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Routh Hurwitz

$$
\begin{array}{lll}
s^3 & 1 & 4 \\
s^2 & 5 & 2k \\
s^1 & 20-2k & \\
s^0 & 2k &
\end{array}
$$

$\Rightarrow$ stable if $0 < 2k$, $20 - 2k < 0$ $\Rightarrow$

$0 < k < 10$

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Routh Hurwitz

s^3	1	4
s^2	5	$2k$
s^1	$20-2k$	
s^0	$2k$	

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If we try to make it a type 2: $C(s) = \frac{k}{s^2}$

Char eq: $s^4 + 5s^3 + 4s^2 + 0 \cdot s + 2k$ $\Rightarrow$ always unstable

missing term

Routh Hurwitz Method

$$a_n \cdot s^n + a_{n-1} \cdot s^{n-1} + \ldots + a_1 \cdot s + a_0 = 0$$

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Given a polynomial in *s*

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Procedure:

Step 1: Build the Routh array.

(a) For rows 1 and 2, build h columns, where h = **Largest integer** [$(n+1)/2$],

Routh Hurwitz Method

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Step 1: Build the Routh array.

(a) For rows 1 and 2, build h columns, where h = **Largest integer** [$(n+1)/2$],

If n is odd:

$$\begin{array}{lcccc} \text{Row1:} & a_n & a_{n-2} & \ldots & a_1 \\ \text{Row2:} & a_{n-1} & a_{n-3} & \ldots & a_0 \end{array}$$

If n is even:

$$\begin{array}{lcccc} \text{Row1:} & a_n & a_{n-2} & \ldots & a_0 \\ \text{Row2:} & a_{n-1} & a_{n-3} & \ldots & 0 \end{array}$$

(b) For row 3 to row $n+1$,

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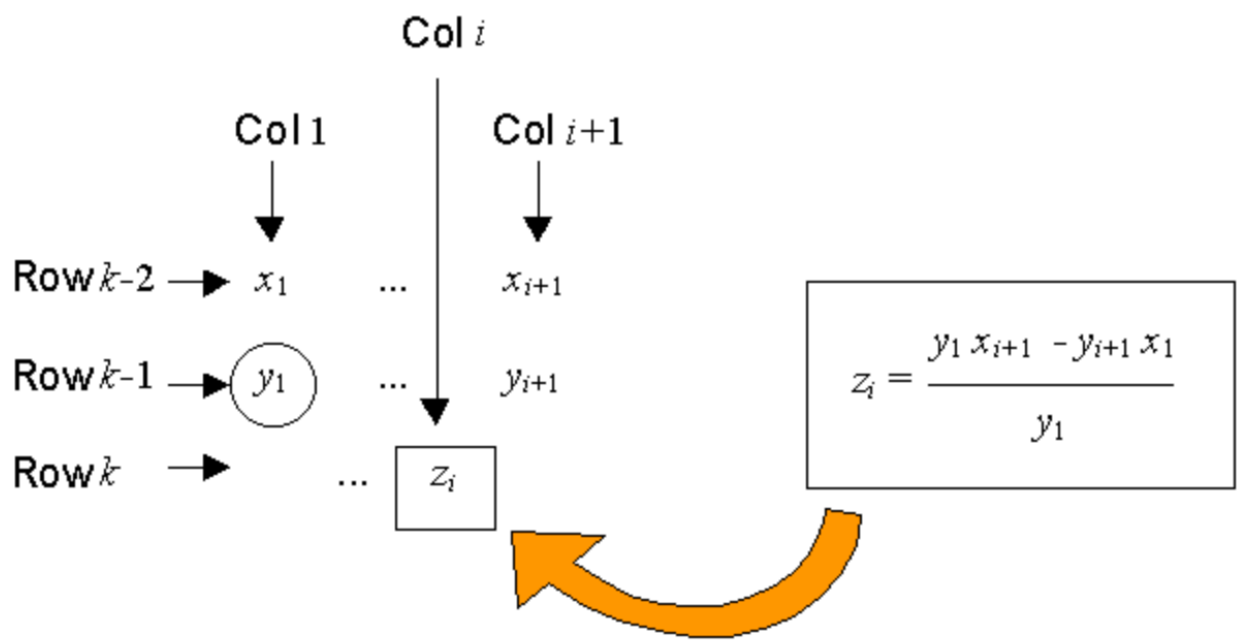

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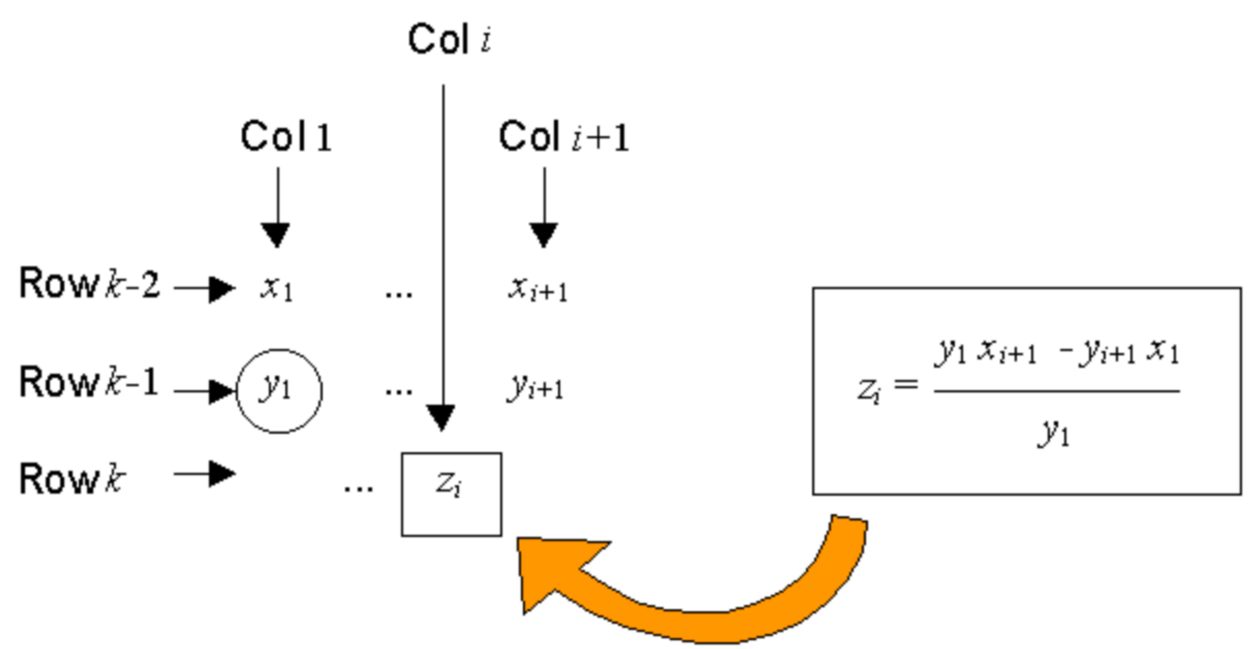

Step 2: <u>Extract the first column of the array and count the number of sign changes.</u>

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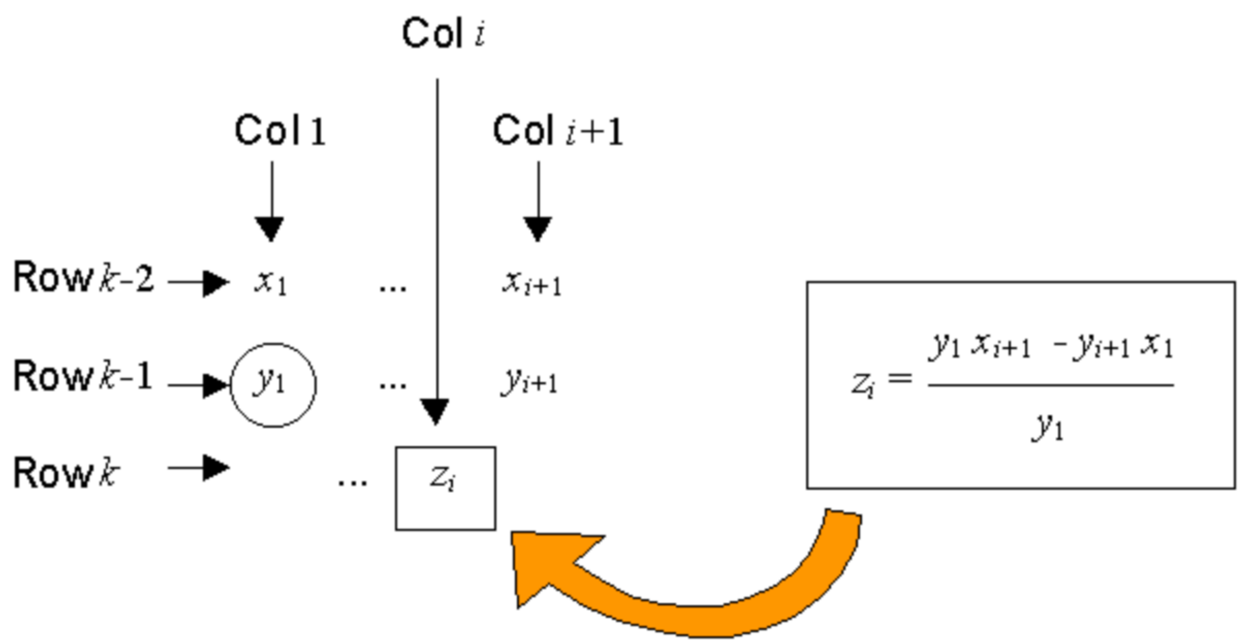

Step 2: <u>Extract the first column of the array and count the number of sign changes.</u>

The number of sign changes gives the number of roots of the polynomial which have positive real parts.

Example:

$$2s^6 + 4s^5 + 2s^4 - s^3 + 2s - 2 = 0$$

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2	2	0	-2
4	-1	2	0
2.5	-1	-2	0
0.6	5.2	0	0
-22.67	-2	0	0
5.142	0	0	0
-2	0	0	0

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There are 3 sign changes in the first column, thus the conclusion is that there are three roots that have positive real parts.

If we want to make it type two we need to combine the $\frac{1}{s^2}$ term with phase lead compensation to stabilize it $\Rightarrow$

$$C(s) = k\frac{(s+a)}{s^2} \quad \Rightarrow \quad \text{Char eq:} \quad s^4 + 5s^3 + 4s^2 + 2ks + 2ka = 0$$

and now we need to find both k and a

Routh - Hurwitz:

s^4	1	4	$2ka$
s^3	5	$2k$	
s^2	$\frac{20-2k}{5}$	$2ka$	
s^1	$\frac{40k-4k^2}{5} - 10ka$		
s^0	$2ka$		

$a > 0$

$k > 0$

need $20 - 2k > 0 \Rightarrow k < 10$

$20 - 2k - 25a > 0$