Digital Control Systems Homework #2

Problem 1

Find the z-transforms of the number sequences generated by sampling the following time functions every T seconds, beginning at t = 0. Express these transforms in closed form.

(a) $e(t) = \exp(-\text{at})$

From this expression, since we are sampling every T seconds we can replace the term t with $t = kT$ and we can include a *unit step* function into the expression since we are concerned with beginning from $t = 0$ onward:

$e(t) = e(\text{kT}) = e^{-\text{akT}} u(k)$

Expressing this in closed form, we get:

$$Z\{e(\text{kT})\} = \sum_{k=0}^{\infty} e(\text{kT}) z^{-k}$$

$$= \sum_{k=0}^{\infty} e^{-akT} z^{-k}$$

$$= \sum_{k=0}^{\infty} (e^{-aT} z^{-1})^k$$

$$E(z) = \frac{1}{1 - e^{-aT} z^{-1}} \text{ or } \frac{z}{z - e^{-aT}}, \qquad \text{for } |e^{-aT} z^{-1}| < 1$$

```
% Prove with MATLAB
syms a k t T
et = exp(-a*t)
```

et = $\mathrm{e}^{-a\,t}$

```
ekT = subs(et,t,k*T)
```

ekT = $\mathrm{e}^{-T\,a\,k}$

```
Ez = ztrans(ekT)
```

Ez =

$$\frac{z}{z - \mathrm{e}^{-T\,a}}$$

(a) $e(t) = \exp(-t + T)u(t - T)$

$e(t) = e^{-t+T} u(t - T)$

Replace the term t with the sampling interval $t \to kT$

$e(kT) = e^{-kT+T}u(kT - T) = e^{-(k-1)T}u((k-1)T)$

Using the z-transform property for a Time-Shift:

$z\{e(k-n)u(k-n)\} = z^{-n}E(z)$

which illustrates that the original z-transform of $e(k)$ is multiplied by a factor of z^{-n} depending on the shift term. The closed-form expression results in:

$$z\{e^{-(k-1)T}u((k-1)T)\} = z^{-1}E(z) = z^{-1}\left[\frac{z}{z - e^{-T}}\right] = \frac{1}{z - e^{-T}}, \qquad \text{for } |e^{-T}z^{-1}| < 1$$

```
% Prove with MATLAB
syms z k t T
et = exp(-t)
```

et = e^{-t}

```
ekT = subs(et,t,k*T)
```

ekT = $e^{-T\,k}$

```
Ez = ztrans(ekT)
```

Ez =

$$\frac{z}{z - e^{-T}}$$

```
Ez = (z^-1) * Ez
```

Ez =

$$\frac{1}{z - e^{-T}}$$

(a) $e(t) = \exp(-t + 5T)u(t - 5T)$

This expression follows the same operations as the previous question, but has a larger delay. Following the same steps previously, we get:

$e(t) = e^{-t+5T}u(t - 5T)$

Replace the term t with the sampling interval $t \to kT$

$e(kT) = e^{-kT+5T}u(kT - 5T) = e^{-(k-5)T}u((k-5)T)$

Again, using the z-transform property for a Time-Shift yields:

$$z\{e^{-(k-5)T}u((k-5)T)\} = z^{-5}E(z) = z^{-5}\left[\frac{z}{z - e^{-T}}\right] = \frac{1}{z^4(z - e^{-T})}, \qquad \text{for } |e^{-T}z^{-1}| < 1$$

```
% Prove with MATLAB
syms k t T
```

```
et = exp(-t)
```

et = e^{-t}

```
ekT = subs(et,t,k*T)
```

ekT = $e^{-T\,k}$

```
Ez = ztrans(ekT)
```

Ez =

$$\frac{z}{z - e^{-T}}$$

```
Ez = (z^-5) * Ez
```

Ez =

$$\frac{1}{z^4\,(z - e^{-T})}$$

Problem 2

A function e(t) is sampled, and the resultant sequence has the z-transform

$$E(z) = \frac{z-b}{z^3 - cz^2 + d}$$

Find the z-transform of exp(akT)$e(kT)$

This problem is asking us to solve for the z-transform of exp(akT)$e(kT)$, which involves using the z-transform property of Complex Translation, that is:

$$z\{\varepsilon^{\mathrm{akT}}e(\mathrm{kT})\} = e(0) + \varepsilon^{\mathrm{aT}}e(T)z^{-1} + \varepsilon^{2\mathrm{aT}}e(2T)z^{-2} + \ \ldots$$

$$= e(0) + e(T)(z\varepsilon^{-\mathrm{aT}})^{-1} + e(2T)(z\varepsilon^{-\mathrm{aT}})^{-2} + \ \ldots$$

or

$$z\{\varepsilon^{akT}e(kT)\} = E(z)_{z\leftarrow z\varepsilon^{-aT}} = E(z\varepsilon^{-aT})$$

Thus, we can solve this problem using this property, which essentially states that we replace all terms of z in $E(z)$ with the term $z\varepsilon^{-\mathrm{aT}}$

$$E(z)\Big|_{z\leftarrow z\varepsilon^{-aT}} = \frac{z-b}{z^3 - cz^2 + d}\Big|_{z\leftarrow z\varepsilon^{-aT}} = \frac{z\varepsilon^{-aT} - b}{z^3\varepsilon^{-aT} - cz^2\varepsilon^{-aT} + d}$$

Thus, using the property of Complex Translation and $E(z)$, we can solve for exp(akT)$e(kT)$

```
syms a b c d T z
% Use property of Complex Translation to replace values of z for ze^-aT
Ez = (z-b)/(z^3 -c*z^2 + d)
```

```
Ez =
```

$$-\frac{b-z}{z^3-c\,z^2+d}$$

```
Ez = subs(Ez,z,z*exp(-a*T))
```

```
Ez =
```

$$-\frac{b-z\,\mathrm{e}^{-T\,a}}{\mathrm{e}^{-3\,T\,a}\,z^3-c\,\mathrm{e}^{-2\,T\,a}\,z^2+d}$$

Problem 3

For the number sequence $\{e(k)\}$,

$$E(z)=\frac{z}{(z-1)^2}$$

(a) Apply the final-value theorem to $E(z)$

The final-value theorem states that $\lim_{k\to\infty} e(k)=\lim_{z\to 1}(z-1)E(z)$ if $e(\infty)$ exists. When examining $E(z)$, we use the transform pair to find the inverse transform is equal to:

$z^{-1}\{E(z)\}=\frac{z}{(z-1)^2}\; = \;k$ then, finding $e(\infty)$ in our case is rather simple since this sequence linearly increases in values, so:

$\lim_{k\to\infty} e(k)=\infty$, which it's limit is unbounded.

Thus, we cannot use the final-value theorem because all the poles of $E(z)$ are not inside the unit circle.

Double-checking in the z-Domain, we find:

$$\lim_{k\to\infty} e(k)=\lim_{z\to 1}(z-1)E(z)$$

$$=\lim_{z\to 1}(z-1)\frac{z}{(z-1)^2}$$

$$=\lim_{z\to 1}\frac{z}{(z-1)}$$

$$=\lim_{z\to 1}\frac{1}{(1-1)}=\frac{1}{0}=\text{indeterminate}$$

```
syms k z
Ez1 = z/(z-1)^2
```

```
Ez1 =
```

$$\frac{z}{(z-1)^2}$$

```
ek1 = iztrans(Ez1,k)
```

ek1 = k

```
ek_final_value = subs(ek1,k,Inf)
```

ek_final_value = ∞

(b) Find the z-transform of $e(k) = k(-1)^k$

For this problem, we can use the transform pair:

$$ka^k \leftrightarrow \frac{az}{(z-a)^2}$$

In our case, $a = -1$, which results in the z-transform:

$$z\{e(k)\} = z\{k(-1)^k\} = -\frac{z}{(z+1)^2}$$

Thus,

$$E(z) = -\frac{z}{(z+1)^2}$$

```
% Prove using MATLAB
syms k z
ek2 = k*(-1)^k
```

ek2 = $(-1)^k\, k$

```
Ez2 = ztrans(ek2)
```

Ez2 =

$$-\frac{z}{(z+1)^2}$$

(c) Explain how parts(a) and (b) are related

From part (a), we showed that the final-value theorem for $E(z)$ was unbounded.

So, now we check if $e(\infty)$ exists for part (b)

$\lim_{k\to\infty} e(k) = k(-1)^k \to \lim_{k\to\infty} \infty(-1)^\infty$ which results in an unbounded result

So, here for the case of part (b), the value of $e(\infty)$ is unbounded as well, which means that it doesn't exist either

So, we conclude that both parts (a) and (b) are related due to both of their time-domai signals at $e(\infty)$ not existing when taking their corresponding limits. This means that we cannot take the final-value theorem approach for these sequences because all of their poles for $E(z)$ are not within the unit circle. In fact, part

(a) has double poles at $z = 1$ and part (b) has double poles at $z = -1$, which is exactly on the unit circle and explains our findings.