Digital Control Systems Homework #6

Problem 1

Problem 7.5-4 of the textbook parts (a) and (c)

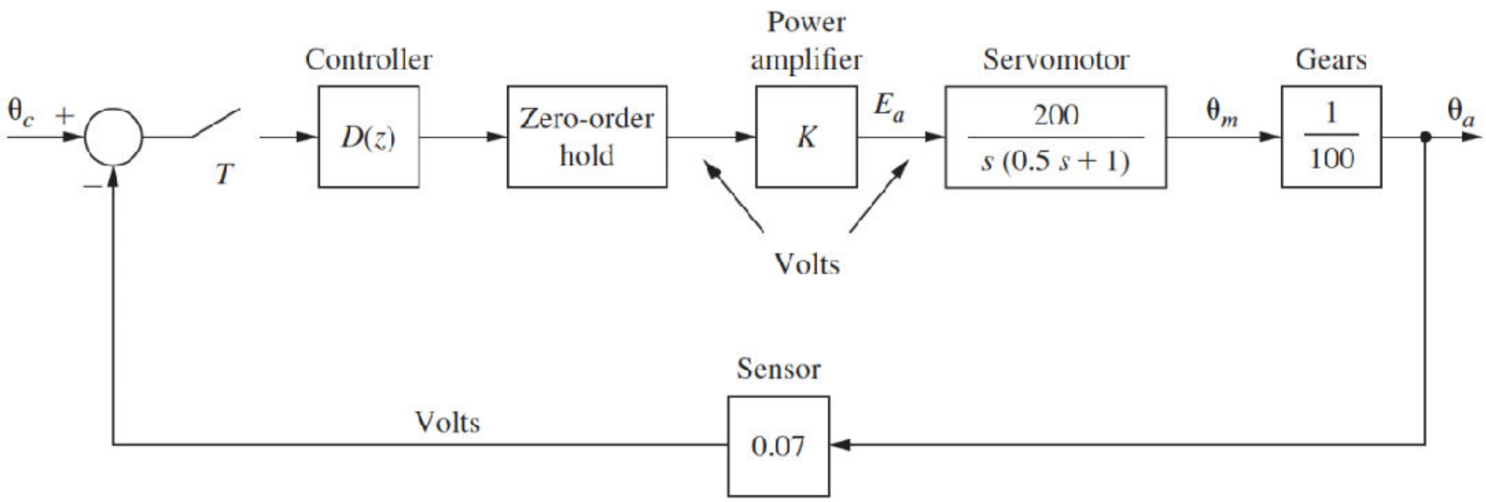


FIGURE P7.5-4 Robot arm joint control system.

(a) Write the closed-loop system characteristic equation

We are given the z-transform of the Open-Loop System which is

$$G(z) = \frac{0.01873z + 0.01752}{(z-1)(z-0.8187)}$$

We determine the closed-llop system characteristic equation by solving: $1 + K\overline{GH}(z) = 0$

$$GH(z) = \frac{0.01873z + 0.01752}{(z-1)(z-0.8187)}(0.07)$$

$$GH(z) = \frac{0.0013111z + 0.0011226}{z^2 - 1.8187 + 0.8187}$$

Then making the substitution for the system characteristic equation is:

$$1 + K\left[\frac{0.0013111z + 0.0011226}{z^2 - 1.8187 + 0.8187}\right] = 0$$

$$z^2 - 1.8187z + 0.8187 + 0.0013111Kz + 0.001226K = 0$$

$$z^2 - (1.8187 - 0.0013111K)z + 0.8187 + 0.001226K = 0$$

Thus, the system characteristic equation is:

$$z^2 - (1.8187 - 0.0013111K)z + 0.8187 + 0.001226K = 0$$

(b) Use Routh-Hurwitz criterion to determine the range of K for stability

Use MATLAB to convert from z-plane to w-plane via bilinear transformation

```
T = 0.1;
numz = [0.0013111 0.0012264];
denomz = [1 -1.8187 0.8187];
GHz = tf(numz,denomz,T)
```

```
GHz =

  0.001311 z + 0.001226
  ----------------------
  z^2 - 1.819 z + 0.8187

Sample time: 0.1 seconds
Discrete-time transfer function.
```

```
Gw = d2c(GHz,'tustin')
```

```
Gw =

  -2.329e-05 s^2 - 0.01349 s + 0.279
  ----------------------------------
            s^2 + 1.994 s

Continuous-time transfer function.
```

Then the characteristic equation in terms of the w-plane is

$$1 + K\overline{GH}(w) = (1 - 0.00002329K)w^2 + (1.994 - 0.01349K)w + 0.279K = 0$$

Then we can construct the Routh array as

$$\begin{bmatrix} w^2 \\ w^1 \\ w^0 \end{bmatrix}\begin{bmatrix} 1 - 0.00002329K \\ 1.994 - 0.01349K \\ 0.279K \end{bmatrix} \rightarrow \begin{bmatrix} K < 42936.9 \\ K < 147.8 \\ K > 0 \end{bmatrix}$$

Thus, K is in the range of $\rightarrow\ 0 < K < 147.8$

(c) Check the results of part (b) using the Jury test

Using the closed-loop system characteristic equation from before,

$z^2 - (1.8187 - 0.0013111K)z + 0.8187 + 0.001226K = 0$

We construct the Jury array as:

$$\begin{bmatrix} z^0 & z^1 & z^2 \\ 0.8187 + 0.001226K & 0.0013111K - 1.8187 & 1 \end{bmatrix}$$

Checking the constraint $Q(1) > 0$ yields

$$(1)^2 - (1.8187 - 0.0013111K)(1) + 0.8187 + 0.0011226K = 0$$

$$0.0024337K > 0 \rightarrow K > 0$$

Checking the constraint $(-1)^2 Q(-1) > 0$ yields

$$(-1)^2 - (1.8187 - 0.0013111K)(-1) + 0.8187 + 0.001226K = 0$$

$$3.6374 - 0.0000851K > 0 \rightarrow K < \frac{3.6374}{0.0000851} \rightarrow K < 42,742.7$$

Checking the constraint $|a_0| < a_2$ yields

$$0.8187 + 0.001226K < 1 \rightarrow K < \frac{0.1813}{0.001226} \rightarrow K < 147.8$$

Thus, the system is stable for $0 < K < 147.8$ which matches the results from part (b)

(d) Determine the location of all roots of the characteristic equation in both the w-plane and the z-plane for the value of $K > 0$ for which the system is marginally stable

From above, it was computed that the system is marginally stable at the value of $K = 147.8$

Substituting this value into the system characteristic equation, we get

$$z^2 - (1.8187 - 0.0013111K)z + 0.8187 + 0.001226K|_{K=147.8} = 0$$

$$z^2 - 1.625 + 1 = 0$$

```
z = roots([1 -1.625 1])
```

```
z = 2×1 complex
   0.8125 + 0.5830i
   0.8125 - 0.5830i
```

```
z_mag = abs(z(1)), z_ang_deg = angle(z(1))*(180/pi), z_ang_rad = angle(z(1))
```

```
z_mag = 1.0000
z_ang_deg = 35.6591
z_ang_rad = 0.6224
```

The roots of this equation in the z-plane are

$$z = 0.8125 \pm j0.583 = 1\angle(\pm 35.66^{\circ}) = 1\angle(\pm 0.6225 \ \ \text{rad/s})$$

Checking for the characteristic equation in the w-plane:

$$(1 - 0.00002329K)w^2 + (1.994 - 0.01349K)w + 0.279K|_{K=147.8} = 0$$

$$0.9966w^2 + 0.000178w + 41.2362 = 0$$

```
w = roots([0.9966 0.000178 41.2362])
```

```
w = 2×1 complex
  -0.0001 + 6.4325i
  -0.0001 - 6.4325i
```

```
w_mag = abs(w(1)), w_ang_deg = angle(w(1))*(180/pi), w_ang_rad = angle(w(1))
```

```
w_mag = 6.4325
w_ang_deg = 90.0008
```

```
w_ang_rad = 1.5708
```

The roots of this equation in the w-plane are

$$w = 0 \pm j0.6433$$

(e) Determine both the s-plane frequency and the w-plane frequency at which the system will oscillate when marginally stable, using results of part (d)

From part (d), the s-plane frequency is

$$1\angle(\pm 0.6225 \ \ \text{rad/s}) = 1\angle(\omega T)$$

$$\omega T = 0.6225 \rightarrow \omega = 0.6225/T = 6.225$$

And the w-plane frequency is

$$w = 0 \pm j0.6433$$

(f) Show that the frequencies in part (e) satisfy (7-10)

Formula (7-10) refers to

$$\omega_w = \frac{2}{T}\tan\left(\frac{\omega T}{2}\right)$$

Substituting values from part (d) into the equation, we get

$$\omega_w = \frac{2}{0.1}\tan\left(\frac{0.6225}{2}\right)$$

$$\omega_w = 20\tan(0.31125)$$

$$\omega_w = 6.434$$

Which this verifies the results from part (d)

Root Locus for System

```
T = 0.1; numz = [0.01873 0.01752]; denomz = [1 -1.8187 0.8187];
Gz = tf(numz,denomz,T)
```

```
Gz =

   0.01873 z + 0.01752
  ----------------------
  z^2 - 1.819 z + 0.8187

Sample time: 0.1 seconds
Discrete-time transfer function.
```

```
GHz = Gz * 0.07
```

```
GHz =

  0.001311 z + 0.001226
  ----------------------
  z^2 - 1.819 z + 0.8187
```

```
Sample time: 0.1 seconds
Discrete-time transfer function.
```

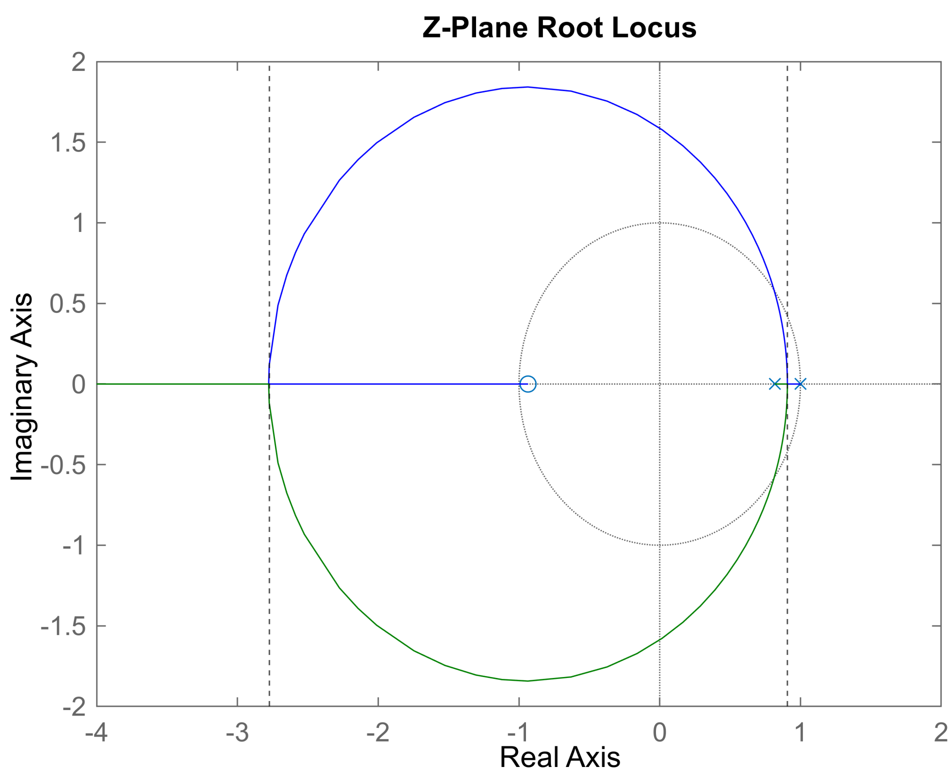
```
zpk(GHz)
```

```
ans =

  0.0013111 (z+0.9354)
  --------------------
    (z-1) (z-0.8187)

Sample time: 0.1 seconds
Discrete-time zero/pole/gain model.
```

```
figure()
rlocus(GHz), hold on
xline(0.9071,'k--')
xline(-2.774,'k--')
title("Z-Plane Root Locus")
axis([-4 2 -2 2])
```



Problem 2

Problem 7.5-5

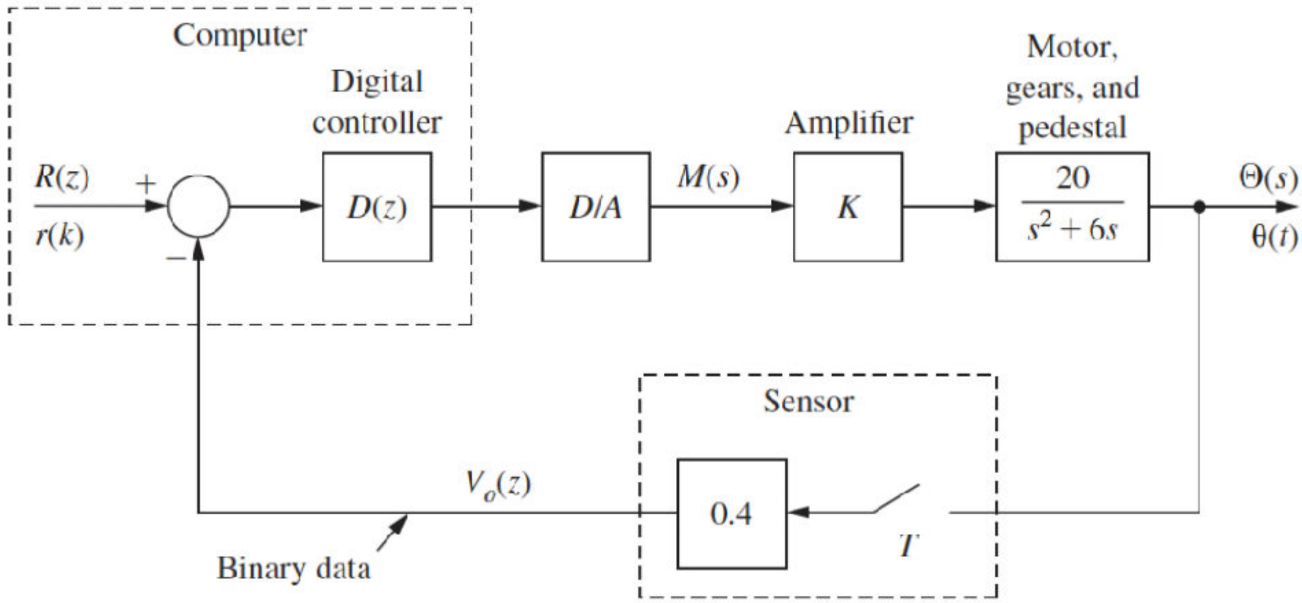


FIGURE P7.5-5 Block diagram for an antenna control system.

(a) Write the closed-loop system characteristic equation

We are given the z-transform of the Open-Loop System which is

$$G(z) = \frac{0.02268z + 0.02052}{(z-1)(z-0.7408)}$$

We determine the closed-llop system characteristic equation by solving: $1 + K\overline{GH}(z) = 0$

$$GH(z) = \frac{0.02268z + 0.02052}{(z-1)(z-0.7408)}(0.4)$$

$$GH(z) = \frac{0.009072z + 0.008208}{z^2 - 1.7408z + 0.7408}$$

Then making the substitution for the system characteristic equation is:

$$1 + K\left[\frac{0.009072z + 0.008208}{z^2 - 1.7408z + 0.7408}\right] = 0$$

$$z^2 - 1.7408z + 0.7408 + 0.009072Kz + 0.008208K = 0$$

$$z^2 + (0.009072K - 1.7408)z + 0.7408 + 0.008208K = 0$$

Thus, the system characteristic equation is:

$$z^2 + (0.009072K - 1.7408)z + 0.7408 + 0.008208K = 0$$

(b) Use Routh-Hurwitz criterion to determine the range of K for stability

Use MATLAB to convert from z-plane to w-plane via bilinear transformation

```
T = 0.05;
numz = [0.009072 0.008208];
denomz = [1 -1.7408 0.7408];
GHz = tf(numz,denomz,T)
```

```
GHz =

  0.009072 z + 0.008208
  ----------------------
  z^2 - 1.741 z + 0.7408

Sample time: 0.05 seconds
Discrete-time transfer function.
```

```
Gw = d2c(GHz,'tustin')
```

```
Gw =

  -0.0002482 s^2 - 0.1886 s + 7.941
  ---------------------------------
      s^2 + 5.956 s + 2.645e-14

Continuous-time transfer function.
```

Then the characteristic equation in terms of the w-plane is

$$1 + K\overline{GH}(w) = (1 - 0.0002482K)w^2 + (5.956 - 0.1886K)w + 7.941K = 0$$

Then we can construct the Routh array as

$$\begin{bmatrix} w^2 \\ w^1 \\ w^0 \end{bmatrix} \begin{bmatrix} 1 - 0.0002482K & \\ 5.956 - 0.1886K & \\ 7.941K & \end{bmatrix} \rightarrow \begin{bmatrix} K < 4029 \\ K < 31.6 \\ K > 0 \end{bmatrix}$$

Thus, K is in the range of $\rightarrow\ 0 < K < 31.6$

(c) Check the results of part (b) using the Jury test

Using the closed-loop system characteristic equation from before,
$z^2 + (0.009072K - 1.7408)z + 0.7408 + 0.008208K = 0$

We construct the Jury array as:

$$\begin{bmatrix} z^0 & z^1 & z^2 \\ 0.7408 + 0.008208K & 0.009072K - 1.7408 & 1 \end{bmatrix}$$

Checking the constraint $Q(1) > 0$ yields

$$(1)^2 + (0.009072K - 1.7408)(1) + 0.7408 + 0.008208K = 0$$

$$0.01728K > 0 \rightarrow K > 0$$

Checking the constraint $(-1)^2 Q(-1) > 0$ yields

$$(-1)^2 + (0.009072K - 1.7408)(-1) + 0.7408 + 0.008208K = 0$$

$$3.4816 - 0.000864K > 0 \rightarrow K < \frac{3.6374}{0.000864} \rightarrow K < 4029.7$$

Checking the constraint $|a_0| < a_2$ yields

$$0.7408 + 0.008208K < 1 \rightarrow K < \frac{0.7408}{0.008208} \rightarrow K < 31.6$$

Thus, the system is stable for $0 < K < 31.6$ which matches the results from part (b)

(d) Determine the location of all roots of the characteristic equation in both the w-plane and the z-plane for the value of $K > 0$ for which the system is marginally stable

From above, it was computed that the system is marginally stable at the value of $K = 31.6$

Substituting this value into the system characteristic equation, we get

$$z^2 + (0.009072K - 1.7408)z + 0.7408 + 0.008208K|_{K=31.6} = 0$$

$$z^2 - 1.4541 + 1 = 0$$

```
z = roots([1 -1.4541 1])
```

```
z = 2×1 complex
   0.7270 + 0.6866i
   0.7270 - 0.6866i
```

```
z_mag = abs(z(1)), z_ang_deg = angle(z(1))*(180/pi), z_ang_rad = angle(z(1))
```

```
z_mag = 1.0000
z_ang_deg = 43.3603
z_ang_rad = 0.7568
```

The roots of this equation in the z-plane are

$$z = 0.727 \pm j0.687 = 1\angle(\pm 43.36^\circ) = 1\angle(\pm 0.7568 \ \text{ rad/s})$$

Checking for the characteristic equation in the w-plane:

$$(1 - 0.0002482K)w^2 + (5.956 - 0.1886K)w + 7.941K|_{K=31.6} = 0$$

$$0.9922w^2 - 0.00376w + 250.94 = 0$$

```
w = roots([0.9922 -0.00376 250.94])
```

```
w = 2×1 complex
   0.0019 +15.9032i
   0.0019 -15.9032i
```

```
w_mag = abs(w(1)), w_ang_deg = angle(w(1))*(180/pi), w_ang_rad = angle(w(1))
```

```
w_mag = 15.9032
w_ang_deg = 89.9932
w_ang_rad = 1.5707
```

The roots of this equation in the w-plane are

$w = 0 \pm j15.903$

(e) Determine both the s-plane frequency and the w-plane frequency at which the system will oscillate when marginally stable, using results of part (d)

From part (d), the s-plane frequency is

$1\angle(\pm 0.7568 \ \ \text{rad/s}) = 1\angle(\omega T)$

$\omega T = 0.7568 \rightarrow \omega = 0.7568/0.05 = 15.136 \ \ \text{rad/s}$

And the w-plane frequency is

$w = 0 \pm j15.903$

(f) Show that the frequencies in part (e) satisfy (7-10)

Formula (7-10) refers to

$$\omega_w = \frac{2}{T}\tan\left(\frac{\omega T}{2}\right)$$

Substituting values from part (d) into the equation, we get

$$\omega_w = \frac{2}{0.05}\tan\left(\frac{0.7568}{2}\right)$$

$$\omega_w = 40\tan(0.3784)$$

$$\omega_w = 15.9023$$

Which this verifies the results from part (d)