# Digital Control Systems Homework #1

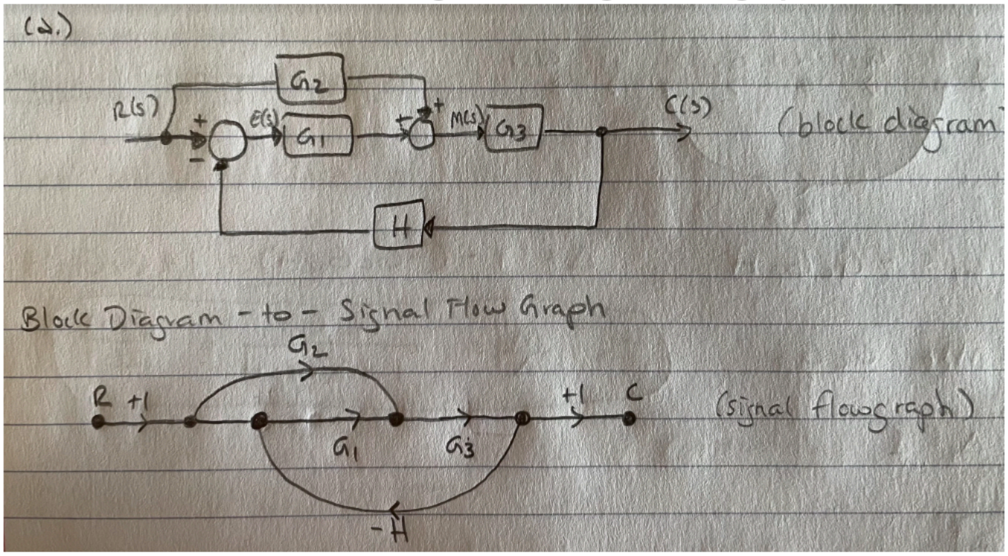
## Problem 1

Calculate the transfer function $\frac{C(s)}{R(s)}$ for the following systems

### Part a.)

```
f = imread('DCS_HW1_P1a.png'); imshow(f);
title('Part A: Block Diagram and Signal Flowgraph')
```

Part A: Block Diagram and Signal Flowgraph




Loops:

- $L_1 = -G_1G_3H$ (touching)

Forward Path Gains:

- $M_1 = G_2G_3$
- $M_2 = G_1G_3$

Then calculate $\Delta$ as:

- $\Delta = 1 - (-G_1G_3H) = 1 + G_1G_3H$

Since the only loop in the system touches all common nodes:

- $\Delta_1 = \Delta_2 = 1$

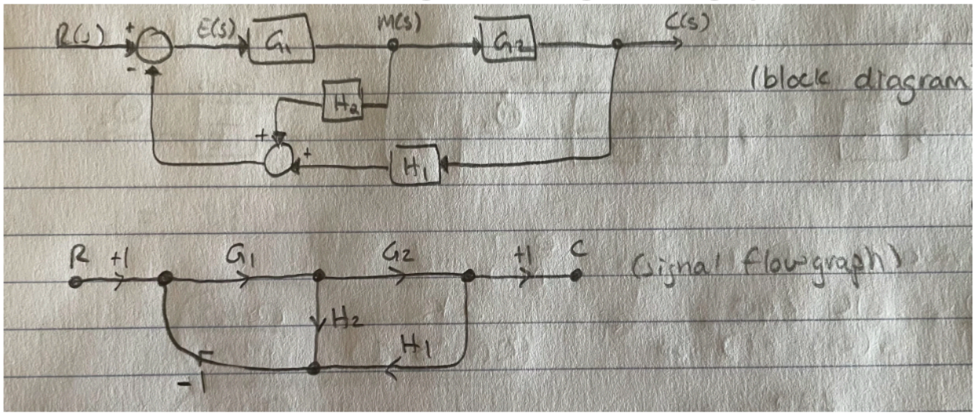
Then, we putting everything together using Mason's Formula, we derive the transfer function to be:

$$T = \frac{1}{\Delta}\sum_{k=1}^{P} M_k\Delta_k = \frac{1}{\Delta}(M_1\Delta_1 + M_2\Delta_2) = \frac{G_2G_3 + G_1G_3}{1 + G_1G_3H} = \frac{C(s)}{R(s)}$$

## Part b.)

```
f = imread('DCS_HW1_P1b.png'); imshow(f);
title('Part B: Block Diagram and Signal Flowgraph')
```

Part B: Block Diagram and Signal Flowgraph




Loops:

- $L_1 = -G_1H_2$
- $L_2 = -G_1G_2H_1$

Forward Path Gains:

- $M_1 = G_1G_2$

Then calculate $\Delta$ as:

- $\Delta = 1 - (-G_1H_2) - (-G_1G_2H_1) = 1 + G_1H_2 + G_1G_2H_1$

Since the only loop in the system touches all common nodes:

- $\Delta_1 = 1$

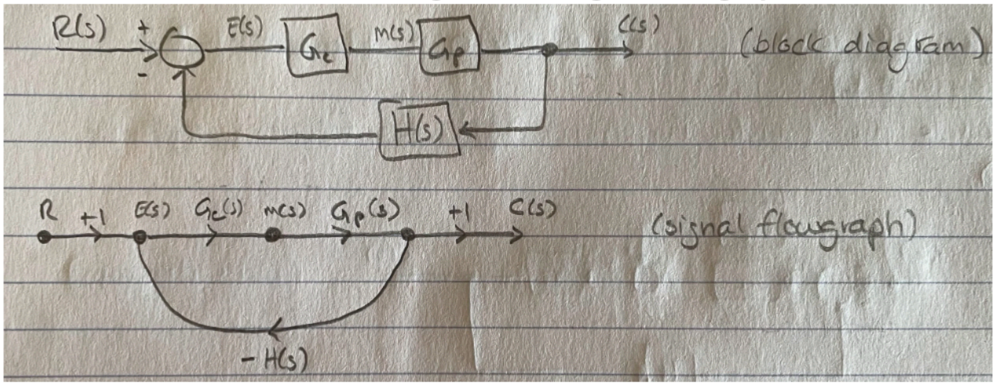
Then, we putting everything together using Mason's Formula, we derive the transfer function to be:

$$T = \frac{1}{\Delta}\sum_{k=1}^{P} M_k\Delta_k = \frac{1}{\Delta}(M_1\Delta_1) = \frac{G_1G_2}{1 + G_1H_2 + G_1G_2H_1} = \frac{C(s)}{R(s)}$$

### Part c.)

```
f = imread('DCS_HW1_P1c.png'); imshow(f);
title('Part C: Block Diagram and Signal Flowgraph')
```

Part C: Block Diagram and Signal Flowgraph




Loops:

- $L_1 = -G_cG_pH$

Forward Path Gains:

- $M_1 = G_cG_p$

Then calculate $\Delta$ as:

- $\Delta = 1 - (-G_cG_pH) = 1 + G_cG_pH$

Since the only loop in the system touches all common nodes:

- $\Delta_1 = 1$

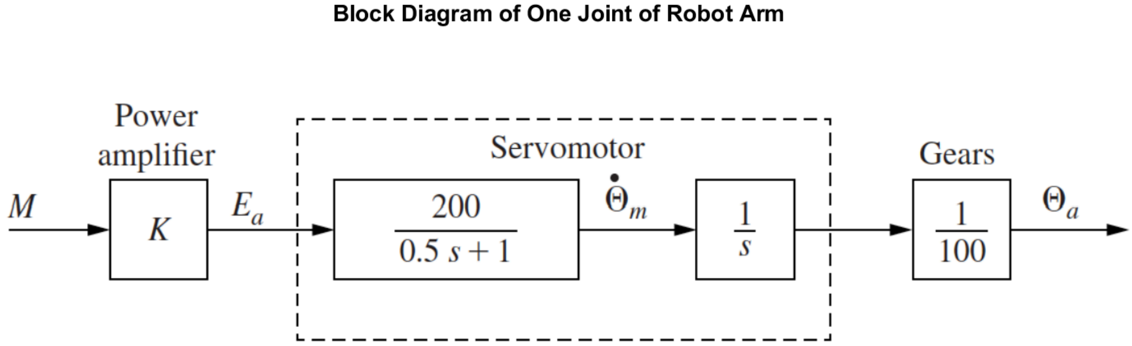
Then, we putting everything together using Mason's Formula, we derive the transfer function to be:

$$T = \frac{1}{\Delta}\sum_{k=1}^{P} M_k\Delta_k = \frac{1}{\Delta}(M_1\Delta_1) = \frac{G_cG_p}{1 + G_cG_pH} = \frac{C(s)}{R(s)}$$

## Problem 2

```
f = imread('DCS_HW1_P2_Diagram.png'); imshow(f)
```

```
title('Block Diagram of One Joint of Robot Arm')
```




## Part a.)

Derive Transfer Functions $\frac{\Theta_a(s)}{M(s)}$ and $\frac{\Theta_a(s)}{E_a(s)}$

$$\frac{\Theta_a(s)}{M(s)} = K \times \left(\frac{200}{0.5s+1}\right) \times \left(\frac{1}{s}\right) \times \left(\frac{1}{100}\right) = \frac{2K}{s(0.5s+1)}$$

$$\frac{\Theta_a(s)}{E_a(s)} = \left(\frac{200}{0.5s+1}\right) \times \left(\frac{1}{s}\right) \times \left(\frac{1}{100}\right) = \frac{2}{s(0.5s+1)}$$

## Part b.)

The Transfer function of just the servomotor can be described as

$$\frac{\dot{\Theta}_m(s)}{E_a(s)} = \frac{200}{0.5s+1}$$

Since we are looking for rated rpm of the motor, we need to find the angular velocity of the motor, and move $E_a(s) = \frac{24}{s}$ to right-side

$$\dot{\Theta}_m(s) = \frac{200}{0.5s+1} \times \frac{24}{s} = \frac{4800}{s(0.5s+1)}$$

simplifying even further we get:

$$\Theta_m(s) = \frac{2}{2} \times \frac{4800}{s(0.5s+1)} = \frac{9600}{s(s+2)}$$

```
s = tf('s');
Theta_m = 400/(s+2) * 24/s
```

```
Theta_m =

    9600
  ---------
```

```
  s^2 + 2 s

Continuous-time transfer function.
```

We can then use Partial Fraction Expansion to break up this Transfer Function, and solve for the inverse Laplace

```
a = [1,2,0]; b = 9600;
[R,P,K] = residue(b,a)
```

```
R = 2×1
       -4800
        4800
P = 2×1
    -2
     0
K =

     []
```

Taking the Partial Fraction Expansion of $\frac{9600}{s(s+2)} = \frac{4800}{s} - \frac{4800}{s+2}$

Then taking the Laplace Transform of these expressions we get:

$$\dot{\theta_m}(t) = L^{-1}\left\{\frac{4800}{s} - \frac{4800}{s+2}\right\} \rightarrow 4800u(t) - 4800e^{-2t}$$

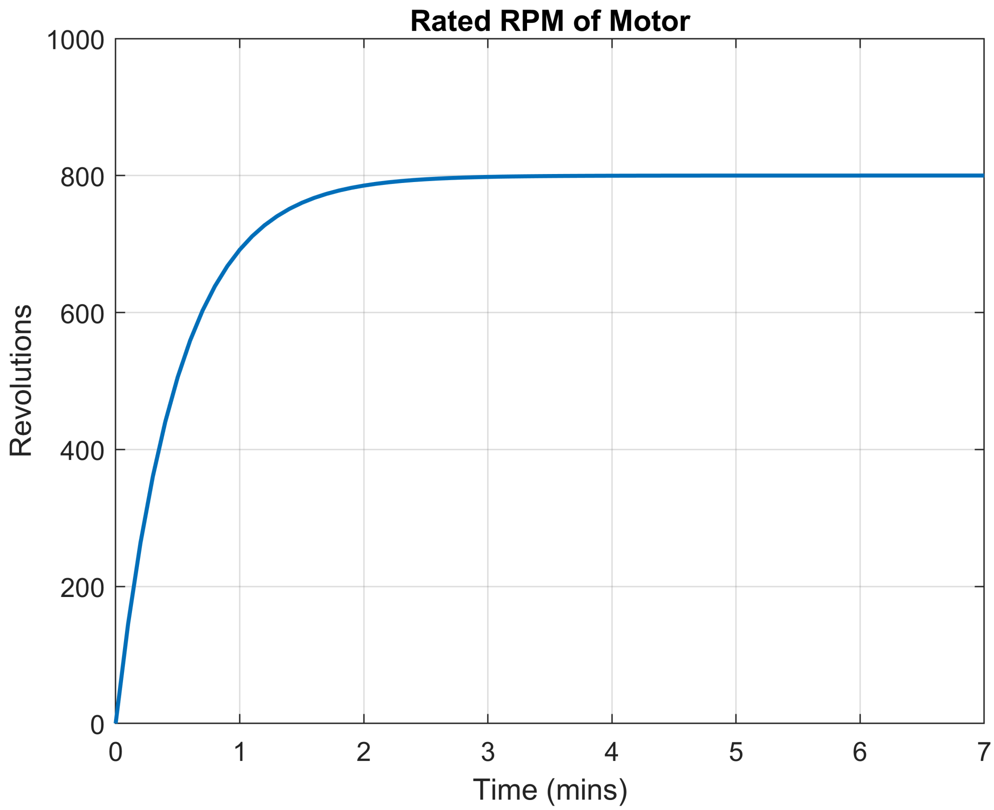
Since $\theta_m$ was given to us in degrees, we manipulate the units to give us rpm (revolutions per minute)

$$\dot{\theta_m}(t) = (4800 - 4800e^{-2t})\left(\frac{deg}{sec}\right) \times \left(\frac{60secs}{minute}\right) \times \left(\frac{revolutions}{360^\circ}\right) = 800 - 800e^{-2t}$$

Then evaluated at steady state gives:

$\dot{\theta_m}(t)|lim_{t\rightarrow\infty} = 800 - 800e^{-2\times\infty} = 800 - 800(0) \approx$ **800 rpm**

```
t = 0:0.1:7;
d_theta_m = 4800 - 4800*exp(-2.*t);
d_theta_m = (60/360).*d_theta_m;
figure
plot(t,d_theta_m,'LineWidth',1.5), title('Rated RPM of Motor')
ylim([0 1000]),yticks(0:200:1000), grid on
xlabel('Time (mins)'), ylabel('Revolutions')
```




### Part c.)

The transfer function between the input voltage $e_a(t)$ and angle of the robot arm $\theta_a(t)$ can be described as:

$$\frac{\Theta_a(s)}{E_a(s)} = \frac{2}{s(0.5s+1)}$$

Again, we need to apply $e_a(t) = 24u(t)$ and represent this in the Laplace Domain and simplify the expression above

$$\Theta_a(s) = \frac{2}{s(0.5s+1)} \times \frac{24}{s} = \frac{96}{s^2(s+2)}$$

Taking the Partial Fraction Expansion of this term we get:

```
Theta_a = 96/(s^2*(s+2))
```

```
Theta_a =

      96
  -----------
  s^3 + 2 s^2

Continuous-time transfer function.
```

```
b = 96; a = [1,2,0,0];
[R,P,K] = residue(b,a)
```

```
R = 3×1
    24
   -24
    48
P = 3×1
    -2
     0
     0
K =

     []
```

Taking the Partial Fraction Expansion of $\frac{96}{s^2(s+2)} = \frac{24}{s-(-2)} - \frac{24}{s-(0)} + \frac{48}{s^2-(0)} = \frac{24}{s+2} - \frac{24}{s} + \frac{48}{s^2}$

Then taking the Laplace Transform of these expressions we get:

$$\theta_a(t) = \quad L^{-1}\left\{\frac{24}{s+2} + \frac{48}{s^2} - \frac{24}{s}\right\} \to -24 + 48t + 24e^{-2t}$$

Then taking the derivative of this expression we derive the maximum rate of movement for the robot arm:

$$\frac{d\dot{\theta}_a(t)}{dt} = \frac{d}{dt}[-24 + 48t + 24e^{-2t}]$$

$$\dot{\theta}_a(t) = [0 + 48 + (-2)24e^{-2t}] = 48 - 48e^{-2t}$$

Then evaluated at steady state gives:

$\dot{\theta}_a(t)|lim_{t\to\infty} = 48 - 48e^{-2\times\infty} = 48 - 0 =$ **48 degrees/second,** which makes sense because the gears for the robot arm make it 1/100th slower in movement

## Part d.)

Find the time required for the arm to be moving at 99% of maximum rate of movement found in Part b.)

Find new max value of rate of movement:

$\dot{\theta}_a(t) = 48 \times 0.99 =$ 47.52 degrees/sec

```
ratedRPM = round(48 * 0.99,2) % Convert ratedRPM to 99% of value
```

```
ratedRPM = 47.5200
```

Substitute the rate of movement for the robot arm and set equivalent to new rotation value

$$47.52 \ = 48 - 48e^{-2t} \leftrightarrow 48e^{-2t} = 48 - 47.52$$

$$48e^{-2t} = 0.48$$

$$e^{-2t} = \frac{0.48}{48} = 0.01$$

$$\ln(e^{-2t}) = \ln(0.01)$$

$-2t = -4.61$

$t = 2.303$ secs

```
t = log((48-48*0.99)/48)/-2
```

```
t = 2.3026
```

From these calculations, it is observed that it takes about **2.3 seconds** for the arm to be moving at 99% of maximum rate of movement

### Part e.)

Suppose the input $m(t)$ is constrained by system hardware to be $\leq 10\ V$. What value should you choose for the gain $K$ and why?

The transfer function for th Power Amplifier is:

$$E_a(s) = K \times M(s) \rightarrow \frac{E_a(s)}{M(s)} = K$$

So, since we want the unit step response for the servomotor to be a value of $24\ V$, then the value of $K$ should be calculated as

$K = \dfrac{E_a(s)}{M(s)} \rightarrow \dfrac{24\ V}{10\ V} = 2.4\ V/V$, this will reassure that if the maximum value of the system hardware is used, than the servomotor's rated rpm will not exceed the $24\ V$ input of $e_a(t)$