Some models that we will use later on:

- Mechanical Translational Systems:

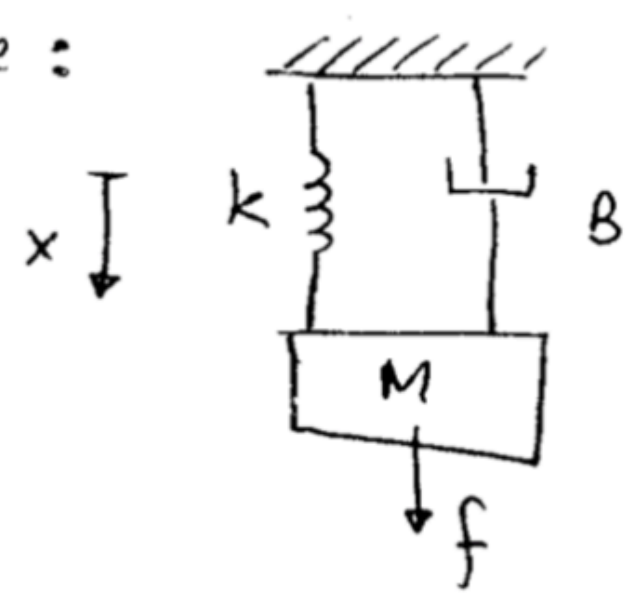
Basic Law: Newton's second law: $M\ddot{x} = \Sigma F$

Elements:

a) Spring $F_k = -kx$

b) Viscous damping and friction $F_b = -b\dot{x}$ (always opposes motion)

- Example:

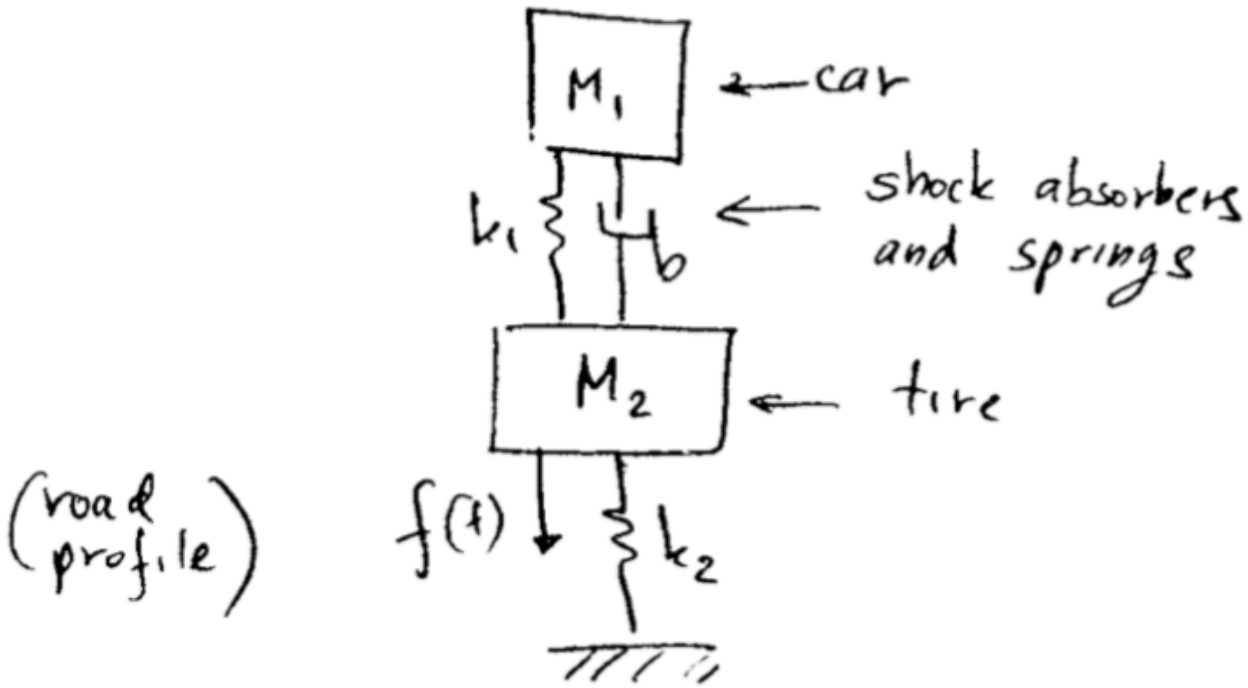


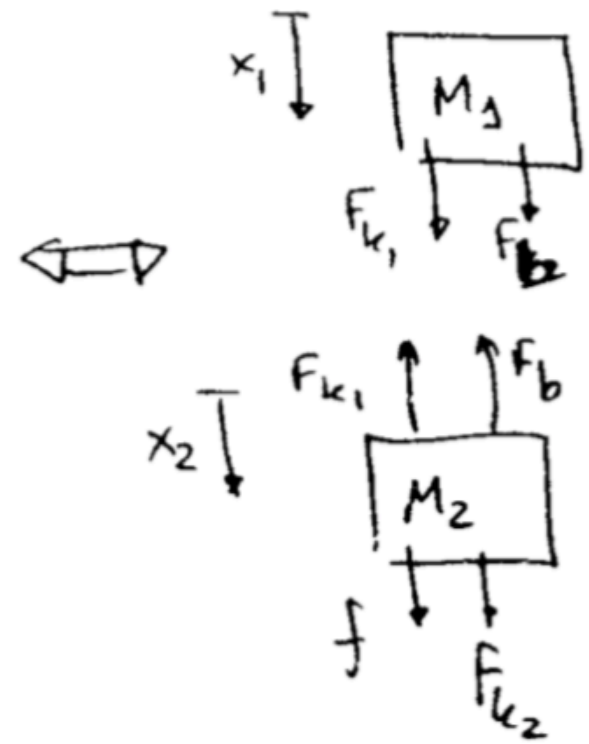
$$M\ddot{x} = -kx - b\dot{x}$$

$$\Rightarrow \boxed{M\ddot{x} + b\dot{x} + kx = 0}$$

Some models that we will use later on:

- Example 2: Simplified model of an automobile suspension:



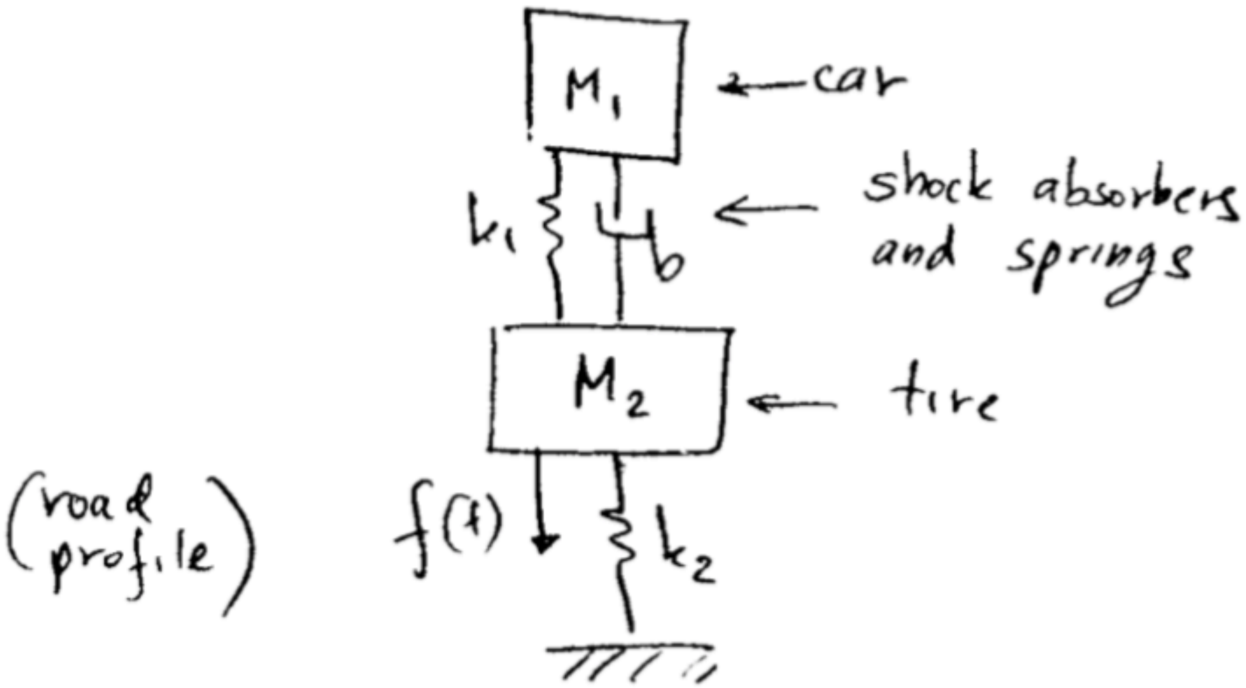


$$M_1\ddot{x}_1 = -k_1(x_1-x_2) - b(\dot{x}_1-\dot{x}_2)$$

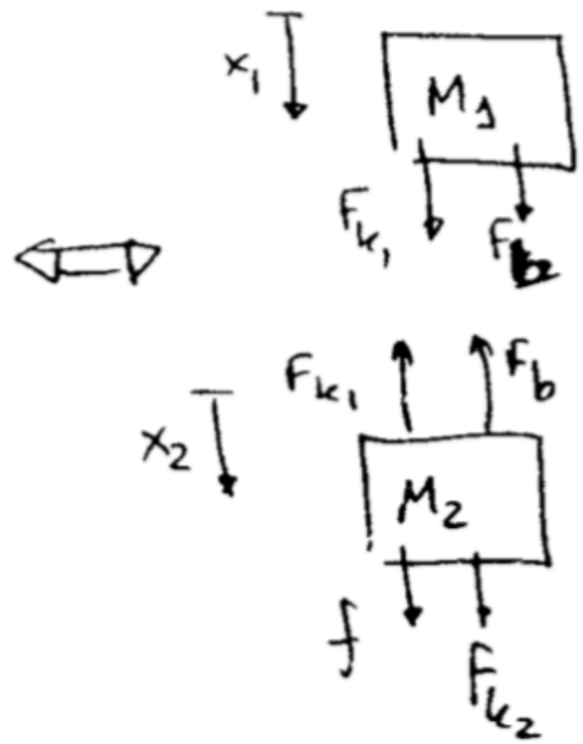
$$M_2\ddot{x}_2 = -k_1(x_2-x_1) - b(\dot{x}_2-\dot{x}_1) - k_2x_2 + f$$

Some models that we will use later on:

- Example 2: Simplified model of an automobile suspension:



$\Longleftrightarrow$



$$M_1 \ddot{x}_1 = -k_1(x_1 - x_2) - b(\dot{x}_1 - \dot{x}_2)$$

$$M_2 \ddot{x}_2 = -k_1(x_2 - x_1) - b_2(\dot{x}_2 - \dot{x}_1) - k_2 x_2 + f$$

Taking Laplace transforms yields:

$$(M_1 s^2 + b_1 s + k_1) x_1 - (bs + k_1) x_2 = 0$$

$$-(bs + k_2) x_2 + (M_2 s^2 + b_1 s + k_1 + k_2) x_2 = F(s)$$

By solving these we can get the transfer functions $G_1(s) = \dfrac{X_1(s)}{F(s)}$ and $G_2(s) = \dfrac{X_2(s)}{F(s)}$

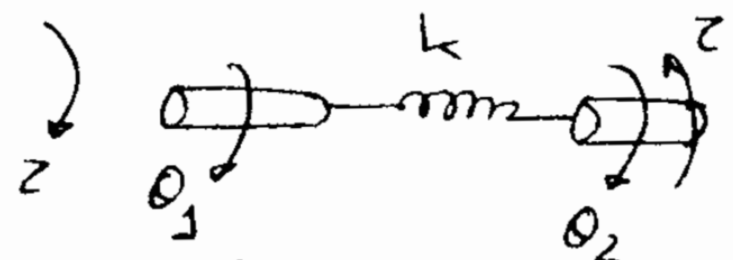
Some models that we will use later on:

- **Mechanical Rotational Systems**

Basic Law: Newton's equation for rotational systems: $J\ddot{\theta} = \Sigma$ Torques

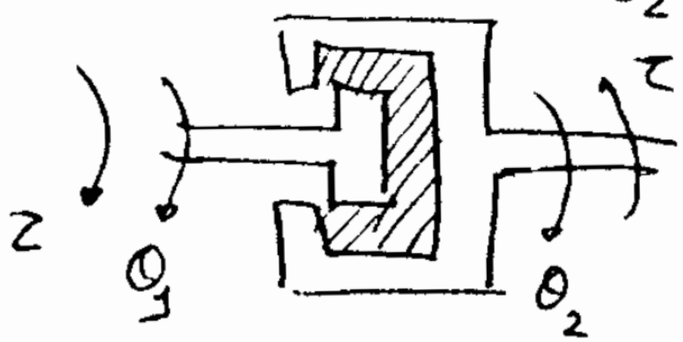
Elements: moment of inertia (similar to mass)
friction
torsion

Torsion spring:



$$\tau = k(\theta_2 - \theta_1)$$

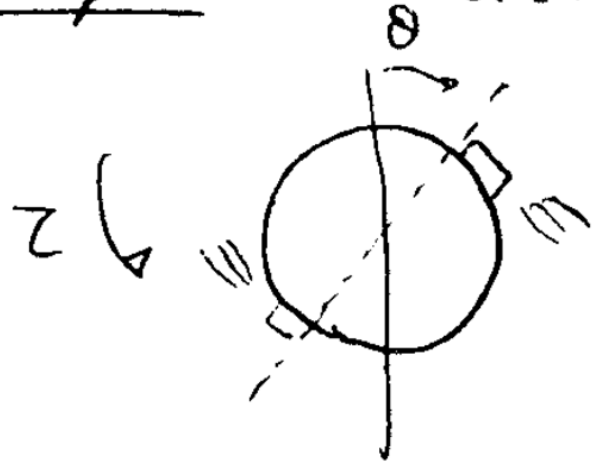
Viscous friction:



$$\tau = B(\dot{\theta}_2 - \dot{\theta}_1)$$

Some models that we will use later on:

- <u>Example</u> Satellite attitude control: (with torque applied by 2 thrusters)



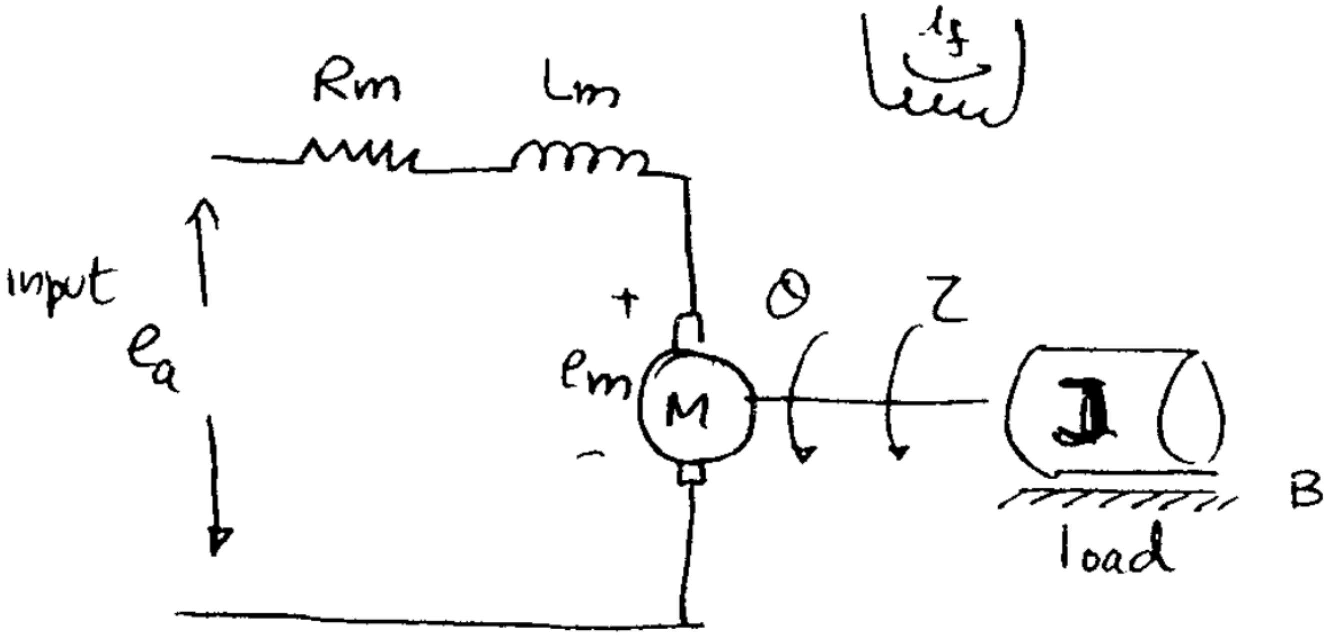
$$J\frac{d^2\theta}{dt^2} = \tau$$

(or, in Laplace domain: $Js^2\Theta = \tau \Rightarrow \Theta = \frac{1}{Js^2}\tau$:
essentially a double integrator)

Some models that we will use later on:

- Electromechanical Systems:

DC motor with independent excitation:



Some models that we will use later on:

1) Electrical equation: $e_a = R_m i_a + L_m \frac{d i_a}{dt} + e_m \Leftrightarrow E_a(s) = (s L_m + R_m) I_a(s) + E_m(s)$

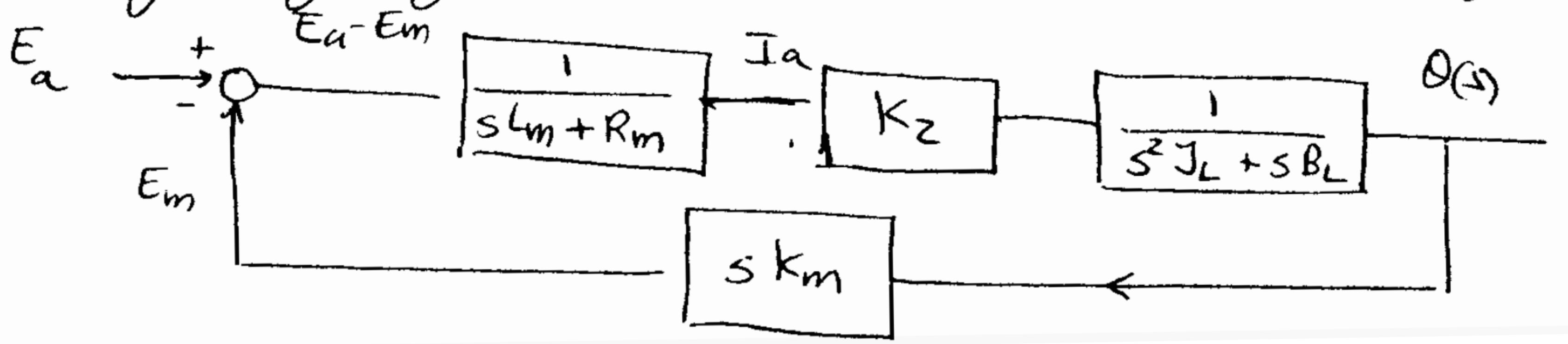
back emf

2) Back emf: $e_m = k_m \dot{\theta} \Leftrightarrow E_m(s) = s k_m \theta(s)$

3) Mechanical equation: $\tau = K_\tau i_a$

4) Newton's second equation: $J_L \frac{d \ddot{\theta}}{dt^2} + B_L \dot{\theta} = \tau \Leftrightarrow (s^2 J_L + s B_L)\, \theta = \tau(s)$

Putting everything together yields the following block diagram:



Some models that we will use later on:

Surprise! The system has built-in feedback (through the back emf).

- Q: How do we find the transfer function from $E_a(s)$ to $\theta(s)$?
- A: We could try solving the 4 simultaneous equations (messy) or applying Mason's formula to the loop above. The latter approach yields:

$$G(s) = \frac{G_1(s)}{1 + s k_m G_1(s)} \quad \text{where} \quad G_1(s) = \frac{K_2}{(sL_m + R_m)(s^2 J_L + s B_L)}$$

so in principle we get a third order system.

Some models that we will use later on:

<u>Common simplifying assumption</u>: neglect L_m $(sL_m \cong 0) \Rightarrow$

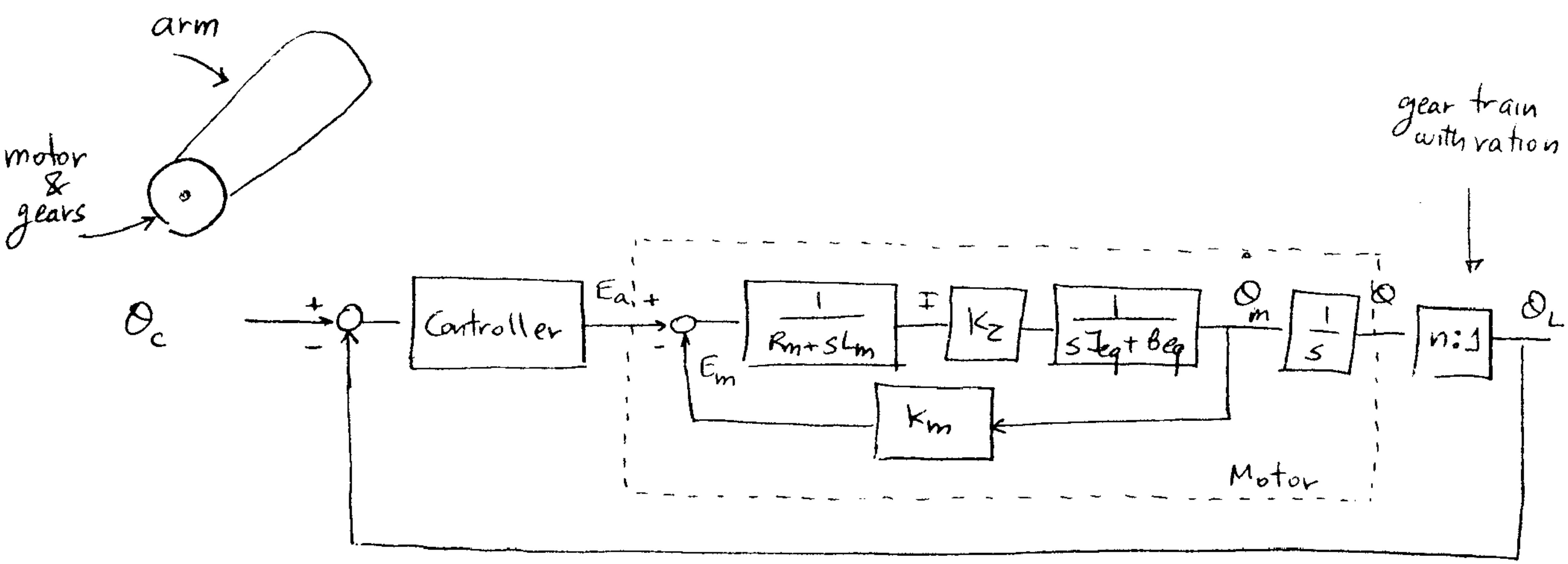
$$G(s) = \frac{\Theta(s)}{E_a(s)} = \frac{\dfrac{k_T}{R_m s\,(sJ_L + B_L)}}{1 + \dfrac{k_T k_m \cancel{s}}{R_m \cancel{s}\,(sJ_L + B_L)}} = \frac{k_T}{\cancel{R_m}\, s\,\cancel{(sJ_L + B_L)}} \cdot \frac{1}{\left(\dfrac{R_m(sJ_L+B_L) + k_T k_m}{\cancel{R_m}\,\cancel{(sJ_L+B_L)}}\right)}$$

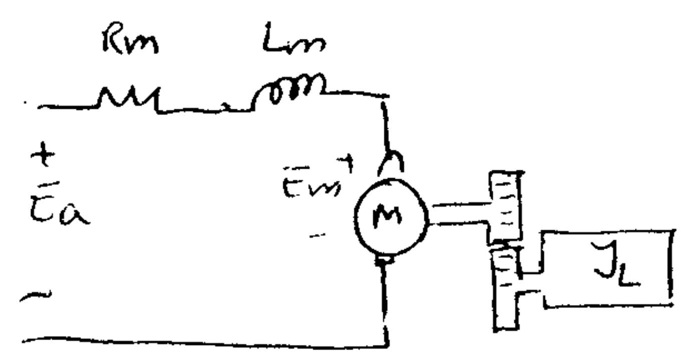
$$G(s) = \left(\frac{k_T}{R_m J_L}\right) \cdot \frac{1}{s\left(s + \dfrac{k_T k_m + B_L}{J_L R_m}\right)} = \boxed{\frac{K}{s\,(s+a)}}$$

(looks like the cascade of a pure integrator and a first order lag)

Some models that we will use later on:

- Example of use: Position control of a single link, single joint, rigid robotic arm. or of the robotic head in room



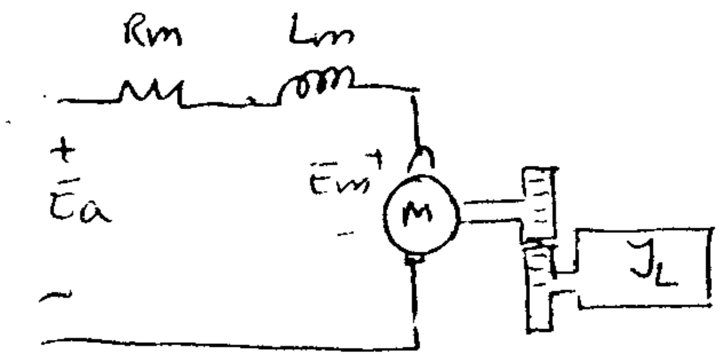


Here J_{eq} = DC motor inertia + (arm. inertia)$\cdot n^2$

$= J_m + J_{arm} \cdot n^2$

$B_{eq} = B_m + B_{arm} \cdot n^2$

Some models that we will use later on:



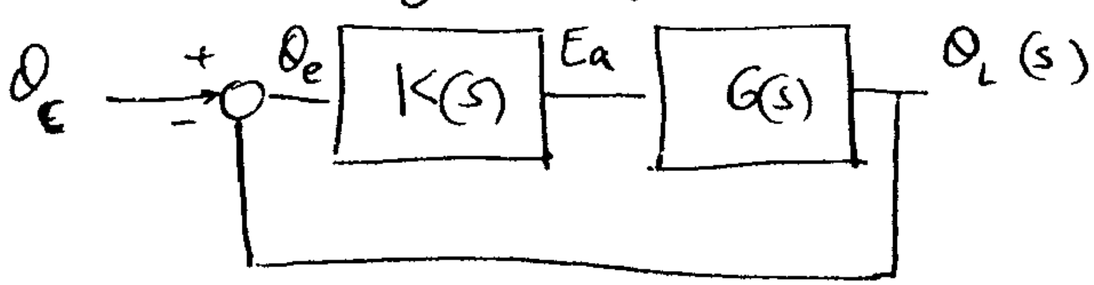
Here J_{eq} = DC motor inertia + (arm. inertia)$\cdot n^2$

$$= J_m + J_{arm} \cdot n^2$$

$$B_{eq} = B_m + B_{arm} \cdot n^2$$

Again, you get a third order system unless you neglect L_m

The block diagram of the closed-loop system is given by:



where K(s) is the transfer function of the controller and G(s) is the T. F. of the arm (including reduction gears)

To find the closed-loop transfer function $\frac{\theta_L}{\theta_c}$ we could, for instance write down the equations:

$$\theta_e = \theta_c - \theta_L, \qquad \theta_L = G(s)\,K(s)\,\theta_e$$

Some models that we will use later on:

Eliminating θ_e yields: $G \cdot K(\theta_c - \theta_L) = \theta_L$ // $GK\,\theta_c = (1+GK)\,\theta_L$

$$\Rightarrow \boxed{\frac{\theta_L}{\theta_c} = \frac{GK}{1+GK}}$$

This is a special case of Mason's formula:

Signal Flow Diagrams and Mason's Formula

They provide an alternative representation of Transfer Function relationships and an alternative (often simpler) to Cramer's rule or block diagram manipulations for computing T. F.

Signal Flow Diagrams and Mason's Formula

Rules:

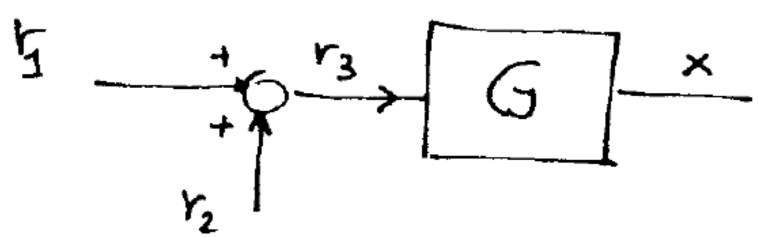
- Each signal is represented by a node
- Each transfer function is represented by a branch (arrow)

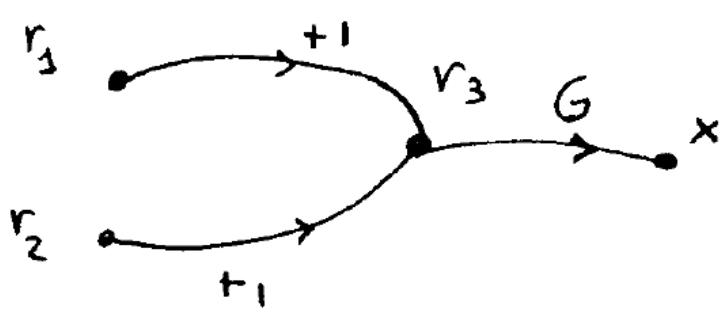
Block Diagram:

Signal flow:

$$x(s) = G(s)\, r(s)$$

- Summing junctions are represented implicitly: all the inputs converging to a node are added together:



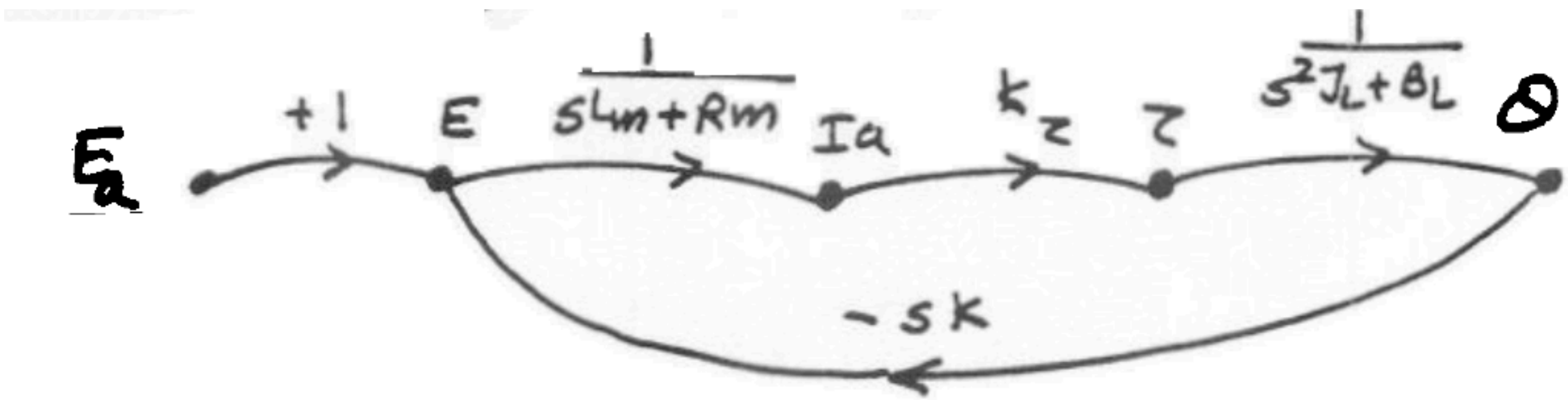


$$r_3 = (+1)\cdot r_1 + (+1)\cdot r_2$$

$$x = G\, r_3$$

Signal Flow Diagrams and Mason's Formula

Question: Find the signal flow graph representation of the DC motor:



Signal Flow Diagrams and Mason's Formula

Some Terminology:

source node: A node that has all signals flowing away from it.

sink node: A node with incoming signals only

Path: Continuous connection of branches between 2 nodes (directed)

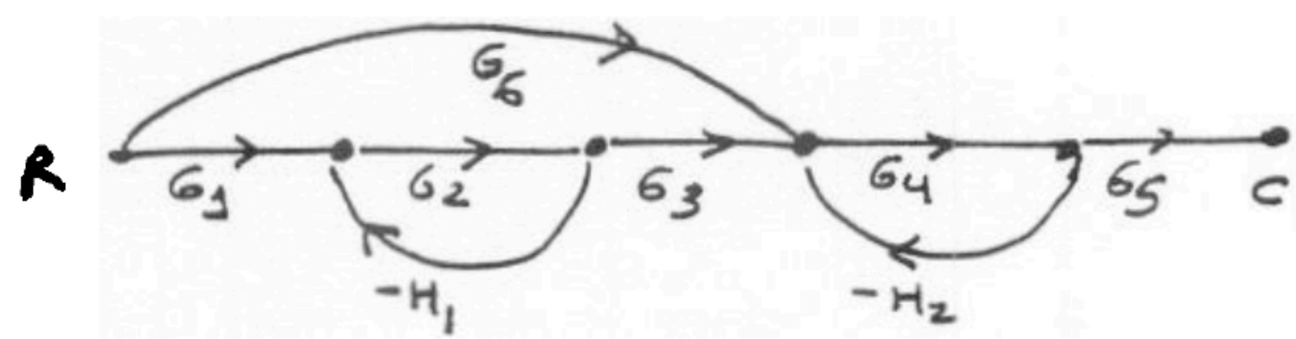
Loop: Closed path (i.e starting node = finishing node)

Path (loop) Gain: Product of all T.F. of all the branches in the path (loop)

Non Touching loops (paths): Loops that do not have any <u>nodes</u> in common.

Signal Flow Diagrams and Mason's Formula

Example:



2 loops: $-G_2 H_1$ (L_1)
$-G_4 H_2$ (L_2)

Path $G_6 G_4 G_5$ does not touch L_1
Path $G_1 G_2 G_3 G_4 G_5$ touches both L_1 and L_2

- Mason's Formula (section 2.4) Provides an alternative to Cramer's rule or elimination for finding Transfer Functions

$$T_{CR} = \frac{1}{\Delta} \sum_{k=1}^{P} M_k \Delta_k = \frac{1}{\Delta} \left(M_1 \Delta_1 + \quad + M_P \Delta_P \right)$$

Signal Flow Diagrams and Mason's Formula

- Mason's Formula (section 2.4) Provides an alternative to Cramer's rule or elimination for finding Transfer Functions

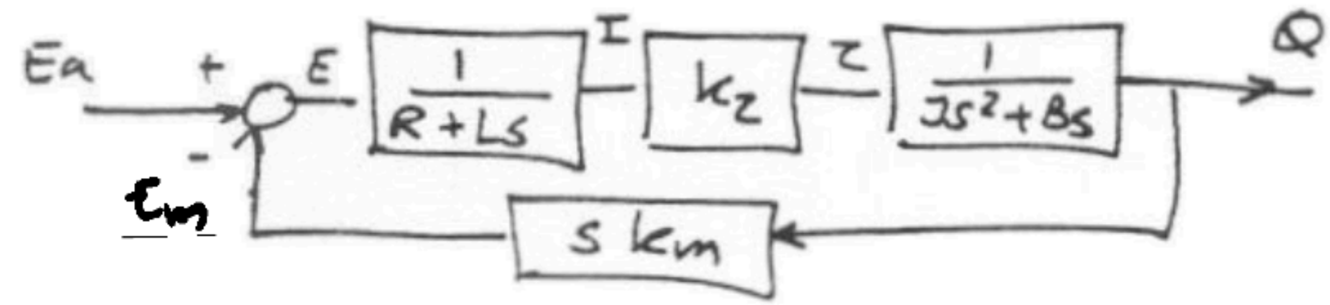
$$T_{CR} = \frac{1}{\Delta}\sum_{k=1}^{P} M_k \Delta_k = \frac{1}{\Delta}\left(M_1\Delta_1 + \dots + M_P\Delta_P\right)$$

Where:

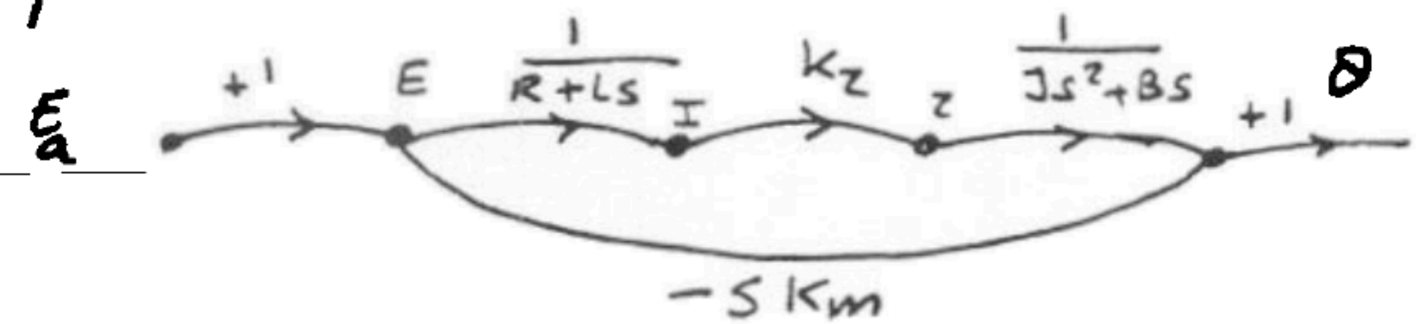
- $\Delta = 1 - \left(\Sigma \text{ gains individual loops}\right) + \Sigma\left(\text{products of non-touching pairs of loops}\right)$
  $- \Sigma\left(\text{products of triplets of non-touching loops}\right)$
  $+ \dots$

- M_k = Gain of the k^{th} path between R and C
- Δ_k = Value of Δ when the nodes in the path M_k are removed from the graph

Example 1 : DC motor:



Signal flow graph:



1 loop: $\quad L_1 = \dfrac{-k_z k_m s}{(R+Ls)(Js+B)s} \quad \Rightarrow \quad \Delta = 1 + \dfrac{k_z k_m s}{(R+Ls)(Js+B)s}$

Only 1 path from E_a to θ: $\quad M_1 = \dfrac{k_z}{(R+Ls)(Js+B)s}$

$\Delta_1 = 1$

$$T_{\theta E_a} = \frac{1}{\Delta} \cdot M_1 \Delta_1 =$$

$$= \frac{\dfrac{k_z}{(R+Ls)(Js+B)s}}{1 + \dfrac{k_z k_m s}{(R+Ls)(Js+B)s}} = \frac{k_z}{(R+Ls)(Js+B)s + k_m k_z s} \quad \#$$