- **Jury's stability test** (section 7.5 book)

Jury's test is similar to Routh Hurwitz in the sense that it counts the number of unstable roots of the (discrete time) char. equation.

You form an array using the coefficients of the polynomial, starting with two rows of length n. From these you compute a succesor row of length n-1, then another one of length n-2 and so on, until we get a row of length 1. Stability is related to the entries of the first column, as follows:

Assume that the characteristic polynomial is given by:

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots \quad a_1 z + a_0, \qquad a_n > 0$$

- <u>Step 1</u>: Form Jury's Array:

$$\begin{array}{cccccc}
z^0 & z^1 & z^2 & \cdots & z^{n-1} & z^n \\
 & & & & & \\
a_0 & a_1 & a_2 & & a_{n-1} & a_n \\
a_n & a_{n-1} & a_{n-2} & & a_1 & a_0 \\
 & & & & & \\
b_0 & b_1 & b_2 & & b_{n-1} & \\
b_{n-1} & b_{n-1} & & & b_0 & \\
 & & & & & \\
c_0 & c_1 & \cdots & & c_{n-2} & \\
c_{n-2} & c_{n-3} & & & c_0 & \\
\vdots & & & & & \\
\ell_0 & \ell_1 & \ell_2 & \ell_3 & & \\
\ell_3 & \ell_2 & \ell_1 & \ell_0 & & \\
 & & & & & \\
m_0 & m_1 & m_2 & & &
\end{array}$$

Remark: the elements of the <u>even numbered</u> rows are the elements of the preceeding row in reverse order

$$b_k = \begin{vmatrix} a_0 & a_{n-k} \\ a_n & a_k \end{vmatrix}$$

$$c_k = \begin{vmatrix} b_0 & b_{n-1-k} \\ b_{n-1} & b_k \end{vmatrix}; \qquad d_k = \begin{vmatrix} c_0 & c_{n-2-k} \\ c_{n-2} & c_k \end{vmatrix}, \ldots$$

$$\boxed{Q(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots a_1 z + a_0, \qquad a_n > 0}$$

Step 2 : Check the following conditions (necessary and sufficient for having all roots in $|z|<1$)

(a) $Q(1) > 0$

(b) $(-1)^n Q(-1) > 0$

(c) $|a_0| < a_n,$

$|b_0| > |b_{n-1}|$

$|c_0| > |c_{n-2}|$

$|d_0| > |d_{n-3}|$

$\vdots$

$|m_0| > |m_2|$

Remark: Check first $\varphi(1) > 0$, $(-1)^n \varphi(-1) > 0$, $a_n > |a_0|$

If any of these conditions fails the system is unstable and there is no need to proceed any further

Example 1: (Laser example with $T = 0.1$)

$$\varphi(z) = 1 + k\,G(z) = z^2 + (0.0085k - 1.5752)\,z + (0.0072k + 0.6065)$$

Conditions: $\varphi(1) = 1 + (0.0085k - 1.5752) + (0.0072k + 0.6065) = 0.0314 + 0.0157\,K > 0$

$$(-1)^2 \varphi(-1) = 3.1817 - 0.0013\,k > 0$$

$$a_2 > |a_0| \Rightarrow |0.0072k + 0.6065| < 1 \Leftrightarrow -1 < 0.0072k + 0.6065 < 1$$

From these conditions we have:

$$k > -2$$
$$k < 2.4475 \cdot 10^3$$
$$k < 54.713$$
$$k > -223.39$$

$$\Rightarrow \boxed{-2 < k < 54.713}$$

same conditions as before

Example 2 (a robust stability example)

Consider the following second order system:

$$P(z) = z^2 + \alpha z + \beta \qquad \text{where } \alpha \ \& \ \beta \text{ are parameters}$$

The Jury array is:

z^0	z^1	z^2
β	α	1
1	α	β
β^2-1	$\alpha(\beta-1)$	

conditions:

$$P(1) > 0 \Rightarrow \alpha + \beta + 1 > 0$$

$$(-1)^2 P(-1) > 0 \Rightarrow \beta + 1 - \alpha > 0$$

$$|a_0| < a_n \Rightarrow |\beta| < 1 \Leftrightarrow -1 < \beta < 1$$

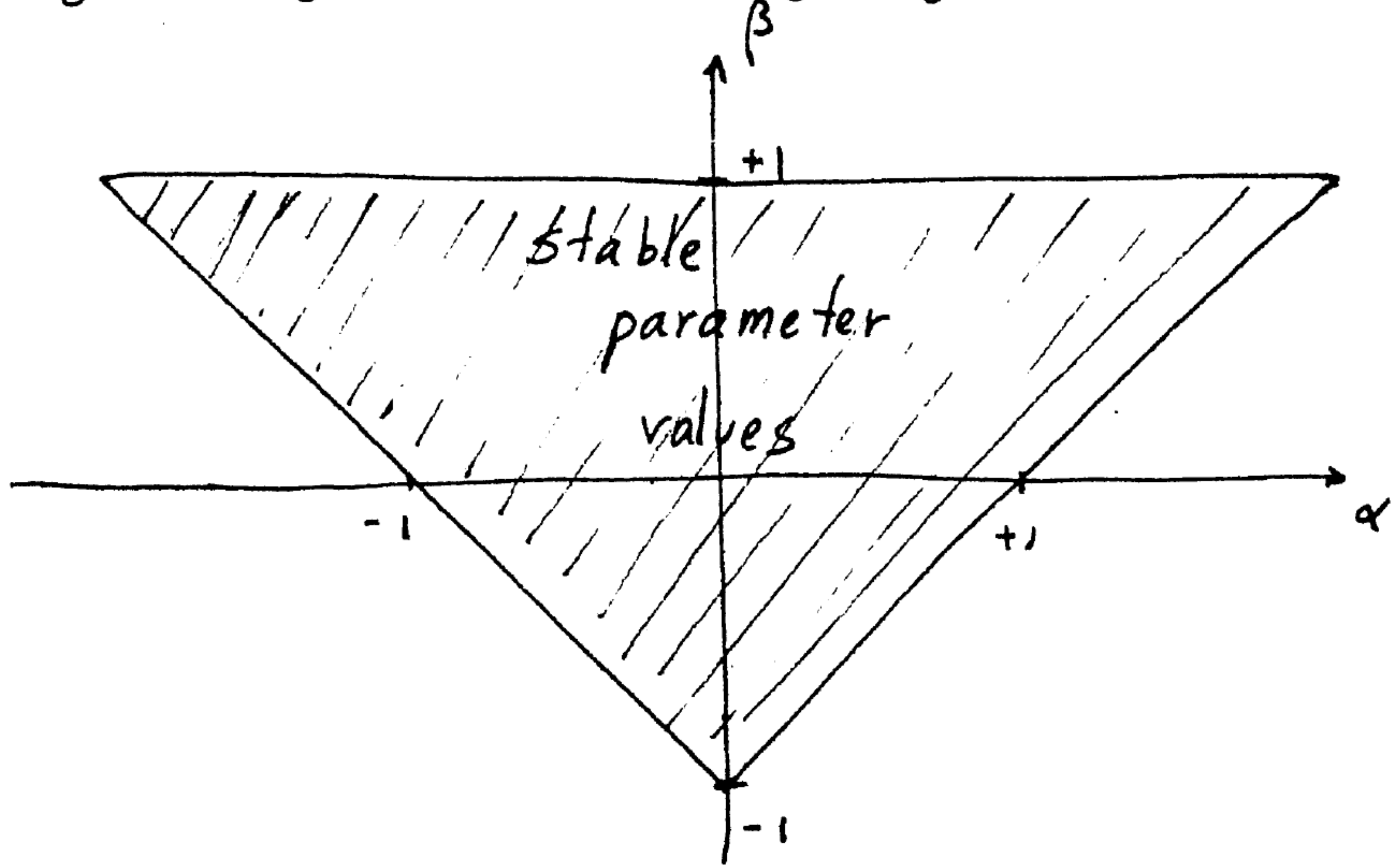
$$|b_0| > |b_{n-1}| \Rightarrow |\beta^2 - 1| > |\alpha(\beta-1)|$$

$$|\cancel{(\beta-1)}(\beta+1)| > |\alpha| \cdot |\cancel{\beta-1}| \quad // \quad |(\beta+1)| > |\alpha|$$

$$\text{or} \quad 1+\beta > \alpha$$

$$1+\beta > -\alpha$$

Graphically we get the following region:



EECE 5610 Digital Control Systems

Lecture 18

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Learning outcomes

By the end of *this* lecture, you should be able to:

- Understand the different notions and forms of stability
- Analyze the stability properties of discrete dynamical systems using:
  - Jury's stability criterion,
  - Nyquist stability criterion, and
  - Lyapunov's stability theory

Updated

- HW#**6** is due **on this Monday**
- Midterm#1 grades are posted
- Midterm#2 grades **are posted**
- Final Exam will be on December **9**

What do we mean by "stability"?

- There are **different notions of stability**:
  - Stability of a particular solution (non-linear and/or time-varying systems)
  - System stability (global property of linear systems)
  - Global stability *vs.* local stability (non-linear systems)

- Also, there are **different forms of stability**: suppose $\boldsymbol{x}_1[k]$ and $\boldsymbol{x}_2[k]$ be solutions to a system with initial conditions $\boldsymbol{x}_1[k_0]$ and $\boldsymbol{x}_2[k_0]$, respectively.
  - **(General) Stability** of a particular solution (non-linear and/or time-varying systems): The solution $\boldsymbol{x}_1[k]$ is stable if for a given $\varepsilon>0$, there exists $\delta(\varepsilon,k_0)>0$, such that:

$$\|\mathbf{x}_2[k_0] - \mathbf{x}_1[k_0]\| < \delta \Rightarrow \|\mathbf{x}_2[k] - \mathbf{x}_1[k]\| < \varepsilon, \quad \forall k \geq k_0$$

  - **Asymptotic stability**: The solution $\boldsymbol{x}_1[k]$ is asymptotically stable if it is stable and if δ can be chosen such that:

$$\|\mathbf{x}_2[k_0] - \mathbf{x}_1[k_0]\| < \delta \Rightarrow \|\mathbf{x}_2[k] - \mathbf{x}_1[k]\| \to 0, \quad \text{as } k \to \infty$$

  - **Bounded Input–Bounded Output (BIBO) stability criterion:** A system is stable if, for finite input, the output is also finite.

Stability of discrete-time LTI systems

- Consider the following discrete-time LTI system:

$$\mathbf{x}[k+1] = \Phi\mathbf{x}[k], \quad \mathbf{x}[0] = \alpha$$

- To investigate the stability of the system above, its initial value is perturbed:

$$\mathbf{x}_0[k+1] = \Phi\mathbf{x}_0[k], \quad \mathbf{x}_0[0] = \alpha_0$$

- Then, the difference $\tilde{\mathbf{x}} = \mathbf{x} - \mathbf{x}_0$ satisfies:

$$\begin{aligned}\tilde{\mathbf{x}}[k+1] &= \mathbf{x}[k+1] - \mathbf{x}_0[k+1] \\ &= \Phi\mathbf{x}[k] - \Phi\mathbf{x}_0[k] \\ &= \Phi\tilde{\mathbf{x}}[k], \quad \tilde{\mathbf{x}}[0] = \alpha - \alpha_0\end{aligned}$$

- This implies that if the solution $\boldsymbol{x}$ is stable, then every other solution is also stable. **For LTI systems, stability is a property of the system and not of a special solution.**

Stability of discrete-time LTI systems

- The solution to the discrete-time LTI system:

$$\mathbf{x}[k+1] = \Phi \mathbf{x}[k], \quad \mathbf{x}[0] = \alpha$$

  is given by

$$\mathbf{x}[k+1] = \Phi \mathbf{x}[k] = \Phi \left(\Phi \mathbf{x}[k-1] \right) = \ldots = \Phi^k \mathbf{x}[0]$$

- **If $\boldsymbol{\Phi}$ is diagonalizable**: $\mathbf{x}[k+1] = \Phi^k \mathbf{x}[0] = V \Lambda^k V^{-1} \mathbf{x}[0]$

- We define $\mathbf{y}[k] = V^{-1} \mathbf{x}[k]$ and hence

$$\mathbf{y}[k+1] = \Lambda^k \mathbf{y}[0], \quad \mathbf{y}[0] = V^{-1} \mathbf{x}[0] = V^{-1} \alpha$$

- **If $\boldsymbol{\Phi}$ is NOT diagonalizable**: the solution is instead a linear combination of the terms

$$p_i[k] \lambda_i^k$$

  where $p_i[k]$ are polynomials in k of order one less than the multiplicity of the corresponding eigenvalue

Stability of discrete-time LTI systems

- To get asymptotic stability, *all solutions* must go to *0* as k increases to infinity

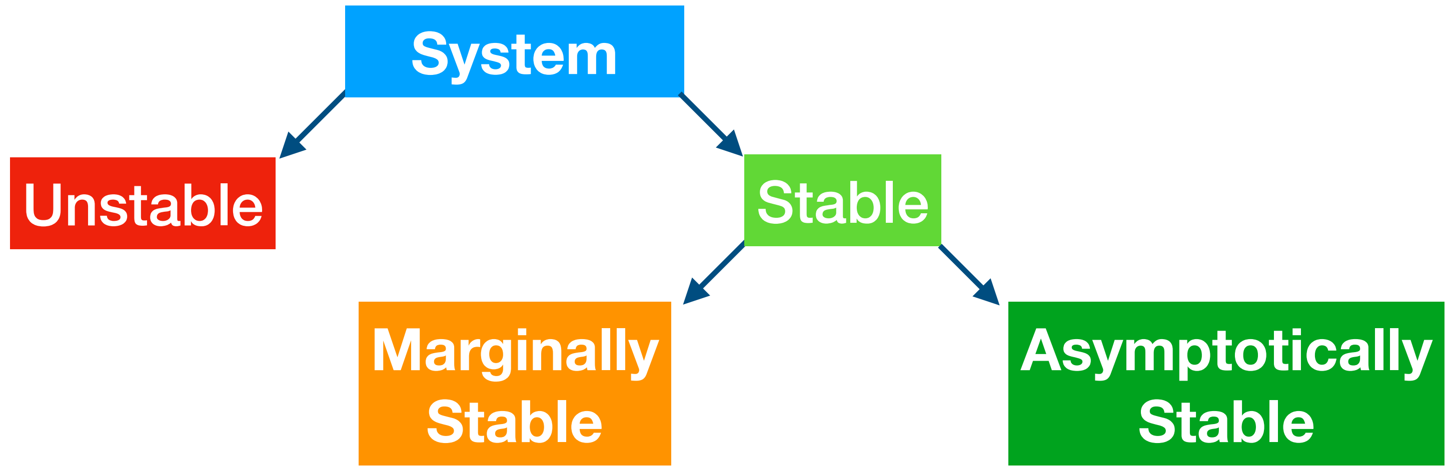
Asymptotic stability theorem: The discrete-time LTI system

$$\mathbf{x}[k+1] = \Phi\mathbf{x}[k], \quad \mathbf{x}[0] = \alpha$$

is asymptotically stable if and only if all eigenvalues of Φ are strictly inside the unit disk, i.e.,

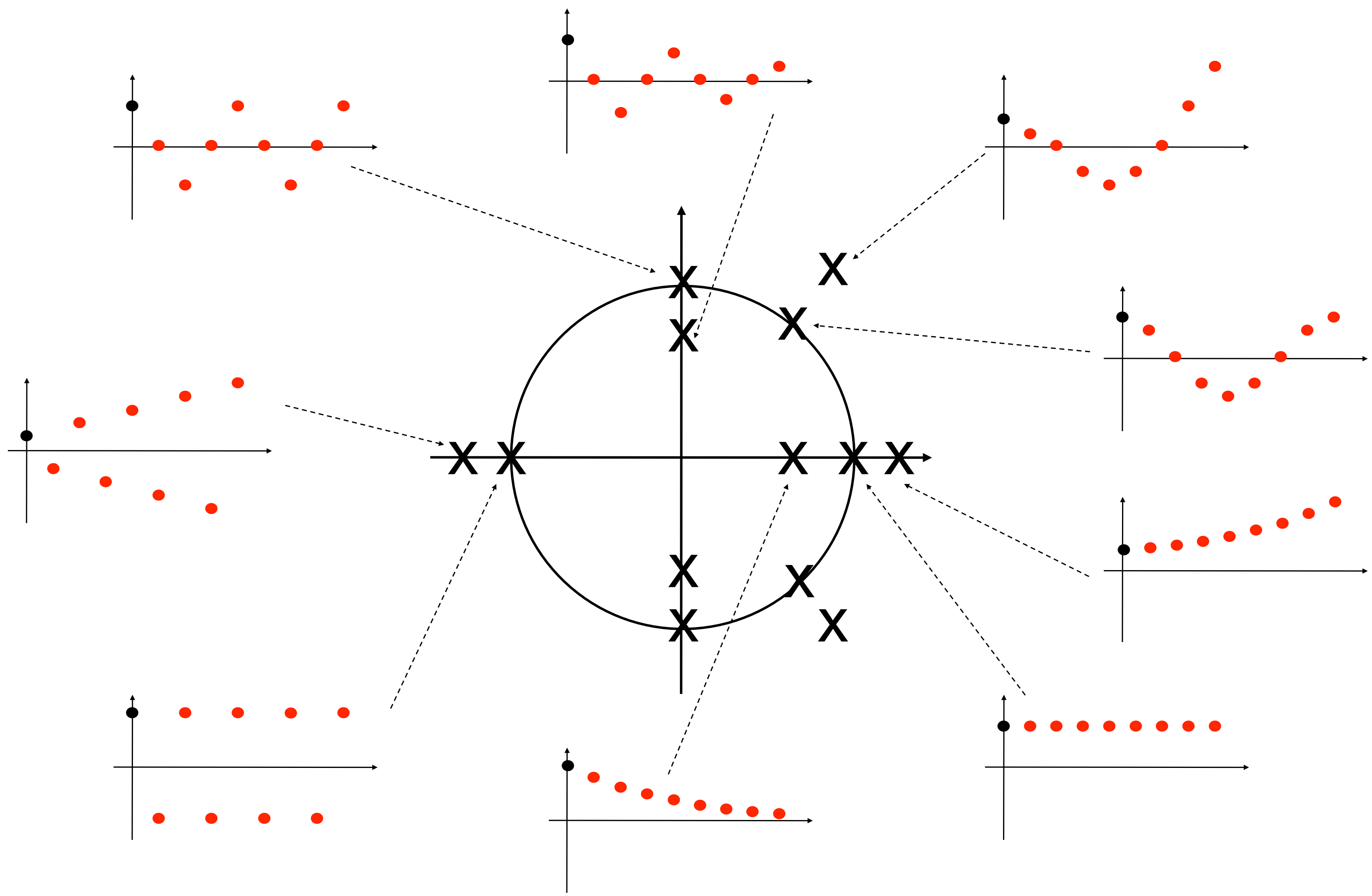
$$|\lambda_i| < 1, \quad i = 1, 2, \ldots, n$$

- If Φ has *unique* eigenvalues on the boundary of the unit circle with all other eigenvalues being located inside → steady-state output will perform oscillations of finite amplitude - system is **marginally stable**



Stability of discrete-time LTI systems

- Distance from origin is a measure of decay rate
- Complex poles just inside unit circle give lightly-damped oscillation. Oscillation is possible for real poles on negative real axis



Time responses as a function of the poles location

Stability tests

- The stability analysis of digital control systems is similar to the stability analysis of analog systems and all the known methods can be applied to digital control systems with some modifications

- There are good numerical algorithms to compute the eigenvalues; in MATLAB:

```
>> eig(A)
```

- Also important to have *algebraic or graphical* methods for investigating stability. The most prevalent methods are:
  - Direct numerical or algebraic computation of the eigenvalues of Φ
  - Methods based on properties of characteristic polynomials (e.g., **Jury's criterion**)
  - The root locus method
  - **The Nyquist stability criterion**
  - The Bode stability criterion
  - **Lyapunov's method**

Jury's stability criterion

- Explicit calculation of the eigenvalues of a matrix cannot be done conveniently by hand for systems of order higher than 2

- In some cases it is easy to calculate the characteristic equation

$$\chi(z) \triangleq |zI - \Phi| = a_0 z^n + a_1 z^{n-1} + \ldots + a_{n-1} z + a_n = 0$$

  and investigate its roots

- It is the discrete time analogue of the Routh–Hurwitz stability criterion:
  - The Jury stability criterion requires that the system poles are located inside the unit circle centered at the origin
  - The Routh-Hurwitz stability criterion requires that the poles are in the left half of the complex plane

Jury's stability criterion

- The following test is useful for determining if the characteristic equation has all its roots inside the unit disc

1st and 2nd rows are the coefficients of $\chi(z)$ in forward and reverse order, respectively

3rd row obtained by multiplying the second row by b_n and subtracting this from the 1st

4th row is the third row in reverse order

$$
\begin{array}{ccccc}
a_0 & a_1 & \cdots & a_{n-1} & a_n \\
a_n & a_{n-1} & \cdots & a_1 & a_0 \\
\hline
a_0^{n-1} & a_1^{n-1} & \cdots & a_{n-1}^{n-1} & \\
a_{n-1}^{n-1} & a_{n-1}^{n-2} & \cdots & a_0^{n-1} & \\
\hline
\vdots & & & & \\
a_0^0 & & & &
\end{array}
$$

$$b_n = \frac{a_n}{a_0}$$

$$b_{n-1} = \frac{a_{n-1}^{n-1}}{a_0^{n-1}}$$

⋮

The scheme is then repeated until there are 2n +1 rows, which includes a single term

where

$$a_i^{k-1} = a_i^k - b_k a_{k-i}^k$$

$$b_k = \frac{a_k^k}{a_0^k}$$

Jury's stability criterion

Jury's stability test: If $a_0 > 0$ then the characteristic polynomial *χ(z)* has all its roots inside the unit circle if and only if

$$a_0^k > 0, \quad k = 0, 1, 2, \ldots, n-1$$

If no $a_0^k = 0$, then the number of negative a_0^k is equal to the roots of *χ(z)* outside the unit circle.

- **Remark:** If $a_0^k > 0, k = 0, 1, 2, \ldots, n-1$, then the condition $a_0^0 > 0$ can be shown to be equivalent to the conditions

$$\begin{aligned} \chi(1) &> 0 \\ (-1)^n \chi(-1) &> 0 \end{aligned}$$

These conditions constitute necessary conditions for stability and hence can be used before forming the table.

In-class exercise

- Consider the system with the following characteristic equation:

$$\chi(z) = 3z^2 + 2z + 1$$

Is the system stable?

Solution:

$$\begin{array}{ccc|ccc|l}
a_0 & a_1 & a_2 & 3 & 2 & 1 & \\
a_2 & a_1 & a_0 & 1 & 2 & 3 & b_2 = \frac{1}{3} \\
\hline
a_0^1 & a_1^1 & & (3 - 1 \times \frac{1}{3} =)\ \frac{8}{3} & (2 - 2 \times \frac{1}{3} =)\ \frac{4}{3} & & \\
a_1^1 & a_0^1 & & \frac{4}{3} & \frac{8}{3} & & b_1 = \frac{4/3}{8/3} = \frac{1}{2} \\
\hline
a_0^0 & & & (\frac{8}{3} - \frac{1}{2} \times \frac{4}{3} =)\ 2 & & &
\end{array}$$

The system is stable: $a_0^0 > 0, \quad a_0^1 > 0, \quad a_0 > 0$

In fact, the roots can easily be solved directly: $-0.3333 \pm 0.4714j$

Example using symbolic calculations

- The Jury criterion can easily be used in the case of a small system, when a computer is not at hand.

- The real power of the Jury criterion comes into action in *symbolic calculations*. Stability can be determined as a function of one or even more parameters.

- For example, consider the system with the following characteristic equation:

$$\chi(z) = z^2 + a_1 z + a_2$$

Example using symbolic calculations

- The Jury criterion can easily be used in the case of a small system, when a computer is not at hand.

- The real power of the Jury criterion comes into action in *symbolic calculations*. Stability can be determined as a function of one or even more parameters.

- For example, consider the system with the following characteristic equation:

$$\chi(z) = z^2 + a_1 z + a_2$$

$$\begin{array}{ccc|ccc|l}
a_0 & a_1 & a_2 & 1 & a_1 & a_2 & \\
a_2 & a_1 & a_0 & a_2 & a_1 & 1 & b_2 = \frac{a_2}{1} = a_2 \\
\hline
a_0^1 & a_1^1 & & 1-(a_2)^2 & a_1(1-a_2) & & \\
a_1^1 & a_0^1 & & a_1(1-a_2) & 1-(a_2)^2 & & b_1 = \frac{a_1(1-a_2)}{1-(a_2)^2} = \frac{a_1}{1+a_2} \\
\hline
a_0^0 & & & 1-(a_2)^2 - \frac{(a_1)^2(1-a_2)}{1+a_2} & & &
\end{array}$$

Example using symbolic calculations

- The system is stable if

$$\begin{cases} 1-(a_2)^2>0 \\ 1-(a_2)^2-\frac{(a_1)^2(1-a_2)}{1+a_2}>0 \end{cases}$$

- Factoring differences of squares in the first inequality

$$1-(a_2)^2=(1-a_2)(1+a_2)>0$$

- Doing algebraic manipulations to the second inequality

$$\begin{aligned} 1-(a_2)^2-\frac{(a_1)^2(1-a_2)}{1+a_2} &= \frac{(1-a_2)(1+a_2)^2-(a_1)^2(1-a_2)}{1+a_2} \\ &= \frac{(1-a_2)}{1+a_2}[(1+a_2)^2-(a_1)^2] \\ &= \frac{(1-a_2)(1+a_2-a_1)(1+a_2+a_1)}{1+a_2}>0 \end{aligned}$$

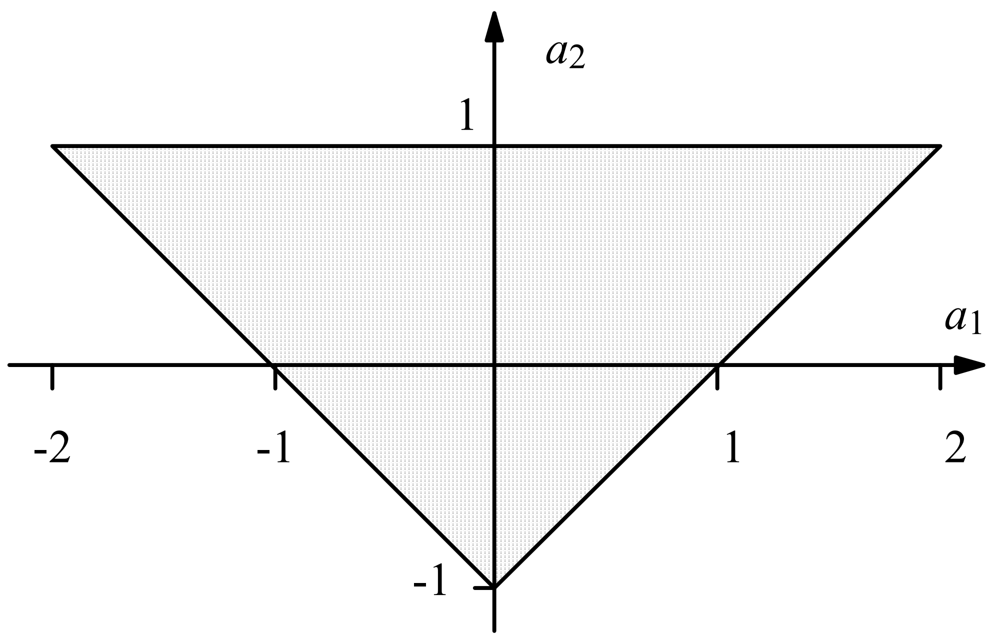
Example using symbolic calculations

- The set of inequalities can be simplified to (triangle rule):

$$\left.\begin{aligned} &(1-a_2)(1+a_2)>0 \\ &\frac{(1-a_2)(1+a_2-a_1)(1+a_2+a_1)}{1+a_2}>0 \end{aligned}\right\} \Rightarrow \begin{cases} (1-a_2)(1+a_2)>0 \\ (1+a_2-a_1)(1+a_2+a_1)>0 \end{cases}$$

$$\Rightarrow \begin{cases} -1<a_2<1 \\ a_1-1<a_2 \\ -a_1-1<a_2 \end{cases} \quad \text{OR} \quad \begin{cases} -1<a_2<1 \\ a_1-1>a_2 \\ -a_1-1>a_2 \end{cases}$$

infeasible



Stability in the frequency domain

- The frequency response of $G(s)$ is: $G(j\omega), \omega \in [0, \infty)$. It can be graphically presented:
  - in the complex plane as the **Nyquist curve**, or,
  - as amplitude/phase curves as a function of frequency - **Bode diagram**

- Correspondingly, for a discrete-time system $H(z)$ the frequency response is $H(e^{j\omega})$, where $\omega h \in [0, \pi)$. This can also be presented graphically as a **discrete-time Nyquist curve** or **discrete-time Bode diagram**

- Note that it is sufficient to consider the map in the interval $\omega h \in [-\pi, \pi)$ because the function $H(e^{j\omega})$ is periodic with period $2\pi/h$

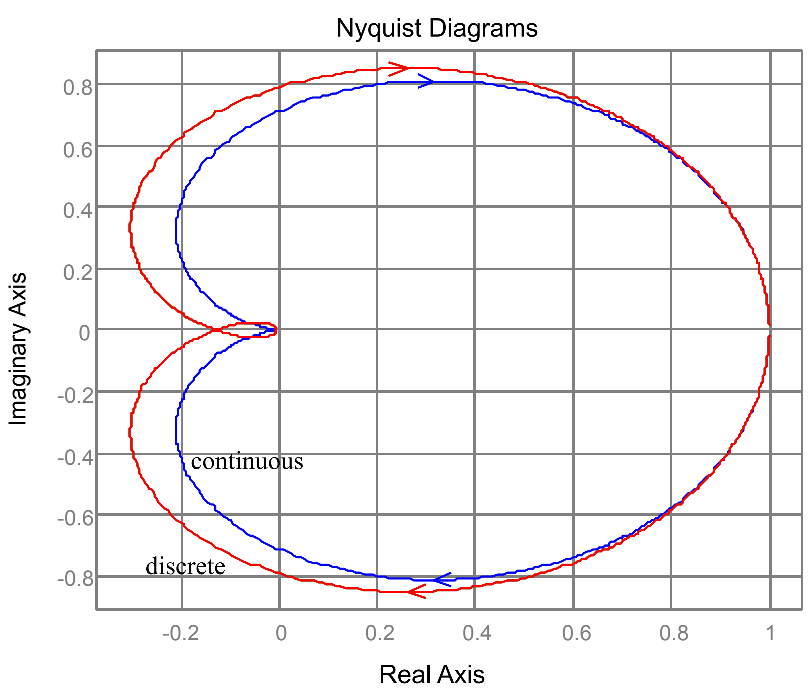
Stability in the frequency domain

- Consider, for example, the continuous-time system with transfer function:

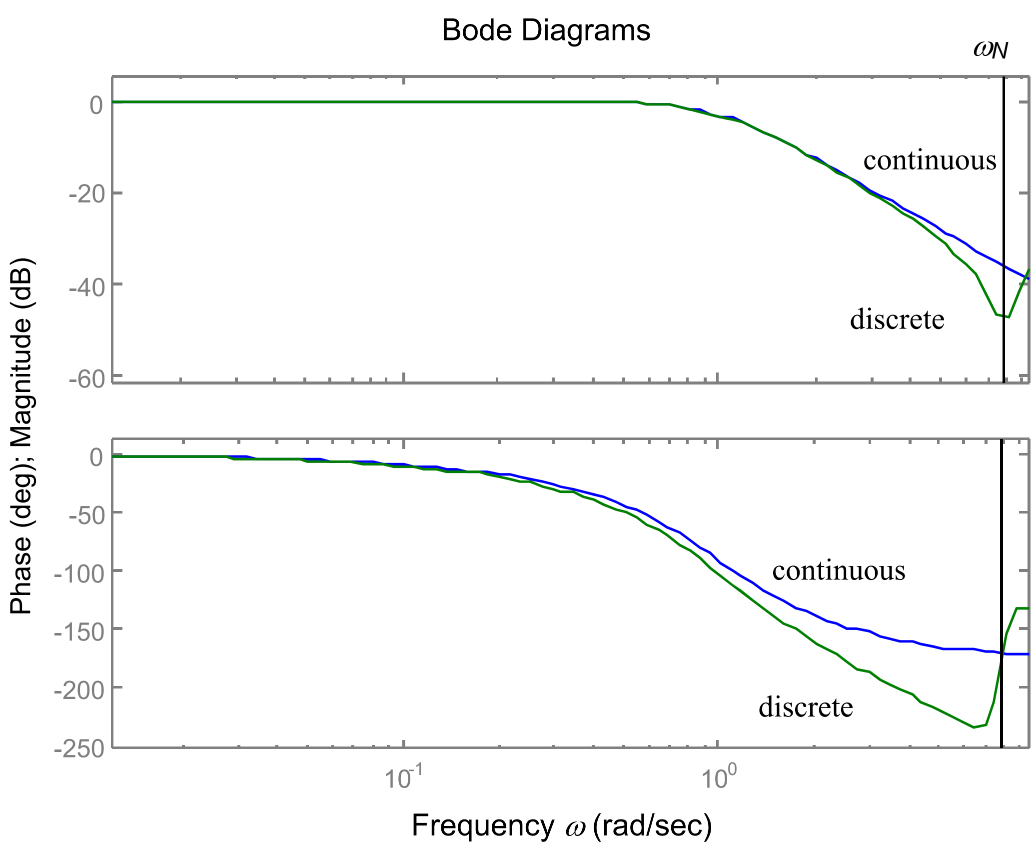
$$G(s) = \frac{1}{s^2 + 1.4s + 1}$$

- Zero-order-hold sampling of the system with h=*0.4* gives the discrete-time system

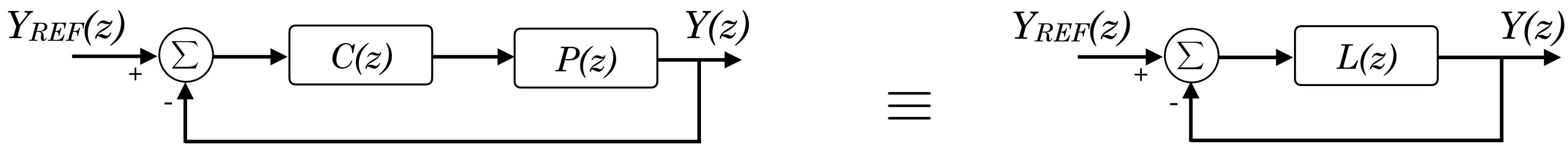
$$G(z) = \frac{0.066z + 0.055}{z^2 - 1.45z + 0.571}$$



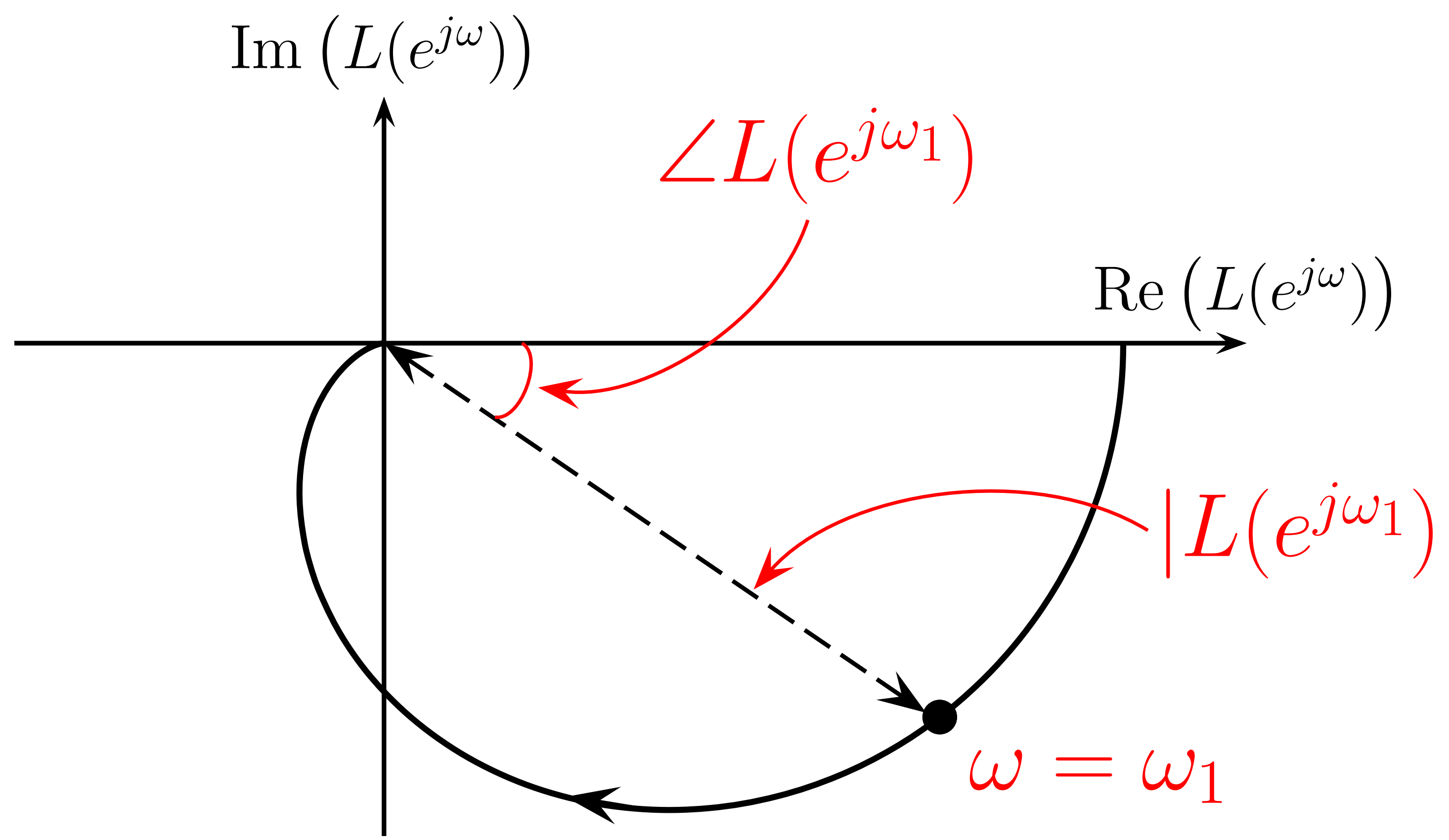

```
>> sys = tf(1,[1 1.4 1]);
>> sysd = c2d(sys,0.4,'zoh');
>> w=logspace(2,1,1000)';

>> figure(1); hold on;
>> nyquist(sys,w)
>> nyquist(sysd,w)

>> figure(2); hold on;
>> bode(sys,w)
>> bode(sysd,w)
```



Nyquist diagram



- **Nyquist diagram:** A plot of the *open-loop* frequency response of $L(z)$, with the imaginary part $Im(L(e^{j\omega}))$ plotted against the real part $Re(L(e^{j\omega}))$ on an Argand diagram (complex plane).



How to find $L(e^{j\omega})$

- As we have already seen, rational z-transform can be written as

$$L(z) = c\frac{(z-\mu_1)(z-\mu_2)\dots(z-\mu_M)}{(z-p_1)(z-p_2)\dots(z-p_N)}$$

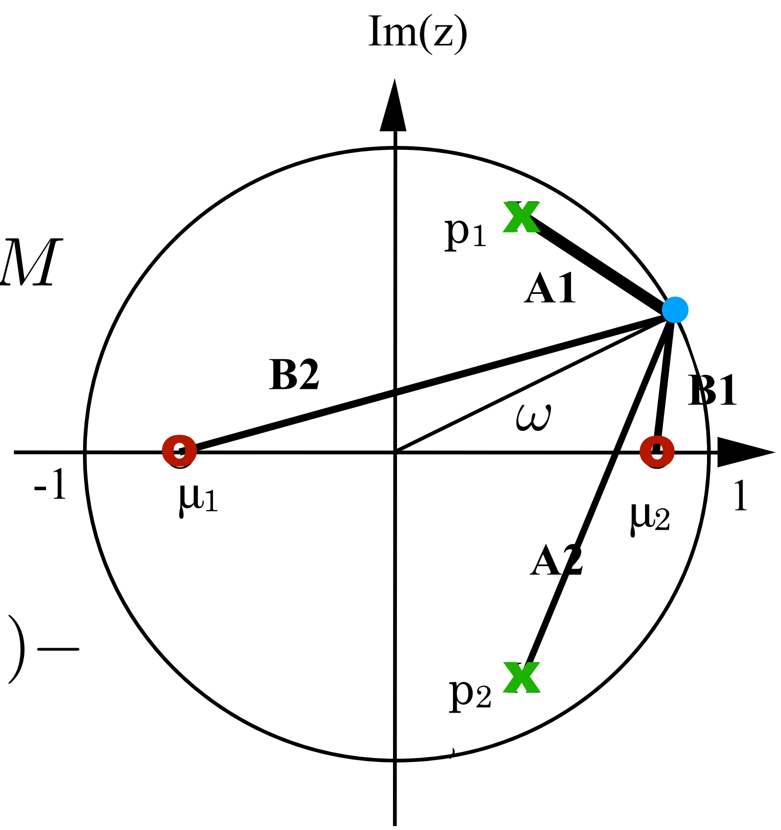
- To compute the frequency response of $L(e^{j\omega})$, we compute $L(z)$ at $z = e^{j\omega}$

- But $z = e^{j\omega}$ represents a point on the circle's perimeter

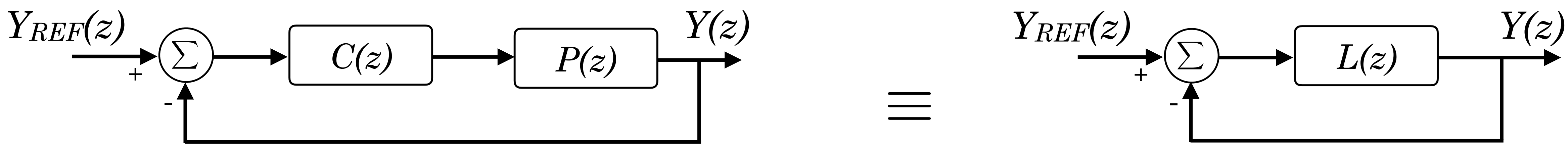
- Let $A_i = |e^{j\omega} - p_i|$ and $B_i = |e^{j\omega} - \mu_i|$

$$|L(e^{j\omega})| = |c|\frac{|e^{j\omega}-\mu_1|\ |e^{j\omega}-\mu_2|\dots|e^{j\omega}-\mu_M|}{|e^{j\omega}-p_1|\ |e^{j\omega}-p_2|\dots|e^{j\omega}-p_N|}, \qquad N \geq M$$
$$= |c|\frac{B_1B_2\dots B_M}{A_1A_2\dots A_N}$$

$$\angle L(e^{j\omega}) = \angle(e^{j\omega}-\mu_1) + \angle(e^{j\omega}-\mu_2) + \dots + \angle(e^{j\omega}-\mu_M) -$$
$$- \angle(e^{j\omega}-p_1) - \angle(e^{j\omega}-p_2) - \dots - \angle(e^{j\omega}-p_N)$$



Discrete-time Nyquist stability criterion



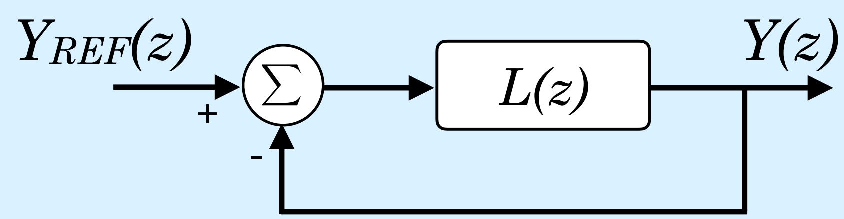
- **Definition (asymptotic stability of a feedback system)**:
  We say that the closed-loop system is asymptotically stable if the closed-loop transfer function $L(z)/(1 + L(z))$ is asymptotically stable.

- Closed-loop poles $\equiv$ poles of $L(z)/(1 + L(z))$ $\equiv$ zeros of $1 + L(z) = 0$

- This corresponds to all the roots lying in the unit circle, i.e., if the characteristic equation has zeros outside the unit circle, the closed loop system is unstable

Discrete-time Nyquist stability criterion

Discrete-time Nyquist stability criterion: The closed-loop system



will be stable if (and only if) the number of clockwise encirclements N of the point -1 by $L(e^{j\omega})$ as ω increases from 0 to 2π is equal to

$$N = Z - P$$

where

Z : # of ***zeros*** of the characteristic equation $1 + L(z)=0$ outside the unit circle

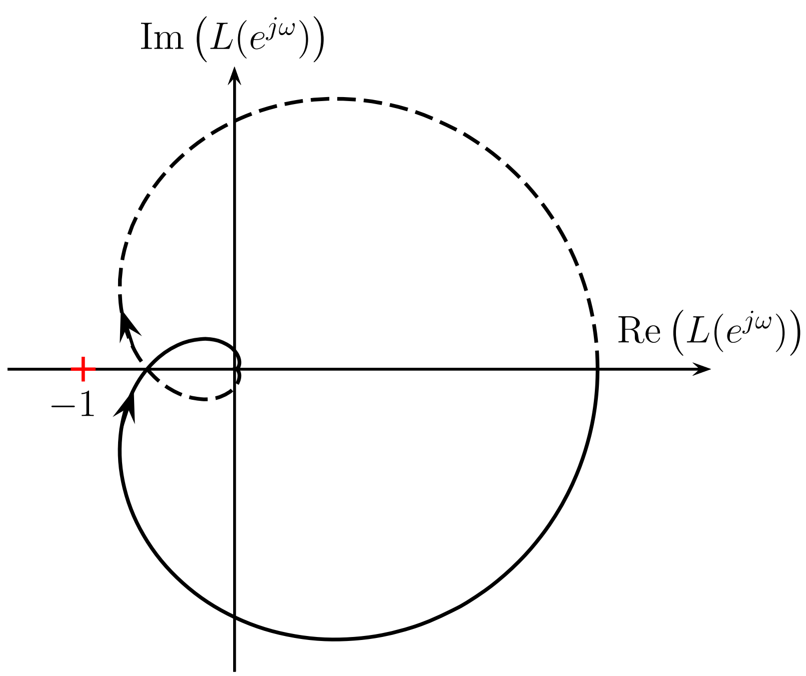
P : # of ***poles*** of the characteristic equation $1 + L(z)=0$ outside the unit circle

- The open loop poles are the same as the poles of the characteristic equation

- The zeros of the characteristic equation determine the stability of the system so that **if the characteristic equation has zeros outside the unit circle, then the closed loop system is unstable.** The stability criterion is thus obtained by setting $Z = 0$ and by demanding that the Nyquist curve encircles the point -1 P times counterclockwise

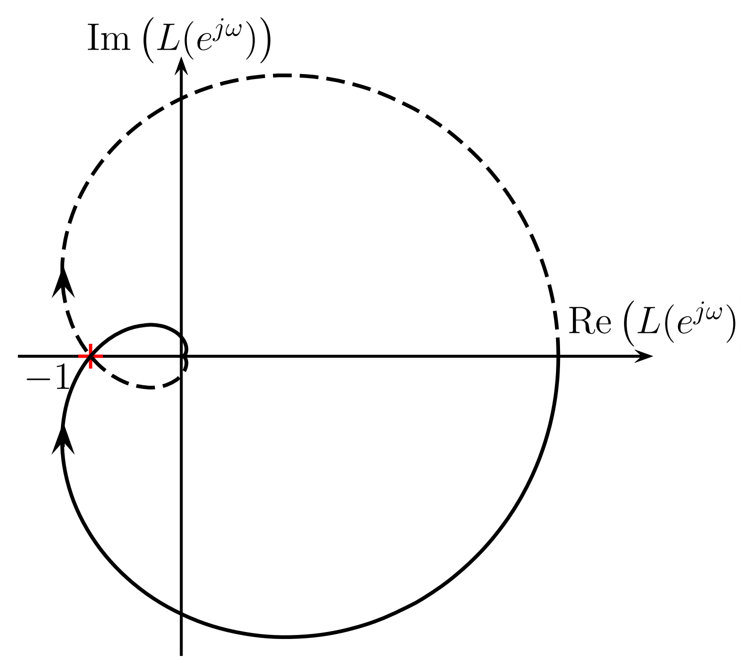
Discrete-time Nyquist stability criterion

- The criterion becomes simple, if the open loop pulse transfer function $L(z)$ has no poles outside the unit circle. Then, the Nyquist curve must not encircle the point -1 at all.



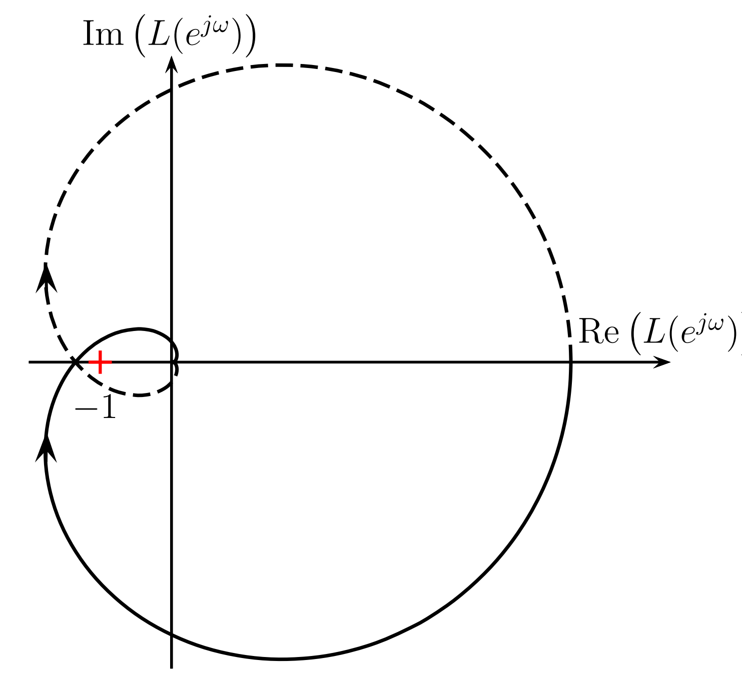
$$\Rightarrow \frac{L(z)}{1+L(z)}$$

Asymptotically Stable



$$\Rightarrow \frac{L(z)}{1+L(z)}$$

Marginally Stable

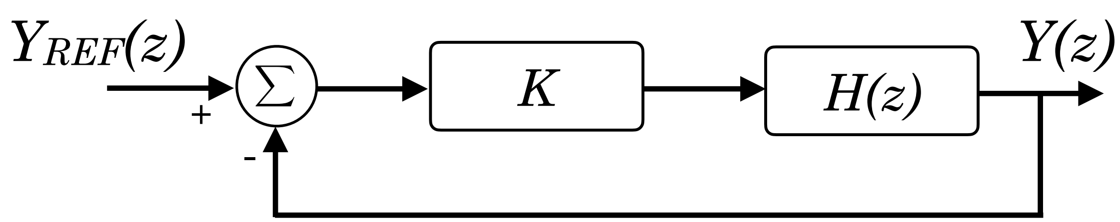


$$\Rightarrow \frac{L(z)}{1+L(z)}$$

Unstable

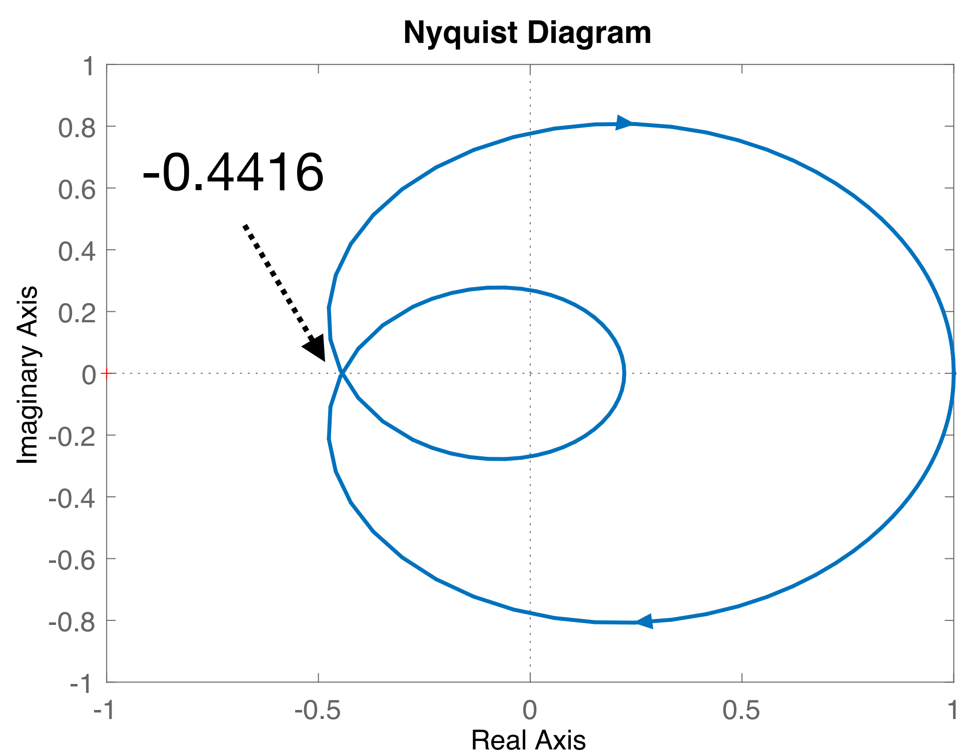
Example

- A discrete process ($h=1$) is controlled with a proportional controller, which has gain K, as shown below



$$H(z) = \frac{0.4}{(z-0.5)(z-0.2)}$$

- The discrete Nyquist diagram is constructed with MATLAB



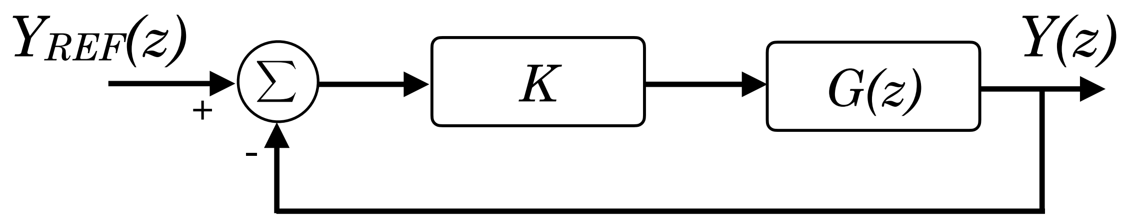

```
>> sysd=zpk([ ],[0.2 0.5],0.4,1);
>> nyquist(sysd)
```

- By zooming in at the intersection wit the real axis, the point is approximately -0.4416
- The magnitude can thus be multiplied with $(1/0.4416)$ to reach the critical point -1
- The controlled system is stable when

$$K < \frac{1}{0.4416} \approx 2.26$$

Example

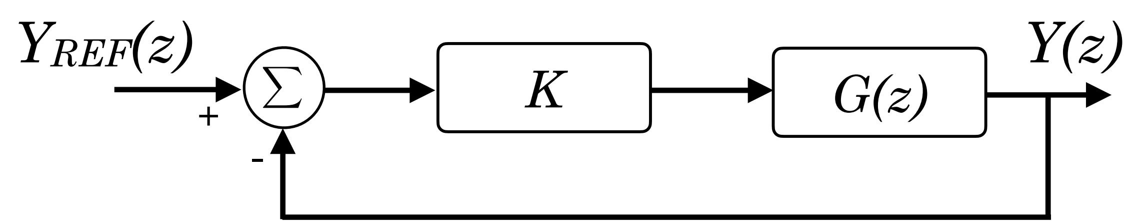
- Find the values of K for which the following system



is stable for $G(z) = \dfrac{1}{z-2}$.

Example

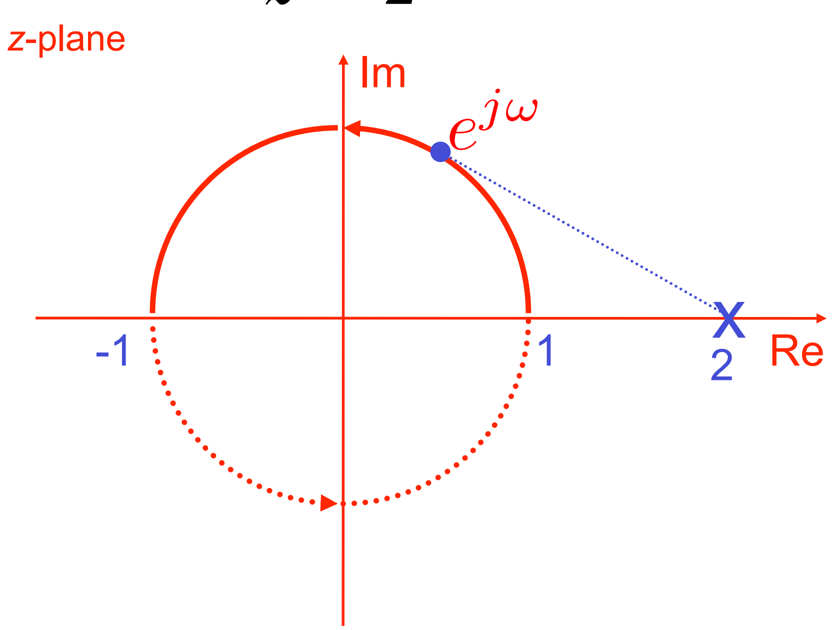
- Find the values of K for which the following system

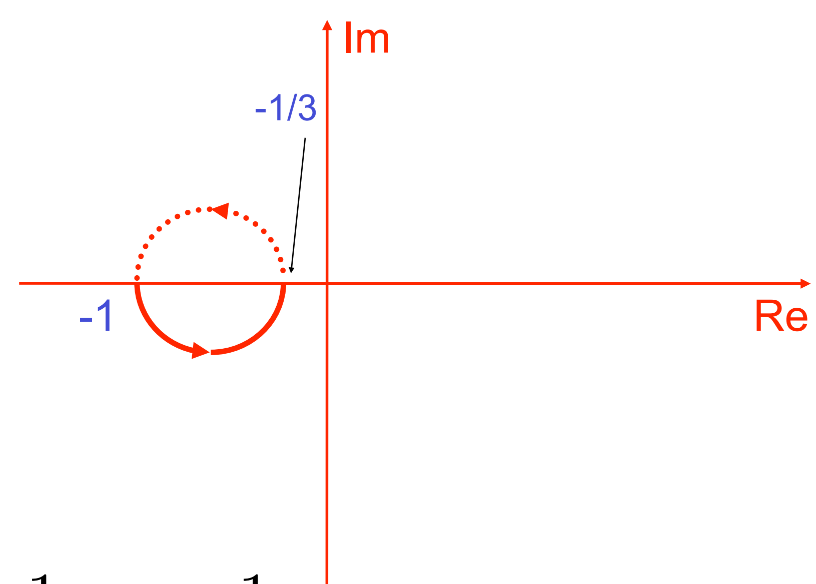


is stable for $G(z) = \dfrac{1}{z-2}$.

Solution:

$$G(z) = \frac{1}{z-2} \Rightarrow G(e^{j\omega}) = \frac{1}{e^{j\omega}-2}$$





One unstable pole

One counter clockwise encirclement of point *-1/K*

Closed-loop stable: $-1 < -\dfrac{1}{K} < -\dfrac{1}{3} \Rightarrow 1 < K < 3$

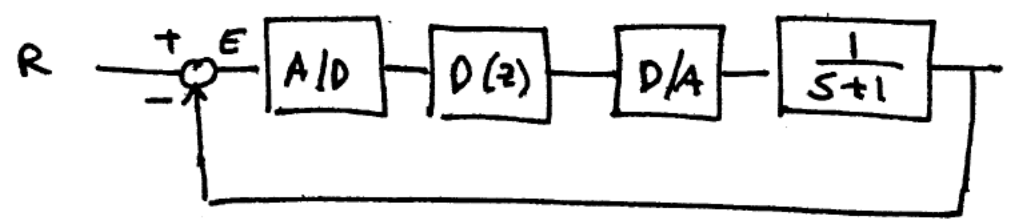
Quiz: Using Jury's stability criterion show that for the following polynomial all roots are inside the unit disc or not?

$$\Phi(z) = z^3 - 1.8z^2 + 1.05z - 0.2$$

Take a photo from your solution and upload it in Canvas (see Quizzes part)
Please clearly mention the system is stable or not in your solution!

Root Locus (section 7.6)

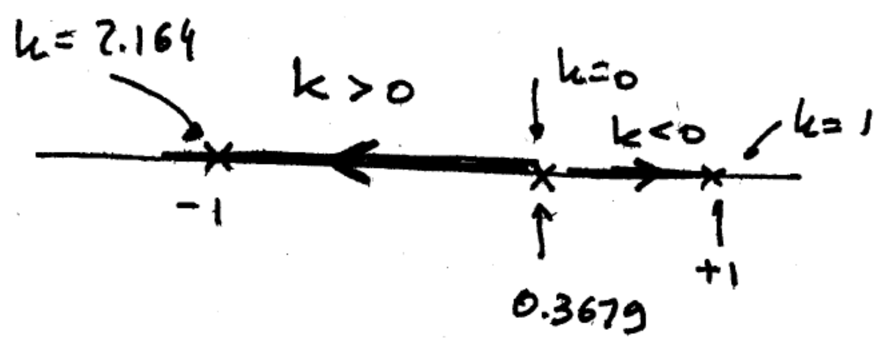
Consider the system:



The characteristic equation is: $1 + D(z)\,\frac{1-e^{-T}}{z-e^{-T}} = 0$

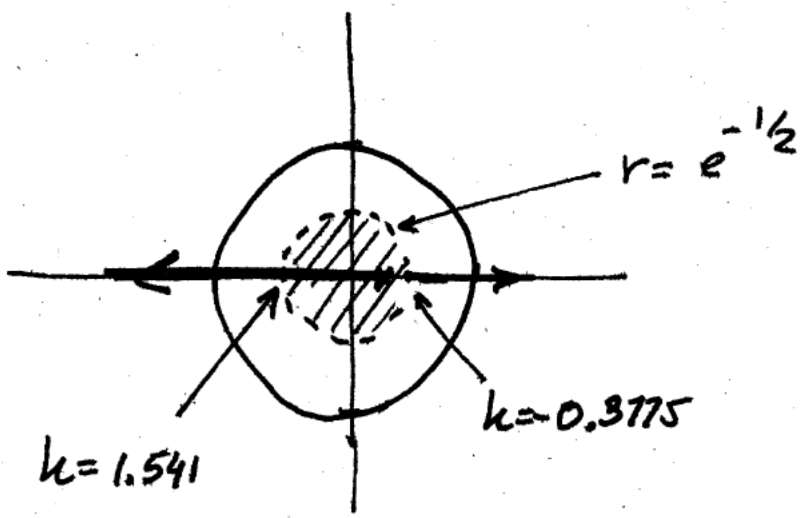
If we take $k = D(z)$, $T=1$ $\Rightarrow$ single pole at $z_0 = e^{-1} - k(1-e^{-1})$

$\cong 0.3679 - 0.6321\,k$



By changing the value of k we change the location of the pole $\Rightarrow$ change Tr, Ts, overshoot, etc.

Suppose that we want $T_s \le 8 \text{ sec} \Rightarrow \tau \le 2 \Rightarrow |r| \le e^{-\frac{1}{2}}$



By looking at the intersection of the region where $|r| \le 1/2$ with the region of achievable closed-loop poles we get the region of admissible values of k

(in this case $-0.38 < k < 1.541$

Now we can select a value of k to optimize some other performance measure such as e_{ss}^{step}: in this case pick $k = 1.541 \Rightarrow e_{ss} \cong \frac{1}{1+k} \cong 0.40$

Suppose we want to achieve $e_{ss} = 0.1 \Rightarrow \frac{1}{1+k} = 0.1 \;//\; k \geq 9$
but this value of k renders the system unstable. $\Rightarrow$ we can try a controller of the form:

$$D(z) = k\frac{(z+a)}{(z-1)}$$

to render the system type I (i.e $e_{ss}^{step} = 0$).

<u>Problem</u>: how do we pick the value of k? First we need to guarantee stability $\Rightarrow$ find char. eq:

$$(z-1)(z-e^{-T}) + k(z+a)(1-e^{-T}) = 0$$

$$z^2 + (0.6321\,k - 1.3679)z + 0.3679 + 0.6321\,ka = 0$$

If we are interested <u>only</u> in stability we can use Jury's test:

$$\varphi(1) > 0 \quad \Rightarrow \quad 0.6321\,k + 0.3679 + 0.6321\,ka > 0$$

$$\varphi(-1) > 0 \qquad 2.7358 + 0.6321 \cdot (1-a) \cdot k > 0$$

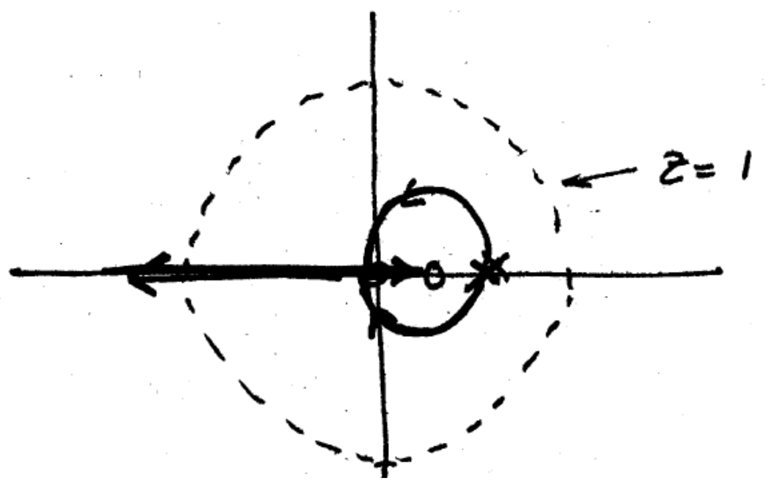
$$|a_0| < a_n \quad \Rightarrow \quad 0.6321\,ka < 1$$

However, if we are also interested in performance, we need to look at the location of the <u>closed-loop</u> poles. In this case, since it is a second order system we can, conceivably, solve explicitly:

$$z_{1,2} = \frac{(1.3679 - 0.6321\,k) \pm \sqrt{(0.6321\,k - 1.3679)^2 - 4(0.3679 + 0.6321\,ka)}}{2}$$

<u>Quite messy</u>. For larger order polynomials this approch is unfeasible

Alternative: Draw a "reasonably" accurate sketch without having to solve the char. equation for each value of K. Once you have this plot (the root locus), you can pick the value of K to meet the design specs.



Definition: The Root Locus of a system is the plot of the roots of the system's characteristic equation as some parameter of the system changes.

A point in the z-plane belongs to the RL iff

$$\boxed{1 + K_{sp}\, G(z) = 0} \iff \begin{cases} |K\,G(z)| = 1 & \text{(Magnitude Criterion)} \\ \angle K\,G(z) = 180 & \text{(angle criterion)} \end{cases}$$

two equations

Remarks :

1) Usually (but not always) K is taken to be positive
2) If we have an "n^{th}" order system the RL has n - branches

3) note that the angle criterion does not depend on k :

if $k>0$, a point $z_1 \in RL \Leftrightarrow \angle G(z_1) = 180 + 2\pi \cdot \ell \quad \ell = 0, \pm 1, \ldots$

$\Rightarrow$ can use the angle criterion to determine whether or not a point z_1 is in the RL. If $z_1 \in RL$ then the actual value of k can be found from the magnitude criterion.

- **Rules for sketching the RL**

These rules are the same as for continuous-time systems (EE 428) with minor modifications:

1) Mark the open loop poles (origin) & zeros (ending)

2) Draw the R.L on the real axis: points to the left of an <u>odd</u> number of real poles & zeros

3) Draw $n-m$ asymptotes leaving at angles

$$\theta = (2\ell+1)\frac{180}{n-m} \qquad \ell = 0, 1 \cdots n-m-1$$

where $n = \#$ open loop poles
$m = \#$ open loop zeros

If $n-m>1$ the asymptotes intersect the real axis at the point

$$\sigma = \frac{\Sigma p_i - \Sigma z_i}{n-m} \qquad \text{(centroid)}$$

4) Break away (break-in) points:

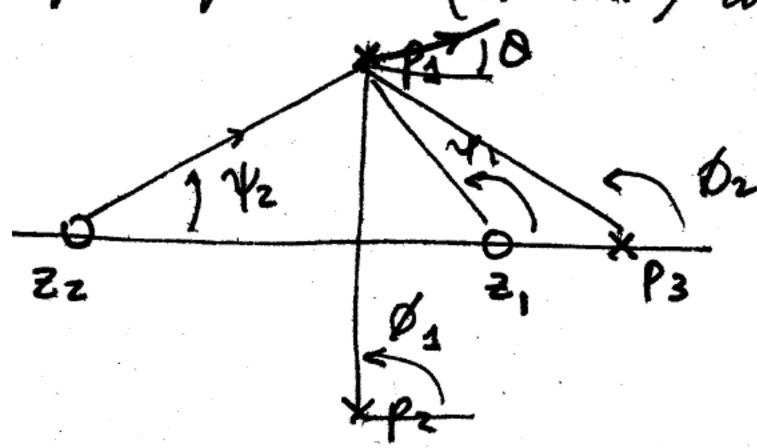
Solve $\frac{d}{dz}\left\{\frac{1}{G(z)}\right\} = 0 \qquad \left(\text{or } \frac{d}{dz}G(z) = 0\right)$

Note: this is only a necessary condition. Need to go back and check that the solutions indeed are in the RL. Solutions not in the RL are discarded

Different from
EECE 5580

5) Unit circle intersect: Use Jury's method to find K. Plug these values of K in the Char. eq. and solve for the roots

6) Departure (arrival) angle:



$$\theta_d = \Sigma \psi_i - \Sigma \phi_i - (2\ell+1)\,180$$

$$\theta_a = \Sigma \phi_i - \Sigma \psi_i + (2\ell+1)\,180$$