EECE5666 (DSP) : Homework-2

Due on February 8, 2022 by 11:59 pm via submission portal.

NAME: Enter your Lastname, Firstname here

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Default Plot Parameters

```
set(0,'defaultfigurepaperunits','points','defaultfigureunits','points');
set(0,'defaultaxesfontsize',10); set(0,'defaultaxeslinewidth',1.5);
set(0,'defaultaxestitlefontsize',1.4,'defaultaxeslabelfontsize',1.2);
```

Problem 2.1

Text Problem 3.45, parts (b) and (d) only (Page 130)

Determine the z-transform and sketch the pole-zero plot with the ROC for each of the following sequences.

(b) $x[n] = (1/2)^n u[n+1] + 3^n u[-n-1]$:

Solution: The given sequence can be written as

$$x[n] = 2(1/2)^{n+1} u[n+1] + 3^n u[-n-1].$$

Now using the shift property and the known z-transform pairs, we obtain

$$X(z) = \frac{2z}{1-\frac{1}{2}z^{-1}} + \frac{-1}{1-3z^{-1}}; \quad \{|z| > 1/2\} \cap \{|z| < 3\}.$$

The overall rational function and zero-pole form can be obtained using the following script:

```
clc; close all; clear;
b1 = 2; a1 = conv([0,1],[1,-1/2]);
b2 = -1; a2 = [1,-3];
B11 = conv(b1,a2); B12 = conv(b2,a1);
b = [B11,0]+B12; Num = ['X(z)_num = [',num2str(round(b,2)),']']; disp(Num);
```

```
X(z)_num = [2          -7          0.5]
```

```
a = conv(a1,a2); Den = ['X(z)_denom = [',num2str(round(a,2)),']']; disp(Den);
```

```
X(z)_denom = [0             1         -3.5          1.5]
```

Hence we have

$$X(z) = \frac{2-7z^{-1}+0.5z^{-2}}{0+z^{-1}-3.5z^{-2}+1.5z^{-3}} = \frac{2-7z^{-1}+0.5z^{-2}}{z^{-1}(1-3.5z^{-1}+1.5z^{-2})}; \quad \{|z| > 1/2\} \cap \{|z| < 3\}.$$

Since the first term is zero in the denominator of the first rational term, the '**roots**' or '**zplane**' functions will not work correctly. Hence we removed the second term, z^{-1}, as a common term in the denominator to obtain the second rational term, along with an addition of a zero at the origin. Now 'zplane' can be used to obtain the correct result.

```
z = [0;roots(b)]; % Zeros of X(z); Note the prepended 0
p = roots(a(2:end)); % Poles of X(z); Note the omition of the first term
Zeros = ['Zeros = [',num2str(round(z.',3)),']']; disp(Zeros);
```

```
Zeros = [0       3.427       0.073]
```

```
Poles = ['Poles = [',num2str(round(p.',3)),']']; disp(Poles);
```

```
Poles = [3         0.5]
```

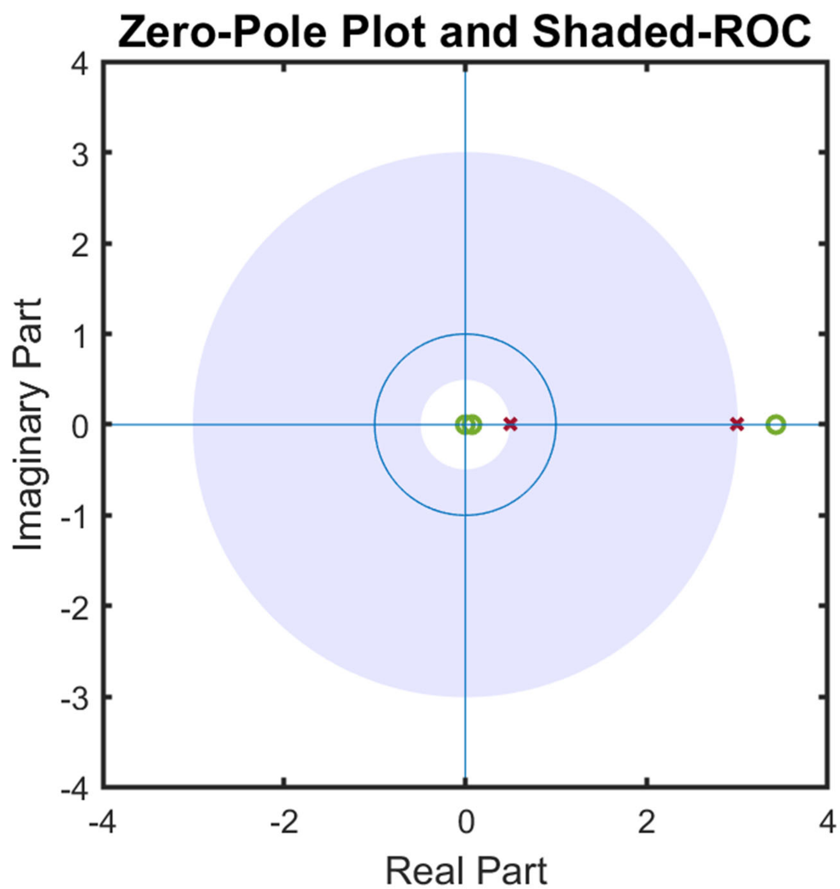
Hence zero-pole representation is

$$X(z) = \frac{\left(1-3.4271z^{-1}\right)\left(1-0.0729z^{-1}\right)}{z^{-1}\left(1-3z^{-1}\right)\left(1-0.5z^{-1}\right)}; \quad \text{ROC: } \{0.5 < |z| < 3\}.$$

The zero-pole plot is obtained using

```
r1 = min(abs(p)); r2 = max(abs(p));
theta = linspace(0,2*pi,1000);
x1 = r1*cos(theta); y1 = r1*sin(theta);
x2 = r2*cos(theta); y2 = r2*sin(theta);
figure('position',[0,0,4,4]*72);
fill(x2,y2,[0.9,0.9,1],'linewidth',0.1,'edgecolor',[0.9,0.9,1]); hold on;
fill(x1,y1,[1 1 1],'linewidth',0.1,'edgecolor',[0.9,0.9,1]);
[Hz,Hp,Hl] = zplane(z,p); axis([-4,4,-4,4]);
```

```
set(Hz,'color',[0.466,0.674,0.1880],'markersize',5,'linewidth',1.5);
set(Hp,'color',[0.635,0.078,0.1840],'markersize',5,'linewidth',1.5);
set(Hl,'linewidth',0.5,'linestyle','-');
title('Zero-Pole Plot and Shaded-ROC');
```



(d) $x[n] = |n|(1/2)^{|n|}$

Solution: This can be written as $x[n] = -n(1/2)^{-n}u[-n-1] + n(1/2)^{n}[n]$. Hence using the multiplication-by-a-ramp property, we have

$$
\begin{aligned}
X(z) &= \frac{2z^{-1}}{(1-2z^{-1})^2} + \frac{\frac{1}{2}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)^2}; \quad \{|z|<2\} \cap \{|z|>1/2\} \\
&= \frac{\frac{5}{2}z^{-1} - 4z^{-2} + \frac{5}{2}z^{-3}}{(1-2z^{-1})^2\left(1-\frac{1}{2}z^{-1}\right)^2}; \quad \{|z|<2\} \cap \{|z|>1/2\} \\
&= \frac{z^{-1}\left(1-(0.8+\jmath 0.6)z^{-1}\right)\left(1-(0.8-\jmath 0.6)z^{-1}\right)}{(1-2z^{-1})^2(1-0.5z^{-1})^2}; \quad \text{ROC: } 0.5<|z|<2.
\end{aligned}
$$

The rational function in the second equality was obtained using the following script:

```
[R1,p1,c1] = residuez([0,2],conv([1,-2],[1,-2])); % PFE 1st term
[R2,p2,c2] = residuez([0,1/2],conv([1,-1/2],[1,-1/2])); % PFE 2nd term
R = [R1;R2]; p = [p1;p2]; % PFE of the entire rational function
[b,a] = residuez(R,p,[]); % Rational function
Num = ['X(z)_num = [',num2str(round(b,2)),']']; disp(Num);
```

```
X(z)_num = [0         2.5          -4         2.5]
```

```
Den = ['X(z)_denom = [',num2str(round(a,2)),']']; disp(Den);
```

```
X(z)_denom = [1          -5        8.25          -5           1]
```

The zeros and poles in the third equality above were obtained using

```
z = [0;roots(b(2:end))]; p = real(roots(a));
Zeros = ['Zeros = [',num2str(round(z.',3)),']']; disp(Zeros);
```

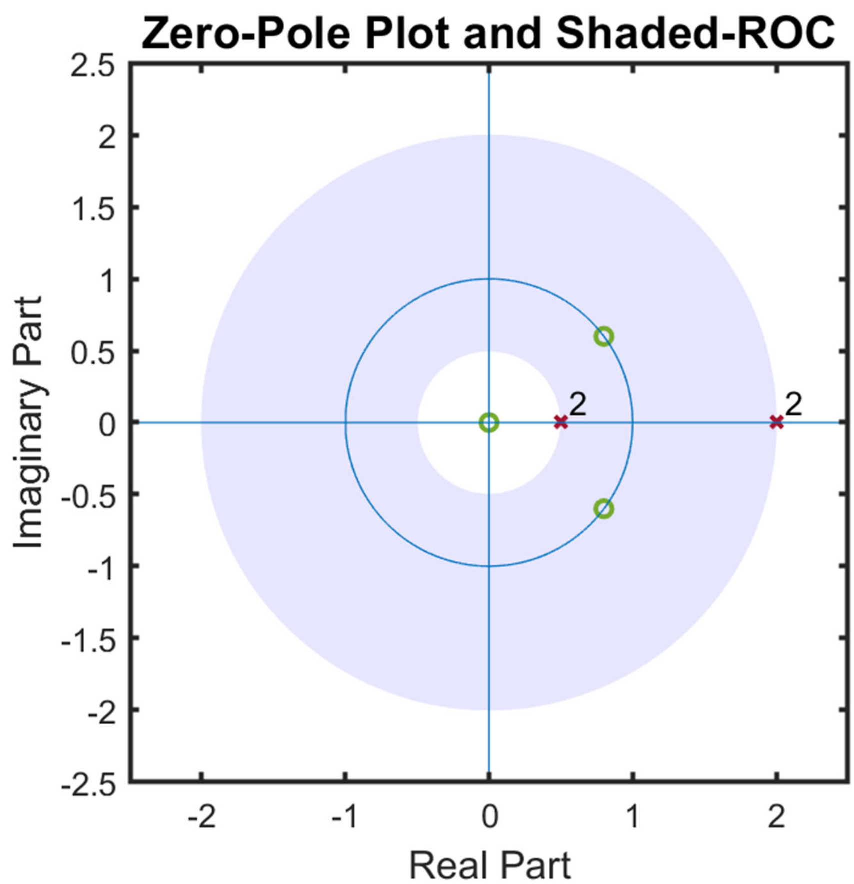
```
Zeros = [0+0i                 0.8+0.6i                 0.8-0.6i]
```

```
Poles = ['Poles = [',num2str(round(p.',3)),']']; disp(Poles);
```

```
Poles = [2           2         0.5         0.5]
```

Finally, the zero-pole plot is obtained using

```
r1 = min(abs(p)); r2 = max(abs(p));
theta = linspace(0,2*pi,1000);
x1 = r1*cos(theta); y1 = r1*sin(theta);
x2 = r2*cos(theta); y2 = r2*sin(theta);
figure('position',[0,0,4,4]*72);
fill(x2,y2,[0.9 0.9 1],'linewidth',0.1,'edgecolor',[0.9,0.9,1]); hold on;
fill(x1,y1,[1 1 1],'linewidth',0.1,'edgecolor',[0.9,0.9,1]);
[Hz,Hp,Hl] = zplane(z,p); axis([-2.5,2.5,-2.5,2.5]);
set(Hz,'color',[0.466,0.674,0.1880],'markersize',5,'linewidth',1.5);
set(Hp,'color',[0.635,0.078,0.1840],'markersize',5,'linewidth',1.5);
set(Hl,'linewidth',0.5,'linestyle','-');
title('Zero-Pole Plot and Shaded-ROC');
```



Problem 2.2

Consider the z-transform expression:

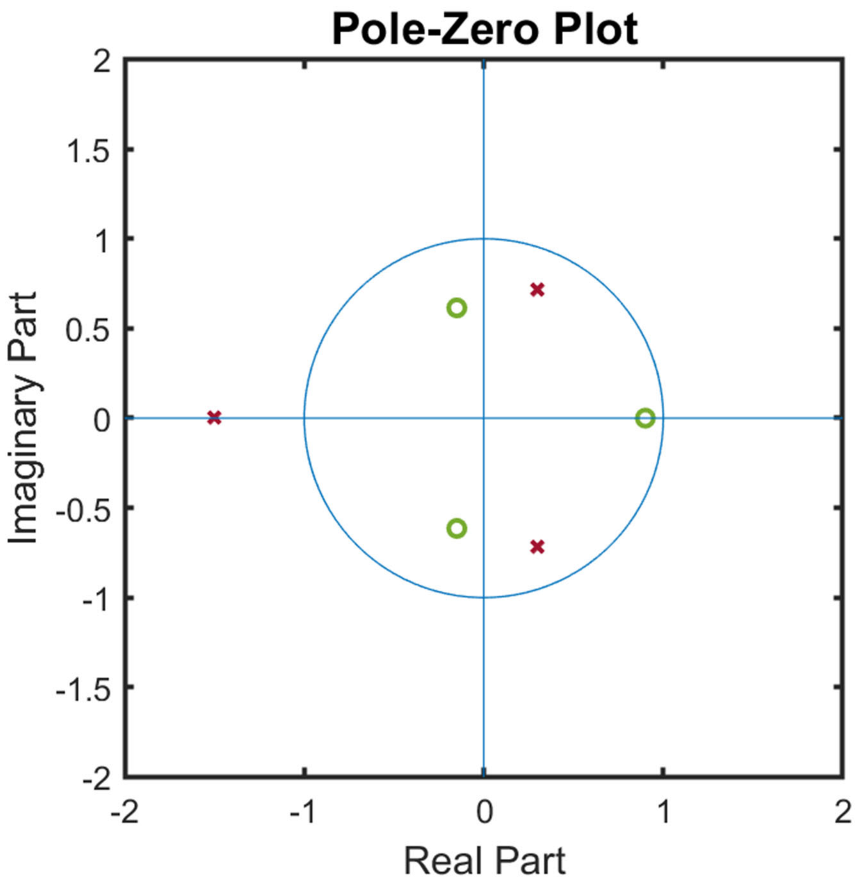
$$X(z) = \frac{(z-0.91)\left(z^2+0.3z+0.4\right)}{(z+1.5)\left(z^2-0.6z+0.6\right)}.$$

(a) Provide a zero-pole plot of $X(z)$.

MATLAB script:

```
clc; close all; clear;
b = conv([1,-0.9],[1,0.3,0.4]);
a = conv([1,-0.6,0.6],[1,1.5]);
figure('position',[0,0,4,4]*72);
[Hz,Hp,Hl] = zplane(b,a); axis([-2,2,-2,2]);
set(Hz,'markersize',5,'color',[0.466,0.674,0.1880],'linewidth',1.5);
set(Hp,'markersize',5,'color',[0.635,0.078,0.1840],'linewidth',1.5);
```

```
set(Hl,'linestyle','-','linewidth',0.5); title('Pole-Zero Plot');
```



(b) List all possible regions of convergence (ROCs) for this z-transform.

Solution: Using the following script

```
Pl = roots(a);
MagPl = ['Pole Magnitudes = [',num2str(round(abs(Pl).',3)),']']; disp(MagPl);
```

```
Pole Magnitudes = [1.5      0.775      0.775]
```

and the pole-zero plot in (a), we observe that there are three possible ROCs, given by

$$\text{ROC-1: } |z| < 0.7746, \qquad \text{ROC-2: } 0.7746 < |z| < 1.5, \qquad \text{ROC-3: } |z| > 1.5\,.$$

(c) Determine the inverse z-transform so that the resulting sequence is absolutely summable. This sequence $x[n]$ should be a real-valued sequence. Provide a **`stem`** plot of $x[n]$.

Solution From part (b) above, the absolutely summable sequence is given by the ROC-2 which contains the unit circle. The residues at the pole locations as well as their magnitudes and angles are computed by the following script:

```
[R,PL,C] = residuez(b,a);
magR = abs(R.'); phaR = angle(R.'); magPL = abs(PL.'); phaPL = angle(PL.');
disp(['Constant = ',num2str(round(C,1))]);
```

```
Constant = -0.4
```

```
disp(['Residue magnitudes: [',num2str(round(magR,4)),']']);
```

```
Residue magnitudes: [0.9387      0.2815      0.2815]
```

```
disp(['Residue angles (deg): [',num2str(round(phaR*180/pi,2)),']']);
```

```
Residue angles (deg): [0        34.97       -34.97]
```

```
disp(['Pole magnitudes: [',num2str(round(magPL,4)),']']);
```

```
Pole magnitudes: [1.5      0.7746      0.7746]
```

```
disp(['Pole angles (pi units): [',num2str(round(phaPL/pi,2)),']']);
```

```
Pole angles (pi units): [1        0.37       -0.37]
```

Therefore, the absolutely summable sequence containing no complex-numbers is given by [see equation (3.40) in the textbook for the third term below]

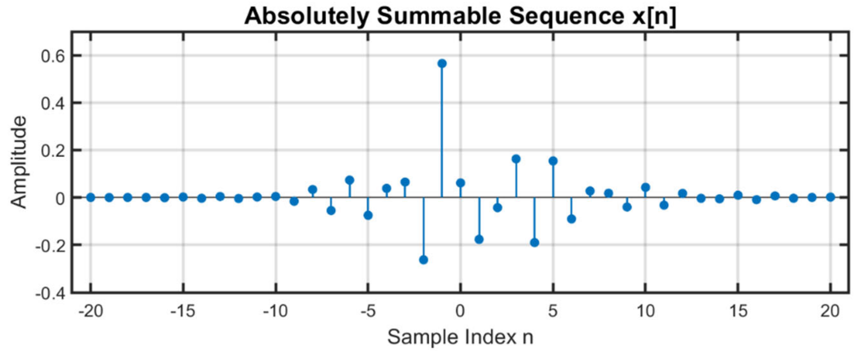
$$x(n) = -0.4\delta(n) - (0.9387)(-1.5)^n u[-n-1] + 2(0.2815)(0.7746)^n \cos(0.3734\pi n + 34.96^\circ)u[n]$$

Note that the sequence contains both causal and anticausal parts. Simplifying,

$$x(n) = -0.4\delta(n) - 0.9387(-1.5)^n u[-n-1] + 0.5630(0.7746)^n \cos\left(0.3734\pi n + 34.96^\circ\right)u[n].$$

Plot of the sequence is computed using the following script.

```
n = -20:20; uac = (n<=-1); uc = (n>=0);
x1 = C*delta(n(1),0,n(end)).';
x2 = -1*magR(1)*(magPL(1).^n).*cos(phaPL(1)*pi*n+phaR(1)).*uac;
x3 = 2*magR(3)*(magPL(3).^n).*cos(phaPL(3)*pi*n+phaR(3)).*uc; x = x1+x2+x3;
figure('position',[0,0,7,3]*72); subplot('position',[0.15,0.2,0.83,0.65]);
stem(n,x,'filled','markersize',4,'linewidth',1); axis([-21,21,-0.4,0.7])
xlabel('Sample Index n'); ylabel('Amplitude');
title('Absolutely Summable Sequence x[n]'); grid;
```



Problem 2.3

Text Problem 3.47, parts (b) and (e), (Page 131)

Given the z-transform pair $x[n] \leftrightarrow X(z) = z^{-1}/(1+0.8z^{-1})$ with ROC: $|z| > 0.8$, use the z-transform properties to determine the z-transform of the following sequences:

(b) $y[n] = x[3-n] = x[-(n-3)]$:

Solution: Using folding and time shifting properties

$$\begin{aligned} Y(z) &= z^{-3}X(1/z) = z^{-3}\frac{(1/z)^{-1}}{1+0.8(1/z)^{-1}}, && \left|\frac{1}{z}\right| > 0.8 \\ &= \frac{z^{-3}z}{1+0.8z}, && \text{ROC: } |z| < 1.25 \\ &= \frac{1.25z^{-3}}{1+1.25z^{-1}}, && \text{ROC: } |z| < 1.25. \end{aligned}$$

(e) $y[n] = x[n] * x[2-n]$:

Solution: Using convolution, time shifting, and folding properties:

$$\begin{aligned} Y(z) &= X(z)\cdot z^{-2}X(1/z) = \frac{z^{-2}}{(1+0.8z)(1+0.8z^{-1})}, && \text{ROC: } 0.8 < |z| < 1.25 \\ &= \frac{1.25z^{-3}}{\left(1+1.25z^{-1}\right)\left(1+0.8z^{-1}\right)}, && \text{ROC: } 0.8 < |z| < 1.25 \\ &= \frac{1.25z^{-3}}{1+2.05z^{-1}+z^{-2}}, && \text{ROC: } 0.8 < |z| < 1.25. \end{aligned}$$

Problem 2.4

An LTI system described by the following impulse response

$$h[n] = n\left(\frac{1}{3}\right)^n u[n] + \left(-\frac{1}{4}\right)^n u[n].$$

(a) Determine the system function representation.

Solution: Taking the z-transform of $h(n)$

$$\begin{aligned} H(z) &= \mathscr{Z}[h[n]] = \mathscr{Z}\left[n(1/3)^n u[n] + (-1/4)^n u[n]\right] \\ &= \frac{(1/3)z^{-1}}{[1-(1/3)z^{-1}]^2} + \frac{1}{1+(1/4)z^{-1}}, && |z| > (1/3) \\ &= \frac{1 - \frac{1}{3}z^{-1} + \frac{7}{36}z^{-2}}{1 - \frac{5}{12}z^{-1} - \frac{1}{18}z^{-2} + \frac{1}{36}z^{-3}}, && |z| > (1/3) \end{aligned}$$

The above rational simplification is done using MATLAB:

MATLAB script:

```
clc; close all; clear;
num1 = [0,1/3]; den1 = conv([1,-1/3],[1,-1/3]);
num2 = 1; den2 = [1,1/4];
NumH = conv(num1,den2) + den1; DenH = conv(den1,den2);
format rat; NumeratorH = NumH
```

```
NumeratorH =
       1              -1/3            7/36
```

```
DenominatorH = DenH
```

```
DenominatorH =
       1              -5/12           -1/18            1/36
```

(b) Determine the difference equation representation.

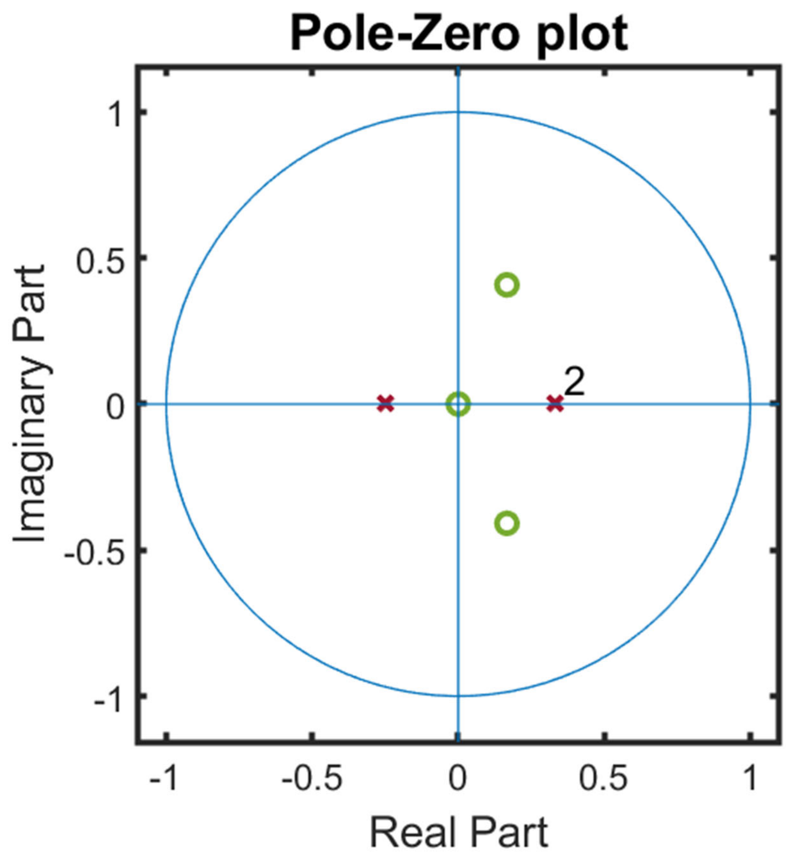
Solution: From $H(z)$ above, the difference equation is

$$y[n] = x[n] - \frac{1}{3}x[n-1] + \frac{7}{36}x[n-2] + \frac{5}{12}y[n-1] + \frac{1}{18}y[n-2] - \frac{1}{36}y[n-3].$$

(c) Determine the pole-zero plot.

Solution: The pole-zero plot is computed using the following script:

```
b = [1 -1/3 7/36]; a = [1 -5/12 -1/18 1/36];
Hf_1 = figure('position',[0,0,3,3]*72);
[Hz,Hp,Hl] = zplane(b,a);
set(Hz,'markersize',5,'color',[0.466,0.674,0.1880],'linewidth',1.5);
set(Hp,'markersize',5,'color',[0.635,0.078,0.1840],'linewidth',1.5);
set(Hl,'linestyle','-','linewidth',0.5); title('Pole-Zero plot');
```



(d) Determine the output sequence $y[n]$ when the input is $x[n] = \left(\frac{1}{4}\right)^n u[n]$.

Solution: Using the z-transform table, we have $X(z) = \mathscr{Z}[(1/4)^n u[n]] = \dfrac{1}{1 - \frac{1}{4}z^{-1}}, |z| > \dfrac{1}{4}$

Now the z-transform of $y[n]$ is

$$
\begin{aligned}
Y(z) &= H(z)X(z) = \left(\frac{1-\frac{1}{3}z^{-1}+\frac{7}{36}z^{-2}}{1-\frac{5}{12}z^{-1}-\frac{1}{18}z^{-2}+\frac{1}{36}z^{-3}}\right)\left(\frac{1}{1-0.25z^{-1}}\right), \quad |z|>\frac{1}{3} \\
&= \frac{-16}{1-\frac{1}{3}z^{-1}}+\frac{4}{\left(1-\frac{1}{3}z^{-1}\right)^2}+\frac{\frac{1}{2}}{1+\frac{1}{4}z^{-1}}+\frac{\frac{25}{2}}{1-\frac{1}{4}z^{-1}}, \quad |z|>\frac{1}{3} \\
&= \frac{-16}{1-\frac{1}{3}z^{-1}}+12z\frac{\frac{1}{3}z^{-1}}{\left(1-\frac{1}{3}z^{-1}\right)^2}+\frac{\frac{1}{2}}{1+\frac{1}{4}z^{-1}}+\frac{\frac{25}{2}}{1-\frac{1}{4}z^{-1}}, \quad |z|>\frac{1}{3}
\end{aligned}
$$

The residue calculations above are done using MATLAB.

```
NumX = 1; DenX = [1,-0.25]; format rational;
NumY = NumH; DenY = conv(DenH,DenX); [R,p,k] = residuez(NumY,DenY);
ResidueY = R.'
```

```
ResidueY =
     -16              4              1/2            25/2
```

```
PolesY = p.'
```

```
PolesY =
       1/3            1/3            -1/4            1/4
```

Hence the output $y[n]$ is

$$
y(n) = -16\left(\frac{1}{3}\right)^n u[n] + 12(n+1)\left(\frac{1}{3}\right)^{n+1} u[n+1] + \frac{1}{2}\left(-\frac{1}{4}\right)^n u[n] + \frac{25}{2}\left(\frac{1}{4}\right)^n u[n].
$$

Problem 2.5

Text Problem 3.57 (Page 132)

Determine the impulse response of the system described by

$$
y[n] + \frac{11}{6}y[n-1] + \frac{1}{2}y[n-2] = 2x[n].
$$

for all possible regions of convergence.

Solution: The system function $H(z) = Y(z)/X(z)$ of the system is obtained by taking the z-transform of the difference equation above and solving for $Y(z)/X(z)$:

```
b = 2; a = [1,11/6,1/2];
[R,p,k] = residuez(b,a);
```

```
format rat;
Residues = R.'
```

```
Residues =
      18/7          -4/7
```

```
Poles = p.'
```

```
Poles =
      -3/2          -1/3
```

Hence

$$H(z) = \frac{Y(z)}{X(z)} = \frac{2}{1+\frac{11}{6}z^{-1}+\frac{1}{2}z^{-2}} = \frac{18/7}{1+\frac{3}{2}z^{-1}} + \frac{-4/7}{1+\frac{1}{3}z^{-1}}.$$

Since there are two poles of $H(z)$, namely, $p_1 = -3/2$ and $p_2 = -1/3$, there are three ROCs.

1. If ROC: $|z| > \frac{3}{2}$, then the impulse response is right-sided:

$$h[n] = \left(\frac{18}{7}\right)\left(-\frac{3}{2}\right)^n u[n] + \left(-\frac{4}{7}\right)\left(-\frac{1}{3}\right)^n u[n].$$

2. If ROC: $|z| < \frac{1}{3}$, then the impulse response is left-sided:

$$h[n] = \left(-\frac{18}{7}\right)\left(-\frac{3}{2}\right)^n u[-n-1] + \left(\frac{4}{7}\right)\left(-\frac{1}{3}\right)^n u[-n-1].$$

3. If ROC: $\frac{1}{3} < |z| < \frac{3}{2}$, then the impulse response is two-sided:

$$h[n] = \left(-\frac{18}{7}\right)\left(-\frac{3}{2}\right)^n u[-n-1] + \left(-\frac{4}{7}\right)\left(-\frac{1}{3}\right)^n u[n].$$

Problem 2.6

Text Problem 3.63 (Page 133)

Consider the following LCCDE

$$y[n] = 2\cos(\omega_0)y[n-1] - y[n-2]$$

with no input but with initial conditions $y[-1] = 0$ and $y[-2] = -A\sin(\omega_0)$.

(a) Show that the solution of the above LCCDE is given by $y[n] = A\sin[(n+1)\omega_0]u[n]$. This system is known as a *digital oscillator*.

Solution: Applying one-sided z-transform to the difference equation, we obtain

$$\begin{aligned} Y^{+}(z) &= 2\cos(\omega_0)\left(y[-1] + z^{-1}Y^{+}(z)\right) - \left(y[-2] + y[-1]z^{-1} + z^{-2}Y^{+}(z)\right) \\ &= 2\cos(\omega_0)\left(z^{-1}Y^{+}(z)\right) - \left(-A\sin(\omega_0) + z^{-2}Y^{+}(z)\right) \\ &= \left(2\cos(\omega_0)\right)z^{-1}Y^{+}(z) - z^{-2}Y^{+}(z) + A\sin(\omega_0) \end{aligned}$$

or

$$Y^{+}(z) = \frac{A\sin(\omega_0)}{1 - 2\cos(\omega_0)z^{-1} + z^{-2}} = z\frac{A\sin(\omega_0)z^{-1}}{1 - 2\cos(\omega_0)z^{-1} + z^{-2}}.$$

The inverse z-transform from the z-transform Table 3.1 is

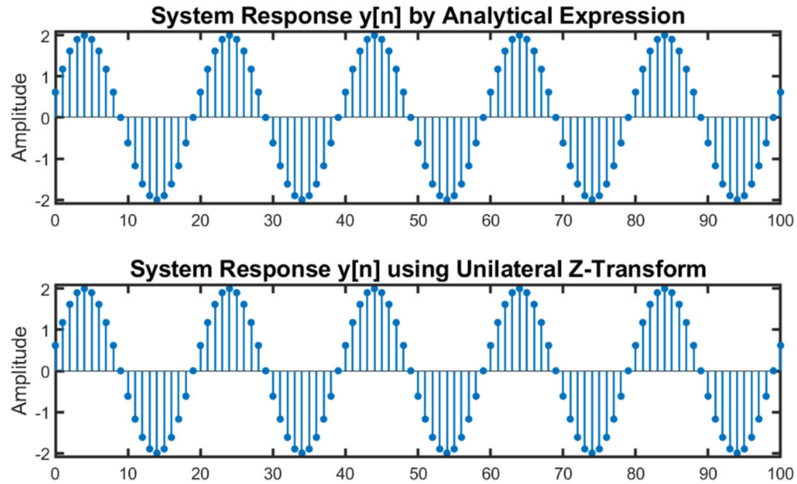
$$y[n] = A\sin\left[(n+1)\omega_0\right]u[n+1] = A\sin\left[(n+1)\omega_0\right]u[n]$$

since at $n = -1$, $\sin(0) = 0$.

(b) For $A = 2$ and $\omega_0 = 0.1\pi$, verify the operation of the above digital oscillator using MATLAB.

MATLAB script: Verification is done using the following script:

```
clc; close all; clear;
n=0:100; A = 2; w0 = 0.1*pi;
yn1 = A*sin((n+1)*w0); % Analytical solution
b = [0 A*sin(w0)]; a = [1 -2*cos(w0) 1]; % System coefficients
yi = [0, -A*sin(w0)]; % Initial Conditions
xIC = filtic(b,a,yi); % Equivalent initial condition input
yn2 = filter(b,a,zeros(1,length(n)),xIC); % Numerical solution
figure('position',[0,0,7,4]*72);
subplot(211); stem(n,yn1,'filled','markersize',3,'linewidth',1);
ylim([min(yn2)-0.1 max(yn2)+0.1]); ylabel('Amplitude');
title('System Response y[n] by Analytical Expression');
subplot(212); stem(n,yn2,'filled','markersize',3,'linewidth',1);
ylim([min(yn2)-0.1 max(yn2)+0.1]); ylabel('Amplitude');
title('System Response y[n] using Unilateral Z-Transform');
```



Problem 2.7

Text Problem 4.38, parts (a) and (d) only, (Page 197)

Determine whether or not each of the following signals is periodic. If a signal is periodic, determine its fundamental period.

(a) $x_1(t) = |\sin(7\pi t)|\cos(11\pi t)$

Solution: Note that since the first term is an absolute value of a sinusoidal function, its fundamental period is one-half of the fundamental period of the sinusoid. Hence $|\sin(7\pi t)|$ is periodic with fundamental period $T_1 = \frac{1}{2}(2\pi)/(7\pi) = \frac{1}{7}$ sec. The second component $\cos(11\pi t)$ is periodic with fundamental period $T_2 = (2\pi)/(11\pi) = \frac{2}{11}$ sec. The smallest integer divisible by $T_1 = \frac{1}{7} = \frac{2}{14}$ and $T_2 = \frac{2}{11}$ is $T = 2$. Thus we will have 14 cycles of the first term and 11 cycles of the second term in $x_1(t)$ which is then periodic with fundamental period $T = 2$ sec.

(d) $x_4[n] = e^{j\pi n/7} + e^{j\pi n/11}$:

Solution: From the fundamental periods of each components,

$$N_1 = \frac{2\pi}{\pi/7} = 14, \quad N_2 = \frac{2\pi}{\pi/11} = 22$$

the least common multiple N is given by

$$N = 2 \times 7 \times 11 = 154.$$

Thus $x_4[n]$ is periodic with fundamental period $N = 154$.

Problem 2.8

Text Problem 4.45, parts (c) and (d) only, (Page 198)

Given that $x[n]$ is a periodic sequence with fundamental period N and Fourier coefficients a_k, determine the Fourier coefficients of the following sequences in terms of a_k.

Note that

$$a_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N}. \qquad (2.8.1)$$

(c) $x_3[n] = 3\cos(2\pi 5n/N)x[-n],\ N > 5$:

Solution: The given sequence $x_3[n]$ can be expressed as

$$x_3[n] = 3\left(\frac{e^{j2\pi 5n/N} + e^{-j2\pi 5n/N}}{2}\right)x[-n] = 1.5e^{j2\pi 5n/N}x[-n] + 1.5e^{j2\pi(-5)n/N}x[-n] \qquad (2.8.2)$$

Consider the first term in $(2.8.2)$ above. Since $N > 5$, its DTFS coefficients are given by

$$\begin{aligned}
\text{DTFS}[(1.5e^{j2\pi 5n/N}x[-n])] &= \frac{1}{N}\sum_{n=0}^{N-1}(1.5e^{j2\pi 5n/N}x[-n])e^{-j2\pi kn/N} \\
&= 1.5\left(\frac{1}{N}\sum_{n=0}^{N-1} x[-n]e^{-j2\pi(k-5)n/N}\right) \\
&= 1.5\left(\frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-j2\pi(5-k)n/N}\right), \quad (n \to -n) \\
&= 1.5a_{5-k} = 1.5a_{-(k-5)}.
\end{aligned}$$

Similarly, the DTFS coefficients of the second term in $(2.8.2)$ are given by

$$\text{DTFS}[(1.5e^{j2\pi(-5)n/N}x[-n])] = 1.5a_{-(k+5)}.$$

Combining the two results, the DTFS coefficients of $x_3[n]$ (denote by b_k) are given by

$$b_k = 1.5(a_{-(k-5)} + a_{-(k+5)}).$$

(d) $x_4[n] = x[n] + x^*[-n]$:

Solution: Consider the DTFS coefficints of $x^*[-n]$

$$\begin{aligned} \text{DTFS}\left[x^*[-n]\right] &= \frac{1}{N}\sum_{n=0}^{N-1} x^*[-n]\mathrm{e}^{-\jmath 2\pi kn/N} = \left(\frac{1}{N}\sum_{n=0}^{N-1} x[-n]\mathrm{e}^{\jmath 2\pi kn/N}\right)^* \\ &= \left(\frac{1}{N}\sum_{n=0}^{N-1} x[n]\mathrm{e}^{-\jmath 2\pi kn/N}\right)^* = a_k^*. \end{aligned}$$

Thus the DTFS coefficients of $x_4[n]$ (denote by b_k) are given by

$$b_k = a_k + a_k^* = 2\mathrm{Re}(a_k).$$

Problem 2.9

Text Problem 4.49, partsa (c) and (d) only, (Page 198)

Determine sequences corresponding to each of the following Fourier transforms.

(c) $X_3(\mathrm{e}^{\jmath\omega}) = \jmath\mathrm{e}^{-\jmath 4\omega}\left[2 + 3\cos(\omega) + \cos(2\omega)\right]$:

Solution: Express the given DTFT as

$$\begin{aligned} X_3(\mathrm{e}^{\jmath\omega}) &= \jmath\mathrm{e}^{-\jmath 4\omega}\left[2 + 1.5\mathrm{e}^{\jmath\omega} + 1.5\mathrm{e}^{-\jmath\omega} + 0.5\mathrm{e}^{\jmath 2\omega} + 0.5\mathrm{e}^{-\jmath 2\omega}\right] \\ &= (\jmath 0.5)\mathrm{e}^{-\jmath 2\omega} + (\jmath 1.5)\mathrm{e}^{-\jmath 3\omega} + (\jmath 2)\mathrm{e}^{-\jmath 4\omega} + (\jmath 1.5)\mathrm{e}^{-\jmath 5\omega} + (\jmath 0.5)\mathrm{e}^{-\jmath 6\omega}. \end{aligned}$$

Hence

$$x_3[n] = (\jmath 0.5)\delta[n-2] + (\jmath 1.5)\delta[n-3] + (\jmath 2)\delta[n-4] + (\jmath 1.5)\delta[n-5] + (\jmath 0.5)\delta[n-6]$$

or

$$x_3[n] = \{\underset{\uparrow}{0}, 0, \jmath 0.5, \jmath 1.5, \jmath 2, \jmath 1.5, \jmath 0.5\}.$$

(d) $X_4(\mathrm{e}^{\jmath\omega}) = \begin{cases} 2, & 0 \le |\omega| \le \pi/8 \\ 1, & \pi/8 \le |\omega| \le 3\pi/4 \\ 0, & 3\pi/4 \le |\omega| \le \pi \end{cases}$

Solution: The given frequency response can be expressed as

$$X_3(\mathrm{e}^{\jmath\omega}) = \begin{cases} 1, & 0 \le |\omega| \le \pi/8 \\ 0, & \pi/8 \le |\omega| \le \pi \end{cases} + \begin{cases} 1, & 0 \le |\omega| \le 3\pi/4 \\ 0, & 3\pi/4 \le |\omega| \le \pi \end{cases}.$$

Using the DTFT pair $(2f_\mathrm{c})\mathrm{sinc}(2f_\mathrm{c}n) \overset{\text{DTFT}}{\longleftrightarrow} \mathrm{rect}\left(\dfrac{\omega}{2\omega_\mathrm{c}}\right)$ where $\omega_\mathrm{c} = 2\pi f_\mathrm{c}$, we obtain

$$x_4[n] = \frac{1}{8}\mathrm{sinc}\left(\frac{n}{8}\right) + \frac{3}{4}\mathrm{sinc}\left(\frac{3n}{4}\right).$$

Problem 2.10

Text Problem 4.53 (Page 199)

Note: There are two corrections in the errata sheet. Please follow them.

Let a sinusoidal pulse be given by $x(n) = (\cos\omega_0 n)\big(u[n] - u[n-M]\big)$.

(a) Using the frequency-shifting property of the DTFT, show that the real-part of DTFT of $x(n)$ is given by

$$\begin{aligned} X_{\mathrm{R}}(e^{j\omega}) &= \frac{1}{2}\cos\left[\frac{(\omega-\omega_0)(M-1)}{2}\right]\left[\frac{\sin\left(\dfrac{(\omega-\omega_0)M}{2}\right)}{\sin\left(\dfrac{\omega-\omega_0}{2}\right)}\right] \\ &\quad +\frac{1}{2}\cos\left[\frac{(\omega+\omega_0)(M-1)}{2}\right]\left[\frac{\sin\left(\dfrac{(\omega+\omega_0)M}{2}\right)}{\sin\left(\dfrac{\omega+\omega_0}{2}\right)}\right] \qquad (2.10.1) \end{aligned}$$

Solution: First note that if the sequence $x[n]$ is a real-valued sequence, then the real part of its DTFT $X_{\mathrm{R}}(e^{j\omega})$ is given by

$$X_{\mathrm{R}}(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\cos(n\omega)$$

Hence for the given sinusoidal pulse, we have

$$\begin{aligned} X_{\mathrm{R}}(e^{j\omega}) &= \sum_{0}^{M-1}\cos(\omega_0 n)\cos(n\omega) \\ &= \frac{1}{2}\sum_{0}^{M-1}\cos[(\omega-\omega_0)n] + \frac{1}{2}\sum_{0}^{M-1}\cos[(\omega+\omega_0)n] \qquad (2.10.2) \end{aligned}$$

Consider the first sum in $(2.10.2)$,

$$\begin{aligned}\sum_{0}^{M-1}\cos[(\omega-\omega_0)n] &= \frac{1}{2}\sum_{0}^{M-1}\{e^{\jmath(\omega-\omega_0)n}+e^{-\jmath(\omega-\omega_0)n}\} = \frac{1}{2}\left\{\frac{1-e^{\jmath(\omega-\omega_0)M}}{1-e^{\jmath(\omega-\omega_0)}}+\frac{1-e^{-\jmath(\omega-\omega_0)M}}{1-e^{-\jmath(\omega-\omega_0)}}\right\}\\
&= \frac{1}{2}\left(\frac{1-\cos(\omega-\omega_0)-\cos[(\omega-\omega_0)M]+\cos[(\omega-\omega_0)(M-1)]}{1-\cos(\omega-\omega_0)}\right)\\
&= \frac{1}{2}\left(\frac{2\sin^2[(\omega-\omega_0)/2]+2\sin[(\omega-\omega_0)/2]\sin[(\omega-\omega_0)(M-1/2)]}{2\sin^2[(\omega-\omega_0)/2]}\right)\\
&= \frac{1}{2}\left(\frac{\sin[(\omega-\omega_0)/2]+\sin[(\omega-\omega_0)(M-1/2)]}{\sin[(\omega-\omega_0)/2]}\right)\\
&= \frac{\cos[(\omega-\omega_0)(M-1)/2]\sin[(\omega-\omega_0)M/2]}{\sin[(\omega-\omega_0)/2]} \qquad (2.10.3)\end{aligned}$$

Similarly,

$$\sum_{0}^{M-1}\cos[(\omega+\omega_0)n] = \frac{\cos[(\omega+\omega_0)(M-1)/2]\sin[(\omega+\omega_0)M/2]}{\sin[(\omega+\omega_0)/2]} \qquad (2.10.4)$$

Substituting $(2.10.3)$ and $(2.10.4)$ in $(2.10.2)$, we obtain the desired result in $(2.10.1)$.

(b) Compute and plot $X_R(e^{\jmath\omega})$ for $\omega_0 = \pi/2$ and $M = 5, 15, 25, 100$. Use the plotting interval of $[-\pi, \pi]$. Comment on your results.

Solution: Using the Dirichlet (or periodic sinc) function [see eq. (4.80) in the textbook]

$$D_M(\omega) = \frac{\sin(M\omega/2)}{M\sin(\omega/2)} \qquad (2.10.5)$$

we can express $(2.10.1)$ as

$$X_R(e^{\jmath\omega}) = \frac{M}{2}\left[\cos\left(\frac{(\omega-\omega_0)(M-1)}{2}\right)D_M(\omega-\omega_0)+\cos\left(\frac{(\omega+\omega_0)(M-1)}{2}\right)D_M(\omega+\omega_0)\right]$$

The following script computes and plots $X_R(e^{\jmath\omega})$ for various M values using the **`diric`** function. The plots are given as normalized plots, that is, $X_R(e^{\jmath\omega})/\max(X_R)$ as a function of ω/π. This way all plots have the same extent in both axes, only their shape changes as a function of M.

```
om0 = pi/2; om = linspace(-1,1,1001)*pi;
omL = om + om0; omH = om - om0;
M = [5,15,25,100];
XR = zeros(4,1001); k = 1;
for l = M
    XR(k,:) = (l/2)*(cos(omH*(l-1)/2).*diric(omH,l) + cos(omL*(l-1)/2).*diric(omL,l));
    k = k+1;
end
XRmax = max(XR,[],2); % Maximum XR for each M
```

```
figure('position',[0,0,7,8]*72);
subplot(4,1,1); plot(om/pi,XR(1,:)/XRmax(1),'linewidth',1); ax = gca;
ax.XAxisLocation = 'origin'; ax.YAxisLocation = 'origin'; box off;
axis([-1.03,1.02,-0.25,1.1]);
ax.XAxis.LineWidth = 0.5; ax.YAxis.LineWidth = 0.5;
xlabel('\omega','verticalalignment','middle',...
    'horizontalalignment','left','position',[1.03,0]);
% ylabel('X_R/M','verticalalignment','top',...
%     'horizontalalignment','center','position',[0,0.75]);
set(gca,'xtick',[-1,-0.5,0,0.5,1],'ytick',[1]);
set(gca,'xticklabel',{'-\pi','-\pi/2',' ','\pi/2','\pi'});
set(gca,'yticklabel',{num2str(XRmax(1))},'ygrid','on');
title('X_R(e^{j\omega}) for M = 5');

subplot(4,1,2); plot(om/pi,XR(2,:)/XRmax(2),'linewidth',1); ax = gca;
ax.XAxisLocation = 'origin'; ax.YAxisLocation = 'origin'; box off;
axis([-1.03,1.02,-0.25,1.1]);
ax.XAxis.LineWidth = 0.5; ax.YAxis.LineWidth = 0.5;
xlabel('\omega','verticalalignment','middle',...
    'horizontalalignment','left','position',[1.03,0]);
% ylabel('X_R/M','verticalalignment','top',...
%     'horizontalalignment','center','position',[0,0.75]);
set(gca,'xtick',[-1,-0.5,0,0.5,1],'ytick',[1]);
set(gca,'xticklabel',{'-\pi','-\pi/2',' ','\pi/2','\pi'});
set(gca,'yticklabel',{num2str(XRmax(2))},'ygrid','on');
title('X_R(e^{j\omega}) for M = 15');

subplot(4,1,3); plot(om/pi,XR(3,:)/XRmax(3),'linewidth',1); ax = gca;
ax.XAxisLocation = 'origin'; ax.YAxisLocation = 'origin'; box off;
axis([-1.03,1.02,-0.25,1.1]);
ax.XAxis.LineWidth = 0.5; ax.YAxis.LineWidth = 0.5;
xlabel('\omega','verticalalignment','middle',...
    'horizontalalignment','left','position',[1.03,0]);
% ylabel('X_R/M','verticalalignment','top',...
%     'horizontalalignment','center','position',[0,0.75]);
set(gca,'xtick',[-1,-0.5,0,0.5,1],'ytick',[1]);
set(gca,'xticklabel',{'-\pi','-\pi/2',' ','\pi/2','\pi'});
set(gca,'yticklabel',{num2str(XRmax(3))},'ygrid','on');
title('X_R(e^{j\omega}) for M = 25');

subplot(4,1,4); plot(om/pi,XR(4,:)/XRmax(4),'linewidth',1); ax = gca;
ax.XAxisLocation = 'origin'; ax.YAxisLocation = 'origin'; box off;
axis([-1.03,1.02,-0.25,1.1]);
ax.XAxis.LineWidth = 0.5; ax.YAxis.LineWidth = 0.5;
xlabel('\omega','verticalalignment','middle',...
    'horizontalalignment','left','position',[1.03,0]);
% ylabel('X_R/M','verticalalignment','top',...
%     'horizontalalignment','center','position',[0,0.75]);
```

```
set(gca,'xtick',[-1,-0.5,0,0.5,1],'ytick',[1]);
set(gca,'xticklabel',{'-\pi','-\pi/2',' ','\pi/2','\pi'});
set(gca,'yticklabel',{num2str(XRmax(4))},'ygrid','on');
title('X_R(e^{j\omega}) for M = 100');
```