EECE-5666 : Midterm-2 Exam Solutions : 2022-SPRG

[40-Points]

Default Plot Parameters:

```
set(0,'defaultfigurepaperunits','points','defaultfigureunits','points');
set(0,'defaultaxesfontsize',10);
set(0,'defaultaxestitlefontsize',1.4,'defaultaxeslabelfontsize',1.2);
```

Problem-1 (15-points)   The Discrete Fourier Transform (DFT)

The following two parts, (a) and (b), are not related to each other.

(a) [9-points] Properties of the DFT

Consider a 4-point sequence $x[n] = \{\alpha, \beta, \delta, \gamma\}$ where $\alpha,\ \beta,\ \delta,$ and γ are (possibly) complex-valued numbers. Let $X[k]$ be the 6-point DFT of $x[n]$.

i. [3-points] Determine the finite-length sequence $y[n]$ whose 6-point DFT is

$$Y[k] = W_3^{2k} X[k].$$

Solution: Since $X[k]$ is a 6-point DFT of $x[n]$, the sequence $x[n]$ is effectively padded by two zeros, that is, $x[n] = \{\alpha, \beta, \delta, \gamma, 0, 0\}$. Using the time-domain circular shift property, we have

$$Y[k] = W_3^{2k} X[k] = W_6^{4k} X[k] \quad \Rightarrow \quad y[n] = x[\langle n-4 \rangle_6]$$

Thus,

$$y[n] = \{\delta, \gamma, 0, 0, \alpha, \beta\}.$$

ii. [3-points] Determine the finite-length sequence $w[n]$ whose 6-point DFT is

$$W[k] = X[\langle k+3 \rangle_6].$$

Solution: Using the frequency-domain circular shift property, we have

$$W[k] = X[\langle k+3 \rangle_6] \quad \Rightarrow \quad w[n] = x[n] W_6^{3n}.$$

Since $W_6^3 = \mathrm{e}^{-\jmath 2\pi 3/6} = \mathrm{e}^{-\jmath\pi} = -1$, we obtain

$$w[n] = x[n](-1)^n = \{\alpha, -\beta, \delta, -\gamma, 0, 0\}.$$

iii. [4-points] Determine the finite-length sequence $v[n]$ whose 3-point DFT is

$$V[k] = X[2k], \quad k = 0, 1, 2.$$

Solution: Since $V[k]$ represents 3 samples of $X[\mathrm{e}^{\jmath\omega}]$ around the unit circle, $v[n]$ represents an aliased version of $x[n]$, that is,

$$\begin{aligned} v[n] &= \sum_{r=-\infty}^{\infty} x[n-3r], && n = 0,1,2 \\ &= \{\alpha,\beta,\delta\} + \{\gamma,0,0\} = \{\alpha+\gamma,\beta,\delta\}, && n = 0,1,2. \end{aligned}$$

(b) [6-points] Let $X[k]$ be an N-point DFT of an N-point sequence $x[n]$.

i. [3-Points] Using the analysis and synthesis equations of the DFT, show that the energy of an N-point sequence satisfies

$$\mathscr{E}_x \triangleq \sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N}\sum_{k=0}^{N-1} |X[k]|^2. \qquad (1b.1)$$

Solution: We begin with the middle term in $(1b.1)$.

$$\begin{aligned} \sum_{n=0}^{N-1} |x[n]|^2 &= \sum_{n=0}^{N-1} x[n]x^*[n] = \sum_{n=0}^{N-1} x[n]\left\{\frac{1}{N}\sum_{k=0}^{N-1} X[k]W_N^{nk}\right\}^* \\ &= \sum_{n=0}^{N-1} x[n]\left\{\frac{1}{N}\sum_{k=0}^{N-1} X^*[k]W_N^{-nk}\right\} = \frac{1}{N}\sum_{k=0}^{N-1} X^*[k]\left\{\sum_{n=0}^{N-1} x[n]W_N^{-nk}\right\} \\ &= \frac{1}{N}\sum_{k=0}^{N-1} X^*[k]X[k] = \frac{1}{N}\sum_{k=0}^{N-1} |X[k]|^2 = \mathscr{E}_x \end{aligned}$$

which proves $(1b.1)$.

ii. [3-Points] Using MATLAB, verify the above relation on the following 9-point sequence

$$x[n] = \{1,2,3,4,5,4,3,2,1\}.$$

MATLAB script:

```
clc; close all; clear;
xn = [1:5,4:-1:1]; N = length(xn); Xk = fft(xn,N);
Exn = sum(xn.*conj(xn)); display(Exn);
```

```
Exn = 85
```

```
EXk = sum(Xk.*conj(Xk))/N; display(EXk);
```

```
EXk = 85.0000
```

This verifies $(1b.1)$.

Problem-2 (13-Points) The Fast Fourier Transform (FFT)

In this problem we will consider a mixed-radix fast Fourier transform algorithm for a $N = 10$ point DFT.

(a) [3-Points] Develop a FFT algorithm which computes five radix-2 DFTs followed by twiddle-factor merging and two radix-5 DFTs. That is, obtain a formula similar to equation (8.50) in the textbook for this 10-point DFT with $N = N_1 N_2$ where $N_1 = 2$ and $N_2 = 5$.

Solution: Let $N = 10 = 2 \times 5$ with $N_1 = 2$ and $N_2 = 5$. Then from (8.46) in the text

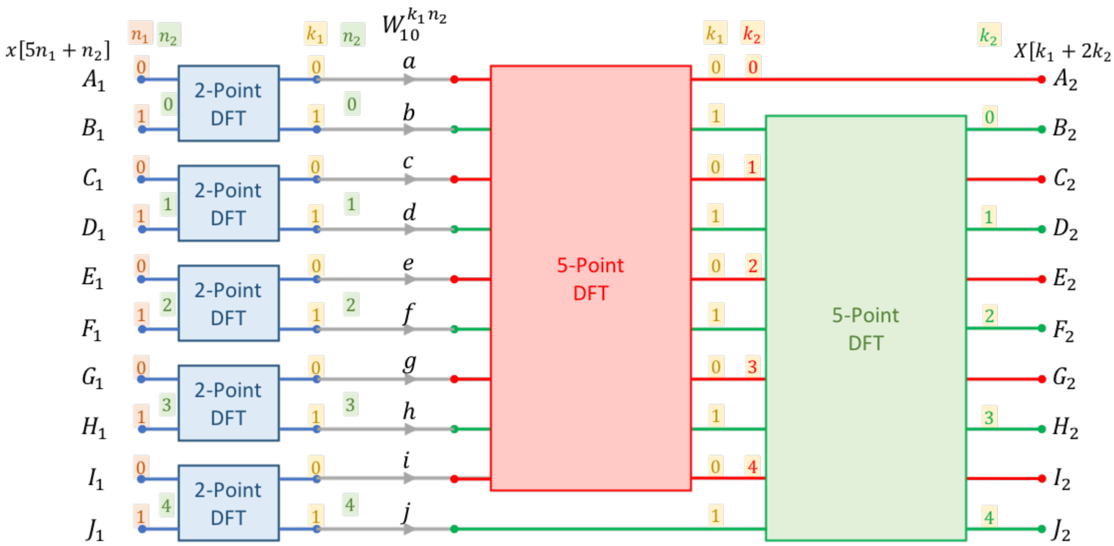
$$\begin{aligned} n &= 5n_1 + n_2; \quad n_1 = 0,1; \quad n_2 = 0,1,2,3,4 \\ k &= k_1 + 2k_2; \quad k_1 = 0,1; \quad k_2 = 0,1,2,3,4 \end{aligned}$$

Then (8.50) becomes

$$X[k_1 + 2k_2] = \underbrace{\sum_{n_2=0}^{4} \left[\underbrace{\left(\sum_{n_1=0}^{1} x[5n_1 + n_2] W_2^{k_1 n_1} \right)}_{X_{n_2}(k_1):\text{2-point DFT}} \underbrace{W_{10}^{k_1 n_2}}_{\text{Twiddle factor}} \right] W_5^{n_2 k_2}}_{\text{5-Point DFT}}. \qquad (2a.1)$$

Thus we have five radix-2 DFTs which are modified by twiddle factor merging followed by two 5-point DFTs.

(b) [6-points] The signal flow graph (SFG) for the formula in $(2a.1)$ can be drawn as follows (with added information for the solution below).



Answer the following subparts based on the above SFG.

i. [2-Point] Determine the input signal samples in the nodes D_1 and G_1. For example, node A_1 has input sample $x[0]$.

Solution: Referring to the above SFG, we have

$$D_1 = x[5(1)+1] = x[6], \quad G_1 = x[5(0+3)] = x[3].$$

ii. [2-Point] Determine twiddle factors d and h.

Solution: Referring to the above SFG, we have

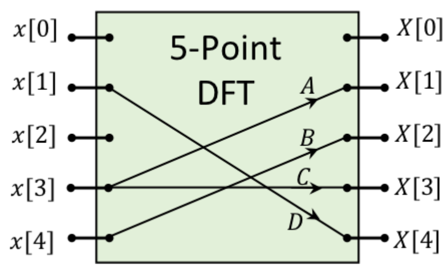
$$d = W_{10}^{(1)(1)} = W_{10}, \quad h = W_{10}^{(1)(3)} = W_{10}^{3}.$$

iii. [2-Point] Determine output signal samples in C_2 and H_2. For example, node A_2 has output sample $X[0]$.

Solution: Referring to the above SFG, we have

$$C_2 = X[0+2(1)] = X[2], \quad H_2 = X[1+2(3)] = X[7].$$

(c) [4-Points] The following diagram shows a partial SFG (to avoid clutter) for the butterfly in radix-5 (i.e., 5-point) DFT. Determine the reduced twiddle factors A, B, C, and D. For example, if the twiddle factor is W_{10}^{12}, then the reduced twiddle factor is $W_{10}^{10+2} = W_{10}^{10}W_{10}^{2} = W_{10}^{2}$.



Answer: The 5-point DFT is given by

$$X[k] = \sum_{k=0}^{4} x[n] W_5^{nk}, \qquad k = 0, 1, 2, 3, 4.$$

Then, referring to the above figure we have

$A = W_5^{nk}\big|_{n=3,k=1} = W_5^3$,

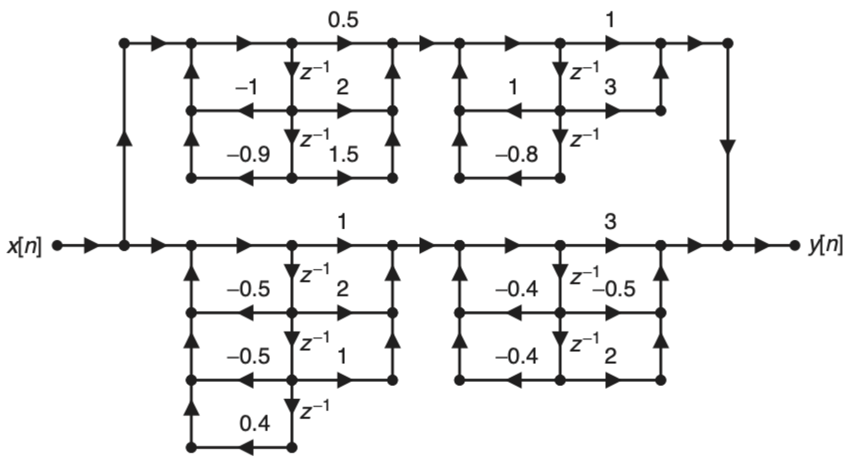
$B = W_5^{nk}\big|_{n=4,k=2} = W_5^8 = W_5^3$,

$C = W_5^{nk}\big|_{n=3,k=3} = W_5^9 = W_5^4$, and

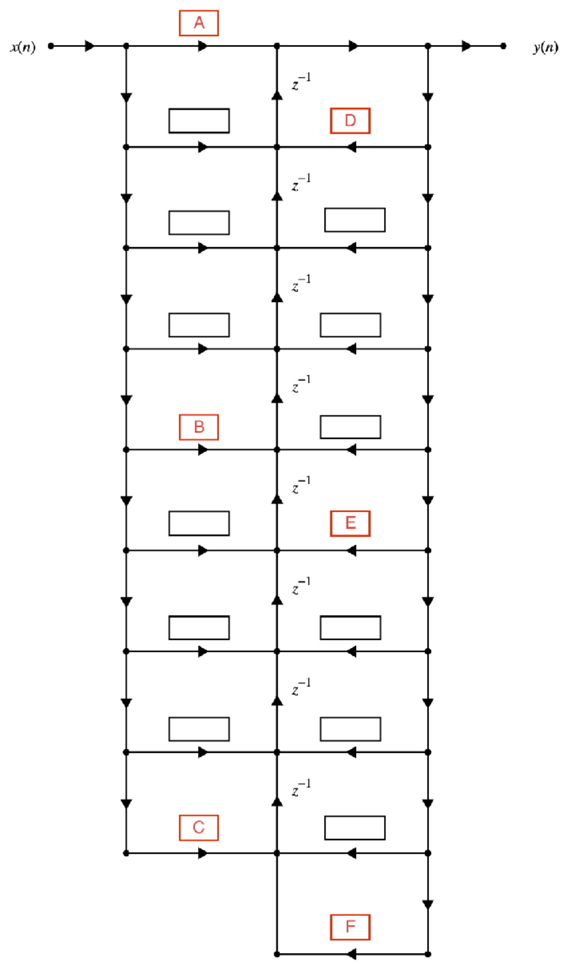
$D = W_5^{nk}\big|_{n=1,k=4} = W_5^4$.

Problem-3 (12-Points)   Digital Filter Structures

The filter structure shown below contains a parallel connection of cascade sections.



(a) [6-Points] The entire filter is to be implemented using a signal flow graph of the form shown below:



The coefficient not shown or labeled are equal to 1. The white rectangular boxes with black borders indicate hidden coefficients. Determine the coefficients A through F hidden behind the red boxes in the above signal flow graph.

MATLAB script for determination: There are two approaches to solve this problem. In the first approach, we do not use any structure conversion functions and emply the **conv** function for polynomial multiplications. In the second approach, we use the structure conversion functions.

Approach-1: Denote the top branch by 'T' and the bottom branch by 'B'. Denote the left structure by 'L' and the right one by 'R'. Denote the numerator polynomial by 'b' and the denominator one by 'a'. Then we have the following variables.

```
clc; clear;
aTL = [1,1,0.9]; bTL = [0.5,2,1.5];          % Top-Left section
aTR = [1,-1,0.8]; bTR = [1,3,0];             % Top-Right section
bT = conv(bTL,bTR); aT = conv(aTL,aTR);      % Top section
aBL = [1,0.5,0.5,-0.4]; bBL = [1,2,1,0];     % Bottom-Left section
aBR = [1,0.4,0.4]; bBR = [3,-0.5,2];         % Bottom-Right section
bB = conv(bBL,bBR); aB = conv(aBL,aBR);      % Bottom section
a = conv(aT,aB); b = conv(bT,aB)+conv(aT,bB); % Entire Direct Form
BA = [b',a']; display(BA);
```

```
BA = 10×2
    3.5000    1.0000
    9.4500    0.9000
   17.3000    1.8000
   22.1500    0.5300
   18.7300    1.4400
   11.0200    0.3780
    3.6700    0.8200
    1.3000   -0.1160
    0.7200    0.0448
         0   -0.1152
```

The required SFG is a transposed direct form II structure. Hence

```
A = b(1); display(A);
```

```
A = 3.5000
```

```
B = b(5); display(B);
```

```
B = 18.7300
```

```
C = b(9); display(C);
```

```
C = 0.7200
```

```
D = -a(2); display(D);
```

```
D = -0.9000
```

```
E = -a(6); display(E);
```

```
E = -0.3780
```

```
F = -a(10); display(F);
```

```
F = 0.1152
```

Enter your coefficient values (up to 4 decimals) below:

$A = 3.5, \quad B = 18.73, \quad C = 0.72, \quad D = -0.9, \quad E = -0.378, \quad F = 0.1152.$

Approach-2: We use the structure conversion functions.

```
% Top Parallel Branch
sosT = [0.5, 2, 1.5, 1,1,0.9; 1,3,0,1,-1,0.8];
[bT,aT] = sos2tf(sosT); % Conversion to direct form
[sosT,CT] = tf2pf(bT,aT); % Conversion to parallel form
% Bottom Parallel Branch
bL = [1,2,1]; aL = [1,0.5,0.5,-0.4]; % 3rd-order (left) section
[sosL] = tf2sos(bL,aL); % conversion to cascade form of left section
sosB = [sosL;3,-0.5,2,1,0.4,0.4]; % Overall cascade form
[bB,aB] = sos2tf(sosB); % Conversion to direct form
[sosB,CB] = tf2pf(bB(1:end-1),aB); % Conversion to parallel form
% Overall Direct Form
sosPF = [sosT;sosB]; % Overall parallel form
[b,a] = pf2tf(sosPF,[]); % Overall direct form
BA = [[b';0],a']; display(BA);
```

```
BA = 10×2
    3.5000    1.0000
    9.4500    0.9000
   17.3000    1.8000
   22.1500    0.5300
   18.7300    1.4400
   11.0200    0.3780
    3.6700    0.8200
    1.3000   -0.1160
    0.7200    0.0448
         0   -0.1152
```

The required SFG is a transposed direct form II structure. Hence

```
A = b(1); display(A);
```

```
A = 3.5000
```

```
B = b(5); display(B);
```

```
B = 18.7300
```

```
C = b(9); display(C);
```

```
C = 0.7200
```

```
D = -a(2); display(D);
```

```
D = -0.9000
```

```
E = -a(6); display(E);
```

```
E = -0.3780
```

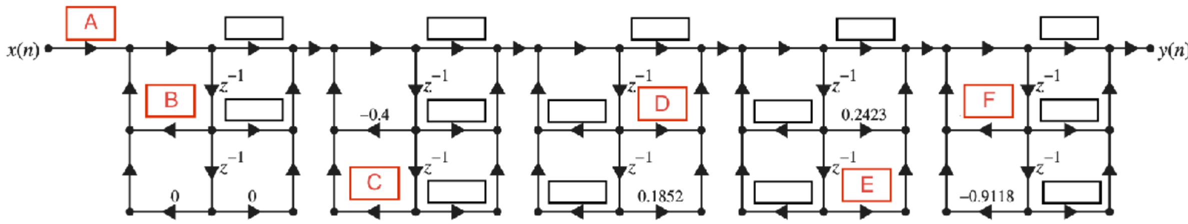
```
F = -a(10); display(F);
```

```
F = 0.1152
```

Enter your coefficient values (up to 4 decimals) below:

$A = 3.5, \quad B = 18.73, \quad C = 0.72, \quad D = -0.9, \quad E = -0.378, \quad F = 0.1152.$

(b) [6-Points] The entire filter is to be implemented using a signal flow graph of the form shown below:



The coefficient not shown or labeled are equal to 1. The white rectangular boxes with black borders indicate hidden coefficients. Determine the coefficients A through F hidden behind the red boxes in the above signal flow graph.

MATLAB script for determination:

```
% (b) Overall Cascade Form
[sosCF,C] = tf2sos(b,a); display(sosCF); display(C);
```

```
sosCF = 5×6
    1.0000    1.0000         0    1.0000   -0.4387         0
    1.0000    0.7486         0    1.0000    0.4000    0.4000
    1.0000   -0.3323    0.1852    1.0000   -1.0000    0.8000
    1.0000    0.2423    1.5819    1.0000    1.0000    0.9000
    1.0000    1.0414    0.9377    1.0000    0.9387    0.9118
C = 3.5000
```

The required SFG is a cascade form with normal direct form II second-order sections. Hence

$A = 3.5, \quad B = 0.4387, \quad C = -0.4, \quad D = -0.3343, \quad E = 1.582, \quad F = -0.9387.$