EECE-5666 : Midterm-2 Exam : 2022-SPRG

130 Minutes between Tuesday 3:30 pm and 6:30 pm

[40-Points]

Instructions:

1. You are required to read the NU Academic Integrity policy (given below) and sign below that the submitted work is your own work. (This is a COE requirement.)
2. You are required to complete this exam using Live Editor.
3. Use of the equation editor to typeset mathematical material such as variables, equations, etc., is strongly recommended. However, in the interest of time, a properly scanned image of a neatly hand-written fragment can be inserted as your work in the required space.
4. After completing this exam, export this Live script to PDF. If you encounter problems in the direct export to PDF, export to Microsoft Word and then print to PDF.
5. Submit the PDF file only. **DO NOT SUBMIT YOUR LIVE EDITOR (.MLX) FILE, IT WILL NOT BE GRADED**.
6. You should complete this exam in two hours with additional 10 minutes for submission activities.
7. The exam is made available for only one attempt.

Academic Integrity Policy

A commitment to the principles of academic integrity is essential to the mission of Northeastern University. The promotion of independent and original scholarship ensures that students derive the most from their educational experience and their pursuit of knowledge. Academic dishonesty violates the most fundamental values of an intellectual community and undermines the achievements of the entire University.

As members of the academic community, students must become familiar with their rights and responsibilities. In each course, they are responsible for knowing the requirements and restrictions regarding research and writing, examinations of whatever kind, collaborative work, the use of study aids, the appropriateness of assistance, and other issues. Students are responsible for learning the conventions of documentation and acknowledgment of sources in their fields. Northeastern University expects students to complete all examinations, tests, papers, creative projects, and assignments of any kind according to the highest ethical standards, as set forth either explicitly or implicitly in this Code or by the direction of instructors. The full academic integrity policy is available at

http://www.northeastern.edu/osccr/academic-integrity-policy/

Declaration

By signing (i.e., entering my name below) and submitting this exam through the submission portal, I declare that I have read the Academic Integrity Policy and that the submitted work is my own work.

Enter your name (Firstname MI Lastname): Tyler B McKean

If you do not enter your name, 2 points will be deducted.

Default Plot Parameters:

```
set(0,'defaultfigurepaperunits','points','defaultfigureunits','points');
set(0,'defaultaxesfontsize',10);
set(0,'defaultaxestitlefontsize',1.4,'defaultaxeslabelfontsize',1.2);
```

Problem-1 (15-points) The Discrete Fourier Transform (DFT)

The following two parts, (a) and (b), are not related to each other.

(a) [9-points] Properties of the DFT

Consider a 4-point sequence $x[n] = \{\alpha, \beta, \delta, \gamma\}$ where α, β, δ, and γ are (possibly) complex-valued numbers. Let $X[k]$ be the 6-point DFT of $x[n]$. In the following sub-parts, you must provide analytical justifications to get full credit.

i. [3-points] Determine the finite-length sequence $y[n]$ whose 6-point DFT is

$$Y[k] = W_3^{2k} X[k].$$

Solution:

In order to determine the finited-length sequence $y[n]$, we have to start by determining the 6-point DFT $X[k]$

$$X[k] = \sum_{n=0}^{5} x[n] W_6^{kn} = \left[\alpha W_6^0, \beta W_6^1, \delta W_6^2, \gamma W_6^3, 0, 0\right] = \left[\alpha, \beta\left(\frac{1}{2} - j\frac{\sqrt{3}}{2}\right), \delta\left(-\frac{1}{2} - j\frac{\sqrt{3}}{2}\right), -\gamma, 0, 0\right]$$

We must then evaluate $Y[k]$ and then take its IDFT to solve for $y[n]$

$$Y[k] = W_3^{2k} X[k] = \left[\alpha W_3^0, \beta W_6^1 W_3^2, \delta W_6^2 W_3^4, -\gamma W_3^6, 0, 0\right] = \left[\alpha, \beta W_6^5, \delta W_3^5, -\gamma, 0, 0\right]$$

Then taking the 6-point IDFT to solve for $y[n]$, we get:

$$y[n] = \frac{1}{N} \sum_{k=0}^{5} Y[k] W_6^{-kn} = \left[\alpha W_6^0, \beta W_6^5 W_6^{-1}, \delta W_6^4 W_6^{-2}, -\gamma W_6^{-3}, 0, 0\right]$$

After simplifying the twiddle factors, we can observe that

$$y[n] = \left[\alpha, \beta W_6^4, \delta W_6^2, -\gamma W_6^{-3}, 0, 0\right] \rightarrow \left[\alpha, \beta\left(\frac{1}{2} + j\frac{\sqrt{3}}{2}\right), \delta\left(-\frac{1}{2} - j\frac{\sqrt{3}}{2}\right), \gamma, 0, 0\right]$$

ii. [3-points] Determine the finite-length sequence $w[n]$ whose 6-point DFT is

$$W[k] = X[\langle k+3 \rangle_6].$$

Solution:

To determine $w[n]$, we first apply the circular left shift of 3 samples to $X[k]$ and then take the resulting IDFT of the sequence:

$$W[k] = X[\langle k+3 \rangle_6] = \left[-\gamma, 0, 0, \alpha, \beta W_6^1, \delta W_6^2\right]$$

$$w[n] = \frac{1}{N}\sum_{k=0}^{5} W[k]W_6^{-kn} = \left[-\gamma W_6^0, 0, 0, \alpha W_6^{-3}, \beta W_6^1 W_6^{-4}, \delta W_6^2 W_6^{-5}\right]$$

Simplifying the twiddle factor terms, we find that the sequence $w[n]$ is

$$w[n] = [-\gamma, 0, 0, -\alpha, -\beta, -\delta]$$

---

iii. [4-points] Determine the finite-length sequence $v[n]$ whose 3-point DFT is

$$V[k] = X[2k], \quad k = 0, 1, 2.$$

Solution:

To determine $v[n]$, first we must solve for $V[k]$ and then proceed to take the 3-point IDFT:

$$V[k] = X[2k] = \left[\alpha, \delta W_6^2, 0\right]$$

Then applying the 3-point IDFT to $V[k]$ to determine $v[n]$, we get

$$v[n] = \frac{1}{N}\sum_{k=0}^{2} V[k]W_3^{-kn} = \left[\alpha W_3^0, \delta W_6^2 W_3^{-1}, 0\right]$$

After simplifying, we observe that

$$v[n] = [\alpha, \delta, 0]$$

(b) [6-points] Let $X[k]$ be an N-point DFT of an N-point sequence $x[n]$.

i. [3-Points] Using the analysis and synthesis equations of the DFT, show that the energy of an N-point sequence satisfies

$$\mathscr{E}_x \triangleq \sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N}\sum_{k=0}^{N-1} |X[k]|^2.$$

Solution:

Let us define the sequence $x[n]$ to be $\{1, 1, 1, 1\}$, thus our value of $N = 4$

The energy of the discrete-time sequence is equal to:

$$E_x = \sum_{n=0}^{3} |x[n]|^2 = 1^2 + 1^2 + 1^2 + 1^2 = 4$$

Now we define the DFT of $x[n]$ to be

$$X[k] = \sum_{n=0}^{3} x[n] W_4^{kn}$$

```
xn = ones(1,4); Xk= fft(xn)
```

```
Xk = 1×4
     4     0     0     0
```

Thus, $X[k] = \{4, 0, 0, 0\}$

Now computing the energy of the DFT we get

$$\frac{1}{4} \sum_{k=0}^{3} |X[k]|^2 = \frac{1}{N} (4^2 + 0^2 + 0^2 + 0^2) = 16/4 = 4$$

Thus, the discrete-time sequence and its 4-point DFT prove the energy equivalence.

---

ii. [3-Points] Using MATLAB, verify the above relation on the following 9-point sequence

$$x[n] = \{\underset{\uparrow}{1}, 2, 3, 4, 5, 4, 3, 2, 1\}.$$

MATLAB script:

```
clc; close all; clear;
x = [1,2,3,4,5,4,3,2,1]; N = 9;
sum(abs(x(:).^2))
```

```
ans = 85
```

```
X = fft(x,9); Xe = 1/N*sum(abs(X(:).^2))
```

```
Xe = 85.0000
```

Thus, the energy is equivalent for the following relation above

---

Problem-2 (13-Points) The Fast Fourier Transform (FFT)

In this problem we will consider a mixed-radix fast Fourier transform algorithm for an $N = 10$ point DFT.

(a) [3-Points] Develop a FFT algorithm which computes five radix-2 DFTs followed by twiddle-factor merging and two radix-5 DFTs. That is, obtain a formula similar to equation (8.50) in the textbook for this 10-point DFT with $N = N_1 N_2$ where $N_1 = 2$ and $N_2 = 5$.

Solution:

The formula for a mixed-radix FFT can be defined as:

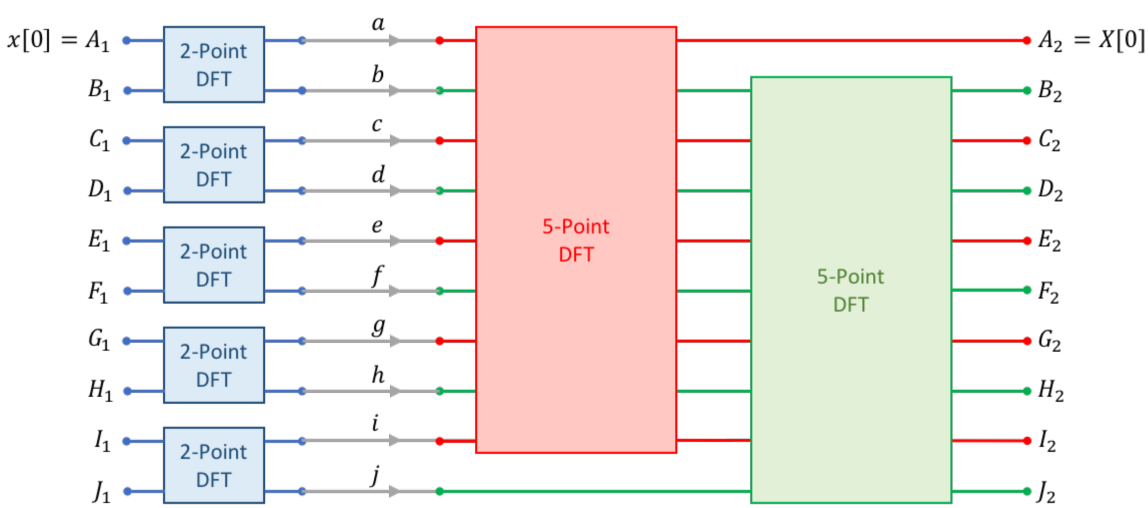
$$X[k_1 + N_1 k_2] = \sum_{n_2=0}^{N_2-1} \left[\left(\sum_{n_1=0}^{N_1-1} x[N_2 n_1 + n_2] W_{N_1}^{k_1 n_1} \right) W_N^{k_1 n_2} \right] W_{N_2}^{k_2 n_2}$$

Filling in a $N = 10$-point DFT with $N_1 = 2$ and $N_2 = 5$, we get

$$X[k_1 + 2k_2] = \sum_{n_2=0}^{4} \left[\left(\sum_{n_1=0}^{1} x[5n_1 + n_2] W_2^{k_1 n_1} \right) W_{10}^{k_1 n_2} \right] W_5^{k_2 n_2}$$

where $n_1 = 0, 1, \quad k_1 = 0, 1, \quad n_2 = 0, 1, 2, 3, 4. \quad k_2 = 0, 1, 2, 3, 4$

(b) [6-points] The signal flow graph (SFG) for the formula in $(2a.1)$ can be drawn as follows.



Answer the following subparts based on the above SFG.

i. [2-Point] Determine the input signal samples in the nodes D_1 and G_1. For example, node A_1 has input sample $x[0]$.

Solution:

From the formula above, we see that the inner DFT operation is performed with an input signal and the next input signal 5 samples away from the current sample.

Thus, for D_1 would be input sample $x[6]$ and G_1 would be input sample $x[3]$

Enter your answers below by clicking next to the $=$ symbol:

$D_1 = x[6]$, $G_1 = x[3]$.

---

ii. [2-Point] Determine twiddle factors d and h.

Solution:

The twiddle factor for d corresponds to the 2nd 2-point DFT

$$\left(\sum_{m=0}^{1} x[5m+1]W_2^{km}\right)W_{10}^{k}$$

so the twiddle factor of d will be equal to W_{10}^{1}

The twiddle factor for h corresponds to the 4th 2-point DFT

$$\left(\sum_{m=0}^{1} x[5m+3]W_2^{km}\right)W_{10}^{3k}$$

so the twiddle factor of h will be equal to W_{10}^{3}

Enter your answers below by clicking next to the $=$ symbol:

$d = W_{10}^{1}$, $h = W_{10}^{3}$.

---

iii. [2-Point] Determine output signal samples in C_2 and H_2. For example, node A_2 has output sample $X[0]$.

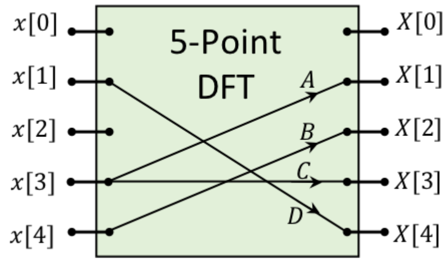
Solution:

The output signal samples for this FFT SFG will be in sequential order, thus $C_2 = X[2]$ and $H_2 = X[7]$.

Enter your answers below by clicking next to the $=$ symbol:

$C_2 = X[2]$, $H_2 = X[7]$.

(c) [4-Points] The following diagram shows a partial SFG (to avoid clutter) for the butterfly in radix-5 (i.e., 5-point) DFT. Determine the reduced twiddle factors A, B, C, and D. For example, if the twiddle factor is W_{10}^{12}, then the reduced twiddle factor is $W_{10}^{10+2} = W_{10}^{10}W_{10}^{2} = W_{10}^{2}$.



Solution:

Here the 5-point DFT can be expressed as:

$$X[n_1 + N_1 k_2] = \left(\sum_{m=0}^{4} x[2m] W_5^{km}\right) + \left(\sum_{m=0}^{4} x[2m+1] W_5^{km}\right)$$

With $A[k] = \left(\sum_{m=0}^{4} x[2m] W_5^{km}\right)$ and $B[k] = \left(\sum_{m=0}^{4} x[2m+1] W_5^{km}\right)$ resulting in

$X[k] = A[k] + B[k] W_{10}^{k}$

$X[k+5] = A[k] + B[k] W_{10}^{k} W_{10}^{5}$

Enter your answers below by clicking next to the $=$ symbol:

$A = \; W_{10}^{7}$,

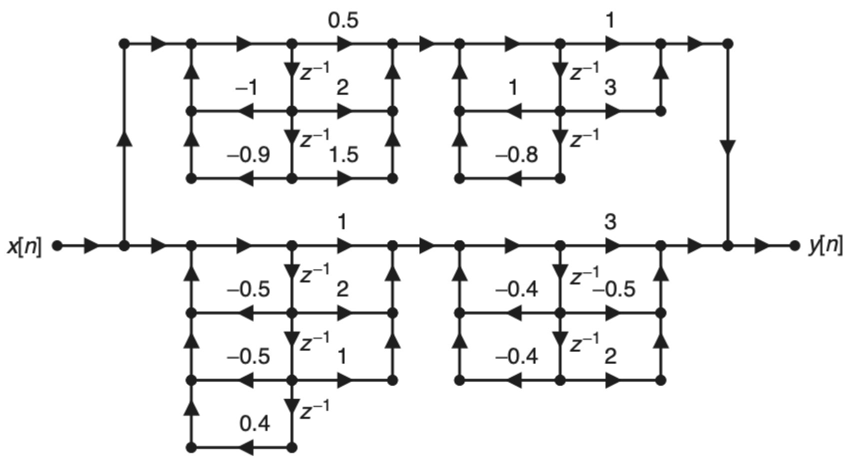
$B = W_{10}^{10} = 1$,

$C = -W_{10}^{9}$, and

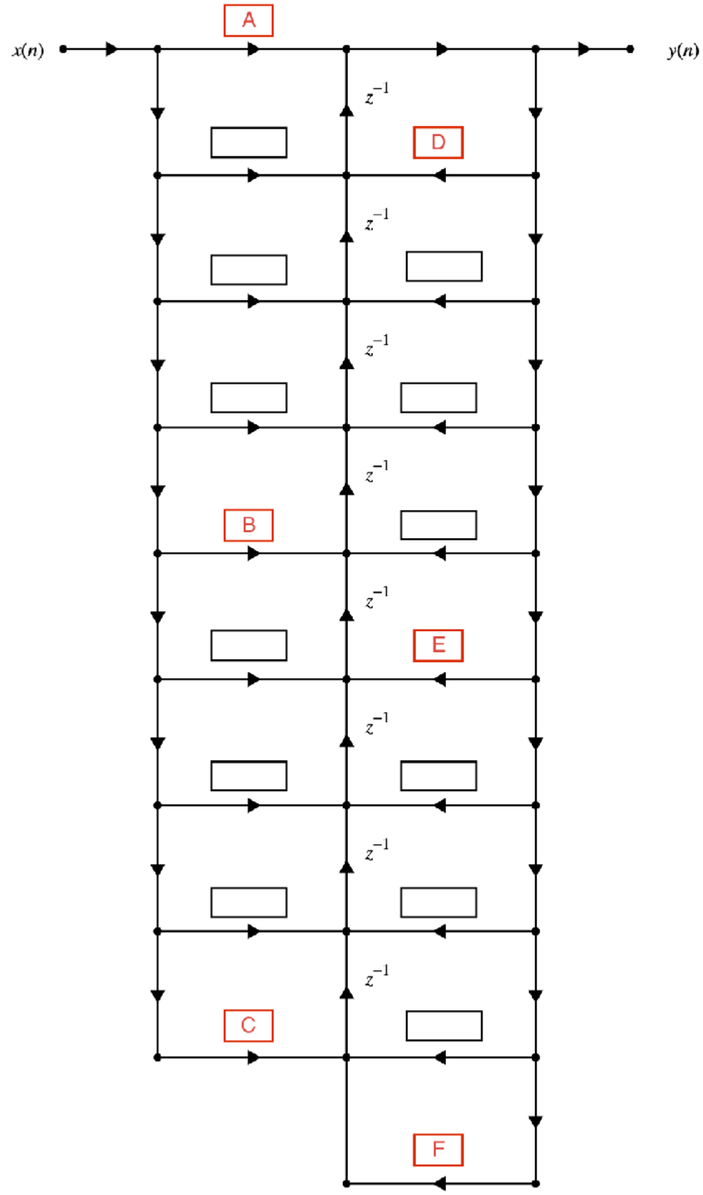
$D = W_{10}^{0} = 1$.

Problem-3 (12-Points) Digital Filter Structures

The filter structure shown below contains a parallel connection of cascade sections.



(a) [6-Points] The entire filter is to be implemented using a signal flow graph of the form shown below:



The coefficient not shown or labeled are equal to 1. The white rectangular boxes with black borders indicate hidden coefficients. Determine the coefficients A through F hidden behind the red boxes in the above signal flow graph.

MATLAB script for determination:

```
clear; close all;
G1 = 1; sos1 = [1 3 0 1 -1 0.8; 0.5 2 1.5 1 1 0.9]; [b1,a1] = sos2tf(sos1,G1)
```

```
b1 = 1×4
    0.5000    3.5000    7.5000    4.5000
a1 = 1×5
    1.0000         0    0.7000   -0.1000    0.7200
```

```
B1 = [1 3 0; 0.5 2 1.5]; A1 = [1 -1 0.8; 1 1 0.9];
[b1,a1] = cas2dir(1,B1,A1), a1 = [a1,0];
```

```
b1 = 1×5
    0.5000    3.5000    7.5000    4.5000         0
a1 = 1×5
    1.0000         0    0.7000   -0.1000    0.7200
```

```
B2 = [1 2 1]; A2 = [1 0.5 0.5 -0.4];
B3 = [3 -0.5 2]; A3 = [1 0.4 0.4];
B = conv(B2,B3); A = conv(A2,A3);
[b2,a2] = cas2dir(1,B,A)
```

```
b2 = 1×5
    3.0000    5.5000    4.0000    3.5000    2.0000
a2 = 1×6
    1.0000    0.9000    1.1000         0    0.0400   -0.1600
```

```
B4 = conv(b1,a2) + conv(b2,a1)
```

```
B4 = 1×10
    3.5000    9.4500   17.3000   22.1500   18.7300   11.0200    3.6700    1.3000 ···
```

```
A4 = conv(a1,a2)
```

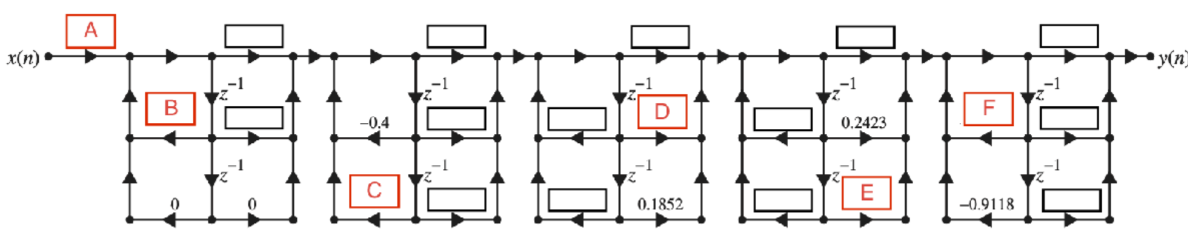
```
A4 = 1×11
    1.0000    0.9000    1.8000    0.5300    1.4400    0.3780    0.8200   -0.1160 ···
```

Enter your coefficient values (up to 4 decimals) below by clicking next to the $=$ symbol:

$A = 3.5$, $B = 18.73$, $C = 0.72$,

$D = -0.9$, $E = -0.378$, $F = 0.1152$.

(b) [6-Points] The entire filter is to be implemented using a signal flow graph of the form shown below:



The coefficient not shown or labeled are equal to 1. The white rectangular boxes with black borders indicate hidden coefficients. Determine the coefficients A through F hidden behind the red boxes in the above signal flow graph.

MATLAB script for determination:

```
[sos,G] = tf2sos(B4,A4)
```

```
sos = 5×6
    1.0000    1.0000         0    1.0000   -0.4387         0
    1.0000    0.7486         0    1.0000    0.4000    0.4000
    1.0000   -0.3323    0.1852    1.0000   -1.0000    0.8000
    1.0000    0.2423    1.5819    1.0000    1.0000    0.9000
    1.0000    1.0414    0.9377    1.0000    0.9387    0.9118
G = 3.5000
```

Enter your coefficient values (up to 4 decimals) below by clicking next to the $=$ symbol:

$A = 3.5$, $B = 0.4387$, $C = -0.4$,

$D = -0.3323$, $E = 1.5819$, $F = -0.9387$.