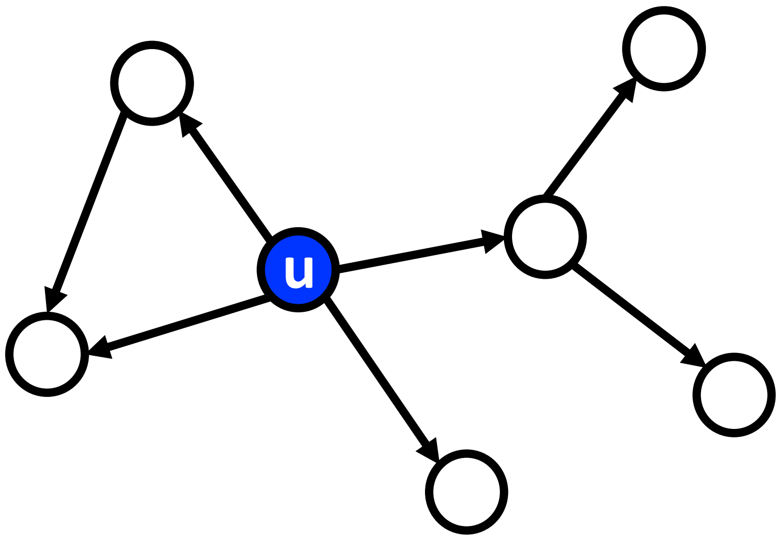
Outline

Network Characteristics – Node

- **Structural Importance**
  - Geometric
  - Connectedness

Structural Importance – Revisited

- Measures of positional importance for each node in the network



Geometric

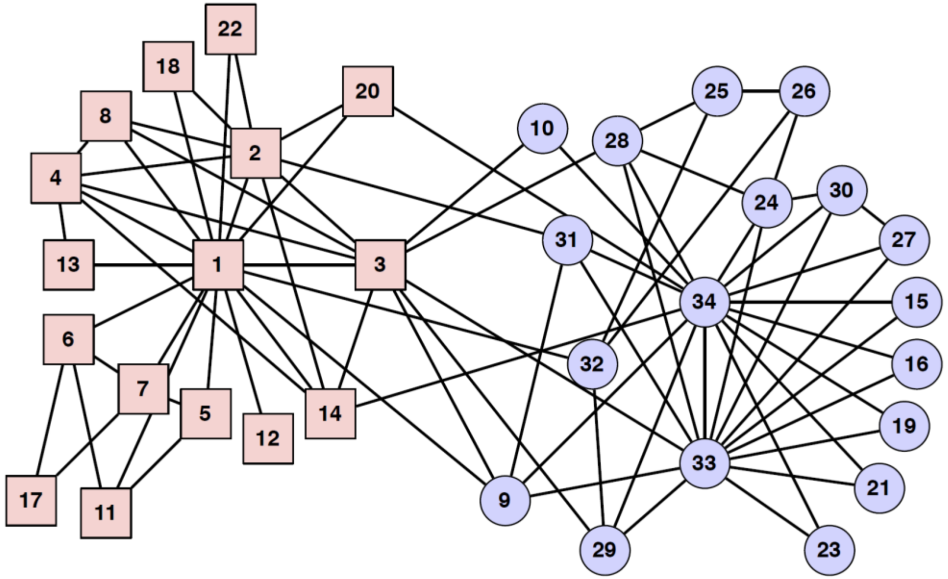
- Closeness Centrality
- Harmonic Centrality
- Betweenness Centrality

Connectedness

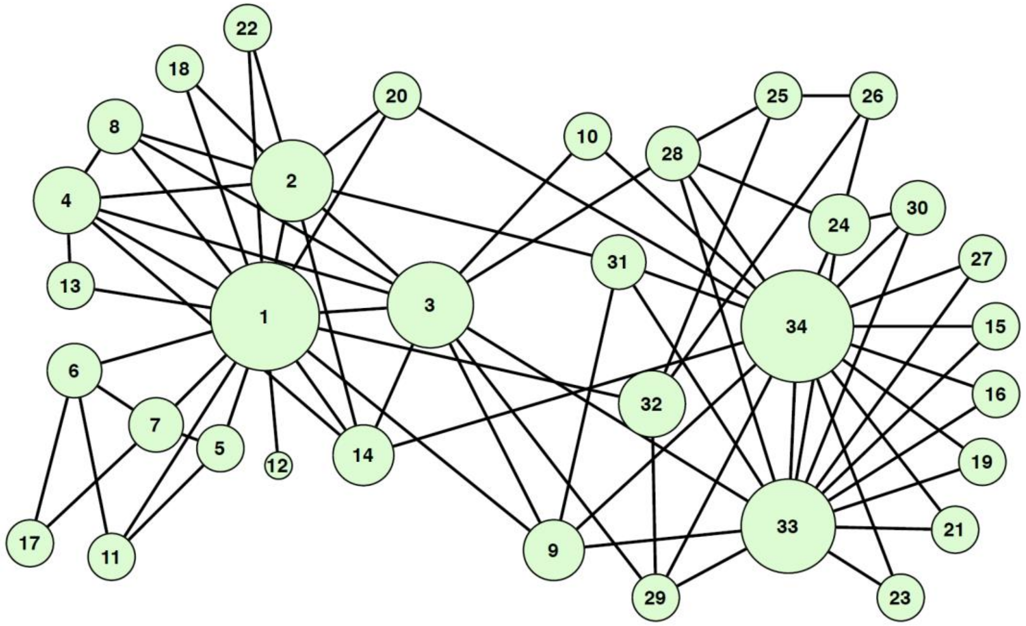
- Degree Centrality
- Eigenvector Centrality
- PageRank

Structural Importance – Degree Centrality

- The simplest measure of importance
  - Degree of a vertex k_i
  - Vertices with larger degrees exert greater effect on the network
  - Highly-connected vertices are likely attributed to large-scale organization of the network



[Zachary's Karate Club]



[Based on Degree Centrality]

Structural Importance – Degree Centrality

- Normalized Degree Centrality
  - Normalized by the maximum possible degree:

$$C_{\mathrm{D}}(i) = \frac{k_i}{|\boldsymbol{V}| - 1}$$

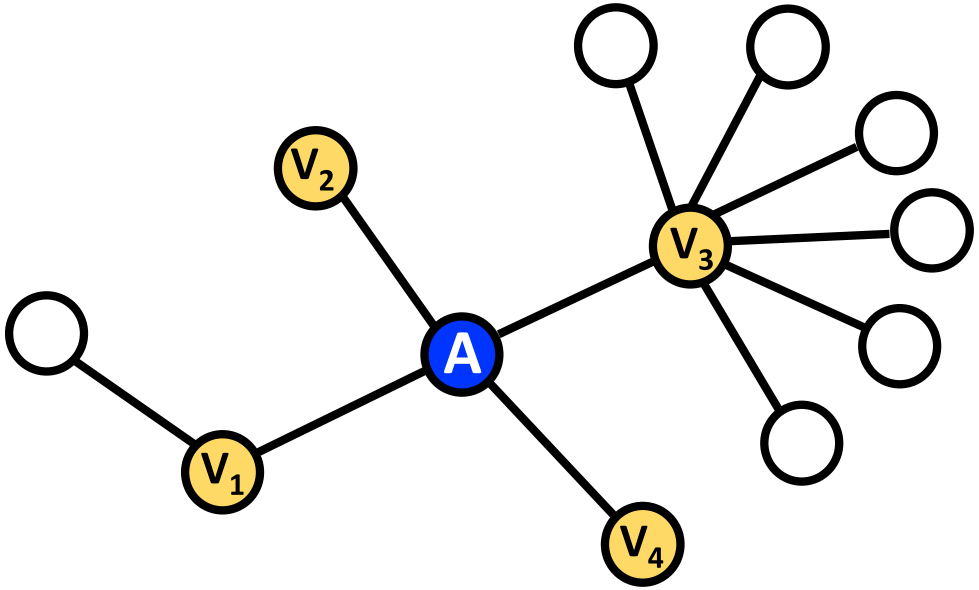
  - Normalized by the maximum degree:

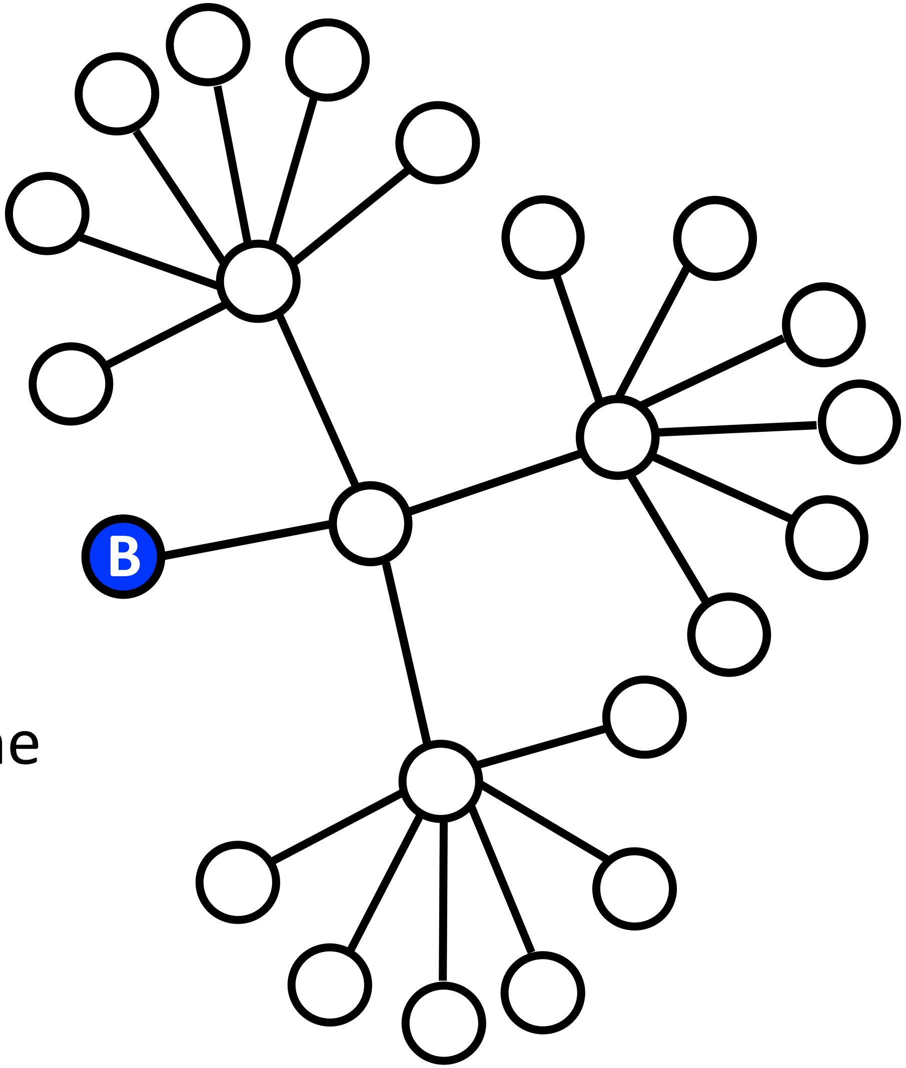
$$C_{\mathrm{D}}(i) = \frac{k_i}{\max_j(k_j)}$$

  - Normalized by the number of edges in the network:

$$C_{\mathrm{D}}(i) = \frac{k_i}{|\boldsymbol{E}|}$$

Structural Importance – More Considerations





- Are importance score $I(\cdot)$ of A's neighbors all same such that $I(v_1) = I(v_2) = I(v_3) = I(v_4)$?
- Then, what do you think about $I(A)$ and $I(B)$?

Structural Importance – Eigenvector Centrality

Not all neighbors are necessarily equivalent.

- ‘How many people you know’ + **‘Who you know’**

  Degree Centrality

- ***Eigenvector Centrality***: an extension of degree centrality
- A node’s importance is proportional to the importance scores of its neighbors:

$$x_i \propto \sum_j A_{ij} x_j$$

Structural Importance – Eigenvector Centrality

- Node's importance: proportional to the importance scores of its neighbors

$$x_i \propto \sum_j A_{ij} x_j \quad \blacktriangleright \quad x_i = \frac{1}{\lambda} \sum_j A_{ij} x_j \quad \blacktriangleright \quad \boldsymbol{x} = \frac{1}{\lambda} \mathbf{A} \boldsymbol{x}$$

$$\boxed{\mathbf{A}\boldsymbol{x} = \lambda \boldsymbol{x}}$$

- $\boldsymbol{x}$ is the vector whose elements are importance scores x_i of nodes in a network.

$$\begin{cases} \boldsymbol{x} \text{ – } \textit{eigenvector} \text{ of the adjacency matrix.} \\ \lambda \text{ – } \textit{eigenvalue} \text{ of the adjacency matrix.} \end{cases}$$

- $n \times n$ matrix:

$$\begin{cases} n \text{ pairs of eigenvectors and eigenvalues} \\ \text{Then, which pair of eigenvector and eigenvalue should we use?} \end{cases}$$

Structural Importance – Eigenvector Centrality

- ***Perron-Frobenius* theorem**:
  - A matrix with all non-negative elements (*e.g.*, adjacency matrix) has only one eigenvector whose elements are all non-negative (*leading eigenvector*) and the corresponding *unique largest real eigenvalue*.

$$\boxed{\mathbf{A}\boldsymbol{x} = \lambda \boldsymbol{x}}$$

  - A node i's centrality (x_i) is the i-th element of the leading eigenvector $\boldsymbol{x}$ of an adjacency matrix $\mathbf{A}$.
  - λ is the corresponding largest eigenvalue.

Recitation – Eigenvector and Eigenvalue

Let's have a recitation session for fundamental *Linear Algebra*.