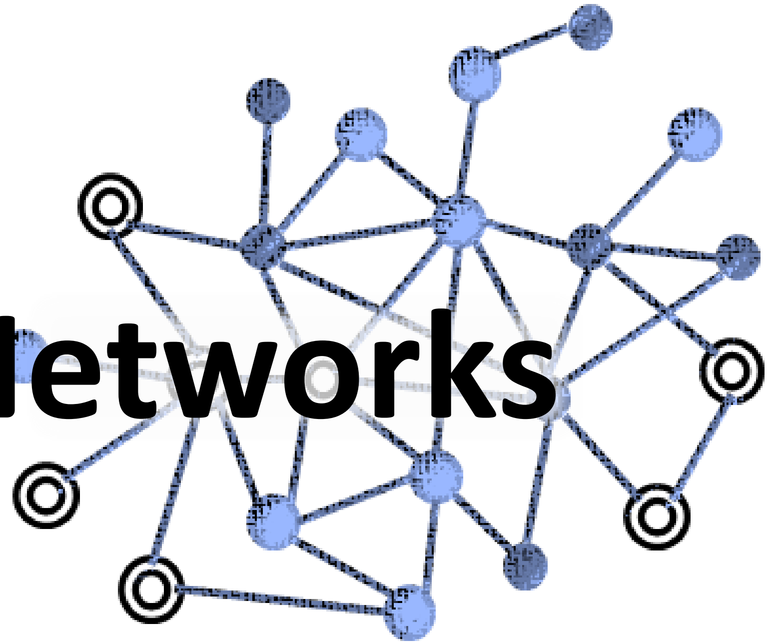
선문대학교
SUN MOON UNIVERSITY

Lecture 03

Social & Information Networks



김 민 경

AI소프트웨어학과

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Outline – Last Lecture

- **Introduction to Networks**
  - Historical Background
  - Definition of a Network
  - Categories of Networks
  - Applications

Outline – This Lecture

- **Basic Mathematics of Networks**
  - Representing Networks
  - Describing Networks – Overview

Mathematics of Networks – Why?

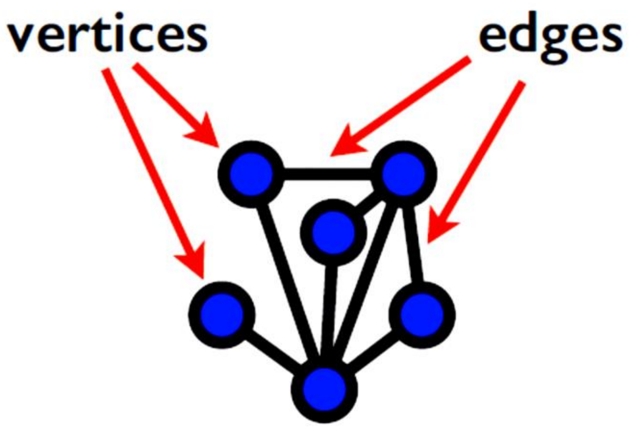
- Basic theoretical tools
  - **Quantify** network structures
  - **Analyze** networks
  - **Reveal** remarkable **patterns** in the real-world networks

Outline

- **Basic Mathematics of Networks**
  - **Representing Networks**
  - Describing Networks – Overview

Mathematical Representation

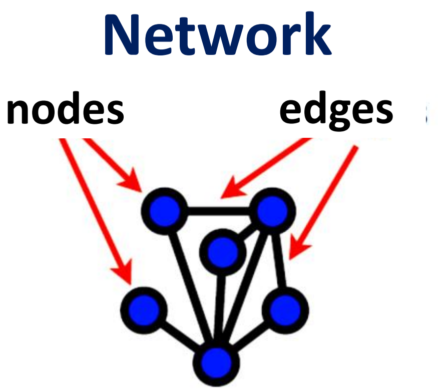
Network Components

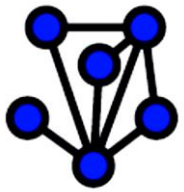


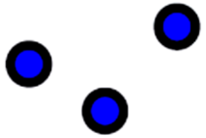
Mathematical literature

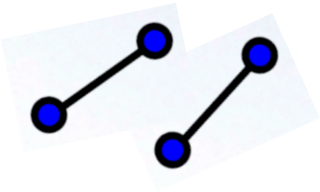
- A Network (Graph)
  - [Object]: node (vertex) $\boldsymbol{N}$
  - [Interaction]: link (edge) $\boldsymbol{E}$
  - [Complex System]: network (graph) $G(\boldsymbol{N}, \boldsymbol{E})$

Examples of Network Components









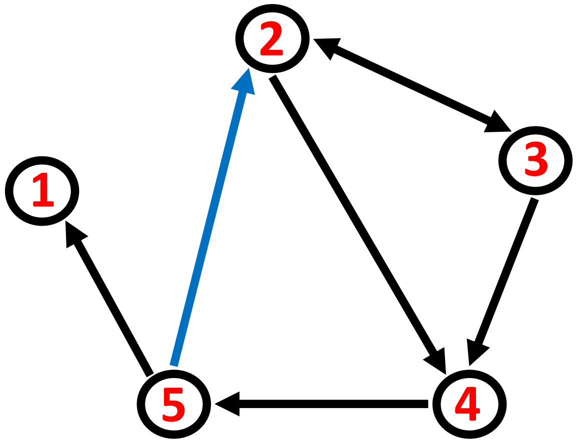
Network	Node	Edge
Internet	Computer or router	Cable or wireless data connection
World Wide Web	Web page	Hyperlink
Citation network	Article, patent, or legal case	Citation
Power grid	Generating station or substation	Transmission line
Friendship network	Person	Friendship
Metabolic network	Metabolite	Metabolic reaction
Neural network	Neuron	Synapse
Food web	Species	Predation

[Table 6.1] in Chap 6.

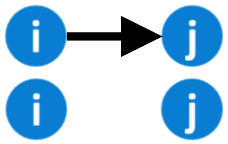
Representing Networks

- Network Data Representation
  - ***Adjacency Matrix***: the fundamental mathematical representation of a network
  - The adjacency matrix $\boldsymbol{A}$ of a network is defined to be the $n \times n$ matrix with elements A_{ij} such that

$$A_{ij} = \begin{cases} 1 & \text{if there is an edge between nodes } i \text{ and } j, \\ 0 & \text{otherwise.} \end{cases}$$



$$\begin{cases} A_{ij} = 1 \\ A_{ij} = 0 \end{cases}$$

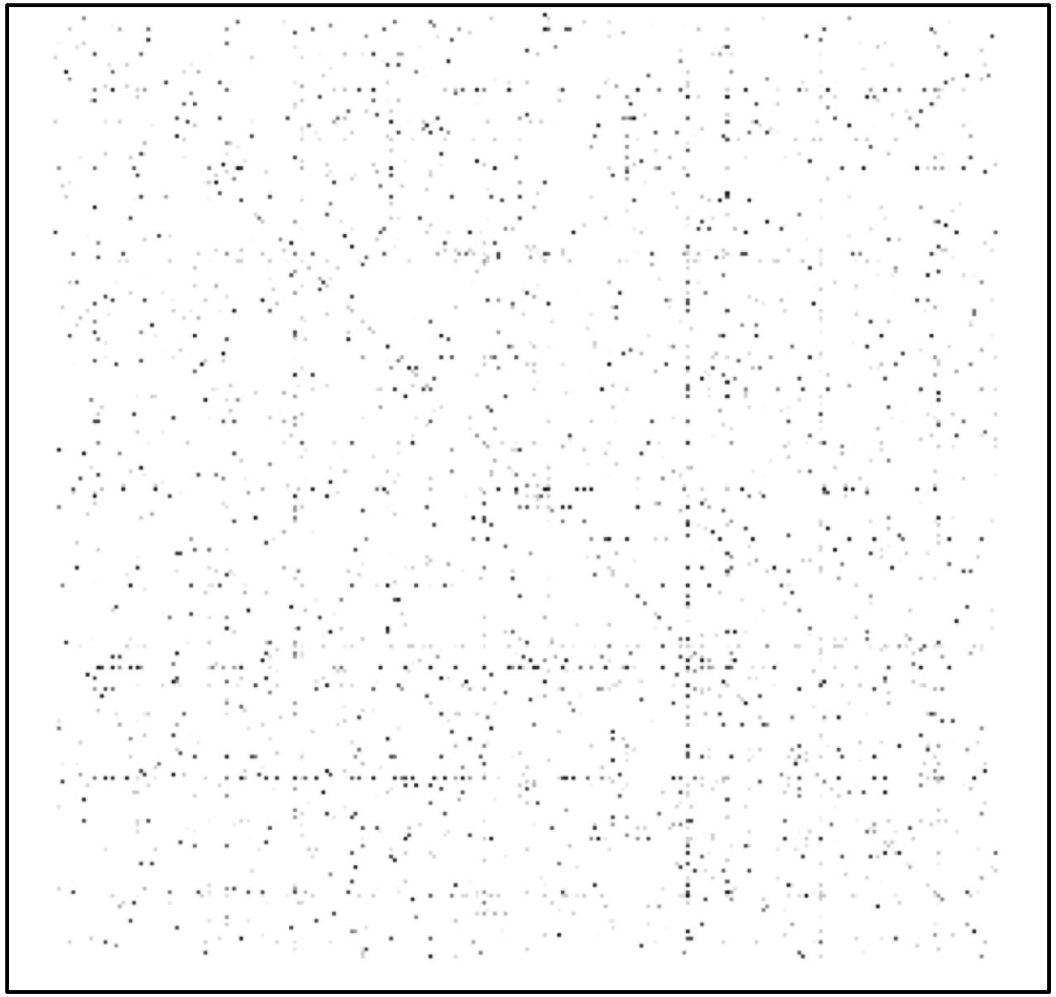


$$\boldsymbol{A} = \begin{array}{c} \\ \\ \\ \\ \\ \text{row } i \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & \boxed{1} & 0 & 0 & 0 \end{bmatrix} \\ \text{column } j \end{array}$$

$$A_{52} = 1$$

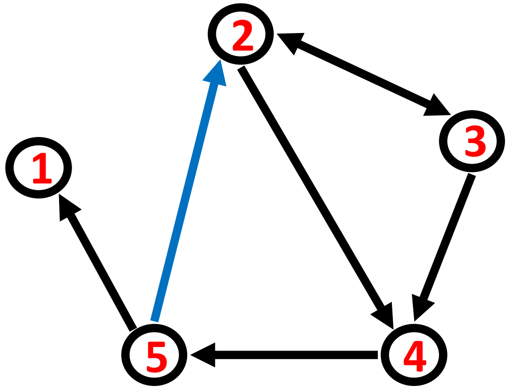
Real-word networks are sparse

- Ex) Adjacency matrix of a real network

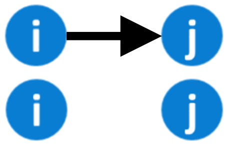


Representing Networks

- Network Data Representation
  - Adjacency Matrix



$$\begin{cases} A_{ij} = 1 \\ A_{ij} = 0 \end{cases}$$



$$\begin{array}{cc} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ A = & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & \boxed{1} & 0 & 0 & 0 \end{bmatrix} \end{array}$$

row i

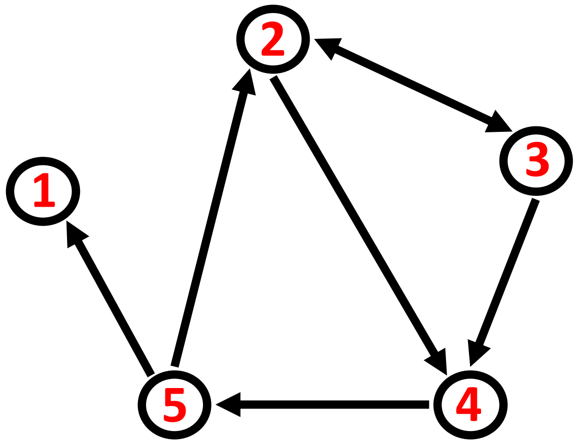
column j

$$A_{52} = 1$$

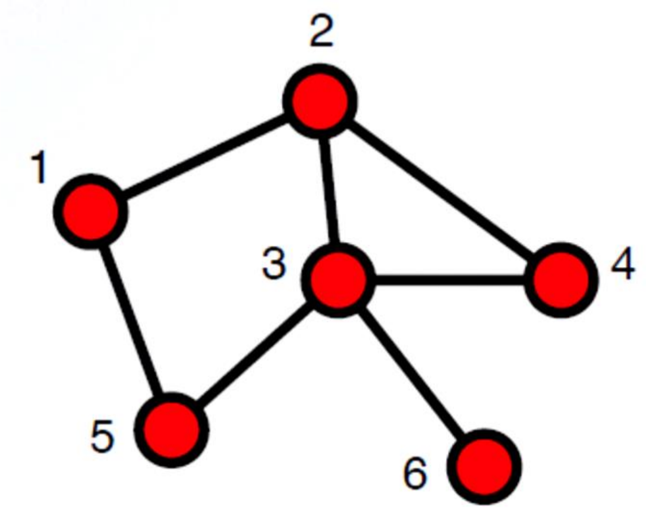
- ***Edge List***: $\{(i, j) \mid A_{ij} = 1\}$
  - (2,3), (2,4)
  - (3,2), (3,4)
  - (4,5)
  - (5,2), (5,1)

Representing Networks

- ***Adjacency list***:
  - Easier to work with if network is
    - Large
    - Sparse
  - Allows us to quickly access all connected nodes of a given node
    - 1:
    - 2: 3, 4
    - 3: 2, 4
    - 4: 5
    - 5: 1, 2



Representing Networks – a simple network

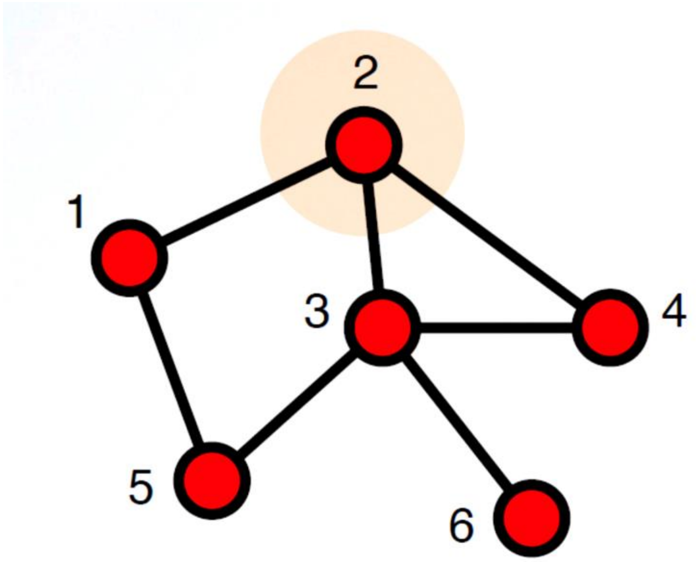


Undirected

Unweighted

No self-loop

Representing Networks – a simple network



adjacency matrix

A	1	2	3	4	5	6
1	0	1	0	0	1	0
2	1	0	1	1	0	0
3	0	1	0	1	1	1
4	0	1	1	0	0	0
5	1	0	1	0	0	0
6	0	0	1	0	0	0

adjacency list

A

$1 \to \{2, 5\}$

$2 \to \{1, 3, 4\}$

$3 \to \{2, 4, 5, 6\}$

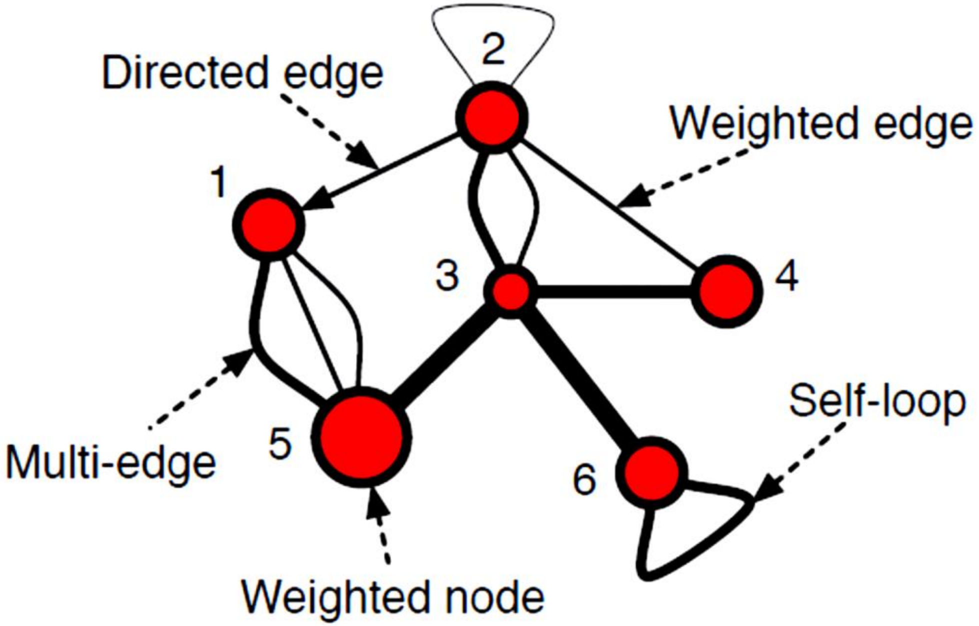
$4 \to \{2, 3\}$

$5 \to \{1, 3\}$

$6 \to \{3\}$

➡ Can you find some characteristics here?

Representing Networks – a less simple network

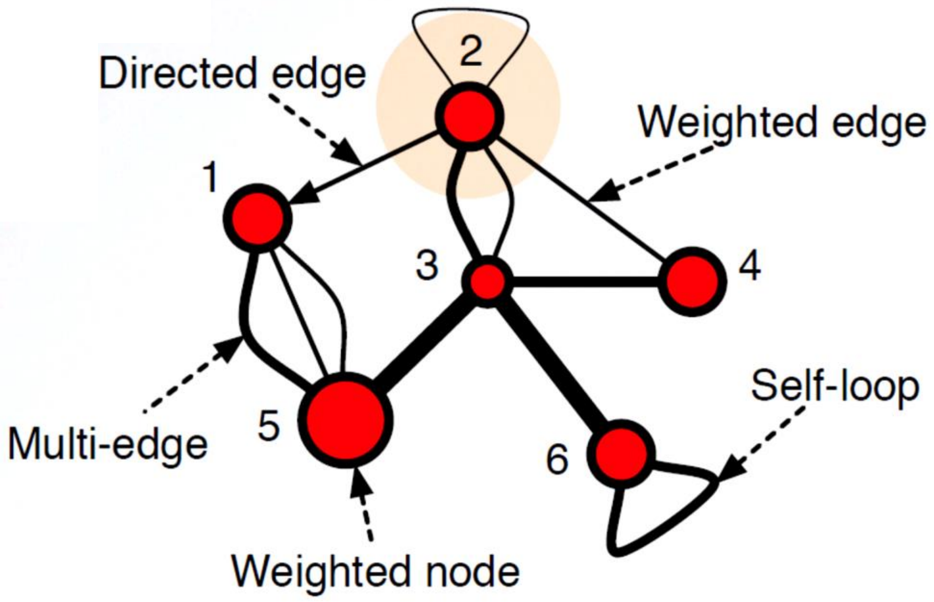


~~Undirected~~

~~Unweighted~~

~~No self-loop~~

Representing Networks – a less simple network



adjacency matrix **{weight}**

A	1	2	3	4	5	6
1	0	0	0	0	$\{1,1,2\}$	0
2	1	$\frac{1}{2}$	$\{2,1\}$	1	0	0
3	0	$\{2,1\}$	0	2	4	4
4	0	1	2	0	0	0
5	$\{1,1,2\}$	0	4	0	0	0
6	0	0	4	0	0	2

adjacency list **{(node, weight)}**

A	
1	$\rightarrow \{(5,1),(5,1),(5,2)\}$
2	$\rightarrow \{(1,1),(2,\frac{1}{2}),(3,2),(3,1),(4,1)\}$
3	$\rightarrow \{(2,2),(2,1),(4,2),(5,4),(6,4)\}$
4	$\rightarrow \{(2,1),(3,2)\}$
5	$\rightarrow \{(1,1),(1,1),(1,2),(3,4)\}$
6	$\rightarrow \{(3,4),(6,2)\}$

Representing Networks – directed networks

- A Directed Network as A Matrix

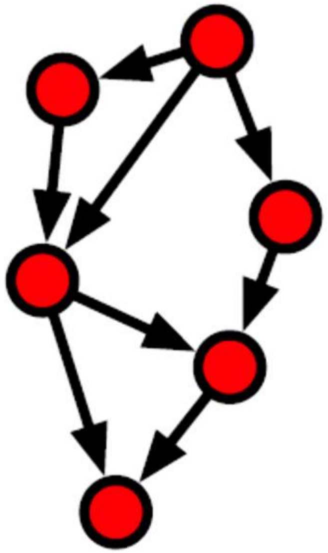
$$A_{ij} \neq A_{ji} \longrightarrow \text{asymmetric}$$

Citation networks

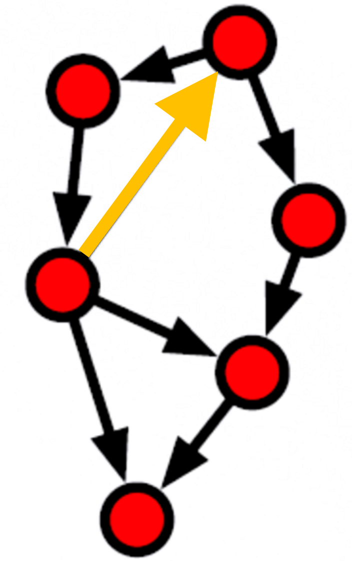
Foodwebs

Epidemiological

…



Directed acyclic graph



Directed graph

WWW

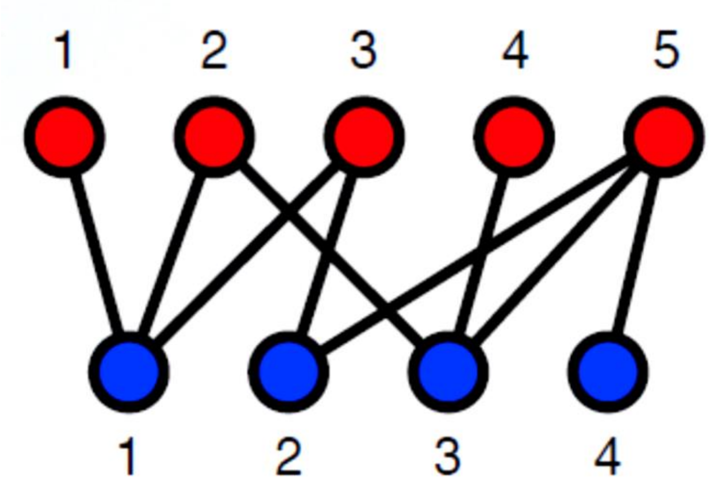
Online friendship

Transportation

…

Representing Networks – bipartite networks

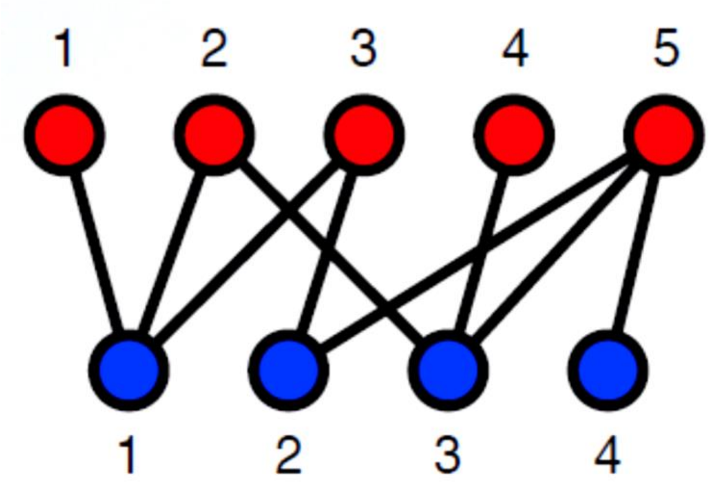
- Examples in the real world



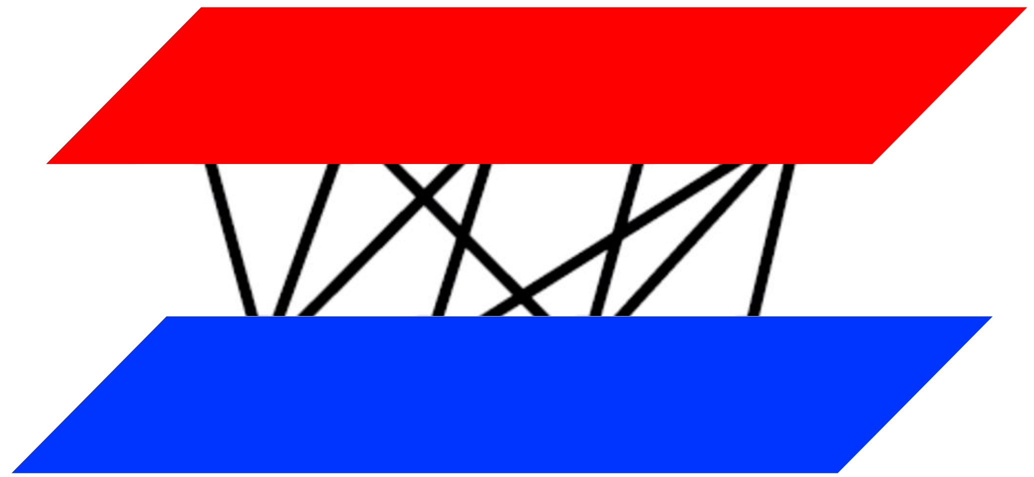
- Authors vs Papers
- Actors vs Movies
- Musicians vs Albums
- Customers vs Products
- People vs Online Communities
- People vs Visiting Locations
- Words vs Documents
- ⋮

Representing Networks – bipartite networks

- What does a bipartite network reflect?



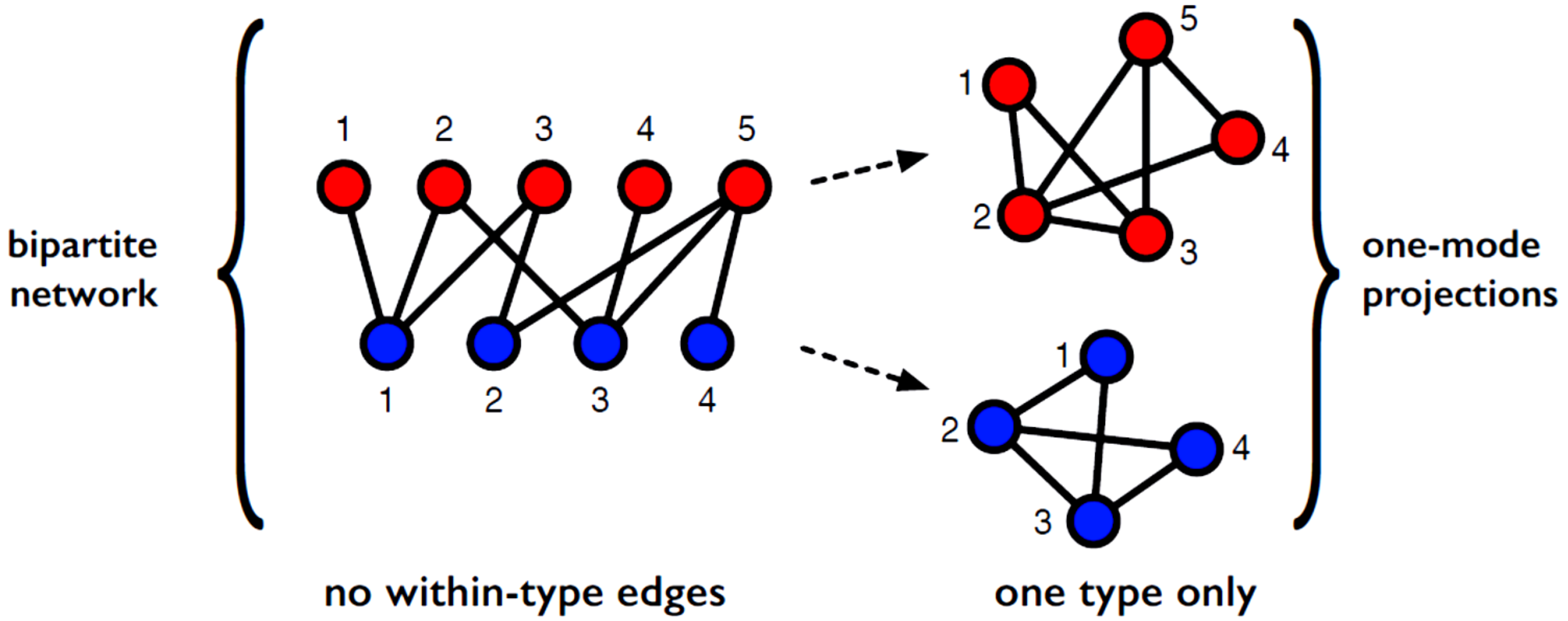
Interplay between two different populations (layers)



Representing Networks – bipartite networks

- What does a bipartite network reflect?

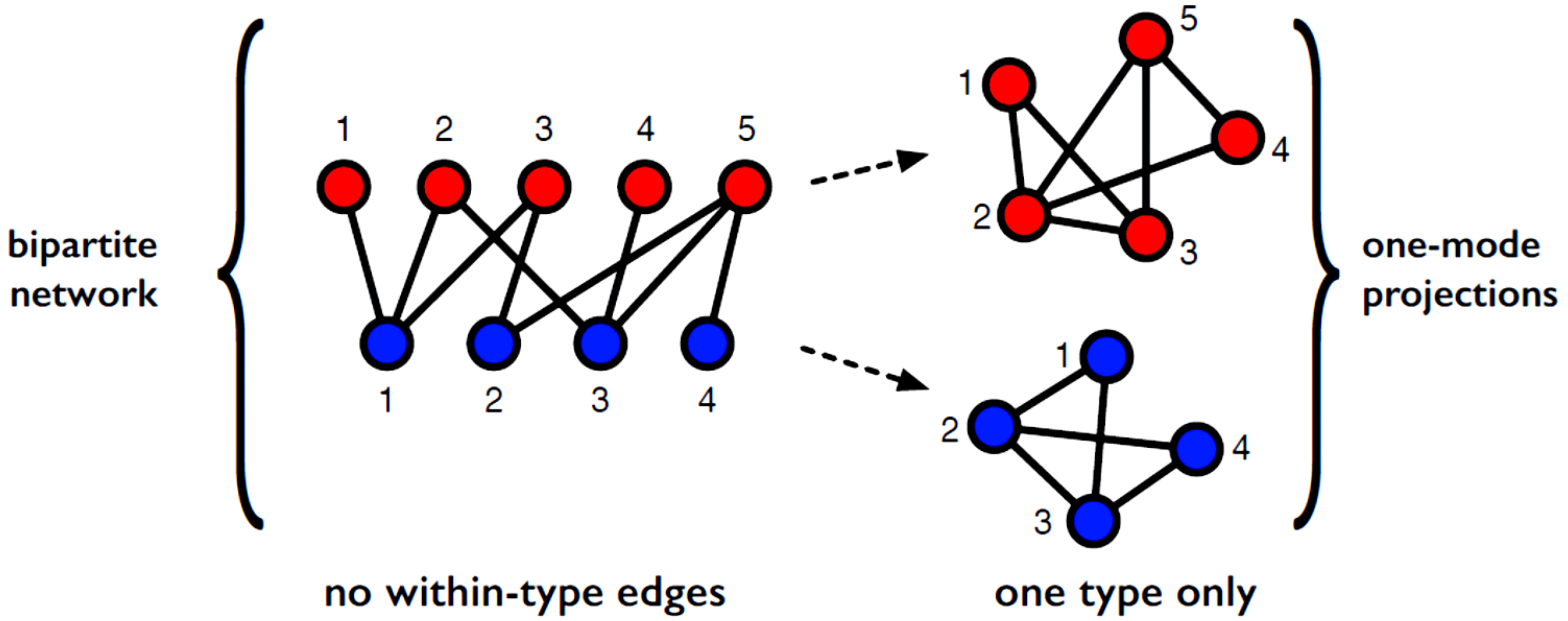
Projections into each population (layer)



Representing Networks – bipartite networks

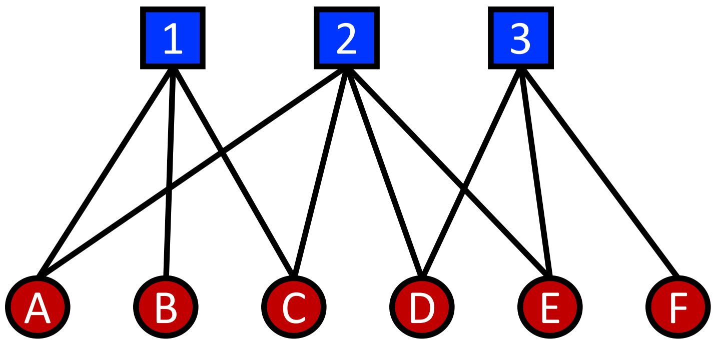
- What does a bipartite network reflect?
  - Reveals hidden(invisible) connectivity within each population

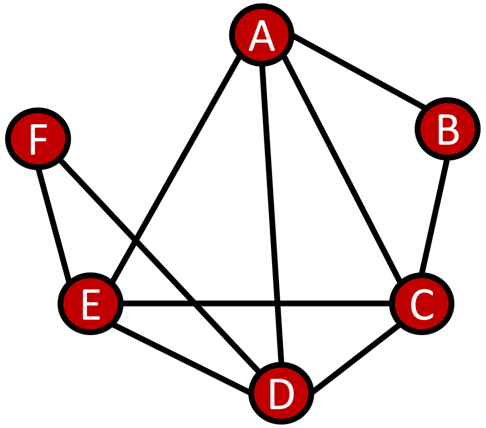
Projections into each population (layer)

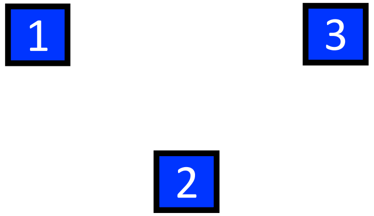


Representing Networks – bipartite networks

- Example)







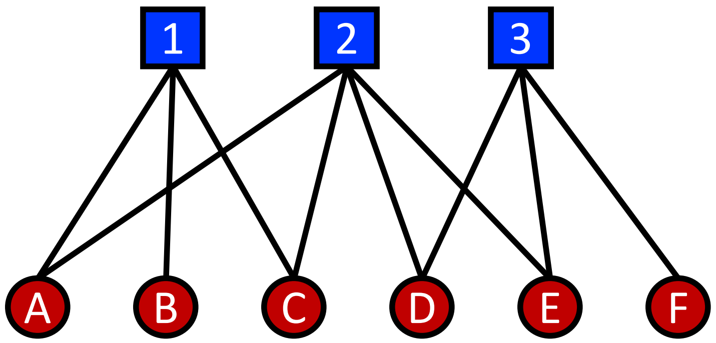
$$
\boldsymbol{X} = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \end{array}
\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \end{array}
$$

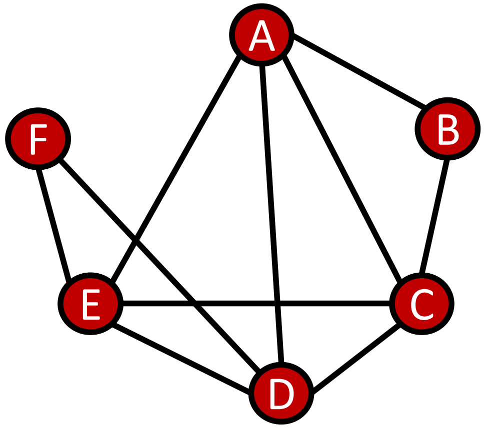
$$
\boldsymbol{X}\boldsymbol{X}^{\mathsf{T}} = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \end{array}
\begin{array}{c} \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{bmatrix} 2 & 1 & 2 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 2 & 1 & 1 & 0 \\ 1 & 0 & 1 & 2 & 2 & 1 \\ 1 & 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}
$$

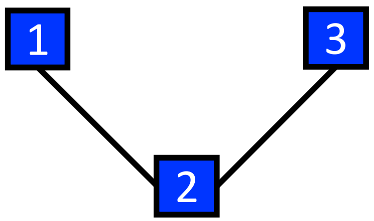
?

Representing Networks – bipartite networks

- Example)







$$X = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

$$XX^{\top} = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \end{array} \begin{array}{c} \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{bmatrix} 2 & 1 & 2 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 2 & 1 & 1 & 0 \\ 1 & 0 & 1 & 2 & 2 & 1 \\ 1 & 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

$$X^{\top} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

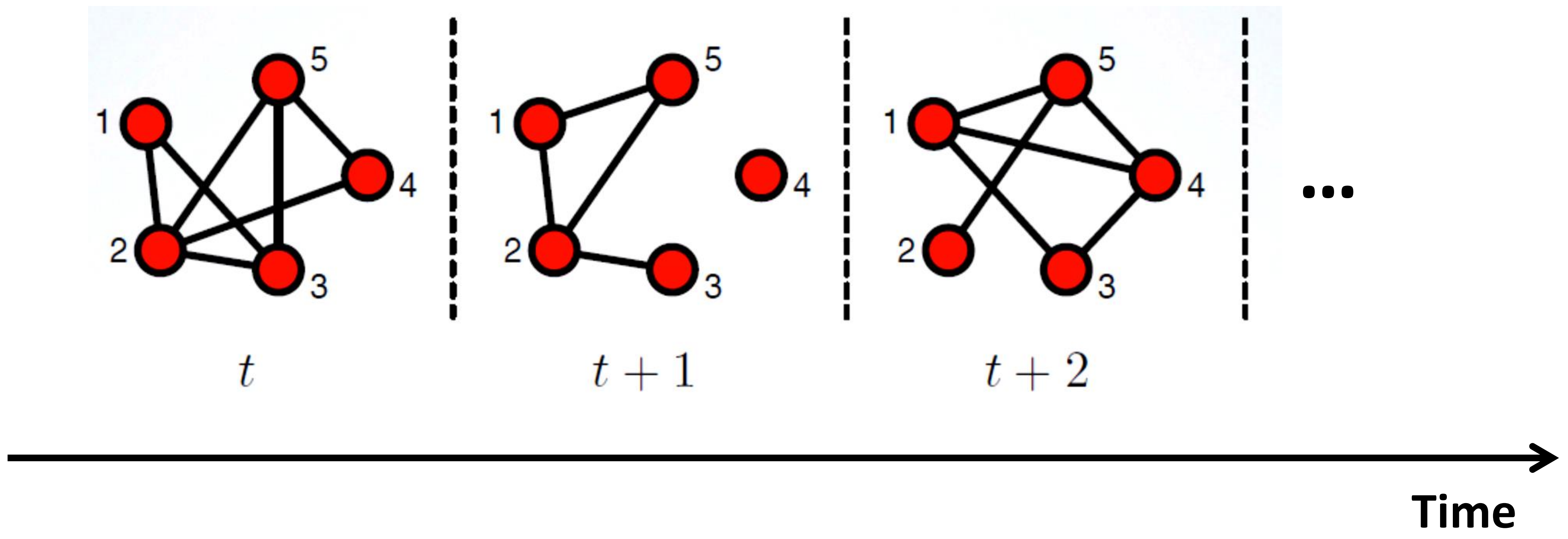
$$X^{\top}X = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 3 & 2 & 0 \\ 2 & 4 & 2 \\ 0 & 2 & 3 \end{bmatrix} \end{array}$$

Representing Networks – temporal networks

- Any network over time

Edges

- Discrete time (snapshots) (i, j, t)
- Continuous time $(i, j, t_s, \Delta t)$



Outline

- **Basic Mathematics of Networks**
  - Representing Networks
  - **Describing Networks – Overview**

Describing Networks

- What do networks look like?
  - How are the **edges organized**?
  - Do **nodes** have different **characteristics**?
  - Are there **patterns**?

What processes lead to form networks?

In this course

Pattern Recognition

Modeling & Simulation

Graduate Level

Describing Networks

- Fundamental Step: describing **network features**

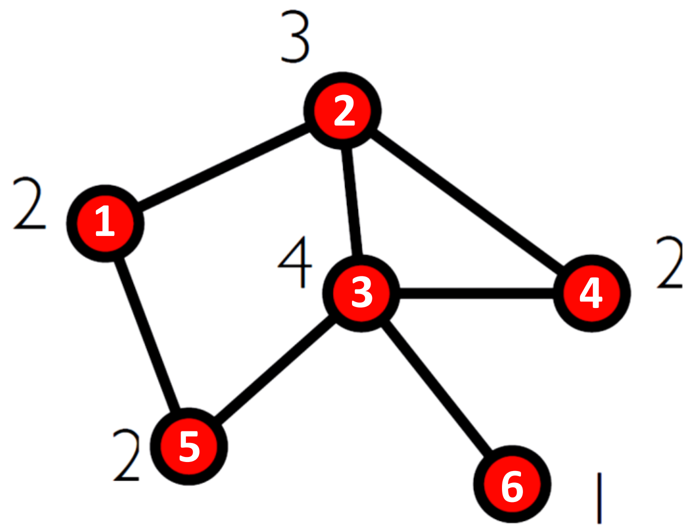
Quantifying Structural Properties:

What does **local-level structure** look like? — **Microscopic View**

What does **large-scale structure** look like? — **Macroscopic View**

How does **structure constrain** network **functionality**? — **Holistic View**

Node Degrees – Undirected Networks



degree:

number of connections k

$$\boxed{k_i} = \sum_j A_{ij}$$

the number of edges adjacent to node *i*

$$A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

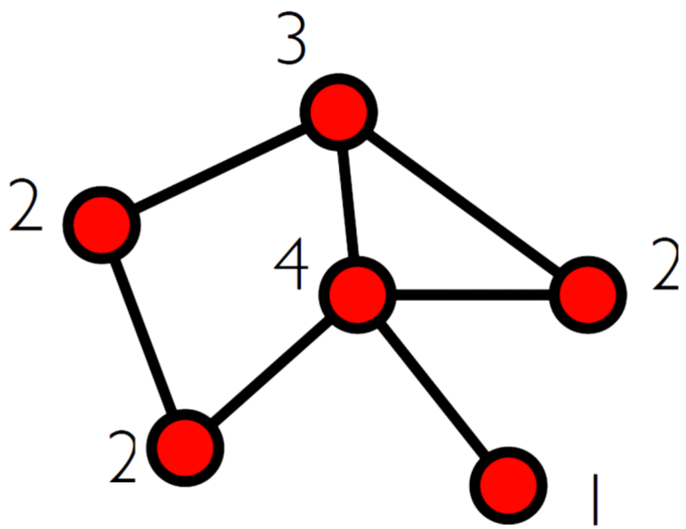
number of edges

$$m = \frac{1}{2}\sum_{i=1}^{n} k_i = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}$$

mean degree

$$\langle k \rangle = \frac{1}{n}\sum_{i=1}^{n} k_i = \frac{2m}{n}$$

Node Degrees – Undirected Networks



degree:

number of connections k

$$\boxed{k_i} = \sum_j A_{ij}$$

the number of edges adjacent to node *i*

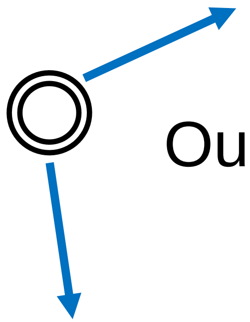
degree sequence $[1, 2, 2, 2, 3, 4]$

degree distribution $\Pr(k) = \left[\left(1, \frac{1}{6}\right), \left(2, \frac{3}{6}\right), \left(3, \frac{1}{6}\right), \left(4, \frac{1}{6}\right)\right]$

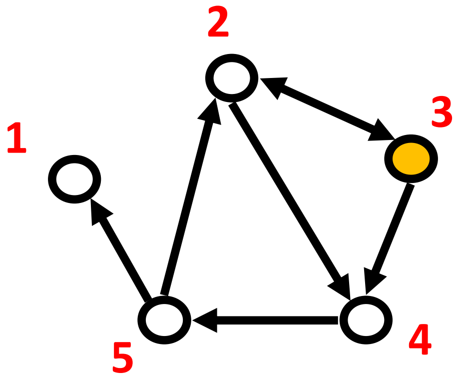
Node Degrees – Directed Networks

- Computing Metrics
  - Outdegree – the number of outgoing links

$$\sum_{j=1}^{n} A_{ij}$$



Outdegree = 2

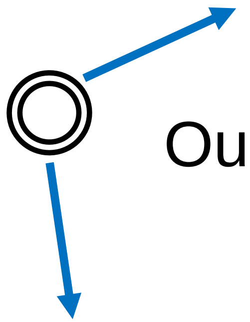


Ex) outdegree for node 3 = ?

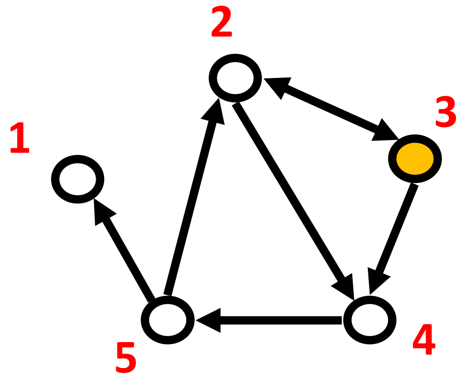
Node Degrees – Directed Networks

- Computing Metrics
  - Outdegree – the number of outgoing links

$$\sum_{j=1}^{n} A_{ij}$$



Outdegree = 2



Ex) outdegree for node 3 is 2, which we obtain by summing the number of non-zero entries in the 3rd ***row***:

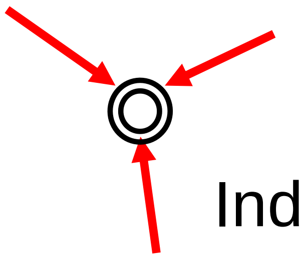
$$\sum_{j=1}^{n} A_{3j} = 2$$

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

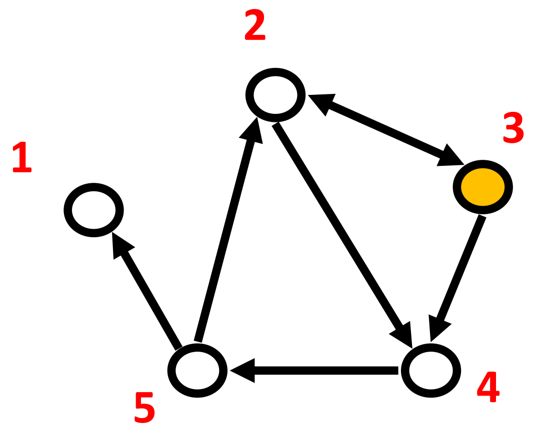
Node Degrees – Directed Networks

- Computing Metrics
  - Indegree – the number of incoming links

$$\sum_{i=1}^{n} A_{ij}$$



Indegree = 3

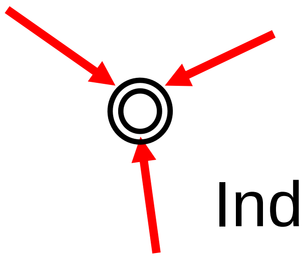


Ex) indegree for node 3 = ?

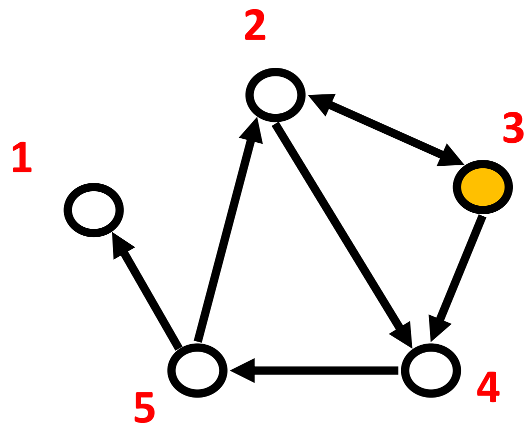
Node Degrees – Directed Networks

- Computing Metrics
  - Indegree – the number of incoming links

$$\sum_{i=1}^{n} A_{ij}$$



Indegree = 3



Ex) indegree for node 3 is 1, which we obtain by summing the number of non-zero entries in the 3rd ***column***:

$$\sum_{i=1}^{n} A_{i3} = 1$$

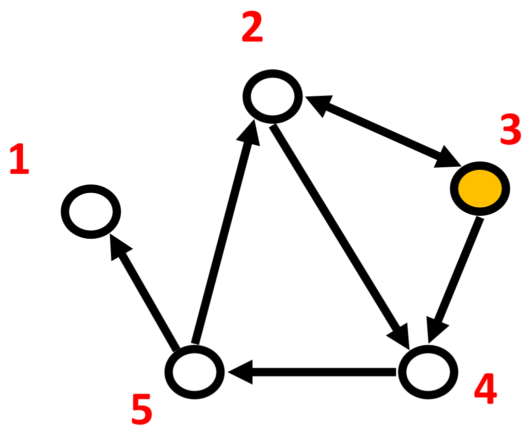
$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Node Degrees – Directed Networks

- Computing Metrics
  - (Total) degree = Indegree + Outdegree

$$k_i = k_i^{\text{in}} + k_i^{\text{out}}$$

$$\langle k \rangle = \frac{1}{n}\sum_{i=1}^{n} k_i = \frac{m}{n}$$



$$\boldsymbol{A} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Ex) degree for node 3 :

$$k_3 = \sum_{i=1}^{n} A_{i3} + \sum_{j=1}^{n} A_{3j} = 1 + 2 = 3$$