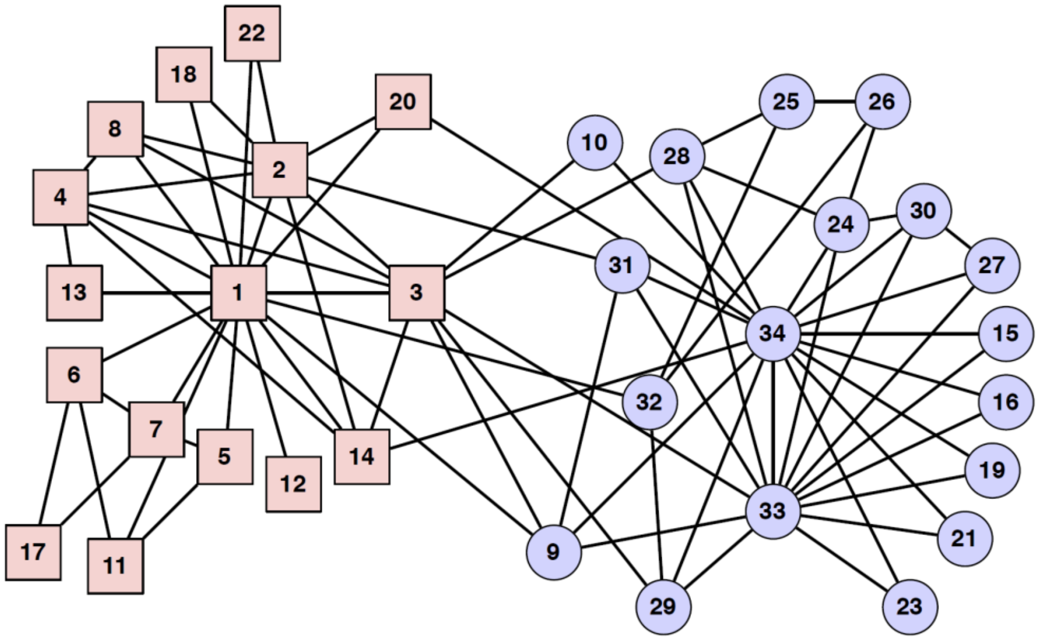
Structural Importance – Eigenvector Centrality

Recursive Notion of Importance

$$x_i^{(t+1)} \propto \sum_j A_{ij} x_j^{(t)}$$

, where $x_i^{(0)} = 1$ for all i.

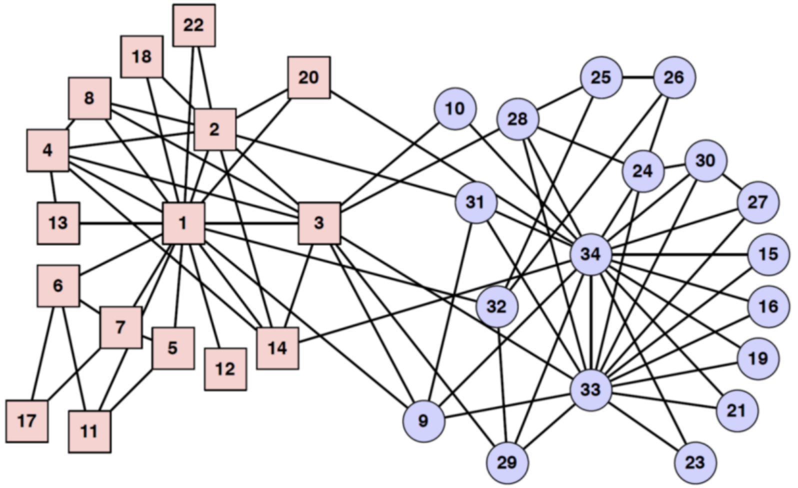


- Values x_i increase with t.
- Relative values are important rather than absolute ones
  - normalize x_i at each step.

Structural Importance – Eigenvector Centrality

Recursive Notion of Importance

$$x_i^{(t+1)} = \frac{1}{\lambda} \sum_j A_{ij} x_j^{(t)}$$



vertex	$x^{(1)}$	$x^{(5)}$	$x^{(10)}$	$x^{(15)}$	$x^{(20)}$	degree, k
1	0.103	0.076	0.071	0.071	0.071	16
2	0.058	0.055	0.053	0.053	0.053	9
3	0.064	0.065	0.064	0.064	0.064	10
4	0.038	0.043	0.042	0.042	0.042	6
5	0.019	0.015	0.015	0.015	0.015	3
6	0.026	0.016	0.016	0.016	0.016	4
7	0.026	0.016	0.016	0.016	0.016	4
8	0.026	0.034	0.034	0.034	0.034	4
9	0.032	0.044	0.046	0.046	0.046	5
10	0.013	0.020	0.021	0.021	0.021	2

Structural Importance – Eigenvector Centrality

- In-Depth:
  - Suppose we take a random vector $\boldsymbol{x}(0)$ and multiply it by a matrix $\mathbf{A}$ whose elements are all non-negative. After t multiplications we get:

$$\boldsymbol{x}(t) = \mathbf{A}^t \boldsymbol{x}(0) \quad ,$$

where $\boldsymbol{x}(0) = \sum_i c_i \boldsymbol{v}_i$ can be written as a linear combination of the eigenvectors $\boldsymbol{v}_i$ of $\mathbf{A}$ for some appropriate choice of constants c_i.

$$\boldsymbol{x}(t) = \mathbf{A}^t \boldsymbol{x}(0) = \lambda^t \boldsymbol{x}(0) = \sum_i c_i \lambda_i^t \boldsymbol{v}_i = \lambda_1^t \sum_i c_i \left(\frac{\lambda_i}{\lambda_1}\right)^t \boldsymbol{v}_i$$

  - When $t \to \infty$, $\boldsymbol{x}(t) \to \lambda_1^t c_1 \boldsymbol{v}_1$

A limiting vector $\boldsymbol{x}(t)$ is proportional to leading eigenvector of an adjacency matrix.

Structural Importance – Eigenvector Centrality (Revisited)

- ***Perron-Frobenius* theorem**:
  - A matrix with all non-negative elements (*e.g.*, adjacency matrix) has only one eigenvector whose elements are all non-negative (*leading eigenvector*) and the corresponding *unique largest real eigenvalue*.

$$\boxed{\mathbf{A}\boldsymbol{x} = \lambda \boldsymbol{x}}$$

  - A node i's centrality (x_i) is the i-th element of the leading eigenvector $\boldsymbol{x}$ of an adjacency matrix $\mathbf{A}$.
  - λ is the corresponding largest eigenvalue.

Exercise with Python – Degree vs Eigenvector Centrality

```python
import networkx as nx
import matplotlib.pyplot as plt

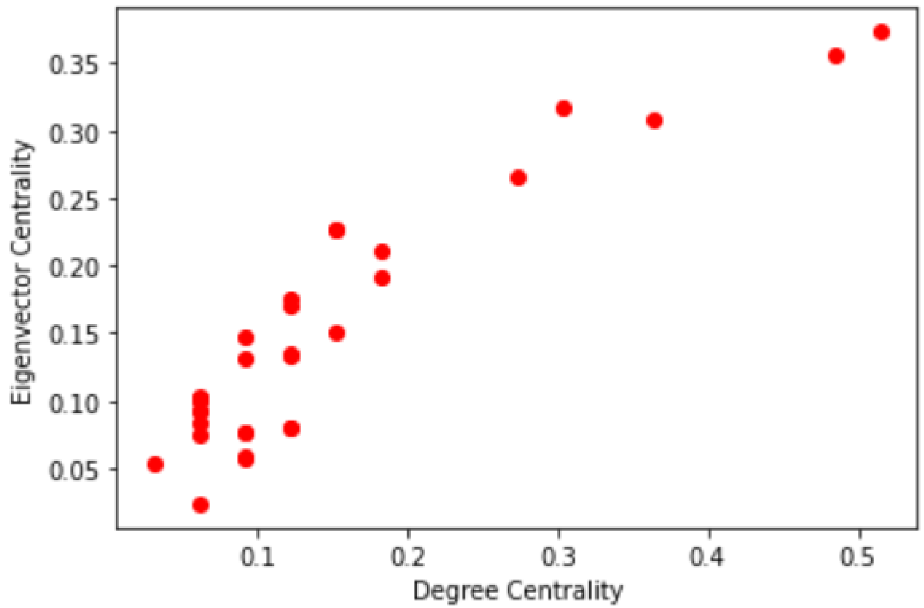
G = nx.karate_club_graph()

dc= nx.degree_centrality(G)       #degree centrality
ec = nx.eigenvector_centrality(G) #eigenvector centrality

x = list(dc.values())
y = list(ec.values())

plt.plot(x, y, 'ro')
plt.xlabel('Degree Centrality')
plt.ylabel('Eigenvector Centrality')
```

```
Text(0, 0.5, 'Eigenvector Centrality')
```



Exercise with Python – Degree vs Eigenvector Centrality

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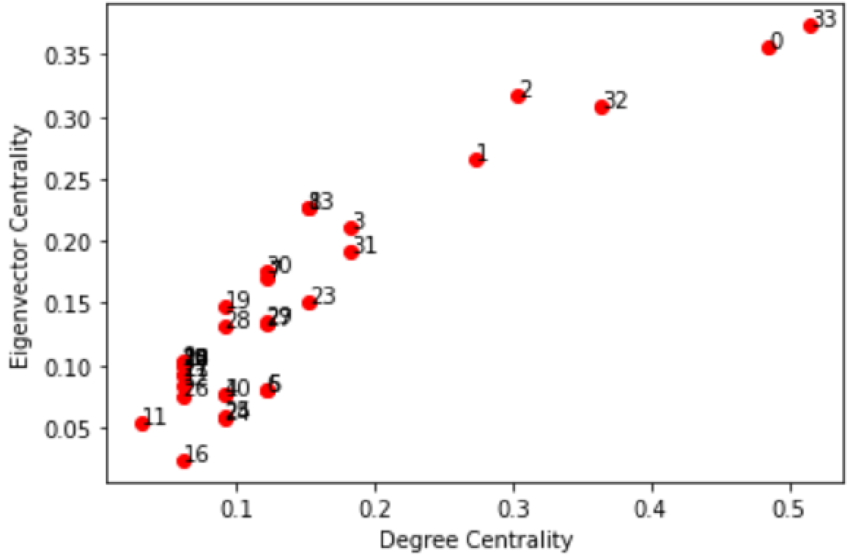
ec = nx.eigenvector_centrality(G) #eigenvector centrality

x = list(dc.values())
y = list(ec.values())

plt.plot(x, y, 'ro')
for i, txt in enumerate(dc.keys()):
    plt.annotate(txt, (x[i], y[i]))

plt.xlabel('Degree Centrality')
plt.ylabel('Eigenvector Centrality')
```

```
Text(0, 0.5, 'Eigenvector Centrality')
```

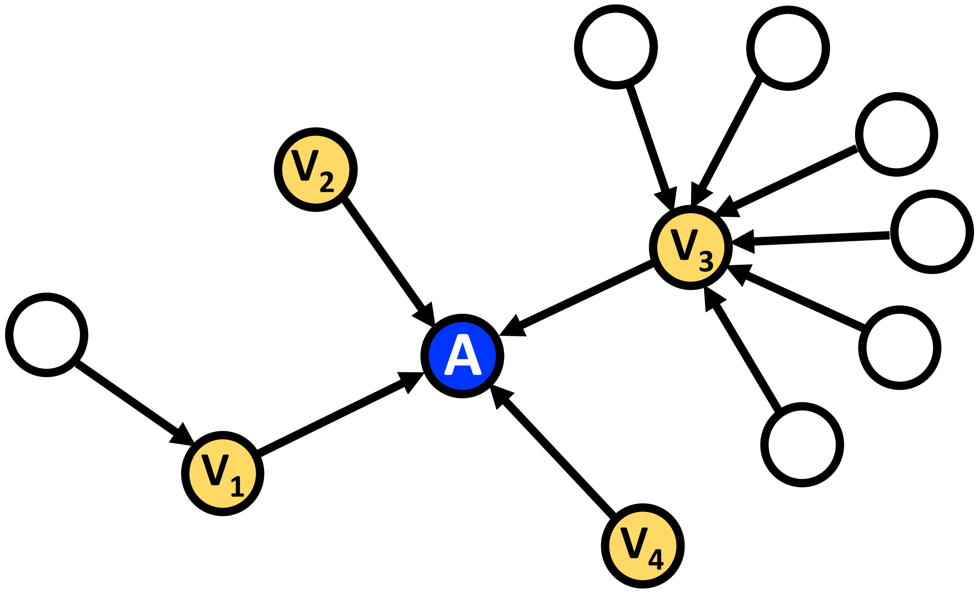


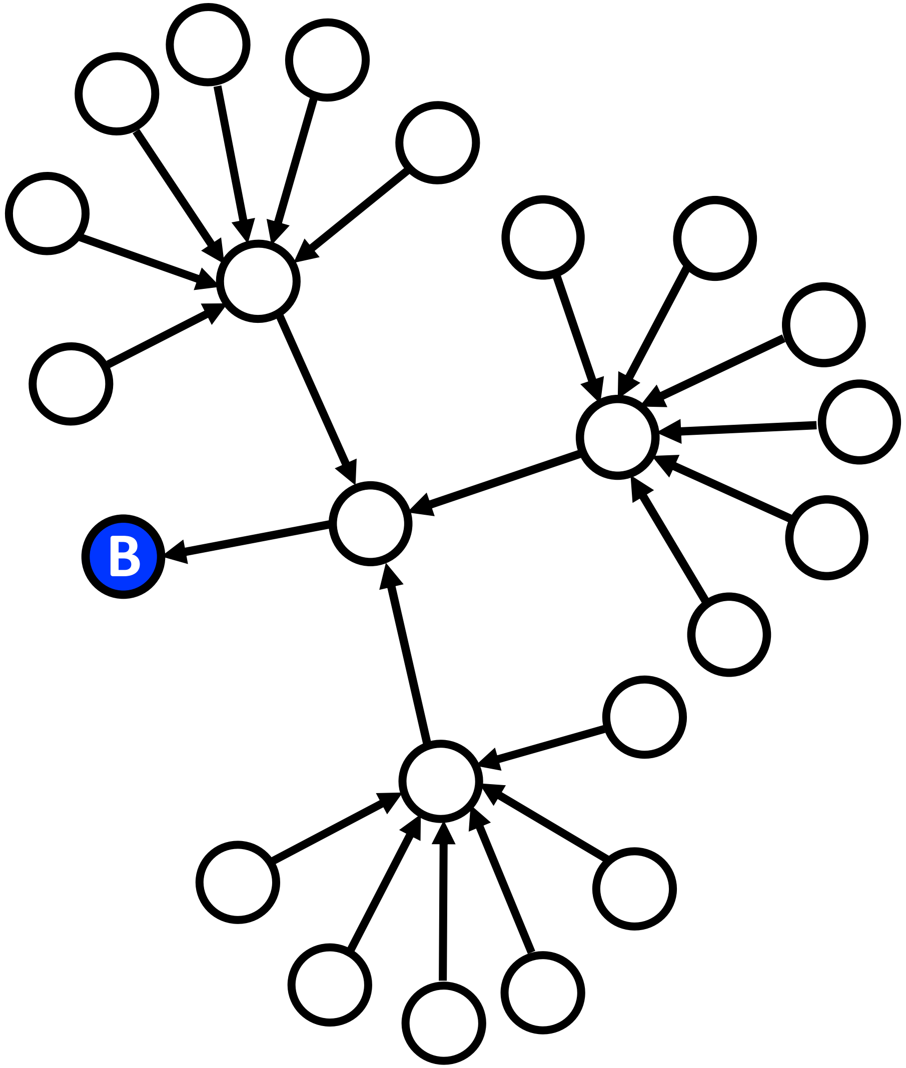
Structural Importance – Eigenvector Centrality

Directed Networks

- Generally consider nodes with larger incoming links more influential

$$x_i = \frac{1}{\lambda} \sum_j A_{ij} x_j \text{ , where } A_{ij} \equiv A_{i \leftarrow j}$$



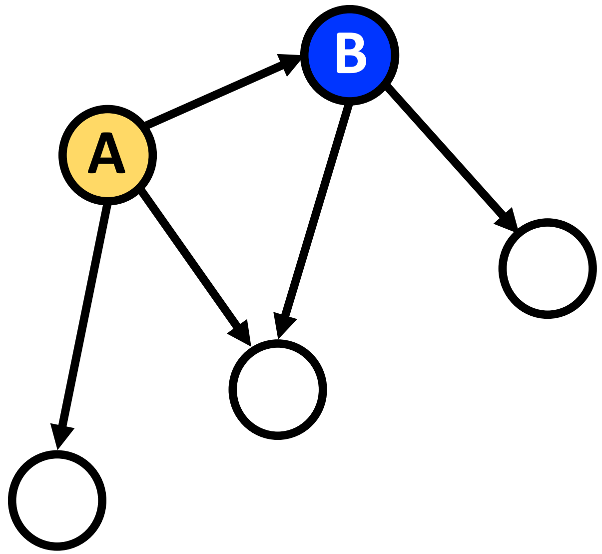


Structural Importance – Eigenvector Centrality

Directed Networks

- What are the eigenvector centrality scores, x_{A} and x_{B}, of nodes A and B?

$$x_i = \frac{1}{\lambda} \sum_j A_{ij} x_j$$



Structural Importance – PageRank

Representative Variant of Eigenvector Centrality

- To resolve two major issues:
  - **(1)** Eigenvector centrality becomes zero for some nodes in a directed network.
  - **(2)** A node may be only one among many neighbors of an influential node.

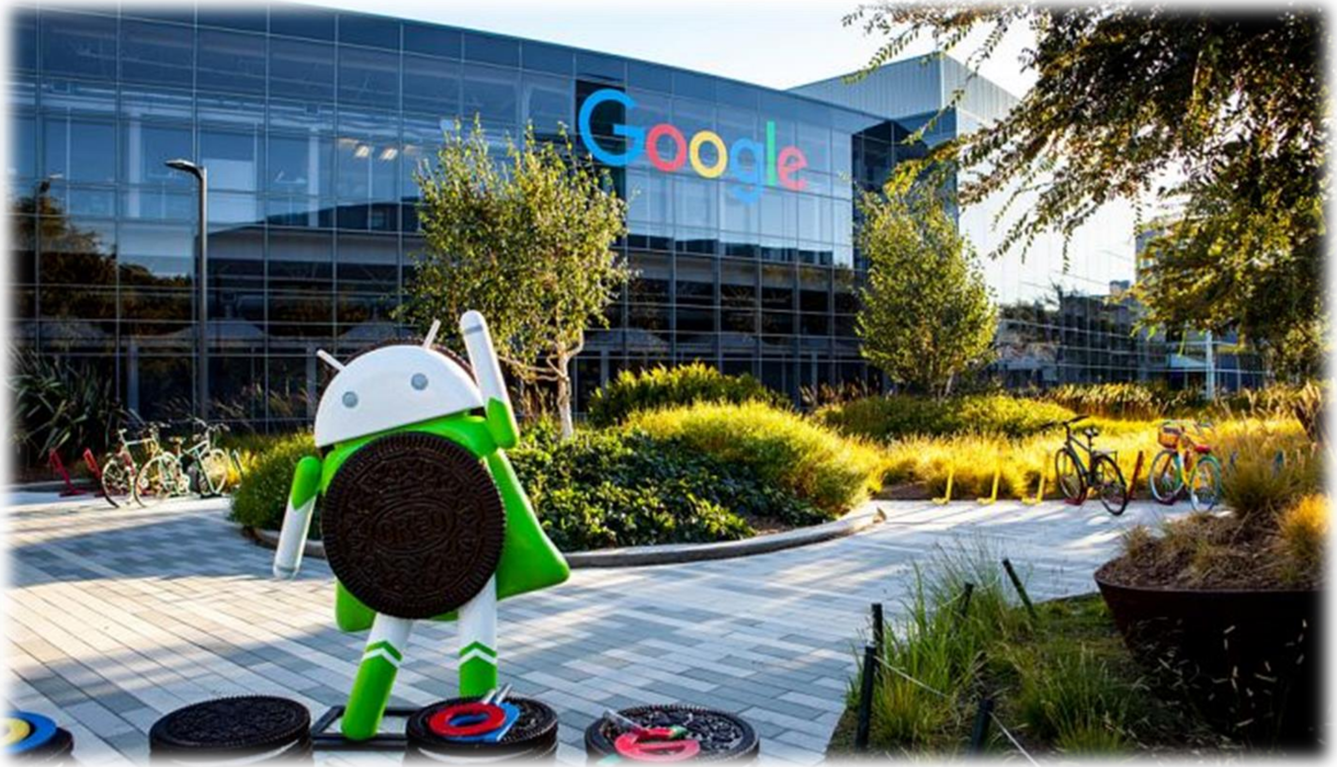
Weight for balancing between the 1st and 2nd terms

(1) Base centrality for free

$$x_i = \boxed{\alpha} \sum_j A_{ij} \frac{x_j}{\boxed{k_j^{\text{out}}}} + \boxed{\beta}$$

(2) Centrality divided by an influential node's outdegree

Structural Importance – PageRank



Googleplex, Mountain View, CA, USA

"Google provides access to the world's information in one click."

- Sergey Brin and Larry Page

Structural Importance – PageRank

All web pages are not equally important.

[Main Idea]

Let's rank web pages by hyperlink structure!