Matrices

Matrices

- Matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} \in \mathbb{R}^{n \times m}$$

Two Views on Matrices

- Collection of Row Vectors

$$\mathbf{A} = \begin{bmatrix} — & \mathbf{r}_1^\top & — \\ — & \mathbf{r}_2^\top & — \\ & \vdots & \\ — & \mathbf{r}_n^\top & — \end{bmatrix}, \quad \mathbf{r}_i \in \mathbb{R}^m$$

- **Collection of Column Vectors**

$$\mathbf{A} = \begin{bmatrix} | & | & & | \\ \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_m \\ | & | & & | \end{bmatrix}, \quad \mathbf{c}_i \in \mathbb{R}^n$$

Transpose

- Given a matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} \in \mathbb{R}^{n \times m}$$

- Its transpose is

$$\mathbf{A}^\top = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1m} & a_{2m} & \cdots & a_{nm} \end{bmatrix} \in \mathbb{R}^{m \times n}$$

Transpose

- Given a matrix

$$\mathbf{A} = \begin{bmatrix} | & | & & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_m \\ | & | & & | \end{bmatrix}, \quad \mathbf{a}_i \in \mathbb{R}^n$$

- Its transpose is

$$\mathbf{A}^\top = \begin{bmatrix} — & \mathbf{a}_1^\top & — \\ — & \mathbf{a}_2^\top & — \\ & \vdots & \\ — & \mathbf{a}_m^\top & — \end{bmatrix}, \quad \mathbf{a}_i \in \mathbb{R}^n$$

Transpose

- E.g.)

$$\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}^\top = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 5 \\ -1 & 4 & 3 \\ 3 & -2 & 7 \end{bmatrix}^\top = \begin{bmatrix} 1 & -1 & 3 \\ 0 & 4 & -2 \\ 5 & 3 & 7 \end{bmatrix}$$

Square Matrices

- A matrix is a square matrix if and only if n=m

$$\mathbf{A} \in \mathbb{R}^{n \times n}$$

- E.g.) 2D

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Symmetric Matrices

- A square matrix is symmetric if and only if

$$\mathbf{A}^\top = \mathbf{A}, \quad \mathbf{A} \in \mathbb{R}^{n\times n}$$

- E.g.) 2D

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$$

Diagonal Matrices

- A symmetric matrix is diagonal if and only if

$$[\mathbf{A}]_{ij} = 0, \quad i \neq j$$

i.e.)

$$\mathbf{A} = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} \in \mathbb{R}^{n \times n}$$

- E.g.) 2D

$$\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

Identity Matrices

- A diagonal matrix is identity if and only if

$$[\mathbf{A}]_{ii} = 1, [\mathbf{A}]_{ij} = 0, \quad i \neq j$$

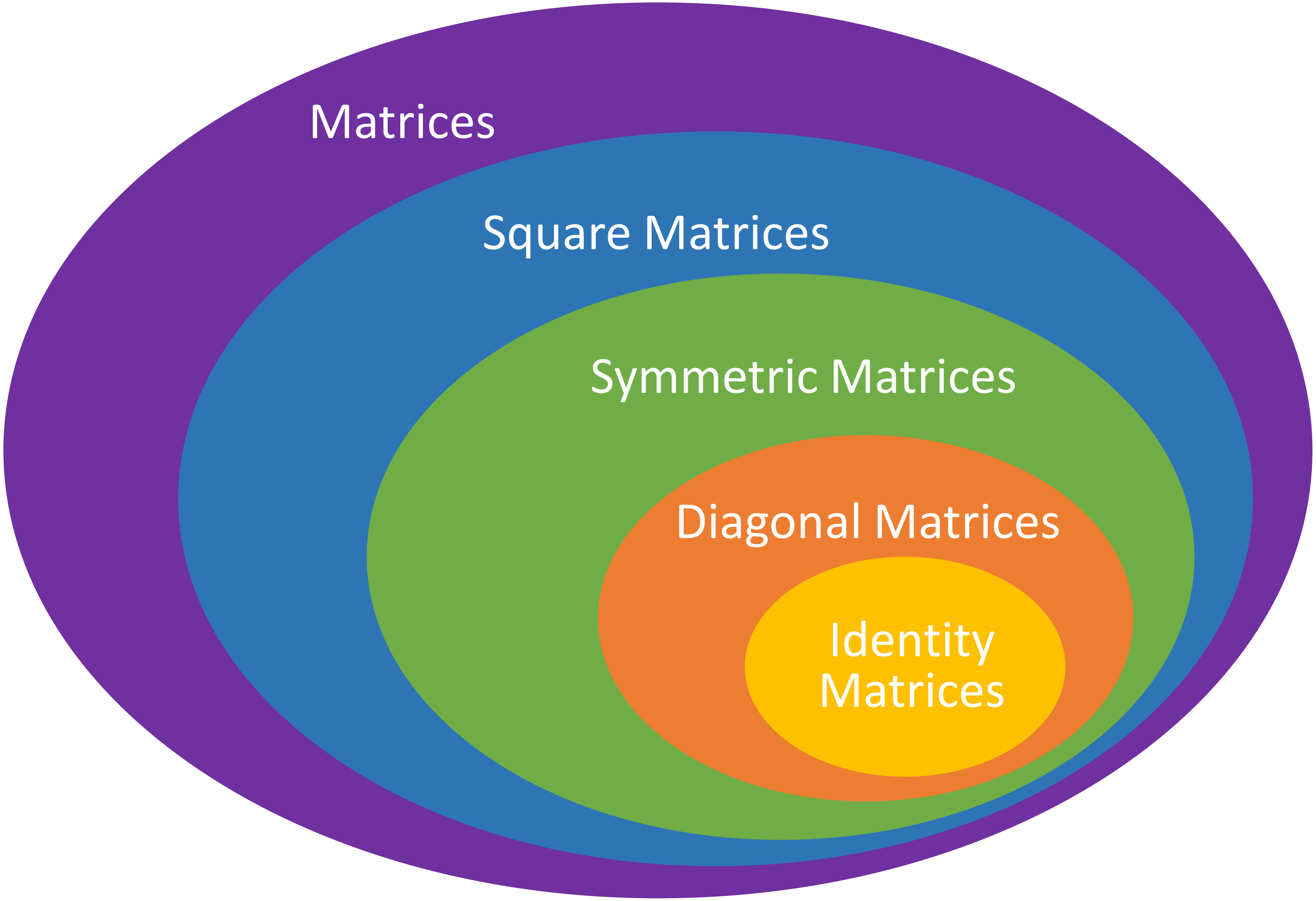
i.e.)

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \in \mathbb{R}^{n \times n}$$

- E.g.) 2D

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Matrix Sets


Matrices
Square Matrices
Symmetric Matrices
Diagonal Matrices
Identity Matrices

Operations on Matrices

Operations on Matrices

$$\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathbb{R}^{n \times m}$$

- **Scalar Multiplication**

$$\mathbf{A} = c\mathbf{B} \;\Rightarrow\; [A]_{ij} = c[B]_{ij}$$

- **Addition**

$$\mathbf{C} = \mathbf{A} + \mathbf{B} \;\Rightarrow\; [C]_{ij} = [A]_{ij} + [B]_{ij}$$

- **Subtraction**

$$\mathbf{C} = \mathbf{A} - \mathbf{B} \;\Rightarrow\; [C]_{ij} = [A]_{ij} - [B]_{ij}$$

Matrix Multiplication

- Multiplication

$$\mathbf{C} = \mathbf{AB} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nk} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{k1} & b_{n2} & \cdots & b_{km} \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{i=1}^{k} a_{1i} b_{i1} & \sum_{i=1}^{k} a_{1i} b_{i2} & \cdots & \sum_{i=1}^{k} a_{1i} b_{im} \\ \sum_{i=1}^{k} a_{2i} b_{i1} & \sum_{i=1}^{k} a_{2i} b_{i2} & \cdots & \sum_{i=1}^{k} a_{2i} b_{im} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=1}^{k} a_{ni} b_{i1} & \sum_{i=1}^{k} a_{ni} b_{i2} & \cdots & \sum_{i=1}^{k} a_{ni} b_{im} \end{bmatrix}$$

Matrix Multiplication

- Multiplication

$$\mathbf{C} = \mathbf{A}\mathbf{B} \;\Rightarrow\; [C]_{ij} = \sum_k [A]_{ik}[B]_{kj}$$

$$\mathbf{A} \in \mathbb{R}^{n\times k},\; \mathbf{B} \in \mathbb{R}^{k\times m},\; \mathbf{C} \in \mathbb{R}^{n\times m}$$

- Non-Commutative

$$\mathbf{AB} \neq \mathbf{BA}$$

- Associative

$$\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB} + \mathbf{AC}$$

Matrix-Vector Multiplication

$$
\mathbf{y} = \mathbf{A}\mathbf{x} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}
$$

$$
= \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m \end{bmatrix}
$$

Matrix-Vector Multiplication: Two Views

1. Dot Product with Row Vectors

$$\mathbf{y} = \mathbf{A}\mathbf{x} = \begin{bmatrix} - & \mathbf{r}_1^\top & - \\ - & \mathbf{r}_2^\top & - \\ & \vdots & \\ - & \mathbf{r}_n^\top & - \end{bmatrix} \mathbf{x} = \begin{bmatrix} \mathbf{r}_1^\top \mathbf{x} \\ \mathbf{r}_2^\top \mathbf{x} \\ \vdots \\ \mathbf{r}_n^\top \mathbf{x} \end{bmatrix}$$

Matrix-Vector Multiplication: Two Views

2. Linear Combination of Column Vectors

$$\mathbf{y} = \mathbf{A}\mathbf{x} = \begin{bmatrix} | & | & & | \\ \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_m \\ | & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$$

$$= x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + \cdots x_m\mathbf{c}_m$$

Matrix-Matrix Multiplication

$$\mathbf{Y} = \mathbf{A}\mathbf{X}$$

$$\begin{bmatrix} | & | & & | \\ \mathbf{y}_1 & \mathbf{y}_2 & \cdots & \mathbf{y}_m \\ | & | & & | \end{bmatrix} = \mathbf{A} \begin{bmatrix} | & | & & | \\ \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_m \\ | & | & & | \end{bmatrix}$$

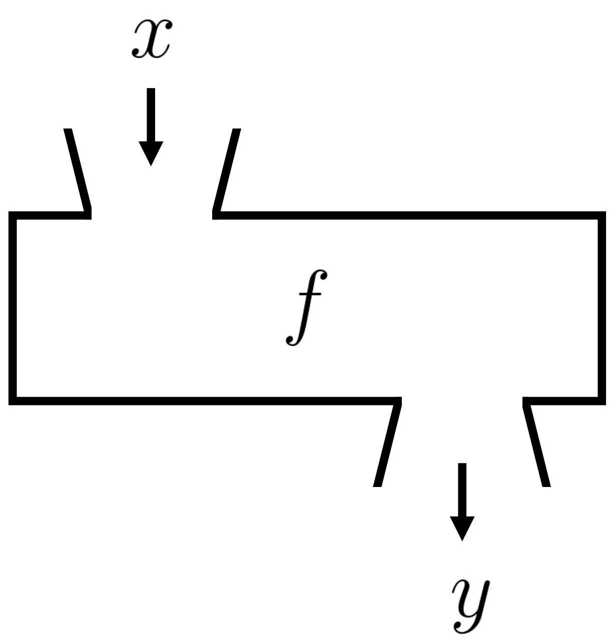
Matrix = Linear Function/Mapping

1D – 1D

Function = Mapping

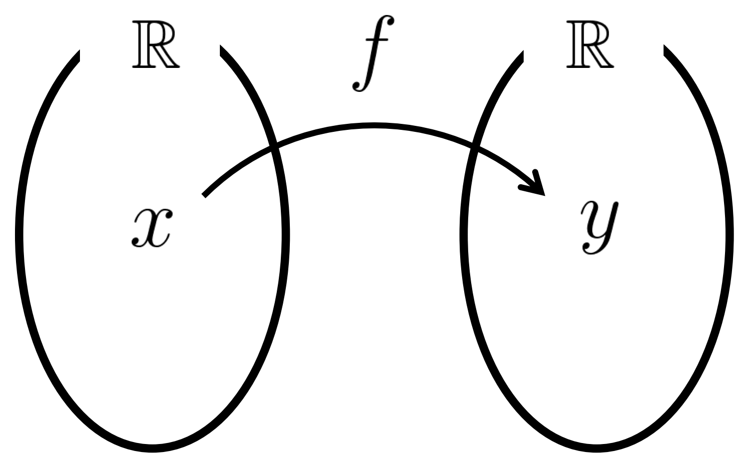
- Function

$$y = f(x)$$



- Mapping

$$f:\ x\ \mapsto\ y$$

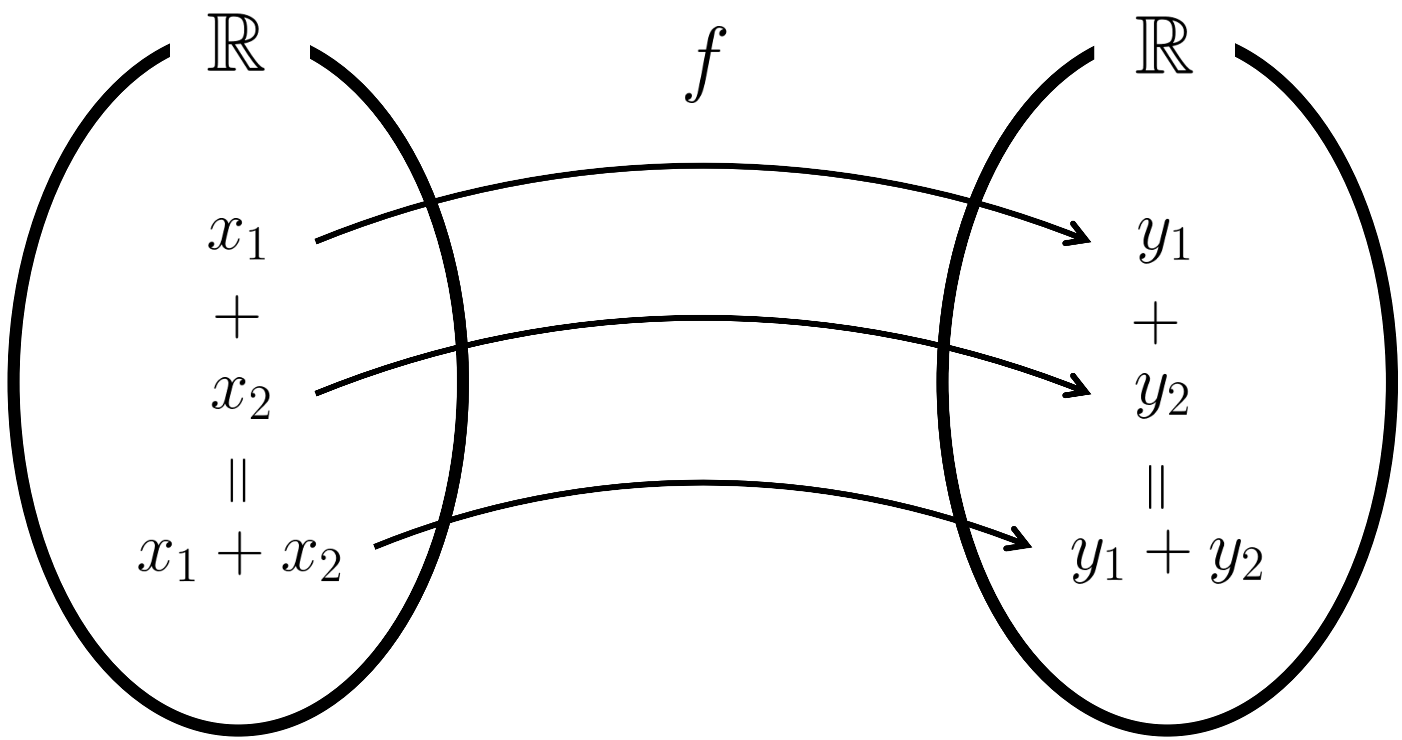


$$x, y \in \mathbb{R}$$

Linear Function

- A function $y = f(x)$ is linear if and only if

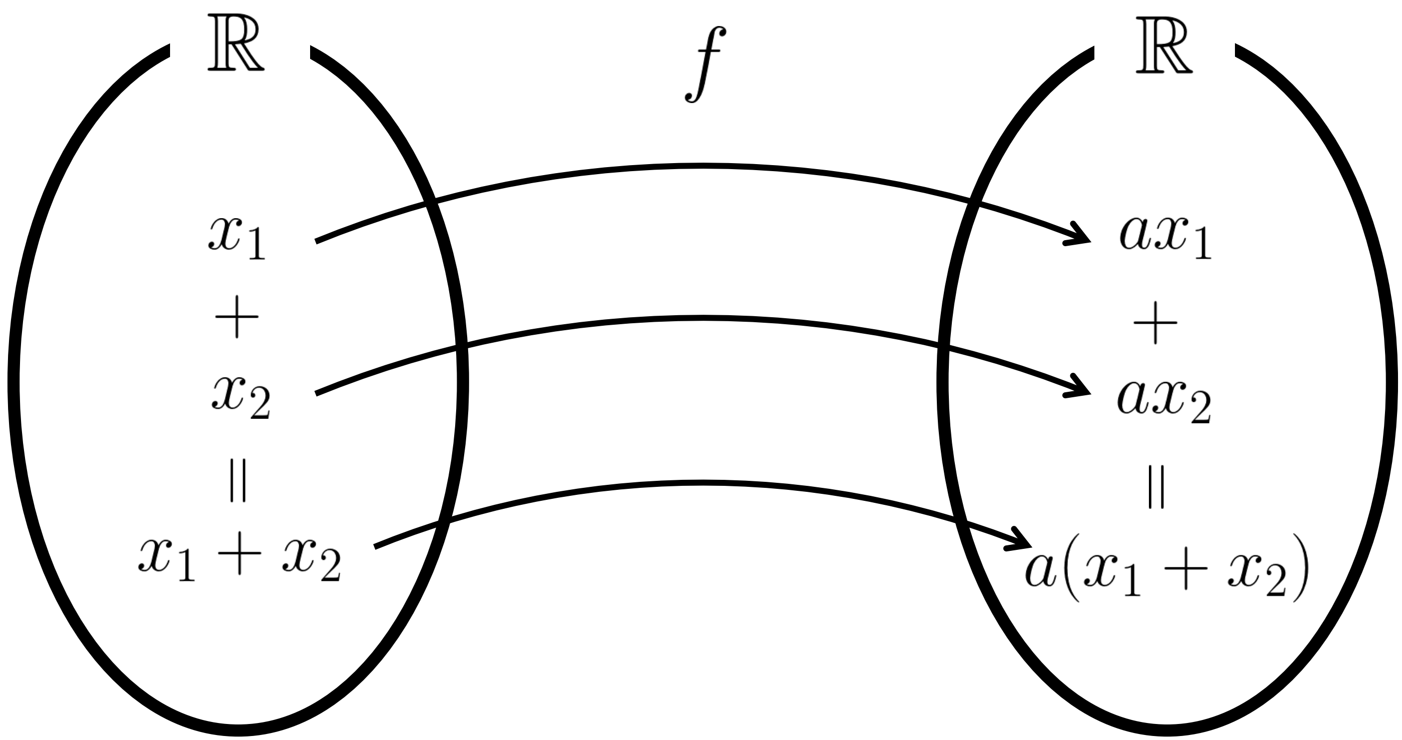
$$f(x_1 + x_2) = f(x_1) + f(x_2)$$



Linear Function

- A function $y = f(x)$ is linear if and only if

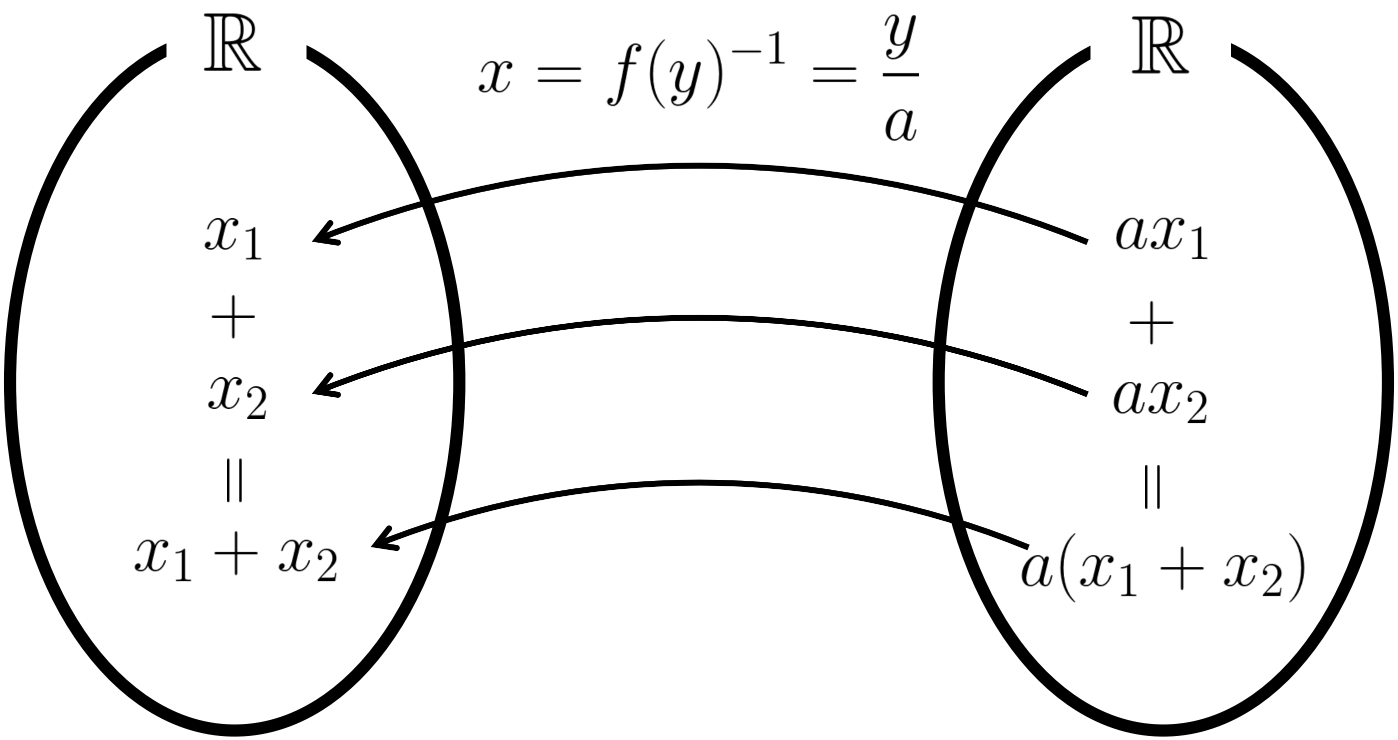
$$f(x) = ax$$



Inverse of Linear Function

- A linear function $y = f(x) = ax$ is invertible if and only if

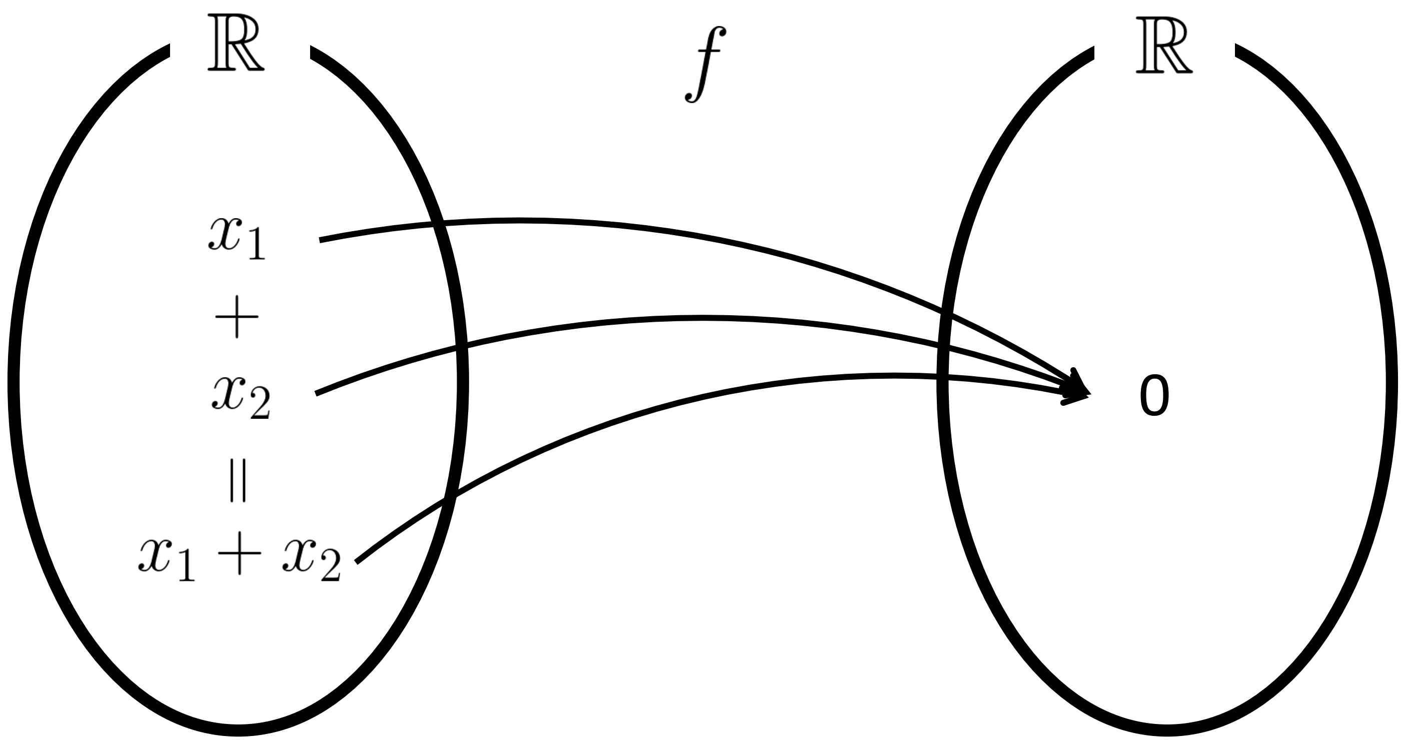
$$a \neq 0$$



Linear Function

- A function $y = f(x)$ is linear if and only if

$$f(x) = ax \quad \text{when } a = 0$$

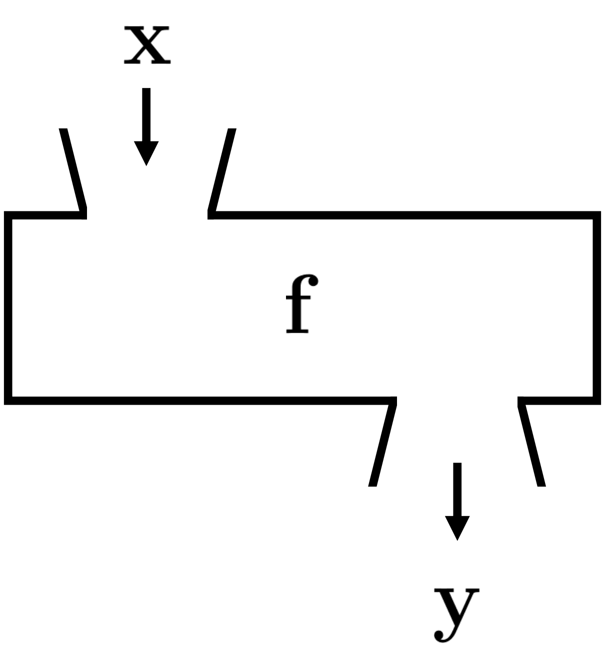


m-D – n-D

Function = Mapping

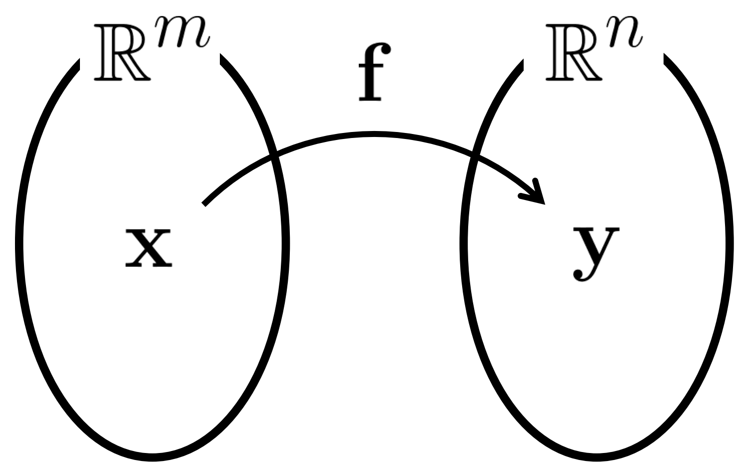
- Function

$$\mathbf{y} = \mathbf{f}(\mathbf{x})$$



- Mapping

$$\mathbf{f} : \mathbf{x} \mapsto \mathbf{y}$$

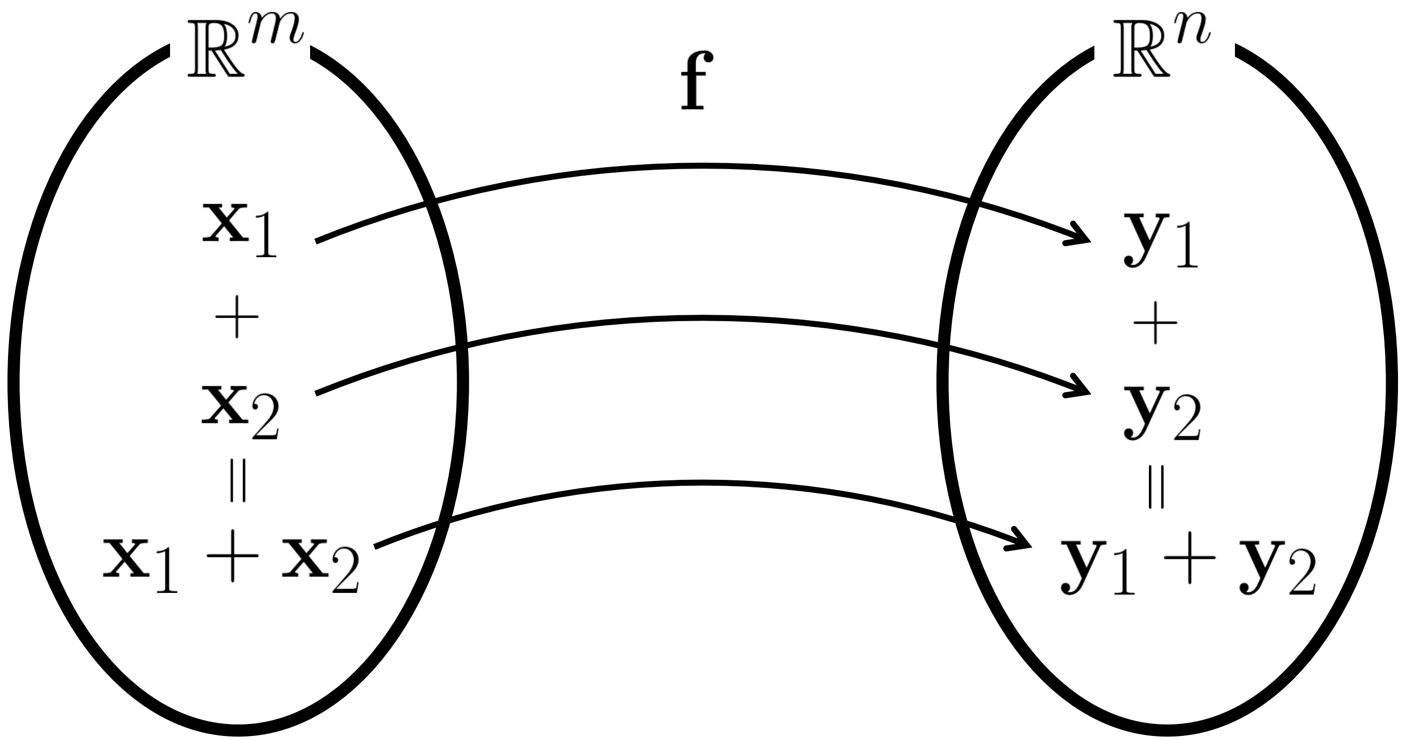


$$\mathbf{x} \in \mathbb{R}^m, \ \mathbf{y} \in \mathbb{R}^n$$

Linear Function

- A function $\mathbf{y} = \mathbf{f}(\mathbf{x})$ is linear if and only if

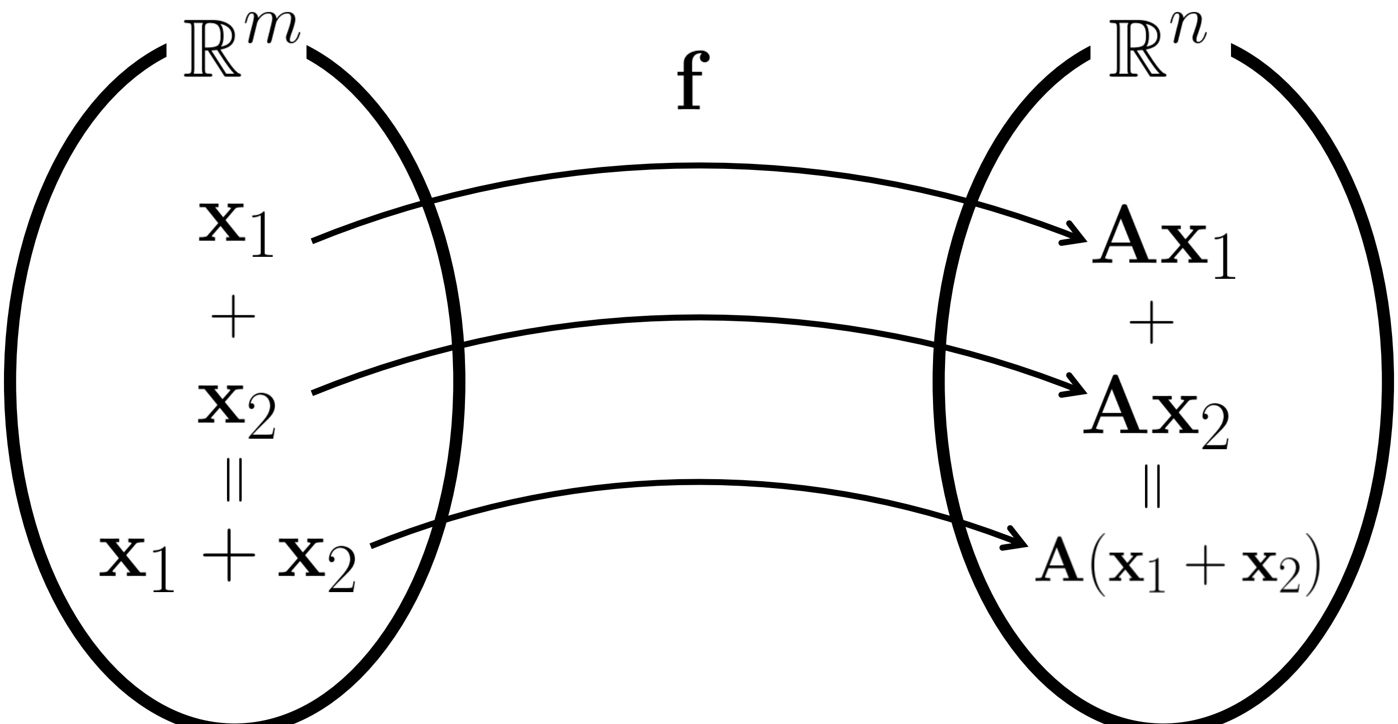
$$\mathbf{f}(\mathbf{x}_1 + \mathbf{x}_2) = \mathbf{f}(\mathbf{x}_1) + \mathbf{f}(\mathbf{x}_2)$$



Linear Function

- A function $\mathbf{y} = \mathbf{f}(\mathbf{x})$ is linear if and only if

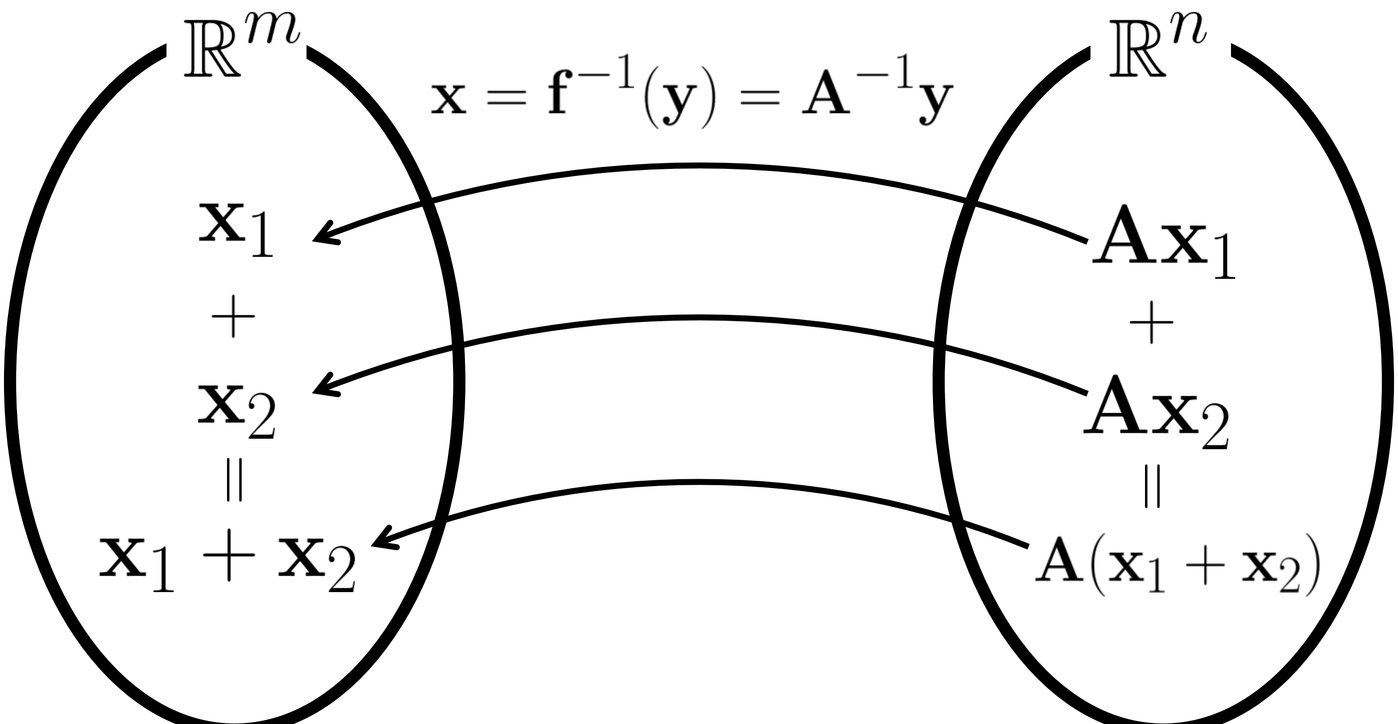
$$\mathbf{f}(\mathbf{x}) = \mathbf{A}\mathbf{x}, \quad \mathbf{A} \in \mathbb{R}^{n \times m}$$



Inverse of Linear Function

- A linear function $\mathbf{y} = \mathbf{f}(\mathbf{x}) = \mathbf{A}\mathbf{x}$ is invertible if and only if

$$\det(\mathbf{A}) \neq 0$$



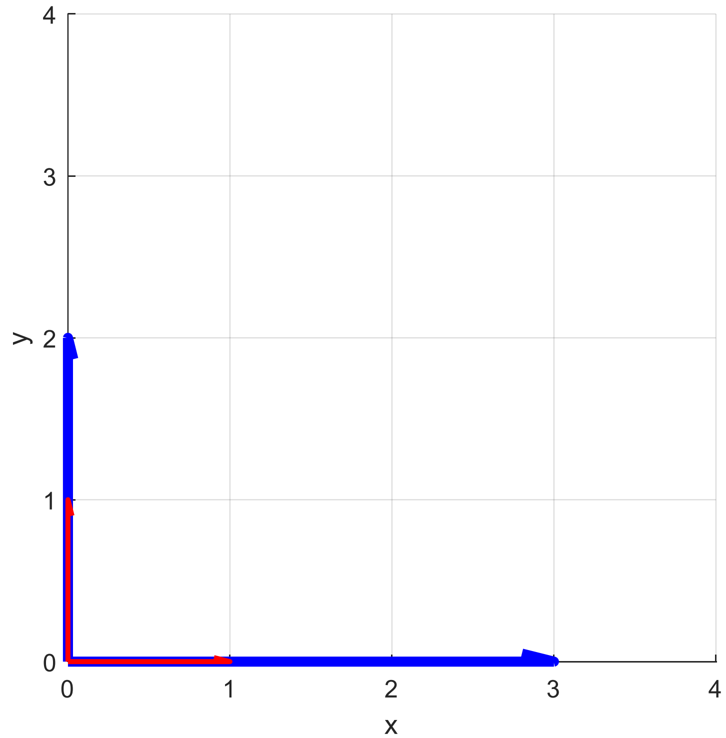
Matrix = Linear Transformation

Example: 2D-2D

Scale

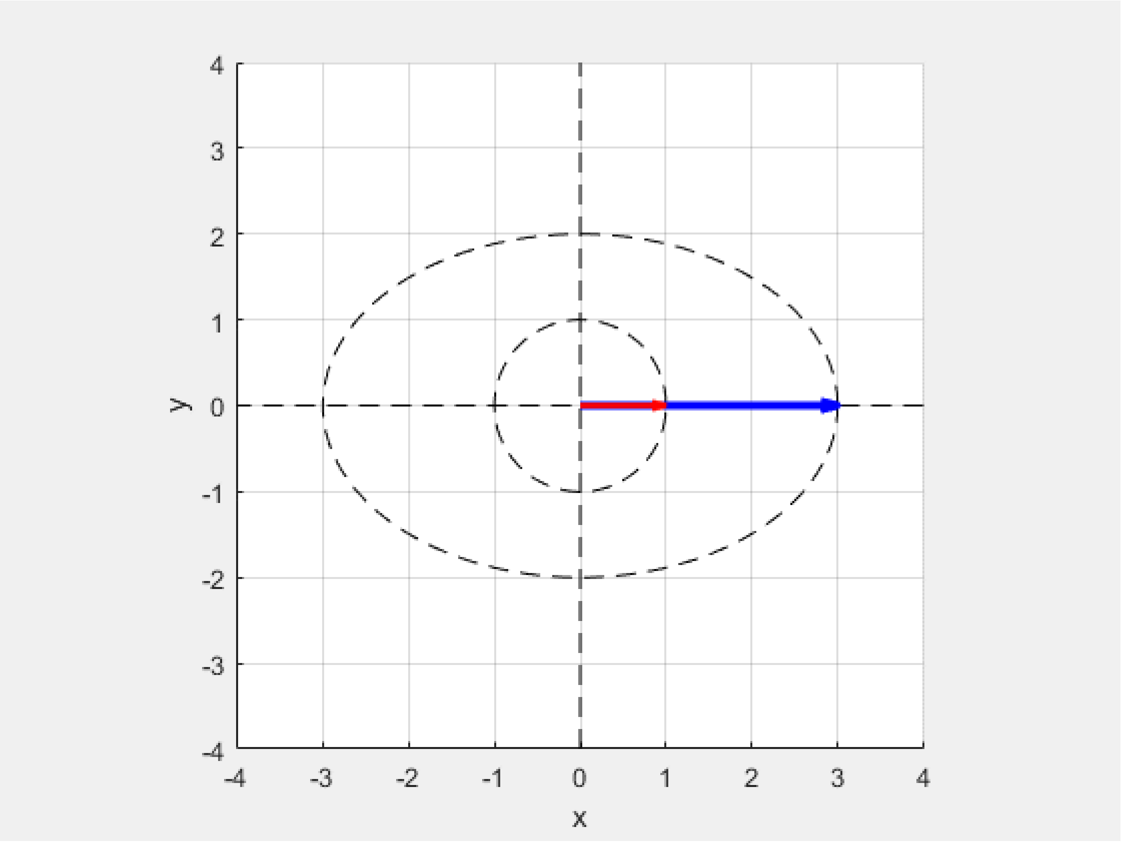
- $\mathbf{A} = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}$ e.g.) $\mathbf{A} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$

- Where do Standard Basis Vectors go?



Scale

- $$\mathbf{A} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

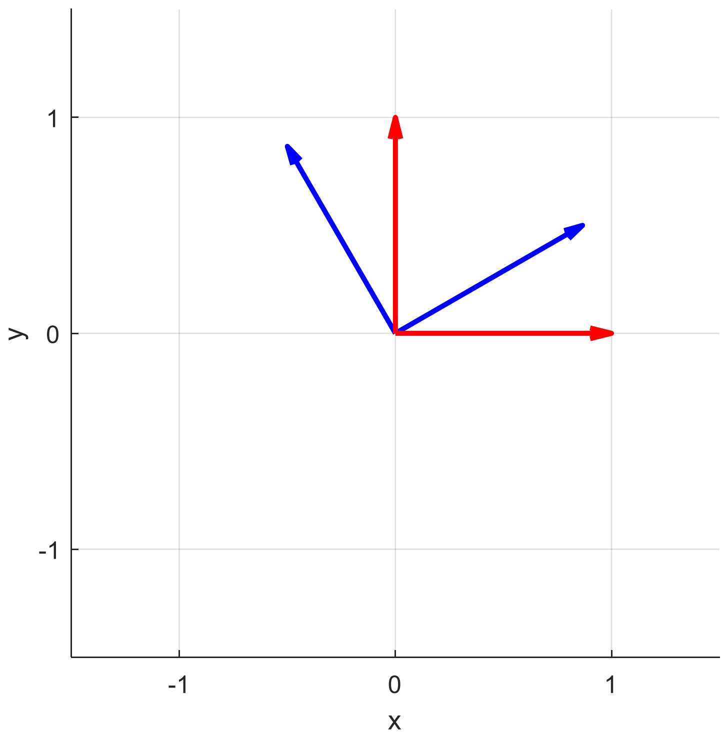


Rotation

$$\theta = 30^\circ$$

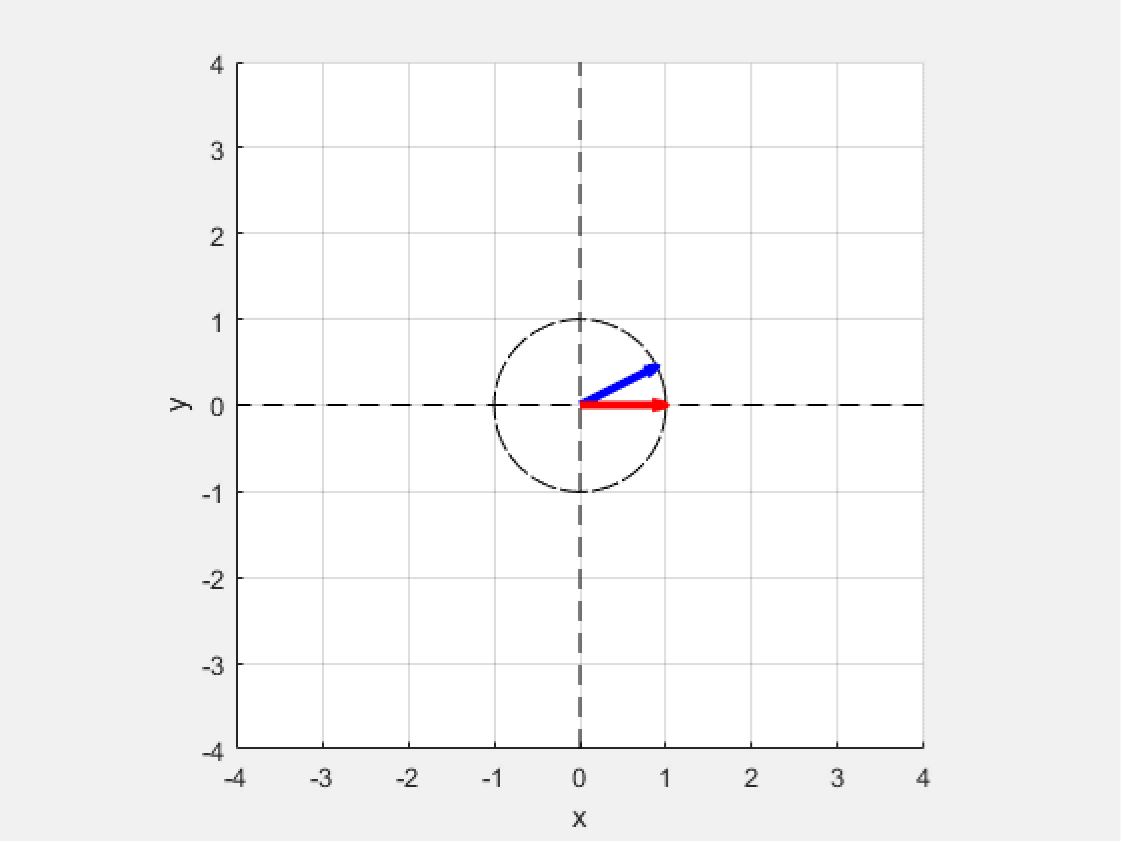
- $\mathbf{A} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ e.g.) $\mathbf{A} = \frac{1}{2}\begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}$

- Where do Standard Basis Vectors go?



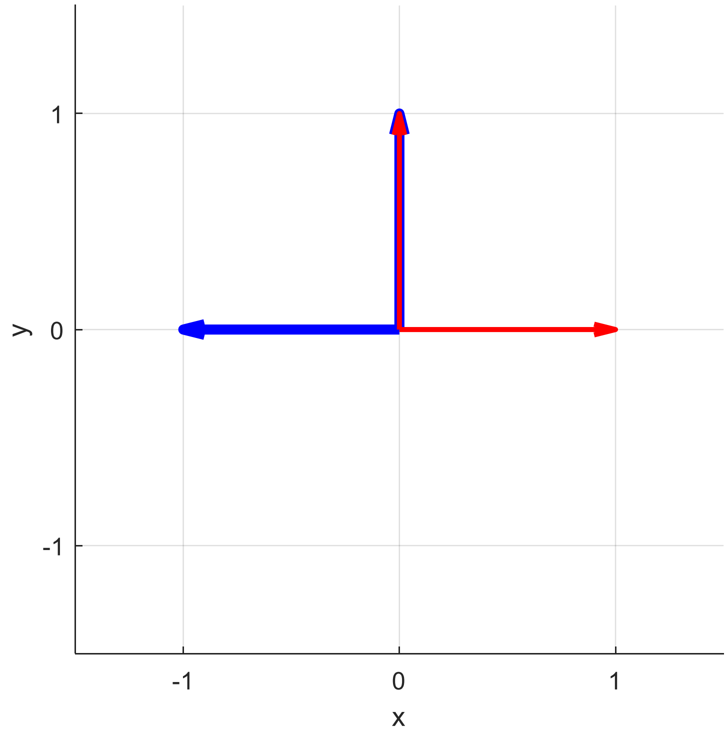
Rotation

- $\mathbf{A} = \frac{1}{2}\begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}$



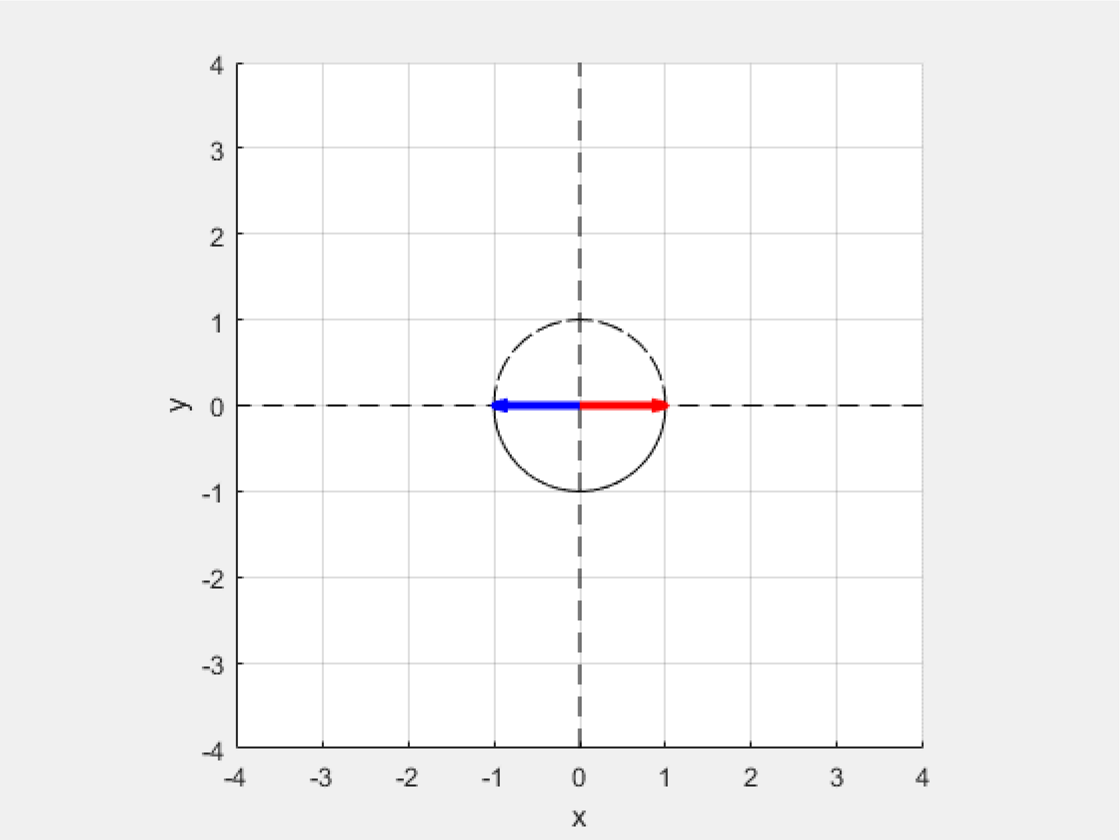
Reflection

- $$\mathbf{A} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$
- Where do Standard Basis Vectors go?



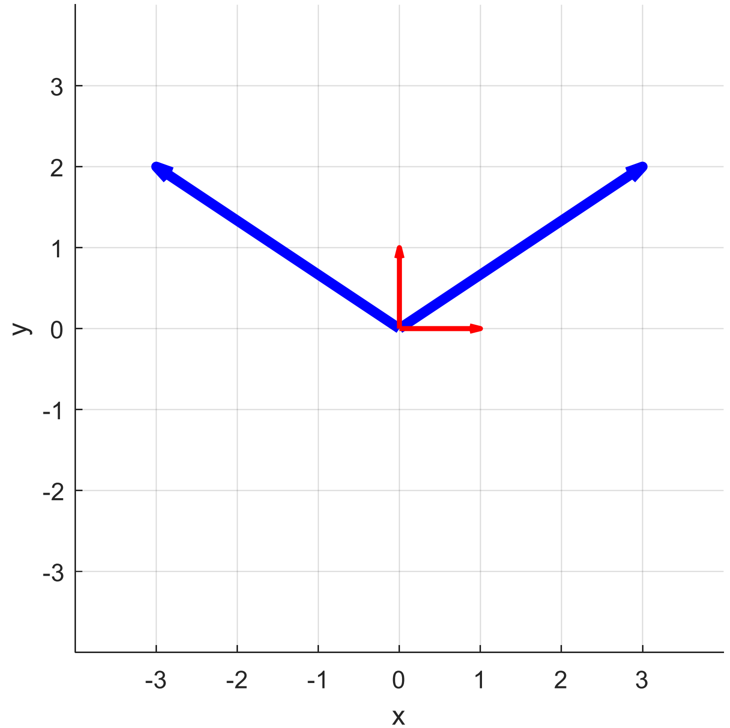
Reflection

- $\mathbf{A} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$



Combinations

- $$\mathbf{A} = \begin{bmatrix} 3 & -3 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 3\sqrt{2} & 0 \\ 0 & 2\sqrt{2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$
- Where do Standard Basis Vectors go?



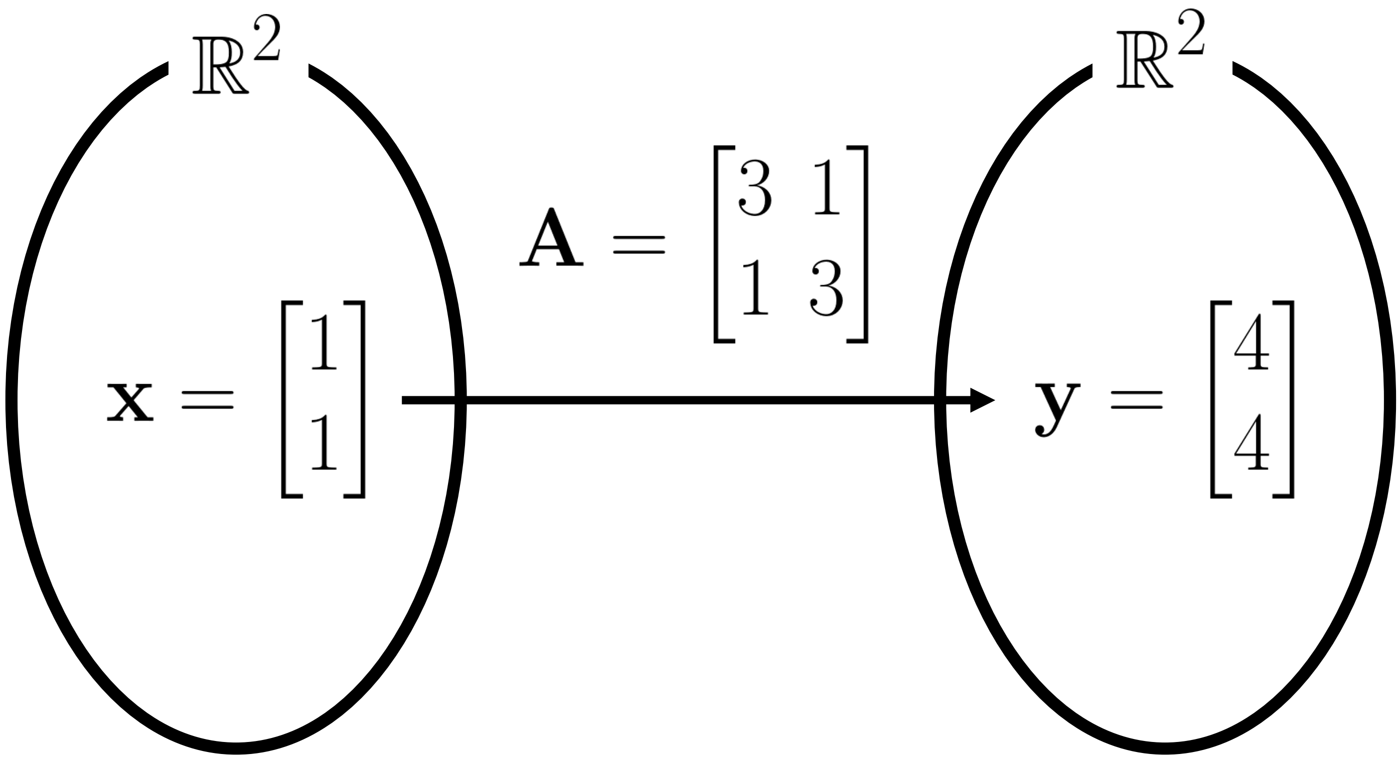
Summary

- Function = Mapping = Transformation
- Matrix = Linear Function/Mapping/Transformation
  - Scale
  - Rotation
  - Reflection
  - Combinations

Eigenvalues and Eigenvectors

Let's take a look at this mapping

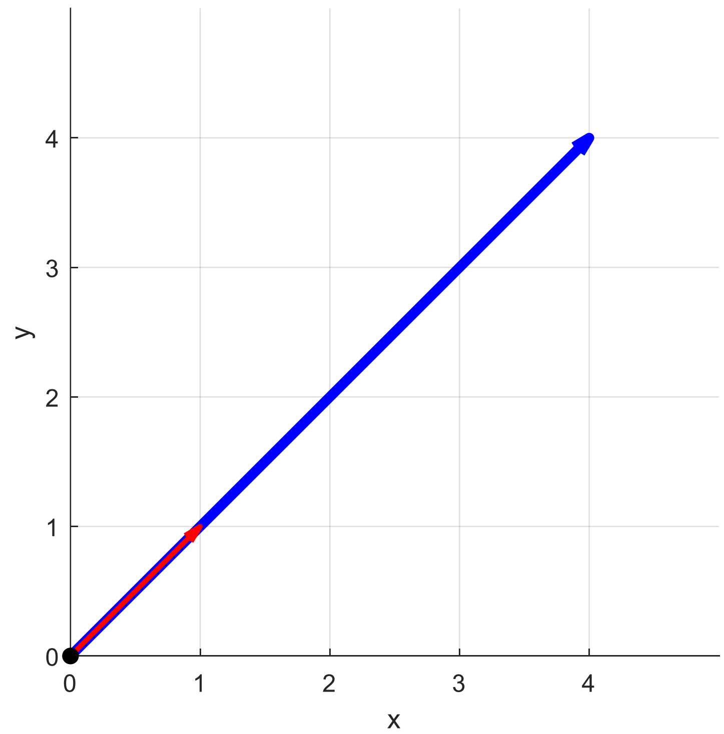
- E.g.) 2D



It only scales the input vector

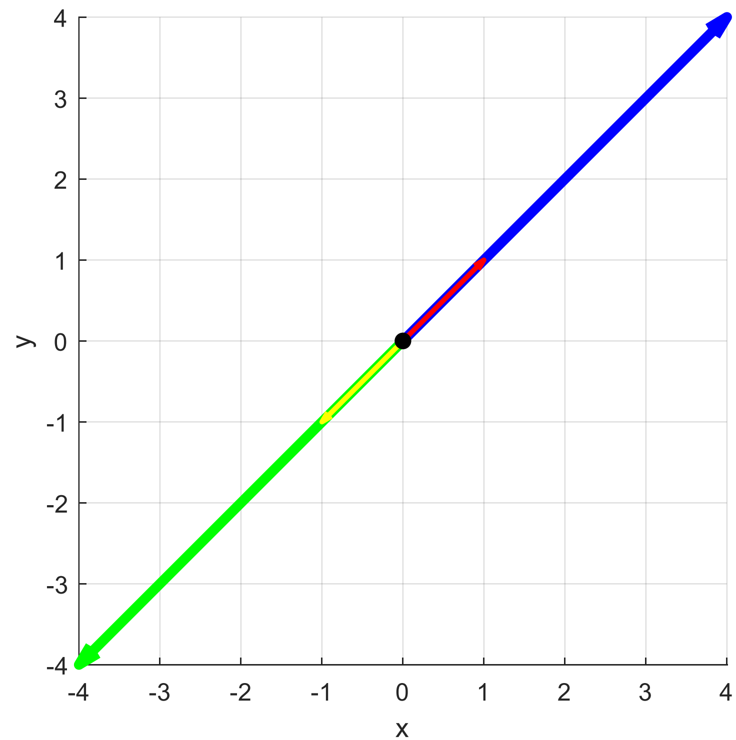
- E.g.) 2D

$$4\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}3&1\\1&3\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}$$



In fact, any vectors on that line

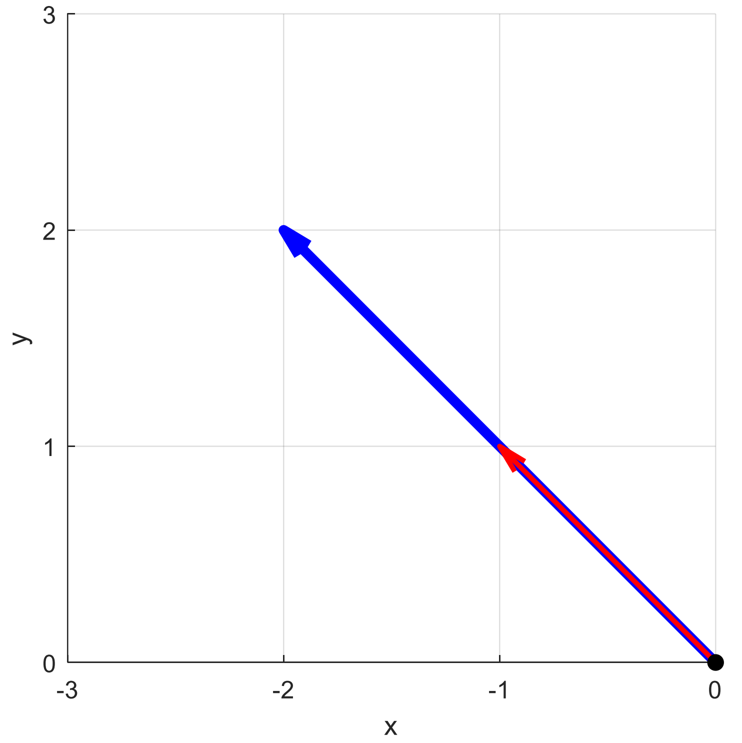
- E.g.) 2D $$4\begin{bmatrix} c \\ c \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} c \\ c \end{bmatrix}, \quad c \in \mathbb{R}$$



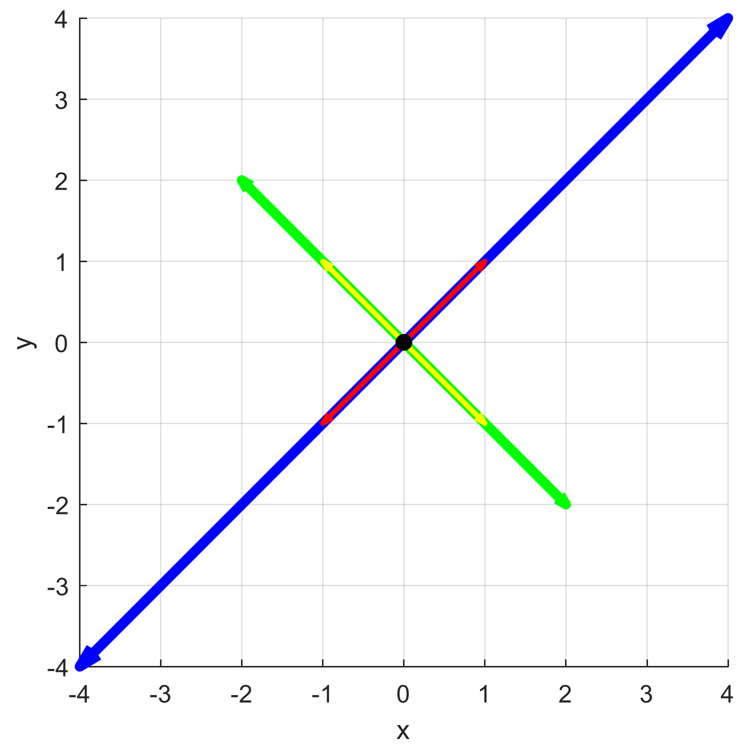
How about this input vector?

- E.g.) 2D

$$2\begin{bmatrix}-1\\1\end{bmatrix}=\begin{bmatrix}3 & 1\\1 & 3\end{bmatrix}\begin{bmatrix}-1\\1\end{bmatrix}$$



Then any vectors on those lines



Wow, they’re very special!

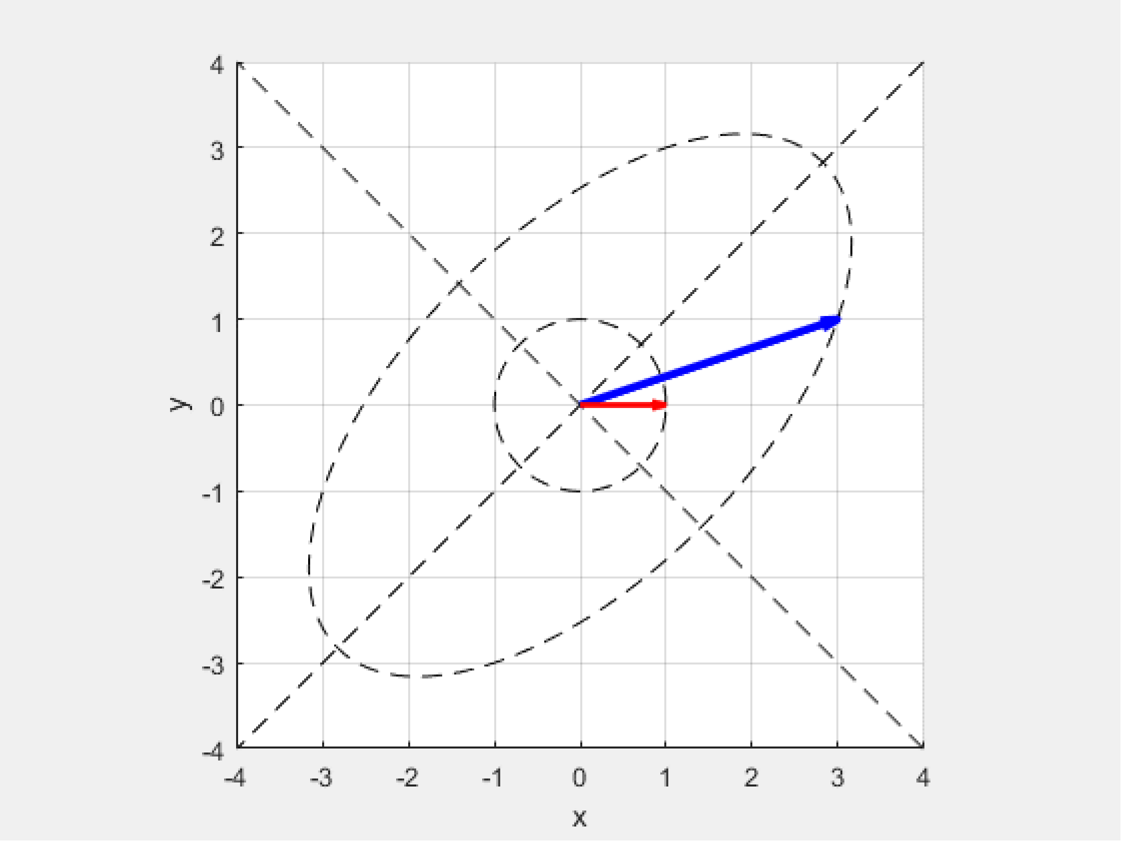
- Let’s call those vectors eigenvectors,

$$\mathbf{v}_1 = c_1 \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \quad \mathbf{v}_2 = c_2 \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

- And how much they scale, eigenvalues.

$$\lambda_1 = 4, \quad \lambda_2 = 2$$

Let’s transform the unit circle!



In fact, this shows all the mapping from R^2 to R^2.

How did you get eigenvalues/vectors?

- Given a square matrix A,

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$$

  - $\mathbf{x}$: eigenvector
  - λ : eigenvalue

How did you get eigenvalues?

- Given a square matrix A,

$$\boxed{\mathbf{A}\mathbf{x} = \lambda\mathbf{x}}$$
$$\Rightarrow \mathbf{A}\mathbf{x} - \lambda\mathbf{x} = \mathbf{0}$$
$$\Rightarrow \mathbf{A}\mathbf{x} - \lambda\mathbf{I}\mathbf{x} = \mathbf{0}$$
$$\Rightarrow (\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$$

- We have two cases.
  - If $(\mathbf{A} - \lambda\mathbf{I})^{-1}$ exists, $\mathbf{x} = \mathbf{0}$ (trivial solution).
  - If $(\mathbf{A} - \lambda\mathbf{I})^{-1}$ does not exist, we get non-trivial solutions.

$$\boxed{\det(\mathbf{A} - \lambda\mathbf{I}) = 0 \quad \text{: Characteristic Equation}}$$

How did you get eigenvalues?

- E.g.) 2D

$$\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow (3-\lambda)(3-\lambda) - 1 = 0$$

$$\Rightarrow (3-\lambda)^2 = 1$$

$$\Rightarrow 3-\lambda = \pm 1$$

$$\Rightarrow \lambda = 4 \text{ or } 2$$

How did you get eigenvectors?

① $\lambda = 4$

$$\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 4 \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Rightarrow \begin{cases} 3x + y = 4x \\ x + 3y = 4y \end{cases}$$

$$\Rightarrow x = y$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ x \end{bmatrix} = x \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\therefore \lambda_1 = 4,\ \mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

How did you get eigenvectors?

② $\lambda = 2$

$$\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 2 \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Rightarrow \begin{cases} 3x + y = 2x \\ x + 3y = 2y \end{cases}$$

$$\Rightarrow x = -y$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ y \end{bmatrix} = y \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\therefore \lambda_2 = 2,\ \mathbf{x}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Q&A