Recitation

Linear Algebra

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Background

- System of Linear Equations

$$\begin{cases} 2x + y = 1 \\ x + 3y = 2 \end{cases} \quad \Rightarrow \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\Rightarrow \mathbf{A}\mathbf{x} = \mathbf{b}$$

Outline

- **Vectors**
  - Operations on Vectors
  - Basis Vectors
- **Matrices**
  - Operations on Matrices
  - Linear Function
  - Linear Transformation
- **Eigenvector and Eigenvalue**

Vectors

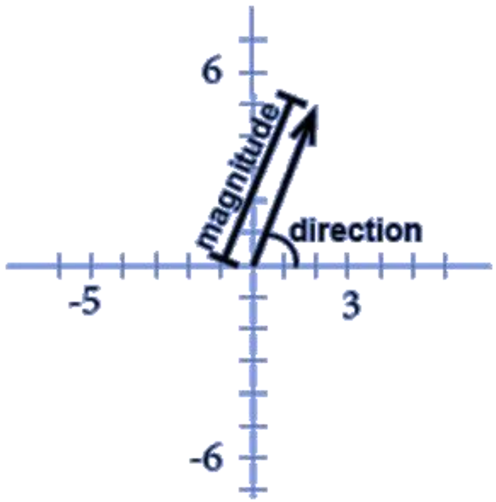
Scalars vs. Vectors

- Scalar: a physical quantity described by its magnitude

$$x \in \mathbb{R}$$

- Vector = Collection of Scalars

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$$



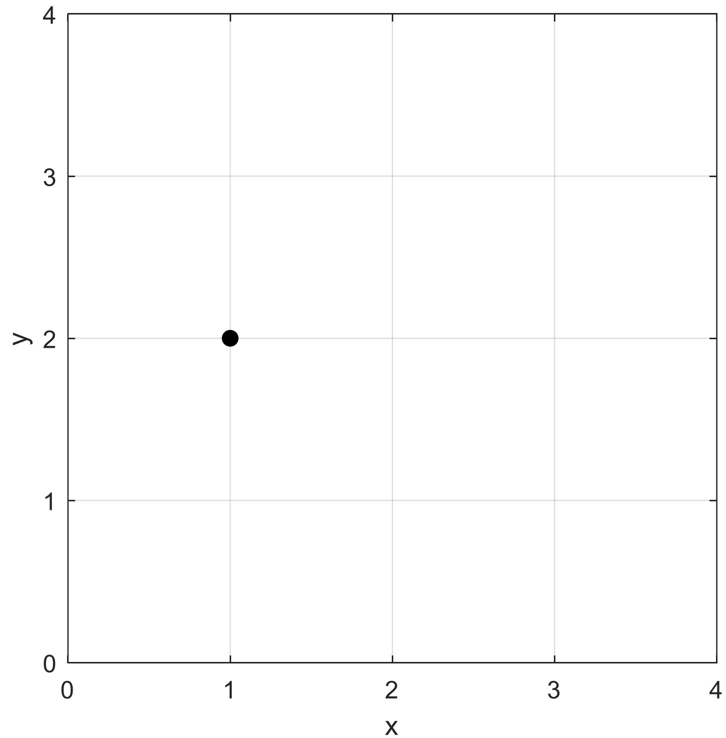
A Point is Represented as a Vector

- Point in n-D

$$\mathbf{x} = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$$

- E.g.) 2D

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$



A Free Vector is Represented as a Vector

- Free Vector in n-D

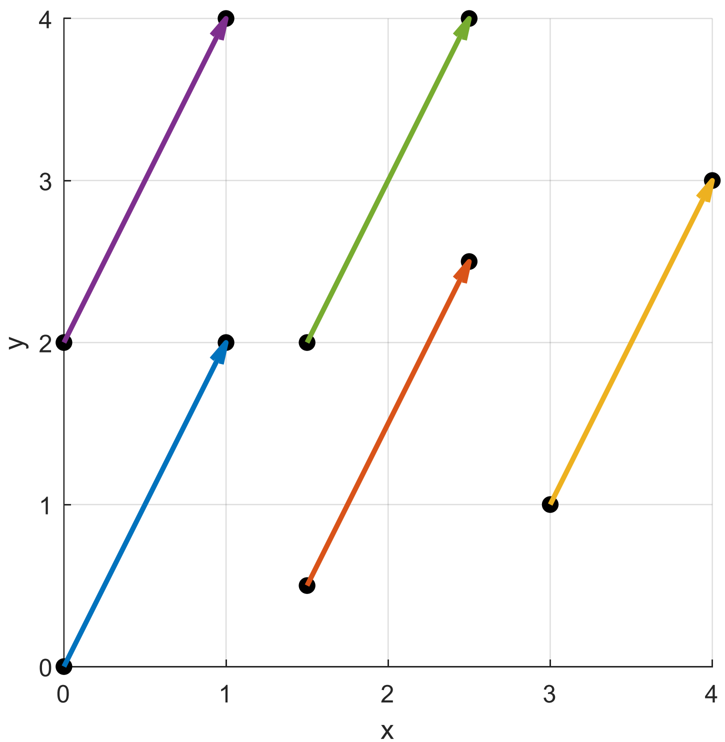
$$\mathbf{x} = [x_1, x_2, \ldots, x_n]^\top \in \mathbb{R}^n$$

- E.g.) 2D

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix} - \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 4 \end{bmatrix} - \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$\vdots$$



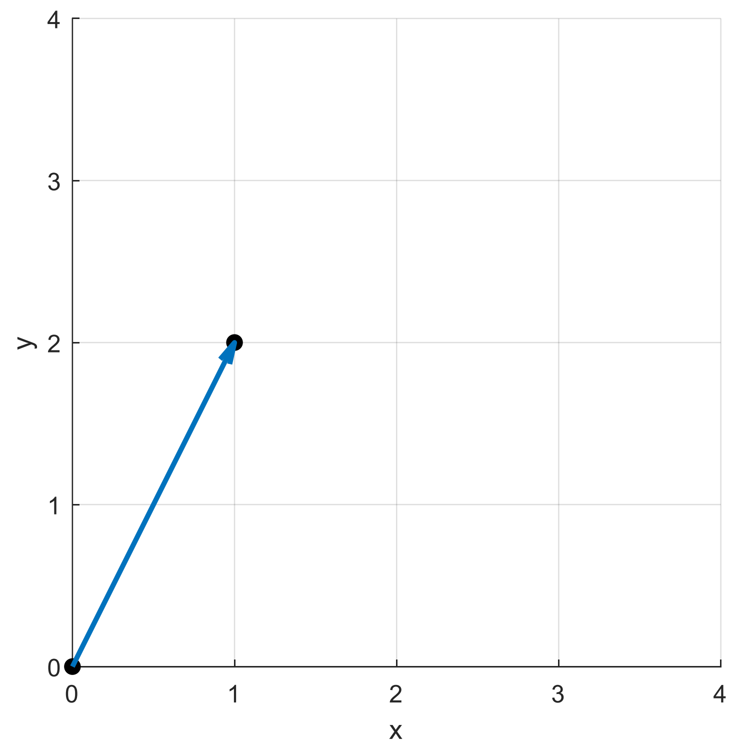
A Fixed Vector is Represented as a Vector

- Fixed Vector in n-D

$$\mathbf{x} = [x_1, x_2, \ldots, x_n]^\top \in \mathbb{R}^n$$

- E.g.) 2D

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$



Points vs. Fixed Vectors vs. Free Vectors

- Vector = Difference between two Points

$$\mathbf{x} = \mathbf{p} - \mathbf{q}$$

Concept	Space	Time
Point	Location	Time
Fixed Vector	Displacement from origin	Duration from origin
Free Vector	Displacement between two locations	Duration between two times

- But, they are represented in the same way in math.

Operations on Vectors

Transpose

- Vector Transpose

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}^\top = [x_1 \; x_2 \; \cdots \; x_n]$$

$$[x_1 \; x_2 \; \cdots \; x_n]^\top = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Norms

- Vector Norm = Length (Magnitude)

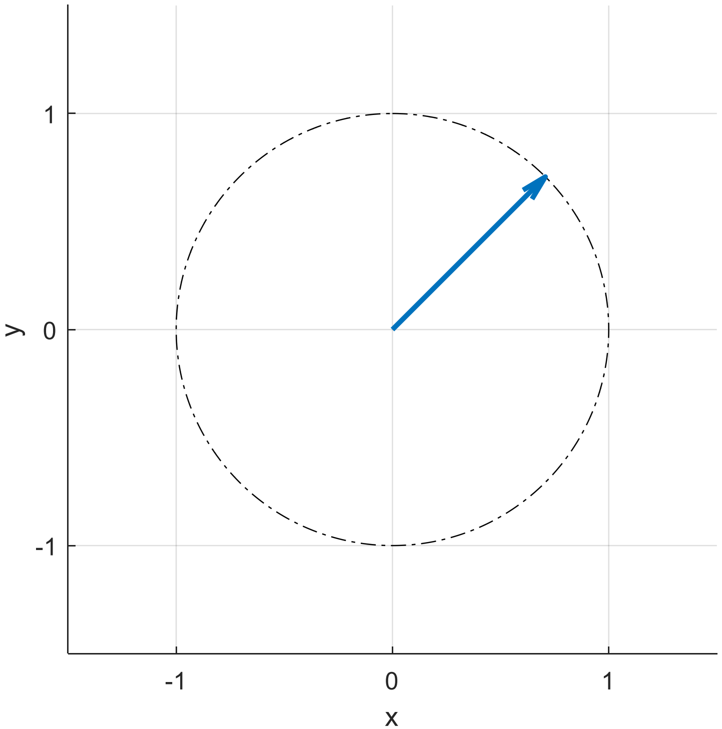
$$|\mathbf{x}| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

- Unit Vector (on Unit Circle)

$$|\hat{\mathbf{x}}| = 1, \quad \hat{\mathbf{x}} = \frac{\mathbf{x}}{|\mathbf{x}|}$$

- E.g.) 2D

$$\mathbf{x} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$



Scalar Multiplications

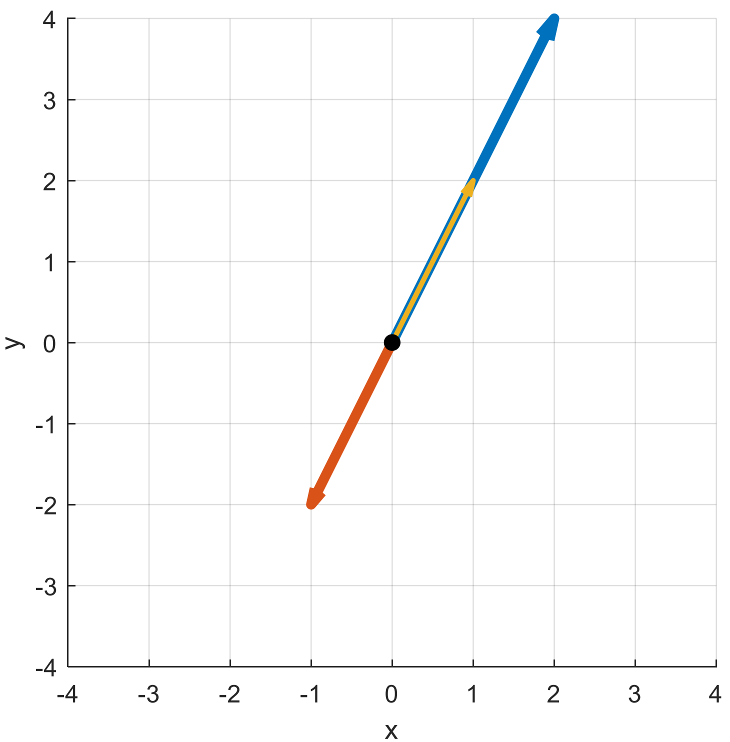
- Scalar Multiplication

$$a\mathbf{x} = (ax_1, \ldots, ax_n)^\top$$

- E.g.) 2D

$$2\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}2\\4\end{bmatrix}$$

$$-1\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}-1\\-2\end{bmatrix}$$



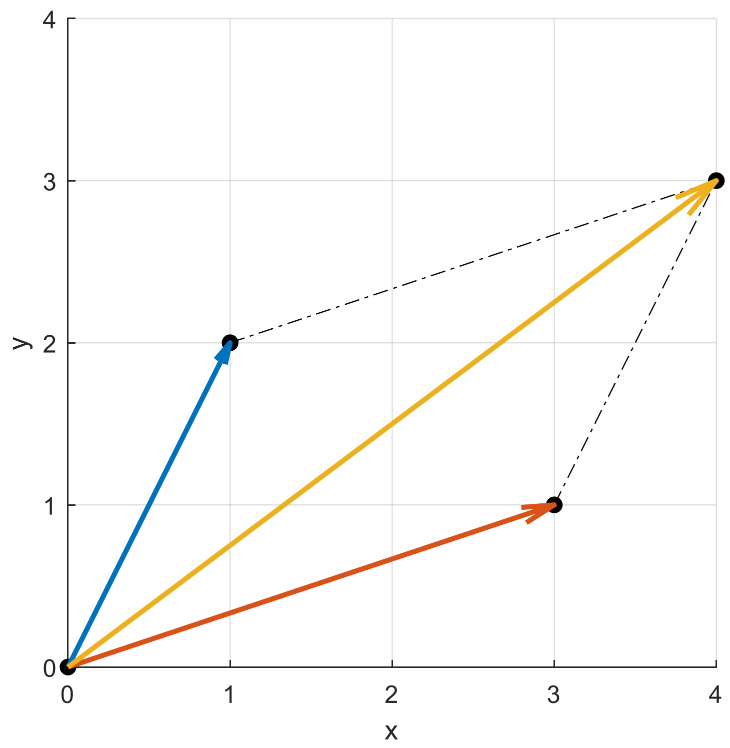
Additions

- Vector Addition

$$\mathbf{x} + \mathbf{y} = (x_1 + y_1, \ldots, x_n + y_n)^\top$$

- E.g.) 2D

$$\begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$



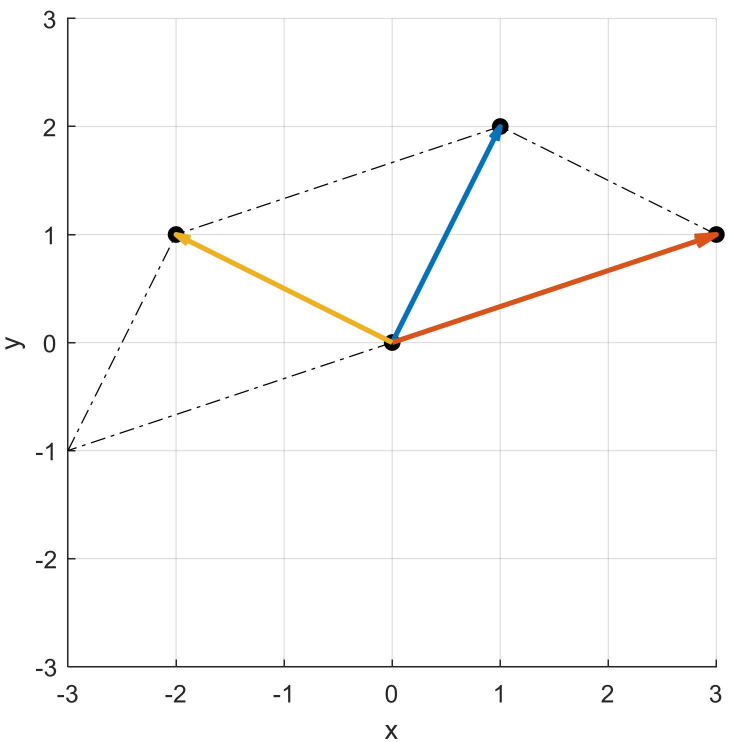
Subtractions

- Vector Subtraction
  (Addition after Scalar Multiplication by -1)

$$\mathbf{x} - \mathbf{y} = (x_1 - y_1, \ldots, x_n - y_n)^\top$$
$$= \mathbf{x} + (-\mathbf{y})$$

- E.g.) 2D

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$



Dot Products

- Given two vectors

$$\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$$

- Dot Product is a **scalar**

$$\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{n} x_i y_i \in \mathbb{R}$$

- E.g.) 2D

$$(4, 2)^\top \cdot (1, 2)^\top = 8$$

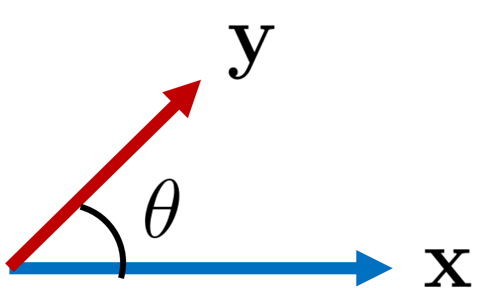
Dot Products

- Commutative

$$\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$$

- Physical Meaning

$$\mathbf{x} \cdot \mathbf{y} = |\mathbf{x}||\mathbf{y}| \cos\theta$$



- Special Case

$$\mathbf{x} \perp \mathbf{y} \;\Leftrightarrow\; \mathbf{x} \cdot \mathbf{y} = 0$$

Basis Vectors

Linear Combinations

- Given vectors,

$$\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_m \in \mathbb{R}^n$$

- Linear Combination is

$$a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + ... + a_m\mathbf{x}_m \in \mathbb{R}^n$$

for scalars

$$a_1, a_2, ..., a_m \in \mathbb{R}$$

Linearly Dependent

- Given a set of vectors and a vector,

$$\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_m \in \mathbb{R}^n, \quad \mathbf{x} \in \mathbb{R}^n$$

- If there exists a linear combination of the set of vectors for the other vector, it is linearly dependent.

$$^\exists a_1, a_2, ..., a_m \in \mathbb{R} \quad \text{s.t.}$$

$$\mathbf{x} = a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + ... + a_m\mathbf{x}_m$$

Linearly Dependent

- E.g.) 2D

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 6 \\ 2 \end{bmatrix}$$

Linearly Independent

- Given a set of vectors and a vector,

$$\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_m \in \mathbb{R}^n, \quad \mathbf{x} \in \mathbb{R}^n$$

- If there exists **no** linear combination of the set of vectors for the other vector, it is linearly independent.

$$\nexists a_1, a_2, ..., a_m \in \mathbb{R} \quad \text{s.t.}$$

$$\mathbf{x} = a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + ... + a_m\mathbf{x}_m$$

Linearly Independent

- E.g.) 3D

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$$

Span

- Given vectors,

$$\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m \in \mathbb{R}^n$$

- All possible linear combinations is their **span**,

$$a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \ldots + a_m\mathbf{x}_m \in \mathbb{R}^n, \ \forall a_1, a_2, \ldots, a_m \in \mathbb{R}$$

Basis

- A *basis* is a set $\boldsymbol{B}$ of elements (vectors) in a vector space $\boldsymbol{V}$, if every element of $\boldsymbol{V}$ is represented as a linear combination of elements of $\boldsymbol{B}$.
  - A vector space is a collection of vectors.
  - The elements of a basis are called basis vectors.

- $\boldsymbol{B}$ is a basis if its elements are linearly independent and every element of $\boldsymbol{V}$ is a linear combination of elements of $\boldsymbol{B}$.
  - A basis is a linearly independent spanning set.

Standard Basis Vectors

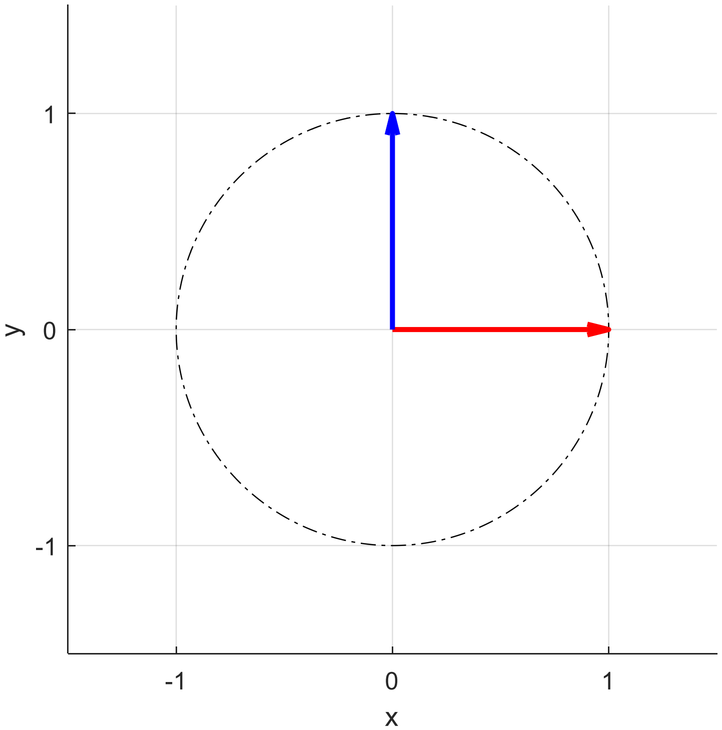
- Standard Basis Vectors

$$\mathbf{e}_i = (0, \ldots, 1, \ldots, 0)^\top, \quad x_i = 1, x_j = 0, i \neq j$$

- E.g.) 2D

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The span of standard basis vectors is the 2D space.



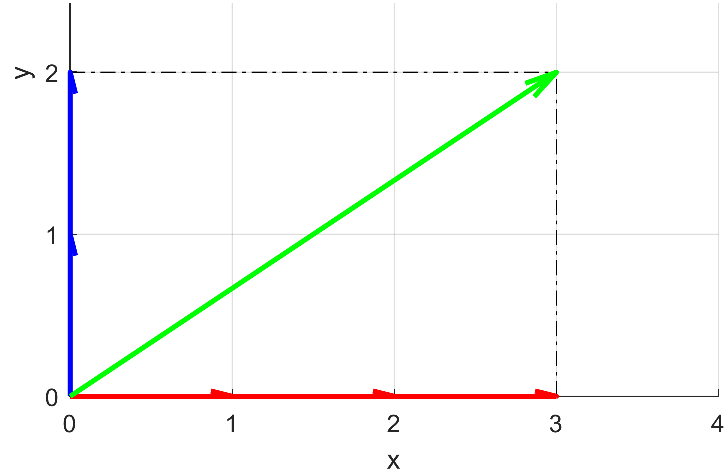
Standard Basis Vectors

- Coordinates = Coefficients for Linear Combination of Standard Basis Vectors

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + x_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} = \sum_{i=1}^{n} x_i \mathbf{e}_i$$

- E.g.) 2D

$$\mathbf{x} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



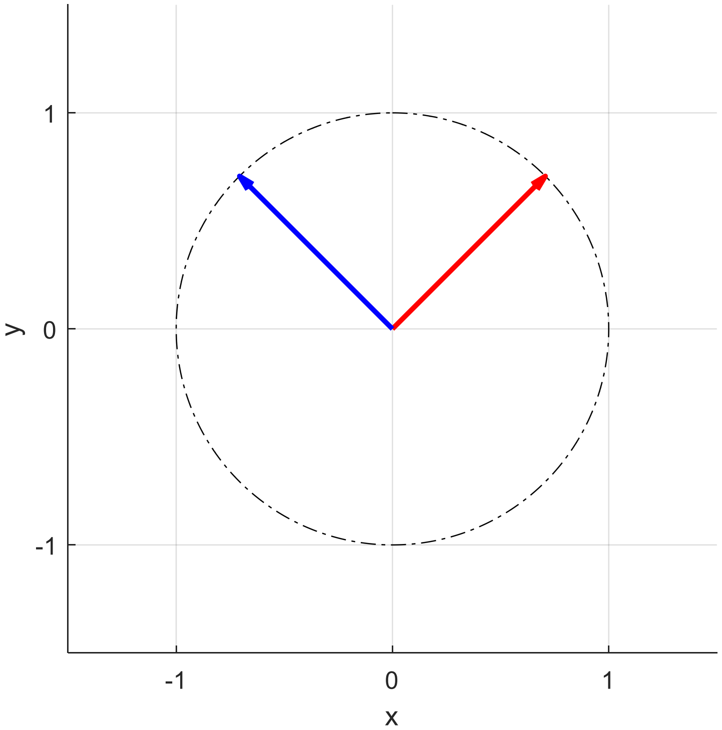
Orthonormal Basis Vectors

- Basis vectors are orthonormal if and only if

$$\mathbf{e}_i \perp \mathbf{e}_j, \quad |\mathbf{e}_i| = 1$$

- E.g.) 2D

$$\mathbf{e}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$



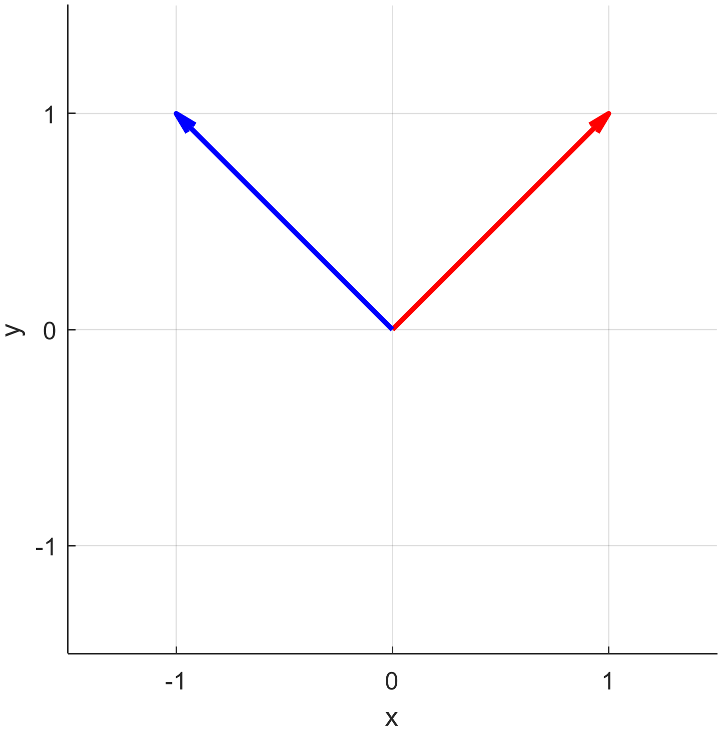
Orthogonal Basis Vectors

- Basis vectors are orthogonal if and only if

$$\mathbf{e}_i \perp \mathbf{e}_j$$

- E.g.) 2D

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

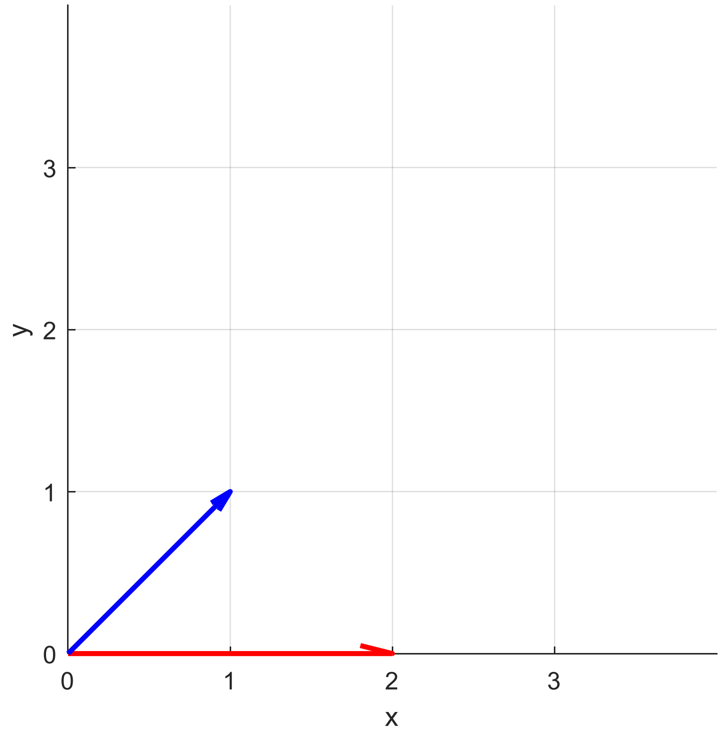


Basis Vectors

- Basis vectors must be linearly independent.

- E.g.) 2D

$$\mathbf{e}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

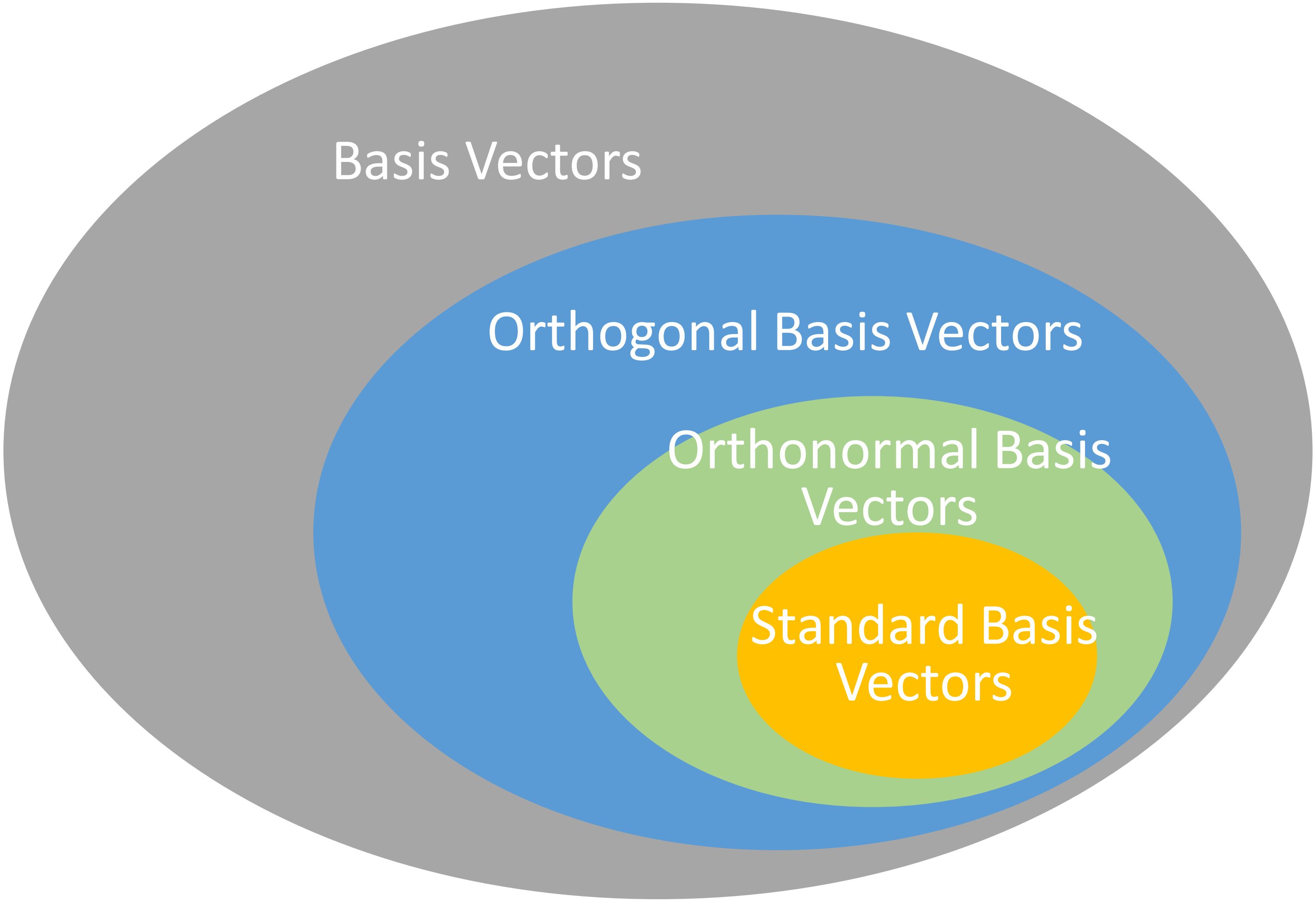


Change of Basis

- E.g.) 2D

$$
\begin{aligned}
\mathbf{x} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} &= 3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \frac{5}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} - \frac{1}{\sqrt{2}} \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \\
&= \frac{5}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\
&= 2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} - 1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\end{aligned}
$$

Basis Vector Sets



Summary – Vectors

- Points vs. Vectors
- Vector Operations
  - Transpose
  - Norm
  - Scalar Multiplication
  - Addition
  - Subtraction
  - Dot Product
- Linear Combination
  - Linearly Dependent
  - Linearly Independent
  - Span
- Basis Vectors
  - Standard Basis Vectors
  - Orthonormal Basis Vectors
  - Orthogonal Basis Vectors
  - Basis Vectors