Invariance of Bar-Natan matrix multifactorizations up to 1-homotopy equivalence

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Abstract

We verify link invariance of a certain construction using matrix factorizations over $\mathbb{Z}[H]$.

1 The construction

Let $\mathcal{D}_0$ be a dotted edge and $\mathcal{D}_1$ be a thick edge.

Consider a base ring $\mathbf{K}$ (e.g. $\mathbf{K} = \mathbb{Z}[\mathsf{G}]$ with grading $\mathsf{gr}(\mathsf{G}) = i^{-2}h^0q^{-2}$). Take some potential $P(X) \in \mathbf{K}[X]$ (or even $\mathbf{K}[\![X]\!]$ if $\mathbf{K}$ is big enough).

We denote by x_0, x_1, x_2 and x_3 the variables of surrounding faces, starting above and going counterclockwise. In $\mathbf{K}[x_0 - x_1, x_2 - x_1, x_3 - x_2]$, it is always possible to factor

$$W = P(x_0 - x_1) + P(x_3 - x_0) - P(x_2 - x_1) - P(x_3 - x_2) = (x_0 - x_2)(x_0 + x_2 - x_1 - x_3)Z$$

for some $Z = Z(x_0 - x_1, x_2 - x_1, x_3 - x_2) \in \mathbf{K}[x_0 - x_1, x_2 - x_1, x_3 - x_2]$. We often drop the inputs of Z to lighten notation.

We assign Koszul matrix factorizations

$$\begin{aligned} M(\mathcal{D}_0) &= K(x_0 - x_2, \quad (x_0 + x_2 - x_1 - x_3)Z) \\ M(\mathcal{D}_1) &= K(Z, \quad (x_0 - x_2)(x_0 + x_2 - x_1 - x_3)). \end{aligned}$$

Given two matrix factorizations of the form

$$M = [\ A \rightleftarrows \hbar B\ ], \qquad M' = [\ A' \rightleftarrows \hbar B'\ ],$$

with A, A', B and B' concentrated in $\hbar$-degree 0, we often specify morphisms by spelling out their components in each $\hbar$-degree as follows:

$$\begin{tikzcd} M \arrow[d, "\hbar^0(\alpha,\beta)"'] \\ M' \end{tikzcd} = \begin{tikzcd} {[A} \arrow[r, shift left] \arrow[d, "\alpha"] & \hbar B] \arrow[l, shift left] \arrow[d, "\beta"] \\ {[A'} \arrow[r, shift left] & \hbar B'] \arrow[l, shift left] \end{tikzcd} \qquad \begin{tikzcd} M \arrow[d, "\hbar^1(\alpha,\beta)"'] \\ M' \end{tikzcd} = \begin{tikzcd} {[A} \arrow[r, shift left] \arrow[dr, "\alpha"] & \hbar B] \arrow[l, shift left] \arrow[dl, "\beta"'] \\ {[A'} \arrow[r, shift left] & \hbar B'] \arrow[l, shift left] \end{tikzcd}$$

Many useful maps can be easily expressed in this notation. For instance, the differential $d_M : M \to M$ itself takes the form

$$d_m = \hbar^1(d_0, d_1)$$

In the case $M' = \hbar M$ we use the special notation

$$s_\hbar = \hbar^1(1, 1) : M \to \hbar M$$

For Koszul matrix factorizations, we will take the convention

$$K_R(a\ ;\ b) = \ q^{\deg_q(a)+3} R \overset{a}{\underset{b}{\rightleftarrows}} \hbar R\ .$$

Here, the grading $\hbar$ is a $\mathbb{Z}/2$ grading. A shift $\hbar M$ on a matrix factorization M has the additional effect of switching the sign of all differentials. This helps us keep track of Koszul sign rules for tensor products of matrix factorizations. Namely, if we tensor a Koszul factorization $K_R(a\ ;\ b)$ with some other matrix factorization M, we can write

$$K_R(a\ ;\ b) \otimes M = \left[\ q^{\deg_q(a)+3} M \underset{b}{\overset{a}{\rightleftarrows}} \hbar M\ \right],$$

which indicates that the tensor product splits, as free R-module, into two copies of M, with the first one shifted in q and $\hbar$ degrees. In terms of this decomposition, the total differential takes the form

$$d_{K_R(a\ ;\ b)\otimes M} = \begin{pmatrix} d_M & b \\ a & -d_M \end{pmatrix},$$

where the negative sign comes precisely from the $\hbar$-shift. Observe also that here a and b are understood to have $\hbar$-degree 1, and they could be more explicitly denoted by $\hbar^1(a,a)$ and $\hbar^1(b,b)$, respectively.

If we want to switch the sign of differentials without altering the $\hbar$ degree, we write $-M$. Using these notations, we have

not sure if I need this

$$\hbar(M \otimes N) = (\hbar M) \otimes N = (-M) \otimes (\hbar N)$$

The most general webs we will be dealing with can be obtained by taking an oriented planar arc diagram T where all inputs have exactly two consecutive incoming arcs and two consecutive outgoing arcs, and then plugging $\mathcal{D}_0$ or $\mathcal{D}_1$ in each input while respecting orientations:

$$\mathcal{D} = T(\mathcal{D}_{i_1}, \ldots, \mathcal{D}_{i_k}),$$

with $i_j = 0$ or 1.

2 Hom spaces

Given two webs $\mathcal{D}_1$ and $\mathcal{D}_2$ with the same boundary, we want a good handle of the space

$$Hom(M(\mathcal{D}_2), M(\mathcal{D}_1)).$$

First, we need to understand closed webs. Let $\mathcal{D}$ be a closed web

When we glue the mirror $\mathcal{D}_1$ and the mirror $\overline{\mathcal{D}}_2$ along the boundary, we obtain a closed web denoted $\mathcal{D}_1 \cup \overline{\mathcal{D}}_2$

Lemma 2.1.

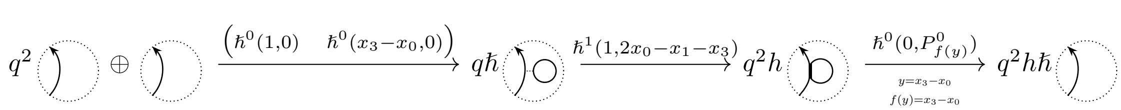
$$\mathrm{Ext}(M(\mathcal{D}_2), M(\mathcal{D}_1)) \cong H(M(\mathcal{D}_1 \cup \overline{\mathcal{D}}_2))$$

Proposition 2.2.

3 Invariance

The invariance of $M(D)$ for a link diagram D under each Reidemeister move will proceed by looking at the explicit matrix multifactorization associated to the relevant local piece of the diagram and simplifying it in two steps. In the first step, one simplifies each resolving web into smaller webs with no internal faces, and then extends into a special 0-deformation retract of the entire multifactorization. There is enough control of the deformation data to compute explicit differentials in the resulting multifactorization, and in particular to identify some identity components in d_1. The second step then consists of simplifying along such identities to obtain a special 1-deformation retract into the desired form.

Reidemeister I moves are illustrated in Figure 1. The purple arrows represent special deformation retracts of vertical matrix factorizations, where each component is obtained from Lemma 3.3 or Lemma 3.4. We then extend, by Lemma **??**, to a special 0-deformation retract of multifactorizations. Notice that we indicate the explicit form of the d_1-component in each of the once-simplified multifactorizations. These are computed as composites:



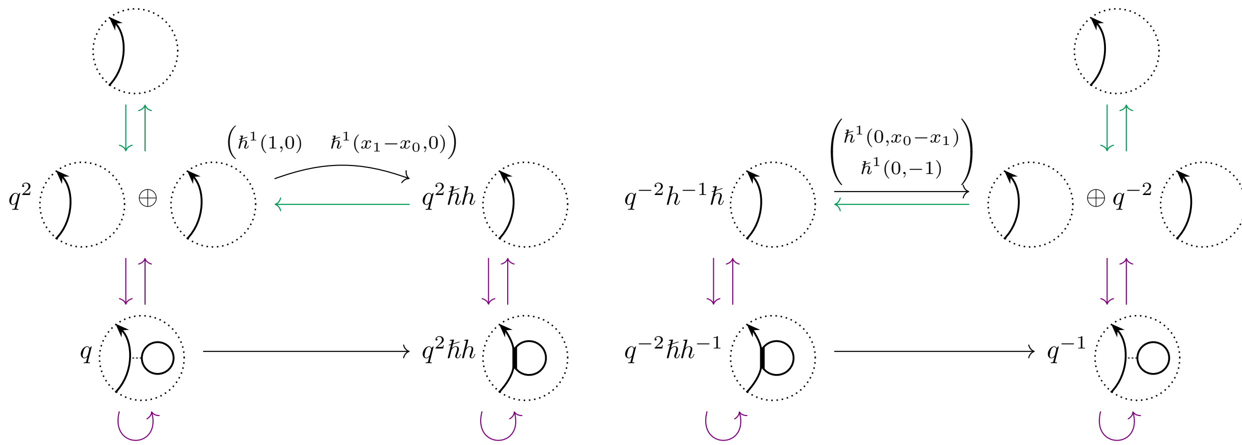
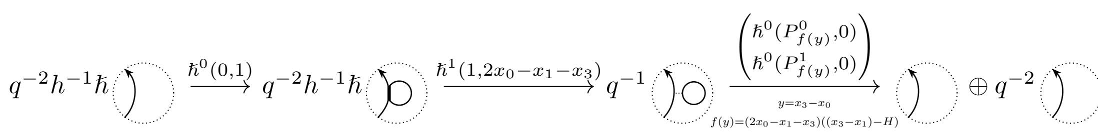
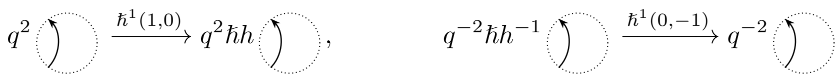


Figure 1: Invariance under positive and negative Reidemeister I moves

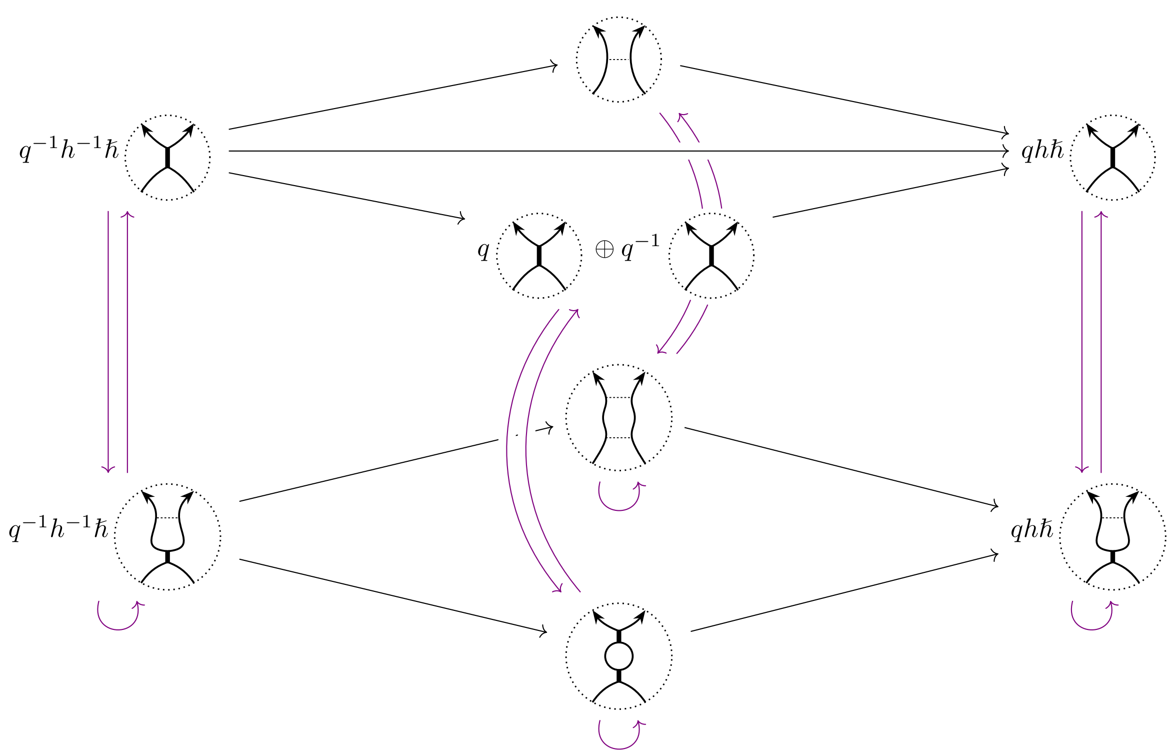


The summands



are easily verified to be 1-contractible. The corresponding special 1-deformation retractions cancelling these summands are represented by the green arrows.

Concerning Reidemeister 2, here is what we do



Definition 3.1. Let (C, D) and (C', D') be multifactorizations. Then a special n-deformation retract from C to C' consists of 0-morphisms $P : C \to C'$ and $I : C' \to C$ such that $PI = 1$, together with a n-homotopy $H : C \to C$ between 1 and IP such that $HI = 0$, $PH = 0$ and $H^2 = 0$. We represent all this data by

$$\begin{tikzcd} (C', D') \arrow[r, shift left, "I"] & (C, D) \arrow[l, shift left, "P"] \arrow[loop right, "H"] \end{tikzcd}$$

The next lemma tells us how to simplify certain Koszul matrix factorizations into special deformation retracts.

Lemma 3.2. *(Adapted from KR How general is R?) Let R be an integral domain. Let $\overline{W} \in R$ and $f, g \in R[y]$ so that f has the form $f = uy^n + \tilde{f}$ wtih $u \in R[y]^\times$ and $\deg_y(\tilde{f}) < n$. Let M be a matrix factorization over $R[y]$ with potential $W = \overline{W} - fg \in R[y]$. Let*

$$\begin{aligned} M' &= M/fM, \\ \overline{M} &= K_{R[y]}(f; g) \otimes_{R[y]} M, \end{aligned}$$

thought of as matrix factorizations over R with potential $\overline{W}$. Then there is a strong deformation retract

$$\begin{tikzcd} M' \arrow[r, shift left, "I"] & \overline{M} \arrow[l, shift left, "P"] \arrow[loop right, "H"] \end{tikzcd}$$

of the form

$$\begin{tikzcd} & M/fM \arrow[dl, bend right, "I_d"'] \arrow[d, shift right, "I_v"'] \\ q^{3-2n}\hbar M \arrow[r, bend left, "f"] & M, \arrow[u, shift right, "P"'] \arrow[l, "H"] \arrow[l, bend left, "g"] \end{tikzcd}$$

The arrows are as follows:

- *P is the usual projection.*
- *$H = -\operatorname{Quo}_f$, the negative of the quotient of division by f.*
- *I consists of a vertical component $I_v = \operatorname{Res}_f$ given by the residue of division by f, and a diagonal component I_d given by the composite $H \circ d_M \circ I_v$.*

carry the modification of the sign of H

In many cases of interest, we additionally have decompositions $M/fM = \bigoplus_{i=0}^{k} M_i$ as matrix factorizations over R. Such an identification, in conjunction with the lemma, will then produce a special deformation retract that simplifies M into a sum of simpler matrix factorizations over a ring with one less variable.

The following is a straightforward corollary.

Lemma 3.3. *In the set up of Lemma 3.2, assume that $M = K_{R[y]}(\overline{\mathbf{a}}, \overline{\mathbf{b}})$ and that $f = y - f_0$, where $\deg_y(f_0) = 0$. Then, under the identification $M/yM \cong K_R\left(\overline{\mathbf{a}}\big|_{y=f_0}, \overline{\mathbf{b}}\big|_{y=f_0}\right)$, we have a special deformation retract*

$$\begin{tikzcd} & K_R\left(\overline{\mathbf{a}}\big|_{y=f_0}, \overline{\mathbf{b}}\big|_{y=f_0}\right) \arrow[dl, bend right] \arrow[d, shift right, "1"'] \\ q\hbar K_{R[y]}(\overline{\mathbf{a}}, \overline{\mathbf{b}}) \arrow[r, bend left, "f"] & K_{R[y]}(\overline{\mathbf{a}}, \overline{\mathbf{b}}), \arrow[u, shift right, "y \mapsto f_0"'] \arrow[l, "-\operatorname{Quo}_f"] \arrow[l, bend left, "g"] \end{tikzcd}$$

The diagonal component of the inclusion is the composite

$$K_R\left(\overline{\mathbf{a}}\big|_{y=f_0}, \overline{\mathbf{b}}\big|_{y=f_0}\right) \xrightarrow{\sum_{j=1}^{k} 1 \otimes \cdots \otimes \hbar^1(-\operatorname{Quo}_f \circ a_j, -\operatorname{Quo}_f \circ b_j) \otimes \cdots \otimes 1} qK_{R[y]}\left(\overline{\mathbf{a}}, \overline{\mathbf{b}}\right) \xrightarrow{s_\hbar} q\hbar K_{R[y]}\left(\overline{\mathbf{a}}, \overline{\mathbf{b}}\right)$$

This can be generalized

Lemma 3.4. *In the set up of Lemma 3.2, assume that $M = K_{R[y]}(\overline{\mathbf{a}}, \overline{\mathbf{b}})$ and that $\deg_y(\mathrm{Res}_f(a_i)) = \deg_y(\mathrm{Res}_f(b_i)) = 0$ for all i. Then the map*

that final degree switch in $\hbar$

$$\bigoplus_{i=0}^{n-1} q^{-2i} K_R\left(\mathrm{Res}_f \overline{\mathbf{a}}, \mathrm{Res}_f \overline{\mathbf{b}}\right) \xrightarrow{(y^i)_i} M/fM$$

is an isomorphism of matrix factorizations over R and, combined with Lemma 3.2, gives a special deformation retract

$$\begin{tikzcd}
 & \bigoplus_{i=0}^{n-1} q^{-2i} K_R\left(\mathrm{Res}_f \overline{\mathbf{a}}, \mathrm{Res}_f \overline{\mathbf{b}}\right) \arrow[dl, bend right] \arrow[d, shift right, "(y^i)_i"'] \\
q^{3-2n}\hbar K_{R[y]}(\overline{\mathbf{a}}, \overline{\mathbf{b}}) \arrow[r, bend left, "f"] & K_{R[y]}(\overline{\mathbf{a}}, \overline{\mathbf{b}}), \arrow[l, "\mathrm{Quo}_f"] \arrow[l, bend left, "g"] \arrow[u, shift right, "\left(\frac{1}{i!}\frac{\partial^i}{\partial y^i}\big|_{y=0} \circ \mathrm{Res}_f\right)_i"']
\end{tikzcd}$$

The diagonal component of the inclusion is the composite

$$\begin{tikzcd}
\bigoplus_{i=0}^{n-1} q^{-2i} K_R\left(\mathrm{Res}_f \overline{\mathbf{a}}, \mathrm{Res}_f \overline{\mathbf{b}}\right) \arrow[r, "\left(\sum_{j=1}^{k} 1\otimes\cdots\otimes\hbar^1(\mathrm{Quo}_f \circ y^i a_j, \mathrm{Quo}_f \circ y^i b_j)\otimes\cdots\otimes 1\right)_i"] & q^{3-2n} K_{R[y]}\left(\overline{\mathbf{a}}, \overline{\mathbf{b}}\right) \arrow[d, "s_\hbar"] \\
 & q^{3-2n}\hbar K_{R[y]}\left(\overline{\mathbf{a}}, \overline{\mathbf{b}}\right)
\end{tikzcd}$$