Invariance of Bar-Natan matrix multifactorizations up to 1-homotopy equivalence

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Abstract

We verify link invariance of a certain construction using matrix factorizations over $\mathbb{Z}[H]$.

1 Matrix factorization conventions

Given two matrix factorizations of the form

$$M = [\ \hbar A \rightleftarrows B\ ], \qquad M' = [\ \hbar A' \rightleftarrows B'\ ],$$

with A, A', B and B' concentrated in $\hbar$-degree 0, we often specify morphisms by spelling out their components in each $\hbar$-degree as follows:

$$\begin{tikzcd} M \arrow[d, "\hbar^0(\alpha,\beta)"'] \\ M' \end{tikzcd} = \begin{tikzcd} {[\,\hbar A} \arrow[r, shift left] \arrow[d, "\alpha"] & B\,] \arrow[l, shift left] \arrow[d, "\beta"] \\ {[\,\hbar A'} \arrow[r, shift left] & B'\,] \arrow[l, shift left] \end{tikzcd} \qquad \begin{tikzcd} M \arrow[d, "\hbar^1(\alpha,\beta)"'] \\ M' \end{tikzcd} = \begin{tikzcd} {[\,A} \arrow[r, shift left] \arrow[dr, bend left, "\alpha"] & \hbar B\,] \arrow[l, shift left] \arrow[dl, bend right, "\beta"'] \\ {[\,A'} \arrow[r, shift left] & \hbar B'\,] \arrow[l, shift left] \end{tikzcd}$$

Many useful maps can be easily expressed in this notation. For instance, the differential $d_M : M \to M$ itself takes the form

$$d_m = \hbar^1(d_1, d_0)$$

In the case $M' = \hbar M$, we will often use the map

$$s_\hbar = \hbar^1(1,1) : M \to \hbar M.$$

For Koszul matrix factorizations, we will take the convention

$$K_R\big(a \quad b\big) = \ q^{\deg_q(a)+3}\hbar R \underset{b}{\overset{a}{\rightleftarrows}} R\ .$$

Here, the grading $\hbar$ is a $\mathbb{Z}/2$ grading. A shift $\hbar M$ on a matrix factorization M has the additional effect of switching the sign of all differentials. This helps us keep track of Koszul sign rules for tensor products of matrix factorizations. Namely, if we tensor a Koszul factorization $K_R\big(a \quad b\big)$ with some other matrix factorization M, we can write

$$K_R\big(a \quad b\big) = \left[\ q^{\deg_q(a)+3}\hbar M \underset{b}{\overset{a}{\rightleftarrows}} M\ \right].$$

This indicates that the tensor product splits, as free R-module, into two shifted copies of M. In terms of this decomposition, the total differential takes the form

Koszul m.f. looks bad as a subindex. fix formatting

$$d_{K_R\big(a \quad b\big)\otimes M} = \begin{pmatrix} -d_M & b \\ a & d_M \end{pmatrix},$$

where the negative sign comes precisely from the $\hbar$-shift. Observe also that here a and b are understood to have $\hbar$-degree 1, and they could be more explicitly denoted by $\hbar^1(a,a)$ and $\hbar^1(b,b)$, respectively.

If we want to switch the sign of differentials in M without altering the $\hbar$ degree, we write $-M$. Using these notations, we have

not sure if I need this

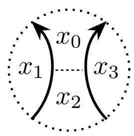
$$\hbar(M \otimes N) = (\hbar M) \otimes N = (-M) \otimes (\hbar N)$$

When $\deg_q(a) + \deg_q(b) = -6$, as is usually the case in $_2$-homology, we have

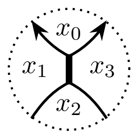
$$K_R\big(a \quad b\big) = q^{\deg_q(a)+3} \hbar K_R\big(-b \quad -a\big)$$

2 Matrix factorizations associated to webs

In the following definitions, the base ring is $\mathbf{K} = \mathbb{Z}[H]$, and the elementary matrix factorizations are over the ring $R = \mathbf{K}[x_0 - x_1, x_2 - x_1, x_3 - x_2]$. The variables x_i and H live in degree q^{-2}.

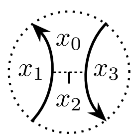
$D_0 =$ 

$\longmapsto \mathcal{M}(D_0) = \left[\; q\hbar R \underset{(x_0+x_2-x_1-x_3)((x_3-x_1)-H)}{\overset{x_0-x_2}{\rightleftarrows}} R \;\right]$

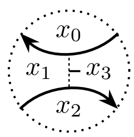
$D_1 =$ 

$\longmapsto \mathcal{M}(D_1) = \left[\; qR \underset{(x_0+x_2-x_1-x_3)(x_0-x_2)}{\overset{(x_3-x_1)-H}{\rightleftarrows}} \hbar R \;\right]$

When working locally, other boundary orientations are also possible. Thus, we also define

$D_2 =$ 

$\longmapsto \mathcal{M}(D_2) = \left[\; q\hbar R \underset{(x_0+x_2-x_1-x_3-H)(x_3-x_1)}{\overset{x_0-x_2}{\rightleftarrows}} R \;\right]$

$D_3 =$ 

$\longmapsto \mathcal{M}(D_3) = \left[\; q\hbar R \underset{(x_0+x_2-x_1-x_3-H)(x_2-x_0)}{\overset{x_1-x_3}{\rightleftarrows}} R \;\right]$

In all three types of thin edges, there is a *negative region* whose variable appears with a negative sign in the linear term of the associated matrix factorization. In the case of D_2 and D_3 there is a 180 degree rotational symmetry. To avoid ambiguity, we indicate the negative region by a little dash.

arrows somehow ruin the symmetry of the pics

The elementary matrix factorizations can be also written in terms of Koszul factorizations:

$$\begin{aligned}
\mathcal{M}(D_0) &= K_R\big(x_0 - x_2 \quad (x_0 + x_2 - x_1 - x_3)((x_3 - x_1) - H)\big), \\
\mathcal{M}(D_1) &= qK_R\big((x_0 + x_2 - x_1 - x_3)(x_0 - x_2) \quad (x_3 - x_1) - H)\big) \\
&= \hbar K_R\big(H - (x_3 - x_1) \quad (x_0 + x_2 - x_1 - x_3)(x_2 - x_0)\big).
\end{aligned}$$

Notice that R is the subring of the polynomial ring in all face variables generated by differences across edges. We refer to R as the *edge ring*, and we extend this nomenclature to more general webs in the expected way. The particular case $R = \mathbf{K}[x_0 - x_1, x_2 - x_1, x_3 - x_2]$ is refered to as the *standard edge ring*. We will often omit the specification of the edge ring when face variables are already indicated in the web diagram.

Now, given a general connected web Γ on the disk, in which we allow any of the local pictures D_i to appear, we can define the associated matrix factorization

$$\mathcal{M}(\Gamma) = \bigotimes_e \mathcal{M}(D_e)$$

over the corresponding edge ring R_Γ. The tensor product runs over all thin and thick edges of Γ and depends, in principle, on an ordering of these edges. However, the operation is symmetric monoidal, which gives canonical isomorphisms between different choices of thin/thick edge orderings.

3 Web simplifications

The next lemma tells us how to simplify certain Koszul matrix factorizations into special deformation retracts.

Lemma 3.1. *(Adapted from KR How general is R?) Let R be an integral domain and y a variable in quantum degree q^{-2}. Let $\overline{W} \in R$ and $f, g \in R[y]$ so that f has the form $f = uy^n + \tilde{f}$ wtih $u \in R^\times$ and $\deg_y(\tilde{f}) < n$. Let M be a matrix factorization over $R[y]$ with potential $W = \overline{W} - fg \in R[y]$. Let*

$$\begin{aligned} M' &= M/fM, \\ \overline{M} &= K_{R[y]}\big(f \quad g\big) \otimes_{R[y]} M, \end{aligned}$$

thought of as matrix factorizations over R with potential $\overline{W}$. Then there is a strong deformation retract

$$\begin{tikzcd} M' \arrow[r, shift left, "I"] & \overline{M} \arrow[l, shift left, "P"] \arrow[loop right, "K"] \end{tikzcd}$$

of the form

$$\begin{tikzcd} & M/fM \arrow[dl, bend right, "I_d"'] \arrow[d, shift right, "I_v"'] \\ q^{3-2n}\hbar M \arrow[r, bend left, "f"] & M, \arrow[u, shift right, "P"'] \arrow[l, "K"] \arrow[l, bend left, "g"] \end{tikzcd}$$

The arrows are as follows:

next in order: change the koszul grading here

- *P is the usual projection.*
- *$K = -\operatorname{Quo}_f$, the negative of the quotient of division by f.*
- *I consists of a vertical component $I_v = \operatorname{Res}_f$ given by the residue of division by f, and a diagonal component I_d given by the composite $-\operatorname{Quo}_f \circ d_M \circ I_v$.*

Proof. The identities $KI = 0$, $PK = 0$, $PI = 1$ and $K^2 = 0$ are straightforward. The identity

$$d_{\overline{M}}K + Kd_{\overline{M}} + 1_{\overline{M}} = IP$$

is verified as follows. Take $(x, y) \in q^{3-2n}\hbar M \oplus M = \overline{M}$. We compute directly, keeping in mind that $\operatorname{Res}_f(z) = z - f\operatorname{Quo}_f(z)$ and $\operatorname{Quo}_f(fz) = z$:

$$\begin{aligned} (d_{\overline{M}}K + Kd_{\overline{M}} + 1_{\overline{M}})(x, y) &= (-\operatorname{Quo}_f(fx) + x - \operatorname{Quo}_f(d_M(y)) - d_M(-\operatorname{Quo}_f(y)), -f\operatorname{Quo}_f(y) + y) \\ &= (-\operatorname{Quo}_f(d_M(y)) + d_M(\operatorname{Quo}_f(y)), \operatorname{Res}_f(y)), \end{aligned}$$

while

$$\begin{aligned} IP(x, y) &= (I_d([y]), I_v([y])) \\ &= (-\operatorname{Quo}_f(d_M(\operatorname{Res}_f(y))), \operatorname{Res}_f(y)) \\ &= (-\operatorname{Quo}_f(d_M(y - f\operatorname{Quo}_f(y))), \operatorname{Res}_f(y)) \\ &= (-\operatorname{Quo}_f(d_M(y)) + \operatorname{Quo}_f(fd_M(\operatorname{Quo}_f(y))), \operatorname{Res}_f(y)) \\ &= (-\operatorname{Quo}_f(d_M(y)) + d_M(\operatorname{Quo}_f(y)), \operatorname{Res}_f(y)). \end{aligned}$$

It is also straightforward to verify that P commutes with differentials. In the case of I, we have

$$\begin{aligned} I(d_{M'}([x])) &= (-\operatorname{Quo}_f(d_M^2(x)) + d_M(\operatorname{Quo}_f(d_M(x))), \operatorname{Res}_f(d_M(x))) \\ &= (d_M(\operatorname{Quo}_f(d_M(x))) - \operatorname{Quo}_f((\overline{W} - fg)x), \operatorname{Res}_f(d_M(x))), \end{aligned}$$

and

$$\begin{aligned}
d_{\overline{M}}(I([x])) &= d_{\overline{M}}(-\operatorname{Quo}_f(d_M(x)) + d_M(\operatorname{Quo}_f(x)), \operatorname{Res}_f(x)) \\
&= (-d_M(-\operatorname{Quo}_f(d_M(x))) - d_M^2(\operatorname{Quo}_f(x)) + g\operatorname{Res}_f(x), \\
&\qquad\quad d_M(\operatorname{Res}_f(x)) - f\operatorname{Quo}_f(d_M(x)) + f d_M(\operatorname{Quo}_f(x))) \\
&= (d_M(\operatorname{Quo}_f(d_M(x))) - (\overline{W} - fg)\operatorname{Quo}_f(x) + g(x - f\operatorname{Quo}_f(x)), \\
&\qquad\quad d_M(\operatorname{Res}_f(x) + f\operatorname{Quo}_f(x)) - f\operatorname{Quo}_f(d_M(x))) \\
&= (d_M(\operatorname{Quo}_f(d_M(x))) - \overline{W}\operatorname{Quo}_f(x) + gx, d_M(x) - f\operatorname{Quo}_f(d_M(x))) \\
&= (d_M(\operatorname{Quo}_f(d_M(x))) - \operatorname{Quo}_f((\overline{W} - fg)x), \operatorname{Res}_f(d_M(x))).
\end{aligned}$$

□

In many cases of interest, we additionally have a decomposition $M/fM = \bigoplus_{i=0}^{k} M_i$ as matrix factorizations over R. Such an identification, together with the preceding lemma, will often produce a special deformation retract that simplifies M into a sum of simpler matrix factorizations over a ring involving one less variable. Observe that there is always a decomposition of free graded R-modules $M/fM = \bigoplus_{i=0}^{\deg f - 1} y^i M/yM$, but it might not be a decomposition of matrix factorizations. The following lemmas describe specific situations where it is.

Lemma 3.2. (Simplification of Koszul factorizations by a linear term) *In the set up of Lemma 3.1, assume that $M = K_{R[y]}\begin{pmatrix}\overline{\mathbf{a}} & \overline{\mathbf{b}}\end{pmatrix}$ and that $f = y - f_0$, where $\deg_y(f_0) = 0$. Then, under the identification $M/yM \cong K_R\begin{pmatrix}\overline{\mathbf{a}}|_{y=f_0} & \overline{\mathbf{b}}|_{y=f_0}\end{pmatrix}$, we have a special deformation retract*

$$\begin{tikzcd}
 & & K_R\begin{pmatrix}\overline{\mathbf{a}}|_{y=f_0} & \overline{\mathbf{b}}|_{y=f_0}\end{pmatrix} \arrow[dll, bend right] \arrow[d, shift right, "1"'] \\
q\hbar K_{R[y]}\begin{pmatrix}\overline{\mathbf{a}} & \overline{\mathbf{b}}\end{pmatrix} \arrow[rr, bend left, "f"] & & K_{R[y]}\begin{pmatrix}\overline{\mathbf{a}} & \overline{\mathbf{b}}\end{pmatrix}, \arrow[ll, "-\operatorname{Quo}_f"'] \arrow[ll, bend left, "g"] \arrow[u, shift right, "y\mapsto f_0"']
\end{tikzcd}$$

The diagonal component of the inclusion is the composite

$$\begin{tikzcd}
K_R\begin{pmatrix}\overline{\mathbf{a}}|_{y=f_0} & \overline{\mathbf{b}}|_{y=f_0}\end{pmatrix} \arrow[r, "\sum_{j=1}^{k} 1\otimes\cdots\otimes\hbar^1(-\operatorname{Quo}_f\circ a_j, -\operatorname{Quo}_f\circ b_j)\otimes\cdots\otimes 1"] & qK_{R[y]}\begin{pmatrix}\overline{\mathbf{a}} & \overline{\mathbf{b}}\end{pmatrix} \arrow[d, "s_\hbar"] \\
 & q\hbar K_{R[y]}\begin{pmatrix}\overline{\mathbf{a}} & \overline{\mathbf{b}}\end{pmatrix}
\end{tikzcd}$$

Lemma 3.3. (Simplification of Koszul factorizations by a higher order polynomial) *In the set up of Lemma 3.1, assume that $M = K_{R[y]}\begin{pmatrix}\overline{\mathbf{a}} & \overline{\mathbf{b}}\end{pmatrix}$ and that $\deg_y(\operatorname{Res}_f(a_i)) = \deg_y(\operatorname{Res}_f(b_i)) = 0$ for all i. Then the map* that final degree switch in $\hbar$

$$\bigoplus_{i=0}^{n-1} q^{-2i} K_R\begin{pmatrix}\operatorname{Res}_f\overline{\mathbf{a}} & \operatorname{Res}_f\overline{\mathbf{b}}\end{pmatrix} \xrightarrow{(y^i)_i} M/fM$$

is an isomorphism of matrix factorizations over R and, combined with Lemma 3.1, gives a special deformation retract

$$\begin{tikzcd}
 & & \bigoplus_{i=0}^{n-1} q^{-2i} K_R\begin{pmatrix}\operatorname{Res}_f\overline{\mathbf{a}} & \operatorname{Res}_f\overline{\mathbf{b}}\end{pmatrix} \arrow[dll, bend right] \arrow[d, shift right, "(y^i)_i"'] \\
q^{3-2n}\hbar K_{R[y]}\begin{pmatrix}\overline{\mathbf{a}} & \overline{\mathbf{b}}\end{pmatrix} \arrow[rr, bend left, "f"] & & K_{R[y]}\begin{pmatrix}\overline{\mathbf{a}} & \overline{\mathbf{b}}\end{pmatrix}, \arrow[ll, "-\operatorname{Quo}_f"'] \arrow[ll, bend left, "g"] \arrow[u, shift right, "\left(\frac{1}{i!}\frac{\partial^i}{\partial y^i}\big|_{y=0}\circ\operatorname{Res}_f\right)_i"']
\end{tikzcd}$$

The diagonal component of the inclusion is the composite

$$\begin{tikzcd} \bigoplus_{i=0}^{n-1} q^{-2i}\hbar K_R\big(\mathrm{Res}_f\,\overline{\mathbf{a}} \qquad \mathrm{Res}_f\,\overline{\mathbf{b}}\big) \arrow[d, "s_\hbar"] & \\ \bigoplus_{i=0}^{n-1} q^{-2i} K_R\big(\mathrm{Res}_f\,\overline{\mathbf{a}} \qquad \mathrm{Res}_f\,\overline{\mathbf{b}}\big) \arrow[r, "\left(\sum_{j=1}^{k} 1\otimes\cdots\otimes\hbar^1(\mathrm{Quo}_f\,\circ y^i a_j,\mathrm{Quo}_f\,\circ y^i b_j)\otimes\cdots\otimes 1\right)_i"] & q^{3-2n}K_{R[y]}\left(\overline{\mathbf{a}},\overline{\mathbf{b}}\right) \end{tikzcd}$$

maybe I want postcomposition with $s_\hbar$

3.1 Thin edge removal

Simplification by linear polynomials 3.2 can be directly applied to remove thin edges that bound an interior region.

Proposition 3.4. *Let D be a four-ended web with a thin edge e such that its negative region is interior. Denote the corresponding edge ring by R_D, and the variables associated to the positive and negative region of e by x_+ and x_-, respectively. Let D' be the web obtained by removal of e, so that its edge ring $R_{D'}$ no longer involves x_-. Then there is a special deformation retract of matrix factorizations over $R_{D'}$*

$$\hbar M(D') \rightleftarrows M(D) \circlearrowleft$$

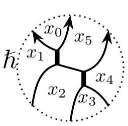
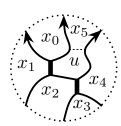
of the form

$$\begin{tikzcd} & \hbar M(D') \arrow[dl, bend right, "\left(\sum_j 1\otimes\cdots\otimes\hbar^1(\mathrm{Quo}_f\,\circ a_j,\mathrm{Quo}_f\,\circ b_j)\otimes\cdots\otimes 1\right)\circ s_\hbar"'] \arrow[d, shift right, "1"'] \\ q\widetilde{M}(D) \arrow[r, bend left, "f=x_+-x_-"] & \hbar\widetilde{M}(D), \arrow[l, "K"] \arrow[l, bend left, "g"] \arrow[u, shift right, "x_-\mapsto x_+"'] \end{tikzcd}$$

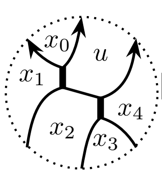
where $\widetilde{M}(D)$ is the Koszul factorization over R_D obtained from all edges in D except e. The only contributions to the sum come from edges with at least one end adjacent to the x_--region.

A similar statement is true if the positive region is interior and the variable x_+ is eliminated instead.

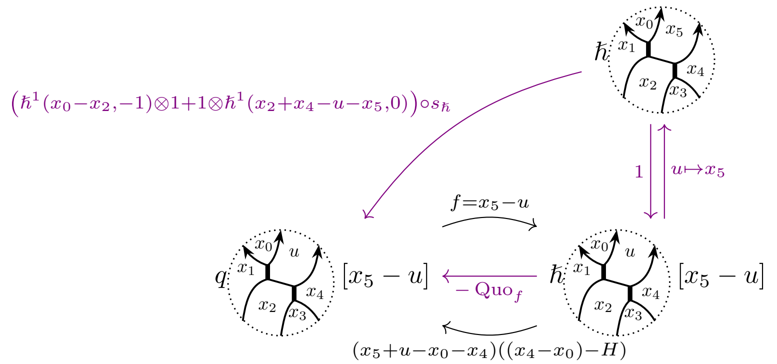
Example 3.5. In the proof of the Reidemeister 3 move we will need a special deformation retract

$\hbar$  $\rightleftarrows$  $\circlearrowleft$

removing the thin edge that separates $x_+ = x_5$ from $x_- = u$. According to the expression above, the retraction map will involve the calculation of Quo_{x_5-u} applied to the various components of

 $[x_5-u] = q^2 K_{R[x_5-u]}\begin{pmatrix} (x_0+x_2-x_1-u)(x_0-x_2) & (u-x_1)-H \\ (u+x_3-x_2-x_4)(u-x_3) & (x_4-x_2)-H \end{pmatrix}.$

The result is



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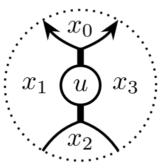
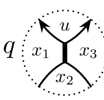
3.2 Circle, bigon and square removal

In the case of bigon removal, we have

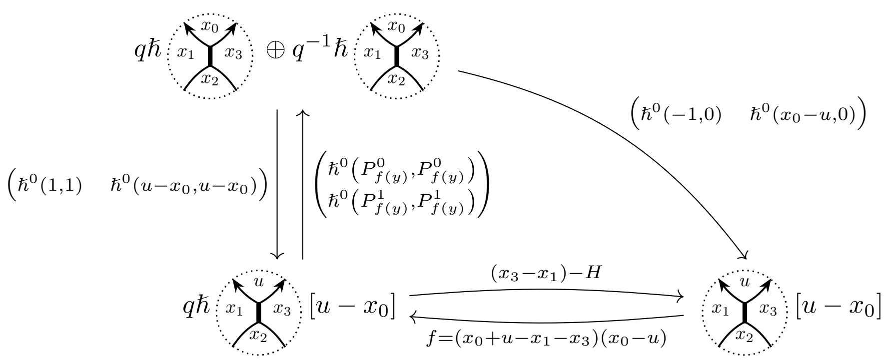


$$= q^2 K_{R[u-x_0]} \begin{pmatrix} (x_0+u-x_1-x_3)(x_0-u) & (x_3-x_1)-H \\ (u+x_2-x_1-x_3)(u-x_2) & (x_3-x_1)-H \end{pmatrix}$$

Letting $y = u - x_0$, $f = (x_0+u-x_1-x_3)(x_0-u)$ and

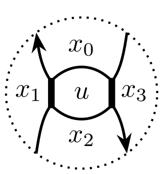
$$M = q^2 K_{R[u-x_0]}\big((u+x_2-x_1-x_3)(u-x_2) \quad (x_3-x_1)-H\big) = q$$  $[u-x_0]$,

we are under the hypotheses of the simplification lemma 3.3. In particular, we have a strong deformation retract



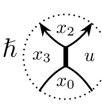
For square removal, we have

these maps still need some checking, esp. the diagonal

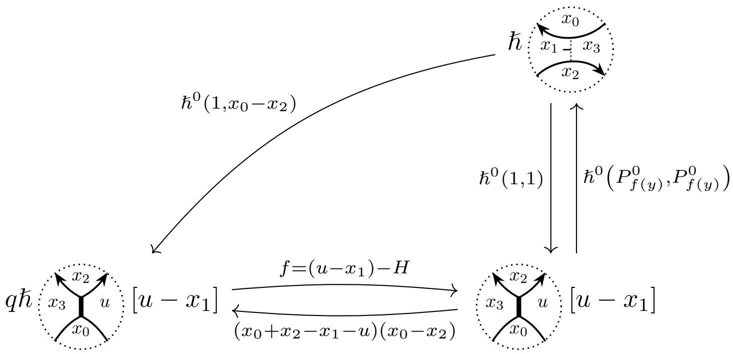


$$= K_{R[u-x_1]} \begin{pmatrix} (u-x_1)-H & (x_0+x_2-x_1-u)(x_0-x_2) \\ H-(u-x_3) & (x_2+x_0-x_3-u)(x_0-x_2) \end{pmatrix}$$

Letting $y = u - x_1$, $f = (u-x_1) - H$ and

$$M = K_{R[u-x_1]}\big(H-(u-x_3) \quad (x_2+x_0-x_3-u)(x_0-x_2)\big) = \hbar$$  $[u-x_1]$,

we are under the hypotheses of the linear simplification lemma 3.2. Thus, we have a strong deformation retract



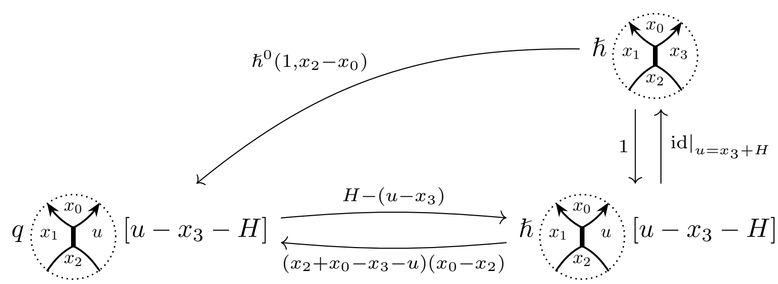
3.3 Triple thick edges

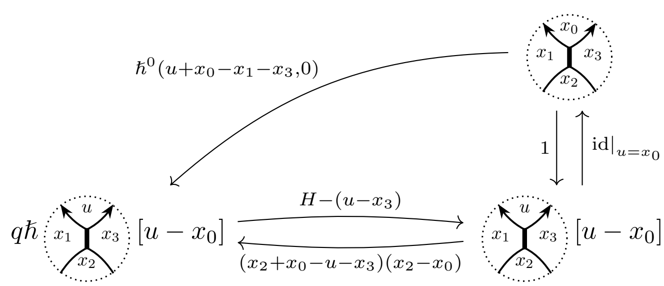
i accidentally ruined the horizontal arrows in the one below

These are the only moves that don't directly follow from the simplification lemmas for Koszul factorizations.

3.4 Other thin edge removals

not too interesting. they also need a double check after all the convention adjustments





4 Multifactorizations and special deformation retracts

define multifactorizations and n-stuff, including the meaning of filtered degree $\geq n$

Definition 4.1. Let (C,D) and (C',D') be multifactorizations. Then a special n-deformation retract from C to C' consists of 0-morphisms $P: C \to C'$ and $I: C' \to C$ such that $PI = 1$, together with a n-homotopy $H: C \to C$ between 1 and IP such that $HI = 0$, $PH = 0$ and $H^2 = 0$. We often represent this data by a triple (I,P,H) or, more suggestively, by a diagram

$$(C',D') \underset{P}{\overset{I}{\rightleftarrows}} (C,D) \circlearrowleft H\,.$$

Proposition 4.2. *Suppose*

$$(C',d') \underset{p}{\overset{i}{\rightleftarrows}} (C,d) \circlearrowleft h$$

is a special 1-deformation retract between matrix multifactorizations C' and C. Let D be another differential in C such that its vertical part coincides with d, i.e. $D_0 = d_0$. Then there exists a special 1-deformation retract

$$(C', D') \underset{P}{\overset{I}{\rightleftarrows}} (C, D) \circlearrowleft H .$$

such that $D'_0 = d'_0$. If h is also a 0-homotopy, so that the original data is a special 0-deformation retract, then the deformed data is also a special 0-deformation retract with the additional properties $P_0 = p_0$, $I_0 = i_0$ and $H_0 = h_0$.

Proof. (Essentially same as Ballinger) Let

$$A = (1 - (D-d)h)^{-1}(D-d) = \sum_{i=0}^{\infty}((D-d)h)^i(D-d).$$

Notice that $D - d$ has filtered degree ≥ 1, while h has filtered degree ≥ -1. In total, A has filtered degree ≥ 1. Take

$$\begin{aligned} D' &= d' + pAi \\ P &= p + pAh \\ I &= i + hAi \\ H &= h + hAh \end{aligned} \tag{1}$$

See (Crainic) for calculations. Notice that the term pAi, i.e. the deformation of the differential, has filtered degree ≥ 1, which guarantees $D'_0 = d'_0$. In the case in which h is a 1-homotopy then pAh and hAi preserve filtration, while hAh has filtered degree ≥ -1, which are necessary conditions for the deformed data (I, P, H) to give a special 1-deformation retract. If h is additionally a 0-homotopy, then all these terms have now filtered degree ≥ 1. In such case, the deformed data (I, P, H) gives a special 0-deformation retract of the desired form. ∎

The following lemma allows us to replace certain subfactorizations in a matrix factorization by their special retracts.

Lemma 4.3. *Let (C, D) be a matrix multifactorization of the form $C = A \oplus B$ and $D = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, so that*

- *(A, α) and (B, δ) are subfactorizations (i.e. $\beta\gamma = 0$ and $\gamma\beta = 0$),*
- *$\beta_0 = 0$ and*
- *$\gamma_0 = 0$.*

Suppose we have special 1-deformation retracts

$$(A', \epsilon) \underset{p_1}{\overset{i_1}{\rightleftarrows}} (A, \alpha) \circlearrowleft h_1 , \qquad (B', \eta) \underset{p_2}{\overset{i_2}{\rightleftarrows}} (B, \delta) \circlearrowleft h_2 .$$

Then there exists a special 1-deformation retract

$$(A' \oplus B', D') \underset{P}{\overset{I}{\rightleftarrows}} (C, D) \circlearrowleft H$$

with $D'_0 = \epsilon_0 \oplus \eta_0$.

Proof. Apply Proposition 4.2 to the special deformation retract

$$(A' \oplus B', \epsilon \oplus \eta) \underset{p_1 \oplus p_2}{\overset{i_1 \oplus i_2}{\rightleftarrows}} (A \oplus B, \alpha \oplus \delta) \circlearrowleft h_1 \oplus h_2 .$$

Here $d = \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}$ and $D = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. Since $\beta_0 = 0$ and $\gamma_0 = 0$, the hypothesis $D_0 = d_0$ is satisfied.

$$A = \sum_{i=0}^{\infty} \begin{pmatrix} \beta h_2 \gamma & \beta \\ \gamma & \gamma h_1 \beta \end{pmatrix} \begin{pmatrix} (h_1 \beta h_2 \gamma)^i & 0 \\ 0 & (h_2 \gamma h_1 \beta)^i \end{pmatrix} . \tag{2}$$

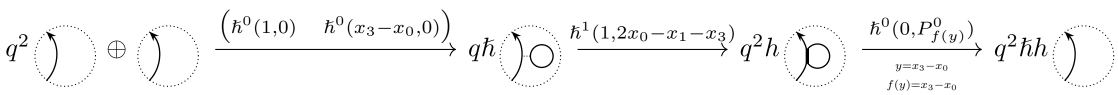
Since $\beta_0 = 0$ and $\gamma_0 = 0$ and h_1 and h_2 have filtered degree ≥ -1, it follows that $A_0 = 0$. This implies $D'_0 = \epsilon_0 \oplus \eta_0$. ∎

We will mostly encounter situations in which β, γ, h_1 or h_2, or short composites thereof, vanish. The expression (2) simplifies significantly in those cases, which in turn gives manageable formulas in (1).

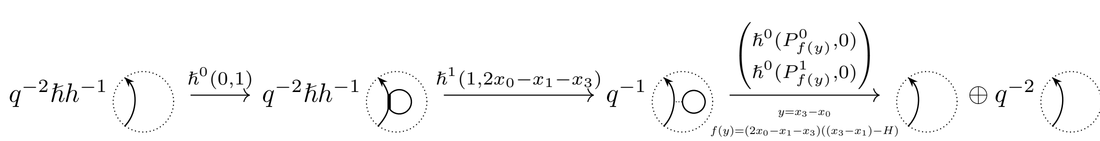
5 Invariance

The invariance of $M(D)$ for a link diagram D under each Reidemeister move will proceed by looking at the explicit matrix multifactorization associated to the relevant local piece of the diagram and simplifying it in two steps. In the first step, one simplifies each resolving web into smaller webs with no internal faces, and then extends into a special 0-deformation retract of the entire multifactorization. There is enough control of the deformation data to compute explicit differentials in the resulting multifactorization, and in particular to identify some identity components in d_1. The second step then consists of simplifying along such identities to obtain a special 1-deformation retract into the desired form.

Reidemeister I moves are illustrated in Figure 1. The purple arrows represent special deformation retracts of vertical matrix factorizations, where each component is obtained from Lemma 3.2 or Lemma 3.3. We then extend, by Lemma 4.2, to a special 0-deformation retract of multifactorizations. Notice that we indicate the explicit form of the d_1-component in each of the once-simplified multifactorizations. These are computed as the composites



in the case of the positive Reidemeister I move, and the composite



for the negative Reidemeister I move.

The summands

are easily verified to be 1-contractible. The corresponding special 1-deformation retractions cancelling these summands are represented by the green arrows.

The same recipe works for Reidemeister II moves. The special 0-deformation retracts represented by violet arrows still come from L

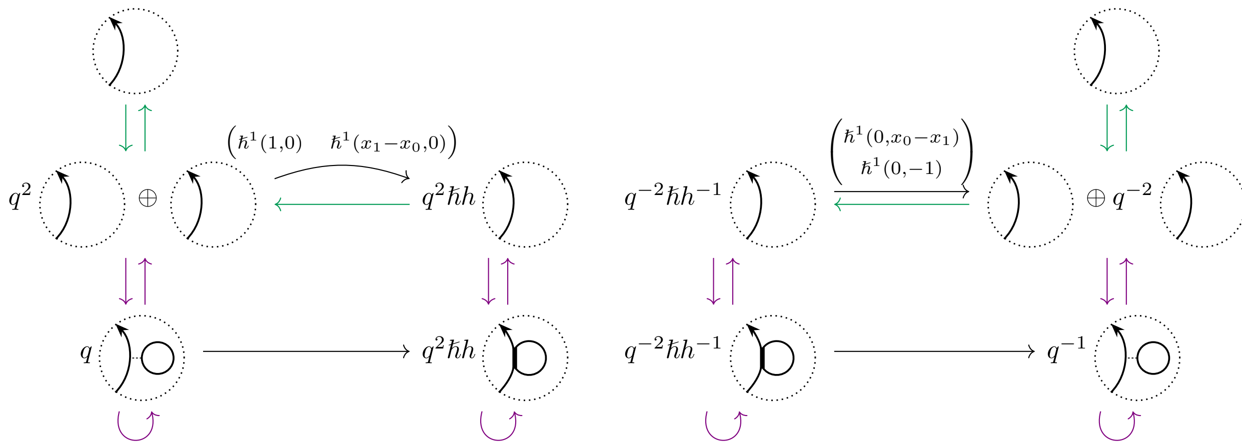
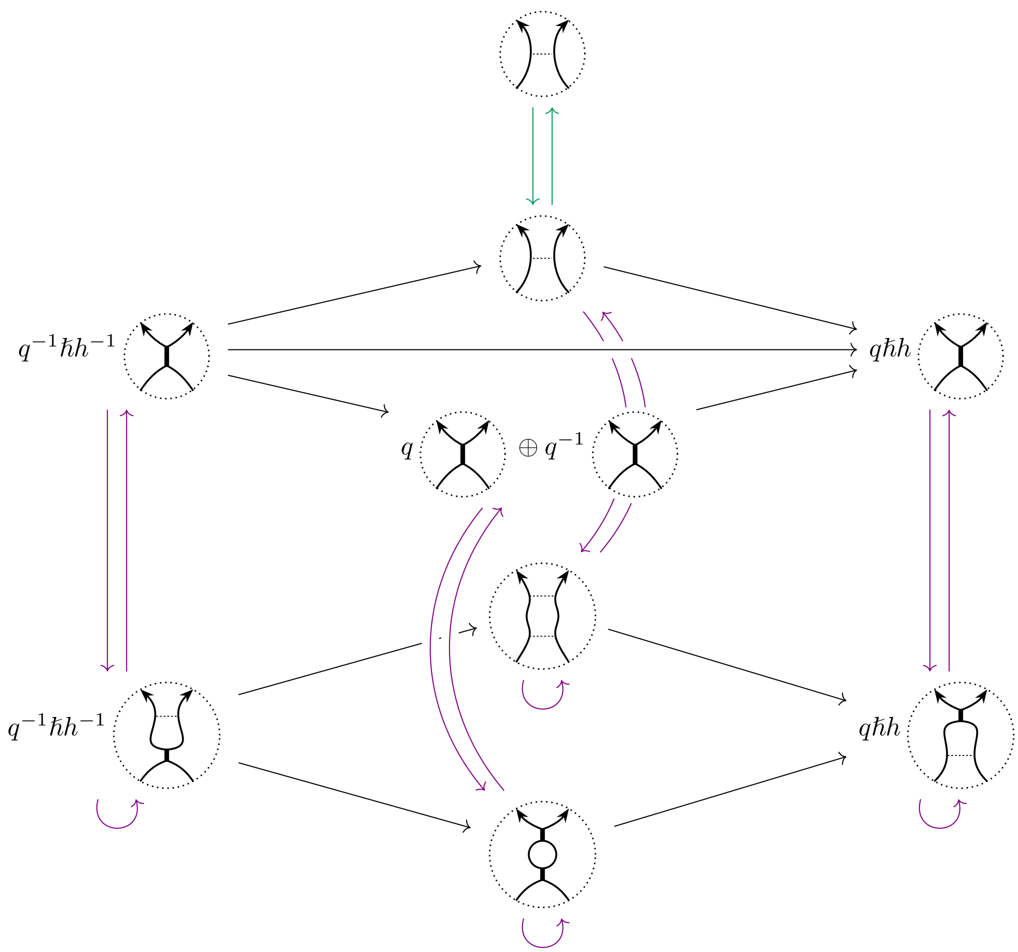
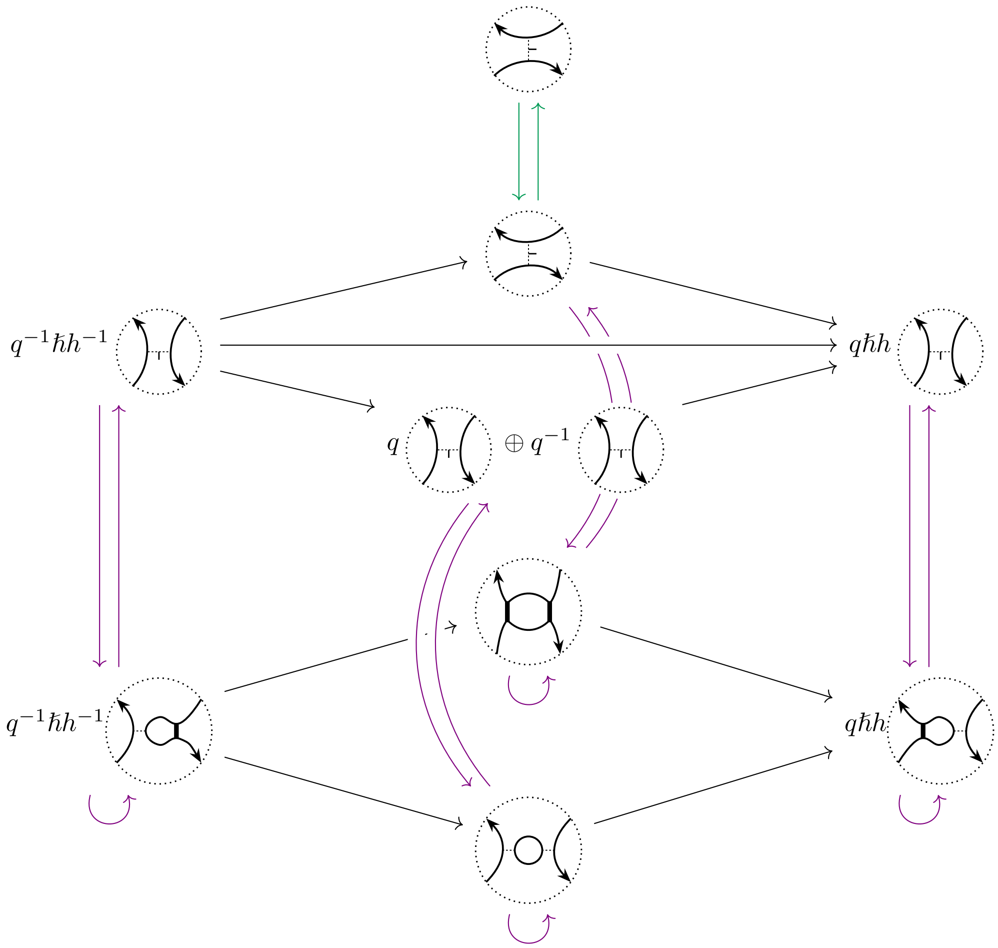


Figure 1: Invariance under positive and negative Reidemeister I moves





5.1 Web simplifications