Invariance of Bar-Natan matrix multifactorizations up to 1-homotopy equivalence

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Abstract

We verify link invariance of a certain construction using matrix factorizations over $\mathbb{Z}[H]$.

1 Matrix factorizations

Let $\mathcal{D}_0$ be a dotted edge and $\mathcal{D}_1$ be a thick edge.

Consider a base ring $\mathbf{K}$ (e.g. $\mathbf{K} = \mathbb{Z}[\mathsf{G}]$ with grading $\mathsf{gr}(\mathsf{G}) = i^{-2}h^0q^{-2}$). Take some potential $P(X) \in \mathbf{K}[X]$ (or even $\mathbf{K}[\![X]\!]$ if $\mathbf{K}$ is big enough).

We denote by x_0, x_1, x_2 and x_3 the variables of surrounding faces, starting above and going counterclockwise. In $\mathbf{K}[x_0 - x_1, x_2 - x_1, x_3 - x_2]$, it is always possible to factor

$$W = P(x_0 - x_1) + P(x_3 - x_0) - P(x_2 - x_1) - P(x_3 - x_2) = (x_0 - x_2)(x_0 + x_2 - x_1 - x_3)Z$$

for some $Z = Z(x_0 - x_1, x_2 - x_1, x_3 - x_2) \in \mathbf{K}[x_0 - x_1, x_2 - x_1, x_3 - x_2]$. We often drop the inputs of Z to lighten notation.

We assign Koszul matrix factorizations

$$\begin{aligned} M(\mathcal{D}_0) &= K(x_0 - x_2, \quad (x_0 + x_2 - x_1 - x_3)Z) \\ M(\mathcal{D}_1) &= K(Z, \quad (x_0 - x_2)(x_0 + x_2 - x_1 - x_3)). \end{aligned}$$

Given two matrix factorizations of the form

$$M = [\ A \rightleftarrows \hbar B\ ], \qquad M' = [\ A' \rightleftarrows \hbar B'\ ],$$

with A, A', B and B' concentrated in $\hbar$-degree 0, we often specify morphisms by spelling out their components in each $\hbar$-degree as follows:

$$\begin{tikzcd} M \arrow[d, "\hbar^0(\alpha,\beta)"'] \\ M' \end{tikzcd} = \begin{tikzcd} {[}\,A \arrow[r, shift left] \arrow[d, "\alpha"] & \hbar B\,{]} \arrow[l, shift left] \arrow[d, "\beta"] \\ {[}\,A' \arrow[r, shift left] & \hbar B'\,{]} \arrow[l, shift left] \end{tikzcd} \qquad \begin{tikzcd} M \arrow[d, "\hbar^1(\alpha,\beta)"'] \\ M' \end{tikzcd} = \begin{tikzcd} {[}\,A \arrow[r, shift left] \arrow[dr, "\alpha"] & \hbar B\,{]} \arrow[l, shift left] \arrow[dl, "\beta"'] \\ {[}\,A' \arrow[r, shift left] & \hbar B'\,{]} \arrow[l, shift left] \end{tikzcd}$$

Many useful maps can be easily expressed in this notation. For instance, the differential $d_M : M \to M$ itself takes the form

$$d_m = \hbar^1(d_0, d_1)$$

In the case $M' = \hbar M$ we use the special notation

$$s_\hbar = \hbar^1(1, 1) : M \to \hbar M.$$

For Koszul matrix factorizations, we will take the convention

$$K_R(a\ ;\ b) = \ q^{\deg_q(a)+3} R \underset{b}{\overset{a}{\rightleftarrows}} \hbar R\ .$$

Here, the grading $\hbar$ is a $\mathbb{Z}/2$ grading. A shift $\hbar M$ on a matrix factorization M has the additional effect of switching the sign of all differentials. This helps us keep track of Koszul sign rules for tensor products of matrix factorizations. Namely, if we tensor a Koszul factorization $K_R(a\ ;\ b)$ with some other matrix factorization M, we can write

$$K_R(a\ ;\ b) \otimes M = \left[\ q^{\deg_q(a)+3} M \underset{b}{\overset{a}{\rightleftarrows}} \hbar M\ \right],$$

which indicates that the tensor product splits, as free R-module, into two copies of M, with the first one shifted in q and $\hbar$ degrees. In terms of this decomposition, the total differential takes the form

$$d_{K_R(a\ ;\ b)\otimes M} = \begin{pmatrix} d_M & b \\ a & -d_M \end{pmatrix},$$

where the negative sign comes precisely from the $\hbar$-shift. Observe also that here a and b are understood to have $\hbar$-degree 1, and they could be more explicitly denoted by $\hbar^1(a,a)$ and $\hbar^1(b,b)$, respectively.

If we want to switch the sign of differentials in M without altering the $\hbar$ degree, we write $-M$. Using these notations, we have

not sure if I need this

$$\hbar(M \otimes N) = (\hbar M) \otimes N = (-M) \otimes (\hbar N)$$

The most general webs we will be dealing with can be obtained by taking an oriented planar arc diagram T where all inputs have exactly two consecutive incoming arcs and two consecutive outgoing arcs, and then plugging $\mathcal{D}_0$ or $\mathcal{D}_1$ in each input while respecting orientations:

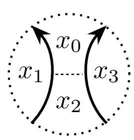
$$\mathcal{D} = T(\mathcal{D}_{i_1}, \dots, \mathcal{D}_{i_k}),$$

with $i_j = 0$ or 1.

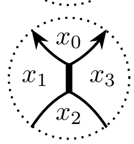
2 Matrix factorizations associated to webs and tangles

In the following definitions, the base ring is $\mathbf{K} = \mathbb{Z}[H]$, and the elementary matrix factorizations are over the ring $R = \mathbf{K}[x_0 - x_1, x_2 - x_1, x_3 - x_2]$.

it seems like the term $H-(x_3-x_1)$ in Caludius' paper needs a sign swap to get the desired potential. I'm swapping the sign

$D_0 =$ 

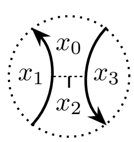
$$\longmapsto \mathcal{M}(D_0) = \left[\ qR \underset{(x_0+x_2-x_1-x_3)((x_3-x_1)-H)}{\overset{x_0-x_2}{\rightleftarrows}} \hbar R\ \right]$$

$D_1 =$ 

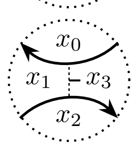
$$\longmapsto \mathcal{M}(D_1) = \left[\ q\hbar R \underset{(x_0+x_2-x_1-x_3)(x_0-x_2)}{\overset{(x_3-x_1)-H}{\rightleftarrows}} R\ \right]$$

When working locally, other boundary orientations are also possible. Thus, we also define

seems like we should swap all the $\hbar$ degrees here. maybe even in the definition of koszul

$D_2 =$ 

$$\longmapsto \mathcal{M}(D_2) = \left[\ qR \underset{(x_0+x_2-x_1-x_3-H)(x_3-x_1)}{\overset{x_0-x_2}{\rightleftarrows}} \hbar R\ \right]$$

$D_3 =$ 

$$\longmapsto \mathcal{M}(D_3) = \left[\ qR \underset{(x_0+x_2-x_1-x_3-H)(x_2-x_0)}{\overset{x_1-x_3}{\rightleftarrows}} \hbar R\ \right]$$

In all three types of thin edges, there is a *negative region* whose variable appears with a negative sign in the linear term of the associated matrix factorization. In the case of D_2 and D_3 there is a 180 degree rotational symmetry. To avoid ambiguity, we indicate the negative region by a little dash.

arrows somehow ruin the symmetry of the pics

The elementary matrix factorizations can be expressed as

$$\begin{aligned} \mathcal{M}(D_0) &= K_R(x_0 - x_2\ ;\ (x_0 + x_2 - x_1 - x_3)((x_3 - x_1) - H)) \\ \mathcal{M}(D_1) &= qK_R((x_0 + x_2 - x_1 - x_3)(x_0 - x_2)\ ;\ (x_3 - x_1) - H) \\ &= \hbar K_R(H - (x_3 - x_1)\ ;\ (x_0 + x_2 - x_1 - x_3)(x_2 - x_0)) \end{aligned}$$

Notice that R is the subring of the polynomial ring in all face variables generated by differences across edges. We refer to R as the *edge ring*, and we extend this nomenclature to more general webs in the expected way. The particular case $R = \mathbf{K}[x_0 - x_1, x_2 - x_1, x_3 - x_2]$ is refered to as the *standard edge ring*. We will often omit the specification of the edge ring when face variables are already indicated in the web diagram.

Now, given a general connected web Γ on the disk, in which we allow any of the local pictures D_i to appear, we can define the associated matrix factorization

$$\mathcal{M}(\Gamma) = \bigotimes_e \mathcal{M}(D_e)$$

over the corresponding edge ring R_Γ, where we choose variables for each region determined by Γ associated a matrix factorization. The tensor product runs over all thin and thick edges of Γ and depends, in principle, on an ordering of these edges. However, the operation is symmetric monoidal, which gives canonical isomorphisms between different choices of thin/thick edge orderings.

3 Hom spaces

4 Multifactorizations and special deformation retracts

define multifactorizatio and n-stuff, including the meaning of filtered degree $\geq n$

Definition 4.1. Let (C, D) and (C', D') be multifactorizations. Then a special n-deformation retract from C to C' consists of 0-morphisms $P : C \to C'$ and $I : C' \to C$ such that $PI = 1$, together with a n-homotopy $H : C \to C$ between 1 and IP such that $HI = 0$, $PH = 0$ and $H^2 = 0$. We often represent this data by a triple (I, P, H) or, more suggestively, by a diagram

$$(C', D') \underset{P}{\overset{I}{\rightleftarrows}} (C, D) \circlearrowleft H \, .$$

Proposition 4.2. *Suppose*

$$(C', d') \underset{p}{\overset{i}{\rightleftarrows}} (C, d) \circlearrowleft h$$

is a special 1-deformation retract between matrix multifactorizations C' and C. Let D be another differential in C such that its vertical part coincides with d, i.e. $D_0 = d_0$. Then there exists a special 1-deformation retract

$$(C', D') \underset{P}{\overset{I}{\rightleftarrows}} (C, D) \circlearrowleft H \, .$$

such that $D'_0 = d'_0$. If h is also a 0-homotopy, so that the original data is a special 0-deformation retract, then the deformed data is also a special 0-deformation retract with the additional properties $P_0 = p_0$, $I_0 = i_0$ and $H_0 = h_0$.

Proof. (Essentially same as Ballinger) Let

$$A = (1 - (D - d)h)^{-1}(D - d) = \sum_{i=0}^{\infty}((D - d)h)^i(D - d).$$

Notice that $D - d$ has filtered degree ≥ 1, while h has filtered degree ≥ -1. In total, A has filtered degree ≥ 1. Take

$$\begin{aligned} D' &= d' + pAi \\ P &= p + pAh \\ I &= i + hAi \\ H &= h + hAh \end{aligned} \tag{1}$$

See (Crainic) for calculations. Notice that the term pAi, i.e. the deformation of the differential, has filtered degree ≥ 1, which guarantees $D'_0 = d'_0$. In the case in which h is a 1-homotopy then pAh and hAi preserve filtration, while hAh has filtered degree ≥ -1, which are necessary conditions for the deformed data (I, P, H) to give a special 1-deformation retract. If h is additionally a 0-homotopy, then all these terms have now filtered degree ≥ 1. In such case, the deformed data (I, P, H) gives a special 0-deformation retract of the desired form. ∎

The following lemma allows us to replace certain subfactorizations in a matrix factorization by their special retracts.

Lemma 4.3. *Let (C, D) be a matrix multifactorization of the form $C = A \oplus B$ and $D = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, so that*

- *(A, α) and (B, δ) are subfactorizations (i.e. $\beta\gamma = 0$ and $\gamma\beta = 0$),*
- *$\beta_0 = 0$ and*
- *$\gamma_0 = 0$.*

Suppose we have special 1-deformation retracts

$$(A', \epsilon) \underset{p_1}{\overset{i_1}{\rightleftarrows}} (A, \alpha) \circlearrowleft h_1\,, \qquad (B', \eta) \underset{p_2}{\overset{i_2}{\rightleftarrows}} (B, \delta) \circlearrowleft h_2\,.$$

Then there exists a special 1-deformation retract

$$(A' \oplus B', D') \underset{P}{\overset{I}{\rightleftarrows}} (C, D) \circlearrowleft H$$

with $D'_0 = \epsilon_0 \oplus \eta_0$.

Proof. Apply Proposition 4.2 to the special deformation retract

$$(A' \oplus B', \epsilon \oplus \eta) \underset{p_1 \oplus p_2}{\overset{i_1 \oplus i_2}{\rightleftarrows}} (A \oplus B, \alpha \oplus \delta) \circlearrowleft h_1 \oplus h_2\,.$$

Here $d = \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}$ and $D = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. Since $\beta_0 = 0$ and $\gamma_0 = 0$, the hypothesis $D_0 = d_0$ is satisfied.

$$A = \sum_{i=0}^{\infty} \begin{pmatrix} \beta h_2 \gamma & \beta \\ \gamma & \gamma h_1 \beta \end{pmatrix} \begin{pmatrix} (h_1 \beta h_2 \gamma)^i & 0 \\ 0 & (h_2 \gamma h_1 \beta)^i \end{pmatrix}. \tag{2}$$

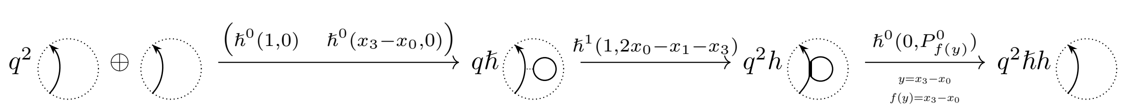
Since $\beta_0 = 0$ and $\gamma_0 = 0$ and h_1 and h_2 have filtered degree ≥ -1, it follows that $A_0 = 0$. This implies $D'_0 = \epsilon_0 \oplus \eta_0$. ∎

We will mostly encounter situations in which β, γ, h_1 or h_2, or short composites there of, vanish. The expression (2) simplifies significantly in those cases, which in turn gives manageable formulas in (1).

5 Invariance

The invariance of $M(D)$ for a link diagram D under each Reidemeister move will proceed by looking at the explicit matrix multifactorization associated to the relevant local piece of the diagram and simplifying it in two steps. In the first step, one simplifies each resolving web into smaller webs with no internal faces, and then extends into a special 0-deformation retract of the entire multifactorization. There is enough control of the deformation data to compute explicit differentials in the resulting multifactorization, and in particular to identify some identity components in d_1. The second step then consists of simplifying along such identities to obtain a special 1-deformation retract into the desired form.

Reidemeister I moves are illustrated in Figure 1. The purple arrows represent special deformation retracts of vertical matrix factorizations, where each component is obtained from Lemma 5.2 or Lemma 5.3. We then extend, by Lemma 4.2, to a special 0-deformation retract of multifactorizations. Notice that we indicate the explicit form of the d_1-component in each of the once-simplified multifactorizations. These are computed as composites:



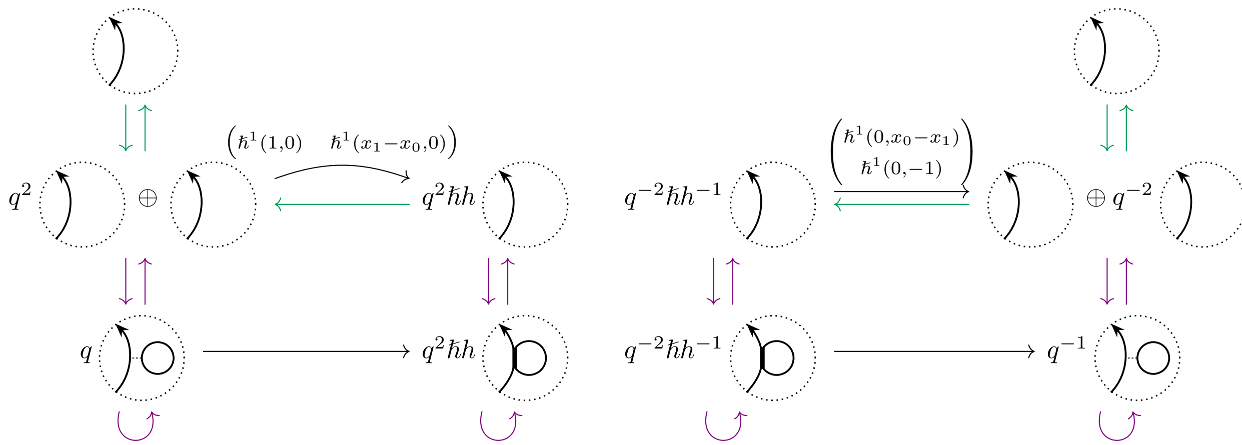
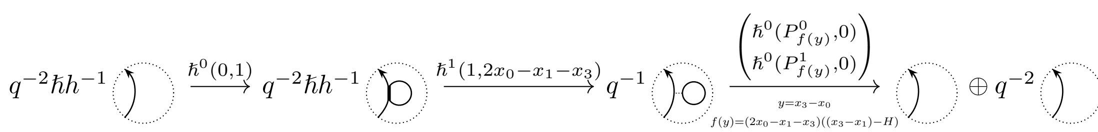
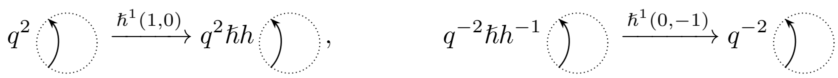


Figure 1: Invariance under positive and negative Reidemeister I moves

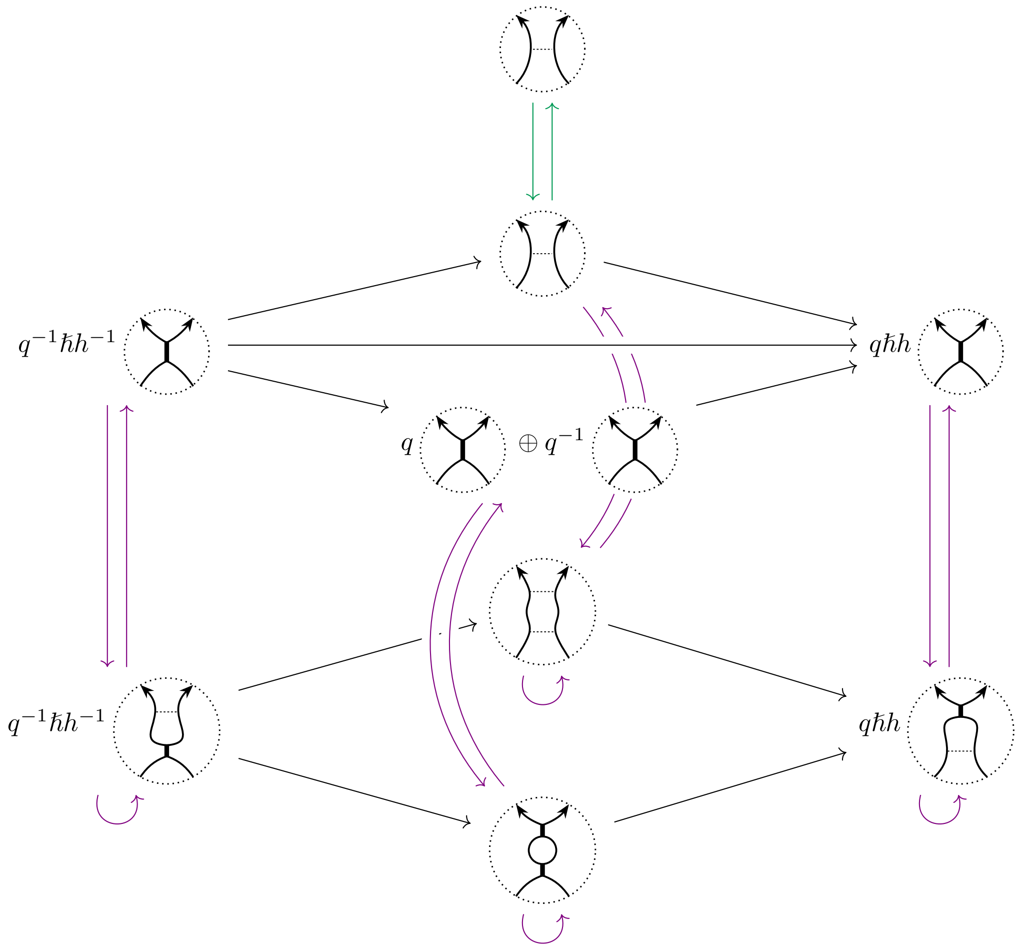


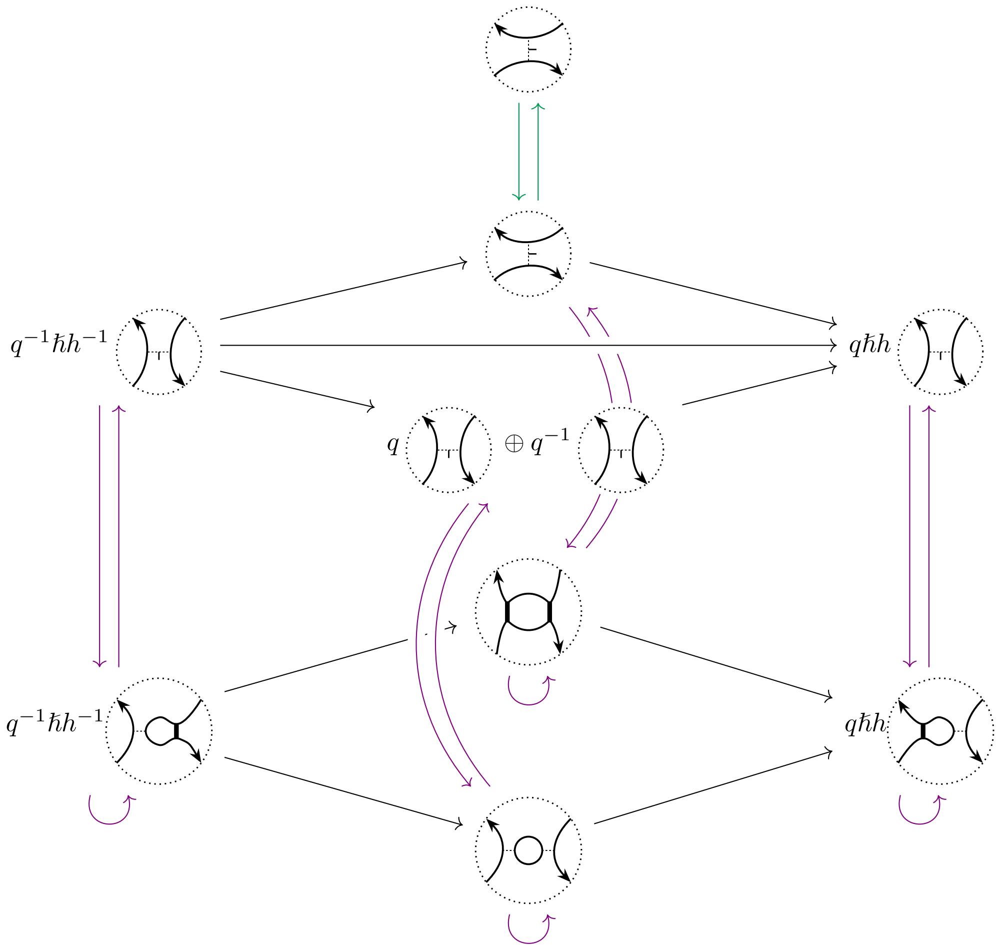
The summands



are easily verified to be 1-contractible. The corresponding special 1-deformation retractions cancelling these summands are represented by the green arrows.

The same recipe works for Reidemeister II moves. The special 0-deformation retracts represented by violet arrows still come from L


$q^{-1}\hbar h^{-1}$
$q\hbar h$
q
$\oplus\, q^{-1}$
$q^{-1}\hbar h^{-1}$
$q\hbar h$



The next lemma tells us how to simplify certain Koszul matrix factorizations into special deformation retracts.

Lemma 5.1. *(Adapted from KR How general is R?) Let R be an integral domain. Let $\overline{W} \in R$ and $f, g \in R[y]$ so that f has the form $f = uy^n + \tilde{f}$ wtih $u \in R[y]^\times$ and $\deg_y(\tilde{f}) < n$. Let M be a matrix factorization over $R[y]$ with potential $W = \overline{W} - fg \in R[y]$. Let*

$$\begin{aligned} M' &= M/fM, \\ \overline{M} &= K_{R[y]}(f; g) \otimes_{R[y]} M, \end{aligned}$$

thought of as matrix factorizations over R with potential $\overline{W}$. Then there is a strong deformation retract

$$M' \underset{P}{\overset{I}{\rightleftarrows}} \overline{M} \circlearrowleft H$$

of the form

$$\begin{tikzcd} & M/fM \arrow[dl, bend right, "I_d"'] \arrow[d, shift right, "I_v"'] \\ q^{3-2n}\hbar M \arrow[r, bend left, "f"] & M, \arrow[u, shift right, "P"'] \arrow[l, "H"] \arrow[l, bend left, "g"] \end{tikzcd}$$

The arrows are as follows:

- *P is the usual projection.*

in order to keep this result clean, it is better to keep

- $H = -\operatorname{Quo}_f$, *the negative of the quotient of division by* f.

carry the modification of the sign of H

- I *consists of a vertical component* $I_v = \operatorname{Res}_f$ *given by the residue of division by* f, *and a diagonal component* I_d *given by the composite* $H \circ d_M \circ I_v$.

Proof. The identities $HI = 0$, $PH = 0$, $PI = 1$ and $H^2 = 0$ are straightforward. The identity $d_{\overline{M}}H + Hd_{\overline{M}} + 1_{\overline{M}} = IP$ is verified as follows. Take $(x, y) \in q^{3-2n}M\hbar \oplus M = \overline{M}$. We compute directly, keeping in mind that $\operatorname{Res}_f(z) = z - f\operatorname{Quo}_f(z)$ and $\operatorname{Quo}_f(fz) = z$:

$$\begin{aligned}(d_{\overline{M}}H + Hd_{\overline{M}} + 1_{\overline{M}})(x, y) &= (-\operatorname{Quo}_f(fx) + x - \operatorname{Quo}_f(d_M(y)) - d_M(-\operatorname{Quo}_f(y)), -f\operatorname{Quo}_f(y) + y)\\ &= (-\operatorname{Quo}_f(d_M(y)) + d_M(\operatorname{Quo}_f(y)), \operatorname{Res}_f(y)),\end{aligned}$$

while

$$\begin{aligned}IP(x, y) &= (I_d([y]), I_v([y]))\\ &= (-\operatorname{Quo}_f(d_M(\operatorname{Res}_f(y))), \operatorname{Res}_f(y))\\ &= (-\operatorname{Quo}_f(d_M(y - f\operatorname{Quo}_f(y))), \operatorname{Res}_f(y))\\ &= (-\operatorname{Quo}_f(d_M(y)) + \operatorname{Quo}_f(fd_M(\operatorname{Quo}_f(y))), \operatorname{Res}_f(y))\\ &= (-\operatorname{Quo}_f(d_M(y)) + d_M(\operatorname{Quo}_f(y)), \operatorname{Res}_f(y)).\end{aligned}$$

It is also straightforward to verify that P commutes with differentials. In the case of I, we have

$$\begin{aligned}I(d_{M'}([x])) &= (-\operatorname{Quo}_f(d_M^2(x)) + d_M(\operatorname{Quo}_f(d_M(x))), \operatorname{Res}_f(d_M(x)))\\ &= (d_M(\operatorname{Quo}_f(d_M(x))) - \operatorname{Quo}_f((\overline{W} - fg)x), \operatorname{Res}_f(d_M(x))),\end{aligned}$$

and

$$\begin{aligned}d_{\overline{M}}(I([x])) &= d_{\overline{M}}(-\operatorname{Quo}_f(d_M(x)) + d_M(\operatorname{Quo}_f(x)), \operatorname{Res}_f(x))\\ &= (d_M(\operatorname{Quo}_f(d_M(x))) - d_M^2(\operatorname{Quo}_f(x)) + g\operatorname{Res}_f(x),\\ &\qquad d_M(\operatorname{Res}_f(x)) - f\operatorname{Quo}_f(d_M(x)) + fd_M(\operatorname{Quo}_f(x)))\\ &= (d_M(\operatorname{Quo}_f(d_M(x))) - (\overline{W} - fg)\operatorname{Quo}_f(x) + g(x - f\operatorname{Quo}_f(x)),\\ &\qquad d_M(\operatorname{Res}_f(x) + f\operatorname{Quo}_f(x)) - f\operatorname{Quo}_f(d_M(x)))\\ &= (d_M(\operatorname{Quo}_f(d_M(x))) - \overline{W}\operatorname{Quo}_f(x) + gx, d_M(x) - f\operatorname{Quo}_f(d_M(x)))\\ &= (d_M(\operatorname{Quo}_f(d_M(x))) - \operatorname{Quo}_f((\overline{W} - fg)x), \operatorname{Res}_f(d_M(x))).\end{aligned}$$

□

In many cases of interest, we additionally have a decomposition $M/fM = \bigoplus_{i=0}^k M_i$ as matrix factorizations over R. Such an identification, together with the preceding lemma, will often produce a special deformation retract that simplifies M into a sum of simpler matrix factorizations over a ring involving one less variable.

The following is a straightforward corollary.

Lemma 5.2. *In the set up of Lemma 5.1, assume that* $M = K_{R[y]}(\overline{\mathbf{a}}, \overline{\mathbf{b}})$ *and that* $f = y - f_0$, *where* $\deg_y(f_0) = 0$. *Then, under the identification* $M/yM \cong K_R\left(\overline{\mathbf{a}}\big|_{y=f_0}, \overline{\mathbf{b}}\big|_{y=f_0}\right)$, *we have a special deformation retract*

$$\begin{tikzcd}
 & K_R\left(\overline{\mathbf{a}}\big|_{y=f_0}, \overline{\mathbf{b}}\big|_{y=f_0}\right) \arrow[dl, bend right] \arrow[d, shift right, "1"'] \\
q\hbar K_{R[y]}(\overline{\mathbf{a}}, \overline{\mathbf{b}}) \arrow[r, bend left, "f"] & K_{R[y]}(\overline{\mathbf{a}}, \overline{\mathbf{b}}), \arrow[l, "-\operatorname{Quo}_f"] \arrow[l, bend left, "g"] \arrow[u, shift right, "y\mapsto f_0"']
\end{tikzcd}$$

The diagonal component of the inclusion is the composite

$$K_R\left(\overline{\mathbf{a}}\big|_{y=f_0}, \overline{\mathbf{b}}\big|_{y=f_0}\right) \xrightarrow{\sum_{j=1}^k 1\otimes\cdots\otimes\hbar^1(-\operatorname{Quo}_f\circ a_j, -\operatorname{Quo}_f\circ b_j)\otimes\cdots\otimes 1} qK_{R[y]}\left(\overline{\mathbf{a}}, \overline{\mathbf{b}}\right) \xrightarrow{s_\hbar} q\hbar K_{R[y]}\left(\overline{\mathbf{a}}, \overline{\mathbf{b}}\right)$$

This can be generalized

Lemma 5.3. *In the set up of Lemma 5.1, assume that $M = K_{R[y]}(\overline{\mathbf{a}}, \overline{\mathbf{b}})$ and that $\deg_y(\mathrm{Res}_f(a_i)) = \deg_y(\mathrm{Res}_f(b_i)) = 0$ for all i. Then the map*

that final degree switch in $\hbar$

$$\bigoplus_{i=0}^{n-1} q^{-2i} K_R\left(\mathrm{Res}_f\,\overline{\mathbf{a}}, \mathrm{Res}_f\,\overline{\mathbf{b}}\right) \xrightarrow{(y^i)_i} M/fM$$

is an isomorphism of matrix factorizations over R and, combined with Lemma 5.1, gives a special deformation retract

$$\begin{tikzcd}
& & \bigoplus_{i=0}^{n-1} q^{-2i} K_R\left(\mathrm{Res}_f\,\overline{\mathbf{a}}, \mathrm{Res}_f\,\overline{\mathbf{b}}\right) \arrow[dll, bend right] \arrow[d, shift right, "(y^i)_i"'] \\
q^{3-2n}\hbar K_{R[y]}(\overline{\mathbf{a}}, \overline{\mathbf{b}}) \arrow[rr, bend left, "f"] & & K_{R[y]}(\overline{\mathbf{a}}, \overline{\mathbf{b}}), \arrow[ll, "\mathrm{Quo}_f"] \arrow[ll, bend left, "g"] \arrow[u, shift right, "\left(\frac{1}{i!}\frac{\partial^i}{\partial y^i}\big|_{y=0}\circ \mathrm{Res}_f\right)_i"']
\end{tikzcd}$$

The diagonal component of the inclusion is the composite

$$\begin{tikzcd}
\bigoplus_{i=0}^{n-1} q^{-2i} K_R\left(\mathrm{Res}_f\,\overline{\mathbf{a}}, \mathrm{Res}_f\,\overline{\mathbf{b}}\right) \arrow[r, "\left(\sum_{j=1}^{k} 1\otimes\cdots\otimes\hbar^1(\mathrm{Quo}_f\circ y^i a_j, \mathrm{Quo}_f\circ y^i b_j)\otimes\cdots\otimes 1\right)_i"] & q^{3-2n} K_{R[y]}\left(\overline{\mathbf{a}}, \overline{\mathbf{b}}\right) \arrow[d, "s_\hbar"] \\
& q^{3-2n}\hbar K_{R[y]}\left(\overline{\mathbf{a}}, \overline{\mathbf{b}}\right)
\end{tikzcd}$$

5.1 MOY moves in $U(1)$-equivariant $_2$

Let us write some explicit special deformation retracts that can be used for computations and proof of invariance.

5.2 Thin edge removal

Lemma 5.2 can be applied to remove dotted edges that bound an interior region.

Proposition 5.4. *Let D be a four-ended web with a thin edge e such that its negative region is interior. Denote the corresponding edge ring by R_D, and the variables associated to the positive and negative region of e by x_+ and x_-, respectively. Let D' be the web obtained by removal of e, so that its edge ring $R_{D'}$ no longer involves x_-. Then there is a special deformation retract of matrix factorizations over $R_{D'}$*

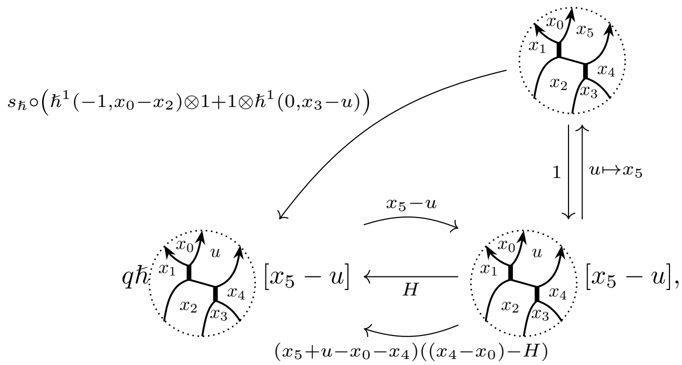
$$M(D') \rightleftarrows M(D) \circlearrowleft$$

of the form

$$\begin{tikzcd}
& M(D') \arrow[dl, bend right, "s_\hbar\circ\sum_j 1\otimes\cdots\otimes\hbar^1(\mathrm{Quo}_f\circ a_j, \mathrm{Quo}_f\circ b_j)\otimes\cdots\otimes 1"'] \arrow[d, shift right, "1"'] \\
q\hbar\widetilde{M}(D) \arrow[r, bend left, "f=x_+-x_-"] & \widetilde{M}(D), \arrow[l, "H"] \arrow[l, bend left, "g"] \arrow[u, shift right, "x_-\mapsto x_+"']
\end{tikzcd}$$

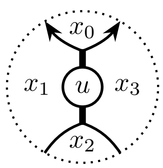
where $\widetilde{M}(D)$ is the Koszul factorization over R_D obtained from all edges in D except e. The only contributions to the sum come from edges with at least one end adjacent to the x_--region.

A similar statement is true if the positive region is interior and the variable x_+ is eliminated instead. Here is an example, which will be of use later.



5.2.1 Bigon removal

In the case of bigon removal, we have

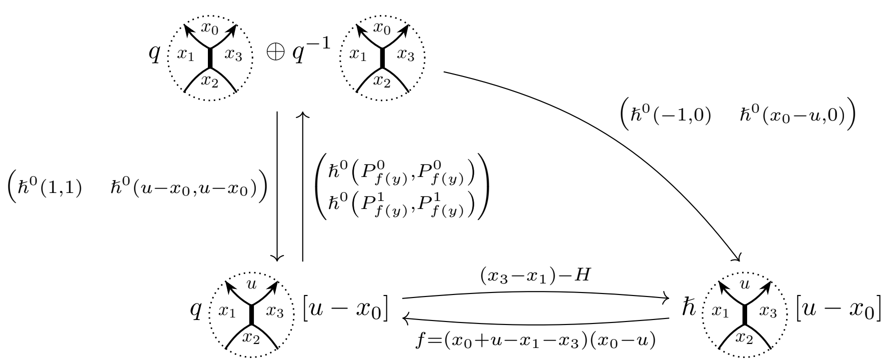


$$= q^2 K_{R[u-x_0]} \begin{pmatrix} (x_0+u-x_1-x_3)(x_0-u) & (x_3-x_1)-H \\ (u+x_2-x_1-x_3)(u-x_2) & (x_3-x_1)-H \end{pmatrix}$$

Letting $y = u - x_0$, $f = (u+x_2-x_1-x_3)(u-x_2)$ and

$$M = K_{R[u-x_0]} \begin{pmatrix} (u+x_2-x_1-x_3)(u-x_2) & (x_3-x_1)-H \end{pmatrix},$$

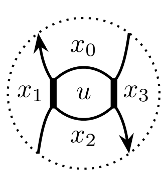
we are under the hypotheses of Lemma 5.3. In particular, we have a strong deformation retract



these maps still need some checking, esp. the diagonal

5.2.2 Thick edge removal

For square removal, we have

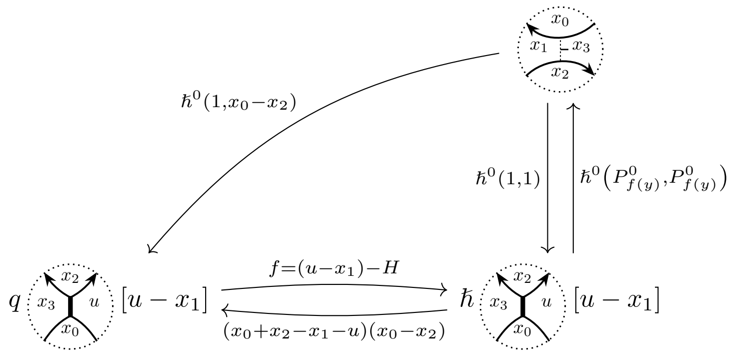


$$= K_{R[u-x_0]} \begin{pmatrix} (u-x_1)-H & (x_0+x_2-x_1-u)(x_0-x_2) \\ H-(u-x_3) & (x_2+x_0-x_3-u)(x_0-x_2) \end{pmatrix}$$

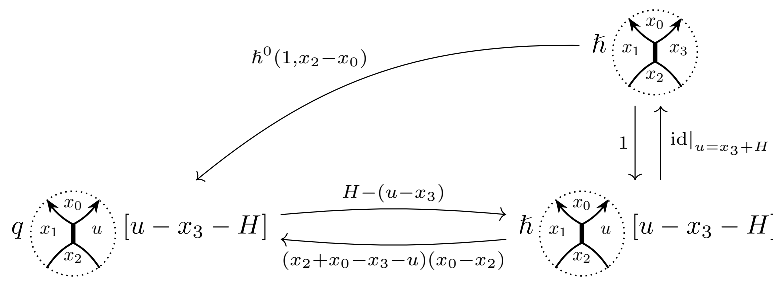
Letting $y = u - x_1$, $f = (u-x_1) - H$ and

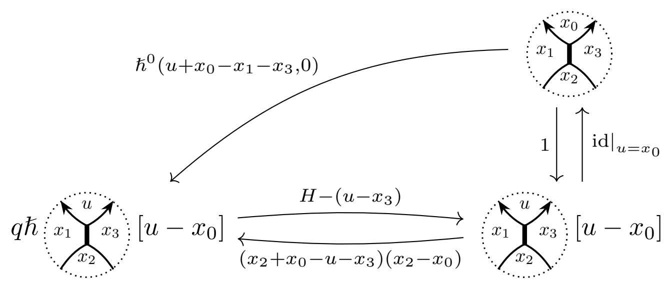
$$M = K_{R[u-x_3]} \begin{pmatrix} H-(u-x_3) & (x_2+x_0-x_3-u)(x_0-x_2) \end{pmatrix},$$

we are under the hypotheses of Lemma 5.2. Thus, we have a strong deformation retract



i accidentally ruined the horizontal arrows in the one below





6 Delooping of four-ended webs