Invariance of Bar-Natan matrix multifactorizations up to 1-homotopy equivalence

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Abstract

We verify link invariance of a certain construction using matrix factorizations over $\mathbb{Z}[H]$.

1 Matrix factorization conventions

Given two matrix factorizations of the form

$$M = [\ \hbar A \rightleftarrows B\ ], \qquad M' = [\ \hbar A' \rightleftarrows B'\ ],$$

with A, A', B and B' concentrated in $\hbar$-degree 0, we often specify morphisms by spelling out their components in each $\hbar$-degree as follows:

$$
\begin{array}{ccc}
M & & [\ \hbar A \rightleftarrows B\ ] \\
\hbar^0(\alpha,\beta)\downarrow & = & \downarrow\alpha \qquad \downarrow\beta \\
M' & & [\ \hbar A' \rightleftarrows B'\ ]
\end{array}
\qquad
\begin{array}{ccc}
M & & [\ \hbar A \rightleftarrows B\ ] \\
\hbar^1(\alpha,\beta)\downarrow & = & \beta \diagdown \quad \diagdown \alpha \\
M' & & [\ \hbar A' \rightleftarrows B'\ ]
\end{array}
$$

Many useful maps can be easily expressed in this notation. In the case $M' = \hbar M$, we will often use the map

$$s_\hbar = \hbar^1(1,1) : M \to \hbar M.$$

For Koszul matrix factorizations, we will take the convention

$$K_R\big(a \quad b\big) = \left[\ q^{\deg_q(a)+3}\hbar R \xrightleftharpoons[b]{a} R\ \right],$$

so that the differential can be written as

$$d_{K_R(a\ b)} = \hbar^1(a,b)$$

More generally, given sequences $\overline{\mathbf{a}} = (a_1,\ldots,a_r)$ and $\overline{\mathbf{b}} = (b_1,\ldots,b_r)$, we have an associated Koszul factorization

$$K_R\big(\overline{\mathbf{a}} \quad \overline{\mathbf{b}}\big) = \bigotimes_i K_R\big(a_i \quad b_i\big)$$

with differential

$$d_{K_R(\overline{\mathbf{a}}\ \overline{\mathbf{b}})} = \sum_{i=1}^{r} \hbar_i^1(a_i,b_i),$$

where $\hbar_i^1(a,b) = 1 \otimes \cdots \otimes \hbar^1(a,b) \otimes \cdots \otimes 1$ is only non-identity in the i-th factor.

A shift $\hbar M$ on a matrix factorization M has the additional effect of switching the sign of all differentials. This helps us keep track of Koszul sign rules for tensor products of matrix factorizations. Namely, if we tensor a Koszul factorization $K_R\big(a \quad b\big)$ with some other matrix factorization M, we can write

$$K_R\big(a \quad b\big) \otimes_R M = \left[\ q^{\deg_q(a)+3}\hbar M \xrightleftharpoons[b]{a} M\ \right].$$

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This indicates that the tensor product splits as a free R-module into two copies of M, the first one with shifted q and $\hbar$ degrees. In terms of this decomposition, the total differential takes the form

$$d_{K_R(a\ b)\otimes M} = \begin{pmatrix} -d_M & b \\ a & d_M \end{pmatrix},$$

where the negative sign comes precisely from the $\hbar$-shift. Here are some useful isomorphisms

$$K_R\begin{pmatrix} \overline{\mathbf{a}} & \overline{\mathbf{b}} \end{pmatrix} \xrightarrow{\ \mathrm{id}+\lambda\hbar_i^1(1,0)\hbar_j^1(1,0)\ } K_R\begin{pmatrix} \overline{\mathbf{a}} + \lambda b_i\overline{\mathbf{e}}_j - \lambda b_j\overline{\mathbf{e}}_i & \overline{\mathbf{b}} \end{pmatrix}$$

$$K_R\begin{pmatrix} \overline{\mathbf{a}} & \overline{\mathbf{b}} \end{pmatrix} \xrightarrow{\ \mathrm{id}+\lambda\hbar_i^1(0,1)\hbar_j^1(1,0)\ } K_R\begin{pmatrix} \overline{\mathbf{a}} - \lambda a_i\overline{\mathbf{e}}_j & \overline{\mathbf{b}} + \lambda b_j\overline{\mathbf{e}}_i \end{pmatrix}$$

If we want to switch the sign of differentials in M without altering the $\hbar$ degree, we write $-M$. Using these notations, we have

not sure if I need this

$$\hbar(M \otimes N) = (\hbar M) \otimes N = (-M) \otimes (\hbar N)$$

When $\deg_q(a) + \deg_q(b) = -6$, as is usually the case in $_2$-homology, we have

$$K_R\begin{pmatrix} a & b \end{pmatrix} = q^{\deg_q(a)+3}\hbar K_R\begin{pmatrix} -b & -a \end{pmatrix}.$$

2 Matrix factorizations associated to webs

In the following definitions, the base ring is $\mathbf{K} = \mathbb{Z}[H]$, and the elementary matrix factorizations are over the ring $R = \mathbf{K}[x_0 - x_1, x_2 - x_1, x_3 - x_2]$. The variables x_i and H live in degree q^{-2}.

$$D_0 = \longmapsto \mathcal{M}(D_0) = \left[\ q\hbar R \underset{(x_0+x_2-x_1-x_3)((x_3-x_1)-H)}{\overset{x_0-x_2}{\rightleftarrows}} R\ \right]$$

$$D_1 = \longmapsto \mathcal{M}(D_1) = \left[\ qR \underset{(x_0+x_2-x_1-x_3)(x_0-x_2)}{\overset{(x_3-x_1)-H}{\rightleftarrows}} \hbar R\ \right]$$

When working locally, other boundary orientations are also possible. Thus, we also define

$$D_2 = \longmapsto \mathcal{M}(D_2) = \left[\ q\hbar R \underset{(x_0+x_2-x_1-x_3-H)(x_3-x_1)}{\overset{x_0-x_2}{\rightleftarrows}} R\ \right]$$

$$D_3 = \longmapsto \mathcal{M}(D_3) = \left[\ q\hbar R \underset{(x_0+x_2-x_1-x_3-H)(x_2-x_0)}{\overset{x_1-x_3}{\rightleftarrows}} R\ \right]$$

In all three types of thin edges, there is a *negative region* whose variable appears with a negative sign in the linear term of the associated matrix factorization. In the case of D_2 and D_3 there is a 180 degree rotational symmetry. To avoid ambiguity, we indicate the negative region by a little dash.

arrows somehow ruin the symmetry of the pics

The elementary matrix factorizations can be also written in terms of Koszul factorizations:

$$\mathcal{M}(D_0) = K_R\begin{pmatrix} x_0 - x_2 & (x_0 + x_2 - x_1 - x_3)((x_3 - x_1) - H) \end{pmatrix}$$
$$= q\hbar K_R\begin{pmatrix} (x_0 + x_2 - x_1 - x_3)(H - (x_3 - x_1)) & x_2 - x_0 \end{pmatrix},$$

$$\mathcal{M}(D_1) = q K_R\begin{pmatrix} (x_0 + x_2 - x_1 - x_3)(x_0 - x_2) & (x_3 - x_1) - H \end{pmatrix}$$
$$= \hbar K_R\begin{pmatrix} H - (x_3 - x_1) & (x_0 + x_2 - x_1 - x_3)(x_2 - x_0) \end{pmatrix}.$$

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Notice that R is the subring of the polynomial ring in all face variables generated by differences across edges. We refer to R as the *edge ring*. If we are only considering differences across boundary-adjacent edges, we call R the *boundary ring*. We extend this nomenclature to more general webs in the expected way. Note that the boundary ring is a subring of the edge ring, and they coincide if and only if there are no interior faces. The particular case $R = \mathbf{K}[x_0 - x_1, x_2 - x_1, x_3 - x_2]$ is refered to as the *standard boundary ring*. We will often omit the specification of the edge or boundary ring when face variables are already indicated in the web diagram.

Now, given a general connected web Γ on the disk, in which we allow any of the local pictures D_i to appear, we can define the associated matrix factorization

$$\mathcal{M}(\Gamma) = \bigotimes_e \mathcal{M}(D_e)$$

over the corresponding edge ring R_Γ. The tensor product runs over all thin and thick edges of Γ and depends, in principle, on an ordering of these edges. However, the operation is symmetric monoidal, which gives canonical isomorphisms between different choices of thin/thick edge orderings.

3   Web simplifications

In this section, we explain how to express matrix factorizations associated to webs in terms of simpler webs.

Example 3.1. As a first example, let's consider the following local picture ⸻⸻⸻⸻⸻ triple edgepic

$$= qK_{R[u-x_1]} \begin{pmatrix} H - (x_4 - x_0) & (x_5 + u - x_4 - x_0)(u - x_5) \\ (u - x_1) - H & (x_0 + x_2 - u - x_1)(x_0 - x_2) \\ (u + x_3 - x_4 - x_2)(u - x_3) & (x_4 - x_2) - H \end{pmatrix}$$

$$\cong qK_{R[u-x_1]} \begin{pmatrix} (u - x_1) - H & (x_0 + x_2 - u - x_1)(x_0 - x_2) \\ H - (x_4 - x_0) & (x_5 + u - x_4 - x_0)(u - x_5) \\ (u + x_3 - x_4 - x_2)(u - x_3) & (x_4 - x_2) - H \end{pmatrix} := N.$$

Here we are taking the convention of numbering the thick edges top to bottom. Swapping rows gives an isomorphic matrix factorization N, which could have also been obtained by a different ordering of the thick edges, but we want to emphasize the fact that we will keep track of these isomorphisms in situations sensitive to the choice order.

Let R be the boundary ring. The matrix factorization N is, in principle, a matrix factorization over the larger edge ring $R[u - x_1]$. However, the potential lives in R, i.e. all the terms involving u cancel out. This means that one can restrict the base ring to R and simply think of N as a matrix factorization over R, albeit one of infinite rank.

The following lemma gives us a recipe to obtain a finite rank special deformation retract of N as a matrix factorization over R by eliminating the first row and taking a quotient with respect to one of its terms.

Lemma 3.2. *(Adapted from KR How general is R?) Let R be an integral domain and y a variable in quantum degree q^{-2}. Let $\overline{W} \in R$ and $f, g \in R[y]$ so that f has the form $f = uy^n + \tilde{f}$ wtih $u \in R^\times$ and $\deg_y(\tilde{f}) < n$. Let M be a matrix factorization over $R[y]$ with potential $W = \overline{W} - fg \in R[y]$. Let*

$$M' = M/fM,$$
$$\overline{M} = K_{R[y]}\begin{pmatrix} f & g \end{pmatrix} \otimes_{R[y]} M,$$

thought of as matrix factorizations over R with potential $\overline{W}$. Then there is a strong deformation retract

$$M' \xrightarrow[\;P\;]{\;I\;} \overline{M} \circlearrowleft K$$

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of the form

$$
\begin{array}{ccc}
 & & M/fM \\
 & \nearrow^{I_d} & \uparrow I_v \Big\Uparrow P \\
q^{3-2n}\hbar M & \underset{\substack{K \\ g}}{\overset{f}{\rightleftarrows}} & M,
\end{array}
$$

The arrows are as follows:

- P is the usual projection.

- $K = -\operatorname{Quo}_f$, the negative of the quotient of division by f.

- I consists of a vertical component $I_v = \operatorname{Res}_f$ given by the residue of division by f, and a diagonal component I_d given by the composite $-\operatorname{Quo}_f \circ d_M \circ I_v$.

Proof. The identities $KI = 0$, $PK = 0$, $PI = 1$ and $K^2 = 0$ are straightforward. The identity

$$
d_{\overline{M}}K + Kd_{\overline{M}} + 1_{\overline{M}} = IP
$$

is verified as follows. Take $(x, y) \in q^{3-2n}\hbar M \oplus M = \overline{M}$. We compute directly, keeping in mind that $\operatorname{Res}_f(z) = z - f\operatorname{Quo}_f(z)$ and $\operatorname{Quo}_f(fz) = z$:

$$
\begin{aligned}
(d_{\overline{M}}K + Kd_{\overline{M}} + 1_{\overline{M}})(x, y) &= (-\operatorname{Quo}_f(fx) + x - \operatorname{Quo}_f(d_M(y)) - d_M(-\operatorname{Quo}_f(y)), -f\operatorname{Quo}_f(y) + y) \\
&= (-\operatorname{Quo}_f(d_M(y)) + d_M(\operatorname{Quo}_f(y)), \operatorname{Res}_f(y)),
\end{aligned}
$$

while

$$
\begin{aligned}
IP(x, y) &= (I_d([y]), I_v([y])) \\
&= (-\operatorname{Quo}_f(d_M(\operatorname{Res}_f(y))), \operatorname{Res}_f(y)) \\
&= (-\operatorname{Quo}_f(d_M(y - f\operatorname{Quo}_f(y))), \operatorname{Res}_f(y)) \\
&= (-\operatorname{Quo}_f(d_M(y)) + \operatorname{Quo}_f(fd_M(\operatorname{Quo}_f(y))), \operatorname{Res}_f(y)) \\
&= (-\operatorname{Quo}_f(d_M(y)) + d_M(\operatorname{Quo}_f(y)), \operatorname{Res}_f(y)).
\end{aligned}
$$

It is also straightforward to verify that P commutes with differentials. In the case of I, we have

$$
\begin{aligned}
I(d_{M'}([x])) &= (-\operatorname{Quo}_f(d_M^2(x)) + d_M(\operatorname{Quo}_f(d_M(x))), \operatorname{Res}_f(d_M(x))) \\
&= (d_M(\operatorname{Quo}_f(d_M(x))) - \operatorname{Quo}_f((\overline{W} - fg)x), \operatorname{Res}_f(d_M(x))),
\end{aligned}
$$

and

$$
\begin{aligned}
d_{\overline{M}}(I([x])) &= d_{\overline{M}}(-\operatorname{Quo}_f(d_M(x)) + d_M(\operatorname{Quo}_f(x)), \operatorname{Res}_f(x)) \\
&= (-d_M(-\operatorname{Quo}_f(d_M(x))) - d_M^2(\operatorname{Quo}_f(x)) + g\operatorname{Res}_f(x), \\
&\qquad d_M(\operatorname{Res}_f(x)) - f\operatorname{Quo}_f(d_M(x)) + fd_M(\operatorname{Quo}_f(x))) \\
&= (d_M(\operatorname{Quo}_f(d_M(x))) - (\overline{W} - fg)\operatorname{Quo}_f(x) + g(x - f\operatorname{Quo}_f(x)), \\
&\qquad d_M(\operatorname{Res}_f(x) + f\operatorname{Quo}_f(x)) - f\operatorname{Quo}_f(d_M(x))) \\
&= (d_M(\operatorname{Quo}_f(d_M(x))) - \overline{W}\operatorname{Quo}_f(x) + gx, d_M(x) - f\operatorname{Quo}_f(d_M(x))) \\
&= (d_M(\operatorname{Quo}_f(d_M(x))) - \operatorname{Quo}_f((\overline{W} - fg)x), \operatorname{Res}_f(d_M(x))).
\end{aligned}
$$

$\square$

Example 3.3. Let us return to example 3.1 and apply the lemma. In this case, we have $y = u - x_1$, $f = (u - x_1) - H$ and

$$
M = qK_{R[u-x_1]}\begin{pmatrix} H - (x_4 - x_0) & (x_5 + u - x_4 - x_0)(u - x_5) \\ (u + x_3 - x_4 - x_2)(u - x_3) & (x_4 - x_2) - H \end{pmatrix},
$$

so that

$$M/fM = qK_R\begin{pmatrix} H - (x_4 - x_0) & (x_5 + x_1 + H - x_4 - x_0)(x_1 + H - x_5) \\ (x_1 + H + x_3 - x_4 - x_2)(x_1 + H - x_3) & (x_4 - x_2) - H \end{pmatrix},$$

is a special deformation retract of $N = \overline{M} = K_{R[u-x_1]}\begin{pmatrix} f & g \end{pmatrix} \otimes_{R[u-x_1]} M$.

Now that we have a smaller factorization M/fM over a smaller ring R, we would like to identify it as the matrix factorization associated to a smaller web. We can do this by a couple of elementary operations.

$$M/fM = qK_R\begin{pmatrix} H - (x_4 - x_0) & (x_5 + x_1 + H - x_4 - x_0)(x_1 + H - x_5) \\ (x_1 + H + x_3 - x_4 - x_2)(x_1 + H - x_3) & (x_4 - x_2) - H \end{pmatrix}$$

$$\cong qK_R\begin{pmatrix} x_0 - x_2 & (x_5 + x_1 + H - x_4 - x_0)(x_1 + H - x_5) \\ (x_0 - x_2)(x_1 + H - x_5) + (x_5 - x_3)(x_5 + x_3 - x_4 - x_2) & (x_4 - x_2) - H \end{pmatrix}$$

$$\cong qK_R\begin{pmatrix} x_0 - x_2 & (x_5 + x_1 - x_2 - x_0)(x_1 + H - x_5) \\ (x_5 - x_3)(x_5 + x_3 - x_4 - x_2) & (x_4 - x_2) - H \end{pmatrix}$$

In our running example, we simplified a Koszul matrix factorization by killing a linear polynomial. This procedure will be applied often, so we record it as a more general lemma.

Lemma 3.4. (Simplification of Koszul factorizations by a linear term) *In the set up of Lemma 3.2, assume that $M = K_{R[y]}\begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix}$ and that $f = y - f_0$, where $\deg_y(f_0) = 0$. Then, under the identification $M/yM \cong K_R\begin{pmatrix} \overline{\mathbf{a}}\big|_{y=f_0} & \overline{\mathbf{b}}\big|_{y=f_0} \end{pmatrix}$, we have a special deformation retract*

$$K_R\begin{pmatrix} \overline{\mathbf{a}}\big|_{y=f_0} & \overline{\mathbf{b}}\big|_{y=f_0} \end{pmatrix}$$

$$q\hbar K_{R[y]}\begin{pmatrix} \overline{\mathbf{a}} & \overline{\mathbf{b}} \end{pmatrix} \xleftarrow{\ -\operatorname{Quo}_f\ } K_{R[y]}\begin{pmatrix} \overline{\mathbf{a}} & \overline{\mathbf{b}} \end{pmatrix},$$

with labels I_v, f, g, 1, $y \mapsto f_0$.

The diagonal component of the inclusion is the composite

$$K_R\begin{pmatrix} \overline{\mathbf{a}}\big|_{y=f_0} & \overline{\mathbf{b}}\big|_{y=f_0} \end{pmatrix} \xrightarrow{\ \sum_{j=1}^{k} 1 \otimes \cdots \otimes \hbar^1(-\operatorname{Quo}_f \circ a_j, -\operatorname{Quo}_f \circ b_j) \otimes \cdots \otimes 1\ } qK_{R[y]}\begin{pmatrix} \overline{\mathbf{a}} & \overline{\mathbf{b}} \end{pmatrix}$$

$$\downarrow{\scriptstyle s_\hbar}$$

$$q\hbar K_{R[y]}\begin{pmatrix} \overline{\mathbf{a}} & \overline{\mathbf{b}} \end{pmatrix}$$

In other cases of interest, we have a decomposition into several summands $M/fM = \bigoplus_{i=0}^{k} M_i$ as matrix factorizations over R, where M_i are simpler factorizations of a desirable form. The following lemma, a higher order generalization of 3.4, gives such decompositions in favorable cases.

In the following, we introduce the shorthand

$$P_{f(y)}^i = \frac{\partial^i}{\partial y^i}\bigg|_{y=0} \circ \operatorname{Res}_f,$$

which is more succinctly described as the map that takes the coefficient of y^i after taking the residue of division by $f(y)$.

Lemma 3.5. (Simplification of Koszul factorizations by a higher order polynomial) *In the set up of Lemma 3.2, assume that $M = K_{R[y]}\begin{pmatrix}\overline{\mathbf{a}} & \overline{\mathbf{b}}\end{pmatrix}$ and that $\deg_y(\mathrm{Res}_f(a_i)) = \deg_y(\mathrm{Res}_f(b_i)) = 0$ for all i. Then the map*

$$\bigoplus_{i=0}^{n-1} q^{-2i} K_R\begin{pmatrix}\mathrm{Res}_f\,\overline{\mathbf{a}} & \mathrm{Res}_f\,\overline{\mathbf{b}}\end{pmatrix} \xrightarrow{(y^i)_i} M/fM$$

is an isomorphism of matrix factorizations over R and, combined with Lemma 3.2, gives a special deformation retract

The diagonal component of the inclusion is the composite

3.1 Thin edge removal

Simplification by linear polynomials 3.4 can be directly applied to remove thin edges that bound an interior region.

Proposition 3.6. *Let D be a four-ended web with a thin edge e such that its negative region is interior. Denote the corresponding edge ring by R_D, and the variables associated to the positive and negative region of e by x_+ and x_-, respectively. Let D' be the web obtained by removal of e, so that its edge ring $R_{D'}$ no longer involves x_-. Then there is a special deformation retract of matrix factorizations over $R_{D'}$*

$$M(D') \rightleftharpoons M(D) \;\circlearrowleft$$
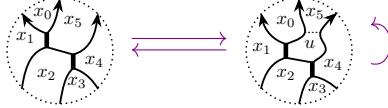
of the form

where $\widetilde{M}(D)$ is the Koszul factorization over R_D obtained from all edges in D except e. The only contributions to the sum come from edges with at least one end adjacent to the x_--region.

A similar statement is true if the positive region is interior and the variable x_+ is eliminated instead.

Example 3.7. In the proof of the Reidemeister 3 move we will need special deformation retracts of the form

removing the thin edge that separates $x_+ = x_5$ from $x_- = u$. According to the expression above, the retraction map will involve the evaluation of $-\operatorname{Quo}_{x_5-u}$ at the various components of

$$[x_5 - u] = q^2 K_{R[x_5-u]}\begin{pmatrix} (x_0 + x_2 - x_1 - u)(x_0 - x_2) & (u - x_1) - H \\ (u + x_3 - x_2 - x_4)(u - x_3) & (x_4 - x_2) - H \end{pmatrix}.$$

The result is

signs need double check here

3.2 Circle removal (delooping)

The usual delooping consists of removing the circle in the local picture

$$= q\hbar K_{R[x_0-x_3]}\big((2x_0 - x_1 - x_3)(H - (x_3 - x_1)) \quad 0\big),$$

where the boundary ring is $R = \mathbf{K}[x_0 - x_1]$. By simplifying with respect to the polynomial $f(y) = (2x_0 - x_1 - x_3)((x_3 - x_1) - H)$ thought of as quadratic in the variable $y = x_0 - x_3$, we are left with the special deformation retract

There is also a version of delooping for thick edges

$$= \hbar K_{R[x_0-x_3]}\big(H - (x_3 - x_1) \quad 0\big).$$

We simplify with respect to $f(y) = H - (x_3 - x_1)$ which is linear in $y = x_0 - x_3$, to get a special deformation retract

Loops may appear in more complicated ways, and there are different ways to deal with them. The following two situations appear in Reidemeister II move, where R is back to being the standard boundary ring. First, we have

$$\left(\begin{array}{c} x_0 \\ x_1 \quad u \quad x_3 \\ x_2 \end{array}\right) = q^2 K_{R[u-x_1]} \begin{pmatrix} (x_0 + x_2 - x_1 - u)((u - x_1) - H) & x_0 - x_2 \\ (x_2 + x_0 - u - x_3)(H - (u - x_3)) & x_0 - x_2 \end{pmatrix}.$$

It is possible to reduce this case to (1), but we opt for the direct approach of simplifying with respect to the polynomial $f(y) = (x_0 + x_2 - x_1 - u)((u - x_1) - H)$, thought of as quadratic in the variable $y = u - x_1$, to obtain a special deformation retract

$$q\hbar \begin{pmatrix} x_0 \\ x_1 \,\cdots\, x_3 \\ x_2 \end{pmatrix} \oplus q^{-1}\hbar \begin{pmatrix} x_0 \\ x_1 \,\cdots\, x_3 \\ x_2 \end{pmatrix} \longleftarrow \begin{pmatrix} x_0 \\ x_1 \quad u \quad x_3 \\ x_2 \end{pmatrix} \quad \text{(3)}$$

given by

$$q\hbar \begin{pmatrix} x_0 \\ x_1 \,\cdots\, x_3 \\ x_2 \end{pmatrix} \oplus q^{-1}\hbar \begin{pmatrix} x_0 \\ x_1 \,\cdots\, x_3 \\ x_2 \end{pmatrix}$$

$$\left(\hbar^0(1,0) \quad \hbar^0(u-x_1,0)\right) \quad \begin{pmatrix} \hbar^0(P^0_{f(y)},0) \\ \hbar^0(P^1_{f(y)},0) \end{pmatrix}$$

$$q\hbar \begin{pmatrix} x_2 \\ x_3 \quad u \\ x_0 \end{pmatrix} [u - x_1] \xrightarrow{\;\; x_0 - x_2 \;\;} \begin{pmatrix} x_2 \\ x_3 \quad u \\ x_0 \end{pmatrix} [u - x_1].$$
$$\xleftarrow{\; f = (x_0 + x_2 - x_1 - u)((u-x_1)-H) \;}$$

Similarly, we also have

$$\left(\begin{array}{c} x_0 \\ x_1 \quad u \quad x_3 \\ x_2 \end{array}\right) = \hbar K_{R[u-x_1]} \begin{pmatrix} H - (u - x_1) & (x_0 + x_2 - x_1 - u)(x_2 - x_0) \\ x_2 - x_0 & (x_2 + x_0 - u - x_3)((u - x_3) - H) \end{pmatrix},$$

which we simplify with respect to the linear polynomial $f(y) = (u - x_1) - H$ in the variable $y = u - x_1$ to obtain

$$\hbar \begin{pmatrix} x_0 \\ x_1 \,\cdots\, x_3 \\ x_2 \end{pmatrix} \longleftarrow \begin{pmatrix} x_0 \\ x_1 \quad u \quad x_3 \\ x_2 \end{pmatrix} \quad \text{(4)}$$

given by

$$\hbar \begin{pmatrix} x_0 \\ x_1 \,\cdots\, x_3 \\ x_2 \end{pmatrix}$$

$$\hbar^0(1,0) \quad \hbar^0(P^0_{f(y)},0)$$

$$q \begin{pmatrix} x_2 \\ x_3 \,\cdots\, u \\ x_0 \end{pmatrix} [u - x_1] \xrightarrow{\; f = (u-x_1)-H \;} \hbar \begin{pmatrix} x_2 \\ x_3 \quad u \\ x_0 \end{pmatrix} [u - x_1]$$
$$\xleftarrow{\; (x_0 + x_2 - x_1 - u)(x_0 - x_2) \;}$$

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3.3 Bigon and square removal

The local picture

$$\begin{array}{c}\vcenter{\hbox{}}\end{array} = q^2 K_{R[u-x_0]} \begin{pmatrix} (x_0 + u - x_1 - x_3)(x_0 - u) & (x_3 - x_1) - H \\ (u + x_2 - x_1 - x_3)(u - x_2) & (x_3 - x_1) - H \end{pmatrix}$$

is known as a *bigon*. Letting $y = u - x_0$, $f = (x_0 + u - x_1 - x_3)(x_0 - u)$ and

$$M = q^2 K_{R[u-x_0]}\big((u + x_2 - x_1 - x_3)(u - x_2) \quad (x_3 - x_1) - H\big) = q \,\vcenter{\hbox{}}\,[u - x_0],$$

we are under the hypotheses of the higher order simplification lemma 3.5. In particular, we have a strong deformation retract

For square removal, we have

$$\vcenter{\hbox{}} = K_{R[u-x_1]} \begin{pmatrix} (u - x_1) - H & (x_0 + x_2 - x_1 - u)(x_0 - x_2) \\ H - (u - x_3) & (x_2 + x_0 - x_3 - u)(x_0 - x_2) \end{pmatrix}$$

Letting $y = u - x_1$, $f = (u - x_1) - H$ and

$$M = K_{R[u-x_1]}\big(H - (u - x_3) \quad (x_2 + x_0 - x_3 - u)(x_0 - x_2)\big) = \hbar \,\vcenter{\hbox{}}\,[u - x_1],$$

we are under the hypotheses of the linear simplification lemma 3.4. Thus, we have a strong deformation retract

these maps still need some checking, esp. the diagonal

i accidentally ruined the horizontal

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4 Multifactorizations and special deformation retracts

Definition 4.1. Let (C, D) and (C', D') be multifactorizations. Then a special n-deformation retract from C to C' consists of 0-morphisms $P : C \to C'$ and $I : C' \to C$ such that $PI = 1$, together with a n-homotopy $H : C \to C$ between 1 and IP such that $HI = 0$, $PH = 0$ and $H^2 = 0$. We often represent this data by a triple (I, P, H) or, more suggestively, by a diagram

$$(C', D') \xleftarrow[P]{\quad I \quad} (C, D) \;\circlearrowleft H \,.$$

Proposition 4.2. *Suppose*

$$(C', d') \xleftarrow[p]{\quad i \quad} (C, d) \;\circlearrowleft h$$

is a special 1-deformation retract between matrix multifactorizations C' and C. Let D be another differential in C such that its vertical part coincides with d, i.e. $D_0 = d_0$. Then there exists a special 1-deformation retract

$$(C', D') \xleftarrow[P]{\quad I \quad} (C, D) \;\circlearrowleft H \,.$$

such that $D'_0 = d'_0$. If h is also a 0-homotopy, so that the original data is a special 0-deformation retract, then the deformed data is also a special 0-deformation retract with the additional properties $P_0 = p_0$, $I_0 = i_0$ and $H_0 = h_0$.

Proof. (Essentially same as Ballinger) Let

$$A = (1 - (D - d)h)^{-1}(D - d) = \sum_{i=0}^{\infty}((D - d)h)^i(D - d).$$

Notice that $D - d$ has filtered degree ≥ 1, while h has filtered degree ≥ -1. In total, A has filtered degree ≥ 1. Take

$$\begin{aligned}
D' &= d' + pAi \\
P &= p + pAh \\
I &= i + hAi \\
H &= h + hAh
\end{aligned} \tag{5}$$

See (Crainic) for calculations. Notice that the term pAi, i.e. the deformation of the differential, has filtered degree ≥ 1, which guarantees $D'_0 = d'_0$. In the case in which h is a 1-homotopy then pAh and hAi preserve filtration, while hAh has filtered degree ≥ -1, which are necessary conditions for the deformed data (I, P, H) to give a special 1-deformation retract. If h is additionally a 0-homotopy, then all these terms have now filtered degree ≥ 1. In such case, the deformed data (I, P, H) gives a special 0-deformation retract of the desired form. $\qquad\square$

The following lemma allows us to replace certain subfactorizations in a matrix factorization by their special retracts.

Lemma 4.3. *Let (C, D) be a matrix multifactorization of the form $C = A \oplus B$ and $D = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, so that*

- *(A, α) and (B, δ) are subfactorizations (i.e. $\beta\gamma = 0$ and $\gamma\beta = 0$),*

- *$\beta_0 = 0$ and*

- *$\gamma_0 = 0$.*

Suppose we have special 1-deformation retracts

$$(A', \epsilon) \xleftarrow[p_1]{\quad i_1 \quad} (A, \alpha) \;\circlearrowleft h_1 \,, \qquad (B', \eta) \xleftarrow[p_2]{\quad i_2 \quad} (B, \delta) \;\circlearrowleft h_2 \,.$$

Then there exists a special 1-deformation retract

$$(A' \oplus B', D') \xleftarrow[P]{I} (C, D) \,\righttoleftarrow\, H$$

with $D'_0 = \epsilon_0 \oplus \eta_0$.

Proof. Apply Proposition 4.2 to the special deformation retract

$$(A' \oplus B', \epsilon \oplus \eta) \xleftarrow[p_1 \oplus p_2]{i_1 \oplus i_2} (A \oplus B, \alpha \oplus \delta) \,\righttoleftarrow\, h_1 \oplus h_2 \, .$$

Here $d = \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}$ and $D = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. Since $\beta_0 = 0$ and $\gamma_0 = 0$, the hypothesis $D_0 = d_0$ is satisfied.

$$A = \sum_{i=0}^{\infty} \begin{pmatrix} \beta h_2 \gamma & \beta \\ \gamma & \gamma h_1 \beta \end{pmatrix} \begin{pmatrix} (h_1 \beta h_2 \gamma)^i & 0 \\ 0 & (h_2 \gamma h_1 \beta)^i \end{pmatrix}. \tag{6}$$

Since $\beta_0 = 0$ and $\gamma_0 = 0$ and h_1 and h_2 have filtered degree ≥ -1, it follows that $A_0 = 0$. This implies $D'_0 = \epsilon_0 \oplus \eta_0$. $\qquad\square$

We will mostly encounter situations in which β, γ, h_1 or h_2, or short composites thereof, vanish. The expression (6) simplifies significantly in those cases, which in turn gives manageable formulas in (5).

5   Invariance

The invariance of $M(D)$ for a link diagram D under each Reidemeister move will proceed by looking at the explicit matrix multifactorization associated to the relevant local piece of the diagram and simplifying it in two steps. In the first step, one simplifies each resolving web into smaller webs with no internal faces, and then extends into a special 0-deformation retract of the entire multifactorization. There is enough control of the deformation data to compute explicit differentials in the resulting multifactorization, and in particular to identify some identity components in d_1. The second step then consists of simplifying along such identities to obtain a special 1-deformation retract into the desired form.

Reidemeister I moves are illustrated in Figure 1. The purple arrows represent special deformation retracts of vertical matrix factorizations, where each component is obtained from Lemma 3.4 or Lemma 3.5. We then extend, by Lemma 4.2, to a special 0-deformation retract of multifactorizations. Notice that we indicate the explicit form of the d_1-component in each of the once-simplified multifactorizations. These are computed as the composites

$$q^2 \,\Big\rangle\, \oplus \,\Big\rangle\, \xrightarrow{\begin{pmatrix} \hbar^0(1,0) & \hbar^0(x_3 - x_0, 0) \end{pmatrix}} q\hbar \,\Big\rangle\bigcirc\, \xrightarrow{\hbar^1(1, 2x_0 - x_1 - x_3)} q^2 h \,\Big\rangle\bigcirc\, \xrightarrow[\substack{y = x_3 - x_0 \\ f(y) = x_3 - x_0}]{\hbar^0(0, P^0_{f(y)})} q^2 \hbar h \,\Big\rangle\,$$

in the case of the positive Reidemeister I move, and the composite

$$q^{-2} \hbar h^{-1} \,\Big\rangle\, \xrightarrow{\hbar^0(0,1)} q^{-2} \hbar h^{-1} \,\Big\rangle\bigcirc\, \xrightarrow{\hbar^1(1, 2x_0 - x_1 - x_3)} q^{-1} \,\Big\rangle\bigcirc\, \xrightarrow[\substack{y = x_3 - x_0 \\ f(y) = (2x_0 - x_1 - x_3)((x_3 - x_1) - H)}]{\begin{pmatrix} \hbar^0(P^0_{f(y)}, 0) \\ \hbar^0(P^1_{f(y)}, 0) \end{pmatrix}} \,\Big\rangle\, \oplus q^{-2} \,\Big\rangle\,$$

for the negative Reidemeister I move.

The summands

$$\left[q^2 \,\Big\rangle\, \xrightarrow{\hbar^1(1,0)} q^2 \hbar h \,\Big\rangle\, \right] \qquad \text{and} \qquad \left[q^{-2} \hbar h^{-1} \,\Big\rangle\, \xrightarrow{\hbar^1(0,-1)} q^{-2} \,\Big\rangle\, \right]$$

are easily verified to be 1-contractible. The corresponding special 1-deformation retractions cancelling these summands are represented by the green arrows.

The same recipe works for Reidemeister II moves. The special 0-deformation retracts represented by violet arrows still come from L
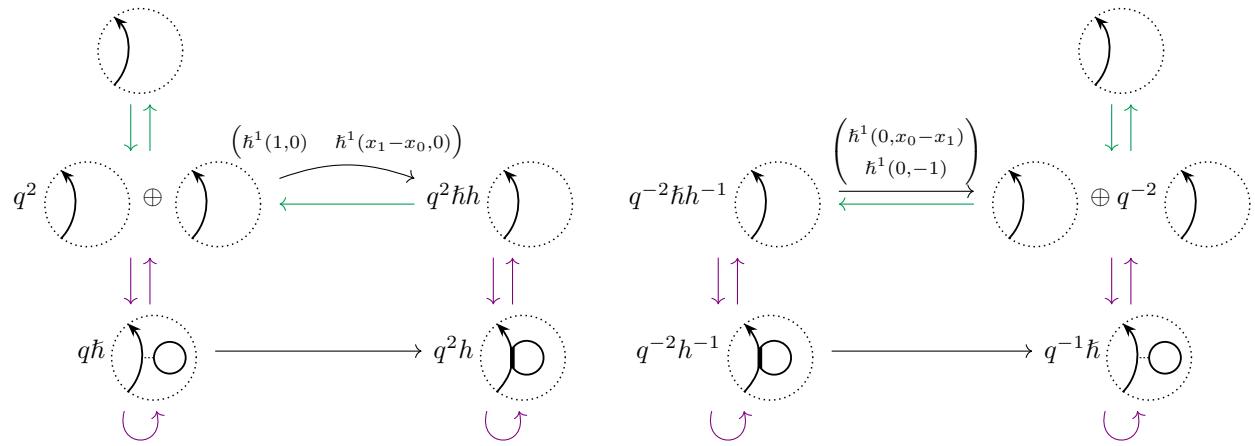
Figure 1: Invariance under positive and negative Reidemeister I moves
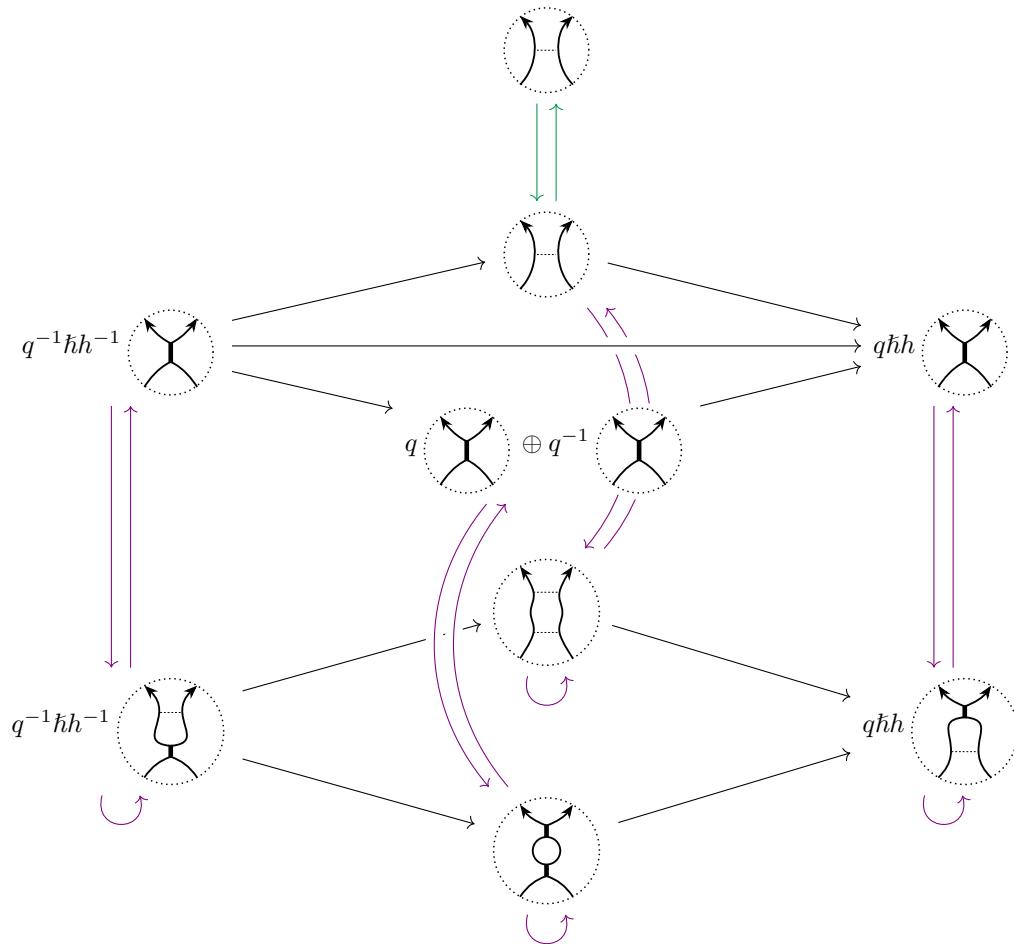
Figure 2: Invariance under positive and negative Reidemeister IIa move
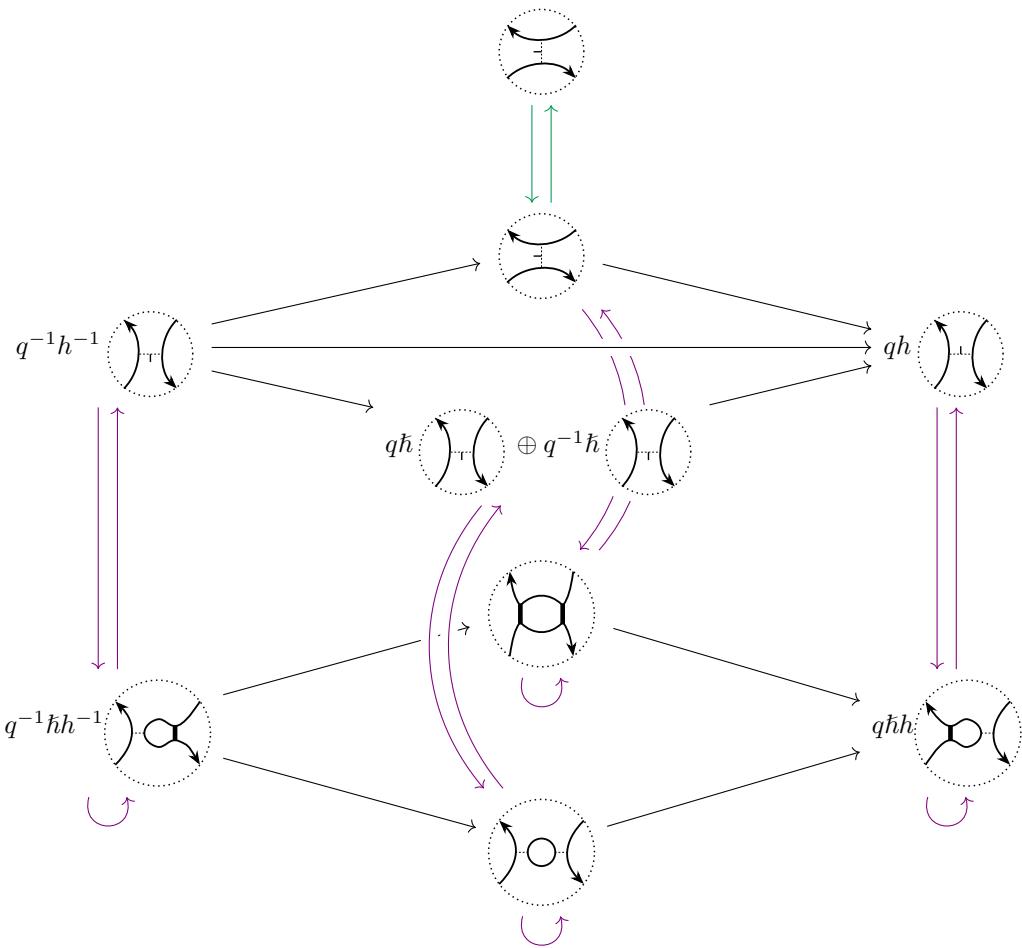
Figure 3: Invariance under Reidemeister IIb move
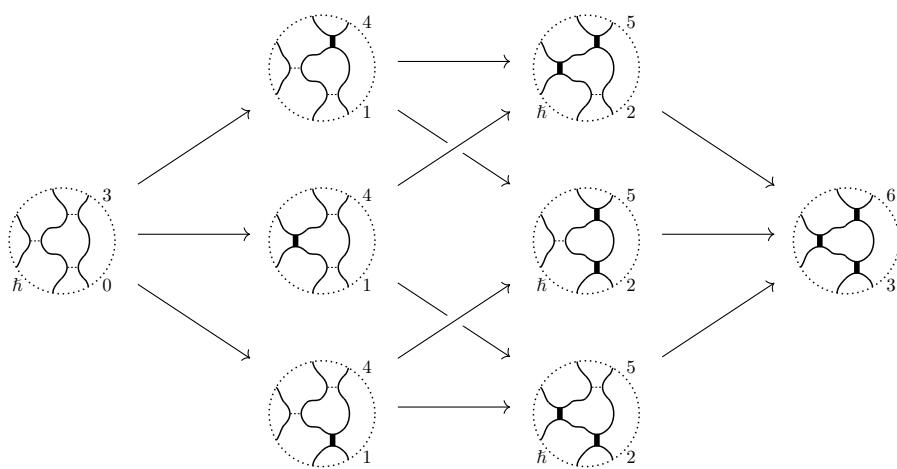

Figure 4: Invariance under Reidemeister III move
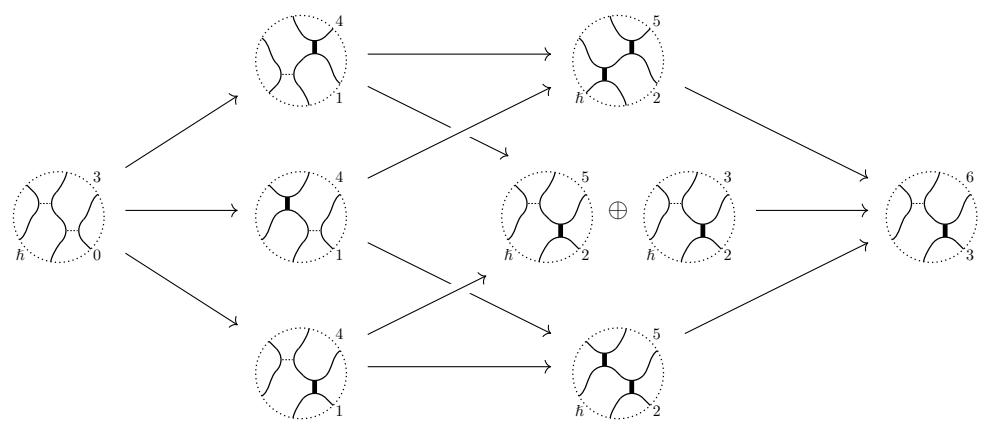
Figure 5: Invariance under Reidemeister III move