# Invariance of Bar-Natan matrix multifactorizations up to 1-homotopy equivalence

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**Abstract**

We verify link invariance of a certain construction using matrix factorizations over $\mathbb{Z}[H]$.

## 1 Matrix factorizations

Let $\mathcal{D}_0$ be a dotted edge and $\mathcal{D}_1$ be a thick edge.

Consider a base ring $\mathbf{K}$ (e.g. $\mathbf{K} = \mathbb{Z}[\mathsf{G}]$ with grading $\mathsf{gr}(\mathsf{G}) = i^{-2}\hbar^0 q^{-2}$). Take some potential $P(X) \in \mathbf{K}[X]$ (or even $\mathbf{K}[\![X]\!]$ if $\mathbf{K}$ is big enough).

We denote by $x_0$, $x_1$, $x_2$ and $x_3$ the variables of surrounding faces, starting above and going counterclockwise. In $\mathbf{K}[x_0 - x_1, x_2 - x_1, x_3 - x_2]$, it is always possible to factor

$$W = P(x_0 - x_1) + P(x_3 - x_0) - P(x_2 - x_1) - P(x_3 - x_2) = (x_0 - x_2)(x_0 + x_2 - x_1 - x_3)Z$$

for some $Z = Z(x_0 - x_1, x_2 - x_1, x_3 - x_2) \in \mathbf{K}[x_0 - x_1, x_2 - x_1, x_3 - x_2]$. We often drop the inputs of $Z$ to lighten notation.

We assign Koszul matrix factorizations

$$\begin{aligned} M(\mathcal{D}_0) &= K(x_0 - x_2, \quad (x_0 + x_2 - x_1 - x_3)Z) \\ M(\mathcal{D}_1) &= K(Z, \quad (x_0 - x_2)(x_0 + x_2 - x_1 - x_3)). \end{aligned}$$

Given two matrix factorizations of the form

$$M = [\ A \rightleftarrows \hbar B\ ], \qquad M' = [\ A' \rightleftarrows \hbar B'\ ],$$

with $A, A', B$ and $B'$ concentrated in $\hbar$-degree 0, we often specify morphisms by spelling out their components in each $\hbar$-degree as follows:

$$\begin{tikzcd} M \arrow[d, "\hbar^0(\alpha,\beta)"'] \\ M' \end{tikzcd} = \begin{tikzcd} {[}A \arrow[r, shift left] \arrow[d, "\alpha"] & \hbar B{]} \arrow[l, shift left] \arrow[d, "\beta"] \\ {[}A' \arrow[r, shift left] & \hbar B'{]} \arrow[l, shift left] \end{tikzcd} \qquad \begin{tikzcd} M \arrow[d, "\hbar^1(\alpha,\beta)"'] \\ M' \end{tikzcd} = \begin{tikzcd} {[}A \arrow[r, shift left] \arrow[dr, "\alpha"] & \hbar B{]} \arrow[l, shift left] \arrow[dl, "\beta"'] \\ {[}A' \arrow[r, shift left] & \hbar B'{]} \arrow[l, shift left] \end{tikzcd}$$

Many useful maps can be easily expressed in this notation. For instance, the differential $d_M : M \to M$ itself takes the form

$$d_m = \hbar^1(d_0, d_1)$$

In the case $M' = \hbar M$ we use the special notation

$$s_\hbar = \hbar^1(1, 1) : M \to \hbar M.$$

For Koszul matrix factorizations, we will take the convention

$$K_R(a\ ;\ b) = \ q^{\deg_q(a)+3} R \underset{b}{\overset{a}{\rightleftarrows}} \hbar R\ .$$

Here, the grading $\hbar$ is a $\mathbb{Z}/2$ grading. A shift $\hbar M$ on a matrix factorization $M$ has the additional effect of switching the sign of all differentials. This helps us keep track of Koszul sign rules for tensor products of matrix factorizations. Namely, if we tensor a Koszul factorization $K_R(a\ ;\ b)$ with some other matrix factorization $M$, we can write

$$K_R(a\ ;\ b) \otimes M = \left[\ q^{\deg_q(a)+3} M \underset{b}{\overset{a}{\rightleftarrows}} \hbar M\ \right],$$

which indicates that the tensor product splits, as free $R$-module, into two copies of $M$, with the first one shifted in $q$ and $\hbar$ degrees. In terms of this decomposition, the total differential takes the form

$$d_{K_R(a\ ;\ b)\otimes M} = \begin{pmatrix} d_M & b \\ a & -d_M \end{pmatrix},$$

where the negative sign comes precisely from the $\hbar$-shift. Observe also that here $a$ and $b$ are understood to have $\hbar$-degree 1, and they could be more explicitly denoted by $\hbar^1(a,a)$ and $\hbar^1(b,b)$, respectively.

If we want to switch the sign of differentials in $M$ without altering the $\hbar$ degree, we write $-M$. Using these notations, we have

not sure if I need this

$$\hbar(M \otimes N) = (\hbar M) \otimes N = (-M) \otimes (\hbar N)$$

The most general webs we will be dealing with can be obtained by taking an oriented planar arc diagram $T$ where all inputs have exactly two consecutive incoming arcs and two consecutive outgoing arcs, and then plugging $\mathcal{D}_0$ or $\mathcal{D}_1$ in each input while respecting orientations:

$$\mathcal{D} = T(\mathcal{D}_{i_1}, \ldots, \mathcal{D}_{i_k}),$$

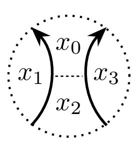
with $i_j = 0$ or 1.

## 2 Hom spaces

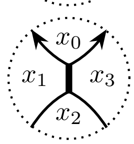
## 3 Matrix factorizations associated to webs and tangles

The base ring is $\mathbf{K} = \mathbb{Z}[H]$, and the elementary matrix factorizations are over $R = \mathbf{K}[x_0 - x_1, x_2 - x_1, x_3 - x_2]$.

it seems like $H-(x_3-x_1)$ in Caludius' paper has the wrong sign if we want to get the correct potential. I'm swapping the sign

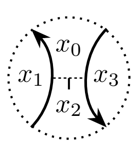
$D_0 =$ 

$\longmapsto M(D_0) = \left[\ qR \underset{(x_0+x_2-x_1-x_3)((x_3-x_1)-H)}{\overset{x_0-x_2}{\rightleftarrows}} \hbar R\ \right]$

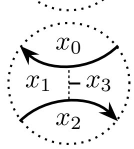
$D_1 =$ 

$\longmapsto M(D_1) = \left[\ q\hbar R \underset{(x_0+x_2-x_1-x_3)(x_0-x_2)}{\overset{(x_3-x_1)-H}{\rightleftarrows}} R\ \right]$

When working locally, other boundary orientations are also possible. Thus, we also define

$D_2 =$ 

$\longmapsto M(D_2) = \left[\ qR \underset{(x_0+x_2-x_1-x_3-H)(x_3-x_1)}{\overset{x_0-x_2}{\rightleftarrows}} \hbar R\ \right]$

$D_3 =$ 

$\longmapsto M(D_3) = \left[\ qR \underset{(x_0+x_2-x_1-x_3-H)(x_2-x_0)}{\overset{x_1-x_3}{\rightleftarrows}} \hbar R\ \right]$

seems like we should swap all the $\hbar$ degrees here. maybe even in the definition of koszul

In all three types of thin edges, there is a *negative region* whose variable appears with a negative sign in the linear term of the associated matrix factorization. In the case of $D_2$ and $D_3$ there is a 180 degree rotational symmetry. To avoid ambiguity, we indicate the negative region by a little dash.

arrows

The elementary matrix factorizations can be expressed as

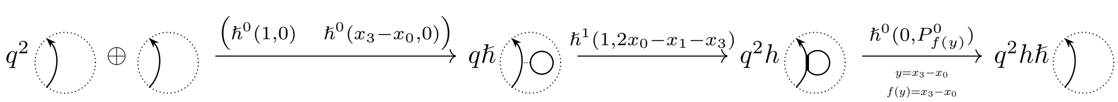
$$\begin{aligned} M(D_0) &= K_R(x_0 - x_2\ ;\ (x_0 + x_2 - x_1 - x_3)((x_3 - x_1) - H)) \\ M(D_1) &= qK_R((x_0 + x_2 - x_1 - x_3)(x_0 - x_2)\ ;\ (x_3 - x_1) - H) \\ &= \hbar K_R(H - (x_3 - x_1)\ ;\ (x_0 + x_2 - x_1 - x_3)(x_2 - x_0)) \end{aligned}$$

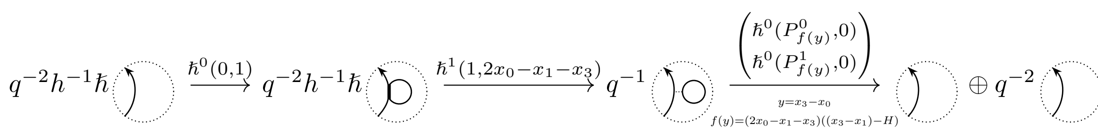
Notice that $R$ is the subring of the polynomial ring in all face variables generated by differences across edges. We refer to $R$ as the *edge ring*, and we extend this nomenclature to more general webs in the expected way. The particular case $R = \mathbf{K}[x_0 - x_1, x_2 - x_1, x_3 - x_2]$ is refered to as the *standard edge ring*. We will often omit the specification of the edge ring when face variables are already indicated in the web diagram.

# 4 Invariance

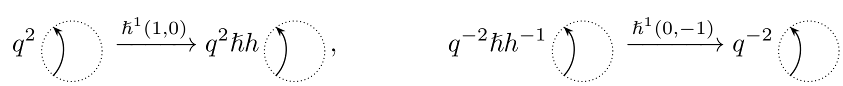
The invariance of $M(D)$ for a link diagram $D$ under each Reidemeister move will proceed by looking at the explicit matrix multifactorization associated to the relevant local piece of the diagram and simplifying it in two steps. In the first step, one simplifies each resolving web into smaller webs with no internal faces, and then extends into a special 0-deformation retract of the entire multifactorization. There is enough control of the deformation data to compute explicit differentials in the resulting multifactorization, and in particular to identify some identity components in $d_1$. The second step then consists of simplifying along such identities to obtain a special 1-deformation retract into the desired form.

Reidemeister I moves are illustrated in Figure 1. The purple arrows represent special deformation retracts of vertical matrix factorizations, where each component is obtained from Lemma 4.3 or Lemma 4.4. We then extend, by Lemma 4.6, to a special 0-deformation retract of multifactorizations. Notice that we indicate the explicit form of the $d_1$-component in each of the once-simplified multifactorizations. These are computed as composites:




The summands




are easily verified to be 1-contractible. The corresponding special 1-deformation retractions cancelling these summands are represented by the green arrows.

Concerning Reidemeister 2, here is what we do now

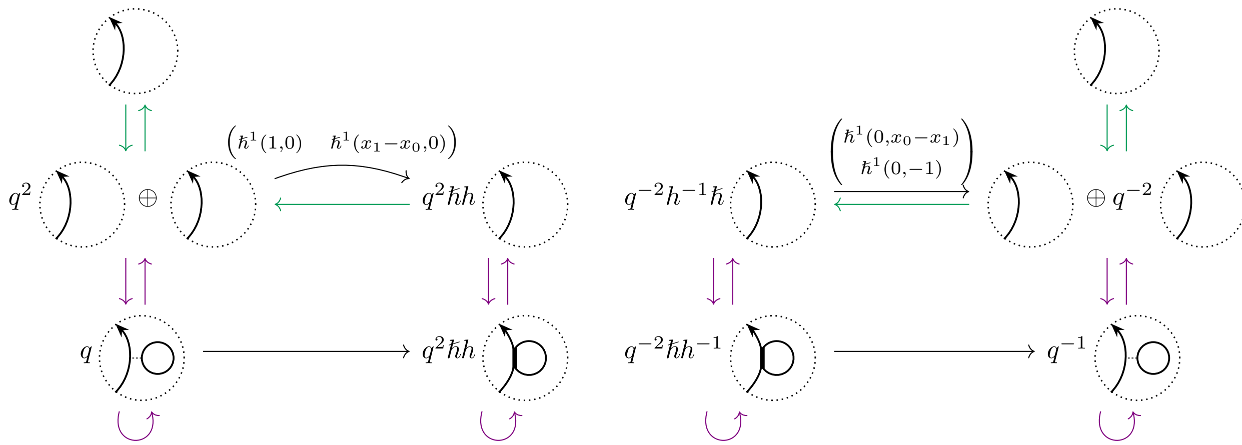
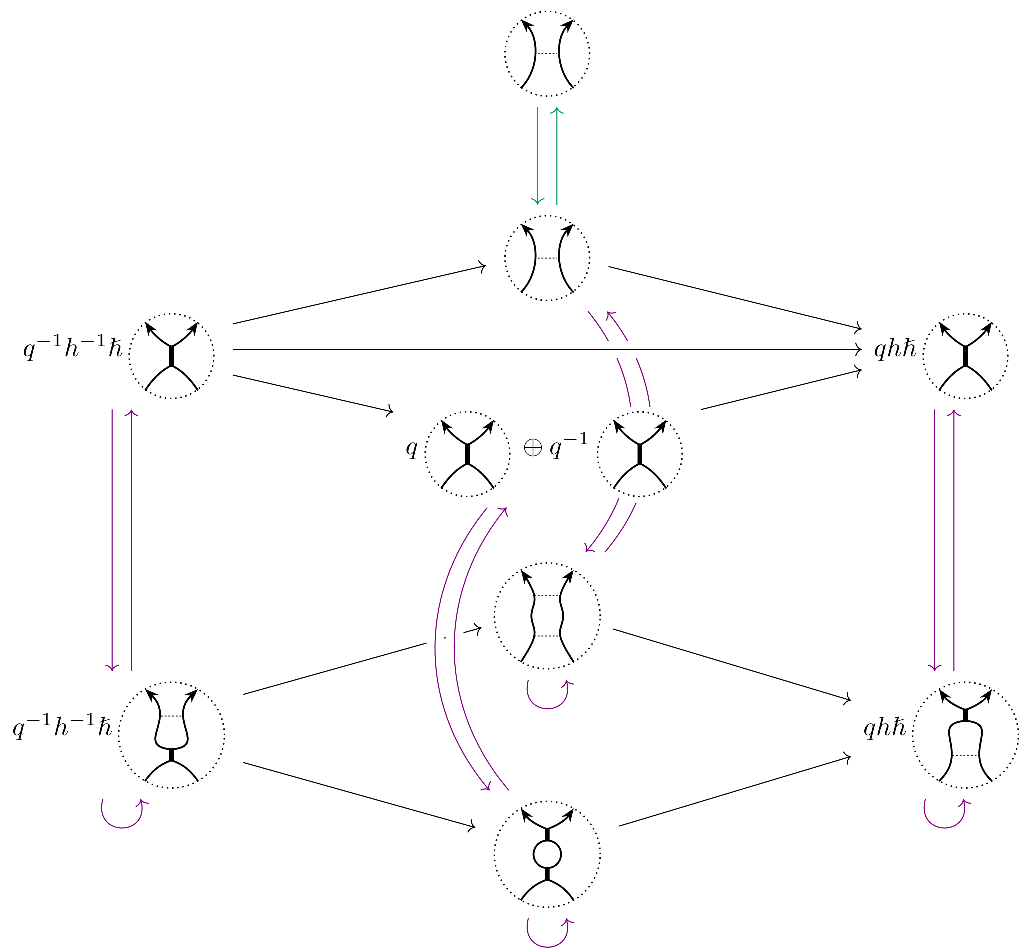
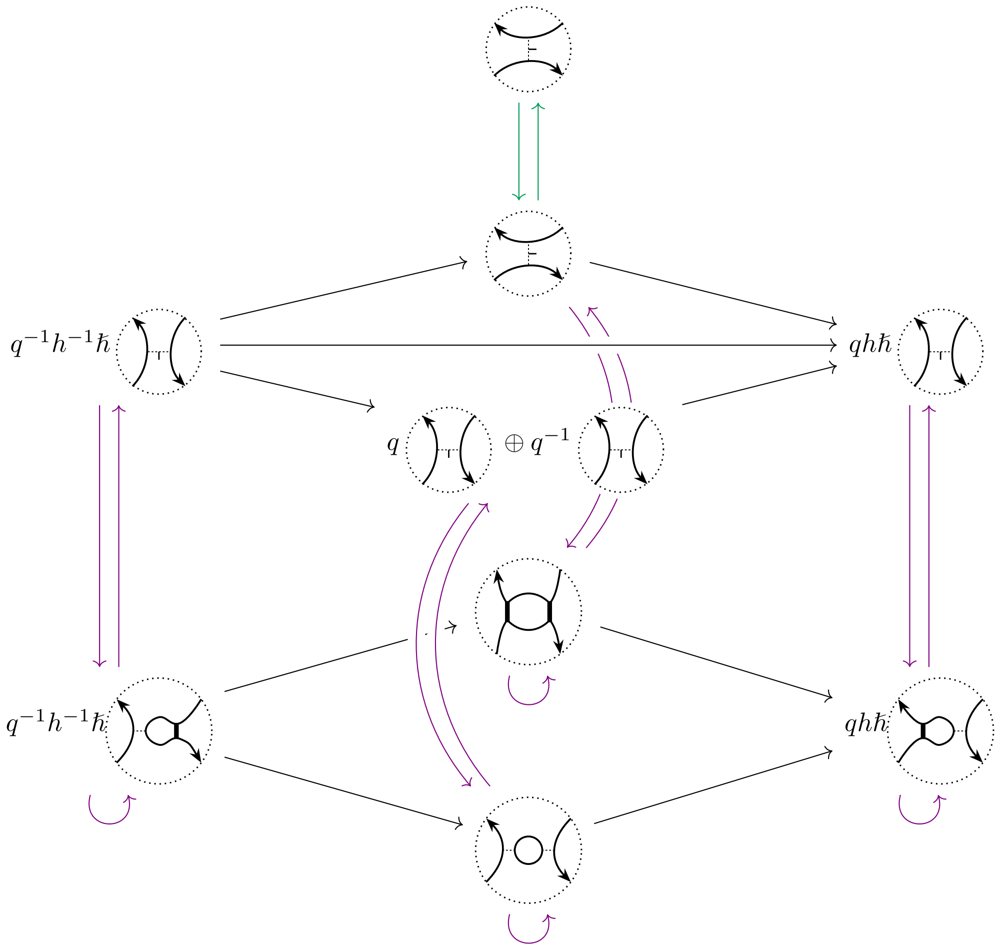



Figure 1: Invariance under positive and negative Reidemeister I moves






**Definition 4.1.** Let $(C, D)$ and $(C', D')$ be multifactorizations. Then a special $n$-deformation retract from $C$ to $C'$ consists of 0-morphisms $P : C \to C'$ and $I : C' \to C$ such that $PI = 1$, together with a $n$-homotopy $H : C \to C$ between 1 and $IP$ such that $HI = 0$, $PH = 0$ and $H^2 = 0$. We represent all this data by

$$(C', D') \underset{P}{\overset{I}{\rightleftarrows}} (C, D) \circlearrowleft H$$

The next lemma tells us how to simplify certain Koszul matrix factorizations into special deformation retracts.

**Lemma 4.2.** *(Adapted from KR How general is R?) Let $R$ be an integral domain. Let $\overline{W} \in R$ and $f, g \in R[y]$ so that $f$ has the form $f = uy^n + \tilde{f}$ wtih $u \in R[y]^\times$ and $\deg_y(\tilde{f}) < n$. Let $M$ be a matrix factorization over $R[y]$ with potential $W = \overline{W} - fg \in R[y]$. Let*

$$\begin{aligned} M' &= M/fM, \\ \overline{M} &= K_{R[y]}(f; g) \otimes_{R[y]} M, \end{aligned}$$

*thought of as matrix factorizations over $R$ with potential $\overline{W}$. Then there is a strong deformation retract*

$$M' \underset{P}{\overset{I}{\rightleftarrows}} \overline{M} \circlearrowleft H$$

*of the form*

$$\begin{tikzcd}
 & M/fM \arrow[dl, bend right, "I_d"'] \arrow[d, shift right, "I_v"'] \\
q^{3-2n}\hbar M \arrow[r, bend left, "f"] & M, \arrow[l, "H"] \arrow[l, bend left, "g"] \arrow[u, shift right, "P"']
\end{tikzcd}$$

*The arrows are as follows:*

in order to keep this result clean, it is better to keep the old convention for Koszul m.f.

- *$P$ is the usual projection.*
- *$H = -\operatorname{Quo}_f$, the negative of the quotient of division by $f$.*
- *$I$ consists of a vertical component $I_v = \operatorname{Res}_f$ given by the residue of division by $f$, and a diagonal component $I_d$ given by the composite $H \circ d_M \circ I_v$.*

carry the modification of the sign of $H$

*Proof.* The identities $HI = 0$, $PH = 0$, $PI = 1$ and $H^2 = 0$ are straightforward. The identity $d_{\overline{M}}H + Hd_{\overline{M}} + 1_{\overline{M}} = IP$ is verified as follows. Take $(x, y) \in q^{3-2n}M\hbar \oplus M = \overline{M}$. We compute directly, keeping in mind that $\operatorname{Res}_f(z) = z - f\operatorname{Quo}_f(z)$ and $\operatorname{Quo}_f(fz) = z$:

$$\begin{aligned}
(d_{\overline{M}}H + Hd_{\overline{M}} + 1_{\overline{M}})(x, y) &= (-\operatorname{Quo}_f(fx) + x - \operatorname{Quo}_f(d_M(y)) - d_M(-\operatorname{Quo}_f(y)), -f\operatorname{Quo}_f(y) + y) \\
&= (-\operatorname{Quo}_f(d_M(y)) + d_M(\operatorname{Quo}_f(y)), \operatorname{Res}_f(y)),
\end{aligned}$$

while

$$\begin{aligned}
IP(x, y) &= (I_d([y]), I_v([y])) \\
&= (-\operatorname{Quo}_f(d_M(\operatorname{Res}_f(y))), \operatorname{Res}_f(y)) \\
&= (-\operatorname{Quo}_f(d_M(y - f\operatorname{Quo}_f(y))), \operatorname{Res}_f(y)) \\
&= (-\operatorname{Quo}_f(d_M(y)) + \operatorname{Quo}_f(fd_M(\operatorname{Quo}_f(y))), \operatorname{Res}_f(y)) \\
&= (-\operatorname{Quo}_f(d_M(y)) + d_M(\operatorname{Quo}_f(y)), \operatorname{Res}_f(y)).
\end{aligned}$$

It is also straightforward to verify that $P$ commutes with differentials. In the case of $I$, we have

$$\begin{aligned}
I(d_{M'}([x])) &= (-\operatorname{Quo}_f(d_M^2(x)) + d_M(\operatorname{Quo}_f(d_M(x))), \operatorname{Res}_f(d_M(x))) \\
&= (d_M(\operatorname{Quo}_f(d_M(x))) - \operatorname{Quo}_f((\overline{W} - fg)x), \operatorname{Res}_f(d_M(x))),
\end{aligned}$$

and

$$\begin{aligned}
d_{\overline{M}}(I([x])) &= d_{\overline{M}}(-\operatorname{Quo}_f(d_M(x)) + d_M(\operatorname{Quo}_f(x)), \operatorname{Res}_f(x)) \\
&= (d_M(\operatorname{Quo}_f(d_M(x))) - d_M^2(\operatorname{Quo}_f(x)) + g\operatorname{Res}_f(x), \\
&\qquad d_M(\operatorname{Res}_f(x)) - f\operatorname{Quo}_f(d_M(x)) + fd_M(\operatorname{Quo}_f(x))) \\
&= (d_M(\operatorname{Quo}_f(d_M(x))) - (\overline{W} - fg)\operatorname{Quo}_f(x) + g(x - f\operatorname{Quo}_f(x)), \\
&\qquad d_M(\operatorname{Res}_f(x) + f\operatorname{Quo}_f(x)) - f\operatorname{Quo}_f(d_M(x))) \\
&= (d_M(\operatorname{Quo}_f(d_M(x))) - \overline{W}\operatorname{Quo}_f(x) + gx, d_M(x) - f\operatorname{Quo}_f(d_M(x))) \\
&= (d_M(\operatorname{Quo}_f(d_M(x))) - \operatorname{Quo}_f((\overline{W} - fg)x), \operatorname{Res}_f(d_M(x))).
\end{aligned}$$

□

In many cases of interest, we additionally have a decomposition $M/fM = \bigoplus_{i=0}^k M_i$ as matrix factorizations over $R$. Such an identification, together with the preceding lemma, will often produce a special deformation retract that simplifies $M$ into a sum of simpler matrix factorizations over a ring involving one less variable.

The following is a straightforward corollary.

**Lemma 4.3.** *In the set up of Lemma 4.2, assume that* $M = K_{R[y]}(\overline{\mathbf{a}}, \overline{\mathbf{b}})$ *and that* $f = y - f_0$, *where* $\deg_y(f_0) = 0$. *Then, under the identification* $M/yM \cong K_R\left(\overline{\mathbf{a}}\big|_{y=f_0}, \overline{\mathbf{b}}\big|_{y=f_0}\right)$, *we have a special deformation retract*

$$\begin{tikzcd}[column sep=large, row sep=large]
 & K_R\left(\overline{\mathbf{a}}\big|_{y=f_0}, \overline{\mathbf{b}}\big|_{y=f_0}\right) \arrow[dl, bend right] \arrow[d, shift right, "1"'] \\
q\hbar K_{R[y]}(\overline{\mathbf{a}}, \overline{\mathbf{b}}) \arrow[r, bend left, "f"] & K_{R[y]}(\overline{\mathbf{a}}, \overline{\mathbf{b}}), \arrow[l, "-\operatorname{Quo}_f"] \arrow[l, bend left, "g"] \arrow[u, shift right, "y \mapsto f_0"']
\end{tikzcd}$$

*The diagonal component of the inclusion is the composite*

$$K_R\left(\overline{\mathbf{a}}\big|_{y=f_0}, \overline{\mathbf{b}}\big|_{y=f_0}\right) \xrightarrow{\sum_{j=1}^k 1\otimes\cdots\otimes\hbar^1(-\operatorname{Quo}_f \circ a_j, -\operatorname{Quo}_f \circ b_j)\otimes\cdots\otimes 1} qK_{R[y]}\left(\overline{\mathbf{a}}, \overline{\mathbf{b}}\right) \xrightarrow{s_\hbar} q\hbar K_{R[y]}\left(\overline{\mathbf{a}}, \overline{\mathbf{b}}\right)$$

This can be generalized

**Lemma 4.4.** *In the set up of Lemma 4.2, assume that* $M = K_{R[y]}(\overline{\mathbf{a}}, \overline{\mathbf{b}})$ *and that* $\deg_y(\operatorname{Res}_f(a_i)) = \deg_y(\operatorname{Res}_f(b_i)) = 0$ *for all* $i$. *Then the map*

that final degree switch in $\hbar$

$$\bigoplus_{i=0}^{n-1} q^{-2i} K_R\left(\operatorname{Res}_f \overline{\mathbf{a}}, \operatorname{Res}_f \overline{\mathbf{b}}\right) \xrightarrow{(y^i)_i} M/fM$$

*is an isomorphism of matrix factorizations over* $R$ *and, combined with Lemma 4.2, gives a special deformation retract*

$$\begin{tikzcd}[column sep=large, row sep=large]
 & \bigoplus_{i=0}^{n-1} q^{-2i} K_R\left(\operatorname{Res}_f \overline{\mathbf{a}}, \operatorname{Res}_f \overline{\mathbf{b}}\right) \arrow[dl, bend right] \arrow[d, shift right, "(y^i)_i"'] \\
q^{3-2n}\hbar K_{R[y]}(\overline{\mathbf{a}}, \overline{\mathbf{b}}) \arrow[r, bend left, "f"] & K_{R[y]}(\overline{\mathbf{a}}, \overline{\mathbf{b}}), \arrow[l, "\operatorname{Quo}_f"] \arrow[l, bend left, "g"] \arrow[u, shift right, "\left(\frac{1}{i!}\frac{\partial^i}{\partial y^i}\big|_{y=0} \circ \operatorname{Res}_f\right)_i"']
\end{tikzcd}$$

*The diagonal component of the inclusion is the composite*

$$\begin{tikzcd}
\bigoplus_{i=0}^{n-1} q^{-2i} K_R\left(\operatorname{Res}_f \overline{\mathbf{a}}, \operatorname{Res}_f \overline{\mathbf{b}}\right) \arrow[r, "\left(\sum_{j=1}^k 1\otimes\cdots\otimes\hbar^1(\operatorname{Quo}_f \circ y^i a_j, \operatorname{Quo}_f \circ y^i b_j)\otimes\cdots\otimes 1\right)_i"] & q^{3-2n} K_{R[y]}\left(\overline{\mathbf{a}}, \overline{\mathbf{b}}\right) \arrow[d, "s_\hbar"] \\
 & q^{3-2n}\hbar K_{R[y]}\left(\overline{\mathbf{a}}, \overline{\mathbf{b}}\right)
\end{tikzcd}$$

### 4.1 MOY moves for rational $_n$

The following explicit instances of the lemma should be thought of as refinements of MOY moves.

Circle removal:

$$\begin{aligned} R &= \mathbb{Q}[\mathsf{G}, x_0 - x_1] \\ M &= R[x_3 - x_0] \\ \overline{W} &= 0 \\ (n+1)(x_3 - x_0)^n - p &= (2x_0 - x_1 - x_3)Z \\ q &= 0 \end{aligned}$$

$$\begin{tikzcd}[column sep=huge]
\bigoplus_{i=0}^{n-1} R \arrow[d, shift right] & \\
R[x_3 - x_0] \arrow[u, shift right] & R[x_3 - x_0] \arrow[l, "H"] \arrow[l, bend left, "(2x_0 - x_1 - x_3)Z"]
\end{tikzcd}$$

Loop removal:

$$
\begin{aligned}
R &= \mathbb{Q}[\mathsf{G}, x_0 - x_1] \\
M &= R[x_3 - x_0] \\
\overline{W} &= 0 \\
(n+1)(x_3 - x_0)^{n-1} - p &= Z \\
q &= 0
\end{aligned}
$$

$$
\begin{tikzcd}
 & \bigoplus_{i=0}^{n-2} R \arrow[d, shift right] \\
R[x_3 - x_0] \arrow[r, bend left, "Z"] & R[x_3 - x_0] \arrow[l, "H"] \arrow[l, bend left, "0"] \arrow[u, shift right]
\end{tikzcd}
$$

Digon removal:

$$
\begin{aligned}
R &= \mathbb{Q}[\mathsf{G}, x_0 - x_1, x_2 - x_1, x_3 - x_2] \\
M &= K(Z(x_0 - x_1, x - x_1, x_3 - x), (x_0 - x)(x_0 + x - x_1 - x_3))_{R[x - x_1]} \\
\tilde{M} &= K(Z(x_0 - x_1, x_2 - x_1, x_3 - x_2), (x_0 - x_2)(x_0 + x_2 - x_1 - x_3))_R \\
\overline{W} &= P(x_0 - x_1) + P(x_3 - x_0) - P(x_2 - x_1) - P(x_3 - x_2) \\
(x - x_1)^2 - p &= (x - x_2)(x + x_2 - x_1 - x_3) \\
q &= Z(x - x_1, x_2 - x_1, x_3 - x_2)
\end{aligned}
$$

$$
\begin{tikzcd}
\tilde{M} \oplus \tilde{M} \arrow[d, shift right] & \\
M \arrow[u, shift right] \arrow[r, bend left, "Z(x - x_1, x_2 - x_1, x_3 - x_2)"] & M \arrow[l, "H"] \arrow[l, bend left, "(x - x_2)(x + x_2 - x_1 - x_3)"]
\end{tikzcd}
$$

Square removal:

$$
\begin{aligned}
R &= \mathbb{Q}[\mathsf{G}, x_0 - x_1, x_2 - x_1, x_2 - x_3] \\
M &= K(Z(x_0 - x_1, x_2 - x_1, x - x_2), (x_0 - x_2)(x_0 + x_2 - x_1 - x))_{R[x - x_0]} \\
M_0 &= K()_R \\
M_1 &= K()_R \\
\overline{W} &= P(x_0 - x_1) - P(x_0 - x_3) - P(x_2 - x_1) + P(x_2 - x_3) \\
(n+1)(x - x_0)^{n-1} - p &= Z(x_2 - x_3, x_0 - x_3, x - x_0) \\
q &= (x_2 - x_0)(x_2 + x_0 - x_3 - x)
\end{aligned}
$$

$$
\begin{tikzcd}
 & M_0 \oplus \bigoplus_{i=0}^{n-3} M_1 \arrow[d, shift right] \\
M \arrow[r, bend left, "Z(x_2 - x_3, x_0 - x_3, x - x_0)"] & M \arrow[l, "H"] \arrow[l, bend left, "(x_2 - x_0)(x_2 + x_0 - x_3 - x)"] \arrow[u, shift right]
\end{tikzcd}
$$

## 4.2 MOY moves in $U(1)$-equivariant $_2$

Let us write some explicit special deformation retracts that can be used for computations and proof of invariance.

## 4.3 Thin edge removal

Lemma 4.3 can be applied to remove dotted edges that bound an interior region.

**Proposition 4.5.** *Let $D$ be a four-ended web with a thin edge $e$ such that its negative region is interior. Denote the corresponding edge ring by $R_D$, and the variables associated to the positive and negative region of $e$ by $x_+$ and $x_-$, respectively. Let $D'$ be the web obtained by removal of $e$, so that its edge ring $R_{D'}$ no longer involves $x_-$. Then there is a special deformation retract of matrix factorizations over $R_{D'}$*

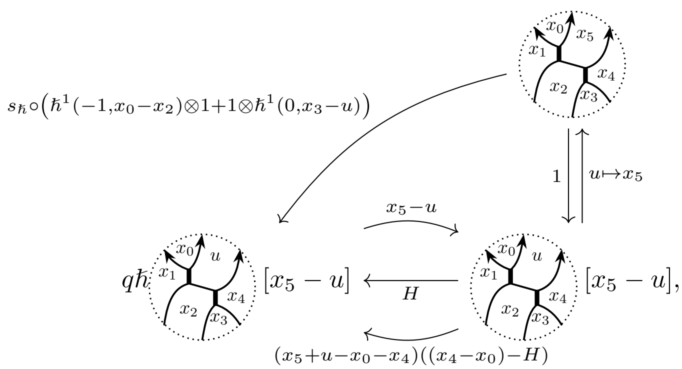
$$M(D') \rightleftarrows M(D) \circlearrowleft$$

*of the form*

$$\begin{tikzcd}
 & M(D') \arrow[dl, bend right, "s_\hbar\circ\sum_j 1\otimes\cdots\otimes\hbar^1(\mathrm{Quo}_f\circ a_j,\mathrm{Quo}_f\circ b_j)\otimes\cdots\otimes 1"'] \arrow[d, shift right, "1"'] \\
q\hbar\widetilde{M}(D) \arrow[r, bend left, "f=x_+-x_-"] & \widetilde{M}(D), \arrow[l, "H"] \arrow[l, bend left, "g"] \arrow[u, shift right, "x_-\mapsto x_+"']
\end{tikzcd}$$

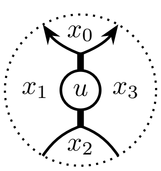
*where $\widetilde{M}(D)$ is the Koszul factorization over $R_D$ obtained from all edges in $D$ except $e$. The only contributions to the sum come from edges with at least one end adjacent to the $x_-$-region.*

A similar statement is true if the positive region is interior and the variable $x_+$ is eliminated instead. Here is an example, which will be of use later.



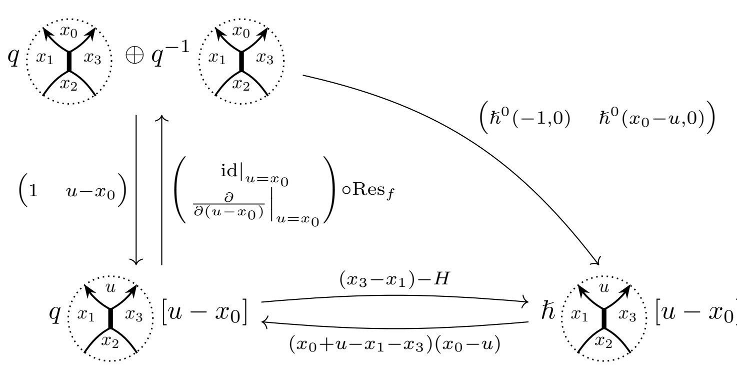

### 4.3.1 Bigon removal

In the case of bigon removal, we have




$$= q^2 K_{R[u-x_0]}\begin{pmatrix}(x_0+u-x_1-x_3)(x_0-u) & (x_3-x_1)-H\\ (u+x_2-x_1-x_3)(u-x_2) & (x_3-x_1)-H\end{pmatrix}$$

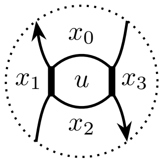
Letting $y=u-x_0$, $f=(u+x_2-x_1-x_3)(u-x_2)$ and $M=K_{R[u-x_0]}((x_0+u-x_1-x_3)(x_0-u)\ ;\ (x_3-x_1)-H)$, we are under the hypotheses of Lemma 4.4. In particular, we have a strong deformation retract




these maps still need some checking, esp. the

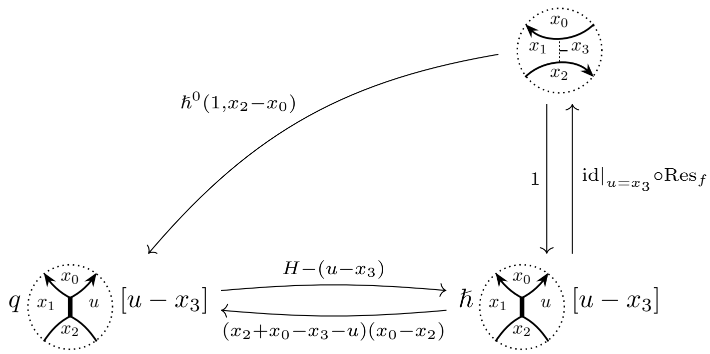
### 4.3.2 Thick edge removal

For square removal, we have

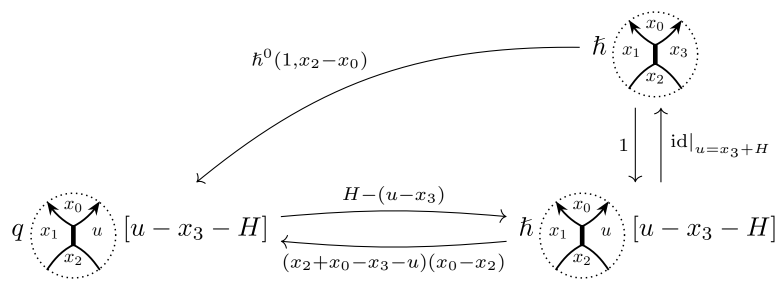


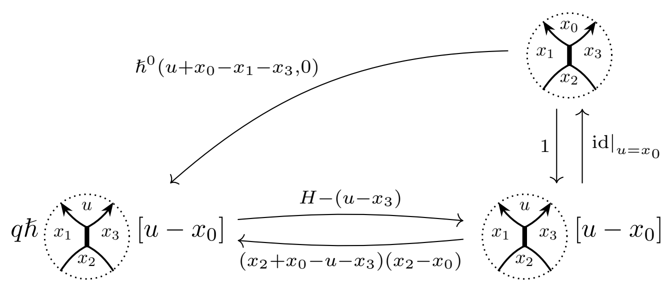

$$= K_{R[u-x_0]}\begin{pmatrix} H-(u-x_1) & (x_0+x_2-x_1-u)(x_2-x_0) \\ H-(u-x_3) & (x_2+x_0-x_3-u)(x_0-x_2) \end{pmatrix}$$

Letting $y = u - x_3$, $f = H - (u - x_3)$ and $M = K_{R[u-x_3]}(H - (u - x_1)\ ;\ (x_0 + x_2 - x_1 - u)(x_2 - x_0))$, we are under the hypotheses of Lemma 4.3. Thus, we have a strong deformation retract




i accidentally ruined the horizontal arrows in the one below




**Proposition 4.6.** *Suppose*

$$(C', d') \underset{p}{\overset{i}{\rightleftarrows}} (C, d) \circlearrowleft h$$

*is a special 1-deformation retract between matrix multifactorizations $C'$ and $C$. Let $D$ be another differential in $C$ such that its vertical part coincides with $d$, i.e. $D_0 = d_0$. Then there exists a special 1-deformation retract*

$$(C', D') \underset{P}{\overset{I}{\rightleftarrows}} (C, D) \circlearrowleft H\,.$$

*such that $D'_0 = d'_0$. If $h$ is also a 0-homotopy, so that the original data is a special 0-deformation retract, then we also get $P_0 = p_0$, $I_0 = i_0$ and $H_0 = h_0$*

*Proof.* (Essentially same as Ballinger) Let

$$A = (1-(D-d)h)^{-1}(D-d) = \sum_{i=0}^{\infty} ((D-d)h)^i (D-d).$$

Notice that $D-d$ raises filtration by at least 1, while $h$ lowers it by at most 1. Thus, $A$ raises filtration degree by at least 1. Take $D' = d' + pAi$, $P = p + pAh$, $I = i + hAi$ and $H = h + hAh$. See (Crainic) for calculations. Notice that the term $pAi$ raises filtration by at least 1. In case $h$ is a 1-homotopy then $pAh$ and $hAi$ preserve filtration and $hAh$ lowers filtration by at most 1, but if $h$ is a 0-homotopy then all these terms raise filtration by at least 1. ☐

The following lemma allows us to replace certain subfactorizations in a matrix factorization by their special retracts.

**Lemma 4.7.** *Let $(C, D)$ be a matrix multifactorization of the form $C = A \oplus B$ and $D = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, so that*

- *$(A, \alpha)$ and $(B, \delta)$ are subfactorizations (satisfied e.g. if $\beta\gamma = 0$ and $\gamma\beta = 0$),*
- *$\beta_0 = 0$ and*
- *$\gamma_0 = 0$.*

*Suppose we have special deformation retracts*

$$\begin{tikzcd} (A', \epsilon) \arrow[r, shift left, "i_1"] & (A, \alpha) \arrow[l, shift left, "p_1"] \arrow[loop right, "h_1"] \end{tikzcd} \qquad \begin{tikzcd} (B', \eta) \arrow[r, shift left, "i_2"] & (B, \delta) \arrow[l, shift left, "p_2"] \arrow[loop right, "h_2"] \end{tikzcd}$$

*in which $h_1$ and $h_2$ are 1-homotopies. Then there exists a special deformation retract*

$$\begin{tikzcd} (A' \oplus B', D') \arrow[r, shift left, "I"] & (C, D) \arrow[l, shift left, "P"] \arrow[loop right, "H"] \end{tikzcd}$$

*in which $H$ is a 1-homotopy and $D'_0 = \epsilon_0 \oplus \eta_0$.*

*Proof.* Apply Proposition 4.6 to the special deformation retract

$$\begin{tikzcd} (A' \oplus B', \epsilon \oplus \eta) \arrow[r, shift left, "i_1 \oplus i_2"] & (A \oplus B, \alpha \oplus \delta) \arrow[l, shift left, "p_1 \oplus p_2"] \arrow[loop right, "h_1 \oplus h_2"] \end{tikzcd}.$$

Here $d = \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}$ and $D = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. Since $\beta_0 = 0$ and $\gamma_0 = 0$, the hypothesis $D_0 = d_0$ is satisfied.

In this situation we have the explicit formula

$$A = \sum_{i=0}^{\infty} \begin{pmatrix} \beta h_2 \gamma & \beta \\ \gamma & \gamma h_1 \beta \end{pmatrix} \begin{pmatrix} (h_1 \beta h_2 \gamma)^i & 0 \\ 0 & (h_2 \gamma h_1 \beta)^i \end{pmatrix}.$$

$$D' = \begin{pmatrix} \epsilon & p\beta \\ \gamma i & \delta + \gamma h \beta \end{pmatrix}, \quad P = \begin{pmatrix} p & 0 \\ \gamma h & 1 \end{pmatrix}, \quad I = \begin{pmatrix} i & h\beta \\ 0 & 1 \end{pmatrix}, \quad H = \begin{pmatrix} h & 0 \\ 0 & 0 \end{pmatrix}.$$

Since $\beta_0 = 0$ and $\gamma_0 = 0$ and $h_1$ and $h_2$ can lower filtration degree at most by 1, thus it is easy to verify that $A_0 = 0$. This implies $D'_0 = \epsilon_0 \oplus \eta_0$. ☐

For the first Reidemeister move, we start with the following multifactorization over the ring $R = \mathbb{Q}[x_0 - x_1]$:

$$\begin{tikzcd} R[x_3 - x_0] \arrow[drr, bend right] & & R[x_3 - x_0] \arrow[ll, bend right, "(2x_0 - x_1 - x_3)Z"'] \arrow[dll, bend left] \\ R[x_3 - x_0] \arrow[rr, bend right, "Z"'] & & R[x_3 - x_0], \end{tikzcd}$$

where $Z = Z(x_0 - x_1, x_0 - x_1, x_3 - x_0)$ Here the cubical filtration degree increases downwards. We can apply Lemma 4.7 to obtain a special 0-deformation retract into a multifactorization $\bigoplus_{i=0}^{n-1} R \to \bigoplus_{i=0}^{n-2} R$, with the differential described by a matrix

$$\begin{pmatrix} 1 & & & a_1 \\ & \ddots & & \vdots \\ & & 1 & a_{n-1} \end{pmatrix}$$

where, up to rescaling, the $a_i$ appear as coefficients of lower order terms in $Z$ with respect to the variable $x_3 - x_0$. The differential increases filtration degree 1, so the contractible summand $\bigoplus_{i=0}^{n-2} R \xrightarrow{1} \bigoplus_{i=0}^{n-2} R$ can be cancelled out by a 1-nulhomotopy. After this, we are left with a single copy of $R$.

A similar argument applies to reduce

$$\begin{tikzcd} R[x_3 - x_0] \arrow[r, bend left, "Z"] \arrow[dr] & R[x_3 - x_0] \arrow[dl] \\ R[x_3 - x_0] & R[x_3 - x_0], \arrow[l, bend left, "(2x_0 - x_1 - x_3)Z"] \end{tikzcd}$$

into $R$ by a special 1-deformation retract.

# 5 Delooping of four-ended webs