Invariance of Bar-Natan matrix multifactorizations up to 1-homotopy equivalence

Tomas Mejia Gomez

November 14, 2025

Abstract

We verify link invariance of a certain construction using matrix factorizations over $\mathbb{Z}[H]$.

1 Matrix factorization conventions

Given two matrix factorizations of the form

$$M = [\ \hbar A \rightleftarrows B\ ], \qquad M' = [\ \hbar A' \rightleftarrows B'\ ],$$

with A, A', B and B' concentrated in $\hbar$-degree 0, we often specify morphisms by spelling out their components in each $\hbar$-degree as follows:

$$\begin{tikzcd} M \arrow[d, "\hbar^0(\alpha,\beta)"'] \\ M' \end{tikzcd} = \begin{tikzcd} {[\,\hbar A} \arrow[r, shift left] \arrow[d, "\alpha"] & B\,] \arrow[l, shift left] \arrow[d, "\beta"] \\ {[\,\hbar A'} \arrow[r, shift left] & B'\,] \arrow[l, shift left] \end{tikzcd} \qquad \begin{tikzcd} M \arrow[d, "\hbar^1(\alpha,\beta)"'] \\ M' \end{tikzcd} = \begin{tikzcd} {[\,\hbar A} \arrow[r, shift left] \arrow[dr, "\alpha"] & B\,] \arrow[l, shift left] \arrow[dl, "\beta"'] \\ {[\,\hbar A'} \arrow[r, shift left] & B'\,] \arrow[l, shift left] \end{tikzcd}$$

Many useful maps can be easily expressed using this notation. In the case $M' = \hbar M$, we will often use the map

$$s_\hbar = \hbar^1(1,1) : M \to \hbar M.$$

1.1 Koszul matrix factorizations

For Koszul matrix factorizations, we will take the convention

$$K_R\big(a \quad b\big) = \left[\ q^{\frac{1}{2}(\deg_q(a)-\deg_q(b))}\hbar R \underset{b}{\overset{a}{\rightleftarrows}} R\ \right].$$

As a module, we think of $K_R\big(a \quad b\big)$ as having a basis $\left\{\widehat{a}, \widehat{b}\right\}$, with $\widehat{a}$ in degree $q^{\frac{1}{2}(\deg_q(a)-\deg_q(b))}\hbar$ and $\widehat{b}$ in degree q^0, so that the differential can be written as

$$d_{K_R(\,a\ b\,)} = \begin{pmatrix} 0 & b \\ a & 0 \end{pmatrix} = \hbar^1(a,b)$$

More generally, given sequences $\overline{\mathbf{a}} = (a_1, \dots, a_r)^t$ and $\overline{\mathbf{b}} = (b_1, \dots, b_r)^t$, we have an associated Koszul factorization

$$K_R\big(\overline{\mathbf{a}} \quad \overline{\mathbf{b}}\big) = \bigotimes_i K_R\big(a_i \quad b_i\big).$$

For any shift of a Koszul matrix factorization, say $q^k h^j \hbar^\sigma K_R\big(\overline{\mathbf{a}} \quad \overline{\mathbf{b}}\big)$, the choice of $\overline{\mathbf{a}}$ and $\overline{\mathbf{b}}$ determines a basis of the entire tensor product: $\left\{\theta_1 \otimes \cdots \otimes \theta_r \mid \theta_i = \widehat{a}_i \text{ or } \widehat{b}_i\right\}$, where the degree of $\widehat{b}_1 \otimes \cdots \otimes \widehat{b}_r$ is $q^k h^j \hbar^\sigma$, while the other basis elements are shifted according to which $\widehat{a}_i$ appear in a given basis element.

Given $p, q \in R$, and two shifted Koszul factorizations of the same length, we will often use maps

$$\hbar_i^\sigma(p,q) : q^k h^j \hbar^\sigma K_R\big(\overline{\mathbf{a}} \quad \overline{\mathbf{b}}\big) \longrightarrow q^{k'} h^{j'} \hbar^{\sigma'} K_R\Big(\overline{\mathbf{a}}' \quad \overline{\mathbf{b}}'\Big)$$

that amount to applying $\hbar^\sigma(p,q)$ at the i-th tensor component. Let us use our basis to make this precise. Here we use the same notation θ_i for the bases on the domain and codomain.

$$\hbar_i^0(p,q)(\theta_1 \otimes \cdots \otimes \theta_r) = \begin{cases} p\theta_1 \otimes \cdots \otimes \widehat{a}_i \otimes \cdots \otimes \theta_r, & \text{if } \theta_i = \widehat{a}_i \\ q\theta_1 \otimes \cdots \otimes \widehat{b}_i \otimes \cdots \otimes \theta_r, & \text{if } \theta_i = \widehat{b}_i \end{cases}$$

$$\hbar_i^1(p,q)(\theta_1 \otimes \cdots \otimes \theta_r) = \begin{cases} (-1)^{\sum_{j<i} \deg_\hbar(\theta_j)} p\theta_1 \otimes \cdots \otimes \widehat{b}_i \otimes \cdots \otimes \theta_r, & \text{if } \theta_i = \widehat{a}_i \\ (-1)^{\sum_{j<i} \deg_\hbar(\theta_j)} q\theta_1 \otimes \cdots \otimes \widehat{a}_i \otimes \cdots \otimes \theta_r, & \text{if } \theta_i = \widehat{b}_i \end{cases}$$

The sign in $\hbar_i^1(p,q)(\theta_1 \otimes \cdots \otimes \theta_r)$ can also be formulated as the parity of $|\{j \mid j < i, \theta_j = \widehat{a}_j\}|$.

With this notation, the differential can be written as

$$d_{q^i h^j \hbar^\sigma K_R(\overline{\mathbf{a}}\ \overline{\mathbf{b}})} = (-1)^\sigma \sum_{i=1}^{r} \hbar_i^1(a_i, b_i).$$

We have following properties, which follow easily from the definitions:

Lemma 1.1. *The following identities hold.*

$$\begin{aligned} &\hbar_i^\sigma(p,q)\hbar_j^\rho(c,d) = (-1)^{\sigma\rho}\hbar_j^\rho(c,d)\hbar_i^\sigma(p,q), && i \neq j \\ &\hbar_i^\sigma(c,d)\hbar_i^0(p,q) = \hbar_i^\sigma(cp,dq) \\ &\hbar_i^\sigma(p,q)\hbar_i^1(c,d) = \hbar_i^{1-\sigma}(qc,pd) \end{aligned}$$

Remark 1.2. The definition of $\hbar_i^\sigma(p,q)$ depends on the basis, which ultimately depends on the $\overline{\mathbf{a}}$ and $\overline{\mathbf{b}}$ used to present the matrix factorization. For example, the same matrix factorization could be written in two different ways:

$$K_R\big(f \quad g\big) = q^{\frac{1}{2}(\deg_q(f) - \deg_q(g))} \hbar K_R\big(-g \quad -f\big),$$

and the roles of $\widehat{a}$ and $\widehat{b}$ switch from one expression to the other. What we would denote $\hbar_1^\sigma(p,q)$ with domain $K_R\big(f \quad g\big)$, becomes $\hbar_1^{1-\sigma}(q,p)$ when the domain is $q^{\frac{1}{2}(\deg_q(f) - \deg_q(g))} \hbar K_R\big(-g \quad -f\big)$ instead.

Lemma 1.3. *Let $\overline{\mathbf{a}}'$ and $\overline{\mathbf{b}}'$ be related to $\overline{\mathbf{a}}$ and $\overline{\mathbf{b}}$ by*

$$a'_j = \begin{cases} a_j, & j \neq i \\ -b_i, & j = i, \end{cases} \qquad b'_j = \begin{cases} b_j, & j \neq i \\ -a_i, & j = i, \end{cases}$$

Then we have an isomorphism

$$\mathrm{id} = \hbar_i^1(1,1) : K_R\big(\overline{\mathbf{a}} \quad \overline{\mathbf{b}}\big) \longrightarrow q^{\frac{1}{2}(\deg_q(a_i) - \deg_q(b_i))} K_R\Big(\overline{\mathbf{a}}' \quad \overline{\mathbf{b}}'\Big).$$

When tensoring a Koszul factorization $K_R\,\big(a \quad b\big)$ with some other matrix factorization M, we sometimes write

$$K_R\big(a \quad b\big) \otimes_R M = \left[\ q^{\frac{1}{2}(\deg_q(a) - \deg_q(b))} \hbar M \underset{b}{\overset{a}{\rightleftarrows}} M\ \right]. \tag{1}$$

This indicates that the tensor product splits as a free R-module into two copies of M, the first one with shifted q and $\hbar$ degrees. In terms of this decomposition, the total differential takes the form

$$d_{K_R(\,a\ b) \otimes M} = \begin{pmatrix} -d_M & b \\ a & d_M \end{pmatrix},$$

where the negative sign comes precisely from the $\hbar$-shift.

Let write down some useful isomorphisms.

Lemma 1.4. *Let $\lambda \in R$. The following are isomorphisms with matrix factorizations.*

$$l_{i,j}(\lambda) = \mathrm{id} + \lambda \hbar_i^1(1,0)\hbar_j^1(1,0) : K_R\big(\overline{\mathbf{a}} \quad \overline{\mathbf{b}}\big) \longrightarrow K_R\big(\overline{\mathbf{a}} - \lambda b_i \overline{\mathbf{e}}_j + \lambda b_j \overline{\mathbf{e}}_i \quad \overline{\mathbf{b}}\big)$$
$$m_{i,j}(\lambda) = \mathrm{id} + \lambda \hbar_i^1(0,1)\hbar_j^1(1,0) : K_R\big(\overline{\mathbf{a}} \quad \overline{\mathbf{b}}\big) \longrightarrow K_R\big(\overline{\mathbf{a}} - \lambda a_i \overline{\mathbf{e}}_j \quad \overline{\mathbf{b}} + \lambda b_j \overline{\mathbf{e}}_i\big)$$

Proof. First, we verify that $l_{i,j}(\lambda)$ is a morphism of matrix factorizations by verifying

$$l_{i,j}(\lambda) \sum_k \hbar_k^1(a_k, b_k) = \left(\lambda(\hbar_i^1(b_j, 0) - \hbar_j^1(b_i, 0)) + \sum_k \hbar_k^1(a_k, b_k) \right) l_{i,j}(\lambda)$$

Since $l_{i,j}(\lambda)$ commutes with $\sum_{k \neq i,j} \hbar_k^1(a_k, b_k)$, it reduces to

$$l_{i,j}(\lambda) \left(\hbar_i^1(a_i, b_i) + \hbar_j^1(a_j, b_j)\right) = \left(\hbar_i^1(a_i + \lambda b_j, b_i) + \hbar_j^1(a_j - \lambda b_i, b_j)\right) l_{i,j}(\lambda),$$

and expanding both sides and cancelling identical terms reduces once again to

$$\begin{aligned} \lambda \hbar_i^1(1,0)\hbar_j^1(1,0) \left(\hbar_i^1(a_i, b_i) + \hbar_j^1(a_j, b_j)\right) = {} & \left(\hbar_i^1(a_i + \lambda b_j, b_i) + \hbar_j^1(a_j - \lambda b_i, b_j)\right) \lambda \hbar_i^1(1,0)\hbar_j^1(1,0) \\ & + \hbar_i^1(\lambda b_j, 0) + \hbar_j^1(-\lambda b_i, 0) \end{aligned}$$

Using the formulas in Lemma 1.1, the left and right hand side become

$$\lambda \left(-\hbar_i^1(0, b_i)\hbar_j^1(1,0) + \hbar_i^1(1,0)\hbar_j^1(0, b_i)\right) = \lambda \left(\hbar_i^1(b_i, 0)\hbar_j^1(1,0) - \hbar_i^1(1,0)\hbar_j^1(b_j, 0) + \hbar_i^1(b_j, 0) - \hbar_j^1(b_i, 0)\right).$$

This last identity is true because

$$\begin{aligned} -\hbar_i^1(0,1)\hbar_j^1(1,0) &= \hbar_i^1(1,0)\hbar_j^1(1,0) - \hbar_j^1(1,0) \\ \hbar_i^1(1,0)\hbar_j^1(0,1) &= -\hbar_i^1(1,0)\hbar_j^1(1,0) + \hbar_i^1(1,0) \end{aligned}$$

To prove that $l_{i,j}(\lambda)$ is an isomorphism, we simply observe that $l_{i,j}(\lambda)^{-1} = l_{i,j}(-\lambda)$. The case of $m_{i,j}$ is completely analogous. $\square$

Lemma 1.5. *If $\overline{\mathbf{a}}'$ and $\overline{\mathbf{b}}'$ are obtained from $\overline{\mathbf{a}}$ and $\overline{\mathbf{b}}$ respectively by swapping the i-th and j-th entries: $a_i \leftrightarrow a_j$ and $b_i \leftrightarrow b_j$. Then we have a swap isomorphism $\tau_{i,j} : K_R\big(\overline{\mathbf{a}} \quad \overline{\mathbf{b}}\big) \to K_R\left(\overline{\mathbf{a}}' \quad \overline{\mathbf{b}}'\right)$ of the form*

$$\tau_{i,j} = \hbar_i^0(1,0)\hbar_j^0(1,0) + \hbar_i^0(0,1)\hbar_j^0(0,1) + \hbar_i^1(1,0)\hbar_j^1(0,1) - \hbar_i^1(0,1)\hbar_j^1(1,0).$$

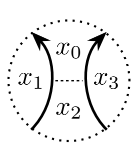
If we want to switch the sign of differentials in M without altering the $\hbar$ degree, we write $-M$. Using these notations, we have

not sure if I need this

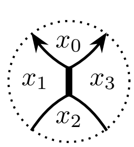
$$\hbar(M \otimes N) = (\hbar M) \otimes N = (-M) \otimes (\hbar N).$$

2 Matrix factorizations associated to webs

In the following definitions, the base ring is $\mathbf{K} = \mathbb{Z}[H]$, and the elementary matrix factorizations are over the ring $R = \mathbf{K}[x_0 - x_1, x_2 - x_1, x_3 - x_2]$. The variables x_i and H live in degree q^{-2}.

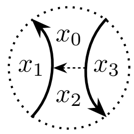
$D_0 =$ 

$$\longmapsto \mathcal{M}(D_0) = \left[\; q\hbar R \underset{(x_0+x_2-x_1-x_3)((x_3-x_1)-H)}{\overset{x_0-x_2}{\rightleftarrows}} R \;\right]$$
$$= K_R\big(x_0 - x_2 \quad (x_0 + x_2 - x_1 - x_3)((x_3 - x_1) - H)\big)$$

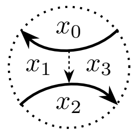
$D_1 =$ 

$$\longmapsto \mathcal{M}(D_1) = \left[\; \hbar R \underset{(x_3-x_1)-H}{\overset{(x_0+x_2-x_1-x_3)(x_0-x_2)}{\rightleftarrows}} qR \;\right]$$
$$= qK_R\big((x_0 + x_2 - x_1 - x_3)(x_0 - x_2) \quad (x_3 - x_1) - H\big)$$

When working locally, other boundary orientations are also possible. Thus, we also define

$D_2 =$ 

$$\longmapsto \mathcal{M}(D_2) = \left[\; q\hbar R \underset{(x_0+x_2-x_1-x_3-H)(x_3-x_1)}{\overset{x_0-x_2}{\rightleftarrows}} R \;\right]$$

$D_3 =$ 

$$\longmapsto \mathcal{M}(D_3) = \left[\; q\hbar R \underset{(x_0+x_2-x_1-x_3-H)(x_2-x_0)}{\overset{x_1-x_3}{\rightleftarrows}} R \;\right]$$

In all types of thin edges, there is a *negative region* whose variable appears with a negative sign in the linear term of the associated matrix factorization. Thus, all thin and thick edges come with an orientation, so that the negative region is the one on the left relative to such orientation. In the case of D_0 and D_1, there are already canonical orientations and in these cases the arrows are omitted. orientations for D_0 and D_1

For D_2 and D_3, there is a 180 degree rotational symmetry preventing a canonical choice of orientation, so the arrow is always indicated.

Notice that R is the subring of the polynomial ring in all face variables generated by differences across edges. We refer to R as the *edge ring*. The subring R' generated by differences across boundary-adjacent edges is called the *boundary ring*, and $R = R'$ if and only if there are no internal faces. We extend this nomenclature to more general webs in the expected way. The particular case $R' = \mathbf{K}[x_0 - x_1, x_2 - x_1, x_3 - x_2]$ is refered to as the *standard boundary ring*. We will often omit the specification of the edge or boundary ring when face variables are already indicated in the web diagram.

Now, let Γ be an arbitrary connected web on the disk. By this, we mean an embedded directed graph, considered up to isotopy, with 1-valent vertices only possibly at the boundary of the disk, and interior 3-valent vertices only possibly at the interior. Each 3-valent vertex must have a neighborhood looking like one of the D_i, possibly with a reversal of thin or thick edge orientation. We associate a matrix factorization

$$\mathcal{M}(\Gamma) = \bigotimes_e \mathcal{M}(D_e)$$

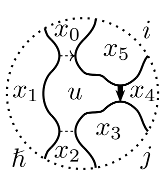
over the corresponding edge ring R_Γ. The tensor product runs over all thin and thick edges of Γ and depends, in principle, on an ordering of these edges. However, the operation is symmetric monoidal, which gives canonical isomorphisms between different choices of thin/thick edge orderings.

For convenience, we often represent Koszul matrix factorizations associated to webs by the web itself with some decorations, namely

- labels indicating the region variables and
- surrounding indices indicating overall degree shifts.

There is an implicit convention for the ordering of the thin and thick edges: an edge on top comes before an edge on bottom, and then left comes before right if their centers are in the same horizontal line.

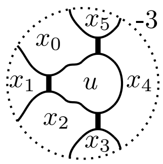
For instance, if Γ is the web with one thin edge and two thick edges as depicted, then

$q^i h^j \hbar \mathcal{M}(\Gamma) =$ 

$$= q^{i+1} h^j \hbar K_{R[u-x_1]} \begin{pmatrix} -(x_0 - u) & -(x_0 + u - x_5 - x_1)((x_5 - x_1) - H) \\ -(x_5 + x_3 - x_4 - u)(x_5 - x_3) & -((x_4 - u) - H) \\ (u - x_2) & (u + x_2 - x_3 - x_1)((x_3 - x_1) - H) \end{pmatrix}$$

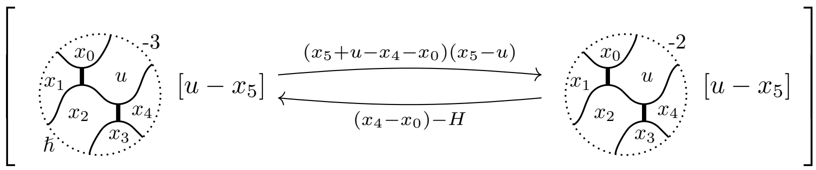
as a matrix factorization with boundary ring $R' = \mathbf{K}[x_0 - x_1, x_2 - x_1, x_3 - x_2, x_4 - x_3, x_4 - x_5]$ and edge ring $R = R'[u - x_1]$. The additional q^1 shift comes from the shift already present in the definition of the matrix factorization associated to the thick edge.

Example 2.1. According to our conventions, we have

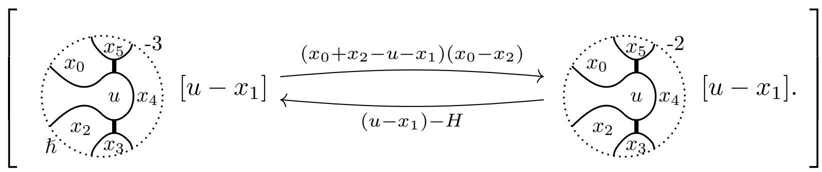
$N_1 :=$ 

$$= K_{R[u-x_1]} \begin{pmatrix} (x_5+u-x_4-x_0)(x_5-u) & (x_4-x_0)-H \\ (x_0+x_2-u-x_1)(x_0-x_2) & (u-x_1)-H \\ (u+x_3-x_4-x_2)(u-x_3) & (x_4-x_2)-H \end{pmatrix}$$

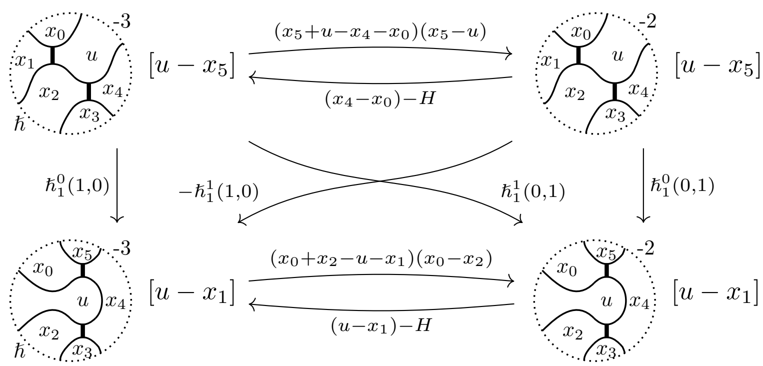
This is the Koszul matrix factorization obtained by tensoring the contributions of each of the three thick edges, with a correcting shift by q^{-3}. Expanding as in (1), we can write

$N_1 =$ 

Let use this type of expansion to represent the swap map between the first two rows of N_1. Denoting by N_2 the matrix factorization obtained after the swap, we can write

$N_2 :=$ 

The swap map $\tau_{1,2} : N_1 \to N_2$ is



3 Web simplifications

In this section, we explain how to express matrix factorizations associated to more complex webs in terms of simpler webs. For the most part, this entails the reduction of webs that have internal faces into webs with less or no internal faces. In particular, these are situations where the edge ring R is larger than the boundary ring R'. Example 2.1, for instance, involved matrix factorizations N_i over $R = R'[u - x_1]$, where u is the variable labeling the internal face. Since the potential itself is an element of the boundary ring R' (all the terms involving u cancel out), it is possible to think of N_i as matrix factorizations over R', albeit one of infinite rank.

The following lemma gives us a recipe to obtain a finite rank special deformation retract in that scenario by eliminating the first row and taking a quotient.

Lemma 3.1. *(Adapted from KR How general is R?) Let S be an integral domain and y a variable in quantum degree q^{-2}. Let $\overline{W} \in S$ and $f, g \in S[y]$ so that f has the form $f = uy^n + \tilde{f}$ witih $u \in S^\times$ and $\deg_y(\tilde{f}) < n$. Let g have degree q^{-2m}. Let M be a matrix factorization over $S[y]$ with potential $W = \overline{W} - fg \in S[y]$. Denote*

$$M' = M/fM,$$
$$\overline{M} = K_{S[y]}\big(f \quad g\big) \otimes_{S[y]} M,$$

thought of as matrix factorizations over S with potential $\overline{W}$. Then there is a strong deformation retract

$$M' \underset{P}{\overset{I}{\rightleftarrows}} \overline{M} \circlearrowleft K$$

of the form

$$\begin{tikzcd}
 & M/fM \arrow[dl, bend right, "I_d"'] \arrow[d, shift right, "I_v"'] \\
q^{m-n}\hbar M \arrow[r, bend left, "f"] & M. \arrow[u, shift right, "P"'] \arrow[l, "K"] \arrow[l, bend left, "g"]
\end{tikzcd}$$

The arrows are as follows:

- *P is the usual projection.*
- *$K = -\operatorname{Quo}_f$, the negative of the quotient of division by f.*
- *I consists of a vertical component $I_v = \operatorname{Res}_f$ given by the residue of division by f, and a diagonal component I_d given by the composite $-\operatorname{Quo}_f \circ d_M \circ I_v$.*

Proof. The identities $KI = 0$, $PK = 0$, $PI = 1$ and $K^2 = 0$ are straightforward. The identity

$$d_{\overline{M}}K + Kd_{\overline{M}} + 1_{\overline{M}} = IP$$

is verified as follows. Take $(x, y) \in q^{m-n}\hbar M \oplus M = \overline{M}$. We compute directly, keeping in mind that $\operatorname{Res}_f(z) = z - f\operatorname{Quo}_f(z)$ and $\operatorname{Quo}_f(fz) = z$:

$$\begin{aligned}
(d_{\overline{M}}K + Kd_{\overline{M}} + 1_{\overline{M}})(x, y) &= (-\operatorname{Quo}_f(fx) + x - \operatorname{Quo}_f(d_M(y)) - d_M(-\operatorname{Quo}_f(y)), -f\operatorname{Quo}_f(y) + y) \\
&= (-\operatorname{Quo}_f(d_M(y)) + d_M(\operatorname{Quo}_f(y)), \operatorname{Res}_f(y)),
\end{aligned}$$

while

$$\begin{aligned}
IP(x, y) &= (I_d([y]), I_v([y])) \\
&= (-\operatorname{Quo}_f(d_M(\operatorname{Res}_f(y))), \operatorname{Res}_f(y)) \\
&= (-\operatorname{Quo}_f(d_M(y - f\operatorname{Quo}_f(y))), \operatorname{Res}_f(y)) \\
&= (-\operatorname{Quo}_f(d_M(y)) + \operatorname{Quo}_f(fd_M(\operatorname{Quo}_f(y))), \operatorname{Res}_f(y)) \\
&= (-\operatorname{Quo}_f(d_M(y)) + d_M(\operatorname{Quo}_f(y)), \operatorname{Res}_f(y)).
\end{aligned}$$

It is also straightforward to verify that P commutes with differentials. In the case of I, we have

$$\begin{aligned}
I(d_{M'}([x])) &= (-\operatorname{Quo}_f(d_M^2(x)) + d_M(\operatorname{Quo}_f(d_M(x))), \operatorname{Res}_f(d_M(x))) \\
&= (d_M(\operatorname{Quo}_f(d_M(x))) - \operatorname{Quo}_f((\overline{W} - fg)x), \operatorname{Res}_f(d_M(x))),
\end{aligned}$$

and

$$\begin{aligned}
d_{\overline{M}}(I([x])) &= d_{\overline{M}}(-\operatorname{Quo}_f(d_M(x)) + d_M(\operatorname{Quo}_f(x)), \operatorname{Res}_f(x)) \\
&= (-d_M(-\operatorname{Quo}_f(d_M(x))) - d_M^2(\operatorname{Quo}_f(x)) + g\operatorname{Res}_f(x), \\
&\qquad d_M(\operatorname{Res}_f(x)) - f\operatorname{Quo}_f(d_M(x)) + fd_M(\operatorname{Quo}_f(x))) \\
&= (d_M(\operatorname{Quo}_f(d_M(x))) - (\overline{W} - fg)\operatorname{Quo}_f(x) + g(x - f\operatorname{Quo}_f(x)), \\
&\qquad d_M(\operatorname{Res}_f(x) + f\operatorname{Quo}_f(x)) - f\operatorname{Quo}_f(d_M(x))) \\
&= (d_M(\operatorname{Quo}_f(d_M(x))) - \overline{W}\operatorname{Quo}_f(x) + gx, d_M(x) - f\operatorname{Quo}_f(d_M(x))) \\
&= (d_M(\operatorname{Quo}_f(d_M(x))) - \operatorname{Quo}_f((\overline{W} - fg)x), \operatorname{Res}_f(d_M(x))).
\end{aligned}$$

□

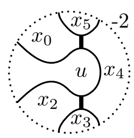
Remark 3.2. There is a slight variation in which f appears in the second column

$$\overline{M} = K_{S[y]}\begin{pmatrix} g & f \end{pmatrix} \otimes_{S[y]} M.$$

In this case, we have $M' = q^{n-m}\hbar M/fM$, and the strong defomation retract has the form

$$\begin{tikzcd}
q^{n-m}\hbar M/fM \arrow[d, shift right, "I_v"'] \arrow[drr, bend left, "I_d"] & & \\
q^{n-m}\hbar M \arrow[u, shift right, "P"'] \arrow[rr, bend left, "g"] \arrow[rr, "K"'] & & M, \arrow[ll, bend left, "f"]
\end{tikzcd}$$

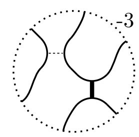
Example 3.3. Let us return to example **??** and apply the lemma to the matrix factorization N_2. Let us denote $y = u - x_1$, $f = (u - x_1) - H$ and

$M =$  $[u - x_1] = K_R\begin{pmatrix} (x_5 + u - x_4 - x_0)(x_5 - u) & (x_4 - x_0) - H \\ (u + x_3 - x_4 - x_2)(u - x_3) & (x_4 - x_2) - H \end{pmatrix},$

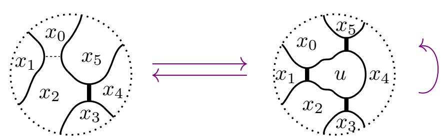
so that $N_2 = K_R\begin{pmatrix} g & f \end{pmatrix} \otimes_R M$. Thus, N_2 admits a deformation retract of the form

$$q^{-1}\hbar M/fM = q^{-1}\hbar K_{R'}\begin{pmatrix} (x_5 + x_1 + H - x_4 - x_0)(x_5 - H - x_1) & (x_4 - x_0) - H \\ (x_1 + H + x_3 - x_4 - x_2)(x_1 + H - x_3) & (x_4 - x_2) - H \end{pmatrix}.$$

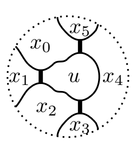
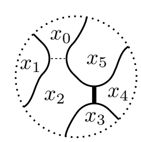
Now that we have a smaller factorization M/fM over a smaller ring R', we would like to identify it as the matrix factorization associated to a web. We can do this through some elementary operations:

$q^{-1}\hbar M/fM \xrightarrow{\xi := \hbar_1^1(1,1)\circ l_{1,2}(x_5 - x_1 - H)\circ m_{1,2}(-1)}$ 

We have enough formulas to calculate an explicit strong deformation retract



but for our purposes, only the retraction(right-to-left) will be needed in this example. This a composite

 $\xrightarrow{\tau_{1,2}} q^3 N_2 \xrightarrow{u \mapsto x_1 + H} q^2\hbar M/fM \xrightarrow{\xi}$ 

We simplified a Koszul matrix factorization by killing a linear polynomial in this example. This procedure will be applied often, so we record it as a more general lemma.

Lemma 3.4. (Simplification of Koszul factorizations by a linear term) *In the set up of Lemma 3.1, assume that* $M = K_{R[y]}\begin{pmatrix} \overline{\mathbf{a}} & \overline{\mathbf{b}} \end{pmatrix}$ *and that* $f = y - f_0$*, where* $\deg_y(f_0) = 0$*. Then, under the identification* $M/yM \cong K_R\begin{pmatrix} \overline{\mathbf{a}}\big|_{y=f_0} & \overline{\mathbf{b}}\big|_{y=f_0} \end{pmatrix}$*, we have a special deformation retract*

$$\begin{tikzcd}
 & & K_R\begin{pmatrix} \overline{\mathbf{a}}\big|_{y=f_0} & \overline{\mathbf{b}}\big|_{y=f_0} \end{pmatrix} \arrow[dll, bend right, "-\sum_{j=1}^{k} \hbar_j^1(\mathrm{Quo}_f \circ a_j, \mathrm{Quo}_f \circ b_j)"'] \arrow[d, shift right, "1"'] \\
q^{m-1}\hbar K_{R[y]}\begin{pmatrix} \overline{\mathbf{a}} & \overline{\mathbf{b}} \end{pmatrix} \arrow[rr, bend left, "f"] & & K_{R[y]}\begin{pmatrix} \overline{\mathbf{a}} & \overline{\mathbf{b}} \end{pmatrix}, \arrow[ll, "-\mathrm{Quo}_f"'] \arrow[ll, bend left, "g"] \arrow[u, shift right, "y \mapsto f_0"']
\end{tikzcd}$$

where g has degree q^{-2m}.

In other cases of interest, we have a decomposition into several summands $M/fM = \bigoplus_{i=0}^{k} M_i$ as matrix factorizations over R, where M_i are simpler factorizations of a desirable form, e.g. coming from simpler webs. The following lemma, a higher order version of 3.4, gives such decompositions in favorable cases.

In the following, we introduce the shorthand

$$P^i_{f(y)} = \frac{\partial^i}{\partial y^i}\bigg|_{y=0} \circ \operatorname{Res}_f,$$

which is more succintly described as the map that takes the coefficient of y^i after taking the residue of division by $f(y)$.

Lemma 3.5. (Simplification of Koszul factorizations by a higher order polynomial) *In the set up of Lemma 3.1, assume that $M = K_{R[y]}\big(\overline{\mathbf{a}} \quad \overline{\mathbf{b}}\big)$ and that $\deg_y(\operatorname{Res}_f(a_i)) = \deg_y(\operatorname{Res}_f(b_i)) = 0$ for all i. Then the map*

$$\bigoplus_{i=0}^{n-1} q^{-2i} K_R\big(\operatorname{Res}_f \overline{\mathbf{a}} \quad \operatorname{Res}_f \overline{\mathbf{b}}\big) \xrightarrow{(y^i)_i} M/fM$$

is an isomorphism of matrix factorizations over R and, combined with Lemma 3.1, gives a special deformation retract

$$\begin{tikzcd}
 & \bigoplus_{i=0}^{n-1} q^{-2i} K_R\big(\operatorname{Res}_f \overline{\mathbf{a}} \qquad \operatorname{Res}_f \overline{\mathbf{b}}\big) \arrow[dl, bend right, "-\left(\sum_{j=1}^{k} \hbar^1_j(\operatorname{Quo}_f \circ y^i a_j, \operatorname{Quo}_f \circ y^i b_j)\right)_i"'] \arrow[d, shift right, "(y^i)_i"'] \\
q^{m-n}\hbar K_{R[y]}\big(\overline{\mathbf{a}} \qquad \overline{\mathbf{b}}\big) \arrow[r, bend left, "f"] & K_{R[y]}\big(\overline{\mathbf{a}} \qquad \overline{\mathbf{b}}\big), \arrow[l, "-\operatorname{Quo}_f"] \arrow[l, bend left, "g"] \arrow[u, shift right, "\left(P^i_{f(y)}\right)_i"']
\end{tikzcd}$$

3.1 Thin edge removal

Simplification by linear polynomials 3.4 can be directly applied to remove thin edges that bound an interior region.

Proposition 3.6. *Let D be a four-ended web with a thin edge e such that its negative region is interior. Denote the corresponding edge ring by R_D, and the variables associated to the positive and negative region of e by x_+ and x_-, respectively. Let D' be the web obtained by removal of e, so that its edge ring $R_{D'}$ no longer involves x_-. Then there is a special deformation retract of matrix factorizations over $R_{D'}$*

$$M(D') \rightleftarrows M(D) \circlearrowright$$

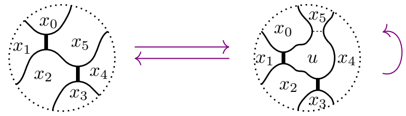
of the form

$$\begin{tikzcd}
 & M(D') \arrow[dl, bend right, "-\left(\sum_j \hbar^1_j(\operatorname{Quo}_f a_j, \operatorname{Quo}_f b_j)\right)"'] \arrow[d, shift right, "1"'] \\
q\hbar\widetilde{M}(D) \arrow[r, bend left, "f=x_+-x_-"] & \widetilde{M}(D), \arrow[l, "K"] \arrow[l, bend left, "g"] \arrow[u, shift right, "x_-\mapsto x_+"']
\end{tikzcd}$$

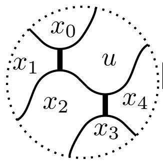
where $\widetilde{M}(D)$ is the Koszul factorization over R_D obtained from all edges in D except e. The only contributions to the sum come from edges with at least one end adjacent to the x_--region.

A similar statement is true if the positive region is interior and the variable x_+ is eliminated instead.

Example 3.7. In the proof of the Reidemeister 3 move we will need a special deformation retract of the form



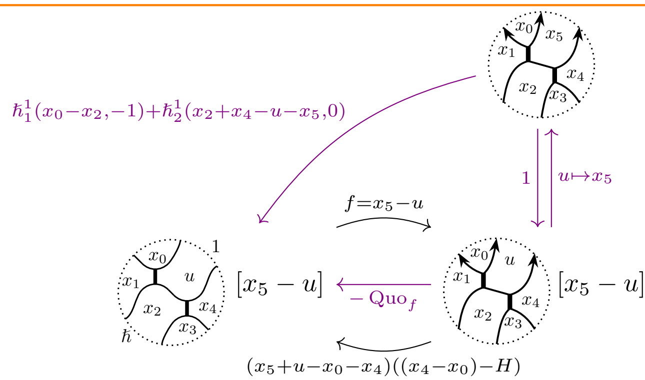
removing the thin edge that separates $x_+ = x_5$ from $x_- = u$. According to the expression above, the retraction map will involve the evaluation of $-\operatorname{Quo}_{x_5-u}$ at the various components of



$$[x_5 - u] = q^2 K_{R[x_5-u]} \begin{pmatrix} (x_0 + x_2 - x_1 - u)(x_0 - x_2) & (u - x_1) - H \\ (u + x_3 - x_2 - x_4)(u - x_3) & (x_4 - x_2) - H \end{pmatrix}.$$

The result is

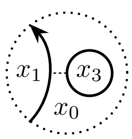
signs need double check here



.

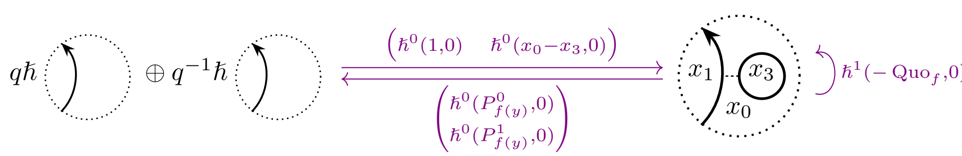
3.2 Circle removal (delooping)

The usual delooping consists of removing the circle in the local picture



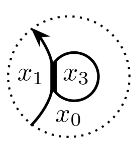
$$= q\hbar K_{R[x_0-x_3]}\big((2x_0 - x_1 - x_3)(H - (x_3 - x_1)) \quad 0\big),$$

where the boundary ring is $R = \mathbf{K}[x_0 - x_1]$. By simplifying with respect to the polynomial $f(y) = (2x_0 - x_1 - x_3)((x_3 - x_1) - H)$ thought of as quadratic in the variable $y = x_0 - x_3$, we are left with the special deformation retract



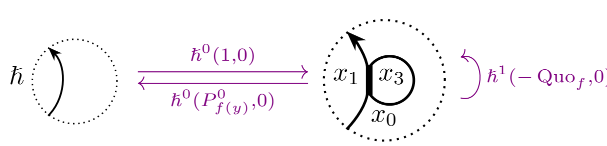
(2)

There is also a version of delooping for thick edges



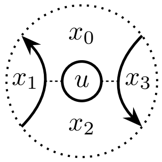
$$= \hbar K_{R[x_0-x_3]}\big(H - (x_3 - x_1) \quad 0\big).$$

We simplify with respect to $f(y) = H - (x_3 - x_1)$ which is linear in $y = x_0 - x_3$, to get a special deformation retract



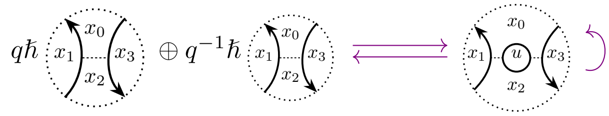
(3)

Loops may appear in more complicated ways, and there are different ways to deal with them. The following two situations appear in Reidemeister II move, where R is back to being the standard boundary ring. First, we have



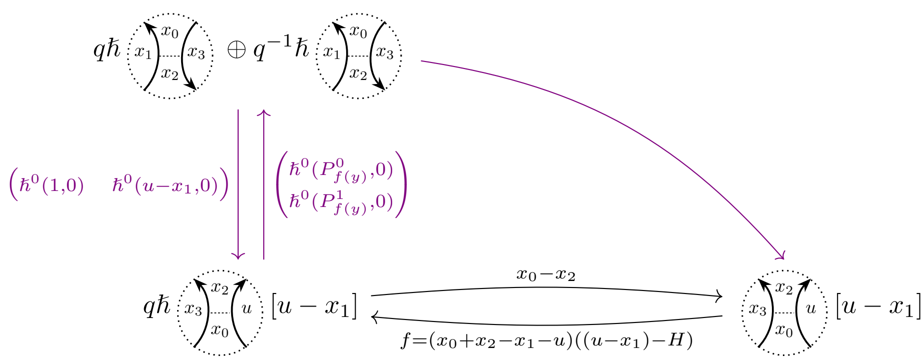
$$= q^2 K_{R[u-x_1]}\begin{pmatrix}(x_0+x_2-x_1-u)((u-x_1)-H) & x_0-x_2\\ (x_2+x_0-u-x_3)(H-(u-x_3)) & x_0-x_2\end{pmatrix}.$$

It is possible to reduce this case to (2), but we opt for the direct approach of simplifying with respect to the polynomial $f(y)=(x_0+x_2-x_1-u)((u-x_1)-H)$, thought of as quadratic in the variable $y=u-x_1$, to obtain a special deformation retract

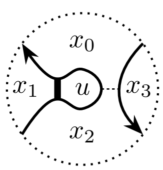


(4)

given by

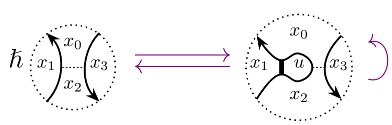


Similarly, we also have



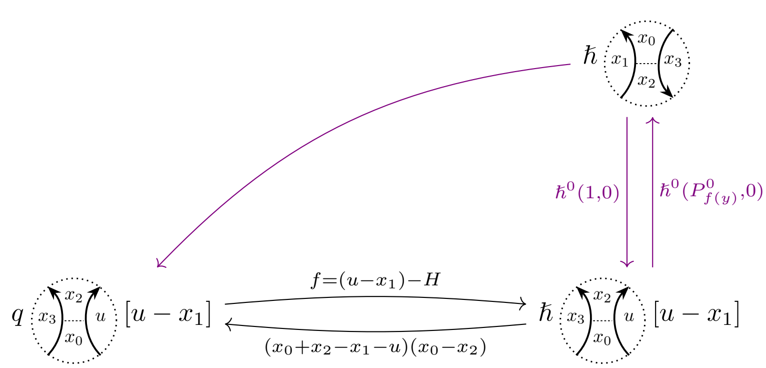
$$= \hbar K_{R[u-x_1]}\begin{pmatrix}H-(u-x_1) & (x_0+x_2-x_1-u)(x_2-x_0)\\ x_2-x_0 & (x_2+x_0-u-x_3)((u-x_3)-H)\end{pmatrix},$$

which we simplify with respect to the linear polynomial $f(y)=(u-x_1)-H$ in the variable $y=u-x_1$ to obtain



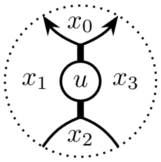
(5)

given by



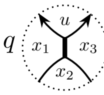
3.3 Bigon and square removal

The local picture

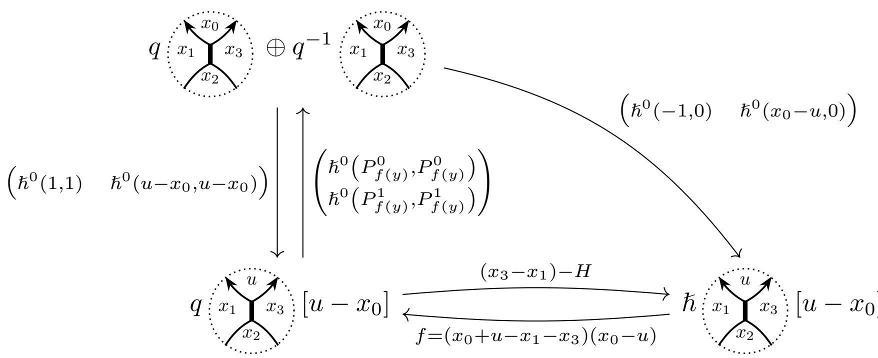


$$= q^2 K_{R[u-x_0]} \begin{pmatrix} (x_0+u-x_1-x_3)(x_0-u) & (x_3-x_1)-H \\ (u+x_2-x_1-x_3)(u-x_2) & (x_3-x_1)-H \end{pmatrix}$$

is known as a *bigon*. Letting $y = u - x_0$, $f = (x_0 + u - x_1 - x_3)(x_0 - u)$ and

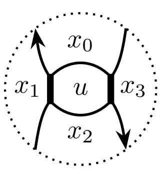
$$M = q^2 K_{R[u-x_0]} \big((u+x_2-x_1-x_3)(u-x_2) \quad (x_3-x_1)-H \big) = q$$  $$[u-x_0],$$

we are under the hypotheses of the higher order simplification lemma 3.5. In particular, we have a strong deformation retract



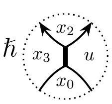
For square removal, we have

these maps still need some checking, esp. the diagonal

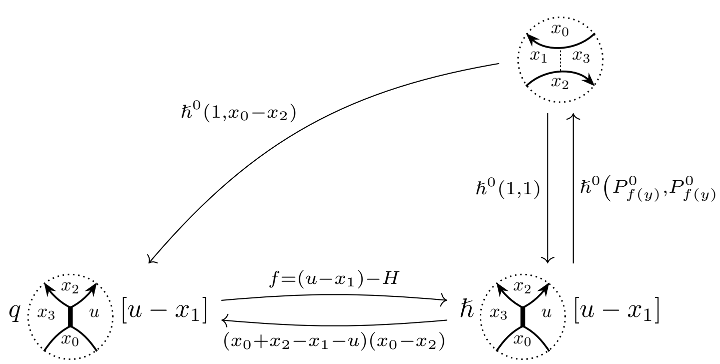


$$= K_{R[u-x_1]} \begin{pmatrix} (u-x_1)-H & (x_0+x_2-x_1-u)(x_0-x_2) \\ H-(u-x_3) & (x_2+x_0-x_3-u)(x_0-x_2) \end{pmatrix}$$

Letting $y = u - x_1$, $f = (u - x_1) - H$ and

$$M = K_{R[u-x_1]} \big(H-(u-x_3) \quad (x_2+x_0-x_3-u)(x_0-x_2) \big) = \hbar$$  $$[u-x_1],$$

we are under the hypotheses of the linear simplification lemma 3.4. Thus, we have a strong deformation retract



i accidentally ruined the horizontal arrows in

4 Multifactorizations and special deformation retracts

Definition 4.1. Let (C, D) and (C', D') be multifactorizations. Then a special n-deformation retract from C to C' consists of 0-morphisms $P : C \to C'$ and $I : C' \to C$ such that $PI = 1$, together with a n-homotopy $H : C \to C$ between 1 and IP such that $HI = 0$, $PH = 0$ and $H^2 = 0$. We often represent this data by a triple (I, P, H) or, more suggestively, by a diagram

define multifactorization and n-stuff, including the meaning of filtered degree $\geq n$

$$(C', D') \underset{P}{\overset{I}{\rightleftarrows}} (C, D) \circlearrowleft H \,.$$

Proposition 4.2. *Suppose*

$$(C', d') \underset{p}{\overset{i}{\rightleftarrows}} (C, d) \circlearrowleft h$$

is a special 1-deformation retract between matrix multifactorizations C' and C. Let D be another differential in C such that its vertical part coincides with d, i.e. $D_0 = d_0$. Then there exists a special 1-deformation retract

$$(C', D') \underset{P}{\overset{I}{\rightleftarrows}} (C, D) \circlearrowleft H \,.$$

such that $D'_0 = d'_0$. If h is also a 0-homotopy, so that the original data is a special 0-deformation retract, then the deformed data is also a special 0-deformation retract with the additional properties $P_0 = p_0$, $I_0 = i_0$ and $H_0 = h_0$.

Proof. (Essentially same as Ballinger) Let

$$A = (1 - (D - d)h)^{-1}(D - d) = \sum_{i=0}^{\infty}((D - d)h)^i(D - d).$$

Notice that $D - d$ has filtered degree ≥ 1, while h has filtered degree ≥ -1. In total, A has filtered degree ≥ 1. Take

$$\begin{aligned} D' &= d' + pAi \\ P &= p + pAh \\ I &= i + hAi \\ H &= h + hAh \end{aligned} \tag{6}$$

See (Crainic) for calculations. Notice that the term pAi, i.e. the deformation of the differential, has filtered degree ≥ 1, which guarantees $D'_0 = d'_0$. In the case in which h is a 1-homotopy then pAh and hAi preserve filtration, while hAh has filtered degree ≥ -1, which are necessary conditions for the deformed data (I, P, H) to give a special 1-deformation retract. If h is additionally a 0-homotopy, then all these terms have now filtered degree ≥ 1. In such case, the deformed data (I, P, H) gives a special 0-deformation retract of the desired form. $\square$

The following lemma allows us to replace certain subfactorizations in a matrix factorization by their special retracts.

Lemma 4.3. *Let (C, D) be a matrix multifactorization of the form $C = A \oplus B$ and $D = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, so that*

- *(A, α) and (B, δ) are subfactorizations (i.e. $\beta\gamma = 0$ and $\gamma\beta = 0$),*
- *$\beta_0 = 0$ and*
- *$\gamma_0 = 0$.*

Suppose we have special 1-deformation retracts

$$(A', \epsilon) \underset{p_1}{\overset{i_1}{\rightleftarrows}} (A, \alpha) \circlearrowleft h_1 \,, \qquad (B', \eta) \underset{p_2}{\overset{i_2}{\rightleftarrows}} (B, \delta) \circlearrowleft h_2 \,.$$

Then there exists a special 1-deformation retract

$$(A' \oplus B', D') \underset{P}{\overset{I}{\rightleftarrows}} (C, D) \circlearrowleft H$$

with $D'_0 = \epsilon_0 \oplus \eta_0$*.*

Proof. Apply Proposition 4.2 to the special deformation retract

$$(A' \oplus B', \epsilon \oplus \eta) \underset{p_1 \oplus p_2}{\overset{i_1 \oplus i_2}{\rightleftarrows}} (A \oplus B, \alpha \oplus \delta) \circlearrowleft h_1 \oplus h_2 \,.$$

Here $d = \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}$ and $D = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. Since $\beta_0 = 0$ and $\gamma_0 = 0$, the hypothesis $D_0 = d_0$ is satisfied.

$$A = \sum_{i=0}^{\infty} \begin{pmatrix} \beta h_2 \gamma & \beta \\ \gamma & \gamma h_1 \beta \end{pmatrix} \begin{pmatrix} (h_1 \beta h_2 \gamma)^i & 0 \\ 0 & (h_2 \gamma h_1 \beta)^i \end{pmatrix}. \tag{7}$$

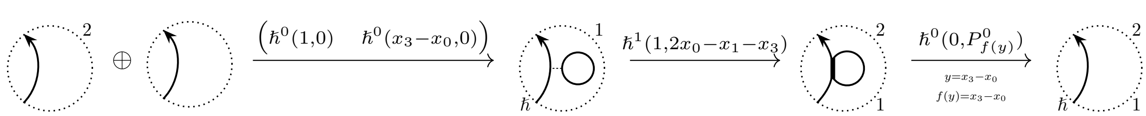
Since $\beta_0 = 0$ and $\gamma_0 = 0$ and h_1 and h_2 have filtered degree ≥ -1, it follows that $A_0 = 0$. This implies $D'_0 = \epsilon_0 \oplus \eta_0$. $\square$

We will mostly encounter situations in which β, γ, h_1 or h_2, or short composites thereof, vanish. The expression (7) simplifies significantly in those cases, which in turn gives manageable formulas in (6).

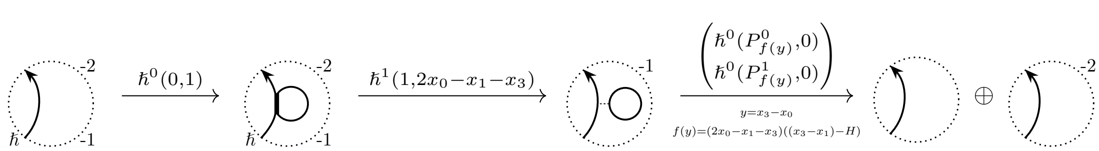
5 Invariance

The invariance of $M(D)$ for a link diagram D under each Reidemeister move will proceed by looking at the explicit matrix multifactorization associated to the relevant local piece of the diagram and simplifying it in two steps. In the first step, one simplifies each resolving web into smaller webs with no internal faces, and then extends into a special 0-deformation retract of the entire multifactorization. There is enough control of the deformation data to compute explicit differentials in the resulting multifactorization, and in particular to identify some identity components in d_1. The second step then consists of simplifying along such identities to obtain a special 1-deformation retract into the desired form.

Reidemeister I moves are illustrated in Figure 1. The purple arrows represent special deformation retracts of vertical matrix factorizations, where each component is obtained from Lemma 3.4 or Lemma 3.5. We then extend, by Lemma 4.2, to a special 0-deformation retract of multifactorizations. Notice that we indicate the explicit form of the d_1-component in each of the once-simplified multifactorizations. These are computed as the composites

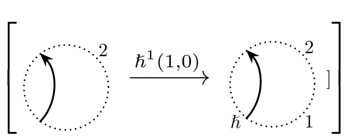


in the case of the positive Reidemeister I move, and the composite

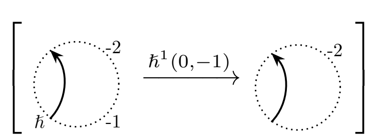


for the negative Reidemeister I move.

The summands



and



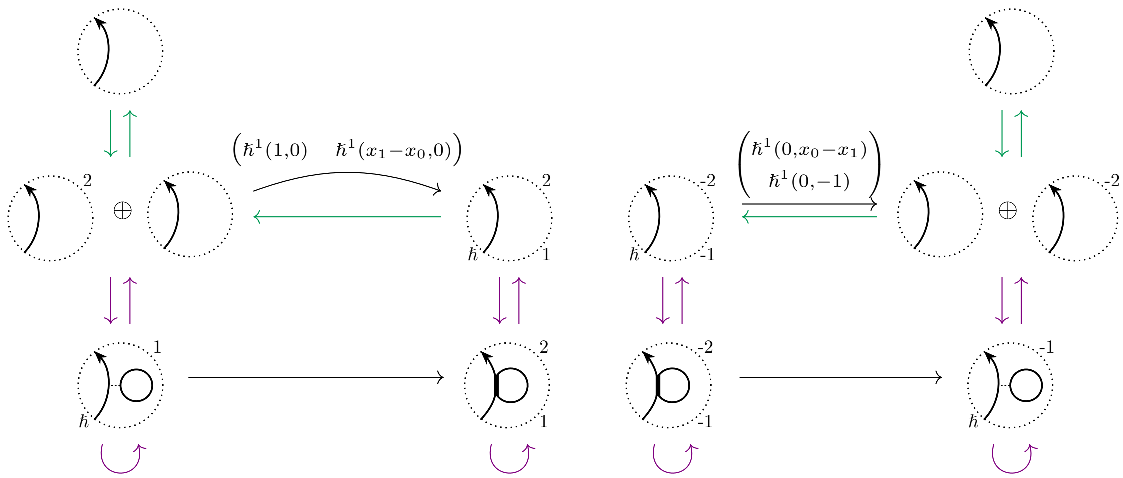


Figure 1: Invariance under positive and negative Reidemeister I moves

are easily verified to be 1-contractible. The corresponding special 1-deformation retractions cancelling these summands are represented by the green arrows.

The same recipe works for Reidemeister II moves. The special 0-deformation retracts represented by violet arrows still come from L

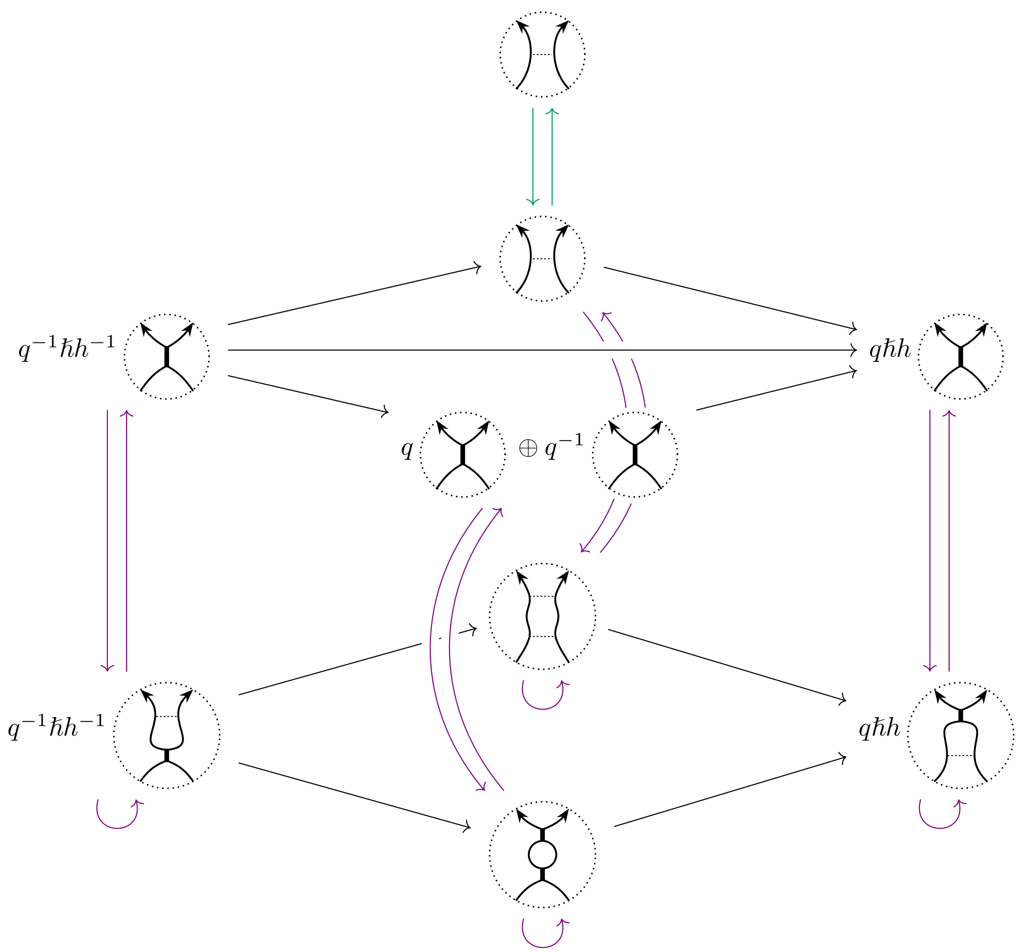


Figure 2: Invariance under positive and negative Reidemeister IIa move

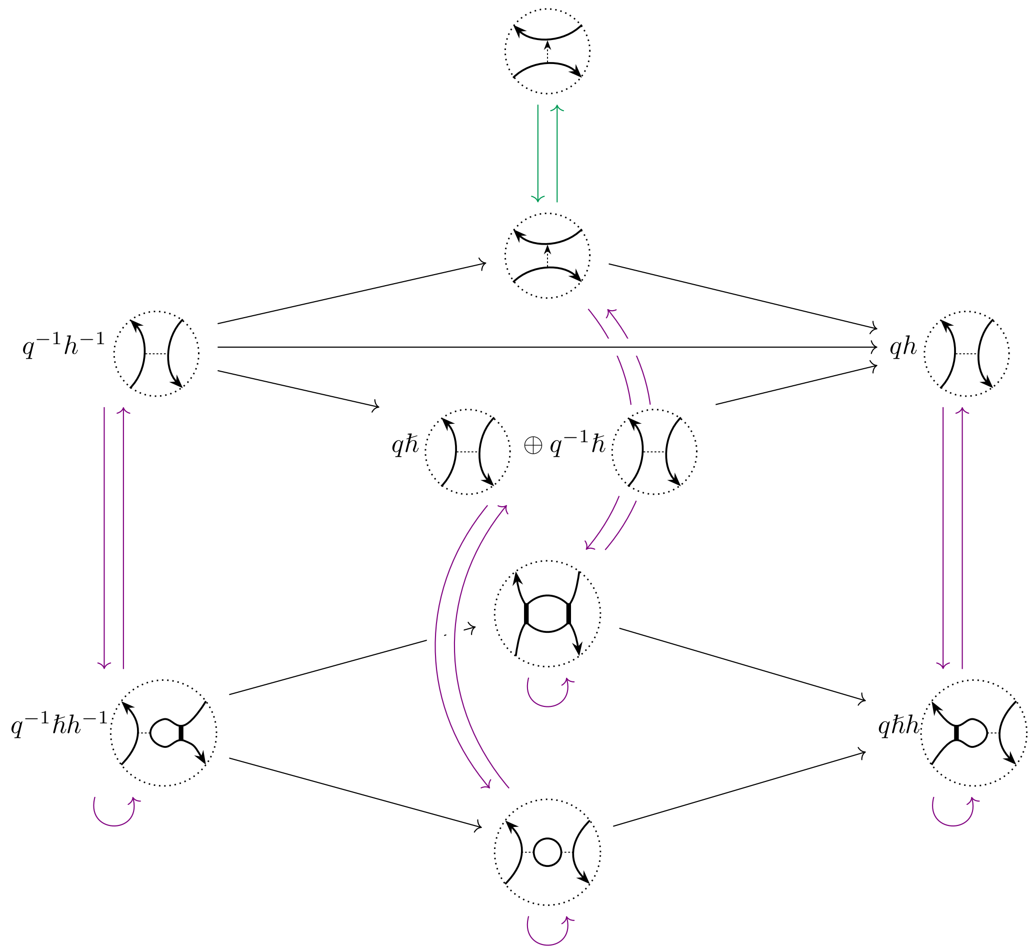


Figure 3: Invariance under Reidemeister IIb move

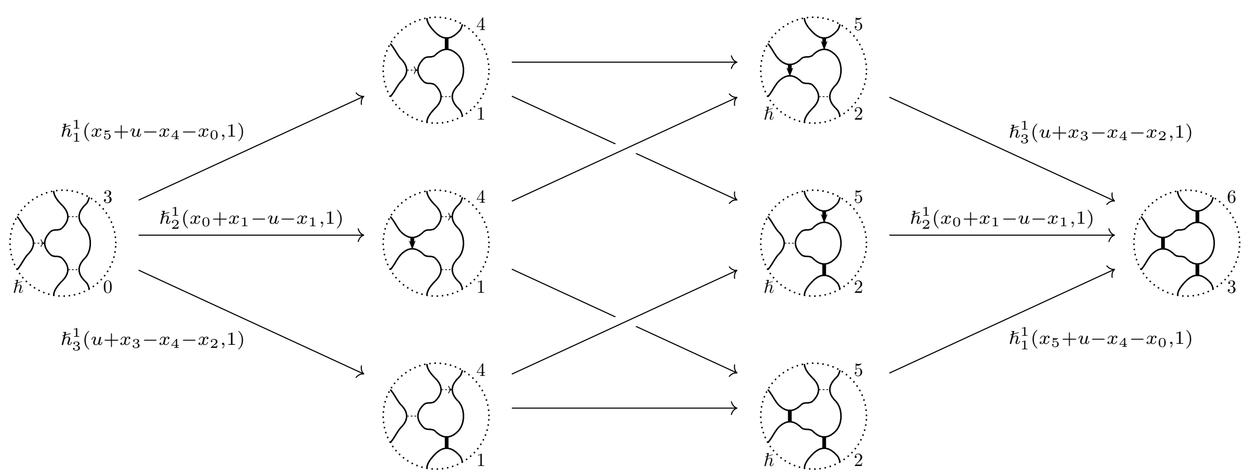


Figure 4: Invariance under Reidemeister III move

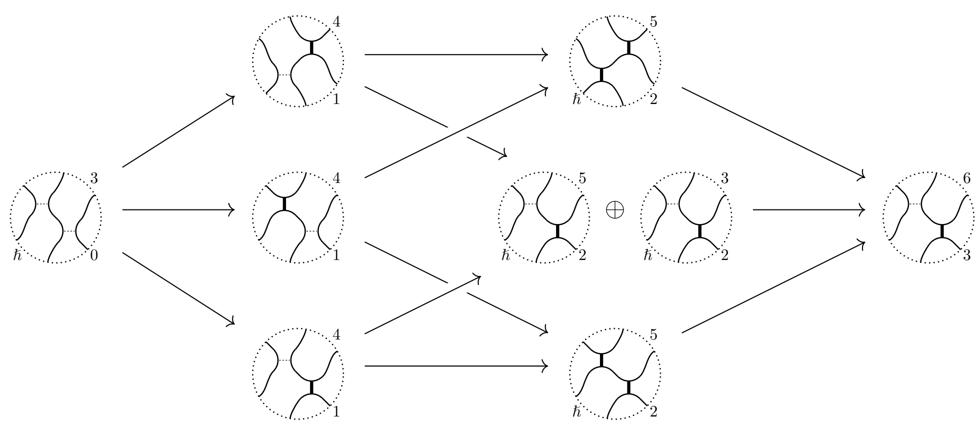


Figure 5: Invariance under Reidemeister III move