Contents

Example-3: Two Inequality Constraints
(Please Check Lecture 11 last part for Ex 1 and Ex 2)

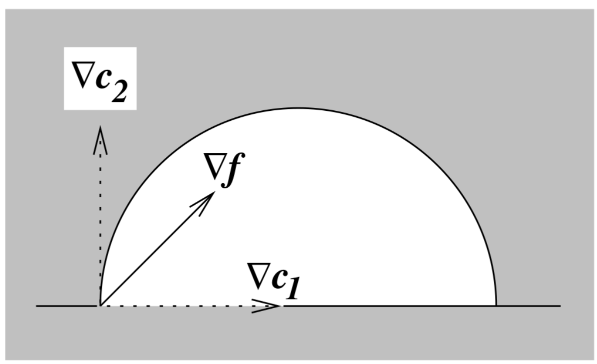

Figure 1: Gradients of the active constraints and objective at the solution

$$min_{x_1,x_2} \quad x1 + x2 \quad s.t. \quad 2 - x_1^2 - x_2^2 \geqslant 0, \quad x_2 \geqslant 0$$

For this case the feasible region is the half-disk as shown in 1. The solution lies at $(-\sqrt{2}, 0)^T$ at which both the constraints are active.

The fist-order feasible descent direction d should satisfy

$$\nabla c_i(x)^T d \geqslant 0, \quad i \,\epsilon\, I = 1,2 \quad , \quad \nabla f(x)^T d < 0$$

The condition $\bigtriangledown c_i(x)^T d \geqslant 0, i \epsilon I = 1,2$ are both satisfied only if d lies in the quadrant defined by $\bigtriangledown c_1(x)$ and $\bigtriangledown c_2(x)$ and also all the vectors in this quadrant does not satisfy $\bigtriangledown f(x)^T d < 0$.

Lagrangian equation is $L(x, \lambda) = f(x) - \lambda_1 c_1(x) - \lambda_2 c_2(x)$ where $\lambda = (\lambda_1, \lambda_2)^T$ is the vector of Lagrangian multipliers.

Now for optimal point which is not at boundary is $\bigtriangledown f(x) = 0$ and for optimal point on the boundary, the conditions are, $\bigtriangledown f(x) = \lambda \bigtriangledown c(x)$ for some $\lambda = (\lambda_1, \lambda_2)^T \geqslant 0$ where $\bigtriangledown c(x)$ is the gradient normal.
Hence the condition becomes

$$\bigtriangledown_x L(x^*, \lambda^*) = 0, \quad for \quad some \quad \lambda^* \geqslant 0. \tag{1}$$

Here the inequality $\lambda^* \geqslant 0$ means all components of λ^* required to be non-negative.
The complementary conditions are

$$\lambda_1^* c_1(x^*) = 0 \quad and \quad \lambda_2^* c_2(x^*) = 0. \tag{2}$$

For all other feasible points we can check from equations (1) and (2) that the optimal conditions are not satisfied 2.

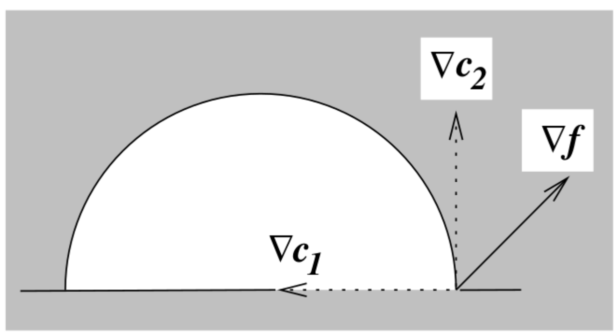

Figure 2: Gradients of the active constraints and objective at a non-optimal point

Some Important Definitions

1. **Linear Independence Constraint Qualification(LICQ)** $\Rightarrow$ Given the point x and the active set A(x) defined for $min_{x \in \mathbb{R}^n} \ f(x)$ s.t. $c_i(x) = 0, i \in \epsilon, c_i(x) \geqslant 0, i \in I$, we say that the LICQ holds if the set of active constraint gradients $\bigtriangledown c_i(x), \ i \in A(x)$ is linearly independent.

2. **Mangasarian-Fromovitz Constraint Qualification(MFCQ)** $\Rightarrow$ Given a point x and the active set A(x) defined for $min_{x \in \mathbb{R}^n}\ f(x)$ s.t. $c_i(x) = 0, i \in \epsilon, c_i(x) \geqslant 0, i \in I$, we say that the MFCQ holds if there exists a vector $w \in IR^n$ such that $\bigtriangledown c_i(x^*)^T w > 0$ for all $i \in A(x) \cap I$ and $\bigtriangledown c_i(x^*)^T w = 0$ for all i $\in \varepsilon$

   Note: If x* $\in \Omega$ satisfies LICQ, then x^* satisfies MFCQ. But MFCQ does not imply LICQ.

3. **Feasible Sequence** $\Rightarrow$ Given a feasible point x, we call z_k a feasible sequence approaching x if $z_k \in \Omega$ for all k sufficiently large and $z_k \rightarrow x$.

   The figure 3 shows the feasible Sequence for $x_1^2 + x_2^2 - 2 = 0$ and figure 4 shows the feasible Sequence for $x_1^2 + x_2^2 - 2 \geqslant 0$.

4. **Tangent Cone** $\Rightarrow$ The vector d is said to be a tangent (or tangent vector) to Ω at a point x if there are a feasible sequence z_k approaching x and a sequence of positive scalars t_k with $t_k \rightarrow 0$ such that

$$lim_{k \rightarrow \infty} \frac{z_k - x}{t_k} = d$$

   The set of all tangents to Ω at x^* is called the tangent cone and is denoted by $T_\Omega(x^*)$.

5. **Feasible direction set** $\Rightarrow$ Given a feasible point x and the active constraint set $A(x)$, the set of linearized feasible directions $F(x)$ is

$$F(x) = \left\{ d \mid \begin{array}{ll} d^T \bigtriangledown c_i(x) = 0, & \text{for } all\ i \in \varepsilon \\ d^T \bigtriangledown c_i(x) \geqslant 0, & \text{for } all\ i \in A(x) \cap I \end{array} \right\}$$

6. **Critical Cone** $\Rightarrow$ Given $F(x^*)$ and some Lagrangian multiplier λ^* satisfying the KKT conditions, the critical cone is defined as

$$C(x^*, \lambda^*) = w \in F(x^*) \mid \bigtriangledown c_i(x^*)^T w = 0, all\ i \in A(x^*) \cap I\ with\ \lambda_i^* > 0$$

   Equivalently,

$$\text{w} \in C(x^*, \lambda^*) \iff \left\{ \begin{array}{l} \bigtriangledown c_i(x^*)^T w = 0, all\ i \in \varepsilon \\ \bigtriangledown c_i(x^*)^T w = 0, all\ i \in A(x^*) \cap I\ with\ \lambda_i^* > 0 \\ \bigtriangledown c_i(x^*)^T w \geqslant 0, all\ i \in A(x^*) \cap I\ with\ \lambda_i^* = 0 \end{array} \right\}$$

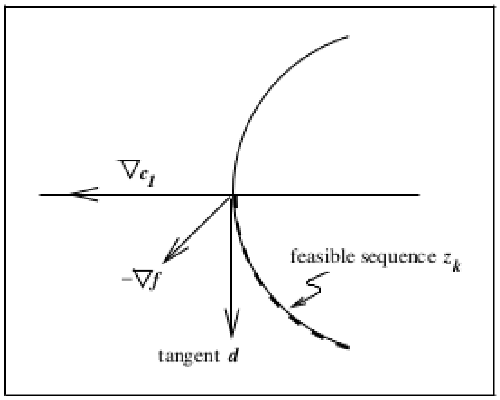

Figure 3: For equality constrained case

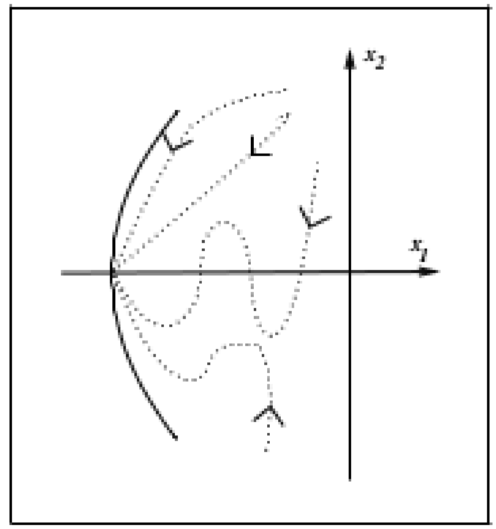

Figure 4: For inequality constrained case

Optimality Conditions

1. **First-Order Necessary Conditions**
   Suppose that x^* is a local solution of $min_{x \in \mathbb{R}^n}\ f(x)$ s.t. $c_i(x) = 0, i \in \epsilon, c_i(x) \geqslant 0, i \in I$, that the functions f and c_i are continuously differentiable , and that the LICQ holds at x^* . Then there is a Lagrange multiplier vector λ^* , with components $\lambda_i^*, i \in \epsilon \cap I$, such that the following conditions are satisfied at (x^*, λ^*)

$$\begin{aligned} \bigtriangledown_x L(x^*, \lambda^*) &= 0 \\ c_i(x^*) &= 0, for\ all\ i \in \varepsilon \\ c_i(x^*) &\geqslant 0, for\ all\ i \in I \\ \lambda_i^* &\geqslant 0, for\ all\ i \in I \\ \lambda_i^* c_i(x^*) &= 0, for\ all\ i \in \varepsilon \cap I \end{aligned}$$

   The above conditions are known as Karush Kuhn Tucker conditions, or KKT conditions for short.

   The condition $\lambda_i^* c_i(x^*) = 0, for\ all\ i \in \varepsilon \cap I$ are complementarity conditions.They imply that either constraint i is active or $\lambda_i^* = 0$, or possibly both.

2. **Second-Order Necessary Conditions**
   Suppose that x^* is a local solution of $min_{x \in \mathbb{R}^n}\ f(x)$ s.t. $c_i(x) = 0, i \in \epsilon, c_i(x) \geqslant 0, i \in I$

and that the LICQ condition is satisfied. Let λ^* be the Lagrange multiplier vector for which the KKT conditions are satisfied. Then

$$w^T \bigtriangledown^2_{xx} L(x^*, \lambda^*)w \geqslant 0, for\ all\ w \in C(x^*, \lambda^*).$$

3. **Second-Order Sufficient Conditions**
   Suppose that for some feasible point $x^* \in \mathrm{I\!R^n}$ there is a Lagrange multiplier vector such that the KKT conditions are satisfied. Suppose also that

$$w^T \bigtriangledown^2_{xx} L(x^*, \lambda^*)w > 0, for\ all\ w \in \mathrm{C(x^*, \lambda^*), w} \neq 0.$$

   Then x^* is a strict local solution for $min_{x \in \mathrm{I\!R^n}}\ f(x)$ s.t. $c_i(x) = 0, i \in \epsilon, c_i(x) \geqslant 0, i \in I$.