Competition Among Walkers on a Torus

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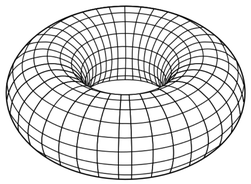
Abstract

- Simulated a single random walker on a large discrete 2D and 3D torus efficiently, tracking its proportional area covered on the torus, $S(t)$.
- Approximated expectation and variance of $S(t)$ and plotted its histogram.
- Investigated the transition time in the behavior of the area covered by $S(t)$ and introduced competition among multiple random walkers on the torus.
- Explore and compare strategies an agent could take to maximize the area of the covered region in competition with random walks. Study effects of different levels of information available to the agent, such as local occupancy and its past.
- Simulated the above in 3 dimensions for these strategies.
- Compared collaborative strategies against solo strategies in both 2D and 3D.

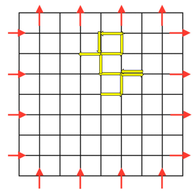
Guiding Question

How can we devise a strategy that maximizes area covered against random walk and other strategies in both two and three dimensional torus?

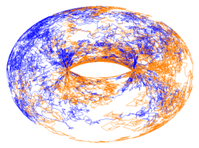
Introduction



A 2D Torus



A 2D Random Walk



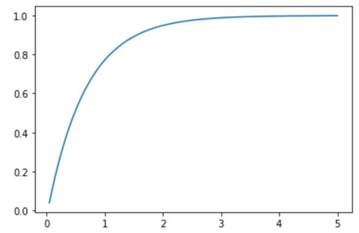
Competition

- A d-dimensional **torus** is defined as $\mathbb{T}^d = (\mathbb{Z}/n\mathbb{Z})^d$.
- A **strategy** is a function that accepts its surrounding vertices and its own memory as parameters and returns a direction.
- A **random walk (RW)** is a sequence of uniformly random steps between vertices, in our case on the torus.
- An agent's **proportional area**, $S(t)$ is the proportion of vertices claimed by the agent.
- **Competition** between agents compares area covered by each agent to see which is larger, while **collaboration** examines the total area covered by both agents.

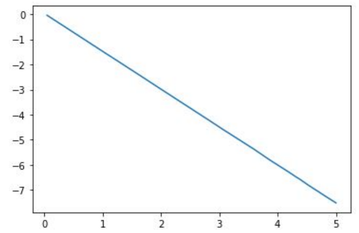
Strategies

- **Greedy biased (GB)**: assign a priority order of directions (e.g. right, down, left, up) and always choose to move to the first unclaimed adjacent vertex in the priority; if none are open, choose at random.
- **Greedy unbiased (GU)**: choose to move to any unclaimed adjacent vertex with equal probability; if none are open, choose at random.
- **Viki**: prioritize choosing to turn left if the vertex is unclaimed, then straight ahead, then right; if none are open, choose at random.

Two Dimensions



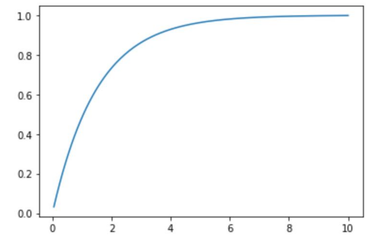
$S(t)$ for RW vs u Value



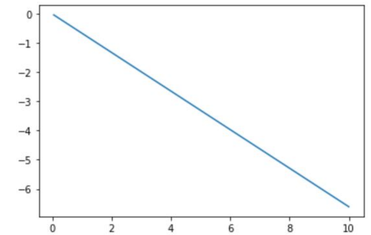
$\log(1 - S(t))$ for RW vs u Value

2D Random Walk Data at times $t = u \cdot n \log n$ ($n = 1001$ and sample size $= 1000$).

Three Dimensions



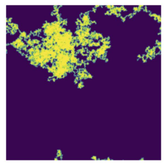
$S(t)$ for RW vs u Value



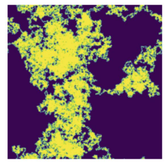
$\log(1 - S(t))$ for RW vs u Value

3D Random Walk Data at times $u \cdot n^3$ ($n = 349$ and sample size $= 250$)

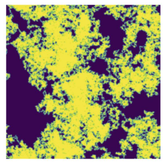
Random Walk Progression ($n = 101$)



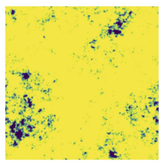
$u = 0.05$



$u = 0.30$



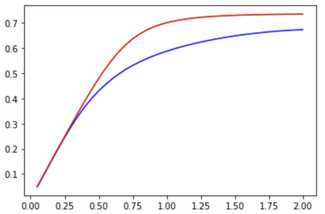
$u = 0.55$



$u = 2.00$

Area covered by a 2D Random Walk at time $u \cdot n^2 \log n$.

3D Competition ($n = 233$, sample size $= 1000$)



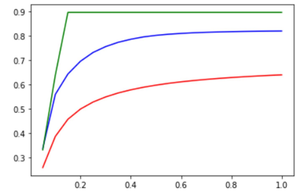
Area Covered against RW vs u value

Strategy	RW	GB	GU
RW	0.50000	0.31957	0.26363
GB	0.68043	0.50000	0.44579
GU	0.73637	0.55421	0.50000

Area Covered Head-to-Head

blue=greedy biased, red=greedy unbiased

2D Competition ($n = 1001$, sample size $= 1000$)



Area Covered against RW vs u value

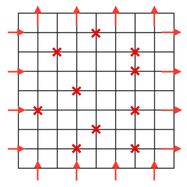
Strategy	RW	GB	GU	Viki
RW	0.50000	0.17867	0.34259	0.10406
GB	0.82133	0.50000	0.73158	0.61132
GU	0.65741	0.26842	0.50000	0.39310
Viki	0.89594	0.38868	0.60690	0.50000

Area Covered Head-to-Head

green=viki, blue=greedy biased, red=greedy unbiased

Competition on 2D Torus with Distortion

A **distortion** of the torus is when a vertex is deleted, meaning an agent cannot move to or claim that vertex. The value of m determines the probability with which any given vertex is deleted at the beginning of the simulation.

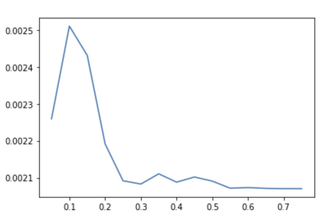


A torus with distortions

Strategy	$m = 0.01$	$m = 0.1$	$m = 0.3$
GB	0.68912	0.61065	0.65890
GU	0.68047	0.65924	0.62900

Area Covered vs RW with Distortions
($n = 1001$, sample size $= 1000$)

Collaboration



3 GU Area - GU Area vs u value
(2D, $n = 1001$, sample size $= 1000$)

Strategies	First	Second
GU vs 3 GU (2D)	0.49896	0.50104
GB vs GU (3D)	0.55532	0.44468
GB vs 3 GB (3D)	0.47672	0.52328
GU vs 3 GU (3D)	0.49992	0.50008
3 GU vs 3 GB (3D)	0.56037	0.43963

3D Collaboration ($n = 251$, sample size $= 250$)

Conjecture

Collaboration between agents claims at least as much area in expectation than a solo agent does.

References

[1] P. Dey and D. Kim, Fluctuation results for size of the vacant set for random walks on discrete torus. Preprint (2021). *Available at https://arxiv.org/abs/2108.06450.*

[2] J. Miller, Painting a graph with competing random walks, Ann. Probab. 41 (2013), no. 2, 636-670.

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