Diffusion Bridge Simulation and Applications

Travis Dunn

May 2020

Supervisor: Greg Rice

A project

In Partial fulfilment of the

requirement for

Master of Mathematics, Statistics

Contents

1 Introduction

Diffusion brigde simulation can be used to enhance the performance of some common tools in quantitative finance.

Roncalli [6] discusses how diffusion bridges can be used to estimate the pathwise maximum of a diffusion process over a compact time interval.

Pellegrino and Sabino [5] give a framework for applying the Least Squares Monte Carlo methodology of Longstaff and Schwartz [3] to the problem of pricing gas or power facilities. They demonstrate how diffusion bridge simulation can be used to improve the computational efficiency of the method.

Bladt and Sørensen [1] show how diffusion bridge simulation can be used to improve the efficiency of likelihood inference for diffusions. Typically, the risk neutral dynamics of an asset are estimated by calibrating a pricing model to fit a set of derivative prices. In contrast, likelihood inference seeks to directly infer the real world dynamics of a diffusion process from a discrete sequence of observations. This problem is more commonly discussed in physics and biology. For quantitative finance practitioners concerned with real world asset dynamics, it is more common to model returns using ARCH and GARCH type time-series methods. The reason for this disconnect is not clear. It could simply by the fact that inference methods for diffusions are more recent and harder to understand.

In this essay, I will discuss the diffusion simulation algorithm of Bladt and Sørensen [1] and use it to apply importance sampling to a Monte carlo quantile estimation problem.

Section 2 sets up the mathematical tools that will be used and introduces the diffusion bridge algorithm. Section 3 proves that the algorithm does indeed sample the distribution of the bridge. Section 4 demonstrates an application of diffusion bridge simulation to a Monte carlo quantile regression problem.

2 Setup

Bladt and Sørensen 2014 [1] give a simple algorithm for generating diffusion bridges that follow a general Stochastic Differential Equation (SDE)

$$dX_t = \alpha(X_t)dt + \sigma(X_t)dW_t, \tag{1}$$

where α and σ satisfy regularity conditions sufficient for the uniqueness and existence of a weak solution.

Let $p_s(x, y)$ be a transition density associated with the SDE in (1). That is, the density of X_{t+s} at y given $X_t = x$. Note that the variable t is arbitrary here since the process is time-homogeneous.

Define the diffusion bridge process Z to be the solution to the SDE in (1) with an initial condition $Z_0 = a$ conditioned on the event $\{Z_T = b\}$. Define the diffusion Y to be the solution to the SDE in (1) with initial condition $Y_0 \sim p_T(b, y)$. Y can be simulated on the time interval $[0, T]$ by first simulating a diffusion V that follows (1) and the initial condition $V_0 = b$ on the time interval $[0, 2T]$ and then using its second half. That is, set $Y_t = V_{T+t}$ for $t \in [0, T]$.

Bladt and Sørensen [1] introduce an approximate diffusion bridge simulation algorithm as follows. Let X^a and X^b be solutions to SDE 1 on the time interval $[0, T]$ with initial conditions $X^a_0 = a$ and $X^b_0 = b$. Define the time-reverse of X^b as $\widetilde{X}^b_t := X^b_{T-t}$ for $t \in [0, T]$. Let τ be the first hitting time of the diffusions X^a and $\widetilde{X}^b$ on the interval $[0, T]$. If the diffusions do not hit each other before time T then as a matter of convention set τ to infinity. Formally, $\tau := \inf\{t \in [0, T] | X^a_t = \widetilde{X}^b_t\}$ where $\inf(\emptyset) = \infty$. Define the diffusion Z^* as X^a before τ and $\widetilde{X}^b$ after τ. Formally,

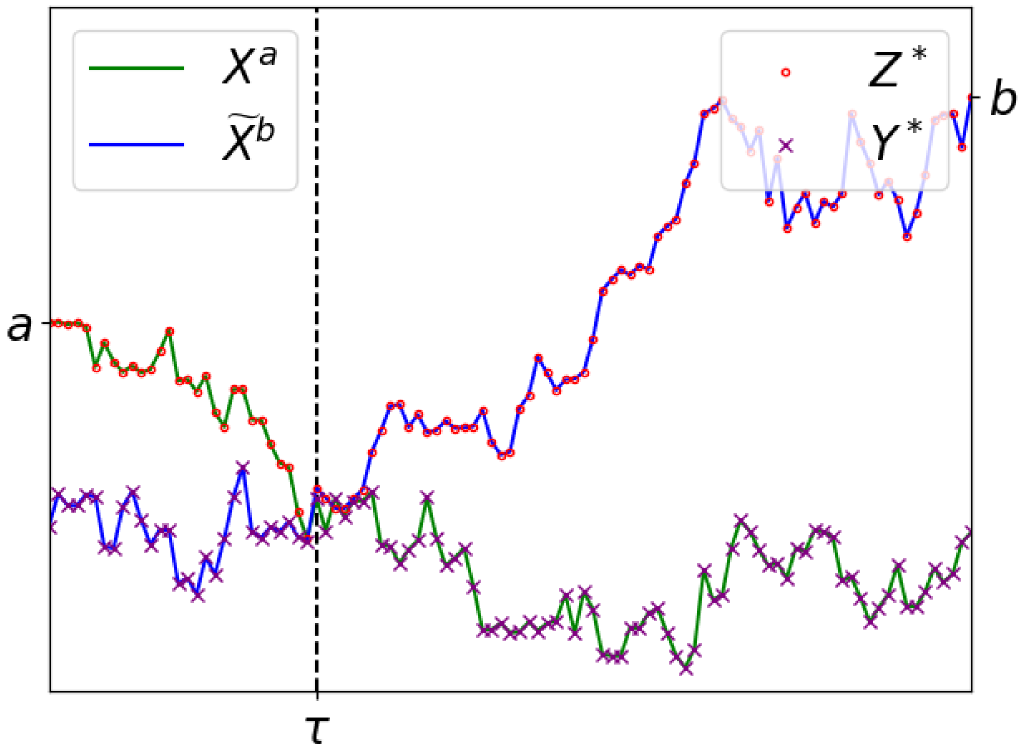

Figure 1: Sample paths of X^a and $\widetilde{X}^b$ reconstructed to yield sample paths from Z^* and Y^*

$$Z_t^* := \begin{cases} X_t^a & 0 \leq t \leq \tau, \\ \widetilde{X}_t^b & \tau < t \leq T. \end{cases} \tag{2}$$

Also, define the diffusion Y^* as $\widetilde{X}^b$ before τ and X^a after τ. Formally,

$$Y_t^* := \begin{cases} \widetilde{X}_t^b & 0 \leq t \leq \tau, \\ X_t^a & \tau < t \leq T. \end{cases} \tag{3}$$

The key consequence of this set-up, which will be proved later, is as follows:

$$Z^* \bigm| \{\tau < \infty\} \sim Z \bigm| \{Z \text{ and } Y \text{ hit before } T\}. \tag{4}$$

See figure 1 for a visual representation of a sample from this setup.

Thus the relationship shown in equation (5) between the pathwise densities of $Z^* \mid \{\tau < \infty\}$ and Z can be established using a version of Bayes Theorem.

$$f_Z(x) = \frac{P(Z \text{ and } Y \text{ hit before T})}{P(Y \text{ hits } x \text{ before } T)} f_{Z^*|\{\tau<\infty\}}(x). \tag{5}$$

Equation (5) is used to design a Markov Chain Monte Carlo (MCMC) whose startionary distribution has density f_Z using a pseudo-marginal Metropolis-Hastings approach.

In the standard Metropolis-Hastings setup, a proposal sample point (in this case path) x' is generated from a transition density $g(x'|\, x_n)$ and is accepted with probability

$$\alpha(x_n, x') = \min\left(1, \frac{P(x')}{P(x_n)} \frac{g(x_n|\, x')}{g(x'|\, x_n)}\right), \tag{6}$$

in the case that the proposal x' is accepted x_{n+1} is set to x', otherwise x_{n+1} is set to the same value as x_n. The initial state x_0 is set arbitrarily. This scheme results in a markov chain $x_0, x_1 ... x_n ...$ with stationary distribution $P(x)$. The distribution $P(x)$ is often reffered to as a target distribution of the algorithm as it is the limiting distribution of the sample obtained from the algorithm.

Often, the explicit form of the target distribution's density is unknown. This is almost always the case in the diffusion bridge setting. Rarely is f_Z known. If an unbiased estimator $\hat{P}_x$ for the value of the target density $P(.)$ at an arbitrary value x can be constructed, then a pseudo-marginal approach can be applied by substituting the estimator $\hat{P}_x$ for the density $P(x)$ in equation (6). The pseudo-marginal approach may take longer to converge but yields the same stationary distribution. Bladt and Sørensen use a modification of this approach which lends itself nicely to the circumstances of the problem.

Towards a pseudo-marginal Metropolis-Hastings algorithm for the bridge Z, set the proposal density $g(x' \mid x_n)$ equal to $f_{Z^*|\{\tau<\infty\}}(x')$ and the target density $P(x)$ to $f_Z(x)$. Notice that the initial value of the chain x_0 need not be selected here since the proposal density is stationary. Substituting equation (5) into the resulting acceptance probability yields

$$\alpha(x_n, x') = \min\Big(1, \frac{f_Z(x')}{f_Z(x_n)} \frac{f_{Z\big|\{\tau<\infty\}}(x_n| \ x')}{f_{Z\big|\{\tau<\infty\}}(x'| \ x_n)}\Big) = \min\Big(1, \frac{P(Y \text{ hits } x' \text{ before } T)}{P(Y \text{ hits } x_n \text{ before } T)}\Big), \quad (7)$$

an approximate psedo-marginal approach can then be applied by estimating the probability that the diffusion Y intersects with a given sample path as follows:

1. Simulate N paths from Y $(y_1, y_2...y_N)$ on the interval $[0, T]$.

2. Take the poportion of paths that intersect the path x i.e.

$$\hat{P}(Y \text{ hits } x \text{ before } T) = \frac{1}{N} \sum_{i=1}^{N} 1_{\{y_i \text{ hits } x\}}. \quad (8)$$

This estimation method would be performed at every proposal step. Bladt and Sørensen claim that the scheme results in a chain whose stationary distribution is approximately but not exactly f_Z. They give no justification as to why this is and then suggest an alternative method that yields the correct stationary distribution. The method consists of estimating the inverse probability $\rho_x = \frac{1}{P(Y \text{ hits } x)}$ as follows

1. Simulate paths from Y on the interval $[0, T]$ until one of them hits x.

2. Define T as the number of samples in the above step. i.e. y_T is the first path in the sample to intersect x.

3. Repeat setps (1) and (2) a total of N times to obtain a sample $(T_1, T_2...T_N)$ and take the mean. i.e.

$$\hat{\rho_x} = \frac{1}{N} \sum_{i=1}^{N} T_i \quad (9)$$

3 Main Result

In this section we will prove and dicuss the central result of the paper, namely that the distribution of Z^* conditional on X^a and $\widetilde{X}^b$ hitting is the same as the distribution of the bridge Z conditional on it being hit by Y . It begins with a lemma stating that the distribution of the time reveresed diffusion $\widetilde{X}$ is equal to that of the original diffusion with the constraint $X_T = b$. The lemma is proved by showing that the transition densities of each process are equal using a balance equation as follows.

The balance equation for the transition density of X,

$$p_t(x, y)m(x) = p_t(y, x)m(y), \tag{10}$$

where m is the density of the speed measure of X. However, this fact is irrelevant to our purposes. As demonstrated below, we only make use of the fact that there exist some function m for which the balance equation holds.

$$\begin{aligned}
p_{X|X_T=b}(x, s; y, t) &= \frac{p_{t-s}(x, y)p_{T-t}(y, b)}{p_{T-s}(x, b)} \\
&= \frac{p_{t-s}(x, y)m(x)p_{T-t}(y, b)}{p_{T-s}(x, b)m(x)} \text{ multiply top and bottom by } m(x) \\
&= \frac{p_{t-s}(y, x)m(y)p_{T-t}(b, y)}{p_{T-s}(b, x)m(b)} \text{ using the balance equation in (10)} \\
&= \frac{p_{t-s}(x, y)p_{T-t}(b, y)m(b)}{p_{T-s}(b, x)m(b)} \text{ using (10) again} \\
&= \frac{p_{t-s}(x, y)p_{T-t}(b, y)}{p_{T-s}(b, x)} \\
&= p_{\widetilde{X}\big|X_0=b}(x, s; y, t)
\end{aligned}$$

The first and last equalities are not immediately obvious. Bladt and Sørensen provide a proof for the latter and refer to Fitzimmons, Pitman and Yor [4] for the former. However Fitzsimmons et al. leave the proof as an excercise for the reader. Thus proof for the expression for the transition density of X given $X_T = b$ is given below. The proof is quite similar to Bladt and Sørensens proof of the expression for the transition density of the time reversed process $\widetilde{X}$.

$$\begin{aligned} p_{X|X_T=b}(x,s;y,t) &= p_{X_t \big| X_s=x, X_T=b}(y) \\ &= \frac{p_{X_t, X_T \big| X_s=x}(y,b)}{p_{X_T|X_s=x}(b)} \\ &= \frac{p_{X_t \big| X_s=x}(y) p_{X_T \big| X_t=y}(b)}{p_{X_T|X_s=x}(b)} \\ &= \frac{p_{t-s}(x,y) p_{T-t}(y,b)}{p_{T-s}(x,b)}. \end{aligned}$$

Note that the trasition densities of these constrained diffusions are not time-homogeneous. This is intuitively clear because as the transition interval $[s,t]$ shifts closer to $[T-(t-s),T]$, the densities must give successively greater likelihood closer to b.

Now we will prove equation (4). Let ρ be the first hitting time of X^a and Y. Define the diffusion Z^{**} as X^a up to ρ and Y after ρ and vice-versa for Y^{**}. Formally,

$$Z_t^{**} := \begin{cases} X_t^a & 0 \le t \le \rho \\ Y_t & \rho < t \le T \end{cases} \quad Y_t^{**} := \begin{cases} Y_t & 0 \le t \le \rho \\ X_t^a & \rho < t \le T \end{cases},$$

on the event $\{\rho < \infty\}$, and $Z_t^{**} = X_t^a, Y_t^{**} = Y_t$ on the event $\{\rho = \infty\}$.

Using the strong Markov property, it can be shown that $Z^* \overset{d}{=} X^a$ and $Y^* \overset{d}{=} Y$. The following financial analogy describes the intuition for this step. Suppose that two individuals Bob and Alice both have trading accounts. Bob purchases one share of MSFT for \$10 while Alice sends an order to purchare one share in an IPO for a new company called Circuit IQ at whatever the initial price is. The initial price of Circuit IQ follows a distribution ν. The dynamics of both stocks are assumed to be Markovian and identical. Alice and Bob agree ahead of time that if the prices of MSFT and CIQ are equal at any point in the future, they will transfer the stocks between them so that Bob owns CIQ and Alice owns MSFT. Clearly this scheme yields the same profit distribution for both individuals as if they had no agreement.

With this in place we note that

$$Y \;\Big|\; \{Y_T = b\} \sim X|\{X_T = b\} \overset{\text{lemma}}{\sim} \widetilde{X}^b. \tag{11}$$

Define Z^* and Y^* as before,

$$Z_t^* := \begin{cases} X_t^a & 0 \leq t \leq \rho \\ \widetilde{X}_t^b & \rho < t \leq T \end{cases} \quad Y_t^* := \begin{cases} \widetilde{X}_t^b & 0 \leq t \leq \rho \\ X_t^a & \rho < t \leq T \end{cases}.$$

Thus, conditional on X^a and Y hitting before T,

$$\begin{aligned} X^a \;\Big|\; \{X_T = b\} &\sim Z^{**} \;\Big|\; \{Z_T^{**} = b\} \\ &\sim Z^{**} \;\Big|\; \{Y_T = b\} \\ &\sim Z^* \text{ by equation (11)} \end{aligned}$$

4 A Financial Application

Given an underlying $S_t, t > 0$ whose dynamics are governed by (1), and its law P, we seek to approximate the τ^{th} quantile function of the path dependent European option payoff $L(S_{0:T})$ conditional on the terminal path value S_T. That is, a real valued function $f_\tau(s)$ such that $P(L(S_{0:T}) \leq f_\tau(s)|S_T = s) = \tau$.

As per Kolkiewicz, 2014 [2] the quantile function can then be used to find the solution h to the risk minimization problem.

$$\inf_h E^P[((L(S_{0:T}) - h(S_T)))^+)^p]. \tag{12}$$

One approach for estimating such an f_τ is using quantile regression methods on a Monte carlo sample of the joint distribution $(S_T, L(S_{0:T}))$ under the law P. This approach depends on estimating the expected value of the tilted absolute error or check function. That is $E[\rho_\tau(L(S_{0:T}) - f_\tau(S_T))]$. Where

$$\rho_\tau(x) := \begin{cases} \tau \cdot x & \text{for } 0 \leq x, \\ (\tau - 1) \cdot x & \text{for } x < 0. \end{cases}$$

I suggest to use a stratified Monte Carlo sampling method to estimate this expectation as follows.

$$\begin{aligned} E[\rho_\tau(L(S_{0:T}) - f_\tau(S_T))] &= E[E[\rho_\tau(L(S_{0:T}) - f_\tau(s))|S_T = s]] && (13) \\ &= E_g[\frac{f(S_T)}{g(S_T)} E_{P|S_T}[\rho_\tau(L(S_{0:T}) - f_\tau(S_T))], && (14) \end{aligned}$$

where $g(S_T)$ is an auxiliary marginal pdf and $f(S_T)$ is the original marginal pdf under P. In order to approximate this expectation we need to be able to sample paths conditional on their terminal value. In other words, for a given terminal value s we should be able to sample $S_{0:T}$ and thus the option payoff from $P \mid S_T = s$. This is where the diffusion bridge algorithm is useful. The accuracy of continuous piecewise linear quantile regressors under naive and stratified sampling will be compared.

The continuous piecewise linear regressors were implemented in tensor flow as neural networks with 32 hidden units and a rectified linear unit activation function. This results in a family of arbitrary coninuous piecewise linear functions with at most 32 pieces. The family is represented in equation (15)

$$f(x) = \sum_{j=1}^{32} w_j^{(2)} (w_j^{(1)} x + b_j^{(1)})^+ + b^{(2)}. \tag{15}$$

In the case where the marginal density $f(\cdot)$ is not available analyticaly, it can be found by solving the Fokker-Planck partial differential equation numerically or approximated by applying a kernel density estimator to a sample of terminal values. The simulation study below use an Ornstein-Uhlenbeck process and its explicit transition density as a matter of convenience.

Define P' to be the distribution implied by sampling S_T under g and then sampling the rest of the path under $P \bigm| S_T$. P' will be refered to as the auxiliary law of S.

As is typical for importance sampling techniques in quantitative finance, the choice of auxhilliary distribution $g(\cdot)$ is critical and must be done hueristically with a specific goal in mind. The goal in this setting is to obtain a Monte carlo sample from $(S_T, L(S_{0:T}))$, such that the fit of the estimated quantile function is improved at extreme values of S_T. With this in mind the most obvious choice for $g(\cdot)$ is a uniform distribution over a compact interval representing the range of terminal values we are interested in. See figure (2) for an example of this method being applied to the pay off of a look back call. Note that the quantile function trained on the sample from P' is closer to the "True" quantile function for extreme values of S_T. A look back call is defined here as an option with the following payoff,

$$L(S_{0:T}) = \max_{0 \leq t \leq T} (S_t) - S_T.$$

The conditioning in (14) can also be applied to the value of the process at $T/2$ and T yielding a second auxiliary measure $P'' = P \bigm| S_{T/2} \sim g_1, S_T \sim g_2$. This method is used below to estimate the quantile function of a convex look back call. That is, an option with

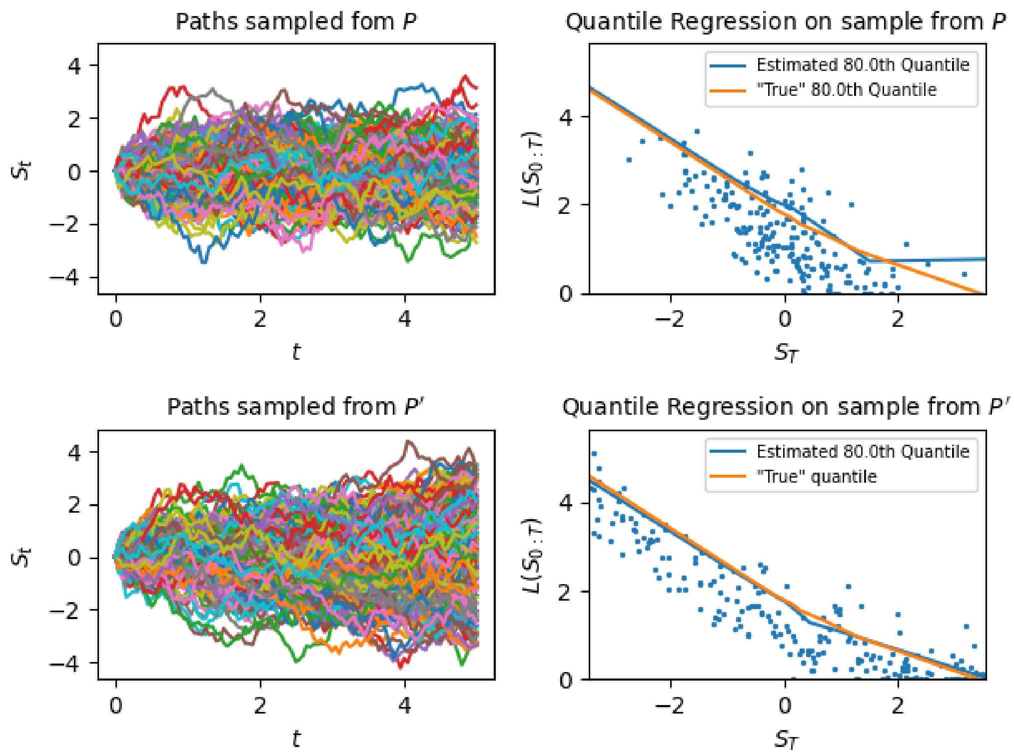

Figure 2: Quantile regressions on samples of size 200 from P and P'. Regressors are dense neural networks with a single 32 unit hidden layer and ReLu activation function, trained with stochastic gradient descent and a batch size of 32 for 200 epochs. The "True" quantile function was obtained by the same training method but on a Monte carlo sample of 4000 paths.

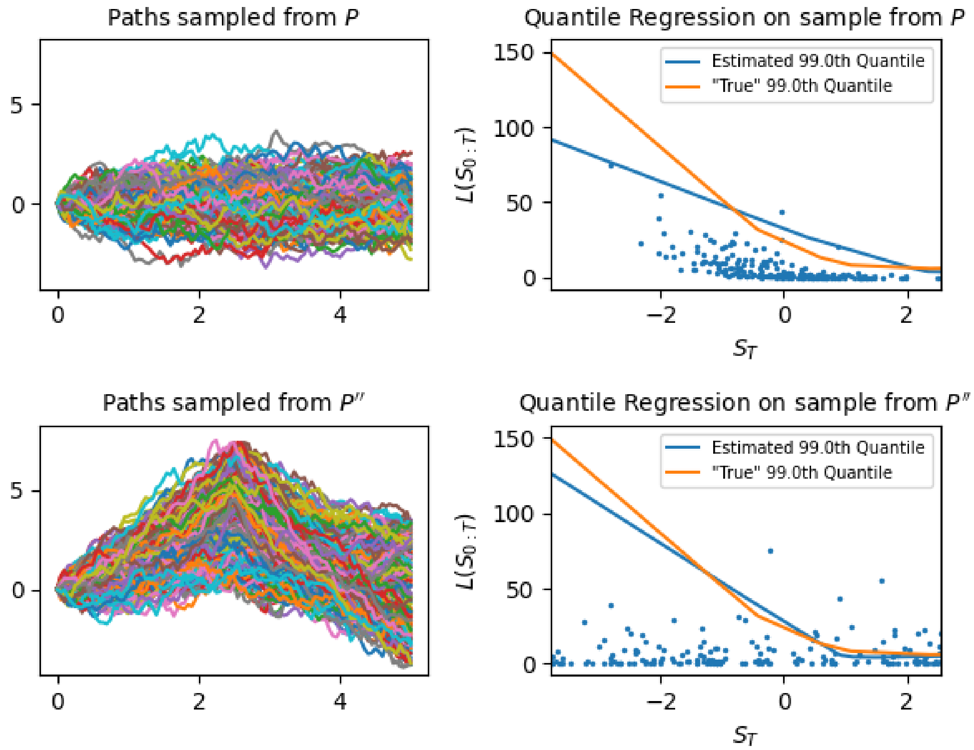

Figure 3: Quantile regression on samples of size 200 from P and P''. Regressors are dense neural networks with a single 32 unit hidden layer and ReLu activation function, trained with stochastic gradient descent and a batch size of 32 for 200 epochs. The "True" quantile function was obtained by the same training method but on a Monte carlo sample of 4000 paths.

the following payoff,

$$L(S_{0:T}) = \left(\max_{0 \leq t \leq T} (S_t) - S_T \right)^3 .$$

Here the auxiliary density g_1 is set to an upward shift of the uniform terminal density g_2 so that paths sampled under P'' will give more information about the tails of the convex look back call. See figure (3) for plots of the regressions. Note that the quantile function trained on the sample from P'' more closely fits the "True" quantile function at extreme values of the option payoff.

References

[1] Mogens Bladt and Michael Sørensen. Simple simultion of diffusion bridges with application to likelihood inference for diffusions. *Bernoulli*, 2014.

[2] Adam Kolkiewicz. Efficient hedging of path dependent options. *International Journal of Theoretical and Applied Finance*, 2014.

[3] Francis A. Longstaff and Eduardo S. Schwartz. Valuing American options by simulation: a simple least-squares approach. *The Review of Financial Studies*, 14(1):113–147, 2001.

[4] Jim Pitman Pat Fitzsimmons and Marc Yor. Markovian bridges: construction, palm interpretation and slicing. *Progress in Probability*, 1993.

[5] T. Pellegrino and P. Sabino. Enhancing least squares monte carlo with diffusion bridges: an application to energy facilities. *Quantitative Finance*, 15(5):761–772, 2015.

[6] Thierry Roncalli. *Handbook of Financial Risk Management.* Chapman and Hall/CRC, Boca Raton, Florida, 2020.