Precalculus
Angle sum formulas memorization

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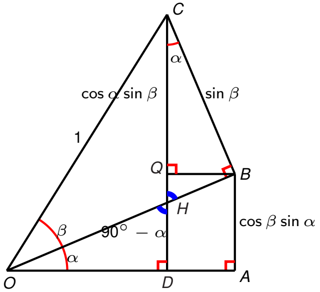
$\sin(\alpha+\beta), \cos(\alpha+\beta)$ via $\sin\alpha, \sin\beta, \cos\alpha, \cos\beta$



$$
\begin{aligned}
\sin(\alpha+\beta) &= \frac{|CD|}{|OC|} = |CD| \\
&= |QD| + |CQ| \\
&= \sin\alpha\cos\beta + \cos\alpha\sin\beta \\
\cos(\alpha+\beta) &= \frac{|OD|}{|OC|} = |OD| \\
&= |OA| - |DA| \\
&= \cos\alpha\cos\beta - \sin\alpha\sin\beta
\end{aligned}
$$

$$
\begin{aligned}
|QD| &= |BA| && \square DABQ \\
&= \sin\alpha|OB| && \triangle OAB \\
&= \sin\alpha\cos\beta|OC| && \triangle OBC \\
&= \sin\alpha\cos\beta \\
|CQ| &= \cos\alpha|CB| && \triangle CQB \\
&= \cos\alpha\sin\beta|OC| && \triangle OBC \\
&= \cos\alpha\sin\beta \\
|OA| &= \cos\alpha|OB| && \triangle OAB \\
&= \cos\alpha\cos\beta|OC| && \triangle OBC \\
&= \cos\alpha\cos\beta \\
|DA| &= |QB| && \square DABQ \\
&= \sin\alpha|CB| && \triangle CQB \\
&= \sin\alpha\sin\beta|OC| && \triangle OBC \\
&= \sin\alpha\sin\beta
\end{aligned}
$$

Trig Functions of Sums and Differences of Angles

Theorem

$$
\begin{aligned}
\sin(\alpha+\beta) &= \sin\alpha\cos\beta+\cos\alpha\sin\beta\\
\sin(\alpha-\beta) &= \sin\alpha\cos\beta-\cos\alpha\sin\beta\\
\cos(\alpha+\beta) &= \cos\alpha\cos\beta-\sin\alpha\sin\beta\\
\cos(\alpha-\beta) &= \cos\alpha\cos\beta+\sin\alpha\sin\beta
\end{aligned}
$$

- We gave a geometric proof of the sum formulas when the two angles are acute and their sum is less than $\pi = 90°$.
- The theorem holds for all angles α, β without any restrictions.
- This can be shown by combining the preceding proof with identities such as $\cos\left(\frac{\pi}{2}-\alpha\right) = \sin\alpha$, $\cos\left(\frac{\pi}{2}+\alpha\right) = -\sin\alpha$.
- There is a theoretically more advanced (but algebraically simpler) proof using Euler's formula (to be studied later/in another course).
- The difference formulas are a consequence of the sum formulas and the fact that $\sin$ is an odd function and $\cos$ is even.

Trig Functions of Differences of Angles

Example

Prove the identities

$$\begin{aligned} \sin(\alpha-\beta) &= \sin\alpha\cos\beta-\cos\alpha\sin\beta \\ \cos(\alpha-\beta) &= \cos\alpha\cos\beta+\sin\alpha\sin\beta \end{aligned}$$

from the (already demonstrated) identities

$$\begin{aligned} \sin(\alpha+\beta) &= \sin\alpha\cos\beta+\cos\alpha\sin\beta \\ \cos(\alpha+\beta) &= \cos\alpha\cos\beta-\sin\alpha\sin\beta \end{aligned}$$

$$\begin{aligned} \sin(\alpha-\beta) &= \sin(\alpha+(-\beta)) \\ &= \sin\alpha\cos(-\beta)+\cos\alpha\sin(-\beta) && \left|\begin{array}{l}\cos \text{ is even },\\ \sin \text{ is odd}\end{array}\right. \\ &= \sin\alpha\cos\beta-\cos\alpha\sin\beta \\ \cos(\alpha-\beta) &= \cos(\alpha+(-\beta)) \\ &= \cos\alpha\cos(-\beta)-\sin\alpha\sin(-\beta) && \left|\begin{array}{l}\cos \text{ is even },\\ \sin \text{ is odd}\end{array}\right. \\ &= \cos\alpha\cos\beta+\cos\alpha\sin\beta \end{aligned}$$