

Precalculus

Trigonometric equations and inequalities

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Outline

1 Trigonometric equations and inequalities

- The Equations $\sin x = A$, $\cos x = B$
- Equations that reduce to $\sin x = A$, $\cos x = B$

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2 Product-to-Sum Formulas

Outline

- 1 Trigonometric equations and inequalities
 - The Equations $\sin x = A$, $\cos x = B$
 - Equations that reduce to $\sin x = A$, $\cos x = B$
- 2 Product-to-Sum Formulas
- 3 Trigonometric inequalities

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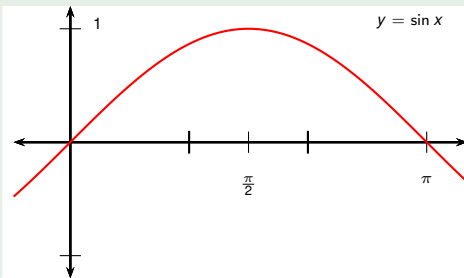
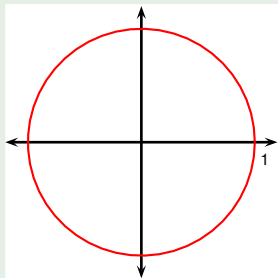
Trigonometric equations

- Some problems will not ask you to prove a trigonometric identity, but rather to solve a trigonometric equation.
- Consider the problem of finding all values of x for which $\sin x = \sin(2x) = 2 \sin x \cos x$.
- This is not a trigonometric identity - the two sides are different.
- However, there are values for x which the above equality holds.

Example

Find all solutions and then find those that lie between -360° and 360° .

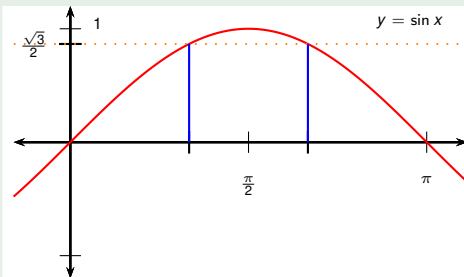
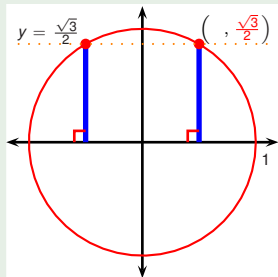
$$\sin \theta = \frac{\sqrt{3}}{2}$$



Example

Find all solutions and then find those that lie between -360° and 360° .

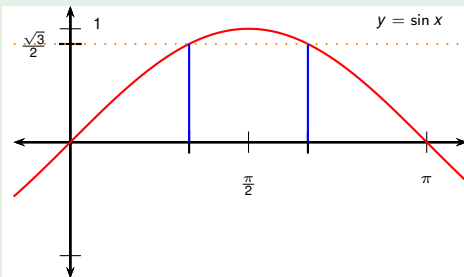
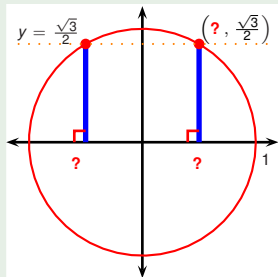
$$\sin \theta = \frac{\sqrt{3}}{2}$$



Example

Find all solutions and then find those that lie between -360° and 360° .

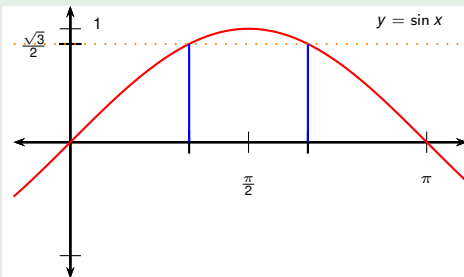
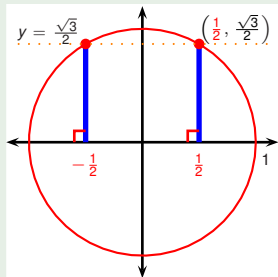
$$\sin \theta = \frac{\sqrt{3}}{2}$$



Example

Find all solutions and then find those that lie between -360° and 360° .

$$\sin \theta = \frac{\sqrt{3}}{2}$$

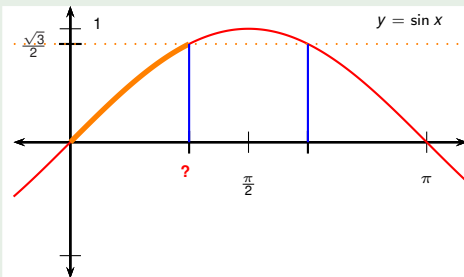
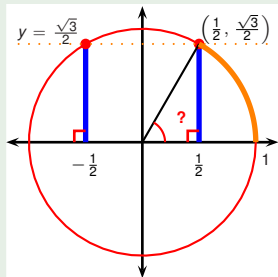


Example

Find all solutions and then find those that lie between -360° and 360° .

$$\sin \theta = \frac{\sqrt{3}}{2}$$

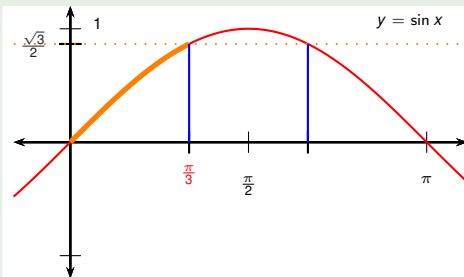
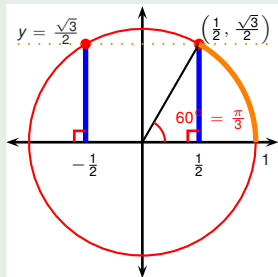
$$\theta = ?$$



Example

Find all solutions and then find those that lie between -360° and 360° .

$$\begin{aligned}\sin \theta &= \frac{\sqrt{3}}{2} \\ \theta &= 60^\circ\end{aligned}$$



Example

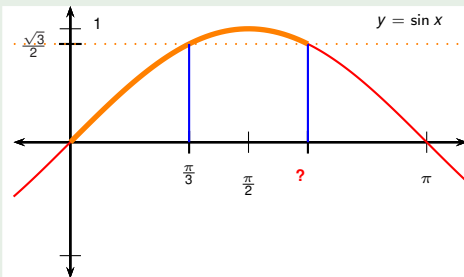
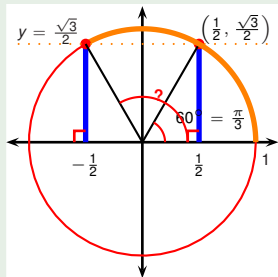
Find all solutions and then find those that lie between -360° and 360° .

$$\sin \theta = \frac{\sqrt{3}}{2}$$

$$\theta = 60^\circ$$

or

?



Example

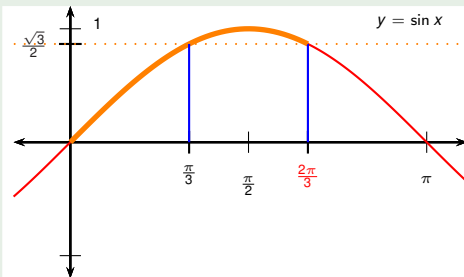
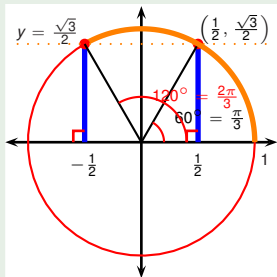
Find all solutions and then find those that lie between -360° and 360° .

$$\sin \theta = \frac{\sqrt{3}}{2}$$

$$\theta = 60^\circ$$

or

$$120^\circ$$



Example

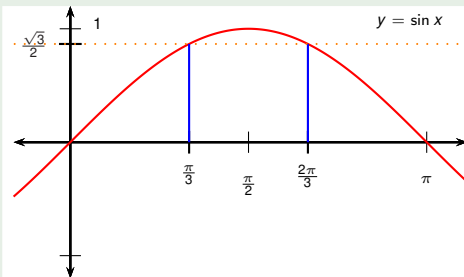
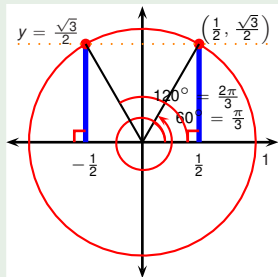
Find all solutions and then find those that lie between -360° and 360° .

$$\sin \theta = \frac{\sqrt{3}}{2}$$

$$\theta = 60^\circ + k \cdot 360^\circ$$

or

$$120^\circ$$



Example

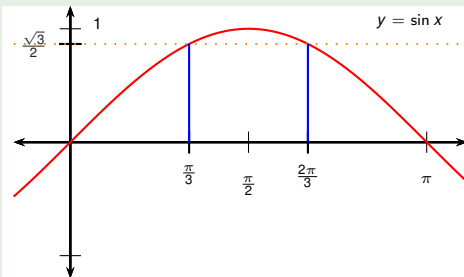
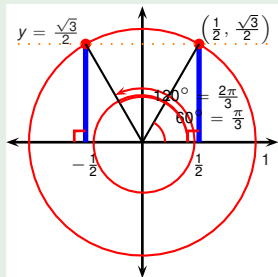
Find all solutions and then find those that lie between -360° and 360° .

$$\sin \theta = \frac{\sqrt{3}}{2}$$

$$\theta = 60^\circ + k \cdot 360^\circ$$

or

$$120^\circ + k \cdot 360^\circ$$



Example

Find all solutions and then find those that lie between -360° and 360° .

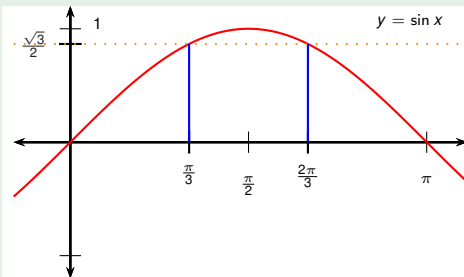
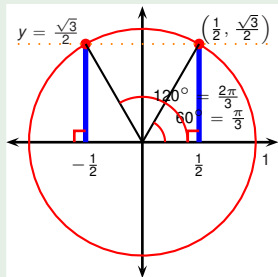
$$\sin \theta = \frac{\sqrt{3}}{2}$$

$$\theta = 60^\circ + k \cdot 360^\circ = \dots -660^\circ,$$

or

$$\dots k = -2$$

$$120^\circ + k \cdot 360^\circ = \dots -600^\circ,$$



Example

Find all solutions and then find those that lie between -360° and 360° .

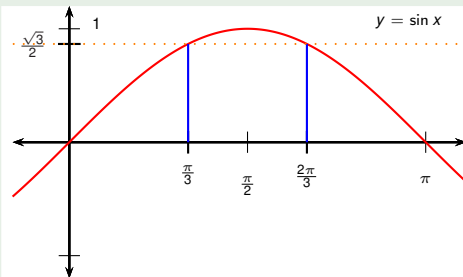
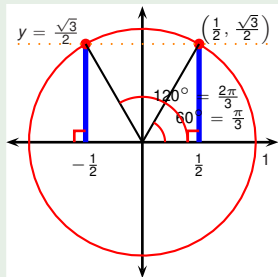
$$\sin \theta = \frac{\sqrt{3}}{2}$$

$$\theta = 60^\circ + k \cdot 360^\circ = \dots -660^\circ, -300^\circ,$$

or

$$\dots k=-2 \quad k=-1$$

$$120^\circ + k \cdot 360^\circ = \dots -600^\circ, -240^\circ,$$



Example

Find all solutions and then find those that lie between -360° and 360° .

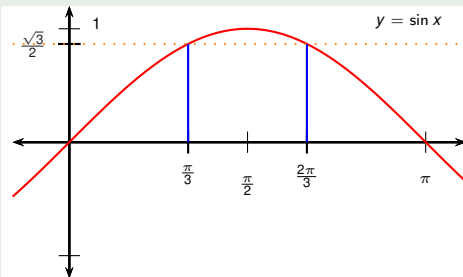
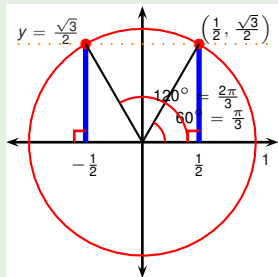
$$\sin \theta = \frac{\sqrt{3}}{2}$$

$$\theta = 60^\circ + k \cdot 360^\circ = \dots -660^\circ, -300^\circ, 60^\circ,$$

or

$$\dots \quad k=-2 \quad k=-1 \quad k=0$$

$$120^\circ + k \cdot 360^\circ = \dots -600^\circ, -240^\circ, 120^\circ,$$



Example

Find all solutions and then find those that lie between -360° and 360° .

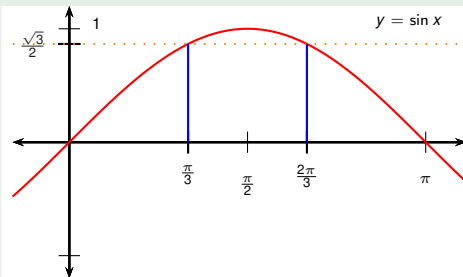
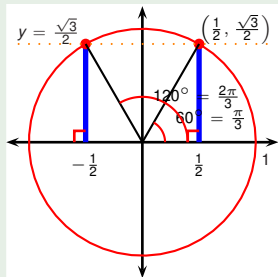
$$\sin \theta = \frac{\sqrt{3}}{2}$$

$$\theta = 60^\circ + k \cdot 360^\circ = \dots -660^\circ, -300^\circ, 60^\circ, 420^\circ, \dots$$

or

$$\dots \quad k=-2 \quad k=-1 \quad k=0 \quad k=1 \quad \dots$$

$$120^\circ + k \cdot 360^\circ = \dots -600^\circ, -240^\circ, 120^\circ, 480^\circ, \dots$$



Example

Find all solutions and then find those that lie between -360° and 360° .

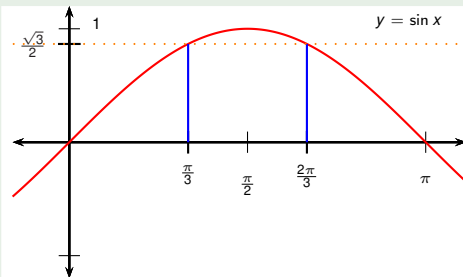
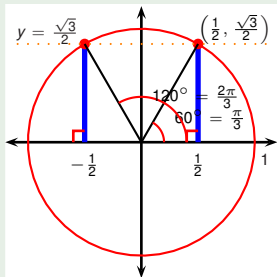
$$\sin \theta = \frac{\sqrt{3}}{2}$$

$$\theta = 60^\circ + k \cdot 360^\circ = \dots -660^\circ, -300^\circ, 60^\circ, 420^\circ, \dots$$

or

$$\dots \quad k=-2 \quad k=-1 \quad k=0 \quad k=1 \quad \dots$$

$$120^\circ + k \cdot 360^\circ = \dots -600^\circ, -240^\circ, 120^\circ, 480^\circ, \dots$$



Example

Find all solutions and then find **those that lie between -360° and 360°** .

$$\sin \theta = \frac{\sqrt{3}}{2}$$

$$\theta = 60^\circ + k \cdot 360^\circ = \dots -660^\circ, -300^\circ, 60^\circ, 420^\circ, \dots$$

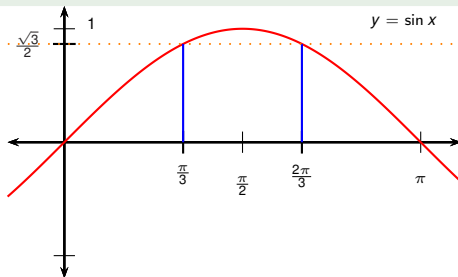
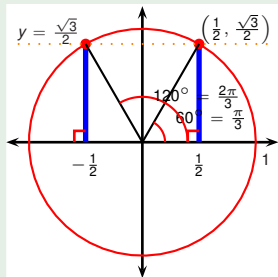
or

$$\dots \quad k=-2 \quad k=-1 \quad k=0 \quad k=1 \quad \dots$$

$$120^\circ + k \cdot 360^\circ = \dots -600^\circ, -240^\circ, 120^\circ, 480^\circ, \dots$$

$$\theta = \dots -660^\circ, -300^\circ, 60^\circ, 420^\circ, \dots$$

$$\dots -600^\circ, -240^\circ, 120^\circ, 480^\circ, \dots$$



Example

Find all solutions and then find **those that lie between -360° and 360°** .

$$\sin \theta = \frac{\sqrt{3}}{2}$$

$$\theta = 60^\circ + k \cdot 360^\circ = \dots -660^\circ, -300^\circ, 60^\circ, 420^\circ, \dots$$

or

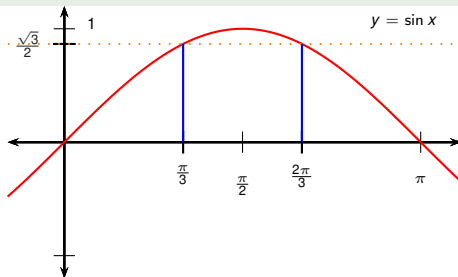
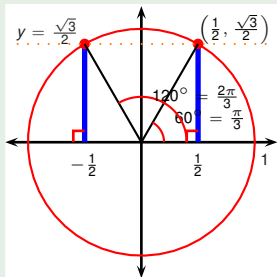
$$\dots \quad k=-2 \quad k=-1 \quad k=0 \quad k=1 \quad \dots$$

$$120^\circ + k \cdot 360^\circ = \dots -600^\circ, -240^\circ, 120^\circ, 480^\circ, \dots$$

$$\theta =$$

$$\cancel{\dots -660^\circ}, -300^\circ, 60^\circ, \cancel{420^\circ}, \cancel{\dots}$$

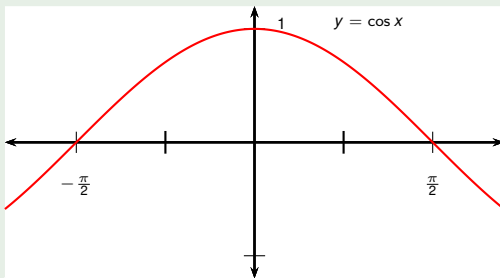
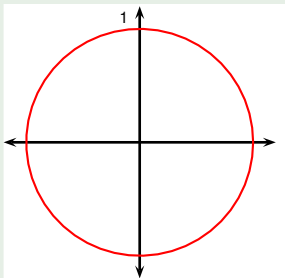
$$\cancel{\dots -600^\circ}, -240^\circ, 120^\circ, \cancel{480^\circ}, \cancel{\dots}$$



Example

Find all solutions and then find those that lie between -180° and 180° .

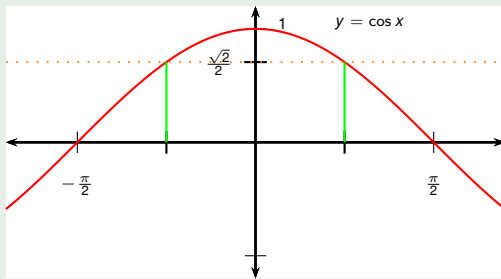
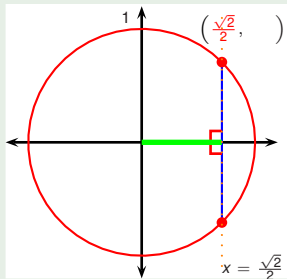
$$\cos \theta = \frac{\sqrt{2}}{2}$$



Example

Find all solutions and then find those that lie between -180° and 180° .

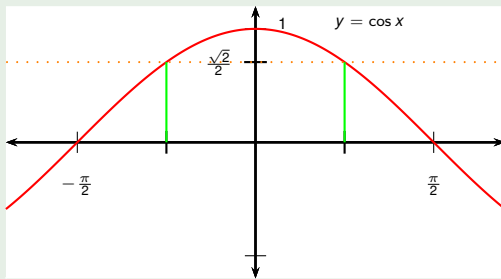
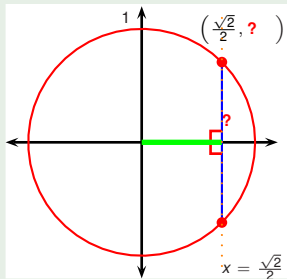
$$\cos \theta = \frac{\sqrt{2}}{2}$$



Example

Find all solutions and then find those that lie between -180° and 180° .

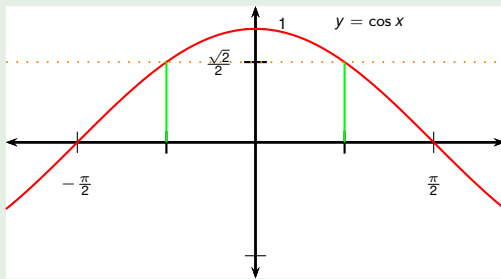
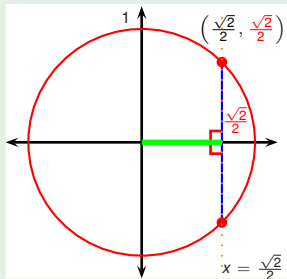
$$\cos \theta = \frac{\sqrt{2}}{2}$$



Example

Find all solutions and then find those that lie between -180° and 180° .

$$\cos \theta = \frac{\sqrt{2}}{2}$$

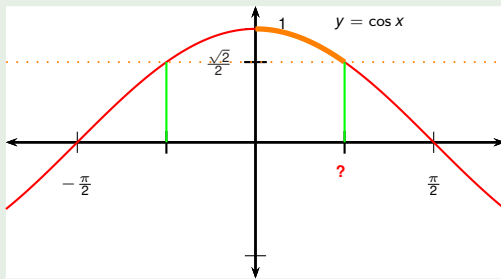
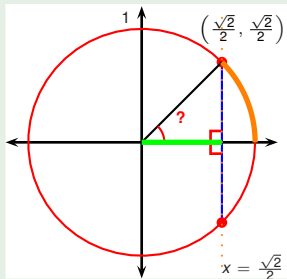


Example

Find all solutions and then find those that lie between -180° and 180° .

$$\cos \theta = \frac{\sqrt{2}}{2}$$

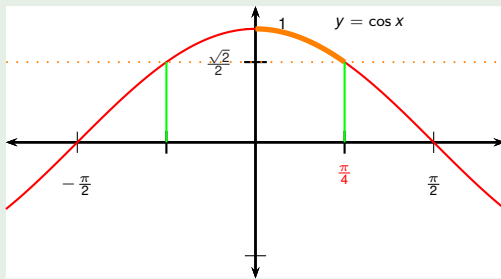
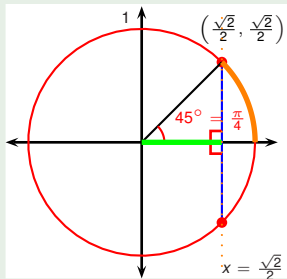
$$\theta = ?$$



Example

Find all solutions and then find those that lie between -180° and 180° .

$$\cos \theta = \frac{\sqrt{2}}{2}$$
$$\theta = 45^\circ$$



Example

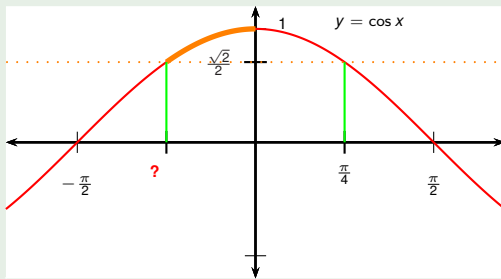
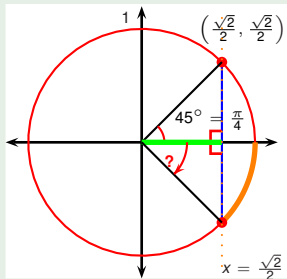
Find all solutions and then find those that lie between -180° and 180° .

$$\cos \theta = \frac{\sqrt{2}}{2}$$

$$\theta = 45^\circ$$

or

?



Example

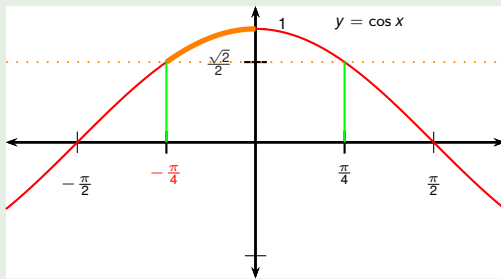
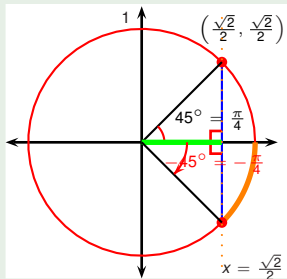
Find all solutions and then find those that lie between -180° and 180° .

$$\cos \theta = \frac{\sqrt{2}}{2}$$

$$\theta = 45^\circ$$

or

$$-45^\circ$$



Example

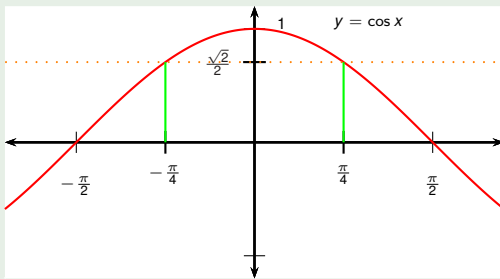
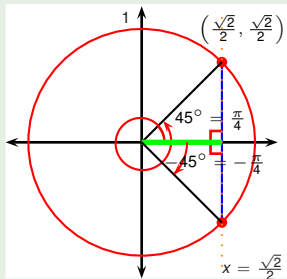
Find all solutions and then find those that lie between -180° and 180° .

$$\cos \theta = \frac{\sqrt{2}}{2}$$

$$\theta = 45^\circ + k \cdot 360^\circ$$

or

$$-45^\circ$$



Example

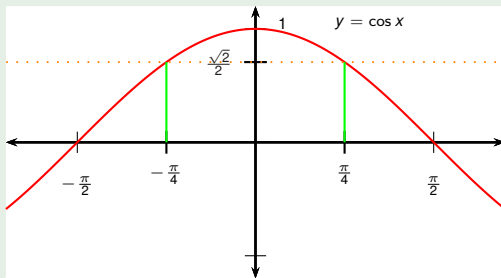
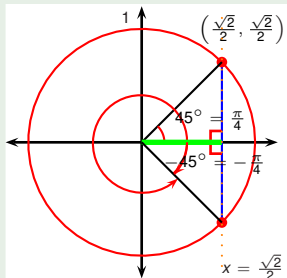
Find all solutions and then find those that lie between -180° and 180° .

$$\cos \theta = \frac{\sqrt{2}}{2}$$

$$\theta = 45^\circ + k \cdot 360^\circ$$

or

$$-45^\circ + k \cdot 360^\circ$$



Example

Find all solutions and then find those that lie between -180° and 180° .

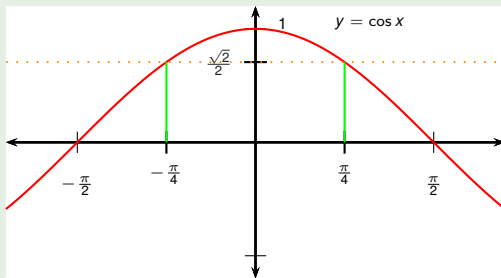
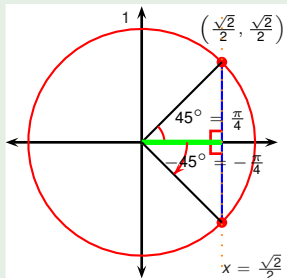
$$\cos \theta = \frac{\sqrt{2}}{2}$$

$$\theta = 45^\circ + k \cdot 360^\circ = \dots - 675^\circ,$$

or

$$\dots k = -2$$

$$-45^\circ + k \cdot 360^\circ = \dots - 765^\circ,$$



Example

Find all solutions and then find those that lie between -180° and 180° .

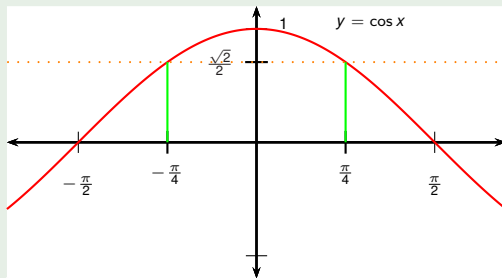
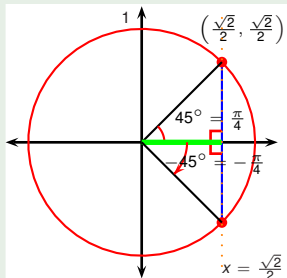
$$\cos \theta = \frac{\sqrt{2}}{2}$$

$$\theta = 45^\circ + k \cdot 360^\circ = \dots - 675^\circ, -315^\circ,$$

or

$$\dots \quad k=-2 \quad k=-1$$

$$-45^\circ + k \cdot 360^\circ = \dots - 765^\circ, -405^\circ,$$



Example

Find all solutions and then find those that lie between -180° and 180° .

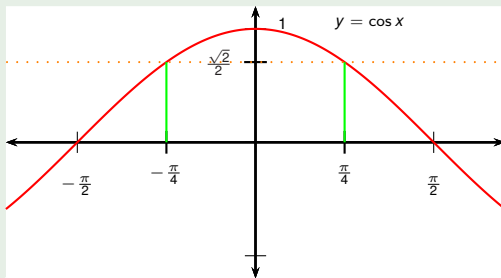
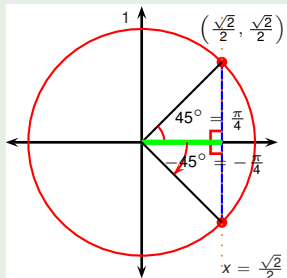
$$\cos \theta = \frac{\sqrt{2}}{2}$$

$$\theta = 45^\circ + k \cdot 360^\circ = \dots - 675^\circ, -315^\circ, 45^\circ,$$

or

$$\dots \quad k=-2 \quad k=-1 \quad k=0$$

$$-45^\circ + k \cdot 360^\circ = \dots - 765^\circ, -405^\circ, -45^\circ,$$



Example

Find all solutions and then find those that lie between -180° and 180° .

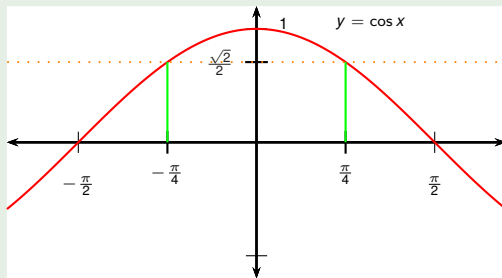
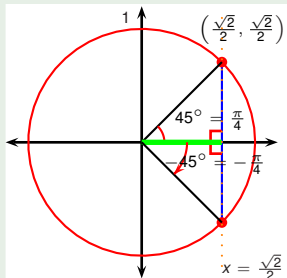
$$\cos \theta = \frac{\sqrt{2}}{2}$$

$$\theta = 45^\circ + k \cdot 360^\circ = \dots - 675^\circ, -315^\circ, 45^\circ, 405^\circ, \dots$$

or

$$\dots \quad k=-2 \quad k=-1 \quad k=0 \quad k=1 \quad \dots$$

$$-45^\circ + k \cdot 360^\circ = \dots - 765^\circ, -405^\circ, -45^\circ, 315^\circ, \dots$$



Example

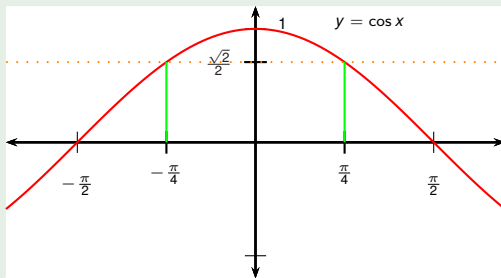
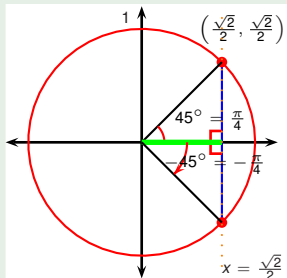
Find all solutions and then find those that lie between -180° and 180° .

$$\cos \theta = \frac{\sqrt{2}}{2}$$

$$\theta = 45^\circ + k \cdot 360^\circ = \dots - 675^\circ, -315^\circ, 45^\circ, 405^\circ, \dots$$

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$$-45^\circ + k \cdot 360^\circ = \dots - 765^\circ, -405^\circ, -45^\circ, 315^\circ, \dots$$



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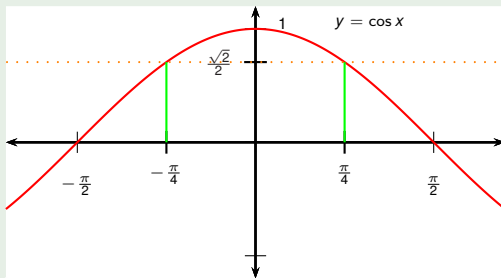
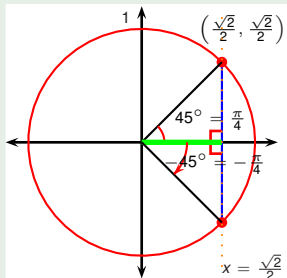
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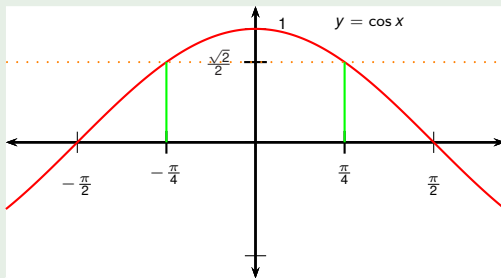
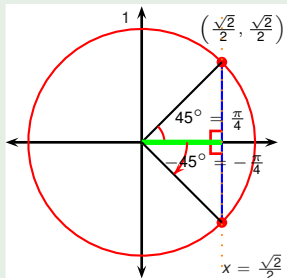
$$\dots \quad k=-2 \quad k=-1 \quad k=0 \quad k=1 \quad \dots$$

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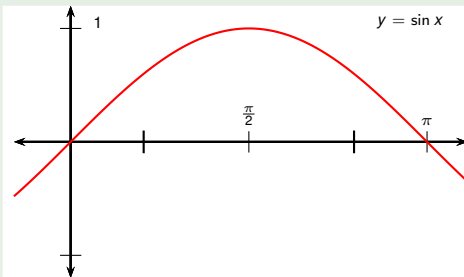
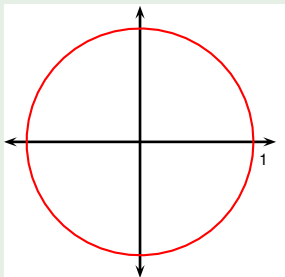
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Example

Find all solutions of the equation.

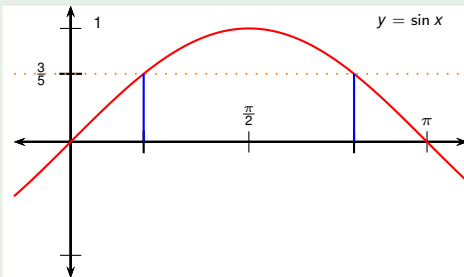
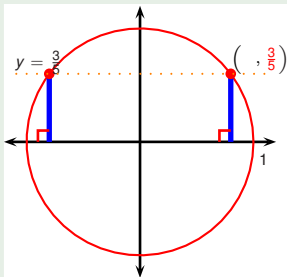
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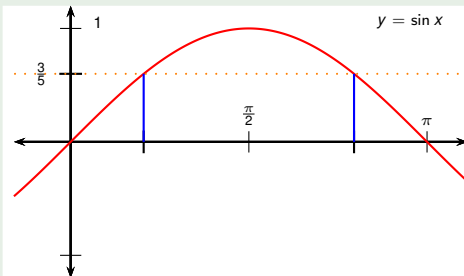
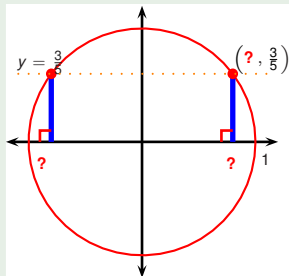
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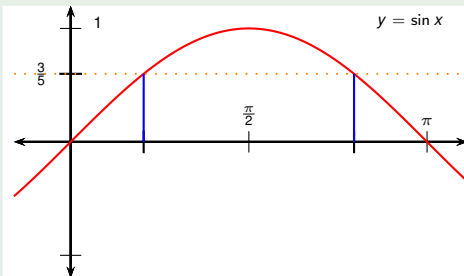
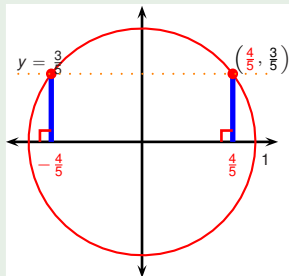
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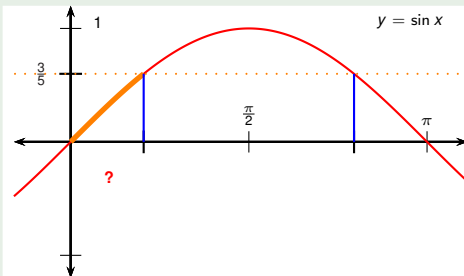
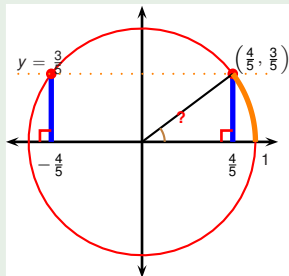


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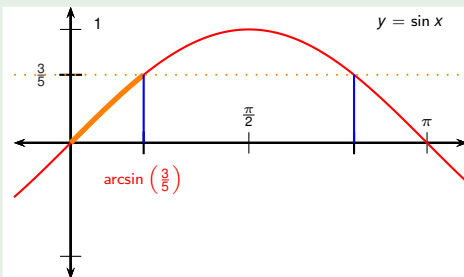
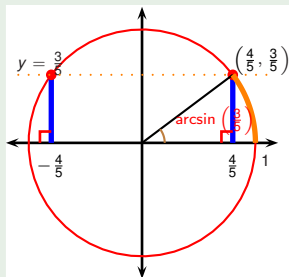
Example

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\arcsin implies radians



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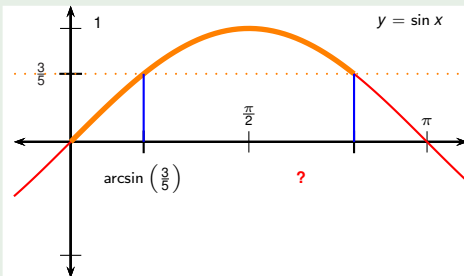
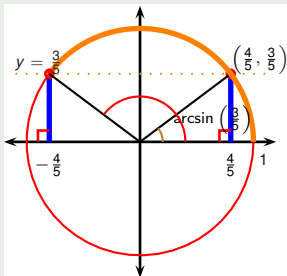
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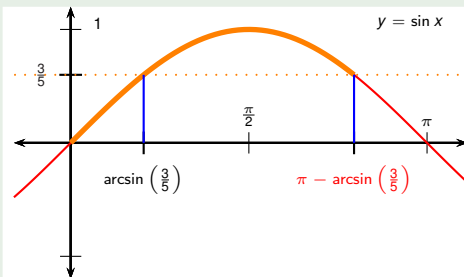
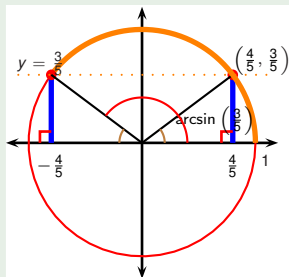
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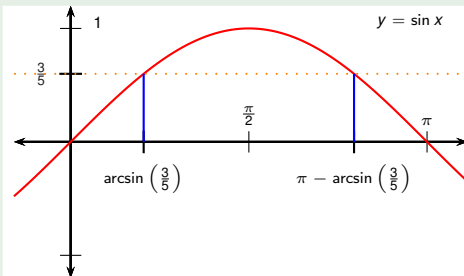
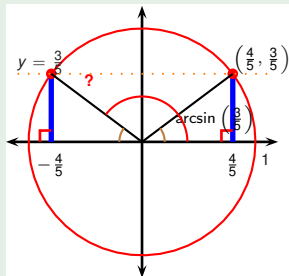
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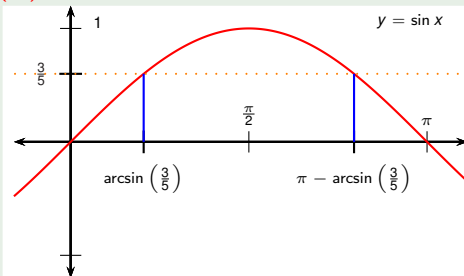
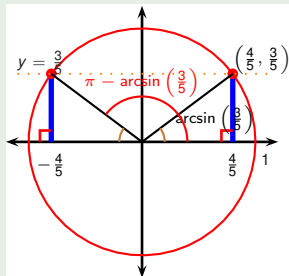
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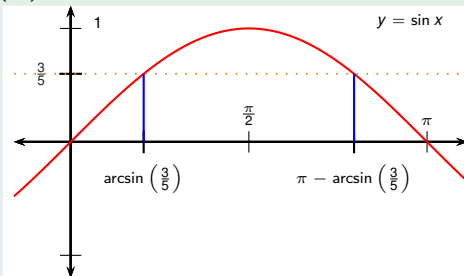
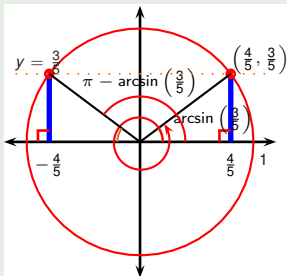
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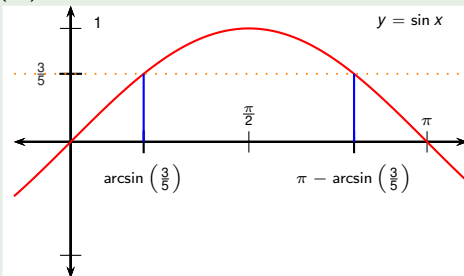
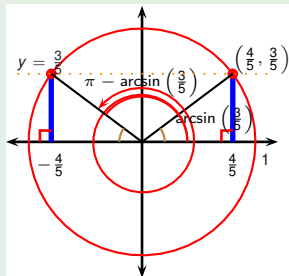
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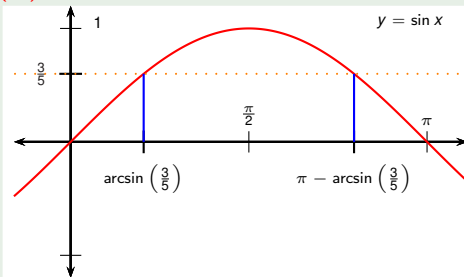
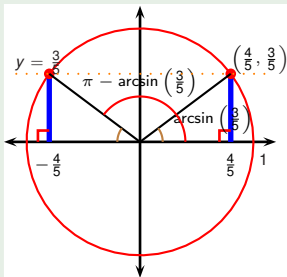
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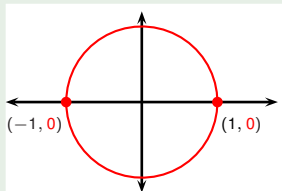
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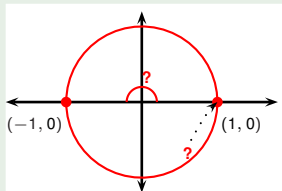
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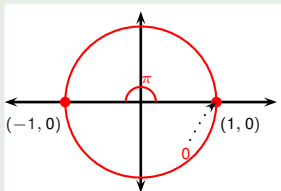
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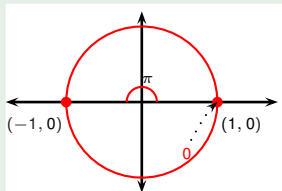
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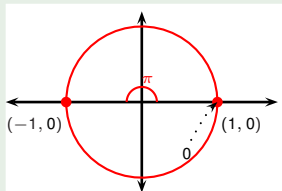
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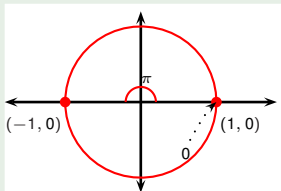
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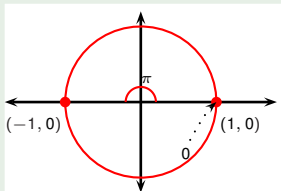
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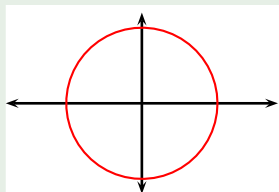
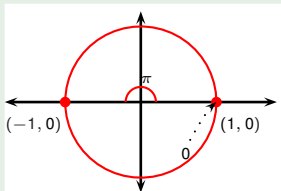
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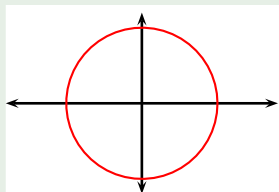
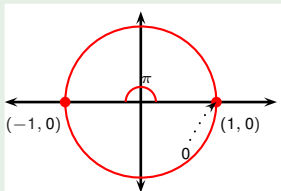
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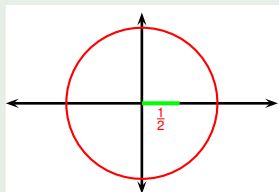
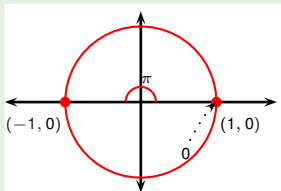
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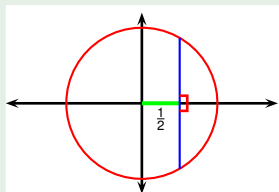
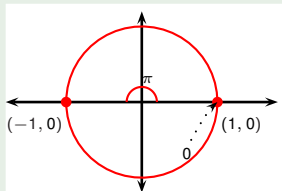
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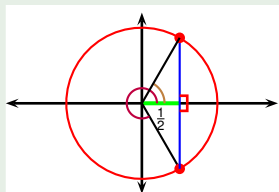
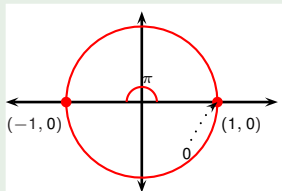
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$$\sin \theta = \sin(2\theta)$$

$$\sin \theta = 2 \sin \theta \cos \theta$$

$$0 = 2 \sin \theta \cos \theta - \sin \theta$$

$$0 = \sin \theta (2 \cos \theta - 1)$$

$$\sin \theta = 0$$

$$\theta = 0 + 2k\pi$$

$$\text{or } \pi + 2k\pi$$

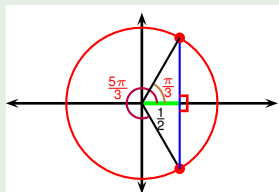
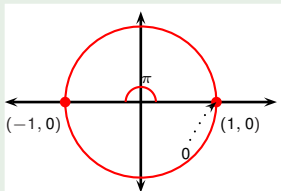
$$\theta = 0 \text{ or } 2\pi \text{ or } \pi$$

or

$$2 \cos \theta - 1 = 0$$

$$\cos \theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{3} + 2k\pi \text{ or } \frac{5\pi}{3} + 2k\pi$$



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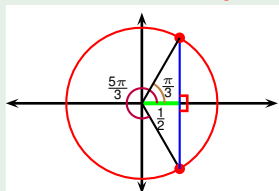
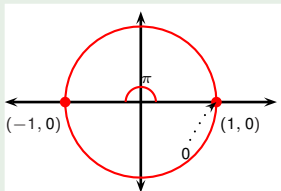
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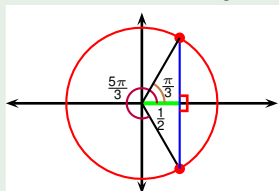
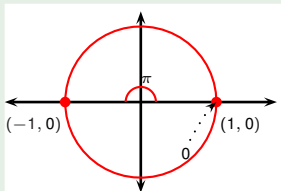
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Example

Find all values of θ in the interval $\theta \in [0, 2\pi]$ for which

$$\cos(2\theta) = \cos \theta$$

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Find all values of θ in the interval $\theta \in [0, 2\pi]$ for which

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?

$$- \cos \theta = 0$$

Example

Find all values of θ in the interval $\theta \in [0, 2\pi]$ for which

$$\begin{aligned} \cos(2\theta) &= \cos \theta \\ -\cos \theta &= 0 \end{aligned}$$

?

Example

Find all values of θ in the interval $\theta \in [0, 2\pi]$ for which

$$\begin{aligned}\cos(2\theta) &= \cos \theta \\ \cos^2 \theta - \sin^2 \theta - \cos \theta &= 0\end{aligned}$$

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Find all values of θ in the interval $\theta \in [0, 2\pi]$ for which

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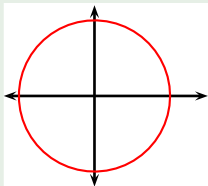
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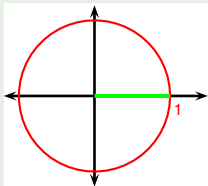
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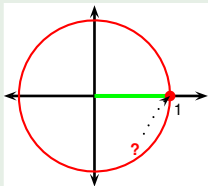
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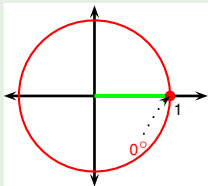
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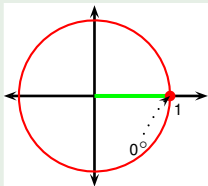
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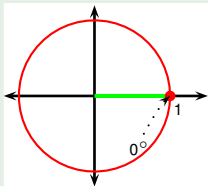
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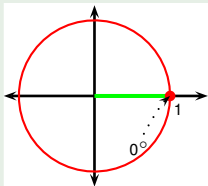
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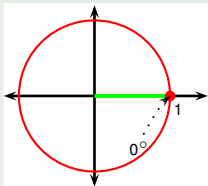
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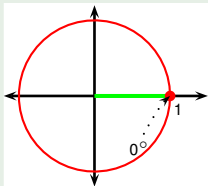
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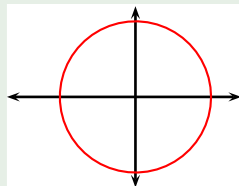
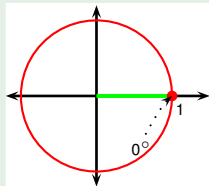
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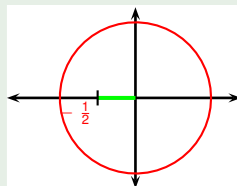
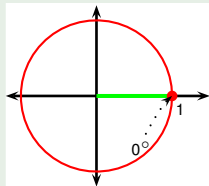
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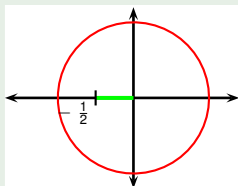
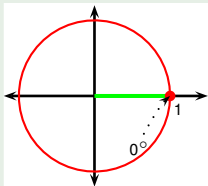
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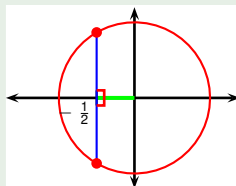
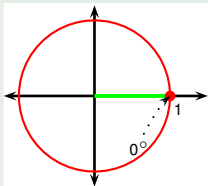
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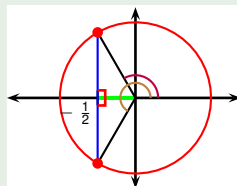
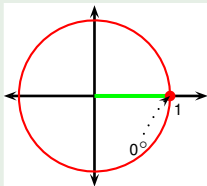
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$$\theta = 0 + 2k\pi$$

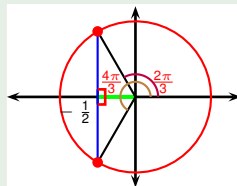
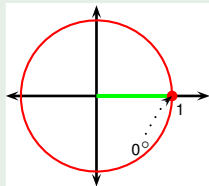
$$\theta = 0 \text{ or } 2\pi$$

or

$$2u + 1 = 0$$

$$\cos \theta = -\frac{1}{2}$$

$$\theta = \frac{2\pi}{3} + 2k\pi \text{ or } \frac{4\pi}{3} + 2k\pi$$



Example

Find all values of θ in the interval $\theta \in [0, 2\pi]$ for which

$$\cos(2\theta) = \cos \theta$$

$$\cos^2 \theta - \sin^2 \theta - \cos \theta = 0 \quad \left| \text{Express via } \cos \theta \right.$$

$$\cos^2 \theta - (1 - \cos^2 \theta) - \cos \theta = 0$$

$$2 \cos^2 \theta - \cos \theta - 1 = 0 \quad \left| \text{Set } \cos \theta = u \right.$$

$$2u^2 - u - 1 = 0$$

$$(u - 1)(2u + 1) = 0$$

$$u - 1 = 0$$

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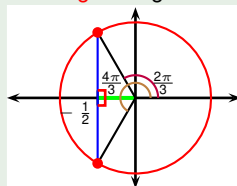
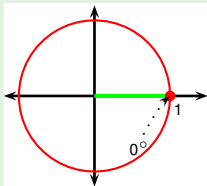
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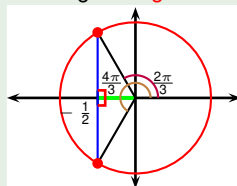
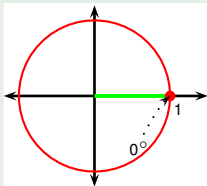
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Strategy for solving trigonometric equations

- Suppose we want to solve an algebraic trigonometric equation.
- More precisely, the equation should be an algebraic expressions of the trigonometric functions of a single variable.
- Here is a general strategy for solving such a problem:
 - Using trig identities, rewrite in terms of $\sin x$ and $\cos x$ only.
 - Suppose $x \in [2n\pi, (2n+1)\pi]$.
 - Set $\sin x = \sqrt{1 - \cos^2 x}$ (allowed due to restrictions on x).
 - Set $\cos x = u$. Solve the resulting algebraic equation for u .
 - For the found solutions for u , solve $\cos x = u$.
 - Check whether your solutions satisfy $x \in [2n\pi, (2n+1)\pi]$.
 - Suppose $x \in [(2n-1)\pi, 2n\pi]$.
 - Set $\sin x = -\sqrt{1 - \cos^2 x}$ (allowed due to restrictions on x).
 - Set $\cos x = u$. Solve the resulting algebraic equation for u .
 - For the found solutions for u , solve $\cos x = u$.
 - Check whether your solutions satisfy $x \in [(2n-1)\pi, 2n\pi]$.
- A similar strategy exists for $u = \sin x$ instead of $u = \cos x$.
- Problems requiring full algorithm may be too hard for Calc exams.

Proposition (Product to sum formulas)

$$\begin{aligned}\sin \alpha \sin \beta &= \frac{1}{2} (\cos(\alpha - \beta) - \cos(\alpha + \beta)) \\ \cos \alpha \cos \beta &= \frac{1}{2} (\cos(\alpha - \beta) + \cos(\alpha + \beta)) \\ \sin \alpha \cos \beta &= \frac{1}{2} (\sin(\alpha + \beta) + \sin(\alpha - \beta))\end{aligned}$$

Proof.



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Proof.

?

$$= \cos(\alpha + \beta)$$



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Proof.

$$\cos \alpha \cos \beta - \sin \alpha \sin \beta = \cos(\alpha + \beta)$$



Proposition (Product to sum formulas)

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Proof.

$$\begin{aligned}\textcolor{red}{?} &= \textcolor{red}{\cos(\alpha - \beta)} \\ \cos \alpha \cos \beta - \sin \alpha \sin \beta &= \cos(\alpha + \beta)\end{aligned}$$



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Proof.

$$\begin{aligned}\cos \alpha \cos \beta + \sin \alpha \sin \beta &= \cos(\alpha - \beta) \\ \cos \alpha \cos \beta - \sin \alpha \sin \beta &= \cos(\alpha + \beta)\end{aligned}$$



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$$\begin{aligned}\sin \alpha \sin \beta &= \frac{1}{2} (\cos(\alpha - \beta) - \cos(\alpha + \beta)) \\ \cos \alpha \cos \beta &= \frac{1}{2} (\cos(\alpha - \beta) + \cos(\alpha + \beta)) \\ \sin \alpha \cos \beta &= \frac{1}{2} (\sin(\alpha + \beta) + \sin(\alpha - \beta))\end{aligned}$$

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Proof.

$$\begin{aligned}+ \quad & \cos \alpha \cos \beta + \sin \alpha \sin \beta = \cos(\alpha - \beta) \\ & \cos \alpha \cos \beta - \sin \alpha \sin \beta = \cos(\alpha + \beta) \\ & \quad ? = \cos(\alpha - \beta) + \cos(\alpha + \beta)\end{aligned}$$



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$$\sin \alpha \sin \beta = \frac{1}{2} (\cos(\alpha - \beta) - \cos(\alpha + \beta))$$

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$$\sin \alpha \cos \beta = \frac{1}{2} (\sin(\alpha + \beta) + \sin(\alpha - \beta))$$

Proof.

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 + \quad & \cos \alpha \cos \beta + \cancel{\sin \alpha \sin \beta} = \cos(\alpha - \beta) \\
 & \cos \alpha \cos \beta - \cancel{\sin \alpha \sin \beta} = \cos(\alpha + \beta) \\
 & \quad \quad \quad 2 \cos \alpha \cos \beta = \cos(\alpha - \beta) + \cos(\alpha + \beta)
 \end{aligned}$$



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Proof.

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Proposition (Product to sum formulas)

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 \sin \alpha \sin \beta &= \frac{1}{2} (\cos(\alpha - \beta) - \cos(\alpha + \beta)) \\
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 \sin \alpha \cos \beta &= \frac{1}{2} (\sin(\alpha + \beta) + \sin(\alpha - \beta))
 \end{aligned}$$

Proof.

$$\begin{array}{rcl}
 + & \cos \alpha \cos \beta + \cancel{\sin \alpha \sin \beta} & = \cos(\alpha - \beta) \\
 & \cos \alpha \cos \beta - \cancel{\sin \alpha \sin \beta} & = \cos(\alpha + \beta) \\
 & \hline
 & 2 \cos \alpha \cos \beta & = \cos(\alpha - \beta) + \cos(\alpha + \beta) \\
 - & \cos \alpha \cos \beta + \sin \alpha \sin \beta & = \cos(\alpha - \beta) \\
 & \cos \alpha \cos \beta - \sin \alpha \sin \beta & = \cos(\alpha + \beta) \\
 & \hline
 & 2 \sin \alpha \sin \beta & = \cos(\alpha - \beta) - \cos(\alpha + \beta)
 \end{array}$$



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Proof.

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Proposition (Product to sum formulas)

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- Product to sum formulas are used when integrating (a topic to be studied later/in another course).

Proposition (Sum to product formulas)

$$\sin \alpha + \sin \beta = 2 \sin \left(\frac{\alpha + \beta}{2} \right) \cos \left(\frac{\alpha - \beta}{2} \right)$$

$$\sin \alpha - \sin \beta = 2 \sin \left(\frac{\alpha - \beta}{2} \right) \cos \left(\frac{\alpha + \beta}{2} \right)$$

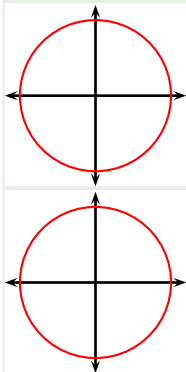
$$\cos \alpha + \cos \beta = 2 \cos \left(\frac{\alpha + \beta}{2} \right) \cos \left(\frac{\alpha - \beta}{2} \right)$$

$$\cos \alpha - \cos \beta = 2 \sin \left(\frac{\alpha + \beta}{2} \right) \sin \left(\frac{\alpha - \beta}{2} \right)$$

Example

Find all solutions in the interval $[0, 2\pi)$.

$$\sin(2x) + \sin(5x) = 0$$

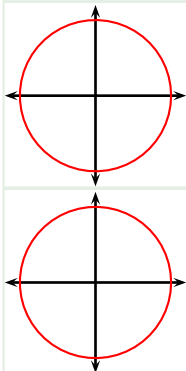


Recall the formula $\sin \alpha + \sin \beta = ?$

Example

Find all solutions in the interval $[0, 2\pi)$.

$$\sin(2x) + \sin(5x) = 0 \quad | \quad \text{use f-l-a}$$

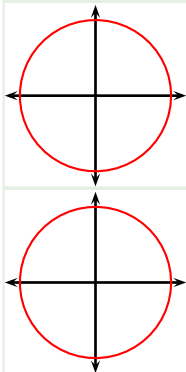


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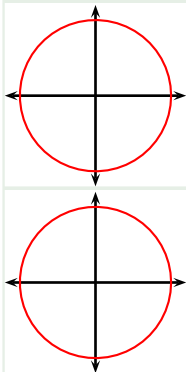
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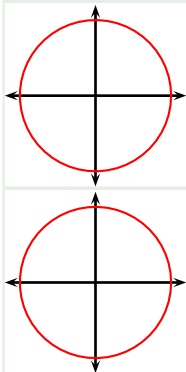


$$\begin{array}{l} \sin(2x) + \sin(5x) = 0 \quad | \text{ use f-l-a} \\ 2 \sin \left(\frac{2x + 5x}{2} \right) \cos \left(\frac{2x - 5x}{2} \right) = 0 \end{array}$$

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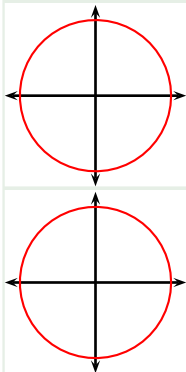
$$\sin(2x) + \sin(5x) = 0 \quad | \quad \text{use f-la}$$

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Recall the formula $\sin \alpha + \sin \beta = 2 \sin \left(\frac{\alpha + \beta}{2} \right) \cos \left(\frac{\alpha - \beta}{2} \right)$

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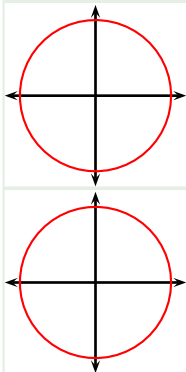


$$\begin{aligned} \sin(2x) + \sin(5x) &= 0 \quad | \text{ use f-l-a} \\ 2 \sin \left(\frac{2x + 5x}{2} \right) \cos \left(\frac{2x - 5x}{2} \right) &= 0 \\ 2 \sin \left(\frac{7}{2}x \right) \cos \left(-\frac{3}{2}x \right) &= 0 \end{aligned}$$

Recall the formula $\sin \alpha + \sin \beta = 2 \sin \left(\frac{\alpha + \beta}{2} \right) \cos \left(\frac{\alpha - \beta}{2} \right)$

Example

Find all solutions in the interval $[0, 2\pi)$.

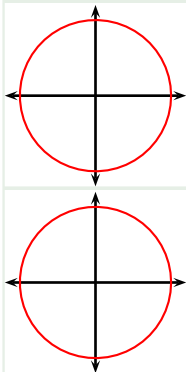


$$\begin{aligned} \sin(2x) + \sin(5x) &= 0 && \text{use f-l-a} \\ 2 \sin \left(\frac{2x + 5x}{2} \right) \cos \left(\frac{2x - 5x}{2} \right) &= 0 \\ 2 \sin \left(\frac{7}{2}x \right) \cos \left(-\frac{3}{2}x \right) &= 0 && \left. \begin{array}{l} \cos \\ \text{is even} \end{array} \right\} \\ 2 \sin \left(\frac{7}{2}x \right) \cos \left(\frac{3}{2}x \right) &= 0 \end{aligned}$$

Recall the formula $\sin \alpha + \sin \beta = 2 \sin \left(\frac{\alpha + \beta}{2} \right) \cos \left(\frac{\alpha - \beta}{2} \right)$

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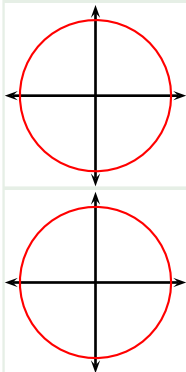


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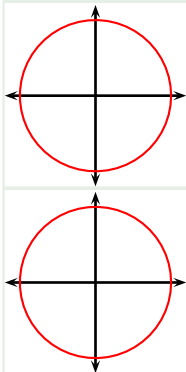


$$\sin(2x) + \sin(5x) = 0 \quad | \quad \text{use f-l-a} \\ 2 \sin \left(\frac{7}{2}x \right) \cos \left(\frac{3}{2}x \right) = 0$$

Recall the formula $\sin \alpha + \sin \beta = 2 \sin \left(\frac{\alpha + \beta}{2} \right) \cos \left(\frac{\alpha - \beta}{2} \right)$

Example

Find all solutions in the interval $[0, 2\pi)$.



$$\sin \left(\frac{7}{2}x \right) = 0$$

$$\sin(2x) + \sin(5x) = 0 \quad | \text{ use f-la}$$
$$2 \sin \left(\frac{7}{2}x \right) \cos \left(\frac{3}{2}x \right) = 0$$

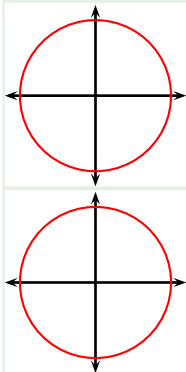
or

$$\cos \left(\frac{3}{2}x \right) = 0$$

Recall the formula $\sin \alpha + \sin \beta = 2 \sin \left(\frac{\alpha + \beta}{2} \right) \cos \left(\frac{\alpha - \beta}{2} \right)$

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Find all solutions in the interval $[0, 2\pi)$.



$$\sin \left(\frac{7}{2}x \right) = 0$$

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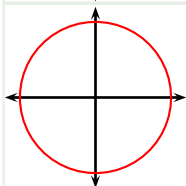
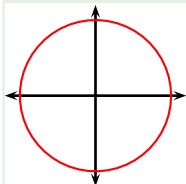
$$\cos \left(\frac{3}{2}x \right) = 0$$

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Recall the formula $\sin \alpha + \sin \beta = 2 \sin \left(\frac{\alpha + \beta}{2} \right) \cos \left(\frac{\alpha - \beta}{2} \right)$

Example

Find all solutions in the interval $[0, 2\pi)$.



$$\sin \left(\frac{7}{2}x \right) = 0$$

$$\frac{7}{2}x = ?$$

$$\sin(2x) + \sin(5x) = 0 \quad | \text{ use f-la}$$

$$2 \sin \left(\frac{7}{2}x \right) \cos \left(\frac{3}{2}x \right) = 0$$

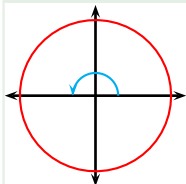
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Example

Find all solutions in the interval $[0, 2\pi)$.

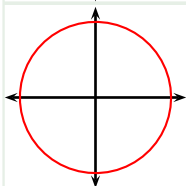


$$\sin \left(\frac{7}{2}x \right) = 0$$

$$\frac{7}{2}x = ?$$

$$\sin(2x) + \sin(5x) = 0 \quad | \text{ use f-l-a}$$

$$2 \sin \left(\frac{7}{2}x \right) \cos \left(\frac{3}{2}x \right) = 0$$



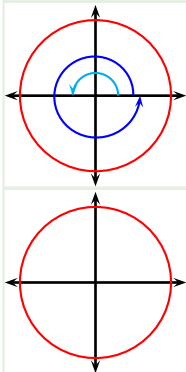
$$\cos \left(\frac{3}{2}x \right) = 0$$

or

Recall the formula $\sin \alpha + \sin \beta = 2 \sin \left(\frac{\alpha + \beta}{2} \right) \cos \left(\frac{\alpha - \beta}{2} \right)$

Example

Find all solutions in the interval $[0, 2\pi)$.



$$\sin \left(\frac{7}{2}x \right) = 0$$

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$$2 \sin \left(\frac{7}{2}x \right) \cos \left(\frac{3}{2}x \right) = 0$$

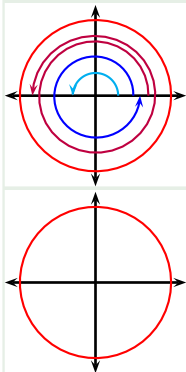
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Example

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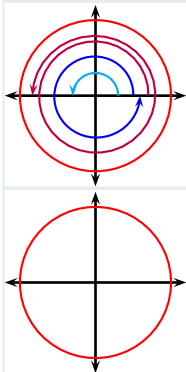
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Example

Find all solutions in the interval $[0, 2\pi)$.



$$\sin \left(\frac{7}{2}x \right) = 0$$

$$\frac{7}{2}x = k\pi$$

$$\sin(2x) + \sin(5x) = 0 \quad | \text{ use f-l-a}$$

$$2 \sin \left(\frac{7}{2}x \right) \cos \left(\frac{3}{2}x \right) = 0$$

k – integer

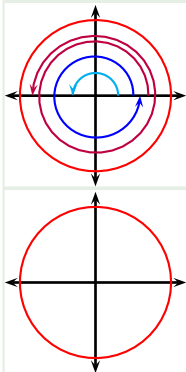
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Example

Find all solutions in the interval $[0, 2\pi)$.



$$\sin(2x) + \sin(5x) = 0 \quad | \text{ use f-l-a}$$

$$2 \sin \left(\frac{7}{2}x \right) \cos \left(\frac{3}{2}x \right) = 0$$

$$\sin \left(\frac{7}{2}x \right) = 0$$

$$\frac{7}{2}x = k\pi$$

$$x = \frac{2k\pi}{7}$$

k – integer

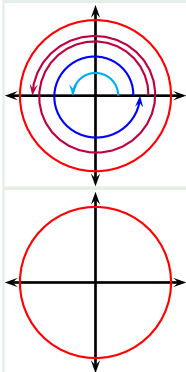
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k – integer

$$x = \dots, -\frac{2\pi}{7}, 0, \frac{2\pi}{7}, \frac{4\pi}{7}, \frac{6\pi}{7}, \frac{8\pi}{7}, \frac{10\pi}{7}, \frac{12\pi}{7}, \frac{14\pi}{7}, \dots$$

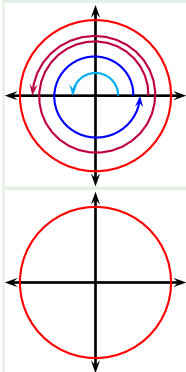
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Example

Find all solutions **in the interval $[0, 2\pi)$** .



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$$\sin \left(\frac{7}{2}x \right) = 0$$

$$\frac{7}{2}x = k\pi$$

$$x = \frac{2k\pi}{7}$$

k – integer

$$x = \cancel{0}, \cancel{\frac{-2\pi}{7}}, 0, \frac{2\pi}{7}, \frac{4\pi}{7}, \frac{6\pi}{7}, \frac{8\pi}{7}, \frac{10\pi}{7}, \frac{12\pi}{7}, \cancel{\frac{14\pi}{7}}, \cancel{2\pi}$$

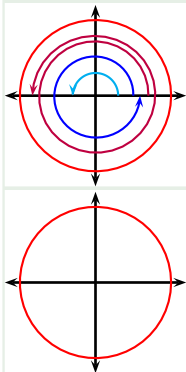
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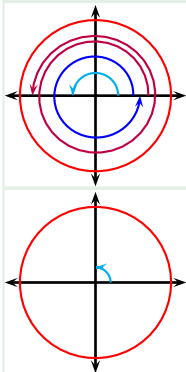
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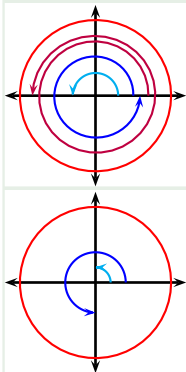
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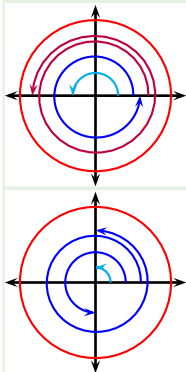
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or

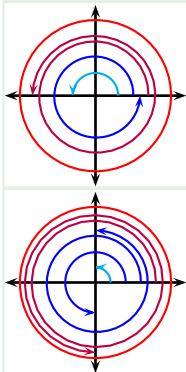
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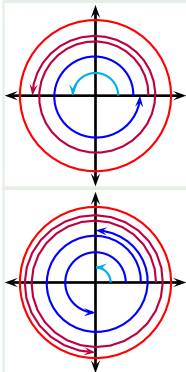
$$\frac{3}{2}x = \frac{\pi}{2} + k\pi$$

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Recall the formula $\sin \alpha + \sin \beta = 2 \sin \left(\frac{\alpha + \beta}{2} \right) \cos \left(\frac{\alpha - \beta}{2} \right)$

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Find all solutions in the interval $[0, 2\pi)$.



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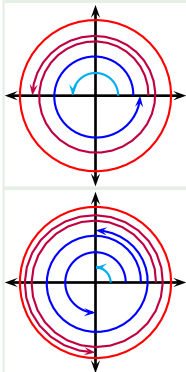
$$\frac{3}{2}x = \frac{\pi}{2} + k\pi = \frac{(2k+1)\pi}{2}$$

k – integer

Recall the formula $\sin \alpha + \sin \beta = 2 \sin \left(\frac{\alpha + \beta}{2} \right) \cos \left(\frac{\alpha - \beta}{2} \right)$

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Find all solutions in the interval $[0, 2\pi)$.



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or

$$\cos \left(\frac{3}{2}x \right) = 0$$

$$\textcolor{red}{2}x = \frac{\pi}{2} + k\pi = \frac{(2k+1)\pi}{2}$$

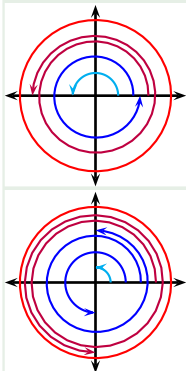
k – integer

$$x = \frac{(2k+1)\pi}{\textcolor{red}{3}}$$

Recall the formula $\sin \alpha + \sin \beta = 2 \sin \left(\frac{\alpha + \beta}{2} \right) \cos \left(\frac{\alpha - \beta}{2} \right)$

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$$x = \cancel{\dots}, \cancel{\frac{-2\pi}{7}}, 0, \frac{2\pi}{7}, \frac{4\pi}{7}, \frac{6\pi}{7}, \frac{8\pi}{7}, \frac{10\pi}{7}, \frac{12\pi}{7}, \cancel{\frac{14\pi}{7}}, \cancel{\dots}$$

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k – integer

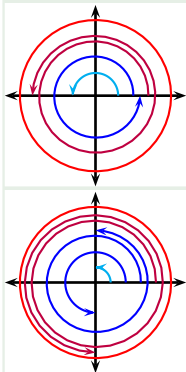
$$x = \frac{(2k+1)\pi}{3}$$

$$x = \dots, -\frac{\pi}{3}, \frac{\pi}{3}, \frac{3\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \dots$$

Recall the formula $\sin \alpha + \sin \beta = 2 \sin \left(\frac{\alpha + \beta}{2} \right) \cos \left(\frac{\alpha - \beta}{2} \right)$

Example

Find all solutions **in the interval $[0, 2\pi)$** .



$$\begin{aligned} \sin(2x) + \sin(5x) &= 0 \quad | \text{ use f-l-a} \\ 2 \sin \left(\frac{7}{2}x \right) \cos \left(\frac{3}{2}x \right) &= 0 \end{aligned}$$

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or

$$\cos \left(\frac{3}{2}x \right) = 0$$

$$\frac{3}{2}x = \frac{\pi}{2} + k\pi = \frac{(2k+1)\pi}{2}$$

k – integer

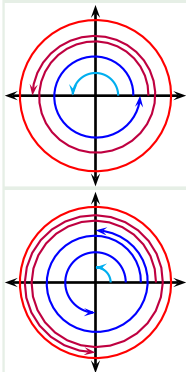
$$x = \frac{(2k+1)\pi}{3}$$

$$x = \cancel{\cdot}, \cancel{\frac{\pi}{3}}, \frac{\pi}{3}, \frac{3\pi}{3}, \frac{5\pi}{3}, \cancel{\frac{7\pi}{3}}, \cancel{\cdot}$$

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Example

Find all solutions in the interval $[0, 2\pi)$.



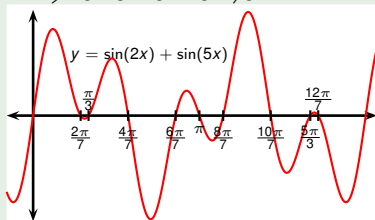
$$\sin(2x) + \sin(5x) = 0 \quad | \quad \text{use formula}$$

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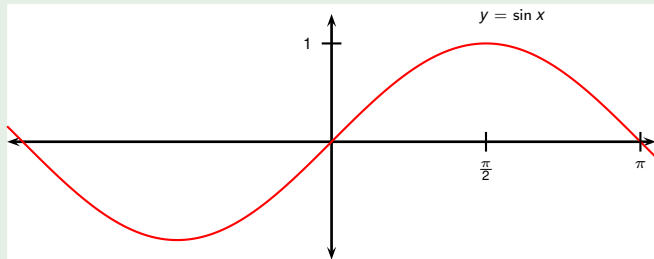
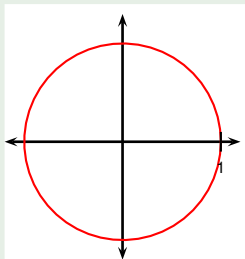
$$x = \cancel{\cdot}, \cancel{-\frac{\pi}{3}}, \frac{\pi}{3}, \frac{3\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \cancel{\cdot}$$



Example

Solve. Among your solutions, find those between -360° and 450° .

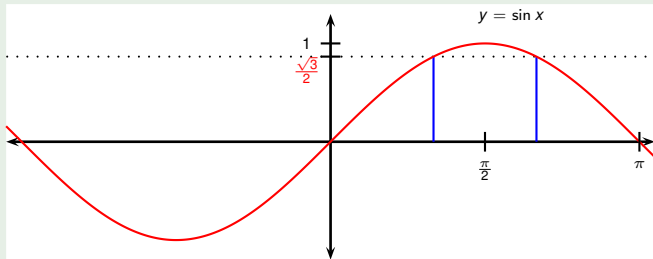
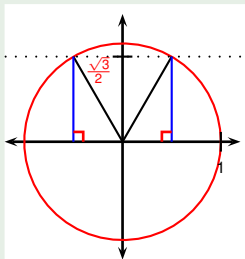
$$\frac{1}{2} \leq \sin \theta < \frac{\sqrt{3}}{2}$$



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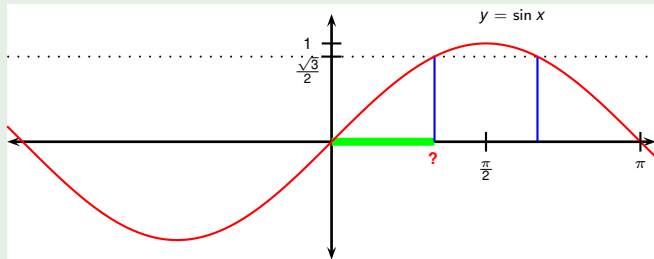
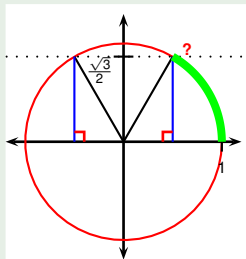
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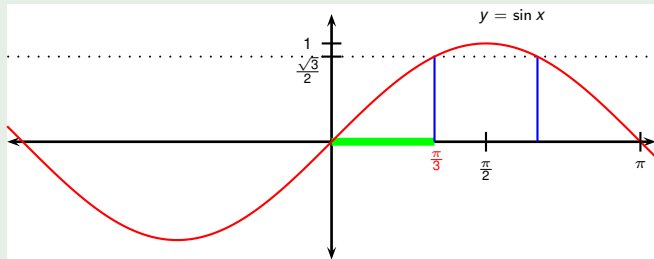
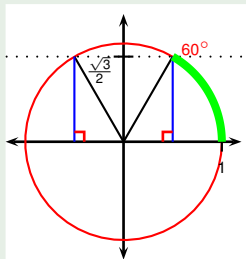
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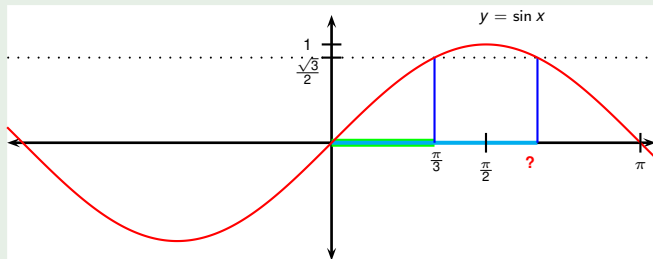
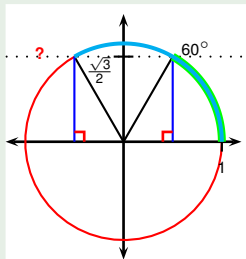
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Example

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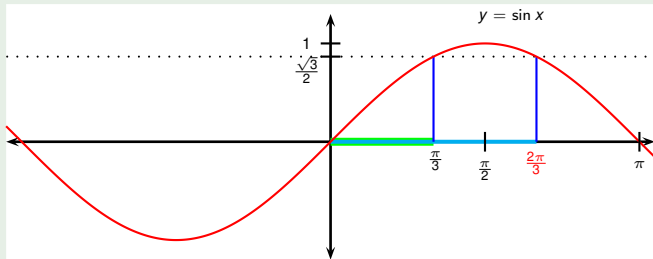
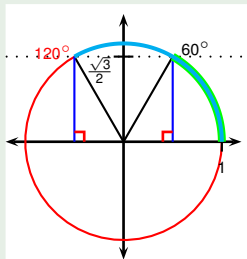
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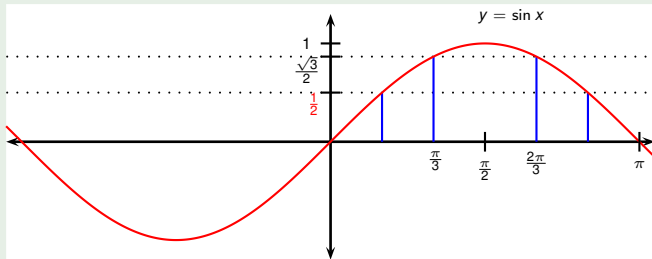
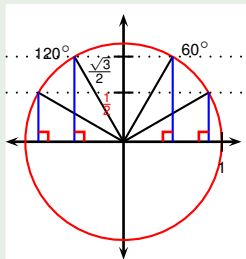
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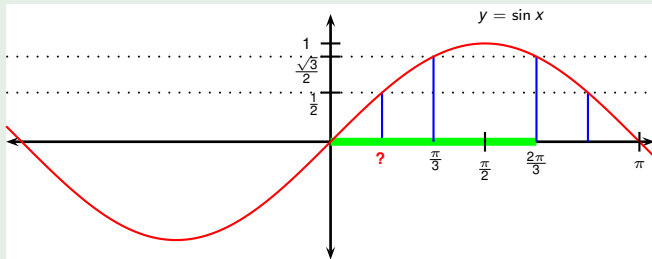
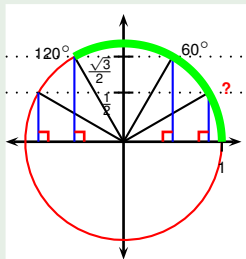
$$\frac{1}{2} \leq \sin \theta < \frac{\sqrt{3}}{2}$$



Example

Solve. Among your solutions, find those between -360° and 450° .

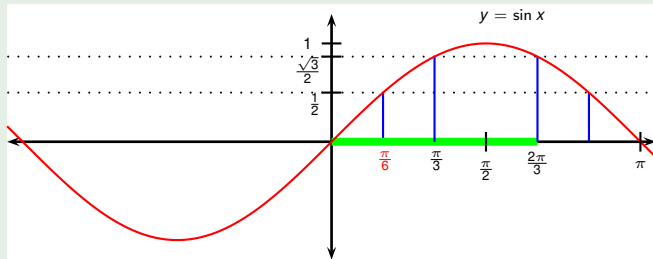
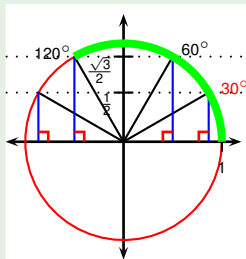
$$\frac{1}{2} \leq \sin \theta < \frac{\sqrt{3}}{2}$$



Example

Solve. Among your solutions, find those between -360° and 450° .

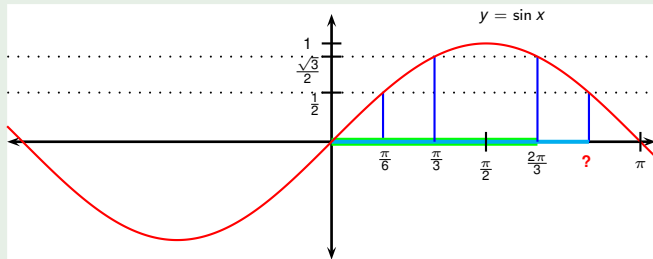
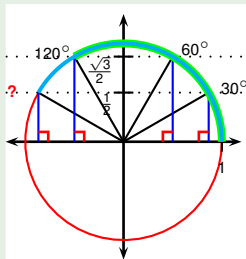
$$\frac{1}{2} \leq \sin \theta < \frac{\sqrt{3}}{2}$$



Example

Solve. Among your solutions, find those between -360° and 450° .

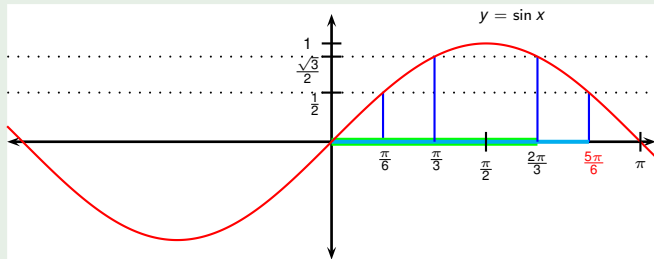
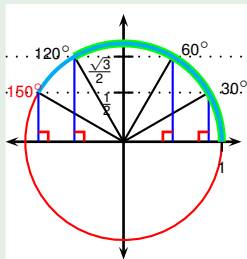
$$\frac{1}{2} \leq \sin \theta < \frac{\sqrt{3}}{2}$$



Example

Solve. Among your solutions, find those between -360° and 450° .

$$\frac{1}{2} \leq \sin \theta < \frac{\sqrt{3}}{2}$$

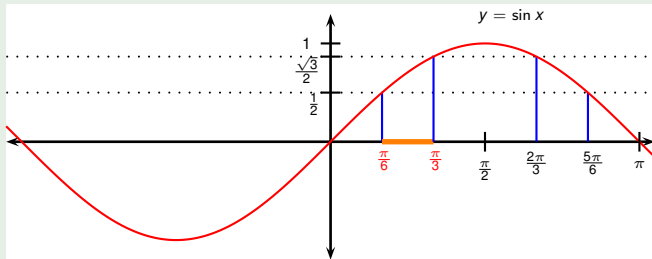
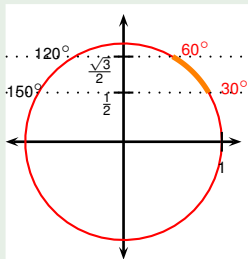


Example

Solve. Among your solutions, find those between -360° and 450° .

$$\frac{1}{2} \leq \sin \theta < \frac{\sqrt{3}}{2}$$

$$x \in [30^\circ, 60^\circ) \quad)$$

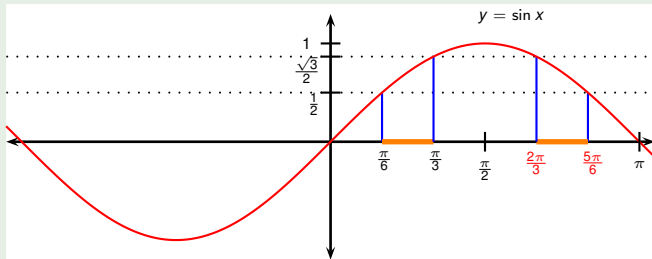
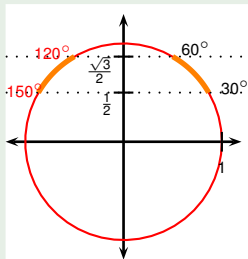


Example

Solve. Among your solutions, find those between -360° and 450° .

$$\frac{1}{2} \leq \sin \theta < \frac{\sqrt{3}}{2}$$

$$x \in [30^\circ, 60^\circ) \cup (120^\circ, 150^\circ]$$

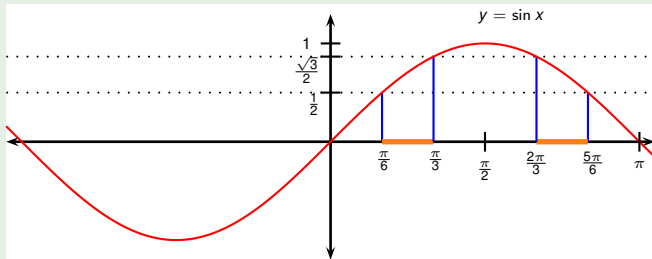
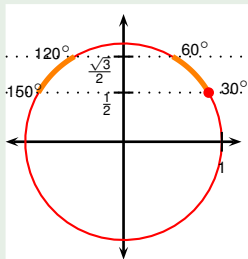


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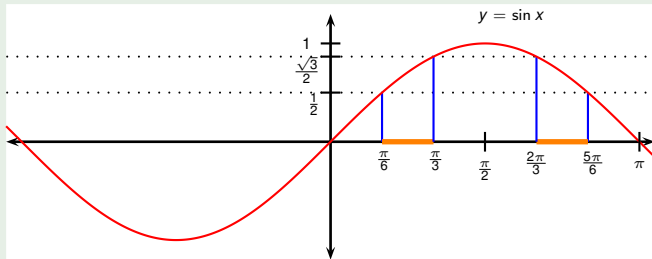
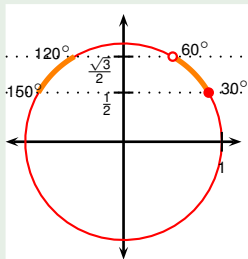


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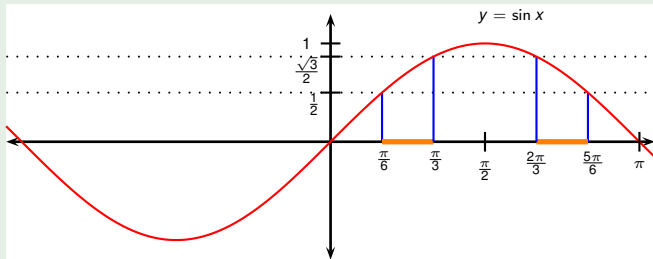
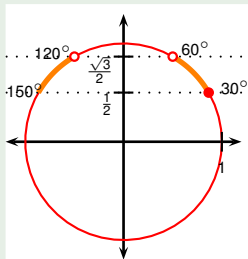


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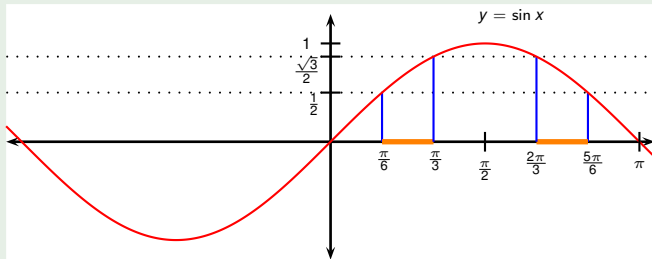
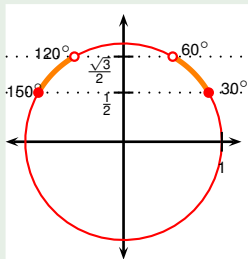


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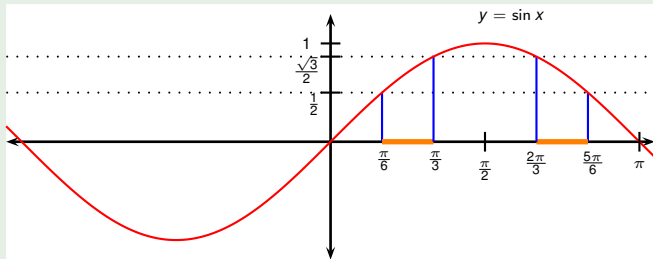
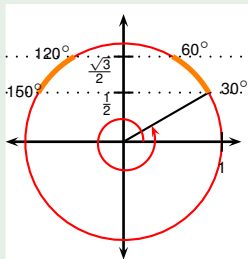


Example

Solve. Among your solutions, find those between -360° and 450° .

$$\frac{1}{2} \leq \sin \theta < \frac{\sqrt{3}}{2}$$

$$x \in [30^\circ + k360^\circ, 60^\circ + k360^\circ) \cup (120^\circ + k360^\circ, 150^\circ + k360^\circ]$$



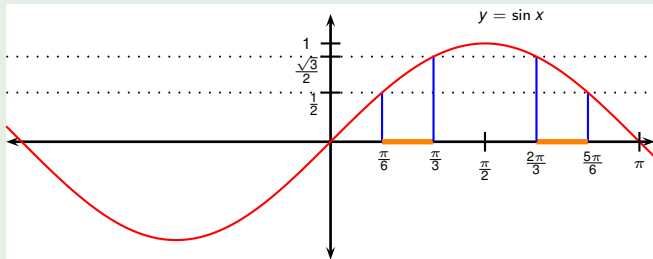
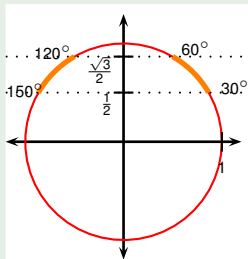
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Solve. Among your solutions, find those between -360° and 450° .

$$\frac{1}{2} \leq \sin \theta < \frac{\sqrt{3}}{2}$$

$$x \in [30^\circ + k360^\circ, 60^\circ + k360^\circ) \cup (120^\circ + k360^\circ, 150^\circ + k360^\circ]$$

$$x \in [30^\circ, 60^\circ) \cup (120^\circ, 150^\circ] \quad | \quad k = 0$$



Example

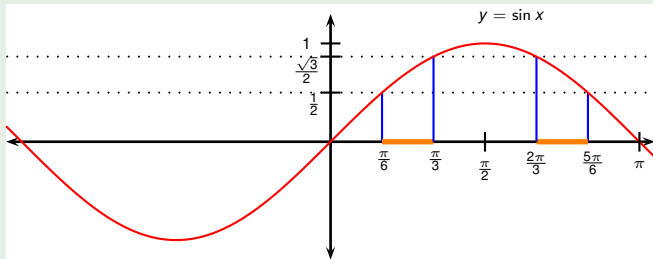
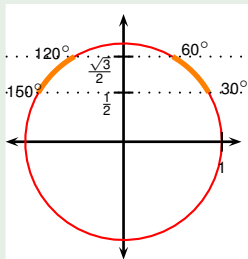
Solve. Among your solutions, find those between -360° and 450° .

$$\frac{1}{2} \leq \sin \theta < \frac{\sqrt{3}}{2}$$

$$x \in [30^\circ + k360^\circ, 60^\circ + k360^\circ) \cup (120^\circ + k360^\circ, 150^\circ + k360^\circ]$$

$$x \in [30^\circ, 60^\circ) \cup (120^\circ, 150^\circ] \quad \left| \begin{array}{l} k = 0 \\ k = 1 \end{array} \right.$$

$$\cup [390^\circ, 420^\circ) \cup (480^\circ, 510^\circ]$$



Example

Solve. Among your solutions, find those between -360° and 450° .

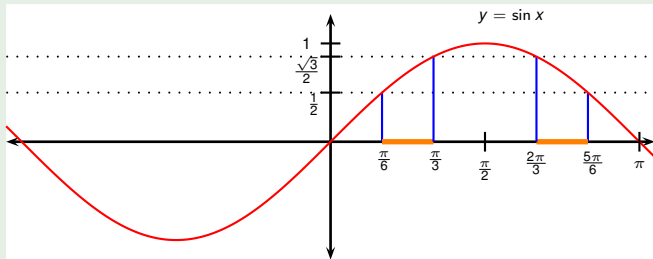
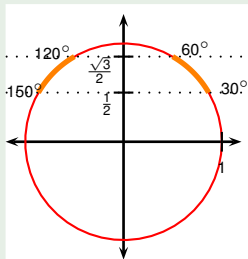
$$\frac{1}{2} \leq \sin \theta < \frac{\sqrt{3}}{2}$$

$$x \in [30^\circ + k360^\circ, 60^\circ + k360^\circ) \cup (120^\circ + k360^\circ, 150^\circ + k360^\circ]$$

$$x \in [30^\circ, 60^\circ) \cup (120^\circ, 150^\circ] \cup [390^\circ, 420^\circ) \cup (480^\circ, 510^\circ]$$

$$\left| \begin{array}{l} k = 0 \\ k = 1 \end{array} \right.$$

...



Example

Solve. Among your solutions, find those between -360° and 450° .

$$\frac{1}{2} \leq \sin \theta < \frac{\sqrt{3}}{2}$$

$$x \in [30^\circ + k360^\circ, 60^\circ + k360^\circ) \cup (120^\circ + k360^\circ, 150^\circ + k360^\circ]$$

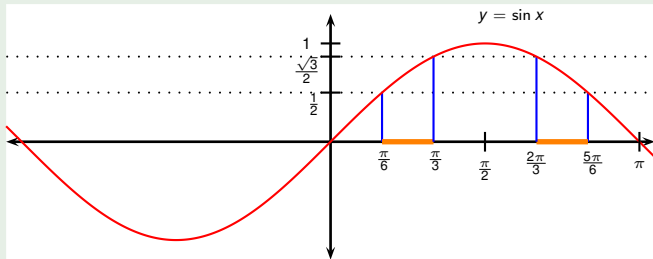
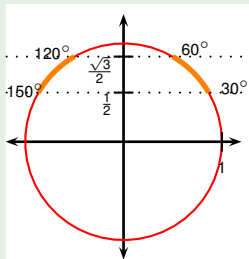
$$x \in \begin{aligned} &[-330^\circ, -300^\circ) \cup (-240^\circ, -210^\circ] \\ &\cup [30^\circ, 60^\circ) \cup (120^\circ, 150^\circ] \\ &\cup [390^\circ, 420^\circ) \cup (480^\circ, 510^\circ] \end{aligned}$$

...

$$k = -1$$

$$k = 0$$

$$k = 1$$



Example

Solve. Among your solutions, find those between -360° and 450° .

$$\frac{1}{2} \leq \sin \theta < \frac{\sqrt{3}}{2}$$

$$x \in [30^\circ + k360^\circ, 60^\circ + k360^\circ) \cup (120^\circ + k360^\circ, 150^\circ + k360^\circ]$$

$$x \in \begin{aligned} & [-690^\circ, -660^\circ) \cup (-600^\circ, -570^\circ] \\ & \cup [-330^\circ, -300^\circ) \cup (-240^\circ, -210^\circ] \\ & \cup [30^\circ, 60^\circ) \cup (120^\circ, 150^\circ] \\ & \cup [390^\circ, 420^\circ) \cup (480^\circ, 510^\circ] \end{aligned}$$

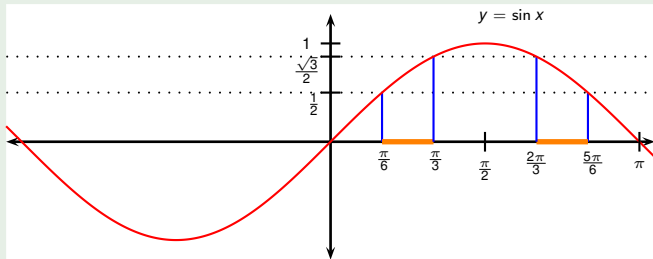
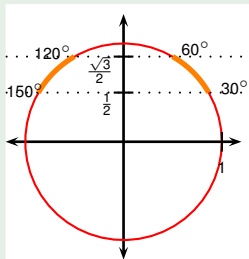
...

$$k = -2$$

$$k = -1$$

$$k = 0$$

$$k = 1$$



Example

Solve. Among your solutions, find those between -360° and 450° .

$$\frac{1}{2} \leq \sin \theta < \frac{\sqrt{3}}{2}$$

$$x \in [30^\circ + k360^\circ, 60^\circ + k360^\circ) \cup (120^\circ + k360^\circ, 150^\circ + k360^\circ]$$

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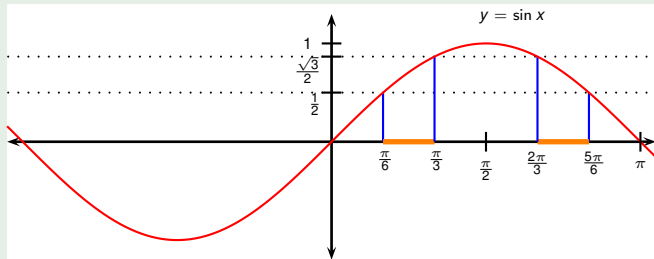
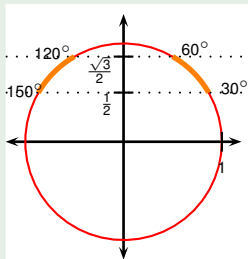
...

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Example

Solve. Among your solutions, find those **between -360° and 450°** .

$$\frac{1}{2} \leq \sin \theta < \frac{\sqrt{3}}{2}$$

$$x \in [30^\circ + k360^\circ, 60^\circ + k360^\circ) \cup (120^\circ + k360^\circ, 150^\circ + k360^\circ]$$

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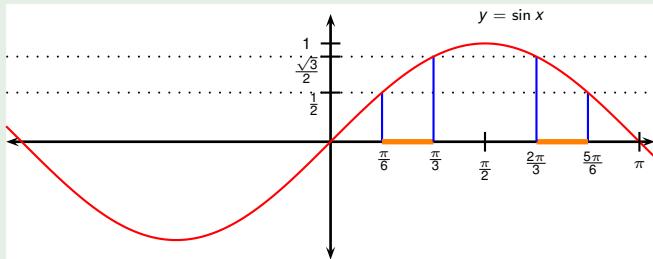
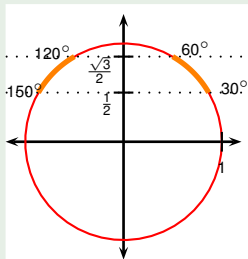
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Example

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$$x \in [30^\circ + k360^\circ, 60^\circ + k360^\circ) \cup (120^\circ + k360^\circ, 150^\circ + k360^\circ]$$

$$\begin{array}{l}
 x \in \quad \cup \quad \cancel{[-690^\circ, -660^\circ)} \cup \quad \cancel{(-600^\circ, -570^\circ]} \\
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 \cup \quad [30^\circ, 60^\circ) \cup \quad (120^\circ, 150^\circ] \\
 \cup \quad [390^\circ, 420^\circ) \cup \quad \cancel{(480^\circ, 510^\circ]}
 \end{array}
 \quad \left| \begin{array}{l} k = -2 \\ k = -1 \\ k = 0 \\ k = 1 \end{array} \right.$$

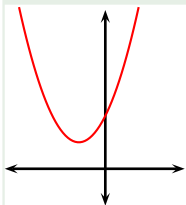
In radians:

$$x \in \left[-\frac{11\pi}{6}, -\frac{5\pi}{3}\right) \cup \left[-\frac{4\pi}{3}, -\frac{7\pi}{6}\right) \cup \left[\frac{\pi}{6}, \frac{\pi}{3}\right) \cup \left[\frac{2\pi}{3}, \frac{5\pi}{6}\right) \cup \left[\frac{13\pi}{6}, \frac{7\pi}{3}\right)$$

Example

- Solve the inequality $2u^2 + 2u + 1 \leq u + 2$.
- Find all solutions of $2 \cos^2 \theta + 2 \cos \theta + 1 \leq \cos \theta + 2$ lying in $[-360^\circ, 360^\circ]$.

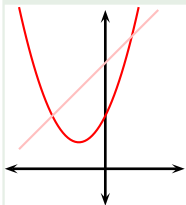
Example



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$$2u^2 + 2u + 1 \leq u + 2$$

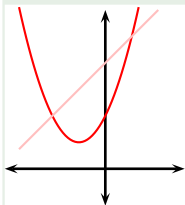
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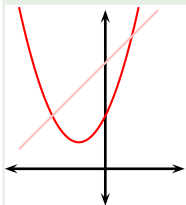
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$$\begin{aligned} 2u^2 + 2u + 1 &\leq u + 2 \\ 2u^2 + u - 1 &\leq 0 \end{aligned}$$

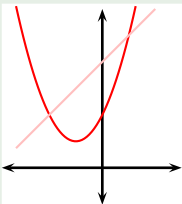
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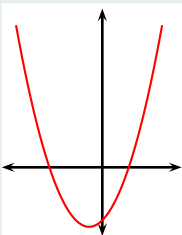
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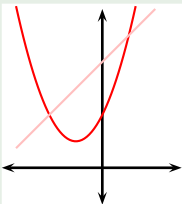


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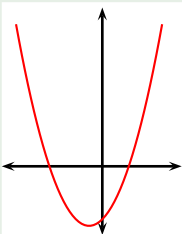


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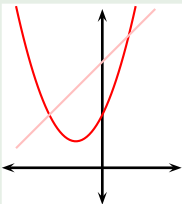
$$2u^2 + 2u + 1 \leq u + 2$$

$$2u^2 + u - 1 \leq 0$$

$$2(\text{?})(\text{?}) \leq 0$$



Example

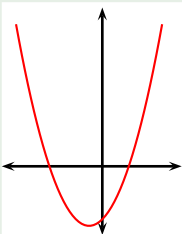


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$$2u^2 + 2u + 1 \leq u + 2$$

$$2u^2 + u - 1 \leq 0$$

$$2 \left(u - \frac{1}{2} \right) (u + 1) \leq 0$$



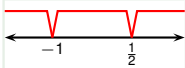
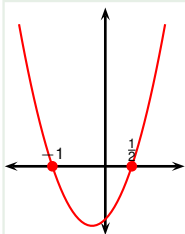
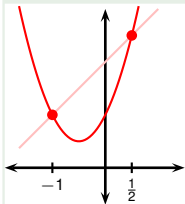
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$$2u^2 + 2u + 1 \leq u + 2$$

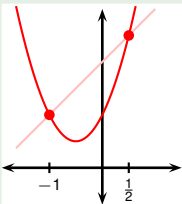
$$2u^2 + u - 1 \leq 0$$

$$2(u - \frac{1}{2})(u + 1) \leq 0$$

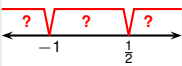
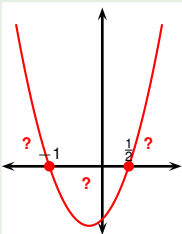


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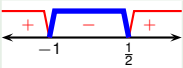
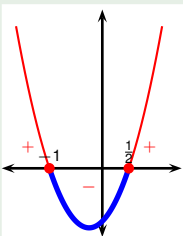
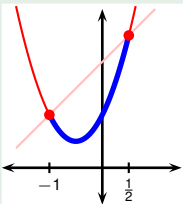


$$\begin{aligned}
 2u^2 + 2u + 1 &\leq u + 2 \\
 2u^2 + u - 1 &\leq 0 \\
 2\left(u - \frac{1}{2}\right)(u + 1) &\leq 0 \\
 u &\in ?
 \end{aligned}$$



Example

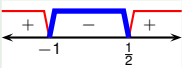
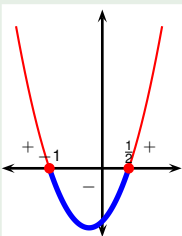
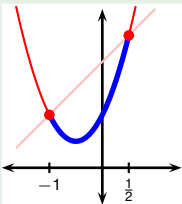
- Solve the inequality $2u^2 + 2u + 1 \leq u + 2$.
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Example

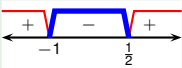
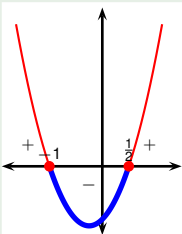
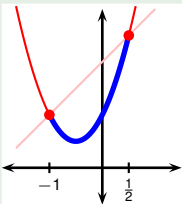
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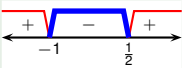
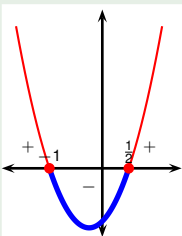
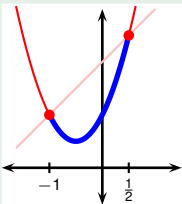
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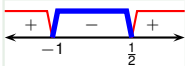
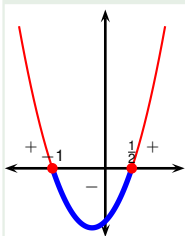
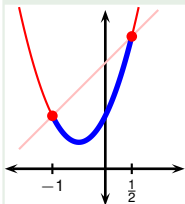
$$u \in \left[-1, \frac{1}{2}\right]$$

$$2\cos^2 \theta + 2\cos \theta + 1 \leq \cos \theta + 2 \quad \text{Set } \cos \theta = u$$

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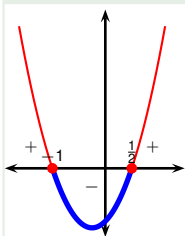
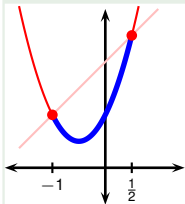
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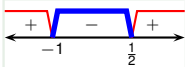
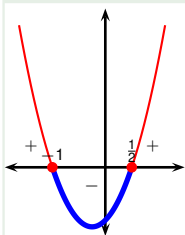
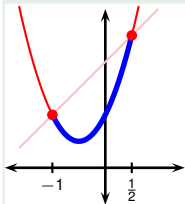
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$$\cos \theta \in \left[-1, \frac{1}{2}\right]$$

(solved above)

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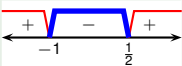
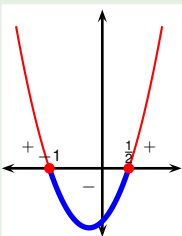
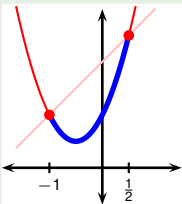


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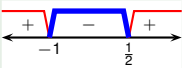
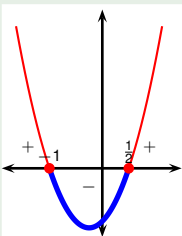
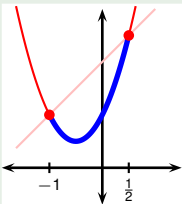


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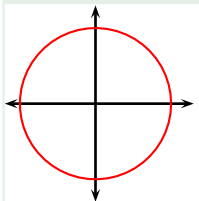
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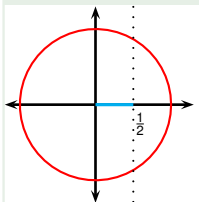


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Example



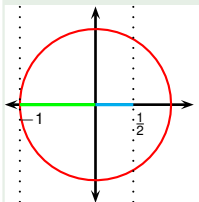
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Example



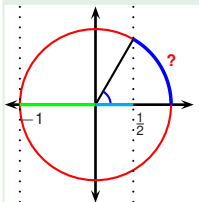
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Example



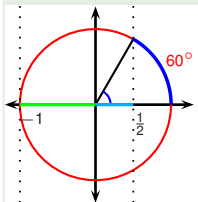
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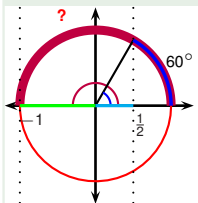


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Example

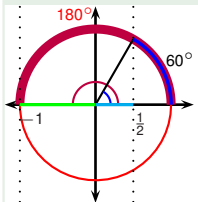


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Example

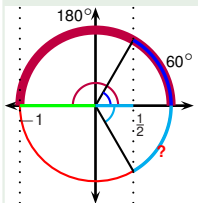


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$$\theta \in \quad [60^\circ \quad , 180^\circ \quad]$$

Example

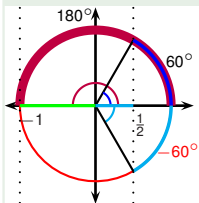


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$$\theta \in [?, \quad , ? \quad] \cup [60^\circ \quad , 180^\circ \quad]$$

Example



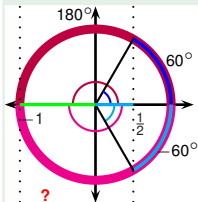
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$$\theta \in [?, -60^\circ] \cup [60^\circ, 180^\circ]$$

Example



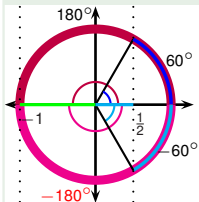
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Example



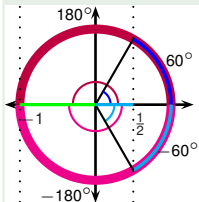
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$$\theta \in [-180^\circ, -60^\circ] \cup [60^\circ, 180^\circ]$$

Example



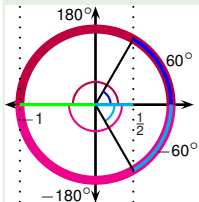
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$$\cos \theta \in \left[-1, \frac{1}{2}\right]$$

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$$\theta \in [-180^\circ + k360^\circ, -60^\circ + k360^\circ] \cup [60^\circ + k360^\circ, 180^\circ + k360^\circ]$$

Example



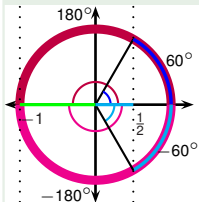
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Example



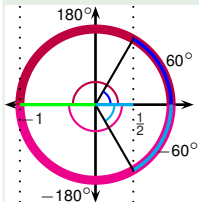
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$$\theta \in [-180^\circ, -60^\circ] \cup [60^\circ, 180^\circ] \quad k = 0$$

Example



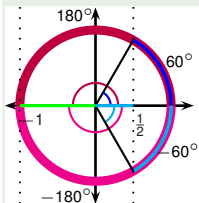
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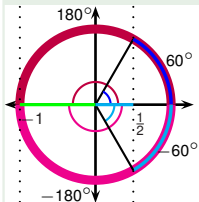
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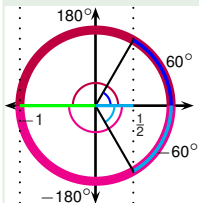
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$$\theta \in \begin{aligned} & [-540^\circ, -420^\circ] \cup [-300^\circ, -180^\circ] & k = -1 \\ & \cup [-180^\circ, -60^\circ] \cup [60^\circ, 180^\circ] & k = 0 \\ & \cup [180^\circ, 300^\circ] \cup [420^\circ, 540^\circ] & k = 1 \\ & \dots \end{aligned}$$

Example



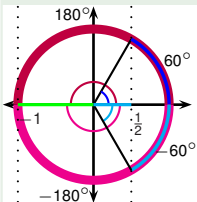
- Solve the inequality $2u^2 + 2u + 1 \leq u + 2$.
- Find all solutions of $2 \cos^2 \theta + 2 \cos \theta + 1 \leq \cos \theta + 2$ lying in $[-360^\circ, 360^\circ]$.

$$\begin{aligned} \cos \theta &\in \left[-1, \frac{1}{2}\right] \\ -1 &\leq \cos \theta \leq \frac{1}{2} \end{aligned}$$

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$$\begin{aligned} \theta \in & \quad \dots \\ & \cup [-540^\circ, -420^\circ] \cup [-300^\circ, -180^\circ] & k = -1 \\ & \cup [-180^\circ, -60^\circ] \cup [60^\circ, 180^\circ] & k = 0 \\ & \cup [180^\circ, 300^\circ] \cup [420^\circ, 540^\circ] & k = 1 \\ & \dots \end{aligned}$$

Example



- Solve the inequality $2u^2 + 2u + 1 \leq u + 2$.
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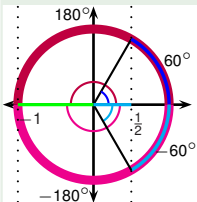
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- Find all solutions of $2 \cos^2 \theta + 2 \cos \theta + 1 \leq \cos \theta + 2$ lying in $[-360^\circ, 360^\circ]$.

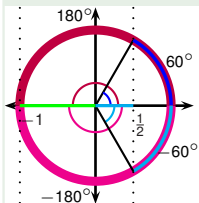
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$$\theta \in [-180^\circ + k360^\circ, -60^\circ + k360^\circ] \cup [60^\circ + k360^\circ, 180^\circ + k360^\circ]$$

$$\begin{aligned} \theta \in & \quad \cup \left[\cancel{-540^\circ}, \cancel{-420^\circ} \right] \cup \left[-300^\circ, -180^\circ \right] & k = -1 \\ & \quad \cup \left[-180^\circ, -60^\circ \right] \cup [60^\circ, 180^\circ] & k = 0 \\ & \quad \cup [180^\circ, 300^\circ] \cup \left[\cancel{420^\circ}, \cancel{540^\circ} \right] & k = 1 \\ \theta \in & \quad \left[-300^\circ, -60^\circ \right] \cup [60^\circ, 300^\circ] \end{aligned}$$

Example



- Solve the inequality $2u^2 + 2u + 1 \leq u + 2$.
- Find all solutions of $2 \cos^2 \theta + 2 \cos \theta + 1 \leq \cos \theta + 2$ lying in $[-360^\circ, 360^\circ]$.

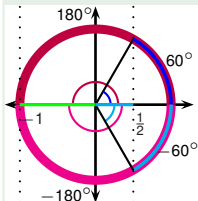
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Example



- Solve the inequality $2u^2 + 2u + 1 \leq u + 2$.
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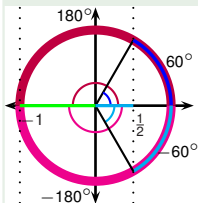
$$-1 \leq \cos \theta \leq \frac{1}{2}$$

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Example

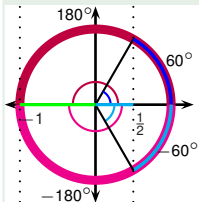


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Example

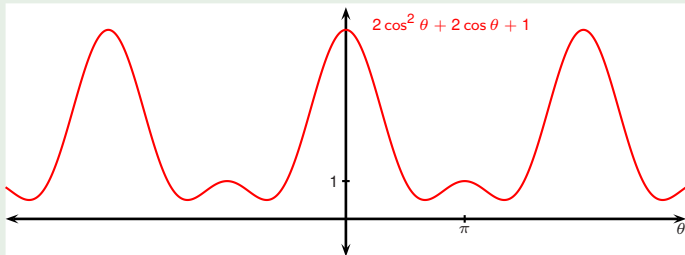


- Solve the inequality $2u^2 + 2u + 1 \leq u + 2$.
- Find all solutions of $2\cos^2 \theta + 2\cos \theta + 1 \leq \cos \theta + 2$ lying in $[-360^\circ, 360^\circ]$.

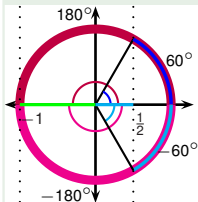
$$\cos \theta \in \left[-1, \frac{1}{2}\right]$$

$$-1 \leq \cos \theta \leq \frac{1}{2}$$

$$\theta \in [-300^\circ, -60^\circ] \cup [60^\circ, 300^\circ]$$



Example

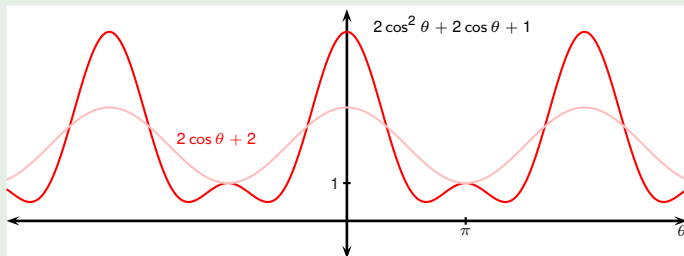


- Solve the inequality $2u^2 + 2u + 1 \leq u + 2$.
- Find all solutions of $2 \cos^2 \theta + 2 \cos \theta + 1 \leq \cos \theta + 2$ lying in $[-360^\circ, 360^\circ]$.

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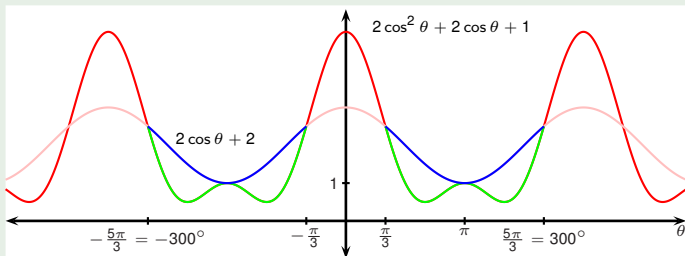
$$-1 \leq \cos \theta \leq \frac{1}{2}$$

$$\theta \in [-300^\circ, -60^\circ] \cup [60^\circ, 300^\circ]$$

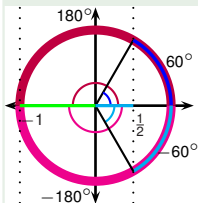


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