Precalculus
Homework
Graphs of trig functions; inverse trig

1. Find each of the following values. Express your answers precisely, not as decimals.

   (a) $\arcsin(\sin 4)$.

   answer: $\pi - 4$

   (b) $\arcsin(\sin 0.5)$.

   answer: 0.5

   (c) $\arcsin(\cos 120^\circ)$.

   answer: $-\frac{\pi}{6}$

   (d) $\arccos(\cos(3))$.

   answer: 3

   (e) $\arccos(\cos(-2))$.

   answer: 2

   (f) $\arccos(\sin(-4))$.

   answer: $\frac{3\pi}{2} - 4 \approx 0.712389$

   (g) $\arctan(\tan 5)$.

   answer: $5 - 2\pi$

   Solution. 1.g $\frac{3\pi}{2} \approx 4.71$ and $2\pi \approx 6.28$, so

   $$\frac{3\pi}{2} < 5 < 2\pi$$
   $$\text{Therefore} \quad -\frac{\pi}{2} < 5 - 2\pi < 0 < \frac{\pi}{2}.$$

   Therefore $5 - 2\pi$ is in the restricted domain of the tangent function. Moreover, the tangent function is π-periodic, so $\tan 5 = \tan(5 - 2\pi)$. Therefore $\arctan(\tan 5) = 5 - 2\pi$.

2. Express as the following as an algebraic expression of x. In other words, "get rid" of the trigonometric and inverse trigonometric expressions.

   (a) $\cos^2(\arctan x)$.

   answer: $\frac{1}{1+x^2}$

   (b) $-\sin^2(\operatorname{arccot} x)$.

   answer: $-\frac{1}{1+x^2}$

   (c) $\dfrac{1}{\cos(\arcsin x)}$.

   answer: $\frac{1}{\sqrt{1-x^2}}$

   (d) $-\dfrac{1}{\sin(\arccos x)}$.

   answer: $-\frac{1}{\sqrt{1-x^2}}$

   Solution. 2.b. We follow the strategy outlined in the end of the solution of Problem 3.c. We set $y = \operatorname{arccot} x$. Then we need to express $-\sin^2 y$ via $\cot y$. That is a matter of algebra:

$$
\begin{aligned}
-\sin^2(\operatorname{arccot} x) &= -\sin^2 y && \text{Set } y = \operatorname{arccot} x \\
&= -\frac{\sin^2 y}{\sin^2 y + \cos^2 y} && \text{use } \sin^2 y + \cos^2 y = 1 \\
&= -\frac{1}{\frac{\sin^2 y + \cos^2 y}{\sin^2 y}} \\
&= -\frac{1}{1 + \cot^2 y} && \text{Substitute back } \cot y = x \\
&= -\frac{1}{1 + x^2} \quad .
\end{aligned}
$$

3. Let $x \in (0, 1)$. Express the following using x and $\sqrt{1 - x^2}$.

(a) $\sin(\arcsin(x))$.

answer: x

(b) $\sin(2 \arcsin(x))$.

answer: $2x\sqrt{1 - x^2}$

(c) $\sin(3 \arcsin(x))$.

answer: $-4x^3 + 3x$

(d) $\sin(\arccos(x))$.

answer: $\sqrt{1 - x^2}$

(e) $\sin(2 \arccos(x))$.

answer: $2x\sqrt{1 - x^2}$

(f) $\sin(3 \arccos(x))$.

answer: $(4x^2 - 1)\sqrt{1 - x^2} = -4\left(\sqrt{1 - x^2}\right)^3 + 3\sqrt{1 - x^2}$

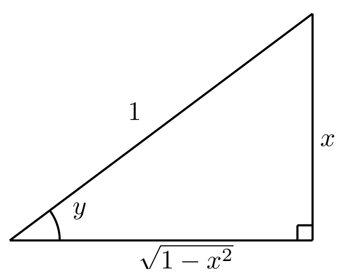
(g) $\cos(2 \arcsin(x))$.

answer: $1 - 2x^2$

(h) $\cos(3 \arccos(x))$.

answer: $4x^3 - 3x$

Solution. 3.b. Let $y = \arcsin x$. Then $\sin y = x$, and we can draw a right triangle with opposite side length x and hypotenuse length 1 to find the other trigonometric ratios of y.



Then $\cos y = \frac{\sqrt{1-x^2}}{1} = \sqrt{1 - x^2}$. Now we use the double angle formula to find $\sin(2 \arcsin x)$.

$$
\begin{aligned}
\sin(2 \arcsin x) &= \sin(2y) \\
&= 2 \sin y \cos y \\
&= 2x\sqrt{1 - x^2}.
\end{aligned}
$$

Solution. 3.c. Use the result of Problem 3.b. This also requires the addition formula for sine:

$$\sin(A + B) = \sin A \cos B + \sin B \cos A,$$

and the double angle formula for cosine:

$$\cos(2y) = \cos^2 y - \sin^2 y.$$

$$
\begin{aligned}
\sin(3\arcsin x) &= \sin(3y) \\
&= \sin(2y+y) \\
&= \sin(2y)\cos y + \sin y\cos(2y) && \text{Use addition formula} \\
&= (2\sin y\cos y)\cos y + \sin y(\cos^2 y - \sin^2 y) && \text{Use double angle formulas} \\
&= 2\sin y\cos^2 y + \sin y\cos^2 y - \sin^3 y \\
&= 3\sin y\cos^2 y - \sin^3 y \\
&= 3\sin y(1-\sin^2 y) - \sin^3 y \\
&= 3x(1-x^2) - x^3 \\
&= 3x - 4x^3.
\end{aligned}
$$

The solution is complete. A careful look at the solution above reveals a strategy useful for problems similar to this one.

(a) Identify the inverse trigonometric expression- $\arcsin x, \arccos x, \arctan x, \ldots$. In the present problem that was $y = \arcsin x$.

(b) The problem is therefore a trigonometric function of y.

(c) Using trig identities and algebra, rewrite the problem as a trigonometric expression involving only the trig function that transforms y to x. In the present problem we rewrote everything using $\sin y$.

(d) Use the fact that $\sin(\arcsin x) = x$, $\cos(\arccos x) = x, \ldots$, etc. to simplify.

Solution. 3.f We use the same strategy outlined in the end of the solution of Problem 3.c. Set $y = \arccos x$ and so $\cos(y) = x$. Therefore:

$$
\begin{aligned}
\sin(3y) &= \sin(2y+y) \\
&= \sin(2y)\cos y + \sin y\cos(2y) \\
&= 2\sin y\cos y\cos y + \sin y(2\cos^2 y - 1) \\
&= 2\sin y\cos^2 y + \sin y(2\cos^2 y - 1) \\
&= \sin y(4\cos^2 y - 1) && \text{use } \begin{array}{rcl} \cos y &=& x \\ \sin y &=& \sqrt{1-x^2} \end{array} \\
&= \sqrt{1-x^2}(4x^2-1) \quad .
\end{aligned}
$$