Calculus II
Integral of rational function with cubic denominator, part 1

Todor Milev

2019

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$Q(x)$ has distinct linear factors

- Suppose $Q(x)$ is a product of distinct linear factors:

$$Q(x) = (a_1x + b_1)(a_2x + b_2)\cdots(a_kx + b_k)$$

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- We show how to find $A_1, A_2, \cdots, A_k$ on examples.