Precalculus

Express $\sin(kx), \cos(kx)$ via $\sin x, \cos x$ using Euler's formula

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2019

Example

Express $\sin(3x)$ and $\cos(3x)$ via $\cos x$ and $\sin x$.

- Recall Euler's formula: $e^{i\alpha} = \cos\alpha + i\sin\alpha$.

Example

Express $\sin(3x)$ and $\cos(3x)$ via $\cos x$ and $\sin x$.

$$\cos(3x) + i\sin(3x) \qquad \Big| \text{ Euler's f-la}$$
$$= e^{3ix}$$

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$$\begin{array}{rl|l} & \cos(3x) + i\sin(3x) & \text{Euler's f-la} \\ = & e^{3ix} & \end{array}$$

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$$
\begin{aligned}
&\cos(3x) + i\sin(3x) && \Big| \text{ Euler's f-la} \\
&= e^{3ix} \\
&= \left(e^{ix}\right)^{3}
\end{aligned}
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= e^{3ix} & \\
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- Recall Euler's formula: $e^{i\alpha} = \cos\alpha + i\sin\alpha$.
- Recall the formula: $(a+b)^3 = \textbf{?}$.

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= \cos^3 x + 3\cos^2 x(i\sin x) + 3\cos x(i\sin x)^2 + ({\color{red}i}\sin x)^{\color{red}3} & \\
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= \cos^3 x + 3i\cos^2 x \sin x + 3{\color{red}i^2}\cos x \sin^2 x + i^3\sin^3 x & \\
= \cos^3 x + 3i\cos^2 x \sin x {\color{red}-} 3\cos x \sin^2 x - i\sin^3 x & \Big| \text{ Use } {\color{red}i^2 = -1}
\end{array}
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\end{array}$$

The real parts of the starting and final expression must be equal; therefore:

$$\cos(3x) = \cos^3 x - 3\cos x\sin^2 x$$

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= \cos^3 x + 3i\cos^2 x\sin x - 3\cos x\sin^2 x - i\sin^3 x & \Big|\ \text{Use } i^2 = -1 \\
= (\cos^3 x - 3\cos x\sin^2 x) + i(3\cos^2 x\sin x - \sin^3 x) &
\end{array}
$$

The real parts of the starting and final expression must be equal; likewise the imaginary parts must be equal; therefore:

$$
\begin{array}{rcl}
\cos(3x) & = & \cos^3 x - 3\cos x\sin^2 x \\
\sin(3x) & = & 3\cos^2 x\sin x - \sin^3 x
\end{array}
$$