Calculus II
Homework
Power series, full lecture

1. Determine the interval of convergence for the following power series.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(x-2)^n}{3\sqrt{n+1}}$.

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{10^n x^n}{n^3}$.

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{10^n (x-1)^n}{n^3}$.

   (d) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{(x+1)^n}{2n+1}$.

   (e) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{(x-3)^n}{2n+1}$.

   (f) $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

   (g) $\displaystyle\sum_{n=0}^{\infty} (n+1)x^n$.

   (h) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}$.

   (i) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$.

   (j) $\displaystyle\sum_{n=1}^{\infty} \binom{\frac{1}{2}}{n} x^n$, where we recall that the binomial coefficient $\displaystyle\binom{q}{n}$ stands for $\displaystyle\frac{q(q-1)\dots(q-n+1)}{n!}$.

2. (a) Find the Maclaurin series for xe^{x^3}.

   (b) Use your series to find the Maclaurin series of $\displaystyle\int xe^{x^3}\mathrm{d}x$.

3. For each of the items below, do the following.

   - Find the Maclaurin series of the function (i.e., the power series representation of the function around $a = 0$).
   - Find the radius of convergence of the series you found in the preceding point. You are not asked to find the entire interval of convergence, but just the radius.

   (a) e^x.

   (b) xe^{-2x}.

   (c) e^{2x}.

   (d) e^{x^2}.

   (e) e^{-3x^2}.

   (f) x^2e^{2x}.

   (g) $\sin x$.

   (h) $\cos x$.

   (i) $\sin(2x)$.

   (j) $\cos(2x)$.

   (k) $\cos^2(x)$.

   (l) $x\sin x$.

4. For each of the items below, do the following.
   - Find the Maclaurin series of the function (i.e., the power series representation of the function around $a = 0$).
   - Find the radius of convergence of the series you found in the preceding point.

   (a) $\frac{1}{3-x}$.

   (b) $\frac{1}{3-2x}$.

   (c) $\frac{1}{2x+3}$.

   (d) $\frac{1}{1+x^2}$.

   (e) $\frac{1}{1-2x^2}$.

   (f) $\frac{1}{x^2-1}$.

   (g) $\frac{\frac{1}{2}}{x-1} - \frac{\frac{1}{2}}{x+1}$.

   (h) $\frac{1}{(1-x)^2}$.

   (i) $\frac{1}{(1-x)^3}$.

   (j) $\ln(1+x)$.

   (k) $\ln(1-x)$.

   (l) $\ln(1-3x)$.

   (m) $\ln(1-3x^2)$.

   (n) $\ln(3-2x^2)$.

   (o) $x\ln(3-2x^2)$.

   (p) $\arctan x$.

   (q) $\arctan(2x)$.

   (r) $\arctan\left(2x^2\right)$.

5. Compute the Maclaurin series of
$$\left(\frac{1}{(1-x)^k}\right) \quad ,$$
where $n \geq 1$ is an integer.

6. Compute the Maclaurin series of
$$(1+x)^q \quad ,$$
where $q \in \mathbb{R}$ is an arbitrary real number.

7. Compute the Maclaurin series of the function.

   (a) $\sqrt{1+x}$.

   (b) $\frac{1}{\sqrt{1+x}}$.

   (c) $\frac{1}{\sqrt{1-x^2}}$.

   (d) $\arcsin x$.

8. Find the Taylor series of the function at the indicated point.

   (a) $\frac{1}{x^2}$ at $a = -1$.

   (b) $\ln\left(\sqrt{x^2 - 2x + 2}\right)$ at $a = 1$.

   (c) Write the Taylor series of the function $\ln x$ around $a = 2$.

9. Find the Taylor series around the indicated point. The answer key has not been proofread, use with caution.

   (a) $\frac{1}{x}$ at $a = 1$.

   (b) $\frac{1}{x^2}$ at $a = 1$.

10. Let $f(x)$ be defined as
$$f(x) := \begin{cases} e^{-\frac{1}{x^2}} & \text{if } x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Prove that if $R(x)$ is an arbitrary rational function,
$$\lim_{\substack{x \to 0 \\ x > 0}} R(x) e^{-\frac{1}{x^2}} = 0$$

   (b) Prove that $f(x)$ is differentiable at 0 and $f'(0) = 0$.

   (c) Prove that the Maclaurin series of $f(x)$ are 0 (but $f(x)$ is clearly a non-zero function).