Calculus I
Integrals reducing to integrals of rational monomials, part 2

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Example

Integrate.

$$\int \left(x^{\frac{3}{2}} - \frac{1}{x^{\frac{1}{3}}} \right)^2 \mathrm{d}x$$

Example

Integrate.

$$\int \left(x^{\frac{3}{2}} - \frac{1}{x^{\frac{1}{3}}} \right)^2 \mathrm{d}x = \int \left(x^{\frac{3}{2}} - x^{-\frac{1}{3}} \right)^2 \mathrm{d}x$$

Example

Integrate.

$$\begin{aligned}\int \left(x^{\frac{3}{2}} - \frac{1}{x^{\frac{1}{3}}}\right)^2 \mathrm{d}x &= \int \left(x^{\frac{3}{2}} - x^{-\frac{1}{3}}\right)^2 \mathrm{d}x \\ &= \int \left(\left(x^{\frac{3}{2}}\right)^2 - 2x^{\frac{3}{2}}x^{-\frac{1}{3}} + \left(x^{-\frac{1}{3}}\right)^2\right) \mathrm{d}x\end{aligned}$$

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&= \int \left(\left(x^{\frac{3}{2}}\right)^{\cancel{2}} - 2x^{\frac{3}{2}}x^{-\frac{1}{3}} + \left(x^{-\frac{1}{3}}\right)^2\right) \mathrm{d}x \\
&= \int \left(x^3 - 2x^{\frac{3}{2}-\frac{1}{3}} + x^{-\frac{2}{3}}\right) \mathrm{d}x
\end{aligned}$$

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&= \int \left(x^3 - 2x^{\frac{3}{2}-\frac{1}{3}} + x^{-\frac{2}{3}}\right) \mathrm{d}x \\
&= \int \left(x^3 - 2x^{\frac{7}{6}} + x^{-\frac{2}{3}}\right) \mathrm{d}x
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&= \int \left(x^3 - 2x^{\frac{3}{2}-\frac{1}{3}} + x^{-\frac{2}{3}}\right) \mathrm{d}x \\
&= \int \left(x^3 - 2x^{\frac{7}{6}} + x^{-\frac{2}{3}}\right) \mathrm{d}x \\
&= \mathbf{?} \quad - 2\mathbf{?} \qquad + \mathbf{?}
\end{aligned}$$

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\int \left(x^{\frac{3}{2}} - \frac{1}{x^{\frac{1}{3}}}\right)^2 \mathrm{d}x &= \int \left(x^{\frac{3}{2}} - x^{-\frac{1}{3}}\right)^2 \mathrm{d}x \\
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&= \frac{x^4}{4} - 2\,\frac{x^{\frac{7}{6}+1}}{\frac{7}{6}+1} + \mathbf{?}
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&= \frac{x^4}{4} - \frac{2x^{\frac{13}{6}}}{\frac{13}{6}} + \frac{x^{\frac{1}{3}}}{\frac{1}{3}} + C
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&= \frac{x^4}{4} - \frac{12x^{\frac{13}{6}}}{13} + 3x^{\frac{1}{3}} + C
\end{aligned}$$

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$$= \int \left(x^3 - 2x^{\frac{3}{2}-\frac{1}{3}} + x^{-\frac{2}{3}}\right) \mathrm{d}x$$

$$= \int \left(x^3 - 2x^{\frac{7}{6}} + x^{-\frac{2}{3}}\right) \mathrm{d}x$$

$$= \frac{x^4}{4} - 2\frac{x^{\frac{7}{6}+1}}{\frac{7}{6}+1} + \frac{x^{\frac{-2}{3}+1}}{\frac{-2}{3}+1} + C$$

$$= \frac{x^4}{4} - \frac{2x^{\frac{13}{6}}}{\frac{13}{6}} + \frac{x^{\frac{1}{3}}}{\frac{1}{3}} + C$$

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