Precalculus
Quadratic polynomials viewed as functions

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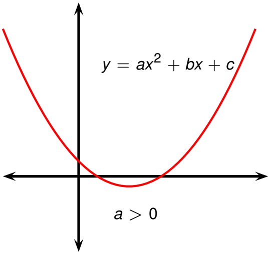
Definition

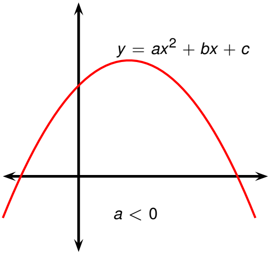
Let a, b, c be real numbers with $a \neq 0$. The function

$$f(x) = ax^2 + bx + c$$

is called a *quadratic function*.

- The graph of a quadratic function is called a parabola.





Example (Completing the square)

Complete the square.

$$
\begin{aligned}
3x^2 - 5x + 1 &= 3\left(x^2 - \frac{5}{3}x\right) + 1\\
&= 3\left(x^2 - 2\cdot\frac{5}{2\cdot 3}x\right) + 1\\
&= 3\left(x^2 - 2\cdot\frac{5}{6}x + \left(\frac{5}{6}\right)^2 - \left(\frac{5}{6}\right)^2\right) + 1\\
&= 3\left(\left(x - \frac{5}{6}\right)^2 - \frac{25}{36}\right) + 1\\
&= 3\left(x - \frac{5}{6}\right)^2 - \frac{25}{12} + 1\\
&= 3\left(x - \frac{5}{6}\right)^2 - \frac{13}{12}.
\end{aligned}
$$

Definition (Completing the square)

Let $a \neq 0$. To *complete the square* means to carry out the following algebraic manipulation.

$$
\begin{aligned}
ax^2+bx+c &= a\left(x^2+\frac{b}{a}x\right)+c\\
&= a\left(x^2+2\cdot\frac{b}{2a}x\right)+c\\
&= a\left(x^2+2\frac{b}{2a}x+\left(\frac{b}{2a}\right)^2-\left(\frac{b}{2a}\right)^2\right)+c\\
&= a\left(\left(x+\frac{b}{2a}\right)^2-\frac{b^2}{4a^2}\right)+c\\
&= a\left(x+\frac{b}{2a}\right)^2-\cancel{a}\cdot\frac{b^2}{4a^{\cancel{2}}}+c\\
&= a\left(x+\frac{b}{2a}\right)^2+c-\frac{b^2}{4a}.
\end{aligned}
$$

Add & subtract $\left(\frac{b}{2a}\right)^2$ use $(A+B)^2 = A^2+2AB+B^2$

Definition (Discriminant of quadratic function)

The quantity $D = b^2 - 4ac$ is called the *discriminant* of the quadratic function $ax^2 + bx + c$.

Let $a \neq 0$ and let $f(x) = ax^2 + bx + c$. Then we have the equality

$$\begin{array}{rcll} f(x) &=& a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a} & \Big| \text{ complete the square} \\ &=& a\left(x - \left(-\frac{b}{2a}\right)\right)^2 - \frac{b^2 - 4ac}{4a} & \\ &=& a\left(x - \left(-\frac{b}{2a}\right)\right)^2 - \frac{D}{4a}. & \end{array}$$

Definition

The expression $f(x) = a(x-h)^2 + k$, where $h = -\frac{b}{2a}$ and $k = -\frac{D}{4a} = -\frac{b^2-4ac}{4a}$ is called the standard form of $ax^2 + bx + c$.

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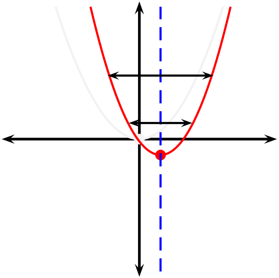

- The graph of $y = x^2$ is a parabola; its shape is assumed known.
- The standard form shows how the graph of an arbitrary quadratic is obtained from the graph of $y = x^2$:
  - ax^2 stretches $y = x^2$ by factor of a and possibly reflects across the x axis.
  - $a(x-h)^2$ shifts $y = ax^2$ by h units right.
  - $a(x-h)^2 + k$ shifts $y = a(x-h)^2 + k$ by k units up.

Definition

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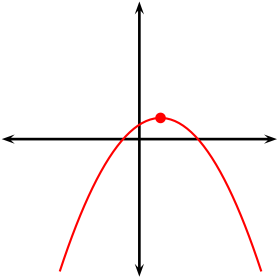
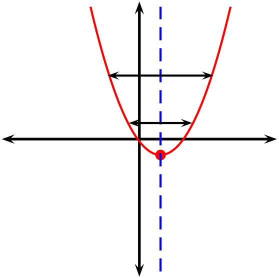


- The graph of a quadratic function is a parabola.
- When $a > 0$ the parabola opens upwards.
- When $a < 0$ the parabola opens downwards.
- When $|a|$ increases, the parabola becomes steeper.
- The point $(h,k) = \left(-\frac{b}{2a}, -\frac{D}{4a}\right)$ is called the vertex of the parabola.
- The parabola is symmetric with respect to the line $x = h = -\frac{b}{2a}$, i.e., the vertical line through its vertex.

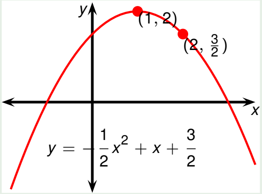
Definition

The expression $f(x) = a(x-h)^2 + k$, where $h = -\frac{b}{2a}$ and $k = -\frac{D}{4a} = -\frac{b^2 - 4ac}{4a}$ is called the standard form of $ax^2 + bx + c$.



- When we change h and k we move the vertex of the parabola without change in steepness.
- Therefore when we change b and c we move the vertex of the parabola without change in steepness.

Example



Write an equation of a parabola with vertex at $(1, 2)$ that passes through the point $\left(2, \frac{3}{2}\right)$.

$$
\begin{array}{rcl|l}
a(x-h)^2 + k &=& y & \text{Standard form} \\
a(x-1)^2 + 2 &=& y & \text{Vertex at } (1,2) \\
a(2-1)^2 + 2 &=& \frac{3}{2} & \text{Passes through } \left(2, \frac{2}{3}\right) \\
a &=& \frac{3}{2} - 2 = -\frac{1}{2} & \\
y &=& -\frac{1}{2}(x-1)^2 + 2 & \text{Final answer} \\
y &=& -\frac{1}{2}x^2 + x + \frac{3}{2} & \text{Alternative answer}
\end{array}
$$

Problem (Quadratic equation formula)

Solve the general quadratic equation

$$
\begin{aligned}
ax^2+bx+c &= 0 && \text{complete the square}\\
a\left(x+\frac{b}{2a}\right)^2-\frac{D}{4a} &= 0 && \text{where } D=b^2-4ac\\
a\left(\left(x+\frac{b}{2a}\right)^2-\frac{D}{4a^2}\right) &= 0 &&\\
a\left(\left(x+\frac{b}{2a}\right)^2-\left(\frac{\sqrt{D}}{2a}\right)^2\right) &= 0 &&\\
a\left(x+\frac{b}{2a}-\frac{\sqrt{D}}{2a}\right)\left(x+\frac{b}{2a}+\frac{\sqrt{D}}{2a}\right) &= 0 && \text{use } A^2-B^2 = (A-B)(A+B)
\end{aligned}
$$

$$
x+\frac{b}{2a}-\frac{\sqrt{D}}{2a} = 0 \quad \text{or} \quad x+\frac{b}{2a}+\frac{\sqrt{D}}{2a} = 0
$$

$$
x = \frac{-b+\sqrt{D}}{2a} \quad \text{or} \quad x = \frac{-b-\sqrt{D}}{2a}.
$$

Theorem

The solutions of the quadratic equation

$$ax^2 + bx + c = 0$$

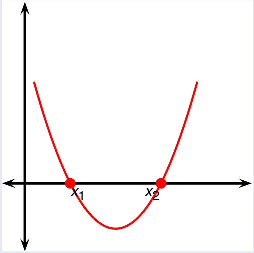
are given by:

$$x = x_1 = \frac{-b + \sqrt{D}}{2a} \quad \text{or} \quad x = x_2 = \frac{-b - \sqrt{D}}{2a} \ ,$$

where $D = b^2 - 4ac$, *or equivalently by:*

$$x_1, x_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Theorem



The solutions of the quadratic equation

$$ax^2 + bx + c = 0$$

are the numbers

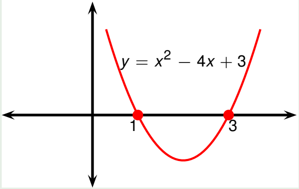
$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \qquad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

- Abbreviated as

$$x_1, x_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{D}}{2a}, \qquad \text{where } D = b^2 - 4ac.$$

- If $D < 0$ then $\sqrt{D}$ is not a real $\Rightarrow$ quadratic has no real solutions.
- If $D = 0$ then $x_1 = x_2$, the equation has only one zero (with multiplicity two). The zero is located at the vertex of the parabola.

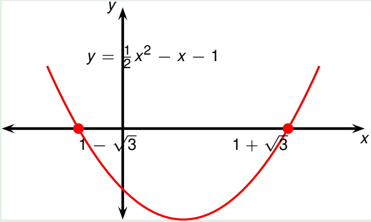
Example



Find the x-intercepts of $x^2 - 4x + 3$.

$$
\begin{aligned}
x_1, x_2 &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot 1 \cdot 3}}{2 \cdot 1} \\
&= \frac{4 \pm \sqrt{4}}{2} \\
&= \frac{4 \pm 2}{2} \\
&= \begin{cases} \frac{4+2}{2} = \frac{6}{2} = 3 \\ \frac{4-2}{2} = \frac{2}{2} = 1 \end{cases}
\end{aligned}
$$

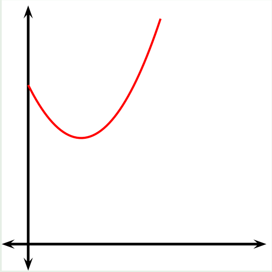
Example



Find the x-intercepts of $\frac{x^2}{2} - x - 1$.

$$\begin{aligned} x_1, x_2 &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \cdot \frac{1}{2} \cdot (-1)}}{\cancel{2} \cdot \frac{1}{\cancel{2}}} \\ &= 1 \pm \sqrt{3} \end{aligned}$$

Example



Find the x-intercepts of $x^2 - 2x + 3$.

$$
\begin{aligned}
x_1, x_2 &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\\
&= \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \cdot 1 \cdot (3)}}{2 \cdot 1}\\
&= \frac{2 \pm \sqrt{-8}}{2}
\end{aligned}
$$

no real solutions

no $x -$ intercepts

Proposition

Let $ax^2 + bx + c$, $a \neq 0$ be a quadratic with discriminant $D = b^2 - 4ac$ and roots x_1 and x_2. Then $D = a^2 (x_1 - x_2)^2$.

Proof.

$$\begin{aligned} a^2(x_1 - x_2)^2 &= a^2 \left(\frac{\cancel{-b} + \sqrt{D}}{2a} - \frac{\cancel{-b} - \sqrt{D}}{2a} \right)^2 \\ &= a^2 \left(\frac{\cancel{2}\sqrt{D}}{\cancel{2}a} \right)^2 \\ &= \cancel{a^2} \frac{D}{\cancel{a^2}} \\ &= D, \quad \text{as desired.} \end{aligned}$$

$\square$

- Discriminant is zero $\Leftrightarrow$ the quadratic has non-distinct roots, hence the discriminant discriminates between the two roots.

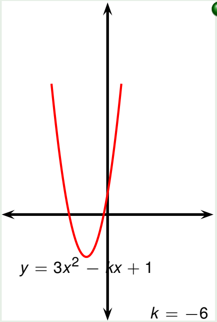
Example

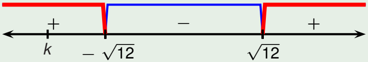
Find the values of the parameter k for which the equation $3x^2 - kx + 1$ has two real distinct roots.

- Quadratic roots: $x_1, x_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.
- $\Rightarrow$ The roots x_1, x_2 are real and distinct when

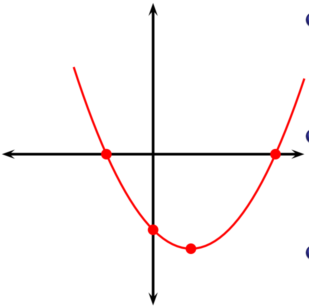
$$\begin{aligned} b^2 - 4ac &> 0 \\ (-k)^2 - 4 \cdot 3 \cdot 1 &> 0 \\ k^2 - 12 &> 0 \\ k^2 - \sqrt{12}^2 &> 0 \\ \left(k - \sqrt{12}\right)\left(k + \sqrt{12}\right) &> 0 \end{aligned}$$

$$k \in \left(-\infty, -\sqrt{12}\right) \cup \left(\sqrt{12}, \infty\right)$$



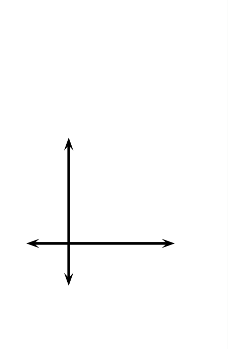


To plot a parabola by hand roughly, we need to do the following.



- Find the vertex of the parabola.
- Find the y intercept.
- Find the x intercept(s) if any.
- Select (or re-select) axes scale so all important points found in the preceding items fit in the plot.
- Plot the parabola freehand, making sure that the parabola passes through all special points you found in the preceding items.
- If $a > 0$ your parabola should open upwards, if $a < 0$ your parabola should open downwards.
- For $|a| > 1$ we should aim to draw the graph steeper than $a = x^2$, for $|a| < 1$ we should aim to draw the graph flatter than $a = x^2$.

Example



Vertex at: $\left(\frac{21}{4}, \frac{171}{8}\right)$
y-intercept at $y = 3$
x-intercepts at
$x = \frac{21-3\sqrt{57}}{4}$,
$x = \frac{21+3\sqrt{57}}{4}$.

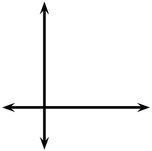
Plot roughly by hand the graph of $f(x) = -\frac{2}{3}x^2 + 7x + 3$.

- The vertex of the parabola is given by:

$$\begin{aligned} x &= -\frac{b}{2a} = -\frac{7}{2\left(-\frac{2}{3}\right)} = \frac{21}{4} \\ y &= f\left(-\frac{b}{2a}\right) = -\frac{D}{4a} = -\frac{\left(b^2 - 4ac\right)}{4a} \\ &= -\frac{7^2 - 4\left(-\frac{2}{\cancel{3}}\right)\cancel{3}}{4\left(-\frac{2}{3}\right)} = \frac{49+8}{\frac{8}{3}} \\ &= \frac{3 \cdot 57}{8} = \frac{171}{8}. \end{aligned}$$

- The y-intercept is $f(0) = 3$.

Example



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$x = \frac{21-3\sqrt{57}}{4}$,
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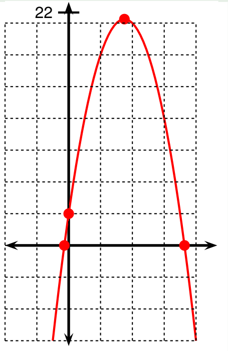
Plot roughly by hand the graph of
$f(x) = -\frac{2}{3}x^2 + 7x + 3$.

- The x intercepts are given by the solutions of

$$\begin{aligned} -\frac{2}{3}x^2 + 7x + 3 &= 0 & \Big| \cdot 3 \\ -2x^2 + 21x + 9 &= 0 \end{aligned}$$

$$\begin{aligned} x &= \frac{-b\pm\sqrt{b^2-4ac}}{2a} \\ &= \frac{-21\pm\sqrt{21^2-4\cdot(-2)\cdot 9}}{2\cdot(-2)} \\ &= \frac{-21\pm\sqrt{441+72}}{-4} \\ &= \frac{21\mp\sqrt{513}}{4} \\ &= \frac{21\mp\sqrt{9\cdot 57}}{4} \\ &= \frac{21\mp\sqrt{9}\sqrt{57}}{4} \\ &= \frac{21\mp 3\sqrt{57}}{4} \end{aligned}$$

Example



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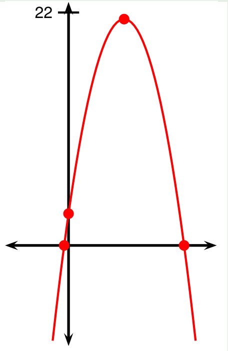
x-intercepts at
$x = \frac{21 - 3\sqrt{57}}{4}$,
$x = \frac{21 + 3\sqrt{57}}{4}$.

Plot roughly by hand the graph of $f(x) = -\frac{2}{3}x^2 + 7x + 3$.

- Select scale to fit the picture:
  - $\frac{21}{4}$ is close to $\frac{20}{4} = 5$.
  - $\frac{171}{8}$ is between the integers 21 and 22.
  - $\frac{21+3\sqrt{57}}{4}$ is close to $\frac{21+3\sqrt{64}}{4} = \frac{21+24}{4} = \frac{45}{4}$ which is close to $\frac{44}{4} = 11$.
  - $\frac{21-3\sqrt{57}}{4}$ is close to $\frac{21-3\sqrt{64}}{4} = \frac{21-24}{4} = -\frac{3}{4}$ which is close to -1.
  - The parabola vertex is less than 22 units high and the parabola opens downwards.
  - Axes height of 22 units appears reasonable.
  - A grid of width 3 units appears reasonable.
  - Plot all relevant points.
  - Finally "connect the dots with a freehand drawing".

Example



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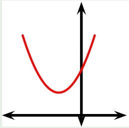
Example



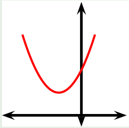
Find the values of the parameter k for which $x^2 + (k+1)x + 2k > 0$ holds for all real x.

- In order for the quadratic to be positive, its graph must lie entirely above the x axis.

- Leading coefficient is positive $\Rightarrow$ graph opens up $\Rightarrow$ is above x axis if it does not intersect it $\Rightarrow$ the quadratic has no real solutions.
- The roots of a quadratic are $x_1, x_2 = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$.

$$\begin{aligned}
b^2 - 4ac &< 0\\
(k+1)^2 - 4\cdot 1\cdot 2k &< 0\\
k^2 + 2k + 1 - 8k &< 0\\
k^2 - 6k + 1 &< 0\\
(k-k_1)(k-k_2) &< 0\\
k_1, k_2 &= \frac{2\cdot 3 \pm \sqrt{4}\sqrt{8}}{2} = \frac{\cancel{2}\left(3\pm\sqrt{8}\right)}{\cancel{2}} = 3\pm\sqrt{8}
\end{aligned}$$

Example



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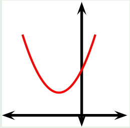
$$
\begin{aligned}
b^2 - 4ac &< 0\\
(k+1)^2 - 4\cdot 1\cdot 2k &< 0\\
(k-k_1)(k-k_2) &< 0\\
k_1, k_2 &= \frac{-(-6)\pm\sqrt{(-6)^2-4\cdot 1\cdot 1}}{2\cdot 1} = \frac{6\pm\sqrt{32}}{2}\\
&= \frac{2\cdot 3\pm\sqrt{4}\sqrt{8}}{2} = \frac{\cancel{2}\left(3\pm\sqrt{8}\right)}{\cancel{2}} = 3\pm\sqrt{8}
\end{aligned}
$$

Example



Find the values of the parameter k for which $x^2 + (k+1)x + 2k > 0$ holds for all real x.

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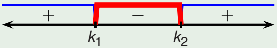
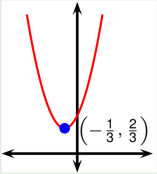
$$\begin{aligned}
b^2 - 4ac &< 0\\
(k+1)^2 - 4\cdot 1 \cdot 2k &< 0\\
(k-k_1)(k-k_2) &< 0\\
k_1, k_2 &= \frac{2\cdot 3 \pm \sqrt{4}\sqrt{8}}{2} = \frac{\cancel{2}\left(3 \pm \sqrt{8}\right)}{\cancel{2}} = 3 \pm \sqrt{8}\\
k &\in (k_1, k_2) = \left(3-\sqrt{8}, 3+\sqrt{8}\right)
\end{aligned}$$



Example



Find the minimum point on the curve $y = 3x^2 + 2x + 1$ by completing the square.

$$\begin{aligned}
3x^2 + 2x + 1 &= 3\left(x^2 + 2 \cdot \frac{1}{3}x + \frac{1}{9} - \frac{1}{9}\right) + 1 \\
&= 3\left(\left(x + \frac{1}{3}\right)^2 - \frac{1}{9}\right) + 1 \\
&= 3\left(x + \frac{1}{3}\right)^2 - \frac{1}{3} + 1 \\
&= 3\left(x - \left(-\frac{1}{3}\right)\right)^2 + \frac{2}{3} \\
\text{Minimum point} &= \left(-\frac{1}{3}, \frac{2}{3}\right)
\end{aligned}$$

Maximum or minimum value of a quadratic function

- Let $f(x) = ax^2 + bx + c$ - quadratic ($a \neq 0$).
- Let D be the discriminant $D = b^2 - 4ac$.

$$f(x) = a\left(x - \left(-\frac{b}{2a}\right)\right)^2 - \frac{D}{4a} \quad \Big| \text{ complete the square}$$

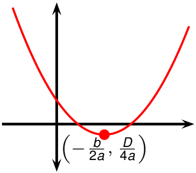
- Therefore if $a > 0$ then $f(x) = a(\text{square}) - \dfrac{D}{4a} \geq -\dfrac{D}{4a}$.
- Similarly if $a < 0$ then $f(x) = a(\text{square}) - \dfrac{D}{4a} \leq -\dfrac{D}{4a}$.

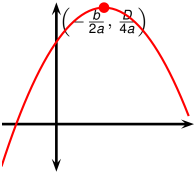
Recall $f(x) = ax^2 + bx + c = a\left(x - \left(-\frac{b}{2a}\right)\right)^2 - \frac{D}{4a}$.

Proposition

Let $f(x) = ax^2 + bx + c$, $a \neq 0$ and let $D = b^2 - 4ac$.

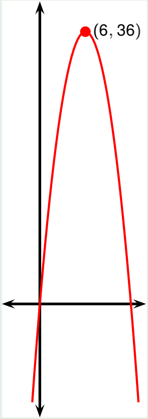
- *If $a > 0$ then $f(x)$ has no maximum and has minimum at $x = -\frac{b}{2a}$.*
- *If $a < 0$ then $f(x)$ has no minimum and has maximum at $x = -\frac{b}{2a}$.*
- *In both cases, the extremal value (either maximum or minimum) is $f\left(-\frac{b}{2a}\right) = -\frac{b^2-4ac}{4a} = -\frac{D}{4a}$.*





Example

Let x, z be two numbers that add to 12. Choose x and z so that the product $x \cdot z$ is maximal.



$$\begin{aligned} x + z &= 12 \\ z &= 12 - x \end{aligned}$$

Maximizing:

$$\begin{aligned} xz &= x(12 - x) \\ &= -x^2 + 12x \end{aligned}$$

Parabola opens down $\Rightarrow$ has maximum, attained at:

$$\begin{aligned} x &= -\frac{b}{2a} \\ &= -\frac{12}{-2} = 6 \\ z &= 12 - x = 12 - 6 = 6 \\ \text{Max. product} &= xz = 6 \cdot 6 = 36. \end{aligned}$$