Calculus I
Homework
Fermat's Theorem and the Mean Value Theorem

1. Use the Intermediate Value theorem and the Mean Value Theorem/Rolle's Theorem to prove that the function has **exactly one** real root.

   (a) $f(x) = x^3 + 4x + 7$.

   (b) $f(x) = x^3 + x^2 + x + 1$.

   (c) $f(x) = \cos^3\left(\frac{x}{3}\right) + \sin x - 3x$.

   Solution. 1.a. $f(-2) = -9$ and $f(1) = 12$. Since $f(x)$ is continuous and has both negative and positive outputs, it must have a zero. In other words, for some c between -2 and 1, $f(c) = 0$. If there were solutions $x = a$ and $x = b$, then we would have $f(a) = f(b)$, and Rolle's Theorem would guarantee that for some x-value, $f'(x) = 0$. However, $f'(x) = 3x^2 + 4$, which always positive and therefore is never 0. Therefore there cannot be 2 or more solutions.

   The above can be stated informally as follows. Note that $f'(x) = 3x^2 + 4$, which is always positive. Therefore, the graph of f is increasing from left to right. So once the graph crosses the x-axis, it can never turn around and cross again, so there can only be a single zero (that is, a single solution to $f(x) = 0$).

   Solution. 1.c. $f(5) = \cos^3\left(\frac{5}{3}\right) + \sin 5 - 15 \leq 2 - 15 = -13 < 0$ (because $\cos a, \sin b \in [-1, 1]$ for arbitrary a, b). Similarly $f(-5) = \cos^3\left(-\frac{5}{3}\right) + \sin(-5) + 15 \geq 15 - 2 > 0$. Therefore by the Intermediate Value Theorem $f(x) = 0$ has at least one solution in the interval $[-5, 5]$.

   Suppose on the contrary to what we are trying to prove, $f(x) = 0$ has two or more solutions; call the first 2 solutions a, b. That means that $f(a) = f(b) = 0$, so by the Mean value theorem, there exists a $c \in (a, b)$ such that $f'(c) = (f(a) - f(b))/(a - b) = (0 - 0)/(a - b) = 0$. On the other hand we may compute:

   $$f'(x) = -3 + \cos x - \cos^2\left(\frac{x}{3}\right)\sin\left(\frac{x}{3}\right) \leq -1 < 0,$$

   where the first inequality follows from the fact that $\sin x, \cos x \in [-1, 1]$. So we got that $f'(c) = 0$ for some c but at the same time $f'(x) < 0$ for all x, which is a contradiction. Therefore $f(x) = 0$ has exactly one solution.

2. Use the Intermediate Value theorem and the Mean Value Theorem/Rolle's Theorem to prove that the function has **exactly one** real root.

   (a) $x^5 + 7x = 2$.

   (b) $x^7 + x^5 + x^3 = 3$.

   (c) $2x - 1 = \sin x$.

   (d) $e^x + 2x = 3$.