Precalculus
Degree lowering formulas

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Proposition (Power-Reducing Formulas)

$$\sin^2\alpha = \frac{1-\cos(2\alpha)}{2} \qquad \cos^2\alpha = \frac{1+\cos(2\alpha)}{2}$$

Proof.

$\square$

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$$\cos(2\alpha) = 1 - 2\sin^2\alpha$$

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$$\begin{aligned}\cos(2\alpha) &= 1 - 2\sin^2\alpha\\ 2\sin^2\alpha &= 1-\cos(2\alpha)\end{aligned}$$

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Corollary

$$\sin\alpha = \pm\sqrt{\frac{1-\cos(2\alpha)}{2}} \qquad \cos\alpha = \pm\sqrt{\frac{1+\cos(2\alpha)}{2}}$$

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Corollary

$$\sin\alpha = \pm\sqrt{\frac{1-\cos(2\alpha)}{2}} \qquad \cos\alpha = \pm\sqrt{\frac{1+\cos(2\alpha)}{2}}$$

Corollary (Half-Angle Formulas)

$$\sin\left(\frac{\beta}{2}\right) = \pm\sqrt{\frac{1-\cos\beta}{2}} \qquad \cos\left(\frac{\beta}{2}\right) = \pm\sqrt{\frac{1+\cos\beta}{2}}$$

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- The power reducing formulas are used to express $\sin^k\alpha$ and $\cos^k\alpha$ via lower powers of the $\sin$ and $\cos$ functions (applied to angles other than α).

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- The power reducing formulas are used to express $\sin^k\alpha$ and $\cos^k\alpha$ via lower powers of the $\sin$ and $\cos$ functions (applied to angles other than α).
- This technique will play a key role in integration (studied later/in another course).

Example

Rewrite $\sin^4 \alpha$ in terms of first powers of the cosines and sines of multiples of the angle α.

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$$\sin^{4} \alpha = \left(\sin^{2} \alpha\right)^{2}$$

Recall the formulas: $\sin^2\beta = \textbf{?}$, $\cos^2\beta = \textbf{?}$.

Example

Rewrite $\sin^4\alpha$ in terms of first powers of the cosines and sines of multiples of the angle α.

$$\begin{aligned}\sin^4\alpha &= \left(\sin^2\alpha\right)^2\\ &= \left(\textbf{?} \qquad\qquad\right)^2\end{aligned}$$

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$$\begin{aligned}\sin^4\alpha &= \left(\sin^2\alpha\right)^2\\ &= \left(\frac{1-\cos(2\alpha)}{2}\right)^2\end{aligned}$$

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Rewrite $\sin^4\alpha$ in terms of first powers of the cosines and sines of multiples of the angle α.

$$\begin{aligned}
\sin^4\alpha &= \left(\sin^2\alpha\right)^2 \\
&= \left(\frac{1-\cos(2\alpha)}{2}\right)^2 \\
&= \frac{1}{4}\left(\textbf{?} \qquad\qquad\qquad\qquad \right)
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$$\begin{aligned}
\sin^4\alpha &= \left(\sin^2\alpha\right)^2 \\
&= \left(\frac{1-\cos(2\alpha)}{2}\right)^2 \\
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Recall the formulas: $\sin^2\beta = \frac{1-\cos(2\beta)}{2}$, $\cos^2\beta = \frac{\cos(2\beta)+1}{2}$.

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&= \left(\frac{1-\cos(2\alpha)}{2}\right)^2 \\
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&= \frac{1}{4}\left(1 - 2\cos(2\alpha) + \frac{\cos(2\cdot 2\alpha)+1}{2}\right)
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&= \frac{1}{4}\left(1-2\cos(2\alpha)+\frac{\cos(2\cdot 2\alpha)}{2}+\frac{1}{2}\right)\\
&= \frac{1}{4}\left(\frac{3}{2}-2\cos(2\alpha)+\frac{\cos(4\alpha)}{2}\right)
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&= \frac{1}{4}\left(\frac{3}{2}-2\cos(2\alpha)+\frac{\cos(4\alpha)}{2}\right)\\
&= \frac{1}{8}\left(3-4\cos(2\alpha)+\cos(4\alpha)\right)
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