Precalculus
, Factorization of polynomials: overview

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2019

Outline

Recall that $i^2 = -1$, $\sqrt{-1} = i$.

Example (Polynomial factorizations)

$$
\begin{aligned}
2x^2 + 3x - 5 &= (2x+5)(x-1) = 2\left(x - \left(-\tfrac{5}{2}\right)\right)(x-1)\\
x^2 + 1 &= x^2 - (-1) = x^2 - i^2 = (x+i)(x-i)\\
x^4 - 1 &= (x^2-1)(x^2+1) = (x-1)(x+1)(x^2+1)\\
&= (x-1)(x+1)(x-i)(x+i)\\
x^4 + 1 &= \left(x^2 - \sqrt{2}x + 1\right)\left(x^2 + \sqrt{2}x + 1\right)\\
&= \left(x - \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)\right)\left(x - \left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right)\right)\\
&\quad \left(x - \left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)\right)\left(x - \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right)\right)
\end{aligned}
$$

Theorem (The Fundamental Theorem of Algebra)

Every polynomial can be factored into product of linear terms

$$p(x) = a_0x^n + a_1x^{n-1} + \dots + a_n = a_0(x - x_1)\dots(x - x_n),$$

where $x_1, \dots, x_n$ are the (not necessarily different) roots of $p(x)$.

- Every pol. of deg. n can be factored as product of n linear factors.
- $x_1, \dots, x_n$ may be complex numbers. Reminder: complex numbers are of the form $p + qi$, where $i^2 = -1$ and $\sqrt{-1} = i$.
- While we can find $x_1, \dots x_n$ with arbitrary precision, there may not exist a formula involving radicals for computing each $x_1, \dots, x_n$.

Corollary

Every real polynomial can be factored into a product of real linear terms and real quadratic terms with no real roots, i.e., factors of form

- *$(x - r)$, where r is real and*
- *$ax^2 + bx + c$ with $b^2 - 4ac < 0$ where a, b, c are real.*

$$a_0x^n + a_1x^{n-1} + \dots + a_n = a_0(x - x_1)\dots(x - x_n)$$
$$= \text{prod. real quadratics no roots \& lin. terms.}$$

Example

$$2x^2 + 3x - 5 = (2x+5)(x-1) = 2\left(x - \left(-\tfrac{5}{2}\right)\right)(x-1) \quad \text{real roots}$$

$$x^2 + 1 = x^2 - (-1) = x^2 - i^2 = (x - (-i))(x - i) \quad \text{complex roots}$$

$$x^4 - 1 = (x^2 - 1)(x^2 + 1) = (x-1)(x+1)(x^2+1)$$
$$= (x-1)(x-(-1))(x-i)(x-(-i)) \quad \text{mixed roots}$$

$$x^4 + 1 = (x^2 - \sqrt{2}x + 1)(x^2 + \sqrt{2}x + 1)$$
$$= \left(x - \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)\right)\left(x - \left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right)\right)$$
$$\left(x - \left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)\right)\left(x - \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right)\right) \quad \text{complex roots}$$

Factoring polynomials in practice

- In theory every polynomial can be factored.

$$a_0x^n + a_1x^{n-1} + \cdots + a_n = a_0(x - x_1)\dots(x - x_n)$$

- Theory guarantees numerical approximations for roots $x_1, \dots, x_n$.
- Can we find algebraic formulas for $x_1, \dots, x_n$?
- No, if using finitely many operations $+, -, *, /, \sqrt[n]{}$.
- First (advanced) proof by Norwegian Niels Henrik Abel(1824) based on work of Italian Paolo Ruffini(1799).
- Yes, with extra operations. Difficult: google Galois Theory to get started.

What does factorization mean?

- Based on context, “to factor a polynomial” means one of:
  - Factor the polynomial over the rational numbers. Use integers/quotients, but no $\sqrt[n]{}$).
  - Factor the polynomial over the real numbers. Use radicals and/or numerical approximations, no use of $i = \sqrt{-1}$.
  - Fully factor the polynomial using complex numbers.

These poly’s are equal	Type of factorization
$x^4 + 1$	factored over rationals
$(x^2 - \sqrt{2}x + 1)(x^2 + \sqrt{2}x + 1)$	factored over the reals
$\left(x - \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)\right)\left(x - \left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)\right)$ $\left(x - \left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right)\right)\left(x - \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right)\right)$	full complex factorization

Factorization over the rationals

- Suppose we want to factor a polynomial using only rational numbers (no $\sqrt[n]{}$ or numerical approximations).
- No guarantee to get:

$$a_0x^n + a_1x^{n-1} + \cdots + a_n = a_0(x - x_1)\dots(x - x_n)$$

- A factorization using rationals may have arbitrarily large factors.
- Efficient algorithms for factoring using rationals exist.
  - Kronecker algorithm (German Leopold Kronecker (1823-1891)).
  - Methods based on finite fields.
  - Lenstra-Lenstra-Lovász algorithm (Dutch, Dutch, Hungarian mathematicians, all contemporary).
- Above methods require computer; no rational roots assumption.
- If we assume rational roots there are practical algorithms by hand.
- We study those for cubics with the aid of scientific calculator.