

# Calculus II

## Differential equation basics

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# Modeling with Differential Equations

- When modeling real-world problems, we often have a relationship between an unknown function and some of its derivatives.
- Such a relationship is called a differential equation.
- It is not always possible to find an explicit solution to a differential equation, but sometimes a graphical or approximate answer can be good enough for applications.

# Models of Population Growth

- One model for population growth assumes that the population grows at a rate proportional to its size.
- In other words, if a certain number of bacteria produce a certain number of offspring in a certain time, then ten times that many bacteria produce ten times that many offspring in the same time.
- This is plausible when the population has unlimited food and environment and no restrictions on its size.
- Name the variables:

*$t = \text{time}$*

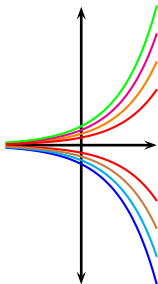
*$P = \text{the number of individuals in the population}$*

- The rate of growth is  $dP/dt$ .
- Then “rate of growth proportional to population size” means

$$\frac{dP}{dt} = kP$$

where  $k$  is the proportionality constant.

$$\frac{dP}{dt} = kP$$



- This is a differential equation.
- Exponential functions satisfy this condition.
- Let  $P(t) = Ce^{kt}$  ( $C$  is a constant). Then

$$\frac{dP}{dt} = \frac{d}{dt}(Ce^{kt}) = Cke^{kt} = kCe^{kt} = kP(t)$$

- Therefore any function of the form  $P(t) = Ce^{kt}$  satisfies the equation. We will see later that there is no other solution.
- Letting  $C$  vary over the real numbers gives a family of solutions.
- Since populations are non-negative, only solutions with  $C > 0$  are relevant.

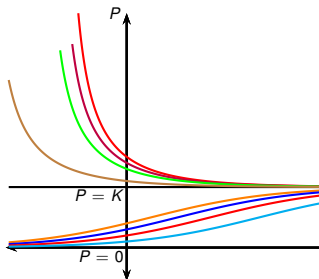
- This model works well under ideal conditions.
- In real life, most populations are constrained by the environment, the amount of food, etc.
- Many populations start by increasing exponentially, but then level off when they approach some upper bound, called the carrying capacity  $K$ .
- To take this into account, make two assumptions:
  - $\frac{dP}{dt} \approx kP$  if  $P$  is small (Initially, the growth rate is proportional to  $P$ ).
  - $\frac{dP}{dt} < 0$  if  $P > K$  ( $P$  decreases if it ever exceeds  $K$ ).
- Here is an expression that takes both assumptions into account:

$$\frac{dP}{dt} = kP \left( 1 - \frac{P}{K} \right)$$

- This is called the logistic differential equation.

$$\frac{dP}{dt} = kP \left( 1 - \frac{P}{K} \right)$$

- What do the solutions look like?
- $P = 0$  and  $P = K$  are special solutions, called equilibrium solutions.
- If  $P > K$ , then  $1 - P/K < 0$ , so  $dP/dt < 0$ , and  $P$  decreases.
- If  $P < K$ , then  $1 - P/K > 0$ , so  $dP/dt > 0$ , and  $P$  increases.
- As  $P \rightarrow K$ ,  $1 - P/K \rightarrow 0$ , so  $dP/dt \rightarrow 0$  and  $P$  levels off.





# A Model for the Motion of a Spring

- Suppose we have an object with mass  $m$  attached to a spring.
- Hooke's Law: if the spring is stretched or compressed  $x$  units from its natural length, then it exerts a force that is proportional to  $x$ .
- Force equals mass times acceleration.
- Acceleration is the second derivative of displacement with respect to time.

$$m \frac{d^2 x}{dt^2} = -kx$$

- This is called a second-order differential equation because it involves second derivatives.
- Sine and cosine functions are solutions.

# General Differential Equations

## Definition (Differential Equation)

A differential equation is an equation that contains an unknown function and some of its derivatives.

## Definition (Order of a Differential Equation)

The order of a differential equation is the highest derivative that appears in it.

## Definition (Solution)

A function  $f$  is called a solution of a differential equation if the equation is satisfied when  $f$  and its derivatives are plugged in.

## Definition (To Solve a Differential Equation)

When we are asked to solve a differential equation we are expected to find all possible solutions.

## Example

Show that every member of the family of functions

$$y = \frac{1 + ce^t}{1 - ce^t}$$

is a solution of the differential equation  $y' = \frac{1}{2}(y^2 - 1)$ .

$$\begin{aligned} \text{LHS} &= \frac{(1 - ce^t)(ce^t) - (1 + ce^t)(-ce^t)}{(1 - ce^t)^2} \\ &= \frac{ce^t - c^2e^{2t} + ce^t + c^2e^{2t}}{(1 - ce^t)^2} = \frac{2ce^t}{(1 - ce^t)^2} \end{aligned}$$

$$\begin{aligned} \text{RHS} &= \frac{1}{2} \left[ \left( \frac{1 + ce^t}{1 - ce^t} \right)^2 - 1 \right] = \frac{1}{2} \left[ \frac{(1 + ce^t)^2 - (1 - ce^t)^2}{(1 - ce^t)^2} \right] \\ &= \frac{1}{2} \left[ \frac{1 + 2ce^t + c^2e^{2t} - 1 + 2ce^t - c^2e^{2t}}{(1 - ce^t)^2} \right] \\ &= \frac{1}{2} \frac{4ce^t}{(1 - ce^t)^2} = \frac{2ce^t}{(1 - ce^t)^2} = \text{LHS} \end{aligned}$$

- Often we don't want to find all solutions (the general solution).
- Instead, we only want to find a single solution that satisfies some additional requirement.
- Often that requirement has the form  $y(t_0) = y_0$ .
- This is called an initial condition.
- This type of problem is called an initial value problem.

## Example

Find a solution of the differential equation  $y' = \frac{1}{2}(y^2 - 1)$  that satisfies the initial condition  $y(0) = 2$ .

Substitute  $t = 0$  and  $y = 2$  into the formula

$$y = \frac{1 + ce^t}{1 - ce^t}$$

from Example 1.

$$\begin{aligned} 2 &= \frac{1 + ce^0}{1 - ce^0} = \frac{1 + c}{1 - c} \\ 2(1 - c) &= 1 + c \\ 2 - 2c &= 1 + c \\ c &= 1/3 \end{aligned}$$

Therefore the solution to the initial-value problem is

$$y = \frac{1 + \frac{1}{3}e^t}{1 - \frac{1}{3}e^t} = \frac{3 + e^t}{3 - e^t}.$$

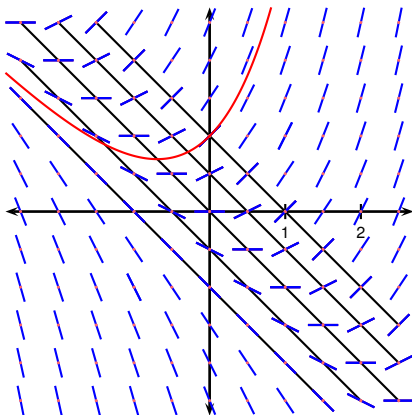
# Direction Fields and Euler's Method

- Often we don't know how to find explicit solutions to a differential equation.
- Nevertheless, we can learn a lot about the solutions using:
  - A graphical approach (direction fields)
  - A numerical approach (Euler's method)
- Today we will discuss direction fields, but not Euler's method.

# Direction Fields

- How do we sketch the graph of the solution to  $y' = x + y$  that satisfies the initial condition  $y(0) = 1$ ?
- Make a table of values of  $y'$ .

Point	$y'$
(1, 0)	1
(-1, 0)	-1
(0, 1)	1
(0, -1)	-1
(0, 0)	0
(1, 1)	2
(1, -1)	0
(-1, 1)	0
(-1, -1)	-2



Line	$y'$
$y = -x$	0
$y = -x + \frac{1}{2}$	$\frac{1}{2}$
$y = -x + 1$	1
$y = -x - \frac{1}{2}$	$-\frac{1}{2}$
$y = -x - 1$	-1

# Separable Equations

In this section, we will discuss a type of differential equation, called a separable equation, for which it is possible to find an explicit solution.

## Definition (Separable Equation)

A separable equation is a first-order equation in which the expression for  $dy/dx$  can be factored as a function of  $x$  times a function of  $y$ . In other words,

$$\frac{dy}{dx} = g(x)f(y).$$

Let  $f(y) = 1/h(y)$ . Then

$$\frac{dy}{dx} = \frac{g(x)}{h(y)}.$$



$$\frac{dy}{dx} = \frac{g(x)}{h(y)}.$$

- To solve, write this in differential form:

$$h(y)dy = g(x)dx$$

- Now integrate:

$$\int h(y)dy = \int g(x)dx$$

- This defines  $y$  implicitly as a function of  $x$ .
- Sometimes we might be able to solve explicitly for  $y$  in terms of  $x$ .

Why does this process yield a function that satisfies the original differential equation? Suppose that  $\int h(y)dy = \int g(x)dx$ . Then we will use the Chain Rule to show that  $y$  satisfies the original equation.

$$\begin{aligned}\int h(y)dy &= \int g(x)dx \\ \frac{d}{dx} \left( \int h(y)dy \right) &= \frac{d}{dx} \left( \int g(x)dx \right) \\ \frac{d}{dy} \left( \int h(y)dy \right) \frac{dy}{dx} &= \frac{d}{dx} \left( \int g(x)dx \right) \\ h(y) \frac{dy}{dx} &= g(x) \\ \frac{dy}{dx} &= \frac{g(x)}{h(y)}\end{aligned}$$

## Example

Solve the differential equation  $\frac{dy}{dx} = \frac{x^2}{y^2}$ , and find the solution that satisfies the initial condition  $y(0) = 2$ .

$$\begin{aligned}y^2 dy &= x^2 dx \\ \int y^2 dy &= \int x^2 dx \\ \frac{y^3}{3} &= \frac{x^3}{3} + C \\ y &= \sqrt[3]{x^3 + 3C} \\ y &= \sqrt[3]{x^3 + K}\end{aligned}$$

To find the solution satisfying the initial condition, set  $2 = y(0) = \sqrt[3]{0^3 + K} = \sqrt[3]{K}$ . Then  $\sqrt[3]{K} = 2$ , so  $K = 8$ .

$$y = \sqrt[3]{x^3 + 8}.$$

## Example

Solve the equation  $y' = x^2 y$ .

$$\frac{dy}{dx} = x^2 y$$

$$\frac{1}{y} dy = x^2 dx \quad y \neq 0$$

$$\int \frac{1}{y} dy = \int x^2 dx$$

$$\ln |y| = \frac{1}{3} x^3 + C$$

$$e^{\ln |y|} = e^{x^3/3 + C}$$

$$|y| = e^C e^{x^3/3}$$

$$y = \pm e^C e^{x^3/3}$$

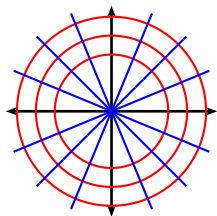
The function  $y = 0$  satisfies the equation. General solution:

$$y = A e^{x^3/3}.$$

# Orthogonal Trajectories

## Definition (Orthogonal Trajectory)

An orthogonal trajectory to a family of curves is a curve that intersects each curve of the family orthogonally (that is, at right angles).



Each member of the family  $y = mx$  of straight lines passing through the origin is an orthogonal trajectory to the family  $x^2 + y^2 = r^2$  of circles centered at the origin.

## Example

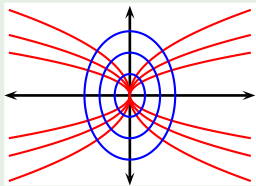
Find the orthogonal trajectories of the family  $x = ky^2$ , where  $k$  is an arbitrary constant. Differentiate implicitly:

$$x = ky^2$$

$$1 = 2ky \frac{dy}{dx}$$

$$1 = 2 \left( \frac{x}{y^2} \right) y \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{y}{2x}$$



An orthogonal trajectory will have a slope that is the negative reciprocal of the slope of the curve.

$$\frac{dy}{dx} = -\frac{2x}{y}$$

$$\int y dy = -\int 2x dx$$

$$\frac{y^2}{2} = -x^2 + C$$

$$x^2 + \frac{y^2}{2} = C$$

The ellipses  $x^2 + \frac{y^2}{2} = C$  are all orthogonal trajectories to  $x = ky^2$ .

# Mixing Problems

- Typical mixing problems involve:
- A tank of fixed capacity.
- A completely mixed solution of some substance in the tank.
- A solution of a certain concentration enters the tank at a fixed rate.
- In the tank, the solution immediately becomes completely stirred.
- The mixture leaves at the other end at a fixed rate (possibly a different rate).
- Let  $y(t)$  denote the amount of substance in the tank at time  $t$ .
- Then  $y'(t)$  denotes the rate at which the substance is being added minus the rate at which it is being removed.
- This often gives a differential equation.

## Example

A tank contains 20 kg of salt dissolved in 5000 L of water. Brine that contains 0.03 kg of salt per liter of water enters the tank at a rate of 25 L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate. How much salt is in the tank after half an hour?

- Let  $y(t)$  denote the amount of salt (in kg) after  $t$  minutes.
- Given:  $y(0) = 20$ . We want to know:  $y(30)$ .

$$\frac{dy}{dt} = (\text{rate in}) - (\text{rate out}) = 0.75 - \frac{y(t)}{200} = \frac{150 - y(t)}{200}$$

$$\text{rate in} = (\text{concentration in})(\text{rate of volume in})$$

$$= \left(0.03 \frac{\text{kg}}{\text{L}}\right) \left(25 \frac{\text{L}}{\text{min}}\right) = 0.75 \frac{\text{kg}}{\text{min}}$$

$$\text{rate out} = (\text{concentration out})(\text{rate of volume out})$$

$$= \left(\frac{y(t)}{5000} \frac{\text{kg}}{\text{L}}\right) \left(25 \frac{\text{L}}{\text{min}}\right) = \frac{y(t)}{200} \frac{\text{kg}}{\text{min}}$$



### Example (Example 6, p. 621)

A tank contains 20 kg of salt dissolved in 5000 L of water. Brine that contains 0.03 kg of salt per liter of water enters the tank at a rate of 25 L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate. How much salt is in the tank after half an hour?

$$\frac{dy}{dt} = \frac{150 - y(t)}{200}$$

$$\int \frac{dy}{150 - y} = \int \frac{dt}{200}$$

$$-\ln |150 - y| = t/200 + C \quad y(0) = 20, \text{ so } C = -\ln 130$$

$$-\ln |150 - y| = t/200 - \ln 130$$

$$|150 - y| = 130e^{-t/200}$$

$$y < 150 = (0.03)(5000), \text{ so } |150 - y| = 150 - y$$

$$y = 150 - 130e^{-t/200}$$

$$y(30) = 150 - 130e^{-30/200} \approx 38.1 \text{ kg}$$

# The Law of Natural Growth

- Recall that differential equations could be used to model population growth.
- The Law of Natural Growth works in ideal cases, where populations are unconstrained by lack of food, or the environment.
- Let  $P(t)$  be the population at time  $t$ .
- Then the Law of Natural Growth says:

$$\frac{dP}{dt} = kP$$

- The constant  $k$  is sometimes called the relative growth rate.

$$\frac{dP}{dt} = kP$$

This is a separable equation, so we can solve it.

$$\begin{aligned}\int \frac{dP}{P} &= \int k dt \\ \ln |P| &= kt + C \\ |P| &= e^C e^{kt} \\ P &= \pm e^C e^{kt}\end{aligned}$$

- Let  $A = \pm e^C$ . Then the solution is  $P = Ae^{kt}$ .
- $A = \pm e^C$  can be any positive or negative number.
- The function  $P = 0$  is also a solution, so  $A$  can be any number.
- $P(0) = Ae^{k \cdot 0} = A$ .

The solution to the initial value problem

$$\begin{aligned}\frac{dP}{dt} &= kP, & P(0) &= P_0 \\ \text{is} && P(t) &= P_0 e^{kt}.\end{aligned}$$

# The Logistic Model

- The Logistic Model works in cases when the population is constrained by its environment.
- Let  $P(t)$  be the population at time  $t$ .
- Then the Logistic Equation is:

$$\frac{dP}{dt} = kP \left( 1 - \frac{P}{K} \right)$$

- The constant  $K$  is called the carrying capacity. It represents how many individuals the environment can sustain in the long run.

$$\frac{dP}{dt} = kP \left(1 - \frac{P}{K}\right)$$

$$\int \frac{1}{P(1 - P/K)} dP = \int k dt$$

$$\int \frac{K}{P(K - P)} dP = \int k dt$$

$$\int \left( \frac{1}{P} + \frac{1}{K - P} \right) dP = \int k dt$$

$$\ln |P| - \ln |K - P| = kt + C$$

$$\ln \left| \frac{K - P}{P} \right| = -kt - C$$

$$\frac{K - P}{P} = \pm e^{-C} e^{-kt} = Ae^{-kt}$$

$$K = P(1 + Ae^{-kt})$$

$$P = \frac{K}{1 + Ae^{-kt}}$$

Plug in  $P(0) = P_0$ :

$$\frac{K - P_0}{P_0} = Ae^{-k \cdot 0} = A.$$

The solution to the initial value problem

$$\frac{dP}{dt} = kP \left( 1 - \frac{P}{K} \right), \quad P(0) = P_0$$

is

$$P = \frac{K}{1 + Ae^{-kt}}, \quad A = \frac{K - P_0}{P_0}.$$

## Example

Write the solution of the initial value problem

$$\frac{dP}{dt} = 0.08P \left( 1 - \frac{P}{1000} \right), \quad P(0) = 100$$

and use it to find when the population reaches 900.

$$P(t) = \frac{1000}{1 + Ae^{-0.08t}}, \quad A = \frac{1000 - 100}{100} = 9$$

$$\text{Therefore} \quad P(t) = \frac{1000}{1 + 9e^{-0.08t}}.$$

$$\begin{aligned} \text{Set } P(t) = 900 : \quad & \frac{1000}{1 + 9e^{-0.08t}} = 900 \\ & 1 + 9e^{-0.08t} = 1000/900 \\ & e^{-0.08t} = \frac{1000/900 - 1}{9} = \frac{1}{81} \\ & -0.08t = -\ln 81 \\ & t = \frac{\ln 81}{0.08} \approx 54.9 \end{aligned}$$