Calculus II
Power series expansion of arctangent, part 1

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Example

Find a power series for $\arctan x$ and state its radius of convergence.

$$\begin{aligned}
\arctan(x) &= \int \mathrm{d}(\arctan x) = \int (\arctan x)' \mathrm{d}x && \Big| \text{ up to const.} \\
&= \int \left(\frac{1}{1+x^2} \right) \mathrm{d}x = \int \left(\frac{1}{1-(-x^2)} \right) \mathrm{d}x \\
&= \int \left(1 - x^2 + x^4 - x^6 + \cdots \right) \mathrm{d}x && \Big| \text{ for } |x| < 1 \\
&= \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \right) + C \\
&= C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}
\end{aligned}$$

- To find C, plug in $x = 0$: $C = 0$.
- Therefore the theorem on integrating power series implies that $\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$, for $|x| < 1$.
- By the same theorem, the radius of convergence remains $R = 1$.