Calculus II

Integrals of the form $\int \tan^m x \sec^n x \mathrm{d}x$, $n, m > 0$, m-odd

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Example

$$\int \tan^5 x \sec^9 x \mathrm{d}x$$

Example

$$\int \tan^5 x \sec^9 x dx = \int \tan^4 x \sec^8 x \tan x \sec x dx$$

Example

$$\int \tan^5 x \sec^9 x \mathrm{d}x = \int \tan^4 x \sec^8 x \tan x \sec x \mathrm{d}x$$
$$= \int \tan^4 x \sec^8 x \mathrm{d}(\textbf{?} \quad)$$

Example

$$\begin{aligned}\int \tan^5 x \sec^9 x \mathrm{d}x &= \int \tan^4 x \sec^8 x \tan x \sec x \mathrm{d}x \\ &= \int \tan^4 x \sec^8 x \mathrm{d}(\sec x)\end{aligned}$$

Example

$$\int \tan^5 x \sec^9 x dx = \int \tan^4 x \sec^8 x \tan x \sec x dx$$
$$= \int \tan^4 x \sec^8 x d(\sec x)$$

Can we rewrite $\tan^4 x$ via $\sec x$?

Example

$$\int \tan^5 x \sec^9 x dx = \int \tan^4 x \sec^8 x \tan x \sec x dx$$
$$= \int \tan^4 x \sec^8 x d(\sec x)$$
$$= \int \left(\tan^2 x\right)^2 \sec^8 x d(\sec x)$$

Can we rewrite $\tan^4 x$ via $\sec x$?

Example

$$\begin{aligned}\int \tan^5 x \sec^9 x \mathrm{d}x &= \int \tan^4 x \sec^8 x \tan x \sec x \mathrm{d}x\\ &= \int \tan^4 x \sec^8 x \mathrm{d}(\sec x)\\ &= \int \left(\tan^2 x\right)^2 \sec^8 x \mathrm{d}(\sec x)\\ &= \int \left(\sec^2 x - 1\right)^2 \sec^8 x \mathrm{d}(\sec x)\end{aligned}$$

Can we rewrite $\tan^4 x$ via $\sec x$?

Example

$$\begin{aligned}\int \tan^5 x \sec^9 x \mathrm{d}x &= \int \tan^4 x \sec^8 x \tan x \sec x \mathrm{d}x\\ &= \int \tan^4 x \sec^8 x \mathrm{d}(\sec x) \\ &= \int \left(\tan^2 x\right)^2 \sec^8 x \mathrm{d}(\sec x)\\ &= \int \left(\sec^2 x - 1\right)^2 \sec^8 x \mathrm{d}(\sec x)\\ &= \int \left(1 - u^2\right)^2 u^8 \mathrm{d}u\end{aligned}$$

Can we rewrite $\tan^4 x$ via $\sec x$?

Set $u = \sec x$

Example

$$\begin{aligned}
\int \tan^5 x \sec^9 x dx &= \int \tan^4 x \sec^8 x \tan x \sec x dx \\
&= \int \tan^4 x \sec^8 x d(\sec x) \\
&= \int \left(\tan^2 x\right)^2 \sec^8 x d(\sec x) \\
&= \int \left(\sec^2 x - 1\right)^2 \sec^8 x d(\sec x) \\
&= \int \left(1 - u^2\right)^2 u^8 du \\
&= \int \left(1 - 2u^2 + u^4\right) u^8 du
\end{aligned}$$

Can we rewrite $\tan^4 x$ via $\sec x$?

Set $u = \sec x$

Example

$$\int \tan^5 x \sec^9 x dx = \int \tan^4 x \sec^8 x \tan x \sec x dx$$

$$= \int \tan^4 x \sec^8 x d(\sec x)$$

Can we rewrite $\tan^4 x$ via $\sec x$?

$$= \int \left(\tan^2 x\right)^2 \sec^8 x d(\sec x)$$

$$= \int \left(\sec^2 x - 1\right)^2 \sec^8 x d(\sec x)$$

Set $u = \sec x$

$$= \int \left(1 - u^2\right)^2 u^8 du$$

$$= \int \left(1 - 2u^2 + u^4\right) u^8 du$$

$$= \int \left(u^8 - 2u^{10} + u^{12}\right) du$$

Example

$$\int \tan^5 x \sec^9 x dx = \int \tan^4 x \sec^8 x \tan x \sec x dx$$

$$= \int \tan^4 x \sec^8 x d(\sec x)$$

Can we rewrite $\tan^4 x$ via $\sec x$?

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Set $u = \sec x$

$$= \int \left(1 - u^2\right)^2 u^8 du$$

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$$= \int \left(u^8 - 2u^{10} + u^{12}\right) du$$

$$= \textbf{?}$$

Example

$$\begin{aligned}
\int \tan^5 x \sec^9 x \mathrm{d}x &= \int \tan^4 x \sec^8 x \tan x \sec x \mathrm{d}x \\
&= \int \tan^4 x \sec^8 x \mathrm{d}(\sec x) \\
&= \int \left(\tan^2 x\right)^2 \sec^8 x \mathrm{d}(\sec x) \\
&= \int \left(\sec^2 x - 1\right)^2 \sec^8 x \mathrm{d}(\sec x) \\
&= \int \left(1 - u^2\right)^2 u^8 \mathrm{d}u \\
&= \int \left(1 - 2u^2 + u^4\right) u^8 \mathrm{d}u \\
&= \int \left(u^8 - 2u^{10} + u^{12}\right) \mathrm{d}u \\
&= \frac{u^9}{9} - 2\frac{u^{11}}{11} + \frac{u^{13}}{13} + C
\end{aligned}$$

Can we rewrite $\tan^4 x$ via $\sec x$?

Set $u = \sec x$

Example

$$\begin{aligned}
\int \tan^5 x \sec^9 x \mathrm{d}x &= \int \tan^4 x \sec^8 x \tan x \sec x \mathrm{d}x \\
&= \int \tan^4 x \sec^8 x \mathrm{d}(\sec x) \\
&= \int \left(\tan^2 x\right)^2 \sec^8 x \mathrm{d}(\sec x) \\
&= \int \left(\sec^2 x - 1\right)^2 \sec^8 x \mathrm{d}(\sec x) \\
&= \int \left(1 - u^2\right)^2 u^8 \mathrm{d}u \\
&= \int \left(1 - 2u^2 + u^4\right) u^8 \mathrm{d}u \\
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Set $u = \sec x$

Example

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\int \tan^5 x \sec^9 x dx &= \int \tan^4 x \sec^8 x \tan x \sec x dx \\
&= \int \tan^4 x \sec^8 x d(\sec x) \\
&= \int \left(\tan^2 x\right)^2 \sec^8 x d(\sec x) \\
&= \int \left(\sec^2 x - 1\right)^2 \sec^8 x d(\sec x) \\
&= \int \left(1 - u^2\right)^2 u^8 du \\
&= \int \left(1 - 2u^2 + u^4\right) u^8 du \\
&= \int \left(u^8 - 2u^{10} + u^{12}\right) du \\
&= \frac{u^9}{9} - 2\frac{u^{11}}{11} + \frac{u^{13}}{13} + C
\end{aligned}$$

Can we rewrite $\tan^4 x$ via $\sec x$?

Set $u = \sec x$

Example

$$\begin{aligned}
\int \tan^5 x \sec^9 x \mathrm{d}x &= \int \tan^4 x \sec^8 x \tan x \sec x \mathrm{d}x \\
&= \int \tan^4 x \sec^8 x \mathrm{d}(\sec x) && \Big| \text{Can we rewrite } \tan^4 x \text{ via } \sec x? \\
&= \int \left(\tan^2 x\right)^2 \sec^8 x \mathrm{d}(\sec x) \\
&= \int \left(\sec^2 x - 1\right)^2 \sec^8 x \mathrm{d}(\sec x) && \Big| \text{Set } u = \sec x \\
&= \int \left(1 - u^2\right)^2 u^8 \mathrm{d}u \\
&= \int \left(1 - 2u^2 + u^4\right) u^8 \mathrm{d}u \\
&= \int \left(u^8 - 2u^{10} + u^{12}\right) \mathrm{d}u \\
&= \frac{u^9}{9} - 2\frac{u^{11}}{11} + \frac{u^{13}}{13} + C
\end{aligned}$$

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$$\int \tan^5 x \sec^9 x dx = \int \tan^4 x \sec^8 x \tan x \sec x dx$$

$$= \int \tan^4 x \sec^8 x d(\sec x)$$

Can we rewrite $\tan^4 x$ via $\sec x$?

$$= \int \left(\tan^2 x\right)^2 \sec^8 x d(\sec x)$$

$$= \int \left(\sec^2 x - 1\right)^2 \sec^8 x d(\sec x)$$

Set $u = \sec x$

$$= \int \left(1 - u^2\right)^2 u^8 du$$

$$= \int \left(1 - 2u^2 + u^4\right) u^8 du$$

$$= \int \left(u^8 - 2u^{10} + u^{12}\right) du$$

$$= \frac{u^9}{9} - 2\frac{u^{11}}{11} + \frac{u^{13}}{13} + C$$

$$= \frac{\sec^9 x}{9} - 2\frac{\sec^{11} x}{11} + \frac{\sec^{13} x}{13} + C \quad .$$