Calculus II
Power series expansion of sine and cosine

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$$\begin{aligned} \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{(-1)^n x^{2n+1}} \right| \\ &= \lim_{n\to\infty} \frac{x^2}{(2n+2)(2n+3)} = 0 \end{aligned}$$

Example

Find the Maclaurin series of $f(x) = \sin x$ and its radius of convergence.

$$\begin{array}{rclrcl} f(x) &=& \sin x & f(0) &=& 0 \\ f'(x) &=& \cos x & f'(0) &=& 1 \\ f''(x) &=& -\sin x & f''(0) &=& 0 \\ f'''(x) &=& -\cos x & f'''(0) &=& -1 \\ f^{(4)}(x) &=& \sin x & f^{(4)}(0) &=& 0 \end{array}$$

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Therefore $R = \infty$. It can be shown that this series sums to $\sin x$.

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&= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}\\
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The series for $\sin x$ converges everywhere, so the series for $\cos x$ does too.