Calculus II
Trigonometry review

Todor Milev

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Outline

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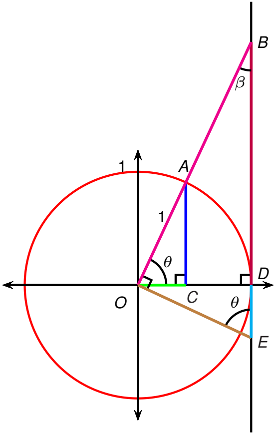
Geometric interpretation of all trigonometric functions



$$\begin{aligned}\beta &= 180^\circ - 90^\circ - \theta\\ &= 90^\circ - \theta\\ \angle OED &= 180^\circ - 90^\circ - \beta\\ &= 90^\circ - (90^\circ - \theta)\\ &= \theta\end{aligned}$$

Fix unit circle, center O, coordinates $(0, 0)$. Let $\angle DOB = \theta$. Let OB intersect the circle at point A. Coordinates of A are $(\cos\theta, \sin\theta)$.

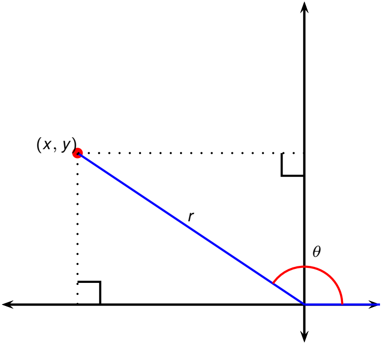
$$\begin{aligned}
\sin\theta &= \frac{\text{opp}}{\text{hyp}} = \frac{|AC|}{|OA|} = \frac{|AC|}{1} = |AC|\\
\cos\theta &= \frac{\text{adj}}{\text{hyp}} = \frac{|OC|}{|OA|} = \frac{|OC|}{1} = |OC|\\
\tan\theta &= \frac{\text{opp}}{\text{adj}} = \frac{|BD|}{|OD|} = \frac{|BD|}{1} = |BD|\\
\cot\theta &= \frac{\text{adj}}{\text{opp}} = \frac{|DE|}{|OD|} = \frac{|DE|}{1} = |DE|\\
\sec\theta &= \frac{\text{hyp}}{\text{adj}} = \frac{|OB|}{|OD|} = \frac{|OB|}{1} = |OB|\\
\csc\theta &= \frac{\text{hyp}}{\text{opp}} = \frac{|OE|}{|DO|} = \frac{|OE|}{1} = |OE|
\end{aligned}$$

Trigonometric Identities

Definition (Trigonometric Identity)

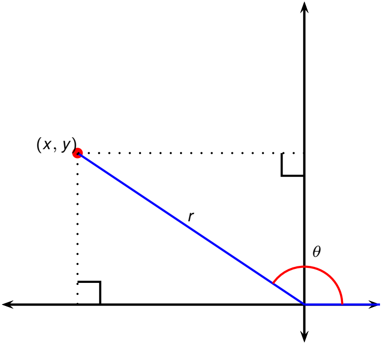
A trigonometric identity is an equality between the trigonometric functions in one or more variables that holds for all values of the involved variables in the domains of all of the expressions.

- By convention, when dealing with trigonometric identities we do not account for the domains of the involved expressions.
- For example, $\frac{\sin\theta}{\sin\theta} = 1$ is considered a valid trigonometric identity, although, when considered as a function, the left hand side is not defined for $\theta \neq 0$.



$$\sin\theta = \frac{y}{r} \quad \csc\theta = \frac{r}{y}$$
$$\cos\theta = \frac{x}{r} \quad \sec\theta = \frac{r}{x}$$
$$\tan\theta = \frac{y}{x} \quad \cot\theta = \frac{x}{y}$$

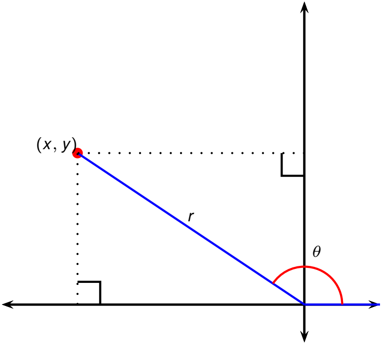
- $\csc\theta = \frac{1}{\sin\theta}$
- $\sec\theta = \frac{1}{\cos\theta}$
- $\cot\theta = \frac{1}{\tan\theta}$
- $\tan\theta = \frac{\sin\theta}{\cos\theta}$
- $\cot\theta = \frac{\cos\theta}{\sin\theta}$



$$\sin\theta = \frac{y}{r} \quad \csc\theta = \frac{r}{y}$$
$$\cos\theta = \frac{x}{r} \quad \sec\theta = \frac{r}{x}$$
$$\tan\theta = \frac{y}{x} \quad \cot\theta = \frac{x}{y}$$

$$\begin{aligned} & \sin^2\theta + \cos^2\theta \\ = \quad & \frac{y^2}{r^2} + \frac{x^2}{r^2} \\ = \quad & \frac{y^2 + x^2}{r^2} \\ = \quad & \frac{r^2}{r^2} \\ = \quad & 1 \end{aligned}$$

Therefore $\sin^2\theta + \cos^2\theta = 1$.



$$\sin\theta = \frac{y}{r} \quad \csc\theta = \frac{r}{y}$$
$$\cos\theta = \frac{x}{r} \quad \sec\theta = \frac{r}{x}$$
$$\tan\theta = \frac{y}{x} \quad \cot\theta = \frac{x}{y}$$

Example ($\tan^2\theta + 1 = \sec^2\theta$)

Prove the identity
$\tan^2\theta + 1 = \sec^2\theta$.

$$\begin{aligned}
\sin^2\theta + \cos^2\theta &= 1 \\
\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} &= \frac{1}{\cos^2\theta} \\
\tan^2\theta + 1 &= \sec^2\theta
\end{aligned}$$

The remaining identities are consequences of the addition formulas:

$$\begin{aligned}\sin(x+y) &= \sin x\cos y + \cos x\sin y\\ \cos(x+y) &= \cos x\cos y - \sin x\sin y\end{aligned}$$

Substitute $-y$ for y, and use the fact that $\sin(-y) = -\sin y$ and $\cos(-y) = \cos y$:

$$\begin{aligned}\sin(x-y) &= \sin x\cos y - \cos x\sin y\\ \cos(x-y) &= \cos x\cos y + \sin x\sin y\end{aligned}$$

The remaining identities are consequences of the addition formulas:

$$\begin{aligned}\sin(x+y) &= \sin x\cos y + \cos x\sin y\\ \cos(x+y) &= \cos x\cos y - \sin x\sin y\end{aligned}$$

To get the double angle formulas, substitute x for y:

$$\begin{aligned}\sin(2x) &= 2\sin x\cos x\\ \cos(2x) &= \cos^2 x - \sin^2 x\end{aligned}$$

Rewrite the second double angle formula in two ways, using $\cos^2 x = 1 - \sin^2 x$ and $\sin^2 x = 1 - \cos^2 x$:

$$\begin{aligned}\cos(2x) &= 2\cos^2 x - 1\\ \cos(2x) &= 1 - 2\sin^2 x\end{aligned}$$

To get the half-angle formulas, solve these equations for $\cos^2 x$ and $\sin^2 x$ respectively.

$$\cos^2 x = \frac{1+\cos(2x)}{2}, \qquad \sin^2 x = \frac{1-\cos(2x)}{2}$$

The remaining identities are consequences of the addition formulas:

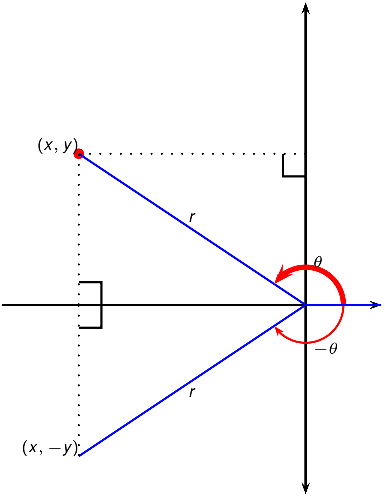
$$
\begin{aligned}
\sin(x+y) &= \sin x \cos y + \cos x \sin y \\
\cos(x+y) &= \cos x \cos y - \sin x \sin y
\end{aligned}
$$

Divide the first equation by the second, and then cancel $\cos x \cos y$ from the top and bottom:

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

Do the same for the subtraction formulas:

$$\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

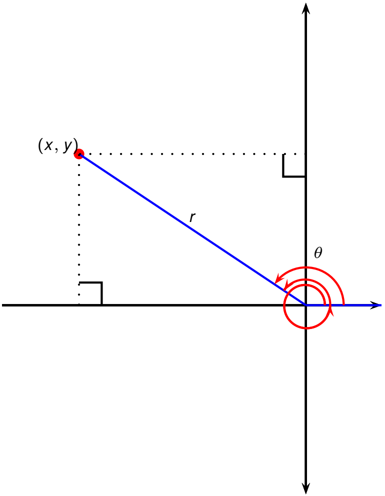


$\sin\theta = \frac{y}{r}$ $\quad$ $\csc\theta = \frac{r}{y}$

$\cos\theta = \frac{x}{r}$ $\quad$ $\sec\theta = \frac{r}{x}$

$\tan\theta = \frac{y}{x}$ $\quad$ $\cot\theta = \frac{x}{y}$

- Positive angles are obtained by rotating counterclockwise.
- Negative angles are obtained by rotating clockwise.
- If (x, y) is on the terminal arm of the angle θ, then $(x, -y)$ is on the terminal arm of $-\theta$.
- $\sin(-\theta) = \frac{-y}{r} = -\frac{y}{r} = -\sin\theta$.
- $\cos(-\theta) = \frac{x}{r} = \cos\theta$.
- sin is an odd function.
- cos is an even function.



$$\sin\theta = \frac{y}{r} \quad \csc\theta = \frac{r}{y}$$
$$\cos\theta = \frac{x}{r} \quad \sec\theta = \frac{r}{x}$$
$$\tan\theta = \frac{y}{x} \quad \cot\theta = \frac{x}{y}$$

- 2π represents a full rotation.
- $\theta + 2\pi$ has the same terminal arm as θ.
- $\theta + 2\pi$ uses the same point (x, y) and the same length r.
- $\sin(\theta + 2\pi) = \sin\theta$.
- $\cos(\theta + 2\pi) = \cos\theta$.
- We say $\sin$ and $\cos$ are 2π-periodic.

Definition (Complex numbers)

The set of complex numbers $\mathbb{C}$ is defined as the set

$$\{a+bi | a, b - \text{real numbers}\},$$

where the number i is a number for which

$$i^2 = -1 \quad .$$

The number i is called the imaginary unit. By definition, $\sqrt{-1} = i$.

- Complex addition/subtraction

$$(a+bi) \pm (c+di) = (a \pm c) + (b \pm d)i \quad .$$

- Complex multiplication

$$\begin{aligned}(a+bi)(c+di) &= ac + adi + bci + bdi^2 = ac + adi + bci - bd \\ &= (ac - bd) + i(ad + bc)\end{aligned}$$

Euler's Formula

Theorem (Euler's Formula)

$$e^{ix} = \cos x + i \sin x,$$

where $e \approx 2.71828$ is Euler's/Napier's constant .

Proof.

Recall $n! = 1 \cdot 2 \cdot 3 \cdot \dots \cdot (n-1) \cdot n$. Borrow from Calc II the f-las:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + \frac{(-1)^n x^{2n}}{(2n)!} + \dots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots$$

Euler's Formula

Theorem (Euler's Formula)

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where $e \approx 2.71828$ *is Euler's/Napier's constant .*

Proof.

Recall $n! = 1 \cdot 2 \cdot 3 \cdot \dots \cdot (n-1) \cdot n$. Borrow from Calc II the f-las:

$$\begin{array}{rclllllll} i\sin x &=& & ix & & -i\frac{x^3}{3!} & & +i\frac{x^5}{5!} & -\dots \\ \cos x &=& 1 & & -\frac{x^2}{2!} & & +\frac{x^4}{4!} & & +\dots \\ \hline e^{ix} &=& 1 & +ix & -\frac{x^2}{2!} & -i\frac{x^3}{3!} & +\frac{x^4}{4!} & +i\frac{x^5}{5!} & -\dots \end{array}$$

Rearrange. Plug-in $z = ix$. Use $i^2 = -1$. Multiply $\sin x$ by i. Add to get $e^{ix} = \cos x + i \sin x$. $\square$

Trigonometric Identities Revisited

- $e^{ix} = \cos x + i \sin x$ (Euler's Formula).
- $e^{ix} e^{iy} = e^{ix+iy} = e^{i(x+y)}$ (exponentiation rule: valid for $\mathbb{C}$).
- $e^0 = 1$ (exponentiation rule).
- $\sin(-x) = -\sin x$, $\cos(-x) = \cos x$ (easy to remember).

Example

$$\begin{array}{rcl} \sin(x+y) & = & \sin x \cos y + \sin y \cos x \\ \cos(x+y) & = & \cos x \cos y - \sin x \sin y \quad . \end{array}$$

Proof.

$$\begin{array}{rcl} e^{i(x+y)} & = & \cos(x+y) + i \sin(x+y) \\ e^{ix} e^{iy} & = & \cos(x+y) + i \sin(x+y) \\ (\cos x + i \sin x)(\cos y + i \sin y) & = & \cos(x+y) + i \sin(x+y) \\ \cos x \cos y - \sin x \sin y + i(\sin x \cos y + \sin y \cos x) & = & \cos(x+y) + i \sin(x+y) \end{array}$$

Compare coefficient in front of i and remaining terms to get the desired equalities. □

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- $\sin(-x) = -\sin x$, $\cos(-x) = \cos x$ (easy to remember).

Example

$$\sin^2 x + \cos^2 x = 1$$

Proof.

$$\begin{aligned}
1 &= e^{0} \\
&= e^{ix-ix} = e^{ix} e^{-ix} = (\cos x + i \sin x)(\cos(-x) + i \sin(-x)) \\
&= (\cos x + i \sin x)(\cos x - i \sin x) = \cos^2 x - i^2 \sin^2 x \\
&= \cos^2 x + \sin^2 x \quad .
\end{aligned}$$

□

Trigonometric Identities Revisited

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Example

$$\begin{aligned} \sin(2x) &= 2 \sin x \cos x \\ \cos(2x) &= \cos^2 x - \sin^2 x \quad . \end{aligned}$$

Proof.

$$\begin{aligned} e^{i(2x)} &= \cos(2x) + i \sin(2x) \\ e^{ix} e^{ix} &= \cos(2x) + i \sin(2x) \\ (\cos x + i \sin x)^2 = (\cos x + i \sin x)(\cos x + i \sin x) &= \cos(2x) + i \sin(2x) \\ \cos^2 x - \sin^2 x + i(2 \sin x \cos x) &= \cos(2x) + i \sin(2x) \end{aligned}$$

Compare coefficient in front of i and remaining terms to get the desired equalities. $\square$

- Recall Euler's formula: $e^{i\alpha} = \cos\alpha + i\sin\alpha$.
- Recall the formula: $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.

Example

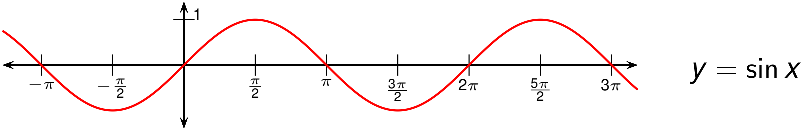
Express $\sin(3x)$ and $\cos(3x)$ via $\cos x$ and $\sin x$.

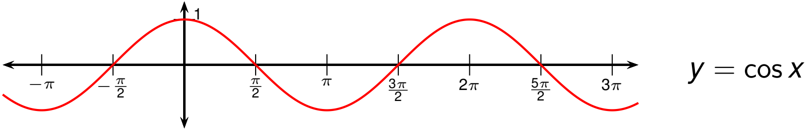
$$
\begin{array}{ll}
\cos(3x) + i\sin(3x) & \Big| \text{ Euler's f-la} \\
= e^{3ix} & \\
= \left(e^{ix}\right)^3 = (\cos x + i\sin x)^3 & \Big| \text{ Euler's f-la} \\
= \cos^3 x + 3\cos^2 x(i\sin x) + 3\cos x(i\sin x)^2 + (i\sin x)^3 & \\
= \cos^3 x + 3i\cos^2 x\sin x + 3i^2\cos x\sin^2 x + i^3\sin^3 x & \\
= \cos^3 x + 3i\cos^2 x\sin x - 3\cos x\sin^2 x - i\sin^3 x & \Big| \text{ Use } i^2 = -1 \\
= (\cos^3 x - 3\cos x\sin^2 x) + i(3\cos^2 x\sin x - \sin^3 x) &
\end{array}
$$

The real parts of the starting and final expression must be equal; likewise the imaginary parts must be equal; therefore:

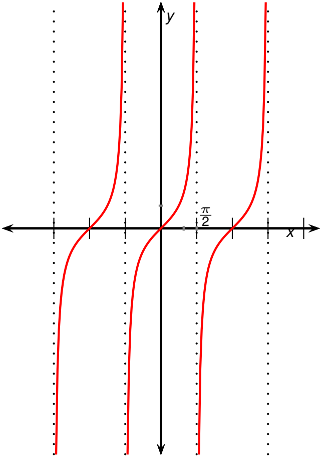
$$
\begin{array}{rcl}
\cos(3x) & = & \cos^3 x - 3\cos x\sin^2 x \\
\sin(3x) & = & 3\cos^2 x\sin x - \sin^3 x
\end{array}
$$

Graphs of the Trigonometric Functions

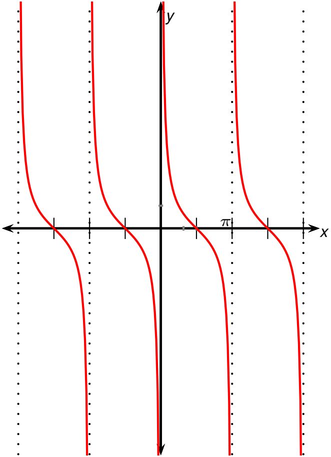




- $\sin x$ has zeroes at $n\pi$ for all integers n.
- $\cos x$ has zeroes at $\frac{\pi}{2} + n\pi$ for all integers n.
- $-1 \leq \sin x \leq 1$.
- $-1 \leq \cos x \leq 1$.
- If we translate the graph of $\cos x$ by $\frac{\pi}{2}$ units to the right we get the graph of $\sin x$. This is a consequence of $\cos\left(x - \frac{\pi}{2}\right) = \sin x$.

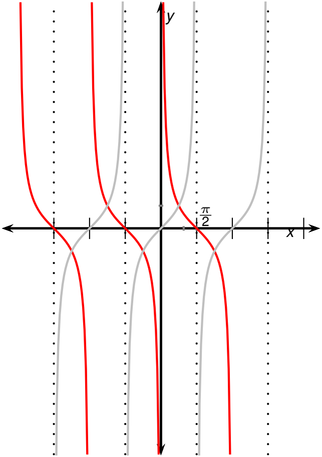


$y = \tan x$



$y = \cot x$

If we move the graph of $\tan x$ by $\frac{\pi}{2}$ units to the left (or right) and reflect across the x axis, we get the graph of $\cot x$. This follows from $\tan\left(x \pm \frac{\pi}{2}\right) = -\cot x$.



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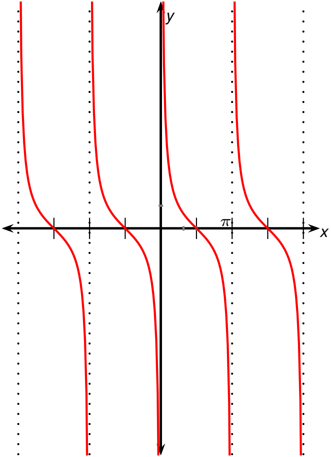

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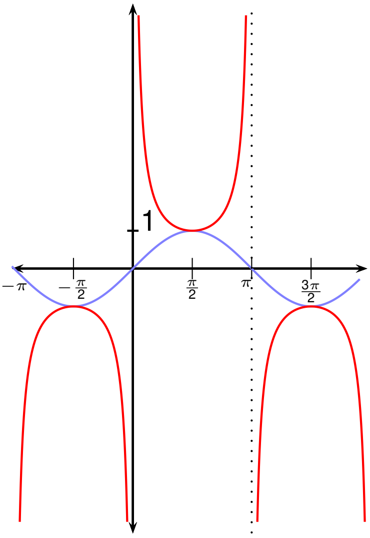
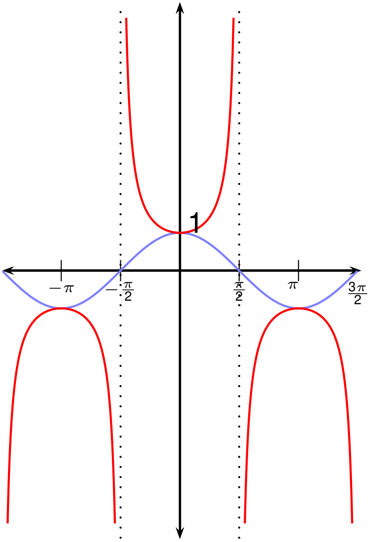
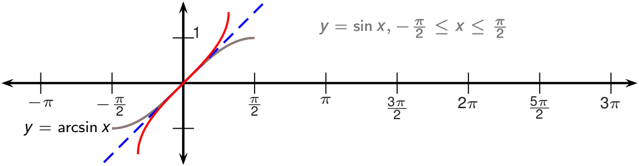

1
$-\pi$
$-\frac{\pi}{2}$
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$

$y = \csc x$

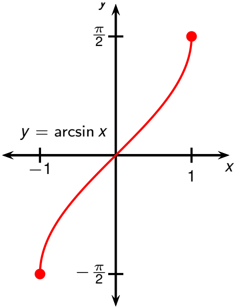

1
$-\pi$
$-\frac{\pi}{2}$
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$

$y = \sec x$

Inverse Trigonometric Functions



- $\sin x$ isn't one-to-one.
- It is if we restrict the domain to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.
- Then it has an inverse function.
- We call it arcsin or $\sin^{-1}$.
- $\arcsin x = y \Leftrightarrow \sin y = x$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.



Observation

- $\arcsin y =$ *the appropriate angle whose sine equals* y.
- *Important: the output angle must lie in the interval* $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Example

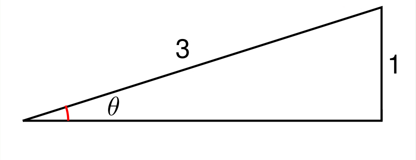
$$\text{Find } \arcsin\left(\frac{1}{2}\right).$$

- $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$.
- $-\frac{\pi}{2} \leq \frac{\pi}{6} \leq \frac{\pi}{2}$.
- Therefore $\arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$.

Example

Find $\tan\left(\arcsin\left(\dfrac{1}{3}\right)\right)$.

- Let $\theta = \arcsin\left(\frac{1}{3}\right)$, so $\sin\theta = \frac{1}{3}$.
- Draw a right triangle with opposite side 1 and hypotenuse 3.
- Let the angle θ be as labeled. Then $\sin\theta = \frac{1}{3}$ and so $\theta = \arcsin\left(\frac{1}{3}\right)$.
- Length of adjacent side $= \sqrt{3^2 - 1^2} = \sqrt{8} = 2\sqrt{2}$.
- Then $\tan\left(\arcsin\left(\frac{1}{3}\right)\right) = \frac{1}{2\sqrt{2}}$.



Example

Find $\arcsin(\sin(1.5))$.

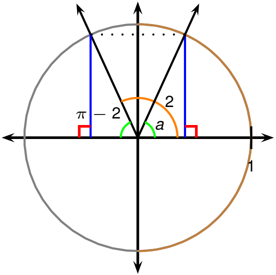
- $\frac{\pi}{2} \approx 1.57$.
- Therefore $-\frac{\pi}{2} \leq 1.5 \leq \frac{\pi}{2}$.
- Therefore $\arcsin(\sin 1.5) = 1.5$.

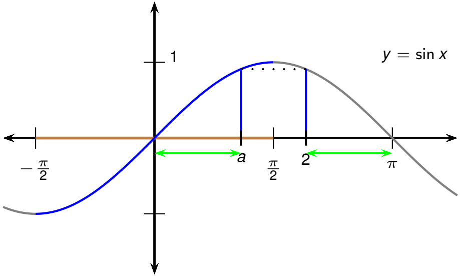
Example

Find $\arcsin(\sin 2)$.

- 2 is not between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.
- We need the angle a between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ for which $\sin 2 = \sin a$.

$$\begin{array}{rrcl} & a & = & \pi - 2. \\ \text{Therefore} & \arcsin(\sin 2) & = & \arcsin(\sin a) \\ & & = & a = \pi - 2. \end{array}$$





Theorem (The Derivative of $\arcsin x$)

$$\frac{\mathrm{d}}{\mathrm{d}x}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}, \qquad -1 < x < 1.$$

Proof.

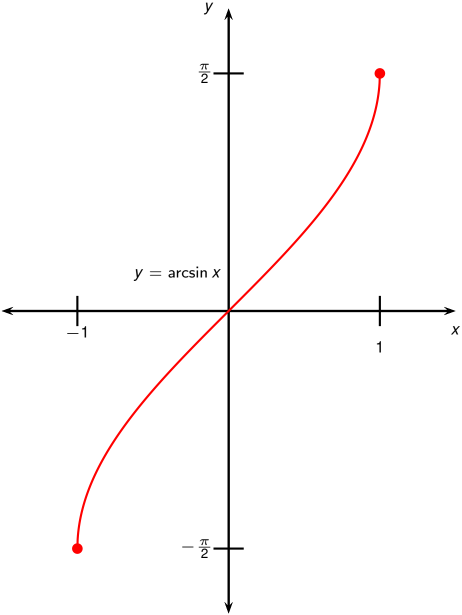
$$\text{Let} \quad y = \arcsin x.$$

$$\text{Then} \quad \sin y = x \quad \text{and} \quad -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}.$$

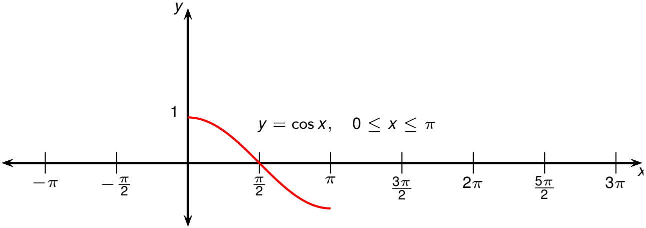
$$\begin{aligned}
\text{Differentiate implicitly:} \quad \cos y \cdot y' &= 1 \\
y' &= \frac{1}{\cos y} \\
&= \frac{1}{\pm\sqrt{1-\sin^2 y}} \\
\text{But } \cos y > 0\text{:} \quad &= \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}.
\end{aligned}$$

$\square$

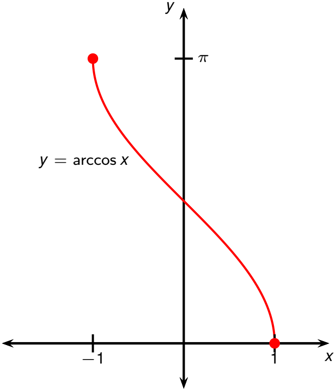
Important facts about arcsin:



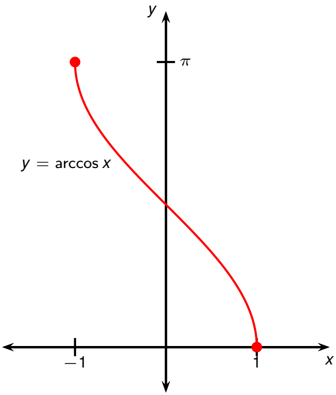
1. Domain: $[-1, 1]$.
2. Range: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.
3. $\arcsin x = y \Leftrightarrow \sin y = x$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$.
4. $\arcsin(\sin x) = x$ for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.
5. $\sin(\arcsin x) = x$ for $-1 \leq x \leq 1$.
6. $\frac{\mathrm{d}}{\mathrm{d}x}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$.



- Same for $\cos x$.
- Restrict the domain to $[0, \pi]$.
- The inverse is called arccos or $\cos^{-1}$.
- $\arccos(x) = y \Leftrightarrow \cos y = x$ and $0 \leq y \leq \pi$.



Important facts about arccos:



1. Domain: $[-1, 1]$.
2. Range: $[0, \pi]$.
3. $\arccos x = y \Leftrightarrow \cos y = x$ and $0 \leq y \leq \pi$.
4. $\arccos(\cos x) = x$ for $0 \leq x \leq \pi$.
5. $\cos(\arccos x) = x$ for $-1 \leq x \leq 1$.
6. $\dfrac{d}{dx}(\arccos x) = -\dfrac{1}{\sqrt{1-x^2}}$.
(The proof is similar to the proof of the formula for the derivative of $\arcsin x$.)

Example

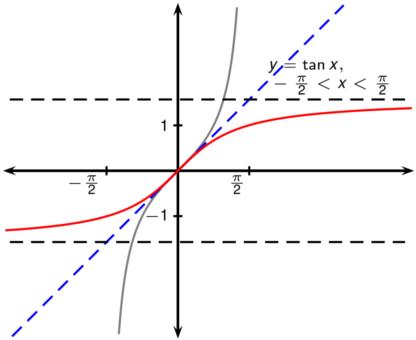
Rewrite $\sin(2\arccos(x))$ as an algebraic expression of x and $\sqrt{1-x^2}$. To simplify $\arccos x$ we try to use $\cos(\arccos x) = x$. Therefore our aim is to rewrite the expression only using the $\cos$ function.

$$
\begin{array}{rll|l}
\sin(2\arccos(x)) & = \sin(2y) & & \text{Set } y = \arccos x \\
& = 2\cos y \sin y & & \text{Express via } \sin y, \cos y \\
& = 2\cos y \left(\pm\sqrt{1-\cos^2 y}\right) & & \text{Express } \sin y \text{ via } \cos y \\
& = 2\cos y \sqrt{1-\cos^2 y} & & \sin y > 0 \text{ because } \\
& & & 0 \leq y \leq \pi \\
& = 2x\sqrt{1-x^2} & & \text{use } x = \cos y
\end{array}
$$

Example

Rewrite $\cos(3\arccos(x))$ as an algebraic expression of x and $\sqrt{1-x^2}$. To simplify $\arccos x$ we try to use $\cos(\arccos x) = x$. Therefore our aim is to rewrite the expression only using the $\cos$ function.

$$
\begin{aligned}
\cos(3\arccos(x)) &= \cos(3y) = \cos(2y+y) && \Big|\ y = \arccos x\\
&= \cos(2y)\cos y - \sin(2y)\sin y && \Big|\ \text{Angle sum f-la}\\
&= (\cos^2 y - \sin^2 y)\cos y && \Big|\ \text{Express via}\\
&\quad -2\sin y\cos y\sin y && \Big|\ \sin y, \cos y\\
&= \cos^3 y - \sin^2 y\cos y - 2\sin^2 y\cos y\\
&= \cos^3 y - 3\sin^2 y\cos y && \Big|\ \text{Express } \sin y \text{ via } \cos y\\
&= \cos^3 y - 3(1-\cos^2 y)\cos y\\
&= 4\cos^3 y - 3\cos y\\
&= 4x^3 - 3x && \Big|\ x = \cos y
\end{aligned}
$$

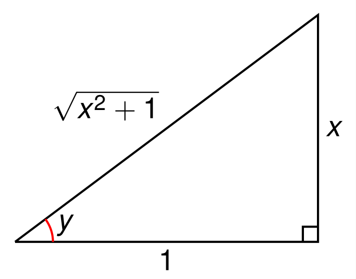


- $\tan x$ isn't one-to-one.
- Restrict the domain to $(-\frac{\pi}{2}, \frac{\pi}{2})$.
- The inverse is called $\tan^{-1}$ or $\arctan$.
- $\arctan x = y \Leftrightarrow \tan y = x$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$.
- Domain of $\arctan$: $(-\infty, \infty)$.
- Range of $\arctan$: $(-\frac{\pi}{2}, \frac{\pi}{2})$.
- $\lim\limits_{x\to\infty} \arctan x = \dfrac{\pi}{2}$.
- $\lim\limits_{x\to-\infty} \arctan x = -\dfrac{\pi}{2}$.

Example

Simplify the expression $\cos(\arctan x)$.

- Let $y = \arctan x$, so $\tan y = x$.
- Draw a right triangle with opposite x and adjacent 1.
- Length of hypotenuse $= \sqrt{1^2 + x^2}$.
- Then $\cos(\arctan x) = \frac{1}{\sqrt{1+x^2}}$.



Example

Evaluate

$$\lim_{x \to 2^+} \arctan\left(\frac{1}{x-2}\right).$$

$$\frac{1}{x-2} \to \infty \qquad \text{as} \qquad x \to 2^+.$$

Therefore

$$\lim_{x \to 2^+} \arctan\left(\frac{1}{x-2}\right) = \frac{\pi}{2}.$$

Theorem (The Derivative of $\arctan x$)

$$\frac{\mathrm{d}}{\mathrm{d}x}(\arctan x) = \frac{1}{1+x^2}.$$

Proof.

$$\begin{aligned}
\text{Let} \quad y &= \arctan x. \\
\text{Then} \quad \tan y &= x. \\
\text{Differentiate implicitly:} \quad \sec^2 y \cdot y' &= 1 \\
y' &= \frac{1}{\sec^2 y} \\
&= \frac{1}{1+\tan^2 y} \\
&= \frac{1}{1+x^2}.
\end{aligned}$$

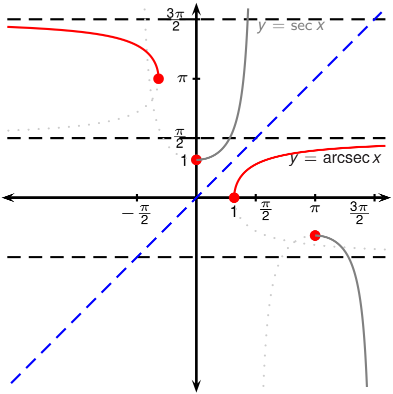
□

The remaining inverse trigonometric functions aren't used as often:

$$
\begin{array}{llllll}
y = \operatorname{arccsc} x & (|x| \geq 1) & \Leftrightarrow & \csc y = x & \text{and} & y \in \left(0, \frac{\pi}{2}\right] \cup \left(\pi, \frac{3\pi}{2}\right] \\
y = \operatorname{arcsec} x & (|x| \geq 1) & \Leftrightarrow & \sec y = x & \text{and} & y \in \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right) \\
y = \operatorname{arccot} x & (|x| \in \mathbb{R}) & \Leftrightarrow & \cot y = x & \text{and} & y \in (0, \pi)
\end{array}
$$

We will however make use of $\operatorname{arcsec} x$: we discuss in detail its domain.

$$y = \operatorname{arcsec} x \quad (|x| \geq 1) \quad \Leftrightarrow \quad \sec y = x \quad \text{and} \quad y \in \mathbf{?} \left[0, \tfrac{\pi}{2}\right) \cup \left[\pi, \tfrac{3\pi}{2}\right)$$



- Plot $\sec x$.
- Restrict domain to make one-to-one: Two common choices: $x \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ and $x \in \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$.
- $x \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ is good because the domain is easiest to remember: an interval without a point. **NOT our choice.**
- $x \in \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$ is good because $\tan x$ is positive on both intervals, resulting in easier differentiation and integration formulas. **Our choice.**

Table of derivatives of inverse trigonometric functions:

$$\frac{\mathrm{d}}{\mathrm{d}x}(\arcsin x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\operatorname{arccsc} x) = -\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\arccos x) = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\operatorname{arcsec} x) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\arctan x) = \frac{1}{1+x^2} \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\operatorname{arccot} x) = -\frac{1}{1+x^2}$$

Example (Chain Rule)

$$
\begin{aligned}
\text{Differentiate} \quad y &= \frac{1}{\arcsin x}. \\
\text{Let} \quad u &= \arcsin x. \\
\text{Then} \quad y &= u^{-1}. \\
\text{Chain Rule:} \quad \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \\
&= \left(-u^{-2}\right)\left(\frac{1}{\sqrt{1-x^2}}\right) \\
&= -\frac{1}{(\arcsin x)^2\sqrt{1-x^2}}.
\end{aligned}
$$

All of the inverse trigonometric derivatives also give rise to integration formulas. These two are the most important:

$$\int \frac{1}{\sqrt{1-x^2}} \mathrm{d}x = \arcsin x + C.$$

$$\int \frac{1}{x^2+1} \mathrm{d}x = \arctan x + C.$$