

Precalculus

Inverse functions

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Outline

1

Inverse Functions

- One-to-one Functions
- The Definition of the Inverse of f

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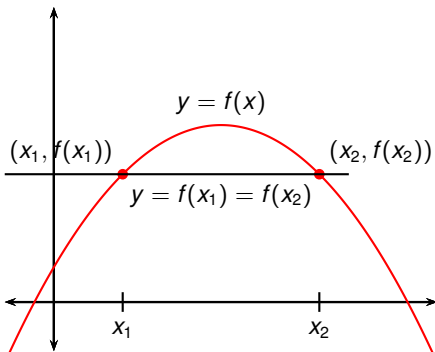
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One-to-one Functions

Definition (One-to-one Function)

A function f is a one-to-one function if it never takes on the same value twice; that is,

$$f(x_1) \neq f(x_2) \text{ whenever } x_1 \neq x_2.$$



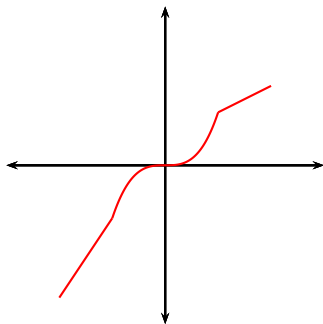
← This function is not one-to-one.

Question: How can we tell from the graph of a function whether it is one-to-one or not?

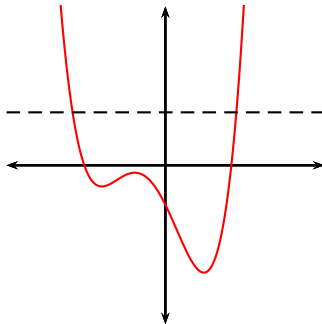
Answer: Use the horizontal line test.

The Horizontal Line Test.

A function is one-to-one if and only if no horizontal line intersects it more than once.



One-to-one



Not one-to-one

The Definition of the Inverse of f

Definition (f^{-1})

Let f be a one-to-one function with domain A and range B . Then the inverse of f is the function f^{-1} that has domain B and range A and is defined by

$$f^{-1}(y) = x \quad \Leftrightarrow \quad f(x) = y$$

for all y in B .

Note:

- Only one-to-one functions have inverses.
- f^{-1} reverses the effect of f .
- domain of f^{-1} = range of f .
- range of f^{-1} = domain of f .

Example ($f(x) = x^3$)

The inverse of $f(x) = x^3$ is $f^{-1}(x) = \sqrt[3]{x}$. This is because if $y = x^3$, then

$$f^{-1}(y) = \sqrt[3]{y} = \sqrt[3]{x^3} = x.$$

The inverse of f is denoted as f^{-1} . This notation is one of the most frequent causes of student confusion. **WARNING:**

$$f^{-1}(x) \text{ does not mean } (f(x))^{-1} = \frac{1}{f(x)}.$$

The notations are different: the superscript -1 has different positions.

- f^{-1} is the compositional inverse of f .
- $\frac{1}{f(x)}$ is the multiplicative inverse of $f(x)$.
- $f^2(x)$ is an abbreviation for $(f(x))^2$, $f^3(x)$ is an abbreviation of $(f(x))^3$, and so on.
- However, $f^{-1}(x)$ is not the abbreviation of $(f(x))^{-1}$ and does not follow this pattern.

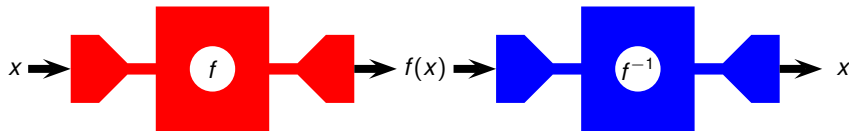
$$f^n(x) = \begin{cases} \text{stands for } (f(x))^n & \text{when } n = 1, 2, 3, \dots \\ \text{stands for inverse of } f \text{ applied to } x & \text{when } n = -1 \\ \text{should be avoided} & \text{when } n \neq -1, 1, 2, 3, \dots \end{cases}$$

To reduce confusion, if possible, use $\frac{1}{f(x)}$ instead of $(f(x))^{-1}$.

$$f^{-1}(y) = x \quad \Leftrightarrow \quad f(x) = y.$$

Therefore

$$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = f^{-1}(y) = x.$$



Switch the roles of x and y :

$$f^{-1}(x) = y \quad \Leftrightarrow \quad f(y) = x.$$

Therefore

$$(f \circ f^{-1})(x) = f(f^{-1}(x)) = f(y) = x.$$

How to Find the Inverse of a One-to-one Function

- 1 Write $y = f(x)$.
- 2 Solve this equation for x in terms of y (if possible).

Example

If $f(x) = x^3 + 2$, find a formula for $f^{-1}(y)$.

$$y = x^3 + 2$$

$$x^3 = y - 2$$

$$x = \sqrt[3]{y - 2}$$

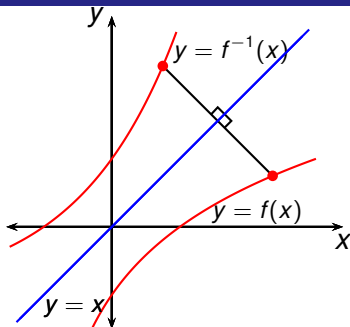
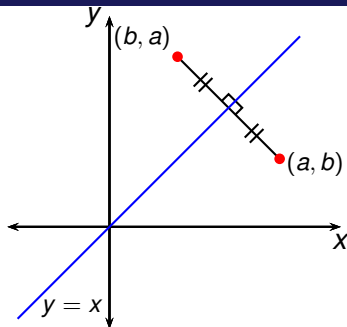
Therefore $x = f^{-1}(y) = \sqrt[3]{y - 2}$. Sometimes we relabel x and y and write $f^{-1}(x) = \sqrt[3]{x - 2}$. Whenever in doubt, do not relabel anything.

Example (Guess and Check)

If $f(x) = 2x + \sin 2x + e^{\frac{x}{2}}$, find $f^{-1}(1)$. You do not need to show that f has an inverse.

$$\begin{aligned} f(\quad) &= 2(\quad) + \sin 2(\quad) + e^{\frac{(\quad)}{2}} \\ &= \\ &= 1. \end{aligned}$$

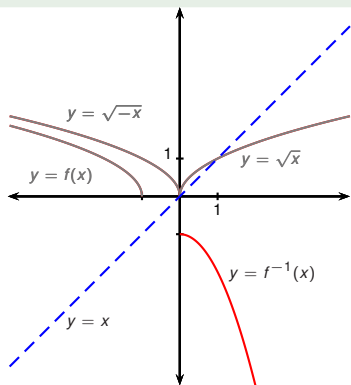
Therefore $f^{-1}(1) =$



Interchanging x and y suggests relation between the graphs of f^{-1} and f :

- Suppose (a, b) is on the graph of f .
- Then $f(a) = b$.
- Then $f^{-1}(b) = a$.
- Then (b, a) is on the graph of f^{-1} .
- (b, a) is the reflection of (a, b) in the line $y = x$.
- Thus the graph of f^{-1} is obtained by reflecting the graph of f in the line $y = x$.

Example

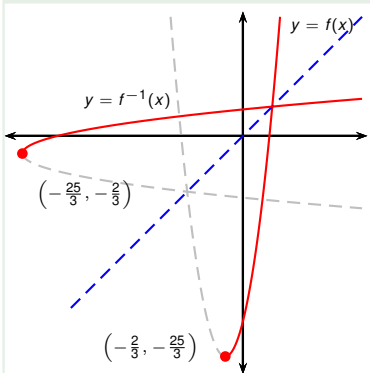


Sketch the graph of $f(x) = \sqrt{-x-1}$ and its inverse function.

- Draw the graph of $y = \sqrt{x}$.
- $y = \sqrt{-x}$ is the reflection of $y = \sqrt{x}$ in the y -axis.
- $y = f(x) = \sqrt{-(x+1)} = \sqrt{-x-1}$ is the shift of $y = \sqrt{-x}$ one unit to the left.
- $y = f^{-1}(x)$ is the reflection of $y = f(x)$ across the line $y = x$.

Example ()

Given: $f(x) = 3x^2 + 4x - 7$ with domain $x \geq -\frac{2}{3}$. Find $f^{-1}(x)$.



$$\begin{aligned} 3x^2 + 4x - 7 &= y \\ 3x^2 + 4x + (-7 - y) &= 0 \end{aligned}$$

That's a quadratic equation in x . Solve:

$$\begin{aligned} &\frac{-4 \pm \sqrt{4^2 - 4 \cdot 3 \cdot (-y - 7)}}{2 \cdot 3} \\ &= -\frac{2 \pm \sqrt{25 + 3y}}{3} = -\frac{2}{3} \pm \frac{\sqrt{25 + 3y}}{3} \end{aligned}$$

Final answer, relabelled:

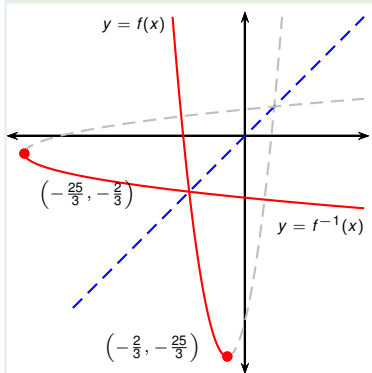
$$f^{-1}(x) = -\frac{2}{3} + \frac{\sqrt{25 + 3x}}{3}$$

We are given $x \geq -\frac{2}{3}$, therefore

$$x = -\frac{2}{3} + \frac{\sqrt{25+3y}}{3} = f^{-1}(y).$$

Example (What if we change the problem to $x \leq -\frac{2}{3}$?)

Given: $f(x) = 3x^2 + 4x - 7$ with domain $x \leq -\frac{2}{3}$. Find $f^{-1}(x)$.



$$\begin{aligned} 3x^2 + 4x - 7 &= y \\ 3x^2 + 4x + (-7 - y) &= 0 \end{aligned}$$

That's a quadratic equation in x . Solve:

$$\begin{aligned} &\frac{-4 \pm \sqrt{4^2 - 4 \cdot 3 \cdot (-y - 7)}}{2 \cdot 3} \\ &= -\frac{2 \pm \sqrt{25 + 3y}}{3} = -\frac{2}{3} \pm \frac{\sqrt{25 + 3y}}{3} \end{aligned}$$

Final answer, relabelled:

$$f^{-1}(x) = -\frac{2}{3} - \frac{\sqrt{25 + 3x}}{3}$$

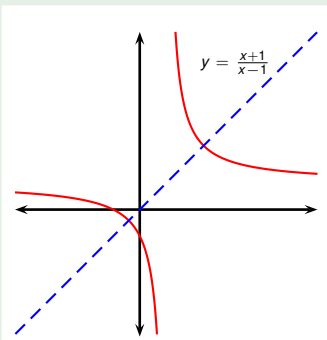
We are given $x \leq -\frac{2}{3}$, therefore

$$x = -\frac{2}{3} - \frac{\sqrt{25+3y}}{3} = f^{-1}(y).$$

Example

Find $f^{-1}(x)$ where $f(x) = \frac{x+1}{x-1}$.

We deal with domains and ranges later:



Answer: $f^{-1}(x) = \frac{x+1}{x-1}$,
 $x \neq 1$.

$$\begin{array}{rcll}
 y & = & \frac{x+1}{x-1} & \left| \begin{array}{l} \text{mult. by } (x-1) \\ \text{div. by } (y-1) \\ \text{relabel } x, y \end{array} \right. \\
 y(x-1) & = & x+1 & \\
 x(y-1) & = & y+1 & \\
 f^{-1}(y) = x & = & \frac{y+1}{y-1} & \\
 f^{-1}(x) & = & \frac{x+1}{x-1} &
 \end{array}$$

We divided by $y-1$ so $y \neq 1$. Therefore the domain of f^{-1} is all real numbers except 1.

Can a non-identity function be its own inverse? Yes, f is.

What does it mean for f to be its own inverse?
 Graph of f is symmetric across $y = x$.