

Calculus II

Integral of rational function with cubic denominator, part 2

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2019

Example

$$\int \frac{x^4 + x^3 - 4x^2 + 4x}{x^3 - x^2 - x + 1} dx = \int \left(x + 2 + \frac{1}{x-1} + \frac{1}{(x-1)^2} - \frac{2}{x+1} \right) dx$$

$$= \frac{x^2}{2} + 2x + \ln|x-1| - \frac{1}{x-1} - 2\ln|x+1| + K$$

- Divide: $\frac{x^4+x^3-4x^2+4x}{x^3-x^2-x+1} = x + 2 + \frac{-x^2+5x-2}{x^3-x^2-x+1} = x + 2 + \frac{-x^2+5x-2}{(x-1)^2(x+1)}$.
- Factor denominator: $x^3 - x^2 - x + 1 = (x-1)^2(x+1)$.
- Set up the partial fraction decomposition:

$$\frac{-x^2 + 5x - 2}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$$

$$-x^2 + 5x - 2 = A(x-1)(x+1) + B(x+1) + C(x-1)^2$$

- Plug-in $x = -1$: $-(-1)^2 + 5(-1) - 2 = C(-1-1)^2 \Rightarrow C = -2$.
- Plug-in $x = 1$: $-(1)^2 + 1 \cdot 5 - 2 = B(1+1) \Rightarrow B = 1$.
- Plug-in $x = 0$:
 $-2 = A(0-1)(0+1) + 1 \cdot (0+1) + (-2)(0-1)^2 \Rightarrow A = 1$.

$Q(x)$ has linear factors with higher multiplicity

- Suppose $Q(x)$ is a product of linear factors, some of which appear with power greater than 1.
- For example suppose the first linear factor has power r , that is, $(a_1x + b_1)^r$ occurs in the factorization of $Q(x)$.
- Then instead of a single term $\frac{A}{a_1x+b_1}$ we use

$$\frac{A_1}{a_1x + b_1} + \frac{A_2}{(a_1x + b_1)^2} + \cdots + \frac{A_r}{(a_1x + b_1)^r}$$

- In a similar fashion we add more partial fractions to account for all other terms of the form $(a_sx + b_s)^t$.