Calculus I
Derivative of rational function, part 2

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Theorem (The Quotient Rule)

If f and g are differentiable and $g(x) \neq 0$, then

$$\begin{array}{rcll} \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f(x)}{g(x)}\right) & = & \frac{\frac{\mathrm{d}}{\mathrm{d}x}(f(x))\,g(x) - f(x)\frac{\mathrm{d}}{\mathrm{d}x}(g(x))}{(g(x))^2} & \textit{(Leibniz notation)} \\ \left(\frac{f(x)}{g(x)}\right)' & = & \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2} & {}'\ \textit{notation} \\ \left(\frac{f}{g}\right)' & = & \frac{f'g - fg'}{g^2} & \textit{abbreviated} \end{array}$$

- The proof of the Quotient Rule is similar to the proof of the Product Rule.
- There is an alternative algebraic proof via the Product Rule, the Power Rule and the (not yet studied) Chain Rule.

Example (Quotient Rule, rational function)

Differentiate $y = \frac{x^5 + 2x}{-x^6 + 2}$.

Quotient Rule:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}}{\mathrm{d}x}\left(x^5 + 2x\right)\left(-x^6 + 2\right) - \left(x^5 + 2x\right)\frac{\mathrm{d}}{\mathrm{d}x}\left(-x^6 + 2\right)}{\left(-x^6 + 2\right)^2}$$

$$= \frac{\left(5x^4 + 2\right)\left(-x^6 + 2\right) - \left(x^5 + 2x\right)\left(-6x^5\right)}{\left(-x^6 + 2\right)^2}$$

$$= \frac{(-5x^{10} - 2x^6 + 10x^4 + 4) - (-6x^{10} - 12x^6)}{(-x^6 + 2)^2}$$

$$= \frac{x^{10} + 10x^6 + 10x^4 + 4}{(-x^6 + 2)^2}.$$