Calculus II
Homework
Trigonometric integrals

1. Integrate.

   (a) $\displaystyle\int \frac{1}{3+\cos x}\mathrm{d}x.$

   answer: $\frac{1}{\sqrt{2}} \arctan\left(\frac{1}{\sqrt{2}} \tan\left(\frac{x}{2}\right)\right) + C$

   (b) $\displaystyle\int \frac{1}{4+\cos x}\mathrm{d}x.$

   answer: $\frac{2}{15}\sqrt{15} \arctan\left(\frac{\sqrt{15}}{5} \tan\left(\frac{x}{2}\right)\right) + C$

   (c) $\displaystyle\int \frac{1}{3+\sin x}\mathrm{d}x.$

   answer: $\frac{1}{\sqrt{2}} \arctan\left(\frac{3\tan\left(\frac{x}{2}\right)+1}{2\sqrt{2}}\right) + C$

   (d) $\displaystyle\int \frac{1}{2+\tan x}\mathrm{d}x.$ (Hint: this integral can be done simply with the substitution $x = \arctan t$.)

   answer: $\frac{1}{5}\ln(\sin x + 2\cos x) + \frac{2}{5}x + C$

   (e) $\displaystyle\int \frac{\mathrm{d}x}{2\sin x - \cos x + 5}.$

   answer: $\frac{\sqrt{5}}{5} \arctan\left(\frac{3}{\sqrt{5}}\left(\tan\left(\frac{\theta}{2}\right) + \frac{1}{3}\right)\right) + C$

Solution. 1.a We use the standard rationalizing substitution $x = 2\arctan t$, $t = \tan\left(\frac{x}{2}\right)$. We recall that from the double angle formulas it follows that

$$\cos(2\arctan t) = \frac{\cos^2(\arctan t) - \sin^2(2\arctan t)}{\cos^2(\arctan t) + \sin^2(\arctan t)} = \frac{1-t^2}{1+t^2} \quad .$$

Therefore we can solve the integral as follows.

$$\begin{array}{rcll}
\displaystyle\int \frac{1}{3+\cos x}\mathrm{d}x & = & \displaystyle\int \frac{1}{3+\cos(2\arctan t)}\mathrm{d}\left(2\arctan t\right) & \Big| \text{ Set } x = 2\arctan t \\
& = & \displaystyle\int \frac{1}{\left(3+\frac{1-t^2}{1+t^2}\right)}\frac{2}{(1+t^2)}\mathrm{d}t & \\
& = & \displaystyle\int \frac{2}{4+2t^2}\mathrm{d}t & \\
& = & \displaystyle\int \frac{1}{2+t^2}\mathrm{d}t & \\
& = & \displaystyle\frac{\sqrt{2}}{2}\arctan\left(\frac{\sqrt{2}}{2}t\right) + C & \\
& = & \displaystyle\frac{\sqrt{2}}{2}\arctan\left(\frac{\sqrt{2}}{2}\tan\left(\frac{x}{2}\right)\right) + C & .
\end{array}$$

Solution. 1.d This integral is of none of the forms that can be integrated quickly. Therefore we can solve it using the standard rationalizing substitution $x = 2\arctan t$, $t = \tan\left(\frac{x}{2}\right)$. This results in somewhat long computations and we invite the reader to try it.

However, as proposed in the hint, the substitution $x = \arctan t$ works much faster:

$$
\begin{aligned}
\int \frac{1}{2+\tan x}\mathrm{d}x &= \int \frac{1}{2+\tan(\arctan t)}\mathrm{d}\left(\arctan t\right) && \text{Substitute } x = \arctan t \\
&= \int \frac{1}{(2+t)}\frac{1}{(1+t^2)}\mathrm{d}t && \text{part. fractions} \\
&= \int \left(\frac{\frac{1}{5}}{(t+2)} + \frac{-\frac{t}{5}+\frac{2}{5}}{(t^2+1)} \right)\mathrm{d}t \\
&= \frac{1}{5}\ln|t+2| - \frac{1}{10}\ln(t^2+1) + \frac{2}{5}\arctan t + C && t = \tan x \\
&= \frac{1}{5}\ln|\tan x + 2| - \frac{1}{10}\ln(\tan^2 x + 1) + \frac{2}{5}x + C \\
&= \frac{1}{5}\ln|\tan x + 2| + \frac{1}{5}\ln|\cos x| + \frac{2}{5}x + C \\
&= \frac{1}{5}\ln|(\tan x + 2)\cos x| + \frac{2}{5}x + C \\
&= \frac{1}{5}\ln|\sin x + 2\cos x| + \frac{2}{5}x + C.
\end{aligned}
$$

Solution. 1.e.

Set $x = 2\arctan t$. As studied, this substitution implies $\cos x = \frac{1-t^2}{1+t^2}$, $\sin x = \frac{2t}{1+t^2}$, $\mathrm{d}x = \frac{2}{1+t^2}\mathrm{d}t$. Therefore

$$
\begin{aligned}
\int \frac{\mathrm{d}x}{2\sin x - \cos x + 5} &= \int \frac{2\mathrm{d}t}{(1+t^2)\left(2\frac{2t}{t^2+1} - \frac{(-t^2+1)}{t^2+1} + 5\right)} && \text{Set } x = 2\arctan t \\
&= \int \frac{\mathrm{d}t}{3t^2+2t+2} \\
&= \int \frac{\mathrm{d}t}{3\left(t^2+\frac{2}{3}t+\frac{1}{9}-\frac{1}{9}+\frac{2}{3}\right)} \\
&= \int \frac{\mathrm{d}t}{3\left(\left(t+\frac{1}{3}\right)^2+\frac{5}{9}\right)} \\
&= \int \frac{\mathrm{d}t}{\frac{5}{3}\left(\left(\frac{3}{\sqrt{5}}\left(t+\frac{1}{3}\right)\right)^2+1\right)} && \begin{array}{rl} & \text{Set} \\ w = & \frac{3}{\sqrt{5}}\left(t+\frac{1}{3}\right) \\ = & \frac{\sqrt{5}}{5}(3t+1) \\ \mathrm{d}w = & \frac{3}{\sqrt{5}}\mathrm{d}t \\ \mathrm{d}t = & \frac{\sqrt{5}}{3}\mathrm{d}w \end{array} \\
&= \int \frac{\frac{\sqrt{5}}{3}\mathrm{d}w}{\frac{5}{3}(w^2+1)} \\
&= \frac{\sqrt{5}}{5}\arctan w + C \\
&= \frac{\sqrt{5}}{5}\arctan\left(\frac{\sqrt{5}}{5}(3t+1)\right) + C \\
&= \frac{\sqrt{5}}{5}\arctan\left(\frac{\sqrt{5}}{5}\left(3\tan\left(\frac{x}{2}\right)+1\right)\right) + C \quad .
\end{aligned}
$$

2. Integrate. The answer key has not been proofread, use with caution.

   (a) $\displaystyle\int \sin(3x)\cos(2x)\mathrm{d}x.$

   answer: $-\frac{1}{10}\cos(5x) - \frac{1}{2}\cos x + C$

   (b) $\displaystyle\int \sin x\cos(5x)\mathrm{d}x.$

   answer: $-\frac{1}{12}\cos(6x) + \frac{1}{8}\cos(4x) + C$

   (c) $\displaystyle\int \cos(3x)\sin(2x)\mathrm{d}x.$

   answer: $-\frac{1}{10}\cos(5x) + \frac{1}{2}\cos x + C$

   (d) $\displaystyle\int \sin(5x)\sin(3x)\mathrm{d}x.$

   answer: $\frac{1}{4}\sin(2x) - \frac{1}{16}\sin(8x) + C$

(e) $\displaystyle\int \cos(x)\cos(3x)\mathrm{d}x$.

answer: $\frac{1}{8}\sin(4x) + \frac{1}{4}\sin(2x) + C$

3. Integrate.

(a) $\displaystyle\int \sin^2 x \cos x \mathrm{d}x$.

answer: $\frac{1}{3}\sin^3 x + C$

(b) $\displaystyle\int \sin^2 x \mathrm{d}x$.

answer: $\frac{x}{2} - \frac{1}{4}\sin(2x) + C$

(c) $\displaystyle\int \cos^3 x \mathrm{d}x$.

answer: $\sin x - \frac{1}{3}\sin^3 x + C$

(d) $\displaystyle\int \sin^3 x \cos^4 x \mathrm{d}x$.

answer: $\frac{1}{7}\cos^7 x - \frac{1}{5}\cos^5 x + C$

4. Integrate.

(a) $\displaystyle\int \sec x \mathrm{d}x$.

answer: $\ln|\sec x + \tan x| = \ln\left|\frac{1+\tan\left(\frac{x}{2}\right)}{1-\tan\left(\frac{x}{2}\right)}\right| + C$

(b) $\displaystyle\int \sec^3 x \mathrm{d}x$.

answer: $\frac{1}{2}(\sec x \tan x + \ln|\sec x + \tan x|) + C$

(c) $\displaystyle\int \tan^3 x \mathrm{d}x$.

answer: $\frac{1}{2}\tan^2 x - \ln|\sec x| + C$

(d) $\displaystyle\int \sec^2 x \tan^2 x \mathrm{d}x$.

answer: $\frac{\tan^3 x}{3} + C$

Solution. 4.a. **Variant I.**

This variant uses the standard method for solving trigonometric integrals with the substitution $x = \arctan(2t)$.

$$
\begin{aligned}
\int \sec x \mathrm{d}x &= \int \sec(2\arctan t)\mathrm{d}(2\arctan t) && \text{Set } x = 2\arctan t\\
&= \int \frac{1}{\cos(2\arctan t)}\frac{2}{1+t^2}\mathrm{d}t && \text{Use } \cos(2z) = \tfrac{1-\tan^2 z}{1+\tan^2 z}\\
&= \int \frac{1}{\frac{1-t^2}{1+t^2}}\frac{2}{1+t^2}\mathrm{d}t\\
&= \int \frac{2}{1-t^2}\mathrm{d}t && \text{part. fractions}\\
&= \int \left(\frac{1}{1-t} + \frac{1}{1+t}\right)\mathrm{d}t\\
&= -\ln|1-t| + \ln|1+t| + C\\
&= \ln\left|\frac{1+t}{1-t}\right| && \text{Subst. } t = \tan\left(\tfrac{x}{2}\right)\\
&= \ln\left|\frac{1+\tan\left(\frac{x}{2}\right)}{1-\tan\left(\frac{x}{2}\right)}\right| + C && \text{Last step: see below}\\
&= \ln|\sec x + \tan x| + C \quad .
\end{aligned}
$$

The expression $\ln\left|\frac{1+\tan\left(\frac{x}{2}\right)}{1-\tan\left(\frac{x}{2}\right)}\right|$ presents a perfectly good answer, which would certainly would qualify for a correct test answer. However, as shown above, it can be rewritten into the shorter form $\ln|\sec x + \tan x|$. Below we quickly prove that $\dfrac{1+\tan\left(\frac{x}{2}\right)}{1-\tan\left(\frac{x}{2}\right)}$ equals $\sec x + \tan x$.

$$
\begin{aligned}
\sec x+\tan x &= \frac{1+\sin x}{\cos x} \\
&= \frac{\cos^2\left(\frac{x}{2}\right)+\sin^2\left(\frac{x}{2}\right)+2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}{\cos^2\left(\frac{x}{2}\right)-\sin^2\left(\frac{x}{2}\right)} \\
&= \frac{\left(\sin\left(\frac{x}{2}\right)+\cos\left(\frac{x}{2}\right)\right)^2}{\left(\cos\left(\frac{x}{2}\right)-\sin\left(\frac{x}{2}\right)\right)\left(\cos\left(\frac{x}{2}\right)+\sin\left(\frac{x}{2}\right)\right)} \\
&= \frac{\left(\sin\left(\frac{x}{2}\right)+\cos\left(\frac{x}{2}\right)\right)\frac{1}{\cos\left(\frac{x}{2}\right)}}{\left(\cos\left(\frac{x}{2}\right)-\sin\left(\frac{x}{2}\right)\right)\frac{1}{\cos\left(\frac{x}{2}\right)}} \\
&= \frac{1+\tan\left(\frac{x}{2}\right)}{1-\tan\left(\frac{x}{2}\right)}
\end{aligned}
\quad\left|\quad
\begin{aligned}
&\text{Use:}\\
\sin x &= 2\sin\left(\tfrac{x}{2}\right)\\
\cos x &= \cos^2\left(\tfrac{x}{2}\right)-\sin^2\left(\tfrac{x}{2}\right)\\
1 &= \cos^2\left(\tfrac{x}{2}\right)+\sin^2\left(\tfrac{x}{2}\right)
\end{aligned}\right.
$$

4.a. **Variant II.** This variant is based on the following observation. For an odd number $m > 0$, we studied a quick technique for integrating $\int \sin^n x \cos^m x \mathrm{d}x$: namely, use the transformation $\cos x \mathrm{d}x = \mathrm{d}(\sin x)$ and change variables $u = \sin x$. This trick relies heavily on the fact that m is odd (as we need to express the remaining even power of $\cos x$ via $\sin x$). However, the positivity of m is not essential: by multiplying top and bottom by $\cos x$ we can make this technique work also for odd negative values of m. We illustrate the technique in the solution below.

$$
\begin{aligned}
\int \sec x \mathrm{d}x &= \int \frac{1}{\cos x}\mathrm{d}x \\
&= \int \frac{\cos x}{\cos^2 x}\mathrm{d}x \\
&= \int \frac{\mathrm{d}(\sin x)}{1-\sin^2 x} && \Big| \text{ Set } u=\sin x \\
&= \int \frac{\mathrm{d}u}{1-u^2} \\
&= \int \frac{\mathrm{d}u}{(1+u)(1-u)} && \Big| \text{ part. fractions} \\
&= \int \left(\frac{\frac{1}{2}}{1+u}+\frac{\frac{1}{2}}{1-u}\right)\mathrm{d}u \\
&= \frac{1}{2}\left(\ln|1+u|-\ln|1-u|\right)+C \\
&= \frac{1}{2}\ln\left|\frac{1+u}{1-u}\right|+C && \Big| \text{ Subst. back } u=\sin x \\
&= \frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right|+C && \Big| \text{ Last step: see below} \\
&= \ln|\sec x+\tan x|+C \quad .
\end{aligned}
$$

The expression $\frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right| + C$ gives a perfectly good answer (which may be the preferred answer depending on the textbook). Let us show however that $\frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right|$ equals $\ln|\sec x + \tan x|$, the answer given in the other variants.

$$
\begin{aligned}
\frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right| &= \frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right| && \Big| \text{ Mult. \& div by } 1+\sin x \\
&= \frac{1}{2}\ln\left|\frac{(1+\sin x)^2}{(1-\sin x)(1+\sin x)}\right| \\
&= \frac{1}{2}\ln\left|\frac{(1+\sin x)^2}{\cos^2 x}\right| && \Big| \text{ use } \tfrac{1}{2}\ln|a|=\ln|a|^{\frac{1}{2}} \\
&= \ln\sqrt{\left|\frac{(1+\sin x)^2}{\cos^2 x}\right|} \\
&= \ln\left|\frac{1+\sin x}{\cos x}\right| \\
&= \ln|\sec x+\tan x| \quad .
\end{aligned}
$$

4.a. **Variant III.** This variant present a quick solution by multiplying and dividing our integrand by the multiplier $\sec x + \tan x$. Of course, the idea of using that multiplier comes from knowing the answer to the problem in advance (which can be obtained, for example, by using the preceding solution variants).

$$
\begin{aligned}
\int \sec x \mathrm{d}x &= \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x}\mathrm{d}x \\
&= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\mathrm{d}x && \left| \begin{array}{l} \mathrm{d}(\tan x) = \sec^2 x\mathrm{d}x \\ \mathrm{d}(\sec x) = \sec x \tan x \mathrm{d}x \\ \text{Set } u = \sec x + \tan x \end{array} \right. \\
&= \int \frac{\mathrm{d}(\sec x + \tan x)}{\sec x + \tan x} \\
&= \int \frac{\mathrm{d}u}{u} \\
&= \ln|u| + C \\
&= \ln|\sec x + \tan x| + C \quad .
\end{aligned}
$$

Solution. 4.b This problem can be solved with the general method by setting $x = 2\arctan t$. However, there are shorter ways to solve the integral, as we show below.

Variant I.

$$
\begin{aligned}
\int \sec^3 x\mathrm{d}x &= \int \frac{1}{\cos^3 x}\mathrm{d}x \\
&= \int \frac{\cos x}{\cos^4 x}\mathrm{d}x && \text{use } \mathrm{d}(\sin x) = \cos x\mathrm{d}x \\
&= \int \frac{1}{\cos^4 x}\mathrm{d}(\sin x) && \text{use } \cos^2 x = 1 - \sin^2 x \\
&= \int \frac{1}{\left(1-\sin^2 x\right)^2}\mathrm{d}(\sin x) && \text{Set } \sin x = u \\
&= \int \frac{1}{\left(1-u^2\right)^2}\mathrm{d}u && \text{split in part. frac.} \\
&= \int \left(\frac{\frac{1}{4}}{u+1} + \frac{\frac{1}{4}}{(u+1)^2} + \frac{-\frac{1}{4}}{u-1} + \frac{\frac{1}{4}}{(u-1)^2} \right)\mathrm{d}u \\
&= \frac{1}{4}\left(\ln|u+1| - \ln|u-1| - \frac{1}{u+1} - \frac{1}{u-1} \right) + C \\
&= \frac{1}{4}\left(\ln\left|\frac{u+1}{u-1}\right| - \frac{2u}{u^2-1} \right) + C \\
&= \frac{1}{4}\left(\ln\left|\frac{\sin x + 1}{\sin x - 1}\right| + \frac{2\sin x}{\cos^2 x} \right) + C.
\end{aligned}
$$

Variant II. This variant uses the preceding problem to get to a solution as follows.

$$
\begin{aligned}
\int \sec^3 x\mathrm{d}x &= \int \sec x\mathrm{d}(\tan x) && \text{int. by parts} \\
&= \sec x \tan x - \int \tan x\mathrm{d}(\sec x) \\
&= \sec x \tan x - \int \sec x \tan^2 x\mathrm{d}x && \tan^2 x = \sec^2 x - 1 \\
&= \sec x \tan x - \int \sec x(\sec^2 x - 1)\mathrm{d}x \\
&= \sec x \tan x - \int \sec^3 x\mathrm{d}x + \int \sec x\mathrm{d}x && \text{Use Problem } 4.a \\
&= \sec x \tan x - \int \sec^3 x\mathrm{d}x + \ln|\sec x + \tan x| && + \textstyle\int \sec^3 x\mathrm{d}x \text{ to both sides} \\
2\int \sec^3 x\mathrm{d}x &= (\sec x \tan x + \ln|\sec x + \tan x|) + C \\
\int \sec^3 x\mathrm{d}x &= \frac{1}{2}(\sec x \tan x + \ln|\sec x + \tan x|) + K \quad .
\end{aligned}
$$