Calculus II
Integrals involving radicals of quadratics, table of substitutions

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2019

- Let R be a rational function in two variables.
- So far, with linear transformations we converted all integrals of the form $\int R(x, \sqrt{ax^2+bx+c})\mathrm{d}x$ to one of the three forms:
  $\int R(x, \sqrt{x^2+1})\mathrm{d}x$, $\int R(x, \sqrt{-x^2+1})\mathrm{d}x$, $\int R(x, \sqrt{x^2-1})\mathrm{d}x$.
- Each of the above integrals can be transformed to a rational trigonometric integral using 3 pairs of substitutions:
  $x = \tan\theta$, $x = \cot\theta$; $x = \sin\theta$, $x = \cos\theta$; $x = \csc\theta$, $x = \sec\theta$.
- We studied that trigonometric integrals are converted to rational function integrals via $\theta = 2\arctan t$.
- The resulting 3 pairs of substitutions are called Euler substitutions:
  $x = \tan(2\arctan t)$, $x = \cot(2\arctan t)$; $x = \sin(2\arctan t)$,
  $x = \cos(2\arctan t)$; $x = \csc(2\arctan t)$, $x = \sec(2\arctan t)$.
- The Euler substitutions directly transform the integral to a rational function integral.
- We will demonstrate that the Euler substitutions are rational.

Trigonometric substitution and Euler substitution

Expression	Substitution	Variable range	Relevant identity
$\sqrt{x^2+1}$	$x=\tan\theta$ $x=\cot\theta$	$\theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ $\theta\in(0,\pi)$	$1+\tan^2\theta=\sec^2\theta$ $1+\cot^2\theta=\csc^2\theta$
$\sqrt{-x^2+1}$	$x=\sin\theta$ $x=\cos\theta$	$\theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ $\theta\in(0,\pi)$	$1-\sin^2\theta=\cos^2\theta$ $1-\cos^2\theta=\cos^2\theta$
$\sqrt{x^2-1}$	$x=\csc\theta$ $x=\sec\theta$	$\theta\in\left[0,\frac{\pi}{2}\right)\cup\left[\pi,\frac{3\pi}{2}\right)$ $\theta\in\left[0,\frac{\pi}{2}\right)\cup\left[\pi,\frac{3\pi}{2}\right)$	$\csc^2\theta-1=\cot^2\theta$ $\sec^2\theta-1=\tan^2\theta$

Euler substitution by applying in addition $\theta = 2\arctan t$

$\sqrt{x^2+1}$	$x=\frac{2t}{1-t^2}$ $x=\frac{1}{2}\left(\frac{1}{t}-t\right)$	$-1<t<1$ $0<t$	(?) (?)
$\sqrt{-x^2+1}$	$x=\frac{2t}{1+t^2}$ $x=\frac{1-t^2}{1+t^2}$	$-1\leq t\leq 1$ $0<t$	(?) (?)
$\sqrt{x^2-1}$	$x=\frac{1}{2}\left(\frac{1}{t}+t\right)$ $x=\frac{1+t^2}{1-t^2}$	$t\in(-\infty,-1)\cup[0,1)$ $t\in(-\infty,-1)\cup[0,1)$	(?) (?)