Calculus II

Definite integrals of the form $\int_p^q \sqrt{ax^2+c}\,\mathrm{d}x$, $a, c > 0$

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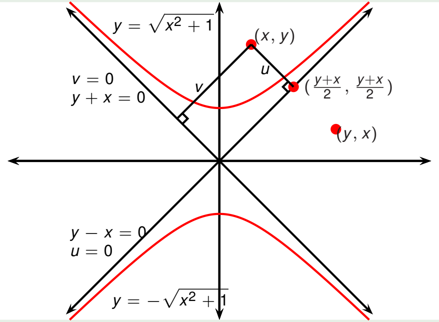
Example

Find the area locked b-n the hyperbolas $y = \pm\sqrt{x^2+1}$ and $x = \pm 2\sqrt{2}$.



Signed distance b-n (x, y) and line $u = 0$ equals

$$\pm\sqrt{\left(x - \frac{(x+y)}{2}\right)^2 + \left(y - \frac{(x+y)}{2}\right)^2} = \pm\sqrt{\frac{1}{2}(y-x)^2} = \pm\frac{\sqrt{2}}{2}(y-x) = u.$$

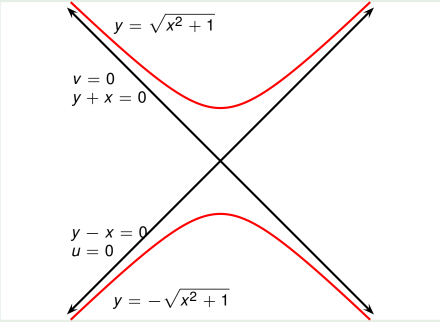
We studied $v = \frac{\frac{1}{2}}{u}$ is called a hyperbola: why do we call $y = \sqrt{x^2+1}$ hyperbola? Compute:

$$\begin{array}{rcl}
\sqrt{x^2+1} &=& y\\
x^2+1 &=& y^2\\
y^2-x^2 &=& 1\\
\frac{\sqrt{2}}{2}(y-x)\frac{\sqrt{2}}{2}(y+x) &=& \frac{1}{2}\\
uv &=& \frac{1}{2}\\
v &=& \frac{\frac{1}{2}}{u},
\end{array}$$

where $\left| \begin{array}{l} u = \frac{\sqrt{2}}{2}(y-x) \\ v = \frac{\sqrt{2}}{2}(y+x) \end{array}\right.$. Consider an arbitrary point (x, y).

Example

Find the area locked b-n the hyperbolas $y = \pm\sqrt{x^2+1}$ and $x = \pm 2\sqrt{2}$.



Signed distance b-n (x, y) and line $u = 0$ equals u. Similarly compute that signed distance b-n (x, y) and the line $v = 0$ equals v. $\Rightarrow y^2 - x^2 = 1$ is the hyperbola $v = \frac{1/2}{u}$ in the (u, v)-plane.

We studied $v = \frac{\frac{1}{2}}{u}$ is called a hyperbola: why do we call $y = \sqrt{x^2+1}$ hyperbola? Compute:

$$\begin{array}{rcl} \sqrt{x^2+1} &=& y\\ x^2+1 &=& y^2\\ y^2-x^2 &=& 1\\ \frac{\sqrt{2}}{2}(y-x)\frac{\sqrt{2}}{2}(y+x) &=& \frac{1}{2}\\ uv &=& \frac{1}{2}\\ v &=& \frac{\frac{1}{2}}{u}, \end{array}$$

where $\left| \begin{array}{l} u = \frac{\sqrt{2}}{2}(y-x)\\ v = \frac{\sqrt{2}}{2}(y+x) \end{array}\right.$. Consider an arbitrary point (x, y).

Example

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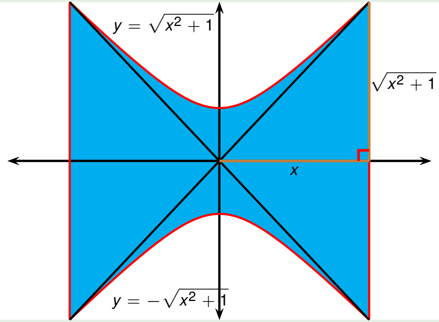

The area in question is:

$$\int_{-2\sqrt{2}}^{2\sqrt{2}} 2\sqrt{x^2+1}\mathrm{d}x$$
$$= 2\left[x\sqrt{x^2+1} + \ln\left(\sqrt{x^2+1}+x\right)\right]_0^{2\sqrt{2}}$$
$$= 2\left(2\sqrt{2}\sqrt{(2\sqrt{2})^2+1} + \ln\left(\sqrt{(2\sqrt{2})^2+1}+2\sqrt{2}\right)\right)$$
$$= 12\sqrt{2} + 2\ln\left(3+2\sqrt{2}\right)$$
$$\approx 20.496$$

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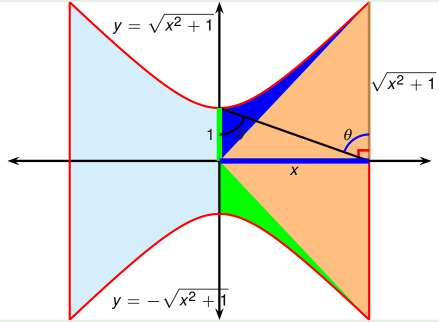
Find the area locked b-n the hyperbolas $y = \pm\sqrt{x^2+1}$ and $x = \pm 2\sqrt{2}$.



- Recall: integral can be solved via $x = \tan\theta$.
- Geometric interpretation of θ?

The area in question is:

$$\int_{-2\sqrt{2}}^{2\sqrt{2}} 2\sqrt{x^2+1}dx$$
$$= 2\left[x\sqrt{x^2+1} + \ln\left(\sqrt{x^2+1}+x\right)\right]_0^{2\sqrt{2}}$$
$$= 2\left(2\sqrt{2}\sqrt{(2\sqrt{2})^2+1} + \ln\left(\sqrt{(2\sqrt{2})^2+1}+2\sqrt{2}\right)\right)$$
$$= 12\sqrt{2} + 2\ln\left(3+2\sqrt{2}\right)$$
$$\approx 20.496$$