

Precalculus

Completing the square

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Example (Completing the square)

Complete the square.

$$\begin{aligned}
 3x^2 - 5x + 1 &= 3 \left(x^2 - \frac{5}{3}x \right) + 1 \\
 &= 3 \left(x^2 - 2 \cdot \frac{5}{2 \cdot 3}x \right) + 1 \\
 &= 3 \left(x^2 - 2 \cdot \frac{5}{6}x + \left(\frac{5}{6} \right)^2 - \left(\frac{5}{6} \right)^2 \right) + 1 \\
 &= 3 \left(\left(x - \frac{5}{6} \right)^2 - \frac{25}{36} \right) + 1 \\
 &= 3 \left(x - \frac{5}{6} \right)^2 - \frac{25}{12} + 1 \\
 &= 3 \left(x - \frac{5}{6} \right)^2 - \frac{13}{12}.
 \end{aligned}$$

Definition (Completing the square)

Let $a \neq 0$. To *complete the square* means to carry out the following algebraic manipulation.

$$\begin{aligned}
 ax^2 + bx + c &= a \left(x^2 + \frac{b}{a}x \right) + c \\
 &= a \left(x^2 + 2 \cdot \frac{b}{2a}x \right) + c \\
 &= a \left(x^2 + 2 \frac{b}{2a}x + \left(\frac{b}{2a} \right)^2 - \left(\frac{b}{2a} \right)^2 \right) + c \quad \left| \begin{array}{l} \text{Add \& subtract} \\ \left(\frac{b}{2a} \right)^2 \\ \text{use} \\ (A+B)^2 = \\ A^2 + 2AB + B^2 \end{array} \right. \\
 &= a \left(\left(x + \frac{b}{2a} \right)^2 - \frac{b^2}{4a^2} \right) + c \\
 &= a \left(x + \frac{b}{2a} \right)^2 - \cancel{a} \cdot \frac{b^2}{4\cancel{a}} + c \\
 &= a \left(x + \frac{b}{2a} \right)^2 + c - \frac{b^2}{4a}.
 \end{aligned}$$