Precalculus
Homework
Trigonometric equations and inequalities

1. Convert from degrees to radians.

   (a) 15°.

   answer: $\frac{\pi}{12} \approx 0.261799388$

   (b) 30°.

   answer: $\frac{\pi}{6} \approx 0.523598776$

   (c) 36°.

   answer: $\frac{\pi}{5} \approx 0.628318531$

   (d) 45°.

   answer: $\frac{\pi}{4} \approx 0.785398163$

   (e) 60°.

   answer: $\frac{\pi}{3} \approx 1.047197551$

   (f) 75°.

   answer: $\frac{5\pi}{12} \approx 1.308997$

   (g) 90°.

   answer: $\frac{\pi}{2}$

   (h) 120°.

   answer: $\frac{2\pi}{3}$

   (i) 135°.

   answer: $\frac{3\pi}{4}$

   (j) 150°.

   answer: $\frac{5\pi}{6}$

   (k) 180°.

   answer: π

   (l) 225°.

   answer: $\frac{5\pi}{4}$

   (m) 270°.

   answer: $\frac{3\pi}{2}$

   (n) 305°.

   answer: $\frac{61\pi}{36} \approx 5.323254$

   (o) 360°.

   answer: 2π

   (p) 405°.

   answer: $\frac{9\pi}{4}$

   (q) 1200°.

   answer: $\frac{20\pi}{3}$

   (r) -900°.

   answer: -5π

   (s) -2014°.

   answer: $-\frac{1007}{90}\pi \approx -35.150931$

2. Convert from radians to degrees. The answer key has not been proofread, use with caution.

   (a) 4π.

   answer: 720°

   (b) $-\frac{7}{6}\pi$.

   answer: -210°

   (c) $\frac{7}{12}\pi$.

   answer: 105°

   (d) $\frac{4}{3}\pi$.

   answer: 240°

   (e) $-\frac{3}{8}\pi$.

   answer: -67.5°

   (f) 2014π.

   answer: 362520°

   (g) 5.

   answer: $\left(\frac{900}{\pi}\right)^\circ \approx 286^\circ$

   (h) -2014.

   answer: -362520°

3. Find the indicated circle arc-length. The answer key has not been proofread, use with caution.

   (a) Circle of radius 3, arc of measure 36°.

   answer: $\frac{3\pi}{5} \approx 1.884956$

   (b) Circle of radius $\frac{1}{2}$, arc of measure 100°.

   answer: $\frac{5\pi}{18} \approx 0.872665$

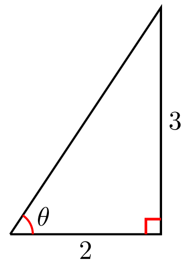
   (c) Circle of radius 1, arc of measure 3 (radians).

   answer: 3

   (d) Circle of radius 3, arc of measure 300°.

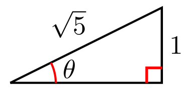
   answer: $5\pi \approx 15.707963$

4. Find the 6 trigonometric functions of the indicated angle in the indicated right triangle.



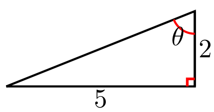
(a)

answer: $\sin\theta = \frac{3}{13}\sqrt{13}, \cos\theta = \frac{2}{13}\sqrt{13}, \tan\theta = \frac{3}{2}, \cot\theta = \frac{2}{3}, \sec\theta = \frac{\sqrt{13}}{2}, \csc\theta = \frac{\sqrt{13}}{3}$



(b)

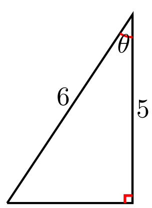
answer: $\sin\theta = \frac{\sqrt{5}}{5}, \cos\theta = \frac{2\sqrt{5}}{5}, \tan\theta = \frac{1}{2}, \cot\theta = 2, \sec\theta = \frac{\sqrt{5}}{2}, \csc\theta = \sqrt{5}$



(c)

answer: $\sin\theta = \frac{5}{\sqrt{29}} = \frac{5\sqrt{29}}{29}, \cos\theta = \frac{2}{\sqrt{29}} = \frac{2\sqrt{29}}{29}, \tan\theta = \frac{2}{5}, \cot\theta = \frac{5}{2}, \sec\theta = \frac{\sqrt{29}}{5}, \csc\theta = \frac{\sqrt{29}}{2}$

(d)



answer: $\sin\theta = \frac{\sqrt{11}}{6}, \cos\theta = \frac{5}{6}, \tan\theta = \frac{\sqrt{11}}{5}, \cot\theta = \frac{5}{\sqrt{11}} = \frac{5\sqrt{11}}{11}, \sec\theta = \frac{6}{5}, \csc\theta = \frac{6}{\sqrt{11}} = \frac{6\sqrt{11}}{11}$

5. Find the exact value of the trigonometric function (using radicals).

(a) $\cos 135°$.

answer:

(b) $\sin 225°$.

answer:

(c) $\cos 495°$.

answer:

(d) $\sin 560°$.

answer:

(e) $\sin\left(\frac{3\pi}{2}\right)$.

answer:

(f) $\cos\left(\frac{11\pi}{6}\right)$.

answer:

(g) $\sin\left(\dfrac{2015\pi}{3}\right)$.

answer:

(h) $\cos\left(\dfrac{17\pi}{3}\right)$.

answer:

6. Find all solutions of the equation in the interval $[0, 2\pi)$. The answer key has not been proofread, use with caution.

(a) $\sin x = -\frac{\sqrt{2}}{2}$.

answer: $x = \frac{5\pi}{4}, \frac{7\pi}{4}$

(b) $\cos x = \frac{\sqrt{3}}{2}$.

answer: $x = \frac{\pi}{6}, \frac{11\pi}{6}$

(c) $\sin(3x) = \frac{1}{2}$.

answer: $x = \frac{\pi}{18}, \frac{5\pi}{18}, \frac{13\pi}{18}, \frac{17\pi}{18}, \frac{25\pi}{18}, \frac{16\pi}{9}$

(d) $\cos(7x) = 0$.

answer: $x = \frac{\pi}{14}, \frac{3\pi}{14}, \frac{5\pi}{14}, \frac{\pi}{2}, \frac{9\pi}{14}, \frac{11\pi}{14}, \frac{13\pi}{14}, \frac{15\pi}{14}, \frac{17\pi}{14}, \frac{19\pi}{14}, \frac{3\pi}{2}, \frac{23\pi}{14}, \frac{25\pi}{14}, \frac{27\pi}{14}$

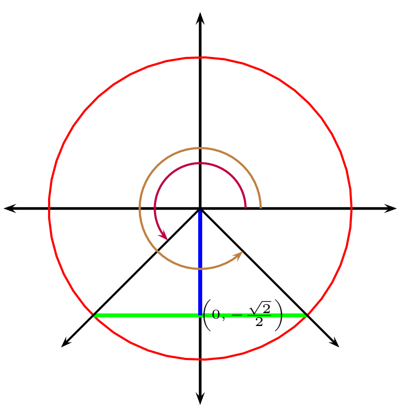
(e) $\cos\left(3x + \frac{\pi}{2}\right) = 0$.

answer: $x = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}$

(f) $\sin\left(5x - \frac{\pi}{3}\right) = 0$.

answer: $x = \frac{\pi}{15}, \frac{4\pi}{15}, \frac{7\pi}{15}, \frac{2\pi}{3}, \frac{13\pi}{15}, \frac{16\pi}{15}, \frac{19\pi}{15}, \frac{22\pi}{15}, \frac{5\pi}{3}, \frac{28\pi}{15}$

Solution. 6.a



$$\sin x = -\frac{\sqrt{2}}{2}$$

Since $\sin x$ is negative it must be either in Quadrant III or IV. Therefore the angle x is coterminal either with $225^\circ = \frac{5\pi}{4}$ (Quadrant III) or $315^\circ = \frac{7\pi}{4}$ (Quadrant IV).

Case 1. x is coterminal with $225^\circ = \frac{5\pi}{4}$. We can compute

$$\begin{array}{rcl|l} x &=& \dfrac{5\pi}{4} + 2k\pi & k \text{ is any integer} \\ x &=& \dfrac{5\pi}{4} + \dfrac{8k\pi}{4} & \\ x &=& \dfrac{5\pi + 8k\pi}{4} & \\ x &=& \dfrac{\pi(5+8k)}{4} & \end{array}$$

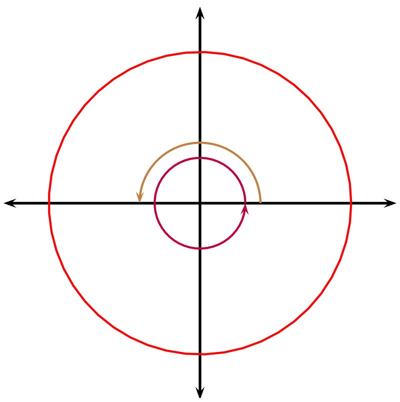
We are looking for solutions in the interval $[0, 2\pi)$ and so we must discard those values of the integer k for which $\frac{\pi(7+8k)}{4}$ is negative or is greater than or equal to 2π. Therefore the only solution in this case is $x = \frac{5\pi}{4}$.

Case 2.

$$
\begin{aligned}
x &= \frac{7\pi}{4} + 2k\pi \\
x &= \frac{7\pi}{4} + \frac{8k\pi}{4} \\
x &= \frac{7\pi + 8k\pi}{4} \\
x &= \frac{\pi(7+8k)}{4}
\end{aligned}
$$

We are looking for solutions in the interval $[0, 2\pi)$ and so we must discard those values of the integer k for which $\frac{\pi(7+8k)}{4}$ is negative or is greater than or equal to 2π. Therefore the only solution in this case is $x = \frac{7\pi}{4}$.

Solution. 6.f



$$
\sin\left(5x - \tfrac{\pi}{3}\right) = 0
$$

Since $\sin 0 = 0$ and $\sin 180^\circ = \sin\pi = 0$, the angle $5x - \frac{\pi}{3}$ must be coterminal with 0 or π.

Case 1. $5x - \frac{\pi}{3}$ is coterminal with 0. We compute

$$
\begin{aligned}
5x - \frac{\pi}{3} &= 0 + 2k\pi \\
5x &= \frac{\pi}{3} + 2k\pi \\
x &= \frac{\frac{\pi}{3} + 2k\pi}{5} \\
x &= \frac{\frac{\pi}{3} + \frac{6k\pi}{3}}{5} \\
x &= \frac{\frac{\pi + 6k\pi}{3}}{5} \\
x &= \frac{\pi + 6k\pi}{15} \\
x &= \frac{\pi(1+6k)}{15} \\
x &= \cancel{\dots}, \frac{\pi[1+6(0)]}{15}, \frac{\pi[1+6(1)]}{15}, \frac{\pi[1+6(2)]}{15}, \frac{\pi(1+12)}{15}, \frac{\pi[1+6(3)]}{15}, \frac{\pi[1+6(4)]}{15}, \cancel{\dots} && \text{Discard other values of } k \text{ as they yield angles outside of } [0, 2\pi) \\
x &= \frac{\pi}{15}, \frac{7\pi}{15}, \frac{13\pi}{15}, \frac{19\pi}{15}, \frac{25\pi}{15}.
\end{aligned}
$$

Case 2.

$$
\begin{aligned}
5x-\frac{\pi}{3} &= \pi+2k\pi\\
5x &= \pi+\frac{\pi}{3}+2k\pi\\
5x &= \frac{4\pi}{3}+2k\pi\\
x &= \frac{\frac{4\pi}{3}+2k\pi}{5}\\
x &= \frac{\frac{4\pi}{3}+\frac{6k\pi}{3}}{5}\\
x &= \frac{\frac{4\pi+6k\pi}{3}}{5}\\
x &= \frac{4\pi+6k\pi}{15}\\
x &= \frac{2\pi(2+3k)}{15}\\
x &= \cancel{\dots}, \frac{2\pi[2+3(0)]}{15}, \frac{2\pi[2+3(1)]}{15}, \frac{2\pi[2+3(2)]}{15}, \frac{2\pi[2+3(3)]}{15}, \frac{2\pi[2+3(4)]}{15}, \cancel{\dots}\\
x &= \frac{4\pi}{15}, \frac{10\pi}{15}, \frac{16\pi}{15}, \frac{22\pi}{15}, \frac{28\pi}{15}.
\end{aligned}
$$

Discard other values of k as they yield angles outside of $[0, 2\pi)$

Our final answer (combined from the two cases) is $x = \frac{\pi}{15}, \frac{4\pi}{15}, \frac{7\pi}{15}, \frac{2\pi}{3}, \frac{13\pi}{15}, \frac{16\pi}{15}, \frac{19\pi}{15}, \frac{22\pi}{15}, \frac{5\pi}{3}$ or $\frac{28\pi}{15}$.

7. Use the known values of $\sin 30°, \cos 30°, \sin 45°, \cos 45°, \sin 60°, \cos 60°, \dots$, the angle sum formulas and the cofunction identities to find an exact value (using radicals) for the trigonometric function.

   (a) The six trigonometric functions of $105° = 45° + 60°$:

   - $\sin(105°)$.

     answer: $\frac{\sqrt{6}+\sqrt{2}}{4}$

   - $\cos(105°)$. Should your answer be a positive or a negative number?

     answer: $\frac{\sqrt{2}-\sqrt{6}}{4}$

   - $\tan(105°)$.

     answer: $-\sqrt{3}-2$

   - $\cot(105°)$.

     answer: $\sqrt{3}-2$

   - $\sec(105°)$.

     answer: $-\sqrt{6}-\sqrt{2}$

   - $\csc(105°)$.

     answer: $\sqrt{6}-\sqrt{2}$

   (b) The six trigonometric functions of $\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}$:

   - $\sin\left(\frac{\pi}{12}\right)$.

     answer: $\frac{\sqrt{6}-\sqrt{2}}{4}$

   - $\cos\left(\frac{\pi}{12}\right)$. Should $\sin\left(\frac{\pi}{12}\right)$ be larger or smaller than $\cos\left(\frac{\pi}{12}\right)$?

     answer: $\frac{\sqrt{2}+\sqrt{6}}{4}$

   - $\tan\left(\frac{\pi}{12}\right)$.

     answer: $-\sqrt{3}+2$

   - $\cot\left(\frac{\pi}{12}\right)$.

     answer: $\sqrt{3}+2$

   - $\sec\left(\frac{\pi}{12}\right)$.

     answer: $\sqrt{6}-\sqrt{2}$

   - $\csc\left(\frac{\pi}{12}\right)$.

     answer: $\sqrt{6}+\sqrt{2}$

8. Simplify to a trigonometric function of the angle θ. The answer key has not been proofread, use with caution.

   (a) $\sin\left(\frac{\pi}{2} - \theta\right)$.

   answer: $\cos\theta$

   (b) $\cos\left(\frac{13\pi}{2} - \theta\right)$.

   answer: $\sin\theta$

   (c) $\tan(\pi - \theta)$

   answer: $-\tan\theta$

   (d) $\cot\left(\frac{3\pi}{2} - \theta\right)$

   answer: $\tan\theta$

   (e) $\csc\left(\frac{3\pi}{2} + \theta\right)$

   answer: $\sec\theta$

9. Using the power-reducing formulas, rewrite the expression in terms of first powers of the cosines and sines of multiples of the angle θ.

   (a) $\sin^4\theta$.

   answer: $\frac{1}{8}\cos(4\theta) - \frac{1}{2}\cos(2\theta) + \frac{3}{8}$

   (b) $\cos^4\theta$.

   answer: $\frac{1}{8}\cos(4\theta) + \frac{1}{2}\cos(2\theta) + \frac{3}{8}$

   (c) $\sin^6\theta$.

   answer: $\sin^6\theta = -\frac{1}{32}\cos(6\theta) + \frac{3}{16}\cos(4\theta) - \frac{15}{32}\cos(2\theta) + \frac{5}{16}$

   (d) $\cos^6\theta$.

   answer: $\cos^6\theta = \frac{1}{32}\cos(6\theta) + \frac{3}{16}\cos(4\theta) + \frac{15}{32}\cos(2\theta) + \frac{5}{16}$

10. Use the sum-to-product formulas to find all solutions of the trigonometric equation in the interval $[0, 2\pi)$.

    Please note that typing a query such as "solve(sin(x)+sin (3x)=0)" at `www.wolframalpha.com` will provide you with a correct answer and a function plot.

    (a) $\sin(x) + \sin(3x) = 0$.

    answer: $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$

    (b) $\cos(x) + \cos(-3x) = 0$.

    answer: $x = \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}$

    (c) $\sin(x) - \sin(3x) = 0$.

    answer: $x = 0, \frac{\pi}{4}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{7\pi}{4}$

    (d) $\cos(2x) - \cos(3x) = 0$.

    answer: $x = 0, \frac{2}{5}\pi, \frac{4\pi}{5}, \frac{6\pi}{5}, \frac{8\pi}{5}$

11. Find the inverse function. You are asked to do the algebra only; you are not asked to determine the domain or range of the function or its inverse.

    (a) $f(x) = 3x^2 + 4x - 7$, where $x \geq -\frac{2}{3}$.

    answer: $f^{-1}(x) = -\frac{2}{3} + \frac{\sqrt{25+3x}}{3}, \quad x \geq -\frac{25}{3}$

    (b) $f(x) = 2x^2 + 3x - 5$, where $x \geq -\frac{3}{4}$.

    answer: $f^{-1}(x) = -\frac{3}{4} + \frac{\sqrt{49+8x}}{4}, \quad x \geq -\frac{49}{8}$

    (c) $f(x) = \dfrac{2x+5}{x-4}$, where $x \neq 4$.

    answer: $f^{-1}(x) = \frac{4x+5}{x-2}, \quad x \neq 2$

    (d) $f(x) = \dfrac{3x+5}{2x-4}$, where $x \neq 2$.

    answer: $f^{-1}(x) = \dfrac{4x+5}{2x-3}, \quad x \neq \dfrac{3}{2}$

    (e) $f(x) = \dfrac{5x+6}{4x+5}$, where $x \neq -\frac{5}{4}$.

    answer: $f^{-1}(x) = \frac{-5x+6}{4x-5}, x \neq \frac{5}{4}$

    (f) $f(x) = \dfrac{2x-3}{-3x+4}$, where $x \neq \frac{4}{3}$..

    answer: $f^{-1}(x) = \frac{4x+3}{3x+2}, x \neq -\frac{2}{3}$

Solution. 11.d This is a concise solution written in form suitable for test taking.

$$
\begin{array}{rcl}
y & = & \dfrac{3x+5}{2x-4} \\
y(2x-4) & = & 3x+5 \\
2xy-4y & = & 3x+5 \\
2xy-3x & = & 4y+5 \\
x(2y-3) & = & 4y+5 \\
x & = & \dfrac{4y+5}{2y-3} \\
\text{Therefore} \quad f^{-1}(y) & = & \dfrac{5+4y}{2y-3} \\
f^{-1}(x) & = & \dfrac{5+4x}{2x-3}.
\end{array}
$$

Solution. 11.e. Set $f(x) = y$. Then

$$
\begin{array}{rcl}
y &=& \dfrac{5x+6}{4x+5} \\
y(4x+5) &=& 5x+6 \\
x(4y-5) &=& -5y+6 \\
x &=& \dfrac{-5y+6}{4y-5}.
\end{array}
$$

Therefore the function $x = g(y) = \dfrac{-5y+6}{4y-5}$ is the inverse of $f(x)$. We write $g = f^{-1}$. The function $g = f^{-1}$ is defined for $y \neq \dfrac{5}{4}$. For our final answer we relabel the argument of g to x:

$$g(x) = f^{-1}(x) = \frac{-5x+6}{4x-5} \quad .$$

Let us check our work. In order for f and g to be inverses, we need that $g(f(x))$ be equal to x.

$$g(f(x)) = \frac{-5f(x)+6}{4f(x)-5} = \frac{-5\frac{(5x+6)}{4x+5}+6}{4\frac{(5x+6)}{4x+5}-5} = \frac{-5(5x+6)+6(4x+5)}{4(5x+6)-5(4x+5)} = \frac{-x}{-1} = x \quad ,$$

as expected.

12. Find the inverse function and its domain.

(a) $y = \ln(x+3)$.

answer: $f^{-1}(x) = e^x - 3$

(b) $y = 4\ln(x-3) - 4$.

answer: $f^{-1}(x) = e^{\frac{x+4}{4}} + 3$

(c) $y = 2\ln(-2x+4) + 1$

answer: $f^{-1}(x) = -\frac{1}{2}e^{\frac{1}{2}x - \frac{1}{2}} + 2$

(d) $f(x) = e^{x^3}$.

answer: $f^{-1}(x) = \sqrt[3]{\ln x}, \quad x > 0$

(e) $y = (\ln x)^2, x \geq 1$.

answer: $f^{-1}(x) = e^{\sqrt{x}}, \quad x \geq 0$

(f) $y = \dfrac{e^x}{1+2e^x}$.

answer: $f^{-1}(x) = \ln\left(\frac{x}{1-2x}\right), \quad x \in \left(0, \frac{1}{2}\right)$

(g) $f(x) = 2^{2x} + 2^x - 2$.

answer: $f^{-1}(x) = \log_2 \frac{-1+\sqrt{9+4x}}{2}, \quad x \geq -2$

Solution. 12.a

$$
\begin{aligned}
y &= \ln(x+3) \\
e^y &= e^{\ln(x+3)} \\
e^y &= x+3 \\
e^y - 3 &= x \\
\text{Therefore} \quad f^{-1}(y) &= e^y - 3.
\end{aligned}
$$

The domain of e^y is all real numbers, so the domain of f^{-1} is all real numbers.

Solution. 12.b

$$
\begin{array}{rcll}
4\ln(x-3) - 4 &=& y & \\
4\ln(x-3) &=& y+4 & \\
\ln(x-3) &=& \dfrac{y+4}{4} & \Big| \text{ exponentiate} \\
e^{\ln(x-3)} &=& e^{\frac{y+4}{4}} & \\
x-3 &=& e^{\frac{y+4}{4}} & \\
f^{-1}(y) = x &=& e^{\frac{y+4}{4}} + 3 & \\
f^{-1}(x) &=& e^{\frac{x+4}{4}} + 3 & \Big| \text{ relabel.}
\end{array}
$$

The domain of f^{-1} is all real numbers (no restrictions on the domain).

Solution. 12.e

$$\begin{array}{rcl|l} y &=& (\ln x)^2 & \text{take } \sqrt{\ } \text{ on both sides, } y \geq 0 \\ \sqrt{y} &=& \ln x & \text{exponentiate} \\ e^{\sqrt{y}} &=& e^{\ln x} = x & \\ f^{-1}(y) &=& e^{\sqrt{y}} & \\ f^{-1}(x) &=& e^{\sqrt{x}} & \end{array}$$

Solution. 12.f

$$\begin{aligned} y &= \frac{e^x}{1+2e^x} \\ y(1+2e^x) &= e^x \\ y &= e^x(1-2y) \\ \frac{y}{1-2y} &= e^x \\ \ln\frac{y}{1-2y} &= \ln e^x \\ \ln\frac{y}{1-2y} &= x \\ \text{Therefore} \quad f^{-1}(y) &= \ln\frac{y}{1-2y}. \end{aligned}$$

The natural logarithm function is only defined for positive input values. Therefore the domain is the set of all y for which

$$\frac{y}{1-2y} > 0.$$

This inequality holds if the numerator and denominator are both positive or both negative. This happens if either

(a) $y > 0$ and $y < \frac{1}{2}$, or
(b) $y < 0$ and $y > \frac{1}{2}$.

The latter option is impossible, so the domain is $\{y \in \mathbb{R} \mid 0 < y < \frac{1}{2}\}$.

13. Find each of the following values. Express your answers precisely, not as decimals.

(a) $\arcsin(\sin 4)$.

answer: $\pi - 4$

(b) $\arcsin(\sin 0.5)$.

answer: 0.5

(c) $\arcsin(\cos 120^\circ)$.

answer: $-\frac{\pi}{6}$

(d) $\arccos(\cos(3))$.

answer: 3

(e) $\arccos(\cos(-2))$.

answer: 2

(f) $\arccos(\sin(-4))$.

answer: $\frac{3\pi}{2} - 4 \approx 0.712389$

(g) $\arctan(\tan 5)$.

answer: $5 - 2\pi$

Solution. 13.g $\frac{3\pi}{2} \approx 4.71$ and $2\pi \approx 6.28$, so

$$\begin{aligned} \frac{3\pi}{2} &< 5 < 2\pi \\ \text{Therefore} \quad -\frac{\pi}{2} &< 5 - 2\pi < 0 < \frac{\pi}{2}. \end{aligned}$$

Therefore $5 - 2\pi$ is in the restricted domain of the tangent function. Moreover, the tangent function is π-periodic, so $\tan 5 = \tan(5 - 2\pi)$. Therefore $\arctan(\tan 5) = 5 - 2\pi$.

14. Express as the following as an algebraic expression of x. In other words, "get rid" of the trigonometric and inverse trigonometric expressions.

(a) $\cos^2(\arctan x)$.

answer: $\frac{1}{1+x^2}$

(b) $-\sin^2(\operatorname{arccot} x)$.

answer: $-\frac{1}{1+x^2}$

(c) $\dfrac{1}{\cos(\arcsin x)}$.

answer: $\frac{1}{\sqrt{1-x^2}}$

(d) $-\dfrac{1}{\sin(\arccos x)}$.

answer: $-\frac{1}{\sqrt{1-x^2}}$

Solution. 14.b. We follow the strategy outlined in the end of the solution of Problem 15.c. We set $y = \operatorname{arccot} x$. Then we need to express $-\sin^2 y$ via $\cot y$. That is a matter of algebra:

$$\begin{aligned}
-\sin^2(\operatorname{arccot} x) &= -\sin^2 y && \text{Set } y = \operatorname{arccot} x \\
&= -\frac{\sin^2 y}{\sin^2 y + \cos^2 y} && \text{use } \sin^2 y + \cos^2 y = 1 \\
&= -\frac{1}{\frac{\sin^2 y + \cos^2 y}{\sin^2 y}} \\
&= -\frac{1}{1+\cot^2 y} && \text{Substitute back } \cot y = x \\
&= -\frac{1}{1+x^2} .
\end{aligned}$$

15. Let $x \in (0,1)$. Express the following using x and $\sqrt{1-x^2}$.

(a) $\sin(\arcsin(x))$.

answer: x

(b) $\sin(2\arcsin(x))$.

answer: $2x\sqrt{1-x^2}$

(c) $\sin(3\arcsin(x))$.

answer: $-4x^3+3x$

(d) $\sin(\arccos(x))$.

answer: $\sqrt{1-x^2}$

(e) $\sin(2\arccos(x))$.

answer: $2x\sqrt{1-x^2}$

(f) $\sin(3\arccos(x))$.

answer: $(4x^2-1)\sqrt{1-x^2} = -4\left(\sqrt{1-x^2}\right)^3 + 3\sqrt{1-x^2}$

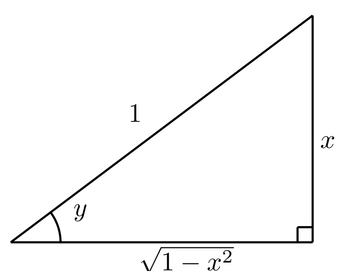
(g) $\cos(2\arcsin(x))$.

answer: $1-2x^2$

(h) $\cos(3\arccos(x))$.

answer: $4x^3-3x$

Solution. 15.b. Let $y = \arcsin x$. Then $\sin y = x$, and we can draw a right triangle with opposite side length x and hypotenuse length 1 to find the other trigonometric ratios of y.



Then $\cos y = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$. Now we use the double angle formula to find $\sin(2\arcsin x)$.

$$\begin{aligned}
\sin(2\arcsin x) &= \sin(2y) \\
&= 2\sin y\cos y \\
&= 2x\sqrt{1-x^2}.
\end{aligned}$$

Solution. 15.c. Use the result of Problem 15.b. This also requires the addition formula for sine:

$$\sin(A+B) = \sin A \cos B + \sin B \cos A,$$

and the double angle formula for cosine:

$$\cos(2y) = \cos^2 y - \sin^2 y.$$

$$\begin{aligned}
\sin(3 \arcsin x) &= \sin(3y) \\
&= \sin(2y + y) \\
&= \sin(2y)\cos y + \sin y \cos(2y) && \text{Use addition formula} \\
&= (2\sin y \cos y)\cos y + \sin y(\cos^2 y - \sin^2 y) && \text{Use double angle formulas} \\
&= 2\sin y \cos^2 y + \sin y \cos^2 y - \sin^3 y \\
&= 3\sin y \cos^2 y - \sin^3 y \\
&= 3\sin y(1 - \sin^2 y) - \sin^3 y \\
&= 3x(1 - x^2) - x^3 \\
&= 3x - 4x^3.
\end{aligned}$$

The solution is complete. A careful look at the solution above reveals a strategy useful for problems similar to this one.

(a) Identify the inverse trigonometric expression- $\arcsin x, \arccos x, \arctan x, \dots$. In the present problem that was $y = \arcsin x$.

(b) The problem is therefore a trigonometric function of y.

(c) Using trig identities and algebra, rewrite the problem as a trigonometric expression involving only the trig function that transforms y to x. In the present problem we rewrote everything using $\sin y$.

(d) Use the fact that $\sin(\arcsin x) = x$, $\cos(\arccos x) = x$, $\dots$, etc. to simplify.

Solution. 15.f We use the same strategy outlined in the end of the solution of Problem 15.c. Set $y = \arccos x$ and so $\cos(y) = x$. Therefore:

$$\begin{aligned}
\sin(3y) &= \sin(2y + y) \\
&= \sin(2y)\cos y + \sin y \cos(2y) \\
&= 2\sin y \cos y \cos y + \sin y(2\cos^2 y - 1) \\
&= 2\sin y \cos^2 y + \sin y(2\cos^2 y - 1) \\
&= \sin y(4\cos^2 y - 1) && \text{use } \begin{array}{rcl} \cos y &=& x \\ \sin y &=& \sqrt{1-x^2} \end{array} \\
&= \sqrt{1-x^2}(4x^2 - 1) \quad .
\end{aligned}$$

16. Find all values of x in the interval $[0, 2\pi]$ that satisfy the equation.

(a) $2\cos x - 1 = 0$.

answer: $x = \frac{\pi}{3}$ or $x = \frac{5\pi}{3}$

(b) $\sin(2x) = \cos x$.

answer: $x = \frac{\pi}{2}$, $x = \frac{3\pi}{2}$, $x = \frac{\pi}{6}$, or $x = \frac{5\pi}{6}$

(c) $\sqrt{3}\sin x = \sin(2x)$.

answer: $x = \frac{\pi}{6}, \frac{11\pi}{6}, 0, \pi, 2\pi$

(d) $2\sin^2 x = 1$.

answer: $x = \frac{\pi}{4}$, $x = \frac{3\pi}{4}$, $x = \frac{5\pi}{4}$, or $x = \frac{7\pi}{4}$

(e) $2 + \cos(2x) = 3\cos x$.

answer: $x = 0$, $x = 2\pi$, $x = \frac{\pi}{3}$, or $x = \frac{5\pi}{3}$

(f) $2\cos x + \sin(2x) = 0$.

answer: $x = \frac{\pi}{2}$, $x = \frac{3\pi}{2}$

(g) $2\cos^2 x - (1 + \sqrt{2})\cos x + \frac{\sqrt{2}}{2} = 0$.

answer: $x = \frac{\pi}{4}, \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{4}$

(h) $|\tan x| = 1$.

answer: $x = \frac{\pi}{4}$, $x = \frac{3\pi}{4}$, $x = \frac{5\pi}{4}$, or $x = \frac{7\pi}{4}$

(i) $3\cot^2 x = 1$.

answer: $x = \frac{\pi}{3}$, $x = \frac{2\pi}{3}$, $x = \frac{4\pi}{3}$, or $x = \frac{5\pi}{3}$

(j) $\sin x = \tan x$.

answer: $x = 0$, $x = \pi$, or $x = 2\pi$

Solution. 16.g Set $\cos x = u$. Then

$$2\cos^2 x - (1 + \sqrt{2})\cos x + \frac{\sqrt{2}}{2} = 0$$

becomes

$$2u^2 - (1 + \sqrt{2})u + \frac{\sqrt{2}}{2} = 0.$$

This is a quadratic equation in u and therefore has solutions

$$\begin{aligned}
u_1, u_2 &= \frac{1+\sqrt{2} \pm \sqrt{(1+\sqrt{2})^2 - 4\sqrt{2}}}{4} \\
&= \frac{1+\sqrt{2} \pm \sqrt{1-2\sqrt{2}+2}}{4} \\
&= \frac{1+\sqrt{2} \pm \sqrt{(1-\sqrt{2})^2}}{4} \\
&= \frac{1+\sqrt{2} \pm (1-\sqrt{2})}{4} = \begin{cases} \frac{1}{2} & \text{or} \\ \frac{\sqrt{2}}{2} \end{cases}
\end{aligned}$$

Therefore $u = \cos x = \frac{1}{2}$ or $u = \cos x = \frac{\sqrt{2}}{2}$, and, as x is in the interval $[0, 2\pi]$, we get $x = \frac{\pi}{3}, \frac{5\pi}{3}$ (for $\cos x = \frac{1}{2}$) or $x = \frac{\pi}{4}, \frac{7\pi}{4}$ (for $\cos x = \frac{\sqrt{2}}{2}$).