Calculus I
Derivatives: linearity, product and quotient rules

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Theorem (The Power Rule (General Version))

If n is any real number, then

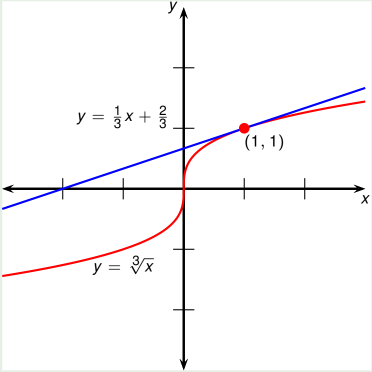
$$\frac{\mathrm{d}}{\mathrm{d}x}(x^n) = nx^{n-1}.$$

Example (Power Rule, negative exponent)

$$
\begin{aligned}
\text{Differentiate} \quad y &= \frac{1}{x}. \\
y &= x^{-1}. \\
\text{Power Rule:} \quad \frac{\mathrm{d}y}{\mathrm{d}x} &= (-1)x^{-2} \\
&= -\frac{1}{x^2}.
\end{aligned}
$$

Example (Calculating the tangent line using the Power Rule)

Find an equation for the tangent line to the cubic $y = \sqrt[3]{x}$ at the point $P = (1, 1)$.



Here $a = 1$ and $f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$.

$$
\begin{aligned}
f'(x) &= \frac{1}{3}x^{\frac{1}{3}-1} \\
&= \frac{1}{3}x^{\frac{-2}{3}} \\
&= \frac{1}{3\sqrt[3]{x^2}}. \\
f'(1) &= \frac{1}{3}.
\end{aligned}
$$

Point-slope form: $y - 1 = \frac{1}{3}(x - 1)$, or $y = \frac{1}{3}x + \frac{2}{3}$.

Example (Power Rule, fractional exponent)

$$\text{Differentiate} \quad y = \sqrt[6]{x^5}.$$

$$y = x^{\frac{5}{6}}.$$

$$\text{Power Rule:} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{5x^{-\frac{1}{6}}}{6}$$

$$= \frac{5}{6\sqrt[6]{x}}.$$

Theorem (The Constant Multiple Rule)

If c is a constant and f is a differentiable function, then

$$\frac{\mathrm{d}}{\mathrm{d}x}[cf(x)] = c\frac{\mathrm{d}}{\mathrm{d}x}f(x).$$

Proof.

$$\begin{aligned}
\text{Let} \quad g(x) &= cf(x). \\
\text{Then} \quad g'(x) &= \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} \\
&= \lim_{h\to 0}\frac{cf(x+h)-cf(x)}{h} \\
&= \lim_{h\to 0}\frac{c(f(x+h)-f(x))}{h} \\
\text{Limit Law 3:} \quad &= c\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= cf'(x).
\end{aligned}$$

□

Example (Constant Multiple Rule, Power Rule)

Find the derivative of $y = \frac{2x^5}{7}$.

$$y = \left(\frac{2}{7}\right)\left(x^5\right).$$

$$\frac{dy}{dx} = \frac{d}{dx}\left[\left(\frac{2}{7}\right)\left(x^5\right)\right]$$

Constant Multiple Rule: $= \left(\frac{2}{7}\right)\frac{d}{dx}\left(x^5\right)$

$$= \left(\frac{2}{7}\right)\left(5x^4\right)$$

$$= \frac{10x^4}{7}.$$

Example (Constant Multiple Rule, Power Rule)

Find the derivative of $u = -x$.

$$u = (-1)(x).$$

$$\frac{du}{dx} = \frac{d}{dx}\left[(-1)(x)\right]$$

Constant Multiple Rule: $= (-1)\frac{d}{dx}(x)$

$$= (-1)(1)$$

$$= -1.$$

Example (Constant Multiple Rule, Power Rule, Negative Exponent)

Find the derivative of $t = \frac{2\pi}{x^4}$.

$$t = (2\pi)\left(x^{-4}\right).$$

$$\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left[(2\pi)\left(x^{-4}\right)\right]$$

Constant Multiple Rule: $= (2\pi)\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{-4}\right)$

$$= (2\pi)\left(-4x^{-5}\right)$$

$$= -\frac{8\pi}{x^5}.$$

Theorem (The Sum Rule)

If f and g are both differentiable, then

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(x)+g(x)] = \frac{\mathrm{d}}{\mathrm{d}x}f(x) + \frac{\mathrm{d}}{\mathrm{d}x}g(x).$$

Proof.

$$\begin{aligned}
\text{Let}\quad F(x) &= f(x)+g(x).\\
\text{Then}\quad F'(x) &= \lim_{h\to 0}\frac{F(x+h)-F(x)}{h}\\
&= \lim_{h\to 0}\frac{[f(x+h)+g(x+h)]-[f(x)+g(x)]}{h}\\
&= \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}+\frac{g(x+h)-g(x)}{h}\right]\\
\text{Limit Law 1:}\quad &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}+\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\\
&= f'(x)+g'(x).
\end{aligned}$$

$\square$

The Sum Rule can be extended to any number of summands. For instance, using the theorem twice, we get

$$(f+g+h)' = [(f+g)+h]' = (f+g)' + h' = f' + g' + h'.$$

By writing $f - g$ as $f + (-1)g$ and applying the Sum Rule and the Constant Multiple Rule, we get

Theorem (The Difference Rule)

If f and g are both differentiable, then

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(x) - g(x)] = \frac{\mathrm{d}}{\mathrm{d}x}f(x) - \frac{\mathrm{d}}{\mathrm{d}x}g(x).$$

The Constant Multiple Rule, the Sum Rule, the Difference Rule, and the Power Rule can be combined to differentiate any polynomial.

Example (Derivative of a Polynomial)

$$
\begin{aligned}
\text{If}\quad y &= x^{16} + 2\sqrt{3}x^7 - 4x^3 + \frac{x}{8} - 5,\\
\text{Then}\quad \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(x^{16} + 2\sqrt{3}x^7 - 4x^3 + \frac{x}{8} - 5\right)\\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(x^{16}\right) + \frac{\mathrm{d}}{\mathrm{d}x}\left(2\sqrt{3}x^7\right) - \frac{\mathrm{d}}{\mathrm{d}x}\left(4x^3\right) + \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x}{8}\right) - \frac{\mathrm{d}}{\mathrm{d}x}(5)\\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(x^{16}\right) + 2\sqrt{3}\frac{\mathrm{d}}{\mathrm{d}x}\left(x^7\right) - 4\frac{\mathrm{d}}{\mathrm{d}x}\left(x^3\right) + \frac{1}{8}\frac{\mathrm{d}}{\mathrm{d}x}(x) - \frac{\mathrm{d}}{\mathrm{d}x}(5)\\
&= (16x^{15}) + 2\sqrt{3}\left(7x^6\right) - 4\left(3x^2\right) + \frac{1}{8}(1) - (0)\\
&= 16x^{15} + 14\sqrt{3}x^6 - 12x^2 + \frac{1}{8}.
\end{aligned}
$$

Example (Difference Rule, Negative Fractional Exponents)

Differentiate $v = \frac{3\sqrt{x} - \sqrt[3]{x}}{x}$.

$$v = 3\frac{\sqrt{x}}{x} - \frac{\sqrt[3]{x}}{x}$$

$$v = 3x^{-\frac{1}{2}} - x^{-\frac{2}{3}}.$$

Difference Rule: $\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(3x^{-\frac{1}{2}}\right) - \frac{\mathrm{d}}{\mathrm{d}x}\left(x^{-\frac{2}{3}}\right)$

Constant Multiple Rule: $= 3\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{-\frac{1}{2}}\right) - \frac{\mathrm{d}}{\mathrm{d}x}\left(x^{-\frac{2}{3}}\right)$

Power Rule: $= 3\left(-\frac{1}{2}x^{-\frac{3}{2}}\right) - \left(-\frac{2}{3}x^{-\frac{5}{3}}\right)$

$$= \frac{2}{3}x^{-\frac{5}{3}} - \frac{3}{2}x^{-\frac{3}{2}}.$$

Derivatives of Exponential Functions

Compute the derivative of $f(x) = a^x$ using the definition:

$$\begin{aligned}
f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h\to 0} \frac{a^{x+h} - a^x}{h} \\
&= \lim_{h\to 0} \frac{a^x a^h - a^x}{h} \\
&= \lim_{h\to 0} \frac{a^x \left(a^h - 1\right)}{h} \\
&= a^x \lim_{h\to 0} \frac{a^h - 1}{h} \\
&= a^x \lim_{h\to 0} \frac{a^h - a^0}{h} \\
&= a^x f'(0).
\end{aligned}$$

We have shown that, if $f(x) = a^x$ is differentiable at 0, then it is differentiable everywhere, and

$$f'(x) = f'(0)a^x.$$

We leave the following theorem without proof.

Theorem

Let a be a positive number and let $f(x) = a^x$. Then the limit

$$f'(0) = \lim_{h\to 0} \frac{a^h - 1}{h}$$

exists.

We will later show that

$$f'(0) = \lim_{h\to 0} \frac{a^h - 1}{h} = \ln(a).$$

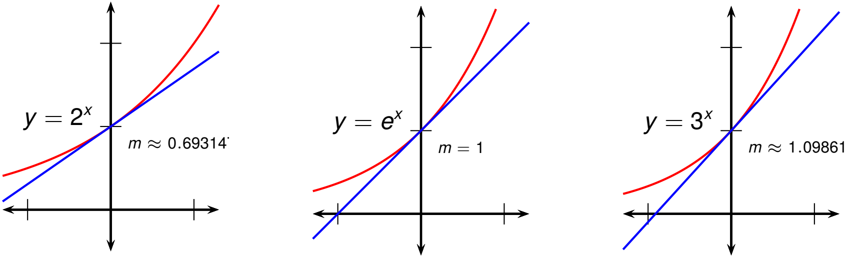
Here, $\ln$ is the natural logarithm function.

$$\text{If} \quad f(x) = a^x, \quad \text{then} \quad f'(x) = f'(0)a^x.$$

The formula above is simplest when $f'(0) = 1$. Since $\lim_{h\to 0} \frac{2^h - 1}{h} \approx 0.69$ and $\lim_{h\to 0} \frac{3^h - 1}{h} \approx 1.10$, we expect there is a number a between 2 and 3 such that $\lim_{h\to 0} \frac{a^h - 1}{h} = 1$.

Definition (e)

e is the number such that $\lim_{h\to 0} \frac{e^h - 1}{h} = 1$.



Definition (Natural Exponential Function)

e^x is called the natural exponential function. Its derivative is

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(e^x\right) = e^x.$$

Example (Derivative of a Polynomial and the Natural Exponential Function)

$$\text{Differentiate} \quad y = e^x + x^7.$$

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}(e^x) + \frac{\mathrm{d}}{\mathrm{d}x}(x^7) \\ &= e^x + 7x^6. \end{aligned}$$

We need a formula for the derivative of the product of two functions. One might guess that the derivative of a product is the product of the derivatives; however, this is wrong.

Example (Not the Product Rule)

Let $f(x) = x$ and $g(x) = x^2$.

$$f'(x) = 1. \qquad (fg)(x) = f(x)g(x) = x^3.$$

$$g'(x) = 2x. \qquad (fg)'(x) = 3x^2.$$

$$f'(x)g'(x) = 2x.$$

Therefore

$$f'(x)g'(x) \neq (fg)'(x) \quad .$$

The correct formula is called the Product Rule.

Theorem (The Product Rule)

If f and g are both differentiable, then

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x).$$

Proof.

Let $F(x) = f(x)g(x)$. Then

$$\begin{aligned}
F'(x) &= \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \left(f(x+h)\frac{g(x+h) - g(x)}{h} + g(x)\frac{f(x+h) - f(x)}{h} \right) \\
&= \lim_{h\to 0} f(x+h) \cdot \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\
&\quad + \lim_{h\to 0} g(x) \cdot \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = f(x)g'(x) + g(x)f'(x). \quad \square
\end{aligned}$$

Example (Product Rule, polynomial times the Natural Exponential Function)

$$\text{Differentiate} \quad f(x) = x^3 e^x.$$

$$\begin{aligned} \text{Product Rule:} \quad f'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(x^3\right)(e^x) + \left(x^3\right)\frac{\mathrm{d}}{\mathrm{d}x}(e^x) \\ &= \left(3x^2\right)(e^x) + \left(x^3\right)(e^x) \\ &= e^x\left(x^3 + 3x^2\right). \end{aligned}$$

Theorem (The Quotient Rule)

If f and g are differentiable and $g(x) \neq 0$, then

$$
\begin{array}{rcll}
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f(x)}{g(x)}\right) &=& \frac{\frac{\mathrm{d}}{\mathrm{d}x}(f(x))\,g(x) - f(x)\frac{\mathrm{d}}{\mathrm{d}x}(g(x))}{(g(x))^2} & \text{(Leibniz notation)} \\
\left(\frac{f(x)}{g(x)}\right)' &=& \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2} & {}'\ \text{notation} \\
\left(\frac{f}{g}\right)' &=& \frac{f'g - fg'}{g^2} & \text{abbreviated}
\end{array}
$$

- The proof of the Quotient Rule is similar to the proof of the Product Rule.
- There is an alternative algebraic proof via the Product Rule, the Power Rule and the (not yet studied) Chain Rule.

Example (Quotient Rule, rational function)

Differentiate $y = \dfrac{x^5 + 2x}{-x^6 + 2}$.

Quotient Rule:

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\frac{\mathrm{d}}{\mathrm{d}x}\left(x^5 + 2x\right)\left(-x^6 + 2\right) - \left(x^5 + 2x\right)\frac{\mathrm{d}}{\mathrm{d}x}\left(-x^6 + 2\right)}{\left(-x^6 + 2\right)^2} \\
&= \frac{\left(5x^4 + 2\right)\left(-x^6 + 2\right) - \left(x^5 + 2x\right)\left(-6x^5\right)}{\left(-x^6 + 2\right)^2} \\
&= \frac{(-5x^{10} - 2x^6 + 10x^4 + 4) - (-6x^{10} - 12x^6)}{(-x^6 + 2)^2} \\
&= \frac{x^{10} + 10x^6 + 10x^4 + 4}{(-x^6 + 2)^2}.
\end{aligned}$$

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

Example

Compute the derivative. Use the quotient rule.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2x-1}\right) &= \frac{(1)'(2x-1) - 1\cdot(2x-1)'}{(2x-1)^2} && \Big| \text{ Product rule}\\ &= \frac{0\cdot(2x-1) - 2}{(2x-1)^2}\\ &= \frac{-2}{(2x-1)^2}\end{aligned}$$