Calculus II
Simplify $\sin(k \arcsin x)$, $\cos(k \arcsin x)$, $\sin(k \arccos x)$, $\cos(k \arccos x)$

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Example

Rewrite $\sin(2\arccos(x))$ as an algebraic expression of x and $\sqrt{1-x^2}$. To simplify $\arccos x$ we try to use $\cos(\arccos x) = x$. Therefore our aim is to rewrite the expression only using the $\cos$ function.

$$
\begin{aligned}
\sin(2\arccos(x)) &= \sin(2y) &&\text{Set } y = \arccos x \\
&= 2\cos y \sin y &&\text{Express via } \sin y, \cos y \\
&= 2\cos y \left(\pm\sqrt{1-\cos^2 y}\right) &&\text{Express } \sin y \text{ via } \cos y \\
&= 2\cos y \sqrt{1-\cos^2 y} &&\text{$\sin y > 0$ because} \\
& &&0 \le y \le \pi \\
&= 2x\sqrt{1-x^2} &&\text{use } x = \cos y
\end{aligned}
$$

Example

Rewrite $\cos(3 \arccos(x))$ as an algebraic expression of x and $\sqrt{1 - x^2}$.
To simplify $\arccos x$ we try to use $\cos(\arccos x) = x$. Therefore our aim
is to rewrite the expression only using the $\cos$ function.

$$
\begin{aligned}
\cos(3 \arccos(x)) &= \cos(3y) = \cos(2y + y) & & y = \arccos x \\
&= \cos(2y) \cos y - \sin(2y) \sin y & & \text{Angle sum f-la} \\
&= (\cos^2 y - \sin^2 y) \cos y & & \text{Express via} \\
& \quad - 2 \sin y \cos y \sin y & & \sin y, \cos y \\
&= \cos^3 y - \sin^2 y \cos y - 2 \sin^2 y \cos y & & \\
&= \cos^3 y - 3 \sin^2 y \cos y & & \text{Express } \sin y \\
& & & \text{via } \cos y \\
&= \cos^3 y - 3(1 - \cos^2 y) \cos y & & \\
&= 4\cos^3 y - 3 \cos y & & \\
&= 4x^3 - 3x & & x = \cos y
\end{aligned}
$$