Calculus I
Homework
Maxima and minima over closed intervals

1. Find the critical points of the function. Identify whether those are local maxima, minima, or neither. The answer key has not been proofread, use with caution.

   (a) $f(x)=\dfrac{x}{1+x^2}$.

   answer: $x=1$, local & global max, $x=-1$, local and global min

   (b) $f(x)=x^3-x^2-x-1$.

   answer: $x=-\frac{1}{3}$, local max, $x=1$, local min

   (c) $f(x)=2x^3-x^2-20x+1$.

   answer: $x=-\frac{5}{3}$, local max $x=2$, local min

   (d) $f(x)=x+\dfrac{1}{x}$.

   answer: $x=-1$, local max, $x=1$, local min

   (e) $f(x)=\dfrac{x-\frac{1}{2}}{x^2-2x+\frac{7}{4}}$.

   answer: $x=-\frac{1}{2}$, local and global min, $x=\frac{3}{2}$, local and global max

2. Find the maximum and minimum values of f on the given interval and the values of x for which they are attained.

   (a) $f(x)=9+3x-x^2$, $x\in[0,4]$.

   answer: $f_{max}=f\left(\frac{3}{2}\right)=\frac{45}{4}$, $f_{min}=f(4)=5$

   (b) $f(x)=5+4x-2x^3$, $x\in[-1,1]$.

   answer: $f_{max}=f\left(\frac{\sqrt{6}}{3}\right)=\frac{8}{9}\sqrt{6}+5$, $f_{min}=f\left(-\frac{\sqrt{6}}{3}\right)=5-\frac{8}{9}\sqrt{6}$

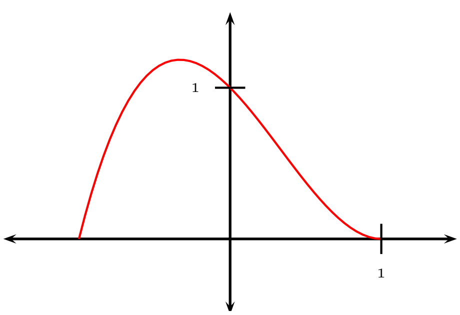
   (c) $f(x)=2x^3-x^2-20x+1$, $x\in[-4,3]$.

   answer: $f_{max}=f\left(-\frac{5}{3}\right)=\frac{602}{27}$, $f_{min}=f(-4)=-63$

   (d) $f(x)=3x^4-4x^3-12x^2+1$, $x\in[-2,3]$.

   answer: $f_{max}=f(-2)=33$, $f_{min}=f(2)=-31$

   (e) $f(x)=x^3-x^2-x+1$, $x\in[-1,1]$.



   (f) $f(x)=x^3-x+1$, $x\in[-2,1]$.

   answer: $f_{max}=f\left(-\frac{\sqrt{3}}{3}\right)=\frac{2}{9}\sqrt{3}+1$, $f_{min}=f(-2)=-5$

   (g) $f(x)=(x^2-1)^3$, $x\in[-1,2]$.

   answer: $f_{max}=f(2)=27$, $f_{min}=f(0)=-1$

   (h) $f(x)=x+\dfrac{1}{x}$, $x\in[0.2,4]$.

   answer: $f_{max}=f(0.2)=\frac{26}{5}=5.2$, $f_{min}=f(1)=2$

   (i) $f(x)=\dfrac{x}{x^2-x+1}$, $x\in[0,3]$.

   answer: $f_{max}=f(1)=1$, $f_{min}=f(-1)=-\frac{1}{3}$

   (j) $f(t)=t\sqrt{4-t^2}$, $x\in[-1,2]$.

   answer: $f_{max}=f\left(\sqrt{2}\right)=2$, $f_{min}=f\left(-\sqrt{2}\right)=-2$

   (k) $f(t)=\sqrt[3]{t}(8-t)$, $x\in[0,8]$.

   answer: $f_{max}=f(2)=6\sqrt[3]{2}$, $f_{min}=f(0)=f(8)=0$

   (l) $f(t)=2\cos t+\sin(2t)$, $x\in[0,\frac{\pi}{2}]$.

   answer: $f_{max}=f\left(\frac{\pi}{6}\right)=\frac{3}{2}\sqrt{3}$, $f_{min}=f\left(\frac{\pi}{2}\right)=0$

   (m) $f(t)=t+\cot\left(\dfrac{t}{2}\right)$, $x\in[\frac{\pi}{4},\frac{7\pi}{4}]$.

   answer: $f_{max}=f\left(\frac{3\pi}{2}\right)=\frac{3\pi}{2}-1$, $f_{min}=f\left(\frac{\pi}{2}\right)=\frac{\pi}{2}+1$

   (n) $f(t)=t+\cot\left(\dfrac{t}{2}\right)$, $x\in[\frac{\pi}{4},\frac{7\pi}{4}]$.

   (o) $f(x)=xe^{3x}$, $x\in\left[-3,\frac{1}{6}\right]$.

   answer: $f_{max}=f\left(\frac{1}{6}\right)=\frac{e^{\frac{1}{2}}}{6}\approx 0.274787$, $f_{min}=f\left(-\frac{1}{3}\right)=-\frac{1}{3e}\approx -0.122626$

   (p) $f(x)=(x-2)(x+1)e^x$, $x\in[-5,2]$.

   answer: $f_{min}=f\left(\frac{\sqrt{13}}{2}-\frac{1}{2}\right)=\left(-\sqrt{13}+2\right)e^{\left(\frac{\sqrt{13}}{2}-\frac{1}{2}\right)}\approx -5.907619$
   $f_{max}=f\left(-\frac{\sqrt{13}}{2}-\frac{1}{2}\right)=\left(\sqrt{13}+2\right)e^{\left(-\frac{\sqrt{13}}{2}-\frac{1}{2}\right)}\approx 0.560448$

   (q) $f(x)=(x+1)e^{-x^2}$, $x\in[-3,3]$.

   answer: $f_{min}=f\left(-\frac{\sqrt{3}}{2}-\frac{1}{2}\right)=\left(-\frac{\sqrt{3}}{2}+\frac{1}{2}\right)e^{-\frac{\sqrt{3}}{2}-1}\approx -0.056638$
   $f_{max}=f\left(\frac{\sqrt{3}}{2}-\frac{1}{2}\right)=\left(\frac{\sqrt{3}}{2}+\frac{1}{2}\right)e^{\frac{\sqrt{3}}{2}-1}\approx 1.194743$

   (r) $f(x)=xe^{2x}$, $x\in\left[-2,\frac{1}{2}\right]$.

   answer: $f_{max}=f\left(\frac{1}{2}\right)=\frac{e}{2}$, $f_{min}=f\left(-\frac{1}{2}\right)=-\frac{1}{2e}$

Solution. 2.e

f is differentiable so its critical points are given by

$$\begin{aligned} f'(x) &= 0 \\ 3x^2 - 2x - 1 &= 0 \\ (x-1)(3x+1) &= 0 \\ x = 1 \quad \text{or} \quad & x = -\frac{1}{3}. \end{aligned}$$

f attains its maximum at minimum at either a critical point or at the end points $-1, 1$ of the interval. From the function plot we see that f attains its minimum at $x=-1$ and $x=1$ and its maximum at $x=-\frac{1}{3}$. Alternatively the minima and maxima follow from the table below.

x	$f(x)$	
-1	0	(minimum)
$-\frac{1}{3}$	$\frac{32}{27}$	(maximum)
1	0	(minimum)

Solution. 2.f By the closed interval method the maximuma/minima over $[-2,1]$ are obtained either at the endpoints or at the critical points of $f(x)$. Since $f'(x)$ is defined over the entire interval, the only critical points are the ones for which $f'(x)=0$.

$$\begin{aligned} f'(x) &= 0 \\ 3x^2 - 1 &= 0 \\ x^2 &= \frac{1}{3} \\ x &= \pm\sqrt{\frac{1}{3}} = \pm\frac{\sqrt{3}}{3} \end{aligned}$$

The maximum and minimum of f over $[-2,1]$ is attained at one of the points $x=-2, -\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, 1$.

x	$f(x)$
-2	$f(-2) = (-2)^3 - (-2) + 1 = -5$
$-\frac{\sqrt{3}}{3}$	$f\left(-\frac{\sqrt{3}}{3}\right) = \left(-\frac{\sqrt{3}}{3}\right)^3 - \left(-\frac{\sqrt{3}}{3}\right) + 1 = \frac{2}{9}\sqrt{3} + 1$
$\frac{\sqrt{3}}{3}$	$f\left(\frac{\sqrt{3}}{3}\right) = \left(\frac{\sqrt{3}}{3}\right)^3 - \left(\frac{\sqrt{3}}{3}\right) + 1 = -\frac{2}{9}\sqrt{3} + 1$
1	$f(1) = (1)^3 - (1) + 1 = 1$

Observation of the table above shows that the maximum of f over $[-2,1]$ is $f\left(-\frac{\sqrt{3}}{3}\right) = \frac{2}{9}\sqrt{3}+1$ attained for $x=-\frac{\sqrt{3}}{3}$ and the minimum is $f(-2)=-5$ (attained for $x=-2$).

Solution. 2.r

By the closed interval method the maximuma/minima over $[-2,\frac{1}{2}]$ are obtained either at the endpoints or at the critical points of $f(x)$. Since $f'(x)$ is defined over the entire interval, the only critical points are the ones for which $f'(x)=0$.

$$\begin{aligned} f'(x) &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(xe^{2x}\right) &= 0 \\ 2xe^{2x} + e^{2x} &= 0 \\ e^{2x}(2x+1) &= 0 \qquad \left|\ e^{2x} \neq 0\right. \\ 2x+1 &= 0 \\ x &= -\frac{1}{2} \end{aligned}$$

The maximum and minimum of f over $\left[-2,\frac{1}{2}\right]$ is attained at one of the points $x=-2,-\frac{1}{2},\frac{1}{2}$.

x	$f(x)$
-2	$f(-2) = -2e^{-4}$
$-\frac{1}{2}$	$f\left(-\frac{1}{2}\right) = -\frac{e^{-1}}{2}$
$\frac{1}{2}$	$f\left(\frac{1}{2}\right) = \frac{e}{2}$

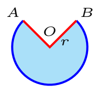
We have $2e^{-4} = \frac{e^{-1}}{2}\left(4e^{-3}\right) < \frac{e^{-1}}{2}$ and therefore $-\frac{e^{-1}}{2} < -2e^{-4}$. Therefore the maximum of f over $\left[-2,\frac{1}{2}\right]$ is $f\left(\frac{1}{2}\right) = \frac{e}{2}$ (attained for $x=\frac{1}{2}$) and the minimum is $f\left(-\frac{1}{2}\right) = -\frac{e^{-1}}{2} = -\frac{1}{2e}$ (attained for $x=-\frac{1}{2}$).

3. (a) Find the dimensions of a rectangle with area 1000 m^2 whose perimeter is as small as possible.

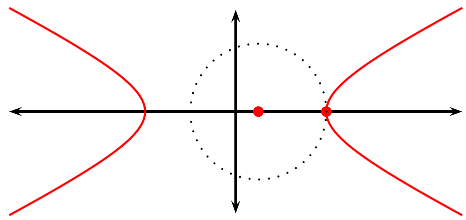
(b) A box with an open top is to be constructed from a square piece of cardboard, 1m wide, by cutting out a square from each of the four corners and bending up the sides. Find the largest volume that such a box can have.

(c) A right circular cylinder is inscribed in a sphere of radius r. Find the largest possible volume of such a cylinder.

(d) A wedge of radius 2 (depicted below) is folded into a cone cup. The volume varies depending on the angle of the wedge. Find the maximal possible volume of the cone cup and the angle of the wedge for which this maximal volume is achieved.

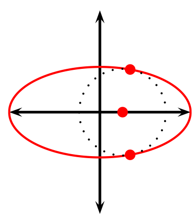


4. (a) What is the x-coordinate of the point on the hyperbola $x^2 - 4y^2 = 16$ that is closest to the point $(1, 0)$?



answer: $x = 4$

(b) What is the x-coordinate of the point on the ellipse $x^2 + 4y^2 = 16$ closest to the point $(1, 0)$?



answer: $x = \frac{4}{3}$

(c) A rectangular box with a square base is being built out of sheet metal. 2 pieces of sheet will be used for the bottom of the box, and a single piece of sheet metal for the 4 sides and the top of the box. What is the largest possible volume of the resulting box that can be obtained with $36m^2$ of metal sheet?

answer: 12 cubic meters.

(d) Recall that the volume of a cylinder is computed as the product of the area of its base by its height. Recall also that the surface area of the wall of a cylinder is given by multiplying the perimeter of the base by the height of the cylinder.

A cylindrical container with an open top is being built from metal sheet. The total surface area of metal used must equal $10m^2$. Let r denote the radius of the base of the cylinder, and h its height. How should one choose h and r so as to get the maximal possible container volume? What will the resulting container volume be?

Solution. 4.a

The distance function between an arbitrary point (x, y) and the point $(1, 0)$ is $d = \sqrt{(x-1)^2 + (y-0)^2}$. On the other hand, when the point (x, y) lies on the hyperbola we have $y^2 = \frac{x^2-16}{4}$. In this way, the problem becomes that of minimizing the distance function

$$dist(x) = \sqrt{(x-1)^2 + y^2} = \sqrt{(x-1)^2 + \frac{x^2-16}{4}} \quad .$$

This is a standard optimization problem: we need to find the critical endpoints, i.e., the points where $dist' = 0$. As the square root function is an increasing function, the function $\sqrt{(x-1)^2 + \frac{x^2-16}{4}}$ achieves its minimum when the function

$$l = dist^2 = (x-1)^2 + \frac{x^2-16}{4}$$

does. l is a quadratic function of x and we can directly determine its minimimum via elementary methods. Alternatively, we find the critical points of l:

$$\begin{aligned} l' &= 0 \\ 2(x-1) + \frac{x}{2} &= 0 \\ \frac{5}{2}x - 2 &= 0 \\ x &= \frac{4}{5} \quad . \end{aligned}$$

On the other hand, $x^2 = 16 + 4y^2$ and therefore $|x| \geq \sqrt{16} = 4$. Therefore $x \in (-\infty, -4] \cup [4, \infty)$. As $x = \frac{4}{5}$ is outside of the allowed range, it follows that our function either attains its minimum at one of the endpoints ± 4 or the function has no minimum at all. It is clear however that as x tends to ∞, so does $dist$. Therefore $dist$ attains its minimum for $x = 4$ or -4 and $y = \pm\sqrt{(\pm 4)^2 - 16} = 0$. Direct check shows that $dist_{|x=4} = \sqrt{(4-1)^2 + \frac{4^2-16}{4}} = 3$ and $dist_{|x=-4} = \sqrt{(-4-1)^2 + \frac{4^2-16}{4}} = 5$ so our function $dist$ has a minimal value of 3 achieved when $x = 4$, which is our final answer. Notice that this answer can be immediately given without computation by looking at the figure drawn for 4.a. Indeed, it is clear that there are no points from the hyperbola lying inside the dotted circle centered at $(1, 0)$. Therefore the point where this circle touches the hyperbola must have the shortest distance to the center of the circle.

Solution. 4.c Let B denote the area of the base of the box, equal to the area of the top. Let W denote the area of the four walls of the box (the four walls are all equal because the base of the box is a square). Then the surface area S of material used will be

$$S = \underbrace{2B}_{\text{two pieces for the bottom}} + \underbrace{4W}_{\text{4 walls}} + \underbrace{B}_{\text{the box lid}} = 3B + 4W \quad .$$

Let x denote the length of the side of the square base and let y denote the height of the box. Then

$$B = x^2$$

and

$$W = xy \quad .$$

As the surface area S is fixed to be 36 square meters, we have that

$$S = 3B + 4W = 36 = 3x^2 + 4xy \quad .$$

As y is positive, the above formula shows that $3x^2 \leq 36$ and so $x \leq \sqrt{12}$. Let us now express y in terms of x:

$$\begin{aligned} 3x^2 + 4xy &= 36 \\ 4xy &= 36 - 3x^2 \\ y &= \frac{36 - 3x^2}{4x} \quad . \end{aligned}$$

The problem asks us to maximize the volume V of the box. The volume of the box equals the area of the base times the height of the box:

$$V = B \cdot y = yx^2 = \frac{(36 - 3x^2)}{4x}x^2 = \frac{36x - 3x^3}{4} \quad .$$

As x is non-negative, it follows that the domain for x is:

$$x \in [0, \sqrt{12}] \quad .$$

To maximize the volume we find the critical points, i.e., the values of x for which V' vanishes:

$$\begin{aligned} 0 &= V' = \left(\frac{36x - 3x^3}{4}\right)' \\ 0 &= \frac{36 - 9x^2}{4} \\ 9x^2 &= 36 \\ x^2 &= 4 \\ x &= \pm 2 \end{aligned}$$

As x measures length, $x = -2$ is not possible (outside of the domain for x). Therefore the only critical point is $x = 2$. Direct check shows that at the endpoints $x = 0$ and $x = \sqrt{12}$, we have that $V = 0$. Therefore the maximal volume is achieved when $x = 2$:

$$V_{max} = V_{|x=2} = \frac{36(2) - 3(2)^3}{4} = 12 \quad .$$