Calculus I
Review of basic functions

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Outline

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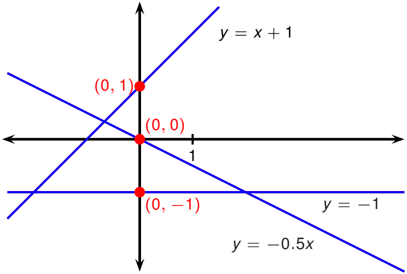
Linear Functions

Definition (Linear Function)

A linear function is a function the graph of which is a line. We can write any linear function in slope-intercept form:

$$f(x) = mx + b.$$

m is called the slope, and b is called the y-intercept.



$f(x)$	Direction	y-intercept
$x + 1$	$\nearrow$	1
$-0.5x + 0$	$\searrow$	0
-1	$\rightarrow$	-1

- $m > 0$ means the graph of f points up ($\nearrow$).
- $m < 0$ means the graph of f points down ($\searrow$).
- $m = 0$ means the graph of f is horizontal ($\rightarrow$).
- b tells us the height of the point where the graph hits the y-axis.

Polynomials

Definition (Polynomial Function)

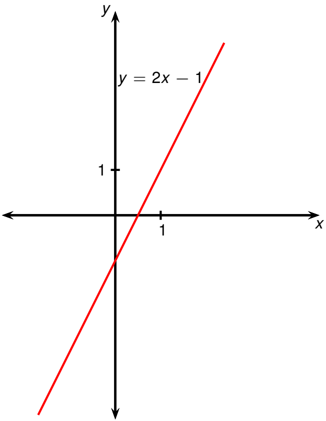
A polynomial function is a function f of the form

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1} + a_n x^n,$$

where n is a non-negative integer and $a_0, \ldots, a_n$ are real numbers, called the coefficients. If $a_n \neq 0$ the integer n is called the degree of f.

$f(x)$	Polynomial?	Degree	a_0	a_1	a_2
$x^4 - x + 1$	Yes	4	1	-1	0
6	Yes	0	6	0	0
$3x^2 - \frac{1}{2}x + \sqrt{x}$	No				
$3x^2 - \frac{1}{2}x + \sqrt{2}$	Yes	2	$\sqrt{2}$	$-\frac{1}{2}$	3
$3x^2 - \frac{1}{2x} + \sqrt{2}$	No				

- Linear functions are polynomial (functions).
- So are quadratic functions. Their graphs are parabolas.
- And there are many more.



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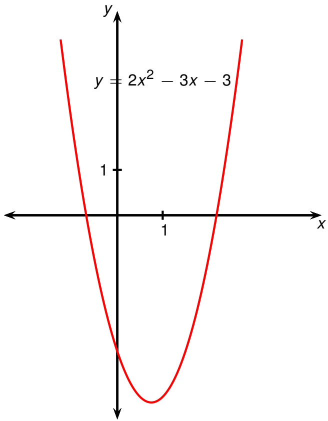

Quadratic

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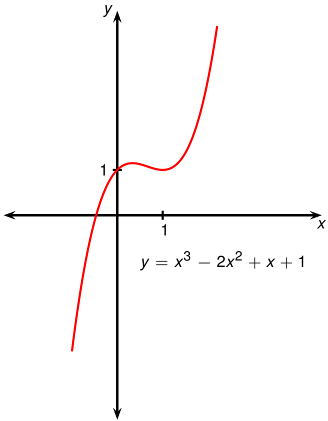

Cubic

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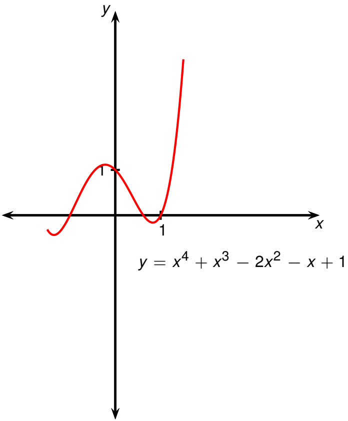

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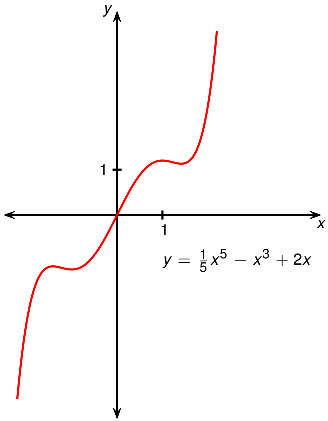


Quintic

Power Functions

Definition (Power Function)

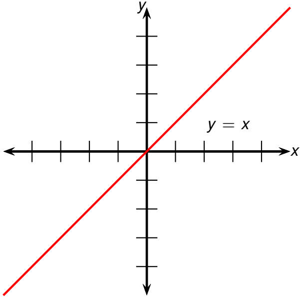
Let $x > 0$, a - arbitrary real number. The power function is defined as

$$f(x) = e^{a \ln x} = x^a \quad .$$

x = base. a = exponent or power. First equality = one of ways to define for non-integer a (we study $\ln x$, e^x later).

If a - positive integer $(1, 2, 3, \ldots)$ then x^a = polynomial function.
$x^n = \underbrace{x \ldots x}_{n \text{ times}}$ when n-integer.

$$\begin{aligned} (x^a)^b &= x^{ab} \\ (xy)^b &= x^b y^b \\ x^{a+b} &= x^a x^b \\ x^{-a} &= \frac{1}{x^a} \end{aligned}$$



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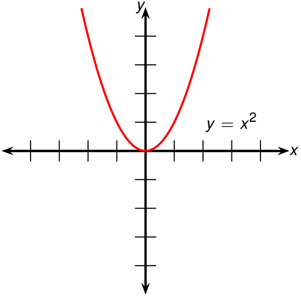

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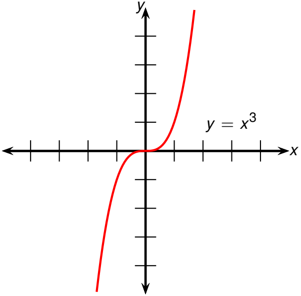

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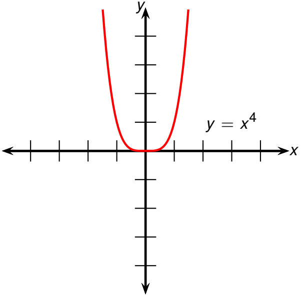

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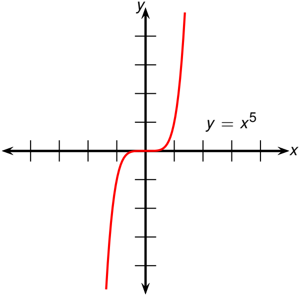
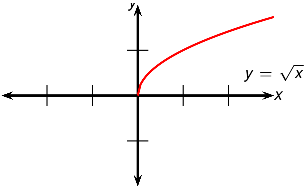
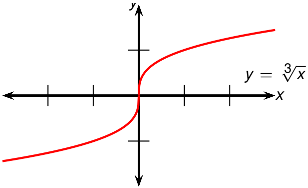


- n - positive integer, $f(x) = x^{\frac{1}{n}} = \sqrt[n]{x}$ = the n^{th} root function. $\sqrt[n]{x} \geq 0$ for $x \geq 0$.
- For $n = 2$, we get the square root $\sqrt{x}$; for $n = 3$ we get the cube root $\sqrt[3]{x}$, and so on.
- Let $x > 0$. For $n = 2m + 1$-odd, we can extend the definition of n^{th} root to negative numbers by $\sqrt[2m+1]{-x} := -\sqrt[2m+1]{x}$.
- In this course, even roots of negative numbers are not defined.
- The graph of $\sqrt{x}$ is the top half of the parabola $x = y^2$. Similarly for $y = \sqrt[2m]{x}$, we graph top of $x = y^{2m}$.
- The graph of the cube root $f(x) = \sqrt[3]{x}$ is the graph of the polynomial $x = y^3$. Similarly for $y = \sqrt[2m+1]{x}$, we graph $x = y^{2m+1}$.





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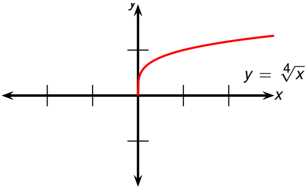
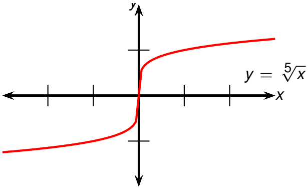
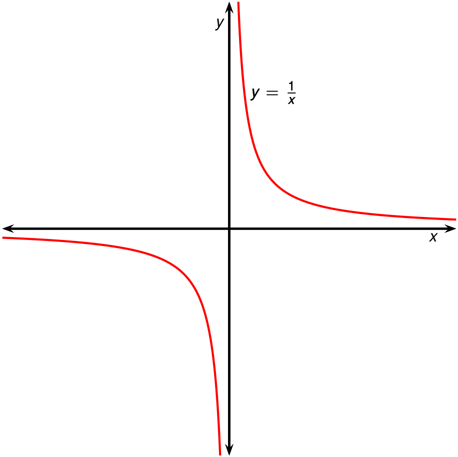




$f(x) = x^{-1} = \frac{1}{x}$ is called the reciprocal function. Its graph has equation $y = \frac{1}{x}$, or $xy = 1$, and is an hyperbola with the coordinate axes as its



asymptotes.

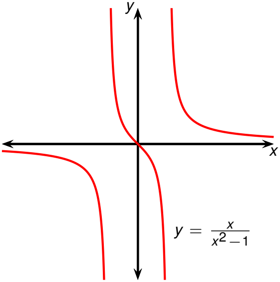
Rational Functions

Definition (Rational Function)

A rational function is a quotient of two polynomials; that is, a function of the form

$$f(x) = \frac{g(x)}{h(x)},$$

where g and h are polynomials.



Example ($x/(x^2 - 1)$)

The function

$$f(x) = \frac{x}{x^2 - 1}$$

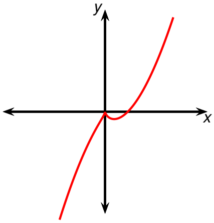
is a rational function.

Algebraic Functions

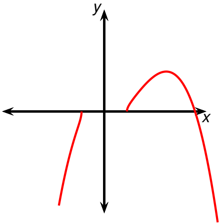
(Algebraic Function)

A function in x that can be constructed using x, constants, and finitely many of the operations $+, -, *, /$, and $\sqrt[n]{\ }$ is an algebraic function.
Outside of present course: function $f(x)$ = algebraic if it satisfies a polynomial equation with polynomial coefficients, i.e., $a_0(x) + a_1(x)f(x) + \cdots + a_n(x)\left(f(x)\right)^n = 0$ for some polynomials $a_i(x)$.

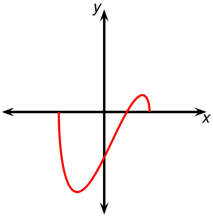
Examples.



$y = (x-1)\sqrt[3]{x^2}$



$y = \frac{1}{5}(4x - x^2)\sqrt{x^2-1}$



$y = (x-1)\sqrt{4-x^2}$

Transcendental Functions

Transcendental functions include many classes of functions.

- Trigonometric functions such as $\cos x, \sin x, \tan x$, etc.
- Exponential functions such as 2^x, $\left(\frac{1}{2}\right)^x$, 5^x, e^x, etc.
- The logarithm function $\ln x$.
- And many more.
- Outside of the present course: by definition, a function is transcendental if it is not algebraic, i.e., if it satisfies no polynomial equation with polynomial coefficients.

Combinations of Functions

Two functions f and g can be combined to form new functions $f+g$, $f-g$, $f\cdot g$, and $\frac{f}{g}$:

$$\begin{array}{rcl|l}
(f+g)(x) &=& f(x)+g(x) & \\
(f-g)(x) &=& f(x)-g(x) & \\
(f\cdot g)(x) &=& f(x)\cdot g(x) & \\
\left(\frac{f}{g}\right)(x) &=& \frac{f(x)}{g(x)} & \text{for } g(x)\neq 0 \quad .
\end{array}$$

Let $\mathrm{Dom}(f)$ denote the domain of f. The function $f+g$ is defined only if both f and g are defined, and similarly for the others. Therefore

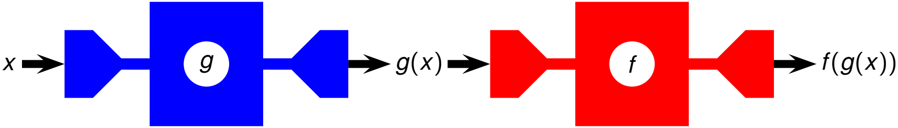
$$\begin{array}{rcl|l}
\mathrm{Dom}(f+g) &=& \mathrm{Dom}(f)\cap\mathrm{Dom}(g) & \cap \text{ stands for} \\
\mathrm{Dom}(f-g) &=& \mathrm{Dom}(f)\cap\mathrm{Dom}(g) & \text{set intersection} \\
\mathrm{Dom}(f\cdot g) &=& \mathrm{Dom}(f)\cap\mathrm{Dom}(g) & \\
\mathrm{Dom}\left(\frac{f}{g}\right) &=& \mathrm{Dom}(f)\cap\mathrm{Dom}(g)\cap\{x|g(x)\neq 0\} & \text{right expr. stands for set where } g(x)\neq 0
\end{array}$$

Definition (Composition of f and g)

If f and g are two functions, then the composition of f and g is written $f \circ g$ and is defined by the formula

$$(f \circ g)(x) = f(g(x)).$$

Imagine f and g as machines taking some input and producing some output. Then $f \circ g$ corresponds to attaching both machines end-to-end so that the output of g becomes the input of f.



The domain of $f \circ g$ is the set of all numbers x in the domain of g such that $g(x)$ is in the domain of f. If the domain of f is A and the domain of g is B, we write this as

$$\{x \in B |\ g(x) \in A\}.$$

Example

Find $f \circ g, g \circ f, g \circ g$ and their domains, where $f(x) = \sqrt{x}$ and $g(x) = \sqrt{3 - x}$.

$$(f \circ g)(x) \quad = \quad f(g(x)) = f\left(\sqrt{3-x}\right) = \sqrt{\sqrt{3-x}} = \sqrt[4]{3-x}$$

Domain:

$$\begin{aligned} 3 - x &\geq 0 \\ -x &\geq -3 \\ x &\leq 3 \\ x &\in (-\infty, 3]. \end{aligned}$$

$$(g \circ f)(x) \quad = \quad g(f(x)) = g(\sqrt{x}) = \sqrt{3 - \sqrt{x}}$$

Domain :

$$\begin{aligned} x &\geq 0 \\ 3 - \sqrt{x} &\geq 0 \\ -\sqrt{x} &\geq -3 \\ \sqrt{x} &\leq 3 \\ x &\leq 9 \\ x &\in [0, 9] \end{aligned}$$

Example

Find $f \circ g, g \circ f, g \circ g$ and their domains, where $f(x) = \sqrt{x}$ and $g(x) = \sqrt{3-x}$.

$$
\begin{array}{rcl}
(g \circ g)(x) & = & g(g(x)) = g\left(\sqrt{3-x}\right) = \sqrt{3-\sqrt{3-x}} \\
\text{Domain :} & & \\
3 - x & \geq & 0 \\
-x & \geq & -3 \\
x & \leq & 3 \\
3-\sqrt{3-x} & \geq & 0 \\
-\sqrt{3-x} & \geq & -3 \\
\sqrt{3-x} & \leq & 3 \\
3 - x & \leq & 9 \\
-x & \leq & 6 \\
x & \geq & -6 \\
x & \in & [-6, 3].
\end{array}
$$

Example

Give simplified f-las for $f \circ g$, $f \circ f$, $g \circ f$, $g \circ g$. Find the implied domains.

$$f(x) = \frac{2x-1}{x+2} \qquad \Big| \quad x \neq -2$$

$$g(x) = \frac{2x+3}{5x-7} \qquad \Big| \quad x \neq \frac{7}{5}$$

$$(f \circ g)(x) = f(g(x)) = f\left(\frac{2x+3}{5x-7}\right) = \frac{2\left(\frac{2x+3}{5x-7}\right)-1}{\frac{2x+3}{5x-7}+2}$$

$$= \frac{\frac{2(2x+3)}{5x-7} - \frac{5x-7}{5x-7}}{\frac{2x+3}{5x-7} + \frac{2(5x-7)}{5x-7}} = \frac{\frac{4x+6-(5x-7)}{\cancel{5x-7}}}{\frac{2x+3+(10x-14)}{\cancel{5x-7}}} = \frac{-x+13}{12x-11} \qquad \Big| \quad x \neq \frac{11}{12}, \frac{7}{5}$$

$$(f \circ f)(x) = f(f(x)) = f\left(\frac{2x-1}{x+2}\right) = \frac{2\left(\frac{2x-1}{x+2}\right)-1}{\frac{2x-1}{x+2}+2}$$

$$= \frac{3x-4}{4x+3} \qquad \Big| \quad x \neq -2, -\frac{3}{4}$$

$$(g \circ f)(x) = \frac{7x+4}{3x-19} \qquad \Big| \quad x \neq -2, \frac{19}{3}$$

$$(g \circ g)(x) = \frac{19x-15}{-25x+64} \qquad \Big| \quad x \neq \frac{7}{5}, \frac{64}{25}$$