Calculus II
Power series expansions related to exponents, part 2

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Example

Find the Taylor series for $f(x) = e^x$ at $a = 3$.

- $f^{(n)}(x) = e^x$.
- $f^{(n)}(3) = e^3$.
- Therefore the Taylor series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(3)}{n!}(x-3)^n = \sum_{n=0}^{\infty} \frac{e^3}{n!}(x-3)^n$$

- To find the radius of convergence, let $a_n = \frac{e^3}{n!}(x-3)^n$.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{e^3(x-3)^{n+1}}{(n+1)!} \cdot \frac{n!}{e^3(x-3)^n} \right| = \lim_{n\to\infty} \frac{|x-3|}{n+1} = 0$$

- Therefore by the Ratio Test the series converges for all x.
- Therefore $R = \infty$.
- Just like the Maclaurin series, this series also represents e^x.

Example

Find the Taylor series for $f(x) = e^x$ at $a = 3$.

$$
\begin{aligned}
e^x &= e^{x-3+3} = e^3 e^{x-3} \\
&= e^3 \sum_{n=0}^{\infty} \frac{(x-3)^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{e^3}{n!}(x-3)^n
\end{aligned}
$$

Recall that $e^y = \sum_{n=0}^{\infty} \frac{y^n}{n!}$

Set $y = x - 3$

The radius of convergence was already computed to be $R = \infty$.