Precalculus
Angles

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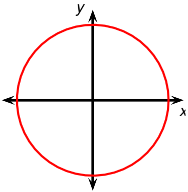
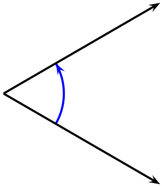
Outline

Definition

The *unit circle* is the circle with radius 1 and center at the center of the coordinate system.

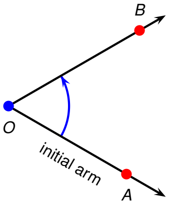


Three Meanings of the Term Angle



- The term "angle" is used to denote three distinct mathematical objects:
  - the (geometric) angle formed by two rays,
  - the angle-measure of such a geometric angle
  - the angle-measure of a rotation.
- All three are referred to as "angle": use context to decide whether "angle" means "angle formed by two rays", "angle measure" or "angle-measure of a rotation".
- Except for a few introductory slides, we take full advantage of this convention.

Geometric angle definition

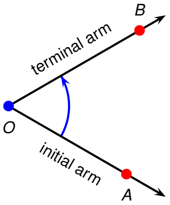


Definition (Geometric angle)

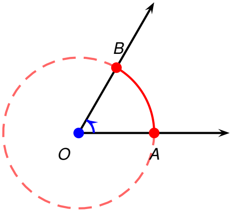
A *geometric angle* (*angle* for short) is the figure formed by two rays, called arms, sharing a common endpoint called the vertex of the angle. The rays are ordered.

- The ray that comes first is called the initial arm (side) of the angle.
- The ray that comes second is called the terminal arm (side) of the angle.
- Angle measures are depicted as arcs pointing from the initial arm towards the terminal arm.
- By convention, the rays are allowed to coincide; the resulting angle is then called the *zero angle*.

Geometric angle definition



- A ray can be identified by its starting point and any other point on the ray.
- Therefore an angle can be identified by its vertex and one point on each of its arms.
- If A is pt. on the first ray and B on the second and O is the vertex, we denote the angle by $\angle AOB$.
- The choice A and B is not unique - for example $\angle AOB$ and $\angle A'OB$ coincide.
- In $\angle AOB$ the ray OA is the initial arm and the ray OB is the terminal arm.
- In $\angle BOA$ the ray OB is the initial arm, the ray OA is the terminal arm, and the angle measure points in the opposite direction.
- In this way $\angle AOB \neq \angle BOA$.



Definition (Radian measure of geometric angle)

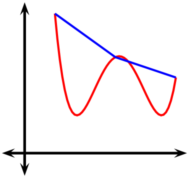
The measure of a geometric angle is a number determined as follows.

- Its magnitude is the length of the short arc cut off by the angle from a radius 1 circle centered at the vertex.
- Whenever traversing the arc from initial arm to terminal results in clockwise motion, take measure with negative sign, else with positive.

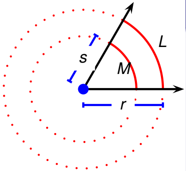
- The unit of this angle measure is called radians, denoted by rad.
- A circle of radius 1 has circumference 2π.
- Convention: half-turn angle is measured with π (rather than $-\pi$).
- Therefore a geometric angle is measured with a number between $(-\pi, \pi]$.
- Angle measures are frequently denoted by greek letters such as $\alpha, \beta, \gamma, \theta, \dots$.

Arc-length of a circle arc

- There is a definition of arc-length of arbitrary smooth curve.
- The definition states that the arc-length of a smooth curve is the limit of the lengths of ever finer straight line approximations.
- The details of how this is done require integrals and we postpone this for later/another course.
- Until then we ask the reader to think of arc-length of a curve as the quantity obtained by “aligning a rope along the curve” and measuring the “length of this rope”.



Arc-length of a circle arc



Proposition

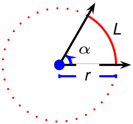
Let two circles have common center and radii s and r. Suppose an arbitrary geometric angle with vertex at the common center of the circles cuts off short arcs of length M and L. Then $\frac{s}{r} = \frac{M}{L}$.

$$\frac{s}{r} = \frac{M}{L} \quad \Big| \text{ Choose } s = 1, \text{ relabel } M = \alpha$$

$$\frac{1}{r} = \frac{\alpha}{L}$$

$$L = \alpha r$$

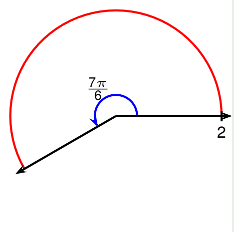
The angle-measure of a geometric angle is the arc-length cut off from a radius 1 circle, therefore we get the following.



Corollary

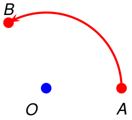
The arc-length cut off by an angle with measure α from a circle of radius r equals αr.

Example

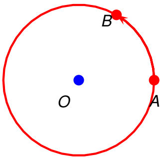


Find the length of an arc of a circle of radius 2 cut off by an angle of measure $\frac{7\pi}{6}$ $(= 210^\circ)$.

$$\text{arc-length} = \alpha r = \frac{7\pi}{6} \cdot 2 = \frac{7\pi}{3} \approx 7.33038 \text{ (units)}$$



- The term rotation refers to two distinct objects:
  - *continuous rotation* (*rotation* for short) - a gradual with respect to time transformation of space and
  - *rotation* - an instantaneous transformation of space. All points transition from their initial to their final positions instantaneously.
- In mathematics, the term rotation usually refers to “instantaneous” rotation.
- In physics, the term rotation usually refers to continuous rotation (time is explicitly parametrized).
- Whether the term rotation refers to continuous rotation or to “instantaneous” rotation should be inferred from context.

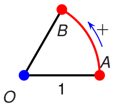


Definition (Continuous rotation)

A continuous rotation about a point (center of rotation), is a continuous motion of points for which:

- All points move in a circular fashion around the center of rotation.
- The distance between each rotated point and the center of rotation does not change.
- The distance between each pair of rotated points is preserved.

- The position of a point under a continuous rotation is assumed to be a function of time.
- The trajectory of a point is an arc of a circle.
- A point can traverse a full circle - more than once. In this case the moving point passes through the same positions more than once.



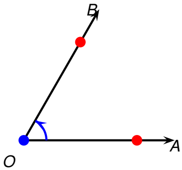
- We say that a continuous rotation is proper if points either move clockwise or counter-clockwise relative to the center, without "changing direction".

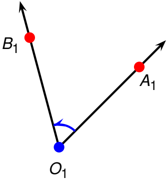
Definition (Radian measure of proper continuous rotation)

- The radian measure of rotation is a number whose magnitude equals the length of the arc traversed by a point divided by the distance of that point from the center of rotation.
- The sign of the radian measure is taken to be negative if the rotation is clockwise, else it is taken to be positive.

- The radian measure (radians for short) does not change when we change the point whose path length we are measuring.
- The radian measure of rotation equals the signed arc-length traveled by point at distance 1 form the center.
- A circle of radius 1 has circumference 2π, therefore a full counter-clockwise turn is measured by 2π radians.

Equivalence of angles





Definition (Congruent angles)

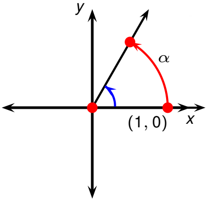
Two geometric angles are congruent (equivalent) if they one can be transformed onto the other with a sequence of translations and rotations.

Proposition

Two geometric angles are congruent if and only if they have equal angle measures.

- Recall “angle” refers to both geometric angle and angle measure (depending on context).
- The expression “the two angles are equal” is to be interpreted as “the angle measures are equal” and therefore “the geometric angles are congruent”.

Angles and the coordinate system



- Given an angle measure α between $(-\pi, \pi]$, there is a conventional way to select a geometric angle with that measure.
    - Select geometric angle's vertex to be the origin.
    - Select the initial arm of the angle on the *x*-axis, pointing in the positive direction.
    - Select the terminal arm by rotating the point $(1, 0)$ on the initial arm by $|\alpha|$ radians: go clockwise if $\alpha < 0$, counter-clockwise if $\alpha > 0$.
    - To rotate the point, move it along the circle with radius 1 for α units of arc-length.
- The construction also works for angle measures greater than π rad/smaller than $-\pi$ rad.
- In this way to every real α we can assign a geometric angle.
- If α is in the interval $(-\pi, \pi]$ the so obtained geometric angle does have measure α, else the measure of the geometric angle differs from α by an even multiple of π.

Degrees and radians

- Degrees is a unit for measuring angles, denoted by $^\circ$.
- The relationship between degrees and radians is:

$$\begin{aligned}\pi \text{ rad} &= 180^\circ\\ 1\text{rad} &= \frac{180^\circ}{\pi} \approx 57.3^\circ\\ 1^\circ &= \frac{\pi}{180}\text{ rad} \approx 0.017 \text{ rad}.\end{aligned}$$

- In other words, a half-turn is measured by πrad or 180°.
- Degrees are useful because the most frequently encountered fractions of a half turn are measured by a whole number of degrees.
- If a measurement unit is not specified, it is implied to be radians. For example, in $\sin 5$, the number 5 stands for 5 radians.

$$t^\circ = \frac{t}{180}\pi \text{ (radians)}.$$

Example

Convert from degrees to radians.

Deg.	45°	36°	-20°	360°	-720°	-225°	2015°
Rad.	$\frac{\pi}{4}$	$\frac{\pi}{5}$	$-\frac{\pi}{9}$	2π	-4π	$-\frac{5\pi}{4}$	$\frac{403}{36}\pi$

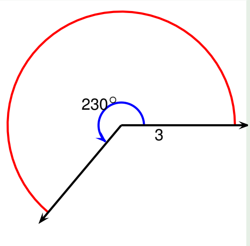
$$x = \frac{x}{\pi}180^\circ.$$

Example

Convert from radians to degrees.

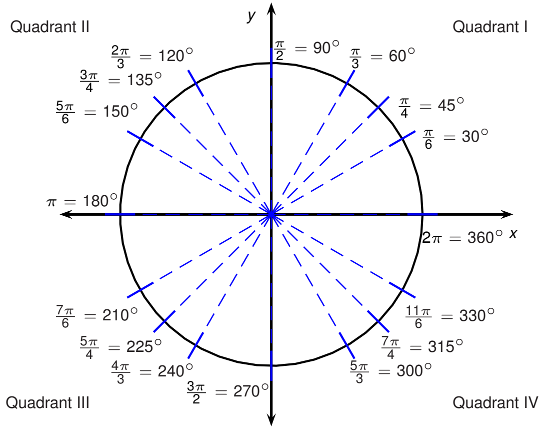
Rad.	$\frac{\pi}{3}$	$\frac{\pi}{10}$	$\frac{11\pi}{6}$	$\frac{7\pi}{4}$	$\frac{\pi}{7}$	$\frac{13\pi}{6}$	$-\frac{5\pi}{4}$	2
Deg.	60°	18°	330°	315°	$\frac{180}{7}^\circ \approx 25.7^\circ$	390°	-225°	$\frac{2}{\pi}180^\circ \approx 114.6^\circ$

Example



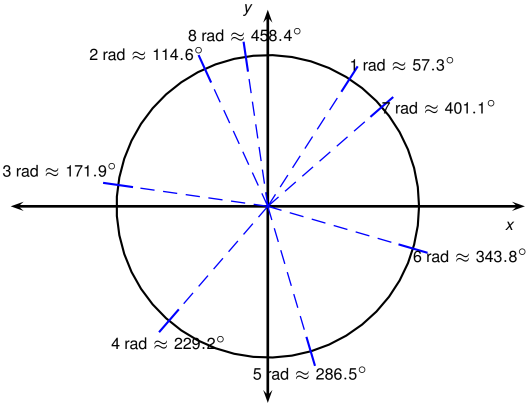
Find the length of an arc of a circle of radius 3 cut off by an angle of measure 230°.

$$
\begin{aligned}
\alpha &= 230^\circ \\
&= 230^\circ \frac{\pi \text{ rad}}{180^\circ} = \frac{23}{18}\pi \text{ rad} && \Big| \text{ Convert to radians} \\
\text{arc-length} &= \alpha r = \frac{23\pi}{18} \cdot 3 = \frac{23\pi}{6} \approx 12.043
\end{aligned}
$$



The most frequently encountered angles are given in the table below.

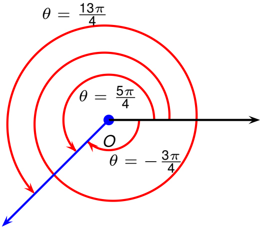
Deg.	0°	30°	45°	60°	90°	120°	135°	150°	180°	270°	360°
Rad.	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π



- Integer quantities of radians are not rational multiples of (the measure of) a half-turn and are not easy to compute with.
- For example to determine in which quadrant is an angle of k radians located one needs to know the numerical value of $\frac{k}{\pi}$, which requires knowledge of π with great numerical accuracy.

Definition (Coterminal Angles)

Two angles (angle measures) are called coterminal if the corresponding geometric angles have the same initial and terminal sides.



Observation

The set of angles coterminal with α consists of the angles $\alpha + 2k\pi$, where k runs over the set of integers. In other words, the angles coterminal with α are the angles:

$$\dots, \alpha - 6\pi, \alpha - 4\pi, \alpha - 2\pi, \alpha, \alpha + 2\pi, \alpha + 4\pi, \alpha + 6\pi, \dots.$$

Example

- Find all angles that are coterminal to $\frac{\pi}{4}$.
- Find all angles in the interval $[-2\pi, \pi]$ that are coterminal to $\frac{\pi}{4}$.

By theory, the angles coterminal with $\frac{\pi}{4}$ are all angles of the form

$$\frac{\pi}{4} + 2k\pi.$$

To find which among the angles $\frac{\pi}{4} + 2k\pi$ lie in the interval $[-2\pi, \pi]$, we write them as an infinite list (we indicate the unboundedness of the list by ellipsis dots) and cross out the angles that lie outside of the desired interval.

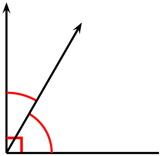
$$\cancel{\dots}, \cancel{\frac{\pi}{4} - 4\pi}, \frac{\pi}{4} - 2\pi, \frac{\pi}{4}, \cancel{\frac{\pi}{4} + 2\pi}, \cancel{\frac{\pi}{4} + 4\pi}, \cancel{\dots}.$$

Our final answer is $-\frac{7\pi}{4}, \frac{\pi}{4}$

Complementary angles

Definition

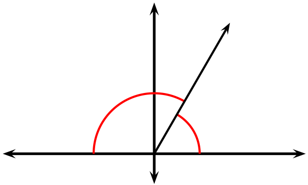
Two positive angles are called complementary when they sum to a right angle, i.e., an angle of measure $\frac{\pi}{2} = 90^\circ$.



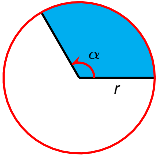
Supplementary angles

Definition

Two positive angles are called supplementary when they sum to $\pi = 180^\circ$.



A sector of a circle is the region cut off from a circle by an angle whose vertex is at the center of the circle.



Proposition (Area of a circle sector)

The area of a circle sector equals

$$\frac{1}{2}\alpha r^2,$$

where α is the angle of the sector.