Calculus II
Integration of rational functions: plan for algorithm

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Integrating arbitrary rational functions

Let $\frac{P(x)}{Q(x)}$ be an arbitrary rational function, i.e., a quotient of polynomials.

Question

Can we integrate $\displaystyle\int \frac{P(x)}{Q(x)} \mathrm{d}x$*?*

- Yes. We will learn how in what follows.
- The algorithm for integration is roughly:
  - We use algebra to split $\frac{P(x)}{Q(x)}$ into smaller pieces ("partial fractions").
  - We use linear substitutions to transform each piece to one of 3 pairs of basic building block integrals.
  - We solve each building block integral and collect the terms.
- We study the algorithm "from the ground up": we start with the building blocks.

The building blocks

Let n be a positive integer.

- (Building block I) The first building block integral is:
$$\int \frac{1}{x^n}\mathrm{d}x \quad .$$
- (Building block II) The second building block integral is:
$$\int \frac{x}{(1+x^2)^n}\mathrm{d}x.$$ (Note: $u = 1 + x^2, x\mathrm{d}x = \frac{1}{2}\mathrm{d}u$ transforms II to I).
- (Building block III) The third building block integral is:
$$\int \frac{1}{(1+x^2)^n}\mathrm{d}x \quad .$$
- The case $n = 1$ is special for each of the building blocks:
$\int \frac{1}{x}\mathrm{d}x$, $\int \frac{x}{1+x^2}\mathrm{d}x$ and $\int \frac{1}{1+x^2}\mathrm{d}x$.
- The case $n = 1$ we call respectively building block Ia, IIa and IIIa.
The case $n > 1$ we call respectively building block Ib, IIb and IIIb.
This “building block” terminology is for our convenience, and is not a part of standard mathematical terminology.