Calculus II
Integral of rational function with cubic denominator, part 1

Todor Milev

2019

Example

Find $\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\mathrm{d}x$.

- $\deg(x^2+2x-1) < \deg(2x^3+3x^2-2x)$: don't divide.
- Factor denominator: $2x^3+3x^2-2x = x(2x-1)(x+2)$.

$$\begin{aligned}
\frac{x^2+2x-1}{x(2x-1)(x+2)} &= \frac{A}{x}+\frac{B}{2x-1}+\frac{C}{x+2}\\
x^2+2x-1 &= A(2x-1)(x+2)+Bx(x+2)+Cx(2x-1)\\
x^2+2x-1 &= (2A+B+2C)x^2+(3A+2B-C)x-2A
\end{aligned}$$

$$\begin{aligned}
2A+\ B+2C &= 1\\
3A+2B-\ C &= 2\\
-2A \qquad &= -1
\end{aligned}$$

Solution:

$A=\frac{1}{2}, B=\frac{1}{5}, C=-\frac{1}{10}$.

$$\begin{aligned}
&\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\mathrm{d}x\\
&= \int \left(\frac{1}{2}\ \frac{1}{x}+\frac{1}{5}\ \frac{1}{2x-1}-\frac{1}{10}\ \frac{1}{x+2}\right)\mathrm{d}x\\
&= \frac{1}{2}\ln|x|+\frac{1}{10}\ln|2x-1|\\
&\qquad -\frac{1}{10}\ln|x+2|+K
\end{aligned}$$

NOTE: There is a quick trick to find A, B, and C.

$$x^2 + 2x - 1 = A(2x-1)(x+2) + Bx(x+2) + Cx(2x-1)$$

To find A, set $x = 0$; to find B, set $x = \frac{1}{2}$; to find C, set $x = -2$.

$$\begin{array}{rcl} 0^2 + 2\cdot 0 - 1 &=& A(2\cdot 0 - 1)(0+2) \\ -1 &=& -2A \\ A &=& \frac{1}{2} \end{array}$$

$$\begin{array}{rcl} \left(\frac{1}{2}\right)^2 + 2\cdot\frac{1}{2} - 1 &=& B\left(\frac{1}{2}\right)\left(\frac{1}{2}+2\right) \\ \frac{1}{4} &=& \frac{5}{4}B \\ B &=& \frac{1}{5} \end{array}$$

$$\begin{array}{rcl} (-2)^2 + 2(-2) - 1 &=& C(-2)(2(-2)-1) \\ -1 &=& 10C \\ C &=& -\frac{1}{10} \end{array}$$

$Q(x)$ has distinct linear factors

- Suppose $Q(x)$ is a product of distinct linear factors:

$$Q(x) = (a_1x + b_1)(a_2x + b_2)\cdots(a_kx + b_k)$$

  where no factor is repeated and no factor is a constant multiple of another.
- Then there exist constants $A_1, A_2, \dots, A_k$ such that

$$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \dots + \frac{A_k}{a_kx + b_k}$$

- We show how to find $A_1, A_2, \cdots, A_k$ on examples.