

Calculus I

Trigonometric derivatives

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Outline

1 Derivatives of Trigonometric Functions

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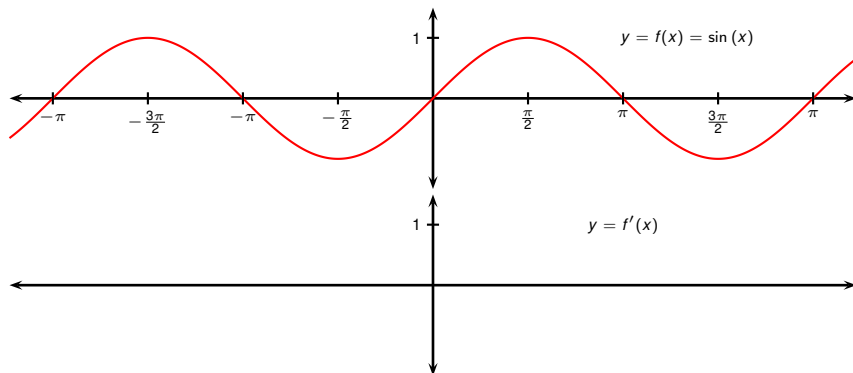
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Derivatives of Trigonometric Functions



What is the derivative of $f(x) = \sin x$? It looks like $\cos x$.

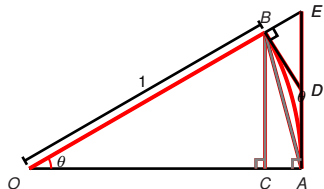
Let $f(x) = \sin x$.

$$\begin{aligned}\text{Then } f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin x}{h} \\&= \lim_{h \rightarrow 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\&= \lim_{h \rightarrow 0} \left(\frac{\sin x \cos h - \sin x}{h} + \frac{\cos x \sin h}{h} \right) \\&= \lim_{h \rightarrow 0} \left(\sin x \left(\frac{\cos h - 1}{h} \right) + \cos x \left(\frac{\sin h}{h} \right) \right) \\&= \lim_{h \rightarrow 0} \sin x \cdot \lim_{h \rightarrow 0} \left(\frac{\cos h - 1}{h} \right) + \lim_{h \rightarrow 0} \cos x \cdot \lim_{h \rightarrow 0} \left(\frac{\sin h}{h} \right) \\&= \sin x \cdot \lim_{h \rightarrow 0} \left(\frac{\cos h - 1}{h} \right) + \cos x \cdot \lim_{h \rightarrow 0} \left(\frac{\sin h}{h} \right)\end{aligned}$$

We need to do more work to find the other two limits.

Claim: $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$

Suppose $0 < \theta < \frac{\pi}{2}$. Write $\sin \theta$ using ratios of side lengths of a triangle.



$$\sin \theta = \frac{|BC|}{|OB|} = |BC| < |AB| < \text{arc} AB = \theta$$

Therefore $\sin \theta < \theta$ and therefore $\frac{\sin \theta}{\theta} < 1$.

$$\begin{aligned} \theta = \text{arc} AB &< |AD| + |DB| < |AD| + |DE| \\ &= |AE| = |OA| \tan \theta = \tan \theta \end{aligned}$$

Therefore $\theta < \tan \theta = \frac{\sin \theta}{\cos \theta}$, so $\cos \theta < \frac{\sin \theta}{\theta}$.

$$\cos \theta < \frac{\sin \theta}{\theta} < 1$$

$\lim_{\theta \rightarrow 0} \cos \theta = 1$ and $\lim_{\theta \rightarrow 0} 1 = 1$, so by the Squeeze Theorem

$\lim_{\theta \rightarrow 0^+} \frac{\sin \theta}{\theta} = 1$. $\frac{\sin \theta}{\theta}$ is even, so the left limit is also 1.

Let $f(x) = \sin x$.

$$\begin{aligned}\text{Then } f'(x) &= \lim_{h \rightarrow 0} \sin x \cdot \lim_{h \rightarrow 0} \left(\frac{\cos h - 1}{h} \right) + \lim_{h \rightarrow 0} \cos x \cdot \lim_{h \rightarrow 0} \left(\frac{\sin h}{h} \right) \\ &= \sin x \cdot \lim_{h \rightarrow 0} \left(\frac{\cos h - 1}{h} \right) + \cos x \cdot 1\end{aligned}$$

We need to find

$$\begin{aligned}\lim_{h \rightarrow 0} \frac{\cos h - 1}{h} &= \lim_{h \rightarrow 0} \frac{(\cos h - 1)}{h} \cdot \frac{(\cos h + 1)}{(\cos h + 1)} = \lim_{h \rightarrow 0} \frac{\cos^2 h - 1}{h(\cos h + 1)} \\ &= \lim_{h \rightarrow 0} \frac{-\sin^2 h}{h(\cos h + 1)} = - \lim_{h \rightarrow 0} \left(\frac{\sin h}{h} \cdot \frac{\sin h}{\cos h + 1} \right) \\ &= - \lim_{h \rightarrow 0} \frac{\sin h}{h} \cdot \lim_{h \rightarrow 0} \frac{\sin h}{\cos h + 1} = -1 \cdot \left(\frac{0}{1 + 1} \right) = 0\end{aligned}$$

Theorem (The Derivative of $\sin x$)

$$\frac{d}{dx}(\sin x) = \cos x$$

Example (Product Rule, Product Rule with Sine)

Differentiate $f(x) = x \sin x$.

$$\begin{aligned}\text{Product Rule: } f'(x) &= \frac{d}{dx}(x)(\sin x) + (x) \frac{d}{dx}(\sin x) \\ &= (1)(\sin x) + (x)(\cos x) \\ &= x \cos x + \sin x.\end{aligned}$$

Example (Quotient Rule, Natural Exponential Function and Sine)

Differentiate $y = \frac{e^x}{2 + \sin x}$.

Quotient Rule:

$$\begin{aligned}\frac{dy}{dx} &= \frac{\frac{d}{dx}(e^x)(2 + \sin x) - (e^x)\frac{d}{dx}(2 + \sin x)}{(2 + \sin x)^2} \\ &= \frac{(e^x)(2 + \sin x) - (e^x)(\cos x)}{(2 + \sin x)^2} \\ &= \frac{2e^x + e^x \sin x - e^x \cos x}{(2 + \sin x)^2} \\ &= \frac{e^x(2 + \sin x - \cos x)}{(2 + \sin x)^2}.\end{aligned}$$

Example (Trigonometric limit)

$$\begin{aligned}\text{Find } \lim_{x \rightarrow 0} \frac{2x}{\sin(9x)} &= \lim_{x \rightarrow 0} \frac{2x}{\sin(9x)} \cdot \frac{9}{9} \\ &= \lim_{x \rightarrow 0} \frac{2}{9} \cdot \frac{9x}{\sin(9x)} \\ &= \lim_{x \rightarrow 0} \frac{2}{9} \cdot \frac{1}{\frac{\sin(9x)}{9x}} = \lim_{\theta \rightarrow 0} \frac{2}{9} \cdot \frac{1}{\frac{\sin \theta}{\theta}}.\end{aligned}$$

$$\text{Let } \theta = 9x.$$

$$\text{As } x \rightarrow 0, \quad \theta \rightarrow 0.$$

$$\begin{aligned}\text{Then } \lim_{x \rightarrow 0} \frac{2x}{\sin(9x)} &= \frac{2}{9} \cdot \frac{1}{\lim_{\theta \rightarrow 0} \left(\frac{\sin \theta}{\theta} \right)} \\ &= \frac{2}{9} \cdot \frac{1}{1} = \frac{2}{9}.\end{aligned}$$

Theorem (The Derivative of $\cos x$)

$$\frac{d}{dx}(\cos x) = -\sin x$$

- This can be proved in a similar fashion as the formula for $\sin x$.
- Alternatively, this can be proved using the derivative of $\sin x$ and (the not yet studied) Implicit Differentiation and Chain Rule.

Example (Product Rule, with Cosine)

Differentiate $f(x) = x \cos x$.

$$\begin{aligned}\text{Product Rule: } f'(x) &= \frac{d}{dx}(x)(\cos x) + (x) \frac{d}{dx}(\cos x) \\ &= (1)(\cos x) + (x)(-\sin x) \\ &= -x \sin x + \cos x.\end{aligned}$$

Theorem (The Derivative of Tangent)

$$\frac{d}{dx}(\tan x) = \sec^2 x.$$

Proof.

$$\text{Let } y = \tan x = \frac{\sin x}{\cos x}.$$

Quotient Rule:

$$\begin{aligned}\frac{dy}{dx} &= \frac{\frac{d}{dx}(\sin x)(\cos x) - (\sin x)\frac{d}{dx}(\cos x)}{(\cos x)^2} \\ &= \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} \\ &= \sec^2 x.\end{aligned}$$



Derivatives of Trigonometric Functions

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

Example (Quotient Rule, Trig)

Differentiate $y = \frac{\sec x}{1 + \tan x}$.

Quotient Rule:

$$\begin{aligned}\frac{dy}{dx} &= \frac{\frac{d}{dx}(\sec x)(1 + \tan x) - (\sec x)\frac{d}{dx}(1 + \tan x)}{(1 + \tan x)^2} \\&= \frac{(\sec x \tan x)(1 + \tan x) - (\sec x)(\sec^2 x)}{(1 + \tan x)^2} \\&= \frac{\sec x(\tan x + \tan^2 x - \sec^2 x)}{(1 + \tan x)^2} \\&= \frac{\sec x(\tan x + (-1))}{(1 + \tan x)^2} = \frac{\sec x(\tan x - 1)}{(1 + \tan x)^2}.\end{aligned}$$

Example (Using the Product Rule twice)

Differentiate:

$$y = \theta e^{\theta} (\tan \theta + \sec \theta).$$

Product Rule:

$$y' = \frac{d}{d\theta} (\theta e^{\theta}) (\tan \theta + \sec \theta) + \theta e^{\theta} \frac{d}{d\theta} (\tan \theta + \sec \theta)$$

Product Rule:

$$\begin{aligned} &= \left(\theta \frac{d}{d\theta} (e^{\theta}) + \frac{d}{d\theta} (\theta) e^{\theta} \right) (\tan \theta + \sec \theta) + \theta e^{\theta} (\sec^2 \theta + \tan \theta \sec \theta) \\ &= (\theta (e^{\theta}) + (1) e^{\theta}) (\tan \theta + \sec \theta) + \theta e^{\theta} (\sec^2 \theta + \tan \theta \sec \theta) \\ &= \theta e^{\theta} \sec \theta (\sec \theta + \tan \theta) + e^{\theta} (\theta + 1) (\tan \theta + \sec \theta) \\ &= (\theta \sec \theta + \theta + 1) e^{\theta} (\tan \theta + \sec \theta). \end{aligned}$$

Example

Find the 27th derivative of $f(x) = \cos x$.

$$f'(x) = -\sin x$$

$$f''(x) = -\cos x$$

$$f'''(x) = \sin x$$

$$f^{(4)}(x) = \cos x$$

$$f^{(5)}(x) = -\sin x$$

- The derivatives repeat in a cycle of length 4.
- $f^{(24)}(x) = \cos x$.
- Differentiate three more times: $f^{(27)}(x) = \sin x$.