Calculus II

Integrals of the form $\int \sqrt{ax^2+bx+c}\,\mathrm{d}x$, quadratic has no real roots.

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Linear substitutions to simplify radicals $\sqrt{ay^2+by+c}$

- Using linear substitutions, radicals of form $\sqrt{ay^2+by+c}$, $a \neq 0$, $b^2-4ac \neq 0$ can be transformed to (multiple of):
  - $\sqrt{x^2+1}$
  - $\sqrt{-x^2+1}$
  - $\sqrt{x^2-1}$.
- We already studied how to do that using completing the square when dealing with rational functions.

Recall: linear substitution is subst. of the form $u = px + q$.

Example

Use linear substitution to transform $\sqrt{x^2 + x + 1}$ to multiple of $\sqrt{u^2 + 1}$.

$$\sqrt{x^2 + x + 1} \quad =$$

Recall: linear substitution is subst. of the form $u = px + q$.

Example

Use linear substitution to transform $\sqrt{x^2 + x + 1}$ to multiple of $\sqrt{u^2 + 1}$.

$$\sqrt{x^2 + x + 1} = \sqrt{x^2 + 2 \cdot \frac{1}{2}x + \textbf{?} - \textbf{?} + 1}$$

Recall: linear substitution is subst. of the form $u = px + q$.

Example

Use linear substitution to transform $\sqrt{x^2 + x + 1}$ to multiple of $\sqrt{u^2 + 1}$.

$$\sqrt{x^2 + x + 1} = \sqrt{x^2 + 2 \cdot \frac{1}{2}x + \frac{1}{4} - \frac{1}{4} + 1}$$

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Example

Use linear substitution to transform $\sqrt{x^2 + x + 1}$ to multiple of $\sqrt{u^2 + 1}$.

$$\begin{aligned}\sqrt{x^2 + x + 1} &= \sqrt{x^2 + 2 \cdot \frac{1}{2}x + \frac{1}{4} - \frac{1}{4} + 1} \\ &= \sqrt{\left(x + \textbf{?}\right)^2 + \textbf{?}}\end{aligned}$$

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Example

Use linear substitution to transform $\sqrt{x^2 + x + 1}$ to multiple of $\sqrt{u^2 + 1}$.

$$\begin{aligned}
\sqrt{x^2 + x + 1} &= \sqrt{x^2 + 2 \cdot \frac{1}{2} x + \frac{1}{4} - \frac{1}{4} + 1} \\
&= \sqrt{\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}} \\
&= \sqrt{\frac{3}{4}\left(\frac{4}{3}\left(x + \frac{1}{2}\right)^2 + 1\right)}
\end{aligned}$$

Recall: linear substitution is subst. of the form $u = px + q$.

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\sqrt{x^2 + x + 1} &= \sqrt{x^2 + 2 \cdot \frac{1}{2} x + \frac{1}{4} - \frac{1}{4} + 1} \\
&= \sqrt{\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}} \\
&= \sqrt{\frac{3}{4}\left(\frac{4}{3}\left(x + \frac{1}{2}\right)^2 + 1\right)} \\
&= \frac{\sqrt{3}}{2}\sqrt{\left(\frac{2}{\sqrt{3}}\left(x + \frac{1}{2}\right)\right)^2 + 1}
\end{aligned}$$

Recall: linear substitution is subst. of the form $u = px + q$.

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Example

Use linear substitution to transform $\sqrt{x^2 + x + 1}$ to multiple of $\sqrt{u^2 + 1}$.

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\sqrt{x^2 + x + 1} &= \sqrt{x^2 + 2 \cdot \frac{1}{2}x + \frac{1}{4} - \frac{1}{4} + 1} \\
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&= \sqrt{\frac{3}{4}\left(\frac{4}{3}\left(x + \frac{1}{2}\right)^2 + 1\right)} \\
&= \frac{\sqrt{3}}{2}\sqrt{\left(\frac{2}{\sqrt{3}}\left(x + \frac{1}{2}\right)\right)^2 + 1} \\
&= \frac{\sqrt{3}}{2}\sqrt{u^2 + 1},
\end{aligned}$$

where $u = \frac{2}{\sqrt{3}}\left(x + \frac{1}{2}\right) = \frac{2\sqrt{3}}{3}x + \frac{\sqrt{3}}{3}$.