Calculus II
Integral of rational function with cubic denominator, part 1

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2019

Example

Find $\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\mathrm{d}x$.

- $\deg(x^2+2x-1) < \deg(2x^3+3x^2-2x)$: don't divide.
- Factor denominator: $2x^3+3x^2-2x = x(2x-1)(x+2)$.

$$\begin{aligned}
\frac{x^2+2x-1}{x(2x-1)(x+2)} &= \frac{A}{x}+\frac{B}{2x-1}+\frac{C}{x+2}\\
x^2+2x-1 &= A(2x-1)(x+2)+Bx(x+2)+Cx(2x-1)\\
x^2+2x-1 &= (2A+B+2C)x^2+(3A+2B-C)x-2A
\end{aligned}$$

$$\begin{aligned}
2A+B+2C &= 1\\
3A+2B-C &= 2\\
-2A &= -1
\end{aligned}$$

Solution:
$A=\frac{1}{2}, B=\frac{1}{5}, C=-\frac{1}{10}$.

$$\begin{aligned}
&\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\mathrm{d}x\\
&= \int\left(\frac{1}{2}\,\frac{1}{x}+\frac{1}{5}\,\frac{1}{2x-1}-\frac{1}{10}\,\frac{1}{x+2}\right)\mathrm{d}x\\
&= \frac{1}{2}\ln|x|+\frac{1}{10}\ln|2x-1|\\
&\qquad -\frac{1}{10}\ln|x+2|+K
\end{aligned}$$

NOTE: There is a quick trick to find A, B, and C.

$$x^2+2x-1 = A(2x-1)(x+2)+Bx(x+2)+Cx(2x-1)$$

To find A, set $x=0$; to find B, set $x=\frac{1}{2}$; to find C, set $x=-2$.

$$\begin{aligned}
0^2+2\cdot 0-1 &= A(2\cdot 0-1)(0+2)\\
-1 &= -2A\\
A &= \frac{1}{2}
\end{aligned}$$

$$\begin{aligned}
\left(\frac{1}{2}\right)^2+2\cdot\frac{1}{2}-1 &= B\left(\frac{1}{2}\right)\left(\frac{1}{2}+2\right)\\
\frac{1}{4} &= \frac{5}{4}B\\
B &= \frac{1}{5}
\end{aligned}$$

$$\begin{aligned}
(-2)^2+2(-2)-1 &= C(-2)(2(-2)-1)\\
-1 &= 10C\\
C &= -\frac{1}{10}
\end{aligned}$$

$Q(x)$ has distinct linear factors

- Suppose $Q(x)$ is a product of distinct linear factors:

$$Q(x) = (a_1 x + b_1)(a_2 x + b_2) \cdots (a_k x + b_k)$$

where no factor is repeated and no factor is a constant multiple of another.

- Then there exist constants $A_1, A_2, \dots, A_k$ such that

$$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1 x + b_1} + \frac{A_2}{a_2 x + b_2} + \dots + \frac{A_k}{a_k x + b_k}$$

- We show how to find $A_1, A_2, \cdots, A_k$ on examples.