Calculus II
Weierstrass substitution theory

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2019

Integrals of the form $\int R(\cos\theta, \sin\theta) \mathrm{d}\theta$, R

Let R be an arbitrary rational function in two variables (quotient of polynomials in two variables).

Question

Can we integrate $\int R(\cos\theta, \sin\theta) \mathrm{d}\theta$?

- Yes. We will learn how in what follows.
- The algorithm for integration is roughly:
  - Apply the substitution $\theta = 2\arctan t$ to transform to integral of rational function.
  - Solve as previously studied.

The rationalizing substitution $\theta = 2\arctan t$

Let R- rational function in two variables. $\int R(\cos\theta, \sin\theta) \mathrm{d}\theta$ can be integrated via the substitution $\theta = 2\arctan t$. How does this transform $\sin\theta$, $\cos\theta$? How does this transform $\mathrm{d}\theta$? How is t expressed via θ?

$$
\begin{aligned}
\sin\theta &= \sin(2\arctan t) = \frac{2\tan(\arctan t)}{1+\tan^2(\arctan t)} = \frac{2t}{1+t^2}\\
\cos\theta &= \cos(2\arctan t) = \frac{1-\tan^2(\arctan t)}{1+\tan^2(\arctan t)} = \frac{1-t^2}{1+t^2}
\end{aligned}
$$

Recall the expression of $\sin(2z), \cos(2z)$ via $\tan z$:

$$
\begin{aligned}
\sin(2z) &= 2\sin z\cos z = \frac{2\sin z\cos z\frac{1}{\cos^2 z}}{(\cos^2 z+\sin^2 z)\frac{1}{\cos^2 z}} = \frac{2\tan z}{1+\tan^2 z} \quad .\\
\cos(2z) &= \cos^2 z - \sin^2 z = \frac{\left(\cos^2 z-\sin^2 z\right)\frac{1}{\cos^2 z}}{\left(\cos^2 z+\sin^2 z\right)\frac{1}{\cos^2 z}} = \frac{1-\tan^2 z}{1+\tan^2 z} .
\end{aligned}
$$

Tl

The rationalizing substitution $\theta = 2\arctan t$

Let R- rational function in two variables. $\int R(\cos\theta, \sin\theta)\mathrm{d}\theta$ can be integrated via the substitution $\theta = 2\arctan t$. How does this transform $\sin\theta$, $\cos\theta$? How does this transform $\mathrm{d}\theta$? How is t expressed via θ?

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\sin\theta &= \sin(2\arctan t) = \frac{2\tan(\arctan t)}{1+\tan^2(\arctan t)} = \frac{2t}{1+t^2}\\
\cos\theta &= \cos(2\arctan t) = \frac{1-\tan^2(\arctan t)}{1+\tan^2(\arctan t)} = \frac{1-t^2}{1+t^2}\\
\mathrm{d}\theta &= 2\mathrm{d}(\arctan t) = \frac{2}{1+t^2}\mathrm{d}t\\
t &= \tan\left(\frac{\theta}{2}\right)
\end{aligned}
$$

Theorem

The substitution given above transforms $\int R(\cos\theta, \sin\theta)\mathrm{d}\theta$ to an integral of a rational function of t.

The integral $\int \sec\theta \mathrm{d}\theta$ appears often in practice. A quicker solution will be shown later, but first we show the standard method.

Example

Set $\theta = 2\arctan t$, $\cos\theta = \dfrac{1-\tan^2(\frac{\theta}{2})}{1+\tan^2\left(\frac{\theta}{2}\right)} = \dfrac{1-t^2}{1+t^2}$, $\mathrm{d}\theta = 2\dfrac{1}{1+t^2}\mathrm{d}t$.

$$
\begin{aligned}
\int \sec\theta \mathrm{d}\theta &= \int \frac{1}{\cos\theta}\mathrm{d}\theta = \int \frac{1}{\left(\frac{1-t^2}{\cancel{1+t^2}}\right)} \frac{2}{\cancel{(1+t^2)}}\mathrm{d}t \\
&= \int \frac{2}{1-t^2}\mathrm{d}t = \int \left(\frac{1}{1-t} + \frac{1}{1+t}\right)\mathrm{d}t \quad \Big| \text{ part. fractions} \\
&= -\ln|1-t| + \ln|1+t| + C \\
&= \ln\left|\frac{1+t}{1-t}\right| + C \\
&= \ln\left|\frac{1+\tan\left(\frac{\theta}{2}\right)}{1-\tan\left(\frac{\theta}{2}\right)}\right| + C
\end{aligned}
$$

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$$\int \sec\theta \mathrm{d}\theta = \ln\left|\frac{1+\tan\left(\frac{\theta}{2}\right)}{1-\tan\left(\frac{\theta}{2}\right)}\right| + C$$

This is a perfectly good answer, however there's a simplification:

$$\begin{aligned}
\tan\theta + \sec\theta &= \frac{\sin\theta + 1}{\cos\theta} = \frac{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) + \sin^2\left(\frac{\theta}{2}\right) + \cos^2\left(\frac{\theta}{2}\right)}{\cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)} \\
&= \frac{\left(\sin\left(\frac{\theta}{2}\right) + \cos\left(\frac{\theta}{2}\right)\right)^{\cancel{2}}}{\left(\cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)\right)\cancel{\left(\cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)\right)}} \\
&= \frac{\sin\left(\frac{\theta}{2}\right) + \cos\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)} = \frac{1+\tan\left(\frac{\theta}{2}\right)}{1-\tan\left(\frac{\theta}{2}\right)} .
\end{aligned}$$

The integral $\int \sec\theta d\theta$ appears often in practice. A quicker solution will be shown later, but first we show the standard method.

Example

Set $\theta = 2\arctan t$, $\cos\theta = \frac{1-\tan^2(\frac{\theta}{2})}{1+\tan^2\left(\frac{\theta}{2}\right)} = \frac{1-t^2}{1+t^2}$, $d\theta = 2\frac{1}{1+t^2}dt$.

$$\int \sec\theta d\theta \quad = \quad \ln\left|\tan\theta + \sec\theta\right| + C$$

This is a perfectly good answer, however there's a simplification:

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\tan\theta + \sec\theta &= \frac{\sin\theta + 1}{\cos\theta} = \frac{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) + \sin^2\left(\frac{\theta}{2}\right) + \cos^2\left(\frac{\theta}{2}\right)}{\cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)} \\
&= \frac{\left(\sin\left(\frac{\theta}{2}\right) + \cos\left(\frac{\theta}{2}\right)\right)^{\cancel{2}}}{\left(\cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)\right)\cancel{\left(\cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)\right)}} \\
&= \frac{\sin\left(\frac{\theta}{2}\right) + \cos\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)} = \frac{1 + \tan\left(\frac{\theta}{2}\right)}{1 - \tan\left(\frac{\theta}{2}\right)} \quad .
\end{aligned}$$