

Precalculus

Angle sum formulas memorization

Todor Milev

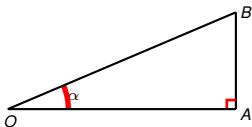
2019

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$$\sin(\alpha + \beta) = ?$$

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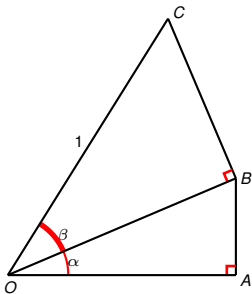
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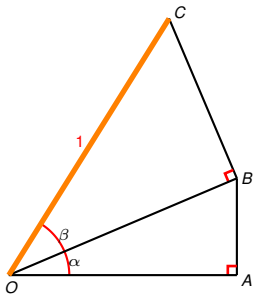
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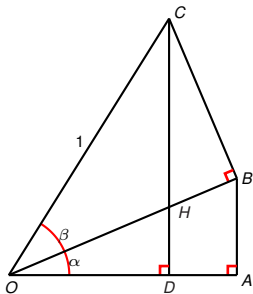
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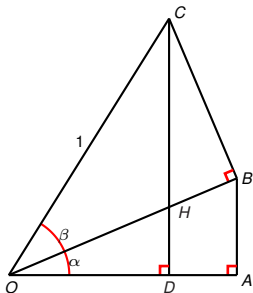
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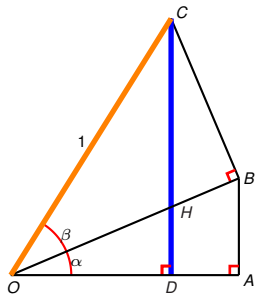
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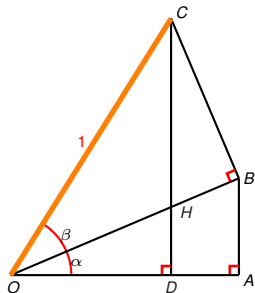
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$$\sin(\alpha + \beta) = \frac{|CD|}{|OC|}$$

$$\cos(\alpha + \beta) = ?$$

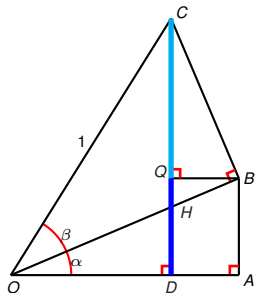
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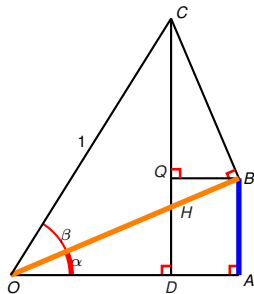
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$$\begin{aligned}\sin(\alpha + \beta) &= \frac{|CD|}{|OC|} = |CD| \\ &= |QD| + |CQ|\end{aligned}$$

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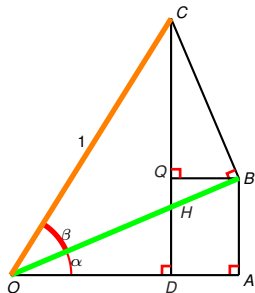
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$$\left| \begin{array}{l} \square DABQ \\ \triangle OAB \end{array} \right|$$

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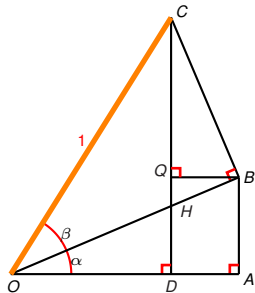


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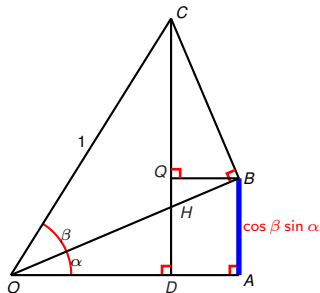


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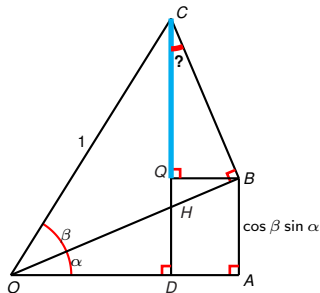


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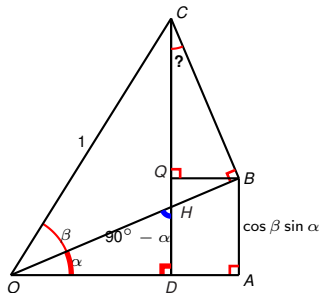
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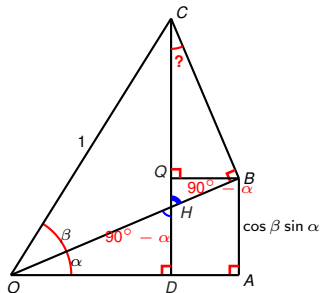


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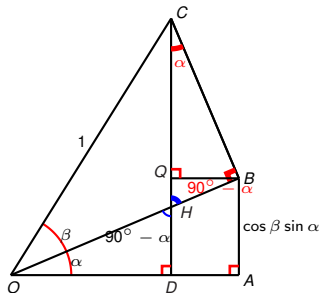


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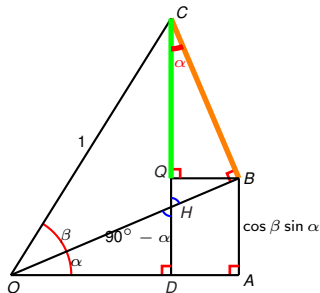


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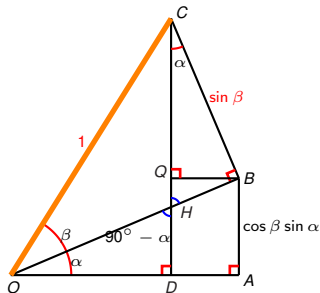


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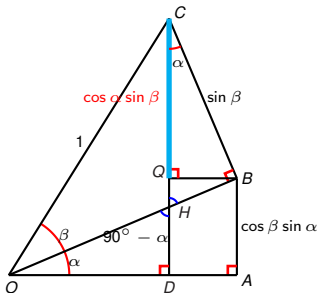


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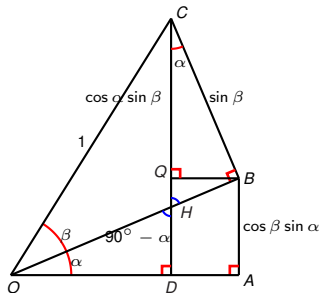


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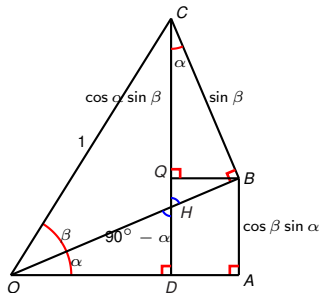
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$$\begin{aligned}\cos(\alpha + \beta) &= \frac{|OD|}{|OC|} = |OD| \\ &= |OA| - |DA| \\ &= \cos \alpha \cos \beta - \sin \alpha \sin \beta\end{aligned}$$

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Trig Functions of Sums and Differences of Angles

Theorem

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

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- There is a theoretically more advanced (but algebraically simpler) proof using Euler's formula (to be studied later/in another course).
- The difference formulas are a consequence of the sum formulas and the fact that \sin is an odd function and \cos is even.

Trig Functions of Differences of Angles

Example

Prove the identities

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$$

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

from the (already demonstrated) identities

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$\sin(\alpha - \beta) = \sin(\alpha + (-\beta))$$

$$= \sin \alpha \cos(-\beta) + \cos \alpha \sin(-\beta) \quad \left| \begin{array}{l} \cos \text{ is even ,} \\ \sin \text{ is odd} \end{array} \right.$$

$$= \sin \alpha \cos \beta - \cos \alpha \sin \beta$$

$$\cos(\alpha - \beta) = \cos(\alpha + (-\beta))$$

$$= \cos \alpha \cos(-\beta) - \sin \alpha \sin(-\beta) \quad \left| \begin{array}{l} \cos \text{ is even ,} \\ \sin \text{ is odd} \end{array} \right.$$

$$= \cos \alpha \cos \beta + \sin \alpha \sin \beta$$