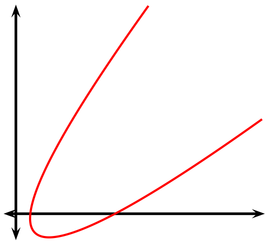
Calculus II
Homework
Tangents and curve length

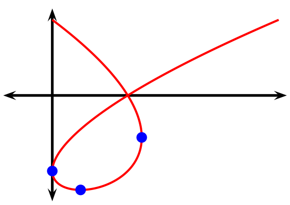
1. Find the values of the parameter t for which the curve has horizontal and vertical tangents.

   (a) $x = t^2 - t + 1, y = t^2 + t - 1$

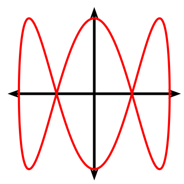
   answer: horiz.: $t = -\frac{1}{2}$, vert.: $t = \frac{1}{2}$

   (b) $x = t^3 - t^2 - t + 1, y = t^2 - t - 1$.

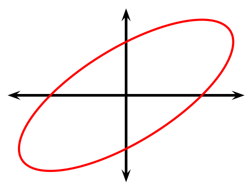
   answer: horiz.: $t = \frac{1}{2}$, vert.: $t = 1, -\frac{1}{3}$

   (c) $x = \cos(t), y = \sin(3t)$

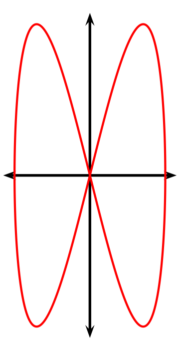
   answer: horiz.: $t = \frac{\pi}{6} + k\pi$, k-integer, vert.:

   (d) $x = \cos(t) + \sin(t)$, $y = \sin(t)$.

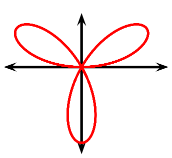
   

2. Show that the parametric curve has multiple tangents at the point and find their slopes.

   (a) $x = \cos t, y = 2\sin(2t)$, two tangents at $(x, y) = (0, 0)$.

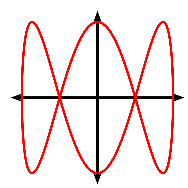
   

   (b) $x = \cos t \sin(3t)$, $y = \sin(t)\sin(3t)$, six tangents at

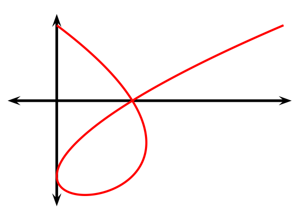
   $(x, y) = (0, 0)$.

   (c) $x = \cos t, y = \sin(3t)$, find the two points at which the curve has double tangent and find the slopes of both pairs

   of tangents.

   (d) $x = t^3 - t^2 - t + 1, y = t^2 - t - 1$, find a point where the curve has double tangent and find the slopes of the tangents.

3. Find the length of the curve.

(a) $y = x^2$, $x \in [1, 2]$.

answer: $L = \sqrt{17} + \frac{1}{4}\log\left(\sqrt{17}+4\right) - \frac{1}{4}\log\left(\sqrt{5}+2\right) - \frac{\sqrt{5}}{2} \approx 3.167841$

(b) $y = \sqrt{x}$, $x \in [1, 2]$.

answer: $L = \frac{1}{8}\left(12\sqrt{2} + \ln(17+12\sqrt{2}) - 4\sqrt{5} - \ln(9+4\sqrt{5})\right) \approx 1.083$

(c) $x = \sqrt{t} - 2t$ and $y = \dfrac{8}{3}t^{\frac{3}{4}}$ from $t = 1$ to $t = 4$.

answer: $L = 7$

(d) $\gamma : \left| \begin{array}{rcl} x(t) &=& \frac{1}{t} + \frac{t^3}{3} \\ y(t) &=& 2t \end{array} \right. , t \in [1, 2]$.

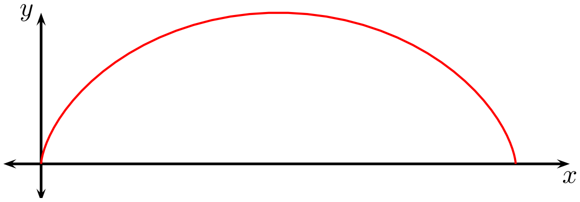
answer: $L = \frac{17}{6}$

(e) $\gamma : \left| \begin{array}{rcl} x(t) &=& \frac{1}{t} + t \\ y(t) &=& 2\ln t \end{array} \right. , t \in [1, 2]$.

answer: $L = \frac{3}{2}$

(f) One arch of the cycloid

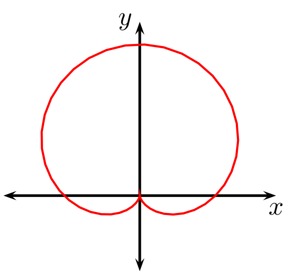
$$\gamma : \left| \begin{array}{rcl} x(t) &=& t - \sin t \\ y(t) &=& 1 - \cos t \end{array} \right. , t \in [0, 2\pi]$$



answer: $L = 8$

(g) The cardioid

$$\gamma : \left| \begin{array}{rcl} x(t) &=& (1 + \sin t)\cos t \\ y(t) &=& (1 + \sin t)\sin t \end{array} \right. , t \in [0, 2\pi]$$



answer: $L = 8$

Solution. 3.a The length of the parametric curve is given by

$$\begin{array}{rcl} L &=& \displaystyle\int_1^2 \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\mathrm{d}x \\ &=& \displaystyle\int_{x=1}^{x=2} \sqrt{1 + 4x^2}\mathrm{d}x \\ &=& \displaystyle\int_{u=2}^{u=4} \sqrt{u^2 + 1}\left(\frac{1}{2}\mathrm{d}u\right) \\ &=& \displaystyle\frac{1}{2}\int_{u=2}^{u=4} \sqrt{u^2+1}\mathrm{d}u \\ &=& \frac{1}{4}\left[u\sqrt{u^2+1} + \ln\left(u + \sqrt{u^2+1}\right)\right]_2^4 \\ &=& \sqrt{17} + \frac{1}{4}\log\left(\sqrt{17}+4\right) - \frac{1}{4}\log\left(\sqrt{5}+2\right) - \frac{\sqrt{5}}{2} \\ &\approx& 3.167841 \end{array}$$

$$\left| \begin{array}{rcl} \text{Substitute } 2x &=& u \\ \mathrm{d}x &=& \frac{1}{2}\mathrm{d}u \end{array} \right.$$

$$\left| \begin{array}{l} \displaystyle\int \sqrt{u^2+1}\mathrm{d}u \\ = \displaystyle\frac{1}{2}\left(u\sqrt{u^2+1} + \ln\left(u + \sqrt{u^2+1}\right)\right) + C \\ \text{previously studied} \end{array} \right.$$

Solution. 3.b

Solution I. The curve can be rewritten in the form $x = y^2$, $y \in [1, \sqrt{2}]$.

$$
\begin{aligned}
L &= \int_1^{\sqrt{2}} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2 + 1}\, \mathrm{d}y \\
&= \int_{y=1}^{y=\sqrt{2}} \sqrt{4y^2+1}\, \mathrm{d}y \\
&= \int_{u=2}^{u=2\sqrt{2}} \sqrt{u^2+1} \left(\frac{1}{2}\mathrm{d}u\right) \\
&= \frac{1}{2}\int \sqrt{u^2+1}\mathrm{d}u
\end{aligned}
$$

Substitute $2y = u$, $\mathrm{d}y = \frac{1}{2}\mathrm{d}u$

$\int \sqrt{u^2+1}\mathrm{d}u$
$= \frac{1}{2}\left(u\sqrt{u^2+1} + \ln\left(u + \sqrt{u^2+1}\right)\right) + C$
previously studied

$$
\begin{aligned}
&= \frac{1}{4}\left[u\sqrt{u^2+1} + \ln\left(u+\sqrt{u^2+1}\right)\right]_2^{2\sqrt{2}} \\
&= \frac{3}{2}\sqrt{2} + \frac{1}{4}\ln\left(2\sqrt{2}+3\right) - \frac{1}{4}\ln\left(\sqrt{5}+2\right) - \frac{\sqrt{5}}{2} \\
&\approx 1.083
\end{aligned}
$$

Solution II. The length of the parametric curve is given by

$$\begin{aligned}
L &= \int_1^2 \sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\mathrm{d}x \\
&= \int_1^2 \sqrt{1+\left(\frac{1}{2\sqrt{x}}\right)^2}\mathrm{d}x \\
&= \int_{x=1}^{x=2} \sqrt{1+\frac{1}{4x}}\mathrm{d}x & \left|\begin{array}{rcl} \text{Substitute } 4x &=& u \\ \mathrm{d}x &=& \frac{1}{4}\mathrm{d}u \end{array}\right. \\
&= \int_{u=4}^{u=8} \sqrt{1+\frac{1}{u}}\left(\frac{1}{4}\mathrm{d}u\right) \\
&= \frac{1}{4}\int_4^8 \sqrt{\frac{u+1}{u}}\mathrm{d}u \\
&= \frac{1}{4}\int_4^8 \sqrt{\frac{u(u+1)}{u^2}}\mathrm{d}u \\
&= \frac{1}{4}\int_4^8 \frac{\sqrt{u^2+u}}{u}\mathrm{d}u \\
&= \frac{1}{4}\int_4^8 \frac{\sqrt{u^2+u+\frac{1}{4}-\frac{1}{4}}}{u}\mathrm{d}u \\
&= \frac{1}{4}\int_4^8 \frac{\sqrt{\left(u+\frac{1}{2}\right)^2-\frac{1}{4}}}{u}\mathrm{d}u \\
&= \frac{1}{4}\int_4^8 \frac{\sqrt{\frac{1}{4}\left((2u+1)^2-1\right)}}{u}\mathrm{d}u \\
&= \frac{1}{8}\int_{u=4}^{u=8} \frac{\sqrt{(2u+1)^2-1}}{u}\mathrm{d}u & \left|\begin{array}{rcl} \text{Substitute } 2u+1 &=& z \\ u &=& \frac{z-1}{2} \\ \mathrm{d}u &=& \frac{1}{2}\mathrm{d}z \end{array}\right. \\
&= \frac{1}{8}\int_{z=9}^{z=17} \frac{\sqrt{z^2-1}}{\frac{z-1}{2}}\frac{1}{2}\mathrm{d}z \\
&= \frac{1}{8}\int_{z=9}^{z=17} \frac{\sqrt{z^2-1}}{z-1}\mathrm{d}z & \left|\begin{array}{rcl} \text{Trig. subst.: } z &=& \sec\theta \\ \sqrt{z^2-1} &=& \tan\theta \\ \mathrm{d}z &=& \tan\theta\sec\theta\mathrm{d}\theta \\ \text{Set } \alpha &=& \operatorname{arcsec}(9) \\ \text{Set } \beta &=& \operatorname{arcsec}(17) \\ \multicolumn{3}{l}{\text{Use } \tan^2\theta = \sec^2\theta - 1} \end{array}\right. \\
&= \frac{1}{8}\int_{\theta=\operatorname{arcsec}(9)}^{\theta=\operatorname{arcsec}(17)} \frac{\tan\theta}{\sec\theta-1}\sec\theta\tan\theta\mathrm{d}\theta \\
&= \frac{1}{8}\int_\alpha^\beta \frac{\tan^2\theta}{\sec\theta-1}\sec\theta\mathrm{d}\theta \\
&= \frac{1}{8}\int_\alpha^\beta \frac{\sec^2\theta-1}{\sec\theta-1}\sec\theta\mathrm{d}\theta \\
&= \frac{1}{8}\int_\alpha^\beta \frac{\cancel{(\sec\theta-1)}(\sec\theta+1)}{\cancel{\sec\theta-1}}\sec\theta\mathrm{d}\theta \\
&= \frac{1}{8}\int_\alpha^\beta \left(\sec^2\theta+\sec\theta\right)\mathrm{d}\theta & \left|\begin{array}{rcl} \multicolumn{3}{l}{\int \sec\theta\mathrm{d}\theta = \ln|\sec\theta+\tan\theta|+C} \\ \multicolumn{3}{l}{\text{previously studied}} \\ \tan\theta &=& \sqrt{\sec^2\theta-1}, \theta\in\left[0,\frac{\pi}{2}\right) \\ \tan\alpha &=& \sqrt{9^2-1}=4\sqrt{5} \\ \tan\beta &=& \sqrt{17^2-1}=12\sqrt{2} \end{array}\right. \\
&= \frac{1}{8}\left[\tan\theta+\ln|\sec\theta+\tan\theta|\right]_\alpha^\beta \\
&= \frac{1}{8}\left(12\sqrt{2}+\ln(17+12\sqrt{2})-4\sqrt{5}-\ln(9+4\sqrt{5})\right) \\
&= \frac{1}{8}\ln\left(12\sqrt{2}+17\right)-\frac{1}{8}\ln\left(4\sqrt{5}+9\right)-\frac{\sqrt{5}}{2}+\frac{3}{2}\sqrt{2} \\
&\approx 1.083 \quad .
\end{aligned}$$

The two answers are both approximately 1.083, so that serves to cross verify our two solutions against one another.

Comparing the two answers we notice that the logarithmic parts in the two answers look different (yet they must be equal). It follows that

$$\frac{1}{8}\ln\left(12\sqrt{2}+17\right)-\frac{1}{8}\ln\left(4\sqrt{5}+9\right)=\frac{1}{4}\ln\left(2\sqrt{2}+3\right)-\frac{1}{4}\ln\left(\sqrt{5}+2\right).$$

A short computation (which computation?), left to the reader, confirms that indeed those two expressions are equal.

Solution. 3.c. The length of the parametric curve is given by

$$L = \int_1^4 \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t \quad .$$

We have that

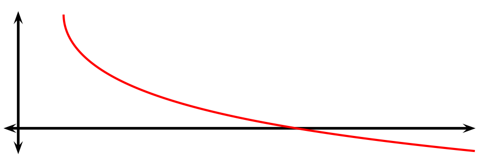
$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= \frac{1}{2\sqrt{t}} - 2 \\
\frac{\mathrm{d}y}{\mathrm{d}t} &= 2t^{-\frac{1}{4}} \\
\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 &= \frac{1}{4t} - \frac{2}{\sqrt{t}} + 4 \\
\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 &= 4t^{-\frac{1}{2}} = \frac{4}{\sqrt{t}} \\
\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 &= \frac{1}{4t} + 2\frac{1}{\sqrt{t}} + 4 = \left(\frac{1}{2\sqrt{t}} + 2\right)^2 \quad .
\end{aligned}$$

$\frac{1}{2\sqrt{t}} + 2$ is positive and $\sqrt{\left(\frac{1}{2\sqrt{t}} + 2\right)^2} = \frac{1}{2\sqrt{t}} + 2$. So the integral becomes

$$L = \int_1^4 \left(\frac{1}{2\sqrt{t}} + 2\right)\mathrm{d}t = \left[\sqrt{t} + 2t\right]_{t=1}^{t=4} = (2+8) - (1+2) = 7 \quad .$$

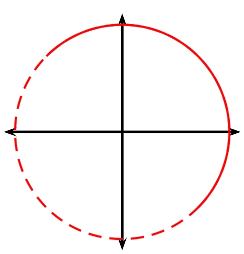
4. Set up an integral that expresses the length of the curve and find the length of the curve.

(a) $\left|\begin{array}{lcl} x(t) &=& e^t + e^{-t} \\ y(t) &=& 5 - 2t \end{array}\right.$ $, t \in [0, 3]$



answer: $e^3 - e^{-3}$

(b) $\left|\begin{array}{lcl} x(t) &=& \sin t + \cos t \\ y(t) &=& \sin t - \cos t \end{array}\right.$ $, t \in [0, \pi]$



answer: $\sqrt{2}\pi$