# Calculus II
# Homework
# Differential equation basics

1. Let $x \in (0,1)$. Express the following using $x$ and $\sqrt{1-x^2}$.

   (a) $\sin(\arcsin(x))$.
   (b) $\sin(2\arcsin(x))$.
   (c) $\sin(3\arcsin(x))$.
   (d) $\sin(\arccos(x))$.
   (e) $\sin(2\arccos(x))$.
   (f) $\sin(3\arccos(x))$.
   (g) $\cos(2\arcsin(x))$.
   (h) $\cos(3\arccos(x))$.

2. Express as the following as an algebraic expression of $x$. In other words, "get rid" of the trigonometric and inverse trigonometric expressions.

   (a) $\cos^2(\arctan x)$.
   (b) $-\sin^2(\operatorname{arccot} x)$.
   (c) $\dfrac{1}{\cos(\arcsin x)}$.
   (d) $-\dfrac{1}{\sin(\arccos x)}$.

3. Rewrite as a rational function of $t$. This problem will be later used to derive the Euler substitutions (an important technique for integrating).

   (a) $\cos(2\arctan t)$.
   (b) $\sin(2\arctan t)$.
   (c) $\tan(2\arctan t)$.
   (d) $\cot(2\arctan t)$.
   (e) $\csc(2\arctan t)$.
   (f) $\sec(2\arctan t)$.
   (g) $\cos(2\operatorname{arccot} t)$.
   (h) $\sin(2\operatorname{arccot} t)$.
   (i) $\tan(2\operatorname{arccot} t)$.
   (j) $\cot(2\operatorname{arccot} t)$.
   (k) $\csc(2\operatorname{arccot} t)$.
   (l) $\sec(2\operatorname{arccot} t)$.

4. Compute the derivative (derive the formula).

   (a) $(\arctan x)'$.
   (b) $(\operatorname{arccot} x)'$.
   (c) $(\arcsin x)'$.
   (d) $(\arccos x)'$.
   (e) Let arcsec denote the inverse of the secant function. Compute $(\operatorname{arcsec} x)'$.

5. (a) Let $a+b \neq k\pi$, $a \neq k\pi + \frac{\pi}{2}$ and $b \neq k\pi + \frac{\pi}{2}$ for any $k \in \mathbb{Z}$ (integers). Prove that

   $$\frac{\tan a + \tan b}{1 - \tan a \tan b} = \tan(a+b) \quad .$$

   (b) Let $x$ and $y$ be real. Prove that, for $xy \neq 1$, we have

   $$\arctan x + \arctan y = \arctan\left(\frac{x+y}{1-xy}\right)$$

   if the left hand side lies between $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

6. Evaluate the indefinite integral. Illustrate the steps of your solutions.

(a) $\int x \sin x \mathrm{d}x$.

(b) $\int x e^{-x} \mathrm{d}x$.

(c) $\int x^2 e^x \mathrm{d}x$.

(d) $\int x \sin(-2x) \mathrm{d}x$.

(e) $\int x^2 \cos(3x) \mathrm{d}x$.

(f) $\int x^2 e^{-2x} \mathrm{d}x$.

(g) $\int x \sin(2x) \mathrm{d}x$.

(h) $\int x \cos(3x) \mathrm{d}x$.

(i) $\int x^2 e^{2x} \mathrm{d}x$.

(j) $\int x^3 e^x \mathrm{d}x$.

7. Evaluate the indefinite integral. Illustrate the steps of your solutions.

(a) $\int x^2 \cos(2x) \mathrm{d}x$.

(b) $\int x^2 e^{ax} \mathrm{d}x$, where $a$ is a constant.

(c) $\int x^2 e^{-ax} \mathrm{d}x$, where $a$ is a constant.

(d) $\int x^2 \frac{(e^{ax} + e^{-ax})^2}{4} \mathrm{d}x$, where $a$ is a constant.

(e) $\int \frac{1}{\cos^2 x} \mathrm{d}x$. (Hint: This problem does not require integration by parts. What is the derivative of $\tan x$?)

(f) $\int (\tan^2 x) \mathrm{d}x$. (Hint: This problem does not require integration by parts. We can use $\tan^2 x = \frac{1}{\cos^2 x} - 1$ and the previous problem. )

(g) $\int x \tan^2 x \mathrm{d}x$. (Hint: $\tan^2 x \mathrm{d}x = \mathrm{d}(F(x))$, where $F(x)$ is the answer from the preceding problem).

(h) $\int e^{-\sqrt{x}} \mathrm{d}x$.

(i) $\int \cos^2 x \ \mathrm{d}x$.

(j) $\int \frac{x}{1+x^2} \mathrm{d}x$ (Hint: use substitution rule, don't use integration by parts)

(k) $\int (\arctan x) \mathrm{d}x$.

(l) $\int (\arcsin x) \mathrm{d}x$.

(m) $\int (\arcsin x)^2 \mathrm{d}x$. (Hint: Try substituting $x = \sin y$.)

(n) $\int \arctan\left(\frac{1}{x}\right) \mathrm{d}x$.

(o) $\int \sin x e^x \mathrm{d}x$

(p) $\int \cos x e^x \mathrm{d}x$

(q) $\int \sin(\ln(x)) \mathrm{d}x$.

(r) $\int \cos(\ln(x)) \mathrm{d}x$.

(s) $\int \ln x \mathrm{d}x$

(t) $\int x \ln x \ \mathrm{d}x$.

(u) $\int \frac{\ln x}{\sqrt{x}} \mathrm{d}x$.

(v) $\int (\ln x)^2 \mathrm{d}x$.

(w) $\int (\ln x)^3 \mathrm{d}x$.

(x) $\int x^2 \cos^2 x \mathrm{d}x$. (This problem is related to Problem 7.d as $\cos x = \frac{e^{ix} + e^{-ix}}{2}$).

8. Compute $\int x^n e^x \mathrm{d}x$, where $n$ is a non-negative integer.

9. Integrate. Illustrate the steps of your solution.

(a) $\int \frac{1}{x+1} \mathrm{d}x$

(b) $\int \frac{x-1}{x+1} \mathrm{d}x$

(c) $\int \frac{1}{(x+1)^2} \mathrm{d}x$

(d) $\int \frac{x}{(x+1)^2} \mathrm{d}x$

(e) $\int \frac{1}{(2x+3)^2} \mathrm{d}x$

(f) $\int \frac{x}{2x^2+3} \mathrm{d}x$

(g) $\int \frac{1}{2x^2+3} \mathrm{d}x$

(h) $\int \frac{x}{2x^2+x+1} \mathrm{d}x$ .

(i) $\int \frac{x}{2x^2+x+3} \mathrm{d}x$

(j) $\int \frac{x}{x^2-x+3} \mathrm{d}x$

(k) $\displaystyle\int \frac{1}{(x^2+1)^2}\mathrm{d}x$

(l) $\displaystyle\int \frac{1}{(x^2+x+1)^2}\mathrm{d}x$

(m) $\displaystyle\int \frac{1}{(x^2+1)^3}\mathrm{d}x$

10. Let $a, b, c, A, B$ be real numbers. Suppose in addition $a \neq 0$ and $b^2 - 4ac < 0$. Integrate

$$\int \frac{Ax+B}{ax^2+bx+c}\mathrm{d}x \quad .$$

The purpose of this exercise is to produce a formula in form ready for implementation in a computer algebra system.

11. Let $a, b, c, A, B$ be real numbers and let $n > 1$ be an integer. Suppose in addition $a \neq 0$ and $b^2 - 4ac < 0$. Let

$$J(n) = \int \frac{1}{\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)^n}\mathrm{d}x \quad .$$

(a) Express the integral

$$\int \frac{Ax+B}{(ax^2+bx+c)^n}\mathrm{d}x$$

via $J(n)$.

(b) Express $J(n)$ recursively via $J(n-1)$

The purpose of this exercise is to produce a formula in form ready for implementation in a computer algebra system.

12. Integrate. Some of the examples require partial fraction decomposition and some do not. Illustrate the steps of your solution.

(a) $\displaystyle\int \frac{1}{4x^2+4x+1}\mathrm{d}x$

(b) $\displaystyle\int \frac{1}{1-x^2}\mathrm{d}x$

(c) $\displaystyle\int \frac{1}{5-x^2}\mathrm{d}x$

(d) $\displaystyle\int \frac{x}{4x^2+x+\frac{1}{16}}\mathrm{d}x$

(e) $\displaystyle\int \frac{x+1}{2x^2+x}\mathrm{d}x$

(f) $\displaystyle\int \frac{x}{4x^2+x+5}\mathrm{d}x$

(g) $\displaystyle\int \frac{x}{4x^2+x-5}\mathrm{d}x$

(h) $\displaystyle\int \frac{x}{3x^2+x-2}\mathrm{d}x$

(i) $\displaystyle\int \frac{x}{3x^2+x+2}\mathrm{d}x$

(j) $\displaystyle\int \frac{x}{2x^2+x+1}\mathrm{d}x$

(k) $\displaystyle\int \frac{x}{2x^2+x-1}\mathrm{d}x$

(l) $\displaystyle\int \frac{1}{x^2+x+1}\mathrm{d}x$

(m) $\displaystyle\int \frac{1}{2x^2+5x+1}\mathrm{d}x$

13. Evaluate the indefinite integral. Illustrate all steps of your solution.

(a) $\displaystyle\int \frac{x^3+4}{x^2+4}\mathrm{d}x$

(b) $\displaystyle\int \frac{4x^2}{2x^2-1}\mathrm{d}x$

(c) $\displaystyle\int \frac{x^3}{x^2+2x-3}\mathrm{d}x$

(d) $\displaystyle\int \frac{x^3}{x^2+3x-4}\mathrm{d}x$

(e) $\displaystyle\int \frac{x^3}{2x^2+3x-5}\mathrm{d}x$

(f) $\displaystyle\int \frac{x^2+1}{(x-3)(x-2)^2}\mathrm{d}x$

(g) $\displaystyle\int \frac{x^4}{(x+1)^2(x+2)}\mathrm{d}x$

(h) $\displaystyle\int \frac{15x^2-4x-81}{(x-3)(x+4)(x-1)}\mathrm{d}x$

(i) $\displaystyle\int \frac{x^4+10x^3+18x^2+2x-13}{x^4+4x^3+3x^2-4x-4}\mathrm{d}x$

Check first that $(x-1)(x+2)^2(x+1) = x^4+4x^3+3x^2-4x-4$.

(j) $\displaystyle\int \frac{x^4}{(x^2+2)(x+2)}\mathrm{d}x$

(k) $\displaystyle\int \frac{x^5}{x^3-1}\mathrm{d}x$

(l) $\displaystyle\int \frac{x^4}{(x^2+2)(x+1)^2}\mathrm{d}x$

(m) $\displaystyle\int \frac{3x^2+2x-1}{(x-1)(x^2+1)}\mathrm{d}x$

(n) $\displaystyle\int \frac{x^2-1}{x(x^2+1)^2}\mathrm{d}x$

14. Integrate
$$\int \frac{x^6 - x^5 + \frac{9}{2}x^4 - 4x^3 + \frac{13}{2}x^2 - \frac{7}{2}x + \frac{11}{4}}{x^5 - x^4 + 3x^3 - 3x^2 + \frac{9}{4}x - \frac{9}{4}}\mathrm{d}x \quad .$$

15. Integrate.

(a) $\displaystyle\int \frac{1}{3+\cos x}\mathrm{d}x.$

(b) $\displaystyle\int \frac{1}{4+\cos x}\mathrm{d}x.$

(c) $\displaystyle\int \frac{1}{3+\sin x}\mathrm{d}x.$

(d) $\displaystyle\int \frac{1}{2+\tan x}\mathrm{d}x.$ (Hint: this integral can be done simply with the substitution $x = \arctan t$.)

(e) $\displaystyle\int \frac{\mathrm{d}x}{2\sin x - \cos x + 5}.$

16. Integrate. The answer key has not been proofread, use with caution.

(a) $\displaystyle\int \sin(3x)\cos(2x)\mathrm{d}x.$

(b) $\displaystyle\int \sin x\cos(5x)\mathrm{d}x.$

(c) $\displaystyle\int \cos(3x)\sin(2x)\mathrm{d}x.$

(d) $\displaystyle\int \sin(5x)\sin(3x)\mathrm{d}x.$

(e) $\displaystyle\int \cos(x)\cos(3x)\mathrm{d}x.$

17. Integrate.

(a) $\displaystyle\int \sin^2 x\cos x\mathrm{d}x.$

(b) $\displaystyle\int \sin^2 x\mathrm{d}x.$

(c) $\displaystyle\int \cos^3 x\mathrm{d}x.$

(d) $\displaystyle\int \sin^3 x\cos^4 x\mathrm{d}x.$

18. Integrate.

(a) $\displaystyle\int \sec x\mathrm{d}x.$

(b) $\displaystyle\int \sec^3 x\mathrm{d}x.$

(c) $\displaystyle\int \tan^3 x\mathrm{d}x.$

(d) $\displaystyle\int \sec^2 x\tan^2 x\mathrm{d}x.$

19. Find a linear substitution (via completing the square) to transform the radical to a multiple of an expression of the form $\sqrt{u^2+1}$, $\sqrt{u^2-1}$ or $\sqrt{1-u^2}$.

(a) $\sqrt{x^2+x+1}.$

(b) $\sqrt{-2x^2+x+1}.$

20. Compute the integral.

(a) $\displaystyle\int \frac{\sqrt{1+x^2}}{x^2}\mathrm{d}x.$

21. Compute the integral using a trigonometric substitution.

(a) $\displaystyle\int \frac{\sqrt{9-x^2}}{x^2}\mathrm{d}x$ .

22. Compute the integral.

    (a) $\displaystyle\int \sqrt{x^2+1}\mathrm{d}x$

    (b) $\displaystyle\int \sqrt{x^2+2}\mathrm{d}x$

    (c) $\displaystyle\int \sqrt{x^2+x+1}\mathrm{d}x$

    (d) $\displaystyle\int \sqrt{(2x^2+2x+1)}\mathrm{d}x$

    (e) $\displaystyle\int \sqrt{(3x^2+2x+1)}\mathrm{d}x$

    (f) $\displaystyle\int \frac{\sqrt{x^2+1}}{x+1}\mathrm{d}x$

23. Let $b^2-4ac<0$ and $a>0$ be (real) numbers. Show that

    $\int \sqrt{(ax^2+bx+c)}\mathrm{d}x = \frac{\sqrt{a}D}{2}\left(\ln\left(\sqrt{\left(\frac{2xa+b}{2\sqrt{D}a}\right)^2+1}+\frac{2xa+b}{2\sqrt{D}a}\right)+\frac{2xa+b}{2\sqrt{D}a}\sqrt{\left(\frac{2xa+b}{2\sqrt{D}a}\right)^2+1}\right)+C,$

    where $D=\dfrac{4ac-b^2}{4a^2}$.

24. Integrate

    (a) $\displaystyle\int \sqrt{1-x^2}\mathrm{d}x$

    (b) $\displaystyle\int \sqrt{2-x^2}\mathrm{d}x$

    (c) $\displaystyle\int \sqrt{-x^2+x+1}\mathrm{d}x$

    (d) $\displaystyle\int \sqrt{2-x-x^2}\mathrm{d}x$

    (e) $\displaystyle\int \frac{\sqrt{1-x^2}}{1+x}\mathrm{d}x$

    (f) $\displaystyle\int \frac{\sqrt{1-x^2}}{2+x}\mathrm{d}x$

25. Integrate

    (a) $\displaystyle\int \sqrt{x^2-1}\mathrm{d}x$

    (b) $\displaystyle\int \sqrt{x^2-2}\mathrm{d}x$

    (c) $\displaystyle\int \sqrt{2x^2+x-1}\mathrm{d}x$

    (d) $\displaystyle\int \sqrt{x^2+x-1}\mathrm{d}x$

26. (a) Express $x, \mathrm{d}x$ and $\sqrt{x^2+1}$ via $\theta$ and $\mathrm{d}\theta$ for the trigonometric substitution $x=\cot\theta$, $\theta\in(0,\pi)$.

    (b) Express $x, \mathrm{d}x$ and $\sqrt{x^2+1}$ via $t$ and $\mathrm{d}t$ for the Euler substitution $x=\cot(2\arctan t)$, $t>0$. Express $t$ via $x$.

27. Let the variables $x$ and $t$ be related via $\sqrt{x^2+1}=x+t$.

    (a) Express $x$ via $t$.

    (b) Express $\sqrt{x^2+1}$ via $t$ alone.

    (c) Express $\mathrm{d}x$ via $t$ and $\mathrm{d}t$.

28. (a) Express $x, \mathrm{d}x$ and $\sqrt{x^2+1}$ via $\theta$ and $\mathrm{d}\theta$ for the trigonometric substitution $x = \tan\theta$, $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

    (b) Express $x, \mathrm{d}x$ and $\sqrt{x^2+1}$ via $t$ and $\mathrm{d}t$ for the Euler substitution $x = \tan(2\arctan t)$, $t \in (-1, 1)$. Express $t$ via $x$.

29. Let the variables $x$ and $t$ be related via $\sqrt{x^2+1} = \frac{x}{t} - 1$.

    (a) Express $x$ via $t$.

    (b) Express $\sqrt{x^2+1}$ via $t$ alone.

    (c) Express $\mathrm{d}x$ via $t$ and $\mathrm{d}t$.

30. (a) Express $x, \mathrm{d}x$ and $\sqrt{1-x^2}$ via $\theta$ and $\mathrm{d}\theta$ for the trigonometric substitution $x = \cos\theta$, $\theta \in [0, \pi]$.

    (b) Express $x, \mathrm{d}x$ and $\sqrt{1-x^2}$ via $t$ and $\mathrm{d}t$ for the Euler substitution $x = \cos(2\arctan t)$, $t \geq 0$. Express $t$ via $x$.

31. Let the variables $x$ and $t$ be related via $\sqrt{-x^2+1} = (1-x)t$.

    (a) Express $x$ via $t$.

    (b) Express $\sqrt{-x^2+1}$ via $t$ alone.

    (c) Express $\mathrm{d}x$ via $t$ and $\mathrm{d}t$.

32. (a) Express $x, \mathrm{d}x$ and $\sqrt{1-x^2}$ via $\theta$ and $\mathrm{d}\theta$ for the trigonometric substitution $x = \sin\theta$, $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

    (b) Express $x, \mathrm{d}x$ and $\sqrt{1-x^2}$ via $t$ and $\mathrm{d}t$ for the Euler substitution $x = \sin(2\arctan t)$, $t \in [-1, 1]$. Express $t$ via $x$.

33. Let the variables $x$ and $t$ be related via $\sqrt{-x^2+1} = 1 - xt$.

    (a) Express $x$ via $t$.

    (b) Express $\sqrt{-x^2+1}$ via $t$ alone.

    (c) Express $\mathrm{d}x$ via $t$ and $\mathrm{d}t$.

34. (a) Express $x, \mathrm{d}x$ and $\sqrt{x^2-1}$ via $\theta$ and $\mathrm{d}\theta$ for the trigonometric substitution $x = \csc\theta$, $\theta \in \left[0, \frac{\pi}{2}\right] \cup \left[\pi, \frac{3\pi}{2}\right)$.

    (b) Express $x, \mathrm{d}x$ and $\sqrt{1-x^2}$ via $t$ and $\mathrm{d}t$ for the Euler substitution $x = \sec(2\arctan t)$, $t \in (-\infty, -1) \cup [1, 0)$. Express $t$ via $x$.

35. Let the variables $x$ and $t$ be related via $\sqrt{x^2-1} = (x+1)t$.

    (a) Express $x$ via $t$.

    (b) Express $\sqrt{x^2-1}$ via $t$ alone.

    (c) Express $\mathrm{d}x$ via $t$ and $\mathrm{d}t$.

36. (a) Express $x, \mathrm{d}x$ and $\sqrt{1-x^2}$ via $\theta$ and $\mathrm{d}\theta$ for the trigonometric substitution $x = \csc\theta$, $\theta \in \left[0, \frac{\pi}{2}\right] \cup \left[\pi, \frac{3\pi}{2}\right)$.

    (b) Express $x, \mathrm{d}x$ and $\sqrt{1-x^2}$ via $t$ and $\mathrm{d}t$ for the Euler substitution $x = \csc(2\arctan t)$, $t \in (-\infty, -1) \cup [0, 1)$. Express $t$ via $x$.

37. Let the variables $x$ and $t$ be related via $\sqrt{x^2-1} = \frac{1}{t} - x$.

    (a) Express $x$ via $t$.

    (b) Express $\sqrt{x^2-1}$ via $t$ alone.

    (c) Express $\mathrm{d}x$ via $t$ and $\mathrm{d}t$.

38. Compute the limits. The answer key has not been fully proofread, use with caution.

    (a) $\displaystyle\lim_{x\to 0} \frac{\sin x}{x}$.

    (b) $\displaystyle\lim_{x\to 0} \frac{x}{\ln(1+x)}$.

    (c) $\displaystyle\lim_{x\to 0} \frac{x^2}{x - \ln(1+x)}$.

    (d) $\displaystyle\lim_{x\to 0} \frac{x^2}{\sin x \ln(1+x)}$.

    (e) $\displaystyle\lim_{x\to 0} \frac{\sin^2 x}{(\ln(1+x))^2}$.

    (f) $\displaystyle\lim_{x\to 0} \frac{\cos x - 1}{\sin x \ln(1+x)}$.

    (g) $\displaystyle\lim_{x\to 0} \frac{\arctan x - x}{x^3}$.

    (h) $\displaystyle\lim_{x\to 0} \frac{\arcsin x - x}{x^3}$.

    (i) $\displaystyle\lim_{x\to 1} \frac{x}{x-1} - \frac{1}{\ln x}$.

    (j) $\displaystyle\lim_{x\to 0} \frac{\cos(nx) - \cos(mx)}{x^2}$.

(k) $\lim\limits_{x \to 0} \dfrac{\arcsin x - x - \frac{1}{6}x^3}{\sin^5 x}$.

(l) $\lim\limits_{x \to 1} \dfrac{\sin(\pi x) \ln x}{\cos(\pi x) + 1}$.

(m) $\lim\limits_{x \to 0} \dfrac{\sin x - x}{\arcsin x - x}$.

(n) $\lim\limits_{x \to 0} \dfrac{\sin x - x}{\arctan x - x}$.

(o) $\lim\limits_{x \to \infty} x \sin\left(\dfrac{2}{x}\right)$.

39. Compute the limit.

(a) $\lim\limits_{x \to \infty} \left(\dfrac{x-2}{x}\right)^x$.

(b) $\lim\limits_{x \to \infty} \left(\dfrac{x-2}{x}\right)^{2x}$

(c) $\lim\limits_{x \to \infty} \left(\dfrac{x}{x+3}\right)^{2x}$

40. Find the limit.

(a) $\lim\limits_{x \to \infty} \left(1 - \dfrac{2}{x}\right)^x$.

(b) $\lim\limits_{x \to 0} (1-x)^{\frac{1}{x}}$.

(c) $\lim\limits_{x \to \infty} \left(\dfrac{x}{x-5}\right)^x$.

(d) $\lim\limits_{x \to \infty} \left(\dfrac{x}{x-2}\right)^{3x+2}$.

41. Determine whether the integral is convergent or divergent. Motivate your answer.

(a) $\displaystyle\int_2^\infty \frac{1}{(x-1)^{\frac{3}{2}}} \mathrm{d}x$.

(b) $\displaystyle\int_{-1}^1 \frac{1}{\sqrt[5]{1+x}} \mathrm{d}x$.

(c) $\displaystyle\int_1^\infty \frac{1}{\sqrt[5]{1+x}} \mathrm{d}x$.

(d) $\displaystyle\int_{-1}^\infty \frac{1}{\sqrt[5]{1+x}} \mathrm{d}x$.

(e) $\displaystyle\int_{-\infty}^0 \frac{1}{2-3x} \mathrm{d}x$.

(f) $\displaystyle\int_{-\infty}^0 \frac{1}{(2-3x)^2} \mathrm{d}x$.

(g) $\displaystyle\int_{-\infty}^0 \frac{1}{(2-3x)^{1.00000001}} \mathrm{d}x$.

(h) $\displaystyle\int_{-2}^{\frac{1}{2}} \frac{1}{2x-1} \mathrm{d}x$.

(i) $\displaystyle\int_{-1}^\infty e^{-3x} \mathrm{d}x$.

(j) $\displaystyle\int_{-\infty}^5 2^x \mathrm{d}x$.

(k) $\displaystyle\int_{-\infty}^\infty x^3 \mathrm{d}x$.

(l) $\displaystyle\int_{-\infty}^\infty x e^{-x^2} \mathrm{d}x$.

(m) $\displaystyle\int_0^\infty \sqrt{x} e^{-\sqrt{x}} \mathrm{d}x$.

(n) $\displaystyle\int_0^\infty \sin^2 x \mathrm{d}x$.

(o) $\displaystyle\int_0^5 \frac{1}{x^2+x-2} \mathrm{d}x$.

(p) $\displaystyle\int_0^\infty \frac{1}{x^2+x+1} \mathrm{d}x$.

(q) $\displaystyle\int_2^\infty \frac{1}{x^2-x-1} \mathrm{d}x$.

(r) $\displaystyle\int_0^\infty \frac{1}{x^2-x-1} \mathrm{d}x$.

(s) $\displaystyle\int_{-\infty}^\infty \frac{x^2}{x^4+2} \mathrm{d}x$.

(t) $\displaystyle\int_{100}^\infty \frac{1}{x \ln x} \mathrm{d}x$.

(u) $\displaystyle\int_{100}^{\infty} \frac{1}{x(\ln x)^2} \mathrm{d}x.$

(v) $\displaystyle\int_{0}^{1} \ln x \mathrm{d}x.$

(w) $\displaystyle\int_{0}^{1} \frac{\ln x}{\sqrt{x}} \mathrm{d}x.$

(x) $\displaystyle\int_{0}^{2} x^3 \ln x \mathrm{d}x.$

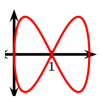
(y) $\displaystyle\int_{0}^{1} \frac{e^{\frac{1}{x}}}{x^2} \mathrm{d}x.$

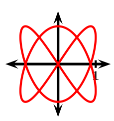
(z) $\displaystyle\int_{-1}^{0} \frac{e^{\frac{1}{x}}}{x^2} \mathrm{d}x.$

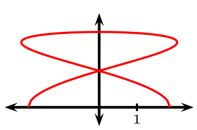
42. Determine whether the integral is convergent or divergent. Motivate your answer. The answer key has not been proofread, use with caution.

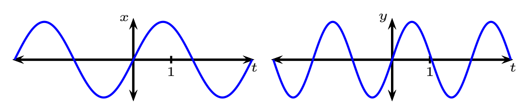
(a) $\displaystyle\int_{0}^{\infty} \sin x^2 \mathrm{d}x$ (This problem is more difficult and may require knowledge of sequences to solve).

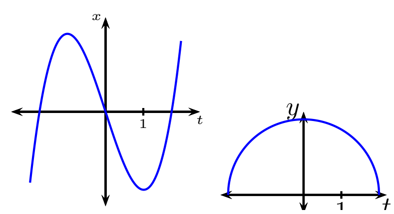
43. Match the graphs of the parametric equations $x = f(t)$, $y = g(t)$ with the graph of the parametric curve $\gamma : \left| \begin{array}{rcl} x & = & f(t) \\ y & = & g(t) \end{array} \right.$

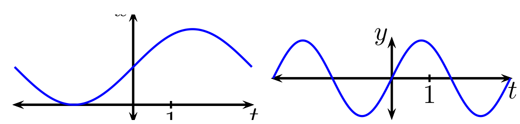
(a) 

(b) 

(c) 

(a) 

(b) 

(c) 

44.

Match the graph of the curve to its graph in polar coordinates and to its polar parametric equations.

(i) $r = 1 + \sin(\theta) + cos(\theta)$

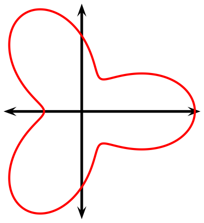
(ii) $r = \theta, \theta \in [-\pi, \pi]$.

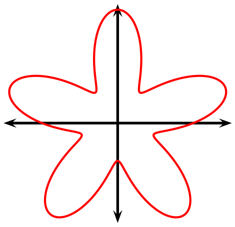
(iii) $r = \cos(3\theta), \theta \in [0, 2\pi]$.

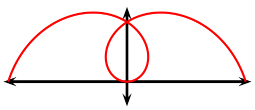
(iv) $r = \frac{1}{4}\sqrt{\theta}, \theta \in [0, 10\pi]$.

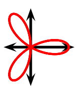
(v) $r = 2 + \sin(5\theta)$.

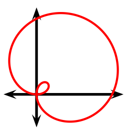
(vi) $r = 2 + cos(3\theta)$.

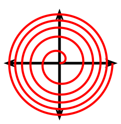
(a) 

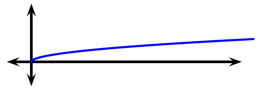
(b) 

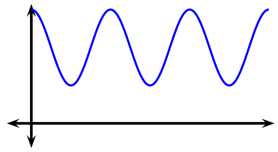
(c) 

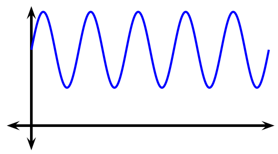
(d) 

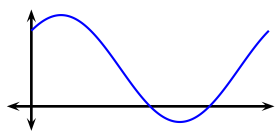
(e) 

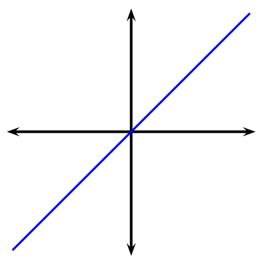
(f) 

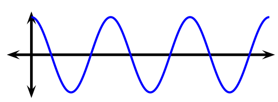
(a) 

(b) 

(c) 

(d) 

(e) 

(f) 

45.

(a) Sketch the curve given in polar coordinates by $r = 2\sin\theta$. What kind of a figure is this curve? Find an equation satisfied by the curve in the $(x, y)$-coordinates.

(b) Sketch the curve given in polar coordinates by $r = 4\cos\theta$. What kind of a figure is this curve? Find an equation satisfied by the curve in the $(x, y)$-coordinates.

(c) Sketch the curve given in polar coordinates by $r = 2\sec\theta$. What kind of a figure is this curve? Find an equation satisfied by the curve in the $(x, y)$-coordinates.

(d) Sketch the curve given in polar coordinates by $r = 2\csc\theta$. What kind of a figure is this curve? Find an equation satisfied by the curve in the $(x, y)$-coordinates.

(e) Sketch the curve given in polar coordinates by $r = 2\sec\left(\theta + \frac{\pi}{4}\right)$. What kind of a figure is this curve? Find an equation satisfied by the curve in the $(x, y)$-coordinates.

(f) Sketch the curve given in polar coordinates by $r = 2\csc\left(\theta + \frac{\pi}{6}\right)$. What kind of a figure is this curve? Find an equation satisfied by the curve in the $(x, y)$-coordinates.

46. Find the values of the parameter $t$ for which the curve has horizontal and vertical tangents.

(a) $x = t^2 - t + 1, y = t^2 + t - 1$

(b) $x = t^3 - t^2 - t + 1, y = t^2 - t - 1$.

(c) $x = \cos(t), y = \sin(3t)$

(d) $x = \cos(t) + \sin(t)\,, y = \sin(t)$.

47. Show that the parametric curve has multiple tangents at the point and find their slopes.

(a) $x = \cos t$, $y = 2\sin(2t)$, two tangents at $(x, y) = (0, 0)$.
(b) $x = \cos t \sin(3t)$, $y = \sin(t)\sin(3t)$, six tangents at $(x, y) = (0, 0)$.
(c) $x = \cos t, y = \sin(3t)$, find the two points at which the curve has double tangent and find the slopes of both pairs of tangents.
(d) $x = t^3 - t^2 - t + 1$, $y = t^2 - t - 1$, find a point where the curve has double tangent and find the slopes of the tangents.

48. Find the length of the curve.

(a) $y = x^2$, $x \in [1, 2]$.
(b) $y = \sqrt{x}$, $x \in [1, 2]$.
(c) $x = \sqrt{t} - 2t$ and $y = \dfrac{8}{3}t^{\frac{3}{4}}$ from $t = 1$ to $t = 4$.
(d) $\gamma : \left| \begin{array}{rcl} x(t) &=& \frac{1}{t} + \frac{t^3}{3} \\ y(t) &=& 2t \end{array} \right. , t \in [1, 2]$ .
(e) $\gamma : \left| \begin{array}{rcl} x(t) &=& \frac{1}{t} + t \\ y(t) &=& 2\ln t \end{array} \right. , t \in [1, 2]$ .
(f) One arch of the cycloid
$$\gamma : \left| \begin{array}{rcl} x(t) &=& t - \sin t \\ y(t) &=& 1 - \cos t \end{array} \right. , t \in [0, 2\pi]$$
(g) The cardioid
$$\gamma : \left| \begin{array}{rcl} x(t) &=& (1 + \sin t)\cos t \\ y(t) &=& (1 + \sin t)\sin t \end{array} \right. , t \in [0, 2\pi]$$

49. Set up an integral that expresses the length of the curve and find the length of the curve.

(a) $\left| \begin{array}{rcl} x(t) &=& e^t + e^{-t} \\ y(t) &=& 5 - 2t \end{array} \right. , t \in [0, 3]$
(b) $\left| \begin{array}{rcl} x(t) &=& \sin t + \cos t \\ y(t) &=& \sin t - \cos t \end{array} \right. , t \in [0, \pi]$

50. Give a geometric definition of the cycloid curve using a circle of radius 1. Using that definition, derive equations for the cycloid curve. Find area locked between one "arch" of the cycloid curve and the $x$ axis.

51. (a) The curve given in polar coordinates by $r = 1 + \sin 2\theta$ is plotted below by computer. Find the area lying outside of this curve and inside of the circle $x^2 + y^2 = 1$.
(b) The curve given in polar coordinates by $r = \cos(2\theta)$ is plotted below by computer. Find the area lying inside the curve and outside of the circle $x^2 + y^2 = \frac{1}{4}$.
(c) Below is a computer generated plot of the curve $r = \sin(2\theta)$. Find the area locked inside one petal of the curve and outside of the circle $x^2 + y^2 = \dfrac{1}{4}$.

52. The answer key has not been proofread, use with caution.

(a) Sketch the graph of the curve given in polar coordinates by $r = 3\sin(2\theta)$ and find the area of one petal.
(b) Sketch the graph of the curve given in polar coordinates by $r = 4 + 3\sin\theta$ and find the area enclosed by the curve.

53. List the first 4 elements of the sequence.

(a) $a_n = \dfrac{(-1)^n}{n}$.
(b) $a_n = \dfrac{1}{n!}$.
(c) $a_n = \cos(\pi n)$.
(d) $a_n = \dfrac{(-1)^n}{2n + 1}$.
(e) $a_n = \dfrac{\sqrt{5}}{5}\left(\left(\dfrac{1 + \sqrt{5}}{2}\right)^n - \left(\dfrac{1 - \sqrt{5}}{2}\right)^n\right)$

54. List the first 5 elements of the sequence.

(a) $a_{n+1} = \frac{1}{2}\left(a_n + \frac{3}{a_n}\right)$, $a_1 = 1$.

(b) $a_n = a_{n-1} + a_{n-2}$, $a_1 = 1$, $a_2 = 1$.

(c) $a_n = \frac{\left(\frac{1}{2} - n\right)}{n} a_{n-1}$, $a_0 = 1$.

(d) $a_n = a_{n-1} + 2n + 1$, $a_0 = 1$.

(e) $a_n := \frac{1}{n} a_{n-1}$, $a_1 = 1$.

55. Give a simple sequence formula that matches the pattern below.

(a) $\left(1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \dots\right)$.

(b) $\left(-1, \frac{1}{5}, -\frac{1}{25}, \frac{1}{125}, -\frac{1}{625}, \frac{1}{3125} \dots\right)$

(c) $\left(-5, 2, -\frac{4}{5}, \frac{8}{25}, -\frac{16}{125}, \frac{32}{625}, \dots\right)$

(d) $(4, 7, 10, 13, 16, 19, \dots)$

(e) $\left(-2, \frac{3}{4}, -\frac{4}{9}, \frac{5}{16}, -\frac{6}{25}, \frac{7}{36}, \dots\right)$

(f) $(0, -1, 0, 1, 0, -1, 0, 1, 0, -1, 0, 1, \dots)$

56. Determine if the sequence is convergent or divergent. If convergent, find the limit of the sequence.

(a) $a_n = n$.

(b) $a_n = 2^n$.

(c) $a_n = 1.0001^n$.

(d) $a_n = 0.999999^n$.

(e) $a_n = n - \sqrt{n+1}\sqrt{n+2}$

(f) $a_n = \frac{\ln n}{n}$.

(g) $a_n = \frac{\ln n}{\sqrt[10]{n}}$.

(h) $a_n = \frac{1}{n}$.

(i) $a_n = \frac{1}{n!}$.

(j) $a_n = \frac{n^n}{n!}$.

(k) $a_n = \cos n$.

(l) $a_n = \cos\left(\frac{1}{n}\right)$

(m) $a_n = \left(\frac{n+1}{n}\right)^n$.

(n) $a_n = \left(\frac{2n+1}{n}\right)^n$.

(o) $a_n = \left(\frac{n+1}{n}\right)^{2n}$.

(p) $a_n = \left(\frac{n+1}{2n}\right)^n$.

57. Express the infinite decimal number as a rational number.

(a) $0.\overline{9} = 0.99999\dots$

(b) $1.\overline{6} = 1.6666\dots$

(c) $1.\overline{3} = 1.3333\dots$

(d) $1.\overline{19} = 1.191919\dots$.

(e) $0.\overline{09} = 0.0909090909\dots$.

(f) $2.\overline{16} = 2.16161616\dots$

(g) $2014.\overline{2014} = 2014.2014201420142014\dots$

58. Express the sum of the series as a rational number.

(a) $\sum_{n=1}^{\infty} \frac{2^n + 3^n}{5^n}$

(b) $\sum_{n=0}^{\infty} \frac{2^n + 5^n}{10^n}$

(c) $\sum_{n=1}^{\infty} \frac{5^n - 3^n}{7^n}$

(d) $\sum_{n=1}^{\infty} \frac{3^{n+1} + 7^{n-1}}{21^n}$

(e) $\sum_{n=0}^{\infty} \frac{2^{n+1} + (-3)^{n-1}}{5^n}$

59. Sum the telescoping series (a sum is "telescoping" if it can be broken into summands so that consecutive terms cancel).

(a) $\sum_{n=0}^{\infty} \frac{-6}{9n^2 + 3n - 2}$ .

(b) $\sum_{n=3}^{\infty} \frac{3}{n^2 - 3n + 2}$ .

(c) $\sum_{n=2}^{\infty} \ln\left(1 - \frac{1}{n^2}\right)$. (Hint: Use the properties of the logarithm to aim for a telescoping series).

60. Use partial fractions to sum the telescoping series (a sum is "telescoping" if it can be broken into summands so that consecutive terms cancel).

(a) $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$

(b) $\sum_{n=2}^{\infty} \frac{2n+1}{n^4+2n^3-n^2-2n}$

(c) $\sum_{n=1}^{\infty} \frac{2n}{n^4-3n^2+1}$

(d) $\sum_{n=3}^{\infty} \frac{n^2+n+2}{n^4-5n^2+4}$

61. Find whether the series is convergent or divergent using an appropriate test. Some of the problems require the alternating series test. The test states the following.

**Alternating series test.** Suppose $b_n \searrow 0$. Then $\sum (-1)^n b_n$ is convergent.

Here, $b_n \searrow 0$ means the following.

- The sequence of numbers $b_n$ is decreasing.
- The sequence decreases to 0, that is,

$$\lim_{n\to\infty} b_n = 0 \quad .$$

(a) $\sum_{n=1}^{\infty} (-1)^n \ln n$.

(b) $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$.

(c) $\sum_{n=2}^{\infty} \frac{n}{\ln n}$

(d) $\sum_{n=2}^{\infty} \frac{\ln n}{n}$

62. Use the integral test, the comparison test or the limit comparison test to determine whether the series is convergent or divergent. Justify your answer.

(a) $\sum_{n=1}^{\infty} \frac{1}{2n+1}$.

(b) $\sum_{n=1}^{\infty} \frac{1}{2n^2+n^3}$.

(c) $\sum_{n=1}^{\infty} \frac{n^2+3}{3n^5+n}$

(d) $\sum_{n=0}^{\infty} \frac{1}{3^n+5}$.

(e) $\sum_{n=2}^{\infty} \frac{1}{n\ln n}$

(f) $\sum_{n=2}^{\infty} \frac{1}{(2n+1)\ln(n)}$.

(g) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

(h) $\sum_{n=2}^{\infty} \frac{1}{(2n+1)(\ln(n))^2}$.

(i) Determine all values of $p$, $q$ $r$ for which the series

$$\sum_{n=30}^{\infty} \frac{1}{n^p(\ln n)^q(\ln(\ln n))^r}$$

is convergent.

63. Establish whether the series is convergent or divergent. Use the ratio or root tests. Show all your work. The answer key has not been proofread, use with caution.

(a) $\sum_{n=0}^{\infty} (-1)^n n^2 3^{-n}$

(b) $\sum_{n=1}^{\infty} \left(\frac{n+1}{4n}\right)^n$

(c) $\sum_{n=1}^{\infty} \left(\frac{4n+1}{n}\right)^n$

(d) $\sum_{n=1}^{\infty} \frac{n^n}{4^n n!}$

(e) $\sum_{n=1}^{\infty} \frac{(4n)^n}{n!}$

64. Except for $x = \pm e$, use the ratio test to determine all real values of $x$ for which

$$\sum_{n=0}^{\infty} x^n \frac{n!}{n^n}$$

is convergent. You are expected to use in your solution the fact that

$$\lim_{x \to 0} \left(1 + \frac{x}{n}\right)^n = e^x \quad .$$

65. Determine the interval of convergence for the following power series.

(a) $\sum_{n=1}^{\infty} \frac{(x-2)^n}{3\sqrt{n+1}}$.

(b) $\sum_{n=1}^{\infty} \frac{10^n x^n}{n^3}$.

(c) $\sum_{n=1}^{\infty} \frac{10^n (x-1)^n}{n^3}$.

(d) $\sum_{n=0}^{\infty} (-1)^n \frac{(x+1)^n}{2n+1}$.

(e) $\sum_{n=0}^{\infty} (-1)^n \frac{(x-3)^n}{2n+1}$.

(f) $\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

(g) $\sum_{n=0}^{\infty} (n+1) x^n$.

(h) $\sum_{n=1}^{\infty} \frac{x^n}{n}$.

(i) $\sum_{n=1}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$.

(j) $\sum_{n=1}^{\infty} \binom{\frac{1}{2}}{n} x^n$, where we recall that the binomial coefficient $\binom{q}{n}$ stands for $\frac{q(q-1)\dots(q-n+1)}{n!}$.

66. (a) Find the Maclaurin series for $xe^{x^3}$.

(b) Use your series to find the Maclaurin series of $\int xe^{x^3} \mathrm{d}x$.

67. For each of the items below, do the following.

- Find the Maclaurin series of the function (i.e., the power series representation of the function around $a = 0$).
- Find the radius of convergence of the series you found in the preceding point. You are not asked to find the entire interval of convergence, but just the radius.

(a) $e^x$.

(b) $xe^{-2x}$.

(c) $e^{2x}$.

(d) $e^{x^2}$.

(e) $e^{-3x^2}$.

(f) $x^2 e^{2x}$.

(g) $\sin x$.

(h) $\cos x$.

(i) $\sin(2x)$.

(j) $\cos(2x)$.

(k) $\cos^2(x)$.

(l) $x \sin x$.

68. For each of the items below, do the following.

   - Find the Maclaurin series of the function (i.e., the power series representation of the function around $a = 0$).
   - Find the radius of convergence of the series you found in the preceding point.

   (a) $\dfrac{1}{3-x}$.

   (b) $\dfrac{1}{3-2x}$.

   (c) $\dfrac{1}{2x+3}$.

   (d) $\dfrac{1}{1+x^2}$.

   (e) $\dfrac{1}{1-2x^2}$.

   (f) $\dfrac{1}{x^2-1}$.

   (g) $\dfrac{\frac{1}{2}}{x-1} - \dfrac{\frac{1}{2}}{x+1}$.

   (h) $\dfrac{1}{(1-x)^2}$.

   (i) $\dfrac{1}{(1-x)^3}$.

   (j) $\ln(1+x)$.

   (k) $\ln(1-x)$.

   (l) $\ln(1-3x)$.

   (m) $\ln(1-3x^2)$.

   (n) $\ln(3-2x^2)$.

   (o) $x\ln(3-2x^2)$.

   (p) $\arctan x$.

   (q) $\arctan(2x)$.

   (r) $\arctan\left(2x^2\right)$.

69. Compute the Maclaurin series of

$$\left(\frac{1}{(1-x)^k}\right) \quad ,$$

   where $n \geq 1$ is an integer.

70. Compute the Maclaurin series of

$$(1+x)^q \quad ,$$

   where $q \in \mathbb{R}$ is an arbitrary real number.

71. Compute the Maclaurin series of the function.

   (a) $\sqrt{1+x}$.

   (b) $\dfrac{1}{\sqrt{1+x}}$.

   (c) $\dfrac{1}{\sqrt{1-x^2}}$.

   (d) $\arcsin x$.

72. Find the Taylor series of the function at the indicated point.

   (a) $\frac{1}{x^2}$ at $a = -1$.

   (b) $\ln\left(\sqrt{x^2-2x+2}\right)$ at $a = 1$.

   (c) Write the Taylor series of the function $\ln x$ around $a = 2$.

73. Find the Taylor series around the indicated point. The answer key has not been proofread, use with caution.

   (a) $\frac{1}{x}$ at $a = 1$.

   (b) $\frac{1}{x^2}$ at $a = 1$.

74. Let $f(x)$ be defined as

$$f(x) := \begin{cases} e^{-\frac{1}{x^2}} & \text{if } x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Prove that if $R(x)$ is an arbitrary rational function,

$$\lim_{\substack{x \to 0 \\ x > 0}} R(x) e^{-\frac{1}{x^2}} = 0$$

   (b) Prove that $f(x)$ is differentiable at 0 and $f'(0) = 0$.

   (c) Prove that the Maclaurin series of $f(x)$ are 0 (but $f(x)$ is clearly a non-zero function).

75. (a) A tank contains 30 kg of salt dissolved in 10000 liters of water and salt solution. Brine that contains 0.05 kg of salt per liter enters the tank at a rate of 10 liters per minute. The solution is kept thoroughly mixed and drains from the tank at the same rate (10 liters per minute). Determine how much salt remains in the tank after 45 minutes.

(b) A tank contains 1000 kg of salt dissolved in 10000 liters of water. Brine that contains 0.05 kg of salt per liter of water enters the tank at a rate of 30 liters per minute. The solution is kept thoroughly mixed and drains from the tank at the same rate (30 liters per minute).

i. Determine how much salt remains in the tank after an hour. The answer key has not been proofread, use with caution.

ii. Determine how much time will be needed in order to have the concentration of salt in the tank reach 0.0501kg/liter. The answer key has not been proofread, use with caution.

76. (a)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 - 1 \quad . \tag{1}$$

i. Find all solutions of the differential equation above.

ii. Find a solution for which $y(0) = -\frac{3}{5}$.

(b) i. Find the general solution to the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 - 4 \quad .$$

Below is a computer-generated plot of the direction field $\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 - 4$, you may use it to get a feeling for what your answer should look like.

ii. Find a solution of the above equation for which $y(0) = -\frac{6}{5}$.

(c) Solve the initial-value differential equation $y' = y^2(1 + x)$, $y(0) = 3$.

(d) Solve the initial-value differential equation problem

$$y' = xe^{-y} \quad , \qquad y(4) = 0.$$

Below is a computer-generated plot of the corresponding direction field, you may use it to get a feeling for what your answer should look like.

(e) Solve the initial-value differential equation problem

$$y' = \frac{\ln x}{xy} \quad , \qquad y(1) = 2.$$

Below is a computer-generated plot of the corresponding direction field, you may use it to get a feeling for what your answer should look like.

(f) i. Solve the initial-value differential equation problem

$$y' = x \tan y \quad , \qquad y(0) = \arcsin\left(\frac{1}{e}\right) \approx 0.376728.$$

ii. Solve the same differential equation with initial condition $y(0) = \pi + \arcsin\left(-\frac{1}{e}\right) \approx 2.764865$.
Below is a computer-generated plot of corresponding direction field, you may use it to get a feeling for what your answer should look like.