

Calculus II

Integrals of the form $\int \sin^n x \cos^m x dx$, both
powers even

Todor Milev

2019

Example

$$\int_0^{\frac{\pi}{2}} \sin^2 x dx$$

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$$\int_0^{\frac{\pi}{2}} \sin^2 x dx = \int_0^{\frac{\pi}{2}} \left(\frac{1 - \cos(2x)}{2} \right) dx$$

express $\sin^2 x$
via $\cos(2x)$

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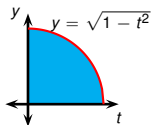
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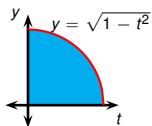
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Set $t = \cos x$, $x \in [0, \frac{\pi}{2}]$.



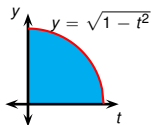
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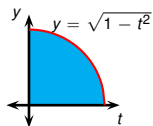
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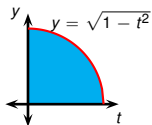
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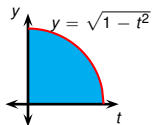
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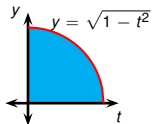
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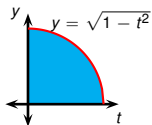


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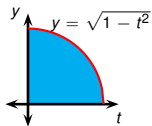
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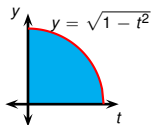
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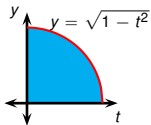
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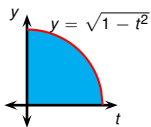
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