Precalculus
Homework
Logarithms basics

1. Convert from degrees to radians.

   (a) 15°.

   answer: $\frac{\pi}{12} \approx 0.261799388$

   (b) 30°.

   answer: $\frac{\pi}{6} \approx 0.523598776$

   (c) 36°.

   answer: $\frac{\pi}{5} \approx 0.628318531$

   (d) 45°.

   answer: $\frac{\pi}{4} \approx 0.785398163$

   (e) 60°.

   answer: $\frac{\pi}{3} \approx 1.047197551$

   (f) 75°.

   answer: $\frac{5\pi}{12} \approx 1.308997$

   (g) 90°.

   answer: $\frac{\pi}{2}$

   (h) 120°.

   answer: $\frac{2\pi}{3}$

   (i) 135°.

   answer: $\frac{3\pi}{4}$

   (j) 150°.

   answer: $\frac{5\pi}{6}$

   (k) 180°.

   answer: π

   (l) 225°.

   answer: $\frac{5\pi}{4}$

   (m) 270°.

   answer: $\frac{3\pi}{2}$

   (n) 305°.

   answer: $\frac{61\pi}{36} \approx 5.323254$

   (o) 360°.

   answer: 2π

   (p) 405°.

   answer: $\frac{9\pi}{4}$

   (q) 1200°.

   answer: $\frac{20\pi}{3}$

   (r) -900°.

   answer: -5π

   (s) -2014°.

   answer: $-\frac{1007}{90}\pi \approx -35.150931$

2. Convert from radians to degrees. The answer key has not been proofread, use with caution.

   (a) 4π.

   answer: 720°

   (b) $-\frac{7}{6}\pi$.

   answer: -210°

   (c) $\frac{7}{12}\pi$.

   answer: 105°

   (d) $\frac{4}{3}\pi$.

   answer: 240°

   (e) $-\frac{3}{8}\pi$.

   answer: -67.5°

   (f) 2014π.

   answer: 362520°

   (g) 5.

   answer: $\left(\frac{900}{\pi}\right)^\circ \approx 286^\circ$

   (h) -2014.

   answer: -362520°

3. Find the indicated circle arc-length. The answer key has not been proofread, use with caution.

   (a) Circle of radius 3, arc of measure 36°.

   answer: $\frac{3\pi}{5} \approx 1.884956$

   (b) Circle of radius $\frac{1}{2}$, arc of measure 100°.

   answer: $\frac{5\pi}{18} \approx 0.872665$

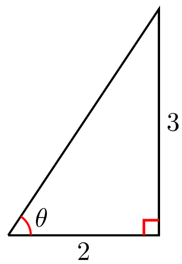
   (c) Circle of radius 1, arc of measure 3 (radians).

   answer: 3

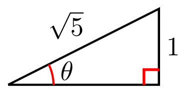
   (d) Circle of radius 3, arc of measure 300°.

   answer: $5\pi \approx 15.707963$

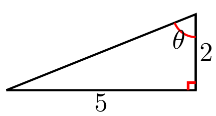
4. Find the 6 trigonometric functions of the indicated angle in the indicated right triangle.

(a) 

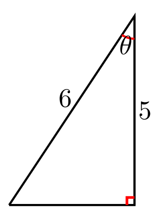
answer: $\sin\theta = \frac{3}{13}\sqrt{13}, \cos\theta = \frac{2}{13}\sqrt{13}, \tan\theta = \frac{3}{2}, \cot\theta = \frac{2}{3}, \sec\theta = \frac{\sqrt{13}}{2}, \csc\theta = \frac{\sqrt{13}}{3}$

(b) 

answer: $\sin\theta = \frac{\sqrt{5}}{5}, \cos\theta = \frac{2\sqrt{5}}{5}, \tan\theta = \frac{1}{2}, \cot\theta = 2, \sec\theta = \frac{\sqrt{5}}{2}, \csc\theta = \sqrt{5}$

(c) 

answer: $\sin\theta = \frac{5}{\sqrt{29}} = \frac{5\sqrt{29}}{29}, \cos\theta = \frac{2}{\sqrt{29}} = \frac{2\sqrt{29}}{29}, \tan\theta = \frac{2}{5}, \cot\theta = \frac{5}{2}, \sec\theta = \frac{\sqrt{29}}{5}, \csc\theta = \frac{\sqrt{29}}{2}$

(d) 

answer: $\sin\theta = \frac{\sqrt{11}}{6}, \cos\theta = \frac{5}{6}, \tan\theta = \frac{\sqrt{11}}{5}, \cot\theta = \frac{5}{\sqrt{11}} = \frac{5\sqrt{11}}{11}, \sec\theta = \frac{6}{5}, \csc\theta = \frac{6}{\sqrt{11}} = \frac{6\sqrt{11}}{11}$

5. Find the exact value of the trigonometric function (using radicals).

(a) $\cos 135°$.

answer:

(b) $\sin 225°$.

answer:

(c) $\cos 495°$.

answer:

(d) $\sin 560°$.

answer:

(e) $\sin\left(\frac{3\pi}{2}\right)$.

answer:

(f) $\cos\left(\frac{11\pi}{6}\right)$.

answer:

(g) $\sin\left(\frac{2015\pi}{3}\right)$.

answer:

(h) $\cos\left(\frac{17\pi}{3}\right)$.

answer:

6. Find all solutions of the equation in the interval $[0, 2\pi)$. The answer key has not been proofread, use with caution.

(a) $\sin x = -\frac{\sqrt{2}}{2}$.

answer: $x = \frac{5\pi}{4}, \frac{7\pi}{4}$

(b) $\cos x = \frac{\sqrt{3}}{2}$.

answer: $x = \frac{\pi}{6}, \frac{11\pi}{6}$

(c) $\sin(3x) = \frac{1}{2}$.

answer: $x = \frac{\pi}{18}, \frac{5\pi}{18}, \frac{13\pi}{18}, \frac{17\pi}{18}, \frac{25\pi}{18}, \frac{16\pi}{9}$

(d) $\cos(7x) = 0$.

answer: $x = \frac{\pi}{14}, \frac{3\pi}{14}, \frac{5\pi}{14}, \frac{\pi}{2}, \frac{9\pi}{14}, \frac{11\pi}{14}, \frac{13\pi}{14}, \frac{15\pi}{14}, \frac{17\pi}{14}, \frac{19\pi}{14}, \frac{3\pi}{2}, \frac{23\pi}{14}, \frac{25\pi}{14}, \frac{27\pi}{14}$

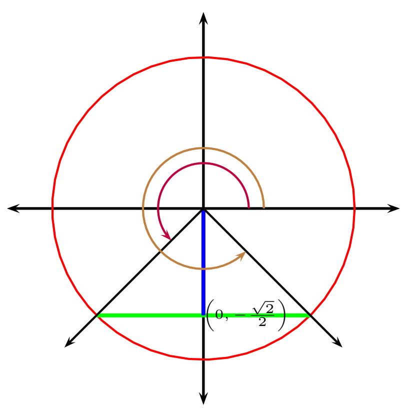
(e) $\cos\left(3x + \frac{\pi}{2}\right) = 0$.

answer: $x = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}$

(f) $\sin\left(5x - \frac{\pi}{3}\right) = 0$.

answer: $x = \frac{\pi}{15}, \frac{4\pi}{15}, \frac{7\pi}{15}, \frac{2\pi}{3}, \frac{13\pi}{15}, \frac{16\pi}{15}, \frac{19\pi}{15}, \frac{22\pi}{15}, \frac{5\pi}{3}, \frac{28\pi}{15}$

Solution. 6.a



$$\sin x = -\frac{\sqrt{2}}{2}$$

Since $\sin x$ is negative it must be either in Quadrant III or IV. Therefore the angle x is coterminal either with $225° = \frac{5\pi}{4}$ (Quadrant III) or $315° = \frac{7\pi}{4}$ (Quadrant IV).

Case 1. x is coterminal with $225° = \frac{5\pi}{4}$. We can compute

$$\begin{array}{rcll} x &=& \frac{5\pi}{4} + 2k\pi & \Big| \; k \text{ is any integer} \\ x &=& \frac{5\pi}{4} + \frac{8k\pi}{4} & \\ x &=& \frac{5\pi + 8k\pi}{4} & \\ x &=& \frac{\pi(5+8k)}{4} & \end{array}$$

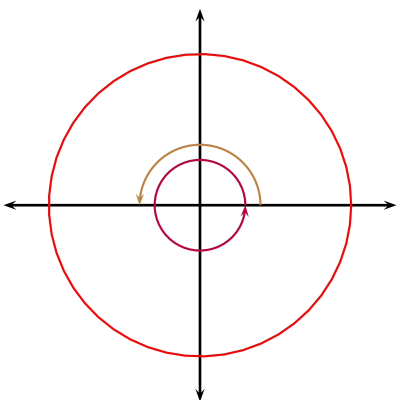
We are looking for solutions in the interval $[0, 2\pi)$ and so we must discard those values of the integer k for which $\frac{\pi(7+8k)}{4}$ is negative or is greater than or equal to 2π. Therefore the only solution in this case is $x = \frac{5\pi}{4}$.

Case 2.

$$\begin{aligned}
x &= \frac{7\pi}{4} + 2k\pi \\
x &= \frac{7\pi}{4} + \frac{8k\pi}{4} \\
x &= \frac{7\pi + 8k\pi}{4} \\
x &= \frac{\pi(7+8k)}{4}
\end{aligned}$$

We are looking for solutions in the interval $[0, 2\pi)$ and so we must discard those values of the integer k for which $\frac{\pi(7+8k)}{4}$ is negative or is greater than or equal to 2π. Therefore the only solution in this case is $x = \frac{7\pi}{4}$.

Solution. 6.f



$$\sin\left(5x - \frac{\pi}{3}\right) = 0$$

Since $\sin 0 = 0$ and $\sin 180^\circ = \sin\pi = 0$, the angle $5x - \frac{\pi}{3}$ must be coterminal with 0 or π.

Case 1. $5x - \frac{\pi}{3}$ is coterminal with 0. We compute

$$\begin{aligned}
5x - \frac{\pi}{3} &= 0 + 2k\pi \\
5x &= \frac{\pi}{3} + 2k\pi \\
x &= \frac{\frac{\pi}{3} + 2k\pi}{5} \\
x &= \frac{\frac{\pi}{3} + \frac{6k\pi}{3}}{5} \\
x &= \frac{\frac{\pi+6k\pi}{3}}{5} \\
x &= \frac{\pi + 6k\pi}{15} \\
x &= \frac{\pi(1+6k)}{15} \\
x &= \cancel{\dots}, \frac{\pi[1+6(0)]}{15}, \frac{\pi[1+6(1)]}{15}, \frac{\pi[1+6(2)]}{15}, \frac{\pi(1+12)}{15}, \frac{\pi[1+6(3)]}{15}, \frac{\pi[1+6(4)]}{15}, \cancel{\dots} \\
x &= \frac{\pi}{15}, \frac{7\pi}{15}, \frac{13\pi}{15}, \frac{19\pi}{15}, \frac{25\pi}{15}.
\end{aligned}$$

Discard other values of k as they yield angles outside of $[0, 2\pi)$

Case 2.

$$
\begin{aligned}
5x - \frac{\pi}{3} &= \pi + 2k\pi \\
5x &= \pi + \frac{\pi}{3} + 2k\pi \\
5x &= \frac{4\pi}{3} + 2k\pi \\
x &= \frac{\frac{4\pi}{3} + 2k\pi}{5} \\
x &= \frac{\frac{4\pi}{3} + \frac{6k\pi}{3}}{5} \\
x &= \frac{\frac{4\pi + 6k\pi}{3}}{5} \\
x &= \frac{4\pi + 6k\pi}{15} \\
x &= \frac{2\pi(2+3k)}{15} \\
x &= \cancel{\dots}, \frac{2\pi[2+3(0)]}{15}, \frac{2\pi[2+3(1)]}{15}, \frac{2\pi[2+3(2)]}{15}, \frac{2\pi[2+3(3)]}{15}, \frac{2\pi[2+3(4)]}{15}, \cancel{\dots} \\
x &= \frac{4\pi}{15}, \frac{10\pi}{15}, \frac{16\pi}{15}, \frac{22\pi}{15}, \frac{28\pi}{15}.
\end{aligned}
$$

Discard other values of k as they yield angles outside of $[0, 2\pi)$

Our final answer (combined from the two cases) is $x = \frac{\pi}{15}, \frac{4\pi}{15}, \frac{7\pi}{15}, \frac{2\pi}{3}, \frac{13\pi}{15}, \frac{16\pi}{15}, \frac{19\pi}{15}, \frac{22\pi}{15}, \frac{5\pi}{3}$ or $\frac{28\pi}{15}$.

7. Use the known values of $\sin 30°, \cos 30°, \sin 45°, \cos 45°, \sin 60°, \cos 60°, \dots$, the angle sum formulas and the cofunction identities to find an exact value (using radicals) for the trigonometric function.

   (a) The six trigonometric functions of $105° = 45° + 60°$:

   - $\sin(105°)$.

     answer: $\frac{\sqrt{6}+\sqrt{2}}{4}$

   - $\cos(105°)$. Should your answer be a positive or a negative number?

     answer: $\frac{\sqrt{2}-\sqrt{6}}{4}$

   - $\tan(105°)$.

     answer: $-\sqrt{3} - 2$

   - $\cot(105°)$.

     answer: $\sqrt{3} - 2$

   - $\sec(105°)$.

     answer: $-\sqrt{6} - \sqrt{2}$

   - $\csc(105°)$.

     answer: $\sqrt{6} - \sqrt{2}$

   (b) The six trigonometric functions of $\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}$:

   - $\sin\left(\frac{\pi}{12}\right)$.

     answer: $\frac{\sqrt{6}-\sqrt{2}}{4}$

   - $\cos\left(\frac{\pi}{12}\right)$. Should $\sin\left(\frac{\pi}{12}\right)$ be larger or smaller than $\cos\left(\frac{\pi}{12}\right)$?

     answer: $\frac{\sqrt{2}+\sqrt{6}}{4}$

   - $\tan\left(\frac{\pi}{12}\right)$.

     answer: $-\sqrt{3} + 2$

   - $\cot\left(\frac{\pi}{12}\right)$.

     answer: $\sqrt{3} + 2$

   - $\sec\left(\frac{\pi}{12}\right)$.

     answer: $\sqrt{6} - \sqrt{2}$

   - $\csc\left(\frac{\pi}{12}\right)$.

     answer: $\sqrt{6} + \sqrt{2}$

8. Simplify to a trigonometric function of the angle θ. The answer key has not been proofread, use with caution.

   (a) $\sin\left(\frac{\pi}{2} - \theta\right)$.

   answer: $\cos\theta$

   (b) $\cos\left(\frac{13\pi}{2} - \theta\right)$.

   answer: $\sin\theta$

   (c) $\tan(\pi - \theta)$

   answer: $-\tan\theta$

   (d) $\cot\left(\frac{3\pi}{2} - \theta\right)$

   answer: $\tan\theta$

   (e) $\csc\left(\frac{3\pi}{2} + \theta\right)$

   answer: $\sec\theta$

9. Using the power-reducing formulas, rewrite the expression in terms of first powers of the cosines and sines of multiples of the angle θ.

   (a) $\sin^4 \theta$.

   answer: $\frac{1}{8}\cos(4\theta) - \frac{1}{2}\cos(2\theta) + \frac{3}{8}$

   (b) $\cos^4 \theta$.

   answer: $\frac{1}{8}\cos(4\theta) + \frac{1}{2}\cos(2\theta) + \frac{3}{8}$

   (c) $\sin^6 \theta$.

   answer: $\sin^6 \theta = -\frac{1}{32}\cos(6\theta) + \frac{3}{16}\cos(4\theta) - \frac{15}{32}\cos(2\theta) + \frac{5}{16}$

   (d) $\cos^6 \theta$.

   answer: $\cos^6 \theta = \frac{1}{32}\cos(6\theta) + \frac{3}{16}\cos(4\theta) + \frac{15}{32}\cos(2\theta) + \frac{5}{16}$

10. Use the sum-to-product formulas to find all solutions of the trigonometric equation in the interval $[0, 2\pi)$.

    Please note that typing a query such as "solve(sin(x)+sin (3x)=0)" at `www.wolframalpha.com` will provide you with a correct answer and a function plot.

    (a) $\sin(x) + \sin(3x) = 0$.

    answer: $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$

    (b) $\cos(x) + \cos(-3x) = 0$.

    answer: $x = \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}$

    (c) $\sin(x) - \sin(3x) = 0$.

    answer: $x = 0, \frac{\pi}{4}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{7\pi}{4}$

    (d) $\cos(2x) - \cos(3x) = 0$.

    answer: $x = 0, \frac{2}{5}\pi, \frac{4\pi}{5}, \frac{6\pi}{5}, \frac{8\pi}{5}$

11. Find the inverse function. You are asked to do the algebra only; you are not asked to determine the domain or range of the function or its inverse.

    (a) $f(x) = 3x^2 + 4x - 7$, where $x \geq -\frac{2}{3}$.

    answer: $f^{-1}(x) = -\frac{2}{3} + \frac{\sqrt{25+3x}}{3}$, $x \geq -\frac{25}{3}$

    (b) $f(x) = 2x^2 + 3x - 5$, where $x \geq -\frac{3}{4}$.

    answer: $f^{-1}(x) = -\frac{3}{4} + \frac{\sqrt{49+8x}}{4}$, $x \geq -\frac{49}{8}$

    (c) $f(x) = \dfrac{2x+5}{x-4}$, where $x \neq 4$.

    answer: $f^{-1}(x) = \frac{4x+5}{x-2}$, $x \neq 2$

    (d) $f(x) = \dfrac{3x+5}{2x-4}$, where $x \neq 2$.

    answer: $f^{-1}(x) = \dfrac{4x+5}{2x-3}$, $x \neq \dfrac{3}{2}$

    (e) $f(x) = \dfrac{5x+6}{4x+5}$, where $x \neq -\frac{5}{4}$.

    answer: $f^{-1}(x) = \frac{-5x+6}{4x-5}$, $x \neq \frac{5}{4}$

    (f) $f(x) = \dfrac{2x-3}{-3x+4}$, where $x \neq \frac{4}{3}$..

    answer: $f^{-1}(x) = \frac{4x+3}{3x+2}$, $x \neq -\frac{2}{3}$

Solution. 11.d This is a concise solution written in form suitable for test taking.

$$
\begin{aligned}
y &= \frac{3x+5}{2x-4} \\
y(2x-4) &= 3x+5 \\
2xy - 4y &= 3x+5 \\
2xy - 3x &= 4y+5 \\
x(2y-3) &= 4y+5 \\
x &= \frac{4y+5}{2y-3} \\
\text{Therefore} \quad f^{-1}(y) &= \frac{5+4y}{2y-3} \\
f^{-1}(x) &= \frac{5+4x}{2x-3}.
\end{aligned}
$$

Solution. 11.e. Set $f(x) = y$. Then

$$\begin{array}{rcl} y &=& \dfrac{5x+6}{4x+5} \\ y(4x+5) &=& 5x+6 \\ x(4y-5) &=& -5y+6 \\ x &=& \dfrac{-5y+6}{4y-5}. \end{array}$$

Therefore the function $x = g(y) = \dfrac{-5y+6}{4y-5}$ is the inverse of $f(x)$. We write $g = f^{-1}$. The function $g = f^{-1}$ is defined for $y \neq \dfrac{5}{4}$. For our final answer we relabel the argument of g to x:

$$g(x) = f^{-1}(x) = \dfrac{-5x+6}{4x-5} \quad .$$

Let us check our work. In order for f and g to be inverses, we need that $g(f(x))$ be equal to x.

$$g(f(x)) = \frac{-5f(x)+6}{4f(x)-5} = \frac{-5\frac{(5x+6)}{4x+5}+6}{4\frac{(5x+6)}{4x+5}-5} = \frac{-5(5x+6)+6(4x+5)}{4(5x+6)-5(4x+5)} = \frac{-x}{-1} = x \quad ,$$

as expected.

12. Find the inverse function and its domain.

(a) $y = \ln(x+3)$.

answer: $f^{-1}(x) = e^x - 3$

(b) $y = 4\ln\,(x-3) - 4$.

answer: $f^{-1}(x) = e^{\frac{x+4}{4}} + 3$

(c) $y = 2\ln\,(-2x+4) + 1$

answer: $f^{-1}(x) = -\frac{1}{2}e^{\frac{1}{2}x - \frac{1}{2}} + 2$

(d) $f(x) = e^{x^3}$.

answer: $f^{-1}(x) = \sqrt[3]{\ln x}, \quad x > 0$

(e) $y = (\ln x)^2, x \geq 1$.

answer: $f^{-1}(x) = e^{\sqrt{x}}, \quad x \geq 0$

(f) $y = \dfrac{e^x}{1+2e^x}$.

answer: $f^{-1}(x) = \ln\left(\frac{x}{1-2x}\right), \quad x \in \left(0, \frac{1}{2}\right)$

(g) $f(x) = 2^{2x} + 2^x - 2$.

answer: $f^{-1}(x) = \log_2 \frac{-1+\sqrt{9+4x}}{2}, \quad x \geq -2$

Solution. 12.a

$$\begin{aligned} y &= \ln(x+3) \\ e^y &= e^{\ln(x+3)} \\ e^y &= x+3 \\ e^y - 3 &= x \\ \text{Therefore} \quad f^{-1}(y) &= e^y - 3. \end{aligned}$$

The domain of e^y is all real numbers, so the domain of f^{-1} is all real numbers.

Solution. 12.b

$$\begin{array}{rcll} 4\ln(x-3) - 4 &=& y & \\ 4\ln(x-3) &=& y+4 & \\ \ln(x-3) &=& \dfrac{y+4}{4} & \Big| \text{ exponentiate} \\ e^{\ln(x-3)} &=& e^{\frac{y+4}{4}} & \\ x-3 &=& e^{\frac{y+4}{4}} & \\ f^{-1}(y) = x &=& e^{\frac{y+4}{4}} + 3 & \\ f^{-1}(x) &=& e^{\frac{x+4}{4}} + 3 & \Big| \text{ relabel.} \end{array}$$

The domain of f^{-1} is all real numbers (no restrictions on the domain).

Solution. 12.e

$$\begin{array}{rcl|l} y &=& (\ln x)^2 & \text{take } \sqrt{\ } \text{ on both sides, } y \geq 0 \\ \sqrt{y} &=& \ln x & \text{exponentiate} \\ e^{\sqrt{y}} &=& e^{\ln x} = x & \\ f^{-1}(y) &=& e^{\sqrt{y}} & \\ f^{-1}(x) &=& e^{\sqrt{x}} & \end{array}$$

Solution. 12.f

$$\begin{aligned} y &= \frac{e^x}{1+2e^x} \\ y(1+2e^x) &= e^x \\ y &= e^x(1-2y) \\ \frac{y}{1-2y} &= e^x \\ \ln\frac{y}{1-2y} &= \ln e^x \\ \ln\frac{y}{1-2y} &= x \\ \text{Therefore} \quad f^{-1}(y) &= \ln\frac{y}{1-2y}. \end{aligned}$$

The natural logarithm function is only defined for positive input values. Therefore the domain is the set of all y for which

$$\frac{y}{1-2y} > 0.$$

This inequality holds if the numerator and denominator are both positive or both negative. This happens if either

(a) $y > 0$ and $y < \frac{1}{2}$, or
(b) $y < 0$ and $y > \frac{1}{2}$.

The latter option is impossible, so the domain is $\{y \in \mathbb{R} \mid 0 < y < \frac{1}{2}\}$.

13. Find each of the following values. Express your answers precisely, not as decimals.

(a) $\arcsin(\sin 4)$.

answer: $\pi - 4$

(b) $\arcsin(\sin 0.5)$.

answer: 0.5

(c) $\arcsin(\cos 120°)$.

answer: $-\frac{\pi}{6}$

(d) $\arccos(\cos(3))$.

answer: 3

(e) $\arccos(\cos(-2))$.

answer: 2

(f) $\arccos(\sin(-4))$.

answer: $\frac{3\pi}{2} - 4 \approx 0.71239$

(g) $\arctan(\tan 5)$.

answer: $5 - 2\pi$

Solution. 13.g $\frac{3\pi}{2} \approx 4.71$ and $2\pi \approx 6.28$, so

$$\begin{aligned} \frac{3\pi}{2} &< 5 < 2\pi \\ \text{Therefore} \quad -\frac{\pi}{2} &< 5 - 2\pi < 0 < \frac{\pi}{2}. \end{aligned}$$

Therefore $5 - 2\pi$ is in the restricted domain of the tangent function. Moreover, the tangent function is π-periodic, so $\tan 5 = \tan(5 - 2\pi)$. Therefore $\arctan(\tan 5) = 5 - 2\pi$.

14. Express as the following as an algebraic expression of x. In other words, "get rid" of the trigonometric and inverse trigonometric expressions.

(a) $\cos^2(\arctan x)$.

answer: $\frac{1}{1+x^2}$

(b) $-\sin^2(\operatorname{arccot} x)$.

answer: $-\frac{1}{1+x^2}$

(c) $\dfrac{1}{\cos(\arcsin x)}$.

answer: $\frac{1}{\sqrt{1-x^2}}$

(d) $-\dfrac{1}{\sin(\arccos x)}$.

answer: $-\frac{1}{\sqrt{1-x^2}}$

Solution. 14.b. We follow the strategy outlined in the end of the solution of Problem 15.c. We set $y = \operatorname{arccot} x$. Then we need to express $-\sin^2 y$ via $\cot y$. That is a matter of algebra:

$$\begin{array}{rcll} -\sin^2(\operatorname{arccot} x) &=& -\sin^2 y & \Big| \text{ Set } y = \operatorname{arccot} x \\ &=& -\dfrac{\sin^2 y}{\sin^2 y + \cos^2 y} & \Big| \text{ use } \sin^2 y + \cos^2 y = 1 \\ &=& -\dfrac{1}{\frac{\sin^2 y + \cos^2 y}{\sin^2 y}} & \\ &=& -\dfrac{1}{1+\cot^2 y} & \Big| \text{ Substitute back } \cot y = x \\ &=& -\dfrac{1}{1+x^2} & . \end{array}$$

15. Let $x \in (0, 1)$. Express the following using x and $\sqrt{1-x^2}$.

(a) $\sin(\arcsin(x))$.

answer: x

(b) $\sin(2\arcsin(x))$.

answer: $2x\sqrt{1-x^2}$

(c) $\sin(3\arcsin(x))$.

answer: $-4x^3 + 3x$

(d) $\sin(\arccos(x))$.

answer: $\sqrt{1-x^2}$

(e) $\sin(2\arccos(x))$.

answer: $2x\sqrt{1-x^2}$

(f) $\sin(3\arccos(x))$.

answer: $(4x^2-1)\sqrt{1-x^2} = -4\left(\sqrt{1-x^2}\right)^3 + 3\sqrt{1-x^2}$

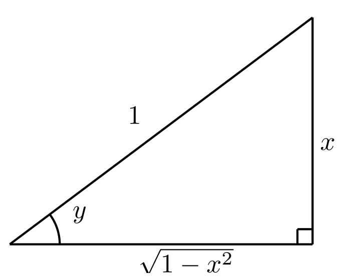
(g) $\cos(2\arcsin(x))$.

answer: $1 - 2x^2$

(h) $\cos(3\arccos(x))$.

answer: $4x^3 - 3x$

Solution. 15.b. Let $y = \arcsin x$. Then $\sin y = x$, and we can draw a right triangle with opposite side length x and hypotenuse length 1 to find the other trigonometric ratios of y.



Then $\cos y = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$. Now we use the double angle formula to find $\sin(2\arcsin x)$.

$$\begin{aligned} \sin(2\arcsin x) &= \sin(2y) \\ &= 2\sin y \cos y \\ &= 2x\sqrt{1-x^2}. \end{aligned}$$

Solution. 15.c. Use the result of Problem 15.b. This also requires the addition formula for sine:

$$\sin(A+B) = \sin A \cos B + \sin B \cos A,$$

and the double angle formula for cosine:

$$\cos(2y) = \cos^2 y - \sin^2 y.$$

$$\begin{aligned}
\sin(3\arcsin x) &= \sin(3y) \\
&= \sin(2y+y) \\
&= \sin(2y)\cos y + \sin y \cos(2y) && \Big|\ \text{Use addition formula} \\
&= (2\sin y \cos y)\cos y + \sin y(\cos^2 y - \sin^2 y) && \Big|\ \text{Use double angle formulas} \\
&= 2\sin y \cos^2 y + \sin y \cos^2 y - \sin^3 y \\
&= 3\sin y \cos^2 y - \sin^3 y \\
&= 3\sin y(1-\sin^2 y) - \sin^3 y \\
&= 3x(1-x^2) - x^3 \\
&= 3x - 4x^3.
\end{aligned}$$

The solution is complete. A careful look at the solution above reveals a strategy useful for problems similar to this one.

(a) Identify the inverse trigonometric expression- $\arcsin x, \arccos x, \arctan x, \dots$. In the present problem that was $y = \arcsin x$.

(b) The problem is therefore a trigonometric function of y.

(c) Using trig identities and algebra, rewrite the problem as a trigonometric expression involving only the trig function that transforms y to x. In the present problem we rewrote everything using $\sin y$.

(d) Use the fact that $\sin(\arcsin x) = x$, $\cos(\arccos x) = x, \dots$, etc. to simplify.

Solution. 15.f We use the same strategy outlined in the end of the solution of Problem 15.c. Set $y = \arccos x$ and so $\cos(y) = x$. Therefore:

$$\begin{aligned}
\sin(3y) &= \sin(2y+y) \\
&= \sin(2y)\cos y + \sin y \cos(2y) \\
&= 2\sin y \cos y \cos y + \sin y(2\cos^2 y - 1) \\
&= 2\sin y \cos^2 y + \sin y(2\cos^2 y - 1) \\
&= \sin y(4\cos^2 y - 1) && \Bigg|\ \text{use} \begin{array}{rcl} \cos y &=& x \\ \sin y &=& \sqrt{1-x^2} \end{array} \\
&= \sqrt{1-x^2}(4x^2-1) \quad .
\end{aligned}$$

16. Find all values of x in the interval $[0, 2\pi]$ that satisfy the equation.

(a) $2\cos x - 1 = 0$.

answer: $x = \frac{\pi}{3}$ or $x = \frac{5\pi}{3}$

(b) $\sin(2x) = \cos x$.

answer: $x = \frac{\pi}{2}$, $x = \frac{3\pi}{2}$, $x = \frac{\pi}{6}$, or $x = \frac{5\pi}{6}$

(c) $\sqrt{3}\sin x = \sin(2x)$.

answer: $x = \frac{\pi}{6}, \frac{11\pi}{6}, 0, \pi, 2\pi$.

(d) $2\sin^2 x = 1$.

answer: $x = \frac{\pi}{4}$, $x = \frac{3\pi}{4}$, $x = \frac{5\pi}{4}$, or $x = \frac{7\pi}{4}$

(e) $2 + \cos(2x) = 3\cos x$.

answer: $x = 0$, $x = 2\pi$, $x = \frac{\pi}{3}$, or $x = \frac{5\pi}{3}$

(f) $2\cos x + \sin(2x) = 0$.

answer: $x = \frac{\pi}{2}$, $x = \frac{3\pi}{2}$

(g) $2\cos^2 x - (1+\sqrt{2})\cos x + \frac{\sqrt{2}}{2} = 0$.

answer: $x = \frac{\pi}{4}, \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{4}$.

(h) $|\tan x| = 1$.

answer: $x = \frac{\pi}{4}$, $x = \frac{3\pi}{4}$, $x = \frac{5\pi}{4}$, or $x = \frac{7\pi}{4}$

(i) $3\cot^2 x = 1$.

answer: $x = \frac{\pi}{3}$, $x = \frac{2\pi}{3}$, $x = \frac{4\pi}{3}$, or $x = \frac{5\pi}{3}$

(j) $\sin x = \tan x$.

answer: $x = 0$, $x = \pi$, or $x = 2\pi$

Solution. 16.g Set $\cos x = u$. Then

$$2\cos^2 x - (1+\sqrt{2})\cos x + \frac{\sqrt{2}}{2} = 0$$

becomes

$$2u^2 - (1+\sqrt{2})u + \frac{\sqrt{2}}{2} = 0.$$

This is a quadratic equation in u and therefore has solutions

$$\begin{aligned}
u_1, u_2 &= \frac{1+\sqrt{2} \pm \sqrt{(1+\sqrt{2})^2 - 4\sqrt{2}}}{4} \\
&= \frac{1+\sqrt{2} \pm \sqrt{1 - 2\sqrt{2} + 2}}{4} \\
&= \frac{1+\sqrt{2} \pm \sqrt{(1-\sqrt{2})^2}}{4} \\
&= \frac{1+\sqrt{2} \pm (1-\sqrt{2})}{4} = \begin{cases} \frac{1}{2} & \text{or} \\ \frac{\sqrt{2}}{2} \end{cases}
\end{aligned}$$

Therefore $u = \cos x = \frac{1}{2}$ or $u = \cos x = \frac{\sqrt{2}}{2}$, and, as x is in the interval $[0, 2\pi]$, we get $x = \frac{\pi}{3}, \frac{5\pi}{3}$ (for $\cos x = \frac{1}{2}$) or $x = \frac{\pi}{4}, \frac{7\pi}{4}$ (for $\cos x = \frac{\sqrt{2}}{2}$).

17. Express each of the following as a single power.

(a) $\dfrac{2^5 \cdot 2^7}{2\sqrt{2}}$

answer: $2^{10.5} = 2^{\frac{21}{2}}$

(b) $\dfrac{3^2 \cdot 3^{-1}}{3^3 \cdot \sqrt{3^3}}$

answer: $3^{-\frac{7}{2}}$

(c) $\dfrac{\pi^3}{\pi^{-1}\sqrt{\pi^5}}$

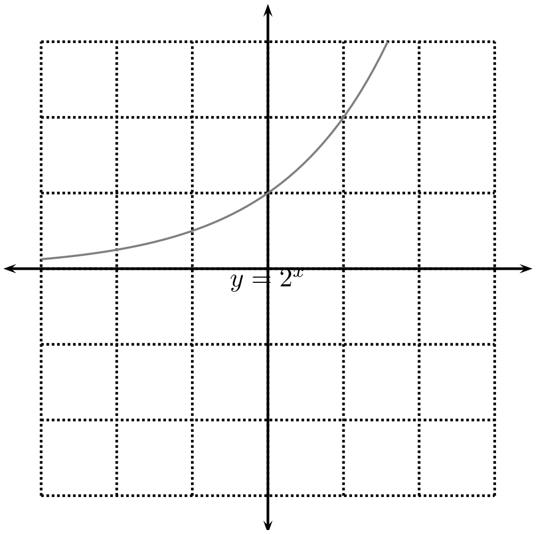
answer: $\pi^{\frac{3}{2}}$

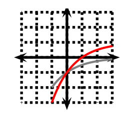
Solution. 17.b.

$$\begin{aligned}
\frac{3^2 \cdot 3^{-1}}{3^3 \cdot \sqrt{3^3}} &= \frac{3^2 \cdot 3^{-1}}{3^3 \cdot (3^3)^{\frac{1}{2}}} \\
&= \frac{3^2 \cdot 3^{-1}}{3^3 \cdot 3^{\frac{3}{2}}} \\
&= \frac{3^{2-1}}{3^{3+\frac{3}{2}}} \\
&= \frac{3^1}{3^{\frac{9}{2}}} \\
&= 3^{1-\frac{9}{2}} \\
&= 3^{-\frac{7}{2}}.
\end{aligned}$$

18. Sketch by hand approximately the given function. The function is obtained by transforming linearly the graph of a known function. The known function has been sketched for you by computer.

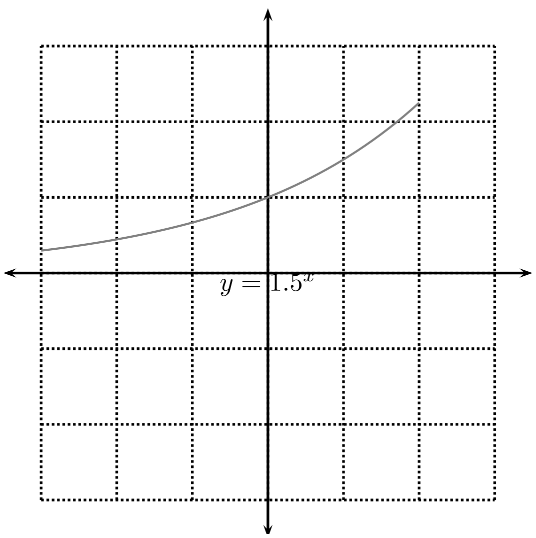
(a) $f(x) = 2^{x+1} - 1$.

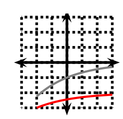




answer:

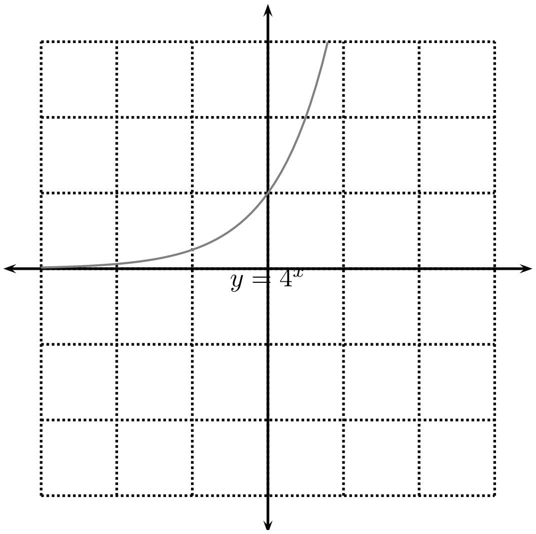
(b) $f(x) = 1.5^{x-2} + 2$.

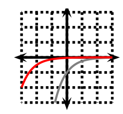




answer:

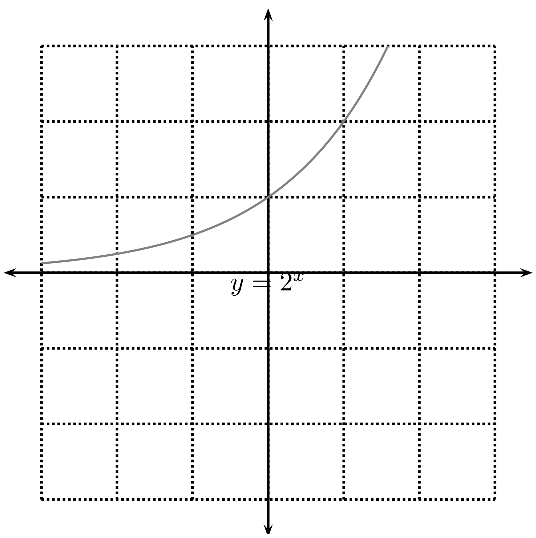
(c) $f(x) = 2^{2x-5}$.

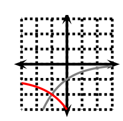




answer:

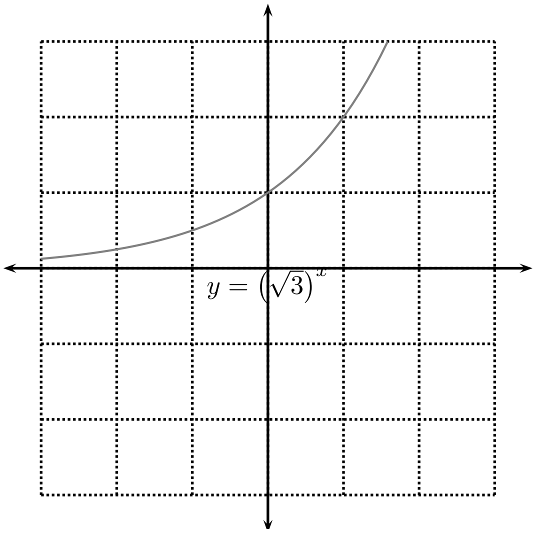
(d) $f(x) = \dfrac{1}{2^{x-1}} + 1$.

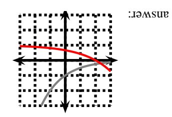




answer:

(e) $f(x) = \dfrac{1}{3^{\frac{1}{2}x+1}} - 1.$





19. (a) A sum is held under a yearly compound interest of 1%. Make an approximation by hand (no calculators allowed) by what factor will have the money increased after 200 years. Can you do the computation in your head?

(b) Decide, without using a calculator, which is more profitable: earning a yearly compound interest of 2% for 150 years or earning yearly simple interest of 11% for 150 years?

Solution. 19.a Each year, the sum increases by a factor of $\left(1 + \frac{1}{100}\right)$. Therefore in 200 years the sum will have increased by

$$\begin{array}{rcll} \left(1+\frac{1}{100}\right)^{200} & = & \left(\left(1+\frac{1}{100}\right)^{100}\right)^2 & \Big| \text{ equals } \left(\left(1+\frac{1}{n}\right)^n\right)^2 \text{ for } n = 100 \\ & \approx & e^2. & \end{array}$$

As a rough estimate for e we can take $e \approx 2.7$, and so $e^2 \approx 2.7^2 = 7.29$. Our sum will have increased approximately 7.3 times. A calculator computation shows that

$$\left(1 + \frac{1}{100}\right)^{200} \approx 7.316018,$$

so our "in the head" estimate is fairly accurate. Notice that the calculator computation is on its own an approximation - it was carried using double floating point precision arithmetics, which does introduce some minimal errors. Such round off errors, of course, are also present in modern banking transactions, so we do not need to adjust for those.

Solution. 19.b Simple interest of 11% per 150 years a profit of

$$0.11 * 150 = 15 + 1.5 = 16.5,$$

or altogether 17.5-fold increase of our initial sum. A 2% compound interest for 150 years yields a

$$\begin{array}{rcl} \left(1+\frac{2}{100}\right)^{150} & = & \left(\left(1+\frac{1}{50}\right)^{50}\right)^3 \\ & \approx & e^3 \end{array}$$

-fold increase of our sum. To establish which of the two options yields more money, we need to compare e^3 to 17.5 (without using a calculator). In the solution of 19.a we established that $e^2 \approx 7.3$, so $e^3 \approx e \cdot 7.3 \approx 2.7 \cdot 7.3 = 2 \cdot 7 + 2 \cdot 0.3 + 0.7 \cdot 7 + 0.7 \cdot 0.3 = 14 + 0.6 + 4.9 + 0.21 = 19.71 \approx 19.7$. We can say that the compound interest results in approximately 19.7-fold increase of the initial sum, so the compound interest is more profitable. A calculator computation shows that

$$\left(1 + \frac{2}{100}\right)^{150} \approx 19.499603 \quad .$$

Our error of approximately 0.2 was not optimal, yet fairly accurate for an "in the head" computation.

20. Use the definition of a logarithm to evaluate each of the following without using a calculator. The answer key has not been proofread, use with caution.

(a) $\log_2 16$.

answer: 4

(b) $\log_3 \left(\frac{1}{9}\right)$.

answer: -2

(c) $\log_{10} 1000$.

answer: 3

(d) $\log_6 36^{-\frac{2}{3}}$.

answer: $-\frac{4}{3}$

(e) $\log_2(8\sqrt{2})$.

answer: $\frac{7}{2}$

(f) $\log_{\frac{1}{2}}(4)$.

answer: -2

(g) $\log_{\frac{1}{9}}(\sqrt{3})$.

answer: $-\frac{1}{4}$

21. Find the exact value of each expression.

(a) $\log_5 125$.

answer: 3

(b) $\log_3 \frac{1}{27}$.

answer: -3

(c) $\ln\left(\frac{1}{e}\right)$.

answer: -1

(d) $\log_{10}\sqrt{10}$.

answer: $\frac{1}{2}$

(e) $e^{\ln 4.5}$.

answer: 4.5

(f) $\log_{10} 0.0001$.

answer: -4

(g) $\log_{1.5} 2.25$.

answer: 2

(h) $\log_5 4 - \log_5 500$.

answer: -3

(i) $\log_2 6 - \log_2 15 + \log_2 20$.

answer: 3

(j) $\log_3 100 - \log_3 18 - \log_3 50$.

answer: -2

(k) $e^{-2\ln 5}$.

answer: $\frac{1}{25}$

(l) $\ln\left(\ln e^{e^{10}}\right)$.

answer: 10

(m) $\log_7\left(\frac{49^x}{343^y}\right)$

answer: $2x - 3y$

Solution. 21.m.

$$\begin{array}{rcl} \log_7\left(\frac{49^x}{343^y}\right) &=& \log_7 49^x - \log_7 343^y \\ &=& x\log_7 49 - y\log_7 343 \\ \text{However } 49 = 7^2 \text{ and } 343 = 7^3, \text{ therefore } \log_7\left(\frac{49^x}{343^y}\right) &=& 2x - 3y. \end{array}$$

22. Using only the $\ln$ operation of your calculator compute the indicated logarithm. Confirm your computation numerically by exponentiation.

(a) $\log_5(13)$.

answer: $\frac{\ln 13}{\ln 5} \approx 1.593693$

(b) $\log_{12}(9)$.

answer: $\frac{\ln 9}{\ln 12} \approx 0.884228$

(c) $\log_{13}(101)$.

answer: $\frac{\ln 101}{\ln 13} \approx 1.799303$

(d) $\log_{10}(2015)$.

answer: $\frac{\ln 2015}{\ln 10} \approx 3.304275$

Solution.

$$\log_5(13) = \frac{\ln 13}{\ln 5} \approx \frac{2.564949357}{1.609437912} \approx 1.593693.$$

As a check of our computations, we compute by calculator: $13 = 5^{\log_5 13} \approx 5^{1.593693} \approx 13.000007508$, and our computations check out.

23. Express each of the following as a single logarithm. If possible, compute the logarithm without using a calculator. The answer key has not been proofread, use with caution.

(a) $\ln 4 + \ln 6 - \ln 5$.

answer: $\ln\left(\frac{24}{5}\right)$

(b) $2\ln 2 - 3\ln 3 + 4\ln 4$.

answer: $\ln\left(\frac{1024}{27}\right)$

(c) $\ln 36 - 2\ln 3 - 3\ln 2$.

answer: $-\ln 2 = \ln\left(\frac{1}{2}\right)$

(d) $\log_2(24) - \log_4 9$.

answer: 3

(e) $\log_7(24) + \log_{\frac{1}{7}} 3 - \log_{49}(64)$.

answer: 0

(f) $\log_3(24) + \log_3\left(\frac{3}{8}\right)$.

answer: 2

Solution. 23.b.

$$\begin{aligned}
2\ln 2 - 3\ln 3 + 4\ln 4 &= \ln 2^2 - \ln 3^3 + \ln 4^4 \\
&= \ln 4 - \ln 27 + \ln 256 \\
&= \ln\left(\frac{4}{27}\right) + \ln 256 \\
&= \ln\left(\frac{4\cdot 256}{27}\right) \\
&= \ln\left(\frac{1024}{27}\right).
\end{aligned}$$

$\frac{1024}{27}$ is not a rational power of e, therefore $\ln\left(\frac{1024}{27}\right)$ is not a rational number and there are no further simplifications of the answer (except possibly a numerical approximation with a calculator or equivalent).

Solution. 23.e

$$\begin{aligned}
\log_7(24) + \log_{\frac{1}{7}}(3) - \log_{49}(64) &= \log_7(24) + \frac{\log_7(3)}{\log_7\left(\frac{1}{7}\right)} - \frac{\log_7(64)}{\log_7(49)} && \text{common base} \\
&= \log_7(24) + \frac{\log_7(3)}{-1} - \frac{\log_7(64)}{2} && \text{simplify logarithms} \\
&= \log_7(24) - \log_7(3) - \frac{1}{2}\log_7(64) \\
&= \log_7\left(\frac{24}{3}\right) - \log_7\left(64^{\frac{1}{2}}\right) && \text{rule: } \log_a x - \log_a y = \log_a\left(\tfrac{x}{y}\right) \\
& && \text{rule: } \log_a x^r = r\log_a x \\
&= \log_7(8) - \log_7\left(\sqrt{64}\right) \\
&= \log_7 8 - \log_7 8 = 0 && \text{alternatively:} \\
&= \log_7\left(\frac{8}{8}\right) \\
&= \log_7(1) \\
&= 0.
\end{aligned}$$

24. Demonstrate the identity(s).

(a) $-\ln\left(\sqrt{1+x^2} - x\right) = \ln\left(x + \sqrt{1+x^2}\right)$