Calculus II
L'Hospital's rule

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Outline

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Example

Find $\lim_{x\to 1} \frac{\ln x}{x-1}$.

- $\lim_{x\to 1} \ln x = 0$.
- $\lim_{x\to 1}(x-1) = 0$.
- We don't get any cancellation between top and bottom.
- We need new techniques.

Theorem (L'Hospital's Rule)

Suppose that f and g are differentiable and $g'(x) \neq 0$ on an open interval that contains a (except possibly at a). Suppose that

$$\lim_{x\to a} f(x) = 0 \qquad \textit{and} \qquad \lim_{x\to a} g(x) = 0$$

$$\textit{or that} \qquad \lim_{x\to a} f(x) = \pm\infty \qquad \textit{and} \qquad \lim_{x\to a} g(x) = \pm\infty$$

(In other words, we have an indeterminate form of type $0/0$ or ∞/∞.) Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Example

Find $\lim_{x\to 1} \frac{\ln x}{x-1}$.

- $\lim_{x\to 1} \ln x = 0$.
- $\lim_{x\to 1}(x-1) = 0$.
- This is an indeterminate form of type $0/0$.
- Apply L'Hospital's rule:

$$\lim_{x\to 1} \frac{\ln x}{x-1} = \lim_{x\to 1} \frac{\frac{\mathrm{d}}{\mathrm{d}x}(\ln x)}{\frac{\mathrm{d}}{\mathrm{d}x}(x-1)} = \lim_{x\to 1} \frac{1/x}{1} = \lim_{x\to 1} \frac{1}{x} = 1.$$

Example

Find $\lim_{x\to\infty} \frac{e^x}{x^2}$.

- $\lim_{x\to\infty} e^x = \infty$.
- $\lim_{x\to\infty} x^2 = \infty$.
- This is an indeterminate form of type ∞/∞.
- Apply L'Hospital's rule:

$$\lim_{x\to\infty} \frac{e^x}{x^2} = \lim_{x\to\infty} \frac{\frac{\mathrm{d}}{\mathrm{d}x}(e^x)}{\frac{\mathrm{d}}{\mathrm{d}x}(x^2)} = \lim_{x\to\infty} \frac{e^x}{2x}$$

- $\lim_{x\to\infty} 2x = \infty$.
- This is an inderminate form of type ∞/∞.
- Apply L'Hospital's rule again:

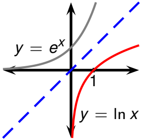
$$\lim_{x\to\infty} \frac{e^x}{x^2} = \lim_{x\to\infty} \frac{e^x}{2x} = \lim_{x\to\infty} \frac{e^x}{2} = \infty.$$

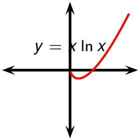
Indeterminate Products

If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \pm\infty$, then it isn't clear what $\lim_{x\to a}(fg)(x)$ will be.
In such a case, write the product *fg* as a quotient:

$$fg = \frac{f}{1/g} \qquad \text{or} \qquad fg = \frac{g}{1/f}.$$

This converts the given limit into an indeterminate form of type $0/0$ or ∞/∞.





Example

Evaluate $\lim_{x \to 0^+} x \ln x$.

- $\lim_{x \to 0^+} \ln x = -\infty$.
- $\lim_{x \to 0^+} x = 0$.
- This is an indeterminate form of type $0(-\infty)$ (or $-\infty/(1/0)$).
- Apply L'Hospital's rule:

$$
\begin{aligned}
\lim_{x \to 0^+} x \ln x &= \lim_{x \to 0^+} \frac{\ln x}{\frac{1}{x}} = \lim_{x \to 0^+} \frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}\left(\frac{1}{x}\right)} \\
&= \lim_{x \to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x \to 0^+} (-x) = 0.
\end{aligned}
$$

Indeterminate Differences

If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$, then the limit

$$\lim_{x\to a} [f(x) - g(x)]$$

is called an indeterminate form of type $\infty - \infty$.
To compute such a limit, try to convert it into a quotient (by using a common denominator, or by rationalizing, or by factoring out a common factor).

Example

Evaluate $\lim_{x\to(\pi/2)^-}(\sec x - \tan x)$.

- $\lim_{x\to(\pi/2)^-} \sec x = \infty$.
- $\lim_{x\to(\pi/2)^-} \tan x = \infty$.
- This is an indeterminate form of type $\infty - \infty$.
- Apply L'Hospital's rule:

$$\begin{aligned}
\lim_{x\to(\pi/2)^-}(\sec x - \tan x) &= \lim_{x\to(\pi/2)^-}\left(\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right)\\
&= \lim_{x\to(\pi/2)^-}\frac{1-\sin x}{\cos x}\\
&\quad \text{(indeterminate form of type } 0/0.\text{)}\\
&= \lim_{x\to(\pi/2)^-}\frac{\frac{d}{dx}(1-\sin x)}{\frac{d}{dx}\cos x}\\
&= \lim_{x\to(\pi/2)^-}\frac{-\cos x}{-\sin x} = 0
\end{aligned}$$

Indeterminate Powers

Several indeterminate forms arise from the limit $\lim_{x\to a} f(x)^{g(x)}$.

$$\begin{array}{lll} \lim_{x\to a} f(x) = 0 \text{ and } \lim_{x\to a} g(x) = 0 & & \text{type } 0^0 \\ \lim_{x\to a} f(x) = \infty \text{ and } \lim_{x\to a} g(x) = 0 & & \text{type } \infty^0 \\ \lim_{x\to a} f(x) = 1 \text{ and } \lim_{x\to a} g(x) = \pm\infty & & \text{type } 1^\infty \end{array}$$

These can all be solved either by taking the natural logarithm:

$$\text{let } y = [f(x)]^{g(x)}, \text{ then } \ln y = g(x) \ln f(x)$$

or by writing the function as an exponential:

$$[f(x)]^{g(x)} = e^{g(x)\ln f(x)}.$$

Example

Find $\lim_{x\to 0^+} x^x$.

- $0^x = 0$ for any $x > 0$.
- $x^0 = 1$ for any $x \neq 0$.
- This is an indeterminate form of type 0^0.
- Write as an exponential:
- $x^x = e^{x \ln x}$.
- Recall that $\lim_{x\to 0^+} x \ln x = 0$.
- Therefore

$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{x \ln x} = e^0 = 1$$

Example

$$
\begin{aligned}
\lim_{x\to\infty}\left(1+\frac{k}{x}\right)^x &= \lim_{x\to\infty} e^{\ln\left(1+\frac{k}{x}\right)^x} && \text{exponent= continuous f-n}\\
&= e^{\lim\limits_{x\to\infty}\ln\left(1+\frac{k}{x}\right)^x} = e^k && \text{limit computed below}\\
\lim_{x\to\infty}\ln\left(1+\frac{k}{x}\right)^x &= \lim_{x\to\infty} x\ln\left(1+\frac{k}{x}\right)\\
&= \lim_{x\to\infty}\frac{\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln\left(1+\frac{k}{x}\right)\right)}{\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x}\right)} && \text{form “}\tfrac{0}{0}\text{”, use L'Hospital}\\
&= \lim_{x\to\infty}\frac{\frac{1}{1+\frac{k}{x}}\left(1+\frac{k}{x}\right)'}{-\frac{1}{x^2}}\\
&= \lim_{x\to\infty}\frac{\frac{1}{1+\frac{k}{x}}\left(-\frac{k}{x^2}\right)}{-\frac{1}{x^2}}\\
&= \lim_{x\to\infty}\frac{k}{1+\frac{k}{x}} = k
\end{aligned}
$$