Calculus II
Integral of rational function with cubic denominator, part 2

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Example

$$\int \frac{x^4+x^3-4x^2+4x}{x^3-x^2-x+1}\mathrm{d}x = \int \left(x+2+\frac{1}{x-1}+\frac{1}{(x-1)^2}-\frac{2}{x+1}\right)\mathrm{d}x$$
$$= \frac{x^2}{2}+2x+\ln|x-1|-\frac{1}{x-1}-2\ln|x+1|+K$$

- Divide: $\frac{x^4+x^3-4x^2+4x}{x^3-x^2-x+1} = x+2+\frac{-x^2+5x-2}{x^3-x^2-x+1} = x+2+\frac{-x^2+5x-2}{(x-1)^2(x+1)}$.
- Factor denominator: $x^3-x^2-x+1 = (x-1)^2(x+1)$.
- Set up the partial fraction decomposition:
$$\begin{aligned} \frac{-x^2+5x-2}{(x-1)^2(x+1)} &= \frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x+1} \\ -x^2+5x-2 &= A(x-1)(x+1)+B(x+1)+C(x-1)^2 \end{aligned}$$
- Plug-in $x=-1$: $-(-1)^2+5(-1)-2 = C(-1-1)^2 \Rightarrow C = -2$.
- Plug-in $x=1$: $-(1)^2+1\cdot 5-2 = B(1+1) \Rightarrow B = 1$.
- Plug-in $x=0$:
  $-2 = A(0-1)(0+1)+1\cdot\ (0+1)+(-2)\ (0-1)^2 \Rightarrow A = 1$.

$Q(x)$ has linear factors with higher multiplicity

- Suppose $Q(x)$ is a product of linear factors, some of which appear with power greater than 1.
- For example suppose the first linear factor has power r, that is, $(a_1x + b_1)^r$ occurs in the factorization of $Q(x)$.
- Then instead of a single term $\frac{A}{a_1x+b_1}$ we use

$$\frac{A_1}{a_1x + b_1} + \frac{A_2}{(a_1x + b_1)^2} + \cdots + \frac{A_r}{(a_1x + b_1)^r}$$

- In a similar fashion we add more partial fractions to account for all other terms of the form $(a_sx + b_s)^t$.