

Calculus II

Series basics

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Outline

1 Series

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<https://github.com/tmilev/freecalc>

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Formal Series

Definition (Formal Series)

A formal series is a list of numbers delimited by the plus sign.

$$a_1 + a_2 + a_3 + a_4 + \cdots + a_n + \cdots$$

- Recall a sequence is a list of numbers.

$$a_1, \quad a_2, \quad a_3, \quad a_4, \quad \dots, \quad a_n, \quad \dots$$

- The $+$ sign indicates our intention to attempt to sum the elements of the formal series.
- Except for the indication of that intention, formal series and sequences are essentially synonymous.
- The sum of a finite sequence/finite formal series is studied in the subject of elementary arithmetics.
- The sum, if convergent, of an infinite sequence/infinite formal series will be defined in the following slides.

Example (The ... and \sum notations for series)

Let A be the sum of the positive even integers between 2 and 124.
Write A using the ... notation and using the \sum notation.

$$\begin{aligned} A &= 2 + 4 + 6 + \dots + 124 \\ &= 2 + 4 + 6 + \dots + 2n + \dots + 124 \\ &= 2 \cdot 1 + 2 \cdot 2 + 2 \cdot 3 + \dots + 2 \cdot n + \dots + 2 \cdot 62 \\ &= \sum_{n=1}^{62} 2n . \end{aligned}$$

- We aim to introduce the \sum notation for series via this example.
- The ... notation is informal but easier to read.
- If the ... are too ambiguous, we should include the general term.
- To make it clearer we should rewrite all elements in the pattern of the general term.
- If that is still ambiguous we should switch to the completely unambiguous \sum notation.

Example (The ... and \sum notations for series)

Let A be the sum of the positive even integers between 2 and 124. Write A using the ... notation and using the \sum notation.

$$\begin{aligned} A &= 2 + 4 + 6 + \dots + 124 \\ &= 2 + 4 + 6 + \dots + 2n + \dots + 124 \\ &= 2 \cdot 1 + 2 \cdot 2 + 2 \cdot 3 + \dots + 2 \cdot n + \dots + 2 \cdot 62 \\ &= \sum_{n=1}^{62} 2n . \end{aligned}$$

- The number n is the index (counter) of the sum.
- \sum tells us to add several copies of the summed term, where in each term the index is replaced by a concrete value.
- The values taken by the index are determined by the boundaries of summation.
- The index varies over all integers starting with the lower boundary and ending with upper boundary.
- In programming, what objects are similar to \sum ?

Example (The ... and \sum notations for series)

Let A be the sum of the positive even integers between 2 and 124.
Write A using the ... notation and using the \sum notation.

$$\begin{aligned} A &= 2 + 4 + 6 + \cdots + 124 \\ &= 2 + 4 + 6 + \cdots + 2n + \cdots + 124 \\ &= 2 \cdot 1 + 2 \cdot 2 + 2 \cdot 3 + \cdots + 2 \cdot n + \cdots + 2 \cdot 62 \\ &= \sum_{n=1}^{62} 2n . \end{aligned}$$

- To go from \sum to ... notation: substitute few values for the index.
Make sure to include the last value.
- To go from ... to \sum notation:
 - figure out a pattern for the general term just as with sequences;
 - select first and last index so that your general term formula reproduces the first and last terms of the sequence.

Example (The ... and \sum notations for series)

Let A be the sum of the positive even integers between 2 and 124. Write A using the ... notation and using the \sum notation.

$$\begin{aligned} A &= 2 + 4 + 6 + \dots + 124 \\ &= 2 + 4 + 6 + \dots + 2n + \dots + 124 \\ &= 2 \cdot 1 + 2 \cdot 2 + 2 \cdot 3 + \dots + 2 \cdot n + \dots + 2 \cdot 62 \\ &= \sum_{n=1}^{62} 2n . \end{aligned}$$

- Bear in mind the ... notation is informal.
 - There are infinitely many formulas that fit any single pattern.
 - Thus it is acceptable to use the ... notation only when we believe there is a single completely obvious pattern that will be recognized by every one.
 - The pattern should be obvious not only to us, but also to our potential readers.
 - If in doubt or seeking complete rigor we should use the \sum notation.

Definition (Arithmetic series)

An arithmetic series is a series whose terms are an arithmetic sequence.

Example (Sum of a small arithmetic series)

The sum of the arithmetic series $7 + 4 + 1 - 2 - 5$ is

Example (Sum of a large arithmetic series)

Find the sum of the arithmetic series

$$7 + 4 + 1 - 2 - 5 - \dots - 53 - 56.$$

Let s denote the sum.

$$\begin{array}{rcccccc}
 s & = & 7 & +4 & +1 & -\dots & -56 \\
 +s & = & -56 & -53 & -50 & -\dots & +7 \\
 \hline
 2s & = & -49 & -49 & -49 & -\dots & -49
 \end{array}$$

$$\text{Therefore } 2s = (-49)(22)$$

$$s = -49 \cdot 22/2 = -539.$$

Theorem (Sum of an arithmetic series)

The sum of a finite arithmetic series is the average of the first and last terms, multiplied by the number of terms. That is,

$$a + (a + d) + (a + 2d) + \cdots + (a + (n - 1)d) = \frac{a + (a + (n - 1)d)}{2} n.$$

The only infinite arithmetic series with a sum is the series of all 0.

Example (Sum of an arithmetic series)

Find the sum of the arithmetic series

$$5 + 10 + 15 + 20 + \cdots + 100.$$

The series contains terms. The average of the first and last terms is
.

Therefore the sum is $\frac{\quad}{2}$.

Definition (Geometric series)

A geometric series is a series whose terms are a geometric sequence.

Example (The sum of a finite geometric series)

Let $r \neq 1$. Find the sum of the geometric series

$$a + ar + ar^2 + ar^3 + \dots + ar^{M-1} = \sum_{n=1}^M ar^{n-1}.$$

Let s denote the sum.

$$\begin{array}{rcl} s & = & a + ar + ar^2 + \dots + ar^{M-1} \\ - \quad rs & = & \quad ar + ar^2 + \dots + ar^{n-1} + ar^M \\ \hline s - rs & = & a - ar^M \\ s & = & \frac{a(1-r^M)}{1-r} \end{array}$$

Theorem (The sum of a finite geometric series)

Let $r \neq 1$. The sum of the finite geometric series $\sum_{n=1}^M ar^{n-1}$ is $a \frac{1-r^M}{1-r}$.

- Does it make sense to add infinitely many numbers?
- Sometimes yes, sometimes no.
- Consider the series $\sum_{n=1}^{\infty} n$.

$$1 + 2 + 3 + 4 + 5 + \cdots + n + \cdots$$

- If we add the terms, we get the partial sums 1,3,6,10,15.
- After the n th term, we get $\frac{n(n+1)}{2}$.
- This goes to ∞ as n gets bigger.
- Now consider the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$.

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots + \frac{1}{2^n} + \cdots$$

- If we add the terms, we get the partial sums $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \frac{31}{32}$.
- After the n th term, we get $1 - \frac{1}{2^n}$.
- This gets closer and closer to 1. We write $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$.

Definition (Partial Sum, Convergent, Divergent, Sum)

Given a series $\sum_{i=1}^{\infty} a_i = a_1 + a_2 + a_3 + \cdots$, let s_n denote the n th partial sum:

$$s_n = \sum_{i=1}^n a_i = a_1 + a_2 + \cdots + a_n$$

If the sequence $\{s_n\}$ is convergent and $\lim_{n \rightarrow \infty} s_n = s$, then we say that the series $\sum_{i=1}^{\infty} a_i$ is convergent, and we write

$$\sum_{i=1}^{\infty} a_i = s.$$

In this case, we call s the sum of the series.

If the sequence $\{s_n\}$ is divergent, then we say that the series $\sum_{i=1}^{\infty} a_i$ is divergent.

Example

An important example is the geometric series

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}, \quad a \neq 0$$

- If $r = 1$, then $s_n = a + a + \cdots + a = na \rightarrow \pm\infty$.
- Since $\lim_{n \rightarrow \infty} s_n$ doesn't exist, the series is divergent when $r = 1$.
- If $r \neq 1$, then

$$\begin{array}{rcl}
 s_n & = & a + ar + ar^2 + \cdots + ar^{n-1} \\
 - \quad rs_n & = & \quad \quad ar + ar^2 + \cdots + ar^{n-1} + ar^n \\
 \hline
 s_n - rs_n & = & a - ar^n \\
 s_n & = & \frac{a(1-r^n)}{1-r}
 \end{array}$$

- If $-1 < r < 1$, then $r^n \rightarrow 0$, so the geometric series is convergent and its sum is $a/(1-r)$.
- If $r > 1$ or $r \leq -1$, then r^n is divergent, so $\sum_{n=1}^{\infty} ar^{n-1}$ diverges.

This theorem summarizes the results of the previous example.

Theorem (Convergence of Geometric Series)

The geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \dots$$

is convergent if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}.$$

If $|r| \geq 1$, the series is divergent.

a is called the first term and r is called the common ratio.

For $|r| < 1$, recall that the sum of a geometric series is

$$a + ar + ar^2 + ar^3 + \dots = a(1 + r + r^2 + r^3 + \dots) = \frac{a}{1 - r}$$

alternatively

$$\sum_{n=1}^{\infty} ar^{n-1} = \sum_{m=0}^{\infty} ar^m = a \sum_{m=0}^{\infty} r^m = \frac{a}{1 - r}$$

Example

Find the sum of the geometric series $-2 + \frac{6}{5} - \frac{18}{25} + \frac{54}{125} - \dots$

- The first term is $a = -2$.
- The common ratio is $r = \frac{\frac{6}{5}}{-2} = -\frac{3}{5}$.
- Therefore the sum is

$$\sum_{n=1}^{\infty} (-2) \left(-\frac{3}{5}\right)^{n-1} = \frac{(-2)}{1 - (-\frac{3}{5})} = -\frac{2}{\frac{8}{5}} = -\frac{5}{4}$$

Example

Write the number $2.\overline{317} = 2.3171717\ldots$ as a quotient of integers.

$$2.3171717\ldots = 2.3 + \frac{17}{10^3} + \frac{17}{10^5} + \frac{17}{10^7} + \cdots$$

- After the first term, we have a geometric series.
- $a = \frac{17}{10^3}$ and $r = \frac{1}{10^2}$.

$$\begin{aligned} 2.3171717\ldots &= 2.3 + \frac{\frac{17}{10^3}}{1 - \frac{1}{10^2}} = 2.3 + \frac{\frac{17}{1000}}{\frac{99}{100}} \\ &= \frac{23}{10} + \frac{17}{990} = \frac{1147}{495} \end{aligned}$$

Example

Show the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent and find its sum.

Is this a geometric series? No, because $\frac{a_{n+1}}{a_n} = \frac{\frac{1}{(n+1)(n+2)}}{\frac{1}{n(n+1)}} = \frac{n}{n+2}$ is not constant. Decompose a_n into partial fractions:

$$a_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

$$\begin{aligned} s_k &= \sum_{n=1}^k \frac{1}{n(n+1)} = \sum_{n=1}^k \left(\frac{1}{n} - \frac{1}{n+1} \right) \\ &= \left(1 - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) + \cdots + \left(\frac{1}{k} - \frac{1}{k+1} \right) \\ &= 1 - \frac{1}{k+1} \end{aligned}$$

$$\text{Therefore } \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{k \rightarrow \infty} s_k = \lim_{k \rightarrow \infty} \left(1 - \frac{1}{k+1} \right) = 1$$

Example

Show that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$ diverges.

$$s_1 = 1$$

$$s_2 = 1 + \frac{1}{2}$$

$$s_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1 + \frac{2}{2}$$

$$s_8 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}$$

$$> 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}$$

$$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1 + \frac{3}{2}$$

$$s_{16} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \dots + \frac{1}{8}\right) + \left(\frac{1}{9} + \dots + \frac{1}{16}\right)$$

$$> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \dots + \frac{1}{8}\right) + \left(\frac{1}{16} + \dots + \frac{1}{16}\right)$$

$$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1 + \frac{4}{2}$$

$$\vdots$$

$$s_{2^n} > 1 + \frac{n}{2}$$

Therefore $s_{2^n} \rightarrow \infty$ as $n \rightarrow \infty$, so $\{s_n\}$ is divergent, so the harmonic series is divergent.