Calculus I
Volumes of solids of revolution

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2019

Outline

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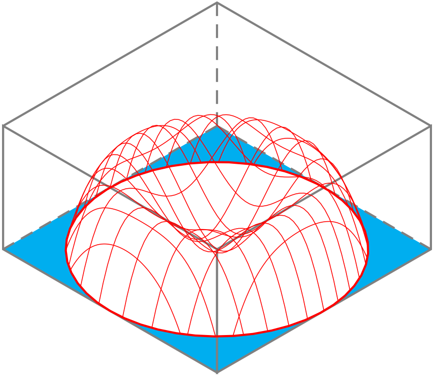
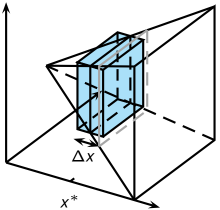
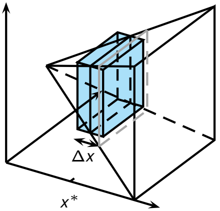
Volumes



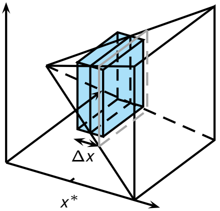
Volumes of solids are found/defined via integration.



- How do we find the volume of a solid S?



- How do we find the volume of a solid S?
- Let P_x be the plane perpendicular to the x-axis and passing through the point x.
- The intersection of P_x with S is called a cross-section.
- Let $A(x)$ be the area of this cross-section.



Approx. volume of slab:

$$A(x^*)\Delta x$$

Approx. volume of S:

$$V \approx \sum_{i=1}^{n} A(x_i^*)\Delta x$$

Exact volume of S:

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\Delta x$$

- How do we find the volume of a solid S?
- Let P_x be the plane perpendicular to the x-axis and passing through the point x.
- The intersection of P_x with S is called a cross-section.
- Let $A(x)$ be the area of this cross-section.
- Consider the part of S between two planes P_{x_1} and P_{x_2}.
- Approximate this part of S:
- Pick a sample point x^* between x_1 and x_2. Use a solid that has the same constant cross-sectional area $A(x^*)$ between x_1 and x_2.
- Let Δx be the distance from x_1 to x_2.

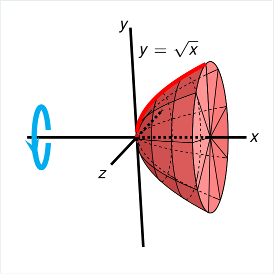
Definition (Volume)

Let S be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area of S in the plane P_x is a continuous function $A(x)$, then the volume of S is

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\Delta x = \int_a^b A(x)\mathrm{d}x$$

Example

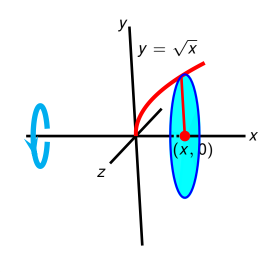
Find the volume of the solid obtained by rotating about the x-axis the region under the curve $y = \sqrt{x}$ from 0 to 1.



Example

Find the volume of the solid obtained by rotating about the x-axis the region under the curve $y = \sqrt{x}$ from 0 to 1.

- The cross-sections of this solid are all circles.
- The circular cross-section through the point $(x, 0)$ has radius $\sqrt{x}$.
- The area of the cross-section is $A(x) =$



Example

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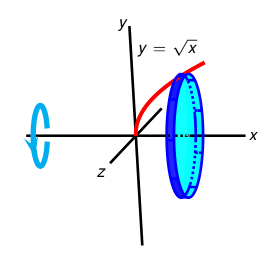
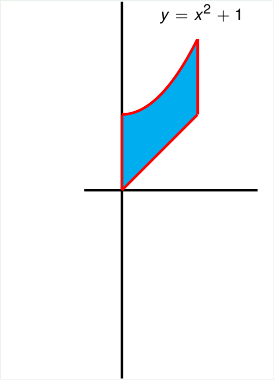
- The cross-sections of this solid are all circles.
- The circular cross-section through the point $(x, 0)$ has radius $\sqrt{x}$.
- The area of the cross-section is $A(x) =$
- The volume of a single approximating section is $A(x)\Delta x$.
- The x coords. of the solid are between 0 and 1, so its volume is



$$V = \int_0^1 A(x)\mathrm{d}x = \int_0^1 \pi x \, \mathrm{d}x$$
$$= \left[\pi \frac{x^2}{2}\right]_0^1 = \frac{\pi}{2} \quad .$$

Example (Typical Cross-Section is a Washer)



Find the volume of the solid obtained by rotating about the x-axis the region bounded by $y = x^2 + 1$, $y = x$, $x = 0$, and $x = 1$.

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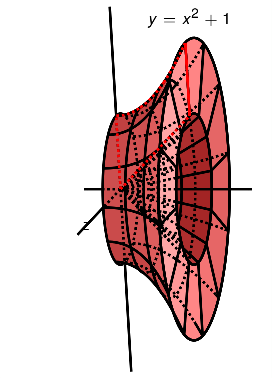

Find the volume of the solid obtained by rotating about the x-axis the region bounded by $y = x^2 + 1$, $y = x$, $x = 0$, and $x = 1$.
Cross-section: washer, center: $(x, 0)$. Area:
$A(x) =$ Area outer disk $-$ Area inner disk
Inner disk radius: x, area: πx^2.
Outer disk radius: $x^2 + 1$, area: $\pi(x^2 + 1)^2$.

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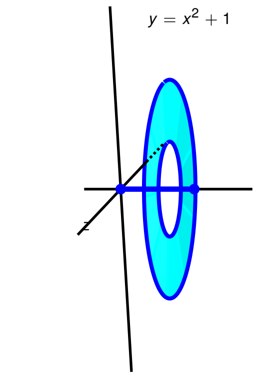

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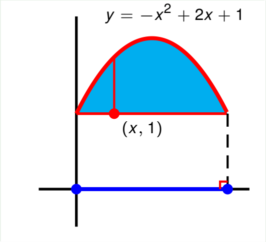
$$
\begin{aligned}
V = & \int_0^1 A(x) \mathrm{d}x = \int_0^1 \left(\pi(x^2+1)^2 - \pi x^2\right) \mathrm{d}x \\
= & \pi \int_0^1 \left(x^4 + x^2 + 1\right) \mathrm{d}x \\
= & \pi \left[\frac{x^5}{5} + \frac{x^3}{3} + x\right]_0^1 \\
= & \pi \left(\frac{1}{5} + \frac{1}{3} + 1\right) = \frac{23}{15}\pi
\end{aligned}
$$

Example (Rotation About a Line Parallel to the x-axis)

Find the volume of the solid obtained by rotating about the line $y = 1$ the region bounded by $y = -x^2 + 2x + 1$ and $y = 1$.



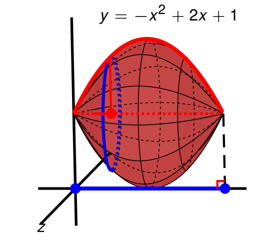
Example (Rotation About a Line Parallel to the x-axis)

Find the volume of the solid obtained by rotating about the line $y = 1$ the region bounded by $y = -x^2 + 2x + 1$ and $y = 1$.

Cross-section: a circle centered at $(x, 1)$,

radius: $(-x^2 + 2x + 1) - 1$,

area: $A(x) = \pi\left((-x^2 + 2x + \cancel{1}) - \cancel{1}\right)^2 = \pi\left(-x^2 + 2x\right)^2$.



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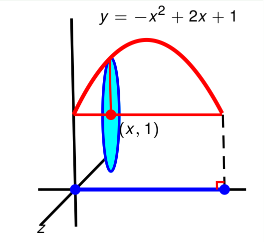
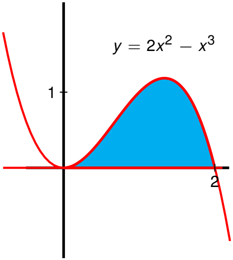
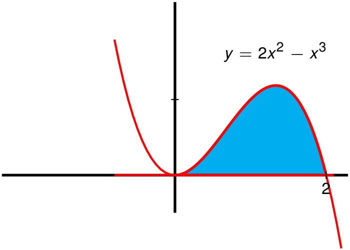


$$
\begin{aligned}
V &= \int_0^2 A(x)\mathrm{d}x = \int_0^2 \pi\left(-x^2 + 2x\right)^2 \mathrm{d}x \\
&= \pi \int_0^2 \left(x^4 - 4x^3 + 4x^2\right)\mathrm{d}x \\
&= \pi \left[\frac{x^5}{5} - x^4 + \frac{4x^3}{3}\right]_0^2 \\
&= \pi \left(\frac{2^5}{5} - 2^4 + 4 \cdot \frac{2^3}{3}\right) \\
&= \pi \left(\frac{32}{5} - 16 + \frac{32}{3}\right) = \frac{16}{15}\pi.
\end{aligned}
$$

Find the volume obtained by rotating the region bounded by $y = 2x^2 - x^3$ and the x-axis around ...





- ... the x-axis.
- Approximate the volume using circular cylinders with radius $2x^2 - x^3$ and height Δx.
- $V = \int_0^2 \pi(2x^2 - x^3)^2 \mathrm{d}x$.
- We understand the problem.

- ... the y-axis.
- Approx. with washers: need inner rad. x_i & outer rad. x_o.

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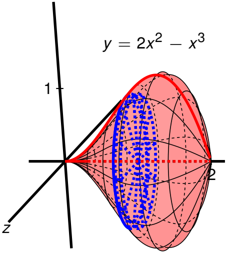

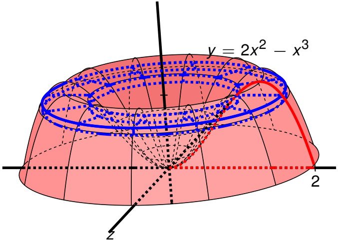

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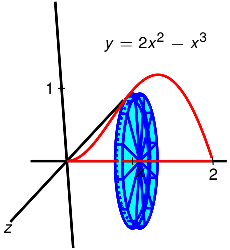
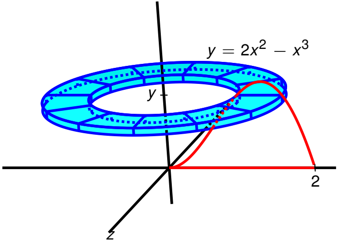


- ... the x-axis.
- Approximate the volume using circular cylinders with radius $2x^2 - x^3$ and height Δx.
- $V = \int_0^2 \pi(2x^2 - x^3)^2 dx$.
- We understand the problem.



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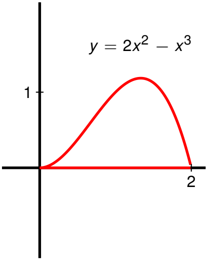
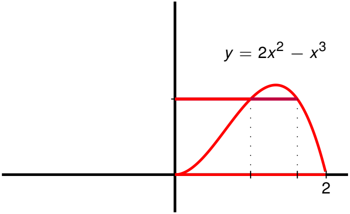
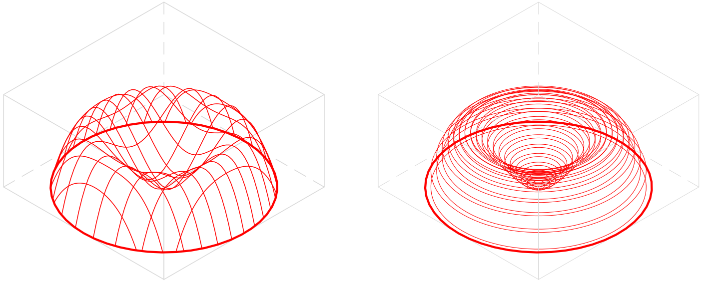


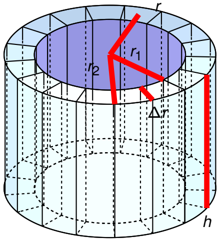


- ... the x-axis.
- Approximate the volume using circular cylinders with radius $2x^2 - x^3$ and height Δx.
- $V = \int_0^2 \pi(2x^2 - x^3)^2 \mathrm{d}x$.
- We understand the problem.

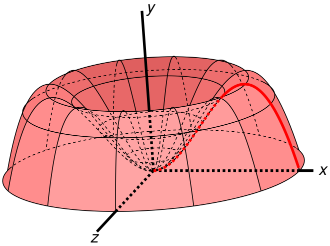
- ... the y-axis.
- Approx. with washers: need inner rad. x_i & outer rad. x_o.
- x_i and x_o: solutions to cubic: $-x^3 + 2x^2 - y = 0$. Solving for x requires lots of algebra.
- We show a simpler technique.



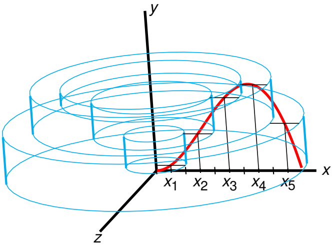
- Consider the solid obtained by rotating around the y-axis the region bounded above by $y = 2x^2 - x^3$ and below by the x-axis.
- Approximate this solid by nested cylindrical shells.
- Cylindrical shells are solids obtained by taking a cylinder and removing from its center another cylinder of equal height but smaller radius.



- Consider a cylindrical shell with:
- outer radius r_2,
- inner radius r_1,
- height h.
- $V_{\text{shell}} = V_{\text{outer cyl.}} - V_{\text{inner cyl.}} = \pi r_2^2 h - \pi r_1^2 h = \pi(r_2 - r_1)(r_2 + r_1)h.$
- Let $\Delta r = r_2 - r_1$.
- Let $r = \dfrac{r_2 + r_1}{2}$.
- Then $V_{\text{shell}} = 2\pi r h \Delta r$.



Consider a solid obtained by rotating the region under $f(x)$ around the y axis.



Consider a solid obtained by rotating the region under $f(x)$ around the y axis. Approximate the volume by cylindrical shells. Select the height of an individual shell to be $h = f(r)$ (r=average outer & inner radius).

$$V_{\text{shell}} = 2\pi r h \Delta r = 2\pi r f(r) \Delta r.$$

Suppose there are n cyclindrical shells and let $x_1, \dots, x_n$ be the averages of outer and inner radii. The shell volume sum is:

$$V_{\text{approx}} = \sum_{i=1}^{n} 2\pi x_i f(x_i) \Delta x.$$

Take the limit as the number of shells goes to ∞ to get

$$V = \lim_{n\to\infty} V_{\text{approx}} = \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi x_i f(x_i) \Delta x = \int_a^b 2\pi x f(x) \mathrm{d}x.$$

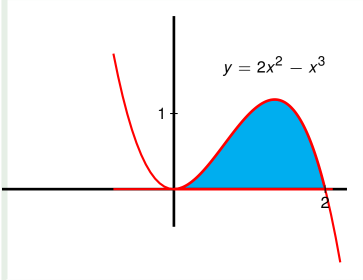
The endpoints of integration are the endpoints of the rotated region.

Definition (Volume by Cylindrical Shells)

The volume of the solid obtained by rotating around the y-axis the region under the curve $y = f(x)$ from a to b is

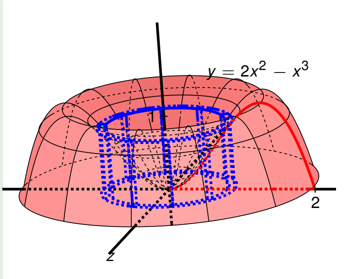
$$V = \int_a^b 2\pi x f(x) \mathrm{d}x.$$

Example



Find the volume of the solid obtained by rotating about the y-axis the region bounded by $y = 2x^2 - x^3$ and the x-axis.

Example



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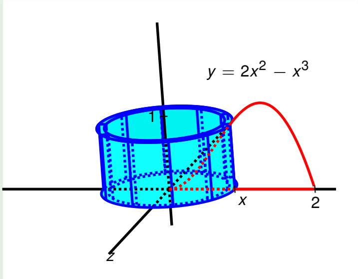


Find the volume of the solid obtained by rotating about the y-axis the region bounded by $y = 2x^2 - x^3$ and the x-axis.

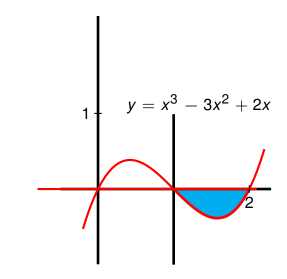
Cylindrical shell: outer radius x; height: $2x^2 - x^3$; circumference: $2\pi x$; infinitesimal volume: $2\pi x(2x^2 - x^3)\mathrm{d}x$.

$$V = \int_0^2 (2\pi x)(2x^2 - x^3)\mathrm{d}x = 2\pi \int_0^2 (2x^3 - x^4)\mathrm{d}x$$

$$= 2\pi \left[\frac{x^4}{2} - \frac{x^5}{5}\right]_0^2 = 2\pi\left(\frac{2^4}{2} - \frac{2^5}{5}\right) = 2\pi\left(8 - \frac{32}{5}\right) = \frac{16}{5}\pi.$$

Example (Rotated About a Line Other Than the y-axis)

Find the volume obtained by rotating about the line $x = 1$ the region to the right of $x = 1$ bounded by $y = x^3 - 3x^2 + 2x$ and the x-axis.



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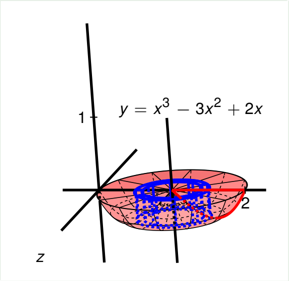
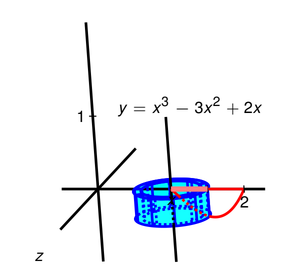


Example (Rotated About a Line Other Than the y-axis)

Find the volume obtained by rotating about the line $x = 1$ the region to the right of $x = 1$ bounded by $y = x^3 - 3x^2 + 2x$ and the x-axis.
Cylindrical shell: outer radius $x - 1$; height: $|x^3 - 3x^2 + 2x| = -(x^3 - 3x^2 + 2x)$; circumference: $2\pi(x - 1)$; infinitesimal volume: $2\pi(x - 1)(-x^3 + 3x^2 - 2x)\mathrm{d}x$.



$$
\begin{aligned}
V &= \int_1^2 2\pi(x-1)(-x^3+3x^2-2x)\mathrm{d}x \\
&= 2\pi\int_1^2 \left(-x^4 + 4x^3 - 5x^2 + 2x\right)\mathrm{d}x \\
&= 2\pi\left[-\frac{x^5}{5} + x^4 - \frac{5x^3}{3} + x^2\right]_1^2 \\
&= 2\pi\left(\left(-\frac{2^5}{5} + 2^4 - \frac{5}{3}\cdot 2^3 + 2^2\right)\right. \\
&\quad \left. - \left(-\frac{1^5}{5} + 1^4 - \frac{5}{3}\cdot 1^3 + 1^2\right)\right) = \frac{4}{15}\pi.
\end{aligned}
$$

	Rotate about ...	
	... a horizontal line	... a vertical line
y is a function of x	Cross-sections $\int \cdot \, \mathrm{d}x$	Cylindrical shells $\int \cdot \, \mathrm{d}x$
x is a function of y	Cylindrical shells $\int \cdot \, \mathrm{d}y$	Cross-sections $\int \cdot \, \mathrm{d}y$

- $\int \cdot \, \mathrm{d}x$ means integrate with respect to x.
- $\int \cdot \, \mathrm{d}y$ means integrate with respect to y.
- Some equations express y as a function of x and x as a function of y. In such cases, you may use either method.