Precalculus
Trigonometric identities theory

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- The discussion here also applies for trigonometric identities in more than one variables.

Types of identites

- In the present course we deal with two basic types of trigonometric identities.
- First, identities that involve operations on the arguments of the trigonometric functions.
  - Example: $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ (this is one of the angle sum identities); $\sin\theta + \sin(-\theta) = 0$.
  - Such identities can be proved using the angle sum formulas and the even/odd function properties of $\sin, \cos$.
- Second, identities that involve trigonometric functions of one variable.
  - Example: $\tan^2\theta + 1 = \sec^2\theta$.
  - Such identities can be proved only using the already demonstrated Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$.
- The Pythagorean identity follows from the angle sum formulas and the even/odd function properties of $\sin, \cos$, so all trigonometric identities follow from those properties alone.

Strategy for proving trigonometric identities

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- Thus, if we pick two variables s and c, and a single relation
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- The method is rather cumbersome for a human and is best suited for computers.

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  - A fraction of θ such that all appearing angles are integer multiples of it will always work.