

Calculus II

Weierstrass substitution, part 2

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Example

Let $\theta = 2 \arctan t$, $\cos \theta = \frac{1-t^2}{1+t^2}$, $\sin \theta = \frac{2t}{1+t^2}$, $z = \frac{3}{\sqrt{5}} \left(t + \frac{1}{3}\right)$.

$$\begin{aligned}
 \int \frac{d\theta}{2 \sin \theta - \cos \theta + 5} &= \int \frac{2dt}{(1+t^2) \left(2 \frac{2t}{t^2+1} - \frac{(1-t^2)}{1+t^2} + 5\right)} \\
 &= \int \frac{2dt}{6t^2 + 4t + 4} \\
 &= \int \frac{dt}{3t^2 + 2t + 2} \\
 \text{(complete square)} &= \int \frac{dt}{3 \left(t^2 + 2t \frac{1}{3} + \frac{1}{9} - \frac{1}{9} + \frac{2}{3}\right)} \\
 &= \frac{1}{3} \int \frac{dt}{\left(t + \frac{1}{3}\right)^2 + \frac{5}{9}} \\
 &= \frac{1}{3} \int \frac{dt}{\frac{5}{9} \left(\frac{9}{5} \left(t + \frac{1}{3}\right)^2 + 1\right)}
 \end{aligned}$$

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$$\begin{aligned}
 \int \frac{d\theta}{2 \sin \theta - \cos \theta + 5} &= \frac{1}{3} \int \frac{dt}{\frac{5}{9} \left(\frac{9}{5} \left(t + \frac{1}{3}\right)^2 + 1 \right)} \\
 &= \frac{3}{5} \int \frac{\frac{\sqrt{5}}{3} d\left(\frac{3}{\sqrt{5}} \left(t + \frac{1}{3}\right)\right)}{\left(\left(\frac{3}{\sqrt{5}} \left(t + \frac{1}{3}\right)\right)^2 + 1\right)} \\
 &= \frac{\sqrt{5}}{5} \int \frac{dz}{z^2 + 1} \\
 &= \frac{\sqrt{5}}{5} \arctan z + C \\
 &= \frac{\sqrt{5}}{5} \arctan \left(\frac{3}{\sqrt{5}} \left(t + \frac{1}{3}\right) \right) + C \\
 &= \frac{\sqrt{5}}{5} \arctan \left(\frac{3}{\sqrt{5}} \left(\tan \left(\frac{\theta}{2} \right) + \frac{1}{3} \right) \right) + C
 \end{aligned}$$