Precalculus
Completing the square

Todor Milev

2019

Example (Completing the square)

Complete the square.

$$3x^2 - 5x + 1$$

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$$3x^2 - 5x + 1 \quad = \quad 3\left(x^2 - \textbf{?}\, x\right) + 1$$

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$$3x^2 - 5x + 1 = 3\left(x^2 - \frac{5}{3}x\right) + 1$$

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$$\begin{aligned} 3x^2 - 5x + 1 &= 3\left(x^2 - \frac{5}{3}x\right) + 1 \\ &= 3\left(x^2 - 2 \cdot \frac{5}{2 \cdot 3}x\right) + 1 \end{aligned}$$

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&= 3\left(x^2 - 2 \cdot \frac{5}{6}x + \mathbf{?} \qquad - \mathbf{?} \qquad\right) + 1
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Definition (Completing the square)

Let $a \neq 0$. To *complete the square* means to carry out the following algebraic manipulation.

$ax^2 + bx + c$

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Add & subtract $\left(\frac{b}{2a}\right)^2$

use $(A+B)^2 = A^2 + 2AB + B^2$

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