Calculus I
Homework
Continuity

1. Find the (implied) domain of $f(x)$. Extend the definition of f at $x = 3$ to make f continuous at 3.

   (a) $f(x) = \dfrac{x^2 - x - 6}{x - 3}$.

   answer: Implied domain: $x \in (-\infty, 3) \cup (3, \infty)$. Extend $f(x)$ to $f(x) = x + 2$.

   (b) $f(x) = \dfrac{x^3 - 27}{x^2 - 9}$.

   answer: Implied domain: $x \in (-\infty, -3) \cup (-3, 3) \cup (3, \infty)$. Extend $f(x)$ to $f(x) = \frac{x^2+3x+9}{x+3}$ with domain $x \in (-\infty, -3) \cup (-3, \infty)$.

2. Use the Intermediate Value Theorem to show that there is a real number solution of the given equation in the specified interval.

   (a) $x^5 + x - 3 = 0$ where $x \in (1, 2)$.
   (b) $\sqrt[4]{x} = 1 - x$ where $x \in \mathbb{R}$ (i.e., x is an arbitrary real number).
   (c) $\cos x = 2x$, where $x \in (0, 1)$.
   (d) $\sin x = x^2 - x - 1$, where $x \in \mathbb{R}$ (i.e., x is an arbitrary real number).
   (e) $\cos x = x^4$, where $x \in \mathbb{R}$ (i.e., x is an arbitrary real number).
   (f) $x^5 - x^2 + x + 3 = 0$, where $x \in \mathbb{R}$.

3.

   (a) i. Solve the equation $x^2 + 13x + 41 = 1$.
   ii. Use the intermediate value theorem to prove that the equation $x^2 + 13x + 41 = \sin x$ has at least two solutions, lying between the two solutions to 3.a.i.

   (b) i. Solve the equation $x^2 - 15x + 55 = 1$.
   ii. Use the intermediate value theorem to prove that the equation $x^2 - 15x + 55 = \cos x$ has at least two solutions, lying between the two solutions to the equation in the preceding item.

Solution. 3.a.i.

$$\begin{array}{rcl} x^2 + 13x + 41 &=& 1 \\ x^2 + 13x + 40 &=& 0 \\ (x+5)(x+8) &=& 0 \end{array} .$$

eqnarray Therefore the two solutions are $x_1 = -5$ and $x_2 = -8$.

3.a.ii. Consider the function

$$f(x) = x^2 + 13x + 41 - \sin x \quad .$$

Our strategy for proving $f(x) = 0$ has a solution consists in finding a number a such that $f(a) < 0$ and a number b such that $f(b) > 0$, and then using the Intermediate Value Theorem (IVT) with $N = 0$.

Let

$$g(x) = x^2 + 13x + 41,$$

and so $f(x) = g(x) - \sin x$. We have no techniques for evaluating $\sin x$ without calculator, but we do have all knowledge necessary to evaluate $g(x)$. Indeed, from high school we know that the lowest point of the parabola $g(x)$ is located at $x = -\frac{13}{2} = -6.5$. Then $g(-6.5) = -1.25$. Therefore

$$f(-6.5) = g(-6.5) - \sin(-6.5) = g(-6.5) + \sin(6.5) = -1.25 + \sin 6.5 \leq -0.25,$$

where for the very last inequality we use the fact that $\sin 6.5 < 1$ (remember $\sin t \leq 1$ for all real values of t).

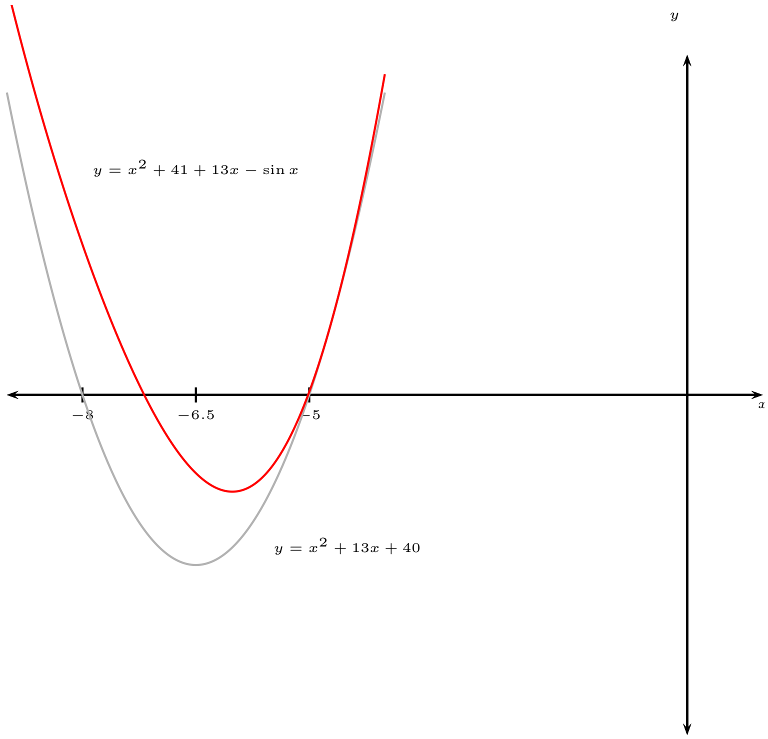
On the other hand,

$$f(-5) = g(-5) - \sin(-5) = 1 + \sin 5 > 0$$

as $\sin 5 > -1$ (remember $\sin t \geq -1$ for all real values of t). Therefore $f(-5) > 0$ and $f(-6.5) < 0$ and by the Intermediate Value Theorem (IVT) $f(x) = 0$ has a solution in the interval $x \in (-6.5, -5)$.

Proving $f(x) = 0$ has a solution in the interval $x \in (-8, -6.5)$ is similar and we leave it to the student.

Below is a computer generated plot of the function with the use of which we can visually verify our answer.



4. For which values of x is f continuous?

   - $f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$
   - $f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ x & \text{if } x \text{ is irrational} \end{cases}$

5. Show that $f(x)$ is continuous at all irrational points and discontinuous at all rational ones.

$$f(x) = \begin{cases} \frac{1}{q^2} & \text{if } x \text{ is rational and } x = \frac{p}{q} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

where in the first item p, q are relatively prime integers (i.e., integers without a common divisor).