Calculus II
Tangents and curve length

Todor Milev

2019

Outline

1 Tangents to Curves
- Tangents to Polar Curves

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1. Tangents to Curves
   - Tangents to Polar Curves

2. Arc Length
   - Arc Length in Polar Coordinates

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Tangents

Let C be the curve $C : \left| \begin{array}{lcl} x & = & f(t) \\ y & = & g(t) \end{array} \right., t \in [a, b]$.

Definition

Suppose $f'(t)$ and $g'(t)$ are not simultaneously equal to 0.

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- We define the line passing through $(f(t), g(t))$ with direction vector equal to the tangent vector to be *tangent line* to C at t. In other words, the tangent line has equation

$$(x - f(t))g'(t) = (y - g(t))f'(t) \quad .$$

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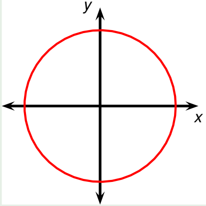
Note. When $f'(t) = g'(t) = 0$, for curves C with additional properties, natural definition(s) of tangent(s) do exist but are beyond Calc II.

Example



Find the tangent to the curve

$$\gamma : \left| \begin{array}{lll} x & = & \cos t \\ y & = & \sin t \end{array} \right., t \in [0, 2\pi) \text{ at } t = \frac{\pi}{4}, t = \frac{2\pi}{3}, t = \pi.$$

Recall $C : \left| \begin{array}{rcl} x & = & f(t) \\ y & = & g(t) \end{array} \right.$, $t \in [a, b]$, tangent vector at t is $(f'(t), g'(t))$.

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If $\frac{\mathrm{d}x}{\mathrm{d}t} \neq 0$, *we have* $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}$.

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$$\begin{array}{rcll} y & = & F(x) & \Big| \text{ apply } \frac{\mathrm{d}}{\mathrm{d}t} \\ \frac{\mathrm{d}y}{\mathrm{d}t} & = & \frac{\mathrm{d}}{\mathrm{d}t}(F(x)) & \Big| \text{ use chain rule} \\ & = & \frac{\mathrm{d}F}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} & \Big| \text{ divide by } x'(t) \\ \frac{\mathrm{d}y}{\mathrm{d}x} & = & \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} & \end{array}$$

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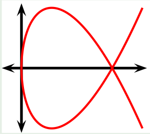
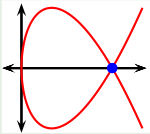


A curve C is defined by $x = t^2, y = t^3 - 3t$.

1. Show C traverses $(x, y) = (3, 0)$ for two values of t; find the tangent slopes for both of these values.
2. Find the points on C where the tangents are horizontal or vertical.
3. Find two intervals where we can write y as a function of x.
4. Determine concavity intervals of the functions found in item 3.

Example



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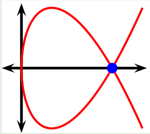

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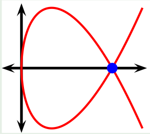

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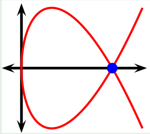

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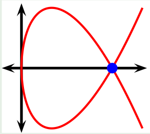

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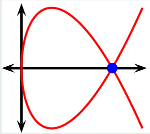

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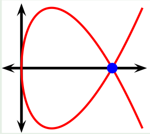

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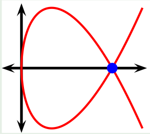

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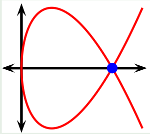

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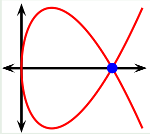

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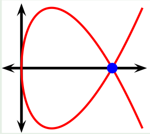

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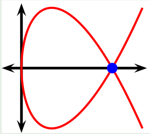

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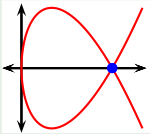

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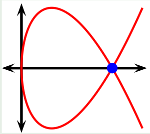

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- $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2-3}{2t}$.
- Plug in $t = \pm\sqrt{3}$: $\frac{dy}{dx}_{|t=\pm\sqrt{3}} = \frac{3(\pm\sqrt{3})^2 - 3}{2(\pm\sqrt{3})} = \pm\frac{6}{2\sqrt{3}} = \pm\sqrt{3}$

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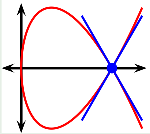

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     Therefore the tangents at $(3, 0)$ have slopes $\pm\sqrt{3}$.

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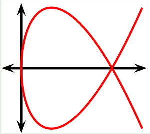

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2. Find the points on C where the tangents are horizontal or vertical.

Horizontal tangent:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 0$$

Vertical tangent:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 0$$

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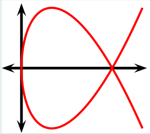

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Horizontal tangent:

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \\ 3t^2 - 3 &= 0 \end{aligned}$$

Vertical tangent:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 0$$

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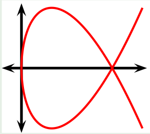

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Horizontal tangent:

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}t} &= 0\\ 3t^2 - 3 &= 0\\ 3(t^2-1) &= 0\end{aligned}$$

Vertical tangent:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 0$$

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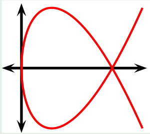

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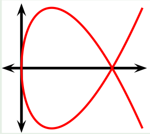

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$\frac{dx}{dt} \neq 0$ when $t = \pm 1$, so there are horizontal tangents when $t = \pm 1$.

Vertical tangent:

$$\frac{dx}{dt} = 0$$

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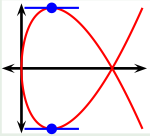

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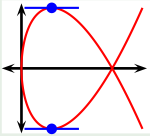
The points are $(1, 2)$ and $(1, -2)$.

Vertical tangent:

$$\frac{dx}{dt} = 0$$

Example



A curve C is defined by $x = t^2, y = t^3 - 3t$.

2. Find the points on C where the tangents are horizontal or vertical.

Horizontal tangent:

$$\begin{aligned} \frac{dy}{dt} &= 0 \\ 3t^2 - 3 &= 0 \\ 3(t^2 - 1) &= 0 \\ t &= \pm 1 \end{aligned}$$

$\frac{dx}{dt} \neq 0$ when $t = \pm 1$, so there are horizontal tangents when $t = \pm 1$. The points are $(1, 2)$ and $(1, -2)$.

Vertical tangent:

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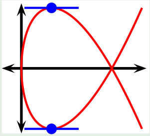

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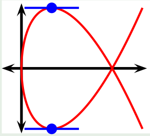

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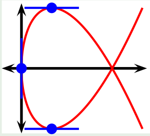
$\frac{dy}{dt} \neq 0$ when $t = 0$, so there is a vertical tangent when $t = 0$.

Example



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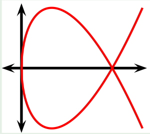
$\frac{\mathrm{d}y}{\mathrm{d}t} \neq 0$ when $t = 0$, so there is a vertical tangent when $t = 0$. The points is $(0, 0)$.

Example



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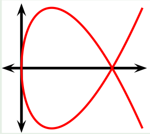
3. Find two intervals where we can write y as a function of x.

Example



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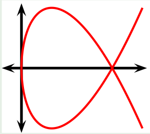
From $x = t^2$ we have that $t = \pm\sqrt{x}$.

Example



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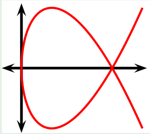
Example



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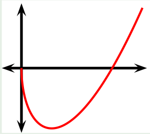
Example



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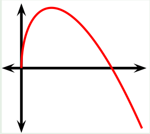
Example



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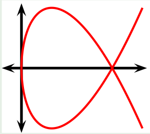
Example



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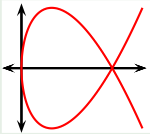
Example



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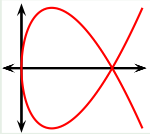

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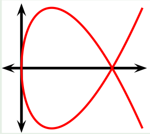
Example



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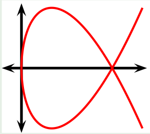

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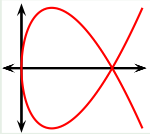

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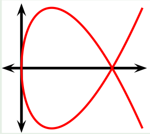

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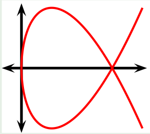

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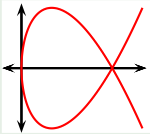

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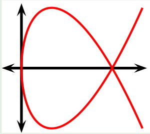

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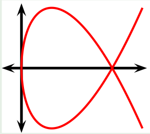

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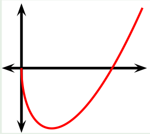

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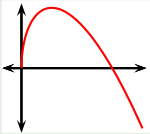
\end{aligned}$$

Therefore y as a function of x (which is a function of t) is concave up when $t > 0$

Example



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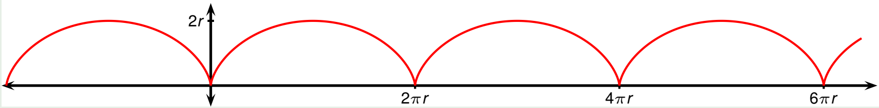
Therefore y as a function of x (which is a function of t) is concave up when $t > 0$ and concave down when $t < 0$.

Example

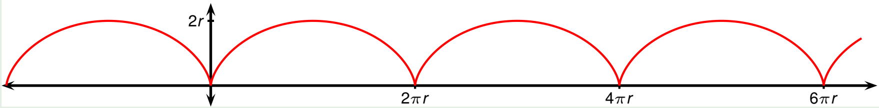
Consider the cycloid $x = r(\theta - \sin\theta)$, $y = r(1 - \cos\theta)$.



1. At what points is the tangent horizontal?
2. At what points is the tangent vertical?

Example

Consider the cycloid $x = r(\theta - \sin\theta)$, $y = r(1 - \cos\theta)$.



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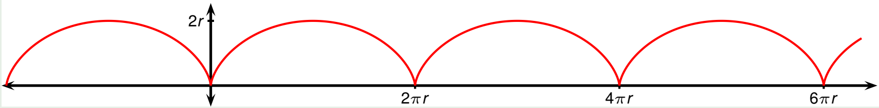

1. At what points is the tangent horizontal?
   - The slope of the tangent is $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}\theta}{\mathrm{d}x/\mathrm{d}\theta}$

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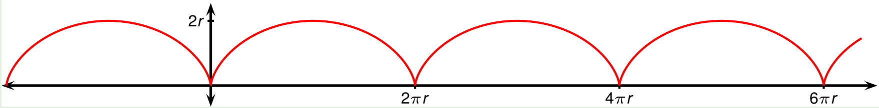

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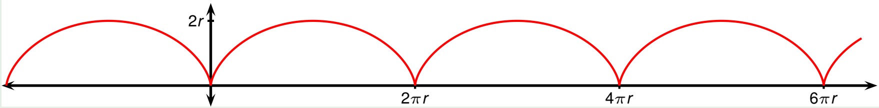

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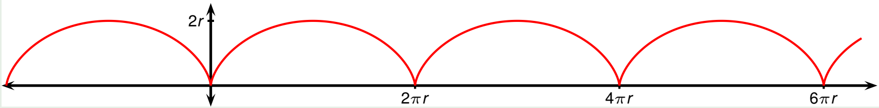

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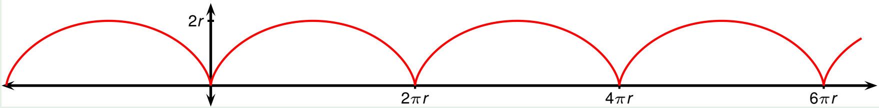

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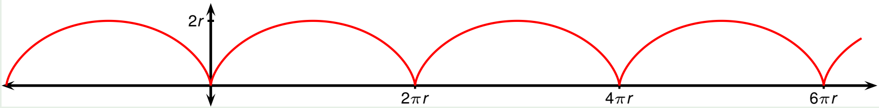

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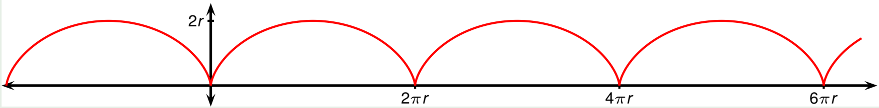

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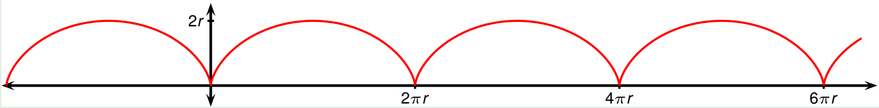

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- $r\sin\theta = dy/d\theta = 0$ if $\theta =$
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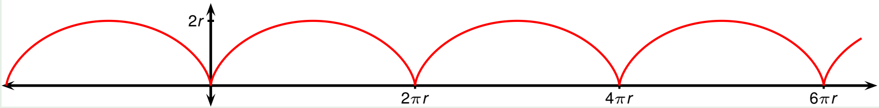

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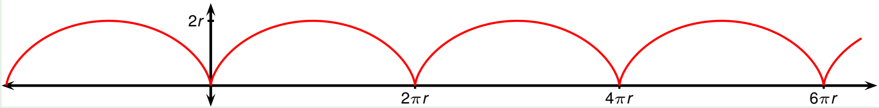

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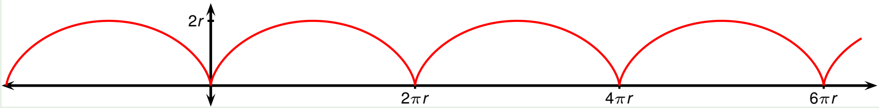

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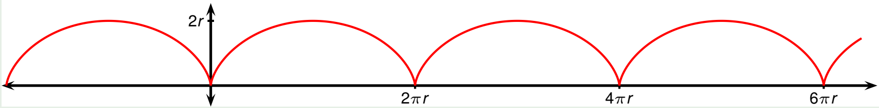

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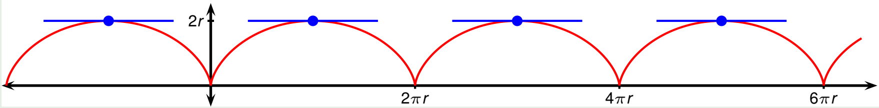

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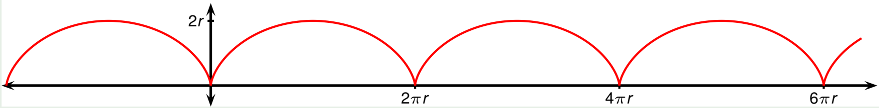

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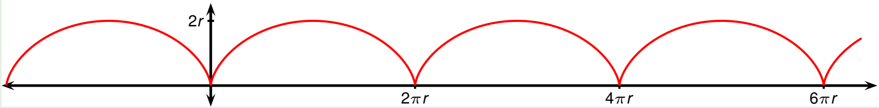

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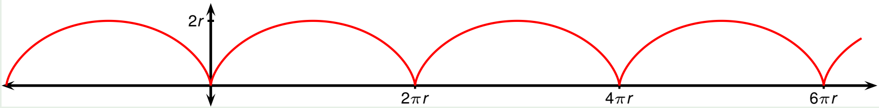

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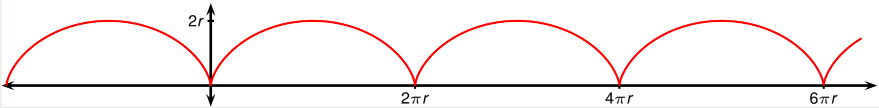

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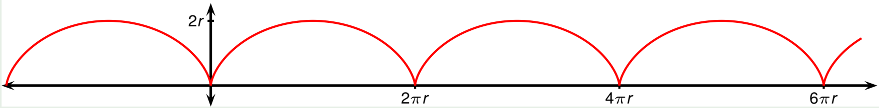

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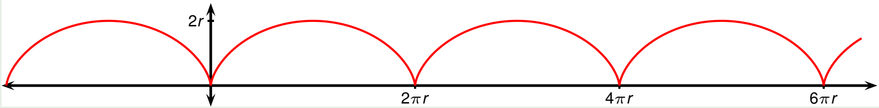

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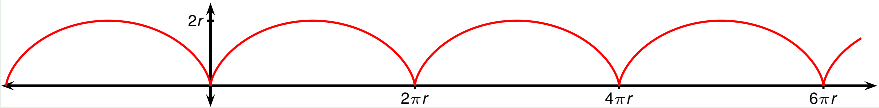

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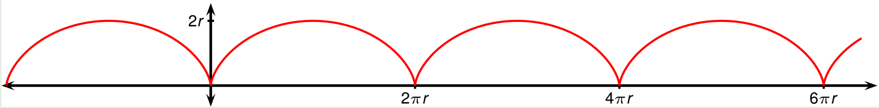

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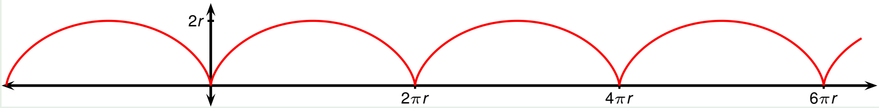

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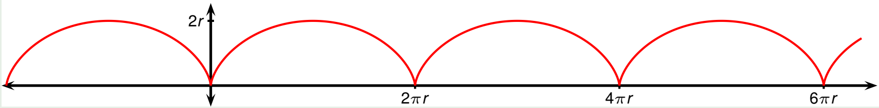

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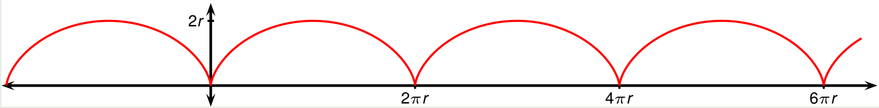

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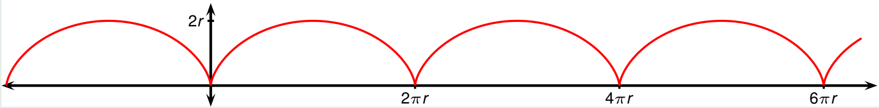

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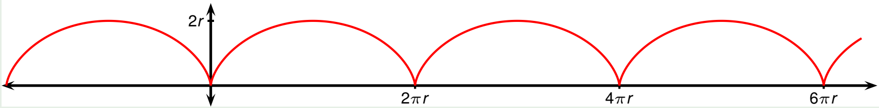

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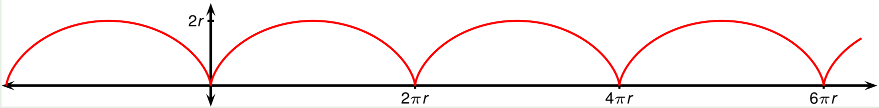

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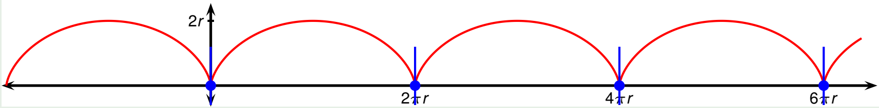

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   - Therefore there is a vertical tangent when $\theta = 2n\pi$.

Tangents to Polar Curves

To find the tangent line to a polar curve $r = f(\theta)$, regard θ as a parameter and write the parametric equations as

$$x = r\cos\theta = f(\theta)\cos\theta \qquad y = r\sin\theta = f(\theta)\sin\theta$$

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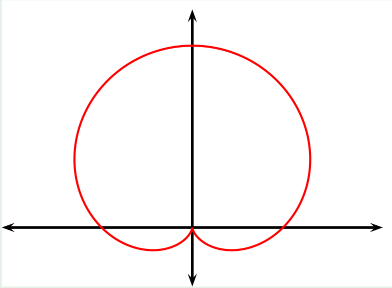
&= \frac{\frac{\mathrm{d}r}{\mathrm{d}\theta}\sin\theta + r\cos\theta}{\frac{\mathrm{d}r}{\mathrm{d}\theta}\cos\theta - r\sin\theta}
\end{aligned}$$

Example

Find the points on $r = 1 + \sin\theta$ where the tangent is horizontal or vertical.



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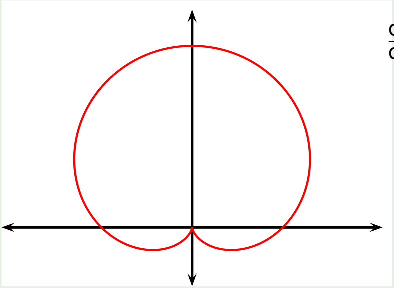

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

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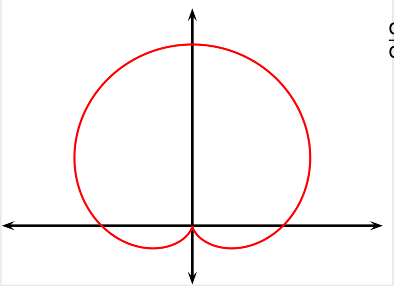
Find the points on $r = 1 + \sin\theta$ where the tangent is horizontal or vertical.



$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} = \frac{\sin\theta + \qquad \cos\theta}{\cos\theta - \qquad \sin\theta}$$

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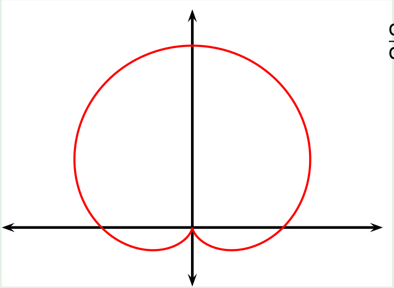

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} = \frac{\sin\theta + (1+\sin\theta)\cos\theta}{\cos\theta - (1+\sin\theta)\sin\theta}$$

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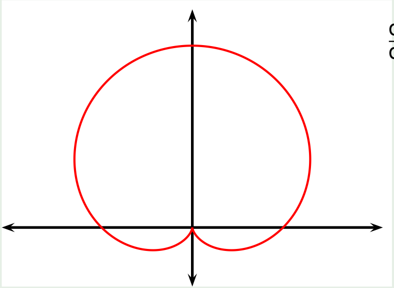

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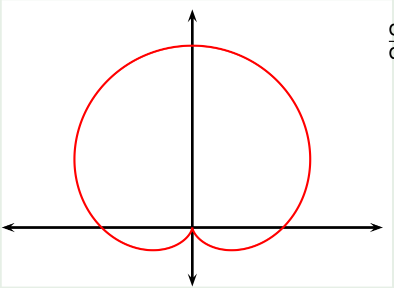

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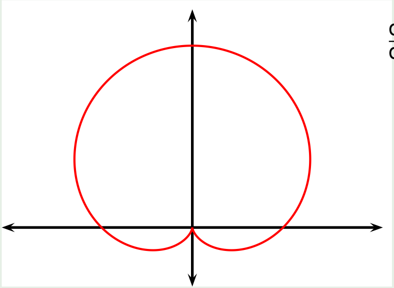

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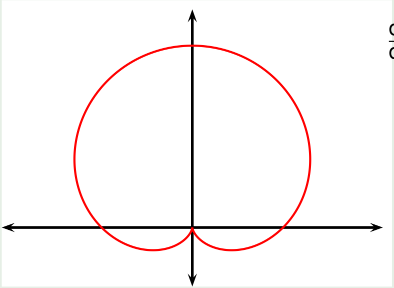

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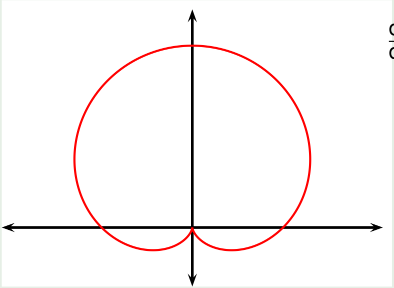

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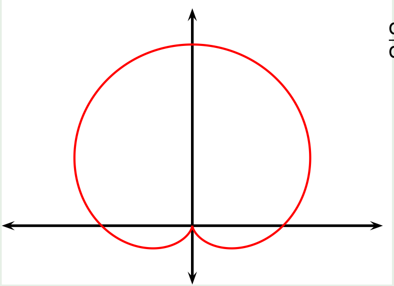

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\frac{\mathrm{d}r}{\mathrm{d}\theta}\sin\theta + r\cos\theta}{\frac{\mathrm{d}r}{\mathrm{d}\theta}\cos\theta - r\sin\theta} = \frac{\cos\theta\sin\theta + (1+\sin\theta)\cos\theta}{\cos\theta\cos\theta - (1+\sin\theta)\sin\theta}\\ &= \frac{\cos\theta(1+2\sin\theta)}{1-2\sin^2\theta - \sin\theta} = \frac{\cos\theta(1+2\sin\theta)}{(1+\sin\theta)(1-2\sin\theta)}\end{aligned}$$

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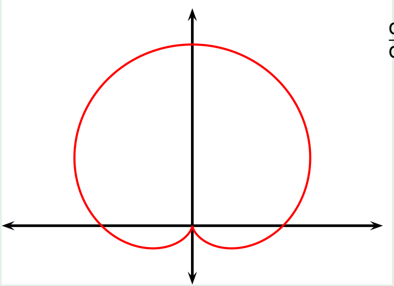

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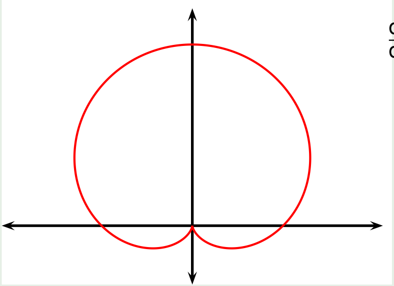

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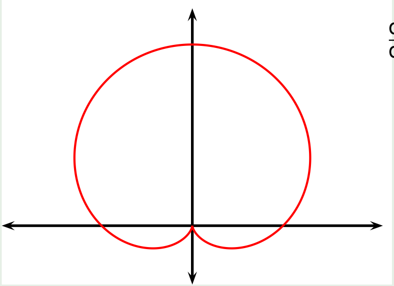

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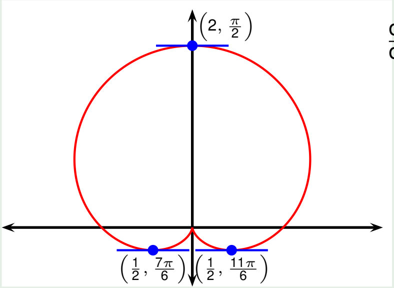

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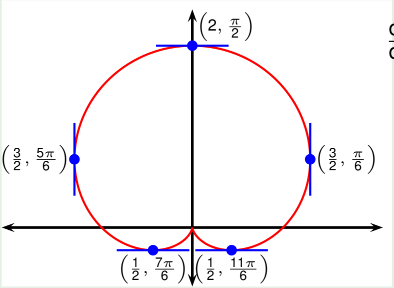

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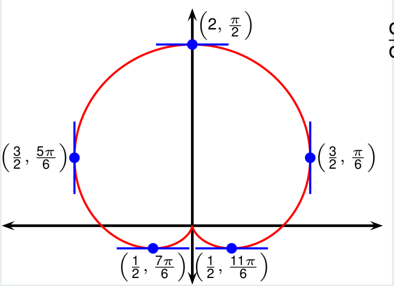

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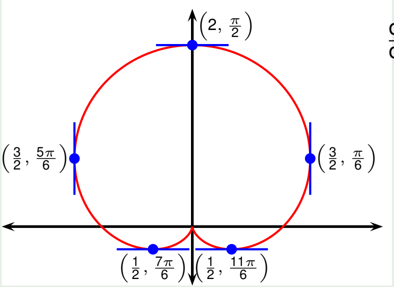

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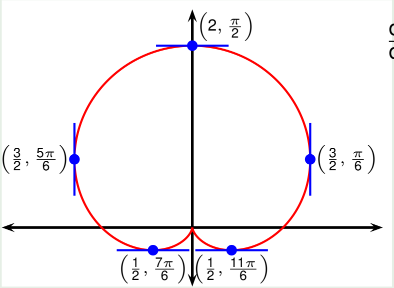

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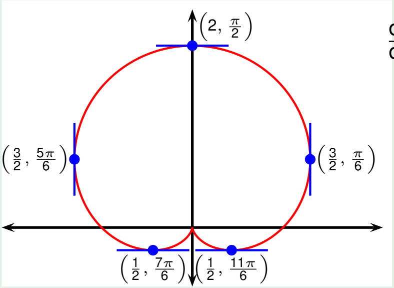

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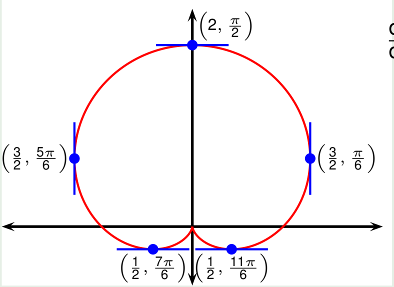

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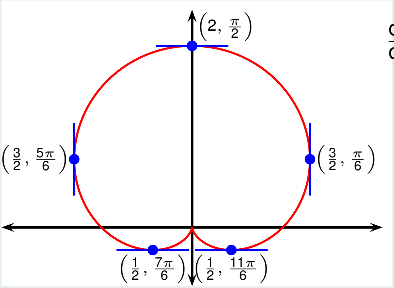

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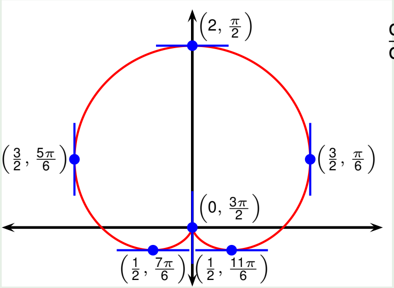

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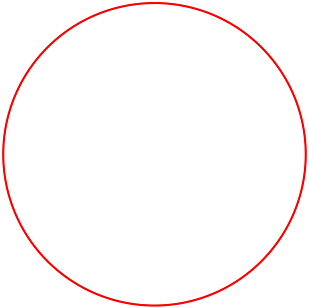
$$\lim_{\theta\to 3\pi/2^-} \frac{dy}{dx} = \lim_{\theta\to 3\pi/2^-} \frac{1+2\sin\theta}{1-2\sin\theta} \cdot \lim_{\theta\to 3\pi/2^-} \frac{\cos\theta}{1+\sin\theta} = -\frac{1}{3} \lim_{\theta\to 3\pi/2^-} \frac{-\sin\theta}{\cos\theta} = \infty$$

Arc Length



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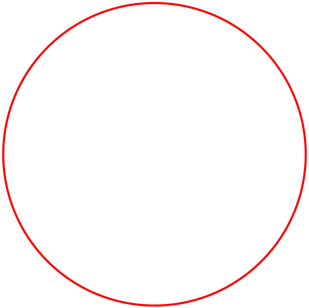

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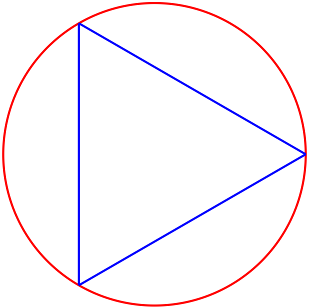

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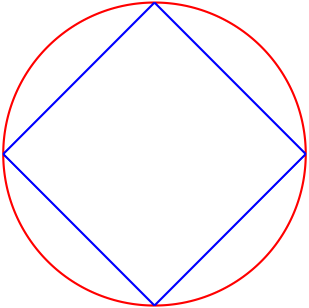

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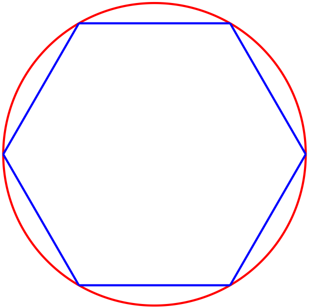

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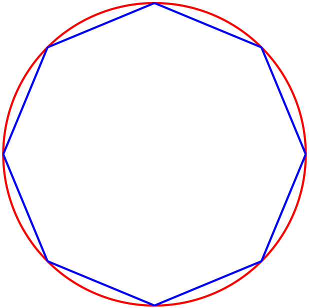

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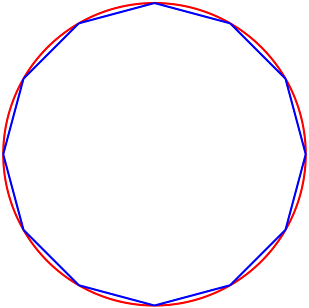

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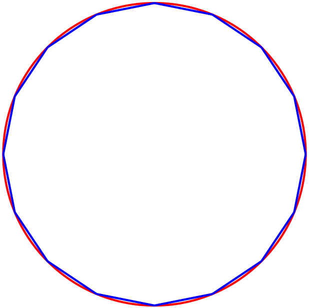

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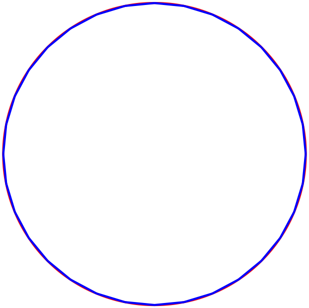

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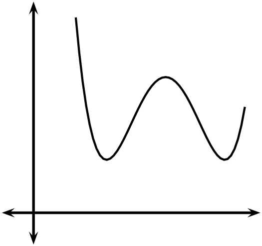

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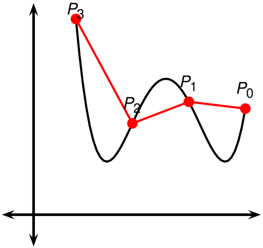

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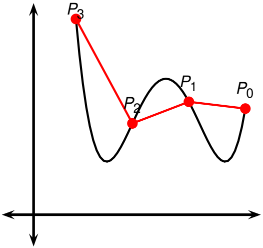

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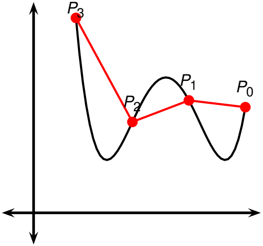

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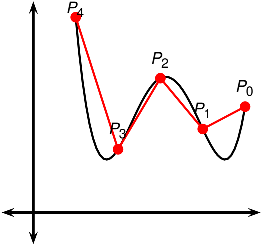

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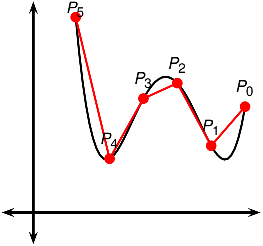

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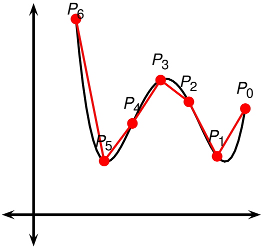

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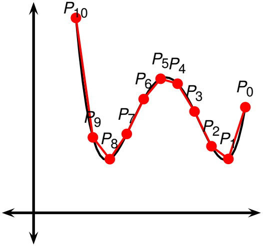

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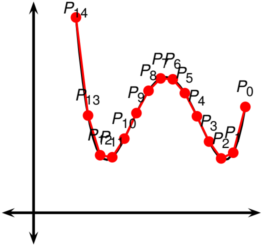

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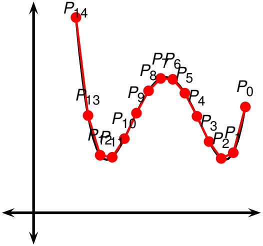

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Let $\gamma : \left| \begin{array}{lcl} x & = & x(t) \\ y & = & y(t) \end{array} \right., t \in [a, b]$.

Definition

Suppose $x'(t)$ and $y'(t)$ (exist and) are continuous on $[a, b]$. Then the length of the curve γ is defined as

$$L(\gamma) = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2} \, \mathrm{d}t$$

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Arc length of graph of a function

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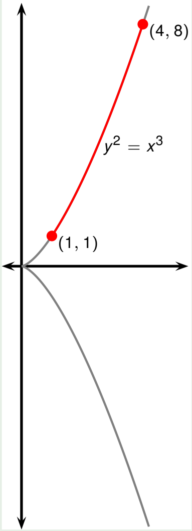
Suppose f' exists and is continuous on $[a, b]$. Then the length of the curve $y = f(x)$, $a \leq x \leq b$, is

$$\begin{aligned} L &= \int_a^b \sqrt{1 + (f'(x))^2}\, \mathrm{d}x \\ &= \int_a^b \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\, \mathrm{d}x \qquad \text{(in Leibniz notation)} \quad . \end{aligned}$$

Example

Find the length of the arc of $y^2 = x^3$ between $(1, 1)$ and $(4, 8)$.



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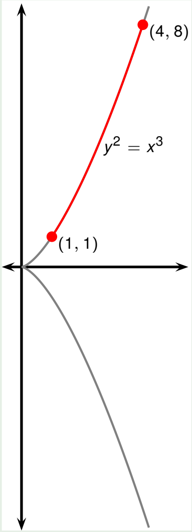

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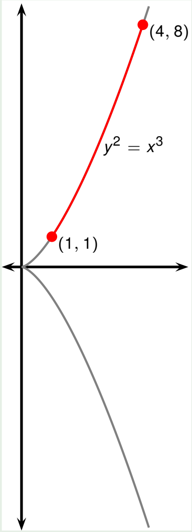

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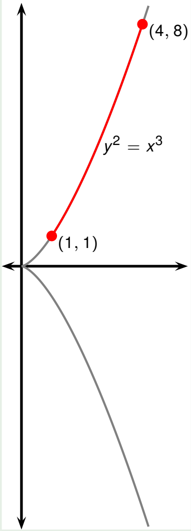

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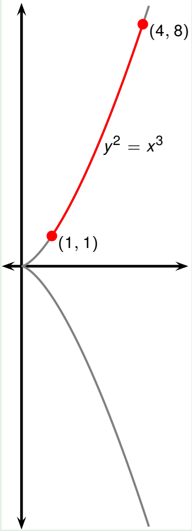

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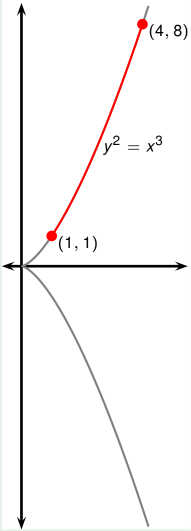

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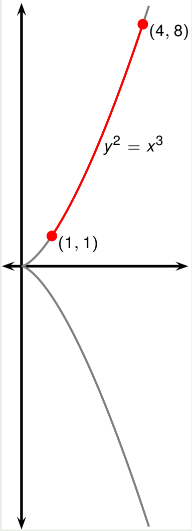

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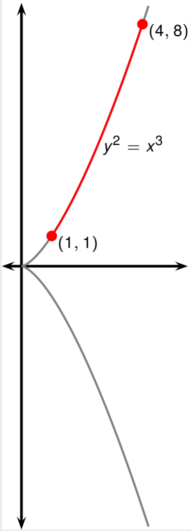

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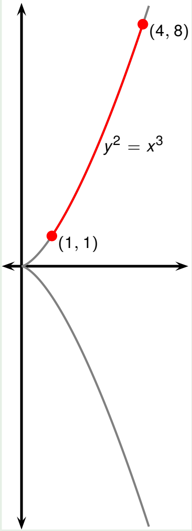

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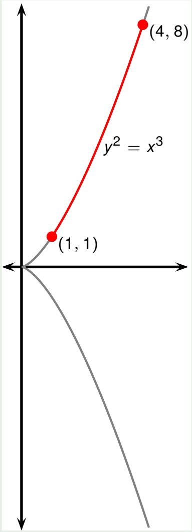

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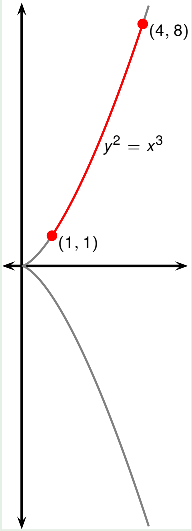

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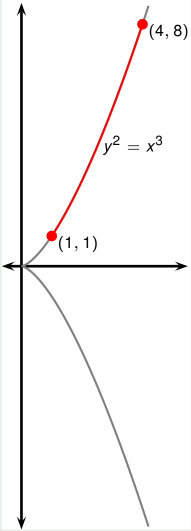

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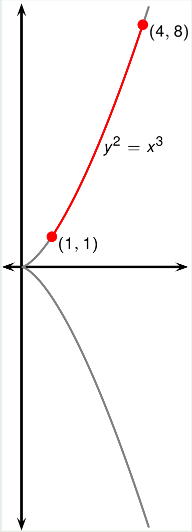

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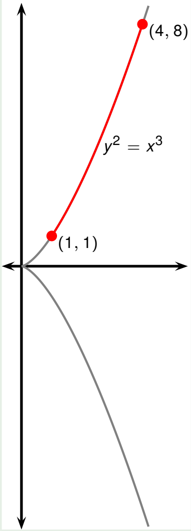

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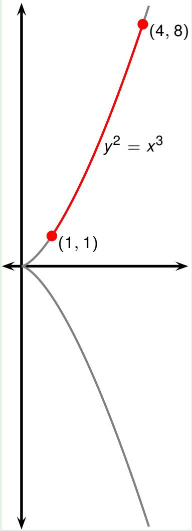

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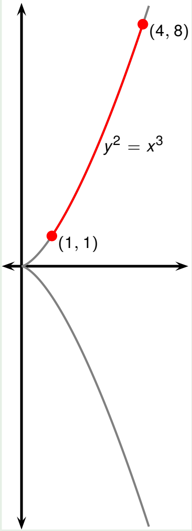

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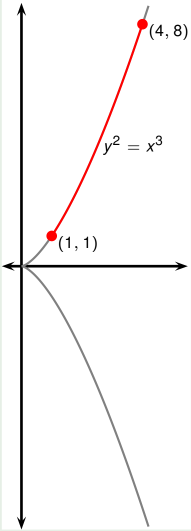

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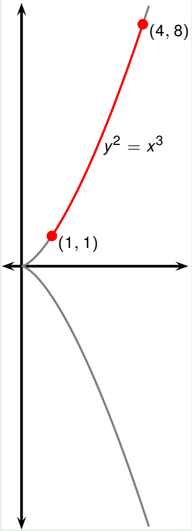
- When $x = 1$, $u = \frac{13}{4}$.
- When $x = 4$, $u = 10$.

$$
\begin{aligned}
L &= \int_1^4 \sqrt{1 + (y')^2}\mathrm{d}x \\
&= \int_1^4 \sqrt{1 + \frac{9}{4}x}\, \mathrm{d}x = \int_{13/4}^{10} \frac{4}{9}\sqrt{u}\, \mathrm{d}u
\end{aligned}
$$

Example

Find the length of the arc of $y^2 = x^3$ between $(1,1)$ and $(4,8)$.



- For the top half of the curve we have:
- $y = x^{3/2}$ and $y' = \frac{3}{2}x^{1/2}$.
- $u = 1 + \frac{9}{4}x$ and $\mathrm{d}u = \frac{9}{4}\mathrm{d}x$.
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$$
\begin{aligned}
L &= \int_1^4 \sqrt{1 + (y')^2}\mathrm{d}x \\
&= \int_1^4 \sqrt{1 + \frac{9}{4}x}\,\mathrm{d}x = \int_{13/4}^{10} \frac{4}{9}\sqrt{u}\,\mathrm{d}u \\
&= \frac{4}{9}\left[\frac{2}{3}u^{3/2}\right]_{13/4}^{10}
\end{aligned}
$$

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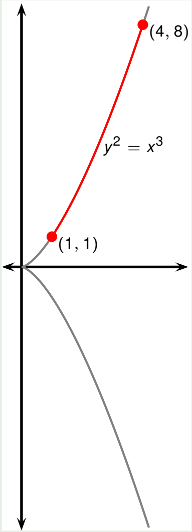
Find the length of the arc of $y^2 = x^3$ between $(1,1)$ and $(4,8)$.



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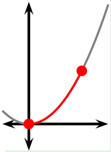
- When $x = 4$, $u = 10$.

$$\begin{aligned} L &= \int_1^4 \sqrt{1 + (y')^2}\mathrm{d}x \\ &= \int_1^4 \sqrt{1 + \frac{9}{4}x}\, \mathrm{d}x = \int_{13/4}^{10} \frac{4}{9}\sqrt{u}\, \mathrm{d}u \\ &= \frac{4}{9}\left[\frac{2}{3}u^{3/2}\right]_{13/4}^{10} = \frac{8}{27}\left(10^{3/2} - \left(\frac{13}{4}\right)^{3/2}\right) \end{aligned}$$

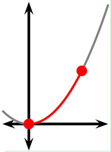
Example



Find the length of the arc of the parabola $y = x^2$ from $(0, 0)$ to $(1, 1)$.

$$L = \int_{x=0}^{x=1} \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} \mathrm{d}x$$

Example



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$$\frac{dy}{dx} = \textbf{?}$$

$$L = \int_{x=0}^{x=1} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_{x=0}^{x=1} \sqrt{1 + \textbf{?}\quad} dx$$

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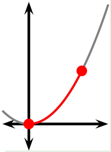

Find the length of the arc of the parabola $y = x^2$ from $(0,0)$ to $(1,1)$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$$

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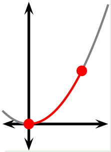

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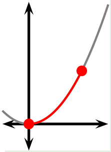
Example



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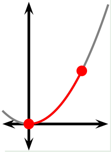

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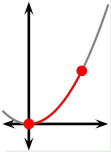

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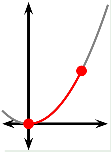

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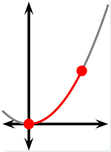

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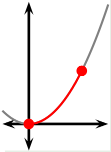

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Set $x = \frac{1}{2}\tan\theta$

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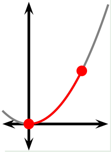
Example



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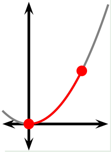

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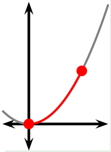
Example



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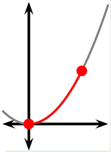
Example



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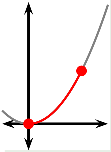
Example



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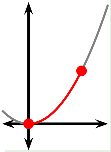

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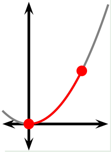
Example



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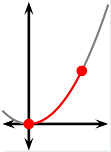
Example



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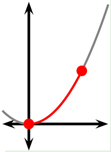

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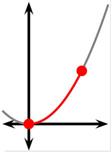

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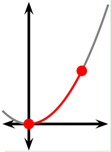

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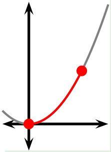

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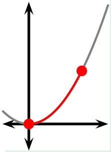
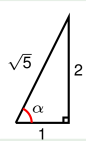
$$= \frac{1}{4}\left(2\cdot \textbf{?} \quad + \ln|\textbf{?} \quad + 2|\right)$$

Example



Find the length of the arc of the parabola $y = x^2$ from $(0,0)$ to $(1,1)$.



$$\frac{dy}{dx} = 2x$$

$$L = \int_{x=0}^{x=1} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_{x=0}^{x=1} \sqrt{1 + 4x^2}\, dx \quad \Big| \quad \text{Set } x = \tfrac{1}{2}\tan\theta$$

$$= \int_{\theta=0}^{\theta=\arctan 2} \sqrt{1 + \tan^2\theta}\; d\left(\frac{1}{2}\tan\theta\right)$$

$$= \int_{\theta=0}^{\theta=\arctan 2} \sec\theta \cdot \frac{1}{2}\sec^2\theta d\theta = \frac{1}{2}\int_0^{\alpha} \sec^3\theta\, d\theta \quad \Big| \quad \text{Set } \alpha = \arctan 2$$

$$= \frac{1}{2} \cdot \left[\frac{1}{2}\left(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right)\right]_{\theta=0}^{\theta=\alpha}$$

$$= \frac{1}{4}\left(\sec\alpha\tan\alpha + \ln|\sec\alpha + \tan\alpha|\right)$$

$$= \frac{1}{4}\left(2 \cdot \sqrt{5} + \ln|\sqrt{5} + 2|\right)$$

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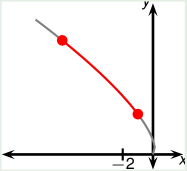

Find the length of the curve γ.

$$\gamma : \left| \begin{array}{lcl} x(t) & = & \sqrt{t} - 2t \\ y(t) & = & \frac{8}{3} t^{\frac{3}{4}} \end{array} \right. , t \in [1, 4] \quad .$$

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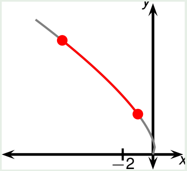

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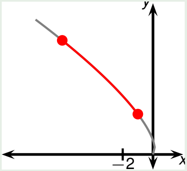

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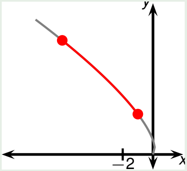

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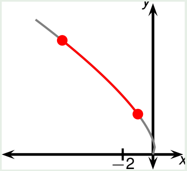

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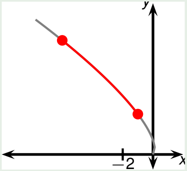

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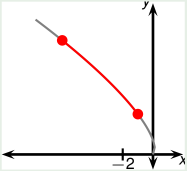

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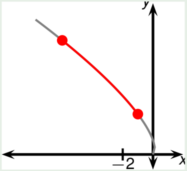

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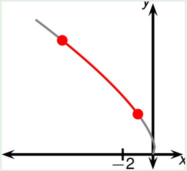

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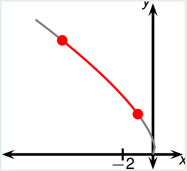

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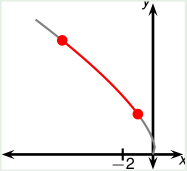

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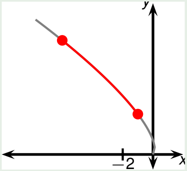

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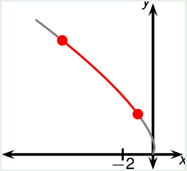

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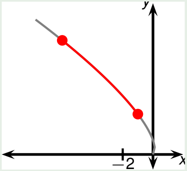

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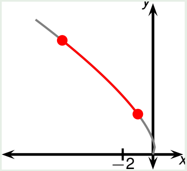

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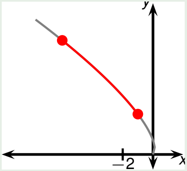

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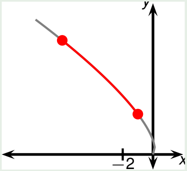

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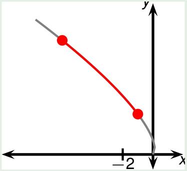

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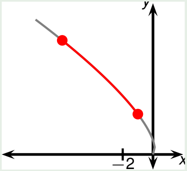

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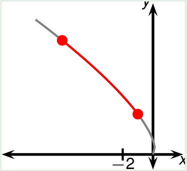
Example



Find the length of the curve γ.

$$\gamma : \left| \begin{array}{lcl} x(t) & = & \sqrt{t} - 2t \\ y(t) & = & \frac{8}{3} t^{\frac{3}{4}} \end{array} \right. , t \in [1, 4] \quad .$$

We have that $x'(t) = \frac{1}{2\sqrt{t}} - 2$ and $y'(t) = \frac{8}{3} \cdot \frac{3}{4} t^{-\frac{1}{4}} = 2t^{-\frac{1}{4}}$.

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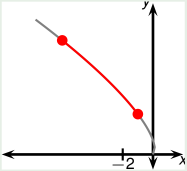
Example



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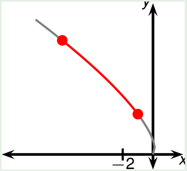
$$L(\gamma) = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2} \mathrm{d}t = \int_1^4 \sqrt{\left(\frac{1}{2\sqrt{t}} - 2\right)^2 + \left(2t^{-\frac{1}{4}}\right)^2} \mathrm{d}t$$

$$= \int_1^4 \sqrt{\frac{1}{4t} - \frac{2}{\sqrt{t}} + 4 + \frac{4}{\sqrt{t}}} \ \mathrm{d}t$$

$$= \int_1^4 \sqrt{\frac{1}{4t} + \frac{2}{\sqrt{t}} + 4} \ \mathrm{d}t = \int_1^4 \sqrt{\left(\frac{1}{2\sqrt{t}} + 2\right)^2} \mathrm{d}t$$

$$= \int_1^4 \left(\frac{1}{2\sqrt{t}} + 2\right) \mathrm{d}t = \left[\sqrt{t} + 2t\right]_1^4 = \sqrt{4} + 2 \cdot 4 - \left(\sqrt{1} + 2 \cdot 1\right)$$

Example



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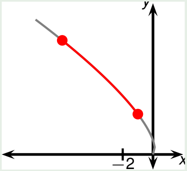
Example



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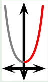
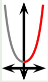
Example ($(a+b)^2$, $(a-b)^2$, $2ab = 1/2$)



Find the length of the arc of $y = \frac{1}{6}e^{3x} + \frac{1}{6}e^{-3x}$ from $x = 0$ to $x = 1$.

Example ($(a+b)^2$, $(a-b)^2$, $2ab = 1/2$)



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$y' =$

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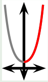

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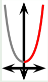

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$$y' = \frac{1}{2}e^{3x} - \frac{1}{2}e^{-3x}.$$

$$(y')^2 = \frac{1}{4}e^{6x} - \frac{1}{4}e^{3x}e^{-3x} - \frac{1}{4}e^{3x}e^{-3x} + \frac{1}{4}e^{-6x}$$

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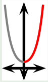

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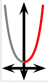

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$$
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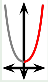
(y')^2 &= \frac{1}{4}e^{6x} - \frac{1}{4}e^{3x}e^{-3x} - \frac{1}{4}e^{3x}e^{-3x} + \frac{1}{4}e^{-6x}\\
&= \frac{1}{4}e^{6x} - \frac{1}{2} + \frac{1}{4}e^{-6x}.
\end{aligned}
$$

Example ($(a+b)^2$, $(a-b)^2$, $2ab = 1/2$)



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&= \frac{1}{4}e^{6x} - \frac{1}{2} + \frac{1}{4}e^{-6x}.\\
L &= \int_0^1 \sqrt{1+(y')^2}\mathrm{d}x
\end{aligned}
$$

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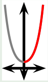

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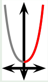
$$\begin{aligned}(y')^2 &= \frac{1}{4}e^{6x} - \frac{1}{4}e^{3x}e^{-3x} - \frac{1}{4}e^{3x}e^{-3x} + \frac{1}{4}e^{-6x}\\ &= \frac{1}{4}e^{6x} - \frac{1}{2} + \frac{1}{4}e^{-6x}.\end{aligned}$$

$$L = \int_0^1 \sqrt{1 + (y')^2}\mathrm{d}x = \int_0^1 \sqrt{1 + \frac{1}{4}e^{6x} - \frac{1}{2} + \frac{1}{4}e^{-6x}}\mathrm{d}x$$

Example ($(a+b)^2$, $(a-b)^2$, $2ab = 1/2$)



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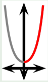
&= \int_0^1 \sqrt{\frac{1}{4}e^{6x} + \frac{1}{2} + \frac{1}{4}e^{-6x}}\mathrm{d}x
\end{aligned}
$$

Example ($(a+b)^2$, $(a-b)^2$, $2ab = 1/2$)



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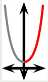
L &= \int_0^1 \sqrt{1+(y')^2}\mathrm{d}x = \int_0^1 \sqrt{1 + \frac{1}{4}e^{6x} - \frac{1}{2} + \frac{1}{4}e^{-6x}}\mathrm{d}x\\
&= \int_0^1 \sqrt{\frac{1}{4}e^{6x} + \frac{1}{2} + \frac{1}{4}e^{-6x}}\mathrm{d}x = \int_0^1 \sqrt{\left(\frac{1}{2}e^{3x} + \frac{1}{2}e^{-3x}\right)^2}\mathrm{d}x
\end{aligned}$$

Example ($(a+b)^2$, $(a-b)^2$, $2ab = 1/2$)



Find the length of the arc of $y = \frac{1}{6}e^{3x} + \frac{1}{6}e^{-3x}$ from $x = 0$ to $x = 1$.

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&= \int_0^1 \sqrt{\frac{1}{4}e^{6x} + \frac{1}{2} + \frac{1}{4}e^{-6x}}\mathrm{d}x = \int_0^1 \sqrt{\left(\frac{1}{2}e^{3x} + \frac{1}{2}e^{-3x}\right)^2}\mathrm{d}x\\
&= \int_0^1 \left(\frac{1}{2}e^{3x} + \frac{1}{2}e^{-3x}\right)\mathrm{d}x
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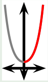
Example ($(a+b)^2$, $(a-b)^2$, $2ab = 1/2$)



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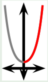
Example ($(a+b)^2$, $(a-b)^2$, $2ab = 1/2$)



Find the length of the arc of $y = \frac{1}{6}e^{3x} + \frac{1}{6}e^{-3x}$ from $x = 0$ to $x = 1$.

$$y' = \frac{1}{2}e^{3x} - \frac{1}{2}e^{-3x}.$$

$$\begin{aligned}
(y')^2 &= \frac{1}{4}e^{6x} - \frac{1}{4}e^{3x}e^{-3x} - \frac{1}{4}e^{3x}e^{-3x} + \frac{1}{4}e^{-6x}\\
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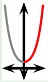
\end{aligned}$$

Example ($(a+b)^2$, $(a-b)^2$, $2ab = 1/2$)



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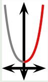
&= \int_0^1 \left(\frac{1}{2}e^{3x} + \frac{1}{2}e^{-3x}\right)\mathrm{d}x = \left[\frac{1}{6}e^{3x} \qquad\qquad \right]_0^1
\end{aligned}
$$

Example ($(a+b)^2$, $(a-b)^2$, $2ab = 1/2$)



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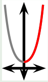
$$
\begin{aligned}
y' &= \frac{1}{2}e^{3x} - \frac{1}{2}e^{-3x}.\\
(y')^2 &= \frac{1}{4}e^{6x} - \frac{1}{4}e^{3x}e^{-3x} - \frac{1}{4}e^{3x}e^{-3x} + \frac{1}{4}e^{-6x}\\
&= \frac{1}{4}e^{6x} - \frac{1}{2} + \frac{1}{4}e^{-6x}.\\
L &= \int_0^1 \sqrt{1+(y')^2}\mathrm{d}x = \int_0^1 \sqrt{1 + \frac{1}{4}e^{6x} - \frac{1}{2} + \frac{1}{4}e^{-6x}}\mathrm{d}x\\
&= \int_0^1 \sqrt{\frac{1}{4}e^{6x} + \frac{1}{2} + \frac{1}{4}e^{-6x}}\mathrm{d}x = \int_0^1 \sqrt{\left(\frac{1}{2}e^{3x} + \frac{1}{2}e^{-3x}\right)^2}\mathrm{d}x\\
&= \int_0^1 \left(\frac{1}{2}e^{3x} + \frac{1}{2}e^{-3x}\right)\mathrm{d}x = \left[\frac{1}{6}e^{3x} - \frac{1}{6}e^{-3x}\right]_0^1
\end{aligned}
$$

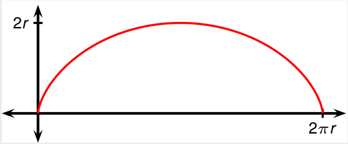
Example ($(a+b)^2$, $(a-b)^2$, $2ab = 1/2$)



Find the length of the arc of $y = \frac{1}{6}e^{3x} + \frac{1}{6}e^{-3x}$ from $x = 0$ to $x = 1$.

$$
\begin{aligned}
y' &= \frac{1}{2}e^{3x} - \frac{1}{2}e^{-3x}.\\
(y')^2 &= \frac{1}{4}e^{6x} - \frac{1}{4}e^{3x}e^{-3x} - \frac{1}{4}e^{3x}e^{-3x} + \frac{1}{4}e^{-6x}\\
&= \frac{1}{4}e^{6x} - \frac{1}{2} + \frac{1}{4}e^{-6x}.\\
L &= \int_0^1 \sqrt{1+(y')^2}\mathrm{d}x = \int_0^1 \sqrt{1 + \frac{1}{4}e^{6x} - \frac{1}{2} + \frac{1}{4}e^{-6x}}\mathrm{d}x\\
&= \int_0^1 \sqrt{\frac{1}{4}e^{6x} + \frac{1}{2} + \frac{1}{4}e^{-6x}}\mathrm{d}x = \int_0^1 \sqrt{\left(\frac{1}{2}e^{3x} + \frac{1}{2}e^{-3x}\right)^2}\mathrm{d}x\\
&= \int_0^1 \left(\frac{1}{2}e^{3x} + \frac{1}{2}e^{-3x}\right)\mathrm{d}x = \left[\frac{1}{6}e^{3x} - \frac{1}{6}e^{-3x}\right]_0^1 = \frac{e^3 - e^{-3}}{6}.
\end{aligned}
$$

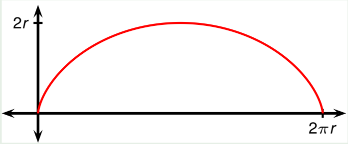
Example



Find the length of one arch of the cycloid

$$x = r(\theta - \sin\theta), \quad y = r(1 - \cos\theta).$$

Example



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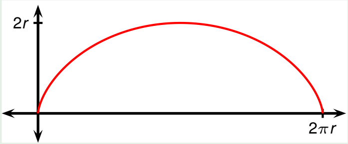
$$x = r(\theta - \sin\theta), \quad y = r(1 - \cos\theta).$$

The first arch is $0 \leq \theta \leq 2\pi$.

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta$$

Example



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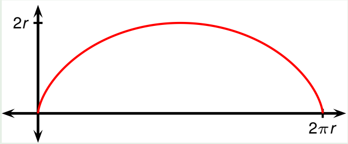
Example



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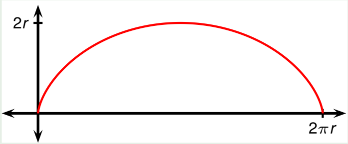

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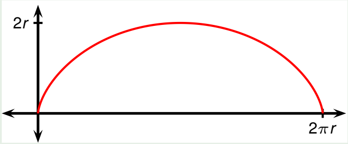

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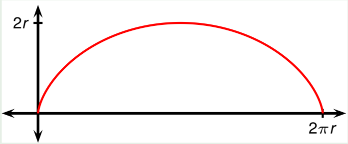

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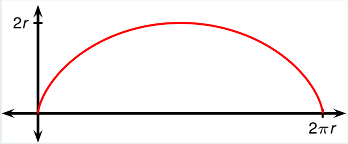
$$= \int_0^{2\pi} \sqrt{r^2(1 - 2\cos\theta + \cos^2\theta + \sin^2\theta)} d\theta$$

Example



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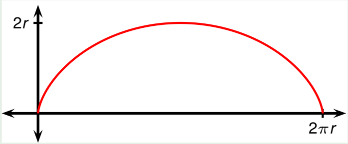
$$= \int_0^{2\pi} \sqrt{r^2(1 - 2\cos\theta + \cos^2\theta + \sin^2\theta)} \mathrm{d}\theta = r\int_0^{2\pi} \sqrt{2(1-\cos\theta)} \mathrm{d}\theta$$

Example



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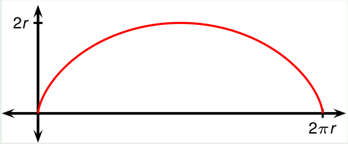
Use the identity $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$.

$$\sqrt{2(1-\cos\theta)}$$

Example



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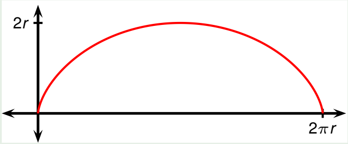
Use the identity $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$. Then

$$\sqrt{2(1-\cos\theta)} = \sqrt{4\sin^2(\theta/2)}$$

Example



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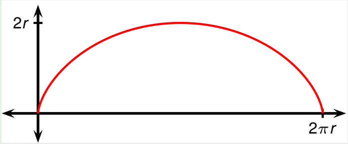
$$\sqrt{2(1 - \cos\theta)} = \sqrt{4\sin^2(\theta/2)} = 2|\sin(\theta/2)|$$

Example



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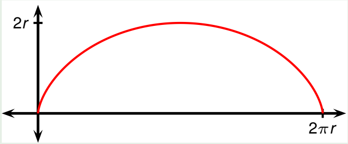

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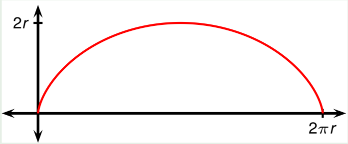
$$L = r\int_0^{2\pi} 2\sin(\theta/2) d\theta$$

Example



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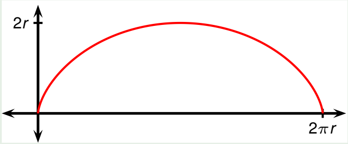
$$L = r\int_0^{2\pi} 2\sin(\theta/2)\mathrm{d}\theta = r\left[-4\cos(\theta/2)\right]_0^{2\pi}$$

Example



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$$L = r\int_0^{2\pi} 2\sin(\theta/2) d\theta = r\left[-4\cos(\theta/2)\right]_0^{2\pi} = 8r$$

Arc Length

To find the arc length of a polar curve $r = f(\theta)$, $a \leq \theta \leq b$, regard θ as a parameter.

The arc length is

$$L = \int_a^b \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2} \mathrm{d}\theta$$

Arc Length

To find the arc length of a polar curve $r = f(\theta)$, $a \leq \theta \leq b$, regard θ as a parameter. Then the derivatives of the parametric equations are

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = \frac{\mathrm{d}r}{\mathrm{d}\theta}\cos\theta - r\sin\theta \qquad \frac{\mathrm{d}y}{\mathrm{d}\theta} = \frac{\mathrm{d}r}{\mathrm{d}\theta}\sin\theta + r\cos\theta$$

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and

$$\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2 =$$

$$+$$

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$$+$$

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Arc Length

To find the arc length of a polar curve $r = f(\theta)$, $a \leq \theta \leq b$, regard θ as a parameter. Then the derivatives of the parametric equations are

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta \qquad \frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta$$

and

$$\begin{aligned}\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 \quad = \quad & \left(\frac{dr}{d\theta}\right)^2\cos^2\theta - 2r\frac{dr}{d\theta}\cos\theta\sin\theta + r^2\sin^2\theta \\ & + \left(\frac{dr}{d\theta}\right)^2\sin^2\theta + 2r\frac{dr}{d\theta}\sin\theta\cos\theta + r^2\cos^2\theta\end{aligned}$$

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and

$$\begin{aligned}\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 &= \left(\frac{dr}{d\theta}\right)^2\cos^2\theta - 2r\frac{dr}{d\theta}\cos\theta\sin\theta + r^2\sin^2\theta \\ &\quad + \left(\frac{dr}{d\theta}\right)^2\sin^2\theta + 2r\frac{dr}{d\theta}\sin\theta\cos\theta + r^2\cos^2\theta \\ &= \left(\frac{dr}{d\theta}\right)^2\end{aligned}$$

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$$L = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta$$

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To find the arc length of a polar curve $r = f(\theta)$, $a \leq \theta \leq b$, regard θ as a parameter. Then the derivatives of the parametric equations are

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta \qquad \frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta$$

and

$$\begin{aligned}
\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 &= \left(\frac{dr}{d\theta}\right)^2 \cos^2\theta - 2r\frac{dr}{d\theta}\cos\theta\sin\theta + r^2\sin^2\theta \\
&\quad + \left(\frac{dr}{d\theta}\right)^2 \sin^2\theta + 2r\frac{dr}{d\theta}\sin\theta\cos\theta + r^2\cos^2\theta \\
&= \left(\frac{dr}{d\theta}\right)^2
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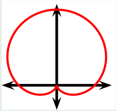
Example



Find the length of the cardioid $r = 1 + \sin\theta$.

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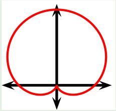

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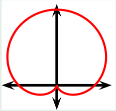

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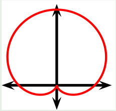

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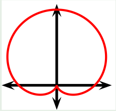

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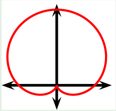

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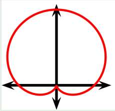

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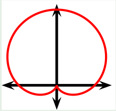

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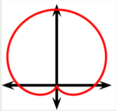

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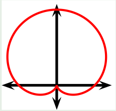

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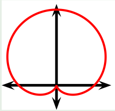

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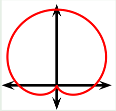

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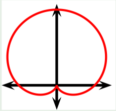

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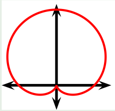

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\end{aligned}
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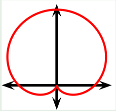
Example



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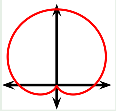

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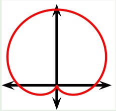

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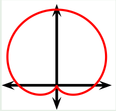

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&= \int_0^{\pi/2} \frac{2\cos\theta}{\sqrt{2-2\sin\theta}}\mathrm{d}\theta + \int_{\pi/2}^{3\pi/2} \frac{-2\cos\theta}{\sqrt{2-2\sin\theta}}\mathrm{d}\theta + \int_{3\pi/2}^{2\pi} \frac{2\cos\theta}{\sqrt{2-2\sin\theta}}\mathrm{d}\theta \\
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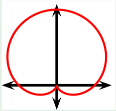
&= -2\left(0-\sqrt{2}\right) + 2\left(\quad - \quad\right) - 2\left(\quad - \quad\right)
\end{aligned}
$$

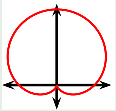
Example



Find the length of the cardioid $r = 1 + \sin\theta$. The full length is given by $0 \leq \theta \leq 2\pi$.

$$
\begin{aligned}
L &= \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta = \int_0^{2\pi} \sqrt{(1+\sin\theta)^2 + \cos^2\theta} d\theta \\
&= \int_0^{2\pi} \sqrt{2+2\sin\theta} \frac{\sqrt{2-2\sin\theta}}{\sqrt{2-2\sin\theta}} d\theta = \int_0^{2\pi} \frac{\sqrt{4-4\sin^2\theta}}{\sqrt{2-2\sin\theta}} d\theta \\
&= \int_0^{2\pi} \frac{\sqrt{4\cos^2\theta}}{\sqrt{2-2\sin\theta}} d\theta = \int_0^{2\pi} \frac{2|\cos\theta|}{\sqrt{2-2\sin\theta}} d\theta \\
&= \int_0^{\pi/2} \frac{2\cos\theta}{\sqrt{2-2\sin\theta}} d\theta + \int_{\pi/2}^{3\pi/2} \frac{-2\cos\theta}{\sqrt{2-2\sin\theta}} d\theta + \int_{3\pi/2}^{2\pi} \frac{2\cos\theta}{\sqrt{2-2\sin\theta}} d\theta \\
&= \left[-2\sqrt{2-2\sin\theta}\right]_0^{\pi/2} + \left[2\sqrt{2-2\sin\theta}\right]_{\pi/2}^{3\pi/2} + \left[-2\sqrt{2-2\sin\theta}\right]_{3\pi/2}^{2\pi} \\
&= -2\left(0-\sqrt{2}\right) + 2\left(2 - \quad\right) - 2\left(\quad - \quad \right)
\end{aligned}
$$

Example



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$$L = \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta = \int_0^{2\pi} \sqrt{(1+\sin\theta)^2 + \cos^2\theta} d\theta$$

$$= \int_0^{2\pi} \sqrt{2+2\sin\theta} \frac{\sqrt{2-2\sin\theta}}{\sqrt{2-2\sin\theta}} d\theta = \int_0^{2\pi} \frac{\sqrt{4-4\sin^2\theta}}{\sqrt{2-2\sin\theta}} d\theta$$

$$= \int_0^{2\pi} \frac{\sqrt{4\cos^2\theta}}{\sqrt{2-2\sin\theta}} d\theta = \int_0^{2\pi} \frac{2|\cos\theta|}{\sqrt{2-2\sin\theta}} d\theta$$

$$= \int_0^{\pi/2} \frac{2\cos\theta}{\sqrt{2-2\sin\theta}} d\theta + \int_{\pi/2}^{3\pi/2} \frac{-2\cos\theta}{\sqrt{2-2\sin\theta}} d\theta + \int_{3\pi/2}^{2\pi} \frac{2\cos\theta}{\sqrt{2-2\sin\theta}} d\theta$$

$$= \left[-2\sqrt{2-2\sin\theta}\right]_0^{\pi/2} + \left[2\sqrt{2-2\sin\theta}\right]_{\pi/2}^{3\pi/2} + \left[-2\sqrt{2-2\sin\theta}\right]_{3\pi/2}^{2\pi}$$

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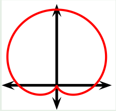
Example



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$$L = \int_0^{2\pi} \sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\mathrm{d}\theta = \int_0^{2\pi} \sqrt{(1+\sin\theta)^2 + \cos^2\theta}\mathrm{d}\theta$$

$$= \int_0^{2\pi} \sqrt{2+2\sin\theta}\frac{\sqrt{2-2\sin\theta}}{\sqrt{2-2\sin\theta}}\mathrm{d}\theta = \int_0^{2\pi} \frac{\sqrt{4-4\sin^2\theta}}{\sqrt{2-2\sin\theta}}\mathrm{d}\theta$$

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$$= \left[-2\sqrt{2-2\sin\theta}\right]_0^{\pi/2} + \left[2\sqrt{2-2\sin\theta}\right]_{\pi/2}^{3\pi/2} + \left[-2\sqrt{2-2\sin\theta}\right]_{3\pi/2}^{2\pi}$$

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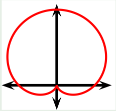
Example



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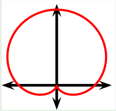
$$
\begin{aligned}
L &= \int_0^{2\pi} \sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\mathrm{d}\theta = \int_0^{2\pi} \sqrt{(1+\sin\theta)^2 + \cos^2\theta}\mathrm{d}\theta \\
&= \int_0^{2\pi} \sqrt{2+2\sin\theta}\frac{\sqrt{2-2\sin\theta}}{\sqrt{2-2\sin\theta}}\mathrm{d}\theta = \int_0^{2\pi} \frac{\sqrt{4-4\sin^2\theta}}{\sqrt{2-2\sin\theta}}\mathrm{d}\theta \\
&= \int_0^{2\pi} \frac{\sqrt{4\cos^2\theta}}{\sqrt{2-2\sin\theta}}\mathrm{d}\theta = \int_0^{2\pi} \frac{2|\cos\theta|}{\sqrt{2-2\sin\theta}}\mathrm{d}\theta \\
&= \int_0^{\pi/2} \frac{2\cos\theta}{\sqrt{2-2\sin\theta}}\mathrm{d}\theta + \int_{\pi/2}^{3\pi/2} \frac{-2\cos\theta}{\sqrt{2-2\sin\theta}}\mathrm{d}\theta + \int_{3\pi/2}^{2\pi} \frac{2\cos\theta}{\sqrt{2-2\sin\theta}}\mathrm{d}\theta \\
&= \left[-2\sqrt{2-2\sin\theta}\right]_0^{\pi/2} + \left[2\sqrt{2-2\sin\theta}\right]_{\pi/2}^{3\pi/2} + \left[-2\sqrt{2-2\sin\theta}\right]_{3\pi/2}^{2\pi} \\
&= -2\left(0-\sqrt{2}\right) + 2(2-0) - 2\left(\quad - \quad\right)
\end{aligned}
$$

Example



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&= \int_0^{\pi/2} \frac{2\cos\theta}{\sqrt{2-2\sin\theta}} d\theta + \int_{\pi/2}^{3\pi/2} \frac{-2\cos\theta}{\sqrt{2-2\sin\theta}} d\theta + \int_{3\pi/2}^{2\pi} \frac{2\cos\theta}{\sqrt{2-2\sin\theta}} d\theta \\
&= \left[-2\sqrt{2-2\sin\theta}\right]_0^{\pi/2} + \left[2\sqrt{2-2\sin\theta}\right]_{\pi/2}^{3\pi/2} + \left[-2\sqrt{2-2\sin\theta}\right]_{3\pi/2}^{2\pi} \\
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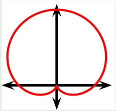
\end{aligned}
$$

Example



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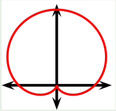
\end{aligned}
$$

Example



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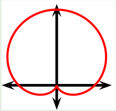
Example



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&= \left[-2\sqrt{2-2\sin\theta}\right]_0^{\pi/2} + \left[2\sqrt{2-2\sin\theta}\right]_{\pi/2}^{3\pi/2} + \left[-2\sqrt{2-2\sin\theta}\right]_{3\pi/2}^{2\pi} \\
&= -2\left(0-\sqrt{2}\right) + 2(2-0) - 2\left(\sqrt{2}-2\right) = 8
\end{aligned}
$$