

Precalculus

Quadratic polynomials viewed as functions

Todor Milev

2019

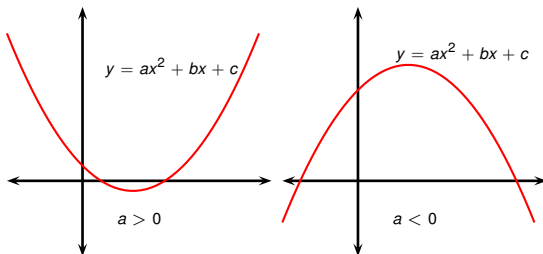
Definition

Let a, b, c be real numbers with $a \neq 0$. The function

$$f(x) = ax^2 + bx + c$$

is called a *quadratic function*.

- The graph of a quadratic function is called a parabola.



Example (Completing the square)

Complete the square.

$$\begin{aligned}3x^2 - 5x + 1 &= 3 \left(x^2 - \frac{5}{3}x \right) + 1 \\&= 3 \left(x^2 - 2 \cdot \frac{5}{2 \cdot 3}x \right) + 1 \\&= 3 \left(x^2 - 2 \cdot \frac{5}{6}x + \left(\frac{5}{6} \right)^2 - \left(\frac{5}{6} \right)^2 \right) + 1 \\&= 3 \left(\left(x - \frac{5}{6} \right)^2 - \frac{25}{36} \right) + 1 \\&= 3 \left(x - \frac{5}{6} \right)^2 - \frac{25}{12} + 1 \\&= 3 \left(x - \frac{5}{6} \right)^2 - \frac{13}{12}.\end{aligned}$$

Definition (Completing the square)

Let $a \neq 0$. To *complete the square* means to carry out the following algebraic manipulation.

$$\begin{aligned}
 ax^2 + bx + c &= a \left(x^2 + \frac{b}{a}x \right) + c \\
 &= a \left(x^2 + 2 \cdot \frac{b}{2a}x \right) + c \\
 &= a \left(x^2 + 2 \frac{b}{2a}x + \left(\frac{b}{2a} \right)^2 - \left(\frac{b}{2a} \right)^2 \right) + c \\
 &= a \left(\left(x + \frac{b}{2a} \right)^2 - \frac{b^2}{4a^2} \right) + c \\
 &= a \left(x + \frac{b}{2a} \right)^2 - \cancel{a} \cdot \frac{b^2}{4\cancel{a}} + c \\
 &= a \left(x + \frac{b}{2a} \right)^2 + c - \frac{b^2}{4a}.
 \end{aligned}$$

Add & subtract
 $\left(\frac{b}{2a} \right)^2$
 use
 $(A + B)^2 =$
 $A^2 + 2AB + B^2$

Definition (Discriminant of quadratic function)

The quantity $D = b^2 - 4ac$ is called the *discriminant* of the quadratic function $ax^2 + bx + c$.

Let $a \neq 0$ and let $f(x) = ax^2 + bx + c$. Then we have the equality

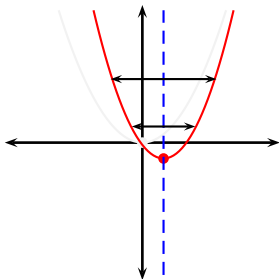
$$\begin{aligned} f(x) &= a \left(x + \frac{b}{2a} \right)^2 + c - \frac{b^2}{4a} \\ &= a \left(x - \left(-\frac{b}{2a} \right) \right)^2 - \frac{b^2 - 4ac}{4a} \\ &= a \left(x - \left(-\frac{b}{2a} \right) \right)^2 - \frac{D}{4a}. \end{aligned} \quad \left| \begin{array}{l} \text{complete the square} \end{array} \right.$$

Definition

The expression $f(x) = a(x - h)^2 + k$, where $h = -\frac{b}{2a}$ and $k = -\frac{D}{4a} = -\frac{b^2 - 4ac}{4a}$ is called the standard form of $ax^2 + bx + c$.

Definition

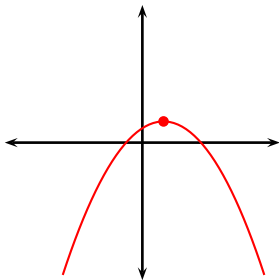
The expression $f(x) = a(x - h)^2 + k$, where $h = -\frac{b}{2a}$ and $k = -\frac{D}{4a} = -\frac{b^2 - 4ac}{4a}$ is called the standard form of $ax^2 + bx + c$.



- The graph of $y = x^2$ is a parabola; its shape is assumed known.
- The standard form shows how the graph of an arbitrary quadratic is obtained from the graph of $y = x^2$:
 - ax^2 stretches $y = x^2$ by factor of a and possibly reflects across the x axis.
 - $a(x - h)^2$ shifts $y = ax^2$ by h units right.
 - $a(x - h)^2 + k$ shifts $y = a(x - h)^2 + k$ by k units up.

Definition

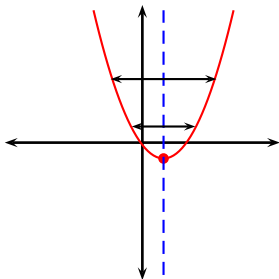
The expression $f(x) = a(x - h)^2 + k$, where $h = -\frac{b}{2a}$ and $k = -\frac{D}{4a} = -\frac{b^2 - 4ac}{4a}$ is called the standard form of $ax^2 + bx + c$.



- The graph of a quadratic function is a parabola.
- When $a > 0$ the parabola opens upwards.
- When $a < 0$ the parabola opens downwards.
- When $|a|$ increases, the parabola becomes steeper.
- The point $(h, k) = \left(-\frac{b}{2a}, -\frac{D}{4a}\right)$ is called the vertex of the parabola.
- The parabola is symmetric with respect to the line $x = h = -\frac{b}{2a}$, i.e., the vertical line through its vertex.

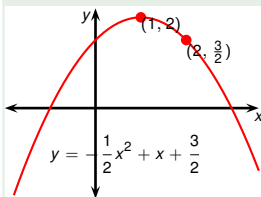
Definition

The expression $f(x) = a(x - h)^2 + k$, where $h = -\frac{b}{2a}$ and $k = -\frac{D}{4a} = -\frac{b^2 - 4ac}{4a}$ is called the standard form of $ax^2 + bx + c$.



- When we change h and k we move the vertex of the parabola without change in steepness.
- Therefore when we change b and c we move the vertex of the parabola without change in steepness.

Example



Write an equation of a parabola with vertex at $(1, 2)$ that passes through the point $(2, \frac{3}{2})$.

$$a(x - h)^2 + k = y$$

Standard form

$$a(x - 1)^2 + 2 = y$$

Vertex at $(1, 2)$

$$a(2 - 1)^2 + 2 = \frac{3}{2}$$

Passes through $(2, \frac{2}{3})$

$$a = \frac{\frac{3}{2}}{1} - 2 = -\frac{1}{2}$$

$$y = -\frac{1}{2}(x - 1)^2 + 2$$

Final answer

$$y = -\frac{1}{2}x^2 + x + \frac{3}{2}$$

Alternative answer

Problem (Quadratic equation formula)

Solve the general quadratic equation

$$ax^2 + bx + c = 0 \quad \left| \begin{array}{l} \text{complete the square} \\ \text{where } D = b^2 - 4ac \end{array} \right.$$

$$a \left(x + \frac{b}{2a} \right)^2 - \frac{D}{4a} = 0$$

$$a \left(\left(x + \frac{b}{2a} \right)^2 - \frac{D}{4a^2} \right) = 0$$

$$a \left(\left(x + \frac{b}{2a} \right)^2 - \left(\frac{\sqrt{D}}{2a} \right)^2 \right) = 0$$

$$a \left(x + \frac{b}{2a} - \frac{\sqrt{D}}{2a} \right) \left(x + \frac{b}{2a} + \frac{\sqrt{D}}{2a} \right) = 0 \quad \left| \begin{array}{l} \text{use } A^2 - B^2 \\ = (A - B)(A + B) \end{array} \right.$$

$$x + \frac{b}{2a} - \frac{\sqrt{D}}{2a} = 0 \quad \text{or} \quad x + \frac{b}{2a} + \frac{\sqrt{D}}{2a} = 0$$

$$x = \frac{-b + \sqrt{D}}{2a} \quad \text{or} \quad x = \frac{-b - \sqrt{D}}{2a}.$$

Theorem

The solutions of the quadratic equation

$$ax^2 + bx + c = 0$$

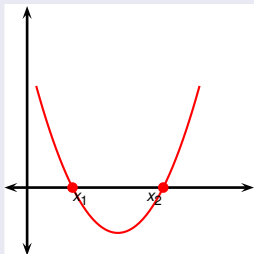
are given by:

$$x = x_1 = \frac{-b + \sqrt{D}}{2a} \quad \text{or} \quad x = x_2 = \frac{-b - \sqrt{D}}{2a},$$

where $D = b^2 - 4ac$, or equivalently by:

$$x_1, x_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Theorem



The solutions of the quadratic equation

$$ax^2 + bx + c = 0$$

are the numbers

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

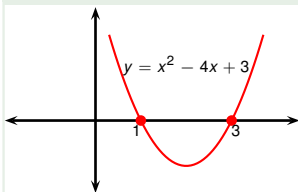
$$x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

- Abbreviated as

$$x_1, x_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{D}}{2a}, \quad \text{where } D = b^2 - 4ac.$$

- If $D < 0$ then \sqrt{D} is not a real \Rightarrow quadratic has no real solutions.
- If $D = 0$ then $x_1 = x_2$, the equation has only one zero (with multiplicity two). The zero is located at the vertex of the parabola.

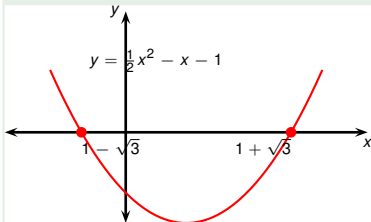
Example



Find the x -intercepts of $x^2 - 4x + 3$.

$$\begin{aligned}
 x_1, x_2 &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
 &= \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot 1 \cdot 3}}{2 \cdot 1} \\
 &= \frac{4 \pm \sqrt{4}}{2} \\
 &= \frac{4 \pm 2}{2} \\
 &= \begin{cases} \frac{4+2}{2} = \frac{6}{2} = 3 \\ \frac{4-2}{2} = \frac{2}{2} = 1 \end{cases}
 \end{aligned}$$

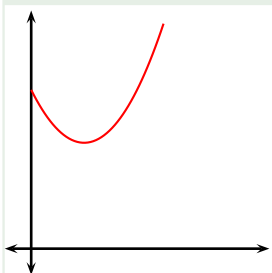
Example



Find the x -intercepts of $\frac{x^2}{2} - x - 1$.

$$\begin{aligned}
 x_1, x_2 &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
 &= \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \cdot \frac{1}{2} \cdot (-1)}}{2 \cdot \frac{1}{2}} \\
 &= 1 \pm \sqrt{3}
 \end{aligned}$$

Example



Find the x-intercepts of $x^2 - 2x + 3$.

$$\begin{aligned}x_1, x_2 &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\&= \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \cdot 1 \cdot (3)}}{2 \cdot 1} \\&= \frac{2 \pm \sqrt{-8}}{2} \\&\quad \text{no real solutions} \\&\quad \text{no } x - \text{intercepts}\end{aligned}$$

Proposition

Let $ax^2 + bx + c$, $a \neq 0$ be a quadratic with discriminant $D = b^2 - 4ac$ and roots x_1 and x_2 . Then $D = a^2(x_1 - x_2)^2$.

Proof.

$$\begin{aligned}
 a^2(x_1 - x_2)^2 &= a^2 \left(\frac{\cancel{b} + \sqrt{D}}{2a} - \frac{\cancel{b} - \sqrt{D}}{2a} \right) \\
 &= a^2 \left(\frac{2\sqrt{D}}{2a} \right)^2 \\
 &= \cancel{a^2} \frac{D}{\cancel{a^2}} \\
 &= D, \quad \text{as desired.}
 \end{aligned}$$



- Discriminant is zero \Leftrightarrow the quadratic has non-distinct roots, hence the discriminant discriminates between the two roots.

Example

Find the values of the parameter k for which the equation $3x^2 - kx + 1$ has two real distinct roots.

- Quadratic roots: $x_1, x_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

- \Rightarrow The roots x_1, x_2 are real and distinct when

$$b^2 - 4ac > 0$$

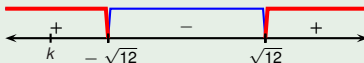
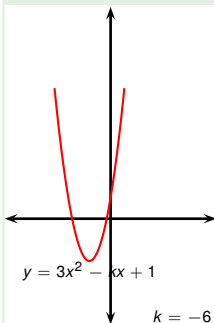
$$(-k)^2 - 4 \cdot 3 \cdot 1 > 0$$

$$k^2 - 12 > 0$$

$$k^2 - \sqrt{12}^2 > 0$$

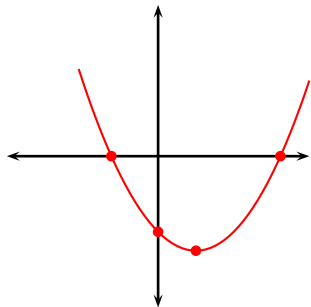
$$(k - \sqrt{12})(k + \sqrt{12}) > 0$$

$$k \in (-\infty, -\sqrt{12}) \cup (\sqrt{12}, \infty)$$

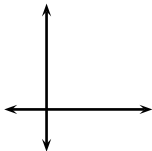


To plot a parabola by hand roughly, we need to do the following.

- Find the vertex of the parabola.
- Find the y intercept.
- Find the x intercept(s) if any.
- Select (or re-select) axes scale so all important points found in the preceding items fit in the plot.
- Plot the parabola freehand, making sure that the parabola passes through all special points you found in the preceding items.
- If $a > 0$ your parabola should open upwards, if $a < 0$ your parabola should open downwards.
- For $|a| > 1$ we should aim to draw the graph steeper than $a = x^2$, for $|a| < 1$ we should aim to draw the graph flatter than $a = x^2$.



Example



Plot roughly by hand the graph of
 $f(x) = -\frac{2}{3}x^2 + 7x + 3$.

- The vertex of the parabola is given by:

$$x = -\frac{b}{2a} = -\frac{7}{2(-\frac{2}{3})} = \frac{21}{4}$$

$$\begin{aligned} y &= f\left(-\frac{b}{2a}\right) = -\frac{D}{4a} = -\frac{(b^2 - 4ac)}{4a} \\ &= -\frac{7^2 - 4\left(-\frac{2}{3}\right)3}{4\left(-\frac{2}{3}\right)} = \frac{49 + 8}{\frac{8}{3}} \\ &= \frac{3 \cdot 57}{8} = \frac{171}{8}. \end{aligned}$$

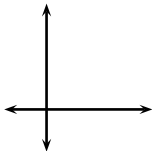
- The y-intercept is $f(0) = 3$.

Vertex at: $\left(\frac{21}{4}, \frac{171}{8}\right)$
 y-intercept at $y = 3$
 x-intercepts at

$$x = \frac{21 - 3\sqrt{57}}{4},$$

$$x = \frac{21 + 3\sqrt{57}}{4}.$$

Example



Plot roughly by hand the graph of
 $f(x) = -\frac{2}{3}x^2 + 7x + 3$.

- The x intercepts are given by the solutions of

$$\begin{aligned} -\frac{2}{3}x^2 + 7x + 3 &= 0 & | \cdot 3 \\ -2x^2 + 21x + 9 &= 0 \end{aligned}$$

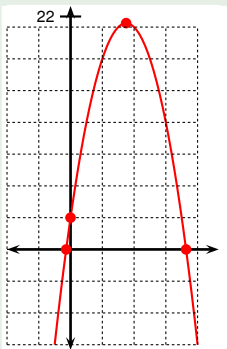
$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-21 \pm \sqrt{21^2 - 4 \cdot (-2) \cdot 9}}{2 \cdot (-2)} \\ &= \frac{-21 \pm \sqrt{441 + 72}}{-4} \\ &= \frac{21 \pm \sqrt{513}}{4} \\ &= \frac{21 \pm \sqrt{9 \cdot 57}}{4} \\ &= \frac{21 \pm \sqrt{9} \sqrt{57}}{4} \\ &= \frac{21 \pm 3\sqrt{57}}{4} \end{aligned}$$

Vertex at: $(\frac{21}{4}, \frac{171}{8})$
 y -intercept at $y = 3$
 x -intercepts at

$$x = \frac{21 - 3\sqrt{57}}{4},$$

$$x = \frac{21 + 3\sqrt{57}}{4}.$$

Example

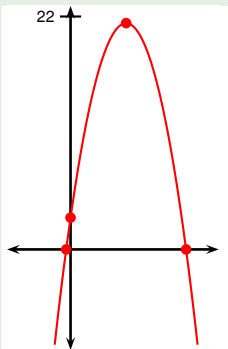


Vertex at: $(\frac{21}{4}, \frac{171}{8})$
 y-intercept at $y = 3$
 x-intercepts at
 $x = \frac{21-3\sqrt{57}}{4}$,
 $x = \frac{21+3\sqrt{57}}{4}$.

Plot roughly by hand the graph of
 $f(x) = -\frac{2}{3}x^2 + 7x + 3$.

- Select scale to fit the picture:
 - $\frac{21}{4}$ is close to $\frac{20}{4} = 5$.
 - $\frac{171}{8}$ is between the integers 21 and 22.
 - $\frac{21+3\sqrt{57}}{4}$ is close to $\frac{21+3\sqrt{64}}{4} = \frac{21+24}{4} = \frac{45}{4}$ which is close to $\frac{44}{4} = 11$.
 - $\frac{21-3\sqrt{57}}{4}$ is close to $\frac{21-3\sqrt{64}}{4} = \frac{21-24}{4} = -\frac{3}{4}$ which is close to -1 .
 - The parabola vertex is less than 22 units high and the parabola opens downwards.
 - Axes height of 22 units appears reasonable.
 - A grid of width 3 units appears reasonable.
 - Plot all relevant points.
 - Finally “connect the dots with a freehand drawing”.

Example

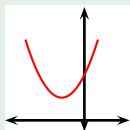


Vertex at: $(\frac{21}{4}, \frac{171}{8})$
 y-intercept at $y = 3$
 x-intercepts at
 $x = \frac{21-3\sqrt{57}}{4},$
 $x = \frac{21+3\sqrt{57}}{4}.$

Plot roughly by hand the graph of
 $f(x) = -\frac{2}{3}x^2 + 7x + 3.$

- Select scale to fit the picture:
 - $\frac{21}{4}$ is close to $\frac{20}{4} = 5.$
 - $\frac{171}{8}$ is between the integers 21 and 22.
 - $\frac{21+3\sqrt{57}}{4}$ is close to $\frac{21+3\sqrt{64}}{4} = \frac{21+24}{4} = \frac{45}{4}$ which is close to $\frac{44}{4} = 11.$
 - $\frac{21-3\sqrt{57}}{4}$ is close to $\frac{21-3\sqrt{64}}{4} = \frac{21-24}{4} = -\frac{3}{4}$ which is close to $-1.$
 - The parabola vertex is less than 22 units high and the parabola opens downwards.
 - Axes height of 22 units appears reasonable.
 - A grid of width 3 units appears reasonable.
 - Plot all relevant points.
 - Finally “connect the dots with a freehand drawing”.

Example



Find the values of the parameter k for which $x^2 + (k + 1)x + 2k > 0$ holds for all real x .

- In order for the quadratic to be positive, its graph must lie entirely above the x axis.
- Leading coefficient is positive \Rightarrow graph opens up \Rightarrow is above x axis if it does not intersect it \Rightarrow the quadratic has no real solutions.
- The roots of a quadratic are $x_1, x_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

$$b^2 - 4ac < 0$$

$$(k + 1)^2 - 4 \cdot 1 \cdot 2k < 0$$

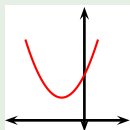
$$k^2 + 2k + 1 - 8k < 0$$

$$k^2 - 6k + 1 < 0$$

$$(k - k_1)(k - k_2) < 0$$

$$k_1, k_2 = \frac{2 \cdot 3 \pm \sqrt{4} \sqrt{8}}{2} = \frac{2(3 \pm \sqrt{8})}{2} = 3 \pm \sqrt{8}$$

Example



Find the values of the parameter k for which $x^2 + (k + 1)x + 2k > 0$ holds for all real x .

- In order for the quadratic to be positive, its graph must lie entirely above the x axis.
- Leading coefficient is positive \Rightarrow graph opens up \Rightarrow is above x axis if it does not intersect it \Rightarrow the quadratic has no real solutions.
- The roots of a quadratic are $x_1, x_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

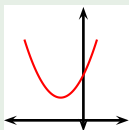
$$b^2 - 4ac < 0$$

$$(k + 1)^2 - 4 \cdot 1 \cdot 2k < 0$$

$$(k - k_1)(k - k_2) < 0$$

$$\begin{aligned} k_1, k_2 &= \frac{-(-6) \pm \sqrt{(-6)^2 - 4 \cdot 1 \cdot 1}}{2 \cdot 1} = \frac{6 \pm \sqrt{32}}{2} \\ &= \frac{2 \cdot 3 \pm \sqrt{4} \sqrt{8}}{2} = \frac{2(3 \pm \sqrt{8})}{2} = 3 \pm \sqrt{8} \end{aligned}$$

Example



Find the values of the parameter k for which $x^2 + (k + 1)x + 2k > 0$ holds for all real x .

- In order for the quadratic to be positive, its graph must lie entirely above the x axis.
- Leading coefficient is positive \Rightarrow graph opens up \Rightarrow is above x axis if it does not intersect it \Rightarrow the quadratic has no real solutions.
- The roots of a quadratic are $x_1, x_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

$$b^2 - 4ac < 0$$

$$(k + 1)^2 - 4 \cdot 1 \cdot 2k < 0$$

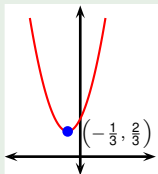
$$(k - k_1)(k - k_2) < 0$$

$$k_1, k_2 = \frac{2 \cdot 3 \pm \sqrt{4} \sqrt{8}}{2} = \frac{2(3 \pm \sqrt{8})}{2} = 3 \pm \sqrt{8}$$

$$k \in (k_1, k_2) = (3 - \sqrt{8}, 3 + \sqrt{8})$$



Example



Find the minimum point on the curve
 $y = 3x^2 + 2x + 1$ by completing the square.

$$3x^2 + 2x + 1 = 3 \left(x^2 + 2 \cdot \frac{1}{3}x + \frac{1}{9} - \frac{1}{9} \right) + 1$$

$$= 3 \left(\left(x + \frac{1}{3} \right)^2 - \frac{1}{9} \right) + 1$$

$$= 3 \left(x + \frac{1}{3} \right)^2 - \frac{1}{3} + 1$$

$$= 3 \left(x - \left(-\frac{1}{3} \right) \right)^2 + \frac{2}{3}$$

$$\text{Minimum point} = \left(-\frac{1}{3}, \frac{2}{3} \right)$$

Maximum or minimum value of a quadratic function

- Let $f(x) = ax^2 + bx + c$ - quadratic ($a \neq 0$).
- Let D be the discriminant $D = b^2 - 4ac$.

$$f(x) = a \left(x - \left(-\frac{b}{2a} \right) \right)^2 - \frac{D}{4a} \quad \left| \text{complete the square} \right.$$

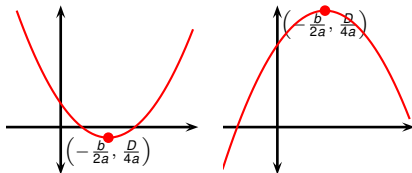
- Therefore if $a > 0$ then $f(x) = a(\text{square}) - \frac{D}{4a} \geq -\frac{D}{4a}$.
- Similarly if $a < 0$ then $f(x) = a(\text{square}) - \frac{D}{4a} \leq -\frac{D}{4a}$.

$$\text{Recall } f(x) = ax^2 + bx + c = a \left(x - \left(-\frac{b}{2a} \right) \right)^2 - \frac{D}{4a}.$$

Proposition

Let $f(x) = ax^2 + bx + c$, $a \neq 0$ and let $D = b^2 - 4ac$.

- If $a > 0$ then $f(x)$ has no maximum and has minimum at $x = -\frac{b}{2a}$.
- If $a < 0$ then $f(x)$ has no minimum and has maximum at $x = -\frac{b}{2a}$.
- In both cases, the extremal value (either maximum or minimum) is $f\left(-\frac{b}{2a}\right) = -\frac{b^2-4ac}{4a} = -\frac{D}{4a}$.



Example

Let x, z be two numbers that add to 12. Choose x and z so that the product $x \cdot z$ is maximal.



$$x + z = 12$$

$$z = 12 - x$$

Maximizing:

$$\begin{aligned} xz &= x(12 - x) \\ &= -x^2 + 12x \end{aligned}$$

Parabola opens down \Rightarrow has maximum, attained at:

$$\begin{aligned} x &= -\frac{b}{2a} \\ &= -\frac{12}{-2} = 6 \end{aligned}$$

$$z = 12 - x = 12 - 6 = 6$$

$$\text{Max. product} = xz = 6 \cdot 6 = 36.$$