Precalculus
Equations formed by setting trigonometric sum equal to 0

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Proposition (Product to sum formulas)

$$\sin \alpha \sin \beta = \frac{1}{2} \left(\cos(\alpha - \beta) - \cos(\alpha + \beta) \right)$$
$$\cos \alpha \cos \beta = \frac{1}{2} \left(\cos(\alpha - \beta) + \cos(\alpha + \beta) \right)$$
$$\sin \alpha \cos \beta = \frac{1}{2} \left(\sin(\alpha + \beta) + \sin(\alpha - \beta) \right)$$

Proof.

$$
\begin{array}{rlcl}
+ & \cos \alpha \cos \beta + \cancel{\sin \alpha \sin \beta} & = & \cos(\alpha - \beta) \\
  & \cos \alpha \cos \beta - \cancel{\sin \alpha \sin \beta} & = & \cos(\alpha + \beta) \\
  & 2 \cos \alpha \cos \beta & = & \cos(\alpha - \beta) + \cos(\alpha + \beta) \\
\hline
- & \cos \alpha \cos \beta + \sin \alpha \sin \beta & = & \cos(\alpha - \beta) \\
  & \cos \alpha \cos \beta - \sin \alpha \sin \beta & = & \cos(\alpha + \beta) \\
  & 2 \sin \alpha \sin \beta & = & \cos(\alpha - \beta) - \cos(\alpha + \beta) \\
\hline
+ & \sin \alpha \cos \beta + \cos \alpha \sin \beta & = & \sin(\alpha + \beta) \\
  & \sin \alpha \cos \beta - \cos \alpha \sin \beta & = & \sin(\alpha - \beta) \\
  & 2 \sin \alpha \cos \beta & = & \sin(\alpha + \beta) + \sin(\alpha + \beta)
\end{array}
$$

$\square$

Proposition (Product to sum formulas)

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$$\cos \alpha \cos \beta = \frac{1}{2} \left(\cos(\alpha - \beta) + \cos(\alpha + \beta)\right)$$
$$\sin \alpha \cos \beta = \frac{1}{2} \left(\sin(\alpha + \beta) + \sin(\alpha - \beta)\right)$$

- Product to sum formulas are used when integrating (a topic to be studied later/in another course).

Proposition (Sum to product formulas)

$$\sin \alpha + \sin \beta = 2 \sin \left(\frac{\alpha + \beta}{2} \right) \cos \left(\frac{\alpha - \beta}{2} \right)$$

$$\sin \alpha - \sin \beta = 2 \sin \left(\frac{\alpha - \beta}{2} \right) \cos \left(\frac{\alpha + \beta}{2} \right)$$
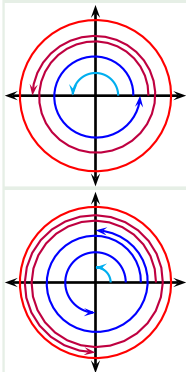
$$\cos \alpha + \cos \beta = 2 \cos \left(\frac{\alpha + \beta}{2} \right) \cos \left(\frac{\alpha - \beta}{2} \right)$$

$$\cos \alpha - \cos \beta = 2 \sin \left(\frac{\alpha + \beta}{2} \right) \sin \left(\frac{\alpha - \beta}{2} \right)$$

Recall the formula $\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

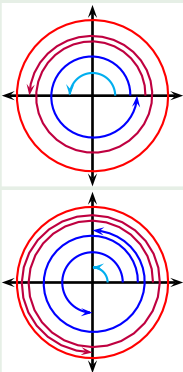## Example

Find all solutions in the interval $[0, 2\pi)$.

$$\sin(2x) + \sin(5x) = 0 \;\Big|\; \text{use f-la}$$

$$2\sin\left(\frac{2x+5x}{2}\right)\cos\left(\frac{2x-5x}{2}\right) = 0$$

$$2\sin\left(\frac{7}{2}x\right)\cos\left(-\frac{3}{2}x\right) = 0 \;\left|\; \begin{array}{l}\cos\\ \text{is even}\end{array}\right.$$

$$2\sin\left(\frac{7}{2}x\right)\cos\left(\frac{3}{2}x\right) = 0$$

Recall the formula $\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

Example

Find all solutions in the interval $[0, 2\pi)$.

$$\sin(2x) + \sin(5x) = 0 \ \Big| \ \text{use f-la}$$
$$2\sin\left(\frac{7}{2}x\right)\cos\left(\frac{3}{2}x\right) = 0$$

$\sin\left(\frac{7}{2}x\right) = 0$

$\quad \frac{7}{2}x = k\pi \qquad\qquad k - \text{integer}$

$\quad x = \frac{2k\pi}{7}$

$\quad x = \dots, \frac{-2\pi}{7}, 0, \frac{2\pi}{7}, \frac{4\pi}{7}, \frac{6\pi}{7}, \frac{8\pi}{7}, \frac{10\pi}{7}, \frac{12\pi}{7}, \frac{14\pi}{7}, \dots$

$$\text{or}$$

$\cos\left(\frac{3}{2}x\right) = 0$

$\quad \frac{3}{2}x = \frac{\pi}{2} + k\pi = \frac{(2k+1)\pi}{2} \qquad\qquad k - \text{integer}$

$\quad x = \frac{(2k+1)\pi}{3}$

$\quad x = \dots, \frac{-\pi}{3}, \frac{\pi}{3}, \frac{3\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \dots$

Recall the formula $\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

Example

Find all solutions in the interval $[0, 2\pi)$.
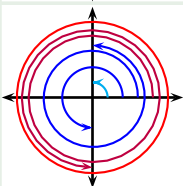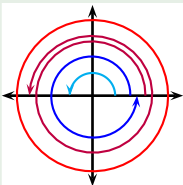
$$\sin(2x) + \sin(5x) = 0 \mid \text{use f-la}$$

$$2\sin\left(\frac{7}{2}x\right)\cos\left(\frac{3}{2}x\right) = 0$$

$$x = \cdots, \frac{-2\pi}{7}, 0, \frac{2\pi}{7}, \frac{4\pi}{7}, \frac{6\pi}{7}, \frac{8\pi}{7}, \frac{10\pi}{7}, \frac{12\pi}{7}, \frac{14\pi}{7}, \cdots$$
$$\text{or}$$
$$x = \cdots, \frac{\pi}{3}, \frac{\pi}{3}, \frac{3\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \cdots$$

$y = \sin(2x) + \sin(5x)$