Calculus II

Integrals of the form $\int \sin^n x \cos^m x \mathrm{d}x$, at least one power is odd

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Example

$$\int \sin^3 x \mathrm{d}x$$

Example

$$\int \sin^3 x \mathrm{d}x = \int \sin^2 x \sin x \mathrm{d}x$$

Example

$$\begin{aligned}\int \sin^3 x\,dx &= \int \sin^2 x \sin x\,dx \\ &= \int \sin^2 x\,d(\textbf{?}\qquad)\end{aligned}$$

Example

$$\begin{aligned}\int \sin^3 x \mathrm{d}x &= \int \sin^2 x \sin x \mathrm{d}x \\ &= \int \sin^2 x \mathrm{d}(-\cos x)\end{aligned}$$

Example

$$\begin{aligned}\int \sin^3 x\mathrm{d}x &= \int \sin^2 x \sin x\mathrm{d}x \\ &= \int \sin^2 x\mathrm{d}(-\cos x) \\ &= \int (-1)\left(\textbf{?}\qquad\qquad\right)\mathrm{d}(\cos x)\end{aligned}$$

Example

$$\begin{aligned}\int \sin^3 x\,dx &= \int \sin^2 x \sin x\,dx \\ &= \int \sin^2 x\, d(-\cos x) \\ &= \int (-1)\left(\text{?}\qquad\qquad\right) d(\cos x)\end{aligned}$$

Can we rewrite $\sin^2 x$ via $\cos x$?

Example

$$\begin{aligned}\int \sin^3 x \mathrm{d}x &= \int \sin^2 x \sin x \mathrm{d}x \\ &= \int \sin^2 x \mathrm{d}(-\cos x) \\ &= \int (-1)\left(1-\cos^2 x\right) \mathrm{d}(\cos x)\end{aligned}$$

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\int \sin^3 x \mathrm{d}x &= \int \sin^2 x \sin x \mathrm{d}x \\
&= \int \sin^2 x \mathrm{d}(-\cos x) \\
&= \int (-1)\left(1-\cos^2 x\right) \mathrm{d}(\cos x) \\
&= \int \left(\cos^2 x - 1\right) \mathrm{d}(\cos x)
\end{aligned}$$

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\int \sin^3 x \mathrm{d}x &= \int \sin^2 x \sin x \mathrm{d}x \\
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&= \int (-1)\left(1-\cos^2 x\right) \mathrm{d}(\cos x) \\
&= \int \left(\cos^2 x - 1\right) \mathrm{d}(\cos x) \qquad \left| \begin{array}{l} \text{Set } u = \cos x \end{array} \right. \\
&= \int \left(u^2 - 1\right) \mathrm{d}u
\end{aligned}$$

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\int \sin^3 x \mathrm{d}x &= \int \sin^2 x \sin x \mathrm{d}x \\
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&= \int (-1)\left(1-\cos^2 x\right) \mathrm{d}(\cos x) \\
&= \int \left(\cos^2 x - 1\right) \mathrm{d}(\cos x) && \left| \begin{array}{l} \text{Set } u = \cos x \end{array} \right. \\
&= \int \left(u^2 - 1\right) \mathrm{d}u \\
&= \frac{u^3}{3} - u + C
\end{aligned}
$$

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\int \sin^3 x \mathrm{d}x &= \int \sin^2 x \sin x \mathrm{d}x \\
&= \int \sin^2 x \mathrm{d}(-\cos x) && \text{Can we rewrite } \sin^2 x \text{ via } \cos x? \\
&= \int (-1)\left(1-\cos^2 x\right) \mathrm{d}(\cos x) \\
&= \int \left(\cos^2 x - 1\right) \mathrm{d}(\cos x) && \text{Set } u = \cos x \\
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&= \int \left(u^2 - 1\right) \mathrm{d}u \\
&= \frac{u^3}{3} - u + C \\
&= \frac{1}{3}\cos^3 x - \cos x + C \quad .
\end{aligned}
$$

Example

$$\int \cos^5 x \sin^2 x \mathrm{d}x$$

Example

$$\int \cos^5 x \sin^2 x dx = \int \cos^4 x \sin^2 x \cos x dx$$

Example

$$\int \cos^5 x \sin^2 x \mathrm{d}x = \int \cos^4 x \sin^2 x \cos x \mathrm{d}x$$
$$= \int \cos^4 x \sin^2 x \mathrm{d}(\textbf{?} \quad)$$

Example

$$\begin{aligned}\int \cos^5 x \sin^2 x \mathrm{d}x &= \int \cos^4 x \sin^2 x \cos x \mathrm{d}x \\ &= \int \cos^4 x \sin^2 x \mathrm{d}(\sin x)\end{aligned}$$

Example

$$\int \cos^5 x \sin^2 x \mathrm{d}x = \int \cos^4 x \sin^2 x \cos x \mathrm{d}x$$
$$= \int \cos^4 x \sin^2 x \mathrm{d}(\sin x)$$

Can we rewrite $\cos^4 x$ via $\sin x$?

Example

$$\begin{aligned}\int \cos^5 x \sin^2 x \mathrm{d}x &= \int \cos^4 x \sin^2 x \cos x \mathrm{d}x \\ &= \int \cos^4 x \sin^2 x \mathrm{d}(\sin x) \\ &= \int \left(\cos^2 x\right)^2 \sin^2 x \mathrm{d}(\sin x)\end{aligned}$$

Can we rewrite $\cos^4 x$ via $\sin x$?

Example

$$\begin{aligned}
\int \cos^5 x \sin^2 x \mathrm{d}x &= \int \cos^4 x \sin^2 x \cos x \mathrm{d}x \\
&= \int \cos^4 x \sin^2 x \mathrm{d}(\sin x) \\
&= \int \left(\cos^2 x\right)^2 \sin^2 x \mathrm{d}(\sin x) \\
&= \int \left(1 - \sin^2 x\right)^2 \sin^2 x \mathrm{d}(\sin x)
\end{aligned}$$

Can we rewrite $\cos^4 x$ via $\sin x$?

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\int \cos^5 x \sin^2 x \mathrm{d}x &= \int \cos^4 x \sin^2 x \cos x \mathrm{d}x \\
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&= \int \left(\cos^2 x\right)^2 \sin^2 x \mathrm{d}(\sin x) \\
&= \int \left(1 - \sin^2 x\right)^2 \sin^2 x \mathrm{d}(\sin x) \quad \left| \text{Set } u = \sin x \right. \\
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&= \int \left(1 - 2u^2 + u^4\right) u^2 \mathrm{d}u
\end{aligned}$$

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&= \int \left(u^2 - 2u^4 + u^6\right) \mathrm{d}u
\end{aligned}$$

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&= ?
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&= \int \left(\cos^2 x\right)^2 \sin^2 x \mathrm{d}(\sin x) \\
&= \int \left(1-\sin^2 x\right)^2 \sin^2 x \mathrm{d}(\sin x) \qquad \left| \text{Set } u = \sin x \right. \\
&= \int \left(1-u^2\right)^2 u^2 \mathrm{d}u \\
&= \int \left(1-2u^2+u^4\right) u^2 \mathrm{d}u \\
&= \int \left(u^2-2u^4+u^6\right) \mathrm{d}u \\
&= \frac{u^3}{3} - 2\frac{u^5}{5} + \frac{u^7}{7} + C
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&= \int \left(1 - u^2\right)^2 u^2 \mathrm{d}u \\
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&= \frac{u^3}{3} - 2\frac{u^5}{5} + \frac{u^7}{7} + C \\
&= \frac{\sin^3 x}{3} - 2\frac{\sin^5 x}{5} + \frac{\sin^7 x}{7} + C \quad .
\end{aligned}
$$