Precalculus
Trigonometric identities theory

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Proof of a Trigonometric Identity

Let F and G be expressions that give a trigonometric identity: $F(\sin\theta, \cos\theta) = G(\sin\theta, \cos\theta)$.

- To prove a trigonometric identity means to show that the two sides of the equality sign are equivalent.
- There are two ways to do this (in the present course the first way will be preferred).
- First method: transform the left and right hand sides to an equal expression. In particular:
  - we can choose to transform the left hand side to the right;
  - we can choose to transform the right hand side to the left;
  - we can choose to transform both sides to a third equivalent expression.
- Second method: start with an already known identity and transform it, by a series of equivalent transformations, to the identity we desire to prove.
- The discussion here also applies for trigonometric identities in more than one variables.

Types of identites

- In the present course we deal with two basic types of trigonometric identities.
- First, identities that involve operations on the arguments of the trigonometric functions.
  - Example: $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ (this is one of the angle sum identities); $\sin\theta + \sin(-\theta) = 0$.
  - Such identities can be proved using the angle sum formulas and the even/odd function properties of $\sin, \cos$.
- Second, identities that involve trigonometric functions of one variable.
  - Example: $\tan^2\theta + 1 = \sec^2\theta$.
  - Such identities can be proved only using the already demonstrated Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$.
- The Pythagorean identity follows from the angle sum formulas and the even/odd function properties of $\sin, \cos$, so all trigonometric identities follow from those properties alone.

Strategy for proving trigonometric identities

An expression is rational trigonometric if it is written using $\sin\theta, \cos\theta$ and the four arithmetic operations.

Question

Is there a general method for proving all rational trigonometric identities in one variable?

- Given a number of variables and relations between them, there is an algorithm to check whether (rational) expressions in those variables are equal under the given relations.
- Thus, if we pick two variables s and c, and a single relation
$$s^2 + c^2 = 1$$
there is a standard method to verify whether two (rational) expressions in s and c are equal.
- The method is rather cumbersome for a human and is best suited for computers.

Strategy for proving trigonometric identities

An expression is rational trigonometric if it is written using $\sin\theta, \cos\theta$ and the four arithmetic operations.

Question

Is there a general method for proving all rational trigonometric identities in one variable?

- Yes.
- For expressions that depend only on $\sin\theta$ and $\cos\theta$, algebra tells us when two expressions in those are equal.
- Problems depending on $\cos\theta, \sin\theta$ alone will always be doable via easy ad-hoc tricks using
$$\sin^2\theta + \cos^2\theta = 1.$$
- The full method will not be needed in this course.
    - The full method: set $s = \sin\theta, c = \cos\theta$.
    - Check whether the two expressions in s, c are equal under the relation $s^2 + c^2 = 1$. (The method lies outside of present scope).

Strategy for proving trigonometric identities

An expression is rational trigonometric if it is written using $\sin\theta, \cos\theta$ and the four arithmetic operations.

Question

Is there a general method for proving all rational trigonometric identities in one variable?

- To prove a general trigonometric identity:
  - Use angle sum/double angle sum formulas to convert all formulas to trig expression depending only on $\sin\theta$, $\cos\theta$.
  - Use $\sin^2\theta + \cos^2\theta = 1$ to show the two formulas are equal (usage: ad-hoc).
  - You may need to use trig functions of angles smaller than θ, for example $\sin\left(\frac{\theta}{2}\right), \cos\left(\frac{\theta}{2}\right)$.
  - A fraction of θ such that all appearing angles are integer multiples of it will always work.