Calculus II
Integrals of the form $\int \arctan(mx)\mathbf{d}x$

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Integration by parts: $\int u \mathrm{d}v = uv - \int v \mathrm{d}u.$

Example

$$\int_0^1 \arctan x \mathrm{d}x \quad =$$

Integration by parts: $\int u \mathrm{d}v = uv - \int v \mathrm{d}u.$

Example

$$\int_0^1 \arctan x \mathrm{d}x = \textbf{?}$$

Integration by parts: $\int u\mathrm{d}v = uv - \int v\mathrm{d}u.$

Example

$$\int_0^1 \arctan x \mathrm{d}x = \left[(\arctan x)x\right]_{x=0}^{x=1} - \int_0^1 x\mathrm{d}\left(\arctan x\right)$$

Integration by parts: $\int u \mathrm{d}v = uv - \int v \mathrm{d}u$.

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$$\begin{aligned}\int_0^1 \arctan x \mathrm{d}x &= \left[(\arctan x)x\right]_{x=0}^{x=1} - \int_0^1 x \mathrm{d}\left(\arctan x\right)\\ &= 1 \cdot \arctan 1 - 0 \cdot \arctan 0 - \int_{x=0}^{x=1} x\mathbf{?}\end{aligned}$$

Integration by parts: $\int u\mathrm{d}v = uv - \int v\mathrm{d}u.$

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$$\begin{aligned}\int_0^1 \arctan x\mathrm{d}x &= \left[(\arctan x)x\right]_{x=0}^{x=1} - \int_0^1 x\mathrm{d}\left(\arctan x\right) \\ &= 1\cdot\arctan 1 - 0\cdot\arctan 0 - \int_{x=0}^{x=1} x\mathbf{?}\end{aligned}$$

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Integration by parts: $\int u\mathrm{d}v = uv - \int v\mathrm{d}u.$

Example

$$
\begin{aligned}
\int_0^1 \arctan x \mathrm{d}x &= \left[(\arctan x)x\right]_{x=0}^{x=1} - \int_0^1 x\mathrm{d}\left(\arctan x\right)\\
&= 1\cdot \arctan 1 - 0\cdot \arctan 0 - \int_{x=0}^{x=1} x\frac{1}{1+x^2}\mathrm{d}x\\
&= \textbf{?} - \int_{x=0}^{x=1} \frac{1}{1+x^2}\mathrm{d}\left(\textbf{?}\quad\right)
\end{aligned}
$$

Integration by parts: $\int u\mathrm{d}v = uv - \int v\mathrm{d}u$.

Example

$$\begin{aligned}\int_0^1 \arctan x\mathrm{d}x &= \left[(\arctan x)x\right]_{x=0}^{x=1} - \int_0^1 x\mathrm{d}\left(\arctan x\right)\\ &= 1\cdot \arctan 1 - 0\cdot\arctan 0 - \int_{x=0}^{x=1} x\frac{1}{1+x^2}\mathrm{d}x\\ &= \frac{\pi}{4} - \int_{x=0}^{x=1}\frac{1}{1+x^2}\mathrm{d}\left(\textbf{?}\quad\right)\end{aligned}$$

Integration by parts: $\int u \mathrm{d}v = uv - \int v \mathrm{d}u$.

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\int_0^1 \arctan x \mathrm{d}x &= \left[(\arctan x)x\right]_{x=0}^{x=1} - \int_0^1 x \mathrm{d}\left(\arctan x\right) \\
&= 1 \cdot \arctan 1 - 0 \cdot \arctan 0 - \int_{x=0}^{x=1} x \frac{1}{1+x^2} \mathrm{d}x \\
&= \frac{\pi}{4} - \int_{x=0}^{x=1} \frac{1}{1+x^2} \mathrm{d}\left(? \quad \right)
\end{aligned}$$

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\int_0^1 \arctan x \mathrm{d}x &= \left[(\arctan x)x\right]_{x=0}^{x=1} - \int_0^1 x \mathrm{d}\left(\arctan x\right) \\
&= 1 \cdot \arctan 1 - 0 \cdot \arctan 0 - \int_{x=0}^{x=1} x \frac{1}{1+x^2} \mathrm{d}x \\
&= \frac{\pi}{4} - \int_{x=0}^{x=1} \frac{1}{1+x^2} \mathrm{d}\left(\frac{x^2}{\color{red}{2}}\right) \\
&= \frac{\pi}{4} - \color{red}{\frac{1}{2}} \int_{x=0}^{x=1} \frac{1}{1+x^2} \mathrm{d}(\quad x^2)
\end{aligned}
$$

Integration by parts: $\int u\mathrm{d}v = uv - \int v\mathrm{d}u$.

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$$\begin{aligned}\int_0^1 \arctan x\mathrm{d}x &= \left[(\arctan x)x\right]_{x=0}^{x=1} - \int_0^1 x\mathrm{d}\left(\arctan x\right)\\ &= 1\cdot\arctan 1 - 0\cdot\arctan 0 - \int_{x=0}^{x=1} x\frac{1}{1+x^2}\mathrm{d}x\\ &= \frac{\pi}{4} - \int_{x=0}^{x=1}\frac{1}{1+x^2}\mathrm{d}\left(\frac{x^2}{2}\right)\\ &= \frac{\pi}{4} - \frac{1}{2}\int_{x=0}^{x=1}\frac{1}{1+x^2}\mathrm{d}(1+x^2)\end{aligned}$$

Integration by parts: $\int u\mathrm{d}v = uv - \int v\mathrm{d}u.$

Example

Set $w = 1 + x^2$.

$$\begin{aligned}\int_0^1 \arctan x \mathrm{d}x &= \left[(\arctan x)x\right]_{x=0}^{x=1} - \int_0^1 x\mathrm{d}\left(\arctan x\right)\\ &= 1\cdot\arctan 1 - 0\cdot\arctan 0 - \int_{x=0}^{x=1} x\frac{1}{1+x^2}\mathrm{d}x\\ &= \frac{\pi}{4} - \int_{x=0}^{x=1}\frac{1}{1+x^2}\mathrm{d}\left(\frac{x^2}{2}\right)\\ &= \frac{\pi}{4} - \frac{1}{2}\int_{x=0}^{x=1}\frac{1}{1+x^2}\mathrm{d}(1+x^2)\end{aligned}$$

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\int_0^1 \arctan x \mathrm{d}x &= \left[(\arctan x)x\right]_{x=0}^{x=1} - \int_0^1 x \mathrm{d}\left(\arctan x\right)\\
&= 1 \cdot \arctan 1 - 0 \cdot \arctan 0 - \int_{x=0}^{x=1} x \frac{1}{1+x^2} \mathrm{d}x\\
&= \frac{\pi}{4} - \int_{x=0}^{x=1} \frac{1}{1+x^2} \mathrm{d}\left(\frac{x^2}{2}\right)\\
&= \frac{\pi}{4} - \frac{1}{2}\int_{x=0}^{x=1} \frac{1}{1+x^2} \mathrm{d}(1+x^2)\\
&= \frac{\pi}{4} - \frac{1}{2}\int_{x=0}^{x=1} \frac{1}{w} \mathrm{d}w
\end{aligned}
$$

Integration by parts: $\displaystyle\int u\mathrm{d}v = uv - \int v\mathrm{d}u.$

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Set $w = 1 + x^2$.

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\int_0^1 \arctan x \mathrm{d}x &= \left[(\arctan x)x\right]_{x=0}^{x=1} - \int_0^1 x\mathrm{d}\left(\arctan x\right)\\
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&= \frac{\pi}{4} - \int_{x=0}^{x=1}\frac{1}{1+x^2}\mathrm{d}\left(\frac{x^2}{2}\right)\\
&= \frac{\pi}{4} - \frac{1}{2}\int_{x=0}^{x=1}\frac{1}{1+x^2}\mathrm{d}(1+x^2)\\
&= \frac{\pi}{4} - \frac{1}{2}\int_{x=0}^{x=1}\frac{1}{w}\mathrm{d}w = \frac{\pi}{4} - \frac{1}{2}\left[\ln|w|\right]_{x=0}^{x=1}
\end{aligned}$$

Integration by parts: $\int u\mathrm{d}v = uv - \int v\mathrm{d}u$.

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Set $w = 1 + x^2$.

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\int_0^1 \arctan x \mathrm{d}x &= \left[(\arctan x)x\right]_{x=0}^{x=1} - \int_0^1 x\mathrm{d}\left(\arctan x\right) \\
&= 1\cdot \arctan 1 - 0\cdot \arctan 0 - \int_{x=0}^{x=1} x\frac{1}{1+x^2}\mathrm{d}x \\
&= \frac{\pi}{4} - \int_{x=0}^{x=1} \frac{1}{1+x^2}\mathrm{d}\left(\frac{x^2}{2}\right) \\
&= \frac{\pi}{4} - \frac{1}{2}\int_{x=0}^{x=1} \frac{1}{1+x^2}\mathrm{d}(1+x^2) \\
&= \frac{\pi}{4} - \frac{1}{2}\int_{x=0}^{x=1} \frac{1}{w}\mathrm{d}w = \frac{\pi}{4} - \frac{1}{2}\left[\ln|w|\right]_{x=0}^{x=1} \\
&= \frac{\pi}{4} - \frac{1}{2}\left[\ln\left(1+x^2\right)\right]_{x=0}^{x=1}
\end{aligned}$$

Integration by parts: $\int u\mathrm{d}v = uv - \int v\mathrm{d}u.$

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\int_0^1 \arctan x \mathrm{d}x &= [(\arctan x)x]_{x=0}^{x=1} - \int_0^1 x\mathrm{d}\left(\arctan x\right)\\
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&= \frac{\pi}{4} - \frac{1}{2}\left[\ln\left(1+x^2\right)\right]_{x=0}^{x=1}\\
&= \frac{\pi}{4} - \frac{1}{2}\left(\ln 2 - \ln 1\right)
\end{aligned}$$

Integration by parts: $\int u dv = uv - \int v du$.

Example

Set $w = 1 + x^2$.

$$\begin{aligned}
\int_0^1 \arctan x dx &= [(\arctan x)x]_{x=0}^{x=1} - \int_0^1 x d(\arctan x) \\
&= 1 \cdot \arctan 1 - 0 \cdot \arctan 0 - \int_{x=0}^{x=1} x \frac{1}{1+x^2} dx \\
&= \frac{\pi}{4} - \int_{x=0}^{x=1} \frac{1}{1+x^2} d\left(\frac{x^2}{2}\right) \\
&= \frac{\pi}{4} - \frac{1}{2}\int_{x=0}^{x=1} \frac{1}{1+x^2} d(1+x^2) \\
&= \frac{\pi}{4} - \frac{1}{2}\int_{x=0}^{x=1} \frac{1}{w} dw = \frac{\pi}{4} - \frac{1}{2}[\ln|w|]_{x=0}^{x=1} \\
&= \frac{\pi}{4} - \frac{1}{2}\left[\ln\left(1+x^2\right)\right]_{x=0}^{x=1} \\
&= \frac{\pi}{4} - \frac{1}{2}(\ln 2 - \ln 1)
\end{aligned}$$

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&= \frac{\pi}{4} - \frac{1}{2}\left[\ln\left(1+x^2\right)\right]_{x=0}^{x=1} \\
&= \frac{\pi}{4} - \frac{1}{2}\left(\ln 2 - \ln 1\right) = \frac{\pi}{4} - \frac{1}{2}\ln 2 \quad .
\end{aligned}$$