Calculus II
Homework
Integration by parts

1. Evaluate the indefinite integral. Illustrate the steps of your solutions.

   (a) $\displaystyle\int x \sin x \mathrm{d}x.$

   answer: $-x\cos x + \sin x + C$

   (b) $\displaystyle\int x e^{-x} \mathrm{d}x.$

   answer: $-(1+x)e^{-x} + C$

   (c) $\displaystyle\int x^2 e^x \mathrm{d}x.$

   answer: $x^2e^x - 2xe^x + 2e^x + C$

   (d) $\displaystyle\int x \sin(-2x) \mathrm{d}x.$

   answer: $\frac{x}{2}\cos(-2x) + \frac{1}{4}\sin(-2x) + C$

   (e) $\displaystyle\int x^2 \cos(3x) \mathrm{d}x.$

   answer: $\frac{x^2}{3}\sin(3x) + \frac{2x}{9}\cos(3x) - \frac{2}{27}\sin(3x) + C$

   (f) $\displaystyle\int x^2 e^{-2x} \mathrm{d}x.$

   answer: $-\frac{x^2e^{-2x}}{2} - \frac{xe^{-2x}}{2} - \frac{e^{-2x}}{4} + C$

   (g) $\displaystyle\int x \sin(2x) \mathrm{d}x.$

   answer: $-\frac{x}{2}\cos(2x) + \frac{1}{4}\sin(2x) + C$

   (h) $\displaystyle\int x \cos(3x) \mathrm{d}x.$

   answer: $\frac{x}{3}\sin(3x) + \frac{1}{9}\cos(3x) + C$

   (i) $\displaystyle\int x^2 e^{2x} \mathrm{d}x.$

   answer: $\frac{x^2}{2}e^{2x} - \frac{x}{2}e^{2x} + \frac{e^{2x}}{4} + C$

   (j) $\displaystyle\int x^3 e^x \mathrm{d}x.$

   answer: $x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C$

Solution. 1.a.

$$\int x \underbrace{\sin x \mathrm{d}x}_{=\mathrm{d}(-\cos x)} = -\int x\mathrm{d}(\cos x) = -x\cos x + \int \cos x \mathrm{d}x = -x\cos x + \sin x + C \quad .$$

Solution. 1.c.

$$\begin{aligned}\int x^2 \underbrace{e^x \mathrm{d}x}_{\mathrm{d}(e^x)} &= \int x^2 \mathrm{d}e^x = x^2e^x - \int e^x 2x\mathrm{d}x = x^2e^x - \int 2x\mathrm{d}e^x \\ &= x^2e^x - 2xe^x + \int 2e^x\mathrm{d}x = x^2e^x - 2xe^x + 2e^x + C \quad .\end{aligned}$$

Solution. 1.f.

$$\begin{aligned}\int x^2 e^{-2x}\mathrm{d}x &= \int x^2 \mathrm{d}\left(\frac{e^{-2x}}{-2}\right) && \Big| \text{ Integrate by parts} \\ &= -\frac{x^2e^{-2x}}{2} - \int \left(\frac{e^{-2x}}{-2}\right)\mathrm{d}\left(x^2\right) \\ &= -\frac{x^2e^{-2x}}{2} + \int xe^{-2x}\mathrm{d}x \\ &= -\frac{x^2e^{-2x}}{2} + \int x\mathrm{d}\left(\frac{e^{-2x}}{-2}\right) && \Big| \text{ Integrate by parts} \\ &= -\frac{x^2e^{-2x}}{2} - \frac{xe^{-2x}}{2} + \frac{1}{2}\int e^{-2x}\mathrm{d}x \\ &= -\frac{x^2e^{-2x}}{2} - \frac{xe^{-2x}}{2} - \frac{e^{-2x}}{4} + C \quad .\end{aligned}$$

2. Evaluate the indefinite integral. Illustrate the steps of your solutions.

(a) $\displaystyle\int x^2\cos(2x)\mathrm{d}x$.

answer: $\frac{1}{2}x^2\sin(2x)+\frac{1}{2}x\cos(2x)-\frac{1}{4}\sin(2x)+C$

(b) $\displaystyle\int x^2e^{ax}\mathrm{d}x$, where a is a constant.

answer: $\frac{1}{a}x^2e^{ax}-\frac{2}{a^2}xe^{ax}+\frac{2}{a^3}e^{ax}+C$

(c) $\displaystyle\int x^2e^{-ax}\mathrm{d}x$, where a is a constant.

answer: $-\frac{1}{a}x^2e^{-ax}-\frac{2}{a^2}xe^{-ax}-\frac{2}{a^3}e^{-ax}+C$

(d) $\displaystyle\int x^2\frac{(e^{ax}+e^{-ax})^2}{4}\mathrm{d}x$, where a is a constant.

answer: $\frac{1}{8}\Big(a^{-1}x^2e^{2ax}-a^{-1}x^2e^{-2ax}-a^{-2}xe^{2ax}-a^{-2}xe^{-2ax}+\frac{1}{2}a^{-3}e^{2ax}-\frac{1}{2}a^{-3}e^{-2ax}+\frac{2}{3}x^3\Big)+C$

(e) $\displaystyle\int\frac{1}{\cos^2 x}\mathrm{d}x$. (Hint: This problem does not require integration by parts. What is the derivative of $\tan x$?)

answer: $\tan x+C$

(f) $\displaystyle\int(\tan^2 x)\mathrm{d}x$. (Hint: This problem does not require integration by parts. We can use $\tan^2 x=\frac{1}{\cos^2 x}-1$ and the previous problem.)

(g) $\displaystyle\int x\tan^2 x\mathrm{d}x$. (Hint: $\tan^2 x\mathrm{d}x=\mathrm{d}(F(x))$, where $F(x)$ is the answer from the preceding problem).

answer: $-\frac{x^2}{2}+x\tan x+\ln|\cos x|+C$

(h) $\displaystyle\int e^{-\sqrt{x}}\mathrm{d}x$.

answer: $-2\sqrt{x}e^{-\sqrt{x}}-2e^{-\sqrt{x}}+C$

(i) $\displaystyle\int\cos^2 x\ \mathrm{d}x$.

answer: $\frac{1}{4}\sin(2x)+\frac{x}{2}+C$

(j) $\displaystyle\int\frac{x}{1+x^2}\mathrm{d}x$ (Hint: use substitution rule, don't use integration by parts)

answer: $\frac{\ln\left(1+x^2\right)}{2}+C$

(k) $\displaystyle\int(\arctan x)\mathrm{d}x$.

answer: $x\arctan x-\frac{\ln\left(1+x^2\right)}{2}+C$

(l) $\displaystyle\int(\arcsin x)\mathrm{d}x$.

answer: $x\arcsin x+\sqrt{1-x^2}+C$

(m) $\displaystyle\int(\arcsin x)^2\mathrm{d}x$. (Hint: Try substituting $x=\sin y$.)

answer: $x(\arcsin x)^2+2\sqrt{1-x^2}\arcsin x-2x+C$

(n) $\displaystyle\int\arctan\left(\frac{1}{x}\right)\mathrm{d}x$.

(o) $\displaystyle\int\sin xe^x\mathrm{d}x$

answer: $\frac{1}{2}\left(e^x\sin x-e^x\cos x\right)+C$

(p) $\displaystyle\int\cos xe^x\mathrm{d}x$

answer: $\frac{1}{2}\left(e^x\cos x+e^x\sin x\right)+C$

(q) $\displaystyle\int\sin(\ln(x))\mathrm{d}x$.

answer: $\frac{x}{2}\left(\sin(\ln x)-\cos(\ln x)\right)+C$

(r) $\displaystyle\int\cos(\ln(x))\mathrm{d}x$.

answer: $\frac{x}{2}\left(\cos(\ln x)+\sin(\ln x)\right)+C$

(s) $\displaystyle\int\ln x\mathrm{d}x$

answer: $x\ln|x|-x+C$

(t) $\displaystyle\int x\ln x\ \mathrm{d}x$.

answer: $\frac{1}{2}x^2\ln|x|-\frac{x^2}{4}+C$

(u) $\displaystyle\int\frac{\ln x}{\sqrt{x}}\mathrm{d}x$.

answer: $2\sqrt{x}(\ln x-2)+C$

(v) $\displaystyle\int(\ln x)^2\mathrm{d}x$.

answer: $x(\ln x)^2-2x\ln x+2x+C$

(w) $\displaystyle\int(\ln x)^3\mathrm{d}x$.

answer: $x(\ln x)^3-3x(\ln x)^2+6x\ln x-6x+C$

(x) $\displaystyle\int x^2\cos^2 x\mathrm{d}x$. (This problem is related to Problem 2.d as $\cos x=\frac{e^{ix}+e^{-ix}}{2}$).

answer: $\frac{1}{4}x^2\sin\ (2x)+\frac{1}{4}x\cos\ (2x)-\frac{1}{8}\sin\ (2x)+\frac{1}{6}x^3$

Solution. 2.g.

$$\begin{array}{rcll}
\int x\tan^2 x\mathrm{d}x &=& \int x\left(\sec^2 x-1\right)\mathrm{d}x & \Big| \text{ use } \sec^2 x-1=\tan^2 x \\
&=& \int x\left(\sec^2 x-1\right)\mathrm{d}x & \\
&=& -\int x\mathrm{d}x+\int x\sec^2 x\mathrm{d}x & \Big| \text{ use } \mathrm{d}(\tan x)=\sec^2 x\mathrm{d}x \\
&=& -\frac{x^2}{2}+\int x\mathrm{d}(\tan x) & \Big| \text{ integrate by parts} \\
&=& -\frac{x^2}{2}+x\tan x-\int \tan x\mathrm{d}x & \\
&=& -\frac{x^2}{2}+x\tan x-\int \frac{\sin x}{\cos x}\mathrm{d}x & \Big| \text{ use } \sin x\mathrm{d}x=-\mathrm{d}(\cos x) \\
&=& -\frac{x^2}{2}+x\tan x+\int \frac{\mathrm{d}(\cos x)}{\cos x} & \Big| \text{ Set } y=\cos x \\
&=& -\frac{x^2}{2}+x\tan x+\int \frac{1}{y}\mathrm{d}y & \\
&=& -\frac{x^2}{2}+x\tan x+\ln|y|+C & \Big| \text{ Substitute back } y=\cos x \\
&=& -\frac{x^2}{2}+x\tan x+\ln|\cos x|+C \quad . &
\end{array}$$

Solution. 2.h.

$$\begin{array}{rcll}
\int e^{-\sqrt{x}}\mathrm{d}x &=& \int 2ye^{-y}\mathrm{d}y & \Bigg| \text{ Subst.: } \begin{array}{rcl} \sqrt{x} &=& y \\ \frac{1}{2\sqrt{x}}\mathrm{d}x &=& \mathrm{d}y \\ \mathrm{d}x &=& 2y\mathrm{d}y \end{array} \\
&=& \int 2y\mathrm{d}\left(-e^{-y}\right) & \Big| \text{ int. by parts} \\
&=& -2ye^{-y}+2\int e^{-y}\mathrm{d}y & \\
&=& -2ye^{-y}-2e^{-y}+C & \\
&=& -2\sqrt{x}e^{-\sqrt{x}}-2e^{-\sqrt{x}}+C \quad . &
\end{array}$$

Solution. 2.i. Later, we shall study general methods for solving trigonometric integrals that will cover this example. Let us however show one way to solve this integral by integration by parts.

$$\begin{array}{rll}
\int \cos^2 x\mathrm{d}x &= x\cos^2 x-\int x\mathrm{d}(\cos^2 x) & \\
&= x\cos^2 x-\int x2\cos x(-\sin x)\mathrm{d}x & \Big| \sin(2x)=2\sin x\cos x \\
&= x\cos^2 x+\int x\sin(2x)\mathrm{d}x & \\
&= x\cos^2 x+\int x\mathrm{d}\left(\frac{-\cos(2x)}{2}\right) & \\
&= x\cos^2 x+x\left(\frac{-\cos(2x)}{2}\right)-\int\left(\frac{-\cos(2x)}{2}\right)\mathrm{d}x & \\
&= \frac{x}{2}\left(2\cos^2 x-\cos(2x)\right)+\frac{\sin(2x)}{4}+C & \Big| \cos(2x)=\cos^2 x-\sin^2 x \\
&= \frac{x}{2}\left(2\cos^2 x-(\cos^2 x-\sin^2 x)\right)+\frac{\sin(2x)}{4}+C & \Big| \cos^2 x+\sin^2 x=1 \\
&= \frac{x}{2}+\frac{\sin(2x)}{4}+C \quad . &
\end{array}$$

Solution. 2.k

$$\begin{array}{rcl}
\int \arctan x\mathrm{d}x &=& x\arctan x-\int x\mathrm{d}(\arctan x) \\
&=& x\arctan x-\int \frac{x}{x^2+1}\mathrm{d}x \\
&=& x\arctan x-\int \frac{\frac{1}{2}\mathrm{d}(x^2)}{x^2+1} \\
&=& x\arctan x-\int \frac{\frac{1}{2}\mathrm{d}(x^2+1)}{x^2+1} \\
&=& x\arctan x-\frac{1}{2}\ln(x^2+1)+C \quad .
\end{array}$$

Solution. 2.m.

$$\begin{array}{rcll}
\int(\arcsin x)^2\mathrm{d}x & = & \int(\arcsin(\sin y))^2\,\mathrm{d}(\sin y) & \text{Set } x=\sin y\\
& = & \int y^2\cos y\mathrm{d}y = \int y^2\mathrm{d}(\sin y) & \text{Integrate by parts}\\
& = & y^2\sin y - \int 2y\sin y\mathrm{d}y & \\
& = & y^2\sin y + \int 2y\mathrm{d}(\cos y) & \text{Integrate by parts}\\
& = & y^2\sin y + 2y\cos y - 2\int\cos y\mathrm{d}y & \\
& = & y^2\sin y + 2y\cos y - 2\sin y + C & \text{Substitute } y=\arcsin x\\
& = & x(\arcsin x)^2 & \\
& & +2\sqrt{1-x^2}\arcsin x - 2x + C \quad . &
\end{array}$$

Solution. 2.o

$$\begin{array}{rcll}
\int\sin x\underbrace{e^x\mathrm{d}x}_{=\mathrm{d}e^x} & = & \sin xe^x - \int e^x\mathrm{d}(\sin x) = \sin xe^x - \int\cos x\underbrace{e^x\mathrm{d}x}_{=\mathrm{d}e^x} & \\
& = & \sin xe^x - e^x\cos x + \int e^x\mathrm{d}(\cos x) & \\
& = & e^x\sin x - e^x\cos x - \int e^x\sin x\mathrm{d}x & \text{add } \int e^x\sin x\mathrm{d}x \text{ to both sides}\\
2\int\sin xe^x\mathrm{d}x & = & \sin xe^x - e^x\cos x & \\
\int\sin xe^x\mathrm{d}x & = & \frac{1}{2}\left(\sin xe^x - e^x\cos x\right) \quad . &
\end{array}$$

Solution. 2.q.

$$\begin{array}{rll}
\int\sin(\ln x)\mathrm{d}x & = x\sin(\ln x) - \int x\mathrm{d}(\sin(\ln x)) & \text{int. by parts}\\
& = x\sin(\ln x) - \int x\left(\cos(\ln x)\right)(\ln x)'\,\mathrm{d}x & \\
& = x\sin(\ln x) - \int\cos(\ln x)\mathrm{d}x & \text{int. by parts}\\
& = x\sin(\ln x) - \left(x\cos(\ln x) - \int x\mathrm{d}(\cos(\ln x))\right) & \\
& = x\sin(\ln x) - x\cos(\ln x) + \int x(-\sin(\ln x))(\ln x)'\mathrm{d}x & \\
& = x\sin(\ln x) - x\cos(\ln x) - \int\sin(\ln x)\mathrm{d}x & \text{add } \int\sin(\ln x)\mathrm{d}x \text{ to both sides}\\
2\int\sin(\ln x)\mathrm{d}x & = x\sin(\ln x) - x\cos(\ln x) & \\
\int\sin(\ln x)\mathrm{d}x & = \frac{x}{2}\left(\sin(\ln x) - \cos(\ln x)\right) \quad . &
\end{array}$$

Solution. 2.s

$$\int\ln x\mathrm{d}x = x\ln x - \int x\mathrm{d}(\ln x) = x\ln x - \int\frac{x}{x}\mathrm{d}x = x\ln x - x + C \quad .$$

Solution. 2.u

$$\begin{aligned}
\int \frac{\ln x}{\sqrt{x}}\mathrm{d}x &= \int (\ln x) 2\mathrm{d}\left(\sqrt{x}\right) && \Big| \text{ integrate by parts} \\
&= (\ln x)2\sqrt{x} - \int 2\sqrt{x}\mathrm{d}(\ln x) \\
&= 2\sqrt{x}\ln x - 2\int \frac{\sqrt{x}}{x}\mathrm{d}x \\
&= 2\sqrt{x}\ln x - 2\int x^{-\frac{1}{2}}\mathrm{d}x \\
&= 2\sqrt{x}\ln x - 4\sqrt{x} + C \\
&= 2\sqrt{x}(\ln x - 2) + C \quad .
\end{aligned}$$

3. Compute $\int x^n e^x \mathrm{d}x$, where n is a non-negative integer.

Solution. 3

$$\begin{aligned}
\int x^n e^x \mathrm{d}x &= \int x^n \mathrm{d}e^x \\
&= x^n e^x - \int e^x \mathrm{d}x^n \\
&= x^n e^x - n\int x^{n-1}e^x\mathrm{d}x \\
&= x^n e^x - n\left(\int x^{n-1}\mathrm{d}e^x\right) \\
&= x^n e^x - n\left(x^{n-1}e^x - \int (n-1)x^{n-2}e^x\mathrm{d}x\right) \\
&= x^n e^x - nx^{n-1}e^x + n(n-1)\int x^{n-2}e^x\mathrm{d}x \\
&= \dots \text{(continue above process)} \dots \\
&= x^n e^x - nx^{n-1}e^x + n(n-1)x^{n-2}e^x + \dots \\
&\quad +(-1)^k n(n-1)(n-2)\dots(n-k+1)x^{n-k}e^x \\
&\quad + \dots + (-1)^n n! e^x + C \\
&= C + \sum_{k=0}^{n} (-1)^n \frac{n!}{(n-k)!} x^{n-k} e^x \quad .
\end{aligned}$$