Calculus I
Exponents and logarithms review

Todor Milev

2019

Outline

1. Exponential Functions
   - Two ways to define exponents
   - Basic properties
   - The Natural Exponential Function

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Properties of exponential expressions.

For integer x, y and bases a, b, we demonstrate the exponent rules by example.

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  - the second alternative definition is easier to compute with.

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$$a^x = \lim_{\substack{y \to x\\ y\text{-rational}}} a^y$$

For example, a^π would be defined as the limit of the sequence $a^{3.14}, a^{3.141}, a^{3.1415}, \dots$.

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- This is the definition assumed in many elementary courses.

Exponent definition using series (approach II)

- The following formula (studied much later) can be used as alternative definition.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots$$

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- For $|x| < 1$ define

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + \frac{(-1)^{n+1} x^n}{n} + \dots$$

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  Infinite sum studied much later.

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  Here $n! = 1 \cdot 2 \cdot 3 \cdot \dots \cdot (n-1) \cdot n$ and is read "n factorial".
- For $|x| < 1$ define

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + \frac{(-1)^{n+1} x^n}{n} + \dots$$

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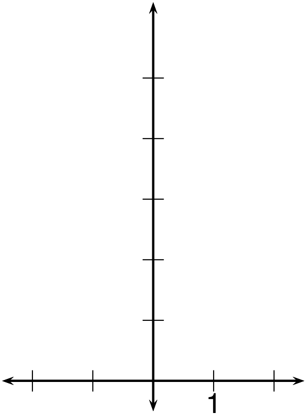
- Pros: this is how e^x and a^x are actually computed (by modern computers and by humans in the past).

Exponential Functions

The function $f(x) = 2^x$ is called an exponential function because the variable x is the exponent.



x	y
2	
1	
0	
-1	
-2	

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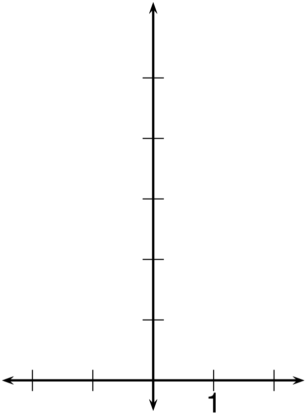

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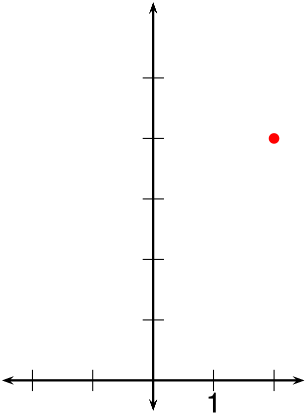

x	y
2	4
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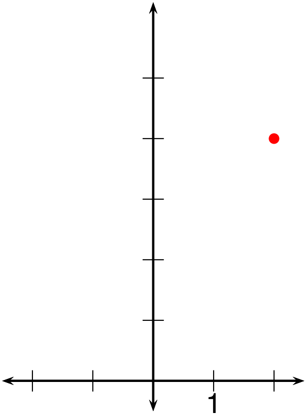

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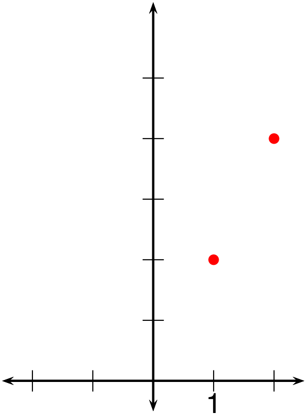

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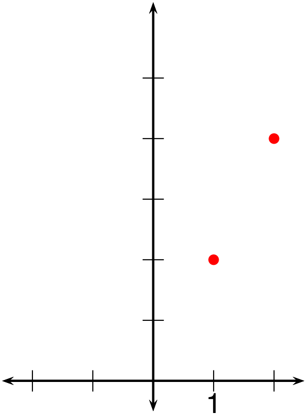

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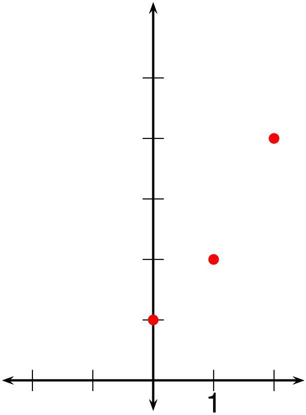

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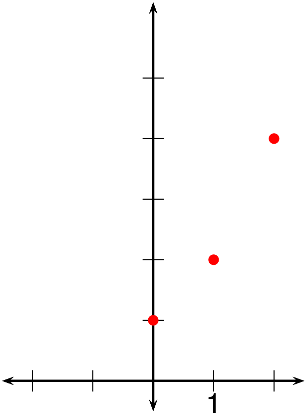

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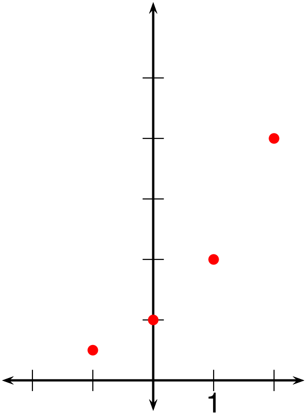

x	y
2	4
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0	1
-1	$\frac{1}{2}$
-2	

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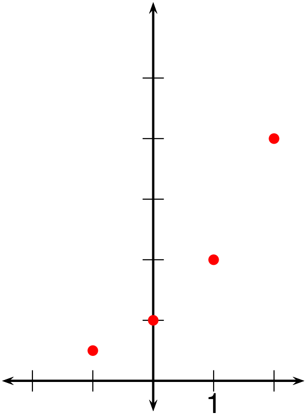

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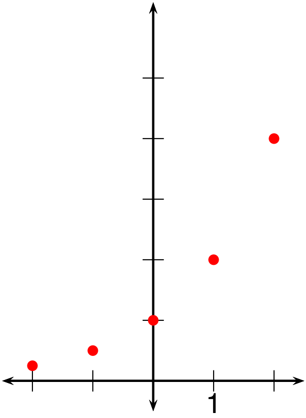

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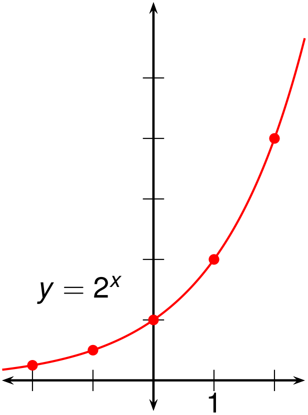

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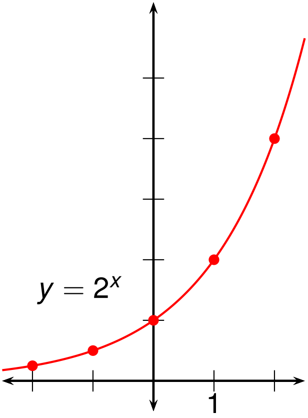

x	y
2	4
1	2
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-1	$\frac{1}{2}$
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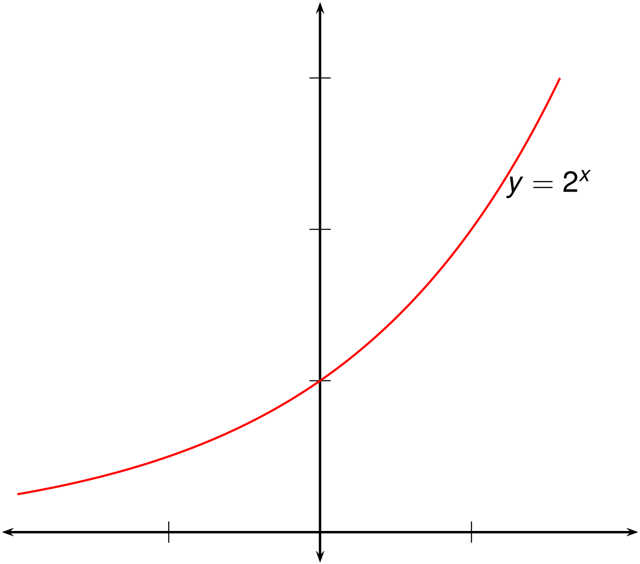
(Exponential Function Terminology)

An exponential function is a function of the form $f(x) = a^x$, where a is a positive constant.

Graphs of various exponential functions.



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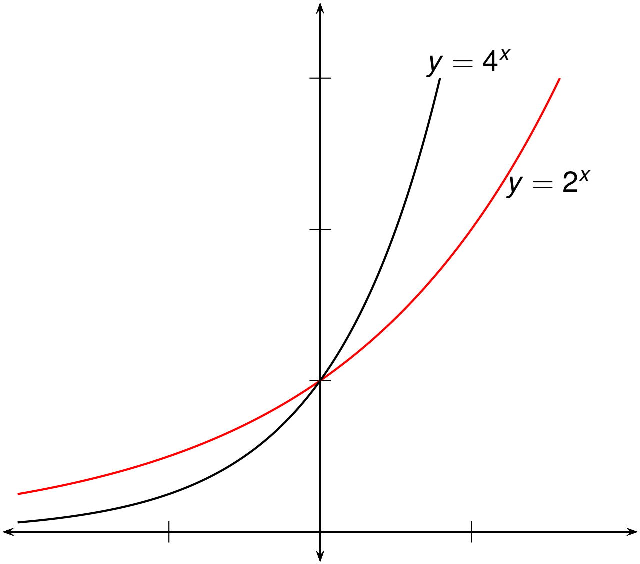

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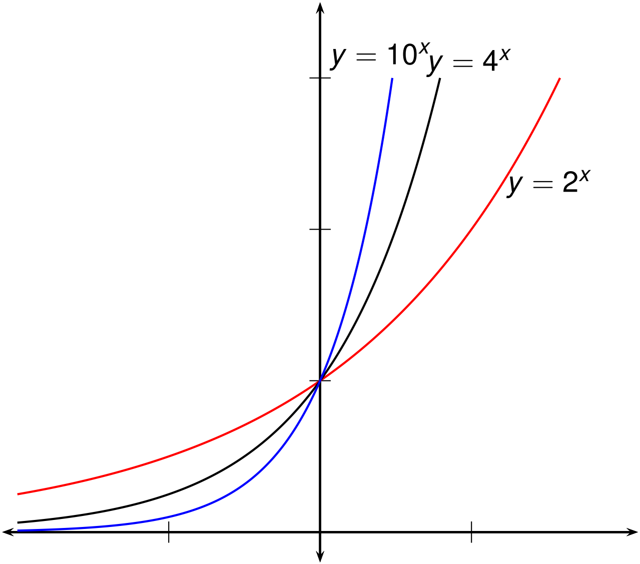

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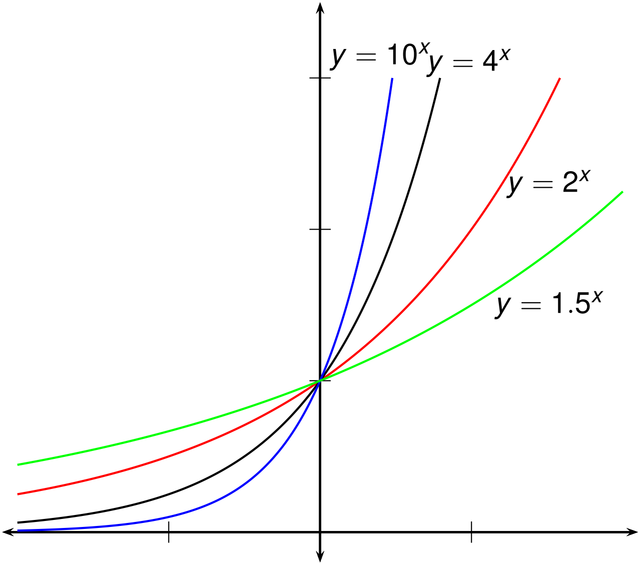

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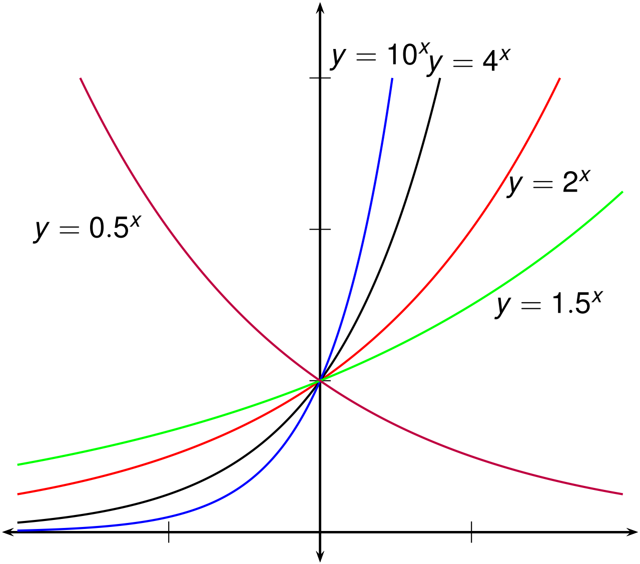

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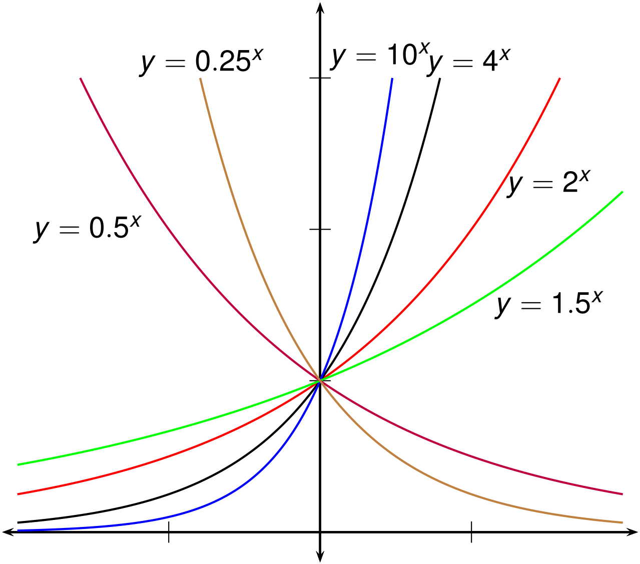

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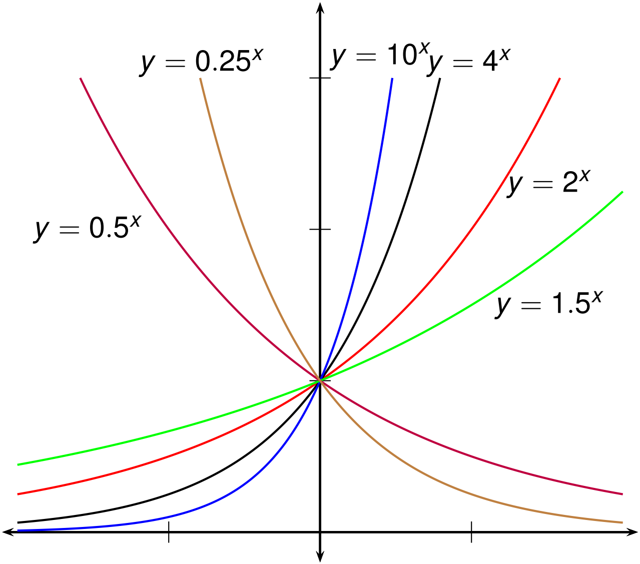

Observations

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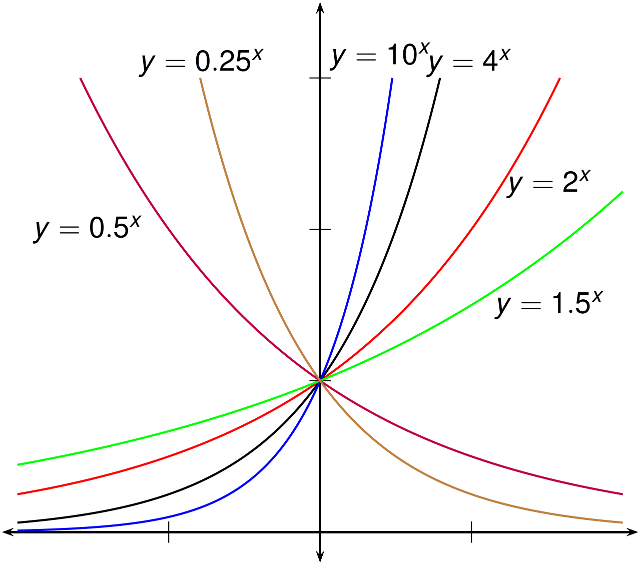

Observations

- a^x is always **?**

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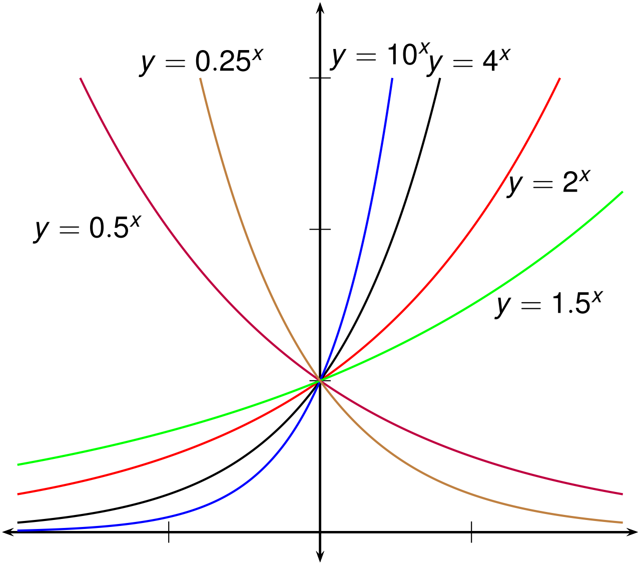

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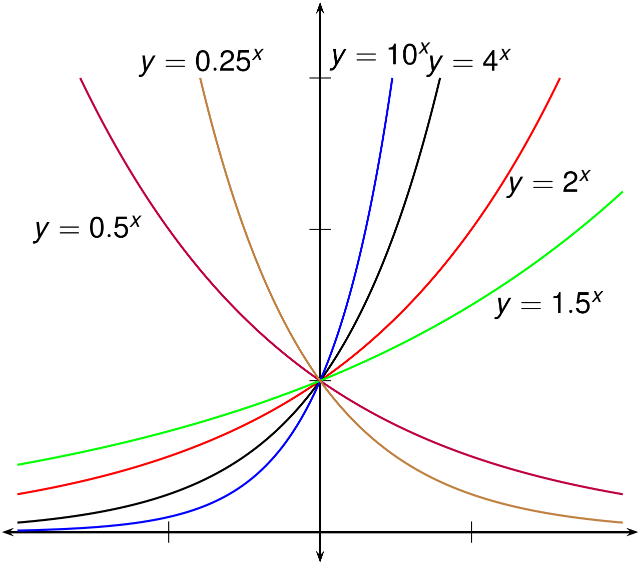

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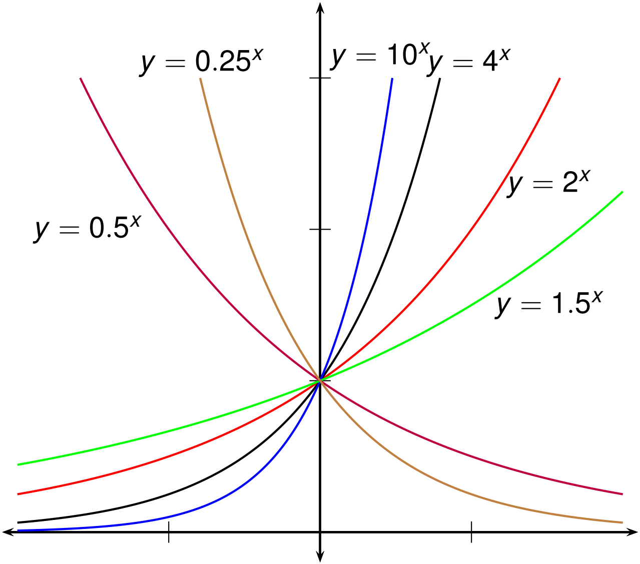

Observations

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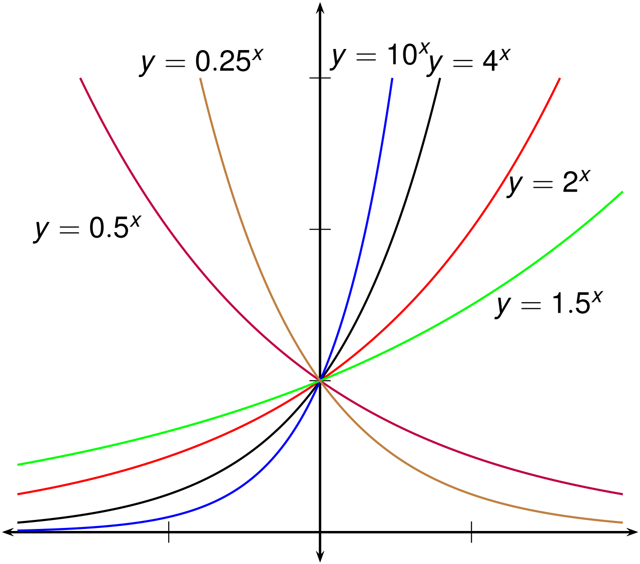

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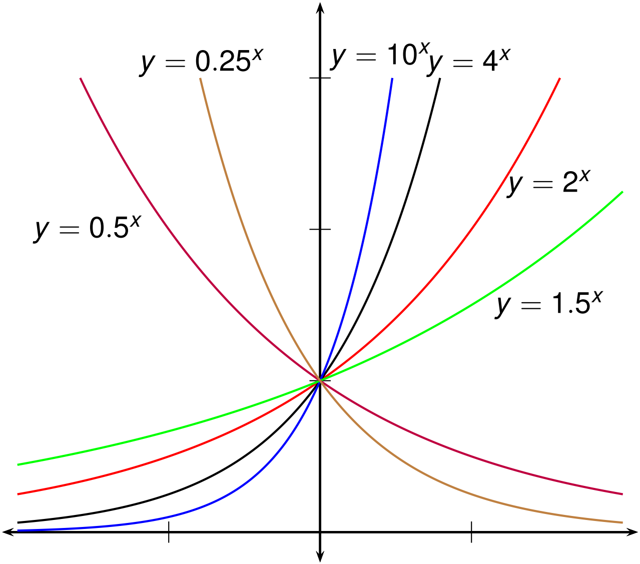

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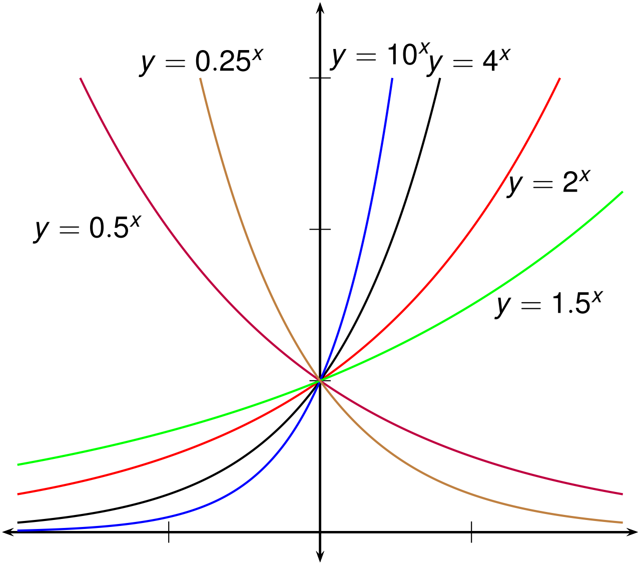

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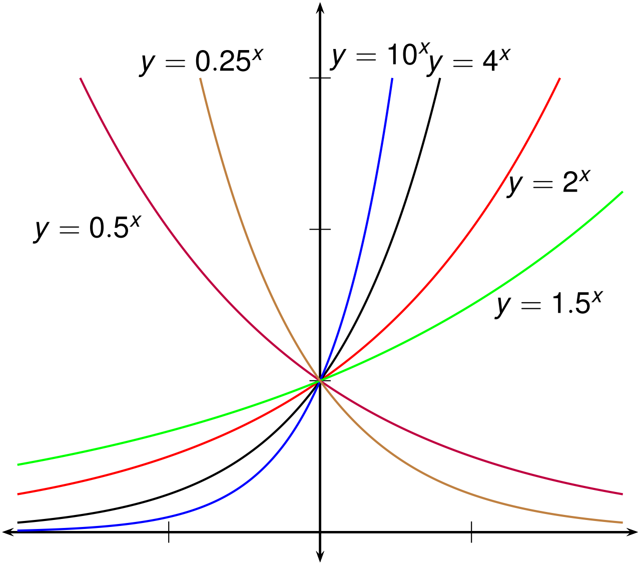

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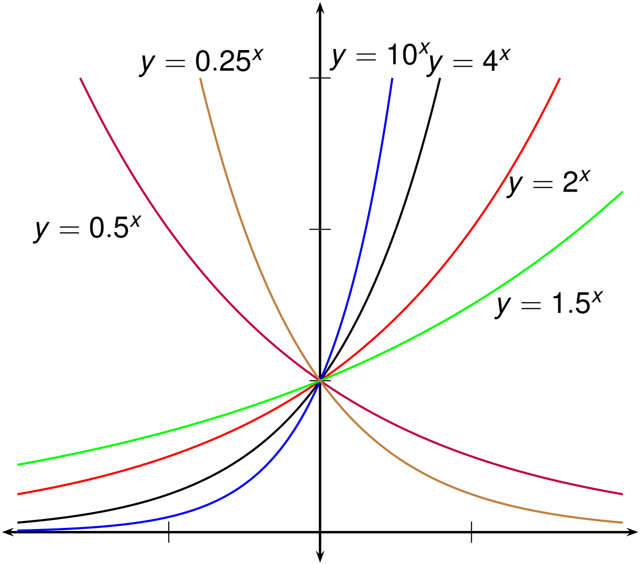

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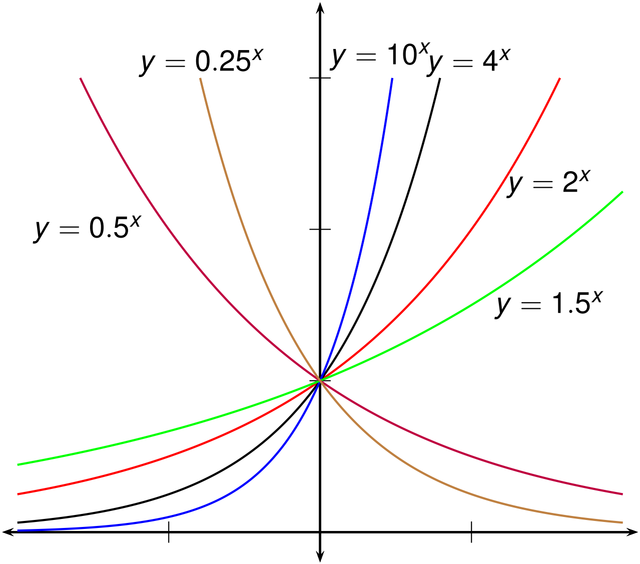

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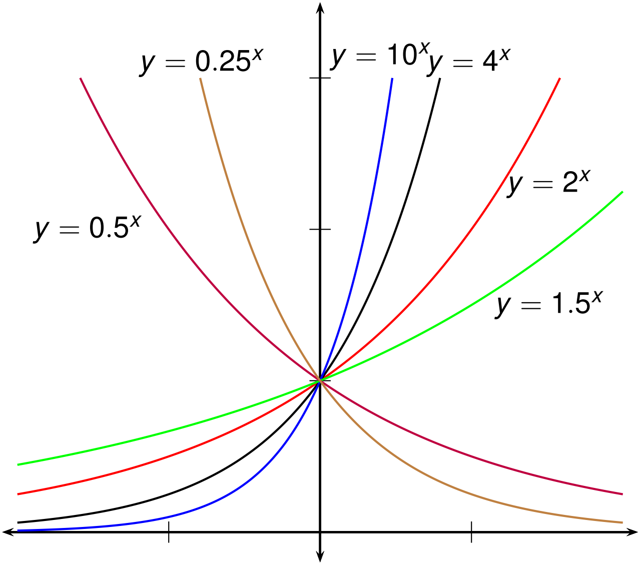

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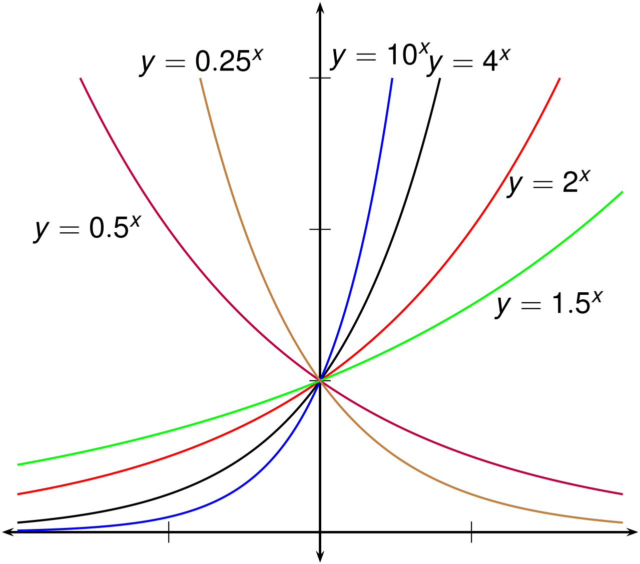

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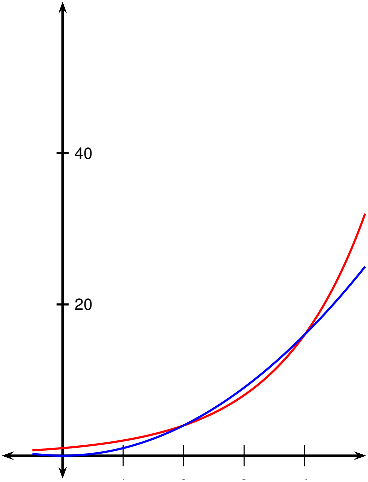
Graphical comparison of $y = 2^x$ with $y = x^2$. Axes have different scales.



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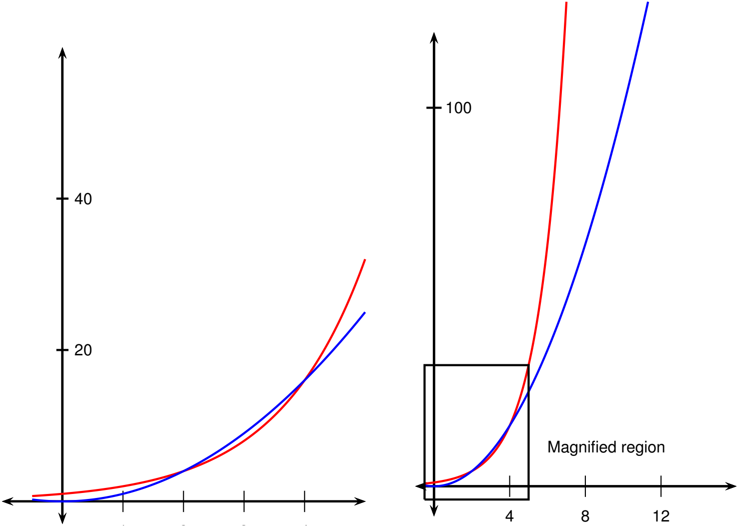
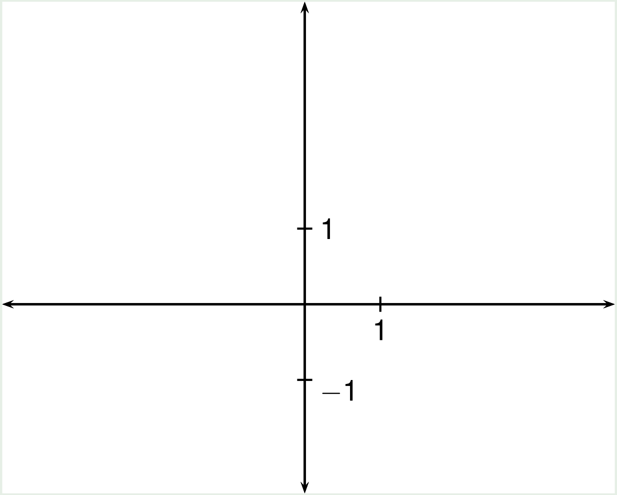


Example

Draw the graph of the function $y = 2^{-x} - 1 = 0.5^x - 1 = \left(\frac{1}{2}\right)^x - 1$.
Assume the graph of $y = 2^x$ given.



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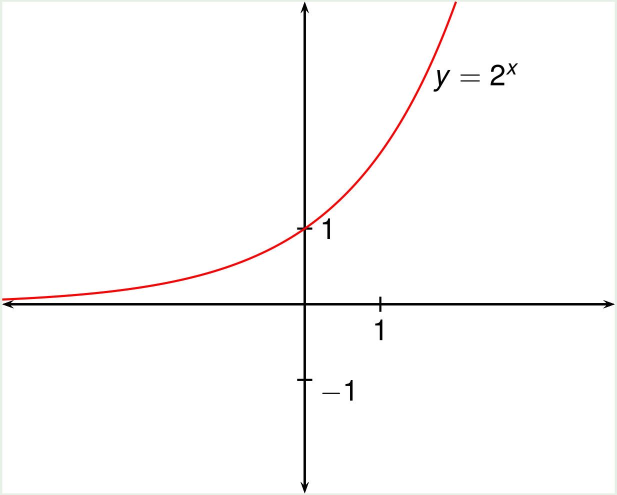

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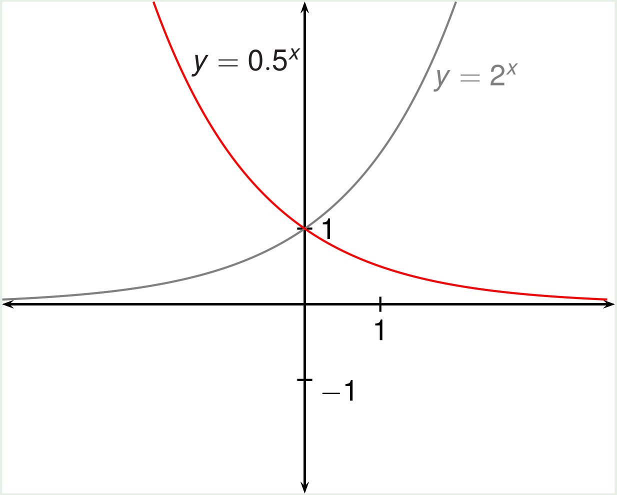

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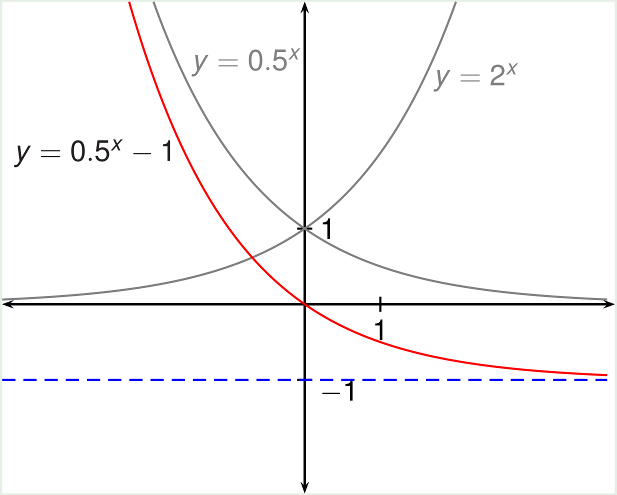

- Plot of 2^x assumed given.
- Plot $f(-x)$ = reflect $f(x)$ across y axis.
- Plot $g(x) - 1$ = shift graph $g(x)$ 1 unit down.

Example (Solve exponential equation without logarithms)

Solve for t.

$$16^{4t} = 8^{t-2}$$

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Example (Solving a quadratic exponential equation)

Solve for x.

$$9^x = 2 \cdot 3^x + 63$$

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$$\begin{aligned} 9^x &= 2\cdot 3^x + 63 \\ 9^x - 2\cdot 3^x - 63 &= 0 \end{aligned}$$

Example (Solving a quadratic exponential equation)

Solve for x.

$$\begin{aligned} 9^x &= 2\cdot 3^x + 63 \\ 9^x - 2\cdot 3^x - 63 &= 0 && \Big|\ \text{Substitute } u = 3^x \\ \textbf{?}\ \ -2u - 63 &= 0 \end{aligned}$$

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9^x - 2\cdot 3^x - 63 &= 0 & \Big|\ \text{Substitute } u = 3^x \\
\textbf{?}\ - 2u - 63 &= 0
\end{aligned}
$$

Example (Solving a quadratic exponential equation)

Solve for x.

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Therefore $x = 2$ is the solution.

Example (Solving an exponential word problem)

A farmer buys 48 chickens and 6 rabbits. The chicken population doubles each year, and the rabbit population doubles every six months. When does the farmer have the same number of chickens as rabbits?

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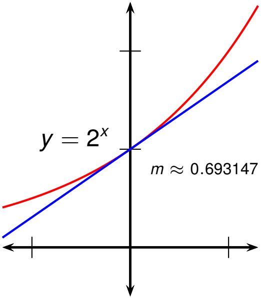
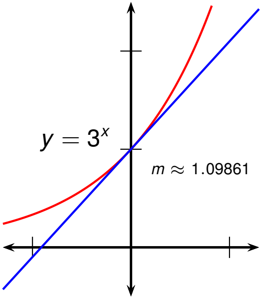
Therefore the chicken and rabbit populations are equal after 3 years.

The Natural Exponential Function

- One base for an exponential function is especially useful.





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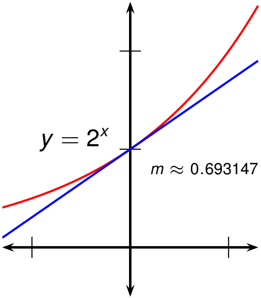
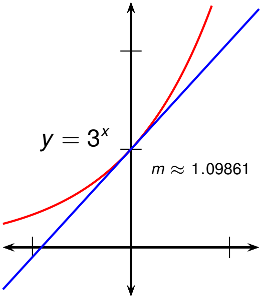
- It has a special property: its tangent line at $x = 0$ has slope $m = 1$.





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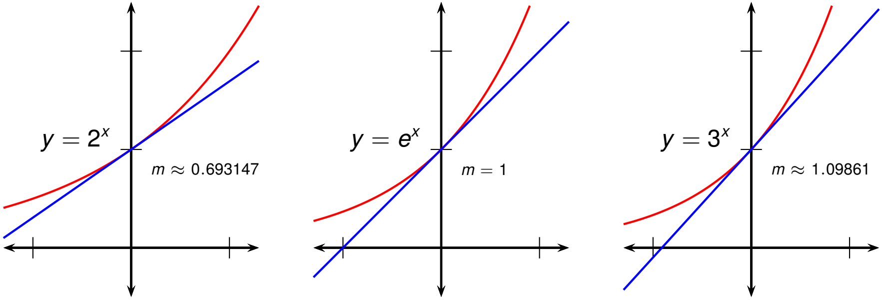

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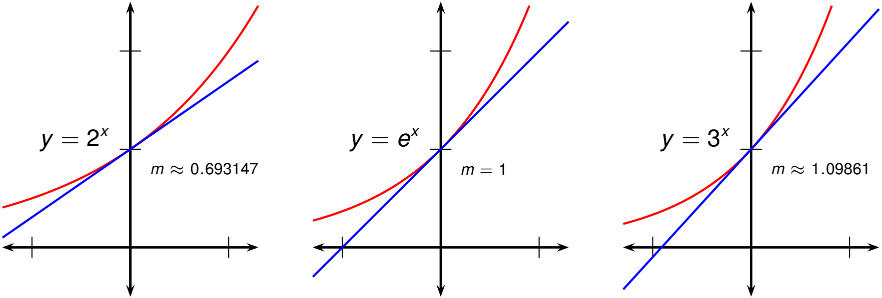

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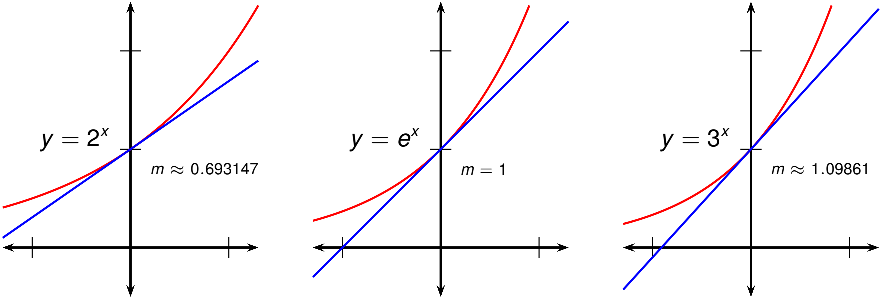
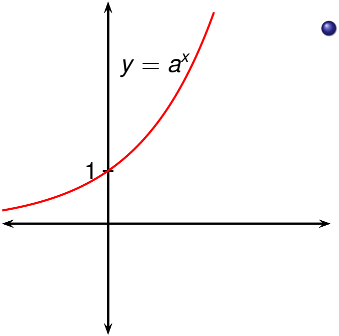
- e is a number between 2 and 3.
- In fact, $e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots \approx 2.71828$.

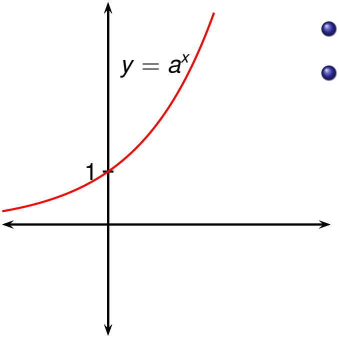


Logarithmic Functions



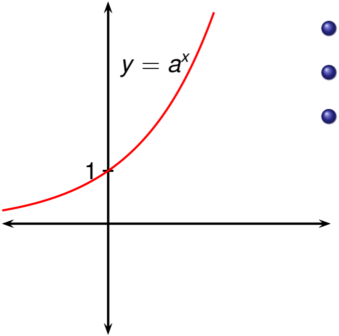
- Suppose $a > 0$, $a \neq 1$.

Logarithmic Functions



- Suppose $a > 0$, $a \neq 1$.
- Let $f(x) = a^x$.

Logarithmic Functions



- Suppose $a > 0$, $a \neq 1$.
- Let $f(x) = a^x$.
- Then f is either increasing or decreasing.

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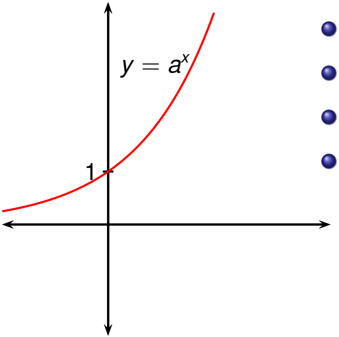

- Suppose $a > 0$, $a \neq 1$.
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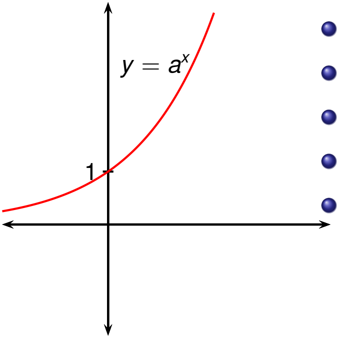
- Therefore f is one-to-one.

Logarithmic Functions



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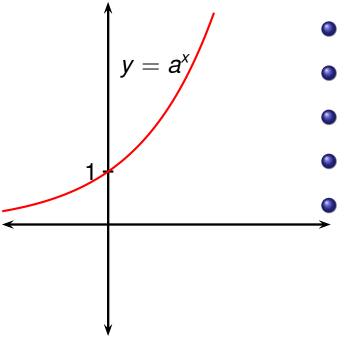

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Definition ($\log_a x$)

The inverse function of $f(x) = a^x$ is called the logarithmic function with base a, and is written $\log_a x$. It is defined by the formula

$$\log_a x = y \qquad \Leftrightarrow \qquad a^y = x.$$

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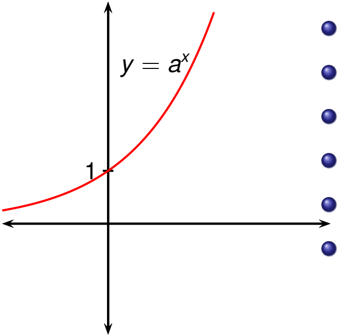

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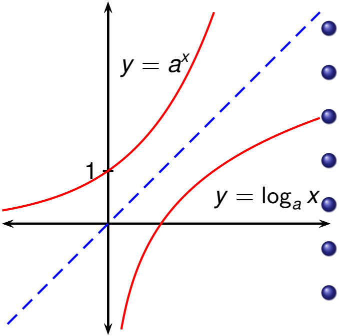

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If $x > 0$, then $\log_a x$ is the exponent to which the base a must be raised to give x.

Example

Evaluate:

1. $\log_3 81 =$
2. $\log_{25} 5 =$
3. $\log_{10} 0.001 =$

If $x > 0$, then $\log_a x$ is the exponent to which the base a must be raised to give x.

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Example

Evaluate:

1. $\log_3 81 = 4$ because $3^4 = 81$.
2. $\log_{25} 5 =$ **?**
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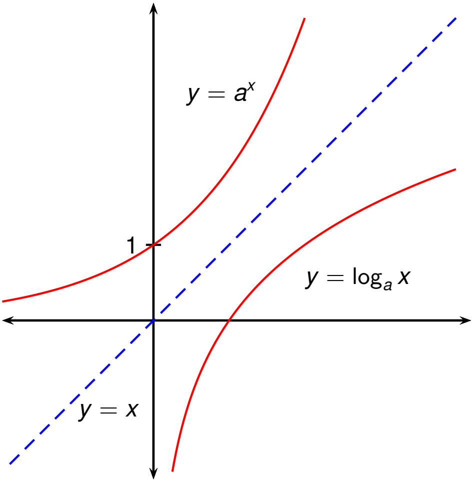

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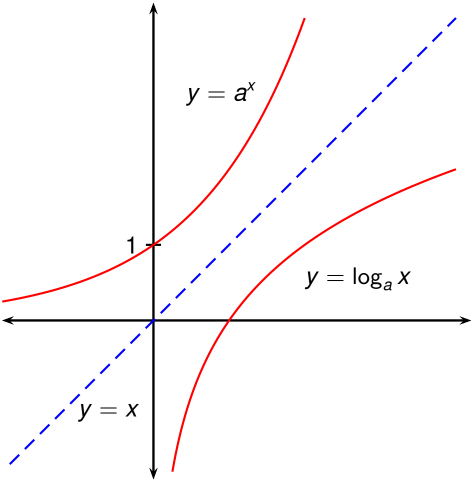

- Suppose $a > 1$.
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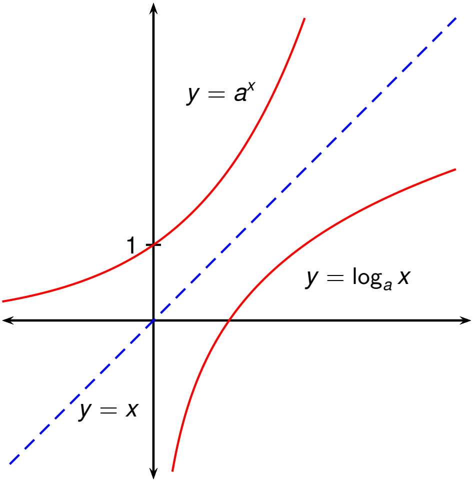

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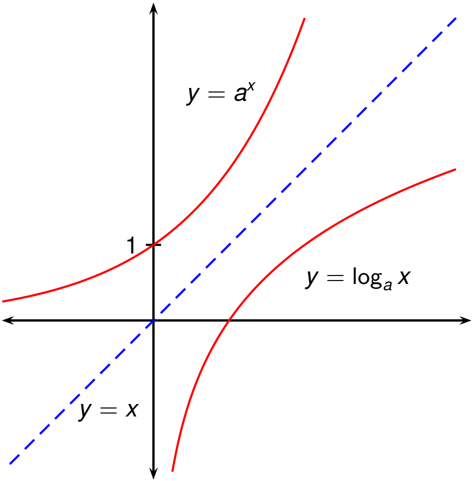

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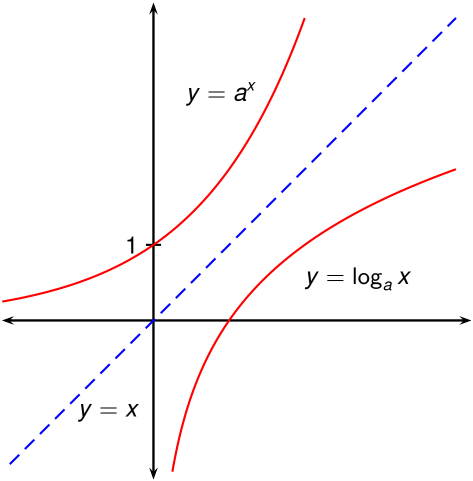

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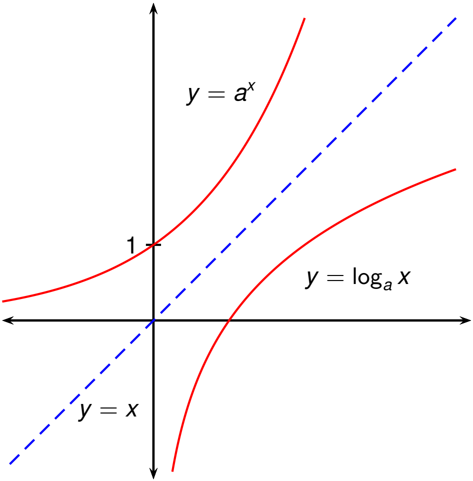

- Suppose $a > 1$.
- Domain of a^x: $\mathbb{R}$.
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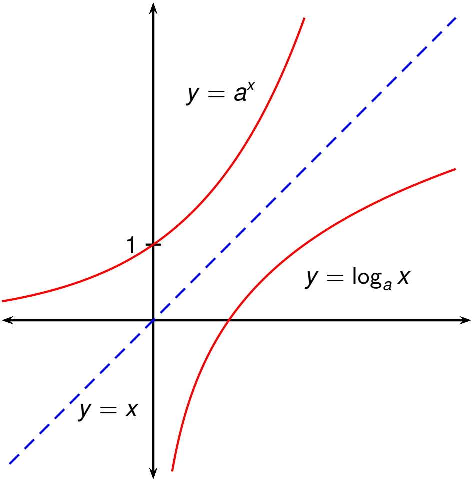

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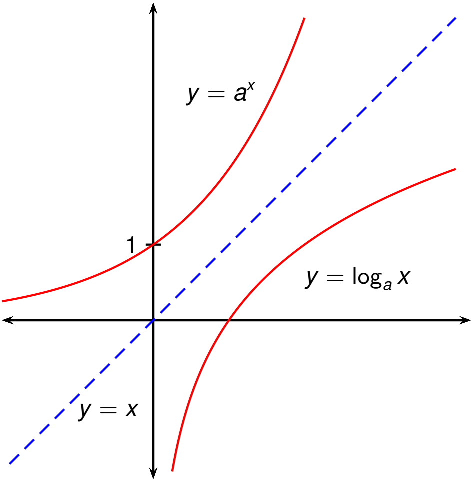

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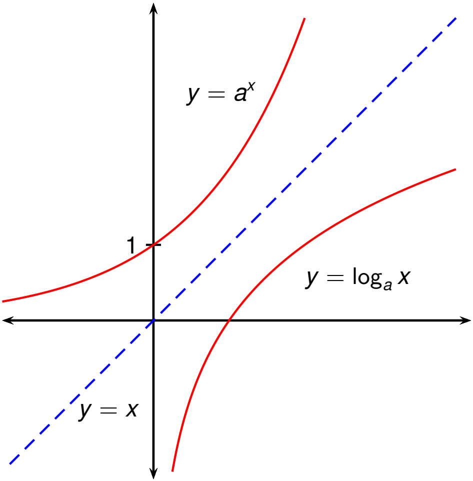

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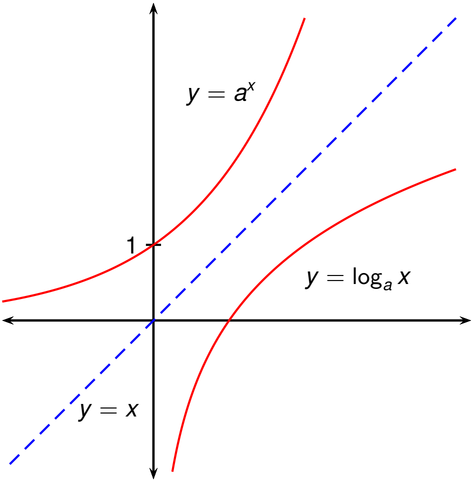

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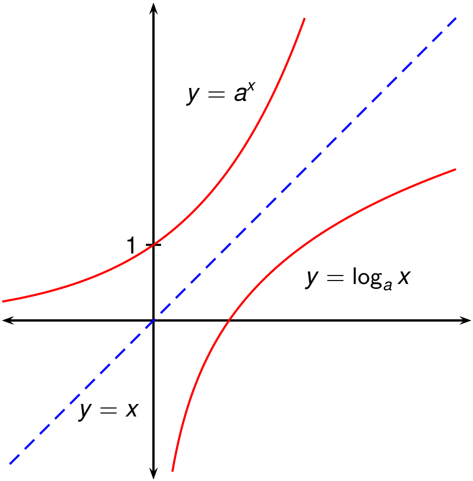

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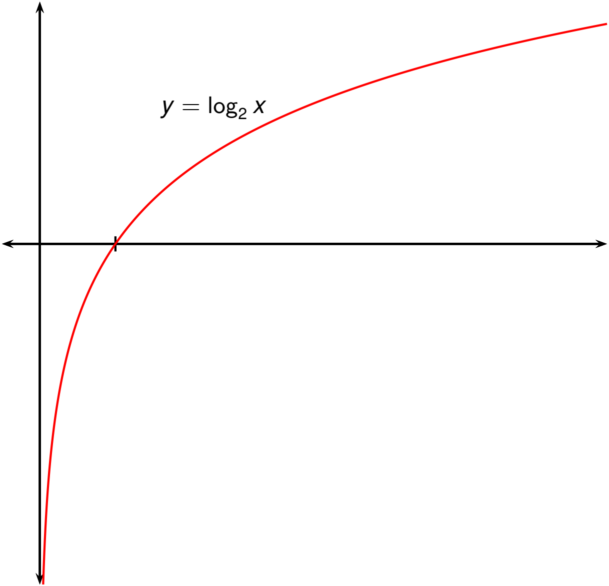
- $\log_a(a^x) = x$ for $x \in \mathbb{R}$.
- $a^{\log_a x} = x$ for $x > 0$.

Graphs of various logarithmic functions with $a > 1$



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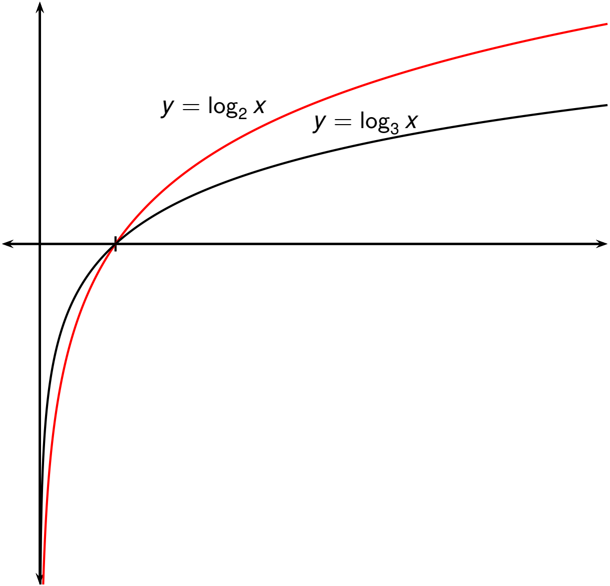

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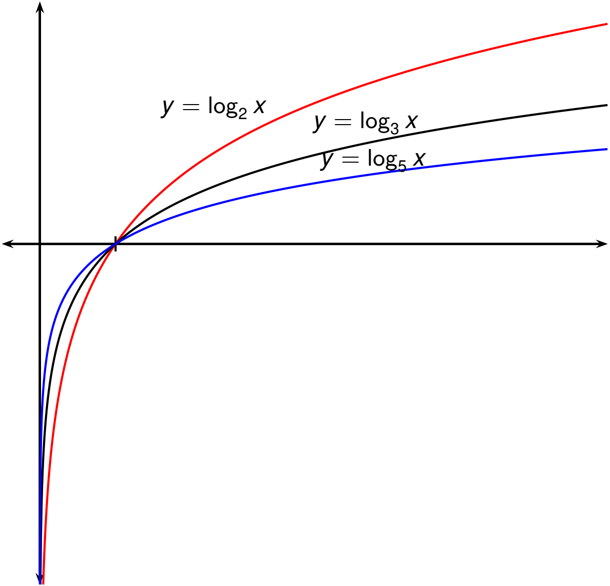

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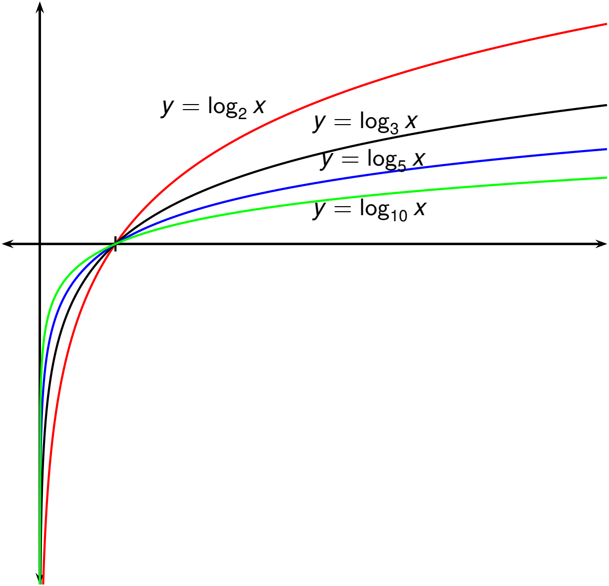

Theorem (Properties of Logarithmic Functions)

If $a > 1$, the function $f(x) = \log_a x$ is a one-to-one, continuous, increasing function with domain $(0, \infty)$ and range $\mathbb{R}$. If $x, y, a, b > 0$ and r is any real number, then

1. $\log_a(xy) = \log_a x + \log_a y$.
2. $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$.
3. $\log_a(x^r) = r \log_a x$.
4. $\log_a(x) = \log_b x \log_a b = \frac{\log_b x}{\log_b a} = \frac{\ln x}{\ln a}$.

Use the properties of logarithms to evaluate the following.

Example

$$\log_4 2 + \log_4 32$$

Example

$$\log_2 80 - \log_2 5$$

Use the properties of logarithms to evaluate the following.

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$$\log_4 2 + \log_4 32 \quad = \quad \log_4(2 \cdot 32)$$

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Use the properties of logarithms to evaluate the following.

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$$\begin{aligned}\log_4 2 + \log_4 32 &= \log_4(2 \cdot 32) \\ &= \log_4(64)\end{aligned}$$

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Use the properties of logarithms to evaluate the following.

Example

$$\begin{aligned} \log_4 2 + \log_4 32 &= \log_4(2 \cdot 32) \\ &= \log_4(64) \\ &= 3 \\ &\quad \text{(because } 4^3 = 64.\text{)} \end{aligned}$$

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Use the properties of logarithms to evaluate the following.

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$$\begin{aligned} \log_2 80 - \log_2 5 &= \log_2\left(\frac{80}{5}\right) \\ &= \log_2(16) \end{aligned}$$

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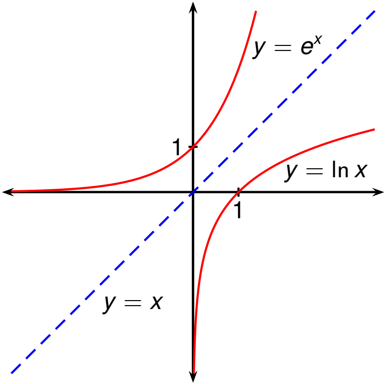
Natural Logarithms

Definition ($\ln x$)

The logarithm with base e is called the natural logarithm, and has a special notation:

$$\log_e x = \ln x.$$



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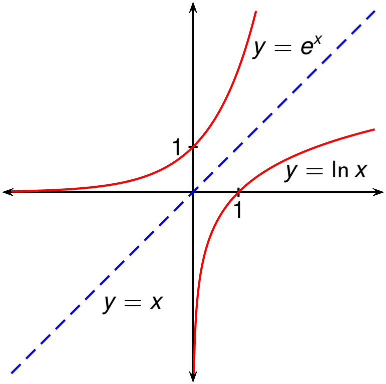

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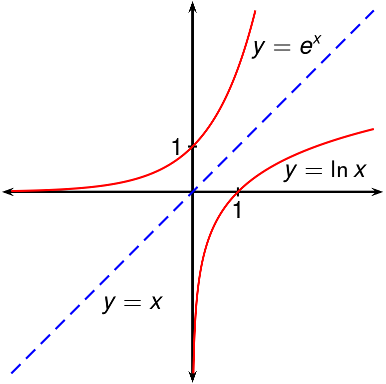

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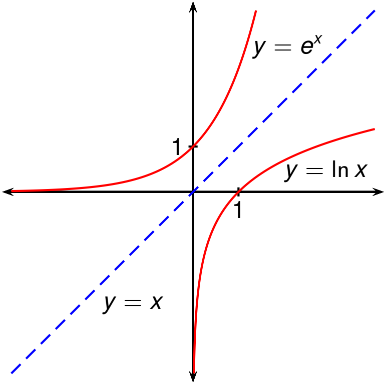

- $\ln x = y \quad \Leftrightarrow \quad e^y = x$.
- $\ln(e^x) = x$ for $x \in \mathbb{R}$.
- $e^{\ln x} = x$ for $x > 0$.

Summary of logarithm notation conventions

	Name	ISO notation	Other notation	Used in
$\log_2(x)$	binary logarithm	$\mathrm{lb}(x)$	$\mathrm{ld}(x)$, $\log(x)$, $\lg(x)$	computer science, information theory, music theory, photography
$\log_e(x)$	natural logarithm	$\ln(x)$	$\log(x)$	mathematics, physics, chemistry, statistics, economics, information theory, and engineering
$\log_{10}(x)$	common logarithm	$\lg(x)$	$\log(x)$	various engineering, logarithm tables, handheld calculators, spectroscopy

Table source: Wikipedia

- Standardized in ISO_31-11 (International Standards Organization).

Example

Solve the equation.

$$e^{5-3x} = 10$$

Example

Solve the equation.

$$\begin{aligned} e^{5-3x} &= 10 && \text{apply } \ln \\ \ln(e^{5-3x}) &= \ln 10 \end{aligned}$$

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Solve the equation.

$$\begin{aligned} e^{5-3x} &= 10 & \text{apply } \ln \\ \ln(e^{5-3x}) &= \ln 10 \\ 5 - 3x &= \ln 10 \\ 3x &= 5 - \ln 10 \end{aligned}$$

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Solve the equation.

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e^{5-3x} &= 10 && \text{apply } \ln \\
\ln(e^{5-3x}) &= \ln 10 \\
5-3x &= \ln 10 \\
3x &= 5-\ln 10 \\
x &= \frac{5-\ln 10}{3}
\end{aligned}$$

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e^{5-3x} &= 10 && \text{apply } \ln\\
\ln(e^{5-3x}) &= \ln 10\\
5-3x &= \ln 10\\
3x &= 5-\ln 10\\
x &= \frac{5-\ln 10}{3}\\
\text{Calculator:}\quad x &\approx 0.8991.
\end{aligned}$$

Example

Solve the equation

$$e^{2x} - 3e^{x} - 4 = 0$$

Example

Solve the equation

$$e^{2x} - 3e^{x} - 4 = 0$$

Set $e^{x} = u$.

Example

Solve the equation

$$e^{2x} - 3e^x - 4 = 0$$

Set $e^x = u$. Then $e^{2x} = \textbf{?}$.

Example

Solve the equation

$$e^{2x} - 3e^x - 4 = 0$$

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$$(\mathbf{?}\qquad)(\mathbf{?}\qquad) = 0$$

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$$u = 4 \qquad \text{or} \qquad u = -1$$

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$$e^x = 4 \quad \text{or} \quad e^x = -1$$

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Example

Solve the equation

$$4^{x+1} - 2^{x+2} - 3 = 0$$

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Example

Solve the equation

$$4^{x+1} - 2^{x+2} - 3 = 0$$

Set $u = 2^x$. Then $4^{x+1} = 4u^2$, $2^{x+2} = \textbf{?}$.

Example

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Set $u = 2^x$. Then $4^{x+1} = 4u^2$, $2^{x+2} = 4u$.

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$$\begin{aligned} 4u^2 - 4u - 3 &= 0 \\ (2u-3)(2u+1) &= 0 \end{aligned}$$

$$2u - 3 = 0 \quad \text{or} \quad 2u + 1 = 0$$

$$u = \frac{3}{2} \quad \text{or} \quad u = -\frac{1}{\ }$$

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$$x = \log_2\left(\frac{3}{2}\right)$$

Example

Solve the equation

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$$\begin{aligned} 4u^2 - 4u - 3 &= 0 \\ (2u-3)(2u+1) &= 0 \end{aligned}$$

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$x = \log_2\left(\frac{3}{2}\right)$ or no real solution

Example

Solve the equation

$$4^{x+1} - 2^{x+2} - 3 = 0$$

Set $u = 2^x$. Then $4^{x+1} = 4u^2$, $2^{x+2} = 4u$.

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$$2u - 3 = 0 \quad \text{or} \quad 2u + 1 = 0$$

$$u = \frac{3}{2} \quad \text{or} \quad u = -\frac{1}{2}$$

$$2^x = \frac{3}{2} \quad \text{or} \quad 2^x = -\frac{1}{2}$$

$$x = \log_2\left(\frac{3}{2}\right) = \frac{\ln(\mathbf{?})}{\ln \mathbf{?}} \quad \text{or} \quad \text{no real solution}$$

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$$4^{x+1} - 2^{x+2} - 3 = 0$$

Set $u = 2^x$. Then $4^{x+1} = 4u^2$, $2^{x+2} = 4u$.

$$\begin{aligned} 4u^2 - 4u - 3 &= 0 \\ (2u-3)(2u+1) &= 0 \\ 2u - 3 = 0 \quad &\text{or} \quad 2u + 1 = 0 \\ u = \frac{3}{2} \quad &\text{or} \quad u = -\frac{1}{2} \\ 2^x = \frac{3}{2} \quad &\text{or} \quad 2^x = -\frac{1}{2} \\ x = \log_2\left(\frac{3}{2}\right) = \frac{\ln\left(\frac{3}{2}\right)}{\ln 2} \quad &\text{or} \quad \text{no real solution} \end{aligned}$$

Example

Solve the equation

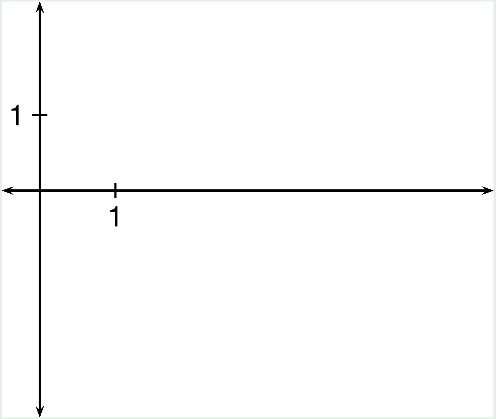
$$4^{x+1} - 2^{x+2} - 3 = 0$$

Set $u = 2^x$. Then $4^{x+1} = 4u^2, 2^{x+2} = 4u$.

$$\begin{aligned} 4u^2 - 4u - 3 &= 0 \\ (2u-3)(2u+1) &= 0 \\ 2u - 3 = 0 \quad &\text{or} \quad 2u + 1 = 0 \\ u = \frac{3}{2} \quad &\text{or} \quad u = -\frac{1}{2} \\ 2^x = \frac{3}{2} \quad &\text{or} \quad 2^x = -\frac{1}{2} \\ x = \log_2\left(\frac{3}{2}\right) = \frac{\ln\left(\frac{3}{2}\right)}{\ln 2} \approx 0.58496 \quad &\text{or} \quad \text{no real solution} \end{aligned}$$

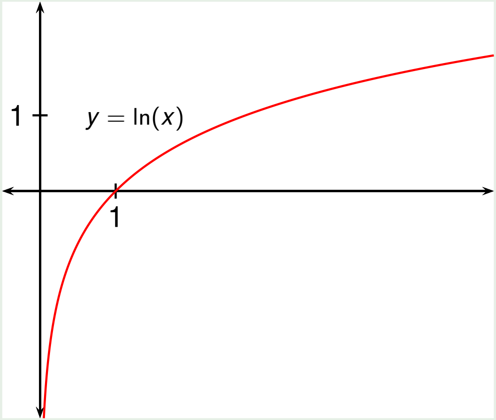
Example

Draw the graph of $y = \ln(x - 2) - 1$.



Example

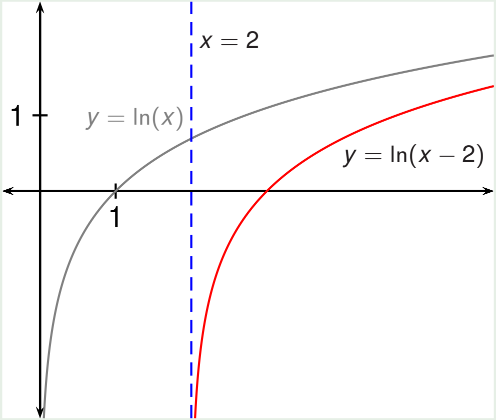
Draw the graph of $y = \ln(x-2) - 1$.



- Graph $y = \ln(x)$ assumed known.

Example

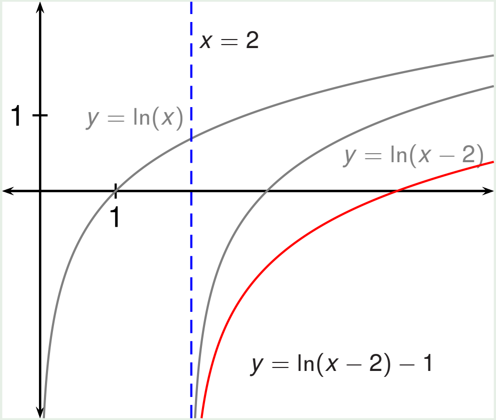
Draw the graph of $y = \ln(x-2) - 1$.



- Graph $y = \ln(x)$ assumed known.
- $f(x-2)$ shifts graph 2 units to the right.

Example

Draw the graph of $y = \ln(x - 2) - 1$.

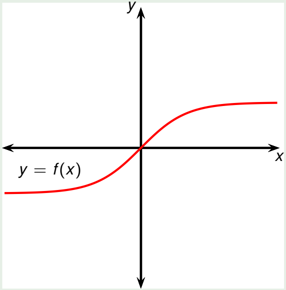


- Graph $y = \ln(x)$ assumed known.
- $f(x - 2)$ shifts graph 2 units to the right.
- $g(x) - 1$ shifts graph 1 unit down.

Example

Find $f^{-1}(x)$ for

$$f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$



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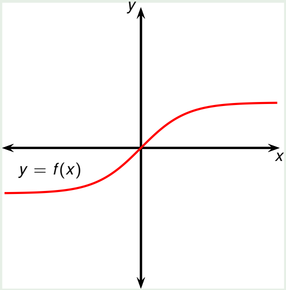


$$\frac{e^x - e^{-x}}{e^x + e^{-x}} = y$$

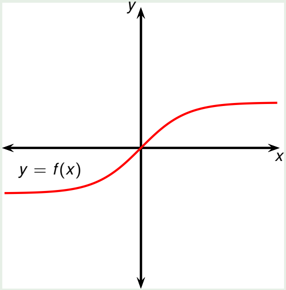
Example

Find $f^{-1}(x)$ for $f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$.

$$\frac{e^x - e^{-x}}{e^x + e^{-x}} = y$$

$$\frac{\left(u - \textbf{?}\right)}{\left(u + \textbf{?}\right)} = y$$

Set $u = e^x$



Example

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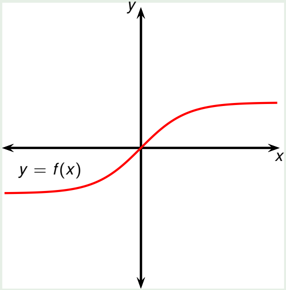
$$f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

$$\frac{e^x - e^{-x}}{e^x + e^{-x}} = y$$
$$\frac{(u - \textbf{?})}{(u + \textbf{?})} = y$$

Set $u = e^x$

$e^{-x} = \textbf{?}$



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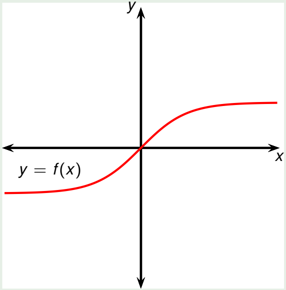
Set $u = e^x$

$$e^{-x} = \frac{1}{e^x} = \frac{1}{u}$$



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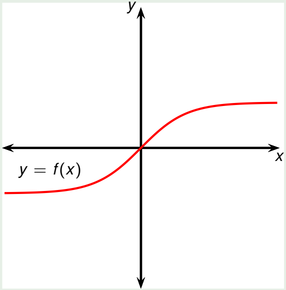

$$\frac{e^x - e^{-x}}{e^x + e^{-x}} = y$$

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Example

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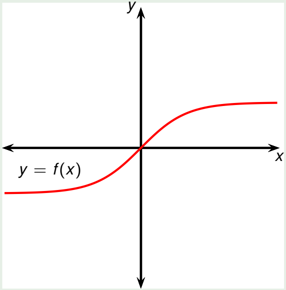
$f(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$.



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$$\frac{u^2 - 1}{u^2 + 1} = y$$

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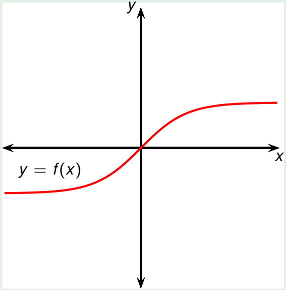

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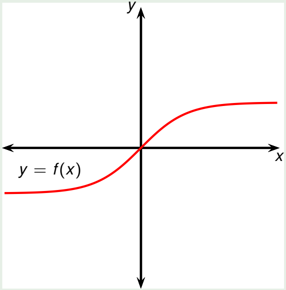

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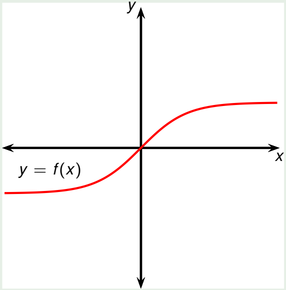

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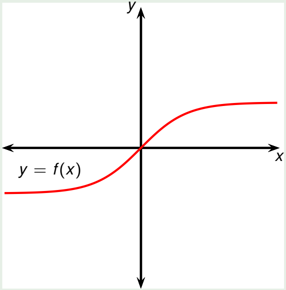

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Example

Find $f^{-1}(x)$ for
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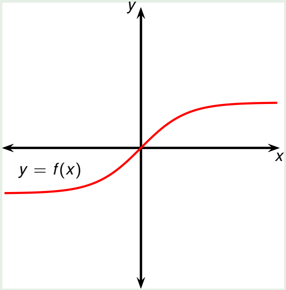

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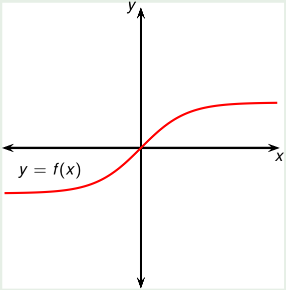
$$u^2 = \frac{1 + y}{1 - y}$$
$$(e^x)^2 = \frac{1 + y}{1 - y}$$

Set $u = e^x$
$e^{-x} = \frac{1}{e^x} = \frac{1}{u}$

Example

Find $f^{-1}(x)$ for
$$f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$



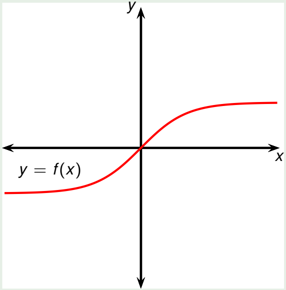
$$\begin{aligned}
\frac{e^x - e^{-x}}{e^x + e^{-x}} &= y \\
\frac{\left(u - \frac{1}{u}\right) u}{\left(u + \frac{1}{u}\right) u} &= y \\
\frac{u^2 - 1}{u^2 + 1} &= y \\
u^2 - 1 &= y(u^2 + 1) \\
u^2(1 - y) &= 1 + y \\
u^2 &= \frac{1 + y}{1 - y} \\
\left(e^{x}\right)^{2} &= \frac{1 + y}{1 - y} \\
e^{2x} &= \frac{1 + y}{1 - y}
\end{aligned}$$

Set $u = e^x$
$e^{-x} = \frac{1}{e^x} = \frac{1}{u}$

Example

Find $f^{-1}(x)$ for

$$f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$



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\frac{e^x - e^{-x}}{e^x + e^{-x}} &= y \\
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u^2(1 - y) &= 1 + y \\
u^2 &= \frac{1 + y}{1 - y} \\
(e^x)^2 &= \frac{1 + y}{1 - y} \\
e^{2x} &= \frac{1 + y}{1 - y} \\
2x &= \ln\left(\frac{1 + y}{1 - y}\right)
\end{aligned}$$

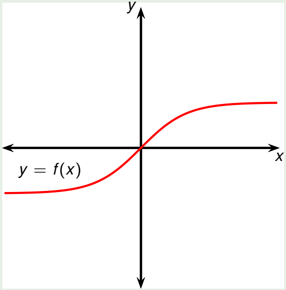
Set $u = e^x$

$e^{-x} = \frac{1}{e^x} = \frac{1}{u}$

Take ln

Example

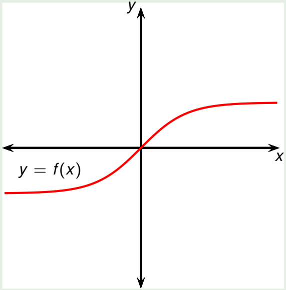
Find $f^{-1}(x)$ for $f(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$.



$$
\begin{aligned}
\frac{e^x - e^{-x}}{e^x + e^{-x}} &= y && \text{Set } u = e^x,\quad e^{-x} = \frac{1}{e^x} = \frac{1}{u}\\
\frac{\left(u - \frac{1}{u}\right)u}{\left(u + \frac{1}{u}\right)u} &= y\\
\frac{u^2 - 1}{u^2 + 1} &= y\\
u^2 - 1 &= y(u^2 + 1)\\
u^2(1 - y) &= 1 + y\\
u^2 &= \frac{1 + y}{1 - y}\\
(e^x)^2 &= \frac{1 + y}{1 - y}\\
e^{2x} &= \frac{1 + y}{1 - y} && \text{Take } \ln\\
{\color{red}2}x &= \ln\left(\frac{1 + y}{1 - y}\right)
\end{aligned}
$$

Example

Find $f^{-1}(x)$ for
$$f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$



answer
$$f^{-1}(y) = \frac{1}{2}\ln\left(\frac{1+y}{1-y}\right)$$

$$\frac{e^x - e^{-x}}{e^x + e^{-x}} = y$$

Set $u = e^x$
$e^{-x} = \frac{1}{e^x} = \frac{1}{u}$

$$\begin{aligned}
\frac{\left(u - \frac{1}{u}\right)u}{\left(u + \frac{1}{u}\right)u} &= y\\
\frac{u^2 - 1}{u^2 + 1} &= y\\
u^2 - 1 &= y(u^2 + 1)\\
u^2(1 - y) &= 1 + y\\
u^2 &= \frac{1+y}{1-y}\\
(e^x)^2 &= \frac{1+y}{1-y}\\
e^{2x} &= \frac{1+y}{1-y}
\end{aligned}$$

Take ln

$$x = \frac{1}{2}\ln\left(\frac{1+y}{1-y}\right)$$

Example

Find $f^{-1}(x)$ for
$f(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$.

$$\frac{e^x - e^{-x}}{e^x + e^{-x}} = y$$

Set $u = e^x$
$e^{-x} = \frac{1}{e^x} = \frac{1}{u}$

$$\frac{\left(u - \frac{1}{u}\right)u}{\left(u + \frac{1}{u}\right)u} = y$$

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$$u^2 - 1 = y(u^2 + 1)$$

$$u^2(1 - y) = 1 + y$$

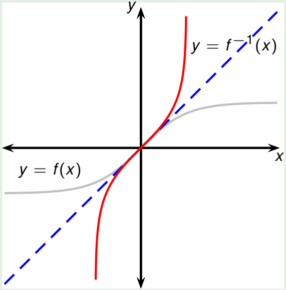
$$u^2 = \frac{1 + y}{1 - y}$$

$$(e^x)^2 = \frac{1 + y}{1 - y}$$

$$e^{2x} = \frac{1 + y}{1 - y}$$

Take ln

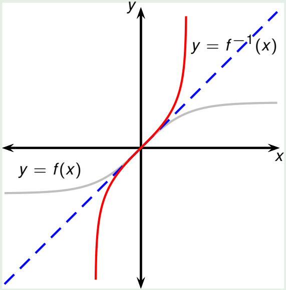
$$x = \frac{1}{2}\ln\left(\frac{1 + y}{1 - y}\right)$$



Final answer, relabeled:
$$f^{-1}(x) = \frac{1}{2}\ln\left(\frac{1 + x}{1 - x}\right)$$

Example

Find $f^{-1}(x)$ for
$$f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$
$f = \tanh$ = hyperbolic tangent function.



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$$f^{-1}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

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Set $u = e^x$
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Take ln

$$x = \frac{1}{2}\ln\left(\frac{1+y}{1-y}\right)$$