Precalculus
Simplify $\sin(k \arcsin x)$, $\cos(k \arcsin x)$, $\sin(k \arccos x)$, $\cos(k \arccos x)$

Todor Milev

2019

Example

Rewrite $\sin(2\arccos(x))$ as an algebraic expression of x and $\sqrt{1-x^2}$.

$$\sin(2\arccos(x))$$

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To simplify $\arccos x$ we try to use $\cos(\arccos x) = x$.

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Rewrite $\sin(2\arccos(x))$ as an algebraic expression of x and $\sqrt{1-x^2}$. To simplify $\arccos x$ we try to use $\cos(\arccos x) = x$. Therefore our aim is to rewrite the expression only using the $\cos$ function.

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$$\sin(2\arccos(x)) = \sin(2y) \qquad \Big| \text{ Set } y = \arccos x$$

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$$\begin{aligned} \sin(2\arccos(x)) &= \sin(2y) \\ &= \textbf{?} \end{aligned}$$

Set $y = \arccos x$
Express via $\sin y, \cos y$

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$$\begin{aligned} \sin(2\arccos(x)) &= \sin(2y) \\ &= 2\cos y \sin y \end{aligned}$$

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$$\begin{aligned} \sin(2\arccos(x)) &= \sin(2y) && \text{Set } y = \arccos x \\ &= 2\cos y \sin y && \text{Express via } \sin y, \cos y \\ &= 2\cos y\left(\pm\sqrt{1-\cos^2 y}\right) && \text{Express } \sin y \text{ via } \cos y \end{aligned}$$

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Rewrite $\sin(2\arccos(x))$ as an algebraic expression of x and $\sqrt{1-x^2}$. To simplify $\arccos x$ we try to use $\cos(\arccos x) = x$. Therefore our aim is to rewrite the expression only using the $\cos$ function.

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\sin(2\arccos(x)) &= \sin(2y) && \text{Set } y = \arccos x \\
&= 2\cos y \sin y && \text{Express via } \sin y, \cos y \\
&= 2\cos y \left(\pm\sqrt{1-\cos^2 y}\right) && \text{Express } \sin y \text{ via } \cos y \\
&= 2\cos y \sqrt{1-\cos^2 y} && \sin y > 0 \text{ because} \\
& && 0 \leq y \leq \pi
\end{aligned}
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$\sin(2\arccos(x)) = \sin(2y)$	Set $y = \arccos x$
$= 2\cos y \sin y$	Express via $\sin y, \cos y$
$= 2\cos y\left(\pm\sqrt{1-\cos^2 y}\right)$	Express $\sin y$ via $\cos y$
$= 2\cos y\sqrt{1-\cos^2 y}$	$\sin y > 0$ because $0 \leq y \leq \pi$

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$= 2x\sqrt{1-x^2}$	use $x = \cos y$

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$$\cos(3\arccos(x)) = \cos(3y) \qquad \Big| \; y = \arccos x$$

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$$\cos(3\arccos(x)) = \cos(3y) = \cos(2y + y) \qquad \left| \; y = \arccos x \right.$$

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Rewrite $\cos(3\arccos(x))$ as an algebraic expression of x and $\sqrt{1-x^2}$. To simplify $\arccos x$ we try to use $\cos(\arccos x) = x$. Therefore our aim is to rewrite the expression only using the $\cos$ function.

$$\cos(3\arccos(x)) = \cos(3y) = \cos(2y+y) \quad \Big| \quad y = \arccos x$$
$$= \mathbf{?} \quad \Big| \quad \text{Angle sum f-la}$$

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$$\begin{aligned} \cos(3\arccos(x)) &= \cos(3y) = \cos(2y+y) && \Big| \; y = \arccos x \\ &= \cos(2y)\cos y - \sin(2y)\sin y && \Big| \; \text{Angle sum f-la} \end{aligned}$$

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&= \cos(2y)\cos y - \sin(2y)\sin y && \Big| \; \text{Angle sum f-la} \\
&= (\mathbf{?} \qquad\qquad\qquad)\cos y && \Big| \; \text{Express via} \\
&\quad -\mathbf{?} \qquad\qquad\qquad \sin y && \Big| \; \sin y, \cos y
\end{aligned}
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= & (\cos^2 y {\color{red}- \sin^2 y}) {\color{red}\cos y} & \text{Express via} \\
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&= \cos^3 y - 3\sin^2 y\cos y && \Big|\ \text{Express } \sin y \text{ via } \cos y\\
&= \cos^3 y - 3(\mathbf{?}\qquad\qquad)\cos y
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&= \cos^3 y - 3\sin^2 y\cos y && \Big|\ \text{Express } \sin y \text{ via } \cos y\\
&= \cos^3 y - 3(1-\cos^2 y)\cos y
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&= \cos^3 y - \sin^2 y\cos y - 2\sin^2 y\cos y\\
&= \cos^3 y - 3\sin^2 y\cos y && \Big|\ \text{Express } \sin y \text{ via } \cos y\\
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&= 4\cos^3 y - 3\cos y
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&= \cos^3 y - 3(1-\cos^2 y)\cos y \\
&= 4\cos^3 y - 3\cos y \\
&= 4x^3 - 3x && \Big|\; x = \cos y
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&\quad -2\sin y\cos y\sin y & & \sin y, \cos y \\
&= \cos^3 y - \sin^2 y\cos y - 2\sin^2 y\cos y & & \\
&= \cos^3 y - 3\sin^2 y\cos y & & \text{Express } \sin y \text{ via } \cos y \\
&= \cos^3 y - 3(1-\cos^2 y)\cos y & & \\
&= 4\cos^3 y - 3\cos y & & \\
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