

## Calculus II

Integrals of the form  $\int \sqrt{ax^2 + c} dx, a, c > 0$

Todor Milev

2019

Euler substitution:  $x = \frac{1}{2} \left( \frac{1}{t} - t \right)$ ,  $\sqrt{x^2 + 1} = \frac{1}{2} \left( \frac{1}{t} + t \right)$ ,  
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$$\int \frac{1}{x^2 \sqrt{x^2 + 9}} dx$$

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$$\theta \in (0, \pi)$$

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 \int \frac{1}{x^2 \sqrt{x^2 + 9}} dx &= \int \frac{1}{x^2 3 \sqrt{\left(\frac{x}{3}\right)^2 + 1}} dx \\
 &= \int \frac{1}{(\textcolor{red}{3} \cot \theta)^2 \textcolor{red}{3} \sqrt{\cot^2 \theta + 1}} d(3 \cot \theta) \\
 &= \int \frac{1}{\textcolor{red}{27} \cot^2 \theta \sqrt{\textcolor{red}{?}}} \left( \textcolor{red}{?} \right) d\theta
 \end{aligned}$$

Set

$$\frac{x}{3} = \cot \theta$$

$$x = 3 \cot \theta$$

$$\theta \in (0, \pi)$$

## Example

$$\begin{aligned}
 \int \frac{1}{x^2 \sqrt{x^2 + 9}} dx &= \int \frac{1}{x^2 3 \sqrt{\left(\frac{x}{3}\right)^2 + 1}} dx \\
 &= \int \frac{1}{(3 \cot \theta)^2 3 \sqrt{\cot^2 \theta + 1}} d(3 \cot \theta) \\
 &= \int \frac{1}{27 \cot^2 \theta \sqrt{?}} \left( ? \right) d\theta
 \end{aligned}$$

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$$\begin{aligned}
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 &= \int \frac{1}{27 \cot^2 \theta \sqrt{\csc^2 \theta}} \left( ? \right) d\theta
 \end{aligned}$$

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## Example

$$\begin{aligned}
 \int \frac{1}{x^2 \sqrt{x^2 + 9}} dx &= \int \frac{1}{x^2 3 \sqrt{\left(\frac{x}{3}\right)^2 + 1}} dx \\
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 \int \frac{1}{x^2 \sqrt{x^2 + 9}} dx &= \int \frac{1}{x^2 3 \sqrt{\left(\frac{x}{3}\right)^2 + 1}} dx \\
 &= \int \frac{1}{(3 \cot \theta)^2 3 \sqrt{\cot^2 \theta + 1}} d(3 \cot \theta) \\
 &= \int \frac{1}{27 \cot^2 \theta \sqrt{\csc^2 \theta}} (-3 \csc^2 \theta) d\theta
 \end{aligned}$$

Set

$$\frac{x}{3} = \cot \theta$$

$$x = 3 \cot \theta$$

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## Example

$$\begin{aligned}
 \int \frac{1}{x^2 \sqrt{x^2 + 9}} dx &= \int \frac{1}{x^2 3 \sqrt{\left(\frac{x}{3}\right)^2 + 1}} dx \\
 &= \int \frac{1}{(3 \cot \theta)^2 3 \sqrt{\cot^2 \theta + 1}} d(3 \cot \theta) \\
 &= \int \frac{1}{\cancel{27} \cot^2 \theta \sqrt{\csc^2 \theta}} (-\cancel{3} \csc^2 \theta) d\theta \\
 &= \frac{\cancel{1}}{\cancel{9}} \int \frac{-\csc^2 \theta}{\cot^2 \theta \csc \theta} d\theta
 \end{aligned}$$

Set

$$\begin{aligned}
 \frac{x}{3} &= \cot \theta \\
 x &= 3 \cot \theta \\
 \theta &\in (0, \pi)
 \end{aligned}$$

## Example

$$\begin{aligned}
 \int \frac{1}{x^2 \sqrt{x^2 + 9}} dx &= \int \frac{1}{x^2 3 \sqrt{\left(\frac{x}{3}\right)^2 + 1}} dx \\
 &= \int \frac{1}{(3 \cot \theta)^2 3 \sqrt{\cot^2 \theta + 1}} d(3 \cot \theta) \\
 &= \int \frac{1}{27 \cot^2 \theta \sqrt{\csc^2 \theta}} (-3 \csc^2 \theta) d\theta \\
 &= \frac{1}{9} \int \frac{-\csc^2 \theta}{\cot^2 \theta \csc \theta} d\theta
 \end{aligned}$$

Set

$$\frac{x}{3} = \cot \theta$$

$$x = 3 \cot \theta$$

$$\theta \in (0, \pi)$$

$$\theta \in (0, \pi) \Rightarrow$$

$$\csc \theta > 0$$

## Example

$$\begin{aligned}
 \int \frac{1}{x^2 \sqrt{x^2 + 9}} dx &= \int \frac{1}{x^2 3 \sqrt{\left(\frac{x}{3}\right)^2 + 1}} dx \\
 &= \int \frac{1}{(3 \cot \theta)^2 3 \sqrt{\cot^2 \theta + 1}} d(3 \cot \theta) \\
 &= \int \frac{1}{27 \cot^2 \theta \sqrt{\csc^2 \theta}} (-3 \csc^2 \theta) d\theta \\
 &= \frac{1}{9} \int \frac{-\csc^2 \theta}{\cot^2 \theta \csc \theta} d\theta \\
 &= \frac{1}{9} \int \frac{-\sin \theta}{\cos^2 \theta} d\theta
 \end{aligned}$$

Set

$$\frac{x}{3} = \cot \theta$$

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 \int \frac{1}{x^2 \sqrt{x^2 + 9}} dx &= \int \frac{1}{x^2 3 \sqrt{\left(\frac{x}{3}\right)^2 + 1}} dx \\
 &= \int \frac{1}{(3 \cot \theta)^2 3 \sqrt{\cot^2 \theta + 1}} d(3 \cot \theta) \\
 &= \int \frac{1}{27 \cot^2 \theta \sqrt{\csc^2 \theta}} (-3 \csc^2 \theta) d\theta \\
 &= \frac{1}{9} \int \frac{-\csc^2 \theta}{\cot^2 \theta \csc \theta} d\theta \\
 &= \frac{1}{9} \int \frac{-\sin \theta}{\cos^2 \theta} d\theta = \frac{1}{9} \int \frac{1}{\cos^2 \theta} d(?)
 \end{aligned}$$

Set

$$\frac{x}{3} = \cot \theta$$

$$x = 3 \cot \theta$$

$$\theta \in (0, \pi)$$

$$\theta \in (0, \pi) \Rightarrow \csc \theta > 0$$

## Example

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 \int \frac{1}{x^2 \sqrt{x^2 + 9}} dx &= \int \frac{1}{x^2 3 \sqrt{\left(\frac{x}{3}\right)^2 + 1}} dx \\
 &= \int \frac{1}{(3 \cot \theta)^2 3 \sqrt{\cot^2 \theta + 1}} d(3 \cot \theta) \\
 &= \int \frac{1}{27 \cot^2 \theta \sqrt{\csc^2 \theta}} (-3 \csc^2 \theta) d\theta \\
 &= \frac{1}{9} \int \frac{-\csc^2 \theta}{\cot^2 \theta \csc \theta} d\theta \\
 &= \frac{1}{9} \int \frac{-\sin \theta}{\cos^2 \theta} d\theta = \frac{1}{9} \int \frac{1}{\cos^2 \theta} d(\cos \theta)
 \end{aligned}$$

Set

$$\frac{x}{3} = \cot \theta$$

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## Example

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 \int \frac{1}{x^2 \sqrt{x^2 + 9}} dx &= \int \frac{1}{x^2 3 \sqrt{\left(\frac{x}{3}\right)^2 + 1}} dx \\
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 &= \int \frac{1}{27 \cot^2 \theta \sqrt{\csc^2 \theta}} (-3 \csc^2 \theta) d\theta \\
 &= \frac{1}{9} \int \frac{-\csc^2 \theta}{\cot^2 \theta \csc \theta} d\theta \\
 &= \frac{1}{9} \int \frac{-\sin \theta}{\cos^2 \theta} d\theta = \frac{1}{9} \int \frac{1}{\cos^2 \theta} d(\cos \theta) \\
 &= \frac{1}{9} \int \frac{du}{u^2}
 \end{aligned}$$

Set

$\frac{x}{3} = \cot \theta$   
 $x = 3 \cot \theta$   
 $\theta \in (0, \pi)$   
 $\theta \in (0, \pi) \Rightarrow \csc \theta > 0$

Set  $u = \cos \theta$



## Example

$$\begin{aligned}
 \int \frac{1}{x^2 \sqrt{x^2 + 9}} dx &= \int \frac{1}{x^2 3 \sqrt{\left(\frac{x}{3}\right)^2 + 1}} dx \\
 &= \int \frac{1}{(3 \cot \theta)^2 3 \sqrt{\cot^2 \theta + 1}} d(3 \cot \theta) \\
 &= \int \frac{1}{27 \cot^2 \theta \sqrt{\csc^2 \theta}} (-3 \csc^2 \theta) d\theta \\
 &= \frac{1}{9} \int \frac{-\csc^2 \theta}{\cot^2 \theta \csc \theta} d\theta \\
 &= \frac{1}{9} \int \frac{-\sin \theta}{\cos^2 \theta} d\theta = \frac{1}{9} \int \frac{1}{\cos^2 \theta} d(\cos \theta) \\
 &= \frac{1}{9} \int \frac{du}{u^2} = ? + C
 \end{aligned}$$

Set

$$\frac{x}{3} = \cot \theta$$

$$x = 3 \cot \theta$$

$$\theta \in (0, \pi)$$

$$\theta \in (0, \pi) \Rightarrow \csc \theta > 0$$

$$\text{Set } u = \cos \theta$$

## Example

$$\begin{aligned}
 \int \frac{1}{x^2 \sqrt{x^2 + 9}} dx &= \int \frac{1}{x^2 3 \sqrt{\left(\frac{x}{3}\right)^2 + 1}} dx \\
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 &= \frac{1}{9} \int \frac{-\csc^2 \theta}{\cot^2 \theta \csc \theta} d\theta \\
 &= \frac{1}{9} \int \frac{-\sin \theta}{\cos^2 \theta} d\theta = \frac{1}{9} \int \frac{1}{\cos^2 \theta} d(\cos \theta) \\
 &= \frac{1}{9} \int \frac{du}{u^2} = -\frac{1}{9u} + C
 \end{aligned}$$

Set

 $\frac{x}{3} = \cot \theta$   
 $x = 3 \cot \theta$   
 $\theta \in (0, \pi)$   
 $\theta \in (0, \pi) \Rightarrow \csc \theta > 0$

Set  $u = \cos \theta$

## Example

$$\begin{aligned}
 \int \frac{1}{x^2 \sqrt{x^2 + 9}} dx &= \int \frac{1}{x^2 3 \sqrt{\left(\frac{x}{3}\right)^2 + 1}} dx \\
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 &= \int \frac{1}{27 \cot^2 \theta \sqrt{\csc^2 \theta}} (-3 \csc^2 \theta) d\theta \\
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 &= \frac{1}{9} \int \frac{-\sin \theta}{\cos^2 \theta} d\theta = \frac{1}{9} \int \frac{1}{\cos^2 \theta} d(\cos \theta) \\
 &= \frac{1}{9} \int \frac{du}{u^2} = -\frac{1}{9u} + C = -\frac{\sec \theta}{9} + C
 \end{aligned}$$

Set

$\frac{x}{3} = \cot \theta$   
 $x = 3 \cot \theta$   
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## Example

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 &= \frac{1}{9} \int \frac{-\csc^2 \theta}{\cot^2 \theta \csc \theta} d\theta \\
 &= \frac{1}{9} \int \frac{-\sin \theta}{\cos^2 \theta} d\theta = \frac{1}{9} \int \frac{1}{\cos^2 \theta} d(\cos \theta) \\
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 \end{aligned}$$

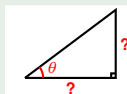
Set

$$\begin{aligned}
 \frac{x}{3} &= \cot \theta \\
 x &= 3 \cot \theta
 \end{aligned}$$

$$\theta \in (0, \pi)$$

$$\theta \in (0, \pi) \Rightarrow \csc \theta > 0$$

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## Example

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 \end{aligned}$$

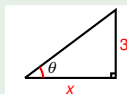
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## Example

$$\begin{aligned}
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 \end{aligned}$$

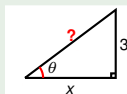
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 \end{aligned}$$

$$\theta \in (0, \pi)$$

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 \int \frac{1}{x^2 \sqrt{x^2 + 9}} dx &= \int \frac{1}{x^2 3 \sqrt{\left(\frac{x}{3}\right)^2 + 1}} dx \\
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 &= \int \frac{1}{27 \cot^2 \theta \sqrt{\csc^2 \theta}} (-3 \csc^2 \theta) d\theta \\
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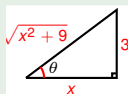
Set

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 x &= 3 \cot \theta
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 \int \frac{1}{x^2 \sqrt{x^2 + 9}} dx &= \int \frac{1}{x^2 3 \sqrt{\left(\frac{x}{3}\right)^2 + 1}} dx \\
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 &= \frac{1}{9} \int \frac{-\sin \theta}{\cos^2 \theta} d\theta = \frac{1}{9} \int \frac{1}{\cos^2 \theta} d(\cos \theta) \\
 &= \frac{1}{9} \int \frac{du}{u^2} = -\frac{1}{9u} + C = -\frac{\sec \theta}{9} + C \\
 &= -\frac{\sqrt{x^2 + 9}}{9x} + C
 \end{aligned}$$

Set

$$\begin{aligned}
 \frac{x}{3} &= \cot \theta \\
 x &= 3 \cot \theta
 \end{aligned}$$

$$\theta \in (0, \pi)$$

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