

Calculus II

L'Hospital's rule

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Outline

- 1 Indeterminate Forms and L'Hospital's Rule
 - Indeterminate Products
 - Indeterminate Differences
 - Indeterminate Powers

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Example

Find $\lim_{x \rightarrow 1} \frac{\ln x}{x-1}$.

- $\lim_{x \rightarrow 1} \ln x = 0$.
- $\lim_{x \rightarrow 1} (x - 1) = 0$.
- We don't get any cancellation between top and bottom.
- We need new techniques.

Theorem (L'Hospital's Rule)

Suppose that f and g are differentiable and $g'(x) \neq 0$ on an open interval that contains a (except possibly at a). Suppose that

$$\lim_{x \rightarrow a} f(x) = 0 \qquad \text{and} \qquad \lim_{x \rightarrow a} g(x) = 0$$

$$\text{or that} \qquad \lim_{x \rightarrow a} f(x) = \pm\infty \qquad \text{and} \qquad \lim_{x \rightarrow a} g(x) = \pm\infty$$

(In other words, we have an indeterminate form of type $0/0$ or ∞/∞ .)
Then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}.$$

Example

Find $\lim_{x \rightarrow 1} \frac{\ln x}{x-1}$.

- $\lim_{x \rightarrow 1} \ln x = 0$.
- $\lim_{x \rightarrow 1} (x - 1) = 0$.
- This is an indeterminate form of type $0/0$.
- Apply L'Hospital's rule:

$$\lim_{x \rightarrow 1} \frac{\ln x}{x-1} = \lim_{x \rightarrow 1} \frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(x-1)} = \lim_{x \rightarrow 1} \frac{1/x}{1} = \lim_{x \rightarrow 1} \frac{1}{x} = 1.$$

Example

Find $\lim_{x \rightarrow \infty} \frac{e^x}{x^2}$.

- $\lim_{x \rightarrow \infty} e^x = \infty$.
- $\lim_{x \rightarrow \infty} x^2 = \infty$.
- This is an indeterminate form of type ∞/∞ .
- Apply L'Hospital's rule:

$$\lim_{x \rightarrow \infty} \frac{e^x}{x^2} = \lim_{x \rightarrow \infty} \frac{\frac{d}{dx}(e^x)}{\frac{d}{dx}(x^2)} = \lim_{x \rightarrow \infty} \frac{e^x}{2x}$$

- $\lim_{x \rightarrow \infty} 2x = \infty$.
- This is an indeterminate form of type ∞/∞ .
- Apply L'Hospital's rule again:

$$\lim_{x \rightarrow \infty} \frac{e^x}{x^2} = \lim_{x \rightarrow \infty} \frac{e^x}{2x} = \lim_{x \rightarrow \infty} \frac{e^x}{2} = \infty.$$

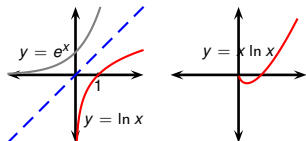
Indeterminate Products

If $\lim_{x \rightarrow a} f(x) = 0$ and $\lim_{x \rightarrow a} g(x) = \pm\infty$, then it isn't clear what $\lim_{x \rightarrow a} (fg)(x)$ will be.

In such a case, write the product fg as a quotient:

$$fg = \frac{f}{1/g} \quad \text{or} \quad fg = \frac{g}{1/f}.$$

This converts the given limit into an indeterminate form of type $0/0$ or ∞/∞ .



Example

Evaluate $\lim_{x \rightarrow 0^+} x \ln x$.

- $\lim_{x \rightarrow 0^+} \ln x = -\infty$.
- $\lim_{x \rightarrow 0^+} x = 0$.
- This is an indeterminate form of type $0(-\infty)$ (or $-\infty/(1/0)$).
- Apply L'Hospital's rule:

$$\begin{aligned}
 \lim_{x \rightarrow 0^+} x \ln x &= \lim_{x \rightarrow 0^+} \frac{\ln x}{\frac{1}{x}} = \lim_{x \rightarrow 0^+} \frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}\left(\frac{1}{x}\right)} \\
 &= \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x \rightarrow 0^+} (-x) = 0.
 \end{aligned}$$

Indeterminate Differences

If $\lim_{x \rightarrow a} f(x) = \infty$ and $\lim_{x \rightarrow a} g(x) = \infty$, then the limit

$$\lim_{x \rightarrow a} [f(x) - g(x)]$$

is called an indeterminate form of type $\infty - \infty$.

To compute such a limit, try to convert it into a quotient (by using a common denominator, or by rationalizing, or by factoring out a common factor).

Example

Evaluate $\lim_{x \rightarrow (\pi/2)^-} (\sec x - \tan x)$.

- $\lim_{x \rightarrow (\pi/2)^-} \sec x = \infty$.
- $\lim_{x \rightarrow (\pi/2)^-} \tan x = \infty$.
- This is an indeterminate form of type $\infty - \infty$.
- Apply L'Hospital's rule:

$$\begin{aligned}\lim_{x \rightarrow (\pi/2)^-} (\sec x - \tan x) &= \lim_{x \rightarrow (\pi/2)^-} \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x} \right) \\&= \lim_{x \rightarrow (\pi/2)^-} \frac{1 - \sin x}{\cos x} \\&\quad \text{(indeterminate form of type } 0/0.) \\&= \lim_{x \rightarrow (\pi/2)^-} \frac{\frac{d}{dx}(1 - \sin x)}{\frac{d}{dx} \cos x} \\&= \lim_{x \rightarrow (\pi/2)^-} \frac{-\cos x}{-\sin x} = 0\end{aligned}$$

Indeterminate Powers

Several indeterminate forms arise from the limit $\lim_{x \rightarrow a} f(x)^{g(x)}$.

$$\lim_{x \rightarrow a} f(x) = 0 \quad \text{and} \quad \lim_{x \rightarrow a} g(x) = 0 \quad \text{type } 0^0$$

$$\lim_{x \rightarrow a} f(x) = \infty \quad \text{and} \quad \lim_{x \rightarrow a} g(x) = 0 \quad \text{type } \infty^0$$

$$\lim_{x \rightarrow a} f(x) = 1 \quad \text{and} \quad \lim_{x \rightarrow a} g(x) = \pm\infty \quad \text{type } 1^\infty$$

These can all be solved either by taking the natural logarithm:

$$\text{let } y = [f(x)]^{g(x)}, \text{ then } \ln y = g(x) \ln f(x)$$

or by writing the function as an exponential:

$$[f(x)]^{g(x)} = e^{g(x) \ln f(x)}.$$

Example

Find $\lim_{x \rightarrow 0^+} x^x$.

- $0^x = 0$ for any $x > 0$.
- $x^0 = 1$ for any $x \neq 0$.
- This is an indeterminate form of type 0^0 .
- Write as an exponential:
- $x^x = e^{x \ln x}$.
- Recall that $\lim_{x \rightarrow 0^+} x \ln x = 0$.
- Therefore

$$\lim_{x \rightarrow 0^+} x^x = \lim_{x \rightarrow 0^+} e^{x \ln x} = e^0 = 1$$

Example

$$\begin{aligned}
 \lim_{x \rightarrow \infty} \left(1 + \frac{k}{x}\right)^x &= \lim_{x \rightarrow \infty} e^{\ln\left(1 + \frac{k}{x}\right)^x} && \text{exponent= continuous f-n} \\
 &= e^{\lim_{x \rightarrow \infty} \ln\left(1 + \frac{k}{x}\right)^x} = e^k && \text{limit computed below} \\
 \lim_{x \rightarrow \infty} \ln\left(1 + \frac{k}{x}\right)^x &= \lim_{x \rightarrow \infty} x \ln\left(1 + \frac{k}{x}\right) \\
 &= \lim_{x \rightarrow \infty} \frac{\frac{d}{dx} (\ln(1 + \frac{k}{x}))}{\frac{d}{dx} (\frac{1}{x})} && \text{form "0/0", use L'Hospital} \\
 &= \lim_{x \rightarrow \infty} \frac{\frac{1}{1 + \frac{k}{x}} \left(1 + \frac{k}{x}\right)'}{-\frac{1}{x^2}} \\
 &= \lim_{x \rightarrow \infty} \frac{\frac{1}{1 + \frac{k}{x}} \left(-\frac{k}{x^2}\right)}{-\frac{1}{x^2}} \\
 &= \lim_{x \rightarrow \infty} \frac{k}{1 + \frac{k}{x}} = k
 \end{aligned}$$