

## Calculus II

# Integrals involving radicals of quadratics, table of substitutions

Todor Milev

2019

- Let  $R$  be a rational function in two variables.
- So far, with linear transformations we converted all integrals of the form  $\int R(x, \sqrt{ax^2 + bx + c})dx$  to one of the three forms:  
 $\int R(x, \sqrt{x^2 + 1})dx$ ,  $\int R(x, \sqrt{-x^2 + 1})dx$ ,  $\int R(x, \sqrt{x^2 - 1})dx$ .
- Each of the above integrals can be transformed to a rational trigonometric integral using 3 pairs of substitutions:  
 $x = \tan \theta$ ,  $x = \cot \theta$ ;  $x = \sin \theta$ ,  $x = \cos \theta$ ;  $x = \csc \theta$ ,  $x = \sec \theta$ .
- We studied that trigonometric integrals are converted to rational function integrals via  $\theta = 2 \arctan t$ .
- The resulting 3 pairs of substitutions are called Euler substitutions:  
 $x = \tan(2 \arctan t)$ ,  $x = \cot(2 \arctan t)$ ;  $x = \sin(2 \arctan t)$ ,  
 $x = \cos(2 \arctan t)$ ;  $x = \csc(2 \arctan t)$ ,  $x = \sec(2 \arctan t)$ .
- The Euler substitutions directly transform the integral to a rational function integral.
- We will demonstrate that the Euler substitutions are rational.

# Trigonometric substitution and Euler substitution

Expression	Substitution	Variable range	Relevant identity
$\sqrt{x^2 + 1}$	$x = \tan \theta$	$\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	$1 + \tan^2 \theta = \sec^2 \theta$
	$x = \cot \theta$	$\theta \in (0, \pi)$	$1 + \cot^2 \theta = \csc^2 \theta$
$\sqrt{-x^2 + 1}$	$x = \sin \theta$	$\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	$1 - \sin^2 \theta = \cos^2 \theta$
	$x = \cos \theta$	$\theta \in (0, \pi)$	$1 - \cos^2 \theta = \sin^2 \theta$
$\sqrt{x^2 - 1}$	$x = \csc \theta$	$\theta \in \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$	$\csc^2 \theta - 1 = \cot^2 \theta$
	$x = \sec \theta$	$\theta \in \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$	$\sec^2 \theta - 1 = \tan^2 \theta$

Euler substitution by applying in addition  $\theta = 2 \arctan t$

$\sqrt{x^2 + 1}$	$x = \frac{2t}{1-t^2}$	$-1 < t < 1$	(?)
	$x = \frac{1}{2} \left( \frac{1}{t} - t \right)$	$0 < t$	(?)
$\sqrt{-x^2 + 1}$	$x = \frac{2t}{1+t^2}$	$-1 \leq t \leq 1$	(?)
	$x = \frac{1-t^2}{1+t^2}$	$0 < t$	(?)
$\sqrt{x^2 - 1}$	$x = \frac{1}{2} \left( \frac{1}{t} + t \right)$	$t \in (-\infty, -1) \cup [0, 1)$	(?)
	$x = \frac{1+t^2}{1-t^2}$	$t \in (-\infty, -1) \cup [0, 1)$	(?)