Calculus II
Homework
Building block integrals

1. Let $x \in (0,1)$. Express the following using x and $\sqrt{1-x^2}$.

   (a) $\sin(\arcsin(x))$.

   answer: x

   (b) $\sin(2\arcsin(x))$.

   answer: $2x\sqrt{1-x^2}$

   (c) $\sin(3\arcsin(x))$.

   answer: $-4x^3+3x$

   (d) $\sin(\arccos(x))$.

   answer: $\sqrt{1-x^2}$

   (e) $\sin(2\arccos(x))$.

   answer: $2x\sqrt{1-x^2}$

   (f) $\sin(3\arccos(x))$.

   answer: $(4x^2-1)\sqrt{1-x^2} = -4\left(\sqrt{1-x^2}\right)^3 + 3\sqrt{1-x^2}$

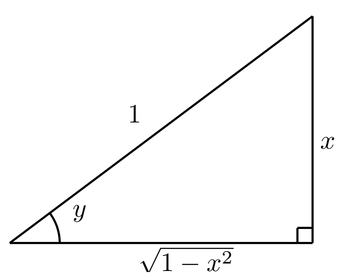
   (g) $\cos(2\arcsin(x))$.

   answer: $1-2x^2$

   (h) $\cos(3\arccos(x))$.

   answer: $4x^3-3x$

Solution. 1.b. Let $y = \arcsin x$. Then $\sin y = x$, and we can draw a right triangle with opposite side length x and hypotenuse length 1 to find the other trigonometric ratios of y.



Then $\cos y = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$. Now we use the double angle formula to find $\sin(2\arcsin x)$.

$$\begin{aligned}\sin(2\arcsin x) &= \sin(2y)\\ &= 2\sin y\cos y\\ &= 2x\sqrt{1-x^2}.\end{aligned}$$

Solution. 1.c. Use the result of Problem 1.b. This also requires the addition formula for sine:

$$\sin(A+B) = \sin A\cos B + \sin B\cos A,$$

and the double angle formula for cosine:

$$\cos(2y) = \cos^2 y - \sin^2 y.$$

$$
\begin{aligned}
\sin(3\arcsin x) &= \sin(3y) \\
&= \sin(2y+y) \\
&= \sin(2y)\cos y + \sin y\cos(2y) && \text{Use addition formula} \\
&= (2\sin y\cos y)\cos y + \sin y(\cos^2 y - \sin^2 y) && \text{Use double angle formulas} \\
&= 2\sin y\cos^2 y + \sin y\cos^2 y - \sin^3 y \\
&= 3\sin y\cos^2 y - \sin^3 y \\
&= 3\sin y(1-\sin^2 y) - \sin^3 y \\
&= 3x(1-x^2) - x^3 \\
&= 3x - 4x^3.
\end{aligned}
$$

The solution is complete. A careful look at the solution above reveals a strategy useful for problems similar to this one.

(a) Identify the inverse trigonometric expression- $\arcsin x, \arccos x, \arctan x, \dots$. In the present problem that was $y = \arcsin x$.

(b) The problem is therefore a trigonometric function of y.

(c) Using trig identities and algebra, rewrite the problem as a trigonometric expression involving only the trig function that transforms y to x. In the present problem we rewrote everything using $\sin y$.

(d) Use the fact that $\sin(\arcsin x) = x$, $\cos(\arccos x) = x, \dots$, etc. to simplify.

Solution. 1.f We use the same strategy outlined in the end of the solution of Problem 1.c. Set $y = \arccos x$ and so $\cos(y) = x$. Therefore:

$$
\begin{aligned}
\sin(3y) &= \sin(2y+y) \\
&= \sin(2y)\cos y + \sin y\cos(2y) \\
&= 2\sin y\cos y\cos y + \sin y(2\cos^2 y - 1) \\
&= 2\sin y\cos^2 y + \sin y(2\cos^2 y - 1) \\
&= \sin y(4\cos^2 y - 1) && \text{use } \begin{array}{rcl}\cos y &=& x \\ \sin y &=& \sqrt{1-x^2}\end{array} \\
&= \sqrt{1-x^2}(4x^2-1) \quad .
\end{aligned}
$$

2. Express as the following as an algebraic expression of x. In other words, "get rid" of the trigonometric and inverse trigonometric expressions.

(a) $\cos^2(\arctan x)$.

answer: $\frac{1}{\sqrt{1-x^2}}$

answer: $\frac{1}{1+x^2}$

(b) $-\sin^2(\operatorname{arccot} x)$.

(d) $-\dfrac{1}{\sin(\arccos x)}$.

answer: $-\frac{1}{1+x^2}$

answer: $-\frac{1}{\sqrt{1-x^2}}$

(c) $\dfrac{1}{\cos(\arcsin x)}$.

Solution. 2.b. We follow the strategy outlined in the end of the solution of Problem 1.c. We set $y = \operatorname{arccot} x$. Then we need to express $-\sin^2 y$ via $\cot y$. That is a matter of algebra:

$$
\begin{aligned}
-\sin^2(\operatorname{arccot} x) &= -\sin^2 y && \text{Set } y = \operatorname{arccot} x \\
&= -\frac{\sin^2 y}{\sin^2 y + \cos^2 y} && \text{use } \sin^2 y + \cos^2 y = 1 \\
&= -\frac{1}{\frac{\sin^2 y + \cos^2 y}{\sin^2 y}} \\
&= -\frac{1}{1+\cot^2 y} && \text{Substitute back } \cot y = x \\
&= -\frac{1}{1+x^2} \quad .
\end{aligned}
$$

3. Rewrite as a rational function of t. This problem will be later used to derive the Euler substitutions (an important technique for integrating).

(a) $\cos(2\arctan t)$.

answer: $\frac{1-t^2}{1+t^2}$.

(b) $\sin(2\arctan t)$.

answer: $\frac{2t}{1+t^2}$

(c) $\tan(2\arctan t)$.

answer: $\frac{2t}{1-t^2}$

(d) $\cot(2\arctan t)$.

answer: $\frac{1}{2}\left(\frac{1}{t}-t\right)$.

(e) $\csc(2\arctan t)$.

answer: $\frac{1}{2}\left(t+\frac{1}{t}\right)$

(f) $\sec(2\arctan t)$.

answer: $\frac{1+t^2}{1-t^2}$

(g) $\cos(2\operatorname{arccot} t)$.

answer: $\frac{t^2-1}{t^2+1}$

(h) $\sin(2\operatorname{arccot} t)$.

answer: $\frac{2t}{t^2+1}$

(i) $\tan(2\operatorname{arccot} t)$.

answer: $\frac{2t}{t^2-1}$

(j) $\cot(2\operatorname{arccot} t)$.

answer: $\frac{1}{2}\left(t-\frac{1}{t}\right)$

(k) $\csc(2\operatorname{arccot} t)$.

answer: $\frac{1}{2}\left(t+\frac{1}{t}\right)$

(l) $\sec(2\operatorname{arccot} t)$.

answer: $\frac{t^2+1}{t^2-1}$

Solution. 3.a Set $z = \arctan t$, and so $\tan z = t$. Then

$$\begin{array}{rcll}
\cos(2\arctan t) & = & \cos(2z) & \\
& = & \dfrac{\cos(2z)}{1} & \text{use double angle formulas} \\
& & & \text{and } 1 = \sin^2 z + \cos^2 z \\
& = & \dfrac{\cos^2 z - \sin^2 z}{\cos^2 z + \sin^2 z} & \text{divide top and bottom by } \cos^2 z \\
& = & \dfrac{(\cos^2 z - \sin^2 z)\frac{1}{\cos^2 z}}{(\sin^2 z + \cos^2 z)\frac{1}{\cos^2 z}} & \\
& = & \dfrac{1-\tan^2 z}{1+\tan^2 z} & \\
& = & \dfrac{1-t^2}{1+t^2} & .
\end{array}$$

Solution. 3.d Set $z = \arctan t$, and so $\tan z = t$. Then

$$\begin{array}{rcll}
\cot(2\arctan t) & = & \cot(2z) & \\
& = & \dfrac{\cos(2z)}{\sin(2z)} & \text{use double angle formulas} \\
& = & \dfrac{\cos^2 z - \sin^2 z}{2\sin z\cos z} & \\
& = & \dfrac{1-\tan^2 z}{2\tan z} & \\
& = & \dfrac{1-t^2}{2t} & \\
& = & \dfrac{1}{2}\left(\dfrac{1}{t}-t\right) & .
\end{array}$$

4. Compute the derivative (derive the formula).

(a) $(\arctan x)'$. answer: $\frac{1}{1+x^2}$

(b) $(\operatorname{arccot} x)'$. answer: $-\frac{1}{1+x^2}$

(c) $(\arcsin x)'$. answer: $\frac{1}{\sqrt{1-x^2}}$

(d) $(\arccos x)'$. answer: $-\frac{1}{\sqrt{1-x^2}}$

(e) Let arcsec denote the inverse of the secant function. Compute $(\operatorname{arcsec} x)'$. answer: $\frac{1}{x\sqrt{x^2-1}}$

5. (a) Let $a+b \neq k\pi$, $a \neq k\pi + \frac{\pi}{2}$ and $b \neq k\pi + \frac{\pi}{2}$ for any $k \in \mathbb{Z}$ (integers). Prove that

$$\frac{\tan a + \tan b}{1 - \tan a \tan b} = \tan(a+b) \quad .$$

(b) Let x and y be real. Prove that, for $xy \neq 1$, we have

$$\arctan x + \arctan y = \arctan\left(\frac{x+y}{1-xy}\right)$$

if the left hand side lies between $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Solution. 5.a We start by recalling the formulas

$$\begin{array}{rcl}\cos(a+b) &=& \cos a\cos b - \sin a \sin b\\ \sin(a+b) &=& \sin a \cos b + \sin b \cos a\end{array} \quad .$$

These formulas have been previously studied; alternatively they follow from Euler's formula and the computation

$$\begin{array}{rcl}\cos(a+b)+i\sin(a+b) &=& e^{i(a+b)} = e^{ia}e^{ib} = (\cos a + i\sin a)(\cos b + i \sin b)\\ &=& \cos a\cos b - \sin a\sin b + i(\sin a\cos b + \sin b\cos a)\end{array} \quad .$$

Now 5.a is done via a straightforward computation:

$$\begin{array}{rcl}\tan(a+b) &=& \dfrac{\sin(a+b)}{\cos(a+b)} = \dfrac{\sin a\cos b + \sin b\cos a}{\cos a\cos b - \sin a \sin b} = \dfrac{(\sin a\cos b + \sin b\cos a)\frac{1}{\cos a\cos b}}{(\cos a\cos b - \sin a\sin b)\frac{1}{\cos a \cos b}}\\ &=& \dfrac{\tan a + \tan b}{1 - \tan a\tan b} \quad .\end{array} \tag{1}$$

5.b is a consequence of 5.a. Let $a = \arctan x$, $b = \arctan y$. Then (**??**) becomes

$$\tan(\arctan x + \arctan y) = \frac{\tan(\arctan x) + \tan(\arctan y)}{1 - \tan(\arctan x)\tan(\arctan y)} = \frac{x+y}{1-xy} \quad ,$$

where we use the fact that $\tan(\arctan w) = w$ for all w. We recall that $\arctan(\tan z) = z$ whenever $z \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Now take $\arctan$ on both sides of the above equality to obtain

$$\arctan x + \arctan y = \arctan\left(\frac{x+y}{1-xy}\right) \quad .$$

6. Evaluate the indefinite integral. Illustrate the steps of your solutions.

(a) $\displaystyle\int x\sin x \mathrm{d}x$.

answer: $-x\cos x + \sin x + C$

(b) $\displaystyle\int x e^{-x}\mathrm{d}x$.

answer: $-(1+x)e^{-x} + C$

(c) $\displaystyle\int x^2 e^{x}\mathrm{d}x$.

answer: $x^2e^x - 2xe^x + 2e^x + C$

(d) $\displaystyle\int x\sin(-2x)\mathrm{d}x$.

answer: $\frac{x}{2}\cos(-2x) + \frac{1}{4}\sin(-2x) + C$

(e) $\displaystyle\int x^2\cos(3x)\mathrm{d}x$.

answer: $\frac{x^2}{3}\sin(3x) + \frac{2x}{9}\cos(3x) - \frac{2}{27}\sin(3x) + C$

(f) $\displaystyle\int x^2 e^{-2x}\mathrm{d}x$.

answer: $-\frac{x^2e^{-2x}}{2} - \frac{xe^{-2x}}{2} - \frac{e^{-2x}}{4} + C$

(g) $\displaystyle\int x\sin(2x)\mathrm{d}x$.

answer: $-\frac{x}{2}\cos(2x) + \frac{1}{4}\sin(2x) + C$

(h) $\displaystyle\int x\cos(3x)\mathrm{d}x$.

answer: $\frac{x}{3}\sin(3x) + \frac{1}{9}\cos(3x) + C$

(i) $\displaystyle\int x^2 e^{2x}\mathrm{d}x$.

answer: $\frac{x^2}{2}e^{2x} - \frac{x}{2}e^{2x} + \frac{e^{2x}}{4} + C$

(j) $\displaystyle\int x^3 e^{x}\mathrm{d}x$.

answer: $x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C$

Solution. 6.a.

$$\int x \underbrace{\sin x\mathrm{d}x}_{=\mathrm{d}(-\cos x)} = -\int x\mathrm{d}(\cos x) = -x\cos x + \int \cos x\mathrm{d}x = -x\cos x + \sin x + C \quad .$$

Solution. 6.c.

$$\begin{array}{rcl}\displaystyle\int x^2 \underbrace{e^x\mathrm{d}x}_{\mathrm{d}(e^x)} &=& \displaystyle\int x^2\mathrm{d}e^x = x^2e^x - \int e^x 2x\mathrm{d}x = x^2e^x - \int 2x\mathrm{d}e^x\\ &=& \displaystyle x^2e^x - 2xe^x + \int 2e^x\mathrm{d}x = x^2e^x - 2xe^x + 2e^x + C \quad .\end{array}$$

Solution. 6.f.

$$\begin{aligned}
\int x^2 e^{-2x}\mathrm{d}x &= \int x^2 \mathrm{d}\left(\frac{e^{-2x}}{-2}\right) && \text{Integrate by parts}\\
&= -\frac{x^2e^{-2x}}{2} - \int \left(\frac{e^{-2x}}{-2}\right)\mathrm{d}\left(x^2\right)\\
&= -\frac{x^2e^{-2x}}{2} + \int xe^{-2x}\mathrm{d}x\\
&= -\frac{x^2e^{-2x}}{2} + \int x\mathrm{d}\left(\frac{e^{-2x}}{-2}\right) && \text{Integrate by parts}\\
&= -\frac{x^2e^{-2x}}{2} - \frac{xe^{-2x}}{2} + \frac{1}{2}\int e^{-2x}\mathrm{d}x\\
&= -\frac{x^2e^{-2x}}{2} - \frac{xe^{-2x}}{2} - \frac{e^{-2x}}{4} + C \quad .
\end{aligned}$$

7. Evaluate the indefinite integral. Illustrate the steps of your solutions.

(a) $\int x^2\cos(2x)\mathrm{d}x$.

answer: $\frac{1}{2}x^2\sin(2x) + \frac{1}{2}x\cos(2x) - \frac{1}{4}\sin(2x) + C$

(b) $\int x^2e^{ax}\mathrm{d}x$, where a is a constant.

answer: $\frac{1}{a}x^2e^{ax} - \frac{2}{a^2}xe^{ax} + \frac{2}{a^3}e^{ax} + C$

(c) $\int x^2e^{-ax}\mathrm{d}x$, where a is a constant.

answer: $-\frac{1}{a}x^2e^{-ax} - \frac{2}{a^2}xe^{-ax} - \frac{2}{a^3}e^{-ax} + C$

(d) $\int x^2\frac{(e^{ax}+e^{-ax})^2}{4}\mathrm{d}x$, where a is a constant.

answer: $\frac{1}{8}\Big(a^{-1}x^2e^{2ax} - a^{-1}x^2e^{-2ax} - a^{-2}xe^{2ax} - a^{-2}xe^{-2ax} + \frac{1}{2}a^{-3}e^{2ax} - \frac{1}{2}a^{-3}e^{-2ax} + \frac{2}{3}x^3\Big) + C$

(e) $\int \frac{1}{\cos^2 x}\mathrm{d}x$. (Hint: This problem does not require integration by parts. What is the derivative of $\tan x$?)

answer: $\tan x + C$

(f) $\int(\tan^2 x)\mathrm{d}x$. (Hint: This problem does not require integration by parts. We can use $\tan^2 x = \frac{1}{\cos^2 x} - 1$ and the previous problem.)

(g) $\int x\tan^2 x\mathrm{d}x$. (Hint: $\tan^2 x\mathrm{d}x = \mathrm{d}(F(x))$, where $F(x)$ is the answer from the preceding problem).

answer: $-\frac{x^2}{2} + x\tan x + \ln|\cos x| + C$

(h) $\int e^{-\sqrt{x}}\mathrm{d}x$.

answer: $-2\sqrt{x}e^{-\sqrt{x}} - 2e^{-\sqrt{x}} + C$

(i) $\int \cos^2 x\ \mathrm{d}x$.

answer: $\frac{1}{4}\sin(2x) + \frac{x}{2} + C$

(j) $\int \frac{x}{1+x^2}\mathrm{d}x$ (Hint: use substitution rule, don't use integration by parts)

answer: $\frac{\ln\left(1+x^2\right)}{2} + C$

(k) $\int(\arctan x)\mathrm{d}x$.

answer: $x\arctan x - \frac{\ln\left(1+x^2\right)}{2} + C$

(l) $\int(\arcsin x)\mathrm{d}x$.

answer: $x\arcsin x + \sqrt{1-x^2} + C$

(m) $\int(\arcsin x)^2\mathrm{d}x$. (Hint: Try substituting $x = \sin y$.)

answer: $x(\arcsin x)^2 + 2\sqrt{1-x^2}\arcsin x - 2x + C$

(n) $\int \arctan\left(\frac{1}{x}\right)\mathrm{d}x$.

(o) $\int \sin xe^x\mathrm{d}x$

answer: $\frac{1}{2}\left(e^x\sin x - e^x\cos x\right) + C$

(p) $\int \cos xe^x\mathrm{d}x$

answer: $\frac{1}{2}\left(e^x\cos x + e^x\sin x\right) + C$

(q) $\int \sin(\ln(x))\mathrm{d}x$.

answer: $\frac{x}{2}\left(\sin(\ln x) - \cos(\ln x)\right) + C$

(r) $\int \cos(\ln(x))\mathrm{d}x$.

answer: $\frac{x}{2}\left(\cos(\ln x) + \sin(\ln x)\right) + C$

(s) $\int \ln x\mathrm{d}x$

answer: $x\ln|x| - x + C$

(t) $\int x\ln x\ \mathrm{d}x$.

answer: $\frac{1}{2}x^2\ln|x| - \frac{x^2}{4} + C$

(u) $\int \frac{\ln x}{\sqrt{x}}\mathrm{d}x$.

answer: $2\sqrt{x}(\ln x - 2) + C$

(v) $\int(\ln x)^2\mathrm{d}x$.

answer: $x(\ln x)^2 - 2x\ln x + 2x + C$

(w) $\int(\ln x)^3\mathrm{d}x$.

answer: $x(\ln x)^3 - 3x(\ln x)^2 + 6x\ln x - 6x + C$

(x) $\int x^2\cos^2 x\mathrm{d}x$. (This problem is related to Problem 7.d as $\cos x = \frac{e^{ix}+e^{-ix}}{2}$).

answer: $\frac{1}{4}x^2\sin\ (2x) + \frac{1}{4}x\cos\ (2x) - \frac{1}{8}\sin\ (2x) + \frac{1}{6}x^3$

Solution. 7.g.

$$
\begin{array}{rcll}
\displaystyle\int x\tan^2 x \mathrm{d}x &=& \displaystyle\int x\left(\sec^2 x - 1\right)\mathrm{d}x & \Big| \text{ use } \sec^2 x - 1 = \tan^2 x \\
&=& \displaystyle\int x\left(\sec^2 x - 1\right)\mathrm{d}x & \\
&=& \displaystyle -\int x\mathrm{d}x + \int x\sec^2 x\mathrm{d}x & \Big| \text{ use } \mathrm{d}(\tan x) = \sec^2 x\mathrm{d}x \\
&=& \displaystyle -\frac{x^2}{2} + \int x\mathrm{d}(\tan x) & \Big| \text{ integrate by parts} \\
&=& \displaystyle -\frac{x^2}{2} + x\tan x - \int \tan x\mathrm{d}x & \\
&=& \displaystyle -\frac{x^2}{2} + x\tan x - \int \frac{\sin x}{\cos x}\mathrm{d}x & \Big| \text{ use } \sin x\mathrm{d}x = -\mathrm{d}(\cos x) \\
&=& \displaystyle -\frac{x^2}{2} + x\tan x + \int \frac{\mathrm{d}(\cos x)}{\cos x} & \Big| \text{ Set } y = \cos x \\
&=& \displaystyle -\frac{x^2}{2} + x\tan x + \int \frac{1}{y}\mathrm{d}y & \\
&=& \displaystyle -\frac{x^2}{2} + x\tan x + \ln|y| + C & \Big| \text{ Substitute back } y = \cos x \\
&=& \displaystyle -\frac{x^2}{2} + x\tan x + \ln|\cos x| + C \quad . &
\end{array}
$$

Solution. 7.h.

$$
\begin{array}{rcll}
\displaystyle\int e^{-\sqrt{x}}\mathrm{d}x &=& \displaystyle\int 2ye^{-y}\mathrm{d}y & \Big| \text{ Subst.: } \begin{array}{rcl} \sqrt{x} &=& y \\ \frac{1}{2\sqrt{x}}\mathrm{d}x &=& \mathrm{d}y \\ \mathrm{d}x &=& 2y\mathrm{d}y \end{array} \\
&=& \displaystyle\int 2y\mathrm{d}\left(-e^{-y}\right) & \Big| \text{ int. by parts} \\
&=& -2ye^{-y} + 2\int e^{-y}\mathrm{d}y & \\
&=& -2ye^{-y} - 2e^{-y} + C & \\
&=& -2\sqrt{x}e^{-\sqrt{x}} - 2e^{-\sqrt{x}} + C \quad . &
\end{array}
$$

Solution. 7.i. Later, we shall study general methods for solving trigonometric integrals that will cover this example. Let us however show one way to solve this integral by integration by parts.

$$
\begin{array}{rll}
\displaystyle\int \cos^2 x\mathrm{d}x &= \displaystyle x\cos^2 x - \int x\mathrm{d}(\cos^2 x) & \\
&= \displaystyle x\cos^2 x - \int x 2\cos x(-\sin x)\mathrm{d}x & \Big| \sin(2x) = 2\sin x\cos x \\
&= \displaystyle x\cos^2 x + \int x\sin(2x)\mathrm{d}x & \\
&= \displaystyle x\cos^2 x + \int x\mathrm{d}\left(\frac{-\cos(2x)}{2}\right) & \\
&= \displaystyle x\cos^2 x + x\left(\frac{-\cos(2x)}{2}\right) - \int\left(\frac{-\cos(2x)}{2}\right)\mathrm{d}x & \\
&= \displaystyle \frac{x}{2}\left(2\cos^2 x - \cos(2x)\right) + \frac{\sin(2x)}{4} + C & \Big| \cos(2x) = \cos^2 x - \sin^2 x \\
&= \displaystyle \frac{x}{2}\left(2\cos^2 x - (\cos^2 x - \sin^2 x)\right) + \frac{\sin(2x)}{4} + C & \Big| \cos^2 x + \sin^2 x = 1 \\
&= \displaystyle \frac{x}{2} + \frac{\sin(2x)}{4} + C \quad . &
\end{array}
$$

Solution. 7.k

$$
\begin{array}{rcl}
\displaystyle\int \arctan x\mathrm{d}x &=& \displaystyle x\arctan x - \int x\mathrm{d}(\arctan x) \\
&=& \displaystyle x\arctan x - \int \frac{x}{x^2+1}\mathrm{d}x \\
&=& \displaystyle x\arctan x - \int \frac{\frac{1}{2}\mathrm{d}(x^2)}{x^2+1} \\
&=& \displaystyle x\arctan x - \int \frac{\frac{1}{2}\mathrm{d}(x^2+1)}{x^2+1} \\
&=& \displaystyle x\arctan x - \frac{1}{2}\ln(x^2+1) + C \quad .
\end{array}
$$

Solution. 7.m.

$$\begin{array}{rcll}
\int (\arcsin x)^2 \mathrm{d}x & = & \int (\arcsin(\sin y))^2 \, \mathrm{d}(\sin y) & \Big| \text{ Set } x = \sin y \\
& = & \int y^2 \cos y \mathrm{d}y = \int y^2 \mathrm{d}(\sin y) & \Big| \text{ Integrate by parts} \\
& = & y^2 \sin y - \int 2y \sin y \mathrm{d}y & \\
& = & y^2 \sin y + \int 2y \mathrm{d}(\cos y) & \Big| \text{ Integrate by parts} \\
& = & y^2 \sin y + 2y \cos y - 2 \int \cos y \mathrm{d}y & \\
& = & y^2 \sin y + 2y \cos y - 2 \sin y + C & \Big| \text{ Substitute } y = \arcsin x \\
& = & x(\arcsin x)^2 & \\
& & +2\sqrt{1-x^2} \arcsin x - 2x + C \quad . &
\end{array}$$

Solution. 7.o

$$\begin{array}{rcll}
\int \sin x \underbrace{e^x \mathrm{d}x}_{=\mathrm{d}e^x} & = & \sin x e^x - \int e^x \mathrm{d}(\sin x) = \sin x e^x - \int \cos x \underbrace{e^x \mathrm{d}x}_{=\mathrm{d}e^x} & \\
& = & \sin x e^x - e^x \cos x + \int e^x \mathrm{d}(\cos x) & \\
& = & e^x \sin x - e^x \cos x - \int e^x \sin x \mathrm{d}x & \Big| \begin{array}{l} \text{add } \int e^x \sin x \mathrm{d}x \\ \text{to both sides} \end{array} \\
2 \int \sin x e^x \mathrm{d}x & = & \sin x e^x - e^x \cos x & \\
\int \sin x e^x \mathrm{d}x & = & \frac{1}{2} \left(\sin x e^x - e^x \cos x \right) \quad . &
\end{array}$$

Solution. 7.q.

$$\begin{array}{rll}
\int \sin(\ln x) \mathrm{d}x & = x \sin(\ln x) - \int x \mathrm{d}(\sin(\ln x)) & \Big| \text{ int. by parts} \\
& = x \sin(\ln x) - \int x \left(\cos(\ln x)\right) (\ln x)' \, \mathrm{d}x & \\
& = x \sin(\ln x) - \int \cos(\ln x) \mathrm{d}x & \Big| \text{ int. by parts} \\
& = x \sin(\ln x) - \left(x \cos(\ln x) - \int x \mathrm{d}(\cos(\ln x)) \right) & \\
& = x \sin(\ln x) - x \cos(\ln x) + \int x(-\sin(\ln x))(\ln x)' \mathrm{d}x & \\
& = x \sin(\ln x) - x \cos(\ln x) - \int \sin(\ln x) \mathrm{d}x & \Big| \begin{array}{l} \text{add } \int \sin(\ln x) \mathrm{d}x \\ \text{to both sides} \end{array} \\
2 \int \sin(\ln x) \mathrm{d}x & = x \sin(\ln x) - x \cos(\ln x) & \\
\int \sin(\ln x) \mathrm{d}x & = \frac{x}{2} \left(\sin(\ln x) - \cos(\ln x) \right) \quad . &
\end{array}$$

Solution. 7.s

$$\int \ln x \mathrm{d}x = x \ln x - \int x \mathrm{d}(\ln x) = x \ln x - \int \frac{x}{x} \mathrm{d}x = x \ln x - x + C \quad .$$

Solution. 7.u

$$
\begin{array}{rcll}
\displaystyle\int \frac{\ln x}{\sqrt{x}}\mathrm{d}x &=& \displaystyle\int (\ln x) 2\mathrm{d}\left(\sqrt{x}\right) & \Big|\ \text{integrate by parts}\\
&=& \displaystyle(\ln x)2\sqrt{x} - \int 2\sqrt{x}\mathrm{d}(\ln x) & \\
&=& \displaystyle 2\sqrt{x}\ln x - 2\int \frac{\sqrt{x}}{x}\mathrm{d}x & \\
&=& \displaystyle 2\sqrt{x}\ln x - 2\int x^{-\frac{1}{2}}\mathrm{d}x & \\
&=& 2\sqrt{x}\ln x - 4\sqrt{x} + C & \\
&=& 2\sqrt{x}(\ln x - 2) + C \quad . &
\end{array}
$$

8. Compute $\displaystyle\int x^n e^x \mathrm{d}x$, where n is a non-negative integer.

Solution. 8

$$
\begin{array}{rcl}
\displaystyle\int x^n e^x \mathrm{d}x &=& \displaystyle\int x^n \mathrm{d}e^x \\
&=& \displaystyle x^n e^x - \int e^x \mathrm{d}x^n \\
&=& \displaystyle x^n e^x - n\int x^{n-1}e^x\mathrm{d}x \\
&=& \displaystyle x^n e^x - n\left(\int x^{n-1}\mathrm{d}e^x\right) \\
&=& \displaystyle x^n e^x - n\left(x^{n-1}e^x - \int (n-1)x^{n-2}e^x\mathrm{d}x\right) \\
&=& \displaystyle x^n e^x - nx^{n-1}e^x + n(n-1)\int x^{n-2}e^x\mathrm{d}x \\
&=& \dots\text{(continue above process)}\dots \\
&=& x^n e^x - nx^{n-1}e^x + n(n-1)x^{n-2}e^x + \dots \\
&& +(-1)^k n(n-1)(n-2)\dots(n-k+1)x^{n-k}e^x \\
&& + \dots + (-1)^n n! e^x + C \\
&=& \displaystyle C + \sum_{k=0}^{n} (-1)^n \frac{n!}{(n-k)!} x^{n-k}e^x \quad .
\end{array}
$$

9. Integrate. Illustrate the steps of your solution.

(a) $\displaystyle\int \frac{1}{x+1}\mathrm{d}x$

answer: $\ln|x+1| + C$

(b) $\displaystyle\int \frac{x-1}{x+1}\mathrm{d}x$

answer: $x - 2\ln|x+1| + C$

(c) $\displaystyle\int \frac{1}{(x+1)^2}\mathrm{d}x$

answer: $-\frac{1}{x+1} + C$

(d) $\displaystyle\int \frac{x}{(x+1)^2}\mathrm{d}x$

answer: $\ln|x+1| + \frac{1}{x+1} + C$

(e) $\displaystyle\int \frac{1}{(2x+3)^2}\mathrm{d}x$

answer: $-\frac{1}{2(2x+3)} + C$

(f) $\displaystyle\int \frac{x}{2x^2+3}\mathrm{d}x$

answer: $\frac{1}{4}\ln\left(2x^2+3\right) + C$

(g) $\displaystyle\int \frac{1}{2x^2+3}\mathrm{d}x$

answer: $\frac{1}{\sqrt{6}}\arctan\left(\sqrt{\frac{2}{3}}x\right)$

(h) $\displaystyle\int \frac{x}{2x^2+x+1}\mathrm{d}x$.

answer: $\frac{1}{4}\ln\left(x^2+\frac{1}{2}x+\frac{1}{2}\right) - \frac{\sqrt{7}}{14}\arctan\left(\frac{4x+1}{\sqrt{7}}\right) + C$

(i) $\displaystyle\int \frac{x}{2x^2+x+3}\mathrm{d}x$

answer: $\frac{1}{4}\ln\left(2x^2+x+3\right) - \frac{1}{2\sqrt{23}}\arctan\left(\frac{4x+1}{\sqrt{23}}\right) + C$

(j) $\displaystyle\int \frac{x}{x^2-x+3}\mathrm{d}x$

answer: $\frac{1}{2}\ln\left|x^2-x+3\right| + \frac{1}{\sqrt{11}}\arctan\left(\frac{x-\frac{1}{2}}{\frac{\sqrt{11}}{2}}\right) + C$

(k) $\displaystyle\int \frac{1}{\left(x^2+1\right)^2}\mathrm{d}x$

answer: $\frac{1}{2}x\left(x^2+1\right)^{-1} + \frac{1}{2}\arctan(x) + C$

(l) $\displaystyle\int \frac{1}{\left(x^2+x+1\right)^2}\mathrm{d}x$

answer: $\frac{2}{3}x\left(x^2+x+1\right)^{-1} + \frac{1}{3}\left(x^2+x+1\right)^{-1} + \frac{4}{9}\sqrt{3}\arctan\left(\frac{2}{3}\sqrt{3}x + \frac{\sqrt{3}}{3}\right)$

(m) $\displaystyle\int \frac{1}{\left(x^2+1\right)^3}\mathrm{d}x$

answer: $\frac{3}{8}x\left(x^2+1\right)^{-1} + \frac{1}{4}x\left(x^2+1\right)^{-2} + \frac{3}{8}\arctan(x) + C$

Solution. 9.h.

$$\begin{aligned}
\int \frac{x}{2x^2+x+1}\mathrm{d}x=&\int \frac{x}{2\left(x^2+2x\frac{1}{4}+\frac{1}{2}\right)}\mathrm{d}x\\
=&\int \frac{x}{2\left(x^2+2x\frac{1}{4}+\frac{1}{16}-\frac{1}{16}+\frac{1}{2}\right)}\mathrm{d}x && \text{complete square in denominator}\\
=&\frac{1}{2}\int \frac{x}{\left(x+\frac{1}{4}\right)^2+\frac{7}{16}}\mathrm{d}x\\
=&\frac{1}{2}\int \frac{x+\frac{1}{4}-\frac{1}{4}}{\left(x+\frac{1}{4}\right)^2+\frac{7}{16}}\mathrm{d}\left(x+\frac{1}{4}\right) && \text{Set } u=x+\tfrac{1}{4}\\
=&\frac{1}{2}\int \frac{u-\frac{1}{4}}{u^2+\frac{7}{16}}\mathrm{d}u\\
=&\frac{1}{2}\left(\int \frac{u}{u^2+\frac{7}{16}}\mathrm{d}u-\frac{1}{4}\int \frac{1}{u^2+\frac{7}{16}}\mathrm{d}u\right)\\
=&\frac{1}{2}\left(\frac{1}{2}\ln\left(u^2+\frac{7}{16}\right)-\frac{1}{4\sqrt{\frac{7}{16}}}\arctan\left(\frac{u}{\sqrt{\frac{7}{16}}}\right)\right)+K\\
=&\frac{1}{4}\ln\left(x^2+\frac{1}{2}x+\frac{1}{2}\right)-\frac{\sqrt{7}}{14}\arctan\left(\frac{4x+1}{\sqrt{7}}\right)+K \quad .
\end{aligned}$$

Solution. 9.l

$$\begin{aligned}
\int \frac{1}{\left(x^2+x+1\right)^2}\mathrm{d}x &= \int \frac{1}{\left(\left(x^2+2x\frac{1}{2}+\frac{1}{4}\right)-\frac{1}{4}+1\right)^2}\mathrm{d}x && \text{complete the square}\\
&= \int \frac{1}{\left(\left(x+\frac{1}{2}\right)^2+\frac{3}{4}\right)^2}\mathrm{d}\left(x+\frac{1}{2}\right) && \text{Set } w=x+\tfrac{1}{2}\\
&= \int \frac{1}{\left(w^2+\frac{3}{4}\right)^2}\mathrm{d}w\\
&= \int \frac{1}{\left(\frac{3}{4}\left(\left(\frac{2w}{\sqrt{3}}\right)^2+1\right)\right)^2}\frac{\sqrt{3}}{2}\mathrm{d}\left(\frac{2w}{\sqrt{3}}\right) && \text{Set } z=\tfrac{2w}{\sqrt{3}}\\
&= \frac{\frac{\sqrt{3}}{2}}{\left(\frac{3}{4}\right)^2}\int \frac{1}{\left(z^2+1\right)^2}\mathrm{d}z\\
&= \frac{8\sqrt{3}}{9}\int \frac{1}{\left(z^2+1\right)^2}\mathrm{d}z \quad .
\end{aligned}$$

The integral $\int \frac{1}{(z^2+1)^2}\mathrm{d}z$ was already studied; it was also given as an exercise in Problem 9.k. We leave the rest of the problem to the reader.

10. Let a, b, c, A, B be real numbers. Suppose in addition $a\neq 0$ and $b^2-4ac<0$. Integrate

$$\int \frac{Ax+B}{ax^2+bx+c}\mathrm{d}x \quad .$$

The purpose of this exercise is to produce a formula in form ready for implementation in a computer algebra system.

Solution. 10.

$$
\begin{aligned}
\int \frac{Ax+B}{ax^2+bx+c}\mathrm{d}x &= \int \frac{Ax+B}{a\left(x^2+2x\frac{b}{2a}+\frac{c}{a}\right)}\mathrm{d}x \\
&= \int \frac{Ax+B}{a\left(x^2+2x\frac{b}{2a}+\frac{b^2}{4a^2}-\frac{b^2}{4a^2}+\frac{c}{a}\right)}\mathrm{d}x && \text{complete square in denominator} \\
&= \frac{1}{a}\int \frac{Ax+B}{\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}}\mathrm{d}x && \text{Set } D=\frac{4ac-b^2}{4a^2} \\
&= \frac{1}{a}\int \frac{A\left(x+\frac{b}{2a}-\frac{b}{2a}\right)+B}{\left(x+\frac{b}{2a}\right)^2+D}\mathrm{d}\left(x+\frac{b}{2a}\right) && \text{Set } u=x+\frac{b}{2a} \\
&= \frac{1}{a}\int \frac{Au+B-\frac{Ab}{2a}}{u^2+D}\mathrm{d}u && \text{Set } C=B-\frac{Ab}{2a} \\
&= \frac{1}{a}\left(A\int \frac{u}{u^2+D}\mathrm{d}u + C\int \frac{1}{u^2+D}\mathrm{d}u\right) \\
&= \frac{1}{a}\left(\frac{A}{2}\ln(u^2+D)+\frac{C}{\sqrt{D}}\arctan\left(\frac{u}{\sqrt{D}}\right)\right)+K \\
&= \frac{1}{a}\left(\frac{A}{2}\ln\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)\right. \\
&\quad \left.+\frac{C}{\sqrt{D}}\arctan\left(\frac{x+\frac{b}{2a}}{\sqrt{D}}\right)\right)+K.
\end{aligned}
$$

The solution is complete. Question to the student: where do we use $b^2-4ac<0$?

11. Let a, b, c, A, B be real numbers and let $n>1$ be an integer. Suppose in addition $a \neq 0$ and $b^2-4ac<0$. Let

$$
J(n)=\int \frac{1}{\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)^n}\mathrm{d}x \quad .
$$

(a) Express the integral

$$
\int \frac{Ax+B}{(ax^2+bx+c)^n}\mathrm{d}x
$$

via $J(n)$.

(b) Express $J(n)$ recursively via $J(n-1)$

The purpose of this exercise is to produce a formula in form ready for implementation in a computer algebra system.

Solution. 11.a.

$$
\begin{aligned}
\int \frac{Ax+B}{(ax^2+bx+c)^n}\mathrm{d}x &= \int \frac{Ax+B}{a^n\left(x^2+2x\frac{b}{2a}+\frac{c}{a}\right)^n}\mathrm{d}x \\
&= \int \frac{Ax+B}{a^n\left(x^2+2x\frac{b}{2a}+\frac{b^2}{4a^2}-\frac{b^2}{4a^2}+\frac{c}{a}\right)^n}\mathrm{d}x && \text{complete square in denominator} \\
&= \frac{1}{a^n}\int \frac{Ax+B}{\left(\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}\right)^n}\mathrm{d}x && \text{Set } D=\frac{4ac-b^2}{4a^2} \\
&= \frac{1}{a^n}\int \frac{A\left(x+\frac{b}{2a}-\frac{b}{2a}\right)+B}{\left(\left(x+\frac{b}{2a}\right)^2+D\right)^n}\mathrm{d}\left(x+\frac{b}{2a}\right) && \text{Set } u=x+\frac{b}{2a} \\
&= \frac{1}{a^n}\int \frac{Au+B-\frac{Ab}{2a}}{(u^2+D)^n}\mathrm{d}u && \text{Set } C=B-\frac{Ab}{2a} \\
&= \frac{1}{a^n}\left(A\int \frac{u}{(u^2+D)^n}\mathrm{d}u + C\int \frac{1}{(u^2+D)^n}\mathrm{d}u\right) \\
&= \frac{1}{a^n}\left(\frac{A}{2(1-n)}\left(u^2+D\right)^{1-n}+CJ(n)\right) \\
&= \frac{1}{a^n}\left(\frac{A}{2(1-n)}\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)^{1-n}+CJ(n)\right)
\end{aligned}
$$

Solution. 11.b. We use all notation and computations from the previous part of the problem. According to theory, in order to solve that integral, we are supposed to integrate by parts the simpler integral

$$
\begin{aligned}
J(n-1) &= \int \frac{1}{\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)^{n-1}}\mathrm{d}x = \int \frac{1}{\left(u^2+D\right)^{n-1}}\mathrm{d}u \qquad \Big|\ \text{int. by parts} \\
&= \frac{u}{\left(u^2+D\right)^{n-1}} - \int u\,\mathrm{d}\left(\frac{1}{\left(u^2+D\right)^{n-1}}\right) \\
&= \frac{u}{\left(u^2+D\right)^{n-1}} + 2(n-1)\int \frac{u^2}{\left(u^2+D\right)^{n}}\mathrm{d}u \\
&= \frac{u}{\left(u^2+D\right)^{n-1}} + 2(n-1)\int \frac{u^2+D-D}{\left(u^2+D\right)^{n}}\mathrm{d}u \\
&= \frac{u}{\left(u^2+D\right)^{n-1}} + 2(n-1)J(n-1) - 2D(n-1)\int \frac{1}{\left(u^2+D\right)^{n}}\mathrm{d}u \\
&= \frac{u}{\left(u^2+D\right)^{n-1}} + 2(n-1)J(n-1) - 2D(n-1)J(n)
\end{aligned}
$$

In the above equality, we rearrange terms to get that

$$
\begin{aligned}
2D(n-1)J(n) &= \frac{u}{\left(u^2+D\right)^{n-1}} + (2n-3)J(n-1) \\
J(n) &= \frac{1}{D}\left(\frac{u}{2(n-1)\left(u^2+D\right)^{n-1}} + \frac{2n-3}{2n-2}J(n-1)\right) \\
&= \frac{1}{D}\left(\frac{x+\frac{b}{2a}}{(2n-2)\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)^{n-1}} + \frac{2n-3}{2n-2}J(n-1)\right) \quad .
\end{aligned}
$$