Precalculus
Completing the square

Todor Milev

2019

Example (Completing the square)

Complete the square.

$$3x^2 - 5x + 1$$

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$$3x^2 - 5x + 1 \quad = \quad 3\left(x^2 - \mathbf{?}\, x\right) + 1$$

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$$3x^2 - 5x + 1 \quad = \quad 3\left(x^2 - \frac{5}{3}x\right) + 1$$

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$$\begin{aligned} 3x^2 - 5x + 1 &= 3\left(x^2 - \frac{5}{3}x\right) + 1 \\ &= 3\left(x^2 - 2 \cdot \frac{5}{2 \cdot 3}x\right) + 1 \end{aligned}$$

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&= 3\left(\mathbf{?}\qquad\qquad-\mathbf{?}\quad\right)+1
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&= 3\left(\left(x - \frac{5}{6}\right)^2 - {\color{red}\frac{25}{36}}\right) + 1
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Definition (Completing the square)

Let $a \neq 0$. To *complete the square* means to carry out the following algebraic manipulation.

$$ax^2 + bx + c$$

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use $(A+B)^2 = A^2 + 2AB + B^2$

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&= a\left(x+\frac{b}{2a}\right)^2+c-\frac{b^2}{4a}.
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