Calculus I
Homework
Limits involving infinity

1. Evaluate the difference quotient and simplify your answer.

   (a) $\dfrac{f(2+h)-f(2)}{h}$, where $f(x)=x^2-x-1$.

   (b) $\dfrac{f(a+h)-f(a)}{h}$, where $f(x)=x^2$.

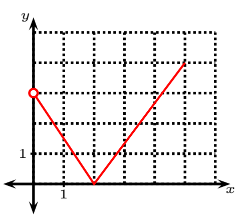
   (c) $\dfrac{f(a+h)-f(a)}{h}$, where $f(x)=x^3$.

   (d) $\dfrac{f(a+h)-f(a)}{h}$, where $f(x)=x^4$.

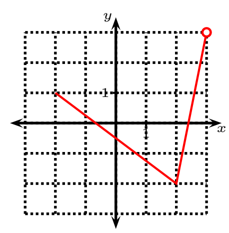
   (e) $\dfrac{f(x)-f(a)}{x-a}$, where $f(x)=\frac{1}{x}$.

   (f) $\dfrac{f(x)-f(1)}{x-1}$, where $f(x)=\frac{x-1}{x+1}$.

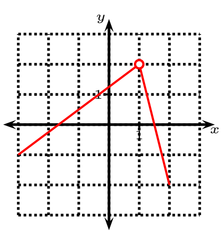
2. Write down a formula for a function whose graphs is given below. The graphs are up to scale. Please note that there is more than one way to write down a correct answer.



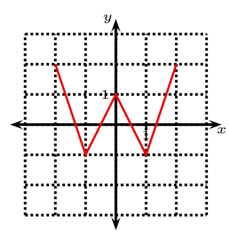
(a)



(c)

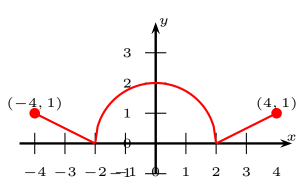


(b)



(d)

3. Write down formulas for function whose graphs are as follows. The graphs are up to scale. All arcs are parts of circles.



(a)

4. Evaluate the difference quotient and simplify your answer.

   (a) $\dfrac{f(2+h)-f(2)}{h}$, where $f(x)=x^2-x-1$.

   (b) $\dfrac{f(a+h)-f(a)}{h}$, where $f(x)=x^2$.

   (c) $\dfrac{f(a+h)-f(a)}{h}$, where $f(x)=x^3$.

   (d) $\dfrac{f(a+h)-f(a)}{h}$, where $f(x)=x^4$.

   (e) $\dfrac{f(x)-f(a)}{x-a}$, where $f(x)=\frac{1}{x}$.

   (f) $\dfrac{f(x)-f(1)}{x-1}$, where $f(x)=\frac{x-1}{x+1}$.

5. Find the implied domain of the function.

(a) $f(x) = \dfrac{x+4}{x^2-4}$.

(b) $f(x) = \dfrac{2x^3-5}{x^2+5x+6}$.

(c) $f(t) = \sqrt[3]{3t-1}$.

(d) $g(t) = \sqrt{5-t} - \sqrt{1+t}$.

(e) $h(x) = \dfrac{1}{\sqrt[6]{x^2-7x}}$.

(f) $f(u) = \dfrac{u+1}{1+\frac{1}{u+1}}$.

(g) $F(x) = \sqrt{10-\sqrt{x}}$.

6. Find the implied domain of the function.

(a) $f(x) = \dfrac{x+4}{x^2-4}$.

(b) $f(x) = \dfrac{2x^3-5}{x^2+5x+6}$.

(c) $f(t) = \sqrt[3]{3t-1}$.

(d) $g(t) = \sqrt{5-t} - \sqrt{1+t}$.

(e) $h(x) = \dfrac{1}{\sqrt[6]{x^2-7x}}$.

(f) $f(u) = \dfrac{u+1}{1+\frac{1}{u+1}}$.

(g) $F(x) = \sqrt{10-\sqrt{x}}$.

7. Compute the composite functions $(f \circ g)(x)$, $(g \circ f)(x)$. Simplify your answer to a single fraction. Find the domain of the composite function.

(a) $f(x) = \dfrac{x+2}{x-2}, g(x) = \dfrac{x-1}{x+2}$.

(b) $f(x) = \dfrac{x+1}{3x-2}, g(x) = \dfrac{x-2}{x-1}$.

(c) $f(x) = \dfrac{2x+1}{3x-1}, g(x) = \dfrac{x-2}{2x-1}$.

(d) $f(x) = \dfrac{x+1}{x-2}, g(x) = \dfrac{x+2}{2x-1}$.

(e) $f(x) = \dfrac{5x+1}{4x-1}, g(x) = \dfrac{4x-1}{3x+1}$.

(f) $f(x) = \dfrac{3x-5}{x-2}, g(x) = \dfrac{x-2}{x-4}$.

(g) $f(x) = \dfrac{x-3}{x+2}, g(y) = \dfrac{y+3}{y-4}$.

8. Find the functions $f \circ g$, $g \circ f$, $f \circ f$ and $g \circ g$ and their implied domains. The answer key has not been proofread, use with caution.

(a) $f(x) = x^2+1, g(x) = x+1$.

(b) $f(x) = \sqrt{x+1}, g(x) = x+1$.

(c) $f(x) = 2x, g(x) = \tan x$.
In this subproblem, you are not required to find the domain.

(d) $f(x) = \dfrac{x+1}{x-1}, g(x) = \dfrac{x-1}{x+1}$.

9. Convert from degrees to radians.

(a) 15°.
(b) 30°.
(c) 36°.
(d) 45°.
(e) 60°.
(f) 75°.
(g) 90°.
(h) 120°.
(i) 135°.
(j) 150°.
(k) 180°.
(l) 225°.
(m) 270°.
(n) 305°.
(o) 360°.
(p) 405°.
(q) 1200°.
(r) -900°.
(s) -2014°.

10. Convert from radians to degrees. The answer key has not been proofread, use with caution.

(a) 4π.
(b) $-\frac{7}{6}\pi$.
(c) $\frac{7}{12}\pi$.
(d) $\frac{4}{3}\pi$.
(e) $-\frac{3}{8}\pi$.
(f) 2014π.
(g) 5.
(h) -2014.

11. Prove the trigonometry identities.

(a) $\sin\theta\cot\theta = \cos\theta$.
(b) $(\sin\theta + \cos\theta)^2 = 1 + \sin(2\theta)$.
(c) $\sec\theta - \cos\theta = \tan\theta\sin\theta$.
(d) $\tan^2\theta - \sin^2\theta = \tan^2\theta\sin^2\theta$.
(e) $\cot^2\theta + \sec^2\theta = \tan^2\theta + \csc^2\theta$.
(f) $2\csc(2\theta) = \sec\theta\csc\theta$.
(g) $\tan(2\theta) = \dfrac{2\tan\theta}{1-\tan^2\theta}$.
(h) $\dfrac{1}{1-\sin\theta} + \dfrac{1}{1+\sin\theta} = 2\sec^2\theta$.
(i) $\tan\alpha + \tan\beta = \dfrac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta}$.
(j) $\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta}$.
(k) $\sin(3\theta) + \sin\theta = 2\sin(2\theta)\cos\theta$.
(l) $\cos(3\theta) = 4\cos^3\theta - 3\cos\theta$.
(m) $1 + \tan^2\theta = \sec^2\theta$.
(n) $1 + \csc^2\theta = \cot^2\theta$.
(o) $2\cos^2(2x) = 2\sin^4\theta + 2\cos^4\theta - \sin^2(2\theta)$.
(p) $\dfrac{1+\tan\left(\frac{\theta}{2}\right)}{1-\tan\left(\frac{\theta}{2}\right)} = \tan\theta + \sec\theta$.

12. Find all values of x in the interval $[0, 2\pi]$ that satisfy the equation.

(a) $2\cos x - 1 = 0$.
(b) $\sin(2x) = \cos x$.
(c) $\sqrt{3}\sin x = \sin(2x)$. .
(d) $2\sin^2 x = 1$.
(e) $2 + \cos(2x) = 3\cos x$.
(f) $2\cos x + \sin(2x) = 0$.
(g) $2\cos^2 x - (1+\sqrt{2})\cos x + \dfrac{\sqrt{2}}{2} = 0$.
.
(h) $|\tan x| = 1$.
(i) $3\cot^2 x = 1$.
(j) $\sin x = \tan x$.

13. Evaluate the limits. Justify your computations.

(a) $\lim\limits_{x\to 2} 2x^2 - 3x - 6$.
(b) $\lim\limits_{x\to -1} \dfrac{x^4 - x}{x^2+2x+3}$.
(c) $\lim\limits_{x\to -1} \dfrac{1}{x^2-3x+2}$.
(d) $\lim\limits_{x\to -2} \sqrt{x^4+16}$.
(e) $\lim\limits_{x\to 8} (1+\sqrt[3]{x})(2-x)$.

14. Evaluate the limit if it exists.

(a) $\lim\limits_{x\to 2} \dfrac{x^2-5x+6}{x-2}$.
(b) $\lim\limits_{x\to 3} \dfrac{x^2-3x}{x^2-2x-3}$.
(c) $\lim\limits_{x\to -2} \dfrac{2x^2+x-6}{x^2-4}$
(d) $\lim\limits_{x\to 2} \dfrac{x^2-5x-6}{x-2}$.
(e) $\lim\limits_{x\to -1} \dfrac{x^2-3x}{x^2-2x-3}$.
(f) $\lim\limits_{x\to -2} \dfrac{x^2-4}{2x^2+5x+2}$.
(g) $\lim\limits_{x\to -1} \dfrac{2x^2+3x+1}{3x^2-2x-5}$.
(h) $\lim\limits_{x\to -4} \dfrac{x^2+7x+12}{x^2+6x+8}$.
(i) $\lim\limits_{h\to 0} \dfrac{(-3+h)^2-9}{h}$.
(j) $\lim\limits_{h\to 0} \dfrac{(-2+h)^3+8}{h}$.
(k) $\lim\limits_{x\to -3} \dfrac{x+3}{x^3+27}$.
(l) $\lim\limits_{x\to 1} \dfrac{x^4-1}{x^3-1}$.
(m) $\lim\limits_{h\to 0} \dfrac{\sqrt{4+h}-2}{h}$.
(n) $\lim\limits_{x\to 3} \dfrac{\sqrt{5x+1}-4}{x-3}$.

(o) $\lim_{x\to -3} \dfrac{\sqrt{x^2+16}-5}{x+3}$.

(p) $\lim_{x\to -3} \dfrac{\frac{1}{3}+\frac{1}{x}}{3+x}$.

(q) $\lim_{x\to -2} \dfrac{x^2+4x+4}{x^4-16}$.

(r) $\lim_{x\to 0} \dfrac{\sqrt{1+x}-\sqrt{1-x}}{x}$.

(s) $\lim_{x\to 0} \left(\dfrac{1}{x}-\dfrac{1}{x^2+x}\right)$.

(t) $\lim_{x\to 9} \dfrac{3-\sqrt{x}}{9x-x^2}$.

(u) $\lim_{h\to 0} \dfrac{(2+h)^{-1}-2^{-1}}{h}$.

(v) $\lim_{x\to 0} \left(\dfrac{1}{x\sqrt{1+x}}-\dfrac{1}{x}\right)$.

(w) $\lim_{h\to 0} \dfrac{(x+h)^3-x^3}{h}$.

(x) $\lim_{h\to 0} \dfrac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h}$.

(y) $\lim_{h\to 0} \dfrac{\frac{1}{(2+h)^2}-\frac{1}{4}}{h}$.

(z) $\lim_{h\to 0} \dfrac{\frac{1}{(1+h)^2}-1}{h}$.

15. Find the (implied) domain of $f(x)$. Extend the definition of f at $x=3$ to make f continuous at 3.

(a) $f(x)=\dfrac{x^2-x-6}{x-3}$.

(b) $f(x)=\dfrac{x^3-27}{x^2-9}$.

16. Use the Intermediate Value Theorem to show that there is a real number solution of the given equation in the specified interval.

(a) $x^5+x-3=0$ where $x\in(1,2)$.

(b) $\sqrt[4]{x}=1-x$ where $x\in\mathbb{R}$ (i.e., x is an arbitrary real number).

(c) $\cos x=2x$, where $x\in(0,1)$.

(d) $\sin x=x^2-x-1$, where $x\in\mathbb{R}$ (i.e., x is an arbitrary real number).

(e) $\cos x=x^4$, where $x\in\mathbb{R}$ (i.e., x is an arbitrary real number).

(f) $x^5-x^2+x+3=0$, where $x\in\mathbb{R}$.

17.

(a) i. Solve the equation $x^2+13x+41=1$.

ii. Use the intermediate value theorem to prove that the equation $x^2+13x+41=\sin x$ has at least two solutions, lying between the two solutions to 17.a.i.

(b) i. Solve the equation $x^2-15x+55=1$.

ii. Use the intermediate value theorem to prove that the equation $x^2-15x+55=\cos x$ has at least two solutions, lying between the two solutions to the equation in the preceding item.

18. For which values of x is f continuous?

- $f(x)=\begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational}\end{cases}$
- $f(x)=\begin{cases} 0 & \text{if } x \text{ is rational} \\ x & \text{if } x \text{ is irrational}\end{cases}$

19. Show that $f(x)$ is continuous at all irrational points and discontinuous at all rational ones.

$$f(x)=\begin{cases} \frac{1}{q^2} & \text{if } x \text{ is rational and } x=\frac{p}{q} \\ 0 & \text{if } x \text{ is irrational}\end{cases}$$

where in the first item p,q are relatively prime integers (i.e., integers without a common divisor).

20. Show the following limits do not exist and compute whether they evaluate to ∞, $-\infty$, or neither.

(a) $\lim_{x\to 3^+} \dfrac{x^2+x-1}{x^2-2x-3}$.

(b) $\lim_{x\to 3^-} \dfrac{x^2+x-1}{x^2-2x-3}$.

(c) $\lim_{x\to 1^+} \dfrac{x^2+1}{\sqrt{x^2+3}-2}$.

(d) $\lim_{x\to 1^-} \dfrac{x^2+1}{\sqrt{x^2+3}-2}$.

(e) $\lim_{x\to 2^+} \dfrac{\sqrt{x^3-8}}{-x^2+x+2}$.

(f) $\lim_{x\to -1^+} \dfrac{\sqrt[3]{x^2+2x+1}}{x^2-2x-3}$.

21. Find the limit or show that it does not exist. If the limit does not exist, indicate whether it is $\pm\infty$, or neither. The answer key has not been proofread, use with caution.

(a) $\lim_{x\to\infty} \frac{x-2}{2x+1}$.

(b) $\lim_{x\to\infty} \frac{1-x^2}{x^3-x-1}$.

(c) $\lim_{x\to-\infty} \frac{x-2}{x^2+5}$.

(d) $\lim_{x\to-\infty} \frac{3x^3+2}{2x^3-4x+5}$.

(e) $\lim_{x\to\infty} \frac{\sqrt{x}+x^2}{\sqrt{x}-x^2}$.

(f) $\lim_{x\to\infty} \frac{3-x\sqrt{x}}{2x^{\frac{3}{2}}-2}$.

(g) $\lim_{x\to\infty} \frac{(2x^2+3)^2}{(x-1)^2(x^2+1)}$.

(h) $\lim_{x\to\infty} \frac{x^2-3}{\sqrt{x^4+3}}$.

(i) $\lim_{x\to-\infty} \frac{\sqrt{x^2+1}}{x+1}$.

(j) $\lim_{x\to\infty} \frac{\sqrt{16x^6-3x}}{x^3+2}$.

(k) $\lim_{x\to-\infty} \frac{\sqrt{16x^6-3x}}{x^3+2}$.

(l) $\lim_{x\to\infty} \frac{\sqrt{3x^2+2x+1}}{x+1}$.

(m) $\lim_{x\to\infty} \sqrt{4x^2+x}-2x$.

(n) $\lim_{x\to-\infty} x+\sqrt{x^2+3x}$.

(o) $\lim_{x\to\infty} \sqrt{x^2+2x}-\sqrt{x^2-2x}$.

(p) $\lim_{x\to-\infty} \sqrt{x^2+x}-\sqrt{x^2-x}$.

(q) $\lim_{x\to\infty} \sqrt{x^2+ax}-\sqrt{x^2+bx}$.

(r) $\lim_{x\to\infty} \cos x$.

(s) $\lim_{x\to\infty} \frac{x^4+x}{x^3-x+2}$.

(t) $\lim_{x\to\infty} \sqrt{x^2+1}$.

(u) $\lim_{x\to-\infty} (x^4+x^5)$.

(v) $\lim_{x\to-\infty} \frac{\sqrt{1+x^6}}{1+x^2}$.

(w) $\lim_{x\to\infty} (x-\sqrt{x})$.

(x) $\lim_{x\to\infty} (x^2-x^3)$.

(y) $\lim_{x\to\infty} x\sin x$.

(z) $\lim_{x\to\infty} \sqrt{x}\sin x$.

22. Find the horizontal and vertical asymptotes of the graph of the function. If a graphing device is available, check your work by plotting the function.

(a) $y=\frac{2x}{\sqrt{x^2+x+3}-3}$.

(b) $y=\frac{3x^2}{\sqrt{x^2+2x+10}-5}$.

(c) $y=\frac{3x+1}{x-2}$.

(d) $y=\frac{x^2-1}{2x^2-3x-2}$.

(e) $y=\frac{2x^2-2x-1}{x^2+x-2}$.

(f) $f(x)=\frac{-5x^2-3x+5}{x^2-2x-3}$

(g) $y=\frac{1+x^4}{x^2-x^4}$.

(h) $y=\frac{x^3-x}{x^2-7x+6}$.

(i) $y=\frac{x-9}{\sqrt{4x^2+3x+3}}$.

(j) $y=\frac{\sqrt{x^2+1}-x}{x}$.

(k) $f(x)=\frac{x}{\sqrt{x^2+3}-2x}$