

Calculus I

Definite integrals and areas between curves

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Outline

1 Integration and symmetry

2 More About Areas

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Symmetry

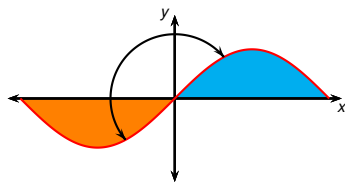
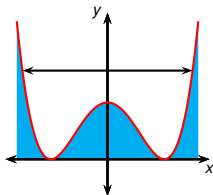
Theorem (Integrals of Symmetric Functions)

Suppose f is continuous on $[-a, a]$.

① If f is even (that is, $f(-x) = f(x)$), then $\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$.

② If f is odd (that is, $f(-x) = -f(x)$), then

$$\int_{-a}^a f(x) dx = \int_0^a f(x) dx + \int_{-a}^0 f(x) dx = 0.$$

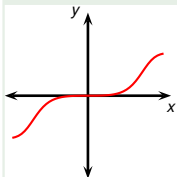


Example

Since $f(x) = x^6 + 1$ satisfies $f(-x) = f(x)$, it is even, and so

$$\begin{aligned}\int_{-2}^2 (x^6 + 1) \, dx &= 2 \int_0^2 (x^6 + 1) \, dx \\ &= 2 \left[\frac{1}{7} x^7 + x \right]_0^2 \\ &= 2 \left(\frac{128}{7} + 2 \right) \\ &= \frac{284}{7}.\end{aligned}$$

Example

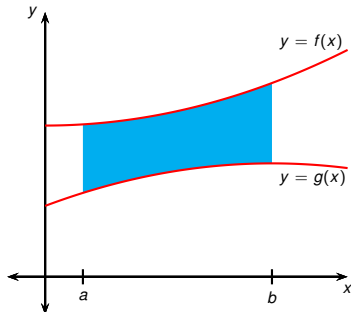


Since $f(x) = \frac{\tan x - x}{1 - 2x^2 + 2x^4}$ satisfies $f(-x) = -f(x)$, it is odd, and so

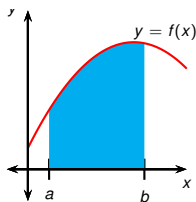
$$\int_{-1}^1 \frac{\tan x - x}{1 - 2x^2 + 2x^4} dx = 0.$$

More About Areas

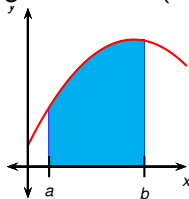
Suppose two curves, $y = f(x)$ and $y = g(x)$, are given. How do we find the area bounded by those curves between the endpoints $x = a$ and $x = b$?



The Area Under a Curve



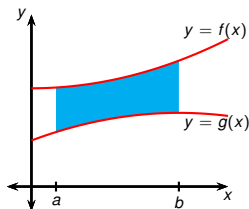
$$\text{rectangle area} = f(x) \cdot \Delta x$$



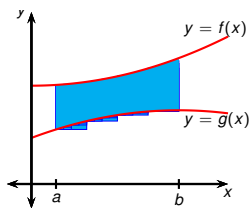
$$\# \text{ rectangles} = n \rightarrow \infty$$

$$A = \int_a^b f(x) dx$$

The Area Between Two Curves



$$\text{rectangle area} = (f(x) - g(x)) \cdot \Delta x$$



$$\# \text{ rectangles} = n \rightarrow \infty$$

$$A = \int_a^b [f(x) - g(x)] dx$$

Definition (The Area Between Two Curves)

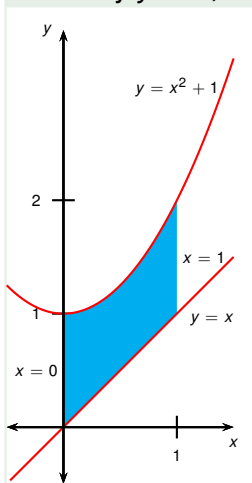
The area between two curves $y = f(x)$ and $y = g(x)$ bounded by the endpoints $x = a$ and $x = b$ is

$$\int_a^b |f(x) - g(x)| dx.$$

Note that we use the absolute value, because in general we don't know which curve is above the other.

Example

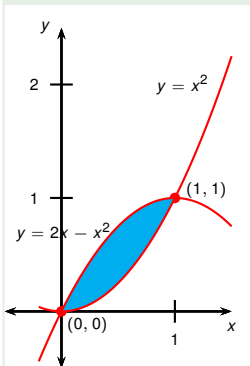
Find the area of the region bounded above by $y = x^2 + 1$, bounded below by $y = x$, and bounded on its sides by $x = 0$ and $x = 1$.



- 1 Graph the functions.
- 2 Identify the region.
- 3 Integrate.

$$\begin{aligned} A &= \int_0^1 |(x^2 + 1) - x| dx \\ &= \int_0^1 (x^2 - x + 1) dx \\ &= \left[\frac{x^3}{3} - \frac{x^2}{2} + x \right]_0^1 \\ &= \frac{1}{3} - \frac{1}{2} + 1 = \frac{5}{6}. \end{aligned}$$

Example



Find the area of the region enclosed by the parabolas $y = x^2$ and $y = 2x - x^2$.

$$x^2 = 2x - x^2$$

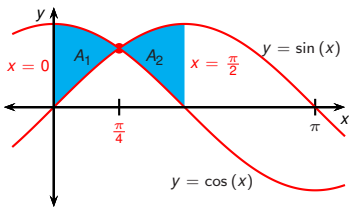
$$0 = 2x - 2x^2 = 2x(1 - x)$$

$$x = 0 \text{ or } 1.$$

$$\begin{aligned} A &= \int_0^1 (2x - 2x^2) dx = 2 \int_0^1 (x - x^2) dx \\ &= 2 \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = 2 \left(\frac{1}{2} - \frac{1}{3} \right) = \frac{1}{3}. \end{aligned}$$

- 1 Find the point of intersection.
- 2 Graph the functions.
- 3 Identify the region.
- 4 Integrate.

Example



Find the area of the region enclosed by the curves $y = \sin x$, $y = \cos x$, $x = 0$ and $x = \pi/2$.

The only point of intersection in the interval $[0, \pi/2]$ is $(\pi/4, 1/\sqrt{2})$.

$$A = A_1 + A_2$$

$$= \int_0^{\pi/4} (\cos x - \sin x) dx$$

$$+ \int_{\pi/4}^{\pi/2} (\sin x - \cos x) dx$$

$$= [\sin x + \cos x]_0^{\pi/4} + [-\cos x - \sin x]_{\pi/4}^{\pi/2}$$

$$= 2\sqrt{2} - 2.$$

- 1 Find the point of intersection.
- 2 Graph the functions.
- 3 Identify the region.
- 4 Integrate.