Calculus II
Convergence of sequences related to the number e as a limit, part 1

Todor Milev

2019

Theorem

If $\lim_{n\to\infty} a_n = L$ *and the function* f *is continuous at* L*, then*

$$\lim_{n\to\infty} f(a_n) = f(L)$$

Example

$$\lim_{x\to\infty}\left(1+\frac{k}{x}\right)^{x}$$

Example

$$\lim_{x\to\infty}\left(1+\frac{k}{x}\right)^x = \quad \lim_{x\to\infty} e^{\ln\left(1+\frac{k}{x}\right)^x}$$

Example

$$\lim_{x\to\infty}\left(1+\frac{k}{x}\right)^x = \lim_{x\to\infty} e^{\ln\left(1+\frac{k}{x}\right)^x}$$

exponent= continuous f-n

$$= e^{\lim\limits_{x\to\infty}\ln\left(1+\frac{k}{x}\right)^x}$$

Example

$$\lim_{x\to\infty}\left(1+\frac{k}{x}\right)^x = \lim_{x\to\infty} e^{\ln\left(1+\frac{k}{x}\right)^x}$$

$$= e^{\lim_{x\to\infty}\ln\left(1+\frac{k}{x}\right)^x}$$

exponent= continuous f-n

$$\lim_{x\to\infty}\ln\left(1+\frac{k}{x}\right)^x$$

Example

$$\begin{aligned}\lim_{x\to\infty}\left(1+\frac{k}{x}\right)^x &= \lim_{x\to\infty} e^{\ln\left(1+\frac{k}{x}\right)^x} \\ &= e^{\lim\limits_{x\to\infty}\ln\left(1+\frac{k}{x}\right)^x}\end{aligned}$$

exponent= continuous f-n

$$\lim_{x\to\infty}\ln\left(1+\frac{k}{x}\right)^{x} = \lim_{x\to\infty} x\ln\left(1+\frac{k}{x}\right)$$

Example

$$\lim_{x\to\infty}\left(1+\frac{k}{x}\right)^x = \lim_{x\to\infty} e^{\ln\left(1+\frac{k}{x}\right)^x}$$

exponent= continuous f-n

$$= e^{\lim\limits_{x\to\infty}\ln\left(1+\frac{k}{x}\right)^x}$$

$$\lim_{x\to\infty}\ln\left(1+\frac{k}{x}\right)^x = \lim_{x\to\infty} {\color{red}x}\ln\left(1+\frac{k}{x}\right)$$

$$= \lim_{x\to\infty}\frac{\left(\ln\left(1+\frac{k}{x}\right)\right)}{{\color{red}\left(\frac{1}{x}\right)}}$$

Example

$$\lim_{x\to\infty}\left(1+\frac{k}{x}\right)^x = \lim_{x\to\infty} e^{\ln\left(1+\frac{k}{x}\right)^x} \quad \Big| \text{ exponent= continuous f-n}$$

$$= e^{\lim\limits_{x\to\infty}\ln\left(1+\frac{k}{x}\right)^x}$$

$$\lim_{x\to\infty}\ln\left(1+\frac{k}{x}\right)^x = \lim_{x\to\infty} x\ln\left(1+\frac{k}{x}\right)$$

$$= \lim_{x\to\infty}\frac{\left(\ln\left(1+\frac{k}{x}\right)\right)}{\left(\frac{1}{x}\right)} \quad \Big| \text{ form “}\frac{0}{0}\text{”,}$$

Example

$$\lim_{x\to\infty}\left(1+\frac{k}{x}\right)^x = \lim_{x\to\infty} e^{\ln\left(1+\frac{k}{x}\right)^x}$$

exponent= continuous f-n

$$= e^{\lim\limits_{x\to\infty} \ln\left(1+\frac{k}{x}\right)^x}$$

$$\lim_{x\to\infty}\ln\left(1+\frac{k}{x}\right)^x = \lim_{x\to\infty} x\ln\left(1+\frac{k}{x}\right)$$

$$= \lim_{x\to\infty}\frac{\frac{d}{dx}\left(\ln\left(1+\frac{k}{x}\right)\right)}{\frac{d}{dx}\left(\frac{1}{x}\right)}$$

form "$\frac{0}{0}$", use L'Hospital

Example

$$\lim_{x\to\infty}\left(1+\frac{k}{x}\right)^x = \lim_{x\to\infty} e^{\ln\left(1+\frac{k}{x}\right)^x}$$

exponent= continuous f-n

$$= e^{\lim\limits_{x\to\infty} \ln\left(1+\frac{k}{x}\right)^x}$$

$$\lim_{x\to\infty}\ln\left(1+\frac{k}{x}\right)^x = \lim_{x\to\infty} x\ln\left(1+\frac{k}{x}\right)$$

$$= \lim_{x\to\infty}\frac{\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln\left(1+\frac{k}{x}\right)\right)}{\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x}\right)}$$

form "$\frac{0}{0}$", use L'Hospital

$$= \lim_{x\to\infty}\frac{?}{?}$$

Example

$$\lim_{x\to\infty}\left(1+\frac{k}{x}\right)^x = \lim_{x\to\infty} e^{\ln\left(1+\frac{k}{x}\right)^x}$$

exponent= continuous f-n

$$= e^{\lim\limits_{x\to\infty}\ln\left(1+\frac{k}{x}\right)^x}$$

$$\lim_{x\to\infty}\ln\left(1+\frac{k}{x}\right)^x = \lim_{x\to\infty} x\ln\left(1+\frac{k}{x}\right)$$

$$= \lim_{x\to\infty}\frac{\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln\left(1+\frac{k}{x}\right)\right)}{\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x}\right)}$$

form "$\frac{0}{0}$", use L'Hospital

$$= \lim_{x\to\infty}\frac{?}{-\frac{1}{x^2}}$$

Example

$$\lim_{x\to\infty}\left(1+\frac{k}{x}\right)^x = \lim_{x\to\infty} e^{\ln\left(1+\frac{k}{x}\right)^x}$$

exponent= continuous f-n

$$= e^{\lim\limits_{x\to\infty}\ln\left(1+\frac{k}{x}\right)^x}$$

$$\lim_{x\to\infty}\ln\left(1+\frac{k}{x}\right)^x = \lim_{x\to\infty} x\ln\left(1+\frac{k}{x}\right)$$

$$= \lim_{x\to\infty}\frac{\frac{d}{dx}\left(\ln\left(1+\frac{k}{x}\right)\right)}{\frac{d}{dx}\left(\frac{1}{x}\right)}$$

form "$\frac{0}{0}$", use L'Hospital

$$= \lim_{x\to\infty}\frac{?}{-\frac{1}{x^2}}$$

Example

$$\lim_{x\to\infty}\left(1+\frac{k}{x}\right)^x = \lim_{x\to\infty} e^{\ln\left(1+\frac{k}{x}\right)^x}$$

exponent= continuous f-n

$$= e^{\lim\limits_{x\to\infty}\ln\left(1+\frac{k}{x}\right)^x}$$

$$\lim_{x\to\infty}\ln\left(1+\frac{k}{x}\right)^x = \lim_{x\to\infty} x\ln\left(1+\frac{k}{x}\right)$$

$$= \lim_{x\to\infty}\frac{\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln\left(1+\frac{k}{x}\right)\right)}{\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x}\right)}$$

form "$\frac{0}{0}$", use L'Hospital

$$= \lim_{x\to\infty}\frac{\frac{1}{1+\frac{k}{x}}\left(1+\frac{k}{x}\right)'}{-\frac{1}{x^2}}$$

Example

$$\lim_{x\to\infty}\left(1+\frac{k}{x}\right)^x = \lim_{x\to\infty} e^{\ln\left(1+\frac{k}{x}\right)^x} \quad \Big| \text{ exponent= continuous f-n}$$

$$= e^{\lim\limits_{x\to\infty} \ln\left(1+\frac{k}{x}\right)^x}$$

$$\lim_{x\to\infty}\ln\left(1+\frac{k}{x}\right)^x = \lim_{x\to\infty} x\ln\left(1+\frac{k}{x}\right)$$

$$= \lim_{x\to\infty}\frac{\frac{d}{dx}\left(\ln\left(1+\frac{k}{x}\right)\right)}{\frac{d}{dx}\left(\frac{1}{x}\right)} \quad \Big| \text{ form “}\tfrac{0}{0}\text{”, use L’Hospital}$$

$$= \lim_{x\to\infty}\frac{\frac{1}{1+\frac{k}{x}}\left(1+\frac{k}{x}\right)'}{-\frac{1}{x^2}}$$

$$= \lim_{x\to\infty}\frac{\frac{1}{1+\frac{k}{x}}\left(-\frac{k}{x^2}\right)}{-\frac{1}{x^2}}$$

Example

$$\lim_{x\to\infty}\left(1+\frac{k}{x}\right)^x = \lim_{x\to\infty} e^{\ln\left(1+\frac{k}{x}\right)^x}$$

exponent= continuous f-n

$$= e^{\lim\limits_{x\to\infty}\ln\left(1+\frac{k}{x}\right)^x}$$

$$\lim_{x\to\infty}\ln\left(1+\frac{k}{x}\right)^x = \lim_{x\to\infty} x\ln\left(1+\frac{k}{x}\right)$$

$$= \lim_{x\to\infty}\frac{\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln\left(1+\frac{k}{x}\right)\right)}{\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x}\right)}$$

form "$\frac{0}{0}$", use L'Hospital

$$= \lim_{x\to\infty}\frac{\frac{1}{1+\frac{k}{x}}\left(1+\frac{k}{x}\right)'}{-\frac{1}{x^2}}$$

$$= \lim_{x\to\infty}\frac{\frac{1}{1+\frac{k}{x}}\left(-\frac{k}{x^2}\right)}{-\frac{1}{x^2}}$$

$$= \lim_{x\to\infty}\frac{k}{1+\frac{k}{x}}$$

Example

$$\lim_{x\to\infty}\left(1+\frac{k}{x}\right)^x = \lim_{x\to\infty} e^{\ln\left(1+\frac{k}{x}\right)^x}$$

exponent= continuous f-n

$$= e^{\lim_{x\to\infty} \ln\left(1+\frac{k}{x}\right)^x}$$

$$\lim_{x\to\infty}\ln\left(1+\frac{k}{x}\right)^x = \lim_{x\to\infty} x\ln\left(1+\frac{k}{x}\right)$$

$$= \lim_{x\to\infty}\frac{\frac{d}{dx}\left(\ln\left(1+\frac{k}{x}\right)\right)}{\frac{d}{dx}\left(\frac{1}{x}\right)}$$

form "$\frac{0}{0}$", use L'Hospital

$$= \lim_{x\to\infty}\frac{\frac{1}{1+\frac{k}{x}}\left(1+\frac{k}{x}\right)'}{-\frac{1}{x^2}}$$

$$= \lim_{x\to\infty}\frac{\frac{1}{1+\frac{k}{x}}\left(-\frac{k}{x^2}\right)}{-\frac{1}{x^2}}$$

$$= \lim_{x\to\infty}\frac{k}{1+\frac{k}{x}} = \textbf{?}$$

Example

$$\lim_{x\to\infty}\left(1+\frac{k}{x}\right)^x = \lim_{x\to\infty} e^{\ln\left(1+\frac{k}{x}\right)^x}$$ | exponent= continuous f-n

$$= e^{\lim\limits_{x\to\infty}\ln\left(1+\frac{k}{x}\right)^x}$$

$$\lim_{x\to\infty}\ln\left(1+\frac{k}{x}\right)^x = \lim_{x\to\infty} x\ln\left(1+\frac{k}{x}\right)$$

$$= \lim_{x\to\infty}\frac{\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln\left(1+\frac{k}{x}\right)\right)}{\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x}\right)}$$ | form “$\frac{0}{0}$”, use L'Hospital

$$= \lim_{x\to\infty}\frac{\frac{1}{1+\frac{k}{x}}\left(1+\frac{k}{x}\right)'}{-\frac{1}{x^2}}$$

$$= \lim_{x\to\infty}\frac{\frac{1}{1+\frac{k}{x}}\left(-\frac{k}{x^2}\right)}{-\frac{1}{x^2}}$$

$$= \lim_{x\to\infty}\frac{k}{1+\frac{k}{x}} = k$$

Example

$$\lim_{x\to\infty}\left(1+\frac{k}{x}\right)^x = \lim_{x\to\infty} e^{\ln\left(1+\frac{k}{x}\right)^x}$$

exponent= continuous f-n

$$= e^{\lim\limits_{x\to\infty}\ln\left(1+\frac{k}{x}\right)^x} = \mathbf{?}$$

$$\lim_{x\to\infty}\ln\left(1+\frac{k}{x}\right)^x = \lim_{x\to\infty} x\ln\left(1+\frac{k}{x}\right)$$

$$= \lim_{x\to\infty}\frac{\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln\left(1+\frac{k}{x}\right)\right)}{\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x}\right)}$$

form "$\frac{0}{0}$", use L'Hospital

$$= \lim_{x\to\infty}\frac{\frac{1}{1+\frac{k}{x}}\left(1+\frac{k}{x}\right)'}{-\frac{1}{x^2}}$$

$$= \lim_{x\to\infty}\frac{\frac{1}{1+\frac{k}{x}}\left(-\frac{k}{x^2}\right)}{-\frac{1}{x^2}}$$

$$= \lim_{x\to\infty}\frac{k}{1+\frac{k}{x}} = k$$

Example

$$\lim_{x\to\infty}\left(1+\frac{k}{x}\right)^x = \lim_{x\to\infty} e^{\ln\left(1+\frac{k}{x}\right)^x}$$

exponent= continuous f-n

$$= e^{\lim\limits_{x\to\infty}\ln\left(1+\frac{k}{x}\right)^x} = e^k$$

$$\lim_{x\to\infty}\ln\left(1+\frac{k}{x}\right)^x = \lim_{x\to\infty} x\ln\left(1+\frac{k}{x}\right)$$

$$= \lim_{x\to\infty}\frac{\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln\left(1+\frac{k}{x}\right)\right)}{\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x}\right)}$$

form "$\frac{0}{0}$", use L'Hospital

$$= \lim_{x\to\infty}\frac{\frac{1}{1+\frac{k}{x}}\left(1+\frac{k}{x}\right)'}{-\frac{1}{x^2}}$$

$$= \lim_{x\to\infty}\frac{\frac{1}{1+\frac{k}{x}}\left(-\frac{k}{x^2}\right)}{-\frac{1}{x^2}}$$

$$= \lim_{x\to\infty}\frac{k}{1+\frac{k}{x}} = k$$

Example

$$\lim_{x\to\infty}\left(1+\frac{k}{x}\right)^x = \lim_{x\to\infty} e^{\ln\left(1+\frac{k}{x}\right)^x}$$

exponent= continuous f-n

$$= e^{\lim\limits_{x\to\infty}\ln\left(1+\frac{k}{x}\right)^x} = e^k$$

limit computed below

$$\lim_{x\to\infty}\ln\left(1+\frac{k}{x}\right)^x = \lim_{x\to\infty} x\ln\left(1+\frac{k}{x}\right)$$

$$= \lim_{x\to\infty}\frac{\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln\left(1+\frac{k}{x}\right)\right)}{\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x}\right)}$$

form "$\frac{0}{0}$", use L'Hospital

$$= \lim_{x\to\infty}\frac{\frac{1}{1+\frac{k}{x}}\left(1+\frac{k}{x}\right)'}{-\frac{1}{x^2}}$$

$$= \lim_{x\to\infty}\frac{\frac{1}{1+\frac{k}{x}}\left(-\frac{k}{x^2}\right)}{-\frac{1}{x^2}}$$

$$= \lim_{x\to\infty}\frac{k}{1+\frac{k}{x}} = k$$