Calculus II
Homework
Differential equation basics

1. (a) A tank contains 30 kg of salt dissolved in 10000 liters of water and salt solution. Brine that contains 0.05 kg of salt per liter enters the tank at a rate of 10 liters per minute. The solution is kept thoroughly mixed and drains from the tank at the same rate (10 liters per minute). Determine how much salt remains in the tank after 45 minutes.

   answer: $500 - 470e^{-\frac{9}{200}} \approx 50.68kg$

   (b) A tank contains 1000 kg of salt dissolved in 10000 liters of water. Brine that contains 0.05 kg of salt per liter of water enters the tank at a rate of 30 liters per minute. The solution is kept thoroughly mixed and drains from the tank at the same rate (30 liters per minute).

   i. Determine how much salt remains in the tank after an hour. The answer key has not been proofread, use with caution.

   answer: $500 + 500e^{-0.18} \approx 917.64kg$

   ii. Determine how much time will be needed in order to have the concentration of salt in the tank reach 0.0501kg/liter. The answer key has not been proofread, use with caution.

   answer: $\frac{1000}{3} \ln 500 \approx 2071.54min \approx 34.53hours$

Solution. 1.a. Let

$$y(t) = \text{salt in the tank after } t \text{ minutes (in kg)} \quad .$$

We are given $y(0) = 30$kg, the initial amount of salt. We are looking to find $y(45)$, the amount of salt after 45 minutes. We have that

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \text{(rate in)} - \text{(rate out)} \quad .$$

The rate of salt entering the tank is constant:

$$\text{(rate in)} = 0.05kg/L \cdot 10L/min = 0.5kg/min \quad .$$

As the solution is thoroughly mixed, at any time the concentration of salt in the tank is

$$\frac{y}{10000} kg/L.$$

Therefore the rate of salt going out of the tank is

$$\text{(rate out)} = \frac{y}{10000} kg/L * 10L/min = \frac{y}{1000} kg/min \quad .$$

Therefore the differential equation for the amount of salt in the tank is

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \underbrace{0.5}_{\text{(rate in)}} - \underbrace{\frac{y}{1000}}_{\text{(rate out)}} \quad .$$

There are two variants for remainder of the solution. Variant I uses indefinite integration and is slightly informal, but is easier to learn and remember. Variant II is rigorous, but more challenging understand and write up. Both solutions are acceptable for full credit in a Calculus exam. Variant I is recommended when taking exams and Variant II is recommended when writing scientific texts.

Variant I

$$
\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}t} &= 0.5 - \frac{y}{1000} \quad . \\
\frac{\mathrm{d}y}{\mathrm{d}t} &= \frac{500-y}{1000} \quad . \\
\frac{1000}{500-y}\frac{\mathrm{d}y}{\mathrm{d}t} &= 1 && \Big| \text{ Use indefinite integration} \\
\int \frac{1000}{500-y}\underbrace{\frac{\mathrm{d}y}{\mathrm{d}t}\mathrm{d}t}_{\mathrm{d}y} &= \int \mathrm{d}t \\
\int \frac{1000}{500-y}\mathrm{d}y &= t + C \\
-1000\int \frac{1}{500-y}\mathrm{d}(500-y) &= t + C \\
-1000\ln|500-y| &= t + C && \Big| \text{ The constant from the second integral is accounted by the constant } C \\
\ln|500-y| &= -\frac{t+C}{1000} \\
|500-y| &= e^{-\frac{t+C}{1000}} && \Big| \text{ Since } 500 - y(0) = 500 - 30 = 470 > 0 \text{ we can drop the absolute values} \\
500-y &= e^{-\frac{t+C}{1000}} \\
y &= 500 - e^{-\frac{t+C}{1000}} && \Big| \text{ Set } D = e^{-\frac{C}{1000}} \\
y &= 500 - De^{-\frac{t}{1000}} \quad .
\end{aligned}
$$

To find the constant D, we observe that

$$
\begin{aligned}
30 &= y(0) = 500 - De^{-\frac{0}{1000}} = 500 - D \\
D &= 470 \quad .
\end{aligned}
$$

Therefore

$$y(t) = 500 - 470e^{-\frac{t}{1000}} \quad ,$$

and the final answer is

$$y(45) = 500 - 470e^{-\frac{45}{1000}} \approx 50.68$$

with measurement unit kg.

Variant II. To find $y(45)$, we integrate from $t = 0$ to $t = 45$:

$$
\begin{array}{rcll}
\displaystyle\int\limits_{t=0}^{45} \frac{1000}{500-y} \underbrace{\frac{\mathrm{d}y}{\mathrm{d}t}\mathrm{d}t}_{\mathrm{d}(y(t))} &=& \displaystyle\int\limits_{t=0}^{45} \mathrm{d}t & \\
\displaystyle\int\limits_{t=0}^{t=45} \frac{1000}{500-y(t)} \mathrm{d}(y(t)) &=& 45 & \Big| \text{ set } z = y(t) \\
-1000 \displaystyle\int\limits_{z=y(0)=30}^{z=y(45)} \frac{1}{500-z} \mathrm{d}(500-z) &=& 45 & \\
-1000 \ln|500-y|\big]_{y(0)=30}^{y(45)} &=& 45 & \\
-1000 \left(\ln|500-y(45)|\right. & & & \\
\left. -\ln|500-30|\right) &=& 45 & \\
\ln\left|\dfrac{470}{500-y(45)}\right| &=& \dfrac{45}{1000} & \\
\ln\left(\dfrac{470}{500-y(45)}\right) &=& \dfrac{45}{1000} & \Big| \text{ see below} \\
\dfrac{470}{500-y(45)} &=& e^{\frac{45}{1000}} & \\
500 - y(45) &=& 470e^{-\frac{9}{200}} & \\
y(45) &=& 500 - 470e^{-\frac{9}{200}} & \\
&\approx& 500 - 470 \cdot 0.955997 & \\
&\approx& 50.681184 \quad , &
\end{array}
$$

where we have used that $\dfrac{470}{500-y(t)} > 0$. The fact that $\dfrac{470}{500-y(t)} > 0$ can be seen as follows. As $500 - y(0) = 470 > 0$ and $y(t)$ is continuous, in order to have $500 - y(t) < 0$ there must exist some x_1 for which $y(x_1) = 500$. However this is impossible since $x = \ln\left|\dfrac{470}{500-y(x)}\right|$.

As the unit of measurement is kg, the final answer to the problem is $\approx 50.68kg$ salt.

2. (a)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 - 1 \quad . \qquad (1)$$

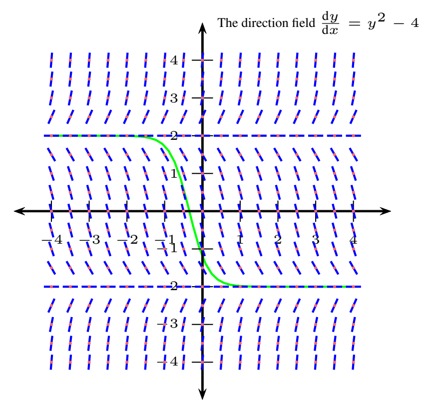
i. Find all solutions of the differential equation above.

ii. Find a solution for which $y(0) = -\frac{3}{5}$.

(b) i. Find the general solution to the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 - 4 \quad .$$

Below is a computer-generated plot of the direction field $\dfrac{\mathrm{d}y}{\mathrm{d}x} = y^2 - 4$, you may use it to get a feeling for what your answer should look like.



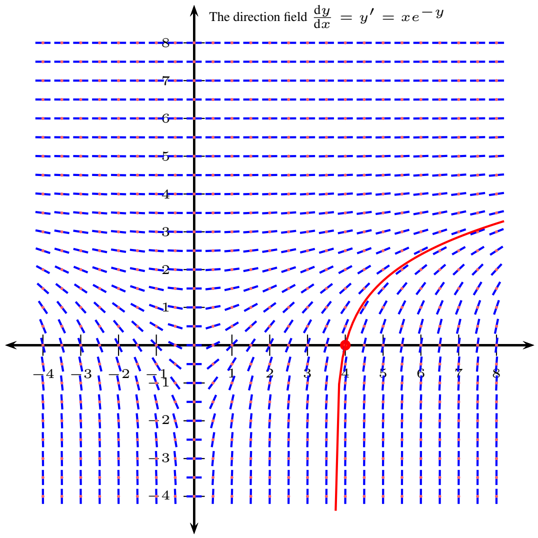
ii. Find a solution of the above equation for which $y(0) = -\frac{6}{5}$.

(c) Solve the initial-value differential equation $y' = y^2(1 + x)$, $y(0) = 3$.

(d) Solve the initial-value differential equation problem

$$y' = xe^{-y} \quad , \qquad y(4) = 0.$$

Below is a computer-generated plot of the corresponding direction field, you may use it to get a feeling for what your answer should look like.

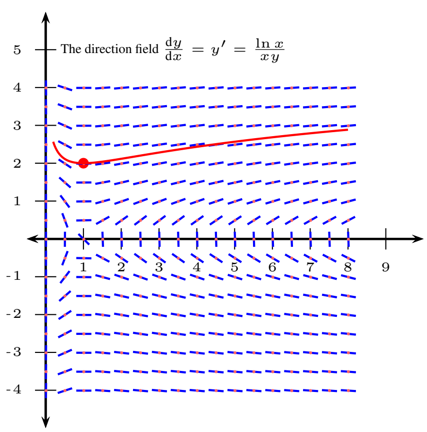


answer: $y(x) = \ln\left(\frac{x^2}{2} - 7\right)$

(e) Solve the initial-value differential equation problem

$$y' = \frac{\ln x}{xy} \quad , \qquad y(1) = 2.$$

Below is a computer-generated plot of the corresponding direction field, you may use it to get a feeling for what your answer should look like.



answer: $y(x) = \sqrt{(\ln x)^2 + 4}$

(f) i. Solve the initial-value differential equation problem

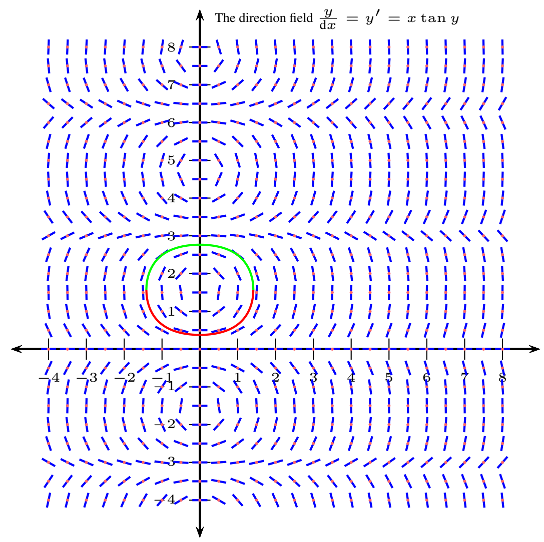
$$y' = x \tan y \quad , \qquad y(0) = \arcsin\left(\frac{1}{e}\right) \approx 0.376728.$$

answer: $y(x) = \arcsin\left(e^{\frac{x^2}{2}-1}\right)$

ii. Solve the same differential equation with initial condition $y(0) = \pi + \arcsin\left(-\frac{1}{e}\right) \approx 2.764865$.

answer: $y(x) = \pi + \arcsin\left(-e^{\frac{x^2}{2}-1}\right)$

Below is a computer-generated plot of corresponding direction field, you may use it to get a feeling for what your answer should look like.



Solution. 2.a.i. There are two variants for solving this problem. The first variant uses indefinite integration and is slightly informal, but easier to apply and remember. The second variant is more rigorous but more difficult to write up. Both solutions are acceptable for full credit in a Calculus exam. Variant I is recommended when taking exams and Variant II is recommended when writing scientific texts.

Variant I

$$
\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= y^2 - 1 && \Big| \text{ Suppose } y^2-1 \neq 0 \\
\frac{\frac{\mathrm{d}y}{\mathrm{d}x}}{y^2-1} &= 1 \\
\int \frac{1}{y^2-1} \underbrace{\frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x}_{=\mathrm{d}y} &= \int \mathrm{d}x \\
\int \frac{\mathrm{d}y}{y^2-1} &= x + C \\
\int \left(\frac{\frac{1}{2}}{y-1} - \frac{\frac{1}{2}}{y+1} \right) \mathrm{d}y &= x + C \\
\frac{1}{2} \ln \left| \frac{y-1}{y+1} \right| &= x + C \\
\ln \left| \frac{y-1}{y+1} \right| &= 2x + 2C \\
\left| \frac{y-1}{y+1} \right| &= e^{2x+2C} \\
\frac{y-1}{y+1} &= \pm e^{2x+2C} \\
y - 1 &= \pm e^{2x+2C}(y+1) \\
y(1 \mp e^{2x+2C}) &= 1 \pm e^{2x+2C} \\
y &= \frac{1 \pm e^{2x+2C}}{1 \mp e^{2x+2C}} \\
y &= \frac{1 \pm e^{2C} e^{2x}}{1 \mp e^{2C} e^{2x}} && \Big| \text{ Set } D = \pm e^{2C} \\
y &= \frac{1 + De^{2x}}{1 - De^{2x}} \quad .
\end{aligned}
$$

The above solution works on condition that $y^2 - 1 \neq 0$. So the only case not covered is that of $y^2 - 1 = 0$, which yields the two solutions $y = \pm 1$.

Our final answer is

$$y(x) = \frac{1 + De^{2x}}{1 - De^{2x}} \quad \text{or} \quad y(x) = -1,$$

where D is an arbitrary real number. Notice that in the above answer, by allowing $D = 0$, we have covered the case $y(x) = 1$. Finally, we note that if we let $D \to \infty$, the solution $y(x) = \frac{1+De^{2x}}{1-De^{2x}}$ tends to the solution $y(x) = -1$ (here we fix a value of x before we let $D \to \infty$).

Variant II

Case 1. Suppose there exists a number x_0 such that $(y(x_0))^2 - 1 \neq 0$. Since y is a differentiable function of x, it is also continuous. Therefore for some t sufficiently close to x_0, all numbers x in the interval between t and x_0 satisfy $y(x)^2 - 1 \neq 0$.

$$\begin{array}{rcll}
\frac{\frac{\mathrm{d}y}{\mathrm{d}x}}{y^2-1} &=& 1 & \\
\int\limits_{x=x_0}^{x=t} \frac{1}{y^2-1} \underbrace{\frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x}_{=\mathrm{d}(y(x))} &=& \int\limits_{x=x_0}^{x=t} \mathrm{d}x & \Big| \text{can integrate as } y(x)^2-1\neq 0 \\
\int\limits_{t=x_0}^{x=t} \frac{\mathrm{d}(y(x))}{(y(x))^2-1} &=& x|_{x=x_0}^{x=t} & \Big| \text{set } z = y(x) \\
\int\limits_{z=y(x_0)}^{z=y(t)} \frac{\mathrm{d}z}{z^2-1} &=& t - x_0 & \\
\int\limits_{z=y(x_0)}^{z=y(t)} \left(\frac{\frac{1}{2}}{z-1} - \frac{\frac{1}{2}}{z+1}\right)\mathrm{d}z &=& t-x_0 & \\
\frac{1}{2}\ln\left|\frac{z-1}{z+1}\right|\Big]_{z=y(x_0)}^{z=y(t)} &=& t-x_0 & \Big| \text{Set } C = 2x_0 - \ln\left|\frac{y(x_0)-1}{y(x_0)+1}\right| \\
\ln\left|\frac{y(t)-1}{y(t)+1}\right| &=& 2t - C & \Big| \text{relabel dummy variable } t \text{ to } x \\
\ln\left|\frac{y(x)-1}{y(x)+1}\right| &=& 2x - C &
\end{array}$$

Set

$$D = e^{-C} \quad .$$

By the assumption of our case, $(y(x_0))^2 - 1 \neq 0$, so there are two remaining cases: $(y(x_0))^2 - 1 > 0$ and $(y(x_0))^2 - 1 < 0$.

Case 1.1. Suppose $\frac{y(x_0)-1}{y(x_0)+1} > 0$. As the function $y(x)$ is differentiable, it is also continuous. Therefore $\frac{y(x)-1}{y(x)+1} > 0$ for all x near x_0. Then we can remove the absolute values in the equality above to get that for all x close to x_0 we have that

$$\begin{array}{rcll}
\ln\left(\frac{y(x)-1}{y(x)+1}\right) &=& 2x - C & \Big| \text{exponentiate, recall } D = e^{-C} \\
\frac{y(x)-1}{y(x)+1} &=& De^{2x} & \\
y(x) - 1 &=& De^{2x}(y(x)+1) & \\
y(x)\left(1 - De^{2x}\right) &=& De^{2x} + 1 & \\
y(x) &=& \frac{1+De^{2x}}{1-De^{2x}} & .
\end{array}$$

The solution $y(x)$ given above satisfies $\frac{y(x)-1}{y(x)+1} = De^{2x}$ for all x. As $D > 0$, this implies that $\frac{y(x)-1}{y(x)+1} > 0$. Therefore the considerations above are valid for all x, rather than only for those x near x_0. Therefore our first case yields the solution

$$y(x) = \frac{1+De^{2x}}{1-De^{2x}} \quad .$$

Case 1.2. Suppose $\frac{y(x_0)-1}{y(x_0)+1} < 0$. Then for all x near x_0 we get $\ln\left|\frac{y(x)-1}{y(x)+1}\right| = \ln\left(\frac{1-y(x)}{y(x)+1}\right)$ and, similarly to Case 1, we get

$$\begin{array}{rcl}
\frac{1-y(x)}{y(x)+1} &=& De^{2x} \\
1 - y(x) &=& De^{2x}(y(x)+1) \\
y(x)\left(1 + De^{2x}\right) &=& 1 - De^{2x} \\
y(x) &=& \frac{1-De^{2x}}{1+De^{2x}} \quad .
\end{array}$$

Since D is a positive constant, we conclude in a fashion analogous to Case 1 that $y(x) < 0$ for all x.

Case 2. Suppose $(y(x_0))^2 - 1 = 0$. Then $y(x_0) = \pm 1$. Clearly the constant functions $y(x) = \pm 1$ are two solutions: if we can plug back $y = \pm 1$ in the original equation we get that $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ and y is a constant function of x. From the preceding two cases we know that if $\frac{y(x)-1}{y(x)+1}$ is defined and not equal to zero for some value of x, then $\frac{y(x)-1}{y(x)+1}$ is defined and not equal to zero for all values of x. Therefore the present case yields only two solutions, the constant functions $y(x) = \pm 1$.

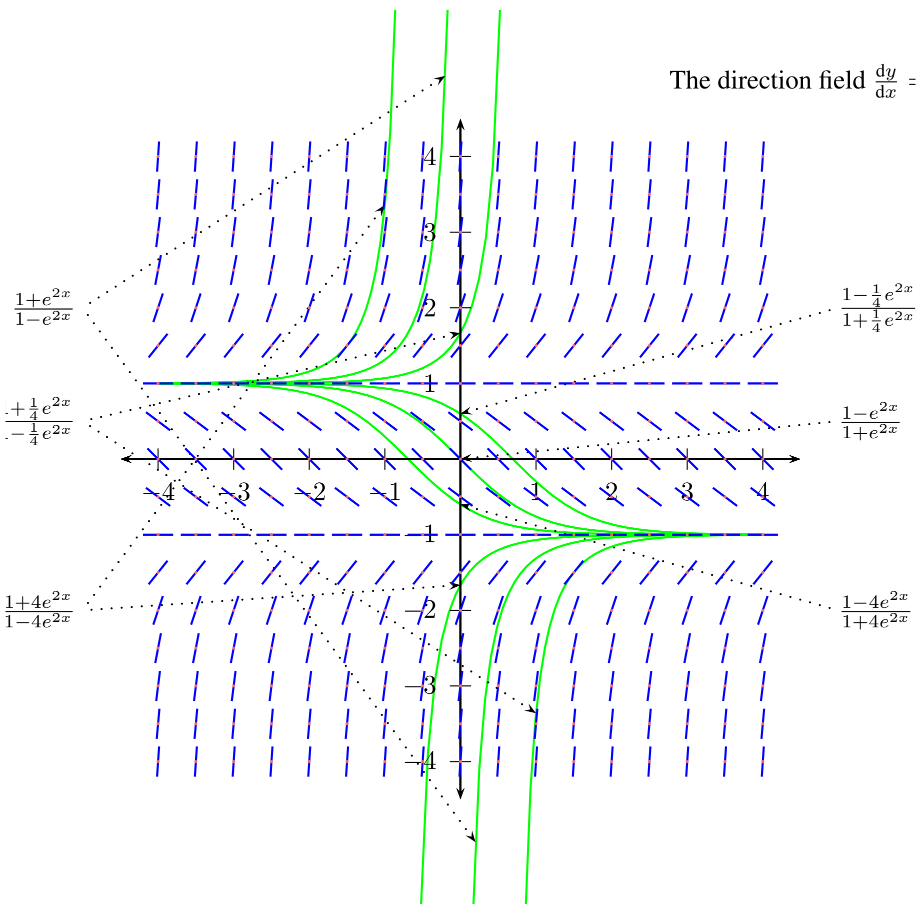
Our final answer is

$$y(x) = \frac{1 + De^{2x}}{1 - De^{2x}} \quad \text{or} \quad y(x) = -1,$$

where D is an arbitrary real number. Notice that in the above answer, we have combined Cases 1.1, 1.2 and the case $y(x) = 1$: by allowing D to be negative we included Case 1.2 and by allowing D to be zero we included the case $y(x) = 1$. Finally, we note that if we let $D \to \infty$, the solution $y(x) = \frac{1+De^{2x}}{1-De^{2x}}$ tends to the solution $y(x) = -1$ (for all values of x).

Solution plots.

We may plot solutions for a few values of D as follows. We overlay the solutions on top of the direction field of the differential equation. The picture tells us a lot about the properties of the solutions of the differential equations.



2.a.ii. From the computer generated picture above, we may visually estimate that $y(x) = \frac{1-4e^{2x}}{1+4e^{2x}}$ intersects the x-axis at $\left(0, -\frac{3}{5}\right)$. Furthermore, we may check directly that for

$$y(x) = \frac{1 - 4e^{2x}}{1 + 4e^{2x}}$$

we have $y(0) = \frac{1-4}{1+5} = -\frac{3}{5}$ and that is a solution to our problem (this however does not prove the solution is unique).

Alternatively, let us give an algebraic solution. As we are given that $y(0) = -\frac{3}{5}$ and so

$$\begin{aligned} -\frac{3}{5} &= y(0) = \frac{1 - De^{2\cdot 0}}{1 + De^{2\cdot 0}} = \frac{1 - D}{1 + D} \\ -\frac{3}{5}(1 + D) &= 1 - D \\ \frac{2}{5}D &= \frac{8}{5} \\ D &= 4 \quad , \end{aligned}$$

which is our final answer.

Solution. 2.c.

This is a concise solution written up in a form suitable for exam taking.

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= y^2(1+x)\\
\frac{\mathrm{d}y}{y^2} &= (1+x)\mathrm{d}x\\
\int\frac{\mathrm{d}y}{y^2} &= \int(1+x)\mathrm{d}x\\
-\frac{1}{y} &= x+\frac{x^2}{2}+C\\
-\frac{1}{3} &= 0+0+C\\
y &= -\frac{1}{\frac{x^2}{2}+x-\frac{1}{3}} = -\frac{3}{3x^2+6x-2} \quad .
\end{aligned}$$

Solution. 2.f.i and 2.f.ii

$$\begin{aligned}
y' &= x\tan y\\
\frac{y'}{\tan y} &= x\\
\frac{(\cos y)y'}{\sin y} &= x && \Big|\ \text{Integrate from 0}\\
\int\limits_{t=0}^{t=x}\frac{\cos(y(t))}{\sin(y(t))}(y'\mathrm{d}t) &= \int\limits_{t=0}^{x} t\mathrm{d}t\\
\int\limits_{t=0}^{t=x}\frac{\cos(y(t))}{\sin(y(t))}\mathrm{d}(y(t)) &= \frac{x^2}{2} && \Big|\ \text{Set } z=y(t)\\
\int\limits_{z=y(0)}^{z=y(x)}\frac{\cos z}{\sin z}\mathrm{d}z &= \frac{x^2}{2}\\
\int\limits_{z=y(0)}^{z=y(x)}\frac{\mathrm{d}(\sin z)}{\sin z} &= \frac{x^2}{2}\\
[\ln|\sin z|]_{y(0)}^{y} &= \frac{x^2}{2}\\
\ln|\sin y|-\ln|\sin(y(0))| &= \frac{x^2}{2}\\
\ln|\sin y| &= \frac{x^2}{2}+\ln|\sin(y(0))|\\
|\sin y| &= e^{\frac{x^2}{2}+\ln|\sin(y(0))|}\\
|\sin y| &= \begin{cases} e^{\frac{x^2}{2}+\ln\left|\sin\left(\arcsin\left(\frac{1}{e}\right)\right)\right|} & \text{for problem 2.f.i}\\ e^{\frac{x^2}{2}+\ln\left|\sin\left(\pi+\arcsin\left(\frac{1}{e}\right)\right)\right|} & \text{for problem 2.f.ii}\end{cases}\\
|\sin y| &= e^{\frac{x^2}{2}+\ln\left(\frac{1}{e}\right)}\\
|\sin y| &= e^{\frac{x^2}{2}-1} && \Big|\ \begin{array}{l} y(0)>0 \text{ for both problems}\\ \text{therefore } \sin y(0)>0\end{array}\\
\sin y &= e^{\frac{x^2}{2}-1} \quad .
\end{aligned}$$

From the elementary properties of the trigonometric functions, we know that $\sin y = \sin\alpha$ implies that either

- $y = \alpha + 2k\pi$, where k is an arbitrary integer or
- $y = (2k+1)\pi - \alpha$, where k is an arbitrary integer.

In other words, if we are given $\sin y$, we know y up to a choice of sign and a choice of an integer k. For our problem, this means that

$$y = \begin{cases} 2k\pi + \arcsin\left(e^{\frac{x^2}{2}-1}\right) & k - \text{integer} \\ \text{or} & \\ (2k+1)\pi - \arcsin\left(e^{\frac{x^2}{2}-1}\right) & k - \text{integer} \end{cases}$$

For problem 2.f.i, the only choice for k and sign which fits the initial condition $y(0) = \arcsin\left(\frac{1}{e}\right)$ is

$$y = \arcsin\left(e^{\frac{x^2}{2}-1}\right) \quad ,$$

which is our final answer.

For problem 2.f.ii, the only choice for k and sign which fits the initial condition $y(0) = \pi + \arcsin\left(-\frac{1}{e}\right) = \pi - \arcsin\left(\frac{1}{e}\right)$ is

$$y = \pi - \arcsin\left(e^{\frac{x^2}{2}-1}\right) \quad ,$$

which is our final answer.