Calculus I
Fermat’s Theorem and the Mean Value Theorem

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Outline

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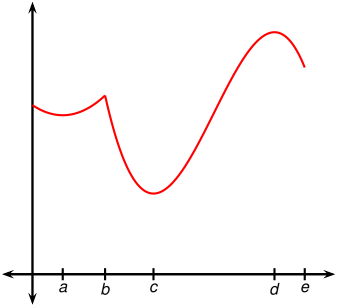
Maximum and Minimum Values

Many real-world problems involve finding minima and maxima (finding minimal costs, maximal profit, shortest time to do a job, etc.). Examples include

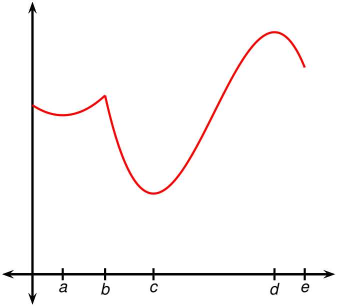
- What shape of can minimizes manufacturing costs?
- What is the maximum acceleration of a space shuttle?
- What is the maximum load an elevator can carry?

Often such questions can be reduced to finding maximum or minimum values of a function. In Calculus I, we study how to minimize and maximize functions in one variable.



Definition (Absolute Maximum or Minimum)

A function f has an absolute maximum (or global maximum) at c if $f(c) \geq f(x)$ for all x in the domain of f. The number $f(c)$ is called the maximum value of f.
Likewise, f has an absolute minimum at c if $f(c) \leq f(x)$ for all x in the domain of f. $f(c)$ is called the minimum value of f.
Maximum and minimum values of f are called extreme values.



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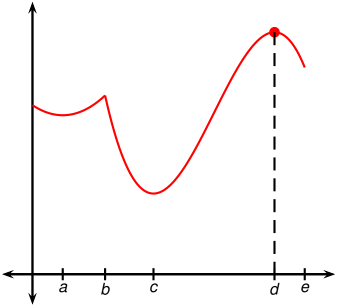

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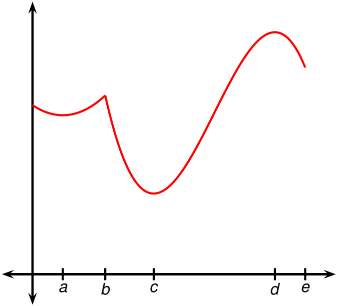

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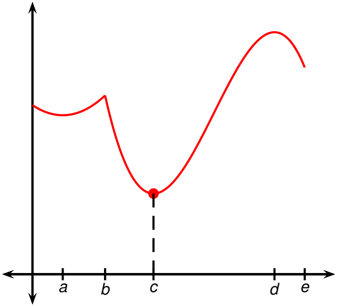

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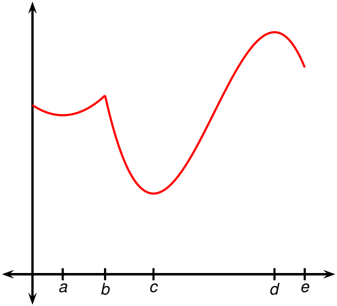

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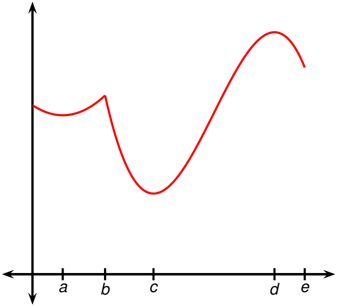

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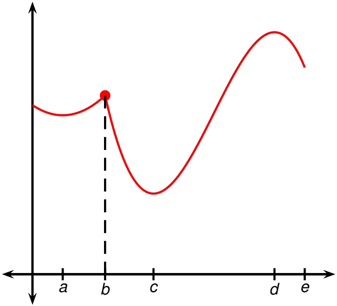

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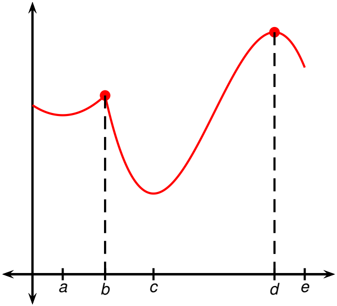

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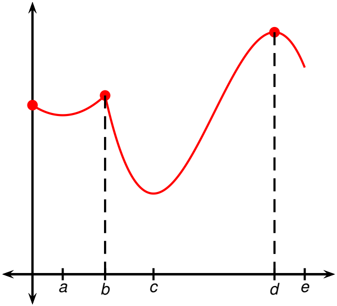

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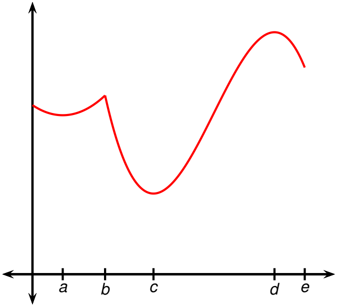

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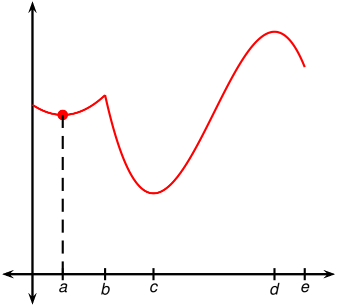

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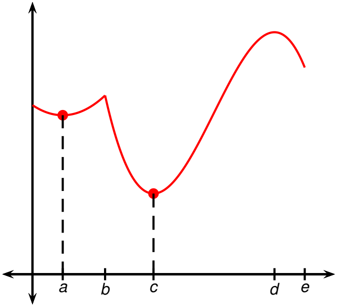

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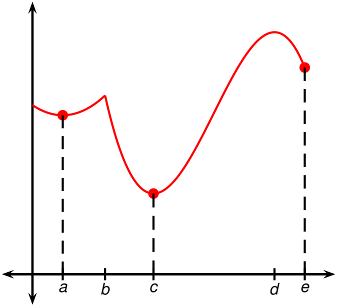

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Question

Is it possible that a function attains its maximum/minimum value for infinitely many values of x*?*

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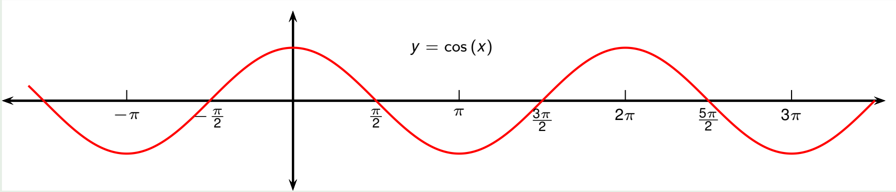
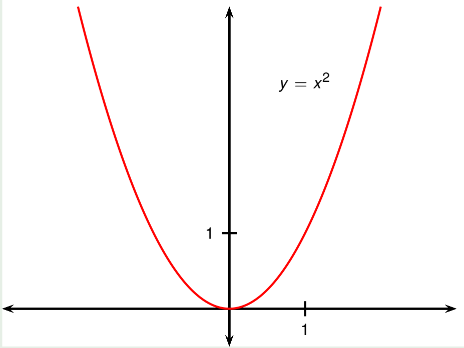
Example

The function $\cos x$ attains its maximum value (=1) infinitely many times, since $\cos(2n\pi) = 1$ for any integer n.
Likewise, it attains its minimum value of -1 infinitely many times, because $\cos((2n+1)\pi) = -1$ for all integers n.



Example

Consider the function $y = x^2$.



- Absolute maximum:
- Absolute minimum:
- Local maximum:
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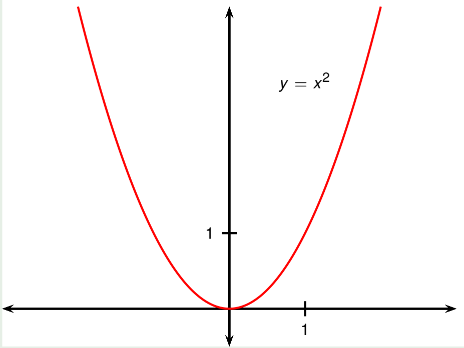

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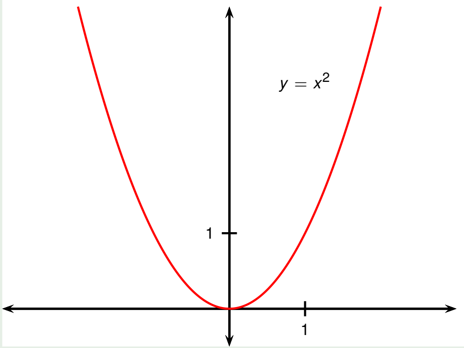

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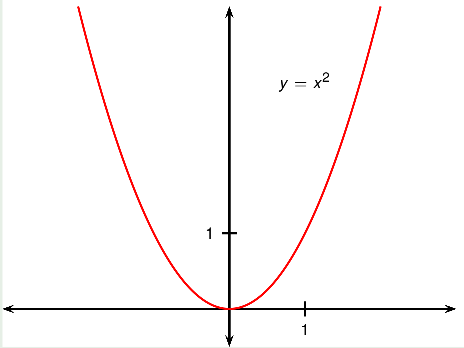

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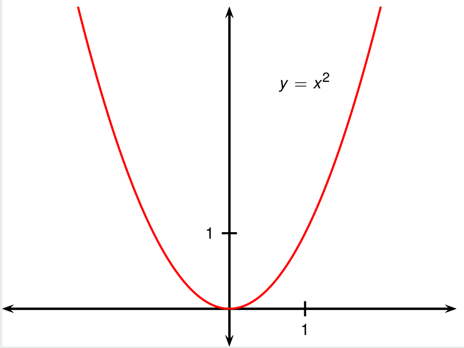

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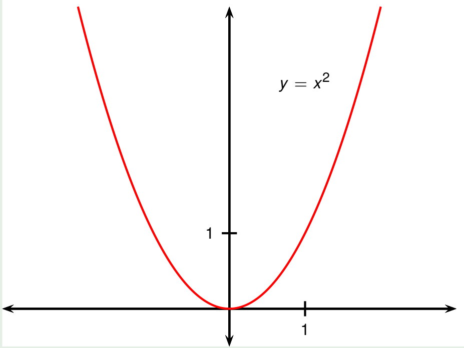

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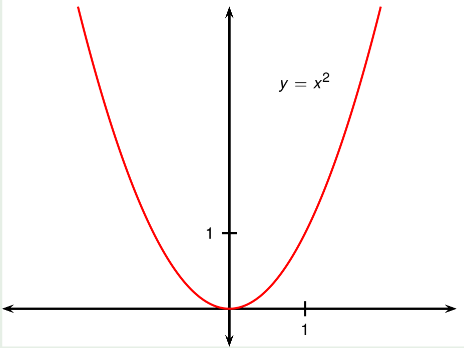

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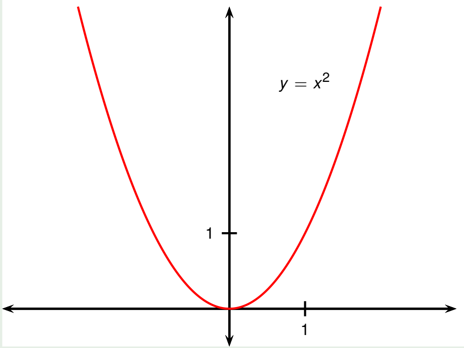

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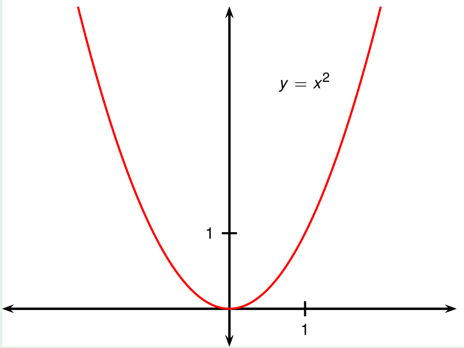

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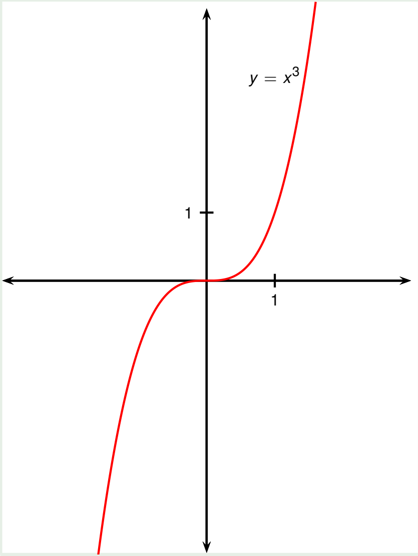

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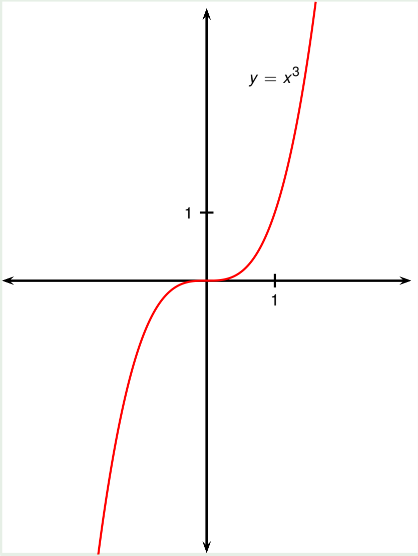

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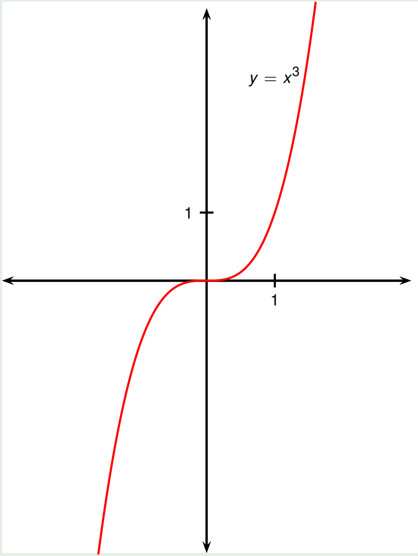

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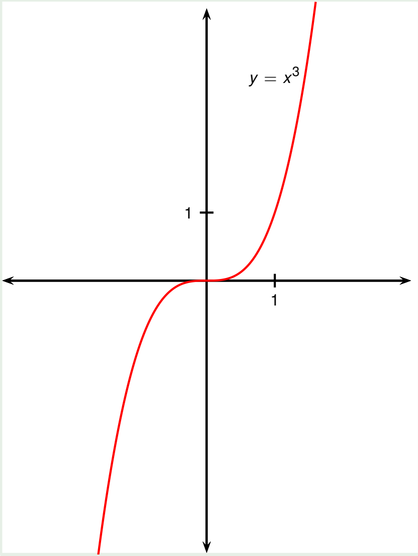

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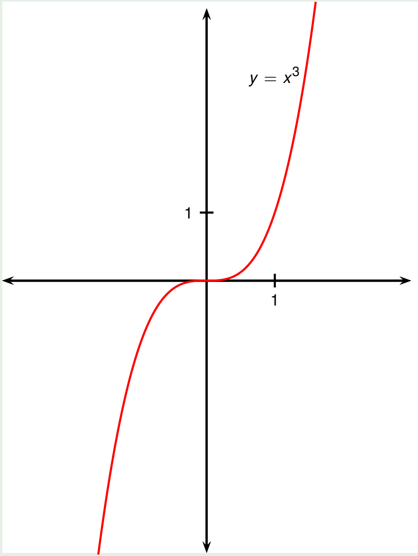

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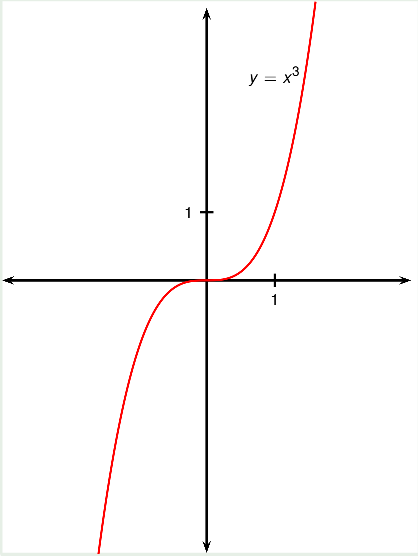

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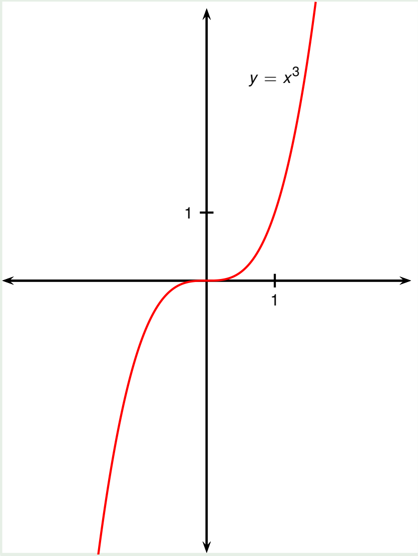

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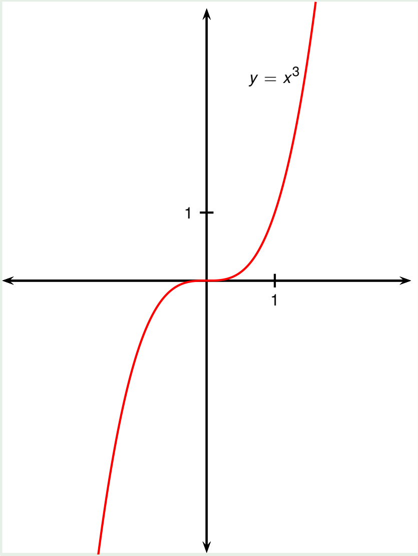

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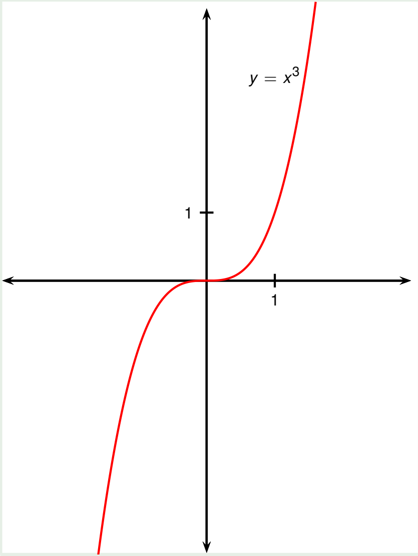

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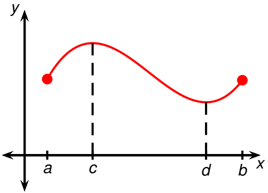
The Extreme Value Theorem

Recall that some functions (such as $y = \cos x$) have extreme values, while other functions (such as $y = x^3$) do not. The next theorem, which we will not prove, gives a condition under which f must have extreme values.

Theorem (The Extreme Value Theorem)

If f is continuous on a closed and bounded interval $[a, b]$, then f attains its maximum and minimum value, each at least once. In other words, there exist numbers c and d in $[a, b]$ such that

$$f(c) \geq f(x) \geq f(d) \qquad \text{for all } x \in [a, b]$$



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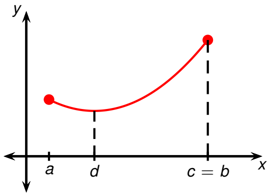

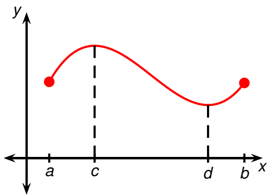

- Extreme values might happen at endpoints.

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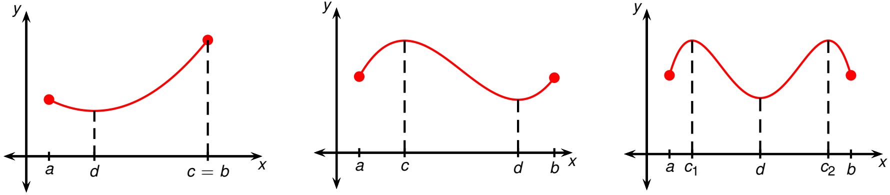

- Extreme values might happen at endpoints.
- Extreme values might happen twice.

Theorem (The Extreme Value Theorem)

If f is continuous on a closed interval $[a, b]$, then f attains its maximum and minimum value, each at least once.

- Do we need all of the hypotheses of the theorem?

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If f is continuous on a closed interval $[a, b]$, then f attains its maximum and minimum value, each at least once.

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- Do we need f to be continuous?
- Do we need the interval to be closed?

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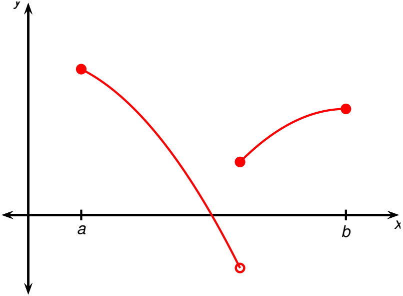

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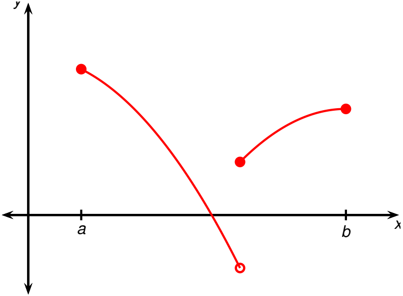

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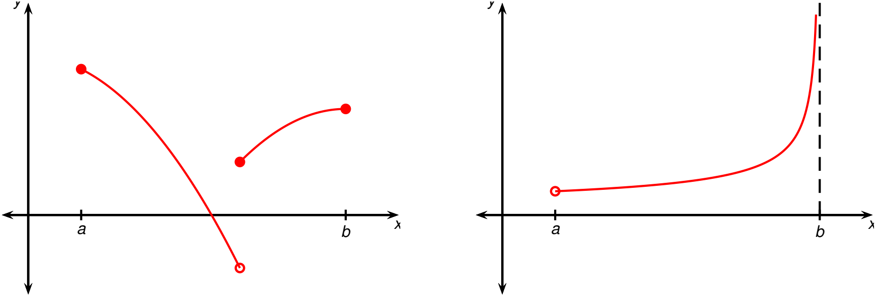

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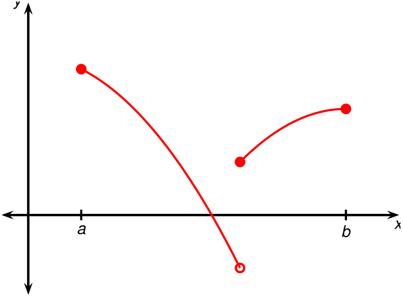

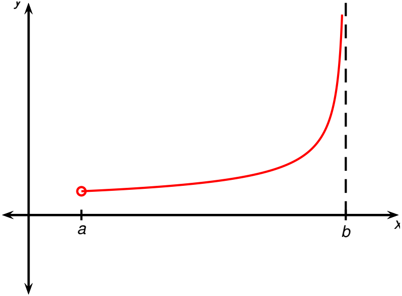

- Do we need all of the hypotheses of the theorem?
- Do we need f to be continuous? Yes.
- Do we need the interval to be closed? Yes.

Fermat's Theorem

The next theorem gives a condition that can help to find local maxima and minima.

Theorem (Fermat's Theorem)

Let f be a function defined in an open interval around c and such that $f'(c)$ exists. If f has a local maximum or minimum at c, then $f'(c) = 0$.

Proof.

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Let f be a function defined in an open interval around c and such that $f'(c)$ exists. If f has a local maximum or minimum at c, then $f'(c) = 0$.

Proof.

- We prove the theorem only when f has a local maximum at c.

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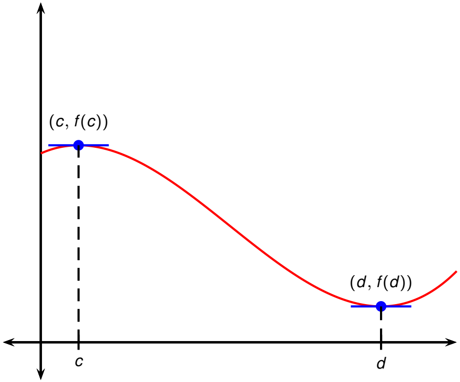

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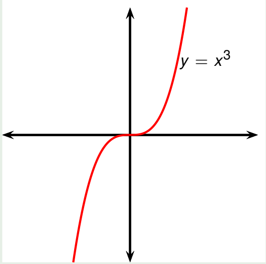

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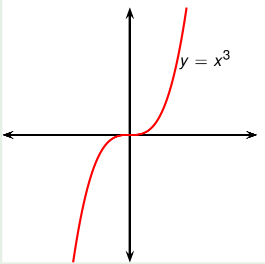

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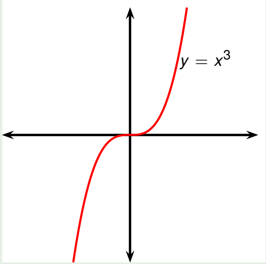

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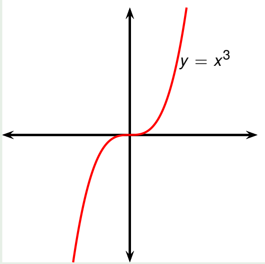

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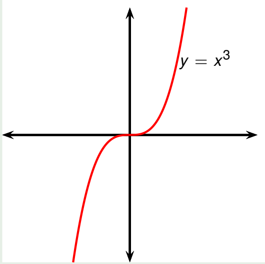

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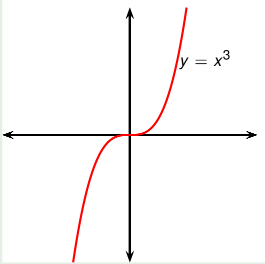

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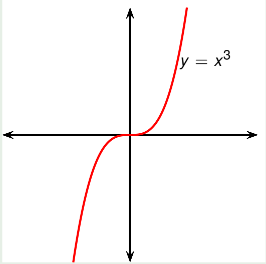

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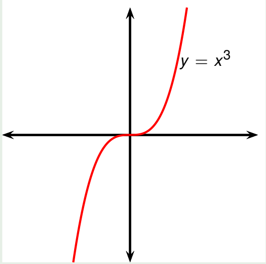

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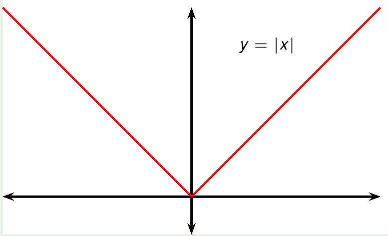

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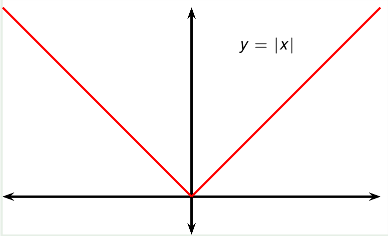

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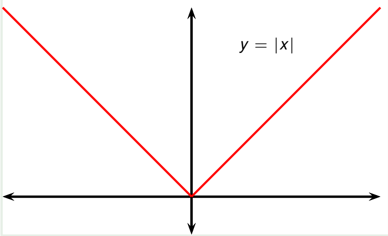

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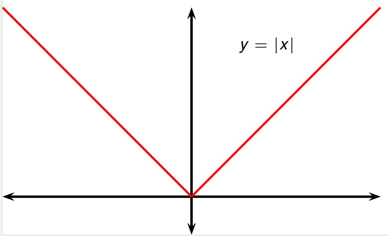

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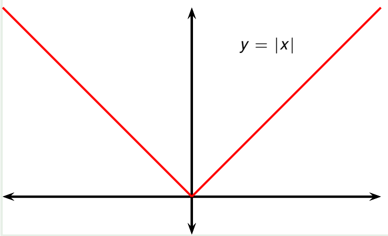

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The Mean Value Theorem

- The first derivative test, the results on concavity and curve sketching, as well as the (soon to be covered) topics of linear approximation and integration depend on an important theorem.
- This is the Mean Value Theorem.
- We will give a complete proof of the Mean Value Theorem.
- We start with a prerequisite result called Rolle's Theorem.

Theorem (Rolle's Theorem)

Let f be a function that satisfies the following three conditions:

- *f is continuous on the closed interval $[a, b]$.*
- *f is differentiable on the open interval (a, b).*
- *$f(a) = f(b)$.*

Then there is a number c in (a, b) such that $f'(c) = 0$.

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The proof breaks down into three cases:

1. f is a horizontal line.
2. $f(x) > f(a)$ for some x in (a, b).
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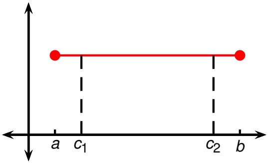

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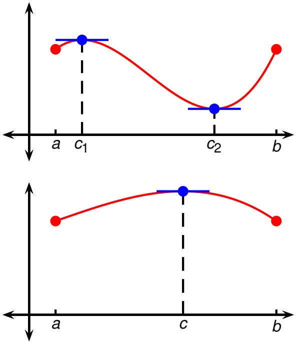

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1. f is a horizontal line.
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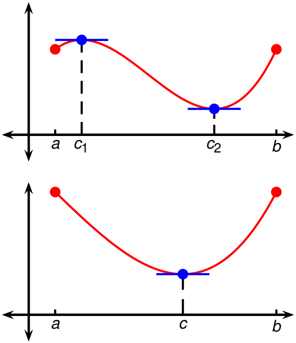

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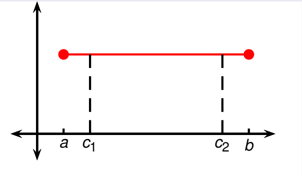

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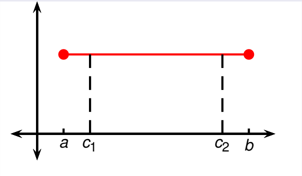

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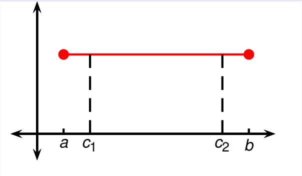

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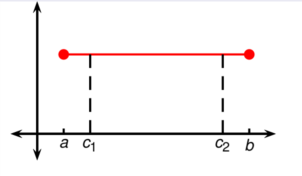

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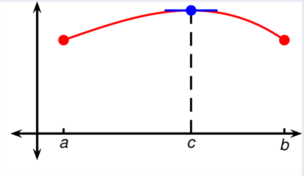

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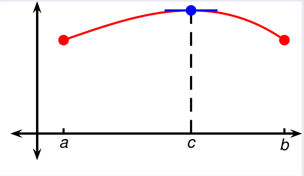

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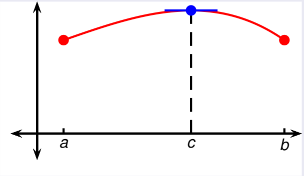

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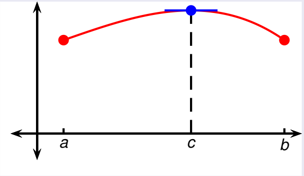

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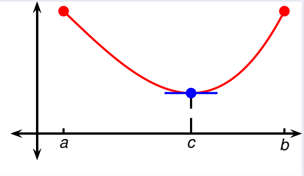

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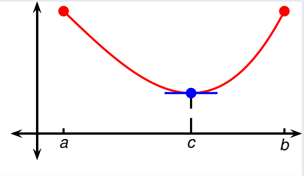

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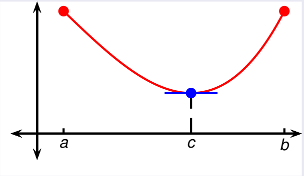

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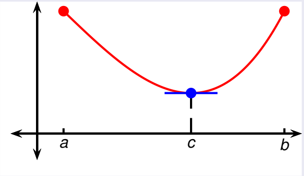

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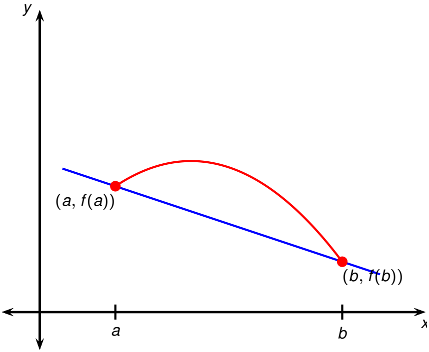

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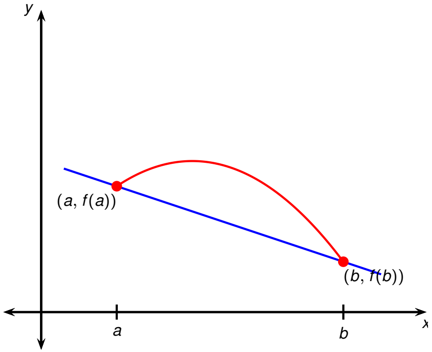

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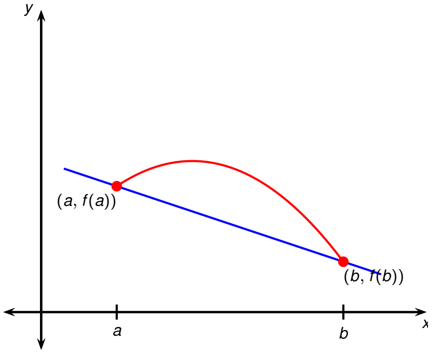

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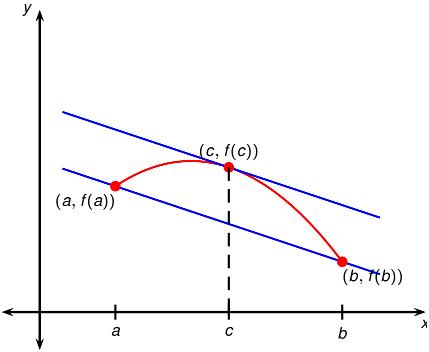

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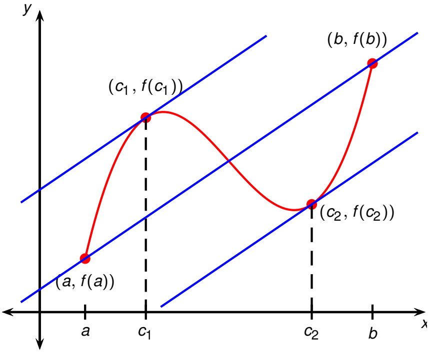

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Therefore f is constant on (a, b). $\square$

Corollary

If $f'(x) = g'(x)$ for all x in an interval (a, b), then $f - g$ is constant on (a, b); that is, $f(x) = g(x) + c$ where c is constant.

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- By the previous theorem, F is constant, so $f - g$ is constant. □