

## Calculus II

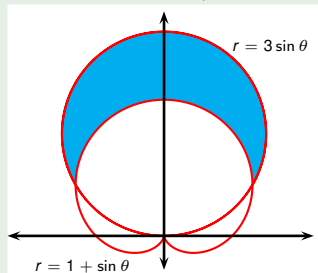
**Area swept by cardioid**  $r = a + b \sin(\theta)$ ,  
 $r = a + b \cos(n\theta)$

Todor Milev

2019

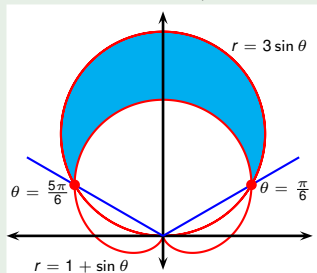
## Example

Find the area that lies within the circle  $r = 3 \sin \theta$  and outside of the cardioid  $r = 1 + \sin \theta$ .



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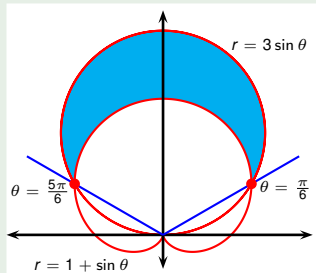
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$$\sin \theta = \frac{1}{2}$$

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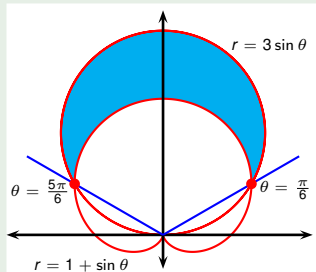
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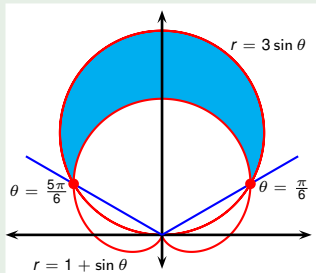
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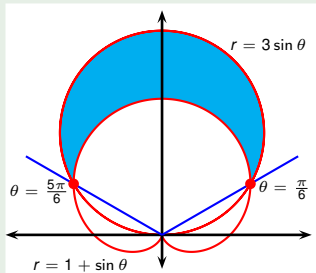
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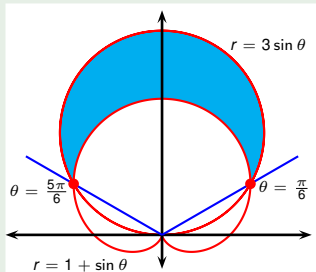
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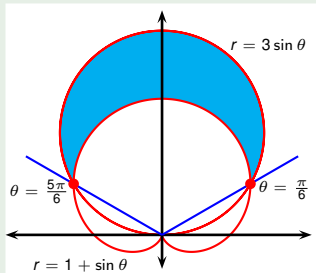
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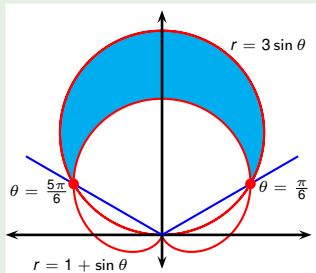
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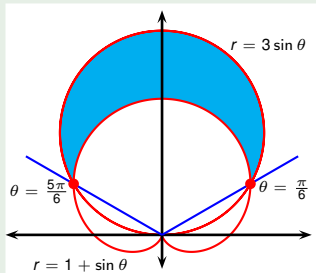
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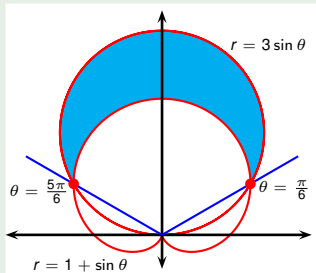
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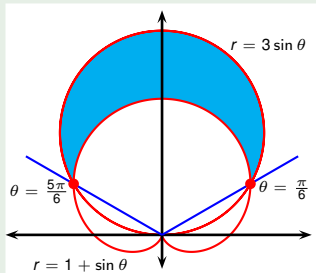
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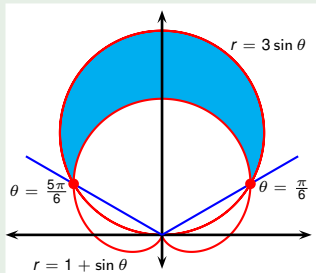
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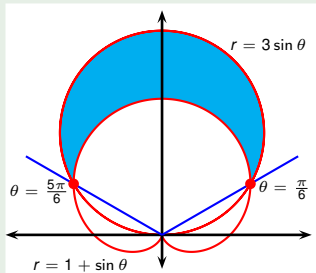
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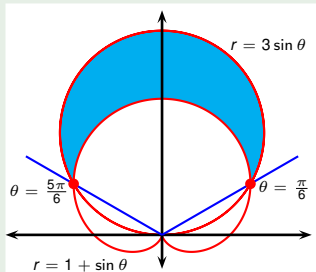
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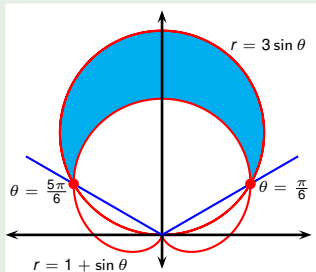
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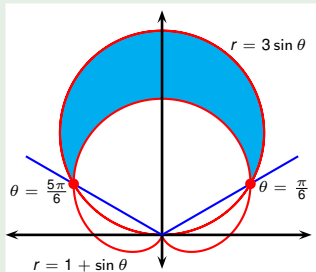
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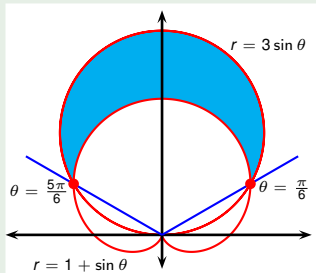
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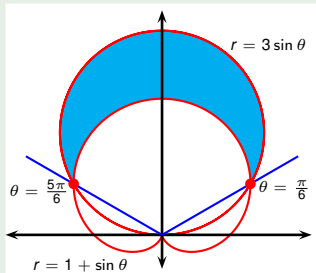
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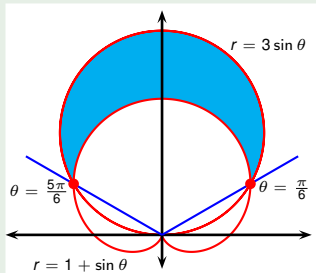
$$\sin \theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$\begin{aligned} A &= \frac{1}{2} \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (3 \sin \theta)^2 d\theta - \frac{1}{2} \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (1 + \sin \theta)^2 d\theta \\ &= \frac{1}{2} \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (9 \sin^2 \theta - (1 + 2 \sin \theta + \sin^2 \theta)) d\theta \\ &= \frac{1}{2} \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (8 \sin^2 \theta - 1 - 2 \sin \theta) d\theta \\ &= \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (8 \sin^2 \theta - 1 - 2 \sin \theta) d\theta \\ &= \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (3 - 4 \cos 2\theta - 2 \sin \theta) d\theta \\ &= [3\theta - 2 \sin 2\theta + 2 \cos \theta]_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \\ &= \left(3\frac{\pi}{2} - 2 \cdot 0 + 2 \cdot 0\right) - \left(3\frac{\pi}{6} - 2\frac{\sqrt{3}}{2} + 2\right) \end{aligned}$$

## Example

Find the area that lies within the circle  $r = 3 \sin \theta$  and outside of the cardioid  $r = 1 + \sin \theta$ .



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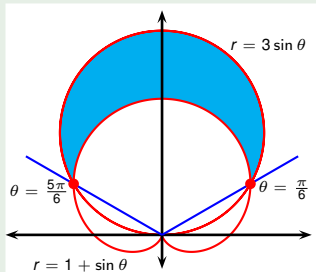
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