Calculus I
Homework
Exponents and logarithms review

1. Find the exact value of each expression.

   (a) $\log_5 125$.

   answer: 3

   (b) $\log_3 \frac{1}{27}$.

   answer: -3

   (c) $\ln\left(\frac{1}{e}\right)$.

   answer: -1

   (d) $\log_{10} \sqrt{10}$.

   answer: $\frac{1}{2}$

   (e) $e^{\ln 4.5}$.

   answer: 4.5

   (f) $\log_{10} 0.0001$.

   answer: 4

   (g) $\log_{1.5} 2.25$.

   answer: 2

   (h) $\log_5 4 - \log_5 500$.

   answer: -3

   (i) $\log_2 6 - \log_2 15 + \log_2 20$.

   answer: 3

   (j) $\log_3 100 - \log_3 18 - \log_3 50$.

   answer: -2

   (k) $e^{-2\ln 5}$.

   answer: $\frac{1}{25}$

   (l) $\ln\left(\ln e^{e^{10}}\right)$.

   answer: 10

2. Use the definition of a logarithm to evaluate each of the following without using a calculator.

   (a) $\log_2 16$

   (b) $\log_3\left(\frac{1}{9}\right)$

   (c) $\log_{10} 1000$

   (d) $\log_6 36^{-\frac{2}{3}}$

   (e) $\log_2(8\sqrt{2})$

   (f) $\log_7\left(\frac{49^x}{343^y}\right)$

   Solution.

$$\begin{array}{rcl} \log_7\left(\frac{49^x}{343^y}\right) & = & \log_7 49^x - \log_7 343^y \\ & = & x\log_7 49 - y\log_7 343 \\ \text{But } 49 = 7^2 \text{ and } 343 = 7^3, \text{ therefore } \log_7\left(\frac{49^x}{343^y}\right) & = & 2x - 3y. \end{array}$$

3. Express each of the following as a single logarithm.

   (a) $\ln 4 + \ln 6 - \ln 5$

   (b) $2\ln 2 - 3\ln 3 + 4\ln 4$

Solution.

$$
\begin{aligned}
2\ln 2 - 3\ln 3 + 4\ln 4 &= \ln 2^2 - \ln 3^3 + \ln 4^4 \\
&= \ln 4 - \ln 27 + \ln 256 \\
&= \ln\left(\frac{4}{27}\right) + \ln 256 \\
&= \ln\left(\frac{4\cdot 256}{27}\right) \\
&= \ln\left(\frac{1024}{27}\right).
\end{aligned}
$$

(c) $\ln 36 - 2\ln 3 - 3\ln 2$

4. Solve each equation for x. If available, use a calculator to give an ($\approx$) answer in decimal notation. If available, use a calculator to verify your approximate solutions.

(a) $e^{7-4x} = 7.$

answer: $\frac{7-\ln 7}{4} \approx 1.263522$

(b) $\ln(2x-9) = 2.$

answer: $\frac{e^2+9}{2} \approx 8.194528$

(c) $\ln(x^2-2) = 3.$

answer: $\pm\sqrt{e^3+2} \approx \pm 4.69925$

(d) $2^{x-3} = 5.$

answer: $\log_2 5 + 3 = \frac{\ln 5}{\ln 2} + 3 \approx 5.321928$

(e) $\ln x + \ln(x-1) = 1.$

answer: $\frac{1}{2}\left(1+\sqrt{1+4e}\right) \approx 2.223$

(f) $e^{2x+1} = t.$

answer: $\frac{\ln t - 1}{2}$

(g) $\log_2(mx) = c.$

answer: $\frac{2^c}{m}$

(h) $e - e^{-2x} = 1.$

answer: $-\frac{1}{2}\ln(e-1) \approx -0.271$

(i) $8(1+e^{-x})^{-1} = 3.$

answer: $-\ln\frac{5}{3} = \ln\frac{3}{5} \approx -0.510826$

(j) $\ln(\ln x) = 1.$

answer: $e^e \approx 15.154$

(k) $e^{e^x} = 10.$

answer: $\ln(\ln 10) \approx 0.834$

(l) $\ln(2x+1) = 3 - \ln x.$

answer: $\frac{-1+\sqrt{1+8e^3}}{4} \approx 2.92878$

(m) $e^{2x} - 4e^x + 3 = 0.$

answer: $x = \ln 3 \approx 1.098612$, $x = 0$

(n) $e^{4x} + 3e^{2x} - 4 = 0.$

answer: $x = 0$

(o) $e^{2x} - e^x - 6 = 0.$

answer: $x = \ln 3$

(p) $4^{3x} - 2^{3x+2} - 5 = 0.$

answer: $x = \frac{\log_2 5}{3}$

(q) $3\cdot 2^x + 2\left(\frac{1}{2}\right)^{x-1} - 7 = 0.$

answer: $x = 0$ or $2 - \log_2 3 = 2 - \frac{\ln 3}{\ln 2}$

Solution. 4.d

$$
\begin{array}{rcl|l}
2^{x-3} & = & 5 & \text{take } \log_2 \\
x-3 & = & \log_2(5) & \text{add 3 to both sides} \\
x & = & \log_2(5)+3 & \text{answer is complete} \\
 & = & \dfrac{\ln 5}{\ln 2}+3 & \text{optional step: convert to } \ln \\
 & \approx & 5.321928095 & \text{calculator}
\end{array}
$$

Solution. 4.h

$$
\begin{array}{rcll}
e - e^{-2x} & = & 1 & \\
e^{-2x} & = & e-1 & \mid \text{apply } \ln \\
\ln e^{-2x} & = & \ln(e-1) & \\
-2x & = & \ln(e-1) & \\
x & = & -\dfrac{1}{2}\ln(e-1) & \\
 & \approx & -0.270662427 & \mid \text{calculator}
\end{array}
$$

Solution. 4.e

$$\begin{array}{rcl} \ln x + \ln(x-1) &=& 1 \\ \ln\left(x^2 - x\right) &=& 1 \\ e^{\ln(x^2-x)} &=& e^1 \\ x^2 - x &=& e \\ x^2 - x - e &=& 0 \\ \text{Quadratic formula:} \quad x &=& \dfrac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-e)}}{2(1)} \\ &=& \dfrac{1 \pm \sqrt{1+4e}}{2}. \end{array}$$

However $\frac{1-\sqrt{1+4e}}{2}$ is negative, so $\ln\left(\frac{1-\sqrt{1+4e}}{2}\right)$ is undefined. Hence the only solution is $x = \frac{1+\sqrt{1+4e}}{2} \approx 2.2229$.

Solution. 4.p

$$\begin{array}{rcl} 4^{3x} - 2^{3x+2} - 5 &=& 0 \\ 4^{3x} - 4 \cdot 2^{3x} - 5 &=& 0 \\ u^2 - 4u - 5 &=& 0 \\ (u-5)(u+1) &=& 0 \\ u = 5 & \text{or} & u = -1 \\ 2^{3x} = 5 & & 2^{3x} = -1 \\ 3x = \log_2(5) & & \text{no real solution} \\ x = \dfrac{\log_2 5}{3} & & \\ \text{Calculator: } x \approx 0.773976 & & \end{array} \qquad \left|\ \text{Set} \begin{array}{rcl} 2^{3x} &=& u \\ 4^{3x} &=& u^2 \end{array}\right.$$

Solution. 4.q

$$\begin{array}{rcl} 3 \cdot 2^x + 2\left(\dfrac{1}{2}\right)^{x-1} - 7 &=& 0 \\ 3 \cdot 2^x + 2\left(\dfrac{1}{2}\right)^{x}\left(\dfrac{1}{2}\right)^{-1} - 7 &=& 0 \\ 3 \cdot 2^x + 4\left(\dfrac{1}{2}\right)^{x} - 7 &=& 0 \\ 3u + \dfrac{4}{u} - 7 &=& 0 \\ 3u^2 - 7u + 4 &=& 0 \\ (u-1)(3u-4) &=& 0 \\ u = 1 & \text{or} & 3u - 4 = 0 \\ 2^x = 1 & & u = \dfrac{4}{3} \\ x = 0 & & 2^x = \dfrac{4}{3} \\ & & x = \log_2 \dfrac{4}{3} = \log_2 4 - \log_2 3 \\ & & x = 2 - \log_2 3 \\ \text{Calculator:} & & x \approx 0.415037 \end{array} \qquad \left|\ \begin{array}{l} \text{Set } 2^x = u \\ \\ \text{Multiply by } u \end{array}\right.$$