Calculus II Review of integration basics

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Outline

- 1 Integration, Review
 - The Evaluation Theorem (FTC part 2)
- Integration Techniques from Calc I, Review
 - Differential Forms, Review
- 3 Integration and Logarithms, Review

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- Latest version of the .tex sources of the slides: https://github.com/tmilev/freecalc
- Should the link be outdated/moved, search for "freecalc project".
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Antiderivatives

Definition (Antiderivative)

A function F is called an antiderivative of f on an interval I if F'(x) = f(x) for all x in I.

Theorem

Let f be a continuous function on [a, b]. Then f is integrable over [a, b].

In other words, $\int_{a}^{b} f(x)dx$ exists for any continuous (over [a, b]) function f.

Theorem (The Evaluation Theorem (FTC part 2))

If f is continuous on [a, b], then

$$\int_a^b f(x) dx = F(b) - F(a),$$

where F is any antiderivative of f.

Indefinite Integrals

- The Evaluation Theorem establishes a connection between antiderivatives and definite integrals.
- It says that $\int_a^b f(x) dx$ equals F(b) F(a), where F is an antiderivative of f.
- We need convenient notation for writing antiderivatives.
- This is what the indefinite integral is.

Definition (Indefinite Integral)

The indefinite integral of f is another way of saying the antiderivative of f, and is written $\int f(x)dx$. In other words,

$$\int f(x) dx = F(x) \qquad \text{means} \qquad F'(x) = f(x).$$

Example

$$\int x^4 \mathrm{d}x = \frac{x^5}{5} + C$$

because

$$\frac{\mathsf{d}}{\mathsf{d}x}\left(\frac{x^5}{5}+C\right)=x^4.$$

- The indefinite integral represents a whole family of functions.
- Example: the general antiderivative of $\frac{1}{x}$ is

$$F(x) = \begin{cases} \ln|x| + C_1 & \text{if} \quad x > 0\\ \ln|x| + C_2 & \text{if} \quad x < 0 \end{cases}$$

- We adopt the convention that the constant participating in an indefinite integral is only valid on one interval.
- $\int \frac{1}{x} dx = \ln |x| + C$, and this is valid either on $(-\infty, 0)$ or $(0, \infty)$.

Differentials

- Recall $\Delta y, \Delta x$ stand for change of x, y. Recall: $\Delta y \approx \frac{dy}{dx} \Delta x$
- $dy = \frac{dy}{dx} dx = dy$
- If we substitute Δy by the formal expression dy and Δx by the formal expression dx, the expression dx appears to "cancel" to give a formal identity.
- Define the differential d and the differential forms dx, d(f(x)) by requesting that d and dx satisfy the transformation law

$$d(f(x)) = f'(x)dx$$

for any differentiable function f(x). In abbreviated notation:

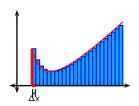
$$df = f' dx$$

Expressions containing expression of the form d(something) are called differential forms.

 \bullet df(x) = f'(x)dx.

- On the previous slide we stated the differential d and the differential forms dx, df(x) are formal expressions related by a transformation law.
- The precise definitions of differential forms and differentials are outside of the scope of Calculus I and II.
- Differential forms "encode" linear approximations which in turn "encode" "infinitesimal" lengths of segments.
- Courses such as "Integration and Manifolds" or "Differential geometry" usually give precise definitions and fill in the details.
- Nonetheless, what we studied is completely sufficient for practical purposes and carrying out computations.
- Do not confuse differentials with derivatives. The correct equality is this.

$$df(x) = f'(x)$$
 $df(x) = f'(x)dx$



- $\int_{a}^{b} f(x) dx$ is the definite integral of f.
- $\int f(x)dx =$ corresponding anti-derivative.
- ∫ stands for the limit of a Riemann sum (sum of approximating rectangles).
- dx "encodes" the base length of "infinitesimally small" approximating rectangle, f(x) is the height.
- f(x)dx is a differential form as discussed already.
- We postponed a formal definition of differential form to another course, but we showed how to compute with those.
- This is consistent: integrals of equal differential forms are equal (follows from Net Change Theorem (subst. rule)).

- All rules for computing with derivatives have analogues for computing with differential forms.
- The rules for computing differential forms are a direct consequence of the corresponding derivative rules and the transformation law d(f(x)) = f'(x)dx.

Let *c* be a constant.

Differential rule Derivative rule
$$d(fg) = gdf + fdg \qquad (fg)' = f'g + fg'$$

$$dc = 0 \qquad (c)' = 0$$

$$d(cf) = c \quad df \qquad (cf)' = cf'$$

$$d(f+g) = df + dg \qquad (f+g)' = f' + g'$$

$$df(g(x)) = f'(g(x))dg(x)$$

$$= f'(g(x))g'(x)dx \qquad (f(g(x)))' = f'(g(x))g'(x)$$

$$df(g) = f'(g)dg$$

$$dx^n = nx^{n-1}dx \qquad (x^n)' = nx^{n-1}$$

$$de^x = e^x dx \qquad (e^x)' = e^x$$

$$d\sin x = \cos x dx \qquad (\sin x)' = \cos x$$

$$d\cos x = -\sin x dx \qquad (\cos x)' = -\sin x$$

$$d\ln x = \frac{1}{x}dx \qquad (\ln x)' = \frac{1}{x}$$

Let *c* be a constant.

Corresponding integration rules. Integration rules justified via the Fundamental Theorem of Calculus

Derivative rule Integration rule $\int d(fg) = \int gdf + \int fdg$ (fg)' = f'g + fg' $\int dc = 0$ (c)' = 0 $\int d(cf) = c \int df$ (cf)'=cf' $\int d(f+g) = \int df + \int dg$ (f+g)'=f'+g' $\int df(g(x)) = \int f'(g(x))dg(x)$ $= \int f'(g(x))g'(x)dx \quad (f(g(x)))' = f'(g(x))g'(x)$ $\int df(g) = \int f'(g)dg$ $dx^n = nx^{n-1}dx$ $(x^n)' = nx^{n-1}$ $de^x = e^x dx$ $(e^x)'=e^x$ $d \sin x = \cos x dx$ $(\sin x)' = \cos x$ $(\cos x)' = -\sin x$ $d\cos x = -\sin x dx$ $(\ln x)' = \frac{1}{y}$ $d \ln x = \frac{1}{y} dx$

We recall from previous slides that

$$\frac{\mathsf{d}}{\mathsf{d}x}(\ln|x|) = \frac{1}{x}.$$

This formula has a special application to integration:

Theorem (The Integral of 1/x)

$$\int \frac{1}{x} dx = \ln|x| + C.$$

This fills in the gap in the rule for integrating power functions:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C, \qquad n \neq -1.$$

Now we know the formula for n = -1 too.