Calculus II
Express $\sin(kx), \cos(kx)$ via $\sin x, \cos x$ using Euler's formula.

Todor Milev

2019

Example

Express $\sin(3x)$ and $\cos(3x)$ via $\cos x$ and $\sin x$.

- Recall Euler's formula: $e^{i\alpha} = \cos\alpha + i\sin\alpha$.

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Express $\sin(3x)$ and $\cos(3x)$ via $\cos x$ and $\sin x$.

$$\begin{aligned} & \cos(3x) + i\sin(3x) && \Big| \text{ Euler's f-la} \\ = \; & e^{3ix} \end{aligned}$$

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\begin{array}{ll}
\cos(3x) + i\sin(3x) & \Big| \text{ Euler's f-la} \\
= e^{3ix} & \\
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- Recall Euler's formula: $e^{i\alpha} = \cos\alpha + i\sin\alpha$.
- Recall the formula: $(a+b)^3 = \textbf{?}$.

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The real parts of the starting and final expression must be equal; therefore:

$$\cos(3x) = \cos^3 x - 3\cos x\sin^2 x$$

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= \cos^3 x + 3i\cos^2 x\sin x - 3\cos x\sin^2 x - i\sin^3 x & \Big|\ \text{Use } i^2 = -1 \\
= (\cos^3 x - 3\cos x\sin^2 x) + i(3\cos^2 x\sin x - \sin^3 x) &
\end{array}
$$

The real parts of the starting and final expression must be equal; likewise the imaginary parts must be equal; therefore:

$$
\begin{array}{rcl}
\cos(3x) & = & \cos^3 x - 3\cos x\sin^2 x \\
\sin(3x) & = & 3\cos^2 x\sin x - \sin^3 x
\end{array}
$$