

Precalculus

Additional trigonometric identity exercises

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Proving the following identities is a good exercise.

$$\textcircled{1} \sin \theta \cot \theta = \cos \theta.$$

$$\textcircled{2} (\sin \theta + \cos \theta)^2 = 1 + \sin(2\theta).$$

$$\textcircled{3} \sec \theta - \cos \theta = \tan \theta \sin \theta.$$

$$\textcircled{4} \tan^2 \theta - \sin^2 \theta = \tan^2 \theta \sin^2 \theta.$$

$$\textcircled{5} \cot^2 \theta + \sec^2 \theta = \tan^2 \theta + \csc^2 \theta.$$

$$\textcircled{6} 2 \csc(2\theta) = \sec \theta \csc \theta.$$

$$\textcircled{7} \tan(2\theta) = \frac{2 \tan \theta}{1 - \tan^2 \theta}.$$

$$\textcircled{8} \frac{1}{1 - \sin \theta} + \frac{1}{1 + \sin \theta} = 2 \sec^2 \theta.$$

$$\textcircled{9} \tan \alpha + \tan \beta = \frac{\sin(\alpha + \beta)}{\cos \alpha \cos \beta}.$$

$$\textcircled{10} \tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}.$$

$$\textcircled{11} \sin(3\theta) + \sin \theta = 2 \sin(2\theta) \cos \theta.$$

$$\textcircled{12} \cos(3\theta) = 4 \cos^3 \theta - 3 \cos \theta.$$

$$\textcircled{13} 1 + \tan^2 \theta = \sec^2 \theta.$$

$$\textcircled{14} 1 + \csc^2 \theta = \cot^2 \theta.$$

$$\textcircled{15} 2 \cos^2(2x) = 2 \sin^4 \theta + 2 \cos^4 \theta - \sin^2(2\theta).$$

$$\textcircled{16} \frac{1 + \tan\left(\frac{\theta}{2}\right)}{1 - \tan\left(\frac{\theta}{2}\right)} = \tan \theta + \sec \theta.$$

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Prove the trigonometric identity.

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$$\left| \begin{array}{l} (A + B)^2 = \\ ? \end{array} \right.$$

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Here we **explicitly permit** the use of the **Pythagorean identities**

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Here we explicitly permit the use of the Pythagorean identities and the double angle f-las:

$$\begin{aligned}\cos^2 \theta + \sin^2 \theta &= 1 \\ \sin(2\theta) &= 2 \sin \theta \cos \theta \\ \cos(2\theta) &= \cos^2 \theta - \sin^2 \theta\end{aligned}$$

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Express $\sin(3x)$ and $\cos(3x)$ via $\cos x$ and $\sin x$.

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$$\sin(3x) = \sin(x + 2x)$$

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Recall the formulas $\sin(\alpha + \beta) = ?$
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$$\begin{aligned}\sin(3x) &= \sin(x + 2x) \\ &= \sin x \cos(2x) + \cos x \sin(2x) \\ &= \sin x (\cos^2 x - \sin^2 x) + \cos x (2 \sin x \cos x) \\ &= \sin x \cos^2 x - \sin^3 x + 2 \sin x \cos^2 x \\ &= 3 \sin x \cos^2 x - \sin^3 x \\ \cos(3x) &= \cos(x + 2x) \\ &= \cos x \cos(2x) - \sin x \sin(2x) \\ &= \cos x (\cos^2 x - \sin^2 x) - \sin x (2 \sin x \cos x) \\ &= \cos^3 x - \cos x \sin^2 x - 2 \cos x \sin^2 x \\ &= \cos^3 x - 3 \cos x \sin^2 x\end{aligned}$$

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