Calculus I
The algebra behind derivatives

Todor Milev

2019

Outline

License to use and redistribute

Rules of differentiation.

We studied the basic rules of differentiation.

- $f(g(x))' = f'(g(x))g'(x)$ (Chain rule).
- $(f * g)' = f'g + fg'$ (Product rule).
- $(f + g)' = f' + g'$ (Sum rule).
- $x' = 1$.
- $(c)' = 0$ if c is a constant (Constant derivative rule).

We studied additional differentiation rules.

- $(e^x)' = e^x$ (Exponent derivative rule).
- $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$ (Quotient rule).
- $(x^r)' = rx^{r-1}$, r-arbitrary real number (Power rule).
- $(\ln x)' = \frac{1}{x}$ (Logarithm derivative rule).
- $(\log_a x)' = \frac{1}{x \ln a}$.
- $(\sin x)' = \cos x$, $(\cos x)' = -\sin x$

We derived the first set of rules by directly computing limits. The second set of rules can be derived from the first set algebraically.

Example

Let c be a constant. Derive the constant multiple rule

$$(cf)' = cf'$$

using the product rule $(fg)' = f'g + fg'$ and the constant derivative rule $(c)' = 0$.

$$(cf)' = (c)'f + cf' = 0f + cf' = cf'$$

as desired.

Example

Let n-positive integer. Derive the positive integer power rules

$$\left(x^2\right)' = 2x, \qquad \left(x^3\right)' = 3x^2, \qquad \left(x^4\right)' = 4x^3, \qquad \dots$$

using the rule $(x)' = 1$ and the product rule.

$$\begin{array}{rcl}
(x)' &=& 1 \\
(x^2)' &=& (x \cdot x)' = x'x + xx' = x + x = 2x \\
(x^3)' &=& (x \cdot x^2)' = x'x^2 + x(x^2)' = x^2 + x(2x) = x^2 + 2x^2 = 3x^2 \\
(x^4)' &=& (x \cdot x^3)' = x'x^3 + x(x^3)' = x^3 + x(3x^2) = x^3 + 3x^3 = 4x^3 \\
&\vdots& \\
(x^n)' &=& \dots = nx^{n-1} \\
(x^{n+1})' &=& (x \cdot x^n)' = x'x^n + x(x^n)' = x^n + x(nx^{n-1}) = (n+1)x^n \\
&\vdots&
\end{array}$$

Example

Let n be a positive integer. Derive the negative integer power rule

$$(x^{-n})' = \left(\frac{1}{x^n}\right)' = -nx^{-n-1} = -\frac{n}{x^{n+1}}$$

using the product rule, the constant derivative rule and the power rule for positive integers.

$$\begin{array}{rcll}
x^n x^{-n} & = & 1 & \left| \ \frac{\mathrm{d}}{\mathrm{d}x} \right. \\
(x^n x^{-n})' & = & (1)' & \\
(x^n)' x^{-n} + x^n (x^{-n})' & = & 0 & \\
n x^{n-1} x^{-n} + x^n (x^{-n})' & = & 0 & \\
\frac{n}{x} + x^n (x^{-n})' & = & 0 & \\
x^n (x^{-n})' & = & -\frac{n}{x} & \left| \ \frac{1}{x^n} \right. \\
(x^{-n})' & = & -\frac{n}{x^{n+1}} &
\end{array}$$

Example

For positive integer q, derive the power rule $\left(x^{\frac{1}{q}}\right)' = \frac{1}{q}x^{\frac{1}{q}-1}$ using the rule $(x)' = 1$, the chain rule and the integer power rule $\frac{\mathrm{d}}{\mathrm{d}u}(u^q) = qu^{q-1}$.

$$
\begin{array}{rcl|l}
\left(x^{\frac{1}{q}}\right)^q &=& x & \frac{\mathrm{d}}{\mathrm{d}x} \\
\left(\left(x^{\frac{1}{q}}\right)^q\right)' &=& 1(x)' & \text{Set } u = x^{\frac{1}{q}} \\
(u^q)' &=& 1 & \\
\frac{\mathrm{d}}{\mathrm{d}u}(u^q)\, u' &=& 1 & \\
q(u)^{q-1}(u)' &=& 1 & \\
q(x^{\frac{1}{q}})^{q-1}\left(x^{\frac{1}{q}}\right)' &=& 1 & \\
qx^{\frac{q-1}{q}}\left(x^{\frac{1}{q}}\right)' &=& 1 & \text{divide by } qx^{\frac{q-1}{q}} \\
\left(x^{\frac{1}{q}}\right)' &=& \frac{1}{qx^{\frac{q-1}{q}}} = \frac{x^{-\frac{q-1}{q}}}{q} = \frac{1}{q}x^{\frac{1}{q}-1} & \text{as desired}
\end{array}
$$

Example

Derive the quotient rules

$$\begin{aligned}\left(\frac{1}{g}\right)' &= -\frac{g'}{g^2}\\ \left(\frac{f}{g}\right)' &= \frac{f'g-fg'}{g^2}\end{aligned}$$

using the chain rule, the negative power rule and the product rule.

$$\left(\frac{1}{g}\right)' = \frac{\mathrm{d}}{\mathrm{d}g}\left(\frac{1}{g}\right)g' = -\frac{1}{g^2}g' \qquad \Big| \text{ as desired}$$

$$\begin{aligned}\left(\frac{f}{g}\right)' &= \left(f\frac{1}{g}\right)' = f'\frac{1}{g} + f\left(\frac{1}{g}\right)' = \frac{f'}{g} + f\left(-\frac{g'}{g^2}\right)\\ &= \frac{f'g-fg'}{g^2} \qquad \Big| \text{ as desired}\end{aligned}$$

Example

Derive the exponent rule $(e^x)' = e^x$ using the Calc II formula below, the infinite (both sides uniformly convergent) sum rule $(f_1 + f_2 + f_3 + \dots)' = f_1' + f_2' + f_3' + \dots$ and the power rule $(x^n)' = nx^{n-1}$.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots,$$

where $n! = 1 \cdot 2 \cdot 3 \cdot \dots \cdot n$. We have that $\frac{n}{n!} = \frac{n}{1 \cdot 2 \cdot \dots \cdot (n-1) \cdot n} = \frac{1}{1 \cdot 2 \cdot \dots \cdot (n-1)} = \frac{1}{(n-1)!}$.

$$\begin{aligned}
(e^x)' &= \left(1 + x + \tfrac{x^2}{2!} + \tfrac{x^3}{3!} + \dots\right)' \\
&= (1)' + (x)' + \frac{\left(x^2\right)'}{2!} + \frac{\left(x^3\right)'}{3!} + \dots + \frac{(x^n)'}{n!} + \dots \\
&= 0 + 1 + \tfrac{2x}{2!} + \tfrac{3x^2}{3!} + \dots + \tfrac{nx^{n-1}}{n!} + \dots \\
&= \quad 1 + \tfrac{x}{1!} + \tfrac{x^2}{2!} + \dots + \tfrac{x^{n-1}}{(n-1)!} + \dots = e^x
\end{aligned}$$

as desired.

Example

Derive the logarithm derivative rules

$$\begin{aligned}(\ln x)' &= \frac{1}{x}\\ (\log_a x)' &= \frac{1}{x\ln a}\end{aligned}$$

using the chain rule, the exponent derivative rule $(e^x)' = e^x$, the rule $(x)' = 1$ and the constant multiple rule $(cf)' = cf'$.

$$\begin{array}{rcl|l}
e^{\ln x} &=& x & \text{set} \quad u = \ln x\\
e^{u} &=& x & \frac{\mathrm{d}}{\mathrm{d}x}\\
\frac{\mathrm{d}}{\mathrm{d}u}(e^u)u' &=& (x)' & \\
e^u u' &=& 1 & \\
e^{\ln x}(\ln x)' &=& 1 & \\
x(\ln x)' &=& 1 & \cdot\frac{1}{x}\\
(\ln x)' &=& \frac{1}{x} & \text{as desired}\\
\hline
(\log_a x)' &=& \left(\frac{\ln x}{\ln a}\right)' = \frac{(\ln x)'}{\ln a} = \frac{1}{x\ln a} & \text{as desired}
\end{array}$$

Example

Derive the power rule

$$(x^r)' = rx^{r-1}, \qquad x > 0$$

using the chain rule, the the rule $(e^x)' = e^x$, the constant multiple derivative rule and the logarithm derivative rule $(\ln x)' = \frac{1}{x}$.

$$\begin{array}{rcll}
(x^r)' & = & \left((e^{\ln x})^r\right)' = \left(e^{r\ln x}\right)' & \Big| \text{ Set } u = r\ln x \\
& = & (e^u)' = \dfrac{\mathrm{d}}{\mathrm{d}u}(e^u)\, u' = e^u u' = & \\
& = & e^{r\ln x}(r\ln x)' = \left(e^{\ln x}\right)^r r(\ln x)' & \\
& = & x^r r\frac{1}{x} = rx^{r-1} & \Big| \text{ as desired}
\end{array}$$

Example

Derive the sine and cosine rules

$$\begin{aligned}(\sin x)' &= \cos x\\ (\cos x)' &= -\sin x\end{aligned}$$

using Euler's formula, the exponent derivative rule, the chain rule, the sum rule and the constant multiple rule. Assume all rules are valid over the complex numbers $\mathbb{C}$.

$$\begin{aligned}
e^{ix} &= \cos x + i\sin x && \Big| \frac{\mathrm{d}}{\mathrm{d}x}\\
\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{ix}\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\cos x + i\sin x\right)\\
e^{ix}(ix)' &= (\cos x)' + i(\sin x)'\\
ie^{ix} &= (\cos x)' + i(\sin x)'\\
i^2\sin x + i\cos x = i(\cos x + i\sin x) &= (\cos x)' + i(\sin x)'\\
-\sin x + i\cos x &= (\cos x)' + i(\sin x)'
\end{aligned}$$

Compare real part and coefficients of i to get the desired equalities.