Precalculus
Trig cofunction identities and angle-sum formulas

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Outline

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Cofunction identities

Proposition (Cofunction identities)

$$\begin{aligned}
\sin\left(\frac{\pi}{2}-\alpha\right) &= \cos\alpha & \sin\left(\frac{\pi}{2}+\alpha\right) &= \cos\alpha \\
\cos\left(\frac{\pi}{2}-\alpha\right) &= \sin\alpha & \cos\left(\frac{\pi}{2}+\alpha\right) &= -\sin\alpha
\end{aligned}$$

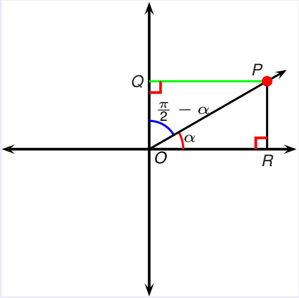
- The proof each formula is broken into 4 cases depending on which quadrant contains α.
- This makes a total of 4 formulas $\times$4 cases per formula = 16 cases.
- We show only a few of the cases.
- The proof provides intuition why the formulas are true.
- The Quadrant I part of the proof serves as a visual aid for memorization.
- There is an algebraically simpler (but theoretically advanced) way to prove the above identities through the angle sum f-las, derived in turn from Euler's formula (studied later/in another course).

Cofunction identities

Proposition (Cofunction identities)

$$\begin{aligned} \sin\left(\frac{\pi}{2}-\alpha\right) &= \cos\alpha & \sin\left(\frac{\pi}{2}+\alpha\right) &= \cos\alpha \\ \cos\left(\frac{\pi}{2}-\alpha\right) &= \sin\alpha & \cos\left(\frac{\pi}{2}+\alpha\right) &= -\sin\alpha \end{aligned}$$

Part of Proof.



We are showing $\sin\left(\frac{\pi}{2}-\alpha\right) = \cos\alpha$ when α is in quadrant I.

$$\begin{aligned} \sin\left(\frac{\pi}{2}-\alpha\right) &= \frac{|PQ|}{|OP|} && \Big|\ \square ORPQ \\ &= \frac{|OR|}{|OP|} \\ &= \cos\alpha && \Big|\ \text{as desired} \end{aligned}$$

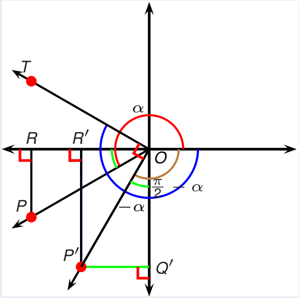
$\square$

Cofunction identities

Proposition (Cofunction identities)

$$
\begin{aligned}
\sin\left(\frac{\pi}{2}-\alpha\right) &= \cos\alpha & \sin\left(\frac{\pi}{2}+\alpha\right) &= \cos\alpha \\
\cos\left(\frac{\pi}{2}-\alpha\right) &= \sin\alpha & \cos\left(\frac{\pi}{2}+\alpha\right) &= -\sin\alpha
\end{aligned}
$$

Part of Proof.



We are showing $\sin\left(\frac{\pi}{2}-\alpha\right)=\cos\alpha$ when α is in Quadrant III. It follows $\frac{\pi}{2}-\alpha$ is in Quadrant III.

$$
\begin{aligned}
\sin\left(\frac{\pi}{2}-\alpha\right) &= -\frac{|P'R'|}{|OP'|} = -\frac{|OQ'|}{|OP'|} && \Big| \ \square OR'P'Q' \\
&= -\frac{|OR|}{|OP|} \\
&= \cos\alpha && \Big| \ \text{as desired}
\end{aligned}
$$

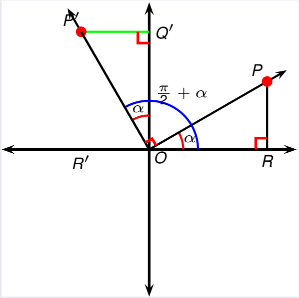
$\square$

Cofunction identities

Proposition (Cofunction identities)

$$\begin{array}{rclrcl}
\sin\left(\frac{\pi}{2}-\alpha\right) & = & \cos\alpha & \sin\left(\frac{\pi}{2}+\alpha\right) & = & \cos\alpha \\
\cos\left(\frac{\pi}{2}-\alpha\right) & = & \sin\alpha & \cos\left(\frac{\pi}{2}+\alpha\right) & = & -\sin\alpha
\end{array}$$

Part of Proof.



We show $\cos\left(\frac{\pi}{2}+\alpha\right) = -\sin\alpha$ when α is in Quadrant I. It follows $\frac{\pi}{2}+\alpha$ is in Quadrant II.

$$\begin{array}{rcll}
\cos\left(\frac{\pi}{2}+\alpha\right) & = & -\frac{|OR'|}{|OP'|} & \Big| \ \square ORPQ \\
& = & -\frac{|P'Q'|}{|OP'|} & \\
& = & -\frac{|PR|}{|OP|} & \\
& = & -\sin\alpha. & \Big| \ \text{as desired}
\end{array}$$

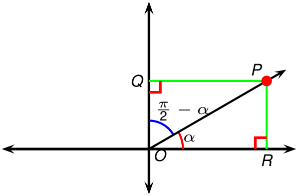
□

Cofunction identities

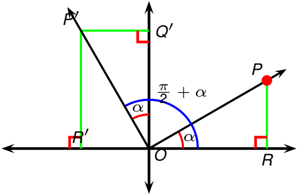
Proposition (Cofunction identities)

$$
\begin{array}{rclrcl}
\sin\left(\frac{\pi}{2}-\alpha\right) & = & \cos\alpha & \sin\left(\frac{\pi}{2}+\alpha\right) & = & \cos\alpha \\
\cos\left(\frac{\pi}{2}-\alpha\right) & = & \sin\alpha & \cos\left(\frac{\pi}{2}+\alpha\right) & = & -\sin\alpha
\end{array}
$$

To memorize the cofunction identities it suffices to memorize the Quadrant I case via the two diagrams below.



$$
\begin{aligned}
\sin\left(\tfrac{\pi}{2}-\alpha\right) &= \frac{|PQ|}{|OP|}\\
\cos\left(\tfrac{\pi}{2}-\alpha\right) &= \frac{|OQ|}{|OP|}
\end{aligned}
$$

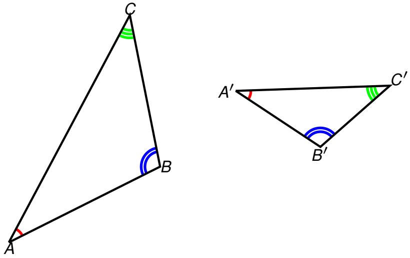


$$
\begin{aligned}
\sin\left(\tfrac{\pi}{2}+\alpha\right) &= \frac{|OQ'|}{|OP|}\\
\cos\left(\tfrac{\pi}{2}+\alpha\right) &= -\frac{|PQ'|}{|OP'|} = \frac{|PR|}{|OP|}
\end{aligned}
$$

Definition (Similar triangles)

We say that a triangle $\triangle ABC$ is similar to a triangle $\triangle A'B'C'$ if the two triangles have equal angles.

- The equal angles are assumed given in the same order for both triangles, that is, $\angle ABC = \angle A'B'C'$, $\angle BCA = \angle B'C'A'$, $\angle CAB = \angle C'A'B'$.

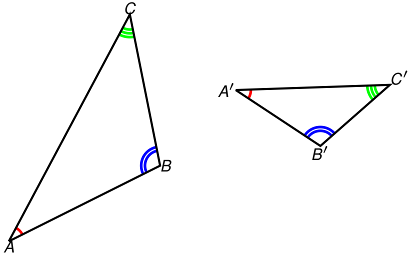


The following statement is proved in the subject of Euclidean (planar) geometry.

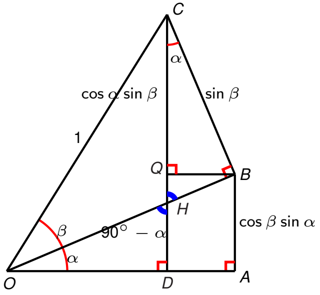
Theorem (Similar triangles have equal side ratios)

Let $\triangle ABC$ and $\triangle A'B'C'$ be two similar triangles. Then the ratios of the lengths of the sides of the two triangles are equal, that is

$$\frac{|AB|}{|BC|} = \frac{|A'B'|}{|B'C'|} \qquad \frac{|BC|}{|CA|} = \frac{|B'C'|}{|C'A'|} \qquad \frac{|CA|}{|AB|} = \frac{|C'A'|}{|A'B'|}$$



$\sin(\alpha+\beta), \cos(\alpha+\beta)$ via $\sin\alpha, \sin\beta, \cos\alpha, \cos\beta$



$$\sin(\alpha+\beta) = \frac{|CD|}{|OC|} = |CD| = |QD| + |CQ| = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$\cos(\alpha+\beta) = \frac{|OD|}{|OC|} = |OD| = |OA| - |DA| = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\begin{aligned}
|QD| &= |BA| && \square DABQ\\
&= \sin\alpha|OB| && \triangle OAB\\
&= \sin\alpha\cos\beta|OC| && \triangle OBC\\
&= \sin\alpha\cos\beta\\
|CQ| &= \cos\alpha|CB| && \triangle CQB\\
&= \cos\alpha\sin\beta|OC| && \triangle OBC\\
&= \cos\alpha\sin\beta\\
|OA| &= \cos\alpha|OB| && \triangle OAB\\
&= \cos\alpha\cos\beta|OC| && \triangle OBC\\
&= \cos\alpha\cos\beta\\
|DA| &= |QB| && \square DABQ\\
&= \sin\alpha|CB| && \triangle CQB\\
&= \sin\alpha\sin\beta|OC| && \triangle OBC\\
&= \sin\alpha\sin\beta
\end{aligned}$$

Trig Functions of Sums and Differences of Angles

Theorem

$$
\begin{aligned}
\sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta\\
\sin(\alpha-\beta) &= \sin\alpha\cos\beta - \cos\alpha\sin\beta\\
\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta\\
\cos(\alpha-\beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta
\end{aligned}
$$

- We gave a geometric proof of the sum formulas when the two angles are acute and their sum is less than $\pi = 90^\circ$.
- The theorem holds for all angles α, β without any restrictions.
- This can be shown by combining the preceding proof with identities such as $\cos\left(\frac{\pi}{2}-\alpha\right) = \sin\alpha$, $\cos\left(\frac{\pi}{2}+\alpha\right) = -\sin\alpha$.
- There is a theoretically more advanced (but algebraically simpler) proof using Euler's formula (to be studied later/in another course).
- The difference formulas are a consequence of the sum formulas and the fact that $\sin$ is an odd function and $\cos$ is even.

Trig Functions of Differences of Angles

Example

Prove the identities

$$\begin{aligned}\sin(\alpha-\beta) &= \sin\alpha\cos\beta-\cos\alpha\sin\beta\\ \cos(\alpha-\beta) &= \cos\alpha\cos\beta+\sin\alpha\sin\beta\end{aligned}$$

from the (already demonstrated) identities

$$\begin{aligned}\sin(\alpha+\beta) &= \sin\alpha\cos\beta+\cos\alpha\sin\beta\\ \cos(\alpha+\beta) &= \cos\alpha\cos\beta-\sin\alpha\sin\beta\end{aligned}$$

$$\begin{aligned}\sin(\alpha-\beta) &= \sin(\alpha+(-\beta))\\ &= \sin\alpha\cos(-\beta)+\cos\alpha\sin(-\beta) && \left|\ \begin{array}{l}\cos \text{ is even },\\ \sin \text{ is odd}\end{array}\right.\\ &= \sin\alpha\cos\beta-\cos\alpha\sin\beta\\ \cos(\alpha-\beta) &= \cos(\alpha+(-\beta))\\ &= \cos\alpha\cos(-\beta)-\sin\alpha\sin(-\beta) && \left|\ \begin{array}{l}\cos \text{ is even },\\ \sin \text{ is odd}\end{array}\right.\\ &= \cos\alpha\cos\beta+\cos\alpha\sin\beta\end{aligned}$$

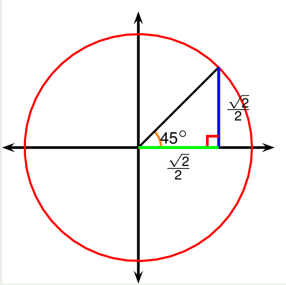
Example

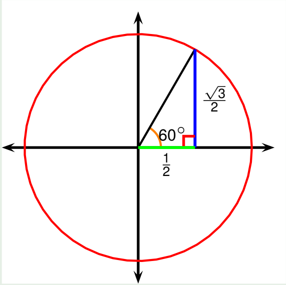
Find the exact value of the trigonometric function using radicals.

$$
\begin{aligned}
\cos(105^\circ) &= \cos(45^\circ + 60^\circ) \\
&= \cos(45^\circ)\cos(60^\circ) - \sin(45^\circ)\sin(60^\circ) \\
&= \frac{\sqrt{2}}{2}\cdot\frac{1}{2} - \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} \\
&= \frac{\sqrt{2}-\sqrt{6}}{4}.
\end{aligned}
$$

we know the trig f-ns of 45° and 60°

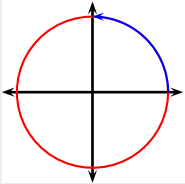
Angle sum f-la





Example

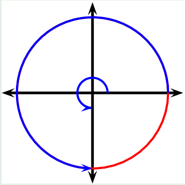
Use the angle sum/difference formulas to simplify.



$$\begin{aligned}\cos\left(\frac{\pi}{2}-x\right) &= \cos\left(\frac{\pi}{2}\right)\cos x+\sin\left(\frac{\pi}{2}\right)\sin x\\ &= 0\cdot\cos(x)+1\cdot\sin x\\ &= \sin x\end{aligned}$$

Example

Use the angle sum/difference formulas to simplify.



$$
\begin{aligned}
\cot\left(\frac{3\pi}{2}+x\right) &= \frac{\cos\left(\frac{3\pi}{2}+x\right)}{\sin\left(\frac{3\pi}{2}+x\right)}\\
&= \frac{\cos\left(\frac{3\pi}{2}\right)\cos x-\sin\left(\frac{3\pi}{2}\right)\sin x}{\sin\left(\frac{3\pi}{2}\right)\cos x+\cos\left(\frac{3\pi}{2}\right)\sin x}\\
&= \frac{0\cdot\cos x-(-1)\sin x}{(-1)\cos x+0\cdot\sin x}\\
&= \frac{\sin x}{-\cos x}=-\frac{\sin x}{\cos x}\\
&= -\tan x
\end{aligned}
$$

Example

Show that $\tan(\pi + x) = \tan x$ using the angle sum formulas.

$$
\begin{aligned}
\tan(\pi + x) &= \frac{\sin(\pi + x)}{\cos(\pi + x)} \\
&= \frac{\sin\pi\cos x + \cos\pi\sin x}{\cos\pi\cos x - \sin\pi\sin x} \\
&= \frac{0\cdot\cos x + (-1)\cdot\sin x}{(-1)\cdot\cos x - 0\cdot\sin x} \\
&= \frac{-\sin x}{-\cos x} \\
&= \frac{\sin x}{\cos x} \\
&= \tan x,
\end{aligned}
$$

as desired.

Proposition ($\tan, \cot$ are π-periodic)

The tangent and cotangent functions are π-periodic, in other words,

$$\begin{aligned}\tan(\theta+\pi) &= \tan\theta\\ \cot(\theta+\pi) &= \cot\theta\end{aligned}$$

Recall the angle sum formula $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$.

Example

Show that the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$ follows from the angle difference formula.

$$\begin{aligned} 1 &= \cos 0 \\ &= \cos(\theta - \theta) \\ &= \cos\theta\cos\theta + \sin\theta\sin\theta \\ &= \cos^2\theta + \sin^2\theta, \end{aligned}$$

as desired.

Example

Prove the angle sum formula $\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta}$.

$$
\begin{aligned}
\tan(\alpha+\beta) &= \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}\\
&= \frac{(\sin\alpha\cos\beta + \cos\alpha\sin\beta)\frac{1}{\cos\alpha\cos\beta}}{(\cos\alpha\cos\beta - \sin\alpha\sin\beta)\frac{1}{\cos\alpha\cos\beta}}\\
&= \frac{\frac{\sin\alpha\cancel{\cos\beta}}{\cos\alpha\cancel{\cos\beta}} + \frac{\cancel{\cos\alpha}\sin\beta}{\cancel{\cos\alpha}\cos\beta}}{\cancel{\frac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta}} - \frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}\\
&= \frac{\frac{\sin\alpha}{\cos\alpha} + \frac{\sin\beta}{\cos\beta}}{1 - \frac{\sin\alpha}{\cos\alpha}\cdot\frac{\sin\beta}{\cos\beta}}\\
&= \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta}
\end{aligned}
$$

Double angle formulas

Proposition (Double angle formulas)

$$
\begin{aligned}
\sin(2\alpha) &= 2\sin\alpha\cos\alpha \\
\cos(2\alpha) &= \cos^2\alpha - \sin^2\alpha \\
&= 2\cos^2\alpha - 1 \\
&= 1 - 2\sin^2\alpha
\end{aligned}
$$

- The double angle formulas play a special role in integration.

Example

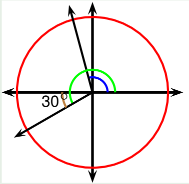
Derive the double-angle formulas.

$$
\begin{aligned}
\sin(2\alpha) &= \sin(\alpha+\alpha) \\
&= \sin\alpha\cos\alpha + \cos\alpha\sin\alpha \\
&= 2\sin\alpha\cos\alpha \\
\cos(2\alpha) &= \cos(\alpha+\alpha) \\
&= \cos\alpha\cos\alpha - \sin\alpha\sin\alpha \\
&= \cos^2\alpha - \sin^2\alpha \\
&= \cos^2\alpha - (1-\cos^2\alpha) \\
&= 2\cos^2\alpha - 1 \\
&= 1 - \sin^2\alpha - \sin^2\alpha \\
&= 1 - 2\sin^2\alpha
\end{aligned}
$$

Recall the half angle formula $\cos\alpha = \pm\sqrt{\frac{1+\cos(2\alpha)}{2}}$.

Example

Using radicals, find the exact value of the trigonometric expression.



$$\begin{aligned}
\cos 105^\circ &= \pm\sqrt{\frac{1+\cos\left(2\cdot 105^\circ\right)}{2}} && \Big|\ \cos 105^\circ < 0\\
&= -\sqrt{\frac{1+\cos\left(210^\circ\right)}{2}}\\
&= -\sqrt{\frac{1-\cos\left(30^\circ\right)}{2}}\\
&= -\sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}} = -\sqrt{\frac{2-\sqrt{3}}{2\cdot 2}}\\
&= -\frac{\sqrt{2-\sqrt{3}}}{2}
\end{aligned}$$

Proposition (Power-Reducing Formulas)

$$\sin^2\alpha = \frac{1-\cos(2\alpha)}{2} \qquad \cos^2\alpha = \frac{1+\cos(2\alpha)}{2}$$

Proof.

$$\begin{aligned} \cos(2\alpha) &= 1-2\sin^2\alpha \\ 2\sin^2\alpha &= 1-\cos(2\alpha) \\ \sin^2\alpha &= \frac{1-\cos(2\alpha)}{2} \end{aligned} \qquad \begin{aligned} \cos(2\alpha) &= 2\cos^2\alpha - 1 \\ 2\cos^2\alpha &= 1+\cos(2\alpha) \\ \cos^2\alpha &= \frac{1+\cos(2\alpha)}{2} \end{aligned}$$

$\square$

Corollary

$$\sin\alpha = \pm\sqrt{\frac{1-\cos(2\alpha)}{2}} \qquad \cos\alpha = \pm\sqrt{\frac{1+\cos(2\alpha)}{2}}$$

Corollary (Half-Angle Formulas)

$$\sin\left(\frac{\beta}{2}\right) = \pm\sqrt{\frac{1-\cos\beta}{2}} \qquad \cos\left(\frac{\beta}{2}\right) = \pm\sqrt{\frac{1+\cos\beta}{2}}$$

Proposition (Power-Reducing Formulas)

$$\sin^2\alpha = \frac{1-\cos(2\alpha)}{2} \qquad \cos^2\alpha = \frac{1+\cos(2\alpha)}{2}$$

- The power reducing formulas are used to express $\sin^k\alpha$ and $\cos^k\alpha$ via lower powers of the $\sin$ and $\cos$ functions (applied to angles other than α).
- This technique will play a key role in integration (studied later/in another course).

Recall the formulas: $\sin^2\beta = \frac{1-\cos(2\beta)}{2}$, $\cos^2\beta = \frac{\cos(2\beta)+1}{2}$.

Example

Rewrite $\sin^4\alpha$ in terms of first powers of the cosines and sines of multiples of the angle α.

$$\begin{aligned}
\sin^4\alpha &= \left(\sin^2\alpha\right)^2\\
&= \left(\frac{1-\cos(2\alpha)}{2}\right)^2\\
&= \frac{1}{4}\left(1-2\cos(2\alpha)+\cos^2(2\alpha)\right)\\
&= \frac{1}{4}\left(1-2\cos(2\alpha)+\frac{\cos(2\cdot 2\alpha)+1}{2}\right)\\
&= \frac{1}{4}\left(1-2\cos(2\alpha)+\frac{\cos(2\cdot 2\alpha)}{2}+\frac{1}{2}\right)\\
&= \frac{1}{4}\left(\frac{3}{2}-2\cos(2\alpha)+\frac{\cos(4\alpha)}{2}\right)\\
&= \frac{1}{8}\left(3-4\cos(2\alpha)+\cos(4\alpha)\right)
\end{aligned}$$