Precalculus
Complex numbers definition; overview of numbers

Todor Milev

2019

Outline

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Definition (Complex numbers)

The set of complex numbers $\mathbb{C}$ is defined as the set

$$\{a + bi | a, b - \text{real numbers}\},$$

where the number i is a number for which

$$i^2 = -1 \quad .$$

The number i is called the imaginary unit.

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Let $u = 2 + 3i$, $v = 5 - 7i$.

Example (Addition)

$u + v =$.

Example (Subtraction)

$u - v =$.

Example (Multiplication)

$u \cdot v \quad =$

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- An integer, or whole number, is one of the numbers:
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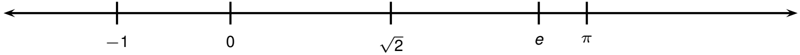
π

e

-1



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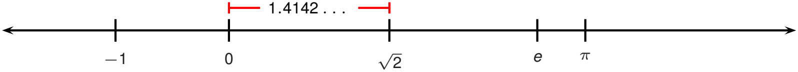

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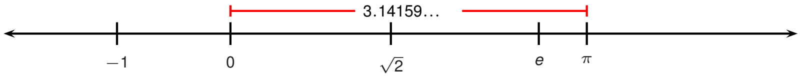

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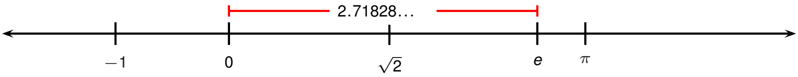

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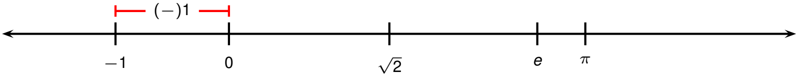

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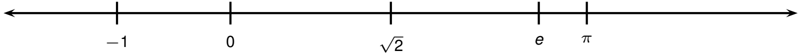

- A number is complex if it equals $a + bi$ with a, b- real, $\sqrt{-1} = i$:
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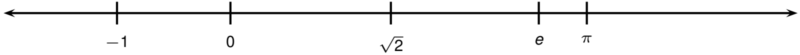

- A number is complex if it equals $a + bi$ with a, b- real, $\sqrt{-1} = i$:
$$2 + 3i, \qquad -i, \qquad 1 + \sqrt{2}i$$
- Geometric interpretation of complex numbers: beyond our scope.