

Calculus II

Convergence of sequences related to the number e as a limit, part 1

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Theorem

If $\lim_{n \rightarrow \infty} a_n = L$ and the function f is continuous at L , then

$$\lim_{n \rightarrow \infty} f(a_n) = f(L)$$

Example

$$\lim_{x \rightarrow \infty} \left(1 + \frac{k}{x}\right)^x$$

Example

$$\lim_{x \rightarrow \infty} \left(1 + \frac{k}{x}\right)^x = \lim_{x \rightarrow \infty} e^{\ln(1 + \frac{k}{x})^x}$$

Example

$$\begin{aligned}\lim_{x \rightarrow \infty} \left(1 + \frac{k}{x}\right)^x &= \lim_{x \rightarrow \infty} e^{\ln\left(1 + \frac{k}{x}\right)^x} \\ &= e^{\lim_{x \rightarrow \infty} \ln\left(1 + \frac{k}{x}\right)^x}\end{aligned}$$

exponent= continuous f-n

Example

$$\begin{aligned}
 \lim_{x \rightarrow \infty} \left(1 + \frac{k}{x}\right)^x &= \lim_{x \rightarrow \infty} e^{\ln\left(1 + \frac{k}{x}\right)^x} & \left| \text{exponent} = \text{continuous f-n} \right. \\
 &= e^{\lim_{x \rightarrow \infty} \ln\left(1 + \frac{k}{x}\right)^x} \\
 &= e^{\lim_{x \rightarrow \infty} \ln \left(1 + \frac{k}{x}\right)^x}
 \end{aligned}$$

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 \lim_{x \rightarrow \infty} \ln \left(1 + \frac{k}{x}\right)^x &= \lim_{x \rightarrow \infty} x \ln \left(1 + \frac{k}{x}\right)
 \end{aligned}$$

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 \lim_{x \rightarrow \infty} \left(1 + \frac{k}{x}\right)^x &= \lim_{x \rightarrow \infty} e^{\ln(1 + \frac{k}{x})^x} & \text{exponent= continuous f-n} \\
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 &= \lim_{x \rightarrow \infty} \frac{(\ln(1 + \frac{k}{x}))}{(\frac{1}{x})}
 \end{aligned}$$

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 \lim_{x \rightarrow \infty} \left(1 + \frac{k}{x}\right)^x &= \lim_{x \rightarrow \infty} e^{\ln(1 + \frac{k}{x})^x} && \left| \text{exponent} = \text{continuous f-n} \right. \\
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 &= \lim_{x \rightarrow \infty} \frac{(\ln(1 + \frac{k}{x}))}{(\frac{1}{x})} && \left| \text{form } \frac{0}{0}, \right.
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 &= \lim_{x \rightarrow \infty} \frac{\frac{d}{dx} (\ln(1 + \frac{k}{x}))}{\frac{d}{dx} (\frac{1}{x})} && \left| \text{form } \frac{0}{0}, \text{ use L'Hospital} \right.
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 &= \lim_{x \rightarrow \infty} \frac{\frac{d}{dx} (\ln(1 + \frac{k}{x}))}{\frac{d}{dx} (\frac{1}{x})} & \text{form "0/0", use L'Hospital} \\
 &\quad ? \\
 &= \lim_{x \rightarrow \infty} \frac{\quad}{\quad} & ?
 \end{aligned}$$

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 &\quad ? \\
 &= \lim_{x \rightarrow \infty} \frac{\quad}{-\frac{1}{x^2}}
 \end{aligned}$$

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 &= \lim_{x \rightarrow \infty} \frac{\text{?}}{-\frac{1}{x^2}}
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 \lim_{x \rightarrow \infty} \left(1 + \frac{k}{x}\right)^x &= \lim_{x \rightarrow \infty} e^{\ln\left(1 + \frac{k}{x}\right)^x} && \left| \text{exponent} = \text{continuous f-n} \right. \\
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 &= \lim_{x \rightarrow \infty} \frac{\frac{d}{dx} \left(\ln\left(1 + \frac{k}{x}\right) \right)}{\frac{d}{dx} \left(\frac{1}{x} \right)} && \left| \text{form } \frac{0}{0}, \text{ use L'Hospital} \right. \\
 &= \lim_{x \rightarrow \infty} \frac{\frac{1}{1 + \frac{k}{x}} \left(1 + \frac{k}{x}\right)'}{-\frac{1}{x^2}}
 \end{aligned}$$

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 \lim_{x \rightarrow \infty} \left(1 + \frac{k}{x}\right)^x &= \lim_{x \rightarrow \infty} e^{\ln\left(1 + \frac{k}{x}\right)^x} && \left| \text{exponent} = \text{continuous f-n} \right. \\
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 &= \lim_{x \rightarrow \infty} \frac{\frac{1}{1 + \frac{k}{x}} \left(-\frac{k}{x^2}\right)}{-\frac{1}{x^2}} \\
 &= \lim_{x \rightarrow \infty} \frac{k}{1 + \frac{k}{x}}
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 &= \lim_{x \rightarrow \infty} \frac{k}{1 + \frac{k}{x}} = ?
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 &= \lim_{x \rightarrow \infty} \frac{k}{1 + \frac{k}{x}} = k
 \end{aligned}$$

Example

$$\lim_{x \rightarrow \infty} \left(1 + \frac{k}{x}\right)^x = \lim_{x \rightarrow \infty} e^{\ln(1 + \frac{k}{x})^x}$$

exponent= continuous f-n

$$= e^{\lim_{x \rightarrow \infty} \ln(1 + \frac{k}{x})^x} = ?$$

$$\lim_{x \rightarrow \infty} \ln \left(1 + \frac{k}{x}\right)^x = \lim_{x \rightarrow \infty} x \ln \left(1 + \frac{k}{x}\right)$$

$$= \lim_{x \rightarrow \infty} \frac{\frac{d}{dx} (\ln(1 + \frac{k}{x}))}{\frac{d}{dx} (\frac{1}{x})}$$

form " $\frac{0}{0}$ ", use L'Hospital

$$= \lim_{x \rightarrow \infty} \frac{\frac{1}{1 + \frac{k}{x}} \left(1 + \frac{k}{x}\right)'}{-\frac{1}{x^2}}$$

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 &= e^{\lim_{x \rightarrow \infty} \ln(1 + \frac{k}{x})^x} = e^k \\
 \lim_{x \rightarrow \infty} \ln \left(1 + \frac{k}{x}\right)^x &= \lim_{x \rightarrow \infty} x \ln \left(1 + \frac{k}{x}\right) \\
 &= \lim_{x \rightarrow \infty} \frac{\frac{d}{dx} (\ln(1 + \frac{k}{x}))}{\frac{d}{dx} (x)} && \left| \text{form "0/0", use L'Hospital} \right. \\
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 &= e^{\lim_{x \rightarrow \infty} \ln\left(1 + \frac{k}{x}\right)^x} = e^k && \text{limit computed below} \\
 \lim_{x \rightarrow \infty} \ln\left(1 + \frac{k}{x}\right)^x &= \lim_{x \rightarrow \infty} x \ln\left(1 + \frac{k}{x}\right) \\
 &= \lim_{x \rightarrow \infty} \frac{\frac{d}{dx} (\ln(1 + \frac{k}{x}))}{\frac{d}{dx} (\frac{1}{x})} && \text{form "0/0", use L'Hospital} \\
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