Calculus I
Homework
The Chain Rule

1. Compute the derivative using the chain rule.

(a) $f(x) = \sqrt{1+x^2}$

answer: $x(x^2+1)^{-\frac{1}{2}}$

(b) $f(x) = \sqrt{3x^2 - x + 2}$.

answer: $\frac{6x-1}{2\sqrt{3x^2-x+2}}$

(c) $f(x) = \dfrac{x}{\sqrt{1+\frac{2}{x^2}}}$.

answer: $\frac{\pm x^2}{\sqrt{x^2+2}}$

(d) $f(x) = \sqrt{1-\sqrt{x}}$.

answer: $-\frac{1}{4}x^{-\frac{1}{2}}\left(-\sqrt{x}+1\right)^{-\frac{1}{2}}$

(e) $y = (\cos x)^{\frac{1}{2}}$

answer: $\frac{1}{2}x^{-\frac{1}{2}}\cos\left(\sqrt{x}\right) = \frac{\cos(\sqrt{x})}{2\sqrt{x}}$

(f) $f(x) = \sin^3 x$.

answer: $3\cos x \sin^2 x$

(g) $y = (1+\cos x)^2$.

answer: $-2\cos x \sin x - 2\sin x = -\sin(2x) - 2\sin x$

(h) $f(x) = \dfrac{1}{\sin^3 x}$.

answer: $-\frac{3\cos x}{\sin^4 x}$

(i) $f(x) = \sqrt[3]{4+3\tan x}$.

answer: $(4+3\tan x)^{-\frac{2}{3}}\sec^2 x$

(j) $f(x) = (\cos x + 3\sin x)^4$.

answer: $4(\cos x + 3\sin x)^3(3\cos x - \sin x)$

(k) $y = \sin\left(\sqrt{x}\right)$

answer: $\frac{\cos\sqrt{x}}{2\sqrt{x}}$

(l) $y = \cos(4x)$

answer: $-4\sin(4x)$

(m) $\sec^2(3x^2)$.

answer: $12\frac{x\sin\left(3x^2\right)}{\left(\cos\left(3x^2\right)\right)^3}$

(n) $\csc^2(3x^2)$.

answer: $-12\frac{x\cos\left(3x^2\right)}{\left(\sin\left(3x^2\right)\right)^3}$

(o) e^{2x}.

answer: $2e^{2x}$

(p) e^{-x^2}

answer: $-2xe^{-x^2}$

(q) $e^{\sqrt{x}}$

answer: $\frac{1}{2\sqrt{x}}e^{\sqrt{x}}$

(r) $f(x) = e^{-\frac{1}{x}}$.

answer: $\frac{e^{-\frac{1}{x}}}{x^2}$

(s) 5^x.

answer: $(\ln 5)5^x$

(t) e^{2^x}.

answer: $e^{2^x}2^x(\ln 2)$

(u) 2^{3^x}.

answer: $2^{3^x}3^x(\ln 2)(\ln 3)$

(v) 3^{2^x}.

answer: $3^{2^x}2^x(\ln 2)(\ln 3)$

(w) $y = \sqrt{\sec(4x)}$

answer: $2\sqrt{\sec(4x)}\tan(4x) = 2(\sec(4x))^{\frac{3}{2}}\sin(4x)$.

(x) $y = x^2\tan(5x)$

answer: $2x\tan(5x) - 5x^2\sec^2(5x)$

(y) $y = \dfrac{1+\sin\left(x^2\right)}{1+\cos\left(x^2\right)}$.

answer: $\frac{2x\left(1+\cos\left(x^2\right)+\sin\left(x^2\right)\right)}{\left(1+\cos\left(x^2\right)\right)^2}$

Solution. 1.b

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{3x^2-x+2}\right) = \frac{(3x^2-x+2)'}{2\sqrt{3x^2-x+2}} = \frac{6x-1}{2\sqrt{3x^2-x+2}}.$$

Solution. 1.c

$$\begin{aligned}\left(\frac{x}{\sqrt{1+\frac{2}{x^2}}}\right)' &= \frac{\sqrt{1+\frac{2}{x^2}}-x\left(\sqrt{1+\frac{2}{x^2}}\right)'}{1+\frac{2}{x^2}} = \frac{\sqrt{1+\frac{2}{x^2}}-x\frac{\frac{1}{2}}{\sqrt{1+\frac{2}{x^2}}}\left(\frac{2}{x^2}\right)'}{1+\frac{2}{x^2}}\\ &= \frac{\sqrt{1+\frac{2}{x^2}}+\frac{2}{x^2\sqrt{1+\frac{2}{x^2}}}}{1+\frac{2}{x^2}} = \frac{x^2\left(1+\frac{2}{x^2}\right)+2}{x^2\left(1+\frac{2}{x^2}\right)^{\frac{3}{2}}} = \frac{x^2+4}{x^2\left(1+\frac{2}{x^2}\right)^{\frac{3}{2}}}\end{aligned}$$

Please note that this problem can be solved also by applying the transformation

$$\frac{x}{\sqrt{1+\frac{2}{x^2}}} = \frac{x}{\sqrt{\frac{x^2+2}{x^2}}} = \frac{x}{\frac{1}{\pm x}\sqrt{x^2+2}} = \frac{\pm x^2}{\sqrt{x^2+2}}$$

before differentiating, however one must not forget the $\pm$ sign arising from $\sqrt{x^2} = \pm x$. Our original approach resulted in more algebra, but did not have the disadvantage of dealing with the $\pm$ sign.

Solution. 1.d

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{1-\sqrt{x}}\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\left(1-x^{\frac{1}{2}}\right)^{\frac{1}{2}}\right) && \Big|\ \text{chain rule}\\ &= \frac{1}{2}\left(1-x^{\frac{1}{2}}\right)^{-\frac{1}{2}}\frac{\mathrm{d}}{\mathrm{d}x}\left(1-x^{\frac{1}{2}}\right)\\ &= -\frac{1}{4}x^{-\frac{1}{2}}\left(1-x^{\frac{1}{2}}\right)^{-\frac{1}{2}}\end{aligned}$$

Solution. 1.e

$$\begin{aligned}\text{Let}\quad & u = \cos x.\\ \text{Then}\quad & y = u^{\frac{1}{2}}.\\ \text{Chain Rule:}\quad & \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}\\ & \quad = \left(\frac{1}{2}u^{-\frac{1}{2}}\right)(-\sin x)\\ & \quad = -\frac{1}{2}\sin x(\cos x)^{-\frac{1}{2}}.\end{aligned}$$

Solution. 1.g

$$\begin{aligned}\text{Let}\quad & u = 1+\cos x.\\ \text{Then}\quad & y = u^2.\\ \text{Chain Rule:}\quad & \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}\\ & \quad = (2u)(-\sin x)\\ & \quad = -2\sin x(1+\cos x)\\ & \quad = -2\sin x - 2\sin x\cos x\\ & \quad = -2\sin x - \sin(2x). \quad \text{(This last step is optional.)}\end{aligned}$$

Solution. 1.k

$$\begin{aligned} \text{Let} \quad & u = \sqrt{x}. \\ \text{Then} \quad & y = \sin u. \\ \text{Chain Rule:} \quad & \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} \\ & = (\cos u)\left(\frac{1}{2}u^{-\frac{1}{2}}\right) \\ & = \frac{\cos\left(\sqrt{x}\right)}{2\sqrt{x}}. \end{aligned}$$

Solution. 1.r

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}x}\left(e^{-\frac{1}{x}}\right) &= e^{-\frac{1}{x}}\tfrac{\mathrm{d}}{\mathrm{d}x}\left(-\tfrac{1}{x}\right) && \Big| \text{ chain rule} \\ &= -e^{-\frac{1}{x}}\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{-1}\right) \\ &= x^{-2}e^{-\frac{1}{x}} \\ &= \frac{e^{-\frac{1}{x}}}{x^2} \end{aligned}$$

Solution. 1.w

$$\begin{aligned} \text{Chain Rule:} \quad \frac{\mathrm{d}y}{\mathrm{d}x} &= \left(\frac{1}{2}(\sec(4x))^{-\frac{1}{2}}\right)\frac{\mathrm{d}}{\mathrm{d}x}(\sec(4x)) \\ \text{Chain Rule:} \quad \frac{\mathrm{d}y}{\mathrm{d}x} &= \left(\frac{1}{2\sqrt{\sec(4x)}}\right)(\sec(4x)\tan(4x))\frac{\mathrm{d}}{\mathrm{d}x}(4x) \\ &= \left(\frac{1}{2\sqrt{\sec(4x)}}\right)(\sec(4x)\tan(4x))(4) \\ &= \frac{2\sec(4x)\tan(4x)}{\sqrt{\sec(4x)}} \end{aligned}$$

There are many ways to simplify this answer, including both of the following.

$$\begin{aligned} &= 2\sqrt{\sec(4x)}\tan(4x). \\ &= 2(\sec(4x))^{\frac{3}{2}}\sin(4x). \end{aligned}$$

Solution. 1.x

$$\text{Product Rule:} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = (x^2)\frac{\mathrm{d}}{\mathrm{d}x}(\tan(5x)) + (\tan(5x))\frac{\mathrm{d}}{\mathrm{d}x}(x^2)$$

Use the Chain Rule to differentiate $\tan(5x)$ in the first term.

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= (x^2)(-5\sec^2(5x) + (\tan(5x))(2x) \\ &= 2x\tan(5x) - 5x^2\sec^2(5x). \end{aligned}$$

Solution. 1.y

$$\text{Quotient Rule:} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\left(1+\cos\left(x^2\right)\right)\frac{\mathrm{d}}{\mathrm{d}y}(1+\sin\left(x^2\right)) - (1+\sin\left(x^2\right))\frac{\mathrm{d}}{\mathrm{d}x}(1+\cos\left(x^2\right))}{(1+\cos\left(x^2\right))^2}$$

By the Chain Rule, $\frac{\mathrm{d}}{\mathrm{d}x}(1+\cos\left(x^2\right)) = -2x\sin\left(x^2\right)$ and $\frac{\mathrm{d}}{\mathrm{d}x}(1+\sin\left(x^2\right)) = 2x\cos\left(x^2\right)$.

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{(1+\cos\left(x^2\right))(2x\cos\left(x^2\right)) - (1+\sin\left(x^2\right))(-2x\sin\left(x^2\right))}{(1+\cos\left(x^2\right))^2} \\ &= \frac{2x\cos\left(x^2\right) + 2x\cos^2\left(x^2\right) + 2x\sin\left(x^2\right) + 2x\sin^2\left(x^2\right)}{(1+\cos\left(x^2\right))^2} \\ &= \frac{2x(\cos^2\left(x^2\right) + \sin^2\left(x^2\right)) + 2x(\cos\left(x^2\right) + \sin\left(x^2\right))}{(1+\cos\left(x^2\right))^2}\end{aligned}$$

By the Pythagorean Identity, $\cos^2\left(x^2\right) + \sin^2\left(x^2\right) = 1$.

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{2x + 2x(\cos\left(x^2\right) + \sin\left(x^2\right))}{(1+\cos\left(x^2\right))^2} \\ &= \frac{2x(1+\cos\left(x^2\right) + \sin\left(x^2\right))}{(1+\cos\left(x^2\right))^2}.\end{aligned}$$

2. Compute the derivative.

(a) $f(x) = (x^4 + 3x^2 - 2)^5$.

answer: $(30x + 20x^3)\left(-2 + 3x^2 + x^4\right)^4$

(b) $f(x) = (4x - x^2)^{100}$.

answer: $(-200x + 400)\left(4x - x^2\right)^{99}$

(c) $f(x) = (2x-3)^4(x^2+x+1)^5$.

answer: $(-7 - 12x + 28x^2)\left(-3 + 2x\right)^3\left(1 + x + x^2\right)^4$

(d) $f(x) = (x^2+1)^3(x^2+2)^6$.

answer: $(24x + 18x^3)\left(1 + x^2\right)^2\left(2 + x^2\right)^5$

(e) $f(x) = (3x-1)^4(2x+1)^{-3}$.

answer: $(3x-1)^3\,\frac{6x+18}{(2x+1)^4}$

(f) $f(x) = \dfrac{1}{1+x^2}$.

answer: $\frac{-2x}{(1+x^2)^2}$

(g) $f(x) = \left(\dfrac{x^2+1}{x^2-1}\right)^3$.

answer: $\frac{-12x}{(x^2-1)^2}\left(\frac{x^2+1}{x^2-1}\right)^2$

(h) $f(x) = (x+1)^{\frac{2}{3}}(2x^2-1)^3$.

answer: $\left(\frac{40}{3}x^2 + 12x - \frac{2}{3}\right)\left(2x^2-1\right)^2(x+1)^{-\frac{1}{3}}$

(i) $f(x) = \sqrt{1-2x}$.

answer: $-(1-2x)^{-\frac{1}{2}}$

(j) $f(x) = \sqrt{\dfrac{x^2+1}{x^2+4}}$.

answer: $\frac{3x}{\left(x^2+4\right)^2}\left(\frac{x^2+1}{x^2+4}\right)^{-\frac{1}{2}}$

(k) $f(x) = 3\cot(2x)$.

answer: $\frac{-6}{(\sin(2x))^2}$

(l) $f(x) = \dfrac{1}{(1+\sec x)^2}$.

answer: $\frac{-2\cos(x)\sin(x)}{(1+\cos(x))^3} = \frac{-\sin(2x)}{(1+\cos(x))^3}$

(m) $f(x) = \sqrt[3]{1+\tan x}$.

answer: $\frac{1}{3}(1+\tan x)^{-\frac{2}{3}}\sec^2 x$

(n) $f(x) = \cos(2+x^3)$.

answer: $-3x^2\sin\left(2+x^3\right)$

(o) $f(x) = \cos\left(\dfrac{1}{x}\right)\sin(x^2)$.

answer: $x^{-2}\sin\left(x^{-1}\right)\sin\left(x^2\right) + 2x\cos\left(x^{-1}\right)\cos\left(x^2\right)$

(p) $f(x) = x\sec(kx)$.

answer: $\frac{\cos(kx)+kx\sin(kx)}{(\cos(kx))^2}$

3. Differentiate.

(a) $f(x) = \sin(\tan(2x))$.

answer: $2\sec^2(2x)\cos(\tan 2x)$

(b) $f(x) = \sec^2(mx)$.

answer: $\frac{2m\sin(mx)}{(\cos(mx))^3}$

(c) $f(x) = \sec^2 x + \tan^2 x$.

answer: $\frac{4\sin x}{\cos^3 x}$

(d) $f(x) = x\sin\left(\dfrac{1}{x}\right)$.

answer: $-x^{-1}\cos\left(x^{-1}\right) + \sin\left(x^{-1}\right)$

(e) $f(x) = \left(\dfrac{1-\cos(2x)}{1+\cos(2x)}\right)^4$.

answer: $\frac{16\sin(2x)}{(\cos(2x)+1)^2}\left(\frac{-\cos(2x)+1}{\cos(2x)+1}\right)^3$

(f) $f(x) = \sqrt{\dfrac{x}{x^2+4}}$.

answer: $\frac{-\frac{1}{2}x^2+2}{(x^2+4)^2}\left(\frac{x}{x^2+4}\right)^{-\frac{1}{2}}$

(g) $f(t) = \cot^2(\sin t)$.

answer: $\frac{-2\cos t\cos(\sin t)}{\sin^3(\sin t)}$

(h) $f(x) = \left(ax + \sqrt{x^2 + b^2}\right)^{-2}$.

answer: $\frac{-2x\left(x^2+b^2\right)^{-\frac{1}{2}}-2a}{\left(\left(x^2+b^2\right)^{\frac{1}{2}}+ax\right)^3}$

(i) $f(x) = \left(x^2 + (1-3x)^5\right)^3$.

answer: $\left(-45(-3x+1)^4+6x\right)\left((-3x+1)^5+x^2\right)^2$
Using computer algebra:
$(-3645x^4+4860x^3-2430x^2+546x-45)\left((-3x+1)^5+x^2\right)^2$
Using computer algebra full expansion:
$-215233605x^{14}+1004423490x^{13}-2176250895x^{12}$
$+2903793624x^{11}-2666357595x^{10}+1782098820x^9$
$-893713176x^8+341444160x^7-99805041x^6$
$+22199676x^5-3697470x^4+447132x^3$
$-37125x^2+1896x-45$

(j) $f(x) = \sin(\sin(\sin x))$.

answer: $\cos x\cos(\sin x)\cos(\sin(\sin x))$

(k) $f(x) = \sqrt{x + \sqrt{x}}$.

answer: $\left(\frac{1}{2}+\frac{1}{4}x^{-\frac{1}{2}}\right)\left(x^{\frac{1}{2}}+x\right)^{-\frac{1}{2}}$

(l) $f(x) = \sqrt{x + \sqrt{x + \sqrt{x}}}$.

answer: $\frac{1}{2}\left(\left(x^{\frac{1}{2}}+x\right)^{\frac{1}{2}}+x\right)^{-\frac{1}{2}}\left(\frac{1}{2}\left(x^{\frac{1}{2}}+x\right)^{-\frac{1}{2}}\left(\frac{1}{2}x^{-\frac{1}{2}}+1\right)+1\right)$

(m) $f(x) = (2r\sin(rx) + n)^p$.

answer: $pr(2r\sin(rx)+n)^{p-1}\cos(rx)$

(n) $f(x) = \cos^4(\sin^3 x)$.

answer: $-12\cos x\sin^2 x\sin(\sin^3 x)\cos^3(\sin^3 x)$

(o) $f(x) = \cos\sqrt{\sin(\tan(\pi x))}$.

answer: $\frac{-\frac{1}{2}\pi\cos(\tan(\pi x))\sin\left(\sqrt{\sin(\tan(\pi x))}\right)}{\sqrt{\sin(\tan(\pi x))}\cos^2(\pi x)}$

(p) $f(x) = \left(x + (x + \sin^2 x)^3\right)^4$.

answer: $4((\sin^2 x+x)^3+x)^3(3(\sin^2 x+x)^2(2\sin x\cos x+1)+1)$

4. Compute the second derivative.

(a) $f(x) = \sin(-5x)$.

answer: $25\sin(5x)$

(b) $f(x) = \cot(2x)$.

answer: $8\cot(2x)\csc^2(2x) = \frac{8\cos(2x)}{\sin^3(2x)}$

(c) $f(x) = e^{-3x}$.

answer: $9e^{-3x}$

(d) $f(x) = e^{\frac{1}{x}}$.

answer: $2e^{x^{-1}}x^{-3}+e^{x^{-1}}x^{-4}$

(e) $f(x) = e^{\sqrt{x}}$.

answer: $e^{x^{\frac{1}{2}}}x^{-\frac{3}{2}}+\frac{1}{4}e^{x^{\frac{1}{2}}}x^{-1}$

(f) $f(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

answer: $\frac{-8\left(-e^{-x}+e^x\right)}{\left(e^{-x}+e^x\right)^3}$

(g) $f(x) = \dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)$

answer: $\frac{1}{2}\left(-\frac{1}{(x+1)^2}+\frac{1}{(-x+1)^2}\right) = \frac{2x}{\left(1-x^2\right)^2}$