Calculus II
Power series, full lecture

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Outline

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Power Series

Definition (Power Series)

A power series is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

where x is a variable and the c_n's are constants called the coefficients of the series.

- For each fixed x, this is a series of constants which either converges or diverges.
- A power series might converge for some values of x and diverge for others.
- The sum of the series is a function
$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$
whose domain is the set of all x for which the series converges.
- f resembles a polynomial, except it has infinitely many terms.

Definition (Power Series Centered at a)

A series of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

is called a power series centered at a or a power series about a or a power series in $(x-a)$.

- We use the convention that $(x-a)^0 = 1$, even if $x = a$.
- If $x = a$, then all terms are 0 for $n \geq 1$, so the series always converges when $x = a$.

Example

For what values of x is the series $\sum_{n=0}^{\infty} n!x^n$ convergent?

- Use the Ratio Test.
- The nth term is $a_n = n!x^n$.
- If $x \neq 0$, then

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(n+1)!x^{n+1}}{n!x^n}\right| \\ &= \lim_{n\to\infty}(n+1)|x| \\ &= \infty\end{aligned}$$

- Therefore by the Ratio Test the series diverges for all $x \neq 0$.
- Therefore the series only converges for $x = 0$.

Example

Find the domain of the Bessel function of order 0 defined by

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

- The nth term is $a_n = \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$.

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{2(n+1)}}{2^{2(n+1)}[(n+1)!]^2}\cdot\frac{2^{2n}(n!)^2}{(-1)^n x^{2n}}\right| \\
&= \lim_{n\to\infty}\frac{x^2}{4(n+1)^2} = 0 < 1
\end{aligned}$$

- Therefore by the Ratio Test the series converges for all x.
- Therefore the domain of the function is $(-\infty, \infty)$, or $\mathbb{R}$.

Example

For what values of x is the series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ convergent?

- Use the Ratio Test.
- The nth term is $a_n = \frac{(x-3)^n}{n}$.

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{(x-3)^{n+1}}{n+1} \cdot \frac{n}{(x-3)^n}\right| \\
&= \lim_{n\to\infty} |x-3| \frac{n}{n+1} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} = \lim_{n\to\infty} |x-3| \frac{1}{1+\frac{1}{n}} = |x-3|
\end{aligned}$$

- Therefore by the Ratio Test the series converges absolutely if $|x-3| < 1$ and diverges if $|x-3| > 1$.

$$|x-3| < 1 \quad \Leftrightarrow \quad -1 < x-3 < 1 \quad \Leftrightarrow \quad 2 < x < 4$$

- If we put $x = 4$ in the series, we get $\sum \frac{1}{n}$, which is divergent.
- If we put $x = 2$ in the series, we get $\sum \frac{(-1)^n}{n}$, which is convergent.
- The series converges if $2 \le x < 4$ and diverges otherwise.

Theorem (Convergence of Power Series)

For a power series $\sum c_n(x-a)^n$*, there are three possibilities:*

1. *The series converges only when* $x = a$*.*
2. *The series converges for all* x*.*
3. *There is a positive number* R *such that the series converges if* $|x-a| < R$ *and diverges if* $|x-a| > R$*.*

Definition (Radius of Convergence)

The number R in case three of the theorem is called the radius of convergence of the power series.

1. In the first case, we say $R = 0$.
2. In the second case, we say $R = \infty$.

Theorem (Convergence of Power Series)

For a power series $\sum c_n(x-a)^n$, *there are three possibilities:*

1. *The series converges only when* $x = a$.
2. *The series converges for all* x.
3. *There is a positive number* R *such that the series converges if* $|x-a| < R$ *and diverges if* $|x-a| > R$.

Definition (Interval of Convergence)

The interval of convergence of a power series is the interval consisting of all numbers x for which the series converges.

1. In the first case, the interval contains the single point a.
2. In the second case, the interval is $(-\infty, \infty)$.
3. In the third case, the inequality $|x-a| < R$ can be rewritten $a - R < x < a + R$.

What happens at the endpoints of the interval $a - R < x < a + R$?

- Anything can happen.
- The series might converge at one endpoint.
- The series might converge at both endpoints.
- The series might diverge at both endpoints.
- Thus we have four possibilities for the endpoints.
  1. $[a - R, a + R)$
  2. $(a - R, a + R]$
  3. $[a - R, a + R]$
  4. $(a - R, a + R)$
- In general, the Ratio Test (or Root Test) should be used to find the radius of convergence R.
- The Ratio and Root Tests will always fail when x is an endpoint $a - R$ or $a + R$, so the endpoints must be checked with another test.

Example

Find the radius of convergence and interval of convergence of the series $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$.

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(-3)^{n+1}x^{n+1}}{\sqrt{n+2}}\cdot\frac{\sqrt{n+1}}{(-3)^n x^n}\right| \\ &= \lim_{n\to\infty} 3|x|\sqrt{\frac{n+1}{n+2}}\cdot\frac{\frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} 3|x|\sqrt{\frac{1+\frac{1}{n}}{1+\frac{2}{n}}} = 3|x|\end{aligned}$$

- Ratio Test: it converges if $3|x| < 1$ and diverges if $3|x| > 1$.
- So it converges if $|x| < \frac{1}{3}$ and diverges if $|x| > \frac{1}{3}$.
- Therefore $R = \frac{1}{3}$.
- If we use $x = \frac{1}{3}$, we get $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$, which is convergent.
- If we use $x = -\frac{1}{3}$, we get $\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}}$, which is divergent.
- The interval of convergence is $(-\frac{1}{3}, \frac{1}{3}]$.

Representations of Functions as Power Series

$$g(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad \text{for } |x| < 1$$

- This is a geometric series with $a = 1$ and $r = x$.
- It is convergent if $|x| < 1$ and divergent otherwise.
- If convergent, the sum is $\frac{1}{1-x}$.
- The domain of $g(x)$ is $|x| < 1$.
- The domain of $f(x) = \frac{1}{1-x}$ is $x \neq 1$.
- In this way $g(x) = \sum_{n=0}^{\infty} x^n$ is a new way to compute/expresses the function $f(x) = \frac{1}{1-x}$ for $|x| < 1$.
- Except for their domains, the functions $g(x)$ and $f(x)$ coincide.

Recall the geometric series formula:

$$\frac{1}{1-y} = \sum_{n=0}^{\infty} y^n = 1 + y + y^2 + y^3 + \dots \qquad \begin{array}{l}\text{if \& only if}\\ |y| < 1\end{array} \quad .$$

Example

Write $\frac{1}{1+x^2}$ as a power series and find the interval of convergence.

$$\begin{aligned} \frac{1}{1+x^2} &= \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} \left(-x^2\right)^n \qquad \left| \begin{array}{l}\text{if \& only if}\\ |-x^2| < 1\end{array} \right. \\ &= 1 + (-x^2) + (-x^2)^2 + (-x^2)^3 + \dots \\ &= 1 \quad - x^2 + x^4 - x^6 + \dots \\ &= \sum_{n=0}^{\infty} (-1)^n x^{2n} \end{aligned}$$

- This converges if and only if $\left| \begin{array}{rcl} |-x^2| & < & 1 \\ |x| & < & 1 \end{array} \right.$.
- Therefore the interval of convergence is $x \in (-1, 1)$.

Example

Find a power series representation for $\frac{1}{x+2}$.

$$\begin{aligned}\frac{1}{2+x} &= \frac{1}{2\left(1+\frac{x}{2}\right)}\\ &= \frac{1}{2}\cdot\frac{1}{\left(1-\left(-\frac{x}{2}\right)\right)} = \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n \quad \Bigg| \quad \text{if \& only if } \left|-\tfrac{x}{2}\right| < 1\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^n\\ &= \frac{1}{2}-\frac{x}{4}+\frac{x^2}{8}-\frac{x^3}{16}+\dots\end{aligned}$$

To find interval of convergence:

$$\begin{aligned}\left|-\frac{x}{2}\right| &< 1\\ |x| &< 2\end{aligned}$$

Therefore the interval of convergence is $x \in (-2, 2)$.

Example

Find a power series representation for $\frac{x^3}{x+2}$.

$$\begin{aligned} \frac{x^3}{x+2} &= x^3 \cdot \frac{1}{x+2} \\ &= x^3 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^n \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^{n+3} \\ &= \frac{x^3}{2} - \frac{x^4}{4} + \frac{x^5}{8} - \frac{x^6}{16} + \cdots \end{aligned}$$

if & only if $|x| < 2$

- Another way to write this is $\frac{x^3}{x+2} = \sum_{n=3}^{\infty} \frac{(-1)^{n-1}}{2^{n-2}} x^n$.
- The interval of convergence is again $x \in (-2, 2)$.

Differentiation and Integration of Power Series

Theorem (Differentiation and Integration of Power Series)

If a power series $\sum c_n(x-a)^n$ *has radius of convergence* $R>0$*, then the function* f *defined by*

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ *and*

1. $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \displaystyle\sum_{n=1}^{\infty} nc_n(x-a)^{n-1}.$
2. $\displaystyle\int f(x)\,\mathrm{d}x = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots$
   $= C + \sum_{n=0}^{\infty} c_n\frac{(x-a)^{n+1}}{n+1}.$

- This is called term-by-term differentiation and integration.
- Another way of saying it is

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left[\sum_{n=0}^{\infty} c_n(x-a)^n\right] &= \sum_{n=0}^{\infty} \frac{\mathrm{d}}{\mathrm{d}x}\left[c_n(x-a)^n\right] \\
\int \left[\sum_{n=0}^{\infty} c_n(x-a)^n\right] \mathrm{d}x &= \sum_{n=0}^{\infty} \int \left[c_n(x-a)^n\right] \mathrm{d}x
\end{aligned}
$$

- We can treat power series like polynomials with infinitely many terms.

Example

Find the derivative of the Bessel function

$$\begin{aligned} J_0(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2} \\ J_0'(x) &= \sum_{n=0}^{\infty} \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{(-1)^n x^{2n}}{2^{2n}(n!)^2} \right) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n 2n x^{2n-1}}{2^{2n}(n!)^2} \end{aligned}$$

- $J_0(x)$ is defined everywhere.
- Therefore its derivative $J_0'(x)$ is also defined everywhere.

Example

Find a power series for $\ln(1-x)$ and state its radius of convergence.

$$\begin{aligned}
\ln(1-x) &= \int \mathrm{d}(\ln(1-x)) = \int (\ln(1-x))'\mathrm{d}x && \Big| \text{ up to const.}\\
&= \int \left(-\frac{1}{1-x}\right)\mathrm{d}x \\
&= -\int \left(1+x+x^2+x^3+\cdots\right)\mathrm{d}x && \Big| \text{ for } |x|<1\\
&= -\left(x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\cdots\right)+C\\
&= C-\sum_{n=1}^{\infty}\frac{x^n}{n}
\end{aligned}$$

- To find C, plug in $x=0$: $C=0$.
- Therefore the theorem on integrating power series implies that
  $$\ln(1-x) = -\sum_{n=1}^{\infty}\frac{x^n}{n}, \text{ for } |x|<1.$$
- By the same theorem, the radius of convergence remains $R=1$.

Example

Find a power series for $\arctan x$ and state its radius of convergence.

$$\begin{aligned}
\arctan(x) &= \int \mathrm{d}(\arctan x) = \int (\arctan x)'\mathrm{d}x && \Big|\ \text{up to const.}\\
&= \int \left(\frac{1}{1+x^2}\right)\mathrm{d}x = \int \left(\frac{1}{1-(-x^2)}\right)\mathrm{d}x \\
&= \int \left(1 - x^2 + x^4 - x^6 + \cdots\right)\mathrm{d}x && \Big|\ \text{for } |x|<1\\
&= \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\right) + C\\
&= C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}
\end{aligned}$$

- To find C, plug in $x=0$: $C=0$.
- Therefore the theorem on integrating power series implies that

$$\arctan x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}, \text{ for } |x|<1.$$

- By the same theorem, the radius of convergence remains $R=1$.

Taylor and Maclaurin Series

- Let f be a function that can be represented by a power series:
- $f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$
- $f(a) = c_0$.
- $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots$
- $f'(a) = c_1$.
- $f''(x) = 2c_2 + 2\cdot 3c_3(x-a) + 3\cdot 4c_4(x-a)^2 + 4\cdot 5c_5(x-a)^3 + \cdots$
- $f''(a) = 2c_2$.
- $f'''(x) = 2\cdot 3c_3 + 2\cdot 3\cdot 4c_4(x-a) + 3\cdot 4\cdot 5c_5(x-a)^2 + \cdots$
- $f'''(a) = 2\cdot 3c_3 = 3!c_3$.
- $f^{(n)}(a) = n!c_n$.
- Therefore $c_n = \frac{f^{(n)}(a)}{n!}$.

Theorem (Coefficients of a Power Series)

If f has a power series representation at a, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n, \qquad |x-a| < R,$$

then its coefficients are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

Here is what we get if we plug these coefficients into the power series:

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots \end{aligned}$$

Definition (Taylor Series)

This series is called the Taylor series of f.

The case when $a = 0$ is special enough to have its own name:

Definition (Maclaurin Series)

The Maclaurin series of f is the Taylor series of f centered at $a = 0$. In other words, it is the series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

Example

Find the Maclaurin series of $f(x) = e^x$ and its radius of convergence.

- $f^{(n)}(x) = e^x$.
- $f^{(n)}(0) = e^0 = 1$.
- Therefore the Maclaurin series is
$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$
- To find the radius of convergence, let $a_n = \frac{x^n}{n!}$.
$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \lim_{n\to\infty} \frac{|x|}{n+1} = 0 < 1$$
- Therefore by the Ratio Test the series converges for all x.
- Therefore $R = \infty$.

Example

Find the sum of the series

$$
\begin{aligned}
\sum_{n=0}^{\infty}(-1)^n\frac{1}{2^n n!} &= 1-\frac{1}{2\cdot 1!}+\frac{1}{4\cdot 2!}-\frac{1}{8\cdot 3!}+\cdots \\
e^x &= \sum_{n=0}^{\infty}\frac{1}{n!}x^n \\
\sum_{n=0}^{\infty}(-1)^n\frac{1}{2^n n!} &= \sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{1}{2}\right)^n \\
&= e^{-\frac{1}{2}} \\
&= \frac{1}{\sqrt{e}}
\end{aligned}
$$

Example

Find the Taylor series for $f(x) = e^x$ at $a = 3$.

- $f^{(n)}(x) = e^x$.
- $f^{(n)}(3) = e^3$.
- Therefore the Taylor series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(3)}{n!}(x-3)^n = \sum_{n=0}^{\infty} \frac{e^3}{n!}(x-3)^n$$

- To find the radius of convergence, let $a_n = \frac{e^3}{n!}(x-3)^n$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{e^3(x-3)^{n+1}}{(n+1)!}\cdot\frac{n!}{e^3(x-3)^n}\right| = \lim_{n\to\infty}\frac{|x-3|}{n+1} = 0$$

- Therefore by the Ratio Test the series converges for all x.
- Therefore $R = \infty$.
- Just like the Maclaurin series, this series also represents e^x.

Example

Find the Taylor series for $f(x) = e^x$ at $a = 3$.

$$\begin{aligned} e^x &= e^{x-3+3} = e^3 e^{x-3} \\ &= e^3 \sum_{n=0}^{\infty} \frac{(x-3)^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{e^3}{n!}(x-3)^n \end{aligned}$$

Recall that $e^y = \sum_{n=0}^{\infty} \frac{y^n}{n!}$

Set $y = x - 3$

The radius of convergence was already computed to be $R = \infty$.

Example

Find the Maclaurin series of $f(x) = \sin x$ and its radius of convergence.

$$\begin{array}{rclrcl} f(x) &=& \sin x & f(0) &=& 0 \\ f'(x) &=& \cos x & f'(0) &=& 1 \\ f''(x) &=& -\sin x & f''(0) &=& 0 \\ f'''(x) &=& -\cos x & f'''(0) &=& -1 \\ f^{(4)}(x) &=& \sin x & f^{(4)}(0) &=& 0 \end{array}$$

The Maclaurin series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Use the Ratio Test to find R.

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{(-1)^n x^{2n+1}} \right| \\ &= \lim_{n\to\infty} \frac{x^2}{(2n+2)(2n+3)} = 0 \end{aligned}$$

Therefore $R = \infty$. It can be shown that this series sums to $\sin x$.

Example

Find the sum of the series

$$\sum_{n=0}^{\infty}(-1)^n\frac{\pi^{2n+1}}{2^{2n+1}(2n+1)!}=\frac{\pi}{2}-\frac{\pi^3}{8\cdot 3!}+\frac{\pi^5}{32\cdot 5!}-\frac{\pi^7}{128\cdot 7!}+\cdots$$

$$\begin{aligned}
\sin x &= \sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)!}x^{2n+1}\\
\sum_{n=0}^{\infty}(-1)^n\frac{\pi^{2n+1}}{2^{2n+1}(2n+1)!} &= \sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)!}\left(\frac{\pi}{2}\right)^{2n+1}\\
&= \sin\frac{\pi}{2}\\
&= 1
\end{aligned}$$

Example

Find the Maclaurin series for $\cos x$.

$$\begin{aligned}
\cos x &= \frac{\mathrm{d}}{\mathrm{d}x}(\sin x)\\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}\right)\\
&= \sum_{n=0}^{\infty}\frac{\mathrm{d}}{\mathrm{d}x}\left((-1)^n\frac{x^{2n+1}}{(2n+1)!}\right)\\
&= \sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)x^{2n}}{(2n+1)!}\\
&= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}\\
&= 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots
\end{aligned}$$

The series for $\sin x$ converges everywhere, so the series for $\cos x$ does too.

Example

Find the Maclaurin series for $x \cos x$.

$$\begin{aligned} x\cos x &= x\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} \\ &= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n)!} \\ &= x-\frac{x^3}{2!}+\frac{x^5}{4!}-\frac{x^7}{6!}+\cdots \end{aligned}$$

Here is a table of some important Maclaurin series we have learned:

Function		Series	R
$\dfrac{1}{1-x}$	$=$	$\displaystyle\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$	1
$\arctan x$	$=$	$\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$	1
e^x	$=$	$\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$	∞
$\sin x$	$=$	$\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$	∞
$\cos x$	$=$	$\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$	∞

Example

Use a power series to find $\lim\limits_{x\to 0}\dfrac{e^x-1-x}{x^2}$.

$$
\begin{aligned}
e^x &= 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\\
e^x-1-x &= \frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots\\
\frac{e^x-1-x}{x^2} &= \frac{1}{2!}+\frac{x}{3!}+\frac{x^2}{4!}+\cdots\\
\lim_{x\to 0}\frac{e^x-1-x}{x^2} &= \lim_{x\to 0}\left(\frac{1}{2}+\frac{x}{3!}+\frac{x^2}{4!}+\cdots\right)=\frac{1}{2} \quad .
\end{aligned}
$$

Example

Use a power series to find $\lim\limits_{x\to 0}\dfrac{x-\sin x}{x^3}$.

$$\begin{aligned}
\sin x &= x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots\\
-\sin x &= -x+\frac{x^3}{3!}-\frac{x^5}{5!}+\frac{x^7}{7!}-\cdots\\
x-\sin x &= \frac{x^3}{3!}-\frac{x^5}{5!}+\frac{x^7}{7!}-\cdots\\
\frac{x-\sin x}{x^3} &= \frac{1}{3!}-\frac{x^2}{5!}+\frac{x^4}{7!}-\cdots\\
\lim_{x\to 0}\frac{x-\sin x}{x^3} &= \lim_{x\to 0}\left(\frac{1}{6}-\frac{x^2}{5!}+\frac{x^4}{7!}-\cdots\right)=\frac{1}{6}
\end{aligned}$$