Calculus I
Fermat's Theorem and the Mean Value Theorem

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Outline

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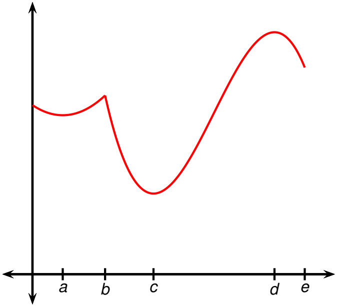
Maximum and Minimum Values

Many real-world problems involve finding minima and maxima (finding minimal costs, maximal profit, shortest time to do a job, etc.). Examples include

- What shape of can minimizes manufacturing costs?
- What is the maximum acceleration of a space shuttle?
- What is the maximum load an elevator can carry?

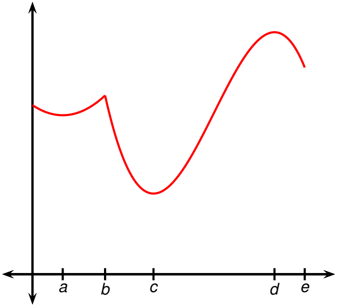
Often such questions can be reduced to finding maximum or minimum values of a function. In Calculus I, we study how to minimize and maximize functions in one variable.



- Absolute maximum at d .
- Absolute minimum at c .

Definition (Absolute Maximum or Minimum)

A function f has an absolute maximum (or global maximum) at c if $f(c) \geq f(x)$ for all x in the domain of f. The number $f(c)$ is called the maximum value of f.
Likewise, f has an absolute minimum at c if $f(c) \leq f(x)$ for all x in the domain of f. $f(c)$ is called the minimum value of f.
Maximum and minimum values of f are called extreme values.



- Absolute maximum at d .
- Absolute minimum at c .
- Local maximum at b, d and 0.
- Local minimum at a, c and e.

Definition (Local Maximum or Minimum)

A function f has a local maximum at c if there exists an open interval containing c such that $f(c) \geq f(x)$ for all x in that interval. Similarly, f has a local minimum at c if there exists an open interval containing c such that $f(c) \leq f(x)$ for all x in that interval.

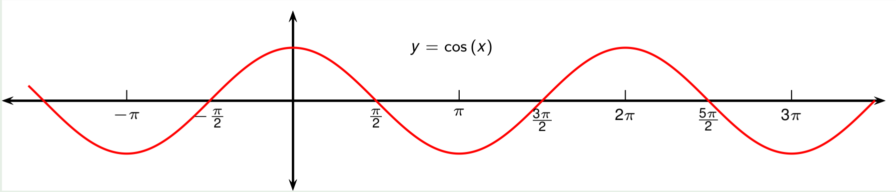
Question

Is it possible that a function attains its maximum/minimum value for infinitely many values of x?

Example

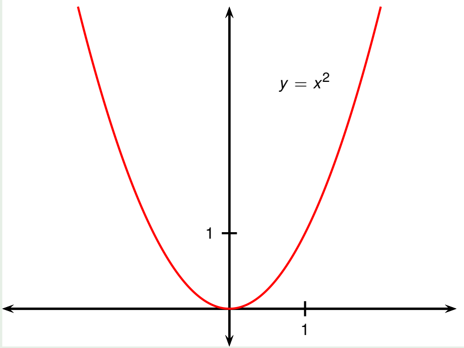
The function $\cos x$ attains its maximum value (=1) infinitely many times, since $\cos(2n\pi) = 1$ for any integer n.

Likewise, it attains its minimum value of -1 infinitely many times, because $\cos((2n+1)\pi) = -1$ for all integers n.



Example

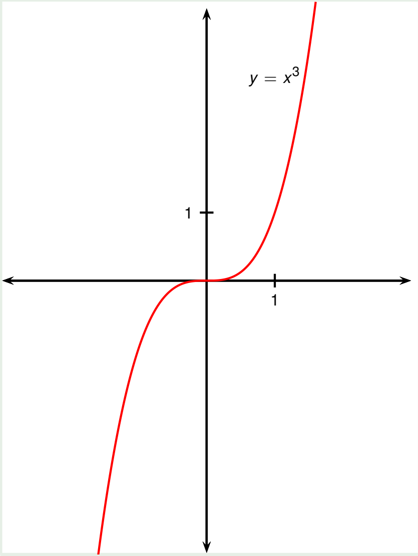
Consider the function $y = x^2$.



- Absolute maximum: None
- Absolute minimum: at 0
- Local maximum: None
- Local minimum: at 0

Example

Consider the function $y = x^3$.



- Absolute maximum: None
- Absolute minimum: None
- Local maximum: None
- Local minimum: None

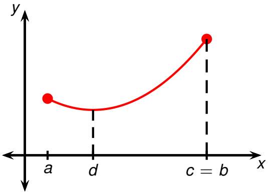
The Extreme Value Theorem

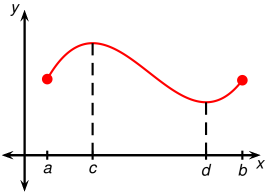
Recall that some functions (such as $y = \cos x$) have extreme values, while other functions (such as $y = x^3$) do not. The next theorem, which we will not prove, gives a condition under which f must have extreme values.

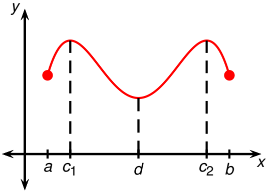
Theorem (The Extreme Value Theorem)

If f is continuous on a closed and bounded interval $[a, b]$, then f attains its maximum and minimum value, each at least once. In other words, there exist numbers c and d in $[a, b]$ such that

$$f(c) \geq f(x) \geq f(d) \qquad \text{for all } x \in [a, b]$$



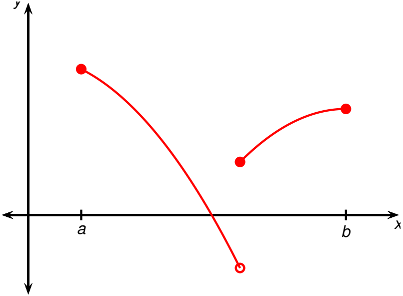


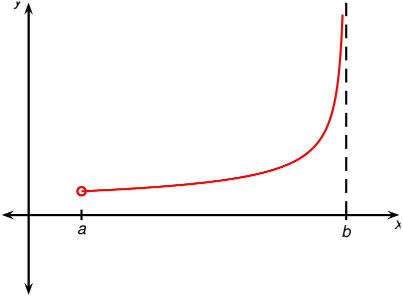


- Extreme values might happen at endpoints.
- Extreme values might happen twice.

Theorem (The Extreme Value Theorem)

If f is continuous on a closed interval $[a, b]$, then f attains its maximum and minimum value, each at least once.





- Do we need all of the hypotheses of the theorem?
- Do we need f to be continuous? Yes.
- Do we need the interval to be closed? Yes.

Fermat's Theorem

The next theorem gives a condition that can help to find local maxima and minima.

Theorem (Fermat's Theorem)

Let f be a function defined in an open interval around c and such that $f'(c)$ exists. If f has a local maximum or minimum at c, then $f'(c) = 0$.

Proof.

- We prove the theorem only when f has a local maximum at c.
- This means that $f(x) \leq f(c)$ for all x close to c.
- If $|h|$ is sufficiently small, then $f(c+h) - f(c) \leq 0$.
- Suppose h is positive, and divide both sides by h:
$$f'(c) = \lim_{h\to 0} \frac{f(c+h) - f(c)}{h} = \lim_{h\to 0^+} \frac{f(c+h) - f(c)}{h} \leq \lim_{h\to 0^+} 0 = 0$$
- Suppose h is negative, and divide both sides by h:
$$f'(c) = \lim_{h\to 0} \frac{f(c+h) - f(c)}{h} = \lim_{h\to 0^-} \frac{f(c+h) - f(c)}{h} \geq \lim_{h\to 0^-} 0 = 0$$
- Therefore $f'(c) \leq 0$ and $f'(c) \geq 0$, so $f'(c) = 0$. □

Theorem (Fermat's Theorem)

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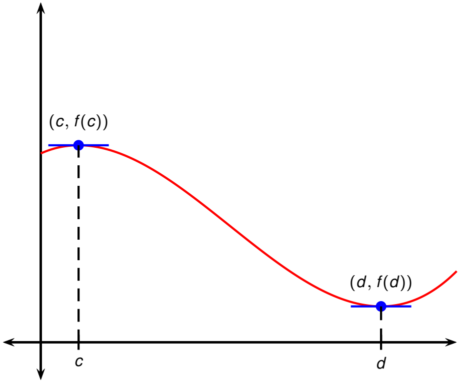
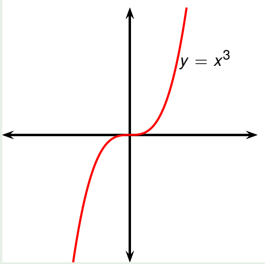


Theorem (Fermat's Theorem)

Let f be a function defined in an open interval around c and such that $f'(c)$ exists. If f has a local maximum or minimum at c, then $f'(c) = 0$.

What does Fermat's Theorem not say?

Example



- Let $f(x) = x^3$.
- Then $f'(x) = 3x^2$.
- $f'(x) = 0$ when $x = 0$.
- But f has no local maximum or minimum at 0!

Fermat's Theorem does not say "if $f'(c) = 0$, then f has a local maximum or a local minimum at c."

Theorem (Fermat's Theorem)

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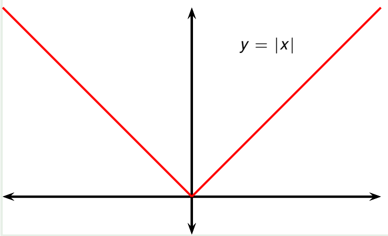


- Let $f(x) = |x|$.
- Then f has a local minimum at 0.
- But $f'(0)$ doesn't exist!

Fermat's Theorem does not say "if f has a local maximum or minimum at c, then $f'(c)$ exists."

The Mean Value Theorem

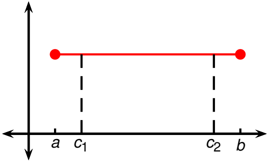
- The first derivative test, the results on concavity and curve sketching, as well as the (soon to be covered) topics of linear approximation and integration depend on an important theorem.
- This is the Mean Value Theorem.
- We will give a complete proof of the Mean Value Theorem.
- We start with a prerequisite result called Rolle's Theorem.

Theorem (Rolle's Theorem)

Let f be a function that satisfies the following three conditions:

- *f is continuous on the closed interval $[a, b]$.*
- *f is differentiable on the open interval (a, b).*
- *$f(a) = f(b)$.*

Then there is a number c in (a, b) such that $f'(c) = 0$.



The proof breaks down into three cases:

1. f is a horizontal line.
2. $f(x) > f(a)$ for some x in (a, b).
3. $f(x) < f(a)$ for some x in (a, b).

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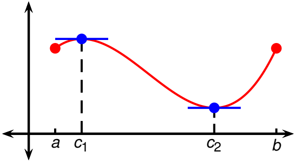

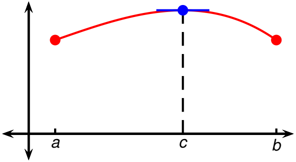

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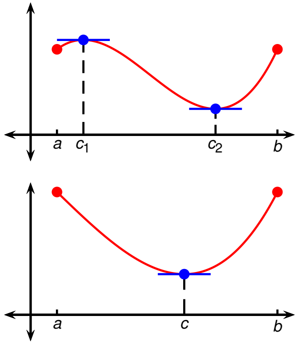

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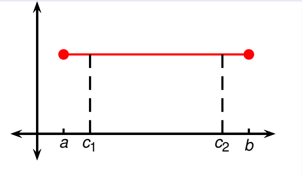
Proof.



1. f is a horizontal line.
   - Then $f'(x) = 0$.
   - Therefore we can take c to be any number in (a, b). □

Theorem (Rolle's Theorem)

Let f be a function that satisfies the following three conditions:

- *f is continuous on the closed interval $[a, b]$.*
- *f is differentiable on the open interval (a, b).*
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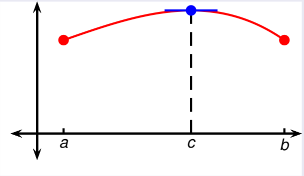
Proof.



2. $f(x) > f(a)$ for some x in (a, b).
   - By the Extreme Value Theorem, f has a maximum in $[a, b]$.
   - Since $f(x) > f(a)$, this value is attained at some c in (a, b).
   - Fermat's Theorem: $f'(c) = 0$. $\square$

Theorem (Rolle's Theorem)

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- *f is continuous on the closed interval $[a, b]$.*
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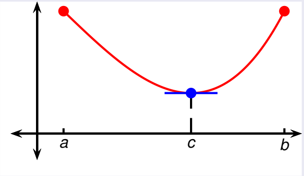
Proof.



3. $f(x) < f(a)$ for some x in (a, b).
   - By the Extreme Value Theorem, f has a minimum in $[a, b]$.
   - Since $f(x) < f(a)$, this value is attained at some c in (a, b).
   - Fermat's Theorem: $f'(c) = 0$. $\square$

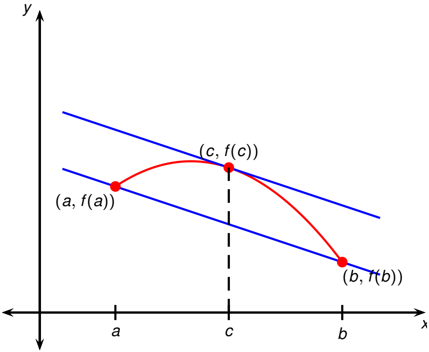
Example

Prove that the function $f(x) = x^3 + 4x - 4$ has exactly one real root.

- First show that it has a real root:
- $f(0) = -4$.
- $f(1) = 1$.
- Therefore by the Intermediate Value Theorem f has a root somewhere between 0 and 1.
- Now suppose that it has more than one root and use Rolle's Theorem to get a contradiction.
- Suppose it has two real roots a and b. Then $f(a) = 0 = f(b)$.
- f is a polynomial, so it is continuous and differentiable everywhere.
- By Rolle's Theorem, there is a c in (a, b) such that $f'(c) = 0$.
- $f'(x) = 3x^2 + 4$.
- Therefore $f'(x)$ is always positive.
- Contradiction.

Theorem (The Mean Value Theorem)

Let f be a function that is continuous on $[a, b]$ and differentiable on (a, b). Then there is a number c in (a, b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.



- Consider the secant line from $(a, f(a))$ to $(b, f(b))$.
- Slope: $m = \frac{f(b)-f(a)}{b-a}$.
- The Mean Value Theorem says that there exists a number c in (a, b) such that the slope of the tangent at c equals m.

Theorem (The Mean Value Theorem)

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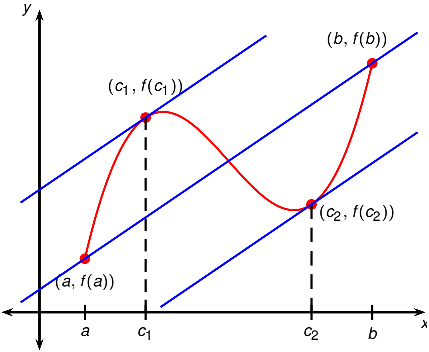

- Consider the secant line from $(a, f(a))$ to $(b, f(b))$.
- Slope: $m = \frac{f(b)-f(a)}{b-a}$.
- The Mean Value Theorem says that there exists a number c in (a, b) such that the slope of the tangent at c equals m.
- More than one number is allowed.

Theorem (The Mean Value Theorem)

Let f be a function that is continuous on $[a, b]$ and differentiable on (a, b). Then there is a number c in (a, b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Proof.

- Let L be the secant line from $(a, f(a))$ to $(b, f(b))$.
- $L(x) = f(a) + \frac{f(b)-f(a)}{b-a}(x - a)$.
- Consider the function $(f - L)(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x - a)$.
- L is linear, so it's continuous and differentiable everywhere.
- $f - L$ is continuous on $[a, b]$ and differentiable on (a, b).
- $(f - L)(a) = f(a) - f(a) - \frac{f(b)-f(a)}{b-a}(a - a) = 0$.
- $(f - L)(b) = f(b) - f(a) - \frac{f(b)-f(a)}{b-a}(b - a) = 0$.
- Rolle's Theorem: There exists c in (a, b) such that $0 = (f - L)'(c) = f'(c) - L'(c) = f'(c) - \frac{f(b)-f(a)}{b-a}$ $\square$

Theorem

If $f'(x) = 0$ for all x in an interval (a, b), then f is constant on (a, b).

Proof.

- Let x_1 and x_2 be any numbers in (a, b) with $x_1 < x_2$.
- f is differentiable on (a, b).
- Therefore f is differentiable on (x_1, x_2) and continuous on $[x_1, x_2]$.
- Mean Value Theorem: There exists c in (x_1, x_2) such that

$$\begin{aligned} f'(c) &= \frac{f(x_2) - f(x_1)}{x_2 - x_1} \\ f'(c)(x_2 - x_1) &= f(x_2) - f(x_1) \\ 0 &= f(x_2) - f(x_1) \\ f(x_1) &= f(x_2) \end{aligned}$$

Therefore f is constant on (a, b). $\square$

Corollary

If $f'(x) = g'(x)$ *for all* x *in an interval* (a, b), *then* $f - g$ *is constant on* (a, b)*; that is,* $f(x) = g(x) + c$ *where* c *is constant.*

Proof.

- Let $F(x) = f(x) - g(x)$.
- Then $F'(x) = f'(x) - g'(x) = 0$ for all x in (a, b).
- By the previous theorem, F is constant, so $f - g$ is constant. $\square$