

Precalculus

Completing the square

Todor Milev

2019

Example (Completing the square)

Complete the square.

$$3x^2 - 5x + 1$$

Example (Completing the square)

Complete the square.

$$3x^2 - 5x + 1 = 3 \left(x^2 - ? x \right) + 1$$

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$$3x^2 - 5x + 1 = 3 \left(x^2 - \frac{5}{3}x \right) + 1$$

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$$\begin{aligned} 3x^2 - 5x + 1 &= 3 \left(x^2 - \frac{5}{3}x \right) + 1 \\ &= 3 \left(x^2 - \color{red}{2} \cdot \frac{5}{\color{red}{2} \cdot 3} x \right) + 1 \end{aligned}$$

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Complete the square.

$$\begin{aligned}
 3x^2 - 5x + 1 &= 3 \left(x^2 - \frac{5}{3}x \right) + 1 \\
 &= 3 \left(x^2 - 2 \cdot \frac{5}{\textcolor{red}{2} \cdot \textcolor{red}{3}} x \right) + 1 \\
 &= 3 \left(x^2 - 2 \cdot \frac{5}{\textcolor{red}{6}} x + \textcolor{red}{?} \quad - \textcolor{red}{?} \right) + 1
 \end{aligned}$$

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Definition (Completing the square)

Let $a \neq 0$. To *complete the square* means to carry out the following algebraic manipulation.

$$ax^2 + bx + c$$

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 \left| \begin{array}{l} \text{Add \& subtract} \\ \left(\frac{b}{2a} \right)^2 \\ \text{use} \\ (A+B)^2 = \\ A^2 + 2AB + B^2 \end{array} \right.$$

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 &= a \left(\left(\color{red}{x} + \frac{b}{2a} \right)^2 - \frac{b^2}{4a^2} \right) + c
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 &= a \left(\left(x + \frac{b}{2a} \right)^2 - \frac{b^2}{4a^2} \right) + c \\
 &= a \left(x + \frac{b}{2a} \right)^2 - \cancel{a} \cdot \frac{b^2}{4\cancel{a}^2} + c \\
 &= a \left(x + \frac{b}{2a} \right)^2 + c - \frac{b^2}{4a}.
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