

# Precalculus

## Trigonometric equations and inequalities

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# Outline

- 1 Trigonometric equations and inequalities
  - The Equations  $\sin x = A$ ,  $\cos x = B$
  - Equations that reduce to  $\sin x = A$ ,  $\cos x = B$
- 2 Product-to-Sum Formulas
- 3 Trigonometric inequalities

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# Trigonometric equations

- Some problems will not ask you to prove a trigonometric identity, but rather to solve a trigonometric equation.
- Consider the problem of finding all values of  $x$  for which  $\sin x = \sin(2x) = 2 \sin x \cos x$ .
- This is not a trigonometric identity - the two sides are different.
- However, there are values for  $x$  which the above equality holds.

## Example

Find all solutions and then find those that lie between  $-360^\circ$  and  $360^\circ$ .

$$\sin \theta = \frac{\sqrt{3}}{2}$$

$$\theta = 60^\circ + k \cdot 360^\circ = \dots -660^\circ, -300^\circ, 60^\circ, 420^\circ, \dots$$

**or**

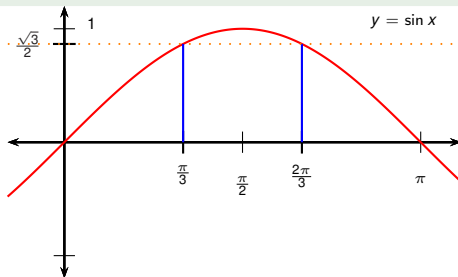
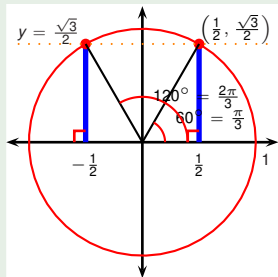
$$\dots \quad k=-2 \quad k=-1 \quad k=0 \quad k=1 \quad \dots$$

$$120^\circ + k \cdot 360^\circ = \dots -600^\circ, -240^\circ, 120^\circ, 480^\circ, \dots$$

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$$\theta = \dots \cancel{-660^\circ}, -300^\circ, 60^\circ, \cancel{420^\circ}, \dots$$

$$\dots \cancel{-600^\circ}, -240^\circ, 120^\circ, \cancel{480^\circ}, \dots$$



Find all solutions and then find those that lie between  $-180^\circ$  and  $180^\circ$ .

$$\cos \theta = \frac{\sqrt{2}}{2}$$

$$\theta = 45^\circ + k \cdot 360^\circ = \dots - 675^\circ, -315^\circ, 45^\circ, 405^\circ, \dots$$

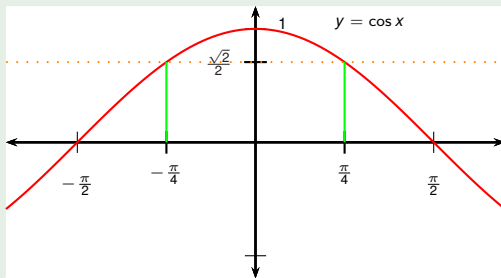
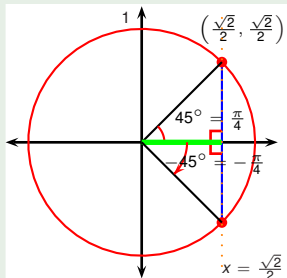
**or**

$$\cdots \quad k=-2 \quad k=-1 \quad k=0 \quad k=1 \quad \cdots$$

$$-45^\circ + k \cdot 360^\circ = \dots -765^\circ, -405^\circ, -45^\circ, 315^\circ, \dots$$

$$\theta = \cancel{0^\circ}, \cancel{-675^\circ}, \cancel{-315^\circ}, 45^\circ, \cancel{405^\circ}, \cancel{765^\circ}$$

~~$\therefore -765^\circ, -405^\circ, -45^\circ, 315^\circ, \therefore$~~



## Example

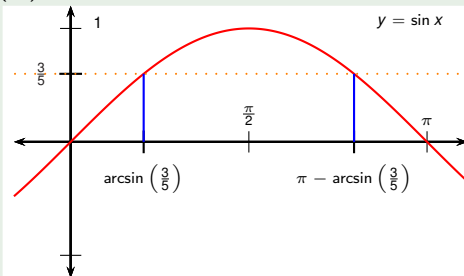
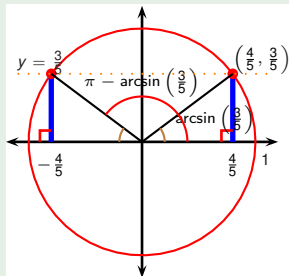
Find all solutions of the equation.

$$\sin \theta = \frac{3}{5}$$

$$\theta = \arcsin\left(\frac{3}{5}\right) + k \cdot (2\pi) \quad \left| \text{arcsin implies radians} \right.$$

or

$$\pi - \arcsin\left(\frac{3}{5}\right) + k \cdot (2\pi)$$



## Example

Find all values of  $\theta$  in the interval  $[0, 2\pi]$  such that  $\sin \theta = \sin(2\theta)$ .

$$\sin \theta = \sin(2\theta)$$

$$\sin \theta = 2 \sin \theta \cos \theta$$

$$0 = 2 \sin \theta \cos \theta - \sin \theta$$

$$0 = \sin \theta (2 \cos \theta - 1)$$

$$\sin \theta = 0$$

$$\theta = 0 + 2k\pi$$

$$\text{or } \pi + 2k\pi$$

$$\theta = 0 \text{ or } 2\pi \text{ or } \pi$$

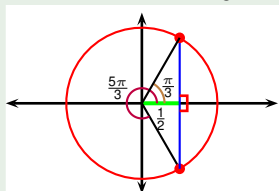
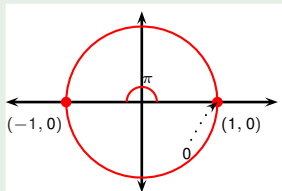
or

$$2 \cos \theta - 1 = 0$$

$$\cos \theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{3} + 2k\pi \text{ or } \frac{5\pi}{3} + 2k\pi$$

$$\theta = \frac{\pi}{3} \text{ or } \frac{5\pi}{3}$$





## Example

Find all values of  $\theta$  in the interval  $\theta \in [0, 2\pi]$  for which

$$\cos(2\theta) = \cos \theta$$

$$\cos^2 \theta - \sin^2 \theta - \cos \theta = 0 \quad \left| \text{Express via } \cos \theta \right.$$

$$\cos^2 \theta - (1 - \cos^2 \theta) - \cos \theta = 0$$

$$2 \cos^2 \theta - \cos \theta - 1 = 0 \quad \left| \text{Set } \cos \theta = u \right.$$

$$2u^2 - u - 1 = 0$$

$$(u - 1)(2u + 1) = 0$$

$$u - 1 = 0$$

$$\cos \theta = 1$$

$$\theta = 0 + 2k\pi$$

$$\theta = 0 \text{ or } 2\pi$$

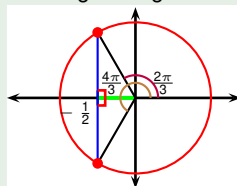
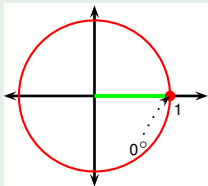
or

$$2u + 1 = 0$$

$$\cos \theta = -\frac{1}{2}$$

$$\theta = \frac{2\pi}{3} + 2k\pi \text{ or } \frac{4\pi}{3} + 2k\pi$$

$$\theta = \frac{2\pi}{3} \text{ or } \frac{4\pi}{3}$$



# Strategy for solving trigonometric equations

- Suppose we want to solve an algebraic trigonometric equation.
- More precisely, the equation should be an algebraic expressions of the trigonometric functions of a single variable.
- Here is a general strategy for solving such a problem:
  - Using trig identities, rewrite in terms of  $\sin x$  and  $\cos x$  only.
  - Suppose  $x \in [2n\pi, (2n+1)\pi]$ .
    - Set  $\sin x = \sqrt{1 - \cos^2 x}$  (allowed due to restrictions on  $x$ ).
    - Set  $\cos x = u$ . Solve the resulting algebraic equation for  $u$ .
    - For the found solutions for  $u$ , solve  $\cos x = u$ .
    - Check whether your solutions satisfy  $x \in [2n\pi, (2n+1)\pi]$ .
  - Suppose  $x \in [(2n-1)\pi, 2n\pi]$ .
    - Set  $\sin x = -\sqrt{1 - \cos^2 x}$  (allowed due to restrictions on  $x$ ).
    - Set  $\cos x = u$ . Solve the resulting algebraic equation for  $u$ .
    - For the found solutions for  $u$ , solve  $\cos x = u$ .
    - Check whether your solutions satisfy  $x \in [(2n-1)\pi, 2n\pi]$ .
- A similar strategy exists for  $u = \sin x$  instead of  $u = \cos x$ .
- Problems requiring full algorithm may be too hard for Calc exams.

## Proposition (Product to sum formulas)

$$\begin{aligned}\sin \alpha \sin \beta &= \frac{1}{2} (\cos(\alpha - \beta) - \cos(\alpha + \beta)) \\ \cos \alpha \cos \beta &= \frac{1}{2} (\cos(\alpha - \beta) + \cos(\alpha + \beta)) \\ \sin \alpha \cos \beta &= \frac{1}{2} (\sin(\alpha + \beta) + \sin(\alpha - \beta))\end{aligned}$$

## Proof.

$$\begin{array}{rcll} + & \cos \alpha \cos \beta + \cancel{\sin \alpha \sin \beta} & = & \cos(\alpha - \beta) \\ & \cos \alpha \cos \beta - \cancel{\sin \alpha \sin \beta} & = & \cos(\alpha + \beta) \\ & 2 \cos \alpha \cos \beta & = & \cos(\alpha - \beta) + \cos(\alpha + \beta) \\ \hline - & \cos \alpha \cos \beta + \sin \alpha \sin \beta & = & \cos(\alpha - \beta) \\ & \cos \alpha \cos \beta - \sin \alpha \sin \beta & = & \cos(\alpha + \beta) \\ & 2 \sin \alpha \sin \beta & = & \cos(\alpha - \beta) - \cos(\alpha + \beta) \\ \hline + & \sin \alpha \cos \beta + \cos \alpha \sin \beta & = & \sin(\alpha + \beta) \\ & \sin \alpha \cos \beta - \cos \alpha \sin \beta & = & \sin(\alpha - \beta) \\ & 2 \sin \alpha \cos \beta & = & \sin(\alpha + \beta) + \sin(\alpha - \beta) \end{array}$$



## Proposition (Product to sum formulas)

$$\begin{aligned}\sin \alpha \sin \beta &= \frac{1}{2} (\cos(\alpha - \beta) - \cos(\alpha + \beta)) \\ \cos \alpha \cos \beta &= \frac{1}{2} (\cos(\alpha - \beta) + \cos(\alpha + \beta)) \\ \sin \alpha \cos \beta &= \frac{1}{2} (\sin(\alpha + \beta) + \sin(\alpha - \beta))\end{aligned}$$

- Product to sum formulas are used when integrating (a topic to be studied later/in another course).

## Proposition (Sum to product formulas)

$$\sin \alpha + \sin \beta = 2 \sin \left( \frac{\alpha + \beta}{2} \right) \cos \left( \frac{\alpha - \beta}{2} \right)$$

$$\sin \alpha - \sin \beta = 2 \sin \left( \frac{\alpha - \beta}{2} \right) \cos \left( \frac{\alpha + \beta}{2} \right)$$

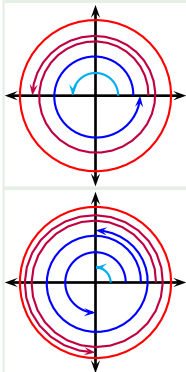
$$\cos \alpha + \cos \beta = 2 \cos \left( \frac{\alpha + \beta}{2} \right) \cos \left( \frac{\alpha - \beta}{2} \right)$$

$$\cos \alpha - \cos \beta = 2 \sin \left( \frac{\alpha + \beta}{2} \right) \sin \left( \frac{\alpha - \beta}{2} \right)$$

Recall the formula  $\sin \alpha + \sin \beta = 2 \sin \left( \frac{\alpha + \beta}{2} \right) \cos \left( \frac{\alpha - \beta}{2} \right)$

## Example

Find all solutions in the interval  $[0, 2\pi)$ .

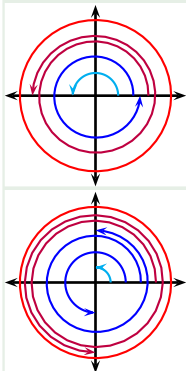


$$\begin{aligned} \sin(2x) + \sin(5x) &= 0 \quad | \text{ use f-l-a} \\ 2 \sin \left( \frac{2x + 5x}{2} \right) \cos \left( \frac{2x - 5x}{2} \right) &= 0 \\ 2 \sin \left( \frac{7}{2}x \right) \cos \left( -\frac{3}{2}x \right) &= 0 \quad | \begin{array}{l} \cos \\ \text{is even} \end{array} \\ 2 \sin \left( \frac{7}{2}x \right) \cos \left( \frac{3}{2}x \right) &= 0 \end{aligned}$$

Recall the formula  $\sin \alpha + \sin \beta = 2 \sin \left( \frac{\alpha + \beta}{2} \right) \cos \left( \frac{\alpha - \beta}{2} \right)$

## Example

Find all solutions in the interval  $[0, 2\pi)$ .



$$\begin{aligned} \sin(2x) + \sin(5x) &= 0 \quad | \text{ use f-l-a} \\ 2 \sin \left( \frac{7}{2}x \right) \cos \left( \frac{3}{2}x \right) &= 0 \end{aligned}$$

$$\sin \left( \frac{7}{2}x \right) = 0$$

$$\frac{7}{2}x = k\pi$$

$$x = \frac{2k\pi}{7}$$

$k$  – integer

$$x = \cancel{\cdot}, \cancel{\frac{2\pi}{7}}, 0, \frac{2\pi}{7}, \frac{4\pi}{7}, \frac{6\pi}{7}, \frac{8\pi}{7}, \frac{10\pi}{7}, \frac{12\pi}{7}, \cancel{\frac{14\pi}{7}}, \cancel{\cdot}$$

or

$$\cos \left( \frac{3}{2}x \right) = 0$$

$$\frac{3}{2}x = \frac{\pi}{2} + k\pi = \frac{(2k+1)\pi}{2}$$

$k$  – integer

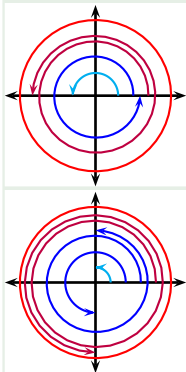
$$x = \frac{(2k+1)\pi}{3}$$

$$x = \cancel{\cdot}, \cancel{\frac{\pi}{3}}, \frac{\pi}{3}, \frac{3\pi}{3}, \frac{5\pi}{3}, \cancel{\frac{7\pi}{3}}, \cancel{\cdot}$$

Recall the formula  $\sin \alpha + \sin \beta = 2 \sin \left( \frac{\alpha + \beta}{2} \right) \cos \left( \frac{\alpha - \beta}{2} \right)$

## Example

Find all solutions in the interval  $[0, 2\pi)$ .



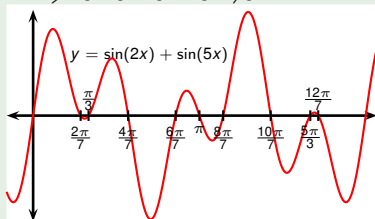
$$\sin(2x) + \sin(5x) = 0 \quad | \quad \text{use formula}$$

$$2 \sin \left( \frac{7}{2}x \right) \cos \left( \frac{3}{2}x \right) = 0$$

$$x = \cancel{\cdot}, \cancel{-\frac{2\pi}{7}}, 0, \frac{2\pi}{7}, \frac{4\pi}{7}, \frac{6\pi}{7}, \frac{8\pi}{7}, \frac{10\pi}{7}, \frac{12\pi}{7}, \frac{14\pi}{7}, \cancel{\cdot}$$

or

$$x = \cancel{\cdot}, \cancel{-\frac{\pi}{3}}, \frac{\pi}{3}, \frac{3\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \cancel{\cdot}$$





## Example

Solve. Among your solutions, find those between  $-360^\circ$  and  $450^\circ$ .

$$\frac{1}{2} \leq \sin \theta < \frac{\sqrt{3}}{2}$$

$$x \in [30^\circ + k360^\circ, 60^\circ + k360^\circ) \cup (120^\circ + k360^\circ, 150^\circ + k360^\circ]$$

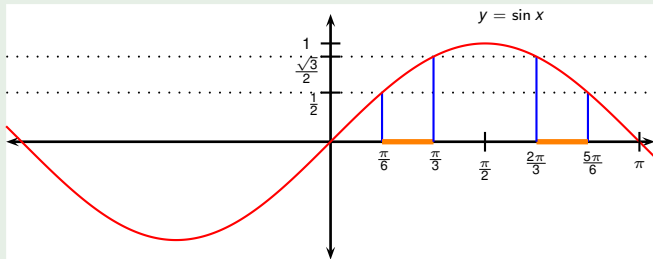
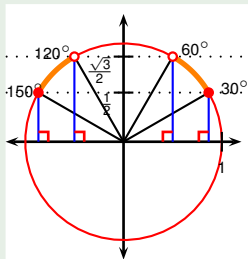
$$\begin{aligned} & \cup [-690^\circ, -660^\circ) \cup (-600^\circ, -570^\circ] \\ & \cup [-330^\circ, -300^\circ) \cup (-240^\circ, -210^\circ] \\ x \in & \cup [30^\circ, 60^\circ) \cup (120^\circ, 150^\circ] \\ & \cup [390^\circ, 420^\circ) \cup (480^\circ, 510^\circ] \end{aligned}$$

$$k = -2$$

$$k = -1$$

$$k = 0$$

$$k = 1$$



## Example

Solve. Among your solutions, find those between  $-360^\circ$  and  $450^\circ$ .

$$\frac{1}{2} \leq \sin \theta < \frac{\sqrt{3}}{2}$$

$$x \in [30^\circ + k360^\circ, 60^\circ + k360^\circ) \cup (120^\circ + k360^\circ, 150^\circ + k360^\circ]$$

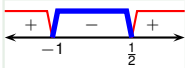
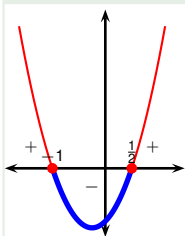
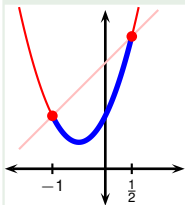
$$\begin{array}{l}
 x \in \quad \cup \quad \cancel{[-690^\circ, -660^\circ)} \cup \quad \cancel{(-600^\circ, -570^\circ]} \\
 \cup \quad [-330^\circ, -300^\circ) \cup \quad (-240^\circ, -210^\circ] \\
 \cup \quad [30^\circ, 60^\circ) \cup \quad (120^\circ, 150^\circ] \\
 \cup \quad [390^\circ, 420^\circ) \cup \quad \cancel{(480^\circ, 510^\circ]}
 \end{array}
 \quad \left| \begin{array}{l} k = -2 \\ k = -1 \\ k = 0 \\ k = 1 \end{array} \right.$$

In radians:

$$x \in \left[-\frac{11\pi}{6}, -\frac{5\pi}{3}\right) \cup \left[-\frac{4\pi}{3}, -\frac{7\pi}{6}\right) \cup \left[\frac{\pi}{6}, \frac{\pi}{3}\right) \cup \left[\frac{2\pi}{3}, \frac{5\pi}{6}\right) \cup \left[\frac{13\pi}{6}, \frac{7\pi}{3}\right)$$

# Example

- Solve the inequality  $2u^2 + 2u + 1 \leq u + 2$ .
- Find all solutions of  $2 \cos^2 \theta + 2 \cos \theta + 1 \leq \cos \theta + 2$  lying in  $[-360^\circ, 360^\circ]$ .



$$2u^2 + 2u + 1 \leq u + 2$$

$$2u^2 + u - 1 \leq 0$$

$$2(u - \frac{1}{2})(u + 1) \leq 0$$

$$u \in [-1, \frac{1}{2}]$$

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$$2\cos^2 \theta + 2 \cos \theta + 1 \leq \cos \theta + 2 \quad \text{Set } \cos \theta = u$$

$$2u^2 + 2u + 1 \leq u + 2$$

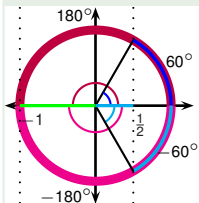
$$u \in [-1, \frac{1}{2}]$$

$$\cos \theta \in [-1, \frac{1}{2}]$$

$$-1 \leq \cos \theta \leq \frac{1}{2}$$

(solved above)

# Example



- Solve the inequality  $2u^2 + 2u + 1 \leq u + 2$ .
- Find all solutions of  $2 \cos^2 \theta + 2 \cos \theta + 1 \leq \cos \theta + 2$  lying in  $[-360^\circ, 360^\circ]$ .

$$\cos \theta \in \left[-1, \frac{1}{2}\right]$$

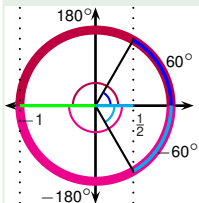
$$-1 \leq \cos \theta \leq \frac{1}{2}$$

$$\theta \in [-180^\circ + k360^\circ, -60^\circ + k360^\circ] \cup [60^\circ + k360^\circ, 180^\circ + k360^\circ]$$

$$\begin{aligned} \theta \in & \quad \cup [-540^\circ, -420^\circ] \cup [-300^\circ, -180^\circ] & k = -1 \\ & \quad \cup [-180^\circ, -60^\circ] \cup [60^\circ, 180^\circ] & k = 0 \\ & \quad \cup [180^\circ, 300^\circ] \cup [420^\circ, 540^\circ] & k = 1 \end{aligned}$$

$$\theta \in [-300^\circ, -60^\circ] \cup [60^\circ, 300^\circ]$$

# Example



- Solve the inequality  $2u^2 + 2u + 1 \leq u + 2$ .
- Find all solutions of  $2 \cos^2 \theta + 2 \cos \theta + 1 \leq \cos \theta + 2$  lying in  $[-360^\circ, 360^\circ]$ .

$$\theta \in [-300^\circ, -60^\circ] \cup [60^\circ, 300^\circ]$$

