Calculus II
Integrals of involving radicals of quadratics

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Outline

1. Integrals of form $\int R(x, \sqrt{ax^2+bx+c})\mathrm{d}x$, R - rational function
    - Transforming to the forms $\sqrt{x^2+1}, \sqrt{-x^2+1}, \sqrt{x^2-1}$
    - Table of Euler and trig substitutions
    - The case $\sqrt{x^2+1}$
    - The case $\sqrt{-x^2+1}$
    - The case $\sqrt{x^2-1}$

2. Rationalizing Substitutions

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Integrals of form $\int R(x, \sqrt{ax^2+bx+c})\mathrm{d}x$, R - rational function

Let $R(x, y)$ be an arbitrary rational expression in two variables (quotient of polynomials in two variables).

Question

Can we integrate $\int R\left(x, \sqrt{ax^2+bx+c}\right)\mathrm{d}x$*?*

- Yes. We will learn how in what follows.
- The algorithm for integration is roughly:
  - Use linear substitution to transform to one of three integrals: $\int R(x, \sqrt{x^2+1})\mathrm{d}x$, $\int R(x, \sqrt{-x^2+1})\mathrm{d}x$, $\int R(x, \sqrt{x^2-1})\mathrm{d}x$.
  - Use trigonometric substitution or Euler substitution to transform to trigonometric or rational function integral (no radicals).
  - Solve as previously studied.
- We motivate why we need such integrals by examples such as computing the area of an ellipse.

Trigonometric Substitution

- To find the area of a circle or ellipse, one needs to compute $\int \sqrt{a^2 - x^2}\mathrm{d}x$.
- For $\int x\sqrt{a^2 - x^2}\mathrm{d}x$, the substitution $u = a^2 - x^2$ would work.
- For $\int \sqrt{a^2 - x^2}\mathrm{d}x$, we need a more elaborate substitution.
- Instead, substitute $x = a\sin\theta$.

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2\sin^2\theta} = \sqrt{a^2(1 - \sin^2\theta)} = \sqrt{a^2\cos^2\theta} = a|\cos\theta|.$$

- With $u = a^2 - x^2$, the new variable is a function of the old one.
- With $x = a\sin\theta$, the old variable is a function of the new one.

Linear substitutions to simplify radicals $\sqrt{ay^2 + by + c}$

- Using linear substitutions, radicals of form $\sqrt{ay^2 + by + c}$, $a \neq 0$, $b^2 - 4ac \neq 0$ can be transformed to (multiple of):
  - $\sqrt{x^2 + 1}$
  - $\sqrt{-x^2 + 1}$
  - $\sqrt{x^2 - 1}$.
- We already studied how to do that using completing the square when dealing with rational functions.

Recall: linear substitution is subst. of the form $u = px + q$.

Example

Use linear substitution to transform $\sqrt{x^2+x+1}$ to multiple of $\sqrt{u^2+1}$.

$$
\begin{aligned}
\sqrt{x^2+x+1} &= \sqrt{x^2 + 2\cdot\frac{1}{2}x + \frac{1}{4} - \frac{1}{4} + 1}\\
&= \sqrt{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}}\\
&= \sqrt{\frac{3}{4}\left(\frac{4}{3}\left(x+\frac{1}{2}\right)^2 + 1\right)}\\
&= \frac{\sqrt{3}}{2}\sqrt{\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right)^2 + 1}\\
&= \frac{\sqrt{3}}{2}\sqrt{u^2+1},
\end{aligned}
$$

where $u = \frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right) = \frac{2\sqrt{3}}{3}x + \frac{\sqrt{3}}{3}$.

Recall: linear substitution is subst. of the form $u = px + q$.

Example

Use linear subst. to transform $\sqrt{-2x^2+x+1}$ to multiple of $\sqrt{-u^2+1}$.

$$\begin{aligned}
\sqrt{-2x^2+x+1} &= \sqrt{-2\left(x^2 - \frac{1}{2}x - \frac{1}{2}\right)} \\
&= \sqrt{-2\left(x^2 - 2\frac{1}{4}x + \frac{1}{16} - \frac{1}{16} - \frac{1}{2}\right)} \\
&= \sqrt{-2\left(\left(x - \frac{1}{4}\right)^2 - \frac{9}{16}\right)} \\
&= \sqrt{\frac{9}{8}\left(-\frac{16}{9}\left(x - \frac{1}{4}\right)^2 + 1\right)} \\
&= \frac{3}{\sqrt{8}}\sqrt{-\left(\frac{4}{3}\left(x - \frac{1}{4}\right)\right)^2 + 1} \\
&= \frac{3}{\sqrt{8}}\sqrt{-u^2+1},
\end{aligned}$$

where $u = \frac{4}{3}\left(x - \frac{1}{4}\right) = \frac{4}{3}x - \frac{1}{3}$.

- Let R be a rational function in two variables.
- So far, with linear transformations we converted all integrals of the form $\displaystyle\int R(x, \sqrt{ax^2+bx+c})\mathrm{d}x$ to one of the three forms: $\int R(x, \sqrt{x^2+1})\mathrm{d}x$, $\int R(x, \sqrt{-x^2+1})\mathrm{d}x$, $\int R(x, \sqrt{x^2-1})\mathrm{d}x$.
- Each of the above integrals can be transformed to a rational trigonometric integral using 3 pairs of substitutions: $x=\tan\theta$, $x=\cot\theta$; $x=\sin\theta$, $x=\cos\theta$; $x=\csc\theta$, $x=\sec\theta$.
- We studied that trigonometric integrals are converted to rational function integrals via $\theta = 2\arctan t$.
- The resulting 3 pairs of substitutions are called Euler substitutions: $x=\tan(2\arctan t)$, $x=\cot(2\arctan t)$; $x=\sin(2\arctan t)$, $x=\cos(2\arctan t)$; $x=\csc(2\arctan t)$, $x=\sec(2\arctan t)$.
- The Euler substitutions directly transform the integral to a rational function integral.
- We will demonstrate that the Euler substitutions are rational.

Trigonometric substitution and Euler substitution

Expression	Substitution	Variable range	Relevant identity
$\sqrt{x^2+1}$	$x=\tan\theta$ $x=\cot\theta$	$\theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ $\theta\in(0,\pi)$	$1+\tan^2\theta=\sec^2\theta$ $1+\cot^2\theta=\csc^2\theta$
$\sqrt{-x^2+1}$	$x=\sin\theta$ $x=\cos\theta$	$\theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ $\theta\in(0,\pi)$	$1-\sin^2\theta=\cos^2\theta$ $1-\cos^2\theta=\cos^2\theta$
$\sqrt{x^2-1}$	$x=\csc\theta$ $x=\sec\theta$	$\theta\in\left[0,\frac{\pi}{2}\right)\cup\left[\pi,\frac{3\pi}{2}\right)$ $\theta\in\left[0,\frac{\pi}{2}\right)\cup\left[\pi,\frac{3\pi}{2}\right)$	$\csc^2\theta-1=\cot^2\theta$ $\sec^2\theta-1=\tan^2\theta$

Euler substitution by applying in addition $\theta=2\arctan t$

$\sqrt{x^2+1}$	$x=\frac{2t}{1-t^2}$ $x=\frac{1}{2}\left(\frac{1}{t}-t\right)$	$-1<t<1$ $0<t$	(?) (?)
$\sqrt{-x^2+1}$	$x=\frac{2t}{1+t^2}$ $x=\frac{1-t^2}{1+t^2}$	$-1\leq t\leq 1$ $0<t$	(?) (?)
$\sqrt{x^2-1}$	$x=\frac{1}{2}\left(\frac{1}{t}+t\right)$ $x=\frac{1+t^2}{1-t^2}$	$t\in(-\infty,-1)\cup[0,1)$ $t\in(-\infty,-1)\cup[0,1)$	(?) (?)

Trigonometric substitution $x = \cot\theta$ for $\sqrt{x^2+1}$

The trigonometric substitution $x = \cot\theta$, $\theta \in (0, \pi)$ for $\sqrt{x^2+1}$:

$$\begin{aligned}
\sqrt{x^2+1} &= \sqrt{\cot^2\theta + 1} \\
&= \sqrt{\frac{\cos^2\theta}{\sin^2\theta} + 1} \\
&= \sqrt{\frac{\cos^2\theta + \sin^2\theta}{\sin^2\theta}} \\
&= \sqrt{\frac{1}{\sin^2\theta}} = \frac{1}{\sqrt{\sin^2\theta}} \\
&= \frac{1}{\sin\theta} = \csc\theta \quad .
\end{aligned}$$

when $\theta \in (0, \pi)$ we have $\sin\theta \geq 0$ and so $\sqrt{\sin^2\theta} = \sin\theta$

Trigonometric substitution $x = \cot\theta$ for $\sqrt{x^2+1}$

The trigonometric substitution $x = \cot\theta$, $\theta \in (0,\pi)$ for $\sqrt{x^2+1}$:

$$\sqrt{x^2+1} = \frac{1}{\sin\theta} = \csc\theta \quad .$$

The differential dx can be expressed via dθ from $x = \cot\theta$. To summarize:

Definition

The trigonometric substitution $x = \cot\theta$, $\theta \in (0,\pi)$ for $\sqrt{x^2+1}$ is given by:

$$\begin{aligned} x &= \cot\theta \\ \sqrt{x^2+1} &= \frac{1}{\sin\theta} = \csc\theta \\ \mathrm{d}x &= -\frac{\mathrm{d}\theta}{\sin^2\theta} = -\csc^2\theta\,\mathbf{?}\mathrm{d}\theta \\ \theta &= \operatorname{arccot} x \quad . \end{aligned}$$

Example

$$
\begin{aligned}
\int \frac{1}{x^2\sqrt{x^2+9}}\mathrm{d}x &= \int \frac{1}{x^2 3\sqrt{\left(\frac{x}{3}\right)^2+1}}\mathrm{d}x \\
&= \int \frac{1}{(3\cot\theta)^2 3\sqrt{\cot^2\theta+1}}\mathrm{d}(3\cot\theta) \\
&= \int \frac{1}{27\cot^2\theta\sqrt{\csc^2\theta}}\left(-3\csc^2\theta\right)\mathrm{d}\theta \\
&= \frac{1}{9}\int \frac{-\csc^2\theta}{\cot^2\theta\csc\theta}\mathrm{d}\theta \\
&= \frac{1}{9}\int \frac{-\sin\theta}{\cos^2\theta}\mathrm{d}\theta = \frac{1}{9}\int \frac{1}{\cos^2\theta}\mathrm{d}(\cos\theta) \\
&= \frac{1}{9}\int \frac{\mathrm{d}u}{u^2} = -\frac{1}{9u}+C = -\frac{\sec\theta}{9}+C \\
&= -\frac{\sqrt{x^2+9}}{9x}+C
\end{aligned}
$$

Set

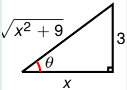
$\frac{x}{3} = \cot\theta$

$x = 3\cot\theta$

$\theta \in (0,\pi)$

$\theta \in (0,\pi) \Rightarrow$
$\csc\theta > 0$

Set $u = \cos\theta$



Euler subst. for $\sqrt{x^2+1}$ corresponding to $x = \cot\theta$

- $x = \cot\theta$ transforms $\mathrm{d}x, x, \sqrt{x^2+1}$ to trig form.
- $\theta = 2\arctan t,\ \ t > 0$ transforms $\mathrm{d}\theta, \cos\theta, \sin\theta$ to rational form.

What if we compose the above? We get the Euler substitution:

$$\begin{aligned} x &= \cot\theta \\ &= \cot(2\arctan t) && \left| \text{Recall: } \cot(2z) = \frac{\cos(2z)}{\sin(2z)} = \frac{1-\tan^2 z}{2\tan z} \right. \\ &= \frac{1-\tan^2(\arctan t)}{2\tan(\arctan t)} \\ &= \frac{1-t^2}{2t} \\ &= \frac{1}{2}\left(\frac{1}{t} - t\right) \quad . \end{aligned}$$

Euler subst. for $\sqrt{x^2+1}$ corresponding to $x = \cot\theta$

- $x = \cot\theta$ transforms $\mathrm{d}x, x, \sqrt{x^2+1}$ to trig form.
- $\theta = 2\arctan t,\ \ t > 0$ transforms $\mathrm{d}\theta, \cos\theta, \sin\theta$ to rational form.

What if we compose the above? We get the Euler substitution:

$$x = \frac{1}{2}\left(\frac{1}{t} - t\right) .$$

We can furthermore compute

$$\begin{aligned}\sqrt{x^2+1} &= \sqrt{\frac{1}{4}\left(\frac{1}{t}-t\right)^2+1}\\ &= \frac{1}{2}\sqrt{\left(\frac{1}{t}-t\right)^2+4} \quad \left| \left(\tfrac{1}{t}-t\right)^2+4 = \left(\tfrac{1}{t}+t\right)^2\right.\end{aligned}$$

Euler subst. for $\sqrt{x^2+1}$ corresponding to $x = \cot\theta$

- $x = \cot\theta$ transforms $\mathrm{d}x, x, \sqrt{x^2+1}$ to trig form.
- $\theta = 2\arctan t, \quad t > 0$ transforms $\mathrm{d}\theta, \cos\theta, \sin\theta$ to rational form.

What if we compose the above? We get the Euler substitution:

$$x = \frac{1}{2}\left(\frac{1}{t} - t\right) \quad .$$

We can furthermore compute

$$\begin{aligned}\sqrt{x^2+1} &= \sqrt{\frac{1}{4}\left(\frac{1}{t}-t\right)^2 + 1} \\ &= \frac{1}{2}\sqrt{\left(\frac{1}{t}+t\right)^2} && \left|\ \sqrt{\left(\tfrac{1}{t}+t\right)^2} = \tfrac{1}{t}+t \text{ because } t > 0\right. \\ &= \frac{1}{2}\left(\frac{1}{t}+t\right) \quad .\end{aligned}$$

Euler subst. for $\sqrt{x^2+1}$ corresponding to $x = \cot\theta$

- $x = \cot\theta$ transforms $\mathrm{d}x, x, \sqrt{x^2+1}$ to trig form.
- $\theta = 2\arctan t,\ \ t > 0$ transforms $\mathrm{d}\theta, \cos\theta, \sin\theta$ to rational form.

What if we compose the above? We get the Euler substitution:

$$x = \frac{1}{2}\left(\frac{1}{t} - t\right) .$$

We can furthermore compute

$$\sqrt{x^2+1} = \frac{1}{2}\left(\frac{1}{t} + t\right) .$$

Finally compute

$$\begin{aligned}
\mathrm{d}x &= \mathrm{d}\left(\frac{1}{2}\left(\frac{1}{t} - t\right)\right) = -\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t \\
t &= \frac{1}{2}\left(\frac{1}{t} + t\right) - \frac{1}{2}\left(\frac{1}{t} - t\right) = \sqrt{x^2+1} - x .
\end{aligned}$$

Euler subst. for $\sqrt{x^2+1}$ corresponding to $x = \cot\theta$

- $x = \cot\theta$ transforms $\mathrm{d}x, x, \sqrt{x^2+1}$ to trig form.
- $\theta = 2\arctan t,\ \ t > 0$ transforms $\mathrm{d}\theta, \cos\theta, \sin\theta$ to rational form.

What if we compose the above? We get the Euler substitution:

Definition

The Euler substitution for $\sqrt{x^2+1}$ corresponding to $x = \cot\theta$ is given by:

$$
\begin{array}{rcl}
x & = & \frac{1}{2}\left(\frac{1}{t} - t\right), \qquad t > 0 \\
\sqrt{x^2+1} & = & \frac{1}{2}\left(\frac{1}{t} + t\right) \\
\mathrm{d}x & = & -\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t \\
t & = & \sqrt{x^2+1} - x \quad .
\end{array}
$$

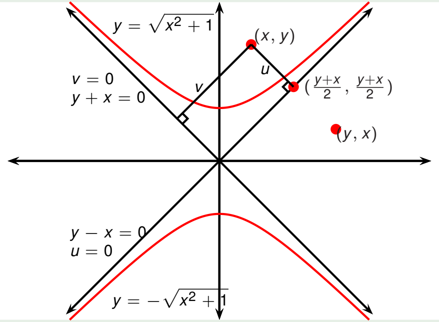
Euler substitution: $x = \frac{1}{2}\left(\frac{1}{t} - t\right)$, $\sqrt{x^2+1} = \frac{1}{2}\left(\frac{1}{t} + t\right)$,
$t = \sqrt{x^2+1} - x$, $\mathrm{d}x = -\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t$. Recall $t > 0$.

Example

$$
\begin{aligned}
\int \sqrt{x^2+1}\,\mathrm{d}x &= -\int \frac{1}{2}\left(\frac{1}{t} + t\right)\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t \\
&= -\frac{1}{4}\int\left(\frac{1}{t^3} + 2\frac{1}{t} + t\right)\mathrm{d}t \\
&= -\frac{1}{4}\left(-\frac{t^{-2}}{2} + 2\ln|t| + \frac{t^2}{2}\right) + C \\
&= \frac{1}{2}\left(\frac{1}{2}\left(t^{-1} - t\right)\frac{1}{2}\left(t^{-1} + t\right)\right) - \frac{1}{2}\ln t + C \\
&= \frac{1}{2}x\sqrt{x^2+1} - \frac{1}{2}\ln\left(\sqrt{x^2+1} - x\right) + C \\
&= \frac{1}{2}x\sqrt{x^2+1} + \frac{1}{2}\ln\frac{\sqrt{x^2+1} + x}{\left(\sqrt{x^2+1} - x\right)\left(\sqrt{x^2+1} + x\right)} + C \\
&= \frac{1}{2}x\sqrt{x^2+1} + \frac{1}{2}\ln\left(\sqrt{x^2+1} + x\right) + C
\end{aligned}
$$

Example

Find the area locked b-n the hyperbolas $y=\pm\sqrt{x^2+1}$ and $x=\pm 2\sqrt{2}$.



Signed distance b-n (x,y) and line $u=0$ equals

$$\pm\sqrt{\left(x-\frac{(x+y)}{2}\right)^2+\left(y-\frac{(x+y)}{2}\right)^2}$$

$$=\pm\sqrt{\frac{1}{2}(y-x)^2}=\pm\frac{\sqrt{2}}{2}(y-x)=u.$$

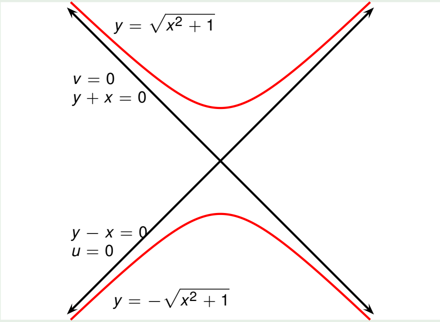
We studied $v=\frac{\frac{1}{2}}{u}$ is called a hyperbola: why do we call $y=\sqrt{x^2+1}$ hyperbola? Compute:

$$\begin{array}{rcl}\sqrt{x^2+1}&=&y\\ x^2+1&=&y^2\\ y^2-x^2&=&1\\ \frac{\sqrt{2}}{2}(y-x)\frac{\sqrt{2}}{2}(y+x)&=&\frac{1}{2}\\ uv&=&\frac{1}{2}\\ v&=&\frac{\frac{1}{2}}{u},\end{array}$$

where $\left|\begin{array}{l}u=\frac{\sqrt{2}}{2}(y-x)\\ v=\frac{\sqrt{2}}{2}(y+x)\end{array}\right.$. Consider an arbitrary point (x,y).

Example

Find the area locked b-n the hyperbolas $y = \pm\sqrt{x^2+1}$ and $x = \pm 2\sqrt{2}$.



Signed distance b-n (x, y) and line $u = 0$ equals u. Similarly compute that signed distance b-n (x, y) and the line $v = 0$ equals v. $\Rightarrow y^2 - x^2 = 1$ is the hyperbola $v = \frac{1/2}{u}$ in the (u, v)-plane.

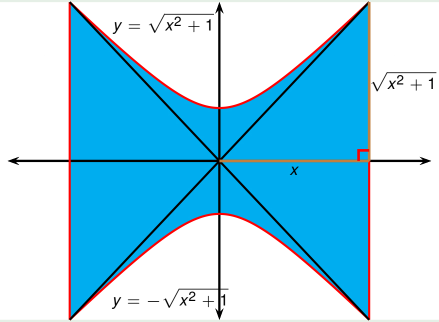
We studied $v = \frac{\frac{1}{2}}{u}$ is called a hyperbola: why do we call $y = \sqrt{x^2+1}$ hyperbola? Compute:

$$\begin{array}{rcl} \sqrt{x^2+1} &=& y \\ x^2+1 &=& y^2 \\ y^2 - x^2 &=& 1 \\ \frac{\sqrt{2}}{2}(y-x)\frac{\sqrt{2}}{2}(y+x) &=& \frac{1}{2} \\ uv &=& \frac{1}{2} \\ v &=& \frac{\frac{1}{2}}{u}, \end{array}$$

where $\left| \begin{array}{l} u = \frac{\sqrt{2}}{2}(y-x) \\ v = \frac{\sqrt{2}}{2}(y+x) \end{array} \right.$. Consider an arbitrary point (x, y).

Example

Find the area locked b-n the hyperbolas $y = \pm\sqrt{x^2+1}$ and $x = \pm 2\sqrt{2}$.



The area in question is:

$$\int\limits_{-2\sqrt{2}}^{2\sqrt{2}} 2\sqrt{x^2+1}\,\mathrm{d}x$$

$$= 2\left[x\sqrt{x^2+1} + \ln\left(\sqrt{x^2+1}+x\right)\right]_0^{2\sqrt{2}}$$

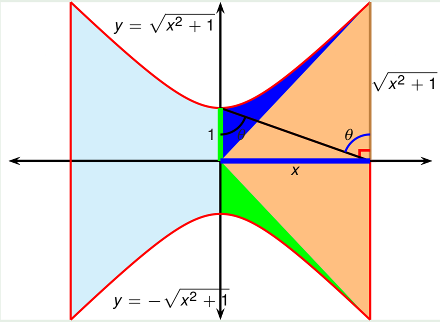
$$= 2\left(2\sqrt{2}\sqrt{(2\sqrt{2})^2+1} + \ln\left(\sqrt{(2\sqrt{2})^2+1}+2\sqrt{2}\right)\right)$$

$$= 12\sqrt{2} + 2\ln\left(3+2\sqrt{2}\right)$$

$$\approx 20.496$$

Example

Find the area locked b-n the hyperbolas $y = \pm\sqrt{x^2+1}$ and $x = \pm 2\sqrt{2}$.



- Recall: integral can be solved via $x = \tan\theta$.
- Geometric interpretation of θ?

The area in question is:

$$\int_{-2\sqrt{2}}^{2\sqrt{2}} 2\sqrt{x^2+1}\,\mathrm{d}x$$

$$= 2\left[x\sqrt{x^2+1} + \ln\left(\sqrt{x^2+1}+x\right)\right]_0^{2\sqrt{2}}$$

$$= 2\left(2\sqrt{2}\sqrt{(2\sqrt{2})^2+1} + \ln\left(\sqrt{(2\sqrt{2})^2+1}+2\sqrt{2}\right)\right)$$

$$= 12\sqrt{2} + 2\ln\left(3+2\sqrt{2}\right)$$

$$\approx 20.496$$

Example

Find $\int \frac{x}{\sqrt{x^2+4}}\mathrm{d}x$.

- We could use the trig substitution $x = 2\tan\theta$.
- But there is an easier way:
- $u = x^2 + 4$.
- $\mathrm{d}u = 2x\mathrm{d}x$.

$$\int \frac{x}{\sqrt{x^2+4}}\mathrm{d}x = \frac{1}{2}\int \frac{\mathrm{d}u}{\sqrt{u}} = \sqrt{u} + C = \sqrt{x^2+4} + C$$

Trigonometric substitution $x = \cos\theta$ for $\sqrt{-x^2+1}$

The trigonometric substitution $x = \cos\theta$, $\theta \in [0, \pi]$ for $\sqrt{-x^2+1}$:

$$\begin{array}{rcl} \sqrt{-x^2+1} & = & \sqrt{1-\cos^2\theta} \\ & = & \sqrt{\sin^2\theta} \\ & = & \sin\theta \quad . \end{array}$$

when $\theta \in [0, \pi]$ we have
$\sin\theta \geq 0$ and so $\sqrt{\sin^2\theta} = \sin\theta$

To summarize:

Definition

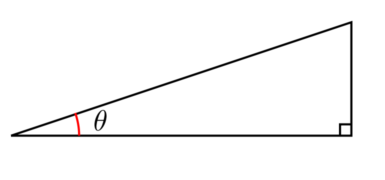
The trigonometric substitution $x = \cos\theta$, $\theta \in [0, \pi]$ for $\sqrt{-x^2+1}$ is given by:

$$\begin{array}{rcl} x & = & \cos\theta \\ \sqrt{-x^2+1} & = & \sin\theta \\ dx & = & -\sin\theta d\theta \\ \theta & = & \arccos x \quad . \end{array}$$

Example

Evaluate $\int \frac{\sqrt{9-x^2}}{x^2}\mathrm{d}x$.

- Let $x = 3\sin\theta$, where $-\pi/2 \leq \theta \leq \pi/2$.
- Then $\mathrm{d}x = 3\cos\theta\mathrm{d}\theta$.

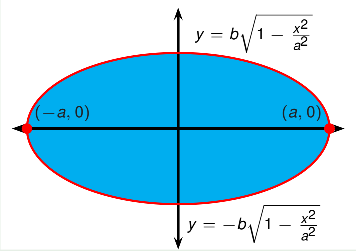


$$\sqrt{9-x^2} = \sqrt{9-9\sin^2\theta} = \sqrt{9\cos^2\theta} = 3|\cos\theta| = 3\cos\theta$$

$$\begin{aligned}
\int \frac{\sqrt{9-x^2}}{x^2}\mathrm{d}x &= \int \frac{3\cos\theta}{9\sin^2\theta}3\cos\theta\mathrm{d}\theta = \int \cot^2\theta\mathrm{d}\theta \\
&= \int (\csc^2\theta - 1)\mathrm{d}\theta = -\cot\theta - \theta + C \\
&= -\frac{\sqrt{9-x^2}}{x} - \arcsin\left(\frac{x}{3}\right) + C
\end{aligned}$$

Example

Find the area enclosed by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $a, b > 0$.



Express y via x:

$$\begin{aligned} \frac{x^2}{a^2}+\frac{y^2}{b^2} &= 1 \\ \frac{y^2}{b^2} &= 1-\frac{x^2}{a^2} \\ y^2 &= b^2\left(1-\frac{x^2}{a^2}\right) \\ y &= \pm b\sqrt{1-\frac{x^2}{a^2}} \end{aligned}$$

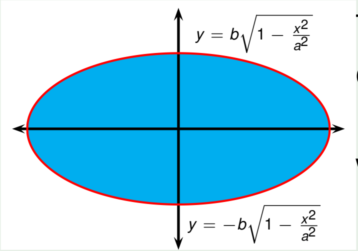
The area in question is

$$\int_{-a}^{a} 2b\sqrt{1-\frac{x^2}{a^2}}\mathrm{d}x$$

$$= 4\int_{0}^{a} b\sqrt{1-\frac{x^2}{a^2}}\mathrm{d}x.$$

Example

Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a, b > 0$.



The area in question is

$$\int_{-a}^{a} 2b\sqrt{1 - \frac{x^2}{a^2}}\mathrm{d}x$$

$$= 4\int_{0}^{a} b\sqrt{1 - \frac{x^2}{a^2}}\mathrm{d}x$$

$$= 4b\frac{a\pi}{4} = \pi ab \quad .$$

Trig subst.: set $x = a\sin\theta$, $\theta \in \left(0, \frac{\pi}{2}\right)$.

Compute: $\sqrt{1 - \frac{x^2}{a^2}} = \sqrt{1 - \frac{a^2\sin^2\theta}{a^2}} = \sqrt{1 - \sin^2\theta} = \cos\theta$. When $x = 0$, $\theta = 0$ and when $x = a$, $\theta = \frac{\pi}{2}$.

$$\begin{aligned}
\int_0^a \sqrt{1 - \frac{x^2}{a^2}}\mathrm{d}x &= \int_0^{\frac{\pi}{2}} \cos\theta \,\mathrm{d}\left(a\sin\theta\right) \\
&= a\int_0^{\frac{\pi}{2}} \cos^2\theta \mathrm{d}\theta \\
&= a\int_0^{\frac{\pi}{2}} \frac{\cos(2\theta) + 1}{2}\mathrm{d}\theta \\
&= a\left[\frac{\sin(2\theta)}{4} + \frac{\theta}{2}\right]_{\theta=0}^{\theta=\frac{\pi}{2}} \\
&= a\left(0 + \frac{\pi}{4} - (0 + 0)\right) \\
&= \frac{a\pi}{4}
\end{aligned}$$

Example

Evaluate $\int \frac{x}{\sqrt{3-2x-x^2}}\mathrm{d}x$.

- Complete the square under the root sign:
- $3-2x-x^2 = 3+1-(x^2+2x+1) = 4-(x+1)^2$
- Substitute $u = x+1$. Then $\mathrm{d}u = \mathrm{d}x$ and $x = u-1$.
- $\int \frac{x}{\sqrt{3-2x-x^2}}\mathrm{d}x = \int \frac{x}{\sqrt{4-(x+1)^2}}\mathrm{d}x = \int \frac{u-1}{\sqrt{4-u^2}}\mathrm{d}u$
- Let $u = 2\sin\theta$, where $-\pi/2 \leq \theta \leq \pi/2$. Then $\mathrm{d}u = 2\cos\theta \mathrm{d}\theta$.
- $\sqrt{4-u^2} = \sqrt{4-4\sin^2\theta} = \sqrt{4\cos^2\theta} = 2|\cos\theta| = 2\cos\theta$

$$\begin{aligned}
&\int \frac{x}{\sqrt{3-2x-x^2}}\mathrm{d}x = \int \frac{u-1}{\sqrt{4-u^2}}\mathrm{d}u = \int \frac{2\sin\theta-1}{2\cos\theta}2\cos\theta\mathrm{d}\theta\\
&= \int(2\sin\theta-1)\mathrm{d}\theta = -2\cos\theta-\theta+C\\
&= -\sqrt{4-u^2}-\sin^{-1}\left(\frac{u}{2}\right)+C\\
&= -\sqrt{3-2x-x^2}-\sin^{-1}\left(\frac{x+1}{2}\right)+C
\end{aligned}$$

Euler subst. for $\sqrt{-x^2+1}$ corresponding to $x = \cos\theta$

- $x = \cos\theta$ transforms $\mathrm{d}x, x, \sqrt{-x^2+1}$ to trig form.
- $\theta = 2\arctan t, \quad t > 0$ transforms $\mathrm{d}\theta, \cos\theta, \sin\theta$ to rational form.

What if we compose the above? We get the Euler substitution:

$$\begin{aligned} x &= \cos\theta \\ &= \cos(2\arctan t) \\ &= \frac{1-\tan^2(\arctan t)}{1+\tan^2(\arctan t)} \\ &= \frac{1-t^2}{1+t^2} \end{aligned} \qquad \left| \; \cos(2z) = \frac{1-\tan^2 z}{1+\tan^2 z} \right.$$

Euler subst. for $\sqrt{-x^2+1}$ corresponding to $x=\cos\theta$

- $x=\cos\theta$ transforms $\mathrm{d}x, x, \sqrt{-x^2+1}$ to trig form.
- $\theta = 2\arctan t,\quad t>0$ transforms $\mathrm{d}\theta, \cos\theta, \sin\theta$ to rational form.

What if we compose the above? We get the Euler substitution:

$$
\begin{array}{rcll}
x &=& \dfrac{1-t^2}{1+t^2} & \\
\hline
\sqrt{-x^2+1} &=& \sqrt{1-\left(\dfrac{1-t^2}{1+t^2}\right)^2} & \\
&=& \sqrt{\dfrac{(1+t^2)^2-(1-t^2)^2}{(1+t^2)^2}} & \left| \ (1+t^2)^2-(1-t^2)^2 = 4t^2 \right. \\
&=& \sqrt{\dfrac{4t^2}{(1+t^2)^2}} & \left| \ \sqrt{4t^2} = 2t \text{ because } t>0 \right. \\
&=& \dfrac{2t}{1+t^2} &
\end{array}
$$

Euler subst. for $\sqrt{-x^2+1}$ corresponding to $x = \cos\theta$

- $x = \cos\theta$ transforms $\mathrm{d}x, x, \sqrt{-x^2+1}$ to trig form.
- $\theta = 2\arctan t, \quad t > 0$ transforms $\mathrm{d}\theta, \cos\theta, \sin\theta$ to rational form.

What if we compose the above? We get the Euler substitution:

$$x = \frac{1-t^2}{1+t^2}$$

$$\sqrt{-x^2+1} = \frac{2t}{1+t^2}$$

$$t = \frac{\sqrt{1-x}}{\sqrt{1+x}}\frac{\sqrt{1+x}}{\sqrt{1+x}} = \frac{\sqrt{-x^2+1}}{x+1} \qquad \text{we use } t > 0$$

$$\begin{aligned} \mathrm{d}x &= \mathrm{d}\left(\frac{1-t^2}{1+t^2}\right) = \mathrm{d}\left(\frac{2-(1+t^2)}{1+t^2}\right) \\ &= \mathrm{d}\left(\frac{2}{1+t^2} - 1\right) = -\frac{4t}{(1+t^2)^2}\mathrm{d}t \end{aligned}$$

Euler subst. for $\sqrt{-x^2+1}$ corresponding to $x = \cos\theta$

- $x = \cos\theta$ transforms $\mathrm{d}x, x, \sqrt{-x^2+1}$ to trig form.
- $\theta = 2\arctan t,\ \ t > 0$ transforms $\mathrm{d}\theta, \cos\theta, \sin\theta$ to rational form.

What if we compose the above? We get the Euler substitution:

Definition

The Euler substitution for $\sqrt{-x^2+1}$ corresponding to $x = \cos\theta$ is given by:

$$\begin{aligned} x &= \frac{1-t^2}{1+t^2}, \qquad t > 0 \\ \sqrt{-x^2+1} &= \frac{2t}{1+t^2} \\ \mathrm{d}x &= -\frac{4t}{(t^2+1)^2}\mathrm{d}t \\ t &= \frac{\sqrt{-x^2+1}}{x+1} \quad . \end{aligned}$$

Trigonometric substitution $x = \sec\theta$ for $\sqrt{x^2-1}$

The trigonometric substitution $x = \sec\theta$, $\theta \in \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$:

$$\begin{aligned}
\sqrt{x^2-1} &= \sqrt{\sec^2\theta - 1} \\
&= \sqrt{\frac{1}{\cos^2\theta} - 1} \\
&= \sqrt{\frac{\sin^2\theta}{\cos^2\theta}} \\
&= \sqrt{\tan^2\theta} \\
&= \tan\theta \quad .
\end{aligned}$$

when $\theta \in \theta \in \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$ we have $\tan\theta \geq 0$ and so $\sqrt{\tan^2\theta} = \tan\theta$

Trigonometric substitution $x = \sec\theta$ for $\sqrt{x^2-1}$

The trigonometric substitution $x = \sec\theta$, $\theta \in \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$:

$$\sqrt{x^2-1} = \tan\theta \quad .$$

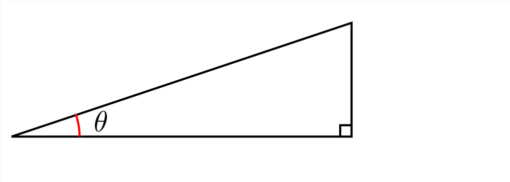
Definition

The trigonometric substitution $x = \sec\theta$, $\theta \in (0,\pi)$ for $\sqrt{x^2+1}$ is given by:

$$\begin{aligned}
x &= \sec\theta = \frac{1}{\cos\theta} \qquad \theta \in \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right) \\
\sqrt{x^2-1} &= \tan\theta \\
\mathrm{d}x &= \frac{\sin\theta}{\cos^2\theta}\mathrm{d}\theta = \sec\theta\tan\theta\mathrm{d}\theta \\
\theta &= \operatorname{arcsec} x \quad .
\end{aligned}$$

Example

Find $\int \frac{dx}{\sqrt{x^2-a^2}}$, $a > 0$.



- $x = a\sec\theta$,
  $0 < \theta < \pi/2$ or
  $\pi < \theta < 3\pi/2$.
- $dx = a\sec\theta\tan\theta d\theta$.
  $\sqrt{x^2-a^2} = \sqrt{a^2\sec^2\theta - a^2} = \sqrt{a^2\tan^2\theta} = a|\tan\theta| = a\tan\theta$

$$
\begin{aligned}
\int \frac{dx}{\sqrt{x^2-a^2}} &= \int \frac{a\sec\theta\tan\theta d\theta}{a\tan\theta} = \int \sec\theta d\theta \\
&= \ln|\sec\theta + \tan\theta| + C = \ln\left|\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}\right| + C \\
&= \ln\left|x + \sqrt{x^2-a^2}\right| + C_1
\end{aligned}
$$

Euler substitution $x = \sec\theta$, $\theta = 2\arctan t$

- $x = \sec\theta$ transforms $\mathrm{d}x, x, \sqrt{x^2-1}$ to trig form.
- $\theta = 2\arctan t$, $t \in (-\infty, -1) \cup [0, 1)$ rationalizes $\mathrm{d}\theta, \cos\theta, \sin\theta$.

What if we compose the above? We get the Euler substitution:

$$
\begin{aligned}
x &= \sec\theta = \frac{1}{\cos\theta} \\
&= \frac{1}{\cos(2\arctan t)} \\
&= \frac{1+\tan^2(\arctan t)}{1-\tan^2(\arctan t)} \\
&= \frac{1+t^2}{1-t^2} = \frac{2-(1-t^2)}{1-t^2} \\
&= -1 + \frac{2}{1-t^2}
\end{aligned}
$$

$$\left|\ \cos(2z) = \frac{1-\tan^2 z}{1+\tan^2 z}\right.$$

Euler substitution $x = \sec\theta$, $\theta = 2\arctan t$

- $x = \sec\theta$ transforms $\mathrm{d}x, x, \sqrt{x^2-1}$ to trig form.
- $\theta = 2\arctan t,\ \ t \in (-\infty, -1) \cup [0, 1)$ rationalizes $\mathrm{d}\theta, \cos\theta, \sin\theta$.

What if we compose the above? We get the Euler substitution:

$$
\begin{aligned}
x &= -1 + \frac{2}{1-t^2} \\
\sqrt{x^2-1} &= \sqrt{\left(\frac{1+t^2}{1-t^2}\right)^2 - 1} \\
&= \sqrt{\frac{(1+t^2)^2-(1-t^2)^2}{(1-t^2)^2}} && \left| \ (1+t^2)^2-(1-t^2)^2 = 4t^2 \right. \\
&= \sqrt{\frac{4t^2}{(1-t^2)^2}} && \left| \ \begin{array}{l} t, 1-t^2 \text{ have same sign} \\ \text{when } t \in (-\infty,-1)\cup[0,1) \end{array} \right. \\
&= \frac{2t}{1-t^2}
\end{aligned}
$$

Euler substitution $x = \sec\theta$, $\theta = 2\arctan t$

- $x = \sec\theta$ transforms $\mathrm{d}x, x, \sqrt{x^2-1}$ to trig form.
- $\theta = 2\arctan t,\ \ t \in (-\infty, -1) \cup [0, 1)$ rationalizes $\mathrm{d}\theta, \cos\theta, \sin\theta$.

What if we compose the above? We get the Euler substitution:

$$x = -1 + \frac{2}{1-t^2}$$

$$\sqrt{x^2-1} = \frac{2t}{1-t^2}$$

$$\begin{aligned}
x &= \frac{1+t^2}{1-t^2}\\
(1-t^2)x &= 1+t^2\\
(1+x)t^2 &= x-1\\
t^2 &= \frac{x-1}{x+1}\\
t &= \pm\sqrt{\frac{x-1}{x+1}}
\end{aligned}$$

Euler substitution $x = \sec\theta$, $\theta = 2\arctan t$

- $x = \sec\theta$ transforms $\mathrm{d}x, x, \sqrt{x^2-1}$ to trig form.
- $\theta = 2\arctan t$, $t \in (-\infty, -1) \cup [0, 1)$ rationalizes $\mathrm{d}\theta, \cos\theta, \sin\theta$.

What if we compose the above? We get the Euler substitution:

$$x = -1 + \frac{2}{1-t^2}$$

$$\sqrt{x^2-1} = \frac{2t}{1-t^2}$$

$$t = \pm\sqrt{\frac{x-1}{x+1}}$$

$$\begin{aligned}\mathrm{d}x &= \mathrm{d}\left(-1 + \frac{2}{1-t^2}\right)\\ &= \frac{4t}{(1-t^2)^2}\mathrm{d}t\end{aligned}$$

Euler substitution $x = \sec\theta$, $\theta = 2\arctan t$

- $x = \sec\theta$ transforms $\mathrm{d}x, x, \sqrt{x^2-1}$ to trig form.
- $\theta = 2\arctan t,\quad t \in (-\infty, -1) \cup [0, 1)$ rationalizes $\mathrm{d}\theta, \cos\theta, \sin\theta$.

What if we compose the above? We get the Euler substitution:

Definition

The Euler substitution for $\sqrt{x^2-1}$ corresponding to $x = \sec\theta$ is given by:

$$\begin{aligned} x &= \frac{1+t^2}{1-t^2}, & t \in (-\infty, -1) \cup [0, 1) \\ \sqrt{x^2-1} &= \frac{2t}{1-t^2} \\ \mathrm{d}x &= \frac{4t}{(1-t^2)^2}\mathrm{d}t \\ t &= \pm\frac{\sqrt{x^2-1}}{x+1} & . \end{aligned}$$

Rationalizing Substitutions

Some non-rational fractions can be changed into rational fractions by means of appropriate substitutions. In particular, when an integrand contains an expression of the form $\sqrt[n]{g(x)}$, the substitution $u = \sqrt[n]{g(x)}$ may be effective.

Example

Let $u = \sqrt{x+4}$. Then $u^2 = x + 4$, so $x = u^2 - 4$ and $\mathrm{d}x = 2u\mathrm{d}u$.

$$
\begin{aligned}
\int \frac{\sqrt{x+4}}{x}\mathrm{d}x &= \int \frac{u}{u^2-4} 2u\mathrm{d}u \\
&= 2\int \frac{u^2}{u^2-4}\mathrm{d}u \\
&= 2\int \left(1 + \frac{4}{u^2-4}\right)\mathrm{d}u && \Big| \text{ long division} \\
&= 2\int \mathrm{d}u + 8\int \frac{\mathrm{d}u}{u^2-4} \\
&= 2\int \mathrm{d}u + 8\int \left(\frac{\frac{1}{4}}{u-2} - \frac{\frac{1}{4}}{u+2}\right)\mathrm{d}u && \Big| \text{ partial fractions} \\
&= 2u + 2(\ln|u-2| - \ln|u+2|) + C \\
&= 2\sqrt{x+4} + 2\ln\left|\frac{\sqrt{x+4}-2}{\sqrt{x+4}+2}\right| + C
\end{aligned}
$$