Calculus II

Integrals of the form $\int \sqrt{ax^2 + c}\mathrm{d}x$, $a, c > 0$

Todor Milev

2019

Euler substitution: $x=\frac{1}{2}\left(\frac{1}{t}-t\right)$, $\sqrt{x^2+1}=\frac{1}{2}\left(\frac{1}{t}+t\right)$,
$t=\sqrt{x^2+1}-x$, $\mathrm{d}x=-\frac{1}{2}\left(\frac{1}{t^2}+1\right)\mathrm{d}t$.

Example

$$\int \sqrt{x^2+1}\,\mathrm{d}x=$$

Euler substitution: $x = \frac{1}{2}\left(\frac{1}{t} - t\right)$, $\sqrt{x^2+1} = \frac{1}{2}\left(\frac{1}{t} + t\right)$,
$t = \sqrt{x^2+1} - x$, $\mathrm{d}x = -\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t$.

Example

$$\int \sqrt{x^2+1}\,\mathrm{d}x = -\int \frac{1}{2}\left(\frac{1}{t} + t\right)\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t$$

Euler substitution: $x = \frac{1}{2}\left(\frac{1}{t} - t\right)$, $\sqrt{x^2+1} = \frac{1}{2}\left(\frac{1}{t} + t\right)$,
$t = \sqrt{x^2+1} - x$, $\mathrm{d}x = -\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t$.

Example

$$\int \sqrt{x^2+1}\,\mathrm{d}x = -\int \frac{1}{2}\left(\frac{1}{t} + t\right)\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t$$

Euler substitution: $x = \frac{1}{2}\left(\frac{1}{t} - t\right)$, $\sqrt{x^2+1} = \frac{1}{2}\left(\frac{1}{t} + t\right)$,
$t = \sqrt{x^2+1} - x$, $\mathrm{d}x = -\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t$.

Example

$$\begin{aligned}\int \sqrt{x^2+1}\,\mathrm{d}x &= -\int \frac{1}{2}\left(\frac{1}{t} + t\right)\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t \\ &= -\frac{1}{4}\int\left(\frac{1}{t^3} + 2\frac{1}{t} + t\right)\mathrm{d}t\end{aligned}$$

Euler substitution: $x = \frac{1}{2}\left(\frac{1}{t} - t\right)$, $\sqrt{x^2+1} = \frac{1}{2}\left(\frac{1}{t} + t\right)$, $t = \sqrt{x^2+1} - x$, $\mathrm{d}x = -\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t$.

Example

$$\begin{aligned}\int \sqrt{x^2+1}\,\mathrm{d}x &= -\int \frac{1}{2}\left(\frac{1}{t} + t\right)\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t\\ &= -\frac{1}{4}\int\left(\frac{1}{t^3} + 2\frac{1}{t} + t\right)\mathrm{d}t\end{aligned}$$

Euler substitution: $x = \frac{1}{2}\left(\frac{1}{t} - t\right)$, $\sqrt{x^2+1} = \frac{1}{2}\left(\frac{1}{t} + t\right)$,
$t = \sqrt{x^2+1} - x$, $\mathrm{d}x = -\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t$.

Example

$$\begin{aligned}\int \sqrt{x^2+1}\,\mathrm{d}x &= -\int \frac{1}{2}\left(\frac{1}{t} + t\right)\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t \\ &= -\frac{1}{4}\int\left(\frac{1}{t^3} + 2\frac{1}{t} + t\right)\mathrm{d}t\end{aligned}$$

Euler substitution: $x = \frac{1}{2}\left(\frac{1}{t} - t\right)$, $\sqrt{x^2+1} = \frac{1}{2}\left(\frac{1}{t} + t\right)$,
$t = \sqrt{x^2+1} - x$, $\mathrm{d}x = -\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t$.

Example

$$\begin{aligned}\int \sqrt{x^2+1}\,\mathrm{d}x &= -\int \frac{1}{2}\left(\frac{1}{t} + t\right)\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t\\ &= -\frac{1}{4}\int\left(\frac{1}{t^3} + 2\frac{1}{t} + t\right)\mathrm{d}t\end{aligned}$$

Euler substitution: $x = \frac{1}{2}\left(\frac{1}{t} - t\right)$, $\sqrt{x^2+1} = \frac{1}{2}\left(\frac{1}{t} + t\right)$,
$t = \sqrt{x^2+1} - x$, $\mathrm{d}x = -\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t$.

Example

$$\begin{aligned}\int \sqrt{x^2+1}\,\mathrm{d}x &= -\int \frac{1}{2}\left(\frac{1}{t} + t\right)\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t\\ &= -\frac{1}{4}\int \left(\frac{1}{t^3} + 2\frac{1}{t} + t\right)\mathrm{d}t\end{aligned}$$

Euler substitution: $x = \frac{1}{2}\left(\frac{1}{t} - t\right)$, $\sqrt{x^2+1} = \frac{1}{2}\left(\frac{1}{t} + t\right)$,
$t = \sqrt{x^2+1} - x$, $\mathrm{d}x = -\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t$.

Example

$$\begin{aligned}\int \sqrt{x^2+1}\,\mathrm{d}x &= -\int \frac{1}{2}\left(\frac{1}{t}+t\right)\frac{1}{2}\left(\frac{1}{t^2}+1\right)\mathrm{d}t\\ &= -\frac{1}{4}\int\left(\frac{1}{t^3}+2\frac{1}{t}+t\right)\mathrm{d}t\\ &= -\frac{1}{4}\left(-\frac{t^{-2}}{2}+2\ln|t|+\frac{t^2}{2}\right)+C\end{aligned}$$

Euler substitution: $x = \frac{1}{2}\left(\frac{1}{t} - t\right)$, $\sqrt{x^2+1} = \frac{1}{2}\left(\frac{1}{t} + t\right)$,
$t = \sqrt{x^2+1} - x$, $\mathrm{d}x = -\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t$.

Example

$$\begin{aligned}
\int \sqrt{x^2+1}\,\mathrm{d}x &= -\int \frac{1}{2}\left(\frac{1}{t} + t\right)\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t \\
&= -\frac{1}{4}\int \left(\frac{1}{t^3} + 2\frac{1}{t} + t\right)\mathrm{d}t \\
&= -\frac{1}{4}\left(-\frac{t^{-2}}{2} + 2\ln|t| + \frac{t^2}{2}\right) + C
\end{aligned}$$

Euler substitution: $x=\frac{1}{2}\left(\frac{1}{t}-t\right)$, $\sqrt{x^2+1}=\frac{1}{2}\left(\frac{1}{t}+t\right)$,
$t=\sqrt{x^2+1}-x$, $\mathrm{d}x=-\frac{1}{2}\left(\frac{1}{t^2}+1\right)\mathrm{d}t$.

Example

$$\begin{aligned}\int\sqrt{x^2+1}\,\mathrm{d}x &= -\int\frac{1}{2}\left(\frac{1}{t}+t\right)\frac{1}{2}\left(\frac{1}{t^2}+1\right)\mathrm{d}t\\ &= -\frac{1}{4}\int\left(\frac{1}{t^3}+2\frac{1}{t}+t\right)\mathrm{d}t\\ &= -\frac{1}{4}\left(-\frac{t^{-2}}{2}+2\ln|t|+\frac{t^2}{2}\right)+C\end{aligned}$$

Euler substitution: $x = \frac{1}{2}\left(\frac{1}{t} - t\right)$, $\sqrt{x^2+1} = \frac{1}{2}\left(\frac{1}{t} + t\right)$,
$t = \sqrt{x^2+1} - x$, $\mathrm{d}x = -\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t$.

Example

$$\begin{aligned}
\int \sqrt{x^2+1}\,\mathrm{d}x &= -\int \frac{1}{2}\left(\frac{1}{t} + t\right)\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t \\
&= -\frac{1}{4}\int \left(\frac{1}{t^3} + 2\frac{1}{t} + t\right)\mathrm{d}t \\
&= -\frac{1}{4}\left(-\frac{t^{-2}}{2} + 2\ln|t| + \frac{t^2}{2}\right) + C \\
&= \frac{1}{2}\left(\frac{1}{2}\left(t^{-1} - t\right)\frac{1}{2}\left(t^{-1} + t\right)\right) - \frac{1}{2}\ln t + C
\end{aligned}$$

Euler substitution: $x = \frac{1}{2}\left(\frac{1}{t} - t\right)$, $\sqrt{x^2+1} = \frac{1}{2}\left(\frac{1}{t} + t\right)$,
$t = \sqrt{x^2+1} - x$, $\mathrm{d}x = -\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t$.

Example

$$\begin{aligned}
\int \sqrt{x^2+1}\,\mathrm{d}x &= -\int \frac{1}{2}\left(\frac{1}{t} + t\right)\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t \\
&= -\frac{1}{4}\int\left(\frac{1}{t^3} + 2\frac{1}{t} + t\right)\mathrm{d}t \\
&= -\frac{1}{4}\left(-\frac{t^{-2}}{2} + 2\ln|t| + \frac{t^2}{2}\right) + C \\
&= \frac{1}{2}\left(\frac{1}{2}\left(t^{-1} - t\right)\frac{1}{2}\left(t^{-1} + t\right)\right) - \frac{1}{2}\ln t + C
\end{aligned}$$

Euler substitution: $x=\frac{1}{2}\left(\frac{1}{t}-t\right)$, $\sqrt{x^2+1}=\frac{1}{2}\left(\frac{1}{t}+t\right)$,
$t=\sqrt{x^2+1}-x$, $\mathrm{d}x=-\frac{1}{2}\left(\frac{1}{t^2}+1\right)\mathrm{d}t$.

Example

$$
\begin{aligned}
\int\sqrt{x^2+1}\,\mathrm{d}x &= -\int\frac{1}{2}\left(\frac{1}{t}+t\right)\frac{1}{2}\left(\frac{1}{t^2}+1\right)\mathrm{d}t\\
&=-\frac{1}{4}\int\left(\frac{1}{t^3}+2\frac{1}{t}+t\right)\mathrm{d}t\\
&=-\frac{1}{4}\left(-\frac{t^{-2}}{2}+2\ln|t|+\frac{t^2}{2}\right)+C\\
&=\frac{1}{2}\left(\frac{1}{2}\left(t^{-1}-t\right)\frac{1}{2}\left(t^{-1}+t\right)\right)-\frac{1}{2}\ln t+C
\end{aligned}
$$

Euler substitution: $x=\frac{1}{2}\left(\frac{1}{t}-t\right)$, $\sqrt{x^2+1}=\frac{1}{2}\left(\frac{1}{t}+t\right)$,
$t=\sqrt{x^2+1}-x$, $\mathrm{d}x=-\frac{1}{2}\left(\frac{1}{t^2}+1\right)\mathrm{d}t$.

Example

$$\begin{aligned}\int\sqrt{x^2+1}\,\mathrm{d}x &= -\int\frac{1}{2}\left(\frac{1}{t}+t\right)\frac{1}{2}\left(\frac{1}{t^2}+1\right)\mathrm{d}t\\ &= -\frac{1}{4}\int\left(\frac{1}{t^3}+2\frac{1}{t}+t\right)\mathrm{d}t\\ &= -\frac{1}{4}\left(-\frac{t^{-2}}{2}+2\ln|t|+\frac{t^2}{2}\right)+C\\ &= \frac{1}{2}\left(\frac{1}{2}\left(t^{-1}-t\right)\frac{1}{2}\left(t^{-1}+t\right)\right)-\frac{1}{2}\ln t+C\end{aligned}$$

Euler substitution: $x = \frac{1}{2}\left(\frac{1}{t} - t\right)$, $\sqrt{x^2+1} = \frac{1}{2}\left(\frac{1}{t} + t\right)$,
$t = \sqrt{x^2+1} - x$, $\mathrm{d}x = -\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t$. Recall $t > 0$.

Example

$$\begin{aligned}
\int \sqrt{x^2+1}\,\mathrm{d}x &= -\int \frac{1}{2}\left(\frac{1}{t} + t\right)\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t \\
&= -\frac{1}{4}\int\left(\frac{1}{t^3} + 2\frac{1}{t} + t\right)\mathrm{d}t \\
&= -\frac{1}{4}\left(-\frac{t^{-2}}{2} + 2\ln|t| + \frac{t^2}{2}\right) + C \\
&= \frac{1}{2}\left(\frac{1}{2}\left(t^{-1} - t\right)\frac{1}{2}\left(t^{-1} + t\right)\right) - \frac{1}{2}\ln t + C
\end{aligned}$$

Euler substitution: $x = \frac{1}{2}\left(\frac{1}{t} - t\right)$, $\sqrt{x^2+1} = \frac{1}{2}\left(\frac{1}{t} + t\right)$,
$t = \sqrt{x^2+1} - x$, $\mathrm{d}x = -\frac{1}{2}\left(\frac{1}{t^2}+1\right)\mathrm{d}t$. Recall $t > 0$.

Example

$$\begin{aligned}\int \sqrt{x^2+1}\,\mathrm{d}x &= -\int \frac{1}{2}\left(\frac{1}{t}+t\right)\frac{1}{2}\left(\frac{1}{t^2}+1\right)\mathrm{d}t\\
&= -\frac{1}{4}\int\left(\frac{1}{t^3}+2\frac{1}{t}+t\right)\mathrm{d}t\\
&= -\frac{1}{4}\left(-\frac{t^{-2}}{2}+2\ln|t|+\frac{t^2}{2}\right)+C\\
&= \frac{1}{2}\left(\frac{1}{2}\left(t^{-1}-t\right)\frac{1}{2}\left(t^{-1}+t\right)\right)-\frac{1}{2}\ln t + C\\
&= \frac{1}{2}x\sqrt{x^2+1}-\frac{1}{2}\ln\left(\sqrt{x^2+1}-x\right)+C\end{aligned}$$

Euler substitution: $x=\frac{1}{2}\left(\frac{1}{t}-t\right)$, $\sqrt{x^2+1}=\frac{1}{2}\left(\frac{1}{t}+t\right)$,
$t=\sqrt{x^2+1}-x$, $\mathrm{d}x=-\frac{1}{2}\left(\frac{1}{t^2}+1\right)\mathrm{d}t$. Recall $t>0$.

Example

$$\begin{aligned}
\int\sqrt{x^2+1}\,\mathrm{d}x &= -\int\frac{1}{2}\left(\frac{1}{t}+t\right)\frac{1}{2}\left(\frac{1}{t^2}+1\right)\mathrm{d}t\\
&=-\frac{1}{4}\int\left(\frac{1}{t^3}+2\frac{1}{t}+t\right)\mathrm{d}t\\
&=-\frac{1}{4}\left(-\frac{t^{-2}}{2}+2\ln|t|+\frac{t^2}{2}\right)+C\\
&=\frac{1}{2}\left(\frac{1}{2}\left(t^{-1}-t\right)\frac{1}{2}\left(t^{-1}+t\right)\right)-\frac{1}{2}\ln t+C\\
&=\frac{1}{2}x\sqrt{x^2+1}-\frac{1}{2}\ln\left(\sqrt{x^2+1}-x\right)+C
\end{aligned}$$

Euler substitution: $x = \frac{1}{2}\left(\frac{1}{t} - t\right)$, $\sqrt{x^2+1} = \frac{1}{2}\left(\frac{1}{t} + t\right)$,
$t = \sqrt{x^2+1} - x$, $\mathrm{d}x = -\frac{1}{2}\left(\frac{1}{t^2}+1\right)\mathrm{d}t$. Recall $t > 0$.

Example

$$
\begin{aligned}
\int \sqrt{x^2+1}\,\mathrm{d}x &= -\int \frac{1}{2}\left(\frac{1}{t}+t\right)\frac{1}{2}\left(\frac{1}{t^2}+1\right)\mathrm{d}t \\
&= -\frac{1}{4}\int\left(\frac{1}{t^3}+2\frac{1}{t}+t\right)\mathrm{d}t \\
&= -\frac{1}{4}\left(-\frac{t^{-2}}{2}+2\ln|t|+\frac{t^2}{2}\right)+C \\
&= \frac{1}{2}\left(\frac{1}{2}\left(t^{-1}-t\right)\frac{1}{2}\left(t^{-1}+t\right)\right)-\frac{1}{2}\ln t + C \\
&= \frac{1}{2}x\sqrt{x^2+1}-\frac{1}{2}\ln\left(\sqrt{x^2+1}-x\right)+C
\end{aligned}
$$

Euler substitution: $x = \frac{1}{2}\left(\frac{1}{t} - t\right)$, $\sqrt{x^2+1} = \frac{1}{2}\left(\frac{1}{t} + t\right)$,
$t = \sqrt{x^2+1} - x$, $\mathrm{d}x = -\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t$. Recall $t > 0$.

Example

$$
\begin{aligned}
\int \sqrt{x^2+1}\,\mathrm{d}x &= -\int \frac{1}{2}\left(\frac{1}{t} + t\right)\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t\\
&= -\frac{1}{4}\int\left(\frac{1}{t^3} + 2\frac{1}{t} + t\right)\mathrm{d}t\\
&= -\frac{1}{4}\left(-\frac{t^{-2}}{2} + 2\ln|t| + \frac{t^2}{2}\right) + C\\
&= \frac{1}{2}\left(\frac{1}{2}\left(t^{-1} - t\right)\frac{1}{2}\left(t^{-1} + t\right)\right) - \frac{1}{2}\ln t + C\\
&= \frac{1}{2}x\sqrt{x^2+1} - \frac{1}{2}\ln\left(\sqrt{x^2+1} - x\right) + C\\
&= \frac{1}{2}x\sqrt{x^2+1} + \frac{1}{2}\ln\frac{1}{\left(\sqrt{x^2+1} - x\right)} + C
\end{aligned}
$$

Euler substitution: $x=\frac{1}{2}\left(\frac{1}{t}-t\right)$, $\sqrt{x^2+1}=\frac{1}{2}\left(\frac{1}{t}+t\right)$, $t=\sqrt{x^2+1}-x$, $\mathrm{d}x=-\frac{1}{2}\left(\frac{1}{t^2}+1\right)\mathrm{d}t$. Recall $t>0$.

Example

$$
\begin{aligned}
\int\sqrt{x^2+1}\,\mathrm{d}x &= -\int\frac{1}{2}\left(\frac{1}{t}+t\right)\frac{1}{2}\left(\frac{1}{t^2}+1\right)\mathrm{d}t\\
&=-\frac{1}{4}\int\left(\frac{1}{t^3}+2\frac{1}{t}+t\right)\mathrm{d}t\\
&=-\frac{1}{4}\left(-\frac{t^{-2}}{2}+2\ln|t|+\frac{t^2}{2}\right)+C\\
&=\frac{1}{2}\left(\frac{1}{2}\left(t^{-1}-t\right)\frac{1}{2}\left(t^{-1}+t\right)\right)-\frac{1}{2}\ln t+C\\
&=\frac{1}{2}x\sqrt{x^2+1}-\frac{1}{2}\ln\left(\sqrt{x^2+1}-x\right)+C\\
&=\frac{1}{2}x\sqrt{x^2+1}+\frac{1}{2}\ln\frac{\sqrt{x^2+1}+x}{\left(\sqrt{x^2+1}-x\right)\left(\sqrt{x^2+1}+x\right)}+C
\end{aligned}
$$

Euler substitution: $x = \frac{1}{2}\left(\frac{1}{t} - t\right)$, $\sqrt{x^2+1} = \frac{1}{2}\left(\frac{1}{t} + t\right)$,
$t = \sqrt{x^2+1} - x$, $\mathrm{d}x = -\frac{1}{2}\left(\frac{1}{t^2}+1\right)\mathrm{d}t$. Recall $t > 0$.

Example

$$
\begin{aligned}
\int \sqrt{x^2+1}\,\mathrm{d}x &= -\int \frac{1}{2}\left(\frac{1}{t}+t\right)\frac{1}{2}\left(\frac{1}{t^2}+1\right)\mathrm{d}t \\
&= -\frac{1}{4}\int\left(\frac{1}{t^3}+2\frac{1}{t}+t\right)\mathrm{d}t \\
&= -\frac{1}{4}\left(-\frac{t^{-2}}{2}+2\ln|t|+\frac{t^2}{2}\right)+C \\
&= \frac{1}{2}\left(\frac{1}{2}\left(t^{-1}-t\right)\frac{1}{2}\left(t^{-1}+t\right)\right)-\frac{1}{2}\ln t+C \\
&= \frac{1}{2}x\sqrt{x^2+1}-\frac{1}{2}\ln\left(\sqrt{x^2+1}-x\right)+C \\
&= \frac{1}{2}x\sqrt{x^2+1}+\frac{1}{2}\ln\frac{\sqrt{x^2+1}+x}{\left(\sqrt{x^2+1}-x\right)\left(\sqrt{x^2+1}+x\right)}+C \\
&= \frac{1}{2}x\sqrt{x^2+1}+\frac{1}{2}\ln\left(\sqrt{x^2+1}+x\right)+C
\end{aligned}
$$

Euler substitution: $x = \frac{1}{2}\left(\frac{1}{t} - t\right)$, $\sqrt{x^2+1} = \frac{1}{2}\left(\frac{1}{t} + t\right)$,
$t = \sqrt{x^2+1} - x$, $\mathrm{d}x = -\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t$. Recall $t > 0$.

Example

$$\begin{aligned}
\int \sqrt{x^2+1}\,\mathrm{d}x &= -\int \frac{1}{2}\left(\frac{1}{t} + t\right)\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t \\
&= -\frac{1}{4}\int \left(\frac{1}{t^3} + 2\frac{1}{t} + t\right)\mathrm{d}t \\
&= -\frac{1}{4}\left(-\frac{t^{-2}}{2} + 2\ln|t| + \frac{t^2}{2}\right) + C \\
&= \frac{1}{2}\left(\frac{1}{2}\left(t^{-1} - t\right)\frac{1}{2}\left(t^{-1} + t\right)\right) - \frac{1}{2}\ln t + C \\
&= \frac{1}{2}x\sqrt{x^2+1} - \frac{1}{2}\ln\left(\sqrt{x^2+1} - x\right) + C \\
&= \frac{1}{2}x\sqrt{x^2+1} + \frac{1}{2}\ln\frac{\sqrt{x^2+1} + x}{\left(\sqrt{x^2+1} - x\right)\left(\sqrt{x^2+1} + x\right)} + C \\
&= \frac{1}{2}x\sqrt{x^2+1} + \frac{1}{2}\ln\left(\sqrt{x^2+1} + x\right) + C
\end{aligned}$$

Example

$$\int \frac{1}{x^2\sqrt{x^2+9}}\mathrm{d}x$$

Example

$$\int \frac{1}{x^2\sqrt{x^2+9}}\mathrm{d}x = \int \frac{1}{x^2 3\sqrt{\left(\frac{x}{3}\right)^2+1}}\mathrm{d}x$$

Example

$$\int \frac{1}{x^2\sqrt{x^2+9}}\mathrm{d}x = \int \frac{1}{x^2 3\sqrt{\left(\frac{x}{3}\right)^2+1}}\mathrm{d}x$$

Set

$$\frac{x}{3} = \cot\theta$$

$$\theta \in (0, \pi)$$

Example

$$\int \frac{1}{x^2\sqrt{x^2+9}}\mathrm{d}x = \int \frac{1}{x^2 3\sqrt{\left(\frac{x}{3}\right)^2+1}}\mathrm{d}x$$

Set

$$\frac{x}{3} = \cot\theta$$

$$x = 3\cot\theta$$

$$\theta \in (0, \pi)$$

Example

$$\int \frac{1}{x^2\sqrt{x^2+9}}\mathrm{d}x = \int \frac{1}{x^2 3\sqrt{\left(\frac{x}{3}\right)^2+1}}\mathrm{d}x$$

$$= \int \frac{1}{(3\cot\theta)^2 3\sqrt{\cot^2\theta+1}}\mathrm{d}(3\cot\theta)$$

Set

$\frac{x}{3} = \cot\theta$

$x = 3\cot\theta$

$\theta \in (0,\pi)$

Example

$$\int \frac{1}{x^2\sqrt{x^2+9}}\mathrm{d}x = \int \frac{1}{x^2 3\sqrt{\left(\frac{x}{3}\right)^2+1}}\mathrm{d}x$$

$$= \int \frac{1}{(3\cot\theta)^2 3\sqrt{\cot^2\theta+1}}\mathrm{d}(3\cot\theta)$$

$$= \int \frac{1}{27\cot^2\theta\sqrt{\textbf{?}}}\left(\textbf{?} \qquad\qquad \right)\mathrm{d}\theta$$

Set

$\frac{x}{3} = \cot\theta$

$x = 3\cot\theta$

$\theta \in (0, \pi)$

Example

$$\int \frac{1}{x^2\sqrt{x^2+9}}\mathrm{d}x = \int \frac{1}{x^2 3\sqrt{\left(\frac{x}{3}\right)^2+1}}\mathrm{d}x$$

$$= \int \frac{1}{(3\cot\theta)^2 3\sqrt{\cot^2\theta+1}}\mathrm{d}(3\cot\theta)$$

$$= \int \frac{1}{27\cot^2\theta\sqrt{\textbf{?}}}\left(\textbf{?} \qquad \right)\mathrm{d}\theta$$

Set

$$\frac{x}{3} = \cot\theta$$

$$x = 3\cot\theta$$

$$\theta \in (0, \pi)$$

Example

$$\int \frac{1}{x^2\sqrt{x^2+9}}\mathrm{d}x = \int \frac{1}{x^2 3\sqrt{\left(\frac{x}{3}\right)^2+1}}\mathrm{d}x$$

$$= \int \frac{1}{(3\cot\theta)^2 3\sqrt{\cot^2\theta+1}}\mathrm{d}(3\cot\theta)$$

$$= \int \frac{1}{27\cot^2\theta\sqrt{\csc^2\theta}}\left(\textbf{?} \qquad\qquad \right)\mathrm{d}\theta$$

Set

$\frac{x}{3} = \cot\theta$

$x = 3\cot\theta$

$\theta \in (0,\pi)$

Example

$$\int \frac{1}{x^2\sqrt{x^2+9}}\mathrm{d}x = \int \frac{1}{x^2 3\sqrt{\left(\frac{x}{3}\right)^2+1}}\mathrm{d}x$$
$$= \int \frac{1}{(3\cot\theta)^2 3\sqrt{\cot^2\theta+1}}\mathrm{d}(3\cot\theta)$$
$$= \int \frac{1}{27\cot^2\theta\sqrt{\csc^2\theta}}\left(\textbf{?} \qquad\qquad \right)\mathrm{d}\theta$$

Set
$\frac{x}{3} = \cot\theta$
$x = 3\cot\theta$

$\theta \in (0, \pi)$

Example

$$\int \frac{1}{x^2\sqrt{x^2+9}}\mathrm{d}x = \int \frac{1}{x^2 3\sqrt{\left(\frac{x}{3}\right)^2+1}}\mathrm{d}x$$

$$= \int \frac{1}{(3\cot\theta)^2 3\sqrt{\cot^2\theta+1}}\mathrm{d}(3\cot\theta)$$

$$= \int \frac{1}{27\cot^2\theta\sqrt{\csc^2\theta}}\left(-3\csc^2\theta\right)\mathrm{d}\theta$$

Set

$\frac{x}{3} = \cot\theta$

$x = 3\cot\theta$

$\theta \in (0,\pi)$

Example

$$\int \frac{1}{x^2\sqrt{x^2+9}}\mathrm{d}x = \int \frac{1}{x^2 3\sqrt{\left(\frac{x}{3}\right)^2+1}}\mathrm{d}x$$

$$= \int \frac{1}{(3\cot\theta)^2 3\sqrt{\cot^2\theta+1}}\mathrm{d}(3\cot\theta)$$

$$= \int \frac{1}{27\cot^2\theta\sqrt{\csc^2\theta}}\left(-3\csc^2\theta\right)\mathrm{d}\theta$$

$$= \frac{1}{9}\int \frac{-\csc^2\theta}{\cot^2\theta\csc\theta}\mathrm{d}\theta$$

Set

$$\frac{x}{3} = \cot\theta$$

$$x = 3\cot\theta$$

$$\theta \in (0,\pi)$$

Example

$$\int \frac{1}{x^2\sqrt{x^2+9}}\mathrm{d}x = \int \frac{1}{x^2 3\sqrt{\left(\frac{x}{3}\right)^2+1}}\mathrm{d}x$$
$$= \int \frac{1}{(3\cot\theta)^2 3\sqrt{\cot^2\theta+1}}\mathrm{d}(3\cot\theta)$$
$$= \int \frac{1}{27\cot^2\theta\sqrt{\csc^2\theta}}\left(-3\csc^2\theta\right)\mathrm{d}\theta$$
$$= \frac{1}{9}\int \frac{-\csc^2\theta}{\cot^2\theta\csc\theta}\mathrm{d}\theta$$

Set
$\frac{x}{3} = \cot\theta$
$x = 3\cot\theta$

$\theta \in (0, \pi)$

$\theta \in (0, \pi) \Rightarrow$
$\csc\theta > 0$

Example

$$\begin{aligned}
\int \frac{1}{x^2\sqrt{x^2+9}}\mathrm{d}x &= \int \frac{1}{x^2 3\sqrt{\left(\frac{x}{3}\right)^2+1}}\mathrm{d}x \\
&= \int \frac{1}{(3\cot\theta)^2 3\sqrt{\cot^2\theta+1}}\mathrm{d}(3\cot\theta) \\
&= \int \frac{1}{27\cot^2\theta\sqrt{\csc^2\theta}}\left(-3\csc^2\theta\right)\mathrm{d}\theta \\
&= \frac{1}{9}\int \frac{-\csc^2\theta}{\cot^2\theta\csc\theta}\mathrm{d}\theta \\
&= \frac{1}{9}\int \frac{-\sin\theta}{\cos^2\theta}\mathrm{d}\theta
\end{aligned}$$

Set

$$\frac{x}{3} = \cot\theta$$

$$x = 3\cot\theta$$

$$\theta \in (0,\pi)$$

$$\theta \in (0,\pi) \Rightarrow \csc\theta > 0$$

Example

$$\int \frac{1}{x^2\sqrt{x^2+9}}\mathrm{d}x = \int \frac{1}{x^2 3\sqrt{\left(\frac{x}{3}\right)^2+1}}\mathrm{d}x$$

$$= \int \frac{1}{(3\cot\theta)^2 3\sqrt{\cot^2\theta+1}}\mathrm{d}(3\cot\theta)$$

$$= \int \frac{1}{27\cot^2\theta\sqrt{\csc^2\theta}}\left(-3\csc^2\theta\right)\mathrm{d}\theta$$

$$= \frac{1}{9}\int \frac{-\csc^2\theta}{\cot^2\theta\csc\theta}\mathrm{d}\theta$$

$$= \frac{1}{9}\int \frac{-\sin\theta}{\cos^2\theta}\mathrm{d}\theta = \frac{1}{9}\int \frac{1}{\cos^2\theta}\mathrm{d}(\textbf{?} \quad)$$

Set

$\frac{x}{3} = \cot\theta$

$x = 3\cot\theta$

$\theta \in (0,\pi)$

$\theta \in (0,\pi) \Rightarrow$
$\csc\theta > 0$

Example

$$\int \frac{1}{x^2\sqrt{x^2+9}}\mathrm{d}x = \int \frac{1}{x^2 3\sqrt{\left(\frac{x}{3}\right)^2+1}}\mathrm{d}x$$

$$= \int \frac{1}{(3\cot\theta)^2 3\sqrt{\cot^2\theta+1}}\mathrm{d}(3\cot\theta)$$

$$= \int \frac{1}{27\cot^2\theta\sqrt{\csc^2\theta}}\left(-3\csc^2\theta\right)\mathrm{d}\theta$$

$$= \frac{1}{9}\int \frac{-\csc^2\theta}{\cot^2\theta\csc\theta}\mathrm{d}\theta$$

$$= \frac{1}{9}\int \frac{-\sin\theta}{\cos^2\theta}\mathrm{d}\theta = \frac{1}{9}\int \frac{1}{\cos^2\theta}\mathrm{d}(\cos\theta)$$

Set
$\frac{x}{3} = \cot\theta$
$x = 3\cot\theta$

$\theta \in (0, \pi)$

$\theta \in (0, \pi) \Rightarrow$
$\csc\theta > 0$

Example

$$
\begin{aligned}
\int \frac{1}{x^2\sqrt{x^2+9}}\mathrm{d}x &= \int \frac{1}{x^2 3\sqrt{\left(\frac{x}{3}\right)^2+1}}\mathrm{d}x \\
&= \int \frac{1}{(3\cot\theta)^2 3\sqrt{\cot^2\theta+1}}\mathrm{d}(3\cot\theta) \\
&= \int \frac{1}{27\cot^2\theta\sqrt{\csc^2\theta}}\left(-3\csc^2\theta\right)\mathrm{d}\theta \\
&= \frac{1}{9}\int \frac{-\csc^2\theta}{\cot^2\theta\csc\theta}\mathrm{d}\theta \\
&= \frac{1}{9}\int \frac{-\sin\theta}{\cos^2\theta}\mathrm{d}\theta = \frac{1}{9}\int \frac{1}{\cos^2\theta}\mathrm{d}(\cos\theta) \\
&= \frac{1}{9}\int \frac{\mathrm{d}u}{u^2}
\end{aligned}
$$

Set

$\frac{x}{3} = \cot\theta$

$x = 3\cot\theta$

$\theta \in (0,\pi)$

$\theta \in (0,\pi) \Rightarrow \csc\theta > 0$

Set $u = \cos\theta$

Example

$$\int \frac{1}{x^2\sqrt{x^2+9}}\mathrm{d}x = \int \frac{1}{x^2 3\sqrt{\left(\frac{x}{3}\right)^2+1}}\mathrm{d}x$$

$$= \int \frac{1}{(3\cot\theta)^2 3\sqrt{\cot^2\theta+1}}\mathrm{d}(3\cot\theta)$$

$$= \int \frac{1}{27\cot^2\theta\sqrt{\csc^2\theta}}\left(-3\csc^2\theta\right)\mathrm{d}\theta$$

$$= \frac{1}{9}\int \frac{-\csc^2\theta}{\cot^2\theta\csc\theta}\mathrm{d}\theta$$

$$= \frac{1}{9}\int \frac{-\sin\theta}{\cos^2\theta}\mathrm{d}\theta = \frac{1}{9}\int \frac{1}{\cos^2\theta}\mathrm{d}(\cos\theta)$$

$$= \frac{1}{9}\int \frac{\mathrm{d}u}{u^2} = \textbf{?} \qquad + C$$

Set
$\frac{x}{3} = \cot\theta$
$x = 3\cot\theta$

$\theta \in (0,\pi)$

$\theta \in (0,\pi) \Rightarrow$
$\csc\theta > 0$

Set $u = \cos\theta$

Example

$$
\begin{aligned}
\int \frac{1}{x^2\sqrt{x^2+9}}\mathrm{d}x &= \int \frac{1}{x^2 3\sqrt{\left(\frac{x}{3}\right)^2+1}}\mathrm{d}x \\
&= \int \frac{1}{(3\cot\theta)^2 3\sqrt{\cot^2\theta+1}}\mathrm{d}(3\cot\theta) \\
&= \int \frac{1}{27\cot^2\theta\sqrt{\csc^2\theta}}\left(-3\csc^2\theta\right)\mathrm{d}\theta \\
&= \frac{1}{9}\int \frac{-\csc^2\theta}{\cot^2\theta\csc\theta}\mathrm{d}\theta \\
&= \frac{1}{9}\int \frac{-\sin\theta}{\cos^2\theta}\mathrm{d}\theta = \frac{1}{9}\int \frac{1}{\cos^2\theta}\mathrm{d}(\cos\theta) \\
&= \frac{1}{9}\int \frac{\mathrm{d}u}{u^2} = -\frac{1}{9u} + C
\end{aligned}
$$

Set
$\frac{x}{3} = \cot\theta$
$x = 3\cot\theta$

$\theta \in (0, \pi)$

$\theta \in (0, \pi) \Rightarrow$
$\csc\theta > 0$

Set $u = \cos\theta$

Example

$$\int \frac{1}{x^2\sqrt{x^2+9}}\mathrm{d}x = \int \frac{1}{x^2 3\sqrt{\left(\frac{x}{3}\right)^2+1}}\mathrm{d}x$$

$$= \int \frac{1}{(3\cot\theta)^2 3\sqrt{\cot^2\theta+1}}\mathrm{d}(3\cot\theta)$$

$$= \int \frac{1}{27\cot^2\theta\sqrt{\csc^2\theta}}\left(-3\csc^2\theta\right)\mathrm{d}\theta$$

$$= \frac{1}{9}\int \frac{-\csc^2\theta}{\cot^2\theta\csc\theta}\mathrm{d}\theta$$

$$= \frac{1}{9}\int \frac{-\sin\theta}{\cos^2\theta}\mathrm{d}\theta = \frac{1}{9}\int \frac{1}{\cos^2\theta}\mathrm{d}(\cos\theta)$$

$$= \frac{1}{9}\int \frac{\mathrm{d}u}{u^2} = -\frac{1}{9u} + C = -\frac{\sec\theta}{9} + C$$

Set

$\frac{x}{3} = \cot\theta$

$x = 3\cot\theta$

$\theta \in (0,\pi)$

$\theta \in (0,\pi) \Rightarrow$

$\csc\theta > 0$

Set $u = \cos\theta$

Example

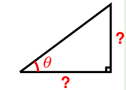
$$\begin{aligned}
\int \frac{1}{x^2\sqrt{x^2+9}}\mathrm{d}x &= \int \frac{1}{x^2 3\sqrt{\left(\frac{x}{3}\right)^2+1}}\mathrm{d}x \\
&= \int \frac{1}{(3\cot\theta)^2 3\sqrt{\cot^2\theta+1}}\mathrm{d}(3\cot\theta) \\
&= \int \frac{1}{27\cot^2\theta\sqrt{\csc^2\theta}}\left(-3\csc^2\theta\right)\mathrm{d}\theta \\
&= \frac{1}{9}\int \frac{-\csc^2\theta}{\cot^2\theta\csc\theta}\mathrm{d}\theta \\
&= \frac{1}{9}\int \frac{-\sin\theta}{\cos^2\theta}\mathrm{d}\theta = \frac{1}{9}\int \frac{1}{\cos^2\theta}\mathrm{d}(\cos\theta) \\
&= \frac{1}{9}\int \frac{\mathrm{d}u}{u^2} = -\frac{1}{9u} + C = -\frac{\sec\theta}{9} + C
\end{aligned}$$

Set
$\frac{x}{3} = \cot\theta$
$x = 3\cot\theta$

$\theta \in (0,\pi)$

$\theta \in (0,\pi) \Rightarrow$
$\csc\theta > 0$

Set $u = \cos\theta$



Example

$$\int \frac{1}{x^2\sqrt{x^2+9}}\mathrm{d}x = \int \frac{1}{x^2 3\sqrt{\left(\frac{x}{3}\right)^2+1}}\mathrm{d}x$$

Set $\frac{x}{3} = \cot\theta$, $x = 3\cot\theta$

$$= \int \frac{1}{(3\cot\theta)^2 3\sqrt{\cot^2\theta+1}}\mathrm{d}(3\cot\theta)$$

$\theta \in (0,\pi)$

$$= \int \frac{1}{27\cot^2\theta\sqrt{\csc^2\theta}}\left(-3\csc^2\theta\right)\mathrm{d}\theta$$

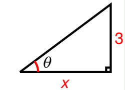
$\theta \in (0,\pi) \Rightarrow \csc\theta > 0$

$$= \frac{1}{9}\int \frac{-\csc^2\theta}{\cot^2\theta\csc\theta}\mathrm{d}\theta$$

$$= \frac{1}{9}\int \frac{-\sin\theta}{\cos^2\theta}\mathrm{d}\theta = \frac{1}{9}\int \frac{1}{\cos^2\theta}\mathrm{d}(\cos\theta)$$

Set $u = \cos\theta$

$$= \frac{1}{9}\int \frac{\mathrm{d}u}{u^2} = -\frac{1}{9u} + C = -\frac{\sec\theta}{9} + C$$



Example

$$\int \frac{1}{x^2\sqrt{x^2+9}}\mathrm{d}x = \int \frac{1}{x^2 3\sqrt{\left(\frac{x}{3}\right)^2+1}}\mathrm{d}x$$

$$= \int \frac{1}{(3\cot\theta)^2 3\sqrt{\cot^2\theta+1}}\mathrm{d}(3\cot\theta)$$

$$= \int \frac{1}{27\cot^2\theta\sqrt{\csc^2\theta}}\left(-3\csc^2\theta\right)\mathrm{d}\theta$$

$$= \frac{1}{9}\int \frac{-\csc^2\theta}{\cot^2\theta\csc\theta}\mathrm{d}\theta$$

$$= \frac{1}{9}\int \frac{-\sin\theta}{\cos^2\theta}\mathrm{d}\theta = \frac{1}{9}\int \frac{1}{\cos^2\theta}\mathrm{d}(\cos\theta)$$

$$= \frac{1}{9}\int \frac{\mathrm{d}u}{u^2} = -\frac{1}{9u} + C = -\frac{\sec\theta}{9} + C$$

Set

$\frac{x}{3} = \cot\theta$

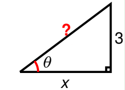
$x = 3\cot\theta$

$\theta \in (0, \pi)$

$\theta \in (0, \pi) \Rightarrow$

$\csc\theta > 0$

Set $u = \cos\theta$



Example

$$
\begin{aligned}
\int \frac{1}{x^2\sqrt{x^2+9}}\mathrm{d}x &= \int \frac{1}{x^2 3\sqrt{\left(\frac{x}{3}\right)^2+1}}\mathrm{d}x \\
&= \int \frac{1}{(3\cot\theta)^2 3\sqrt{\cot^2\theta+1}}\mathrm{d}(3\cot\theta) \\
&= \int \frac{1}{27\cot^2\theta\sqrt{\csc^2\theta}}\left(-3\csc^2\theta\right)\mathrm{d}\theta \\
&= \frac{1}{9}\int \frac{-\csc^2\theta}{\cot^2\theta\csc\theta}\mathrm{d}\theta \\
&= \frac{1}{9}\int \frac{-\sin\theta}{\cos^2\theta}\mathrm{d}\theta = \frac{1}{9}\int\frac{1}{\cos^2\theta}\mathrm{d}(\cos\theta) \\
&= \frac{1}{9}\int\frac{\mathrm{d}u}{u^2} = -\frac{1}{9u}+C = -\frac{\sec\theta}{9}+C
\end{aligned}
$$

Set

$\frac{x}{3} = \cot\theta$

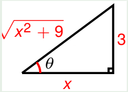
$x = 3\cot\theta$

$\theta \in (0,\pi)$

$\theta \in (0,\pi) \Rightarrow$

$\csc\theta > 0$

Set $u = \cos\theta$



Example

$$
\begin{aligned}
\int \frac{1}{x^2\sqrt{x^2+9}}\mathrm{d}x &= \int \frac{1}{x^2 3\sqrt{\left(\frac{x}{3}\right)^2+1}}\mathrm{d}x \\
&= \int \frac{1}{(3\cot\theta)^2 3\sqrt{\cot^2\theta+1}}\mathrm{d}(3\cot\theta) \\
&= \int \frac{1}{27\cot^2\theta\sqrt{\csc^2\theta}}\left(-3\csc^2\theta\right)\mathrm{d}\theta \\
&= \frac{1}{9}\int \frac{-\csc^2\theta}{\cot^2\theta\csc\theta}\mathrm{d}\theta \\
&= \frac{1}{9}\int \frac{-\sin\theta}{\cos^2\theta}\mathrm{d}\theta = \frac{1}{9}\int \frac{1}{\cos^2\theta}\mathrm{d}(\cos\theta) \\
&= \frac{1}{9}\int \frac{\mathrm{d}u}{u^2} = -\frac{1}{9u} + C = -\frac{\sec\theta}{9} + C \\
&= -\frac{\sqrt{x^2+9}}{9x} + C
\end{aligned}
$$

Set
$\frac{x}{3} = \cot\theta$
$x = 3\cot\theta$

$\theta \in (0,\pi)$

$\theta \in (0,\pi) \Rightarrow$
$\csc\theta > 0$

Set $u = \cos\theta$