

## Precalculus

**Express  $\sin(kx)$ ,  $\cos(kx)$  via  $\sin x$ ,  $\cos x$  using Euler's formula**

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## Example

Express  $\sin(3x)$  and  $\cos(3x)$  via  $\cos x$  and  $\sin x$ .

- Recall Euler's formula:  $e^{i\alpha} = \cos \alpha + i \sin \alpha$ .

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The real parts of the starting and final expression must be equal;  
therefore:

$$\cos(3x) = \cos^3 x - 3\cos x \sin^2 x$$

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The real parts of the starting and final expression must be equal; likewise the imaginary parts must be equal; therefore:

$$\begin{aligned}
 \cos(3x) &= \cos^3 x - 3\cos x \sin^2 x \\
 \sin(3x) &= 3\cos^2 x \sin x - \sin^3 x
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