Calculus II
Area locked by curve

Todor Milev

2019

Outline

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Areas

- The area under a curve $y = F(x)$ from a to b is

$$A = \int_a^b F(x) \mathrm{d}x$$

- Suppose the curve has parametric equations $x = f(t)$, $y = g(t)$, $\alpha \leq t \leq \beta$.

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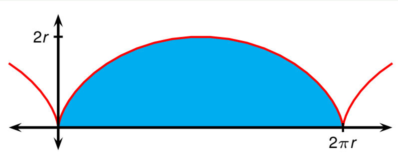
- How do we know where to put α and β?
- When $x = a$, t will be either α or β. When $x = b$, t will take the other value.

Example



Find the area under one arch of the cycloid

$$x = r(\theta - \sin\theta), \qquad y = r(1 - \cos\theta)$$

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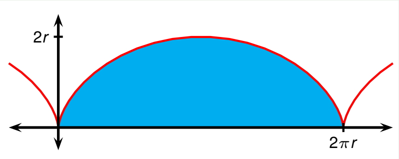

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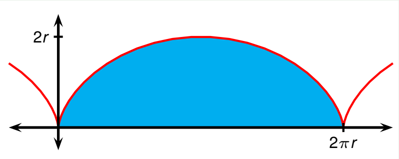

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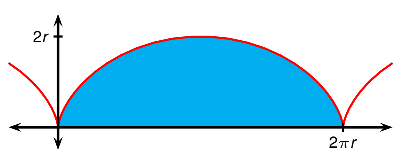

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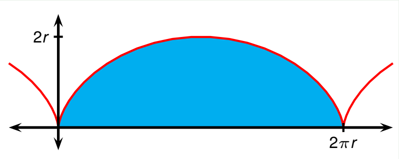

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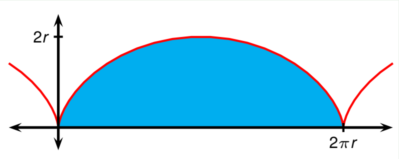

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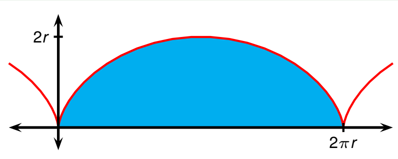

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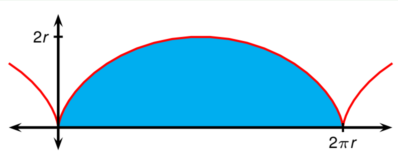

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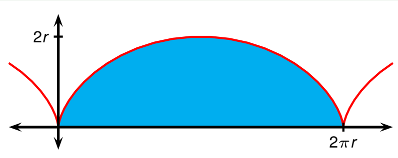

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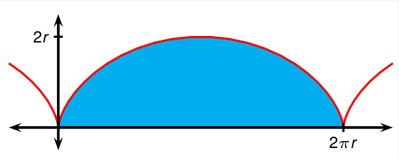

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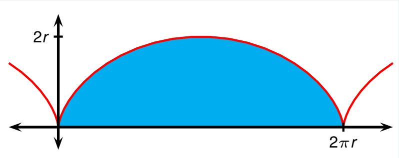

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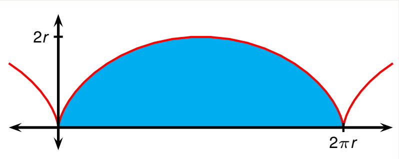

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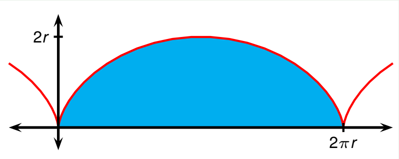

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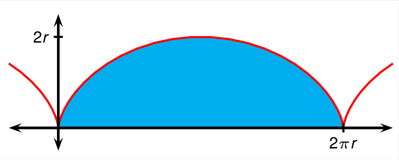

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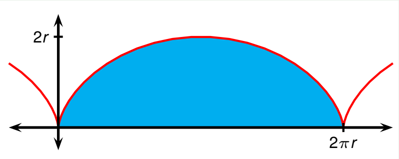

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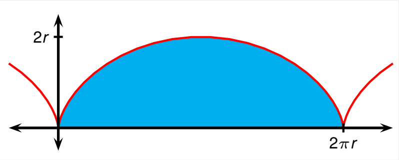

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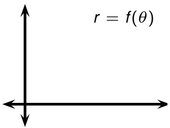
\end{aligned}$$

Areas in Polar Coordinates

Suppose we have a polar curve $r = f(\theta)$, $a \leq \theta \leq b$.

Definition

We say that the figure obtained as the union of the segments connecting the origin with the points of the curve is the figure *swept* by the curve as θ varies from a to b.



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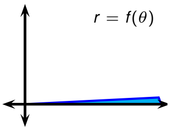

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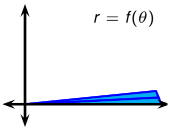

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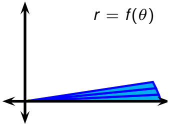

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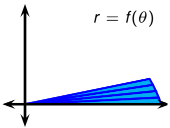

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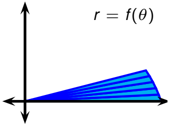

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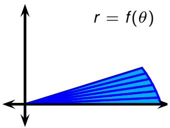

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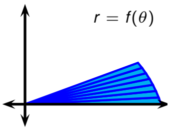

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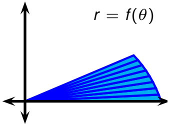

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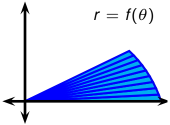

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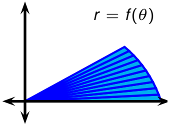

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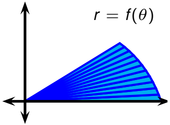

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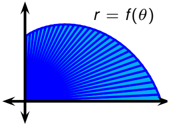

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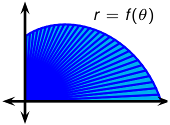
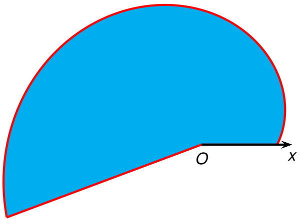


Theorem

Suppose no two points on the curve lie on the same ray from the origin. Then the area swept by the curve equals $A = \int_a^b \frac{1}{2} (f(\theta))^2 \mathrm{d}\theta$.

Area swept by a polar curve: justification



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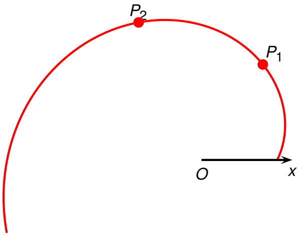

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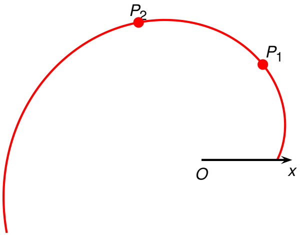

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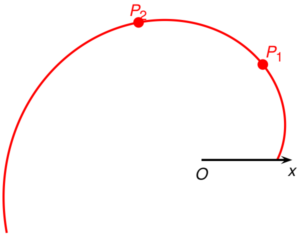

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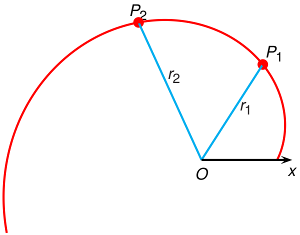
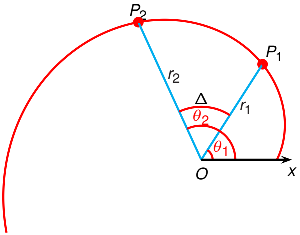
Area swept by a polar curve: justification



Split $[a, b]$ into N equal segments via points $a = \theta_0 \leq \theta_1 \leq \dots \leq \theta_{N-1} \leq \theta_N = b$. The length of each segment is $\Delta = \frac{b-a}{N}$. Let $r_i = f(\theta_i)$. Then each θ_i gives a point P_i with polar coordinates (r_i, θ_i).

Area swept by a polar curve: justification



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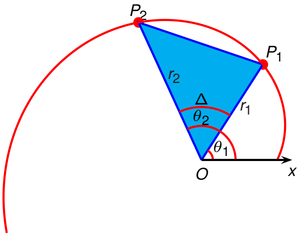
Area swept by a polar curve: justification



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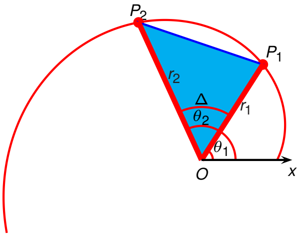
The area swept by the curve is approximated by sum of areas of triangles given by connecting the origin with two consecutive vertices. Consider one such triangle, say, OP_1P_2.

Area swept by a polar curve: justification



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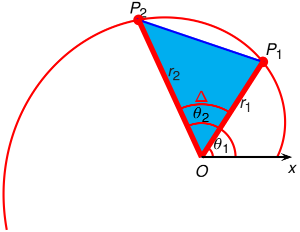
Area swept by a polar curve: justification



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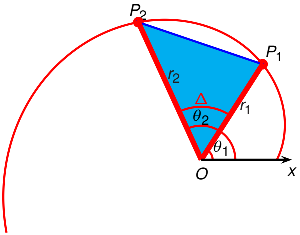
Area swept by a polar curve: justification



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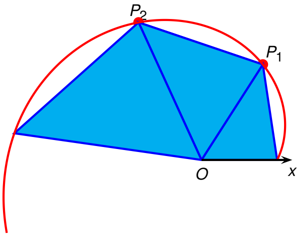
Area swept by a polar curve: justification



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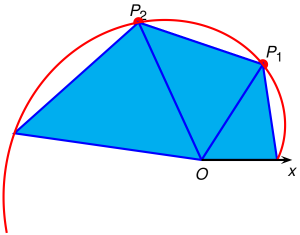
Therefore the area swept by the curve is approximated by the sum:

$$\sum_{i=0}^{N-1} \frac{f(\theta_i)f(\theta_{i+1})\sin\Delta}{2}$$

Area swept by a polar curve: justification



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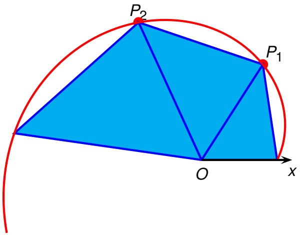
Area swept by a polar curve: justification



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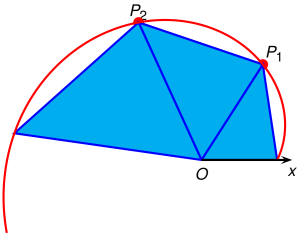
Area swept by a polar curve: justification



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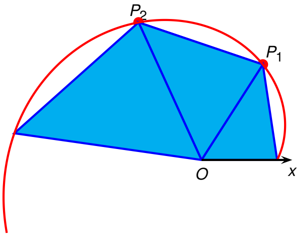
Area swept by a polar curve: justification



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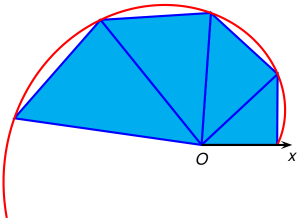
Area swept by a polar curve: justification



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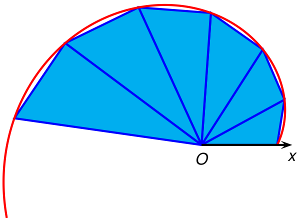
Area swept by a polar curve: justification



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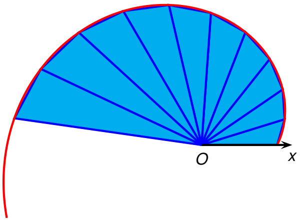
Area swept by a polar curve: justification



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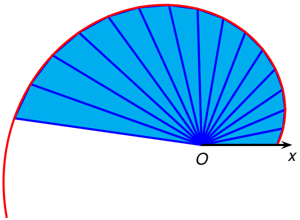
Area swept by a polar curve: justification



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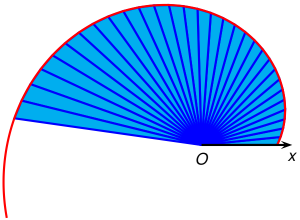
Area swept by a polar curve: justification



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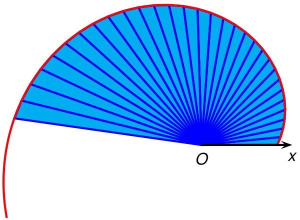
Area swept by a polar curve: justification



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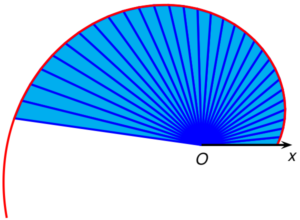
Area swept by a polar curve: justification



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$$
\begin{aligned}
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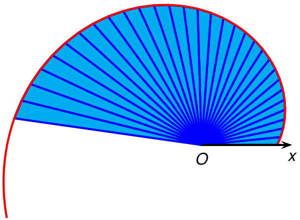
&= \lim_{\Delta\to 0} \frac{\sin\Delta}{\Delta} \lim_{\Delta\to 0} \sum_{i=0}^{N-1} \frac{f(\theta_i)f(\theta_i+\Delta)\Delta}{2} = \mathbf{?} \cdot \lim_{\Delta\to 0} \sum_{i=0}^{N-1} \frac{f(\theta_i)f(\theta_i)\Delta}{2}
\end{aligned}
$$

Area swept by a polar curve: justification



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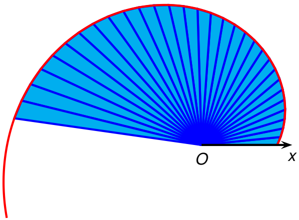
Area swept by a polar curve: justification



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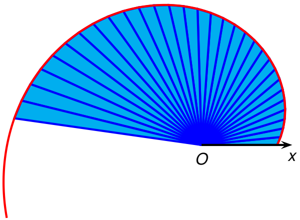
\text{(can be proved)} &= \lim_{\Delta\to 0} \frac{\sin\Delta}{\Delta} \lim_{\Delta\to 0} \sum_{i=0}^{N-1} \frac{f(\theta_i) f(\theta_i + \Delta)\Delta}{2} = 1 \cdot \lim_{\Delta\to 0} \sum_{i=0}^{N-1} \frac{f(\theta_i) f(\theta_i)\Delta}{2}
\end{aligned}
$$

Area swept by a polar curve: justification



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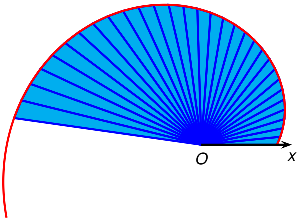
Area swept by a polar curve: justification



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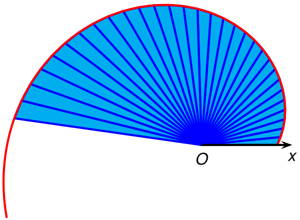
&= \lim_{\Delta\to 0} \sum_{i=0}^{N-1} \frac{f^2(\theta_i)\Delta}{2} = ?
\end{aligned}
$$

Area swept by a polar curve: justification



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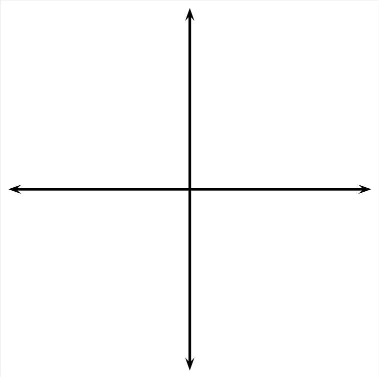
\begin{aligned}
A &= \lim_{\Delta\to 0} \sum_{i=0}^{N-1} \frac{f(\theta_i)f(\theta_{i+1})\sin\Delta}{2} = \lim_{\Delta\to 0} \frac{\sin\Delta}{\Delta} \sum_{i=0}^{N-1} \frac{f(\theta_i)f(\theta_i+\Delta)\Delta}{2} \\
&= \lim_{\Delta\to 0} \frac{\sin\Delta}{\Delta} \lim_{\Delta\to 0} \sum_{i=0}^{N-1} \frac{f(\theta_i)f(\theta_i+\Delta)\Delta}{2} = 1 \cdot \lim_{\Delta\to 0} \sum_{i=0}^{N-1} \frac{f(\theta_i)f(\theta_i)\Delta}{2} \\
\text{(Riemann sum)} &= \lim_{\Delta\to 0} \sum_{i=0}^{N-1} \frac{f^2(\theta_i)\Delta}{2} = \int_a^b \frac{f^2(\theta)}{2} \mathrm{d}\theta
\end{aligned}
$$

Example

Find the area enclosed by one loop of the four-leaved rose $r = \cos 2\theta$.



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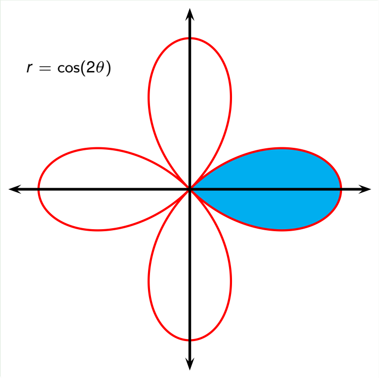

The region enclosed by the right loop corresponds to points whose θ polar coordinate lies in the interval

$\leq \theta \leq$ **?** .

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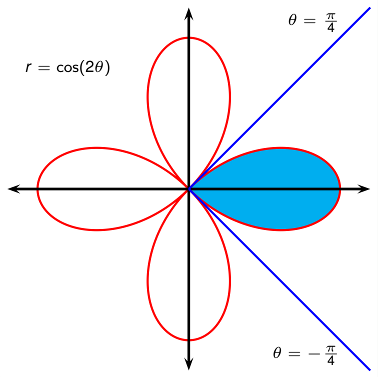
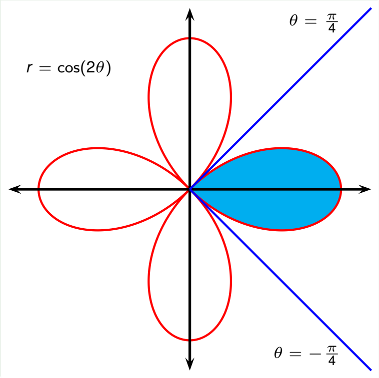


The region enclosed by the right loop corresponds to points whose θ polar coordinate lies in the interval $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$.

Example

Find the area enclosed by one loop of the four-leaved rose $r = \cos 2\theta$.



$$A = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2} r^2 \mathrm{d}\theta$$

The region enclosed by the right loop corresponds to points whose θ polar coordinate lies in the interval $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$.

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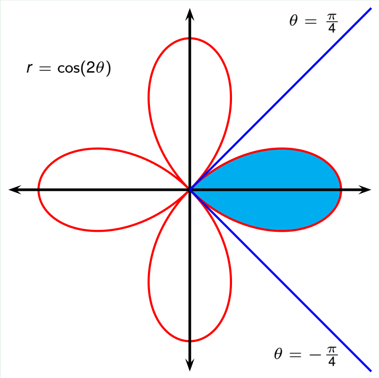

$$\begin{aligned} A &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2} r^2 \mathrm{d}\theta \\ &= \frac{1}{2} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^2(2\theta) \mathrm{d}\theta \end{aligned}$$

The region enclosed by the right loop corresponds to points whose θ polar coordinate lies in the interval $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$.

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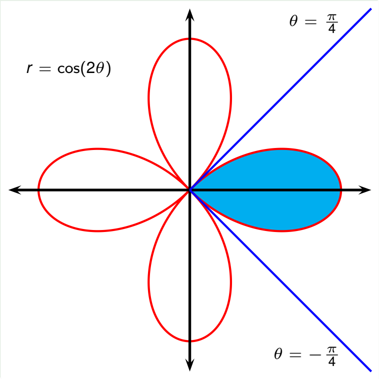
Find the area enclosed by one loop of the four-leaved rose $r = \cos 2\theta$.



$$
\begin{aligned}
A &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2} r^2 d\theta \\
&= \frac{1}{2} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^2(2\theta) d\theta \\
&= \int_{0}^{\frac{\pi}{4}} \cos^2(2\theta) d\theta
\end{aligned}
$$

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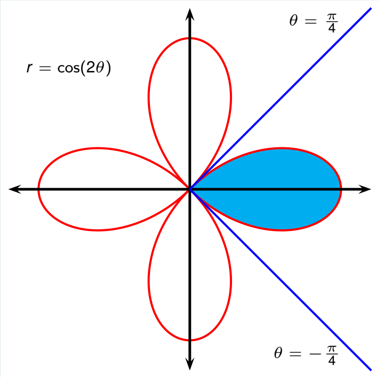

$$\begin{aligned} A &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2} r^2 \mathrm{d}\theta \\ &= \frac{1}{2} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^2(2\theta) \mathrm{d}\theta \\ &= \int_0^{\frac{\pi}{4}} \cos^2(2\theta) \mathrm{d}\theta \\ &= \int_0^{\frac{\pi}{4}} \frac{1}{2}(1 + \cos(4\theta)) \mathrm{d}\theta \end{aligned}$$

The region enclosed by the right loop corresponds to points whose θ polar coordinate lies in the interval $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$.

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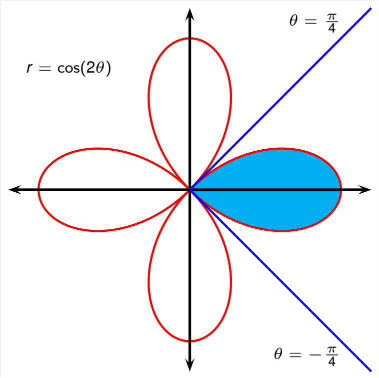
Find the area enclosed by one loop of the four-leaved rose $r = \cos 2\theta$.



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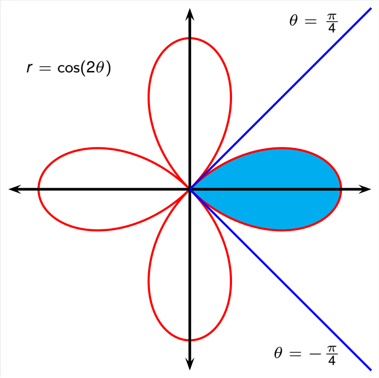

$$
\begin{aligned}
A &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2} r^2 \mathrm{d}\theta \\
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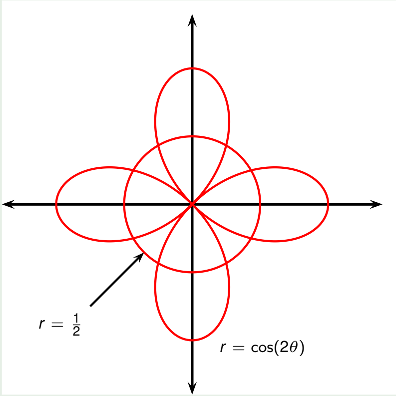
&= \int_{0}^{\frac{\pi}{4}} \cos^2(2\theta) \mathrm{d}\theta \\
&= \int_{0}^{\frac{\pi}{4}} \frac{1}{2}(1 + \cos(4\theta)) \mathrm{d}\theta \\
&= \frac{1}{2}\left[\theta + \frac{1}{4}\sin(4\theta)\right]_0^{\frac{\pi}{4}} \\
&= \frac{\pi}{8}
\end{aligned}
$$

The region enclosed by the right loop corresponds to points whose θ polar coordinate lies in the interval $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$.

Example

Find all points of intersection of the polar curves $r = \frac{1}{2}$ and $r = \cos(2\theta)$.



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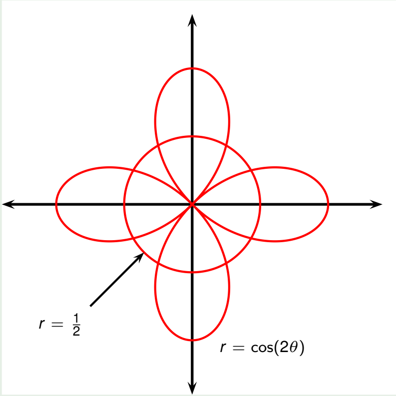

$$\cos 2\theta = \frac{1}{2}$$

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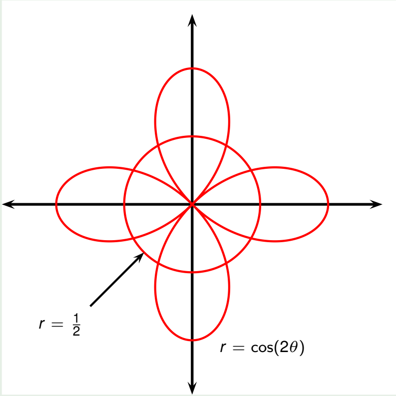

$$\cos 2\theta = \frac{1}{2}$$

$$2\theta = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}$$

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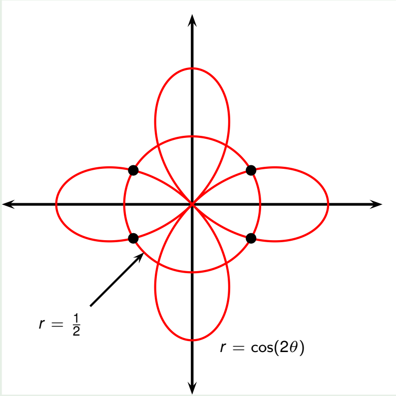

$$\begin{aligned}\cos 2\theta &= \frac{1}{2}\\ 2\theta &= \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}\\ \theta &= \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}\end{aligned}$$

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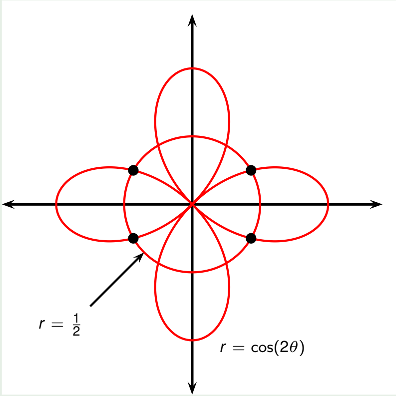

$$\begin{aligned} \cos 2\theta &= \frac{1}{2} \\ 2\theta &= \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3} \\ \theta &= \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6} \end{aligned}$$

- This only gives four points.

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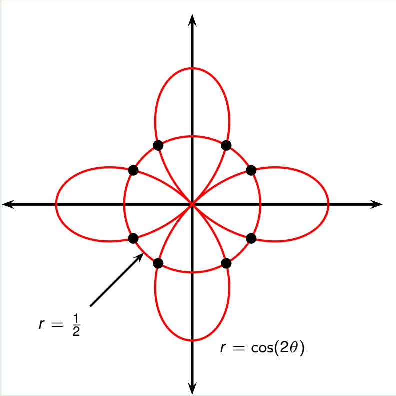

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- There are actually eight.

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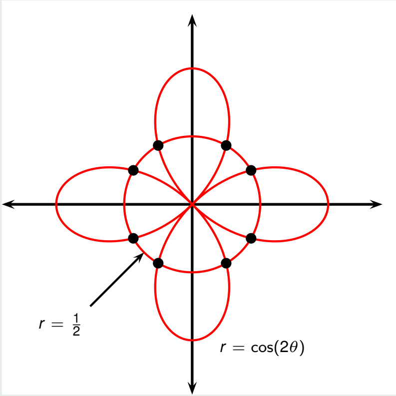

$$\begin{aligned} \cos 2\theta &= \frac{1}{2} \\ 2\theta &= \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3} \\ \theta &= \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6} \end{aligned}$$

- This only gives four points.
- There are actually eight.
- The circle $r = \frac{1}{2}$ also has polar equation $r = -\frac{1}{2}$.

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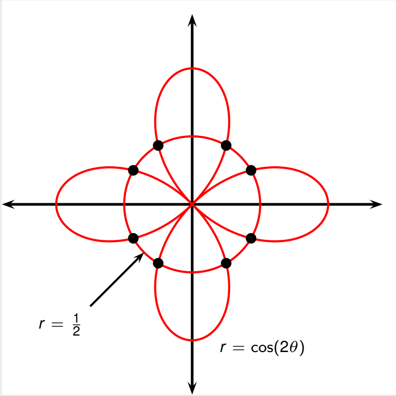
Find all points of intersection of the polar curves $r = \frac{1}{2}$ and $r = \cos(2\theta)$.



$$\begin{aligned} \cos 2\theta &= \frac{1}{2} \\ 2\theta &= \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3} \\ \theta &= \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6} \end{aligned}$$

- This only gives four points.
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- The circle $r = \frac{1}{2}$ also has polar equation $r = -\frac{1}{2}$.
- To find all eight points, solve $\cos(2\theta) = \frac{1}{2}$ and $\cos(2\theta) = -\frac{1}{2}$.

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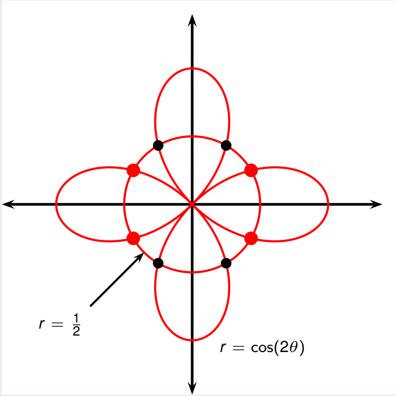

$$\begin{aligned}\cos 2\theta &= \frac{1}{2}\\ 2\theta &= \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}\\ \theta &= \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}\end{aligned}$$

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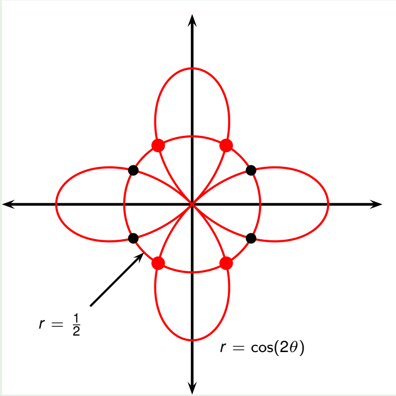

- This only gives four points.
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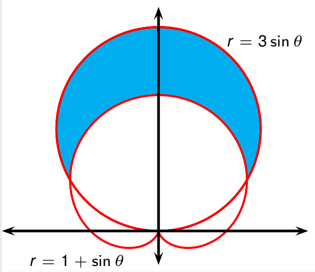
Example

Find the area that lies within the circle $r = 3\sin\theta$ and outside of the cardioid $r = 1 + \sin\theta$.



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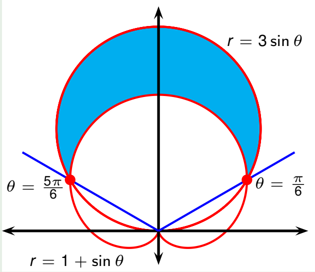

The curves meet if

$$\begin{aligned} 3\sin\theta &= 1 + \sin\theta \\ \sin\theta &= \frac{1}{2} \\ \theta &= \frac{\pi}{6}, \frac{5\pi}{6} \end{aligned}$$

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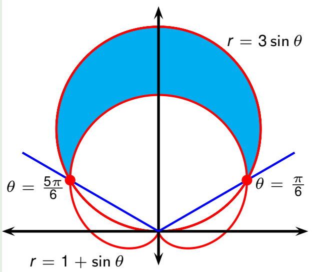

$$A = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (3\sin\theta)^2 d\theta - \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (1+\sin\theta)^2 d\theta$$

The curves meet if

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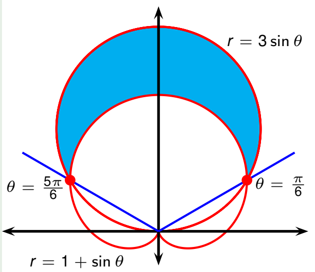

$$
\begin{aligned}
A &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (3\sin\theta)^2 d\theta - \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (1+\sin\theta)^2 d\theta \\
&= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(9\sin^2\theta - (1 + 2\sin\theta + \sin^2\theta)\right) d\theta
\end{aligned}
$$

The curves meet if

$$
\begin{aligned}
3\sin\theta &= 1 + \sin\theta \\
\sin\theta &= \frac{1}{2} \\
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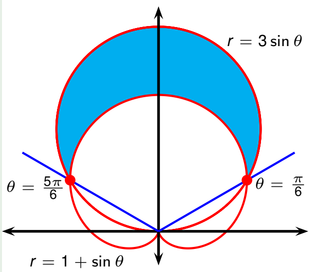

$$
\begin{aligned}
A &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (3\sin\theta)^2 d\theta - \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (1+\sin\theta)^2 d\theta \\
&= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(9\sin^2\theta - (1 + 2\sin\theta + \sin^2\theta)\right) d\theta \\
&= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(8\sin^2\theta - 1 - 2\sin\theta\right) d\theta
\end{aligned}
$$

The curves meet if

$$
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3\sin\theta &= 1 + \sin\theta \\
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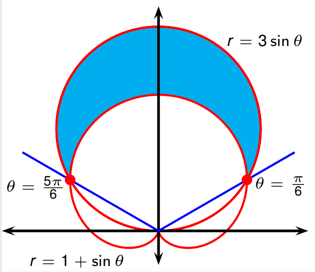

$$
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&= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(8\sin^2\theta - 1 - 2\sin\theta\right) d\theta \\
&= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(8\sin^2\theta - 1 - 2\sin\theta\right) d\theta
\end{aligned}
$$

The curves meet if

$$
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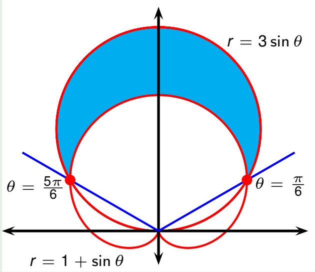


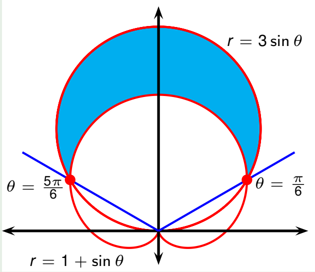
The curves meet if

$$\begin{aligned} 3\sin\theta &= 1 + \sin\theta \\ \sin\theta &= \frac{1}{2} \\ \theta &= \frac{\pi}{6}, \frac{5\pi}{6} \end{aligned}$$

$$\begin{aligned} A &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (3\sin\theta)^2 \mathrm{d}\theta - \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (1+\sin\theta)^2 \mathrm{d}\theta \\ &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(9\sin^2\theta - (1 + 2\sin\theta + \sin^2\theta)\right) \mathrm{d}\theta \\ &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(8\sin^2\theta - 1 - 2\sin\theta\right) \mathrm{d}\theta \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(8\sin^2\theta - 1 - 2\sin\theta\right) \mathrm{d}\theta \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(3 - 4\cos 2\theta - 2\sin\theta\right) \mathrm{d}\theta \end{aligned}$$

Example

Find the area that lies within the circle $r = 3\sin\theta$ and outside of the cardioid $r = 1 + \sin\theta$.



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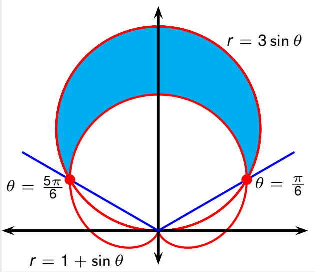
$$\begin{aligned} A &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (3\sin\theta)^2 d\theta - \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (1+\sin\theta)^2 d\theta \\ &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(9\sin^2\theta - (1 + 2\sin\theta + \sin^2\theta)\right) d\theta \\ &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(8\sin^2\theta - 1 - 2\sin\theta\right) d\theta \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(8\sin^2\theta - 1 - 2\sin\theta\right) d\theta \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (3 - 4\cos 2\theta - 2\sin\theta)\, d\theta \\ &= \left[3\theta - 2\sin 2\theta + 2\cos\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} \end{aligned}$$

Example

Find the area that lies within the circle $r = 3\sin\theta$ and outside of the cardioid $r = 1 + \sin\theta$.



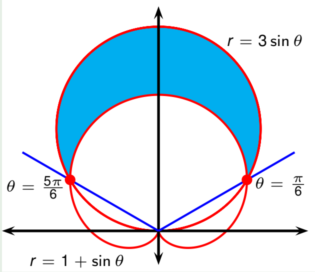
The curves meet if

$$\begin{aligned} 3\sin\theta &= 1 + \sin\theta \\ \sin\theta &= \frac{1}{2} \\ \theta &= \frac{\pi}{6}, \frac{5\pi}{6} \end{aligned}$$

$$\begin{aligned} A &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (3\sin\theta)^2 d\theta - \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (1+\sin\theta)^2 d\theta \\ &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(9\sin^2\theta - (1 + 2\sin\theta + \sin^2\theta)\right) d\theta \\ &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(8\sin^2\theta - 1 - 2\sin\theta\right) d\theta \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(8\sin^2\theta - 1 - 2\sin\theta\right) d\theta \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (3 - 4\cos 2\theta - 2\sin\theta)\, d\theta \\ &= \left[3{\color{red}\theta} - 2\sin 2\theta + 2\cos\theta\right]_{\frac{\pi}{6}}^{\color{red}\frac{\pi}{2}} \\ &= \left(3 \quad - 2\cdot \quad + 2\cdot \quad\right) - \left(3 \quad - 2 \quad + 2 \quad\right) \end{aligned}$$

Example

Find the area that lies within the circle $r = 3\sin\theta$ and outside of the cardioid $r = 1 + \sin\theta$.



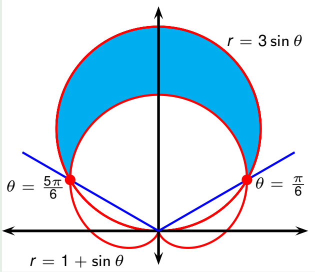
The curves meet if

$$\begin{aligned} 3\sin\theta &= 1 + \sin\theta \\ \sin\theta &= \frac{1}{2} \\ \theta &= \frac{\pi}{6}, \frac{5\pi}{6} \end{aligned}$$

$$\begin{aligned} A &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (3\sin\theta)^2 d\theta - \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (1+\sin\theta)^2 d\theta \\ &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(9\sin^2\theta - (1 + 2\sin\theta + \sin^2\theta)\right) d\theta \\ &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(8\sin^2\theta - 1 - 2\sin\theta\right) d\theta \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(8\sin^2\theta - 1 - 2\sin\theta\right) d\theta \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (3 - 4\cos 2\theta - 2\sin\theta)\, d\theta \\ &= \left[3\theta - 2\sin 2\theta + 2\cos\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} \\ &= \left(3\frac{\pi}{2} - 2\cdot \quad + 2\cdot \quad\right) - \left(3 \quad - 2 \quad + 2 \quad\right) \end{aligned}$$

Example

Find the area that lies within the circle $r = 3\sin\theta$ and outside of the cardioid $r = 1 + \sin\theta$.



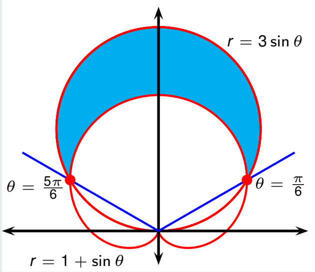
The curves meet if

$$\begin{aligned} 3\sin\theta &= 1 + \sin\theta \\ \sin\theta &= \frac{1}{2} \\ \theta &= \frac{\pi}{6}, \frac{5\pi}{6} \end{aligned}$$

$$\begin{aligned} A &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (3\sin\theta)^2 \mathrm{d}\theta - \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (1+\sin\theta)^2 \mathrm{d}\theta \\ &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(9\sin^2\theta - (1 + 2\sin\theta + \sin^2\theta)\right) \mathrm{d}\theta \\ &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(8\sin^2\theta - 1 - 2\sin\theta\right) \mathrm{d}\theta \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(8\sin^2\theta - 1 - 2\sin\theta\right) \mathrm{d}\theta \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (3 - 4\cos 2\theta - 2\sin\theta) \mathrm{d}\theta \\ &= \left[3\theta - 2\sin 2\theta + 2\cos\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} \\ &= \left(3\frac{\pi}{2} - 2\cdot \quad + 2\cdot \quad\right) - \left(3 \quad - 2 \quad + 2 \quad\right) \end{aligned}$$

Example

Find the area that lies within the circle $r = 3\sin\theta$ and outside of the cardioid $r = 1 + \sin\theta$.



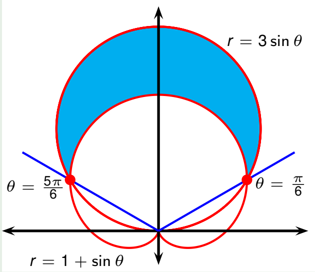
The curves meet if

$$\begin{aligned} 3\sin\theta &= 1 + \sin\theta \\ \sin\theta &= \frac{1}{2} \\ \theta &= \frac{\pi}{6}, \frac{5\pi}{6} \end{aligned}$$

$$\begin{aligned} A &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (3\sin\theta)^2 d\theta - \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (1+\sin\theta)^2 d\theta \\ &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(9\sin^2\theta - (1 + 2\sin\theta + \sin^2\theta)\right) d\theta \\ &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(8\sin^2\theta - 1 - 2\sin\theta\right) d\theta \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(8\sin^2\theta - 1 - 2\sin\theta\right) d\theta \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (3 - 4\cos 2\theta - 2\sin\theta)\, d\theta \\ &= \left[3\theta - 2\color{red}{\sin 2\theta} + 2\cos\theta\right]_{\frac{\pi}{6}}^{\color{red}{\frac{\pi}{2}}} \\ &= \left(3\frac{\pi}{2} - 2\cdot\color{red}{0} + 2\cdot\quad\right) - \left(3\quad - 2\quad + 2\quad\right) \end{aligned}$$

Example

Find the area that lies within the circle $r = 3\sin\theta$ and outside of the cardioid $r = 1 + \sin\theta$.



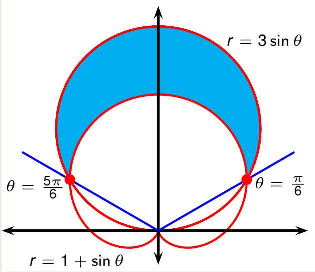
The curves meet if

$$\begin{aligned} 3\sin\theta &= 1 + \sin\theta \\ \sin\theta &= \frac{1}{2} \\ \theta &= \frac{\pi}{6}, \frac{5\pi}{6} \end{aligned}$$

$$\begin{aligned} A &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (3\sin\theta)^2 d\theta - \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (1 + \sin\theta)^2 d\theta \\ &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(9\sin^2\theta - (1 + 2\sin\theta + \sin^2\theta)\right) d\theta \\ &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(8\sin^2\theta - 1 - 2\sin\theta\right) d\theta \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(8\sin^2\theta - 1 - 2\sin\theta\right) d\theta \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (3 - 4\cos 2\theta - 2\sin\theta) d\theta \\ &= \left[3\theta - 2\sin 2\theta + 2\cos\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} \\ &= \left(3\frac{\pi}{2} - 2\cdot 0 + 2\cdot \quad\right) - \left(3 \quad - 2 \quad + 2 \quad\right) \end{aligned}$$

Example

Find the area that lies within the circle $r = 3\sin\theta$ and outside of the cardioid $r = 1 + \sin\theta$.



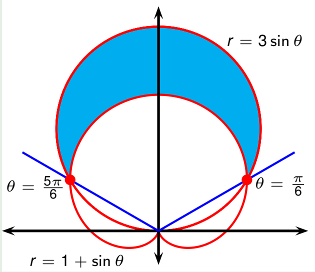
The curves meet if

$$\begin{aligned} 3\sin\theta &= 1 + \sin\theta \\ \sin\theta &= \frac{1}{2} \\ \theta &= \frac{\pi}{6}, \frac{5\pi}{6} \end{aligned}$$

$$\begin{aligned} A &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (3\sin\theta)^2 d\theta - \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (1+\sin\theta)^2 d\theta \\ &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(9\sin^2\theta - (1 + 2\sin\theta + \sin^2\theta)\right) d\theta \\ &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(8\sin^2\theta - 1 - 2\sin\theta\right) d\theta \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(8\sin^2\theta - 1 - 2\sin\theta\right) d\theta \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (3 - 4\cos 2\theta - 2\sin\theta)\, d\theta \\ &= \left[3\theta - 2\sin 2\theta + 2{\color{red}\cos\theta}\right]_{\frac{\pi}{6}}^{\color{red}\frac{\pi}{2}} \\ &= \left(3\frac{\pi}{2} - 2\cdot 0 + 2\cdot {\color{red}0}\right) - \Big(3 \quad - 2 \quad\quad + 2 \quad\quad\Big) \end{aligned}$$

Example

Find the area that lies within the circle $r = 3\sin\theta$ and outside of the cardioid $r = 1 + \sin\theta$.



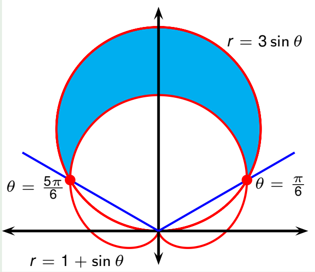
The curves meet if

$$\begin{aligned} 3\sin\theta &= 1 + \sin\theta \\ \sin\theta &= \frac{1}{2} \\ \theta &= \frac{\pi}{6}, \frac{5\pi}{6} \end{aligned}$$

$$\begin{aligned} A &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (3\sin\theta)^2 d\theta - \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (1+\sin\theta)^2 d\theta \\ &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(9\sin^2\theta - (1 + 2\sin\theta + \sin^2\theta)\right) d\theta \\ &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(8\sin^2\theta - 1 - 2\sin\theta\right) d\theta \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(8\sin^2\theta - 1 - 2\sin\theta\right) d\theta \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (3 - 4\cos 2\theta - 2\sin\theta) d\theta \\ &= [3\theta - 2\sin 2\theta + 2\cos\theta]_{\frac{\pi}{6}}^{\frac{\pi}{2}} \\ &= \left(3\frac{\pi}{2} - 2\cdot 0 + 2\cdot 0\right) - \left(3 \quad - 2 \quad + 2 \quad\right) \end{aligned}$$

Example

Find the area that lies within the circle $r = 3\sin\theta$ and outside of the cardioid $r = 1 + \sin\theta$.



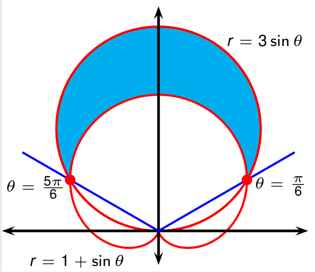
The curves meet if

$$\begin{aligned} 3\sin\theta &= 1 + \sin\theta \\ \sin\theta &= \frac{1}{2} \\ \theta &= \frac{\pi}{6}, \frac{5\pi}{6} \end{aligned}$$

$$\begin{aligned} A &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (3\sin\theta)^2 \mathrm{d}\theta - \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (1+\sin\theta)^2 \mathrm{d}\theta \\ &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(9\sin^2\theta - (1 + 2\sin\theta + \sin^2\theta)\right) \mathrm{d}\theta \\ &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(8\sin^2\theta - 1 - 2\sin\theta\right) \mathrm{d}\theta \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(8\sin^2\theta - 1 - 2\sin\theta\right) \mathrm{d}\theta \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (3 - 4\cos 2\theta - 2\sin\theta)\, \mathrm{d}\theta \\ &= \left[3\theta - 2\sin 2\theta + 2\cos\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} \\ &= \left(3\frac{\pi}{2} - 2\cdot 0 + 2\cdot 0\right) - \left(3\frac{\pi}{6} - 2 \quad\quad + 2 \quad\quad\right) \end{aligned}$$

Example

Find the area that lies within the circle $r = 3\sin\theta$ and outside of the cardioid $r = 1 + \sin\theta$.



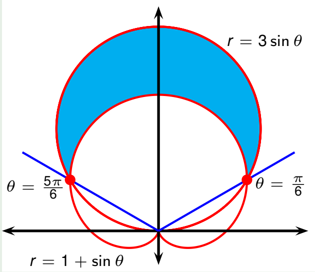
The curves meet if

$$\begin{aligned} 3\sin\theta &= 1 + \sin\theta \\ \sin\theta &= \frac{1}{2} \\ \theta &= \frac{\pi}{6}, \frac{5\pi}{6} \end{aligned}$$

$$\begin{aligned} A &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (3\sin\theta)^2 d\theta - \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (1+\sin\theta)^2 d\theta \\ &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(9\sin^2\theta - (1 + 2\sin\theta + \sin^2\theta)\right) d\theta \\ &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(8\sin^2\theta - 1 - 2\sin\theta\right) d\theta \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(8\sin^2\theta - 1 - 2\sin\theta\right) d\theta \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (3 - 4\cos 2\theta - 2\sin\theta)\, d\theta \\ &= \left[3\theta - 2\sin 2\theta + 2\cos\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} \\ &= \left(3\frac{\pi}{2} - 2\cdot 0 + 2\cdot 0\right) - \left(3\frac{\pi}{6} - 2 \qquad + 2 \qquad\right) \end{aligned}$$

Example

Find the area that lies within the circle $r = 3\sin\theta$ and outside of the cardioid $r = 1+\sin\theta$.



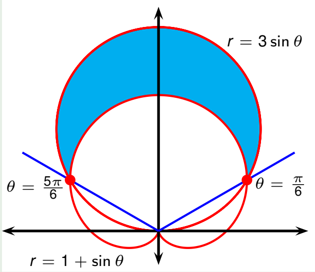
The curves meet if

$$\begin{aligned} 3\sin\theta &= 1+\sin\theta \\ \sin\theta &= \frac{1}{2} \\ \theta &= \frac{\pi}{6}, \frac{5\pi}{6} \end{aligned}$$

$$\begin{aligned} A &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (3\sin\theta)^2 d\theta - \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (1+\sin\theta)^2 d\theta \\ &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(9\sin^2\theta - (1+2\sin\theta+\sin^2\theta)\right) d\theta \\ &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(8\sin^2\theta - 1 - 2\sin\theta\right) d\theta \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(8\sin^2\theta - 1 - 2\sin\theta\right) d\theta \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (3 - 4\cos 2\theta - 2\sin\theta)\, d\theta \\ &= \left[3\theta - 2\sin 2\theta + 2\cos\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} \\ &= \left(3\frac{\pi}{2} - 2\cdot 0 + 2\cdot 0\right) - \left(3\frac{\pi}{6} - 2\frac{\sqrt{3}}{2} + 2 \qquad \right) \end{aligned}$$

Example

Find the area that lies within the circle $r = 3\sin\theta$ and outside of the cardioid $r = 1 + \sin\theta$.



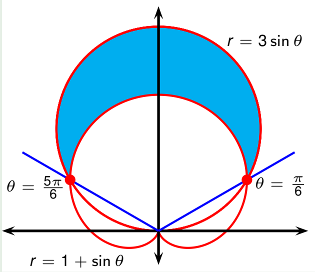
The curves meet if

$$\begin{aligned} 3\sin\theta &= 1 + \sin\theta \\ \sin\theta &= \frac{1}{2} \\ \theta &= \frac{\pi}{6}, \frac{5\pi}{6} \end{aligned}$$

$$\begin{aligned} A &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (3\sin\theta)^2 \mathrm{d}\theta - \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (1+\sin\theta)^2 \mathrm{d}\theta \\ &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(9\sin^2\theta - (1 + 2\sin\theta + \sin^2\theta)\right) \mathrm{d}\theta \\ &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(8\sin^2\theta - 1 - 2\sin\theta\right) \mathrm{d}\theta \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(8\sin^2\theta - 1 - 2\sin\theta\right) \mathrm{d}\theta \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (3 - 4\cos 2\theta - 2\sin\theta)\, \mathrm{d}\theta \\ &= \left[3\theta - 2\sin 2\theta + 2\cos\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} \\ &= \left(3\frac{\pi}{2} - 2\cdot 0 + 2\cdot 0\right) - \left(3\frac{\pi}{6} - 2\frac{\sqrt{3}}{2} + 2 \qquad\right) \end{aligned}$$

Example

Find the area that lies within the circle $r = 3\sin\theta$ and outside of the cardioid $r = 1 + \sin\theta$.



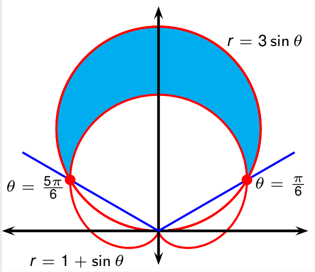
The curves meet if

$$\begin{aligned} 3\sin\theta &= 1 + \sin\theta \\ \sin\theta &= \frac{1}{2} \\ \theta &= \frac{\pi}{6}, \frac{5\pi}{6} \end{aligned}$$

$$\begin{aligned} A &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (3\sin\theta)^2 d\theta - \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (1+\sin\theta)^2 d\theta \\ &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(9\sin^2\theta - (1 + 2\sin\theta + \sin^2\theta)\right) d\theta \\ &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(8\sin^2\theta - 1 - 2\sin\theta\right) d\theta \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(8\sin^2\theta - 1 - 2\sin\theta\right) d\theta \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (3 - 4\cos 2\theta - 2\sin\theta)\, d\theta \\ &= \left[3\theta - 2\sin 2\theta + 2\cos\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} \\ &= \left(3\frac{\pi}{2} - 2\cdot 0 + 2\cdot 0\right) - \left(3\frac{\pi}{6} - 2\frac{\sqrt{3}}{2} + 2\frac{\sqrt{3}}{2}\right) \end{aligned}$$

Example

Find the area that lies within the circle $r = 3\sin\theta$ and outside of the cardioid $r = 1 + \sin\theta$.



The curves meet if

$$\begin{aligned} 3\sin\theta &= 1+\sin\theta \\ \sin\theta &= \frac{1}{2} \\ \theta &= \frac{\pi}{6}, \frac{5\pi}{6} \end{aligned}$$

$$\begin{aligned} A &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (3\sin\theta)^2 d\theta - \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (1+\sin\theta)^2 d\theta \\ &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(9\sin^2\theta - (1 + 2\sin\theta + \sin^2\theta)\right) d\theta \\ &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(8\sin^2\theta - 1 - 2\sin\theta\right) d\theta \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(8\sin^2\theta - 1 - 2\sin\theta\right) d\theta \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (3 - 4\cos 2\theta - 2\sin\theta)\, d\theta \\ &= \left[3\theta - 2\sin 2\theta + 2\cos\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} \\ &= \left(3\frac{\pi}{2} - 2\cdot 0 + 2\cdot 0\right) - \left(3\frac{\pi}{6} - 2\frac{\sqrt{3}}{2} + 2\frac{\sqrt{3}}{2}\right) \\ &= \pi \end{aligned}$$