

Calculus II

Weierstrass substitution theory

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Integrals of the form $\int R(\cos \theta, \sin \theta) d\theta$, R

Let R be an arbitrary rational function in two variables (quotient of polynomials in two variables).

Question

Can we integrate $\int R(\cos \theta, \sin \theta) d\theta$?

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 - Apply the substitution $\theta = 2 \arctan t$ to transform to integral of rational function.

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Question

Can we integrate $\int R(\cos \theta, \sin \theta) d\theta$?

- Yes. We will learn how in what follows.
- The algorithm for integration is roughly:
 - Apply the substitution $\theta = 2 \arctan t$ to transform to integral of rational function.
 - Solve as previously studied.

The rationalizing substitution $\theta = 2 \arctan t$

Recall the expression of $\sin(2z)$, $\cos(2z)$ via $\tan z$:

$$\sin(2z) = ?$$

$$\cos(2z)$$

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$$\cos(2z) = \cos^2 z - \sin^2 z$$

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Let R - rational function in two variables. $\int R(\cos \theta, \sin \theta) d\theta$ can be integrated via the substitution $\theta = 2 \arctan t$. How does this transform $\sin \theta, \cos \theta$?

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$$d\theta$$

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Theorem

The substitution given above transforms $\int R(\cos \theta, \sin \theta) d\theta$ to an integral of a rational function of t .

The integral $\int \sec \theta d\theta$ appears often in practice. A quicker solution will be shown later, but first we show the standard method.

Example

$$\int \sec \theta d\theta$$

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$$\int \sec \theta d\theta = \int \frac{1}{\cos \theta} d\theta$$

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Set $\theta = 2 \arctan t$, $\cos \theta = \frac{1 - \tan^2(\frac{\theta}{2})}{1 + \tan^2(\frac{\theta}{2})} = \frac{1 - t^2}{1 + t^2}$.

$$\int \sec \theta d\theta = \int \frac{1}{\cos \theta} d\theta = \int \frac{1}{\left(\frac{1-t^2}{1+t^2}\right)} \frac{2}{(1+t^2)} dt$$

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$$\begin{aligned} \int \sec \theta d\theta &= \int \frac{1}{\cos \theta} d\theta = \int \frac{1}{\left(\frac{1-t^2}{1+t^2}\right)} \frac{2}{\cancel{1+t^2}} dt \\ &= \int \frac{2}{1-t^2} dt \end{aligned}$$

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