Precalculus
Euler's formula memorization

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Euler's Formula

Theorem (Euler's Formula)

$$e^{ix} = \cos x + i \sin x,$$

where $e \approx 2.71828$ is Euler's/Napier's constant .

Proof.

Recall $n! = 1 \cdot 2 \cdot 3 \cdot \dots \cdot (n-1) \cdot n$. Borrow from Calc II the f-las:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + \frac{(-1)^n x^{2n}}{(2n)!} + \dots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots$$

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Proof.

Recall $n! = 1 \cdot 2 \cdot 3 \cdot \dots \cdot (n-1) \cdot n$. Borrow from Calc II the f-las:

$$\begin{array}{rclcccccccc}
i \sin x & = & & ix & & -i\frac{x^3}{3!} & & +i\frac{x^5}{5!} & -\dots \\
\cos x & = & 1 & & -\frac{x^2}{2!} & & +\frac{x^4}{4!} & & +\dots \\
\hline
e^{ix} & = & 1 & +ix & -\frac{x^2}{2!} & -i\frac{x^3}{3!} & +\frac{x^4}{4!} & +i\frac{x^5}{5!} & -\dots
\end{array}$$

Rearrange. Plug-in $z = ix$. Use $i^2 = -1$. Multiply $\sin x$ by i. Add to get $e^{ix} = \cos x + i \sin x$. $\square$

Trigonometric Identities Revisited

- $e^{ix} = \cos x + i \sin x$ (Euler's Formula).
- $e^{ix} e^{iy} = e^{ix+iy} = e^{i(x+y)}$ (exponentiation rule: valid for $\mathbb{C}$).
- $e^0 = 1$ (exponentiation rule).
- $\sin(-x) = -\sin x$, $\cos(-x) = \cos x$ (easy to remember).

Example

$$\begin{aligned} \sin(x+y) &= \sin x \cos y + \sin y \cos x \\ \cos(x+y) &= \cos x \cos y - \sin x \sin y \quad . \end{aligned}$$

Proof.

$$\begin{aligned} e^{i(x+y)} &= \cos(x+y) + i \sin(x+y) \\ e^{ix} e^{iy} &= \cos(x+y) + i \sin(x+y) \\ (\cos x + i \sin x)(\cos y + i \sin y) &= \cos(x+y) + i \sin(x+y) \\ \cos x \cos y - \sin x \sin y + i(\sin x \cos y + \sin y \cos x) &= \cos(x+y) + i \sin(x+y) \end{aligned}$$

Compare coefficient in front of i and remaining terms to get the desired equalities. $\square$

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Example

$$\sin^2 x + \cos^2 x = 1$$

Proof.

$$\begin{aligned} 1 &= e^0 \\ &= e^{ix-ix} = e^{ix} e^{-ix} = (\cos x + i \sin x)(\cos(-x) + i \sin(-x)) \\ &= (\cos x + i \sin x)(\cos x - i \sin x) = \cos^2 x - i^2 \sin^2 x \\ &= \cos^2 x + \sin^2 x \quad . \end{aligned}$$

□

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Example

$$\begin{aligned} \sin(2x) &= 2 \sin x \cos x \\ \cos(2x) &= \cos^2 x - \sin^2 x \quad . \end{aligned}$$

Proof.

$$\begin{aligned} e^{i(2x)} &= \cos(2x) + i \sin(2x) \\ e^{ix} e^{ix} &= \cos(2x) + i \sin(2x) \\ (\cos x + i \sin x)^2 = (\cos x + i \sin x)(\cos x + i \sin x) &= \cos(2x) + i \sin(2x) \\ \cos^2 x - \sin^2 x + i(2 \sin x \cos x) &= \cos(2x) + i \sin(2x) \end{aligned}$$

Compare coefficient in front of i and remaining terms to get the desired equalities. $\square$