Calculus II

Integrals of the form $\displaystyle\int \frac{Ax+B}{ax^2+bx+c}\mathrm{d}x$, denominator has no real roots

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Linear substitutions leading to blocks IIa and IIIa

Building block IIa: $\int \frac{x}{1+x^2} \mathrm{d}x = \frac{1}{2} \ln(1+x^2) + C$.

Building block IIIa: $\int \frac{1}{1+x^2} \mathrm{d}x = \arctan x + C$.

- Let $ax^2 + bx + c$ have no real roots.

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- Let $ax^2 + bx + c$ have no real roots.
- We can find p, q so that the linear substitution $u = px + q$ transforms the quadratic to:

$$ax^2 + bx + c = r(u^2 + 1)$$

  (where r is some number to be determined).

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- In this way, integrals of the form $\displaystyle\int \frac{Ax + B}{ax^2 + bx + c}\mathrm{d}x$ are transformed to combinations of building blocks IIa and IIIa.
- We show examples; the general case is analogous and we leave it to the student.

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$$\int \frac{x}{x^2 + x + 1} \mathrm{d}x =$$

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No real roots $\Rightarrow$ complete the square.

$$\int \frac{x}{x^2+x+1} \mathrm{d}x = \int \frac{x}{x^2 + 2\cdot\frac{1}{2}x + \frac{1}{4} - \frac{1}{4} + 1} \mathrm{d}x$$

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&= \int \frac{\color{red}{u} - \frac{1}{2}}{u^2 + \frac{3}{4}}\mathrm{d}u \\
&= \int \frac{\color{red}{u}}{u^2 + \frac{3}{4}}\mathrm{d}u - \frac{1}{2}\int \frac{1}{u^2+\frac{3}{4}}\mathrm{d}u
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Linear substitutions leading to blocks IIa and IIIa

Building block IIa: $\int \frac{x}{1+x^2}\mathrm{d}x = \frac{1}{2}\ln(1+x^2) + C$.
Building block IIIa: $\int \frac{1}{1+x^2}\mathrm{d}x = \arctan x + C$.

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