Calculus I
Limits

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Outline

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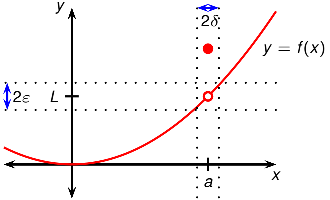
The Limit of a Function

Definition (The Limit of a Function)

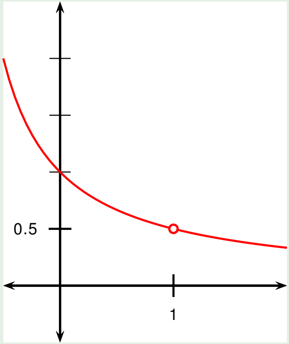
We write

$$\lim_{x\to a} f(x) = L$$

and say "the limit of $f(x)$, as x approaches a, equals L," if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a (on either side of a) but not equal to a.
Equivalent formulation. $\lim_{x\to a} f(x) = L$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - L| < \varepsilon$ for all x with $0 < |x - a| < \delta$.



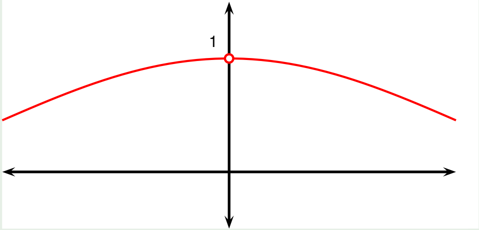
Example



- Guess the value of $\lim_{x \to 1} \frac{x-1}{x^2-1}$.
- Notice that $\frac{x-1}{x^2-1}$ is not defined at 1.
- It is defined for values of x near 1.
- We guess that the limit is 0.5.
- In this case, our guess is correct.

x	$f(x)$	x	$f(x)$
0.5	0.666667	1.5	0.400000
0.9	0.526316	1.1	0.476190
0.99	0.502513	1.01	0.497512
0.999	0.500250	1.001	0.499750
0.9999	0.500025	1.0001	0.499975

Example



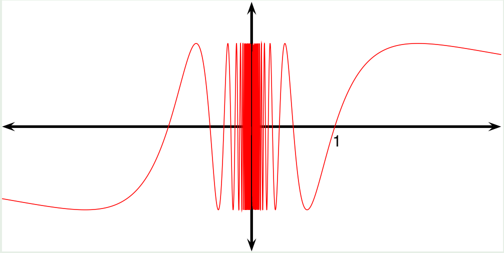
- Guess the value of $\lim_{x\to 0} \frac{\sin x}{x}$.
- Notice that $\frac{\sin x}{x}$ is not defined at 0.
- It is defined for all other values of x near 0.
- We guess that the limit is 1.
- In this case, our guess is correct.

x	$f(x)$	x	$f(x)$
± 1.0	0.841471	± 0.1	0.998334
± 0.5	0.958851	± 0.05	0.999583
± 0.4	0.973546	± 0.01	0.999983
± 0.3	0.985067	± 0.005	0.999995
± 0.2	0.993347	± 0.001	0.999999

Example

- Guess the value of $\lim_{x\to 0} \sin\left(\frac{\pi}{x}\right)$.
- Notice that $\sin\left(\frac{\pi}{x}\right)$ is not defined at 0.
- It is defined for values of x near 0.
- We may guess that the limit is 0.
- Such a guess would be wrong.

x	$f(x)$	x	$f(x)$
1	$\sin \pi = 0$	$\frac{1}{2}$	$\sin(2\pi) = 0$
$\frac{1}{3}$	$\sin(3\pi) = 0$	$\frac{1}{4}$	$\sin(4\pi) = 0$
0.1	$\sin(10\pi) = 0$	0.01	$\sin(100\pi) = 0$

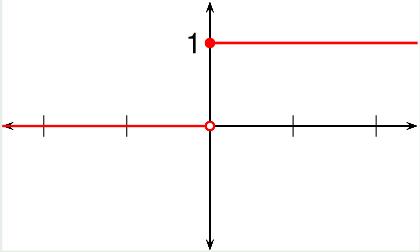


One-sided Limits

Example

The Heaviside function H is defined by

$$H(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases} .$$



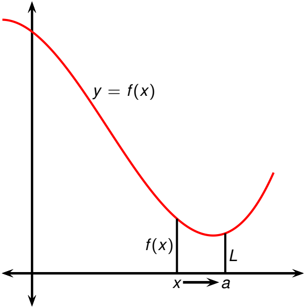
- As x approaches 0 from the left, $H(x)$ approaches 0.
- As x approaches 0 from the right, $H(x)$ approaches 1.
- There is no single number that $H(x)$ approaches as x approaches 0.
- Therefore $\lim\limits_{x \to 0} H(x)$ doesn't exist.

Definition (Left-hand Limit)

We write

$$\lim_{x \to a^-} f(x) = L \qquad \text{or} \qquad \lim_{\substack{x \to a \\ x < a}} f(x) = L$$

and say the left-hand limit of $f(x)$ as x approaches a is equal to L if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to and less than a.

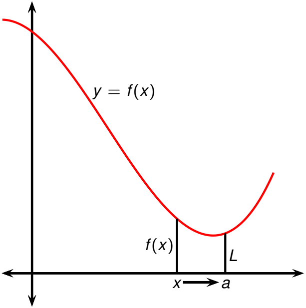


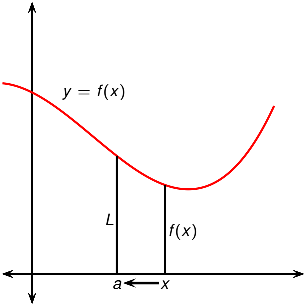
Definition (Right-hand Limit)

We write

$$\lim_{x \to a^+} f(x) = L \qquad \text{or} \qquad \lim_{\substack{x \to a \\ x > a}} f(x) = L$$

and say the right-hand limit of $f(x)$ as x approaches a is equal to L if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to and greater than a.





We can define a right-hand limit similarly.

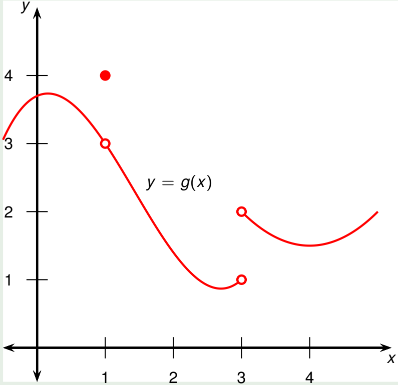
By comparing definitions, we can see that

$$\lim_{x\to a} f(x) = L \text{ if and only if } \lim_{x\to a^-} f(x) = L \text{ and } \lim_{x\to a^+} f(x) = L.$$

Example

The graph of a function g is shown to the right. Use it to state the values (if they exist) of the following:

$$\begin{array}{r|r}
\lim_{x\to 1^-} g(x) = 3 & \lim_{x\to 3^-} g(x) = 1 \\
\lim_{x\to 1^+} g(x) = 3 & \lim_{x\to 3^+} g(x) = 2 \\
\lim_{x\to 1} g(x) = 3 & \lim_{x\to 3} g(x) = \text{DNE}
\end{array}$$



Calculating Limits Using Limit Laws

Theorem (Limit Laws)

*Suppose that c is a constant and that the limits $\lim\limits_{x\to a} f(x)$ and $\lim\limits_{x\to a} g(x)$ exist ($\pm\infty$ **not allowed**). Then*

1. $\lim\limits_{x\to a}[f(x)+g(x)] = \lim\limits_{x\to a} f(x) + \lim\limits_{x\to a} g(x)$.
2. $\lim\limits_{x\to a}[f(x)-g(x)] = \lim\limits_{x\to a} f(x) - \lim\limits_{x\to a} g(x)$.
3. $\lim\limits_{x\to a}[cf(x)] = c\lim\limits_{x\to a} f(x)$.
4. $\lim\limits_{x\to a}[f(x)g(x)] = \lim\limits_{x\to a} f(x)\cdot \lim\limits_{x\to a} g(x)$.
5. $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}$ *if* $\lim\limits_{x\to a} g(x) \neq 0$.

Here are some other useful limit laws:

6. $\lim\limits_{x \to a} [f(x)]^n = [\lim\limits_{x \to a} f(x)]^n$
7. $\lim\limits_{x \to a} c = c$.
8. $\lim\limits_{x \to a} x = a$.
9. $\lim\limits_{x \to a} x^n = a^n$.
10. $\lim\limits_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$, if $a > 0$.
11. $\lim\limits_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x \to a} f(x)}$, if $\lim_{x \to a} f(x) > 0$.

Example

Evaluate the limit and justify each step:

$$
\begin{aligned}
& \lim_{x\to 5}(2x^2 - 3x + 4) \\
=\ & \lim_{x\to 5}(2x^2 - 3x) + \lim_{x\to 5} 4 && \text{Law} \\
=\ & \lim_{x\to 5}(2x^2) - \lim_{x\to 5}(3x) + \lim_{x\to 5} 4 && \text{Law} \\
=\ & 2\lim_{x\to 5} x^2 - 3\lim_{x\to 5} x + \lim_{x\to 5} 4 && \text{Law} \\
=\ & 2\cdot 5^2 - 3\cdot 5 + 4 && \text{Laws} \\
=\ & 39.
\end{aligned}
$$

Example (Limit Laws)

Evaluate the limit and justify each step:

$$\lim_{x\to 3} \frac{x+2}{\sqrt{x-1}(x+1)^2}$$

$$= \frac{\lim_{x\to 3}(x+2)}{\lim_{x\to 3}\left(\sqrt{x-1}(x+1)^2\right)} \qquad \text{Law}$$

$$= \frac{\lim_{x\to 3}(x+2)}{\lim_{x\to 3}\sqrt{x-1}\cdot\lim_{x\to 3}\left((x+1)^2\right)} \qquad \text{Law}$$

$$= \frac{\lim_{x\to 3}(x+2)}{\sqrt{\lim_{x\to 3}(x-1)}\left(\lim_{x\to 3}(x+1)\right)^2} \qquad \text{Laws}$$

$$= \frac{\lim_{x\to 3}x+\lim_{x\to 3}2}{\sqrt{\lim_{x\to 3}x-\lim_{x\to 3}1}\left(\lim_{x\to 3}x+\lim_{x\to 3}1\right)^2} \qquad \text{Laws}$$

$$= \frac{3+2}{\sqrt{3-1}\,(3+1)^2} = \frac{5}{16\sqrt{2}}. \qquad \text{Laws}$$

Recall that every function which can be using the four arithmetic operations $(+, -, *, /)$ and radicals $\sqrt[n]{\ }$ is an algebraic function.

Theorem (Direct Substitution)

Let f be an algebraic function. Let the point a be in its domain (i.e., $f(a)$ is defined). Then $\lim\limits_{x\to a} f(x) = f(a)$.

This theorem is a partial case of the following theorem.

Theorem (Can be taken as definition)

Let f be a continuous function. Let the point a be in its domain (i.e., $f(a)$ is defined). Then $\lim\limits_{x\to a} f(x) = f(a)$.

Continuous functions will be defined later in this lecture.

Example (Limit with Direct Substitution)

$$\text{Find} \quad \lim_{x\to 3} \frac{x+2}{\sqrt{x-1}(x+1)^2}$$

$$\text{Plug in 3:} \quad \frac{(3)+2}{\sqrt{(3)-1}((3)+1)^2} = \frac{5}{16\sqrt{2}}$$

$$\text{Therefore} \quad \lim_{x\to 3} \frac{x+2}{\sqrt{x-1}(x+1)^2} = \frac{5}{16\sqrt{2}}.$$

Example (Limit in Which Direct Substitution Doesn't Work)

$$\text{Find} \quad \lim_{x\to 3} \frac{x^3 - 3x^2 + x - 3}{x^2 - 7x + 12}$$

$$\text{Plug in 3:} \quad \frac{(3)^3 - 3(3)^2 + (3) - 3}{(3)^2 - 7(3) + 12} = \frac{0}{0}$$

Zero over zero is undefined, so we can't use direct substitution.

When computing a limit as x approaches a, we don't care what happens when $x = a$. This gives the following useful fact:

$$\text{If} \quad f(x) = g(x)$$

when $x \neq a$,

$$\text{then} \quad \lim_{x\to a} f(x) = \lim_{x\to a} g(x),$$

provided the limit exists.

We can use this fact to find $\lim_{x\to a} f(x)$ when $f(a)$ has the form $\frac{0}{0}$. In such a case, we use algebra to find a function $g(x)$ that agrees with $f(x)$ at all points except $x = a$. Here are some common techniques.

1. Factoring.
2. Using a conjugate radical.
3. Finding a common denominator.
4. Using Taylor/Maclaurin series expansion. Studied in Calc II.

Example (Limit with Factoring)

$$\text{Find} \quad \lim_{x\to 3} \frac{x^3 - 3x^2 + x - 3}{x^2 - 7x + 12}$$

$$\text{Plug in 3:} \quad \frac{(3)^3 - 3(3)^2 + (3) - 3}{(3)^2 - 7(3) + 12} = -$$

Zero over zero is undefined, so we can't use direct substitution.

$$\begin{aligned}
\text{Factor:} \quad \lim_{x\to 3} \frac{x^3 - 3x^2 + x - 3}{x^2 - 7x + 12} &= \lim_{x\to 3} \underline{\qquad\qquad\qquad} \\
&= \lim_{x\to 3} \frac{x^2 + 1}{x - 4} \\
\text{Plug in 3:} \quad \lim_{x\to 3} \frac{x^3 - 3x^2 + x - 3}{x^2 - 7x + 12} &= \frac{(3)^2 + 1}{(3) - 4} \\
&= \frac{10}{-1} \\
&= -10.
\end{aligned}$$

Example

$$\text{Find} \quad \lim_{t\to 0} \frac{\sqrt{t^2+9}-3}{t^2}$$

$$\text{Plug in 0:} \quad \frac{\sqrt{(0)^2+9}-3}{(0)^2} = \frac{0}{0}$$

Zero over zero is undefined, so we can't use direct substitution.
Multiply top & bottom by (minus) the conjugate radical:

$$\begin{aligned}
\lim_{t\to 0} \frac{\sqrt{t^2+9}-3}{t^2} &= \lim_{t\to 0} \frac{\sqrt{t^2+9}-3}{t^2} \cdot \frac{\sqrt{t^2+9}+3}{\sqrt{t^2+9}+3} \\
&= \lim_{t\to 0} \frac{(t^2+9)-9}{t^2\left(\sqrt{t^2+9}+3\right)} = \lim_{t\to 0} \frac{\cancel{t^2}}{\cancel{t^2}(\sqrt{t^2+9}+3)} \\
&= \lim_{t\to 0} \frac{1}{\sqrt{t^2+9}+3}
\end{aligned}$$

$$\text{Plug in 0:} \quad =$$

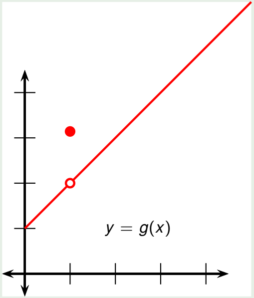
Example

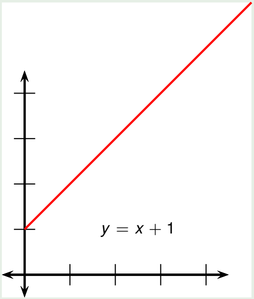
Find $\lim_{x\to 1} g(x)$, where

$$g(x) = \begin{cases} x+1 & \text{if} \quad x \neq 1 \\ \pi & \text{if} \quad x = 1 \end{cases}$$

g agrees with the function $f(x) = x + 1$ at every point except for $x = 1$.

$$\lim_{x\to 1} g(x) = \lim_{x\to 1}(x+1) = 2.$$





Example (Limit with Factoring)

Find $\displaystyle\lim_{h\to 0} \frac{(3+h)^2-9}{h}$

Plug in 0: $\displaystyle\frac{(3+(0))^2-9}{(0)} = \frac{0}{0}$

Zero over zero is undefined, so we can't use direct substitution.

$$\lim_{h\to 0} \frac{(3+h)^2-9}{h} = \lim_{h\to 0} \frac{9+6h+h^2-9}{h} = \lim_{h\to 0} \frac{6h+h^2}{h}$$

Factor: $$= \lim_{h\to 0} \frac{\cancel{h}(6+h)}{\cancel{h}}$$

$$= \lim_{h\to 0} (6+h)$$

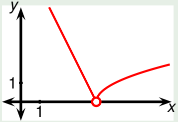
Plug in 0: $$= (6+(0)) = 6.$$

Recall:

$$\lim_{x\to a} f(x) = L \qquad \text{if and only if} \qquad \lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x).$$

We can use this to find the limit of a piecewise defined function, or show that it doesn't exist.

Example



$$f(x) = \begin{cases} \sqrt{x-4} & \text{if} \quad x > 4 \\ 8-2x & \text{if} \quad x < 4 \end{cases}$$

Determine whether $\lim\limits_{x\to 4} f(x)$ exists.

$$\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} \sqrt{x-4} = \sqrt{4-4} = 0$$

$$\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} (8-2x) = 8 - 2\cdot 4 = 0$$

The left and right hand limits are equal. Therefore the limit exists and

$$\lim_{x\to 4} f(x) = 0.$$

Theorem

If $f(x) \leq g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as x approaches a, then

$$\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x).$$

Theorem (The Squeeze Theorem)

Suppose $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at a) and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

Then

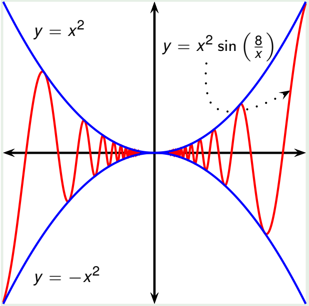
$$\lim_{x \to a} g(x) = L.$$

Example

Show that $\lim_{x\to 0} x^2 \sin\left(\frac{8}{x}\right) = 0.$

WRONG: $$\lim_{x\to 0} x^2 \sin\left(\frac{8}{x}\right) = \lim_{x\to 0} x^2 \cdot \lim_{x\to 0} \sin\left(\frac{8}{x}\right)$$

Doesn't work because $\lim_{x\to 0} \sin\left(\frac{8}{x}\right)$ doesn't exist.



$$\begin{array}{rcccl} -1 & \leq & \sin\left(\frac{8}{x}\right) & \leq & 1. \\ -x^2 & \leq & x^2 \sin\left(\frac{8}{x}\right) & \leq & x^2. \end{array}$$

$$\lim_{x\to 0} x^2 = 0 \quad \text{and} \quad \lim_{x\to 0}(-x^2) = 0.$$

Therefore by the Squeeze Theorem

$$\lim_{x\to 0} x^2 \sin\left(\frac{8}{x}\right) = 0.$$