

# Precalculus

## Trigonometric identities theory

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- The discussion here also applies for trigonometric identities in more than one variables.

# Types of identities

- In the present course we deal with two basic types of trigonometric identities.
- First, identities that involve operations on the arguments of the trigonometric functions.
  - Example:  $\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$  (this is one of the angle sum identities);  $\sin \theta + \sin(-\theta) = 0$ .
  - Such identities can be proved using the angle sum formulas and the even/odd function properties of  $\sin$ ,  $\cos$ .
- Second, identities that involve trigonometric functions of one variable.
  - Example:  $\tan^2 \theta + 1 = \sec^2 \theta$ .
  - Such identities can be proved only using the already demonstrated Pythagorean identity  $\sin^2 \theta + \cos^2 \theta = 1$ .
- The Pythagorean identity follows from the angle sum formulas and the even/odd function properties of  $\sin$ ,  $\cos$ , so all trigonometric identities follow from those properties alone.

# Strategy for proving trigonometric identities

An expression is rational trigonometric if it is written using  $\sin \theta$ ,  $\cos \theta$  and the four arithmetic operations.

## Question

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there is a standard method to verify whether two (rational) expressions in  $s$  and  $c$  are equal.
- The method is rather cumbersome for a human and is best suited for computers.



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  - A fraction of  $\theta$  such that all appearing angles are integer multiples of it will always work.