

# Precalculus

## Inverse functions

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2019

# Outline

1

## Inverse Functions

- One-to-one Functions
- The Definition of the Inverse of  $f$

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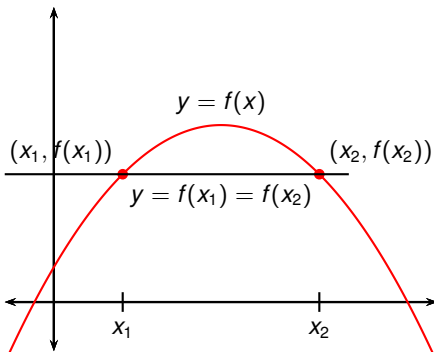
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# One-to-one Functions

## Definition (One-to-one Function)

A function  $f$  is a one-to-one function if it never takes on the same value twice; that is,

$$f(x_1) \neq f(x_2) \text{ whenever } x_1 \neq x_2.$$



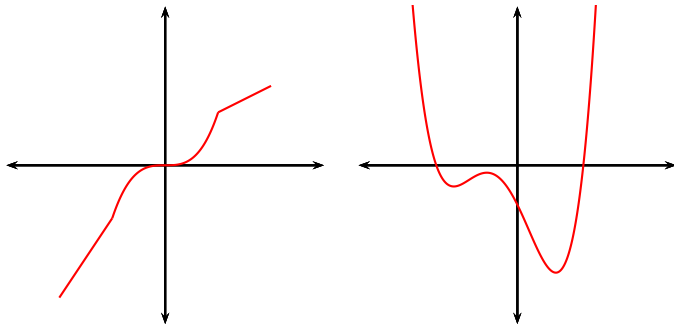
← This function is not one-to-one.

Question: How can we tell from the graph of a function whether it is one-to-one or not?

Answer: Use the horizontal line test.

### The Horizontal Line Test.

A function is one-to-one if and only if no horizontal line intersects it more than once.

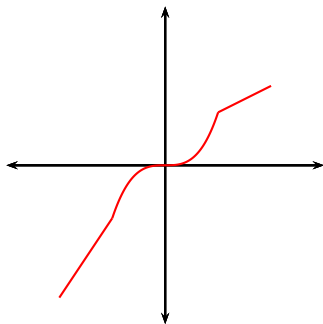


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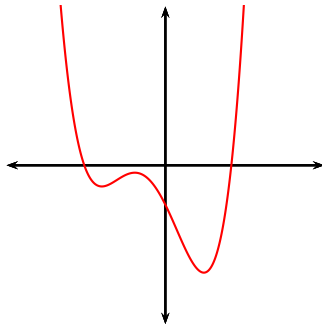
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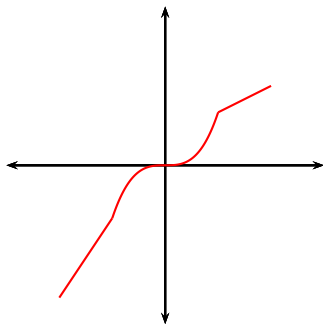


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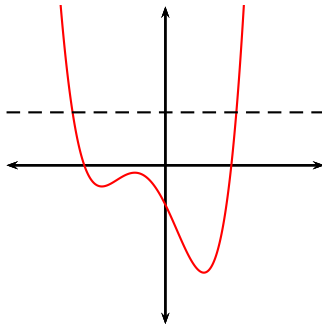
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# The Definition of the Inverse of $f$

## Definition ( $f^{-1}$ )

Let  $f$  be a one-to-one function with domain  $A$  and range  $B$ . Then the inverse of  $f$  is the function  $f^{-1}$  that has domain  $B$  and range  $A$  and is defined by

$$f^{-1}(y) = x \quad \Leftrightarrow \quad f(x) = y$$

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## Example ( $f(x) = x^3$ )

The inverse of  $f(x) = x^3$  is  $f^{-1}(x) = \sqrt[3]{x}$ . This is because if  $y = x^3$ , then

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*No one blamed English language of being logical.*

-Bjarne Stroustrup, creator of the programming language C++

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To reduce confusion, if possible, use  $\frac{1}{f(x)}$  instead of  $(f(x))^{-1}$ .

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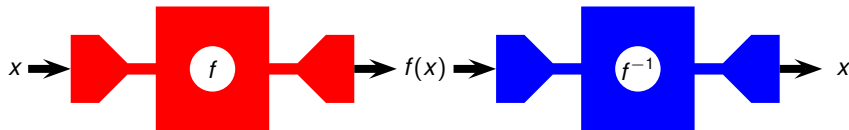
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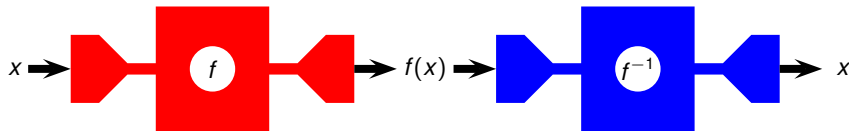




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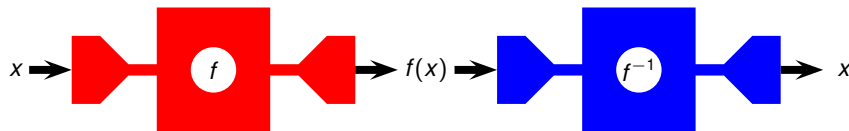
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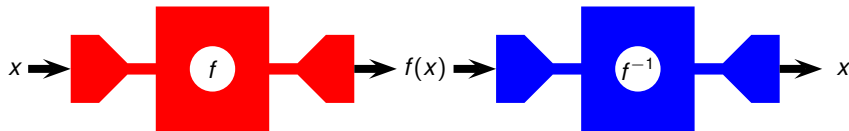
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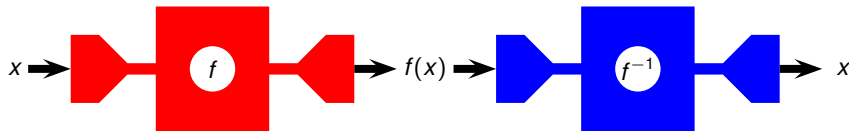
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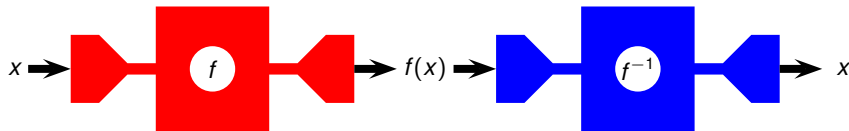
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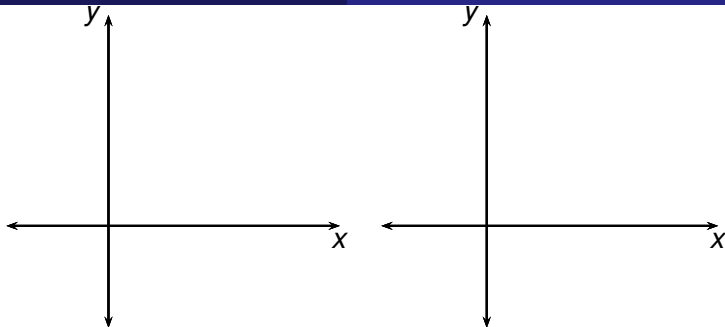


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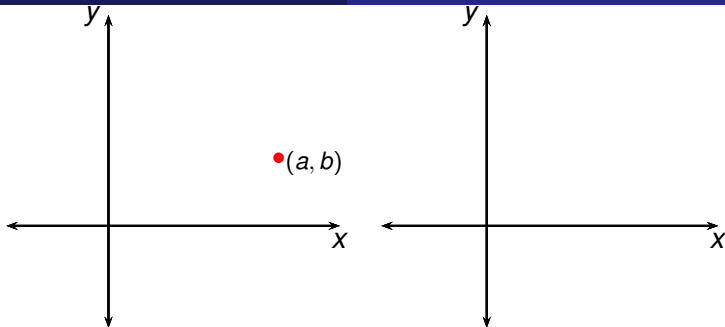
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Therefore  $f^{-1}(1) = 0$ .

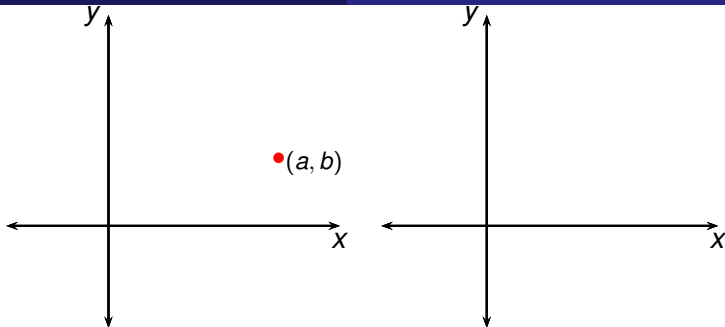


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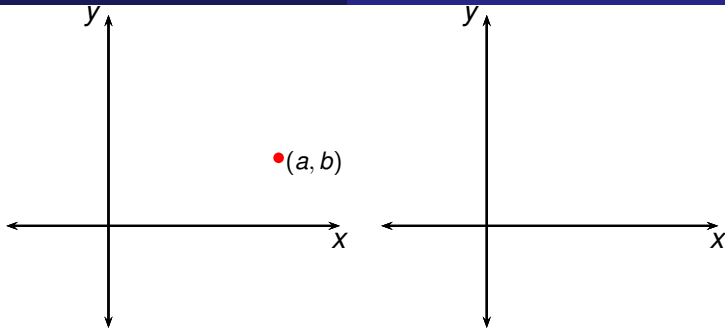
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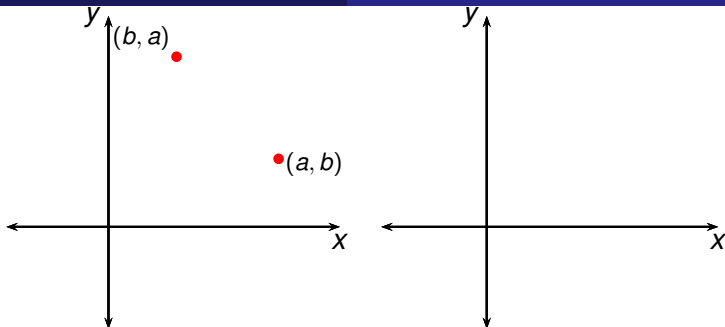
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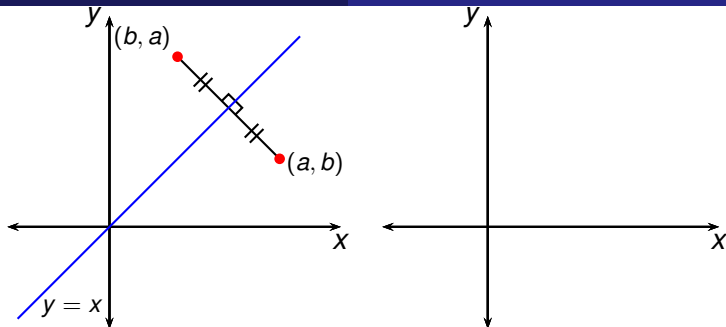
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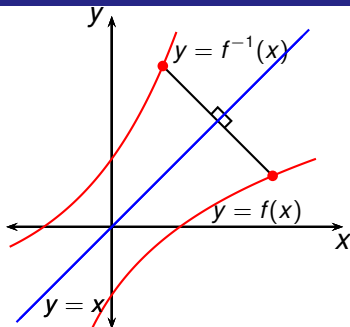
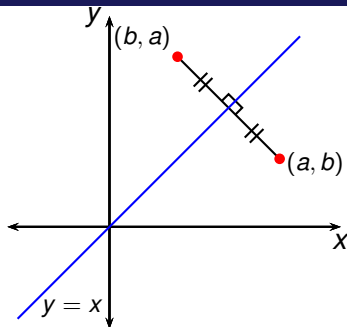
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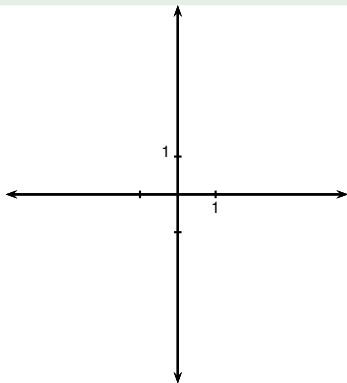


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- Thus the graph of  $f^{-1}$  is obtained by reflecting the graph of  $f$  in the line  $y = x$ .

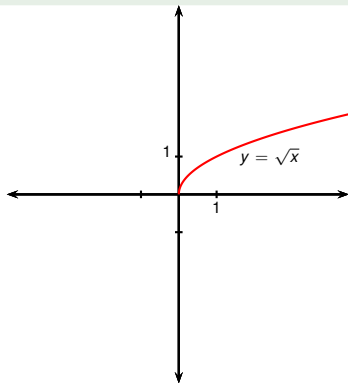


## Example



Sketch the graph of  $f(x) = \sqrt{-x - 1}$  and its inverse function.

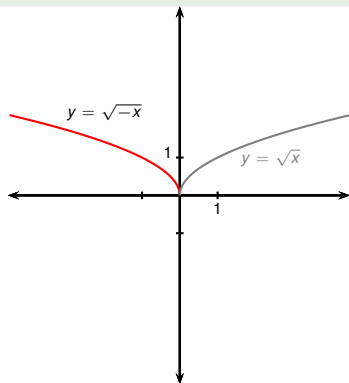
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Sketch the graph of  $f(x) = \sqrt{-x - 1}$  and its inverse function.

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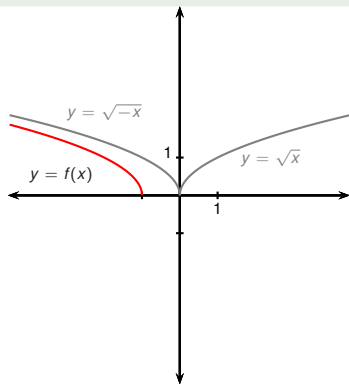
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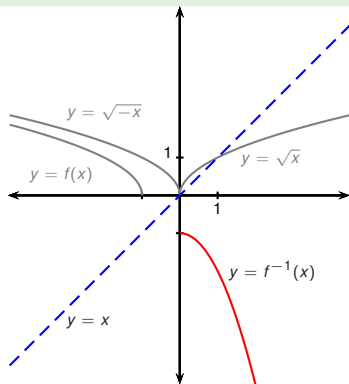
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## Example

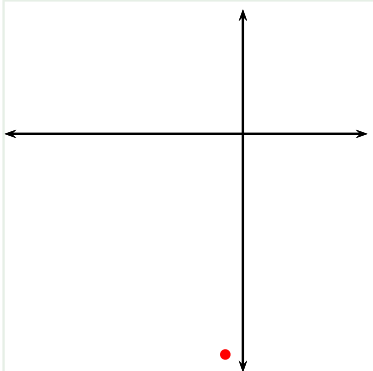


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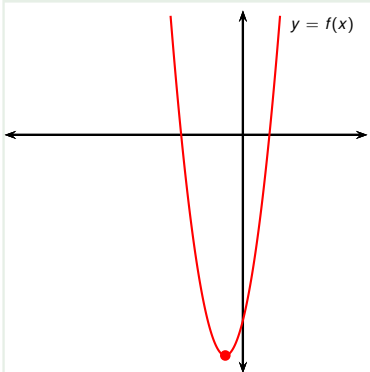
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Given:  $f(x) = 3x^2 + 4x - 7$  with domain  $x \geq -\frac{2}{3}$ . Find  $f^{-1}(x)$ .



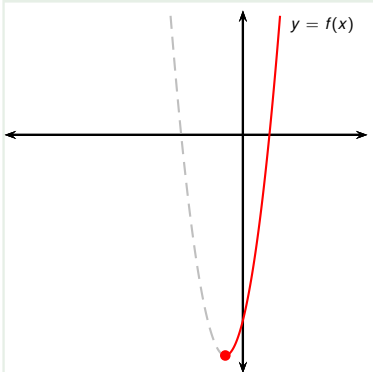
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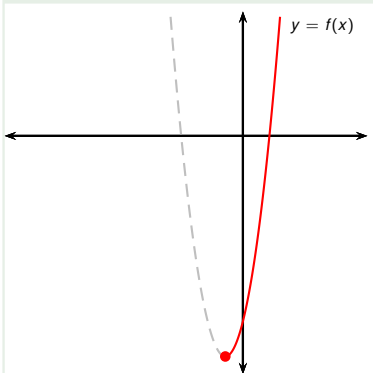
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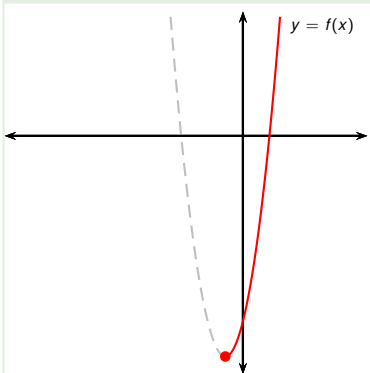
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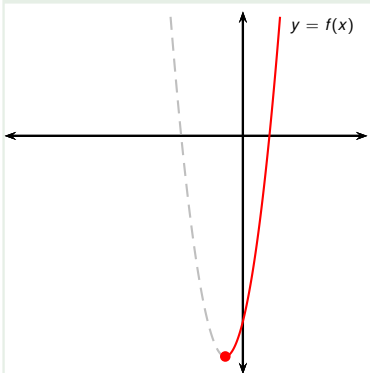
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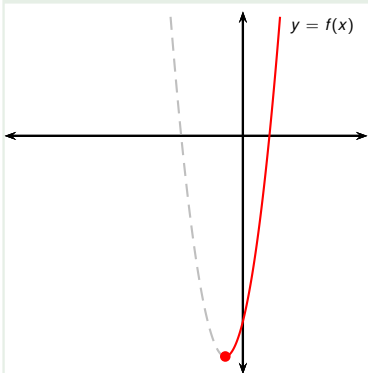
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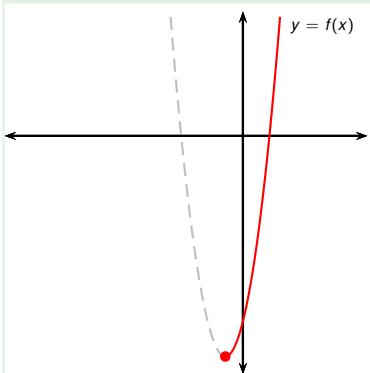
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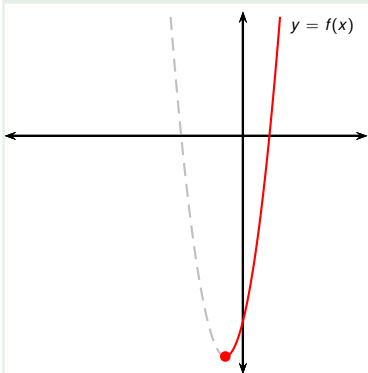
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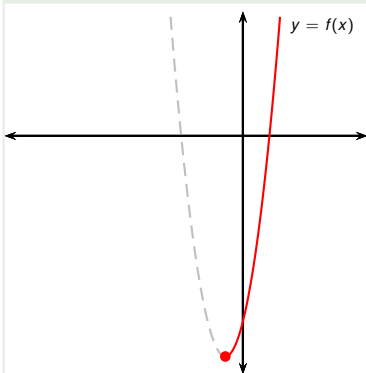
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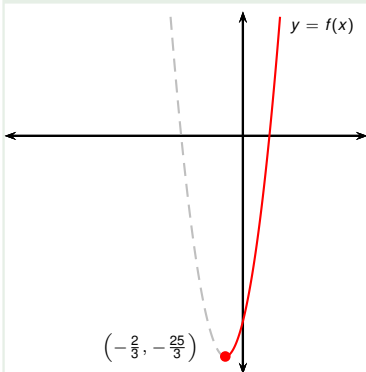
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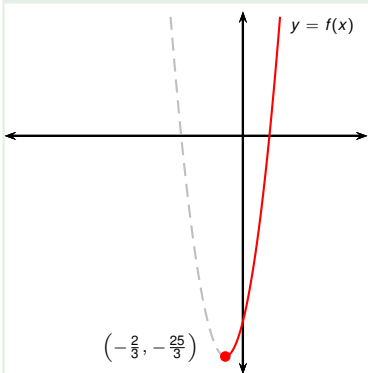
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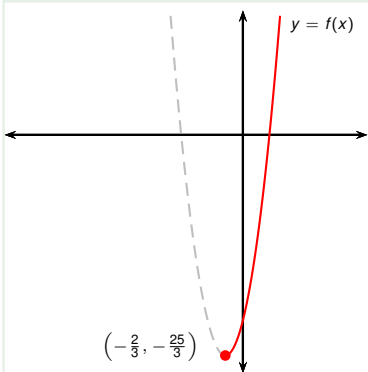
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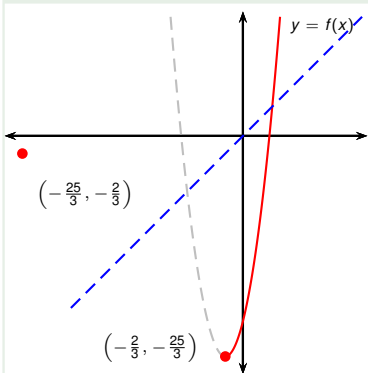
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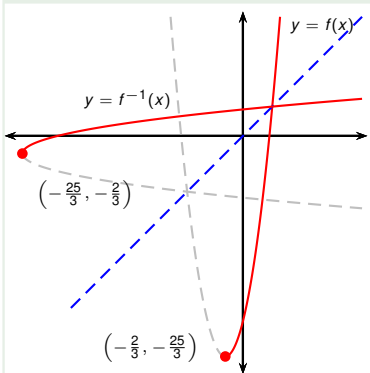
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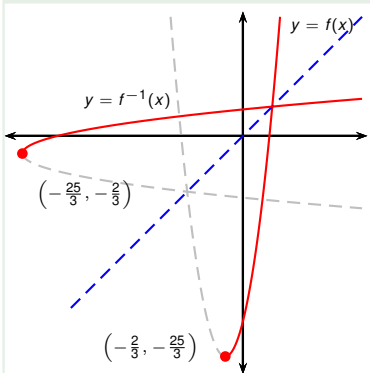
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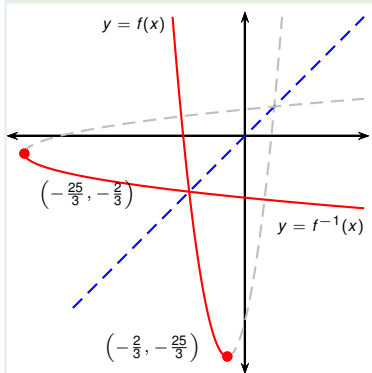
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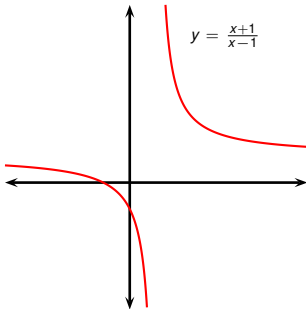
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## Example

Find  $f^{-1}(x)$  where  $f(x) = \frac{x+1}{x-1}$ .

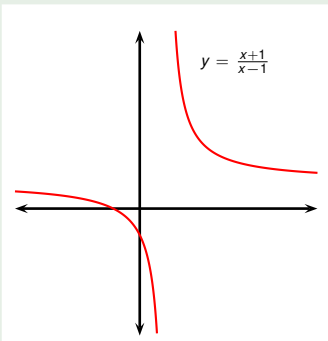


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We deal with domains and ranges later:

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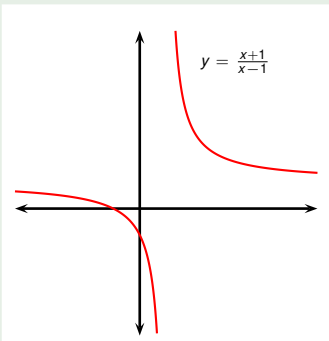


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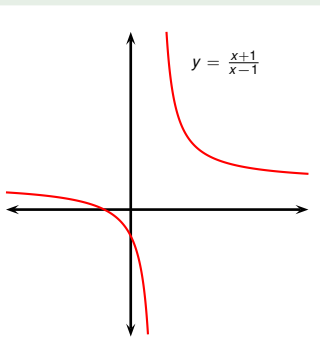
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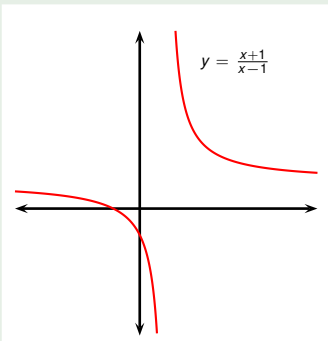


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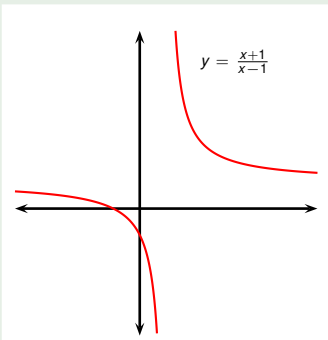
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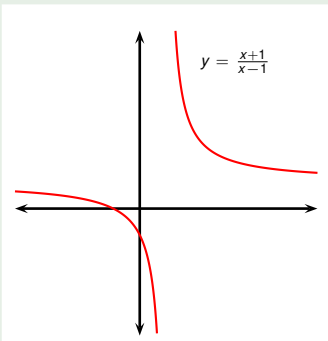


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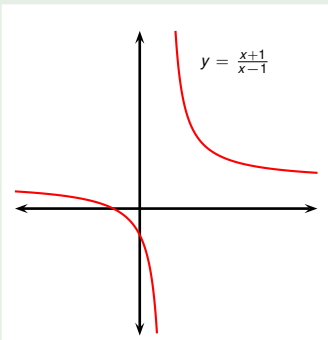
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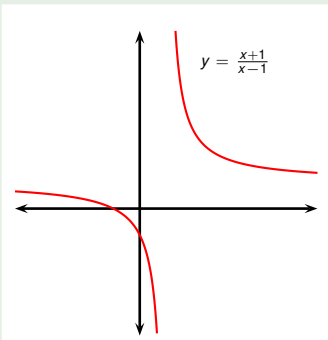


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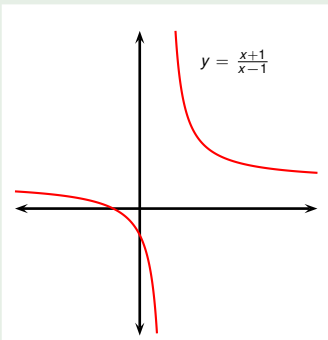


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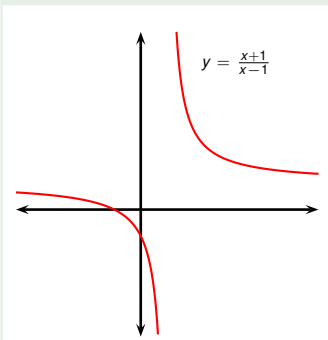
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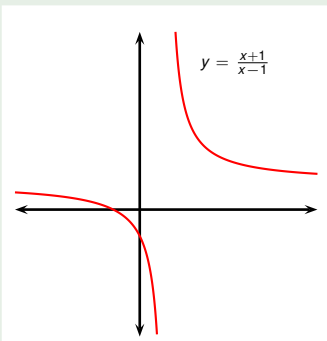
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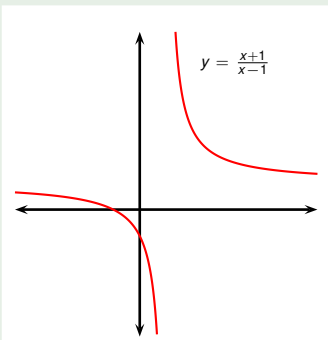
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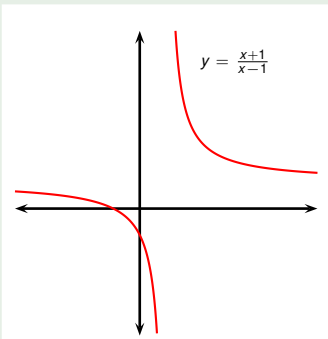
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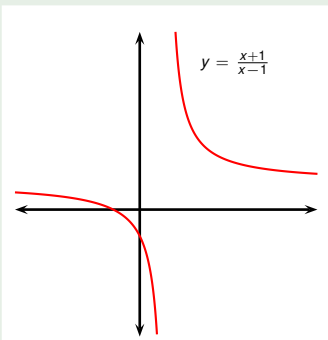
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**Can a non-identity function be its own inverse?**

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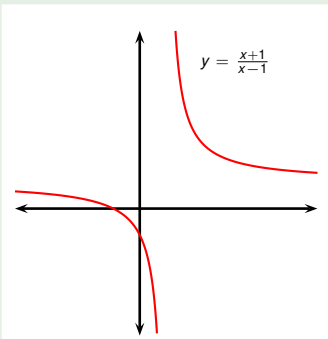
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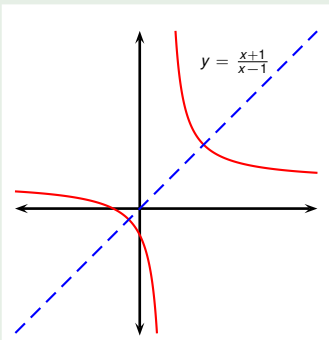
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**Graph of  $f$  is symmetric across  $y = x$ .**