Precalculus
Equations formed by setting trigonometric sum equal to 0

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Proposition (Product to sum formulas)

$$\begin{aligned}
\sin\alpha\sin\beta &= \frac{1}{2}\left(\cos(\alpha-\beta)-\cos(\alpha+\beta)\right)\\
\cos\alpha\cos\beta &= \frac{1}{2}\left(\cos(\alpha-\beta)+\cos(\alpha+\beta)\right)\\
\sin\alpha\cos\beta &= \frac{1}{2}\left(\sin(\alpha+\beta)+\sin(\alpha-\beta)\right)
\end{aligned}$$

Proof.

$$\begin{array}{clcl}
 & \cos\alpha\cos\beta + \cancel{\sin\alpha\sin\beta} & = & \cos(\alpha-\beta)\\
+ & \cos\alpha\cos\beta - \cancel{\sin\alpha\sin\beta} & = & \cos(\alpha+\beta)\\
 & 2\cos\alpha\cos\beta & = & \cos(\alpha-\beta)+\cos(\alpha+\beta)\\
\hline
 & \cos\alpha\cos\beta + \sin\alpha\sin\beta & = & \cos(\alpha-\beta)\\
- & \cos\alpha\cos\beta - \sin\alpha\sin\beta & = & \cos(\alpha+\beta)\\
 & 2\sin\alpha\sin\beta & = & \cos(\alpha-\beta)-\cos(\alpha+\beta)\\
\hline
 & \sin\alpha\cos\beta + \cos\alpha\sin\beta & = & \sin(\alpha+\beta)\\
+ & \sin\alpha\cos\beta - \cos\alpha\sin\beta & = & \sin(\alpha-\beta)\\
 & 2\sin\alpha\cos\beta & = & \sin(\alpha+\beta)+\sin(\alpha+\beta)
\end{array}$$

$\square$

Proposition (Product to sum formulas)

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\end{aligned}$$

- Product to sum formulas are used when integrating (a topic to be studied later/in another course).

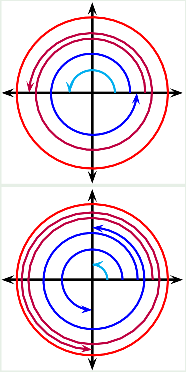
Proposition (Sum to product formulas)

$$
\begin{aligned}
\sin\alpha + \sin\beta &= 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)\\
\sin\alpha - \sin\beta &= 2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)\\
\cos\alpha + \cos\beta &= 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)\\
\cos\alpha - \cos\beta &= 2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)
\end{aligned}
$$

Recall the formula $\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

Example

Find all solutions in the interval $[0, 2\pi)$.



$$\sin(2x) + \sin(5x) = 0 \quad \Big| \text{ use f-la}$$

$$2\sin\left(\frac{2x+5x}{2}\right)\cos\left(\frac{2x-5x}{2}\right) = 0$$

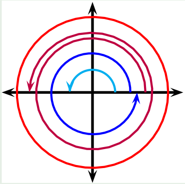
$$2\sin\left(\frac{7}{2}x\right)\cos\left(-\frac{3}{2}x\right) = 0 \quad \Big| \begin{array}{l}\cos \\ \text{is even}\end{array}$$

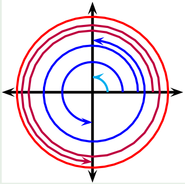
$$2\sin\left(\frac{7}{2}x\right)\cos\left(\frac{3}{2}x\right) = 0$$

Recall the formula $\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

Example

Find all solutions in the interval $[0, 2\pi)$.





$$\sin(2x) + \sin(5x) = 0 \quad \Big| \text{ use f-la}$$

$$2\sin\left(\frac{7}{2}x\right)\cos\left(\frac{3}{2}x\right) = 0$$

$$\sin\left(\tfrac{7}{2}x\right) = 0$$

$$\tfrac{7}{2}x = k\pi \qquad k - \text{integer}$$

$$x = \frac{2k\pi}{7}$$

$$x = \cancel{\ldots}, \cancel{\frac{-2\pi}{7}}, 0, \frac{2\pi}{7}, \frac{4\pi}{7}, \frac{6\pi}{7}, \frac{8\pi}{7}, \frac{10\pi}{7}, \frac{12\pi}{7}, \cancel{\frac{14\pi}{7}}, \cancel{\ldots}$$

or

$$\cos\left(\tfrac{3}{2}x\right) = 0$$

$$\tfrac{3}{2}x = \tfrac{\pi}{2} + k\pi = \frac{(2k+1)\pi}{2} \qquad k - \text{integer}$$

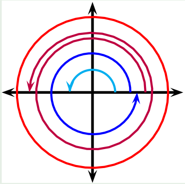
$$x = \frac{(2k+1)\pi}{3}$$

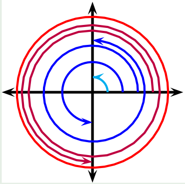
$$x = \cancel{\ldots}, \cancel{-\frac{\pi}{3}}, \frac{\pi}{3}, \frac{3\pi}{3}, \frac{5\pi}{3}, \cancel{\frac{7\pi}{3}}, \cancel{\ldots}$$

Recall the formula $\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

Example

Find all solutions in the interval $[0, 2\pi)$.





$$\sin(2x) + \sin(5x) = 0 \quad \Big| \text{ use f-la}$$

$$2\sin\left(\frac{7}{2}x\right)\cos\left(\frac{3}{2}x\right) = 0$$

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or

$$x = \cancel{\ldots}, \cancel{-\frac{\pi}{3}}, \frac{\pi}{3}, \frac{3\pi}{3}, \frac{5\pi}{3}, \cancel{\frac{7\pi}{3}}, \cancel{\ldots}$$