Calculus I
Derivatives basics

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Outline

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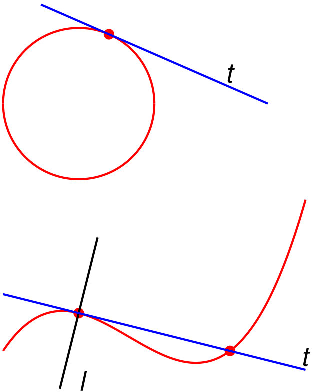
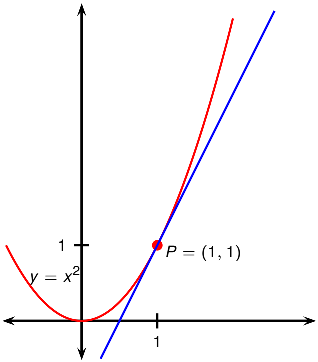
The Tangent Problem

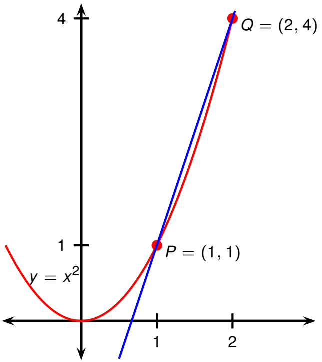


- A tangent is a line that touches a curve.
- Moreover, a tangent should have the same "direction" as the curve at the point of contact.
- For a circle, a tangent is a line that intersects the circle at exactly one point.
- For more general curves, this definition isn't good enough.
- The line l intersects the curve at exactly one point, but it doesn't look like a tangent.
- The line t does look like a tangent, but it intersects the curve at two points.



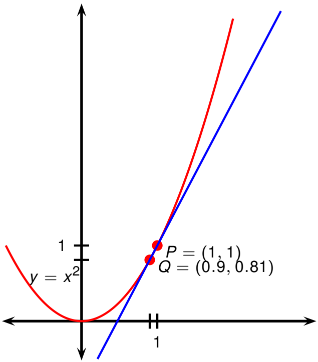
x	m_{PQ}	x	m_{PQ}
2		0	
1.5		0.5	
1.25		0.75	
1.1		0.9	
1.01		0.99	

- Find the tangent to $y = x^2$ at $(1, 1)$.
- Tangent has equation $y - 1 = m(x - 1)$, where m is its slope.
- If we know the slope, we know the line.
- If we know two points, we can find the slope. We know one point, P; we need another point.



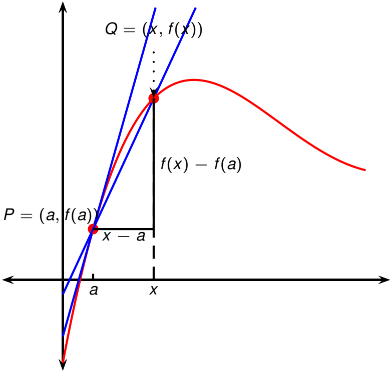
x	m_{PQ}	x	m_{PQ}
2	3	0	
1.5		0.5	
1.25		0.75	
1.1		0.9	
1.01		0.99	

- Find the tangent to $y = x^2$ at $(1, 1)$.
- Tangent has equation $y - 1 = m(x - 1)$, where m is its slope.
- If we know the slope, we know the line.
- If we know two points, we can find the slope. We know one point, P; we need another point.
- Choose a nearby point $Q = (x, x^2)$ on the parabola and find the slope m_{PQ} of the secant line PQ.



x	m_{PQ}	x	m_{PQ}
2	3	0	1
1.5	2.5	0.5	1.5
1.25	2.25	0.75	1.75
1.1	2.1	0.9	1.9
1.01	2.01	0.99	1.99

- Find the tangent to $y = x^2$ at $(1, 1)$.
- Tangent has equation $y - 1 = m(x - 1)$, where m is its slope.
- If we know the slope, we know the line.
- If we know two points, we can find the slope. We know one point, P; we need another point.
- Choose a nearby point $Q = (x, x^2)$ on the parabola and find the slope m_{PQ} of the secant line PQ.
- The closer x is to 1, the closer m_{PQ} is to 2.
- This suggests the slope of the tangent should be 2.



- How to find the tangent line to the curve $y = f(x)$ at $P = (a, f(a))$?
- Consider nearby point $Q = (x, f(x))$.
- Compute slope of secant line PQ: $m_{PQ} = \frac{f(x)-f(a)}{x-a}$.
- As x approaches a, the point Q approaches P.

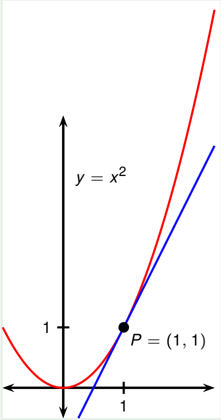
Definition (Non-vertical tangent line)

Let $P = (a, f(a))$. Suppose the limit $m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ exists. Define the tangent to $y = f(x)$ at P to be the line passing through P with slope m, in other words, the line with equation $y - f(a) = m(x - a)$.

Note. Even if the limit does not exist a reasonable notion of a tangent line may still exist.

Example

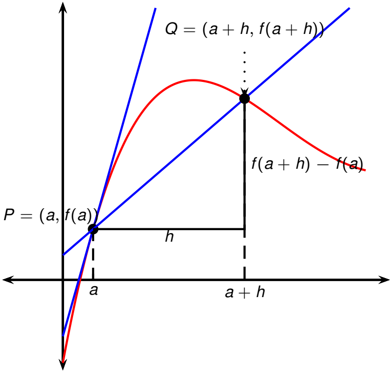
Find an equation for the tangent line to the parabola $y = x^2$ at the point $P = (1, 1)$.



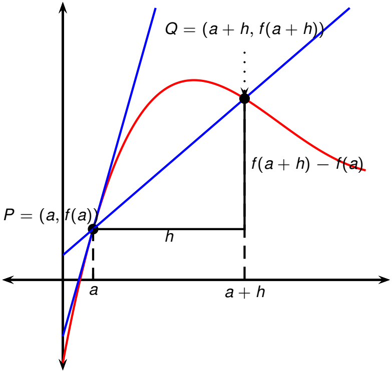
Here $a = 1$ and $f(x) = x^2$.

$$\begin{aligned} m &= \lim_{x\to 1} \frac{f(x) - f(1)}{x - 1} \\ &= \lim_{x\to 1} \frac{x^2 - 1}{x - 1} \\ &= \lim_{x\to 1} \frac{\cancel{(x-1)}(x+1)}{\cancel{x-1}} \\ &= \lim_{x\to 1} (x + 1) = 1 + 1 = 2 \end{aligned}$$

Point-slope form: $y - 1 = 2(x - 1)$, or finally $y = 2x - 1$.



- There is an equivalent expression for the slope of the tangent.
- Again let x tend to a.



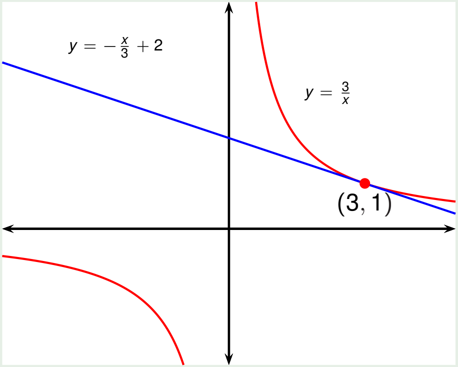
- There is an equivalent expression for the slope of the tangent.
- Again let x tend to a.
- However, think in terms of $h = x - a$.
- Then $x = a + h$ and the slope of the secant line PQ is $m_{PQ} = \frac{f(a+h)-f(a)}{h}$.
- The limit can now be written in terms of the quantity h.

Tangent slope - equivalent expression:

$$m = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}.$$

Example

Find an equation for the tangent line to the hyperbola $y = \frac{3}{x}$ at the point $(3, 1)$.



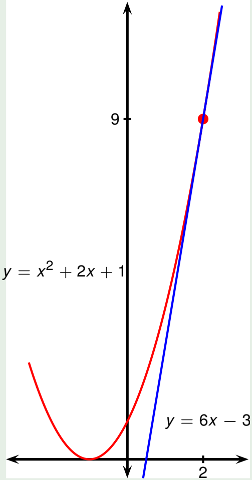
Point-slope form: $y - 1 = -\frac{1}{3}(x - 3)$, or finally $y = -\frac{x}{3} + 2$.

Here $a = 3$ and $f(x) = \frac{3}{x}$.

$$\begin{aligned} m &= \lim_{h\to 0} \frac{f(3+h) - f(3)}{h} \\ &= \lim_{h\to 0} \frac{\frac{3}{3+h} - 1}{h} \\ &= \lim_{h\to 0} \frac{\frac{3-(3+h)}{3+h}}{h} \\ &= \lim_{h\to 0} \frac{-\cancel{h}}{\cancel{h}(3+h)} \\ &= \lim_{h\to 0} -\frac{1}{3+h} = -\frac{1}{3} \end{aligned}$$

Example (Tangent line to a polynomial)

Find an equation for the tangent line to the parabola $y = x^2 + 2x + 1$ at the point $P = (2, 9)$.



Here $a = 2$ and $f(x) = x^2 + 2x + 1$.

$$\begin{aligned} m &= \lim_{x\to 2} \frac{f(x) - f(2)}{x - 2} \\ &= \lim_{x\to 2} \frac{(x^2 + 2x + 1) - 9}{x - 2} \\ &= \lim_{x\to 2} \frac{x^2 + 2x - 8}{x - 2} \\ &= \lim_{x\to 2} \frac{\cancel{(x-2)}(x+4)}{\cancel{x-2}} \\ &= \lim_{x\to 2} (x + 4) = 6. \end{aligned}$$

The tangent line: $y - 9 = 6(x - 2)$, or finally $y = 6x - 3$.

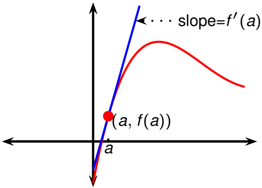
Derivatives

Definition (Derivative)

The derivative of a function f at a number a, denoted by $f'(a)$, is

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

if the limit exists.



- The two alternative formulas result in equivalent definitions.
- Equivalent formulation. The derivative $f'(a)$ is the slope of the tangent line to $y = f(x)$ at $(a, f(a))$, provided that tangent line exists and is non-vertical.

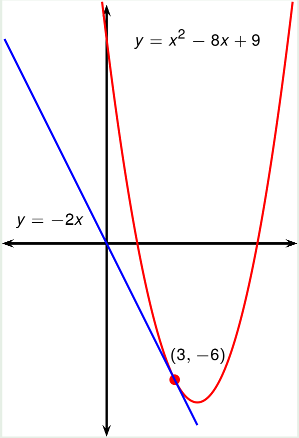
Example

Find the derivative of the function $f(x) = x^2 - 8x + 9$ at the number a.

$$
\begin{aligned}
f'(a) &= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \\
&= \lim_{h\to 0} \frac{(a+h)^2 - 8(a+h) + 9 - \left(a^2 - 8a + 9\right)}{h} \\
&= \lim_{h\to 0} \frac{\cancel{a^2} + 2ah + h^2 - \cancel{8a} - 8h + \cancel{9} - \left(\cancel{a^2} - \cancel{8a} + \cancel{9}\right)}{h} \\
&= \lim_{h\to 0} \frac{2ah + h^2 - 8h}{h} \\
&= \lim_{h\to 0} \frac{\cancel{h}(2a + h - 8)}{\cancel{h}} \\
&= \lim_{h\to 0} (2a + h - 8) = 2a - 8.
\end{aligned}
$$

Example

Find an equation for the tangent line to the parabola $y = x^2 - 8x + 9$ at the point $P = (3, -6)$.



- The slope of the tangent is the derivative $f'(3)$.
- From the previous example, $f'(a) = 2a - 8$.
- Therefore $f'(3) = 2 \cdot 3 - 8 = -2$.
- Point-slope form: $y - (-6) = -2(x - 3)$.
- Slope y-intercept form: $y = -2x$.

Other Notations for Derivative

If $y = f(x)$ is a function, there are many ways to write its derivative.

$$f'(x) = y' = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}f(x) = Df(x) = D_x f(x)$$

- $\frac{\mathrm{d}}{\mathrm{d}x}$ are called differentiation operators because they indicate the operation of differentiation, which is the process of calculating the derivative.
- $\mathrm{d}y/\mathrm{d}x$ is called Leibniz notation; it means the same as $f'(x)$.
- If we want to indicate the value of the derivative $\mathrm{d}y/\mathrm{d}x$ in Leibniz notation at a point a, we write

$$\frac{\mathrm{d}y}{\mathrm{d}x}_{|x=a} \qquad \text{or} \qquad \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_a \qquad \text{or} \qquad \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right]_a$$

The Derivative as a Function

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

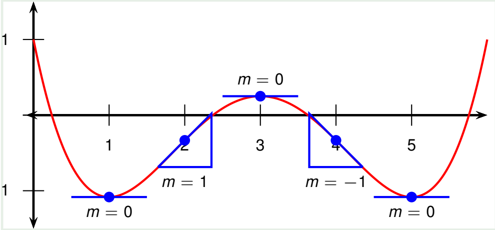
- Change the point of view by letting the number a vary.
- Replace a with the variable x to get:

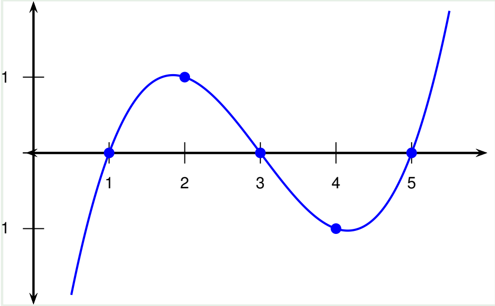
$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

- f' is regarded a function in its own right, called the derivative of f.
- The domain of f' is $\{x | f'(x) \text{ exists } \}$.
- The domain of f' may be smaller than the domain of f.

Example

The graph of a function f appears below. Use it to sketch the graph of the derivative f'.

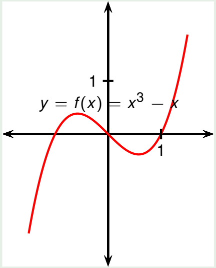


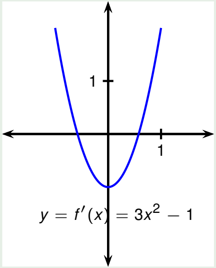


- Find the points where the tangent is horizontal ($m = 0$).
- That is where f' is 0.
- Where the slope of the tangent to f is 1, f' is 1.
- Where the slope of the tangent to f is -1, f' is -1.
- Where the slope of the curve is negative, f' is negative.
- Where the slope of the curve is positive, f' is positive.

Example

If $f(x) = x^3 - x$, find formula for $f'(x)$.





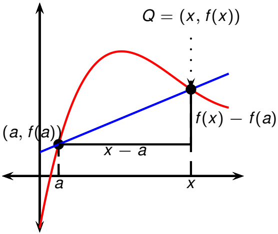
$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^3 - (x+h) - \left(x^3 - x\right)}{h} \\
&= \lim_{h\to 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 - \cancel{x} - h - \cancel{x^3} + \cancel{x}}{h} \\
&= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3 - h}{h} \\
&= \lim_{h\to 0} \frac{\cancel{h}(3x^2 + 3xh + h^2 - 1)}{\cancel{h}} \\
&= \lim_{h\to 0} (3x^2 + 3xh + h^2 - 1) \\
&= 3x^2 - 1
\end{aligned}
$$

Velocities

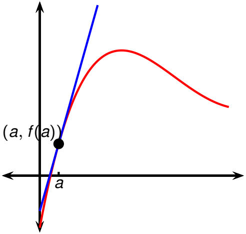
Example

Suppose a ball is dropped from the upper deck of the CN Tower, 450m above the ground. What is the velocity of the ball after 5 seconds?

- We need to know what “instantaneous” velocity is.
- Let $f(x)$ denote the displacement of an object at time x.



Slope of secant
= average velocity



Slope of tangent
= instantaneous velocity

Example

Suppose a ball is dropped from the upper deck of the CN Tower, 450m above the ground. What is the velocity of the ball after 5 seconds?

- The distance $f(x)$ (in meters) that the ball has fallen at time x (in seconds) follows Galileo's law: $f(x) = 4.9x^2$.
- Let $v(a)$ be its velocity at time a.

$$\begin{aligned} v(a) &= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{4.9(a+h)^2 - 4.9a^2}{h} \\ &= \lim_{h\to 0} \frac{4.9(a^2 + 2ah + h^2) - 4.9a^2}{h} \\ &= \lim_{h\to 0} \frac{4.9(2ah + h^2)}{h} \\ &= \lim_{h\to 0} 4.9(2a + h) = 9.8a \end{aligned}$$

Therefore the velocity after 5s is $v(5) = 9.8(5) = 49$m/s.

Definition (Differentiable at a point)

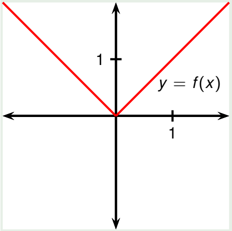
A function f is differentiable at a if $f'(a)$ exists.

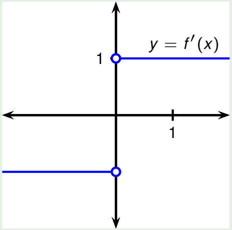
Definition (Differentiable on an interval)

A function f is differentiable on an open interval (a, b) (allowing $a = -\infty, b = \infty$) if it is differentiable at every number in the interval.

Example

Where is the function $f(x) = |x|$ differentiable?





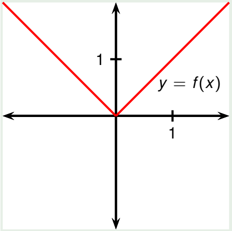
- Suppose $x > 0$.
- Then $|x| = x$.
- If $|h| < x$ it follows that $x + h > 0$.
- Then for $|h| < x$ we have $|x + h| = x + h$.

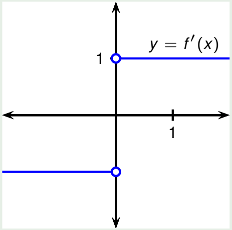
$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{|x+h| - |x|}{h} \\
&= \lim_{h\to 0} \frac{(x+h) - x}{h} \\
&= \lim_{h\to 0} \frac{h}{h} = 1
\end{aligned}
$$

Therefore f is differentiable for any $x > 0$.

Example

Where is the function $f(x) = |x|$ differentiable?





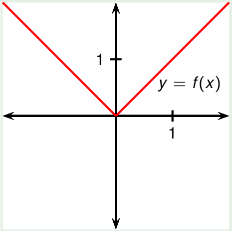
- Suppose $x < 0$.
- Then $|x| = -x$.
- If $|h| < |x|$ it follows that $x + h < 0$.
- Then $|x + h| = -(x + h)$.

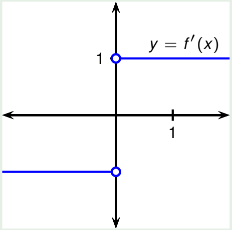
$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{|x+h| - |x|}{h} \\ &= \lim_{h \to 0} \frac{-(x+h) + x}{h} \\ &= \lim_{h \to 0} \frac{-h}{h} = -1 \end{aligned}$$

Therefore f is differentiable for any $x < 0$.

Example

Where is the function $f(x) = |x|$ differentiable?





If $f'(0)$ exists, then

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{|0+h| - |0|}{h}.$$

Does this limit exist?

$$\lim_{h\to 0^+} \frac{|0+h| - |0|}{h} = \lim_{h\to 0^+} \frac{|h|}{h} = \lim_{h\to 0^+} \frac{h}{h} = 1$$

$$\lim_{h\to 0^-} \frac{|0+h| - |0|}{h} = \lim_{h\to 0^-} \frac{|h|}{h} = \lim_{h\to 0^-} \frac{-h}{h} = -1$$

Therefore f is not differentiable at 0.

$$f'(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

Theorem (Differentiability Implies Continuity)

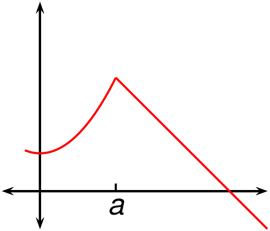
If f is differentiable at a, then f is continuous at a.

Proof.

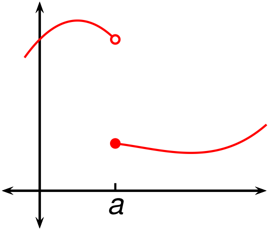
$$\begin{aligned}
\lim_{x\to a} f(x) &= \lim_{x\to a} f(a) + \lim_{x\to a}[f(x) - f(a)] \\
&= \lim_{x\to a} f(a) + \lim_{x\to a} \frac{f(x)-f(a)}{x-a}(x-a) \\
&= \lim_{x\to a} f(a) + \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \cdot \lim_{x\to a}(x-a) \\
&= \lim_{x\to a} f(a) + f'(a)\cdot 0 \\
&= f(a)
\end{aligned}$$

Therefore f is continuous at a. □

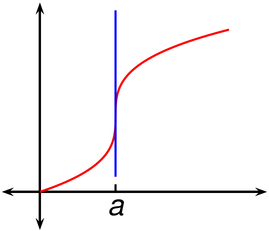
How Can a Function Fail to be Differentiable?



corner



discontinuity



vertical tangent

. . . and many other ways. . .

Higher Derivatives

- Let f be a differentiable function.
- Suppose f' is also differentiable.
- Call the derivative of f' by f''. Call the derivative of f'' by f''' (if it exists) and so on.
- f'' is called second derivative, f''' -third derivative, and so on.
  - f' measures the rate of change of f.
  - Therefore f'' measures the rate of change of the rate of change of f, and so on for the other derivatives.
    - Suppose f measures distance traveled per unit time.
    - f' - the rate of change of distance - is called velocity.
    - f'' - the rate of change of velocity - is called acceleration.

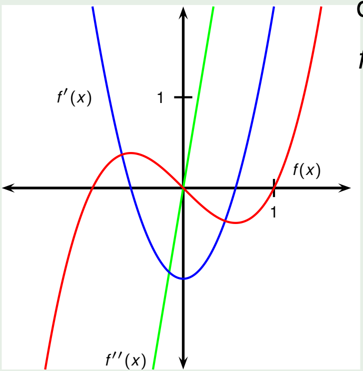
Notation for Higher Derivatives

Name		$y = f(x)$	Leibniz notation	$y = f(x)$
first derivative	$f'(x)$	y'	$\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\mathrm{d}f}{\mathrm{d}x}(x)$	$\frac{\mathrm{d}y}{\mathrm{d}x}$
second derivative	$f''(x)$	y''	$\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} = \frac{\mathrm{d}^2 f}{\mathrm{d}x^2}(x)$	$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$
third derivative	$f'''(x)$	y'''	$\frac{\mathrm{d}^3 f}{\mathrm{d}x^3} = \frac{\mathrm{d}^3 f}{\mathrm{d}x^3}(x)$	$\frac{\mathrm{d}^3 y}{\mathrm{d}x^3}$
$\vdots$				
n^{th} derivative	$f^{(n)}(x)$	$y^{(n)}$	$\frac{\mathrm{d}^n f}{\mathrm{d}x^n} = \frac{\mathrm{d}^n f}{\mathrm{d}x^n}(x)$	$\frac{\mathrm{d}^n y}{\mathrm{d}x^n}$

Note: Do not confuse the superscript in the notation for n^{th} derivative with exponent. The parenthesis indicate we mean derivatives rather than exponents.

Example

If $f(x) = x^3 - x$, find $f''(x)$.



In a previous exercise we found that the first derivative is $f'(x) = 3x^2 - 1$.

$$
\begin{aligned}
f''(x) &= \lim_{h\to 0} \frac{f'(x+h) - f'(x)}{h} \\
&= \lim_{h\to 0} \frac{3(x+h)^2 - 1 - \left(3x^2 - 1\right)}{h} \\
&= \lim_{h\to 0} \frac{\cancel{3x^2} + 6xh + 3h^2 - \cancel{1} - \cancel{3x^2} + \cancel{1}}{h} \\
&= \lim_{h\to 0} \frac{6xh + 3h^2}{h} \\
&= \lim_{h\to 0} \frac{\cancel{h}(6x + 3h)}{\cancel{h}} \\
&= \lim_{h\to 0} (6x + 3h) = 6x
\end{aligned}
$$

Differentiation Formulas

Let c be a constant and consider the constant function $f(x) = c$. Let us calculate the derivative of f:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{c-c}{h} = \lim_{h\to 0} 0 = 0.$$

Theorem (Derivative of a Constant Function)

$$\frac{\mathrm{d}}{\mathrm{d}x}(c) = 0$$

Power Functions

Now consider functions of the form $f(x) = x^n$, where n is a positive integer. For $f(x) = x$, the graph is the line $y = x$, which has slope 1. So

$$\frac{\mathrm{d}}{\mathrm{d}x}(x) = 1.$$

What about $n = 2$ and $n = 3$?

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}(x^2)\\
=&\lim_{h\to 0}\frac{(x+h)^2 - x^2}{h}\\
=&\lim_{h\to 0}\frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^2}}{h}\\
=&\lim_{h\to 0}\frac{\cancel{h}(2x+h)}{\cancel{h}}\\
=&\lim_{h\to 0}(2x+h) = 2x.
\end{aligned}$$

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}(x^3)\\
=&\lim_{h\to 0}\frac{(x+h)^3 - x^3}{h}\\
=&\lim_{h\to 0}\frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 - \cancel{x^3}}{h}\\
=&\lim_{h\to 0}\frac{\cancel{h}(3x^2 + 3xh + h^2)}{\cancel{h}}\\
=&\lim_{h\to 0}(3x^2 + 3xh + h^2) = 3x^2.
\end{aligned}$$

Theorem (The Power Rule)

If n is a positive integer, then $$\frac{\mathrm{d}}{\mathrm{d}x}(x^n) = nx^{n-1}.$$

Proof.

Use this formula (which you can verify):

$$x^n - a^n = (x-a)(x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1}).$$

Let $f(x) = x^n$. Then

$$\begin{aligned}
f'(a) &= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a} \frac{x^n - a^n}{x-a} \\
&= \lim_{x\to a} \frac{\cancel{(x-a)}(x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1})}{\cancel{x-a}} \\
&= \lim_{x\to a} (x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1}) \\
&= a^{n-1} + a^{n-2}a + \cdots + aa^{n-2} + a^{n-1} = na^{n-1}.
\end{aligned}$$

$\square$

Example (Power Rule)

If $f(x) = x^5$,
Then $f'(x) = 5x^4$.

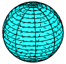
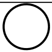
If $y = x^{1000}$,
Then $y' = 1000x^{999}$.

If $u = t^{22}$,
Then $\dfrac{du}{dt} = 22t^{21}$.

$$\frac{d}{dr}(r^3) = 3r^2.$$

The Relation between Ball Volume and Surface Area

There is a relationship between the surface area and the volume of a ball (in any dimension).

Dimension	Set of pts. at dist. $\leq r$ from origin	Inside measure name	Measure f-la	Boundary name	Boundary measure formula	Derivative of inside measure
3	 ball	volume	$\frac{4}{3}\pi r^3$	 sphere	$4\pi r^2$	$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2$
2	 disk, circle	circle area	πr^2	 circle (circumference)	$2\pi r$	$\frac{\mathrm{d}}{\mathrm{d}r}\left(\pi r^2\right) = 2\pi r$
1	 interval	length	$2r$	 endpts.	2	$\frac{\mathrm{d}}{\mathrm{d}r}(2r) = 2$