Calculus I
Derivatives basics

Todor Milev

2019

Outline

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The Tangent Problem

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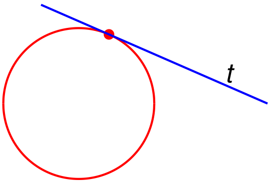

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- For a circle, a tangent is a line that intersects the circle at exactly one point.

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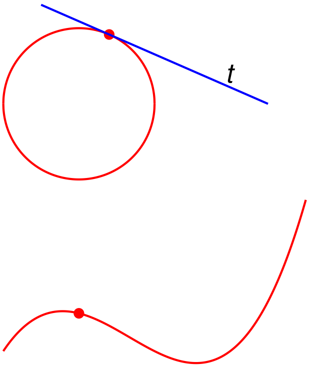

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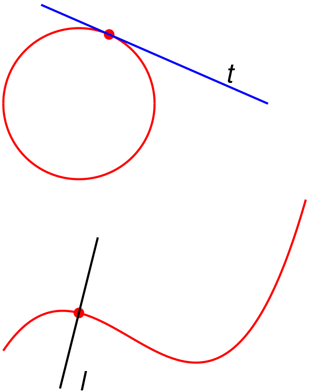

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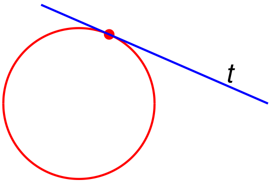

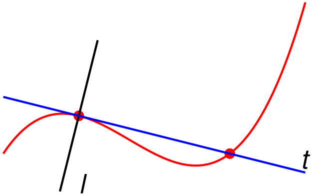

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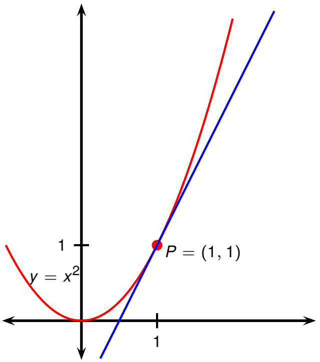
- The line t does look like a tangent, but it intersects the curve at two points.



x	m_{PQ}	x	m_{PQ}
2		0	
1.5		0.5	
1.25		0.75	
1.1		0.9	
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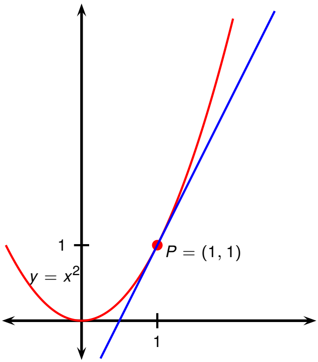

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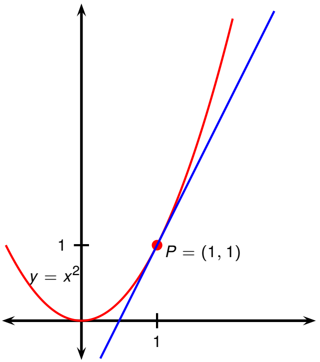

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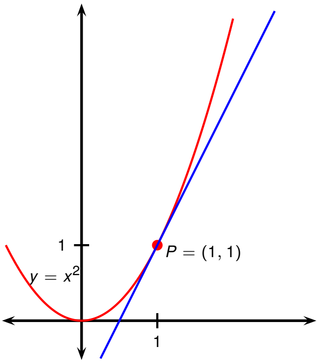

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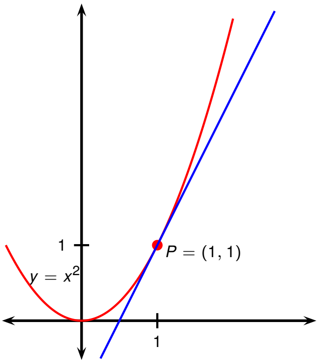

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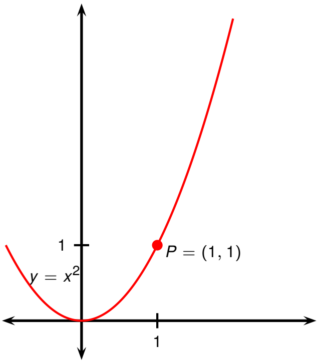

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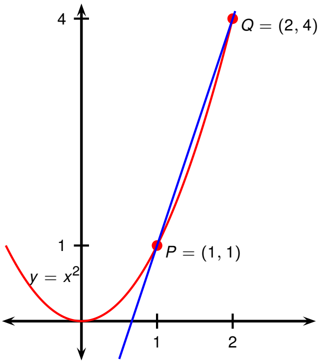

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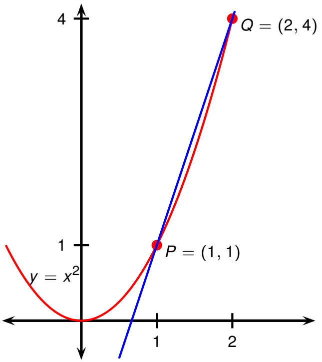

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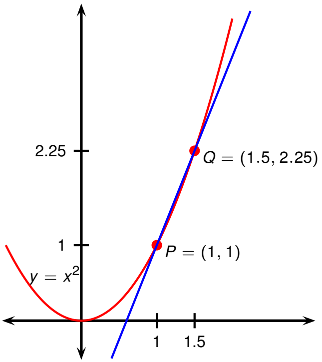

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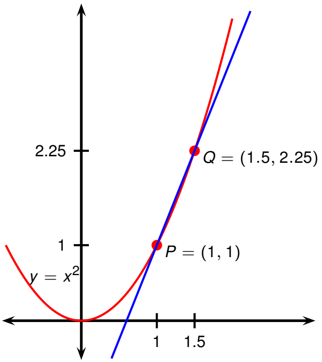

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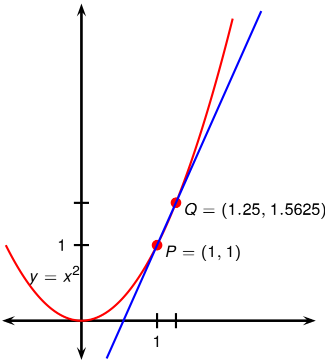

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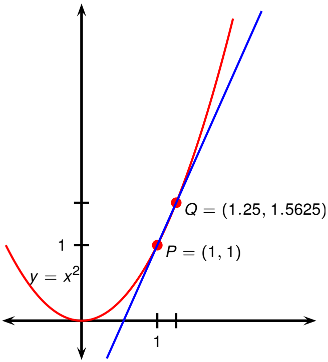

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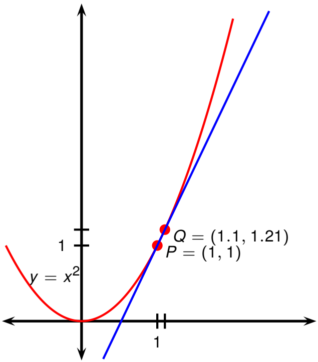

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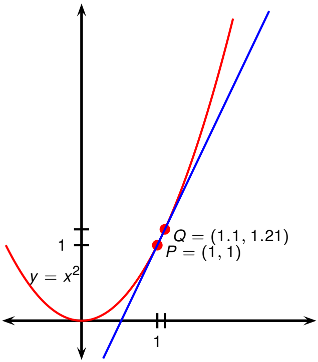

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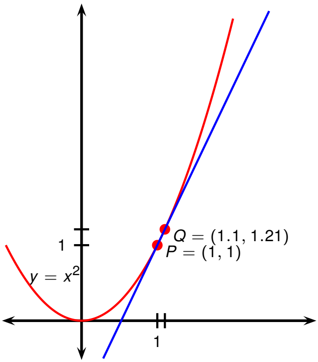

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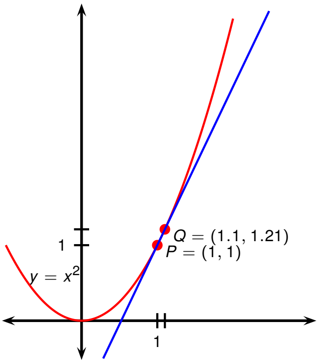

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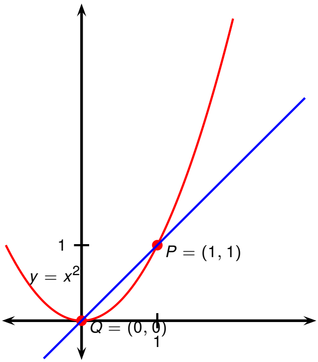

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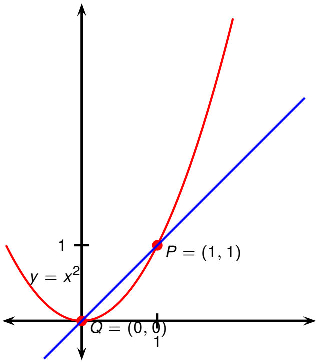

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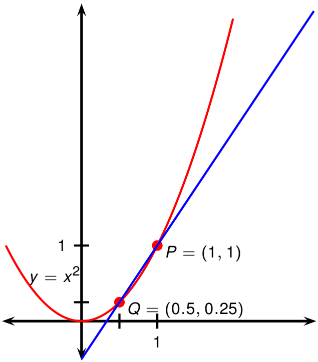

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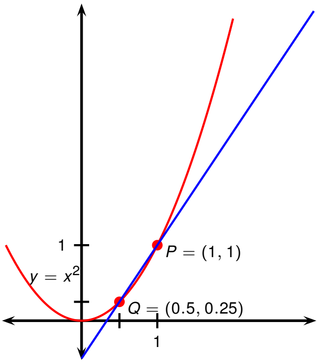

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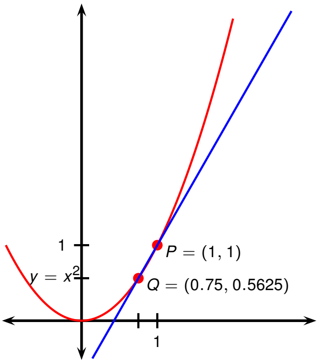

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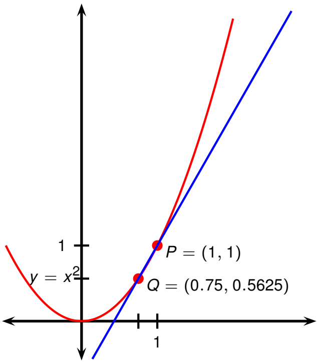

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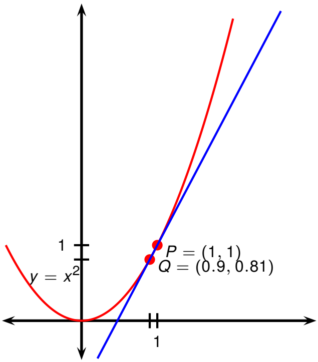

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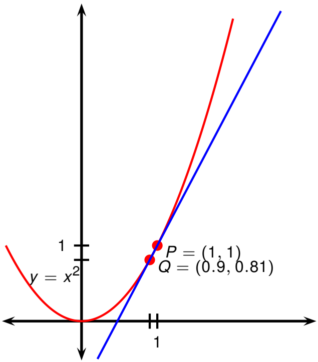

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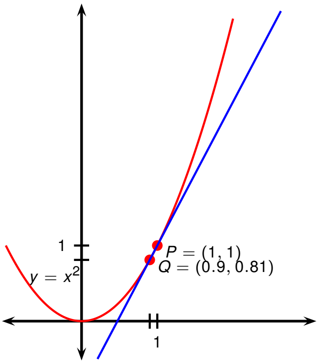

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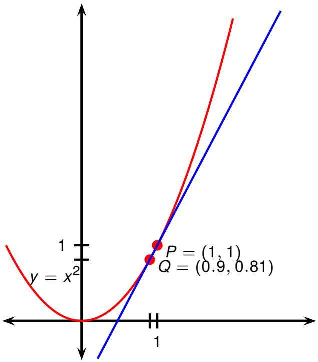

x	m_{PQ}	x	m_{PQ}
2	3	0	1
1.5	2.5	0.5	1.5
1.25	2.25	0.75	1.75
1.1	2.1	0.9	1.9
1.01	2.01	0.99	1.99

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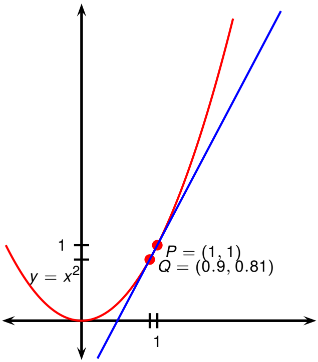

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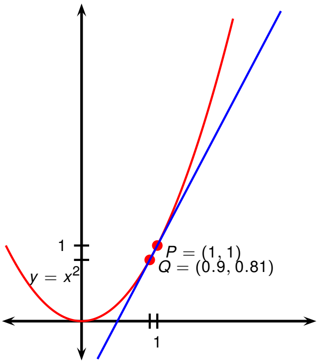

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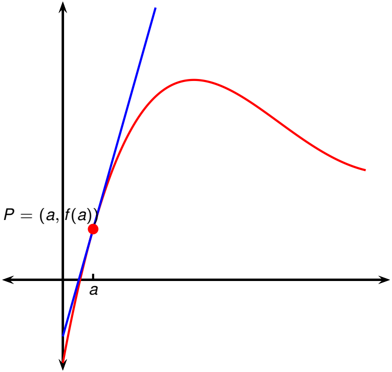
- This suggests the slope of the tangent should be 2.



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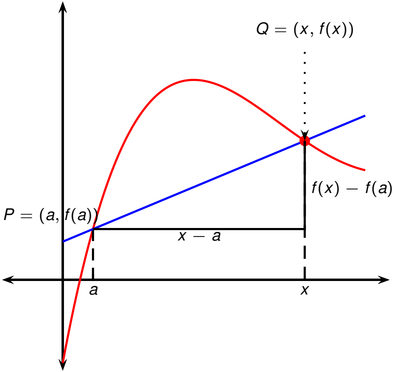

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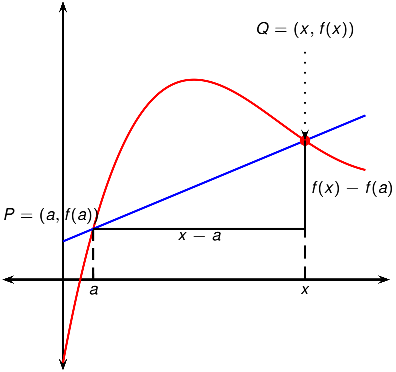

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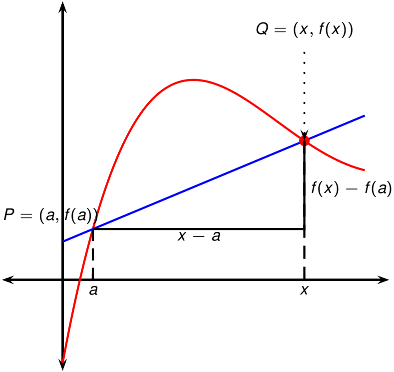

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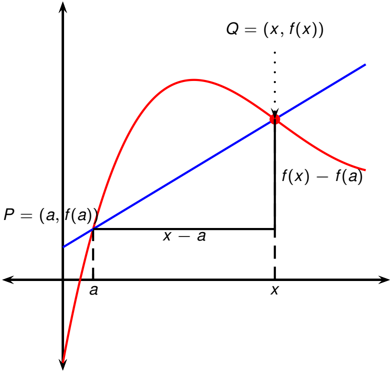

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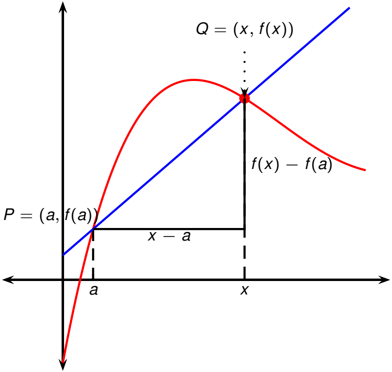

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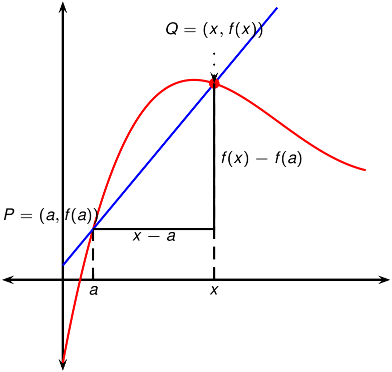

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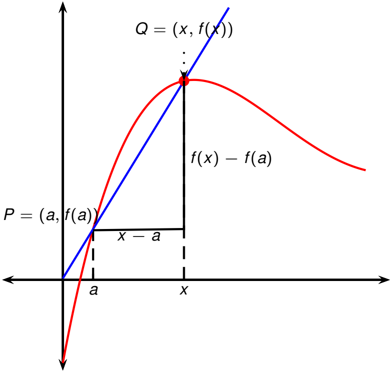

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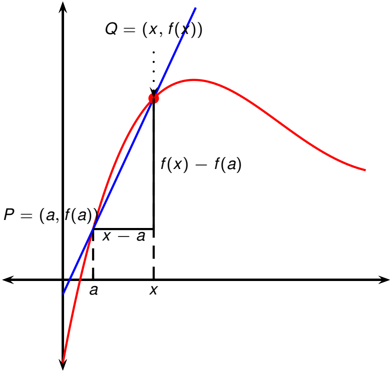

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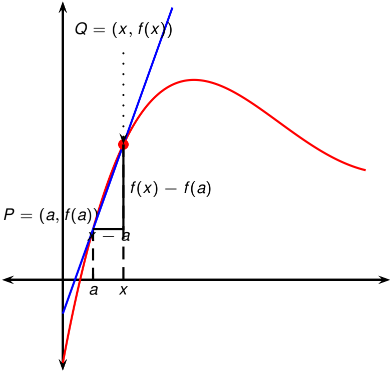

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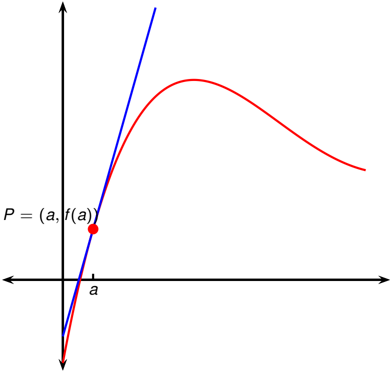

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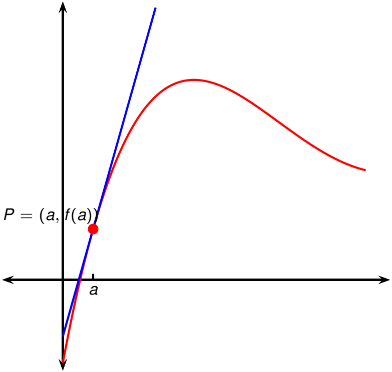

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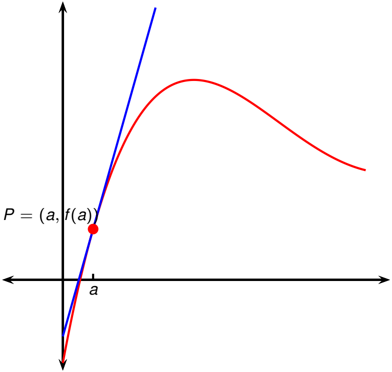

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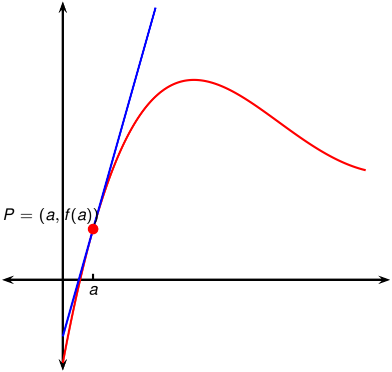

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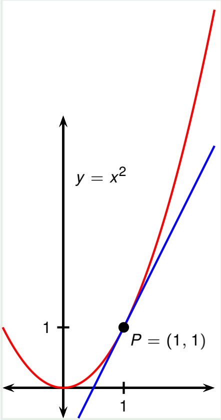
Note. Even if the limit does not exist a reasonable notion of a tangent line may still exist.

Example

Find an equation for the tangent line to the parabola $y = x^2$ at the point $P = (1, 1)$.



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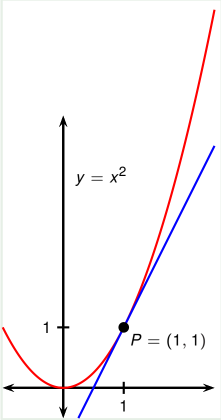

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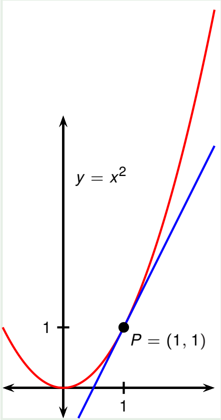

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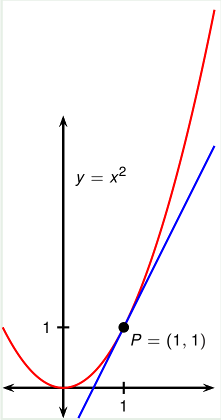

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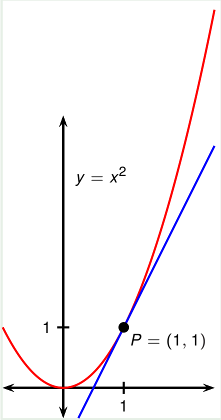

Here $a = 1$ and $f(x) = x^2$.

$$\begin{aligned} m &= \lim_{x\to 1} \frac{f(x) - f(1)}{x - 1} \\ &= \lim_{x\to 1} \frac{x^2 - 1}{x - 1} \\ &= \lim_{x\to 1} \frac{(\textbf{?} \quad)(\textbf{?} \quad)}{x - 1} \end{aligned}$$

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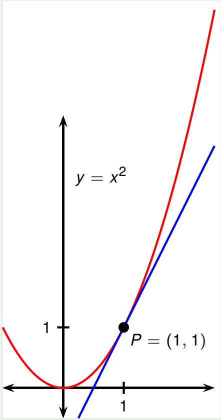

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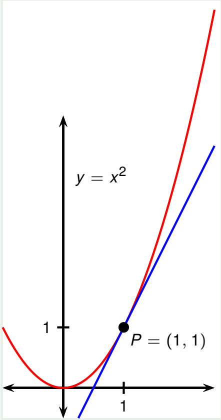

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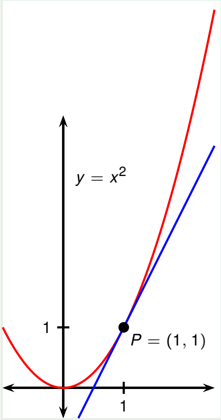

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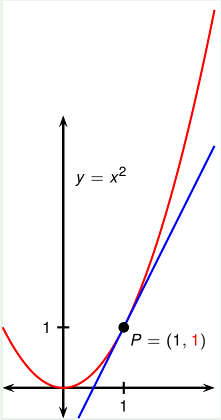

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Point-slope form: $y - 1 = 2(x - 1)$

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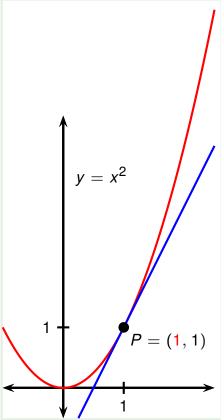

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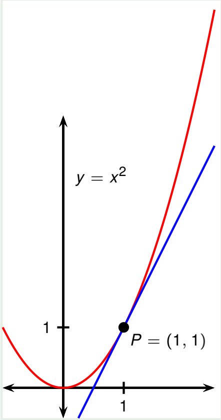

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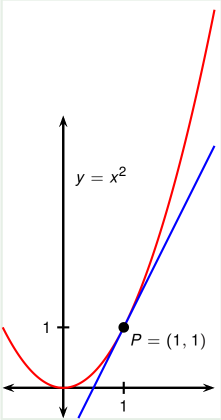

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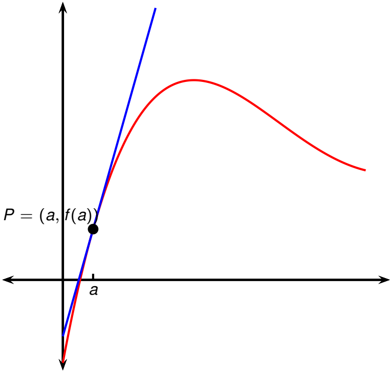
Point-slope form: $y - 1 = 2(x - 1)$, or finally $y = 2x - 1$.



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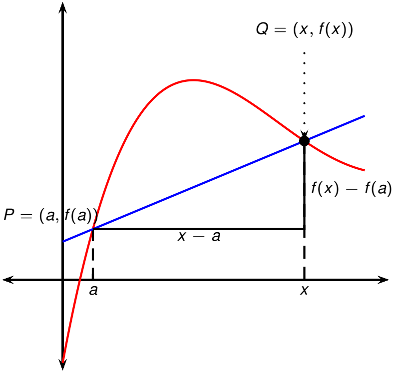

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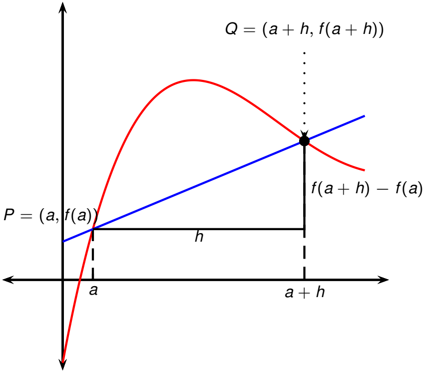

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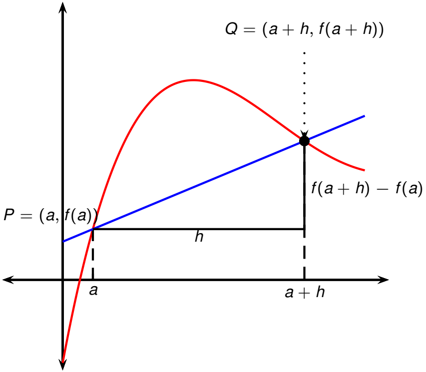

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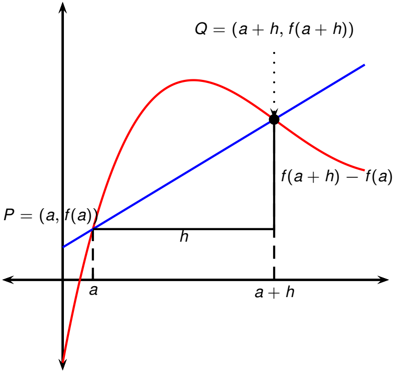

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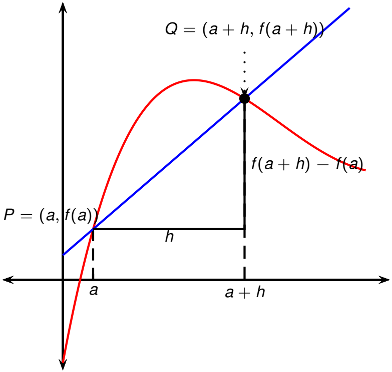

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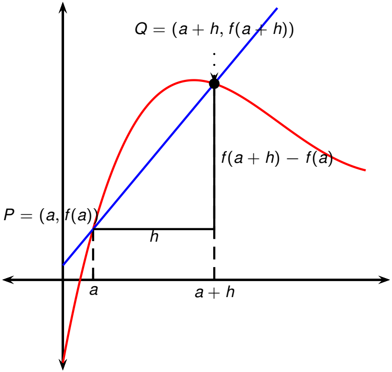

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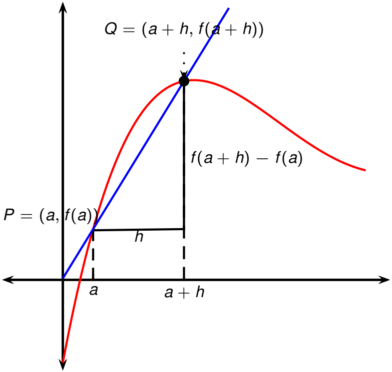

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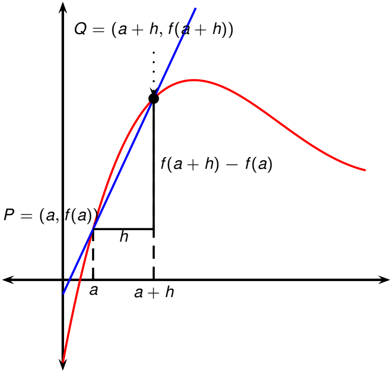

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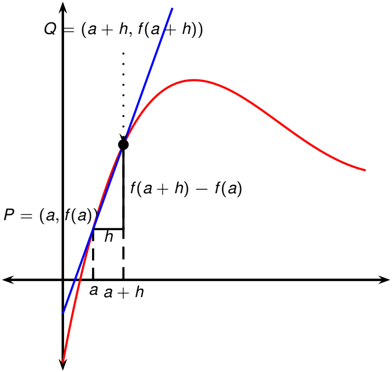

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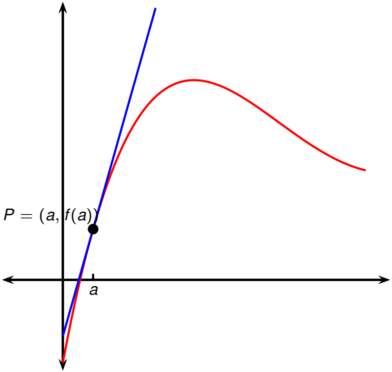

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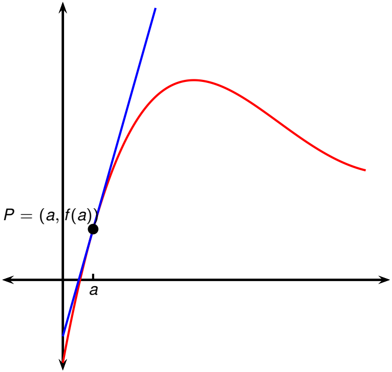

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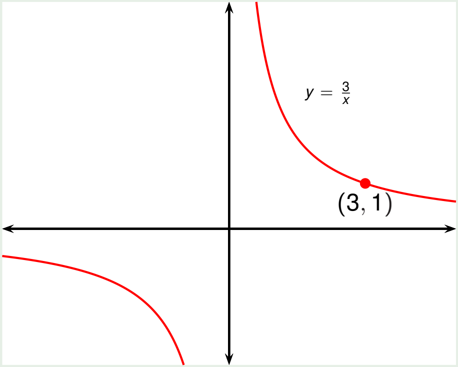
Tangent slope - equivalent expression:

$$m = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}.$$

Example

Find an equation for the tangent line to the hyperbola $y = \frac{3}{x}$ at the point $(3, 1)$.



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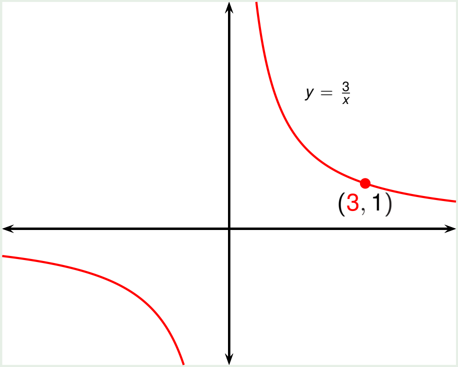

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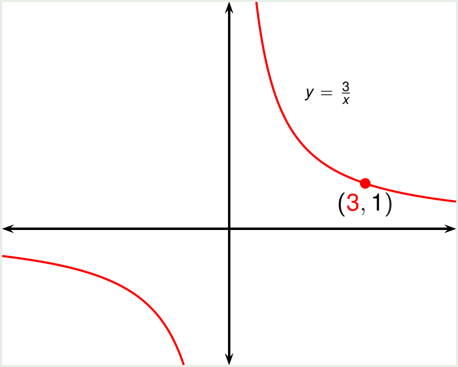

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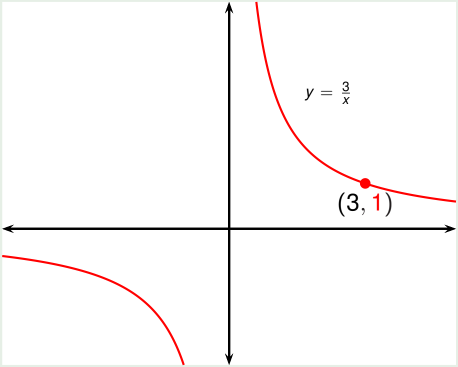

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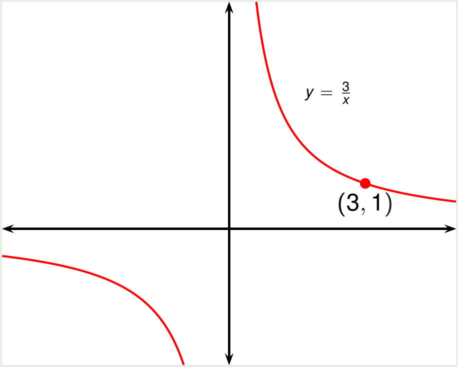

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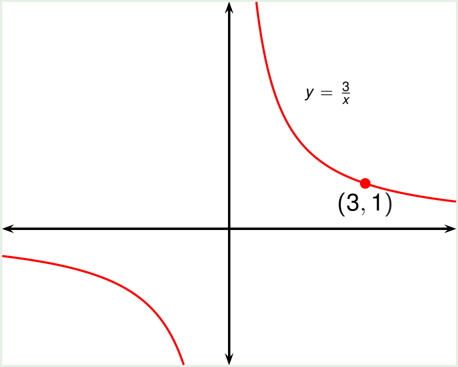

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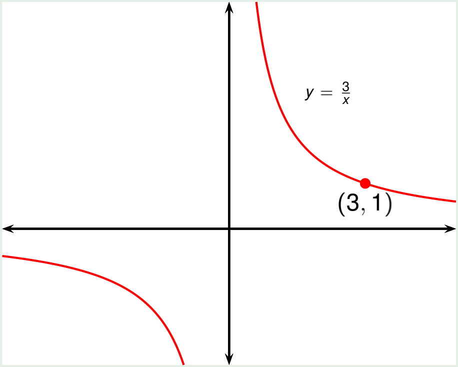

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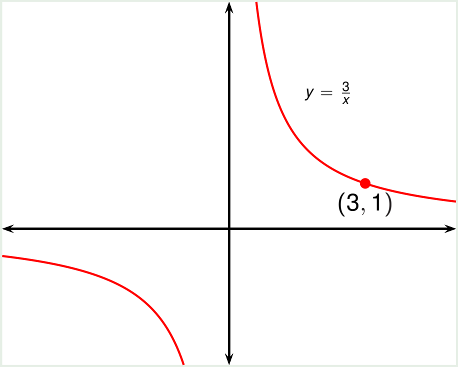

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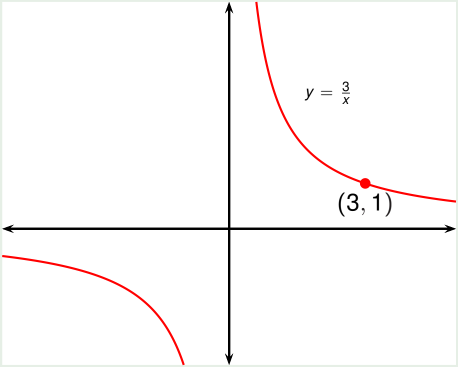

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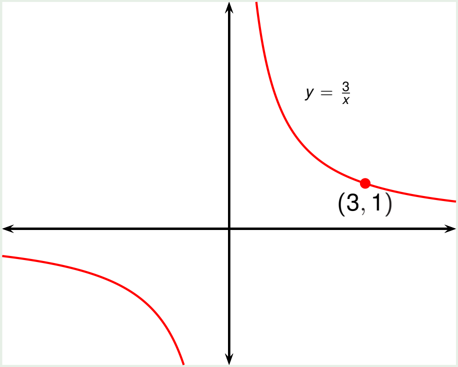

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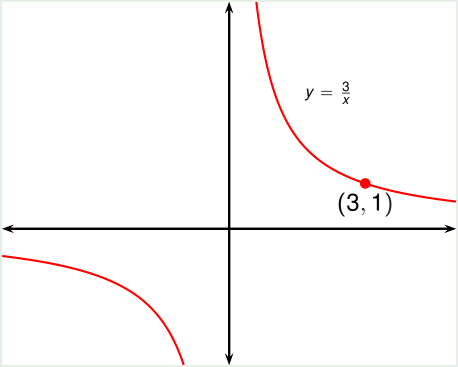

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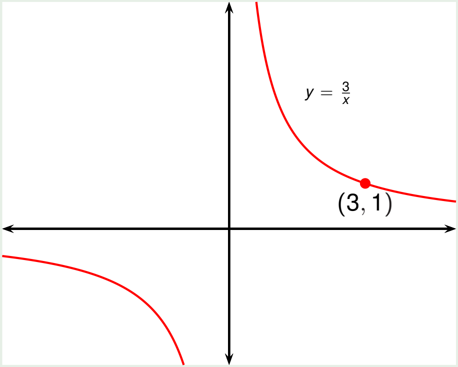

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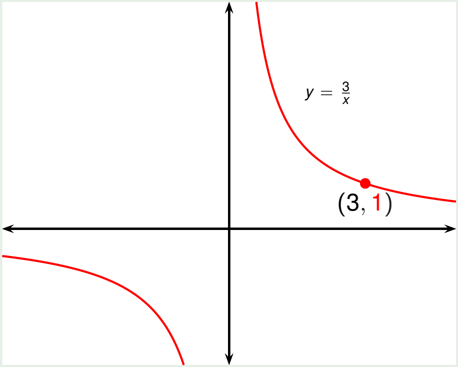

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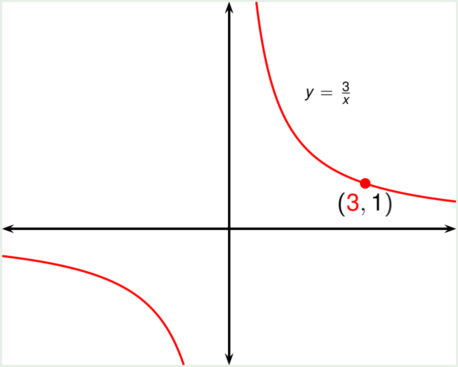

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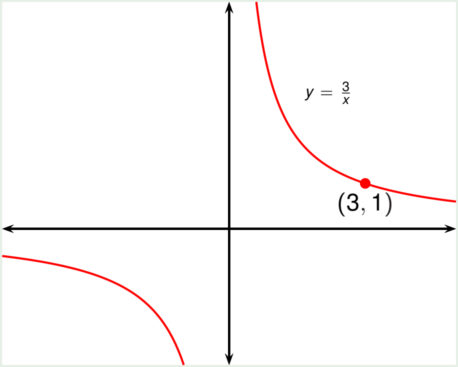

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Point-slope form: $y - 1 = -\frac{1}{3}(x - 3)$

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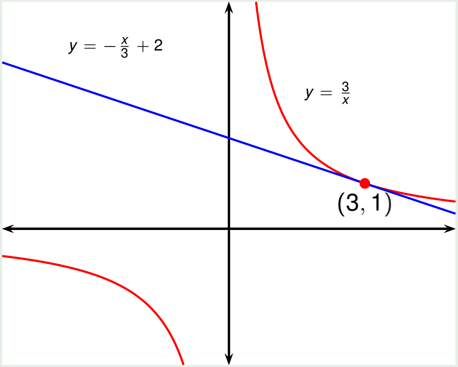

Point-slope form: $y - 1 = -\frac{1}{3}(x - 3)$, or finally $y = -\frac{x}{3} + 2$.

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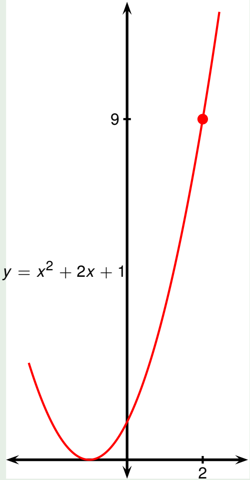
Example (Tangent line to a polynomial)

Find an equation for the tangent line to the parabola $y = x^2 + 2x + 1$ at the point $P = (2, 9)$.



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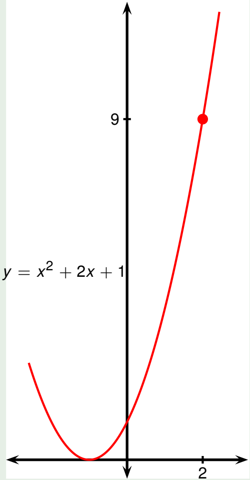

Here $a =$ **?** and $f(x) = x^2 + 2x + 1$.

$$m \qquad \lim_{x \to \mathbf{?}} \frac{f(x) - f(\mathbf{?})}{x - \mathbf{?}}$$

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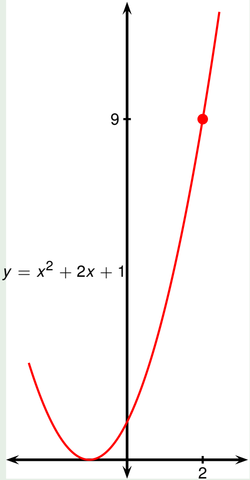

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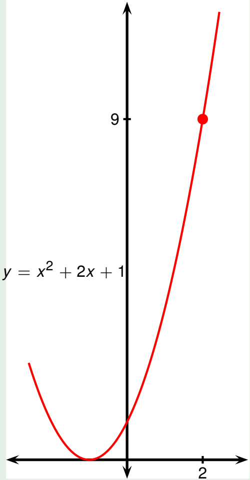

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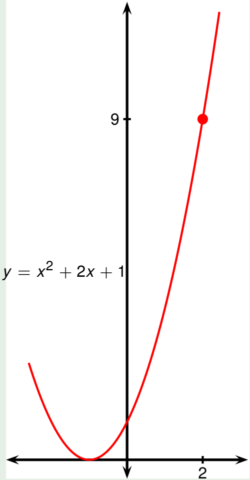

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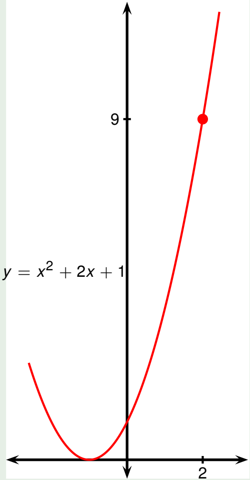

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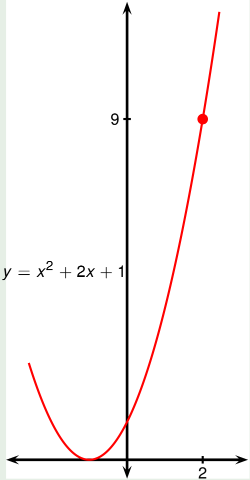

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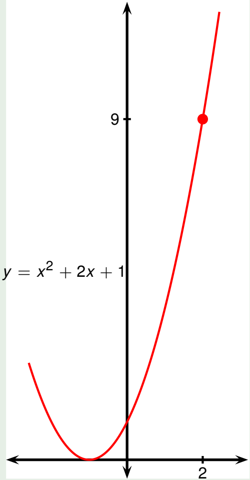

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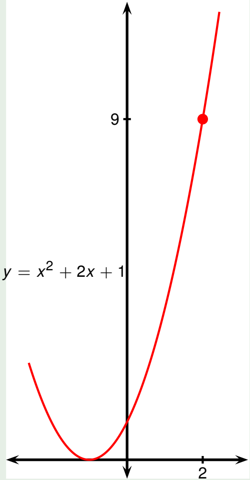

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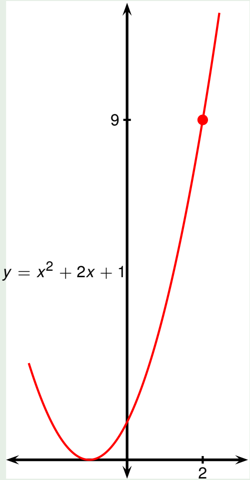

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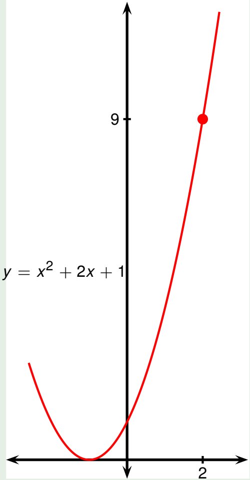

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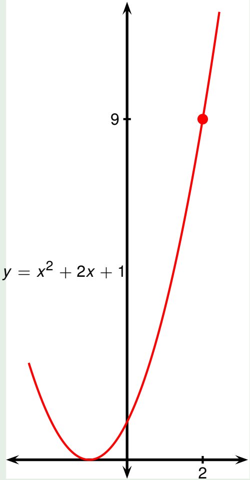
&= \lim_{x\to 2} \frac{x^2+2x-8}{x-2}\\
&= \lim_{x\to 2} \frac{\cancel{(x-2)}(x+4)}{\cancel{x-2}}\\
&= \lim_{x\to 2} (x+4) = 6.
\end{aligned}
$$

Example (Tangent line to a polynomial)

Find an equation for the tangent line to the parabola $y = x^2 + 2x + 1$ at the point $P = (2, 9)$.



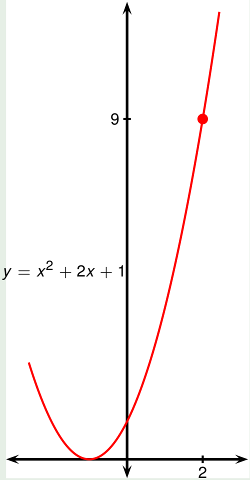
Here $a = 2$ and $f(x) = x^2 + 2x + 1$.

$$\begin{aligned} m &= \lim_{x\to 2} \frac{f(x) - f(2)}{x - 2} \\ &= \lim_{x\to 2} \frac{(x^2 + 2x + 1) - 9}{x - 2} \\ &= \lim_{x\to 2} \frac{x^2 + 2x - 8}{x - 2} \\ &= \lim_{x\to 2} \frac{\cancel{(x - 2)}(x + 4)}{\cancel{x - 2}} \\ &= \lim_{x\to 2} (x + 4) = 6. \end{aligned}$$

The tangent line: $y - 9 = 6(x - 2)$, or finally $y = 6x - 3$.

Example (Tangent line to a polynomial)

Find an equation for the tangent line to the parabola $y = x^2 + 2x + 1$ at the point $P = ({\color{red}2}, 9)$.



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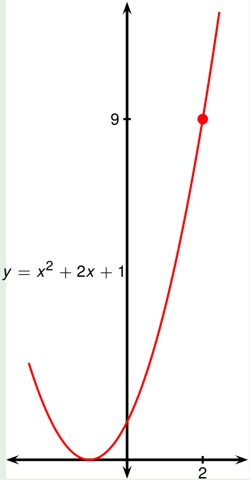

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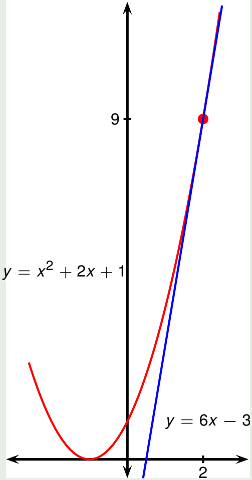

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Derivatives

Definition (Derivative)

The derivative of a function f at a number a, denoted by $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

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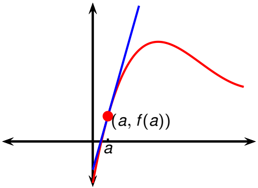

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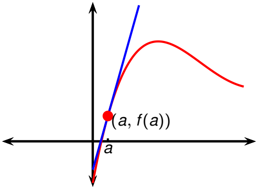

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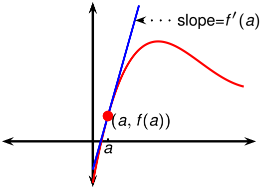

- The two alternative formulas result in equivalent definitions.
- Equivalent formulation. The derivative $f'(a)$ is the slope of the tangent line to $y = f(x)$ at $(a, f(a))$, provided that tangent line exists and is non-vertical.

Example

Find the derivative of the function $f(x) = x^2 - 8x + 9$ at the number a.

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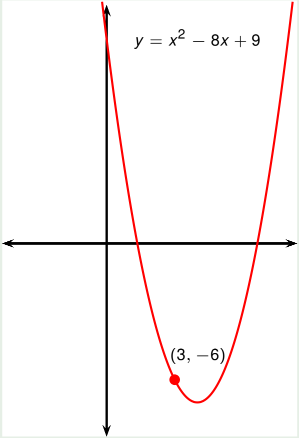
Example

Find an equation for the tangent line to the parabola $y = x^2 - 8x + 9$ at the point $P = (3, -6)$.



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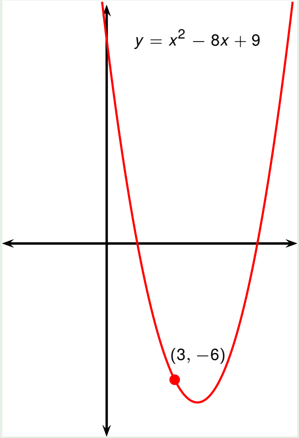

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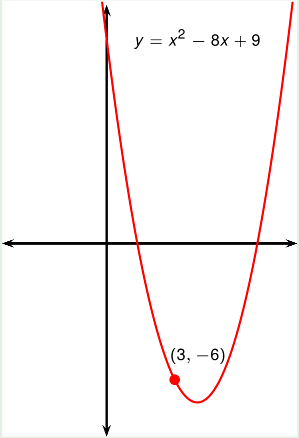

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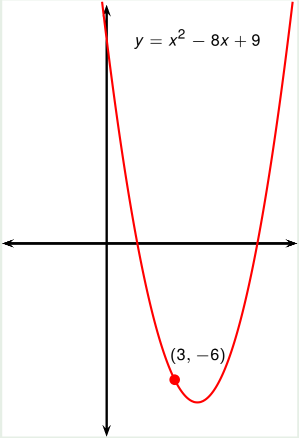

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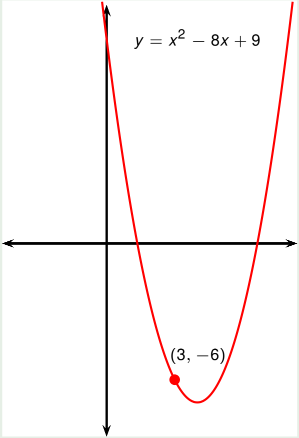

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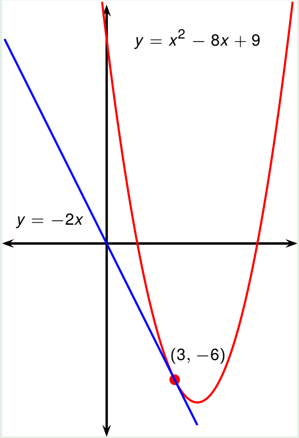

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- Point-slope form: $y - (-6) = -2(x - 3)$.
- Slope y-intercept form: $y = -2x$.

Other Notations for Derivative

If $y = f(x)$ is a function, there are many ways to write its derivative.

$$f'(x) = y' = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}f(x) = Df(x) = D_x f(x)$$

- $\frac{\mathrm{d}}{\mathrm{d}x}$ are called differentiation operators because they indicate the operation of differentiation, which is the process of calculating the derivative.
- $\mathrm{d}y/\mathrm{d}x$ is called Leibniz notation; it means the same as $f'(x)$.
- If we want to indicate the value of the derivative $\mathrm{d}y/\mathrm{d}x$ in Leibniz notation at a point a, we write

$$\frac{\mathrm{d}y}{\mathrm{d}x}_{|x=a} \qquad \text{or} \qquad \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_a \qquad \text{or} \qquad \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right]_a$$

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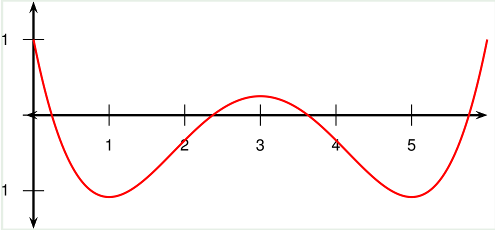
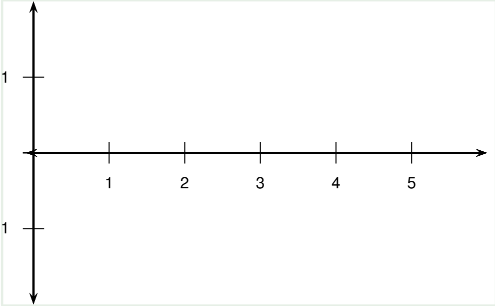
- f' is regarded a function in its own right, called the derivative of f.
- The domain of f' is $\{x | f'(x) \text{ exists } \}$.
- The domain of f' may be smaller than the domain of f.

Example

The graph of a function f appears below. Use it to sketch the graph of the derivative f'.





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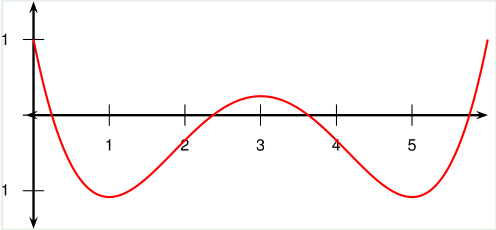

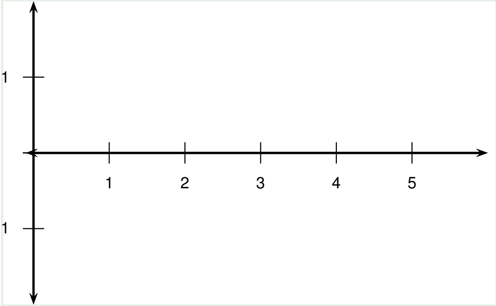

- Find the points where the tangent is horizontal ($m = 0$).

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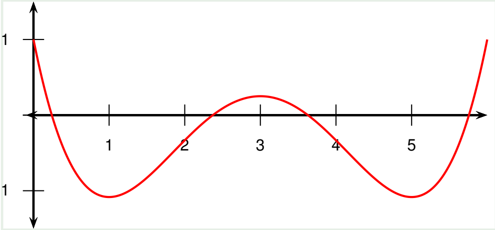

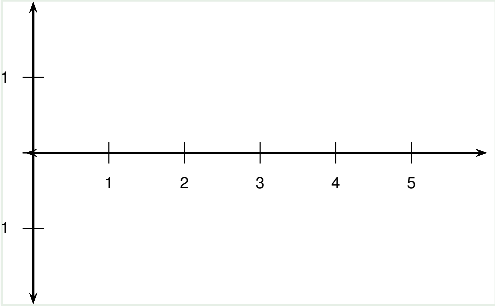

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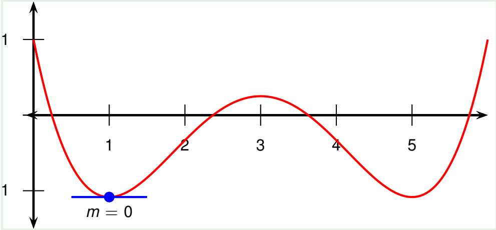

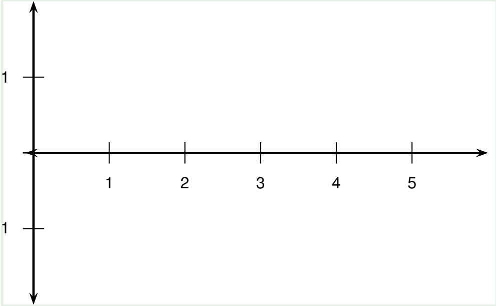

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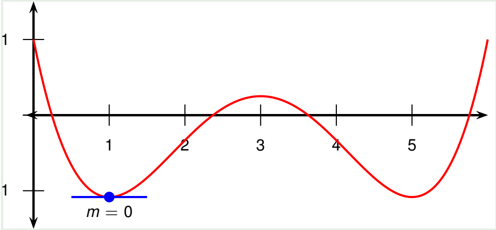

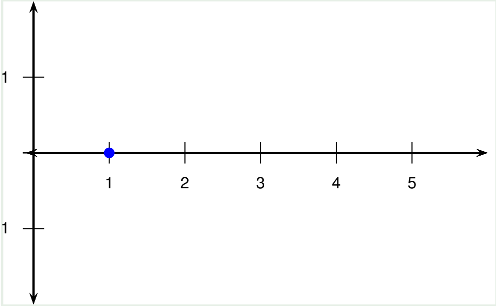

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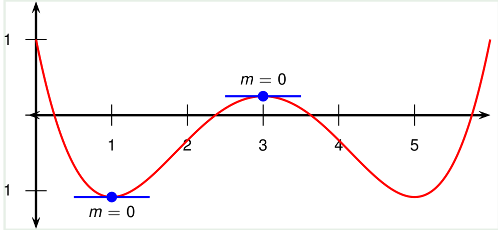

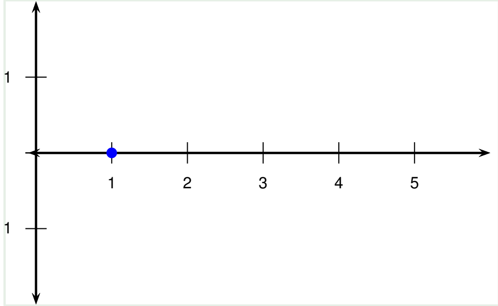

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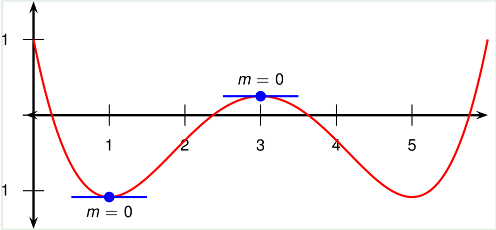

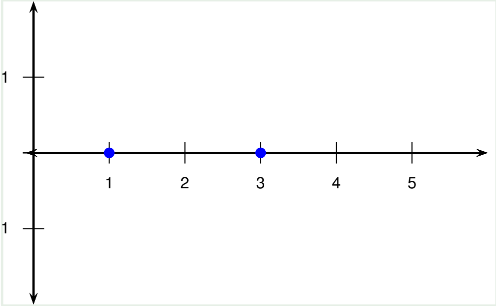

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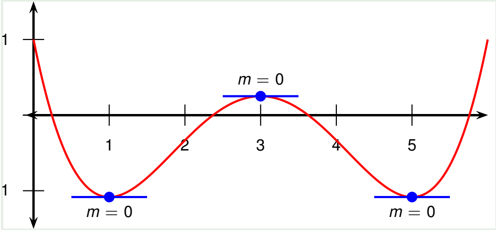

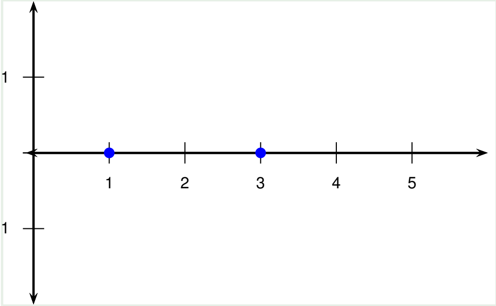

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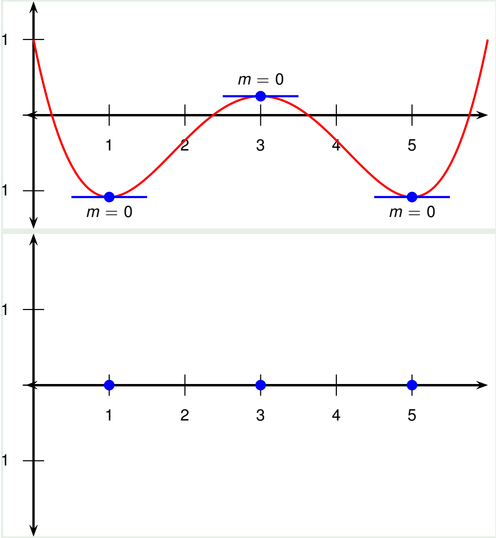

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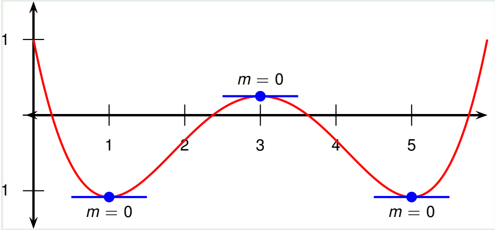

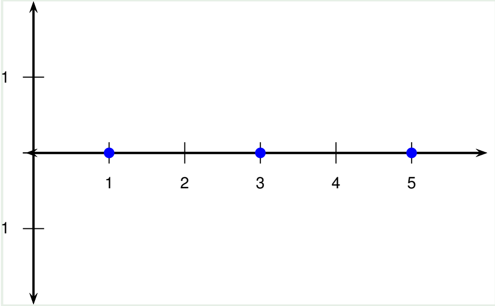

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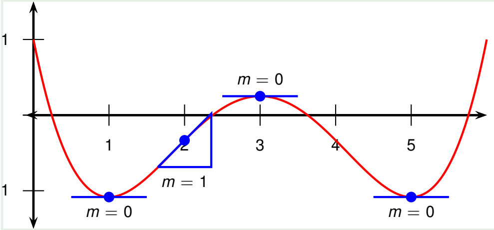

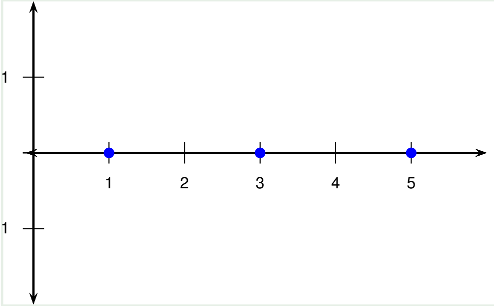

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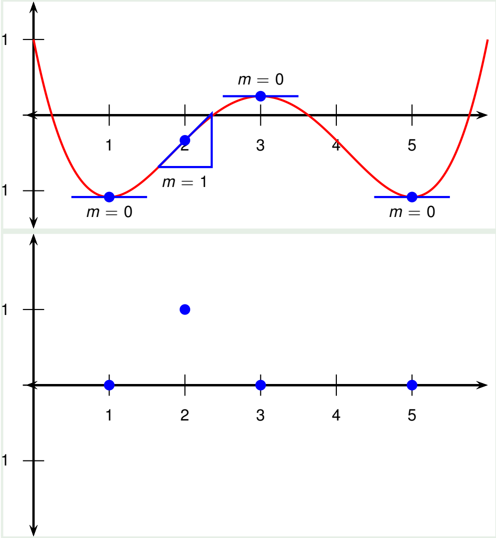

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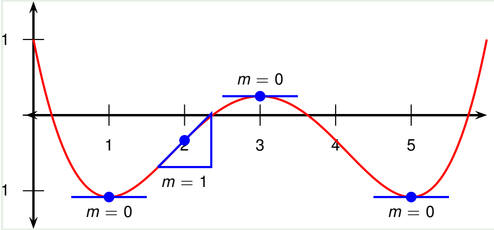

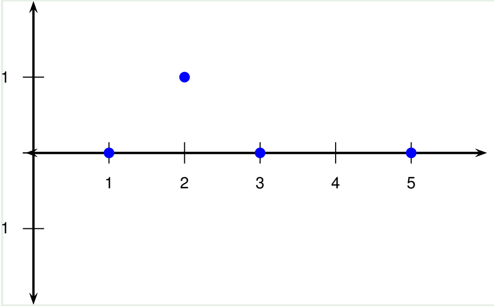

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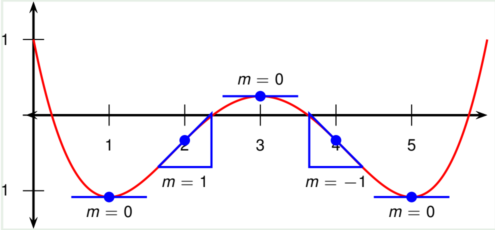

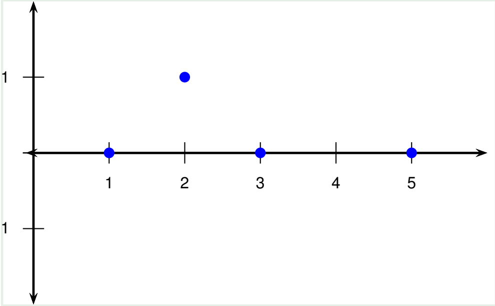

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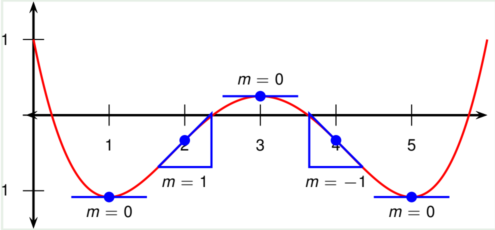

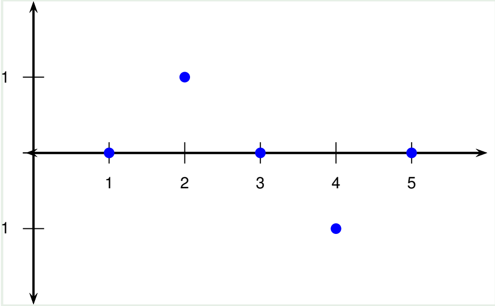

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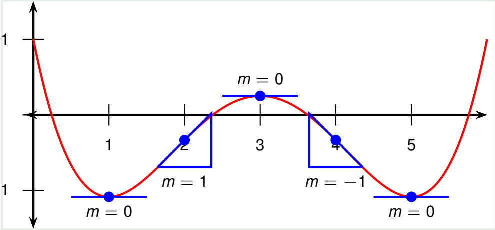

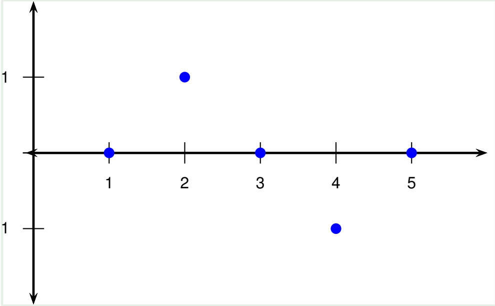

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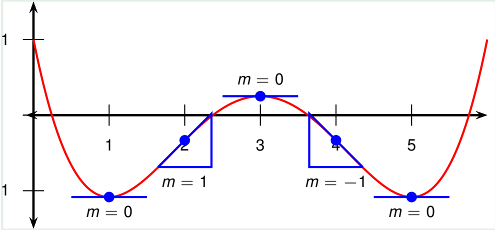

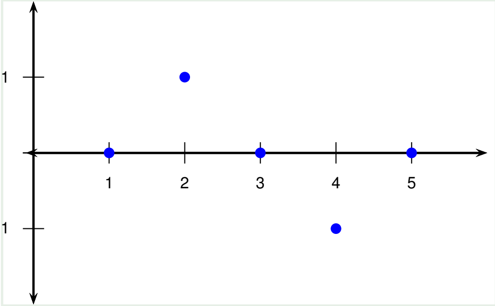

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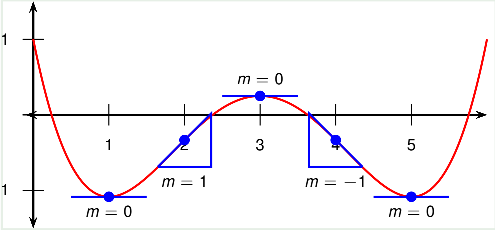

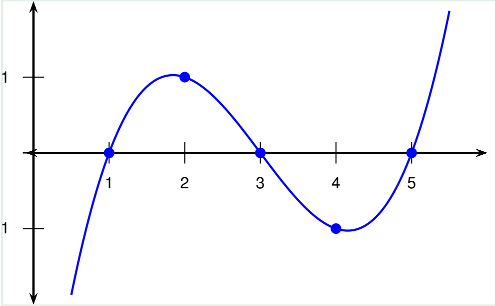

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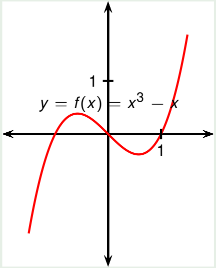

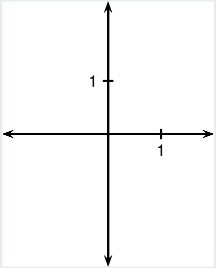

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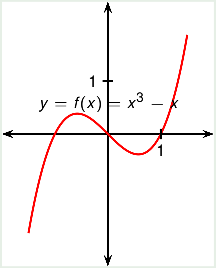

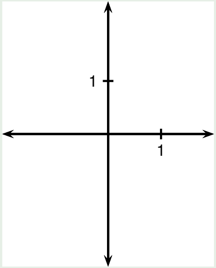

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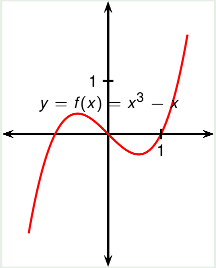

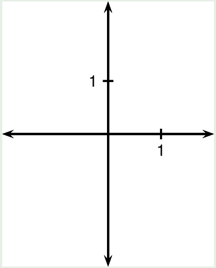

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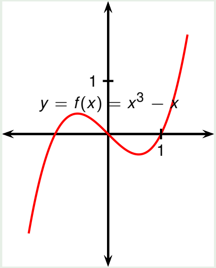

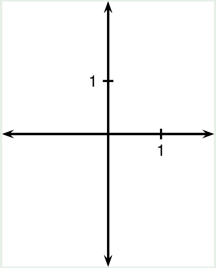

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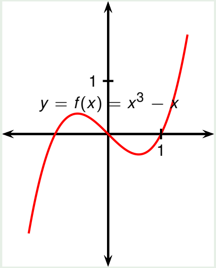

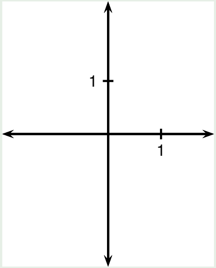

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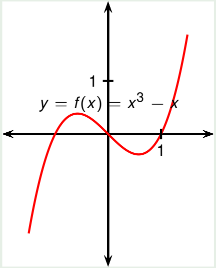

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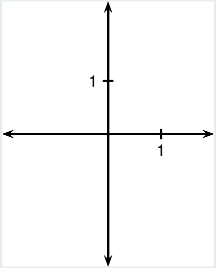

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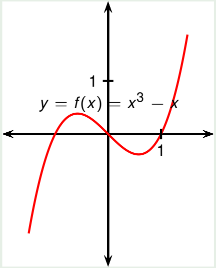

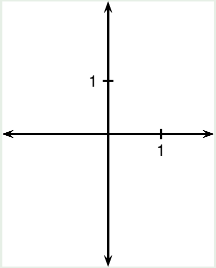

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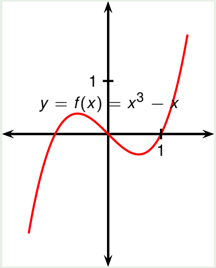

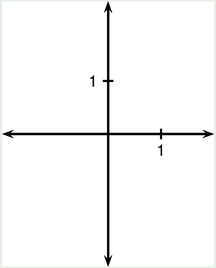

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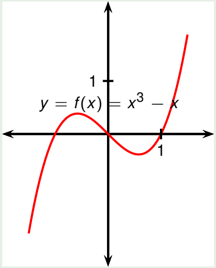

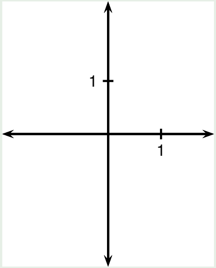

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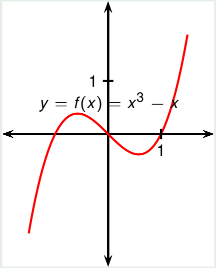

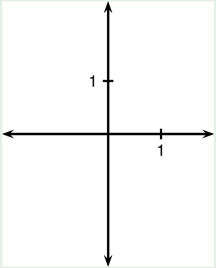

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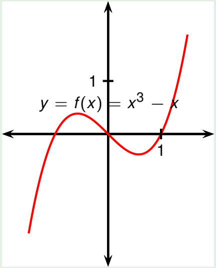

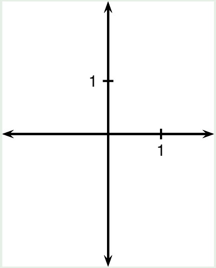

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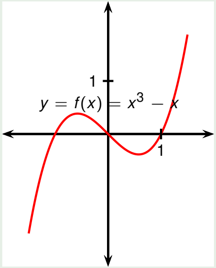

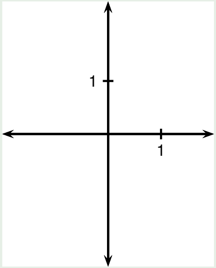

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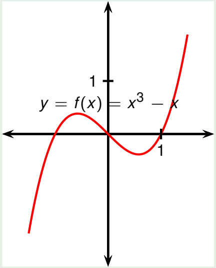

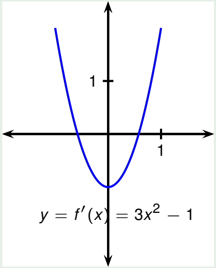

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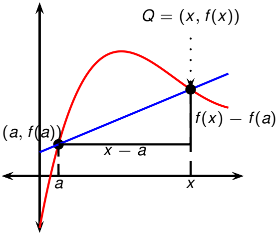

Slope of secant
$=$ average velocity

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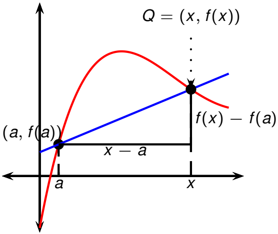
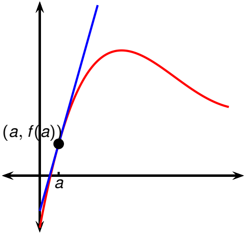


Slope of secant
= average velocity



Slope of tangent
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Example

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Therefore the velocity after 5s is $v(5) = 9.8(5) = 49$m/s.

Definition (Differentiable at a point)

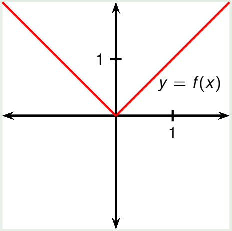
A function f is differentiable at a if $f'(a)$ exists.

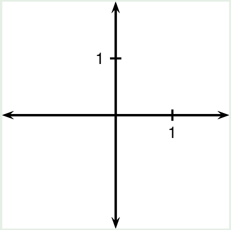
Definition (Differentiable on an interval)

A function f is differentiable on an open interval (a, b) (allowing $a = -\infty, b = \infty$) if it is differentiable at every number in the interval.

Example

Where is the function $f(x) = |x|$ differentiable?





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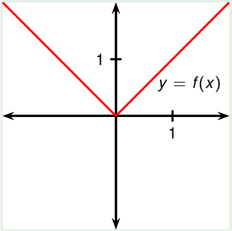

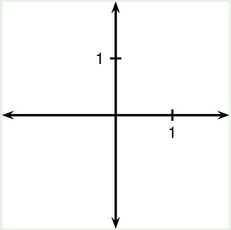

- Suppose $x > 0$.

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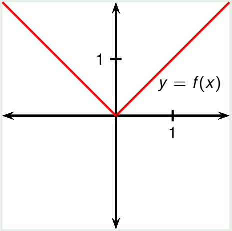

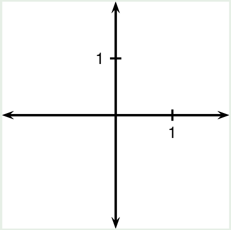

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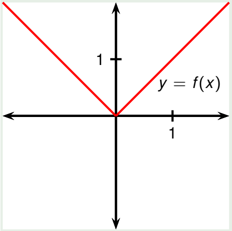

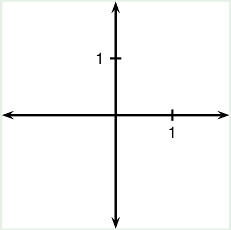

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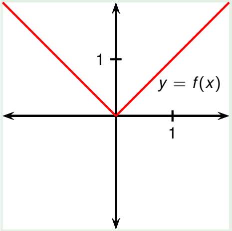

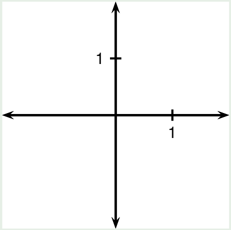

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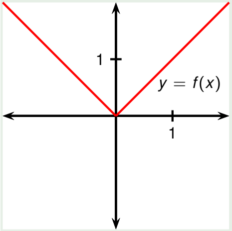

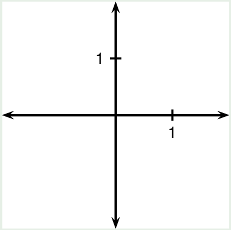

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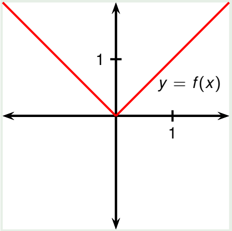

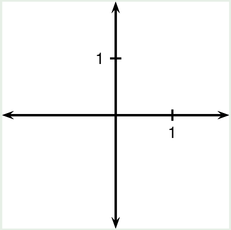

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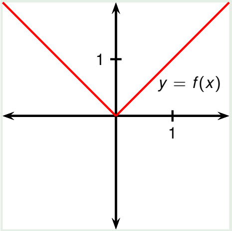

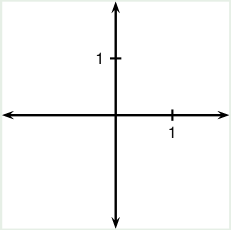

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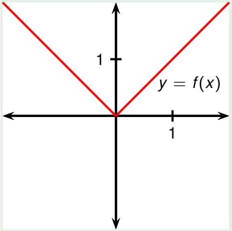

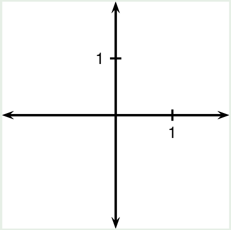

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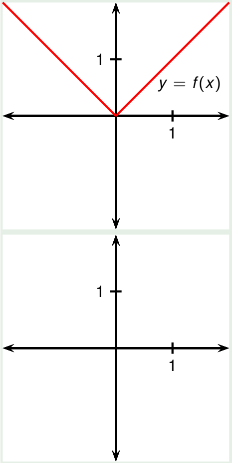

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Therefore f is differentiable for any $x > 0$.

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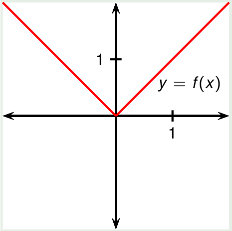

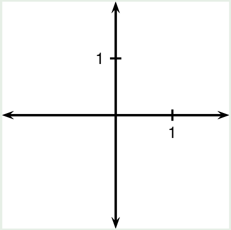

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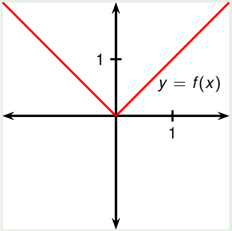

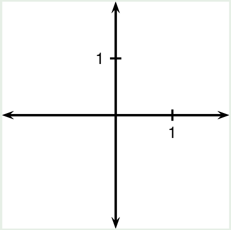

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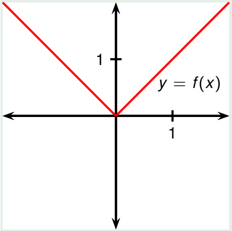

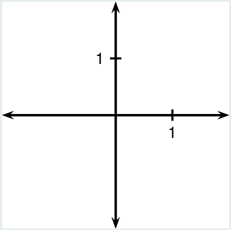

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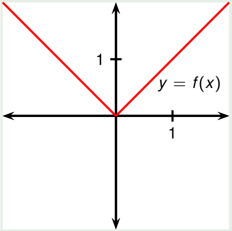

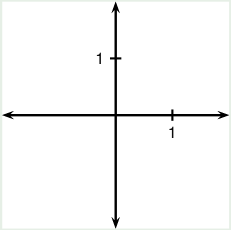

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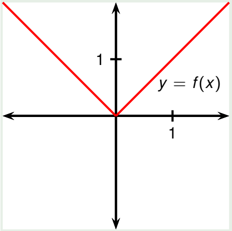

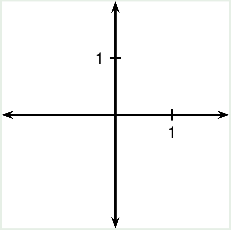

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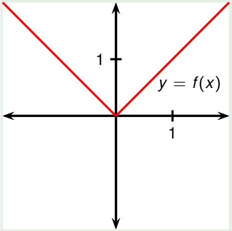

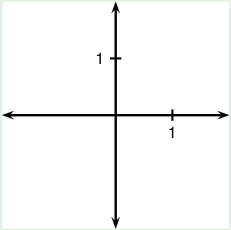

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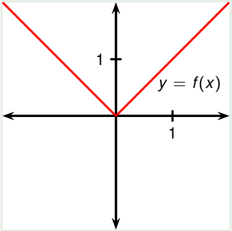

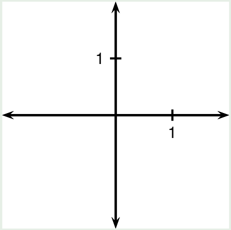

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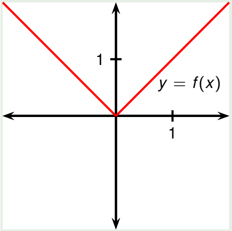

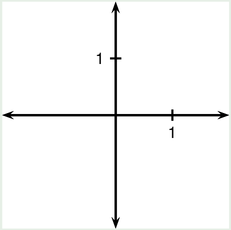

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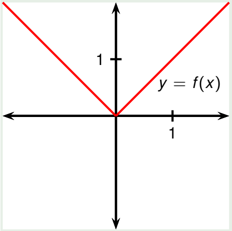

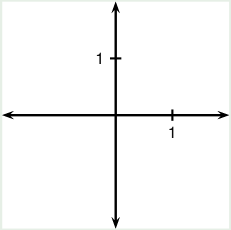

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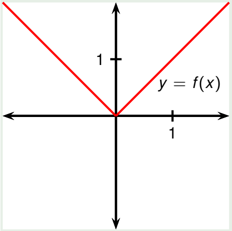

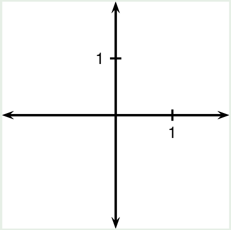

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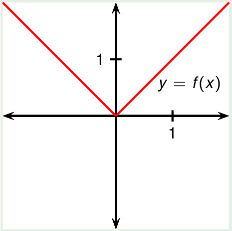

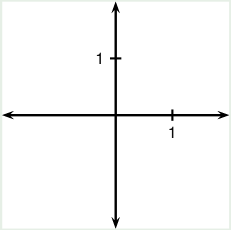

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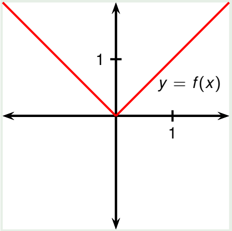

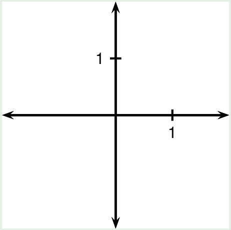

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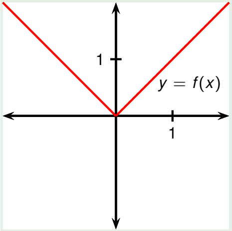

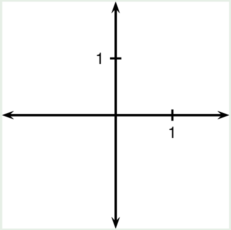

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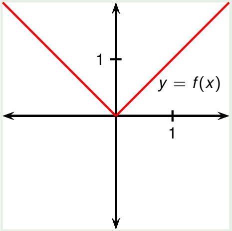

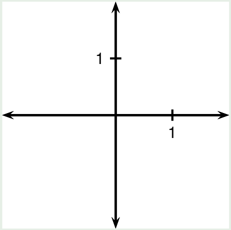

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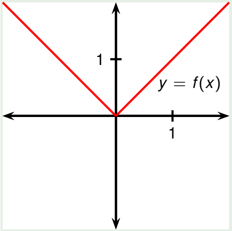

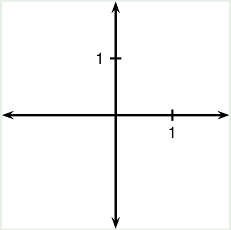

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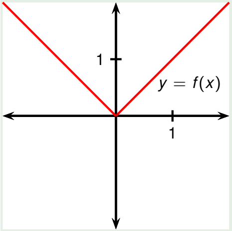

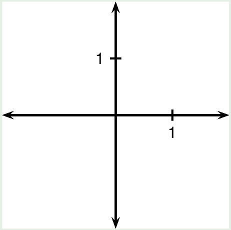

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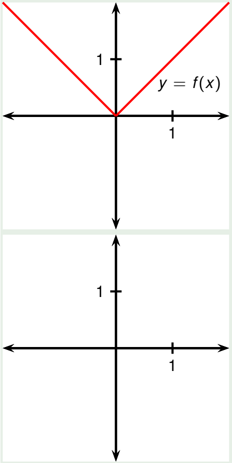

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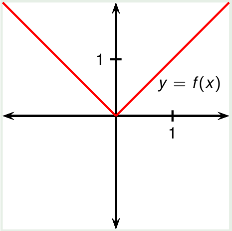

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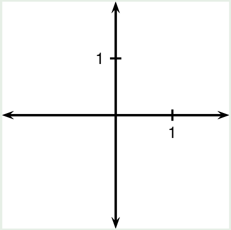

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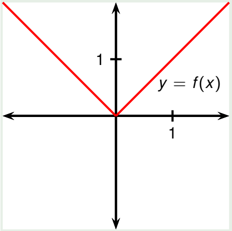

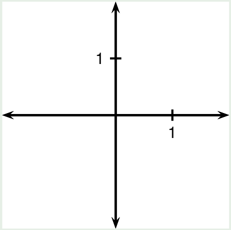

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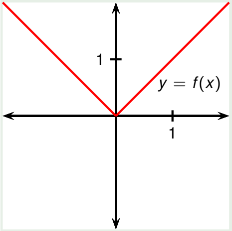

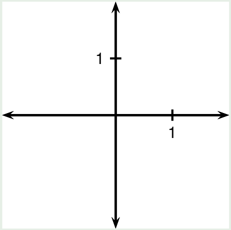

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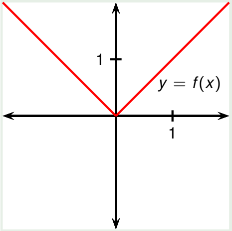

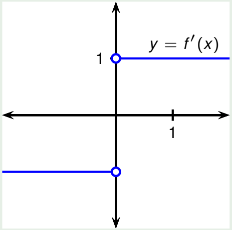

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$$f'(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

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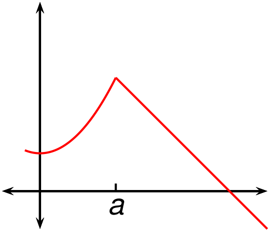

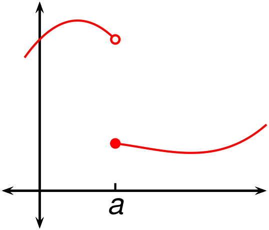

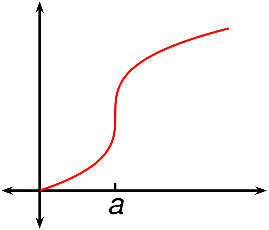

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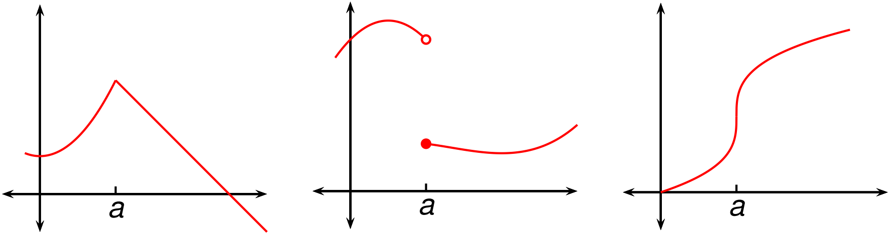

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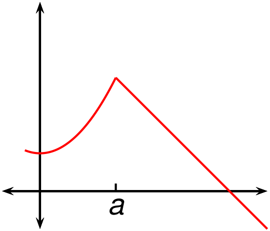

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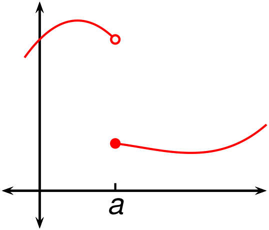

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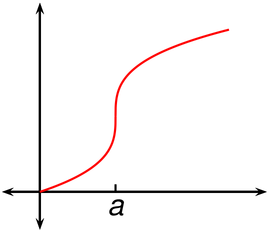

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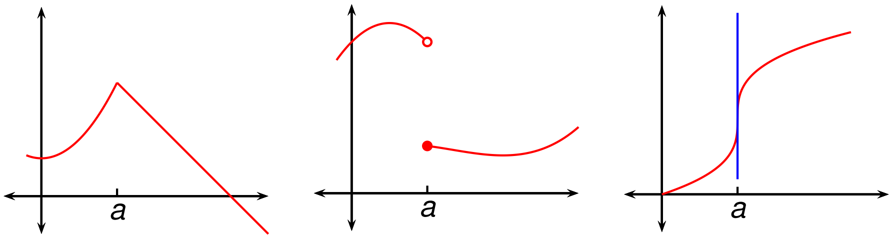

corner | discontinuity | vertical tangent

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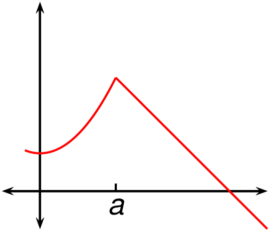

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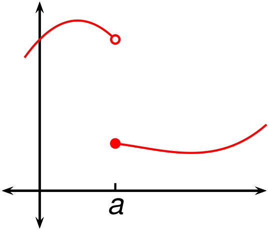

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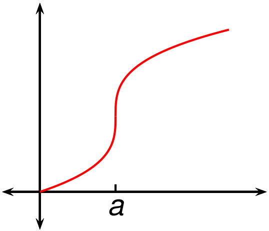

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Notation for Higher Derivatives

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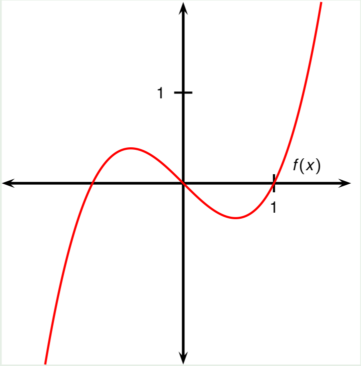
Note: Do not confuse the superscript in the notation for n^{th} derivative with exponent. The parenthesis indicate we mean derivatives rather than exponents.

Example

If $f(x) = x^3 - x$, find $f''(x)$.



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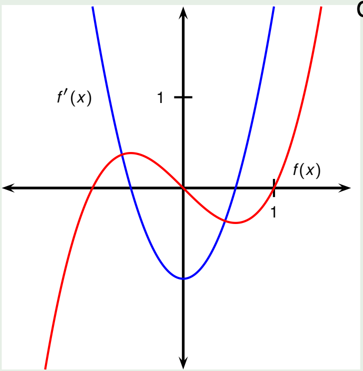

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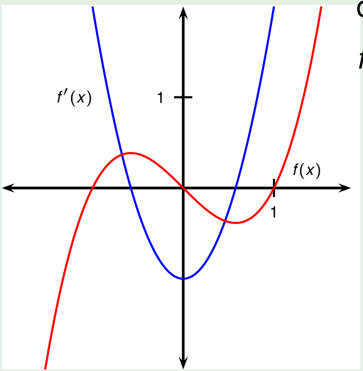

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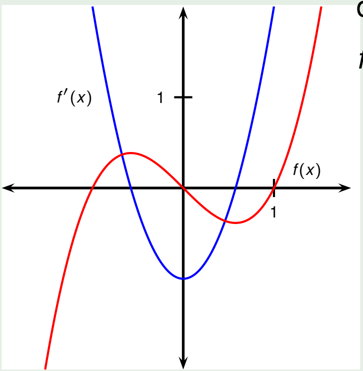

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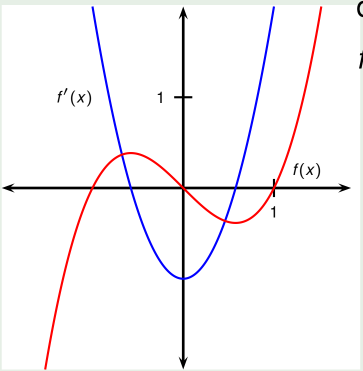

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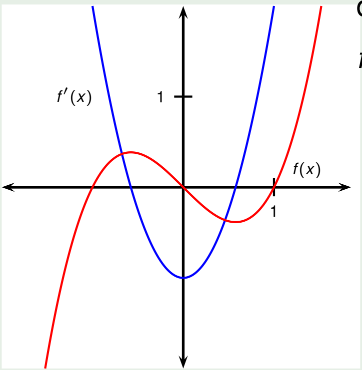

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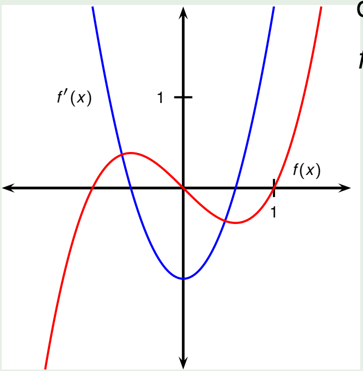

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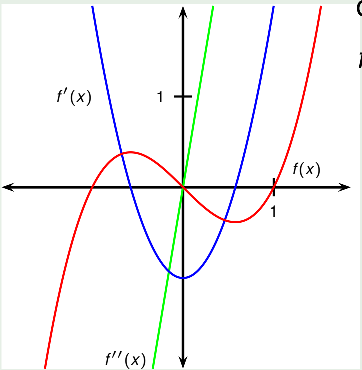

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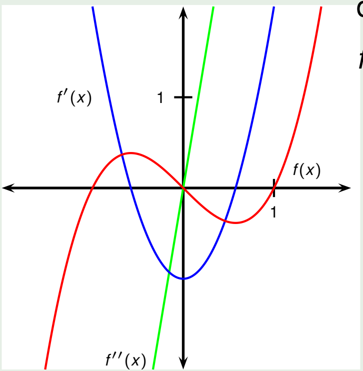

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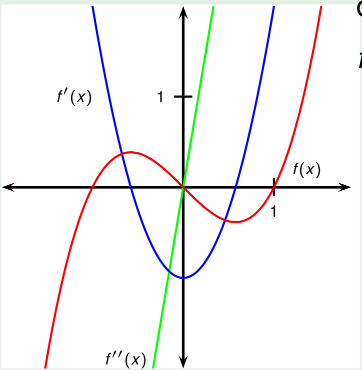

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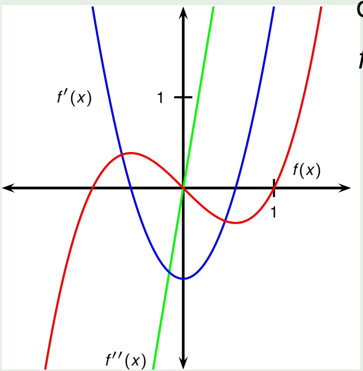

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Theorem (Derivative of a Constant Function)

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Power Functions

Now consider functions of the form $f(x) = x^n$, where n is a positive integer. For $f(x) = x$, the graph is the line $y = x$, which has slope 1. So

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If n is a positive integer, then $$\frac{\mathrm{d}}{\mathrm{d}x}(x^n) = nx^{n-1}.$$

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Let $f(x) = x^n$. Then

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Theorem (The Power Rule)

If n is a positive integer, then $$\frac{\mathrm{d}}{\mathrm{d}x}(x^n) = nx^{n-1}.$$

Proof.

Use this formula (which you can verify):

$$x^n - a^n = (x-a)(x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1}).$$

Let $f(x) = x^n$. Then

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

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$$\begin{aligned} f'(a) &= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a} \frac{x^n - a^n}{x-a} \\ &= \lim_{x\to a} \frac{(x-a)(x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1})}{x-a} \end{aligned}$$

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Theorem (The Power Rule)

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□

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□

Example (Power Rule)

If $f(x) = x^5$,

Then $f'(x) =$

If $y = x^{1000}$,

Then $y' =$

If $u = t^{22}$,

Then $\frac{du}{dt} =$

$\frac{d}{dr}(r^3) =$

Example (Power Rule)

If $f(x) = x^5$,
Then $f'(x) = \textbf{?}$

If $y = x^{1000}$,
Then $y' =$

If $u = t^{22}$,
Then $\frac{\mathrm{d}u}{\mathrm{d}t} =$

$\frac{\mathrm{d}}{\mathrm{d}r}(r^3) =$

Example (Power Rule)

If $f(x) = x^5$,

Then $f'(x) = 5x^4.$

If $y = x^{1000}$,

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Then $\frac{\mathrm{d}u}{\mathrm{d}t} =$

$\frac{\mathrm{d}}{\mathrm{d}r}(r^3) =$

Example (Power Rule)

If $f(x) = x^5$,
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If $y = x^{1000}$,
Then $y' = \mathbf{?}$

If $u = t^{22}$,
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Example (Power Rule)

If $f(x) = x^5$,
Then $f'(x) = 5x^4$.

If $y = x^{1000}$,
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If $u = t^{22}$,
Then $\dfrac{du}{dt} =$

$\dfrac{d}{dr}(r^3) =$

Example (Power Rule)

If $f(x) = x^5$,

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Example (Power Rule)

If $f(x) = x^5$,
Then $f'(x) = 5x^4$.

If $y = x^{1000}$,
Then $y' = 1000x^{999}$.

If $u = t^{22}$,
Then $\frac{\mathrm{d}u}{\mathrm{d}t} = 22t^{21}$.

$\frac{\mathrm{d}}{\mathrm{d}r}(r^3) =$

Example (Power Rule)

If $f(x) = x^5$,
Then $f'(x) = 5x^4$.

If $y = x^{1000}$,
Then $y' = 1000x^{999}$.

If $u = t^{22}$,
Then $\dfrac{du}{dt} = 22t^{21}$.

$\dfrac{d}{dr}(r^3) = \mathbf{?}$

Example (Power Rule)

$$\text{If} \quad f(x) = x^5,$$
$$\text{Then} \quad f'(x) = 5x^4.$$

$$\text{If} \quad y = x^{1000},$$
$$\text{Then} \quad y' = 1000x^{999}.$$

$$\text{If} \quad u = t^{22},$$
$$\text{Then} \quad \frac{\mathrm{d}u}{\mathrm{d}t} = 22t^{21}.$$

$$\frac{\mathrm{d}}{\mathrm{d}r}(r^3) = 3r^2.$$

The Relation between Ball Volume and Surface Area

There is a relationship between the surface area and the volume of a ball (in any dimension).

Dimension	Set of pts. at dist. $\leq r$ from origin	Inside measure name	Measure f-la	Boundary name	Boundary measure formula	Derivative of inside measure
3						
2						
1						

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3						
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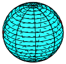
3	 ball					
2						
1						

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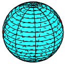
3	 ball	?				
2						
1						

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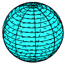
3	 ball	volume				
2						
1						

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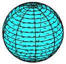
3	 ball	volume	?			
2						
1						

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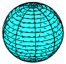
3	 ball	volume	$\frac{4}{3}\pi r^3$			
2						
1						

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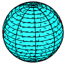
Dimension	Set of pts. at dist. $\leq r$ from origin	Inside measure name	Measure f-la	Boundary name	Boundary measure formula	Derivative of inside measure
3	 ball	volume	$\frac{4}{3}\pi r^3$			
2						
1						

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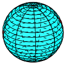
3	 ball	volume	$\frac{4}{3}\pi r^3$	 sphere		
2						
1						

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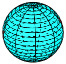
3	 ball	volume	$\frac{4}{3}\pi r^3$	 sphere	**?**	
2						
1						

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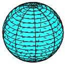
3	 ball	volume	$\frac{4}{3}\pi r^3$	 sphere	$4\pi r^2$	
2						
1						

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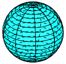
3	 ball	volume	$\frac{4}{3}\pi r^3$	 sphere	$4\pi r^2$	$\frac{d}{dr}\left(\frac{4}{3}\pi r^3\right) = ?$
2						
1						

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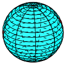
3	 ball	volume	$\frac{4}{3}\pi r^3$	 sphere	$4\pi r^2$	$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2$
2						
1						

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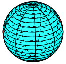
3	 ball	volume	$\frac{4}{3}\pi r^3$	 sphere	$4\pi r^2$	$\frac{d}{dr}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2$
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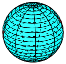
3	 ball	volume	$\frac{4}{3}\pi r^3$	 sphere	$4\pi r^2$	$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2$
2	 disk, circle					
1						

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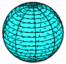
3	 ball	volume	$\frac{4}{3}\pi r^3$	 sphere	$4\pi r^2$	$\frac{d}{dr}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2$
2	 disk, circle	**?**				
1						

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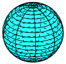
Dimension	Set of pts. at dist. $\leq r$ from origin	Inside measure name	Measure f-la	Boundary name	Boundary measure formula	Derivative of inside measure
3	 ball	volume	$\frac{4}{3}\pi r^3$	 sphere	$4\pi r^2$	$\frac{d}{dr}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2$
2	 disk, circle	circle area				
1						

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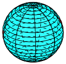
3	 ball	volume	$\frac{4}{3}\pi r^3$	 sphere	$4\pi r^2$	$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2$
2	 disk, circle	circle area	**?**			
1						

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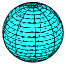
Dimension	Set of pts. at dist. $\leq r$ from origin	Inside measure name	Measure f-la	Boundary name	Boundary measure formula	Derivative of inside measure
3	 ball	volume	$\frac{4}{3}\pi r^3$	 sphere	$4\pi r^2$	$\frac{d}{dr}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2$
2	 disk, circle	circle area	πr^2			
1						

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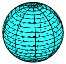
Dimension	Set of pts. at dist. $\leq r$ from origin	Inside measure name	Measure f-la	Boundary name	Boundary measure formula	Derivative of inside measure
3	 ball	volume	$\frac{4}{3}\pi r^3$	 sphere	$4\pi r^2$	$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2$
2	 disk, circle	circle area	πr^2	?		
1						

The Relation between Ball Volume and Surface Area

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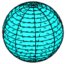
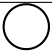
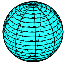
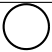
Dimension	Set of pts. at dist. $\leq r$ from origin	Inside measure name	Measure f-la	Boundary name	Boundary measure formula	Derivative of inside measure
3	 ball	volume	$\frac{4}{3}\pi r^3$	 sphere	$4\pi r^2$	$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2$
2	 disk, circle	circle area	πr^2	 circle (circumference)		
1						

The Relation between Ball Volume and Surface Area

There is a relationship between the surface area and the volume of a ball (in any dimension).

Dimension	Set of pts. at dist. $\leq r$ from origin	Inside measure name	Measure f-la	Boundary name	Boundary measure formula	Derivative of inside measure
3	 ball	volume	$\frac{4}{3}\pi r^3$	 sphere	$4\pi r^2$	$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2$
2	 disk, circle	circle area	πr^2	 circle (circumference)	**?**	
1						

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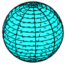
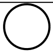
Dimension	Set of pts. at dist. $\leq r$ from origin	Inside measure name	Measure f-la	Boundary name	Boundary measure formula	Derivative of inside measure
3	 ball	volume	$\frac{4}{3}\pi r^3$	 sphere	$4\pi r^2$	$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2$
2	 disk, circle	circle area	πr^2	 circle (circumference)	$2\pi r$	
1						

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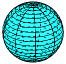
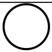
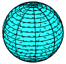
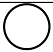
Dimension	Set of pts. at dist. $\leq r$ from origin	Inside measure name	Measure f-la	Boundary name	Boundary measure formula	Derivative of inside measure
3	 ball	volume	$\frac{4}{3}\pi r^3$	 sphere	$4\pi r^2$	$\frac{d}{dr}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2$
2	 disk, circle	circle area	πr^2	 circle (circumference)	$2\pi r$	$\frac{d}{dr}\left(\pi r^2\right) = ?$
1						

The Relation between Ball Volume and Surface Area

There is a relationship between the surface area and the volume of a ball (in any dimension).

Dimension	Set of pts. at dist. $\leq r$ from origin	Inside measure name	Measure f-la	Boundary name	Boundary measure formula	Derivative of inside measure
3	 ball	volume	$\frac{4}{3}\pi r^3$	 sphere	$4\pi r^2$	$\frac{d}{dr}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2$
2	 disk, circle	circle area	πr^2	 circle (circumference)	$2\pi r$	$\frac{d}{dr}\left(\pi r^2\right) = 2\pi r$
1						

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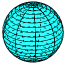
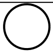
3	 ball	volume	$\frac{4}{3}\pi r^3$	 sphere	$4\pi r^2$	$\frac{d}{dr}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2$
2	 disk, circle	circle area	πr^2	 circle (circumference)	$2\pi r$	$\frac{d}{dr}\left(\pi r^2\right) = 2\pi r$
1						

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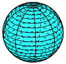
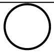
3	 ball	volume	$\frac{4}{3}\pi r^3$	 sphere	$4\pi r^2$	$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2$
2	 disk, circle	circle area	πr^2	 circle (circumference)	$2\pi r$	$\frac{\mathrm{d}}{\mathrm{d}r}\left(\pi r^2\right) = 2\pi r$
1	 interval					

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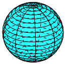
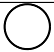
3	 ball	volume	$\frac{4}{3}\pi r^3$	 sphere	$4\pi r^2$	$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2$
2	 disk, circle	circle area	πr^2	 circle (circumference)	$2\pi r$	$\frac{\mathrm{d}}{\mathrm{d}r}\left(\pi r^2\right) = 2\pi r$
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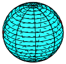
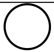
3	 ball	volume	$\frac{4}{3}\pi r^3$	 sphere	$4\pi r^2$	$\frac{d}{dr}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2$
2	 disk, circle	circle area	πr^2	 circle (circumference)	$2\pi r$	$\frac{d}{dr}\left(\pi r^2\right) = 2\pi r$
1	 interval	length				

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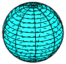
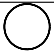
3	 ball	volume	$\frac{4}{3}\pi r^3$	 sphere	$4\pi r^2$	$\frac{d}{dr}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2$
2	 disk, circle	circle area	πr^2	 circle (circumference)	$2\pi r$	$\frac{d}{dr}\left(\pi r^2\right) = 2\pi r$
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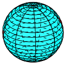
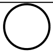
3	 ball	volume	$\frac{4}{3}\pi r^3$	 sphere	$4\pi r^2$	$\frac{d}{dr}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2$
2	 disk, circle	circle area	πr^2	 circle (circumference)	$2\pi r$	$\frac{d}{dr}\left(\pi r^2\right) = 2\pi r$
1	 interval	length	$2r$			

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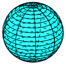
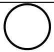
3	 ball	volume	$\frac{4}{3}\pi r^3$	 sphere	$4\pi r^2$	$\frac{d}{dr}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2$
2	 disk, circle	circle area	πr^2	 circle (circumference)	$2\pi r$	$\frac{d}{dr}\left(\pi r^2\right) = 2\pi r$
1	 interval	length	$2r$	?		

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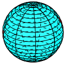
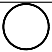
3	 ball	volume	$\frac{4}{3}\pi r^3$	 sphere	$4\pi r^2$	$\frac{d}{dr}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2$
2	 disk, circle	circle area	πr^2	 circle (circumference)	$2\pi r$	$\frac{d}{dr}\left(\pi r^2\right) = 2\pi r$
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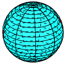
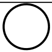
3	 ball	volume	$\frac{4}{3}\pi r^3$	 sphere	$4\pi r^2$	$\frac{d}{dr}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2$
2	 disk, circle	circle area	πr^2	 circle (circumference)	$2\pi r$	$\frac{d}{dr}\left(\pi r^2\right) = 2\pi r$
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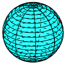
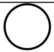
3	 ball	volume	$\frac{4}{3}\pi r^3$	 sphere	$4\pi r^2$	$\frac{d}{dr}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2$
2	 disk, circle	circle area	πr^2	 circle (circumference)	$2\pi r$	$\frac{d}{dr}\left(\pi r^2\right) = 2\pi r$
1	 interval	length	$2r$	 endpts.	2	

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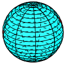
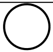
3	 ball	volume	$\frac{4}{3}\pi r^3$	 sphere	$4\pi r^2$	$\frac{d}{dr}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2$
2	 disk, circle	circle area	πr^2	 circle (circumference)	$2\pi r$	$\frac{d}{dr}\left(\pi r^2\right) = 2\pi r$
1	 interval	length	$2r$	 endpts.	2	$\frac{d}{dr}(2r) = $ **?**

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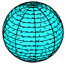
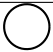
3	 ball	volume	$\frac{4}{3}\pi r^3$	 sphere	$4\pi r^2$	$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2$
2	 disk, circle	circle area	πr^2	 circle (circumference)	$2\pi r$	$\frac{\mathrm{d}}{\mathrm{d}r}\left(\pi r^2\right) = 2\pi r$
1	 interval	length	$2r$	 endpts.	2	$\frac{\mathrm{d}}{\mathrm{d}r}(2r) = 2$