

Precalculus

Euler's formula and trigonometric identities

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Outline

1 Trigonometric Identities

- Trigonometric Identities and Complex Numbers
- Trigonometric Identities without Complex Numbers
- Trig Identities Using $\sin^2 \theta + \cos^2 \theta = 1$
- Trig Identities Using the Angle Sum Formulas
- Trig Identities Exercises

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Euler's Formula

Theorem (Euler's Formula)

$$e^{ix} = \cos x + i \sin x,$$

where $e \approx 2.71828$ is Euler's/Napier's constant .

Proof.

Recall $n! = 1 \cdot 2 \cdot 3 \cdot \dots \cdot (n-1) \cdot n$. Borrow from Calc II the f-las:



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$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \dots$$

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Rearrange.



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Rearrange. **Plug-in** $z = ix$.



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Rearrange. Plug-in $z = ix$. Use $i^2 = -1$. Multiply $\sin x$ by i . **Add to get**
 $e^{ix} = \cos x + i \sin x.$



Trigonometric Identities Revisited

- $e^{ix} = \cos x + i \sin x$ (Euler's Formula).
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All trigonometric formulas can be easily derived using the above formulas.

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- By convention, when dealing with trigonometric identities we do not account for the domains of the involved expressions.
- For example, $\frac{\sin \theta}{\sin \theta} = 1$ is considered a valid trigonometric identity, although, when considered as a function, the left hand side is not defined for $\theta \neq 0$.

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- To prove a trigonometric identity means to show that the two sides of the equality sign are equivalent.

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- The discussion here also applies for trigonometric identities in more than one variables.

Types of identities

- In the present course we deal with two basic types of trigonometric identities.
- First, identities that involve operations on the arguments of the trigonometric functions.
 - Example: $\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$ (this is one of the angle sum identities); $\sin \theta + \sin(-\theta) = 0$.
 - Such identities can be proved using the angle sum formulas and the even/odd function properties of \sin , \cos .
- Second, identities that involve trigonometric functions of one variable.
 - Example: $\tan^2 \theta + 1 = \sec^2 \theta$.
 - Such identities can be proved only using the already demonstrated Pythagorean identity $\sin^2 \theta + \cos^2 \theta = 1$.
- The Pythagorean identity follows from the angle sum formulas and the even/odd function properties of \sin , \cos , so all trigonometric identities follow from those properties alone.

Example

Demonstrate the trigonometric identity $\csc^2 \theta - 1 = \cot^2 \theta$.

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We transform the left hand side to the right one.

$$\begin{aligned}\csc^2 \theta - 1 &= \frac{1}{\sin^2 \theta} - 1 \\ &= \frac{1 - \sin^2 \theta}{\sin^2 \theta} \\ &= \frac{\cos^2 \theta}{\sin^2 \theta} \\ &= \cot^2 \theta \quad \Bigg| \quad \text{as desired.}\end{aligned}$$

Example

Verify the trigonometric identity $2 \csc^2 \alpha = \frac{1}{1 - \cos \alpha} + \frac{1}{1 + \cos \alpha}$

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We transform the right hand side to the left.

$$\begin{aligned}\frac{1}{1 - \cos \alpha} + \frac{1}{1 + \cos \alpha} &= \frac{(1 + \cos \alpha)}{(1 - \cos \alpha)(1 + \cos \alpha)} + \frac{(1 - \cos \alpha)}{(1 - \cos \alpha)(1 + \cos \alpha)} \\&= \frac{1 + \cos \alpha + 1 - \cos \alpha}{1 - \cos^2 \alpha} \\&= \frac{2}{\sin^2 \alpha} \\&= 2 \csc^2 \alpha\end{aligned}$$

as desired.

Example

Verify the identity $\ln(\sec \theta - 1) + \ln(\sec \theta + 1) - 2 \ln(\sec \theta) = 2 \ln(\sin \theta)$, where $0 < \theta < \frac{\pi}{2}$.

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Verify the identity $\ln(\sec \theta - 1) + \ln(\sec \theta + 1) - 2 \ln(\sec \theta) = 2 \ln(\sin \theta)$, where $0 < \theta < \frac{\pi}{2}$. We transform the left hand side to the right.

$$\begin{aligned} & \ln(\sec \theta - 1) + \ln(\sec \theta + 1) - 2 \ln(\sec \theta) \\ = & \ln((\sec \theta - 1)(\sec \theta + 1)) - \ln(\sec^2 \theta) \\ = & \ln(\sec^2 \theta - 1) - \ln(\sec^2 \theta) \\ = & \ln\left(\frac{\sec^2 \theta - 1}{\sec^2 \theta}\right) \\ = & \ln\left(1 - \frac{1}{\sec^2 \theta}\right) \\ = & \ln(1 - \cos^2 \theta) \\ = & \ln(\sin^2 \theta) \\ = & 2 \ln(\sin \theta) \end{aligned}$$

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Example

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$$\begin{aligned}\tan x + \cot x &= \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} \\&= \frac{\sin^2 x}{\sin x \cos x} + \frac{\cos^2 x}{\sin x \cos x} \\&= \frac{\sin^2 x + \cos^2 x}{\sin x \cos x} \\&= \frac{1}{\sin x \cos x} \\&= \frac{1}{\sin x} \frac{1}{\cos x} \\&= \csc x \sec x,\end{aligned}$$

as desired.

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Prove the trigonometric identity.

$$(\sin \theta + \cos \theta)^2 = 1 + \sin(2\theta)$$

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$$(\sin \theta + \cos \theta)^2 = ?$$

$$(A + B)^2 = ?$$

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Express $\sin(3x)$ and $\cos(3x)$ via $\cos x$ and $\sin x$.

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as desired.

Strategy for proving trigonometric identities

An expression is rational trigonometric if it is written using $\sin \theta$, $\cos \theta$ and the four arithmetic operations.

Question

Is there a general method for proving all rational trigonometric identities in one variable?

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there is a standard method to verify whether two (rational) expressions in s and c are equal.
- The method is rather cumbersome for a human and is best suited for computers.

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 - The full method: set $s = \sin \theta$, $c = \cos \theta$.
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 - You may need to use trig functions of angles smaller than θ , for example $\sin \left(\frac{\theta}{2}\right)$, $\cos \left(\frac{\theta}{2}\right)$.

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 - You may need to use trig functions of angles smaller than θ , for example $\sin \left(\frac{\theta}{2}\right)$, $\cos \left(\frac{\theta}{2}\right)$.
 - A fraction of θ such that all appearing angles are integer multiples of it will always work.

Proving the following identities is a good exercise.

$$\textcircled{1} \sin \theta \cot \theta = \cos \theta.$$

$$\textcircled{2} (\sin \theta + \cos \theta)^2 = 1 + \sin(2\theta).$$

$$\textcircled{3} \sec \theta - \cos \theta = \tan \theta \sin \theta.$$

$$\textcircled{4} \tan^2 \theta - \sin^2 \theta = \tan^2 \theta \sin^2 \theta.$$

$$\textcircled{5} \cot^2 \theta + \sec^2 \theta = \tan^2 \theta + \csc^2 \theta.$$

$$\textcircled{6} 2 \csc(2\theta) = \sec \theta \csc \theta.$$

$$\textcircled{7} \tan(2\theta) = \frac{2 \tan \theta}{1 - \tan^2 \theta}.$$

$$\textcircled{8} \frac{1}{1 - \sin \theta} + \frac{1}{1 + \sin \theta} = 2 \sec^2 \theta.$$

$$\textcircled{9} \tan \alpha + \tan \beta = \frac{\sin(\alpha + \beta)}{\cos \alpha \cos \beta}.$$

$$\textcircled{10} \tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}.$$

$$\textcircled{11} \sin(3\theta) + \sin \theta = 2 \sin(2\theta) \cos \theta.$$

$$\textcircled{12} \cos(3\theta) = 4 \cos^3 \theta - 3 \cos \theta.$$

$$\textcircled{13} 1 + \tan^2 \theta = \sec^2 \theta.$$

$$\textcircled{14} 1 + \csc^2 \theta = \cot^2 \theta.$$

$$\textcircled{15} 2 \cos^2(2x) = 2 \sin^4 \theta + 2 \cos^4 \theta - \sin^2(2\theta).$$

$$\textcircled{16} \frac{1 + \tan\left(\frac{\theta}{2}\right)}{1 - \tan\left(\frac{\theta}{2}\right)} = \tan \theta + \sec \theta.$$