Calculus I
Homework
Derivatives: linearity, product and quotient rules

1. Compute the derivative.

   (a) $f(x) = 2^{2015}$.

   answer: 0

   (b) $f(x) = \pi^{2015}$.

   answer: 0

   (c) $f(x) = 2 - \frac{2}{3}x$.

   answer: $-\frac{2}{3}$

   (d) $f(x) = \frac{3}{4}x^8$.

   answer: $6x^7$

   (e) $f(x) = x^3 - 4x + 6$.

   answer: $-4 + 3x^2$

   (f) $f(t) = \frac{1}{2}t^6 - 3t^4 + t$.

   answer: $3t^5 - 12t^3 + 1$

   (g) $g(x) = x^2(1 - 2x)$.

   answer: $2x - 6x^2$

   (h) $h(x) = (x - 2)(2x + 3)$.

   answer: $4x - 1$

   (i) $f(x) = 2x^{-\frac{3}{4}}$.

   answer: $-\frac{3}{2}x^{-\frac{7}{4}}$

   (j) $f(x) = cx^{-6}$.

   answer: $-6cx^{-7}$

   (k) $A(x) = -\frac{12}{x^5}$.

   answer: $60x^{-6}$

   Solution. 1.g Approach 1. Uncover the parenthesis, and then differentiate:
   $\left(x^2(1-2x)\right)' = \left(x^2 - 2x^3\right)' = 2x - 6x^2$
   Approach 2. Use first the product rule and then simplify:
   $$\begin{array}{rcl}\left(x^2(1-2x)\right)' &=& (x^2)'(1-2x) + x^2(1-2x)' \\ &=& 2x(1-2x) + x^2(-2) \\ &=& 2x - 4x^2 - 2x^2 \\ &=& 2x - 6x^2.\end{array}$$
   Of course, both approaches lead to the same answer.

2. (a) Given that $f(0) = 5$, $f'(0) = -1$, $g(0) = -4$, $g'(0) = 1$ and $h(x) = f(x)g(x)$, find the derivative $h'(0)$.

   answer: $h'(0) = (-1) \cdot (-4) + 5 \cdot 1 = 9$

   (b) Given that $f(2) = -3$, $f'(2) = 2$, $g(2) = 5$, $g'(2) = 1$ and $h(x) = f(x)g(x)$, find the derivative $h'(2)$.

   (c) Given that $f(0) = 5$, $f'(0) = -1$, $g(0) = -4$, $g'(0) = 1$ and $h(x) = \frac{f(x)}{g(x)}$, find the derivative $h'(0)$.

   answer: $h'(0) = \frac{(-1)\cdot(-4) - 5 \cdot 1}{(-4)^2} = -\frac{1}{16}$

   (d) Given that $f(1) = 2$, $f'(1) = -1$, $g(1) = -3$, $g'(1) = 1$, $h(1) = 0$, $h'(1) = 1$ and $j(x) = f(x)g(x)h(x)$, find the derivative $j'(1)$.

   answer: $j'(1) = (-1) \cdot (-3) \cdot 0 + 2 \cdot 1 \cdot 0 + 2 \cdot (-3) \cdot 1 = -6$

   Solution. 2.b
   $$\begin{array}{rcll}h'(x) &=& (f(x)g(x))' = f'(x)g(x) + f(x)g'(x) & \Big| \text{ product rule} \\ h'(2) &=& f'(2)g(2) + f(2)g'(2) = 2 \cdot 5 + (-3)1 = 7. & \end{array}$$

3. Compute the derivative.

(a) $y = x^{\frac{5}{3}} - x^{\frac{2}{3}}$.

answer: $\frac{5}{3}x^{\frac{2}{3}} - 2/3x^{-\frac{1}{3}}$

(b) $f(x) = \sqrt{x} - x$.

answer: $-1 + \frac{1}{2}x^{-\frac{1}{2}}$

(c) $y = \sqrt{x}(x-1)$.

answer: $\frac{3}{2}x^{\frac{1}{2}} - \frac{1}{2}x^{-\frac{1}{2}}$

(d) $f(x) = (2x+1)^2$.

answer: $4 + 8x$

(e) $f(x) = 4\pi x^2$.

answer: $8\pi x$

(f) $y = \dfrac{x^2+4x+3}{\sqrt{x}}$.

answer: $2x^{-\frac{1}{2}} + \frac{3}{2}x^{\frac{1}{2}} - \frac{3}{2}x^{-\frac{3}{2}}$

(g) $y = \dfrac{\sqrt{x}+x}{x^2}$.

answer: $-x^{-2} - \frac{3}{2}x^{-\frac{5}{2}}$

(h) $f(x) = \left(x + x^{-1}\right)^3$.

answer: $3x^2 + 3 - 3x^{-2} - 3x^{-4}$

(i) $f(x) = \sqrt{2}x + \sqrt{5x}$.

answer: $\sqrt{2} + \frac{\sqrt{5}}{2}x^{-\frac{1}{2}} = \sqrt{2} + \frac{\sqrt{5}}{2\sqrt{x}}$

(j) $y = \sqrt[5]{x} + 4\sqrt{x^5}$.

answer: $10x^{\frac{3}{2}} + \frac{1}{5}x^{-\frac{4}{5}}$

(k) $y = \left(\sqrt{x} + \dfrac{1}{\sqrt[3]{x}}\right)^2$.

answer: $1 + \frac{1}{3}x^{-\frac{5}{6}} - \frac{2}{3}x^{-\frac{5}{3}}$

(l) $f(x) = (1+2x^2)(x-x^2)$.

answer: $1 - 2x + 6x^2 - 8x^3$

(m) $f(x) = \dfrac{x^4 - 5x^3 + \sqrt{x}}{x^2}$.

answer: $-5 + 2x - \frac{3}{2}x^{-\frac{5}{2}}$

(n) $f(x) = (2x^3+3)(x^4-2x)$.

answer: $-6 - 4x^3 + 14x^6$

(o) $f(x) = (1+x+x^2)(2-x^4)$.

answer: $2 + 4x - 4x^3 - 5x^4 - 6x^5$

(p) $g(y) = \left(\dfrac{1}{y^2} - \dfrac{3}{y^4}\right)(y+5y^3)$.

answer: $5 + 9y^{-4} + 14y^{-2}$

(q) $f(x) = (x^3-2x)(x^{-4}+x^{-2})$.

answer: $1 + 6x^{-4} + x^{-2}$

(r) $f(x) = \dfrac{1+2x}{3-4x}$.

answer: $10(3-4x)^{-2}$

Solution. 3.k

$$
\begin{aligned}
\left(\left(\sqrt{x} + \frac{1}{\sqrt[3]{x}}\right)^2\right)' &= \left(\left(x^{\frac{1}{2}} + x^{-\frac{1}{3}}\right)^2\right)' \\
&= \left(\left(x^{\frac{1}{2}}\right)^2 + 2x^{\frac{1}{2}}x^{-\frac{1}{3}} + \left(x^{-\frac{1}{3}}\right)^2\right)' \\
&= \left(x + 2x^{\frac{1}{6}} + x^{-\frac{2}{3}}\right)' \\
&= 1 + 2\cdot\frac{1}{6}x^{\frac{1}{6}-1} + \left(-\frac{2}{3}\right)x^{-\frac{2}{3}-1} \\
&= 1 + \frac{1}{3}x^{-\frac{5}{6}} - \frac{2}{3}x^{-\frac{5}{3}}.
\end{aligned}
$$

4. Compute the derivative (with respect to the implied variable).

(a) $f(x) = \dfrac{x-3}{x+3}$.

answer: $6(3+x)^{-2}$

(b) $y = \dfrac{x^3}{1-x^2}$.

answer: $\frac{3x^2-x^4}{(1-x^2)^2}$

(c) $y = \dfrac{x+1}{x^3+x-2}$.

answer: $\frac{-3-3x^2-2x^3}{(-2+x+x^3)^2}$

(d) $y = \dfrac{x-1}{x^3+x-2}$.

answer: $\frac{-2x-1}{\left(x^2+x+2\right)^2}$

(e) $f(x) = \dfrac{x+1}{x^3+1}$.

answer: $\frac{-2x+1}{(x^2-x+2)^2}$

(f) $y = \dfrac{x^3 - 2x\sqrt{x}}{x}$.

answer: $2x - x^{-\frac{1}{2}}$

(g) $y = \dfrac{t}{(t-1)^2}$.

answer: $-\frac{t+1}{(t-1)^3}$

(h) $y = \dfrac{t^2+2}{t^4-3t^2+1}$.

answer: $\frac{14t-8t^3-2t^5}{(1-3t^2+t^4)^2}$

(i) $g(t) = \dfrac{t-\sqrt{t}}{t^{\frac{1}{3}}}$.

answer: $-\frac{1}{6}t^{-\frac{5}{6}} + \frac{2}{3}t^{-\frac{1}{3}}$

(j) $y = ax^2 + bx + c$.

answer: $b + 2ax$

(k) $y = A + \dfrac{B}{x} + \dfrac{C}{x^2}$.

answer: $\frac{-Bx-2C}{x^3}$

(l) $f(t) = \dfrac{2t}{2+\sqrt{t}}$.

answer: $\dfrac{4+t^{\frac{1}{2}}}{\left(2+t^{\frac{1}{2}}\right)^2}$

(m) $y = \dfrac{cx}{1+cx}$.

answer: $c(1+cx)^{-2}$

(n) $y = \sqrt[3]{t}(t^2+t+t^{-1})$.

answer: $-\frac{2}{3}t^{-\frac{5}{3}} + \frac{4}{3}t^{\frac{1}{3}} + \frac{7}{3}t^{\frac{4}{3}}$

(o) $y = \dfrac{u^6-2u^3+5}{u^2}$.

answer: $-2-10u^{-3}+4u^3$

(p) $f(x) = \dfrac{ax+b}{cx+d}$.

answer: $\dfrac{ad-bc}{(d+cx)^2}$

(q) $f(x) = \dfrac{1+x}{1+\frac{2}{x}}$.

answer: $\dfrac{x^2+4x+2}{(2+x)^2}$

(r) $f(x) = \dfrac{1+x}{1+\frac{3}{x}}$.

answer: $\dfrac{x^2+6x+3}{(3+x)^2}$

(s) $f(x) = \dfrac{x}{x+\frac{c}{x}}$.

answer: $\dfrac{2xc}{(c+x^2)^2}$

Solution. 4.g This can be differentiated more efficiently using the chain rule, however let us show how the problem can be solved directly using the quotient rule.

$$\begin{aligned}
\left(\frac{t}{(t-1)^2}\right)' &= \frac{(t)'(t-1)^2 - t\left((t-1)^2\right)'}{(t-1)^4} \\
&= \frac{(t-1)^2 - t\left(t^2-2t+1\right)'}{(t-1)^4} \\
&= \frac{(t-1)^2 - t\,(2t-2)}{(t-1)^4} \\
&= \frac{\cancel{(t-1)}\,((t-1)-2t)}{(t-1)^{\cancel{4}\,3}} \\
&= \frac{-t-1}{(t-1)^3} \\
&= -\frac{t+1}{(t-1)^3}
\end{aligned}$$

Solution. 4.e

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x+1}{x^3+1}\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\cancel{x+1}}{\cancel{(x+1)}(x^2-x+1)}\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x^2-x+1}\right)
\end{aligned}$$

Variant I: use quotient rule.

$$\begin{aligned}
&= \frac{\frac{\mathrm{d}}{\mathrm{d}x}(1)\cdot(x^2-x+1) - 1\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2-x+1\right)}{\left(x^2-x+1\right)^2} \\
&= \frac{-2x+1}{\left(x^2-x+1\right)^2}
\end{aligned}$$

Variant I: use chain rule.

$$\begin{aligned}
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(\left(x^2-x+1\right)^{-1}\right) \\
&= -(x^2-x+1)^{-2}\frac{\mathrm{d}}{\mathrm{d}x}(x^2-x+1) \\
&= -(x^2-x+1)^{-2}(2x-1) \\
&= \frac{-2x+1}{\left(x^2-x+1\right)^2}.
\end{aligned}$$