Calculus II
Power series expansion of sine and cosine

Todor Milev

2019

Example

Find the Maclaurin series of $f(x) = \sin x$ and its radius of convergence.

$$\begin{array}{rclrcl} f(x) &=& \sin x & f(0) &=& 0 \\ f'(x) &=& \cos x & f'(0) &=& 1 \\ f''(x) &=& -\sin x & f''(0) &=& 0 \\ f'''(x) &=& -\cos x & f'''(0) &=& -1 \\ f^{(4)}(x) &=& \sin x & f^{(4)}(0) &=& 0 \end{array}$$

The Maclaurin series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Use the Ratio Test to find R.

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{(-1)^n x^{2n+1}} \right| \\ &= \lim_{n\to\infty} \frac{x^2}{(2n+2)(2n+3)} = 0 \end{aligned}$$

Therefore $R = \infty$. It can be shown that this series sums to $\sin x$.

Example

Find the Maclaurin series for $\cos x$.

$$\begin{aligned}
\cos x &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\sin x\right)\\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}\right)\\
&= \sum_{n=0}^{\infty}\frac{\mathrm{d}}{\mathrm{d}x}\left((-1)^n\frac{x^{2n+1}}{(2n+1)!}\right)\\
&= \sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)x^{2n}}{(2n+1)!}\\
&= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}\\
&= 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots
\end{aligned}$$

The series for $\sin x$ converges everywhere, so the series for $\cos x$ does too.