Calculus I
Linearization and differentials

Todor Milev

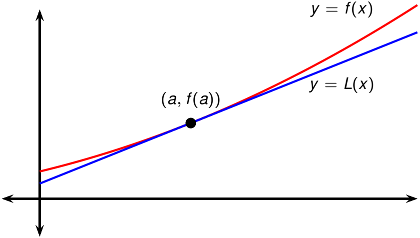
2019

Outline

License to use and redistribute

Linear Approximations and Differentials

- Main idea: A curve is very close to its tangent line at the point of tangency.
- We can use the tangent line at $(a, f(a))$ as an approximation to the curve $y = f(x)$.
- This approximation works well as long as x is near a.



Definition (Linearization of f at a)

The linear function whose graph is the tangent line at $(a, f(a))$ is called the linearization of f at a. Its equation is

$$L(x) = f(a) + f'(a)(x - a).$$

Definition (Linear Approximation of $f(x)$ near a)

The approximation

$$f(x) \approx f(a) + f'(a)(x - a)$$

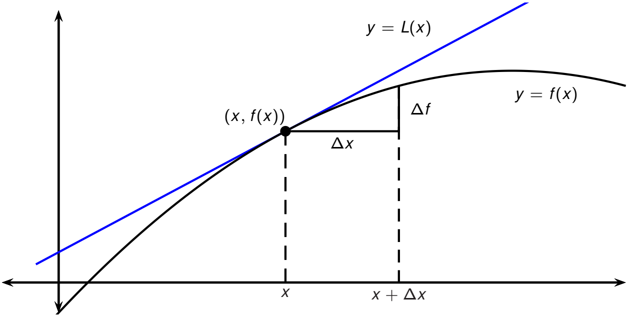
is called the linear approximation of f at a.

Let $y = f(x)$, $\Delta y := f(x) - f(a)$, and $\Delta x := x - a$.

Definition (Linear approx. $y = f(x)$ near a, alternative notation)

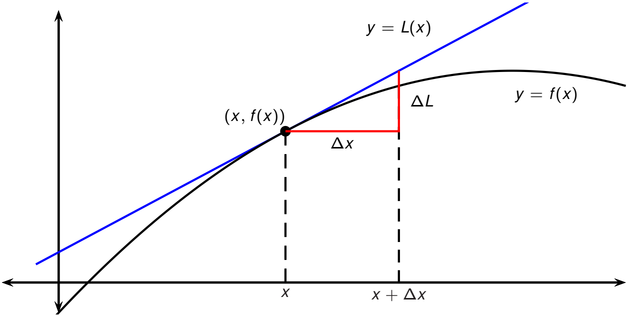
$$\Delta y \approx \frac{dy}{dx}\Delta x \quad .$$

Linear approximations



Function	f	L
Run	Δx	Δx
Rise	Δf	ΔL
Formula	$\Delta f = f(x + \Delta x) - f(x)$	$\Delta L = (\Delta x) f'(x)$

Linear approximations



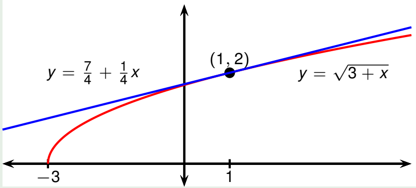
Function	f	L
Run	Δx	Δx
Rise	Δf	ΔL
Formula	$\Delta f = f(x + \Delta x) - f(x)$	$\Delta L = (\Delta x) f'(x)$

Example

Find the linearization of the function $f(x) = \sqrt{x+3}$ at $a = 1$ and use it to approximate the numbers $\sqrt{3.98}$ and $\sqrt{4.05}$. Are these approximations overestimates or underestimates?

- $f'(x) = \frac{1}{2\sqrt{x+3}}$.
- $f(1) = \sqrt{1+3} = 2$.
- $f'(1) = \frac{1}{2\sqrt{1+3}} = \frac{1}{4}$.
- Linearization:

$$\begin{aligned} L(x) &= 2 + \frac{1}{4}(x-1) \\ &= \frac{7}{4} + \frac{x}{4} \end{aligned}$$



The graph of the linearization is above the curve, so these are overestimates.

- $\sqrt{3.98} = f(0.98) \approx \frac{7}{4} + \frac{0.98}{4} = 1.995$.
- $\sqrt{4.05} = f(1.05) \approx \frac{7}{4} + \frac{1.05}{4} = 2.0125$.

Example

Compute Δy and $\Delta L = f'(x)\Delta x$ if $y = f(x) = x^3 + x^2 - 2x + 1$ and x changes from 2 to 2.05.

- $f(2) = 2^3 + 2^2 - 2(2) + 1 = 9.$
- $f(2.05) = (2.05)^3 + (2.05)^2 - 2(2.05) + 1 = 9.717625.$
- $\Delta y = f(2.05) - f(2) = 9.717625 - 9 = 0.717625.$
- $f'(x) = 3x^2 + 2x - 2.$
- $\Delta y \simeq \Delta L = f'(x)\Delta x = f'(x)\Delta x = (3x^2 + 2x - 2)\Delta x.$
- When $x = 2$ and $\Delta x = 0.05$, we have:
- $\Delta L = (3(2)^2 + 2(2) - 2)(0.05) = 0.7.$
- Therefore $\Delta Ly = 0.7$, an approximation of $\Delta y = 0.717625$.

Differentials

- Recall $\Delta y, \Delta x$ stand for change of x, y. Recall: $\Delta y \approx \frac{\mathrm{d}y}{\mathrm{d}x}\Delta x$
- $\mathrm{d}y = \frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x = \mathrm{d}y$
- If we substitute Δy by the formal expression $\mathrm{d}y$ and Δx by the formal expression $\mathrm{d}x$, the expression $\mathrm{d}x$ appears to "cancel" to give a formal identity.
- Define the *differential* d and the *differential forms* $\mathrm{d}x$, $\mathrm{d}(f(x))$ by requesting that d and $\mathrm{d}x$ satisfy the transformation law

$$\mathrm{d}(f(x)) = f'(x)\mathrm{d}x$$

for any differentiable function $f(x)$. In abbreviated notation:

$$\mathrm{d}f = f'\mathrm{d}x$$

Expressions containing expression of the form d(*something*) are called differential forms.

- $\mathrm{d}f(x) = f'(x)\mathrm{d}x$.
- On the previous slide we stated the differential d and the differential forms $\mathrm{d}x, \mathrm{d}f(x)$ are formal expressions related by a transformation law.
- The precise definitions of differential forms and differentials are outside of the scope of Calculus I and II.
- Differential forms “encode” linear approximations which in turn “encode” “infinitesimal” lengths of segments.
- Courses such as “Integration and Manifolds” or “Differential geometry” usually give precise definitions and fill in the details.
- Nonetheless, what we studied is completely sufficient for practical purposes and carrying out computations.
- **Do not confuse differentials with derivatives.** The correct equality is this.

$$\xcancel{\mathrm{d}f(x) = f'(x)} \qquad\qquad \mathrm{d}f(x) = f'(x)\mathrm{d}x$$

Example

Compute the differential (via dx).

$$\mathrm{d}\left(x^2\right) = \left(x^2\right)' \mathrm{d}x = 2x\mathrm{d}x \quad .$$

Example

Compute the differential (via dx).

$$\mathrm{d}\left(\sqrt{x}\right) = \left(\sqrt{x}\right)'\mathrm{d}x = \frac{1}{2\sqrt{x}}\mathrm{d}x \quad .$$

- All rules for computing with derivatives have analogues for computing with differential forms.
- The rules for computing differential forms are a direct consequence of the corresponding derivative rules and the transformation law $\mathrm{d}(f(x)) = f'(x)\mathrm{d}x$.

Rule name: product rule. constant derivative rule. sum rule. chain rule. power rule. exponent derivative rule.

Differential rule	Derivative rule	
$d(fg) = g df + f dg$	$(fg)' = f'g + fg'$	
$dc = 0 = 0dx$	$(c)' = 0$	c-const.
$d(cf) = c df$	$(cf)' = cf'$	c-const.
$d(f+g) = df + dg$	$(f+g)' = f' + g'$	
$df(g(x)) = f'(g(x))dg(x)$		
$= f'(g(x))g'(x)dx$	$(f(g(x)))' = f'(g(x))g'(x)$	
$df(g) = f'(g)dg$		
$d(x^n) = nx^{n-1}dx$	$(x^n)' = nx^{n-1}$	
$d(e^x) = e^x dx$	$(e^x)' = e^x$	
$d(\sin x) = \cos x dx$	$(\sin x)' = \cos x$	
$d(\cos x) = -\sin x dx$	$(\cos x)' = -\sin x$	
$(d \ln x) = \frac{1}{x}dx$	$(\ln x)' = \frac{1}{x}$	

Differentials are especially efficient at "encoding" the chain rule.

Example

Compute the differential $\mathrm{d}\left(\ln\left(1+\sqrt{1+x^2}\right)\right)$.

Set $u = 1+\sqrt{1+x^2}$. Set $v = 1+x^2$.

$$\begin{aligned}
\mathrm{d}\left(\ln\left(1+\sqrt{1+x^2}\right)\right) &= \mathrm{d}\left(\ln u\right) = \frac{1}{u}\mathrm{d}u = \frac{1}{u}\mathrm{d}\left(1+\sqrt{1+x^2}\right) = \\
&= \frac{1}{u}\mathrm{d}\left(\sqrt{1+x^2}\right) = \frac{1}{u}\mathrm{d}\left(v^{\frac{1}{2}}\right) = \frac{1}{u}\frac{1}{2}v^{-\frac{1}{2}}\mathrm{d}v \\
&= \frac{1}{2uv^{\frac{1}{2}}}\mathrm{d}\left(1+x^2\right) = \frac{\cancel{2}x}{\cancel{2}uv^{\frac{1}{2}}}\mathrm{d}x = \frac{x}{uv^{\frac{1}{2}}}\mathrm{d}x \\
&= \frac{x}{\left(1+\sqrt{1+x^2}\right)\sqrt{1+x^2}}\mathrm{d}x
\end{aligned}$$