

# Calculus II

## Area locked by curve

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# Outline

- 1 Areas Locked by Curves
- 2 Areas in Polar Coordinates

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# Areas

- The area under a curve  $y = F(x)$  from  $a$  to  $b$  is

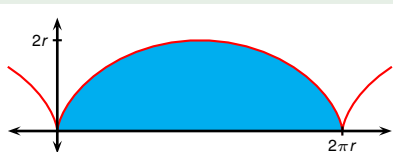
$$A = \int_a^b F(x)dx$$

- Suppose the curve has parametric equations  $x = f(t)$ ,  $y = g(t)$ ,  $\alpha \leq t \leq \beta$ .
- Then use the Substitution Rule to find the area:

$$A = \int_a^b ydx = \int_{\alpha}^{\beta} g(t)f'(t)dt$$

- How do we know where to put  $\alpha$  and  $\beta$ ?
- When  $x = a$ ,  $t$  will be either  $\alpha$  or  $\beta$ . When  $x = b$ ,  $t$  will take the other value.

## Example



Find the area under one arch of the cycloid

$$x = r(\theta - \sin \theta), \quad y = r(1 - \cos \theta)$$

One arch is given by  $0 \leq \theta \leq 2\pi$ .

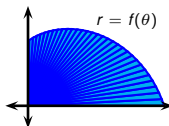
$$\begin{aligned}
 A &= \int_0^{2\pi r} y dx = \int_0^{2\pi} r(1 - \cos \theta) r(1 - \cos \theta) d\theta \\
 &= r^2 \int_0^{2\pi} (1 - \cos \theta)^2 d\theta = r^2 \int_0^{2\pi} (1 - 2 \cos \theta + \cos^2 \theta) d\theta \\
 &= r^2 \int_0^{2\pi} \left( 1 - 2 \cos \theta + \frac{1}{2}(1 + \cos 2\theta) \right) d\theta \\
 &= r^2 \left[ \frac{3}{2}\theta - 2 \sin \theta + \frac{1}{4} \sin 2\theta \right]_0^{2\pi} = r^2 \left( \frac{3}{2} \cdot 2\pi \right) = 3\pi r^2
 \end{aligned}$$

# Areas in Polar Coordinates

Suppose we have a polar curve  $r = f(\theta)$ ,  $a \leq \theta \leq b$ .

## Definition

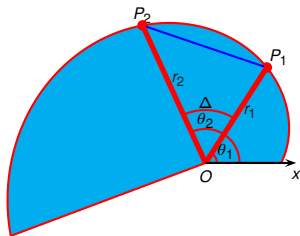
We say that the figure obtained as the union of the segments connecting the origin with the points of the curve is the figure *swept* by the curve as  $\theta$  varies from  $a$  to  $b$ .



## Theorem

Suppose no two points on the curve lie on the same ray from the origin. Then the area swept by the curve equals  $A = \int_a^b \frac{1}{2} (f(\theta))^2 d\theta$ .

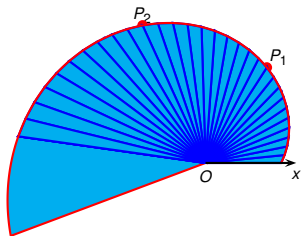
# Area swept by a polar curve: justification



Split  $[a, b]$  into  $N$  equal segments via points  $a = \theta_0 \leq \theta_1 \leq \dots \leq \theta_{N-1} \leq \theta_N = b$ . The length of each segment is  $\Delta = \frac{b-a}{N}$ . Let  $r_i = f(\theta_i)$ . Then each  $\theta_i$  gives a point  $P_i$  with polar coordinates  $(r_i, \theta_i)$ .

The area swept by the curve is approximated by sum of areas of triangles given by connecting the origin with two consecutive vertices. Consider one such triangle, say,  $OP_1P_2$ . By Euclidean geometry, the area of  $\triangle OP_1P_2$  is  $\frac{|OP_1||OP_2|\sin \Delta}{2} = \frac{r_1 r_2 \sin \Delta}{2} = \frac{f(\theta_1)f(\theta_2) \sin \Delta}{2}$ .

# Area swept by a polar curve: justification



Split  $[a, b]$  into  $N$  equal segments via points  $a = \theta_0 \leq \theta_1 \leq \dots \leq \theta_{N-1} \leq \theta_N = b$ . The length of each segment is  $\Delta = \frac{b-a}{N}$ . Let  $r_i = f(\theta_i)$ . Then each  $\theta_i$  gives a point  $P_i$  with polar coordinates  $(r_i, \theta_i)$ .

Therefore the area swept by the curve equals the limit of the sum:

$$A = \lim_{\Delta \rightarrow 0} \sum_{i=0}^{N-1} \frac{f(\theta_i)f(\theta_{i+1})\sin \Delta}{2} = \lim_{\Delta \rightarrow 0} \frac{\sin \Delta}{\Delta} \sum_{i=0}^{N-1} \frac{f(\theta_i)f(\theta_{i+1})\Delta}{2}$$

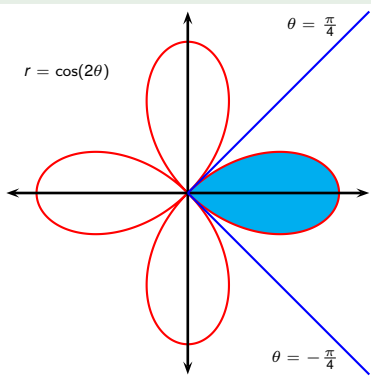
$$\text{(can be proved)} = \lim_{\Delta \rightarrow 0} \frac{\sin \Delta}{\Delta} \lim_{\Delta \rightarrow 0} \sum_{i=0}^{N-1} \frac{f(\theta_i)f(\theta_{i+1})\Delta}{2} = 1 \cdot \lim_{\Delta \rightarrow 0} \sum_{i=0}^{N-1} \frac{f(\theta_i)f(\theta_i)\Delta}{2}$$

$$\text{(Riemann sum)} = \lim_{\Delta \rightarrow 0} \sum_{i=0}^{N-1} \frac{f^2(\theta_i)\Delta}{2} = \int_a^b \frac{f^2(\theta)}{2} d\theta$$



## Example

Find the area enclosed by one loop of the four-leaved rose  $r = \cos 2\theta$ .

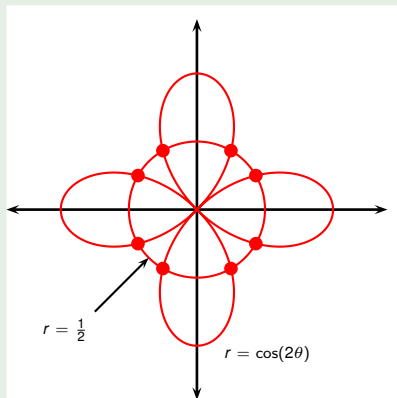


The region enclosed by the right loop corresponds to points whose  $\theta$  polar coordinate lies in the interval  $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$ .

$$\begin{aligned}
 A &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2} r^2 d\theta \\
 &= \frac{1}{2} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^2(2\theta) d\theta \\
 &= \int_0^{\frac{\pi}{4}} \cos^2(2\theta) d\theta \\
 &= \int_0^{\frac{\pi}{4}} \frac{1}{2} (1 + \cos(4\theta)) d\theta \\
 &= \frac{1}{2} \left[ \theta + \frac{1}{4} \sin(4\theta) \right]_0^{\frac{\pi}{4}} \\
 &= \frac{\pi}{8}
 \end{aligned}$$

## Example

Find all points of intersection of the polar curves  $r = \frac{1}{2}$  and  $r = \cos(2\theta)$ .



$$\cos 2\theta = \frac{1}{2}$$

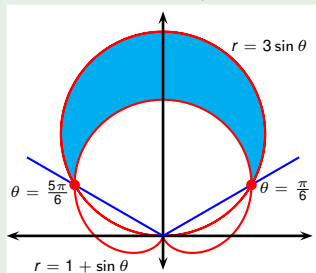
$$2\theta = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}$$

$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

- This only gives four points.
- There are actually eight.
- The circle  $r = \frac{1}{2}$  also has polar equation  $r = -\frac{1}{2}$ .
- To find all eight points, solve  $\cos(2\theta) = \frac{1}{2}$  and  $\cos(2\theta) = -\frac{1}{2}$ .

## Example

Find the area that lies within the circle  $r = 3 \sin \theta$  and outside of the cardioid  $r = 1 + \sin \theta$ .



The curves meet if

$$3 \sin \theta = 1 + \sin \theta$$

$$\sin \theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$\begin{aligned} A &= \frac{1}{2} \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (3 \sin \theta)^2 d\theta - \frac{1}{2} \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (1 + \sin \theta)^2 d\theta \\ &= \frac{1}{2} \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (9 \sin^2 \theta - (1 + 2 \sin \theta + \sin^2 \theta)) d\theta \\ &= \frac{1}{2} \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (8 \sin^2 \theta - 1 - 2 \sin \theta) d\theta \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (8 \sin^2 \theta - 1 - 2 \sin \theta) d\theta \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (3 - 4 \cos 2\theta - 2 \sin \theta) d\theta \\ &= [3\theta - 2 \sin 2\theta + 2 \cos \theta]_{\frac{\pi}{6}}^{\frac{\pi}{2}} \\ &= \left(3 \frac{\pi}{2} - 2 \cdot 0 + 2 \cdot 0\right) - \left(3 \frac{\pi}{6} - 2 \frac{\sqrt{3}}{2} + 2 \frac{\sqrt{3}}{2}\right) \\ &= \pi \end{aligned}$$