Calculus I
Homework
Continuity

1. Evaluate the difference quotient and simplify your answer.

(a) $\dfrac{f(2+h)-f(2)}{h}$, where $f(x) = x^2 - x - 1$.

answer: $h+3$

(b) $\dfrac{f(a+h)-f(a)}{h}$, where $f(x) = x^2$.

answer: $h+2a$

(c) $\dfrac{f(a+h)-f(a)}{h}$, where $f(x) = x^3$.

answer: h^2+3a^2+3ah

(d) $\dfrac{f(a+h)-f(a)}{h}$, where $f(x) = x^4$.

answer: $6a^2h+4ah^2+h^3+4a^3$

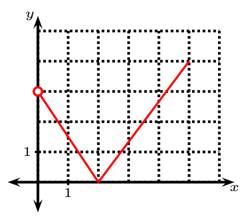
(e) $\dfrac{f(x)-f(a)}{x-a}$, where $f(x) = \frac{1}{x}$.

answer: $-\frac{1}{ax}$.

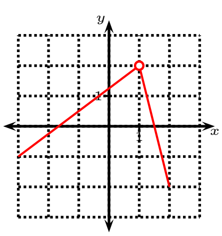
(f) $\dfrac{f(x)-f(1)}{x-1}$, where $f(x) = \frac{x-1}{x+1}$.

answer: $\frac{1}{x+1}$.

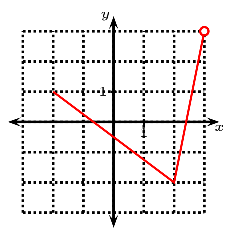
2. Write down a formula for a function whose graphs is given below. The graphs are up to scale. Please note that there is more than one way to write down a correct answer.

(a) 

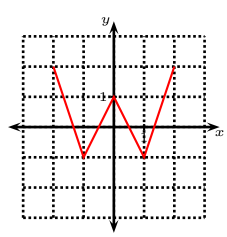
answer: $y = \begin{cases} -\frac{3}{2}x + 3 & \text{if } 0 < x \leq 2 \\ \frac{4}{3}x - \frac{8}{3} & \text{if } 2 < x \leq 6 \end{cases}$

(b) 

answer: $y = \begin{cases} \frac{3}{4}x + \frac{5}{4} & \text{if } -3 \leq x < 1 \\ -4x + 6 & \text{if } 1 < x \leq 2 \end{cases}$

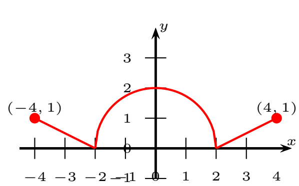
(c) 

answer: $y = \begin{cases} -\frac{3}{4}x - \frac{1}{2} & \text{if } -2 \leq x < 2 \\ 5x - 12 & \text{if } 2 \leq x < 3 \end{cases}$

(d) 

answer: $y = \begin{cases} -3x - 4 & \text{if } -2 \leq x < -1 \\ 2x + 1 & \text{if } -1 \leq x < 0 \\ -2x + 1 & \text{if } 0 \leq x < 1 \\ 3x - 4 & \text{if } 1 \leq x \leq 2 \end{cases}$

3. Write down formulas for function whose graphs are as follows. The graphs are up to scale. All arcs are parts of circles.

(a) 

4. Evaluate the difference quotient and simplify your answer.

(a) $\dfrac{f(2+h)-f(2)}{h}$, where $f(x)=x^2-x-1$.

answer: $h+3$

(b) $\dfrac{f(a+h)-f(a)}{h}$, where $f(x)=x^2$.

answer: $h+2a$

(c) $\dfrac{f(a+h)-f(a)}{h}$, where $f(x)=x^3$.

answer: h^2+3a^2+3ah

(d) $\dfrac{f(a+h)-f(a)}{h}$, where $f(x)=x^4$.

answer: $6a^2h+4ah^2+h^3+4a^3$

(e) $\dfrac{f(x)-f(a)}{x-a}$, where $f(x)=\frac{1}{x}$.

answer: $-\frac{1}{ax}$.

(f) $\dfrac{f(x)-f(1)}{x-1}$, where $f(x)=\frac{x-1}{x+1}$.

answer: $\frac{1}{x+1}$.

5. Find the implied domain of the function.

(a) $f(x)=\dfrac{x+4}{x^2-4}$.

answer: $x\neq\pm 2$, alternatively: $x\in(-\infty,-2)\cup(-2,2)\cup(2,\infty)$

(b) $f(x)=\dfrac{2x^3-5}{x^2+5x+6}$.

answer: $x\neq -2,-3$, alternatively: $x\in(-\infty,-3)\cup(-3,-2)\cup(-2,\infty)$

(c) $f(t)=\sqrt[3]{3t-1}$.

answer: $x\in\mathbb{R}$ (the domain is all real numbers)

(d) $g(t)=\sqrt{5-t}-\sqrt{1+t}$.

answer: $x\in[-1,5]$.

(e) $h(x)=\dfrac{1}{\sqrt[6]{x^2-7x}}$.

answer: $x\in(-\infty,0)\cup(7,\infty)$.

(f) $f(u)=\dfrac{u+1}{1+\frac{1}{u+1}}$.

answer: $u\neq -1,-2$ or $u\in(-\infty,-2)\cup(-2,-1)\cup(-1,\infty)$.

(g) $F(x)=\sqrt{10-\sqrt{x}}$.

answer: $x\in[0,100]$

6. Find the implied domain of the function.

(a) $f(x)=\dfrac{x+4}{x^2-4}$.

answer: $x\neq\pm 2$, alternatively: $x\in(-\infty,-2)\cup(-2,2)\cup(2,\infty)$

(b) $f(x)=\dfrac{2x^3-5}{x^2+5x+6}$.

answer: $x\neq -2,-3$, alternatively: $x\in(-\infty,-3)\cup(-3,-2)\cup(-2,\infty)$

(c) $f(t)=\sqrt[3]{3t-1}$.

answer: $x\in\mathbb{R}$ (the domain is all real numbers)

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answer: $x\in[-1,5]$.

(e) $h(x)=\dfrac{1}{\sqrt[6]{x^2-7x}}$.

answer: $x\in(-\infty,0)\cup(7,\infty)$.

(f) $f(u)=\dfrac{u+1}{1+\frac{1}{u+1}}$.

answer: $u\neq -1,-2$ or $u\in(-\infty,-2)\cup(-2,-1)\cup(-1,\infty)$.

(g) $F(x)=\sqrt{10-\sqrt{x}}$.

answer: $x\in[0,100]$

7. Compute the composite functions $(f\circ g)(x)$, $(g\circ f)(x)$. Simplify your answer to a single fraction. Find the domain of the composite function.

(a) $f(x)=\dfrac{x+2}{x-2}, g(x)=\dfrac{x-1}{x+2}$.

answer: $(f\circ g)(x)=\frac{3+3x}{-5-x}$ $x\neq -2,-5$; $(g\circ f)(x)=\frac{4}{-2+3x}$ $x\neq 2,\frac{2}{3}$

(b) $f(x)=\dfrac{x+1}{3x-2}, g(x)=\dfrac{x-2}{x-1}$.

answer: $(f\circ g)(x)=\frac{-3+2x}{-4+x}$ $x\neq 4,1$; $(g\circ f)(x)=\frac{5-5x}{3-2x}$ $x\neq\frac{2}{3},\frac{3}{2}$

(c) $f(x)=\dfrac{2x+1}{3x-1}, g(x)=\dfrac{x-2}{2x-1}$.

answer: $(f\circ g)(x)=\frac{-5+4x}{-5+x}$ $x\neq 5,\frac{1}{2}$; $(g\circ f)(x)=\frac{3-4x}{3+x}$ $x\neq -3,\frac{1}{3}$

(d) $f(x)=\dfrac{x+1}{x-2}, g(x)=\dfrac{x+2}{2x-1}$.

answer: $(f\circ g)(x)=\frac{1+3x}{4-3x}$ $x\neq\frac{4}{3},\frac{1}{2}$; $(g\circ f)(x)=\frac{-3+3x}{4+x}$ $x\neq -4,2$

(e) $f(x)=\dfrac{5x+1}{4x-1}, g(x)=\dfrac{4x-1}{3x+1}$.

answer: $(f\circ g)(x)=\frac{-4+23x}{-5+13x}$ $x\neq\frac{5}{13},-\frac{1}{3}$; $(g\circ f)(x)=\frac{5+16x}{2+19x}$ $x\neq -\frac{2}{19},\frac{1}{4}$

(f) $f(x) = \dfrac{3x-5}{x-2}$, $g(x) = \dfrac{x-2}{x-4}$.

answer: $(f \circ g)(x) = \frac{-2x+14}{-x+6}$ $\quad x \neq 6, 4$; $(g \circ f)(x) = \frac{x-1}{-x+3}$ $\quad x \neq 3, 2$

(g) $f(x) = \dfrac{x-3}{x+2}$, $g(y) = \dfrac{y+3}{y-4}$.

answer: $(f \circ g)(x) = \frac{-2x+15}{3x-5}$ $\quad x \neq \frac{5}{3}, 4$; $(g \circ f)(x) = \frac{4x+3}{-3x-11}$ $\quad x \neq -\frac{11}{3}, -2$

8. Find the functions $f \circ g$, $g \circ f$, $f \circ f$ and $g \circ g$ and their implied domains. The answer key has not been proofread, use with caution.

   (a) $f(x) = x^2 + 1$, $g(x) = x + 1$.

   answer: Domain, all 4 cases: $x \in \mathbb{R}$ (all reals) in some order: $(1+x)^2+1$, $(x)^2+2$, $((x)^2+1)^2+1$, $2+x$

   (b) $f(x) = \sqrt{x+1}$, $g(x) = x + 1$.

   answer: Domain of $f \circ g$ is $x \geq -2$. Domain of $g \circ f$ is $x \geq -1$ Domain of $f \circ f$ is $x \geq -1$. Domain of $g \circ g$ is all reals ($x \in \mathbb{R}$). in some order: $\sqrt{2+x}$, $1+\sqrt{1+x}$, $\sqrt{1+\sqrt{1+x}}$, $2+x$

   (c) $f(x) = 2x$, $g(x) = \tan x$.

   In this subproblem, you are not required to find the domain.

   answer: Domain $f \circ f$: all reals ($x \in \mathbb{R}$). Domain $g \circ f$: $x \neq (2k+1)\frac{\pi}{2}$ for all $k \in \mathbb{Z}$ Domain $f \circ g$: $x \neq (4k+1)\frac{\pi}{4}$, $x \neq (4k+3)\frac{\pi}{4}$ for all $k \in \mathbb{Z}$ Domain $g \circ g$: $x \neq (2k+1)\frac{\pi}{2}$ and $x \neq k\pi + \arctan\left(\frac{\pi}{2}\right)$ for all $k \in \mathbb{Z}$ in some order: $2\tan x$, $\tan(2x)$, $4x$, $\tan(\tan x)$

   (d) $f(x) = \dfrac{x+1}{x-1}$, $g(x) = \dfrac{x-1}{x+1}$.

   answer: Domain $f \circ f$: $x \neq 1$. Domain $g \circ g$: $x \neq 0, x \neq -1$ Domain $f \circ g$: $x \neq -1$. Domain $g \circ f$: $x \neq 0, x \neq 1$ in some order: $-x$, $\frac{1}{x}$, x, $-\frac{1}{x}$

9. Convert from degrees to radians.

   (a) $15°$. answer: $\frac{\pi}{12} \approx 0.261799388$
   (b) $30°$. answer: $\frac{\pi}{6} \approx 0.523598776$
   (c) $36°$. answer: $\frac{\pi}{5} \approx 0.628318531$
   (d) $45°$. answer: $\frac{\pi}{4} \approx 0.785398163$
   (e) $60°$. answer: $\frac{\pi}{3} \approx 1.047197551$
   (f) $75°$. answer: $\frac{5\pi}{12} \approx 1.308997$
   (g) $90°$. answer: $\frac{\pi}{2}$
   (h) $120°$. answer: $\frac{2\pi}{3}$
   (i) $135°$. answer: $\frac{3\pi}{4}$
   (j) $150°$. answer: $\frac{5\pi}{6}$
   (k) $180°$. answer: π
   (l) $225°$. answer: $\frac{5\pi}{4}$
   (m) $270°$. answer: $\frac{3\pi}{2}$
   (n) $305°$. answer: $\frac{61\pi}{36} \approx 5.323254$
   (o) $360°$. answer: 2π
   (p) $405°$. answer: $\frac{9\pi}{4}$
   (q) $1200°$. answer: $\frac{20\pi}{3}$
   (r) $-900°$. answer: -5π
   (s) $-2014°$. answer: $-\frac{1007}{90}\pi \approx -35.150931$

10. Convert from radians to degrees. The answer key has not been proofread, use with caution.

   (a) 4π. answer: $720°$
   (b) $-\frac{7}{6}\pi$. answer: $-210°$
   (c) $\frac{7}{12}\pi$. answer: $105°$
   (d) $\frac{4}{3}\pi$. answer: $240°$
   (e) $-\frac{3}{8}\pi$. answer: $-67.5°$
   (f) 2014π. answer: $362520°$
   (g) 5. answer: $\left(\frac{900}{\pi}\right)° \approx 286°$
   (h) -2014. answer: $-362520°$

11. Prove the trigonometry identities.

   (a) $\sin\theta \cot\theta = \cos\theta$.
   (b) $(\sin\theta + \cos\theta)^2 = 1 + \sin(2\theta)$.
   (c) $\sec\theta - \cos\theta = \tan\theta \sin\theta$.
   (d) $\tan^2\theta - \sin^2\theta = \tan^2\theta \sin^2\theta$.
   (e) $\cot^2\theta + \sec^2\theta = \tan^2\theta + \csc^2\theta$.
   (f) $2\csc(2\theta) = \sec\theta \csc\theta$.
   (g) $\tan(2\theta) = \dfrac{2\tan\theta}{1-\tan^2\theta}$.
   (h) $\dfrac{1}{1-\sin\theta} + \dfrac{1}{1+\sin\theta} = 2\sec^2\theta$.
   (i) $\tan\alpha + \tan\beta = \dfrac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta}$.

(j) $\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$.

(k) $\sin(3\theta) + \sin\theta = 2\sin(2\theta)\cos\theta$.

(l) $\cos(3\theta) = 4\cos^3\theta - 3\cos\theta$.

(m) $1 + \tan^2\theta = \sec^2\theta$.

(n) $1 + \csc^2\theta = \cot^2\theta$.

(o) $2\cos^2(2x) = 2\sin^4\theta + 2\cos^4\theta - \sin^2(2\theta)$.

(p) $\dfrac{1+\tan\left(\frac{\theta}{2}\right)}{1-\tan\left(\frac{\theta}{2}\right)} = \tan\theta + \sec\theta$.

12. Find all values of x in the interval $[0, 2\pi]$ that satisfy the equation.

(a) $2\cos x - 1 = 0$.

answer: $x = \frac{\pi}{3}$ or $x = \frac{5\pi}{3}$

(b) $\sin(2x) = \cos x$.

answer: $x = \frac{\pi}{2}$, $x = \frac{3\pi}{2}$, $x = \frac{\pi}{6}$, or $x = \frac{5\pi}{6}$

(c) $\sqrt{3}\sin x = \sin(2x)$.

answer: $x = \frac{\pi}{6}, \frac{11\pi}{6}, 0, \pi, 2\pi$

(d) $2\sin^2 x = 1$.

answer: $x = \frac{\pi}{4}$, $x = \frac{3\pi}{4}$, $x = \frac{5\pi}{4}$, or $x = \frac{7\pi}{4}$

(e) $2 + \cos(2x) = 3\cos x$.

answer: $x = 0$, $x = 2\pi$, $x = \frac{\pi}{3}$, or $x = \frac{5\pi}{3}$

(f) $2\cos x + \sin(2x) = 0$.

answer: $x = \frac{\pi}{2}$, $x = \frac{3\pi}{2}$

(g) $2\cos^2 x - (1+\sqrt{2})\cos x + \dfrac{\sqrt{2}}{2} = 0$.

answer: $x = \frac{\pi}{4}, \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{4}$

(h) $|\tan x| = 1$.

answer: $x = \frac{\pi}{4}$, $x = \frac{3\pi}{4}$, $x = \frac{5\pi}{4}$, or $x = \frac{7\pi}{4}$

(i) $3\cot^2 x = 1$.

answer: $x = \frac{\pi}{3}$, $x = \frac{2\pi}{3}$, $x = \frac{4\pi}{3}$, or $x = \frac{5\pi}{3}$

(j) $\sin x = \tan x$.

answer: $x = 0$, $x = \pi$, or $x = 2\pi$

Solution. 12.g Set $\cos x = u$. Then

$$2\cos^2 x - (1+\sqrt{2})\cos x + \frac{\sqrt{2}}{2} = 0$$

becomes

$$2u^2 - (1+\sqrt{2})u + \frac{\sqrt{2}}{2} = 0.$$

This is a quadratic equation in u and therefore has solutions

$$\begin{aligned} u_1, u_2 &= \frac{1+\sqrt{2} \pm \sqrt{(1+\sqrt{2})^2 - 4\sqrt{2}}}{4} \\ &= \frac{1+\sqrt{2} \pm \sqrt{1 - 2\sqrt{2} + 2}}{4} \\ &= \frac{1+\sqrt{2} \pm \sqrt{(1-\sqrt{2})^2}}{4} \\ &= \frac{1+\sqrt{2} \pm (1-\sqrt{2})}{4} = \left\{ \begin{array}{l} \frac{1}{2} \quad \text{or} \\ \frac{\sqrt{2}}{2} \end{array} \right. \end{aligned}$$

Therefore $u = \cos x = \frac{1}{2}$ or $u = \cos x = \frac{\sqrt{2}}{2}$, and, as x is in the interval $[0, 2\pi]$, we get $x = \frac{\pi}{3}, \frac{5\pi}{3}$ (for $\cos x = \frac{1}{2}$) or $x = \frac{\pi}{4}, \frac{7\pi}{4}$ (for $\cos x = \frac{\sqrt{2}}{2}$).

13. Evaluate the limits. Justify your computations.

(a) $\lim\limits_{x\to 2} 2x^2 - 3x - 6$.

answer: -4

(b) $\lim\limits_{x\to -1} \dfrac{x^4 - x}{x^2 + 2x + 3}$.

answer: 1

(c) $\lim\limits_{x\to -1} \dfrac{1}{x^2 - 3x + 2}$.

answer: $\frac{1}{6}$

(d) $\lim\limits_{x\to -2} \sqrt{x^4 + 16}$.

answer: $\sqrt{32}$

(e) $\lim\limits_{x\to 8} (1 + \sqrt[3]{x})(2 - x)$.

answer: -18

14. Evaluate the limit if it exists.

(a) $\lim\limits_{x\to 2} \dfrac{x^2 - 5x + 6}{x - 2}$.

answer: -1

(b) $\lim\limits_{x\to 3} \dfrac{x^2 - 3x}{x^2 - 2x - 3}$.

answer: $\frac{3}{4}$

(c) $\lim\limits_{x\to -2} \dfrac{2x^2 + x - 6}{x^2 - 4}$

answer: $\frac{7}{4}$

(d) $\lim\limits_{x\to 2} \dfrac{x^2 - 5x - 6}{x - 2}$.

answer: DNE

(e) $\lim\limits_{x\to -1} \dfrac{x^2 - 3x}{x^2 - 2x - 3}$.

answer: DNE

(f) $\displaystyle\lim_{x\to -2}\frac{x^2-4}{2x^2+5x+2}$. answer: $\frac{4}{3}$

(g) $\displaystyle\lim_{x\to -1}\frac{2x^2+3x+1}{3x^2-2x-5}$. answer: $\frac{1}{8}$

(h) $\displaystyle\lim_{x\to -4}\frac{x^2+7x+12}{x^2+6x+8}$. answer: $\frac{1}{2}$

(i) $\displaystyle\lim_{h\to 0}\frac{(-3+h)^2-9}{h}$. answer: -6

(j) $\displaystyle\lim_{h\to 0}\frac{(-2+h)^3+8}{h}$. answer: 12

(k) $\displaystyle\lim_{x\to -3}\frac{x+3}{x^3+27}$. answer: $\frac{1}{27}$

(l) $\displaystyle\lim_{x\to 1}\frac{x^4-1}{x^3-1}$. answer: $\frac{4}{3}$

(m) $\displaystyle\lim_{h\to 0}\frac{\sqrt{4+h}-2}{h}$. answer: $\frac{1}{4}$

(n) $\displaystyle\lim_{x\to 3}\frac{\sqrt{5x+1}-4}{x-3}$. answer: $\frac{5}{8}$

(o) $\displaystyle\lim_{x\to -3}\frac{\sqrt{x^2+16}-5}{x+3}$. answer: $-\frac{3}{5}$

(p) $\displaystyle\lim_{x\to -3}\frac{\frac{1}{3}+\frac{1}{x}}{3+x}$. answer: $-\frac{1}{9}$

(q) $\displaystyle\lim_{x\to -2}\frac{x^2+4x+4}{x^4-16}$. answer: 0

(r) $\displaystyle\lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{x}$. answer: 1

(s) $\displaystyle\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{x^2+x}\right)$. answer: 1

(t) $\displaystyle\lim_{x\to 9}\frac{3-\sqrt{x}}{9x-x^2}$. answer: $\frac{1}{54}$

(u) $\displaystyle\lim_{h\to 0}\frac{(2+h)^{-1}-2^{-1}}{h}$. answer: $-\frac{1}{4}$

(v) $\displaystyle\lim_{x\to 0}\left(\frac{1}{x\sqrt{1+x}}-\frac{1}{x}\right)$. answer: $-\frac{1}{2}$

(w) $\displaystyle\lim_{h\to 0}\frac{(x+h)^3-x^3}{h}$. answer: $3x^2$

(x) $\displaystyle\lim_{h\to 0}\frac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h}$. answer: $-\frac{2}{x^3}$

(y) $\displaystyle\lim_{h\to 0}\frac{\frac{1}{(2+h)^2}-\frac{1}{4}}{h}$. answer: $-\frac{1}{4}$

(z) $\displaystyle\lim_{h\to 0}\frac{\frac{1}{(1+h)^2}-1}{h}$. answer: -2

Solution. 14.a

$$\begin{array}{rcl|l}\displaystyle\lim_{x\to 2}\frac{x^2-5x+6}{x-2} &=& \displaystyle\lim_{x\to 2}\frac{(x-3)\cancel{(x-2)}}{\cancel{x-2}} & \text{factor and cancel}\\ &=& 2-3=-1 & \end{array}$$

Solution. 14.c

$$\begin{array}{rcl|l}\displaystyle\lim_{x\to -2}\frac{2x^2+x-6}{x^2-4} &=& \displaystyle\lim_{x\to -2}\frac{(2x-3)\cancel{(x+2)}}{(x-2)\cancel{(x+2)}} & \text{factor and cancel}\\ &=& \displaystyle\frac{(2(-2)-3)}{-2-2} & \text{substitute}\\ &=& \displaystyle\frac{7}{4} & \end{array}$$

Solution. 14.f

$$\begin{array}{rcl|l}\displaystyle\lim_{x\to 2}\frac{x^2-4}{2x^2+5x+2} &=& \displaystyle\lim_{x\to -2}\frac{(x-2)\cancel{(x+2)}}{(2x+1)\cancel{(x+2)}} & \text{factor and cancel}\\ &=& \displaystyle\frac{(-2)-2}{2(-2)+1}=\frac{4}{3}. & \end{array}$$

Solution. 14.g

$$\begin{array}{rcl|l}\displaystyle\lim_{x\to -1}\frac{2x^2+3x+1}{3x^2-2x-5} &=& \displaystyle\lim_{x\to -1}\frac{(2x+1)\cancel{(x+1)}}{(3x-5)\cancel{(x+1)}} & \text{factor and cancel}\\ &=& \displaystyle\frac{2(-1)+1}{3(-1)-5}=\frac{1}{8}. & \end{array}$$

Solution. 14.h.

$$\begin{aligned}\lim_{x\to -4}\frac{x^2+7x+12}{x^2+6x+8} &= \lim_{x\to -4}\frac{(x+3)\cancel{(x+4)}}{(x+2)\cancel{(x+4)}} &&\Big|\ \text{factor}\\ &= \frac{-4+3}{-4+2}=-\frac{1}{2}.\end{aligned}$$

Solution. 14.x

$$\begin{aligned}\lim_{h\to 0}\frac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h} &= \lim_{h\to 0}\frac{x^2-(x+h)^2}{hx^2(x+h)^2}=\lim_{h\to 0}\frac{x^2-(x^2+2xh+h^2)}{hx^2(x+h)^2}\\ &= \lim_{h\to 0}\frac{\cancel{h}(-2x+h)}{\cancel{h}x^2(x+h)^2}=\frac{-2x+0}{x^2(x+0)^2}=-\frac{2}{x^3}.\end{aligned}$$

Solution. 14.y.

Variant I.

$$\begin{aligned}\lim_{h\to 0}\frac{\frac{1}{(2+h)^2}-\frac{1}{4}}{h} &= \lim_{h\to 0}\frac{\frac{4-(2+h)^2}{4(2+h)^2}}{h}\\ &= \lim_{h\to 0}\frac{4-(4+4h+h^2)}{4h(2+h)^2}\\ &= \lim_{h\to 0}\frac{-4h-h^2}{4h(2+h)^2}\\ &= \lim_{h\to 0}\frac{\cancel{h}(-4-h)}{4\cancel{h}(2+h)^2} &&\Big|\ \text{substitute } h=0\\ &= \frac{-4-0}{4(2+0)^2}\\ &= -\frac{1}{4}\end{aligned}$$

Variant II.

$$\begin{aligned}\lim_{h\to 0}\frac{\frac{1}{(2+h)^2}-\frac{1}{4}}{h} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x^2}\right)_{|x=2}\\ &= \left(\frac{-2}{x^3}\right)_{|x=2}\\ &= -\frac{1}{4}\end{aligned}$$

Solution. 14.z.

Variant I.

$$\begin{aligned}\lim_{h\to 0}\frac{\frac{1}{(1+h)^2}-1}{h} &= \lim_{h\to 0}\frac{\frac{1-(1+h)^2}{(1+h)^2}}{h}\\ &= \lim_{h\to 0}\frac{1-(1+2h+h^2)}{h(1+h)^2}\\ &= \lim_{h\to 0}\frac{-2h-h^2}{h(1+h)^2}\\ &= \lim_{h\to 0}\frac{\cancel{h}(-2-h)}{\cancel{h}(1+h)^2} &&\Big|\ \text{substitute } h=0\\ &= \frac{-2-0}{(1+0)^2}\\ &= -2.\end{aligned}$$

Variant II.

$$\begin{aligned}\lim_{h\to 0}\frac{\frac{1}{(1+h)^2}-1}{h} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x^2}\right)_{|x=1} &&\Big|\ \text{derivative definition}\\ &= \left(\frac{-2}{x^3}\right)_{|x=1}\\ &= -2.\end{aligned}$$

15. Find the (implied) domain of $f(x)$. Extend the definition of f at $x=3$ to make f continuous at 3.

(a) $f(x) = \dfrac{x^2 - x - 6}{x - 3}$.

answer: Implied domain: $x \in (-\infty, 3) \cup (3, \infty)$. Extend $f(x)$ to $\bar{f}(x) = x + 2$.

(b) $f(x) = \dfrac{x^3 - 27}{x^2 - 9}$.

answer: Implied domain: $x \in (-\infty, -3) \cup (-3, 3) \cup (3, \infty)$. Extend $f(x)$ to $\bar{f}(x) = \frac{x^2+3x+9}{x+3}$ with domain $x \in (-\infty, -3) \cup (-3, \infty)$.

16. Use the Intermediate Value Theorem to show that there is a real number solution of the given equation in the specified interval.

(a) $x^5 + x - 3 = 0$ where $x \in (1, 2)$.
(b) $\sqrt[4]{x} = 1 - x$ where $x \in \mathbb{R}$ (i.e., x is an arbitrary real number).
(c) $\cos x = 2x$, where $x \in (0, 1)$.
(d) $\sin x = x^2 - x - 1$, where $x \in \mathbb{R}$ (i.e., x is an arbitrary real number).
(e) $\cos x = x^4$, where $x \in \mathbb{R}$ (i.e., x is an arbitrary real number).
(f) $x^5 - x^2 + x + 3 = 0$, where $x \in \mathbb{R}$.

17.

(a) i. Solve the equation $x^2 + 13x + 41 = 1$.
ii. Use the intermediate value theorem to prove that the equation $x^2 + 13x + 41 = \sin x$ has at least two solutions, lying between the two solutions to 17.a.i.

(b) i. Solve the equation $x^2 - 15x + 55 = 1$.
ii. Use the intermediate value theorem to prove that the equation $x^2 - 15x + 55 = \cos x$ has at least two solutions, lying between the two solutions to the equation in the preceding item.

Solution. 17.a.i.

$$\begin{aligned} x^2 + 13x + 41 &= 1 \\ x^2 + 13x + 40 &= 0 \\ (x+5)(x+8) &= 0 \quad . \end{aligned}$$

eqnarray Therefore the two solutions are $x_1 = -5$ and $x_2 = -8$.

17.a.ii. Consider the function

$$f(x) = x^2 + 13x + 41 - \sin x \quad .$$

Our strategy for proving $f(x) = 0$ has a solution consists in finding a number a such that $f(a) < 0$ and a number b such that $f(b) > 0$, and then using the Intermediate Value Theorem (IVT) with $N = 0$.

Let

$$g(x) = x^2 + 13x + 41,$$

and so $f(x) = g(x) - \sin x$. We have no techniques for evaluating $\sin x$ without calculator, but we do have all knowledge necessary to evaluate $g(x)$. Indeed, from high school we know that the lowest point of the parabola $g(x)$ is located at $x = -\frac{13}{2} = -6.5$. Then $g(-6.5) = -1.25$. Therefore

$$f(-6.5) = g(-6.5) - \sin(-6.5) = g(-6.5) + \sin(6.5) = -1.25 + \sin 6.5 \leq -0.25,$$

where for the very last inequality we use the fact that $\sin 6.5 < 1$ (remember $\sin t \leq 1$ for all real values of t).

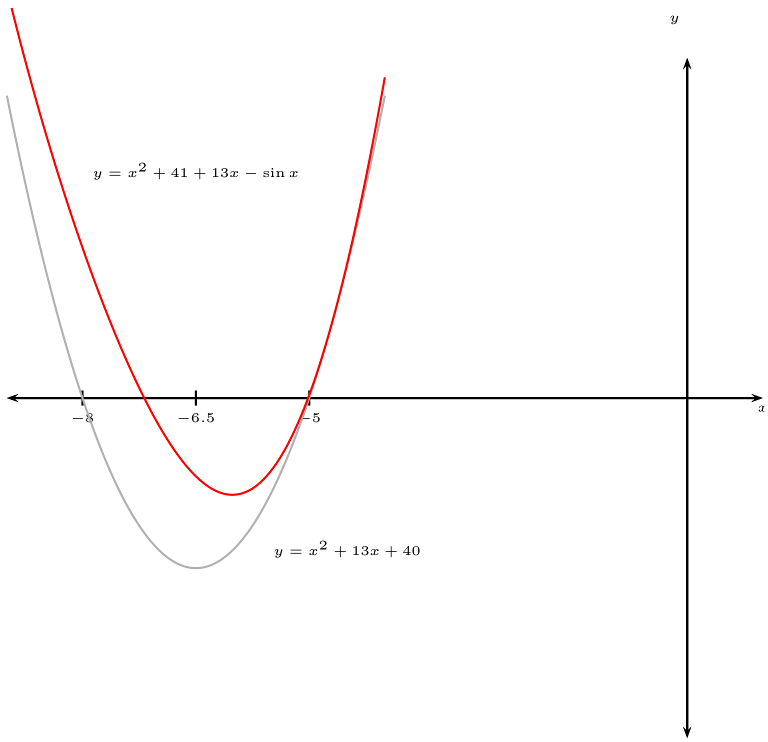
On the other hand,

$$f(-5) = g(-5) - \sin(-5) = 1 + \sin 5 > 0$$

as $\sin 5 > -1$ (remember $\sin t \geq -1$ for all real values of t). Therefore $f(-5) > 0$ and $f(-6.5) < 0$ and by the Intermediate Value Theorem (IVT) $f(x) = 0$ has a solution in the interval $x \in (-6.5, -5)$.

Proving $f(x) = 0$ has a solution in the interval $x \in (-8, -6.5)$ is similar and we leave it to the student.

Below is a computer generated plot of the function with the use of which we can visually verify our answer.



18. For which values of x is f continuous?

- $f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$
- $f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ x & \text{if } x \text{ is irrational} \end{cases}$

19. Show that $f(x)$ is continuous at all irrational points and discontinuous at all rational ones.

$$f(x) = \begin{cases} \frac{1}{q^2} & \text{if } x \text{ is rational and } x = \frac{p}{q} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

where in the first item p, q are relatively prime integers (i.e., integers without a common divisor).