Precalculus
Trigonometry and triangles

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Outline

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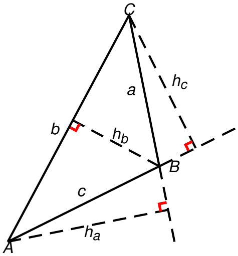
Triangle area = $\frac{1}{2}$base · height

Let $\triangle ABC$ have side lengths a, b, c and height lengths h_a, h_b, h_c, as indicated - side a is opposite to vertex A and h_a starts at A, and so on.

Proposition (Triangle area)

$$Area(\triangle ABC) = \frac{1}{2} height \cdot base = \frac{1}{2} h_a a = \frac{1}{2} h_b b = \frac{1}{2} h_c c.$$



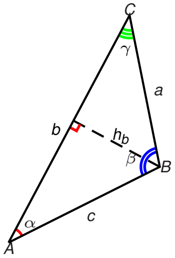
Triangle area from two sides and angle between them

Let $\triangle ABC$ have sides lengths a, b, c angles α, β, γ, as indicated: α is opposite to a, β is opposite to b, γ is opposite to c.

Proposition ($\triangle$ area from two sides and angle between them)

The area of a triangle is half the product of the lengths of two of its sides times the sine of the angle between them. In other words,

$$Area(\triangle ABC) = \frac{ab\sin\gamma}{2} = \frac{bc\sin\alpha}{2} = \frac{ca\sin\beta}{2}$$



Proof.

$$\begin{aligned}\text{Area}(\triangle ABC) &= \frac{\text{base}\cdot\text{height}}{2} = \frac{bh_b}{2}\\ &= \frac{ba\sin\gamma}{2}.\end{aligned}$$

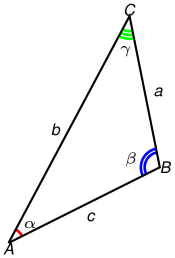
The proof of the other two cases is similar. $\square$

Law of sines

Let $\triangle ABC$ have sides lengths a, b, c angles α, β, γ, as indicated: α is opposite to a, β is opposite to b, γ is opposite to c.

Proposition (Law of Sines)

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}.$$

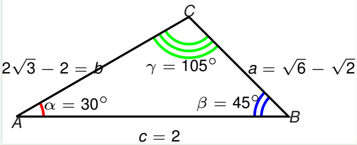


Proof.

$$\begin{array}{rcl|l} \text{Area}(\triangle ABC) = \dfrac{ab\sin\gamma}{2} & = & \dfrac{bc\sin\alpha}{2} & \text{Div. by } \frac{b}{2} \\ a\sin\gamma & = & c\sin\alpha & \\ \dfrac{a}{\sin\alpha} & = & \dfrac{c}{\sin\gamma}. & \end{array}$$

The remaining cases are similar. $\square$

Example

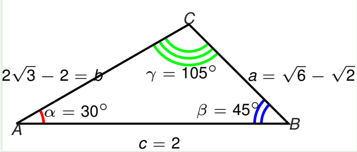


A triangle has a side of length $2cm$; the two angles adjacent to it are 30° and 45°.

- Find the other two sides of the triangle.
- Find the area of the triangle.

- Let the known side be $c = 2cm$.
- Let the known angles 30°, 45° be arranged as in the figure, and let the third angle be $\gamma = 180^\circ - 30^\circ - 45^\circ = 180^\circ - 75^\circ = 105^\circ$.
- Label the unknown sides a, b as indicated.

Example

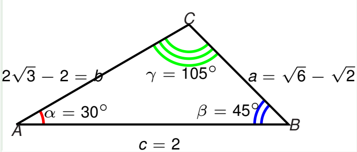


A triangle has a side of length $2cm$; the two angles adjacent to it are 30° and 45°.

- Find the other two sides of the triangle.
- Find the area of the triangle.

$$
\begin{aligned}
\sin 105^\circ &= \sin(60^\circ+45^\circ)=\sin 60^\circ\cos 45^\circ+\cos 60^\circ \sin 45^\circ\\
&= \frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2}+\frac{1}{2}\frac{\sqrt{2}}{2}=\frac{\sqrt{6}+\sqrt{2}}{4}\\
\frac{a}{\sin\alpha} &= \frac{c}{\sin\gamma} && |\text{Law of sines}\\
a &= \frac{c\sin\alpha}{\sin\gamma}=\frac{2\sin 30^\circ}{\sin 105^\circ}=\frac{\cancel{2}\cdot\frac{1}{\cancel{2}}}{\frac{\sqrt{6}+\sqrt{2}}{4}}=\frac{4(\sqrt{6}-\sqrt{2})}{(\sqrt{6}+\sqrt{2})(\sqrt{6}-\sqrt{2})}\\
&= \frac{4(\sqrt{6}-\sqrt{2})}{6-2}=\sqrt{6}-\sqrt{2}
\end{aligned}
$$

Example



A triangle has a side of length $2cm$; the two angles adjacent to it are 30° and 45°.

- Find the other two sides of the triangle.
- Find the area of the triangle.

$$\frac{b}{\sin\beta} = \frac{c}{\sin\gamma} \qquad |\text{Law of sines}$$

$$b = \frac{c\sin\beta}{\sin\gamma} = \frac{2\sin 45^\circ}{\sin 105^\circ} = \frac{\cancel{2}\frac{\sqrt{2}}{\cancel{2}}}{\frac{\sqrt{6}+\sqrt{2}}{4}} = \frac{4\sqrt{2}\left(\sqrt{6}-\sqrt{2}\right)}{\left(\sqrt{6}+\sqrt{2}\right)\left(\sqrt{6}-\sqrt{2}\right)}$$

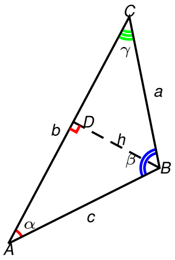
$$= \frac{\cancel{4}\sqrt{2}(\sqrt{6}-\sqrt{2})}{\cancel{4}} = 2\sqrt{3}-2$$

$$\text{Area} = \frac{bc\sin\alpha}{2} = \frac{(2\sqrt{3}-2)\cancel{2}\frac{1}{\cancel{2}}}{2} = \sqrt{3}-1 \quad \text{cm}^2$$

Let $\triangle ABC$ have sides lengths a, b, c angles α, β, γ, as indicated.

Proposition (Law of Cosines)

$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab\cos\gamma \\ a^2 &= b^2 + c^2 - 2bc\cos\alpha \\ b^2 &= c^2 + a^2 - 2ca\cos\beta \end{aligned}$$



Proof if $\gamma < 90^\circ$.

Drop a perpendicular h from B to AC.

$$|CD| = a\cos\gamma$$

$$h = a\sin\gamma$$

$$|AD| = b - |CD| = b - a\cos\gamma$$

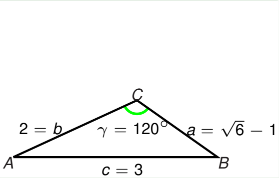
$$c^2 = |AD|^2 + h^2$$

Pyth. thm. $\triangle BDA$

$$\begin{aligned} &= (b - a\cos\gamma)^2 + (a\sin\gamma)^2 \\ &= b^2 - 2ab\cos\gamma + a^2\cos^2\gamma + a^2\sin^2\gamma \\ &= b^2 - 2ab\cos\gamma + a^2. \end{aligned}$$

□

Example



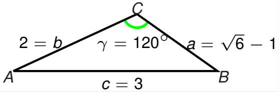
The longest side of a triangle has length 3 and the angle opposite to it is 120°. Another side of that triangle has length 2.

- Find the length of the third side.
- Find the area of the triangle.

$$
\begin{aligned}
a^2 + b^2 - 2ab\cos\gamma &= c^2 && \text{Law of cosines}\\
a^2 + 2^2 - 2a \cdot 2 \cdot \cos 120^\circ &= 3^2 && \text{Solve for } a:\\
a^2 - 4a\left(-\frac{1}{2}\right) - 5 &= 0\\
a^2 + 2a - 5 &= 0
\end{aligned}
$$

$$
\begin{aligned}
a &= \frac{-2 \pm \sqrt{2^2 - 4 \cdot (-5) \cdot 1}}{2 \cdot 1}\\
&= \frac{-2 \pm \sqrt{24}}{2} = \frac{-2 + 2\sqrt{6}}{2} && a > 0\\
&= -1 + \sqrt{6}
\end{aligned}
$$

Example



The longest side of a triangle has length 3 and the angle opposite to it is 120°. Another side of that triangle has length 2.

- Find the length of the third side.
- Find the area of the triangle.

$$a^2 + b^2 - 2ab\cos\gamma = c^2 \quad \Big| \text{ Law of cosines}$$

$$a^2 + 2^2 - 2a\cdot 2\cdot\cos 120^\circ = 3^2 \quad \Big| \text{ Solve for } a:$$

$$a = \frac{-2 \pm \sqrt{2^2 - 4\cdot(-5)\cdot 1}}{2\cdot 1} = -1 + \sqrt{6}$$

$$\textbf{Area} = \frac{ab\sin\gamma}{2} = \frac{\left(\sqrt{6}-1\right)\cancel{2}}{\cancel{2}}\frac{\sqrt{3}}{2} = \frac{3\sqrt{2}-\sqrt{3}}{2}$$