Precalculus
Simplify $\sin(k \arcsin x)$, $\cos(k \arcsin x)$, $\sin(k \arccos x)$, $\cos(k \arccos x)$

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2019

Example

Rewrite $\sin(2\arccos(x))$ as an algebraic expression of x and $\sqrt{1-x^2}$.

$$\sin(2\arccos(x))$$

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To simplify $\arccos x$ we try to use $\cos(\arccos x) = x$.

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Rewrite $\sin(2\arccos(x))$ as an algebraic expression of x and $\sqrt{1-x^2}$. To simplify $\arccos x$ we try to use $\cos(\arccos x) = x$. Therefore our aim is to rewrite the expression only using the $\cos$ function.

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$$\sin(2\arccos(x)) = \sin(2y) \qquad \Big| \text{ Set } y = \arccos x$$

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$$\begin{aligned} \sin(2\arccos(x)) &= \sin(2y) \\ &= \textbf{?} \end{aligned}$$

Set $y = \arccos x$
Express via $\sin y, \cos y$

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$$\begin{aligned} \sin(2\arccos(x)) &= \sin(2y) \\ &= 2\cos y \sin y \end{aligned}$$

Set $y = \arccos x$
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\sin(2\arccos(x)) &= \sin(2y) && \text{Set } y = \arccos x \\
&= 2\cos y \sin y && \text{Express via } \sin y, \cos y \\
&= 2\cos y\left(\pm\sqrt{1-\cos^2 y}\right) && \text{Express } \sin y \text{ via } \cos y
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&= 2\cos y\left(\pm\sqrt{1-\cos^2 y}\right) && \text{Express } \sin y \text{ via } \cos y\\
&= 2\cos y\sqrt{1-\cos^2 y} && \sin y > 0 \text{ because } 0 \leq y \leq \pi
\end{aligned}$$

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$$\begin{aligned} \sin(2\arccos(x)) &= \sin(2y) && \text{Set } {\color{red} y = \arccos x} \\ &= 2\cos y \sin y && \text{Express via } \sin y, \cos y \\ &= 2\cos y \left(\pm\sqrt{1-\cos^2 y}\right) && \text{Express } \sin y \text{ via } \cos y \\ &= 2\cos y\sqrt{1-\cos^2 y} && {\color{red} \sin y > 0 \text{ because}} \\ & && {\color{red} 0 \leq y \leq \pi} \end{aligned}$$

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Rewrite $\sin(2\arccos(x))$ as an algebraic expression of x and $\sqrt{1-x^2}$. To simplify $\arccos x$ we try to use $\cos(\arccos x) = x$. Therefore our aim is to rewrite the expression only using the $\cos$ function.

$\sin(2\arccos(x)) = \sin(2y)$	Set $y = \arccos x$
$= 2\cos y \sin y$	Express via $\sin y, \cos y$
$= 2\cos y\left(\pm\sqrt{1-\cos^2 y}\right)$	Express $\sin y$ via $\cos y$
$= 2\cos y\sqrt{1-\cos^2 y}$	$\sin y > 0$ because $0 \leq y \leq \pi$
$= 2x\sqrt{1-x^2}$	use $x = \cos y$

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$$\cos(3\arccos(x)) = \cos(3y) \qquad \Big| \; y = \arccos x$$

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$$\cos(3\arccos(x)) = \cos(3y) = \cos(2y + y) \qquad \Big|\ y = \arccos x$$

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$$\cos(3\arccos(x)) = \cos(3y) = \cos(2y + y) \qquad \Big|\ y = \arccos x$$

$$= \mathbf{?} \qquad \Big|\ \text{Angle sum f-la}$$

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$$\begin{aligned} \cos(3\arccos(x)) &= \cos(3y) = \cos(2y+y) && \Big|\; y = \arccos x \\ &= \cos(2y)\cos y - \sin(2y)\sin y && \Big|\; \text{Angle sum f-la} \end{aligned}$$

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$$\begin{aligned}\cos(3\arccos(x)) &= \cos(3y) = \cos(2y+y) \\ &= \cos(2y)\cos y - \sin(2y)\sin y \\ &= (\textbf{?} \qquad\qquad)\cos y \\ &\quad -\textbf{?} \qquad\qquad \sin y\end{aligned}$$

$y = \arccos x$
Angle sum f-la
Express via
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&= (\cos^2 y - \sin^2 y)\cos y && \Big|\; \text{Express via} \\
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\cos(3\arccos(x)) &= \cos(3y) = \cos(2y+y) && \Big|\; y = \arccos x \\
&= \cos(2y)\cos y - \color{red}{\sin(2y)}\color{black}\sin y && \Big|\; \text{Angle sum f-la} \\
&= (\cos^2 y - \sin^2 y)\cos y && \Big|\; \text{Express via} \\
&\quad - \color{red}{\mathbf{?}} \color{black}\qquad\qquad\qquad \sin y && \Big|\; \sin y, \cos y
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&= ({\color{red}\cos^2 y} - \sin^2 y){\color{red}\cos y} && \Big|\ \text{Express via} \\
&\quad - 2\sin y\cos y\sin y && \Big|\ \sin y, \cos y \\
&= {\color{red}\cos^3 y} - \sin^2 y\cos y - 2\sin^2 y\cos y
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$$\begin{array}{rl|l} \cos(3\arccos(x)) = & \cos(3y) = \cos(2y+y) & y = \arccos x \\ = & \cos(2y)\cos y - \sin(2y)\sin y & \text{Angle sum f-la} \\ = & (\cos^2 y {\color{red} - \sin^2 y}) {\color{red}\cos y} & \text{Express via} \\ & -2\sin y\cos y\sin y & \sin y, \cos y \\ = & \cos^3 y {\color{red} - \sin^2 y\cos y} - 2\sin^2 y\cos y & \end{array}$$

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&\quad -2\sin y \cos y \sin y && \quad \left| \; \sin y, \cos y \right.\\
&= \cos^3 y - \sin^2 y \cos y - 2\sin^2 y \cos y\\
&= \cos^3 y - 3\sin^2 y \cos y
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&= \cos^3 y - \sin^2 y \cos y - 2\sin^2 y \cos y \\
&= \cos^3 y - 3\color{red}{\sin^2 y}\cos y && \Big| \; \color{red}{\text{Express } \sin y \text{ via } \cos y} \\
&= \cos^3 y - 3(\color{red}{\mathbf{?}} \qquad\qquad)\cos y
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&= \cos^3 y - \sin^2 y \cos y - 2\sin^2 y \cos y \\
&= \cos^3 y - 3\sin^2 y \cos y && \Big| \; \text{Express } \sin y \text{ via } \cos y \\
&= \cos^3 y - 3(1 - \cos^2 y)\cos y
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&= \cos^3 y - 3\sin^2 y \cos y && \Big| \; \text{Express } \sin y \text{ via } \cos y \\
&= \cos^3 y - 3(1 - \cos^2 y)\cos y \\
&= 4\cos^3 y - 3\cos y
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&= \cos^3 y - 3\sin^2 y\cos y && \Big|\ \text{Express } \sin y \text{ via } \cos y \\
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& -2\sin y\cos y\sin y & \Big| \; \sin y, \cos y \\
= & \cos^3 y - \sin^2 y\cos y - 2\sin^2 y\cos y & \\
= & \cos^3 y - 3\sin^2 y\cos y & \Big| \; \text{Express } \sin y \text{ via } \cos y \\
= & \cos^3 y - 3(1-\cos^2 y)\cos y & \\
= & 4\cos^3 y - 3\cos y & \\
= & 4x^3 - 3x & \Big| \; x = \cos y
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& \quad - 2\sin y \cos y \sin y & \Big| \; \sin y, \cos y \\
& = \cos^3 y - \sin^2 y \cos y - 2\sin^2 y \cos y & \\
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