

# Calculus I

## Integral substitution rule

Todor Milev

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# Outline

- 1 The Substitution Rule
  - Substitution rule and definite Integrals

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# The Substitution Rule

- How do we integrate  $\int 2x\sqrt{1+x^2} \, dx$ ?
- Introduce a new variable  $u = 1 + x^2$ .
- Then  $du = d(1 + x^2) = (1 + x^2)' \, dx = 2x \, dx$ .
- Substitute into the integral:

$$\int 2x\sqrt{1+x^2} \, dx = \int \sqrt{u} \, du = \int u^{\frac{1}{2}} \, du = \frac{2}{3} u^{\frac{3}{2}} + C = \frac{2}{3} (1+x^2)^{\frac{3}{2}} + C$$

- Is this procedure justified?
- Take the derivative using the Chain Rule:

$$\frac{d}{dx} \left( \frac{2}{3} (1+x^2)^{\frac{3}{2}} + C \right) = \frac{d}{dx} \left( \frac{2}{3} u^{\frac{3}{2}} \right) = \frac{2}{3} \cdot \frac{3}{2} u^{\frac{1}{2}} \frac{du}{dx} = \sqrt{1+x^2} (2x)$$

## Theorem (The Substitution Rule)

*Let  $u = g(x)$  be a differentiable function whose range is an interval  $I$  and let  $f$  be a function continuous on  $I$ . Then*

$$\int f(g(x))g'(x) \, dx = \int f(u) \, du$$

This is the integration counterpart of the Chain Rule.

## Example (Substitution Rule)

Find  $\int x^3 \cos(x^4 + 3) dx$ .

Let  $u = x^4 + 3$ .

Then  $du = 4x^3 dx$

$$x^3 dx = \frac{1}{4} du.$$

Substitute: 
$$\begin{aligned} \int x^3 \cos(x^4 + 3) dx &= \int \frac{1}{4} \cos u du \\ &= \frac{1}{4} \sin u + C \\ &= \frac{1}{4} \sin(x^4 + 3) + C. \end{aligned}$$

## Example (Substitution Rule)

Find  $\int \sqrt{2x+1} dx$ .

Let  $u = 2x + 1$ .

Then  $du = 2dx$

$$dx = \frac{1}{2} du.$$

Substitute: 
$$\begin{aligned} \int \sqrt{2x+1} dx &= \int \frac{1}{2} \sqrt{u} du \\ &= \frac{1}{2} \cdot \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C \\ &= \frac{1}{3} (2x+1)^{\frac{3}{2}} + C. \end{aligned}$$

## Example (Substitution Rule)

Find  $\int \frac{x}{\sqrt{3-4x^2}} dx.$

Let  $u = 3 - 4x^2.$

Then  $du = -8x dx$

$$x dx = -\frac{1}{8} du.$$

Substitute: 
$$\begin{aligned} \int \frac{x}{\sqrt{3-4x^2}} dx &= \int \left(-\frac{1}{8}\right) \frac{1}{\sqrt{u}} du \\ &= -\frac{1}{8} \cdot \frac{u^{\frac{1}{2}}}{\frac{1}{2}} + C \\ &= -\frac{1}{4} \sqrt{3-4x^2} + C. \end{aligned}$$



## Example (Substitution Rule)

Find  $\int e^{3x} dx$ .

Let  $u = 3x$ .

Then  $du = 3dx$

$$dx = \frac{1}{3} du.$$

$$\begin{aligned}\text{Substitute: } \int e^{3x} dx &= \int \frac{1}{3} e^u du \\ &= \frac{1}{3} e^u + C \\ &= \frac{1}{3} e^{3x} + C.\end{aligned}$$

## Example (Substitution Rule, more factors)

Evaluate  $\int 3x^5 \sqrt{1+x^3} dx = \int 3x^2 x^3 \sqrt{1+x^3} dx.$

$$\text{Let } u = 1 + x^3.$$

$$\text{Then } du = 3x^2 dx.$$

$$x^3 = u - 1.$$

$$\begin{aligned} \int 3x^2 x^3 \sqrt{1+x^3} dx &= \int (u-1) \sqrt{u} du \\ &= \int \left( u^{\frac{3}{2}} - u^{\frac{1}{2}} \right) du \\ &= \left( \frac{u^{\frac{5}{2}}}{\frac{5}{2}} - \frac{u^{\frac{3}{2}}}{\frac{3}{2}} \right) + C \\ &= \frac{2}{5} (1+x^3)^{\frac{5}{2}} - \frac{2}{3} (1+x^3)^{\frac{3}{2}} + C. \end{aligned}$$

# Definite Integrals

There are two ways to find a definite integral with the Substitution Rule:

- ① First evaluate the indefinite integral, then use the FTC.

$$\begin{aligned}\int_0^4 \sqrt{2x+1} \, dx &= \left[ \int \sqrt{2x+1} \, dx \right]_0^4 = \left[ \frac{1}{3}(2x+1)^{3/2} \right]_0^4 \\ &= \frac{1}{3}(2 \cdot 4 + 1)^{3/2} - \frac{1}{3}(2 \cdot 0 + 1)^{3/2} \\ &= \frac{1}{3}(9)^{3/2} - \frac{1}{3}(1)^{3/2} = \frac{1}{3}(27 - 1) = \frac{26}{3}\end{aligned}$$

- ② Change the limits of integration when the variable is changed.

## Theorem (The Substitution Rule for Definite Integrals)

*If  $g'$  is continuous on  $[a, b]$  and  $f$  is continuous on the range of  $g$ , then letting  $u = g(x)$  we get*

$$\int_a^b f(g(x))g'(x) \, dx = \int_{g(a)}^{g(b)} f(u) \, du$$

## Example

Find  $\int_0^4 \sqrt{2x+1} \, dx$ .

- Let  $u = 2x + 1$ .
- Then  $du = 2dx$ .
- Therefore  $dx = \frac{1}{2}du$ .
- When  $x = 0$ ,  $u = 1$ .
- When  $x = 4$ ,  $u = 9$ .

$$\begin{aligned}\int_{x=0}^{x=4} \sqrt{2x+1} \, dx &= \int_{u=1}^{u=9} \frac{1}{2} \sqrt{u} \, du = \int_1^9 \frac{1}{2} u^{\frac{1}{2}} du \\ &= \left[ \frac{1}{\frac{3}{2}} \cdot \frac{2}{3} (u)^{\frac{3}{2}} \right]_1^9 \\ &= \frac{1}{3} (9)^{\frac{3}{2}} - \frac{1}{3} (1)^{\frac{3}{2}} = \frac{1}{3} (27 - 1) = \frac{26}{3}\end{aligned}$$

## Example

Find  $\int_1^2 \frac{dx}{(2-3x)^2}$ .

- Let  $u = 2 - 3x$ .
- Then  $du = -3 dx$ .
- Therefore  $dx = -\frac{1}{3}du$ .
- When  $x = 1$ ,  $u = -1$ .
- When  $x = 2$ ,  $u = -4$ .

$$\begin{aligned}\int_{x=1}^{x=2} \frac{dx}{(2-3x)^2} &= -\frac{1}{3} \int_{u=-1}^{u=-4} \frac{du}{u^2} = -\frac{1}{3} \int_{-1}^{-4} u^{-2} du \\ &= -\frac{1}{3} \cdot \left[ -\frac{1}{u} \right]_{-1}^{-4} = \frac{1}{3} \left[ \frac{1}{u} \right]_{-1}^{-4} \\ &= \frac{1}{3} \left( \frac{1}{-4} - \frac{1}{-1} \right) = \frac{1}{3} \left( 1 - \frac{1}{4} \right) = \frac{1}{4}.\end{aligned}$$