

Calculus II

Integral of rational function with cubic denominator, part 3

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- The for each quadratic factor we need to add a partial fraction of the form

$$\frac{Ax + B}{ax^2 + bx + c}.$$