Calculus I
Derivatives: linearity, product and quotient rules

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2019

Theorem (The Power Rule (General Version))

If n is any real number, then

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^n) = nx^{n-1}.$$

Example (Power Rule, negative exponent)

$$\text{Differentiate} \quad y = \frac{1}{x}.$$

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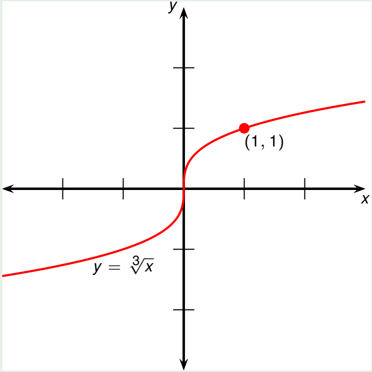
$$= -\frac{1}{x^2}.$$

Example (Calculating the tangent line using the Power Rule)

Find an equation for the tangent line to the cubic $y = \sqrt[3]{x}$ at the point $P = (1, 1)$.



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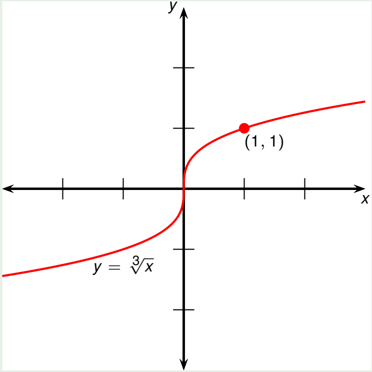

Here $a = 1$ and $f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$.

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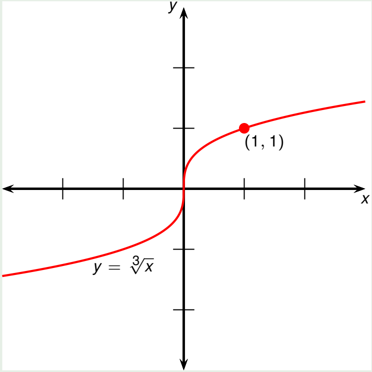

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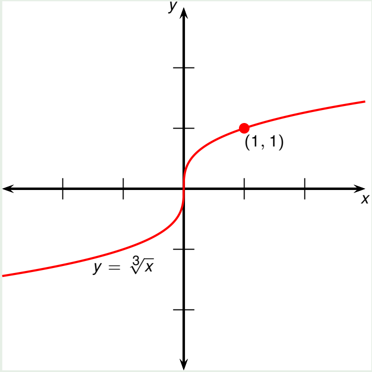

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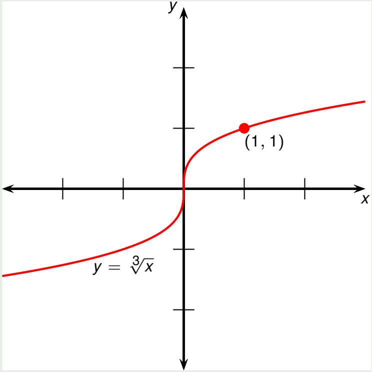

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$$\begin{aligned} f'(x) &= \frac{1}{3}x^{\frac{1}{3}-1} \\ &= \frac{1}{3}x^{\frac{-2}{3}} \\ &= \frac{1}{3\sqrt[3]{x^2}}. \end{aligned}$$

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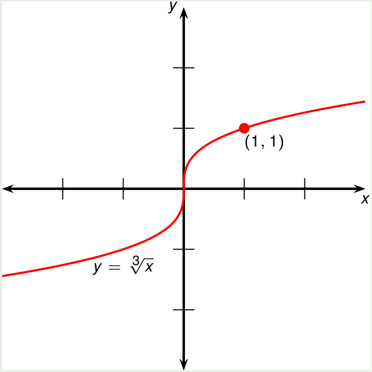

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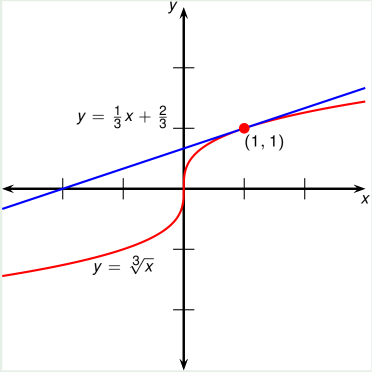

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f'(1) &= \frac{1}{3}.
\end{aligned}
$$

Point-slope form: $y - 1 = \frac{1}{3}(x - 1)$, or $y = \frac{1}{3}x + \frac{2}{3}$.

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Power Rule: $\quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{5x^{-\frac{1}{6}}}{6}$

$$= \frac{5}{6\sqrt[6]{x}}.$$

Theorem (The Constant Multiple Rule)

If c is a constant and f is a differentiable function, then

$$\frac{\mathrm{d}}{\mathrm{d}x}[cf(x)] = c\frac{\mathrm{d}}{\mathrm{d}x}f(x).$$

Proof.

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&= \lim_{h\to 0}\frac{c(f(x+h)-f(x))}{h}
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&= \lim_{h\to 0}\frac{{\color{red}c}(f(x+h)-f(x))}{h} \\
\text{Limit Law 3:} \quad &= {\color{red}c}\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}
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Find the derivative of $y = \dfrac{2x^5}{7}$.

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Constant Multiple Rule: $= \left(\frac{2}{7}\right)\frac{\mathrm{d}}{\mathrm{d}x}\left(x^5\right)$

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Constant Multiple Rule:
$$= \left(\frac{2}{7}\right)\frac{d}{dx}\left(x^5\right)$$

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$$= \frac{10x^4}{7}.$$

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$$= (-1)(\mathbf{?})$$

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Constant Multiple Rule: $$= (-1)\,\frac{\mathrm{d}}{\mathrm{d}x}\,(x)$$

$$= (-1)\,(1)$$

Example (Constant Multiple Rule, Power Rule)

$$
\begin{aligned}
\text{Find the derivative of} \quad u &= -x.\\
u &= (-1)(x).\\
\frac{\mathrm{d}u}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}\left[(-1)(x)\right]\\
\text{Constant Multiple Rule:} \quad &= (-1)\frac{\mathrm{d}}{\mathrm{d}x}(x)\\
&= (-1)(1)\\
&= -1.
\end{aligned}
$$

Example (Constant Multiple Rule, Power Rule, Negative Exponent)

Find the derivative of $t = \dfrac{2\pi}{x^4}$.

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Constant Multiple Rule: $= (2\pi)\dfrac{\mathrm{d}}{\mathrm{d}x}\left(x^{-4}\right)$

$$= (2\pi)\left(-4x^{-5}\right)$$

$$= -\frac{8\pi}{x^5}.$$

Theorem (The Sum Rule)

If f and g are both differentiable, then

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(x)+g(x)]=\frac{\mathrm{d}}{\mathrm{d}x}f(x)+\frac{\mathrm{d}}{\mathrm{d}x}g(x).$$

Proof.

$\square$

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& = \lim_{h\to 0} \left[\frac{f(x+h)-f(x)}{h} + \frac{g(x+h)-g(x)}{h}\right] \\
\text{Limit Law 1:} \quad & = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}
\end{aligned}$$

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&= f'(x)+g'(x).
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\end{aligned}$$

$\square$

The Sum Rule can be extended to any number of summands. For instance, using the theorem twice, we get

$$(f+g+h)' = [(f+g)+h]' = (f+g)'+h' = f'+g'+h'.$$

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By writing $f-g$ as $f+(-1)g$ and applying the Sum Rule and the Constant Multiple Rule, we get

Theorem (The Difference Rule)

If f and g are both differentiable, then

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(x)-g(x)] = \frac{\mathrm{d}}{\mathrm{d}x}f(x) - \frac{\mathrm{d}}{\mathrm{d}x}g(x).$$

The Constant Multiple Rule, the Sum Rule, the Difference Rule, and the Power Rule can be combined to differentiate any polynomial.

Example (Derivative of a Polynomial)

$$\text{If} \quad y = x^{16} + 2\sqrt{3}x^7 - 4x^3 + \frac{x}{8} - 5,$$

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&= (16x^{15}) + 2\sqrt{3}\left(7x^6\right) - 4\left(3x^2\right) + \frac{1}{8}(1) - (0) \\
&= 16x^{15} + 14\sqrt{3}x^6 - 12x^2 + \frac{1}{8}.
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Differentiate $\quad v = \dfrac{3\sqrt{x} - \sqrt[3]{x}}{x}.$

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&= \lim_{h\to 0} \frac{a^x a^h - a^x}{h} \\
&= \lim_{h\to 0} \frac{{\color{red}a^x}\left(a^h - 1\right)}{h} \\
&= {\color{red}a^x} \lim_{h\to 0} \frac{a^h - 1}{h} \\
&= {\color{red}a^x} \lim_{h\to 0} \frac{a^h - a^0}{h}
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Derivatives of Exponential Functions

Compute the derivative of $f(x) = a^x$ using the definition:

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\begin{aligned}
f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h\to 0} \frac{a^{x+h} - a^x}{h} \\
&= \lim_{h\to 0} \frac{a^x a^h - a^x}{h} \\
&= \lim_{h\to 0} \frac{a^x \left(a^h - 1\right)}{h} \\
&= a^x \lim_{h\to 0} \frac{a^h - {\color{red}1}}{h} \\
&= a^x \lim_{h\to 0} \frac{a^h - {\color{red}a^0}}{h}
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&= a^x f'(0).
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We have shown that, if $f(x) = a^x$ is differentiable at 0, then it is differentiable everywhere, and

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Theorem

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exists.

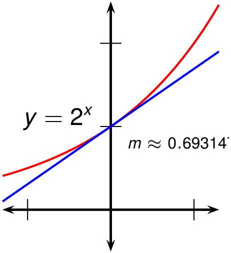
We will later show that

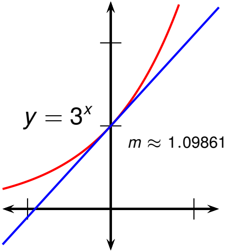
$$f'(0) = \lim_{h \to 0} \frac{a^h - 1}{h} = \ln(a).$$

Here, $\ln$ is the natural logarithm function.

$$\text{If} \quad f(x) = a^x, \quad \text{then} \quad f'(x) = f'(0)a^x.$$

The formula above is simplest when $f'(0) = 1$. Since $\lim_{h\to 0} \frac{2^h-1}{h} \approx 0.69$ and $\lim_{h\to 0} \frac{3^h-1}{h} \approx 1.10$, we expect there is a number a between 2 and 3 such that $\lim_{h\to 0} \frac{a^h-1}{h} = 1$.





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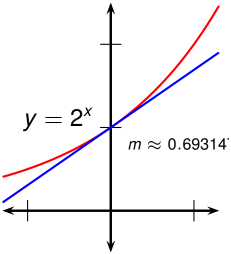
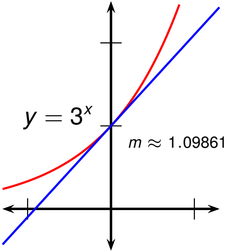
Definition (e)

e is the number such that $\lim\limits_{h\to 0} \frac{e^h-1}{h} = 1$.





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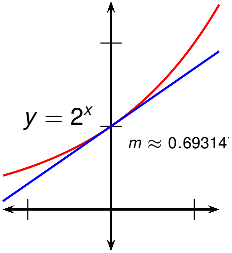

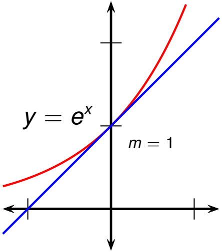

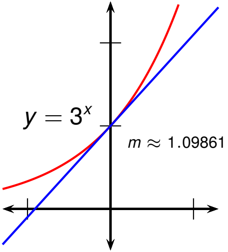

Definition (Natural Exponential Function)

e^x is called the natural exponential function. Its derivative is

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(e^x\right) = e^x.$$

Example (Derivative of a Polynomial and the Natural Exponential Function)

Differentiate $y = e^x + x^7.$

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We need a formula for the derivative of the product of two functions. One might guess that the derivative of a product is the product of the derivatives; however, this is wrong.

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Let $f(x) = x$ and $g(x) = x^2$.

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The correct formula is called the Product Rule.

Theorem (The Product Rule)

If f and g are both differentiable, then

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x).$$

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&= \lim_{h\to 0} \frac{\color{red}{f(x+h)}g(x+h) - \color{red}{f(x+h)}g(x) + f(x+h)g(x) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \left(\color{red}{f(x+h)}\frac{g(x+h) - g(x)}{h} + g(x)\frac{f(x+h) - f(x)}{h} \right)
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If f and g are both differentiable, then

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x).$$

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&= e^x\left(x^3 + 3x^2\right).
\end{aligned}$$

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If f and g are differentiable and $g(x) \neq 0$, then

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f(x)}{g(x)}\right) = \frac{\frac{\mathrm{d}}{\mathrm{d}x}\left(f(x)\right)g(x) - f(x)\frac{\mathrm{d}}{\mathrm{d}x}\left(g(x)\right)}{\left(g(x)\right)^2} \quad \Bigg| \quad \textit{(Leibniz notation)}$$

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- The proof of the Quotient Rule is similar to the proof of the Product Rule.
- There is an alternative algebraic proof via the Product Rule, the Power Rule and the (not yet studied) Chain Rule.

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&= \frac{\left(5x^4 + 2\right)\left(-x^6 + 2\right) - \left(x^5 + 2x\right)\left(-6x^5\right)}{\left(-x^6 + 2\right)^2} \\
&= \frac{(-5x^{10} - 2x^6 + 10x^4 + 4) - (-6x^{10} - 12x^6)}{(-x^6 + 2)^2}
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$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

Example

Compute the derivative. Use the quotient rule.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2x-1}\right)$$

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