

# Precalculus

## Logarithms basics

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# Outline

## 1 Logarithmic Functions

- Logarithm basics
- Natural Logarithms
- Shifting graphs of logarithmic functions

## 2 Basic Operations with Logarithms

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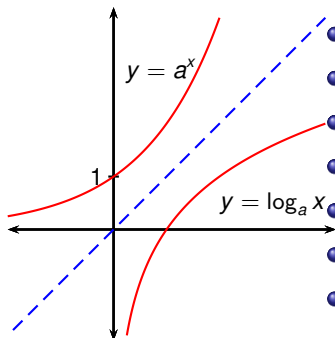
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# Logarithmic Functions



- Suppose  $a > 0$ ,  $a \neq 1$ .
- Let  $f(x) = a^x$ .
- Then  $f$  is either increasing or decreasing.
- Therefore  $f$  is one-to-one.
- Therefore  $f$  has an inverse function,  $f^{-1}$ .
- The graph shows  $y = a^x$  for  $a > 1$ .
- The graph of  $y = \log_a x$  is the reflection of this in the line  $y = x$ .

## Definition ( $\log_a x$ )

The inverse function of  $f(x) = a^x$  is called the logarithmic function with base  $a$ , and is written  $\log_a x$ . It is defined by the formula

$$\log_a x = y \quad \Leftrightarrow \quad a^y = x.$$

If  $x > 0$ , then  $\log_a x$  is the exponent to which the base  $a$  must be raised to give  $x$ .

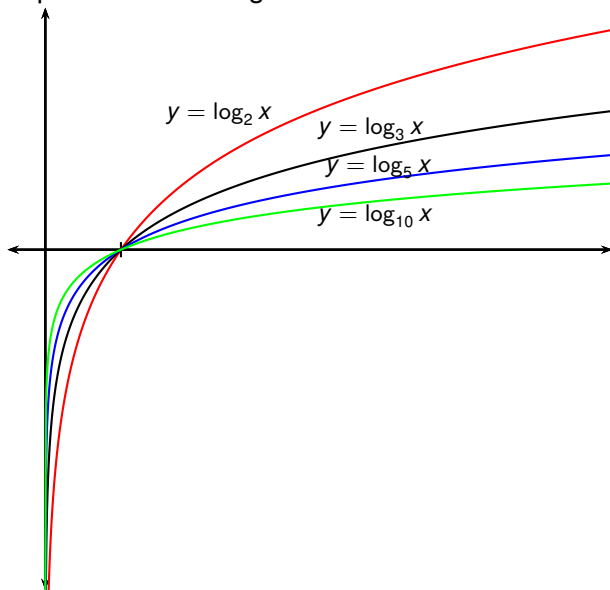
## Example

Evaluate:

- 1  $\log_3 81 = 4$  because  $3^4 = 81$ .
- 2  $\log_{25} 5 = \frac{1}{2}$  because  $25^{\frac{1}{2}} = \sqrt{25} = 5$ .
- 3  $\log_{10} 0.001 = -3$  because  $10^{-3} = 0.001$ .



- Suppose  $a > 1$ .
- Domain of  $a^x$ :  $\mathbb{R}$ .
- Range of  $a^x$ :  $(0, \infty)$ .
- Domain of  $\log_a x$ :  $(0, \infty)$ .
- Range of  $\log_a x$ :  $\mathbb{R}$ .
- $\log_a(a^x) = x$  for  $x \in \mathbb{R}$ .
- $a^{\log_a x} = x$  for  $x > 0$ .

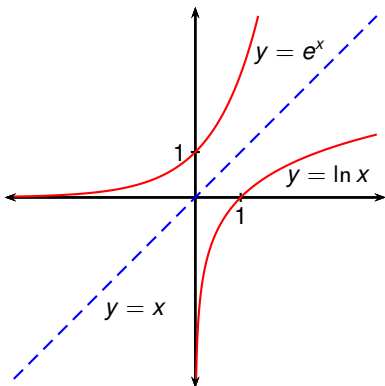
Graphs of various logarithmic functions with  $a > 1$ 

# Natural Logarithms

## Definition ( $\ln x$ )

The logarithm with base  $e$  is called the natural logarithm, and has a special notation:

$$\log_e x = \ln x.$$



- $\ln x = y \quad \Leftrightarrow \quad e^y = x.$
- $\ln(e^x) = x$  for  $x \in \mathbb{R}.$
- $e^{\ln x} = x$  for  $x > 0.$



What does  $\log x$  stand for? **WARNING:** there are **two different** accepted uses for  $\log x$ .

- In some texts/applications  $\log x$  stands for

$$\log x = \log_{10} x \quad .$$

- Used in many engineering texts.
- Used in many natural sciences texts.
- Used in many high school textbooks.
- Used in old math textbooks.

- In other texts/applications  $\log x$  stands for (the principal branch of the) **complex logarithm**

$$\log x = \begin{cases} \ln x = \log_e x & \text{if } x > 0 \\ \ln(-x) + \pi i & \text{if } x < 0 \\ ? & \text{for } x \notin \mathbb{R} \end{cases} .$$

- Used in mathematical, many computer science texts.
- Used in many natural science texts.
- Used in most computer algebra systems.
- This is the notation accepted by most mathematicians.
- $\log$  and  $\ln$  have different domains but else coincide:  $\ln$  is defined for positive reals, and  $\log$  - for non-zero complex.

- *In the present course we shall abstain from using the notation  $\log x$ .*
- *When we need logarithms base 10 we will always write  $\log_{10}$ .*
- Within this course, we request that the student abstain from using  $\log x$  and use instead the unambiguous  $\log_{10} x$ .
- Outside of this course, we recommend that the student continue avoiding the notation  $\log$ .
- Should our recommendation contradict the commonly accepted conventions in the field of study of the student, we expect the student to honor the conventions of their fields of study.

# Summary of logarithm notation conventions

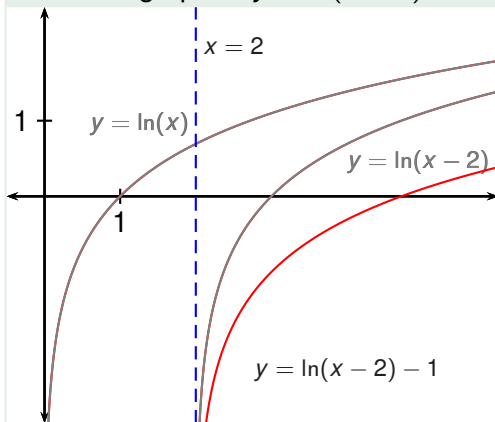
	Name	ISO notation	Other notation	Used in
$\log_2(x)$	binary logarithm	$\text{lb}(x)$	$\text{ld}(x)$ , $\log(x)$ , $\text{lg}(x)$	computer science, information theory, music theory, photography
$\log_e(x)$	natural logarithm	$\ln(x)$	$\log(x)$	mathematics, physics, chemistry, statistics, economics, information theory, and engineering
$\log_{10}(x)$	common logarithm	$\text{lg}(x)$	$\log(x)$	various engineering, logarithm tables, handheld calculators, spectroscopy

Table source: Wikipedia

- Standardized in ISO\_31-11 (International Standards Organization).

## Example

Draw the graph of  $y = \ln(x - 2) - 1$ .



- Graph  $y = \ln(x)$  assumed known.
- $f(x - 2)$  shifts graph 2 units to the right.
- $g(x) - 1$  shifts graph 1 unit down.

## Theorem (Properties of Logarithmic Functions)

*If  $a > 1$ , the function  $f(x) = \log_a x$  is a one-to-one, continuous, increasing function with domain  $(0, \infty)$  and range  $\mathbb{R}$ . If  $x, y, a, b > 0$  and  $r$  is any real number, then*

$$\textcircled{1} \log_a(xy) = \log_a x + \log_a y.$$

$$\textcircled{2} \log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y.$$

$$\textcircled{3} \log_a(x^r) = r \log_a x.$$

$$\textcircled{4} \log_a(x) = \log_b x \log_a b = \frac{\log_b x}{\log_b a} = \frac{\ln x}{\ln a}.$$

Recall that  $\log_a(x) = \log_b x \log_a b = \frac{\log_b x}{\log_b a} = \frac{\ln x}{\ln a}$ .

## Example

Using only the  $\ln$  and arithmetic operations of your calculator, compute  $\log_5(13)$ . Confirm your answer by exponentiation.

$$\log_5(13) = \frac{\ln 13}{\ln 5} \approx \frac{2.564949357}{1.609437912} \approx 1.593693.$$

As a check of our computations, we compute by calculator:

$13 = 5^{\log_5 13} \approx 5^{1.593693} \approx 13.000007508$ , and our computations check out.

Use the properties of logarithms to evaluate the following.

### Example

$$\begin{aligned}\log_4 2 + \log_4 32 &= \log_4(2 \cdot 32) \\ &= \log_4(64) \\ &= 3 \\ &\quad (\text{because } 4^3 = 64.)\end{aligned}$$

### Example

$$\begin{aligned}\log_2 80 - \log_2 5 &= \log_2 \left( \frac{80}{5} \right) \\ &= \log_2(16) \\ &= 4 \\ &\quad (\text{because } 2^4 = 16.)\end{aligned}$$

## Example

Compute the exact value of the expression as a rational number.

$$\begin{aligned}\log_7 \sqrt[3]{49} &= \log_7 \left( 49^{\frac{1}{3}} \right) \\ &= \frac{1}{3} \log_7 49 \\ &= \frac{1}{3} \log_7 7^2 \\ &= \frac{2}{3} \log_7 7 \\ &= \frac{2}{3}\end{aligned}$$



## Example

Fully expand the expression to a sum of logarithms. Your answer should not contain logarithms of products or logarithms of exponents.

$$\begin{aligned}\ln\left(\frac{y\sqrt{1+x}}{z^2}\right) &= \ln\left(y\sqrt{1+x}\right) - \ln\left(z^2\right) \\ &= \ln y + \ln\sqrt{1+x} - 2\ln z \\ &= \ln y + \frac{1}{2}\ln(1+x) - 2\ln z\end{aligned}$$

The inverse hyperbolic function  $\operatorname{arcsinh} = \ln \left( x + \sqrt{1 + x^2} \right)$  is used when studying hyperbolas (types of curves in the plane).

### Example

Demonstrate that  $-\ln \left( \sqrt{1 + x^2} - x \right) = \ln \left( x + \sqrt{1 + x^2} \right)$ .

$$\begin{aligned}
 -\ln \left( \sqrt{1 + x^2} - x \right) &= \ln \left( \frac{1}{\sqrt{x^2 + 1} - x} \right) && \left| \text{rationalize} \right. \\
 &= \ln \left( \frac{\left( \sqrt{x^2 + 1} + x \right)}{\left( \sqrt{x^2 + 1} - x \right) \left( \sqrt{x^2 + 1} + x \right)} \right) \\
 &= \ln \left( \frac{\sqrt{x^2 + 1} + x}{x^2 + 1 - x^2} \right) \\
 &= \ln \left( \sqrt{x^2 + 1} + x \right) .
 \end{aligned}$$

## Proposition (Additional Properties of Logarithmic Functions)

*If  $a, b > 0$ , then*

$$\textcircled{1} \log_{\frac{1}{a}} x = -\log_a x$$

$$\textcircled{2} \log_a b = \frac{1}{\log_b a}.$$

$$\textcircled{3} \log_{a^k} b = \frac{1}{k} \log_a b.$$

## Example

Compute as a rational number, without using calculator.

$$\log_{\frac{1}{\sqrt[3]{49}}} \sqrt{343} =$$

## Example

Compute as a rational number, without using calculator.

$$\begin{aligned}
 \log_7(24) + \log_{\frac{1}{7}}(3) - \log_{49}(64) &= \log_7(24) + \frac{\log_7(3)}{\log_7(\frac{1}{7})} - \frac{\log_7(64)}{\log_7(49)} \\
 &= \log_7(24) + \frac{\log_7(3)}{-1} - \frac{\log_7(64)}{2} \\
 &= \log_7(24) - \log_7(3) - \frac{1}{2} \log_7(64)
 \end{aligned}$$

$$\left[ \begin{array}{l} \log_a x - \log_a y = \log_a \left( \frac{x}{y} \right) \\ \log_a x^r = r \log_a x \end{array} \right]$$

$$\begin{aligned}
 &= \log_7 \left( \frac{24}{3} \right) - \log_7 \left( 64^{\frac{1}{2}} \right) \\
 &= \log_7(8) - \log_7(\sqrt{64}) \\
 &= \log_7 8 - \log_7 8 = 0
 \end{aligned}$$

$$\text{[alternatively:]} \quad = \log_7 \left( \frac{8}{8} \right) = \log_7(1) = 0.$$

## Example

Prove the logarithmic properties.

$$\textcircled{1} \log_a(xy) = \log_a x + \log_a y.$$

$$\textcircled{2} \log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y.$$

$$\textcircled{3} \log_a(x^r) = r \log_a x.$$

$$\textcircled{4} \log_a(x) = \log_b x \log_a b = \frac{\log_b x}{\log_b a} = \frac{\ln x}{\ln a}.$$