

Precalculus

Polynomial systems basics

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2019

Outline

- 1 Overview of polynomial systems
- 2 Ad hoc methods for solving polynomial systems

Systems of polynomial equations

Definition

A collection of one or more simultaneous polynomial equations in one or more variables is called a *polynomial system*.

- The definition includes usual one-variable polynomial equations such as:

$$x^2 + 2x - 3 = 0$$

- Typical polynomial systems have more than one variable/equation:

$$\begin{array}{l|l} y^2 + xy - 4y - 2x + 4 & = 0 \\ y^2x - 2yx - y - 2x + 4 & = 0. \end{array}$$

Here we have 2 variables (x, y), 2 equations.

- The number of variables and equations need not be equal:

$$\begin{array}{l|l} x + y + z + w & = 2 \\ y + z^2 & = 1 \\ y + zw^2 & = 1. \end{array}$$

Here we have 4 variables (x, y, z, w), 3 equations.

Systems of polynomial equations

Definition

A collection of one or more simultaneous polynomial equations in one or more variables is called a *polynomial system*.

- Polynomial systems may have no solutions:
$$\begin{cases} x = 0 \\ xy = 1. \end{cases}$$

The first equation implies $x = 0$, but then the left hand side of the second equation must equal 0.

- Polynomial systems may finitely many solutions:
$$\begin{cases} x = 0 \\ y + x = 1. \end{cases}$$

The first equation implies $x = 0$, and then the second equation implies $y = 1$.

- Polynomial systems may have infinitely many solutions:

$$\begin{cases} x = 0 \\ y + z = 1. \end{cases}$$
 If we set $x = 0$, $y = 1 - z$, we produce infinitely many solutions for every possible value of z .

Systems of polynomial equations

Definition

A collection of one or more simultaneous polynomial equations in one or more variables is called a *polynomial system*.

- The branch of mathematics that studies exact solutions of polynomial systems is called Algebraic Geometry.
- The practical aspects of solving such systems are covered under the subject of Elimination Theory.
- Solving polynomial systems is an indispensable mathematical tool used in other branches of science and mathematics.
- Polynomial systems also have direct practical applications, for example kinematics - the configurations of a robotic arm can be parametrized with polynomials.

Systems of polynomial equations

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- Whether a system has finitely or infinitely many solutions and what are they can be computed with a computer algorithm.
- Algorithm: find so-called Gröbner basis (named after the Austrian W. Gröbner, 1899-1980) using the Buchberger algorithm (named after the Austrian B. Buchberger, 1942-).
- These algorithms are too advanced to cover here.
- Not well-suited for pen and paper computations: can get notoriously large and require computers/super-computers.
- A system doable by hand would typically be solved milliseconds on a modern computer.
- A system doable by hand would typically be solved easily using ad-hoc techniques.

Example

Solve the polynomial system. $\left| \begin{array}{rcl} x - 4y & = & 5 \\ y^2 + xy & = & 10 \end{array} \right.$

$$x = 5 + 4y$$

Solve for x in first eq-n.

$$y^2 + xy = 10$$

Substitute x away

$$y^2 + (5 + 4y)y = 10$$

$$y^2 + 5y + 4y^2 - 10 = 0$$

$$5y^2 + 5y - 10 = 0$$

Divide by 5

$$y^2 + y - 2 = 0$$

$$(y + 2)(y - 1) = 0$$

$$y = -2 \text{ or } y = 1$$

$$x = 5 + 4y$$

$$x = 5 + 4y$$

$$= 5 + 4(-2) = -3$$

$$= 5 + 4 \cdot 1 = 9$$

Final answer: $x = -3, y = -2$ or $x = 9, y = 1$.

Example

Solve the polynomial system.
$$\left| \begin{array}{rcl} x - 4y & = & 5 \\ y^2 + xy & = & 10 \end{array} \right.$$

Final answer: $x = -3, y = -2$ or $x = 9, y = 1$.

Check answer $x = -3, y = -2$:

$$\left| \begin{array}{rcl} x - 4y & = & (-3) - 4(-2) = 5 \\ y^2 + xy & = & (-2)^2 + (-3)(-2) = 10 \end{array} \right.$$

Check answer $y = 1, x = 9$:

$$\left| \begin{array}{rcl} x - 4y & = & 9 - 4 \cdot 1 = 5 \\ y^2 + xy & = & 1^2 + 9 \cdot 1 = 10. \end{array} \right.$$

Example

The sum of two numbers x and y is 25 and the sum of their squares is 313. Given that $y \geq x$, find x and y .

$$x + y = 25 \quad | \text{Solve for } y$$

$$y = 25 - x$$

$$x^2 + y^2 = 313$$

$$x^2 + (25 - x)^2 = 313$$

$$x^2 + (25^2 - 2 \cdot 25 \cdot x + x^2) - 313 = 0 \quad | (a - b)^2 = a^2 - 2ab + b^2$$

$$2x^2 - 50x + 625 - 313 = 0$$

$$2x^2 - 50x + 312 = 0 \quad | \text{Divide by 2}$$

$$x^2 - 25x + 156 = 0$$

$$\begin{aligned} x &= \frac{-(-25) \pm \sqrt{25^2 - 4 \cdot 1 \cdot 156}}{2 \cdot 1} \\ &= \frac{25 \pm \sqrt{625 - 624}}{2} \\ &= \frac{25 \pm 1}{2} = \begin{cases} \frac{25+1}{2} = 13 \\ \frac{25-1}{2} = 12 \end{cases} \end{aligned}$$

Example

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$$x + y = 25 \quad | \text{Solve for } y$$

$$y = 25 - x$$

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$$x^2 + (25 - x)^2 = 313$$

$$x = \frac{-(-25) \pm \sqrt{25^2 - 4 \cdot 1 \cdot 156}}{2 \cdot 1}$$

$$= \frac{25 \pm \sqrt{625 - 624}}{2}$$

$$= \frac{25 \pm 1}{2} = \begin{cases} \frac{25+1}{2} = 13 \\ \frac{25-1}{2} = 12 \end{cases}$$

$$y = 25 - x = \begin{cases} 25 - 13 = 12 \\ 25 - 12 = 13 \end{cases}$$

The two solution candidates are $x = 12, y = 13$ and $x = 13, y = 12$. Since $y \geq x$, one of the solutions needs to be discarded and our final answer is $x = 12, y = 13$.