Calculus II
Homework
Integration by parts

1. Let $x \in (0,1)$. Express the following using x and $\sqrt{1-x^2}$.

   (a) $\sin(\arcsin(x))$.

   answer: x

   (b) $\sin(2\arcsin(x))$.

   answer: $2x\sqrt{1-x^2}$

   (c) $\sin(3\arcsin(x))$.

   answer: $-4x^3+3x$

   (d) $\sin(\arccos(x))$.

   answer: $\sqrt{1-x^2}$

   (e) $\sin(2\arccos(x))$.

   answer: $2x\sqrt{1-x^2}$

   (f) $\sin(3\arccos(x))$.

   answer: $(4x^2-1)\sqrt{1-x^2} = -4\left(\sqrt{1-x^2}\right)^3 + 3\sqrt{1-x^2}$

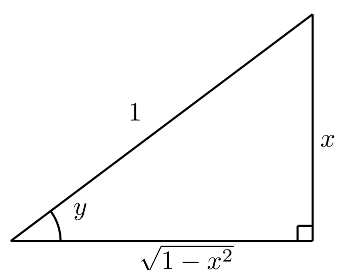
   (g) $\cos(2\arcsin(x))$.

   answer: $1-2x^2$

   (h) $\cos(3\arccos(x))$.

   answer: $4x^3-3x$

Solution. 1.b. Let $y = \arcsin x$. Then $\sin y = x$, and we can draw a right triangle with opposite side length x and hypotenuse length 1 to find the other trigonometric ratios of y.



Then $\cos y = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$. Now we use the double angle formula to find $\sin(2\arcsin x)$.

$$\begin{aligned}\sin(2\arcsin x) &= \sin(2y)\\ &= 2\sin y\cos y\\ &= 2x\sqrt{1-x^2}.\end{aligned}$$

Solution. 1.c. Use the result of Problem 1.b. This also requires the addition formula for sine:

$$\sin(A+B) = \sin A\cos B + \sin B\cos A,$$

and the double angle formula for cosine:

$$\cos(2y) = \cos^2 y - \sin^2 y.$$

$$
\begin{aligned}
\sin(3\arcsin x) &= \sin(3y) \\
&= \sin(2y+y) \\
&= \sin(2y)\cos y + \sin y\cos(2y) && \text{Use addition formula} \\
&= (2\sin y\cos y)\cos y + \sin y(\cos^2 y - \sin^2 y) && \text{Use double angle formulas} \\
&= 2\sin y\cos^2 y + \sin y\cos^2 y - \sin^3 y \\
&= 3\sin y\cos^2 y - \sin^3 y \\
&= 3\sin y(1-\sin^2 y) - \sin^3 y \\
&= 3x(1-x^2) - x^3 \\
&= 3x - 4x^3.
\end{aligned}
$$

The solution is complete. A careful look at the solution above reveals a strategy useful for problems similar to this one.

(a) Identify the inverse trigonometric expression- $\arcsin x, \arccos x, \arctan x, \dots$. In the present problem that was $y = \arcsin x$.

(b) The problem is therefore a trigonometric function of y.

(c) Using trig identities and algebra, rewrite the problem as a trigonometric expression involving only the trig function that transforms y to x. In the present problem we rewrote everything using $\sin y$.

(d) Use the fact that $\sin(\arcsin x) = x$, $\cos(\arccos x) = x, \dots$, etc. to simplify.

Solution. 1.f We use the same strategy outlined in the end of the solution of Problem 1.c. Set $y = \arccos x$ and so $\cos(y) = x$. Therefore:

$$
\begin{aligned}
\sin(3y) &= \sin(2y+y) \\
&= \sin(2y)\cos y + \sin y\cos(2y) \\
&= 2\sin y\cos y\cos y + \sin y(2\cos^2 y - 1) \\
&= 2\sin y\cos^2 y + \sin y(2\cos^2 y - 1) \\
&= \sin y(4\cos^2 y - 1) && \text{use } \begin{array}{rcl}\cos y &=& x \\ \sin y &=& \sqrt{1-x^2}\end{array} \\
&= \sqrt{1-x^2}(4x^2-1) \quad .
\end{aligned}
$$

2. Express as the following as an algebraic expression of x. In other words, "get rid" of the trigonometric and inverse trigonometric expressions.

   (a) $\cos^2(\arctan x)$. answer: $\frac{1}{1+x^2}$

   (b) $-\sin^2(\operatorname{arccot} x)$. answer: $-\frac{1}{1+x^2}$

   (c) $\dfrac{1}{\cos(\arcsin x)}$. answer: $\frac{1}{\sqrt{1-x^2}}$

   (d) $-\dfrac{1}{\sin(\arccos x)}$. answer: $-\frac{1}{\sqrt{1-x^2}}$

Solution. 2.b. We follow the strategy outlined in the end of the solution of Problem 1.c. We set $y = \operatorname{arccot} x$. Then we need to express $-\sin^2 y$ via $\cot y$. That is a matter of algebra:

$$
\begin{aligned}
-\sin^2(\operatorname{arccot} x) &= -\sin^2 y && \text{Set } y = \operatorname{arccot} x \\
&= -\frac{\sin^2 y}{\sin^2 y + \cos^2 y} && \text{use } \sin^2 y + \cos^2 y = 1 \\
&= -\frac{1}{\frac{\sin^2 y + \cos^2 y}{\sin^2 y}} \\
&= -\frac{1}{1+\cot^2 y} && \text{Substitute back } \cot y = x \\
&= -\frac{1}{1+x^2} \quad .
\end{aligned}
$$

3. Rewrite as a rational function of t. This problem will be later used to derive the Euler substitutions (an important technique for integrating).

(a) $\cos(2\arctan t)$.

answer: $\frac{1-t^2}{1+t^2}$.

(b) $\sin(2\arctan t)$.

answer: $\frac{2t}{1+t^2}$

(c) $\tan(2\arctan t)$.

answer: $\frac{2t}{1-t^2}$

(d) $\cot(2\arctan t)$.

answer: $\frac{1}{2}\left(\frac{1}{t}-t\right)$.

(e) $\csc(2\arctan t)$.

answer: $\frac{1}{2}\left(t+\frac{1}{t}\right)$

(f) $\sec(2\arctan t)$.

answer: $\frac{1+t^2}{1-t^2}$

(g) $\cos(2\operatorname{arccot} t)$.

answer: $\frac{t^2-1}{t^2+1}$

(h) $\sin(2\operatorname{arccot} t)$.

answer: $\frac{2t}{t^2+1}$

(i) $\tan(2\operatorname{arccot} t)$.

answer: $\frac{2t}{t^2-1}$

(j) $\cot(2\operatorname{arccot} t)$.

answer: $\frac{1}{2}\left(t-\frac{1}{t}\right)$

(k) $\csc(2\operatorname{arccot} t)$.

answer: $\frac{1}{2}\left(t+\frac{1}{t}\right)$

(l) $\sec(2\operatorname{arccot} t)$.

answer: $\frac{t^2+1}{t^2-1}$

Solution. 3.a Set $z = \arctan t$, and so $\tan z = t$. Then

$$\begin{array}{rcll}
\cos(2\arctan t) & = & \cos(2z) & \\
& = & \dfrac{\cos(2z)}{1} & \text{use double angle formulas and } 1=\sin^2 z+\cos^2 z \\
& = & \dfrac{\cos^2 z-\sin^2 z}{\cos^2 z+\sin^2 z} & \text{divide top and bottom by } \cos^2 z \\
& = & \dfrac{(\cos^2 z-\sin^2 z)\frac{1}{\cos^2 z}}{(\sin^2 z+\cos^2 z)\frac{1}{\cos^2 z}} & \\
& = & \dfrac{1-\tan^2 z}{1+\tan^2 z} & \\
& = & \dfrac{1-t^2}{1+t^2} & .
\end{array}$$

Solution. 3.d Set $z = \arctan t$, and so $\tan z = t$. Then

$$\begin{array}{rcll}
\cot(2\arctan t) & = & \cot(2z) & \\
& = & \dfrac{\cos(2z)}{\sin(2z)} & \text{use double angle formulas} \\
& = & \dfrac{\cos^2 z-\sin^2 z}{2\sin z\cos z} & \\
& = & \dfrac{1-\tan^2 z}{2\tan z} & \\
& = & \dfrac{1-t^2}{2t} & \\
& = & \dfrac{1}{2}\left(\dfrac{1}{t}-t\right) & .
\end{array}$$

4. Compute the derivative (derive the formula).

(a) $(\arctan x)'$. answer: $\frac{1}{1+x^2}$

(b) $(\operatorname{arccot} x)'$. answer: $-\frac{1}{1+x^2}$

(c) $(\arcsin x)'$. answer: $\frac{1}{\sqrt{1-x^2}}$

(d) $(\arccos x)'$. answer: $-\frac{1}{\sqrt{1-x^2}}$

(e) Let arcsec denote the inverse of the secant function. Compute $(\operatorname{arcsec} x)'$. answer: $\frac{1}{x\sqrt{x^2-1}}$

5. (a) Let $a+b \neq k\pi$, $a \neq k\pi+\frac{\pi}{2}$ and $b \neq k\pi+\frac{\pi}{2}$ for any $k \in \mathbb{Z}$ (integers). Prove that

$$\frac{\tan a+\tan b}{1-\tan a\tan b} = \tan(a+b) \quad .$$

(b) Let x and y be real. Prove that, for $xy \neq 1$, we have

$$\arctan x + \arctan y = \arctan\left(\frac{x+y}{1-xy}\right)$$

if the left hand side lies between $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Solution. 5.a We start by recalling the formulas

$$\begin{array}{rcl}\cos(a+b) &=& \cos a\cos b - \sin a \sin b\\ \sin(a+b) &=& \sin a\cos b + \sin b\cos a \end{array} \quad .$$

These formulas have been previously studied; alternatively they follow from Euler's formula and the computation

$$\begin{array}{rcl}\cos(a+b)+i\sin(a+b) &=& e^{i(a+b)} = e^{ia}e^{ib} = (\cos a + i\sin a)(\cos b + i \sin b)\\ &=& \cos a\cos b - \sin a\sin b + i(\sin a\cos b + \sin b\cos a) \end{array} \quad .$$

Now 5.a is done via a straightforward computation:

$$\begin{array}{rcl}\tan(a+b) &=& \dfrac{\sin(a+b)}{\cos(a+b)} = \dfrac{\sin a\cos b+\sin b\cos a}{\cos a\cos b - \sin a\sin b} = \dfrac{(\sin a\cos b+\sin b\cos a)\frac{1}{\cos a\cos b}}{(\cos a\cos b - \sin a\sin b)\frac{1}{\cos a\cos b}}\\ &=& \dfrac{\tan a+\tan b}{1-\tan a\tan b} \quad . \end{array} \tag{1}$$

5.b is a consequence of 5.a. Let $a = \arctan x$, $b = \arctan y$. Then (**??**) becomes

$$\tan(\arctan x + \arctan y) = \frac{\tan(\arctan x) + \tan(\arctan y)}{1 - \tan(\arctan x)\tan(\arctan y)} = \frac{x+y}{1-xy} \quad ,$$

where we use the fact that $\tan(\arctan w) = w$ for all w. We recall that $\arctan(\tan z) = z$ whenever $z \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Now take $\arctan$ on both sides of the above equality to obtain

$$\arctan x + \arctan y = \arctan\left(\frac{x+y}{1-xy}\right) \quad .$$

6. Evaluate the indefinite integral. Illustrate the steps of your solutions.

(a) $\displaystyle\int x\sin x \mathrm{d}x$.

answer: $-x\cos x + \sin x + C$

(b) $\displaystyle\int xe^{-x}\mathrm{d}x$.

answer: $-(1+x)e^{-x} + C$

(c) $\displaystyle\int x^2e^{x}\mathrm{d}x$.

answer: $x^2e^x - 2xe^x + 2e^x + C$

(d) $\displaystyle\int x\sin(-2x)\mathrm{d}x$.

answer: $\frac{x}{2}\cos(-2x) + \frac{1}{4}\sin(-2x) + C$

(e) $\displaystyle\int x^2\cos(3x)\mathrm{d}x$.

answer: $\frac{x^2}{3}\sin(3x) + \frac{2x}{9}\cos(3x) - \frac{2}{27}\sin(3x) + C$

(f) $\displaystyle\int x^2e^{-2x}\mathrm{d}x$.

answer: $-\frac{x^2e^{-2x}}{2} - \frac{xe^{-2x}}{2} - \frac{e^{-2x}}{4} + C$

(g) $\displaystyle\int x\sin(2x)\mathrm{d}x$.

answer: $-\frac{x}{2}\cos(2x) + \frac{1}{4}\sin(2x) + C$

(h) $\displaystyle\int x\cos(3x)\mathrm{d}x$.

answer: $\frac{x}{3}\sin(3x) + \frac{1}{9}\cos(3x) + C$

(i) $\displaystyle\int x^2e^{2x}\mathrm{d}x$.

answer: $\frac{x^2}{2}e^{2x} - \frac{x}{2}e^{2x} + \frac{e^{2x}}{4} + C$

(j) $\displaystyle\int x^3e^{x}\mathrm{d}x$.

answer: $x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C$

Solution. 6.a.

$$\int x \underbrace{\sin x\mathrm{d}x}_{=\mathrm{d}(-\cos x)} = -\int x\mathrm{d}(\cos x) = -x\cos x + \int \cos x\mathrm{d}x = -x\cos x + \sin x + C \quad .$$

Solution. 6.c.

$$\begin{array}{rcl}\displaystyle\int x^2 \underbrace{e^x\mathrm{d}x}_{\mathrm{d}(e^x)} &=& \displaystyle\int x^2\mathrm{d}e^x = x^2e^x - \int e^x 2x\mathrm{d}x = x^2e^x - \int 2x\mathrm{d}e^x\\ &=& \displaystyle x^2e^x - 2xe^x + \int 2e^x\mathrm{d}x = x^2e^x - 2xe^x + 2e^x + C \end{array} \quad .$$

Solution. 6.f.

$$
\begin{aligned}
\int x^2 e^{-2x}\mathrm{d}x &= \int x^2 \mathrm{d}\left(\frac{e^{-2x}}{-2}\right) && \text{Integrate by parts}\\
&= -\frac{x^2e^{-2x}}{2} - \int \left(\frac{e^{-2x}}{-2}\right)\mathrm{d}\left(x^2\right)\\
&= -\frac{x^2e^{-2x}}{2} + \int xe^{-2x}\mathrm{d}x\\
&= -\frac{x^2e^{-2x}}{2} + \int x\mathrm{d}\left(\frac{e^{-2x}}{-2}\right) && \text{Integrate by parts}\\
&= -\frac{x^2e^{-2x}}{2} - \frac{xe^{-2x}}{2} + \frac{1}{2}\int e^{-2x}\mathrm{d}x\\
&= -\frac{x^2e^{-2x}}{2} - \frac{xe^{-2x}}{2} - \frac{e^{-2x}}{4} + C \quad .
\end{aligned}
$$

7. Evaluate the indefinite integral. Illustrate the steps of your solutions.

(a) $\int x^2\cos(2x)\mathrm{d}x$.

answer: $\frac{1}{2}x^2\sin(2x) + \frac{1}{2}x\cos(2x) - \frac{1}{4}\sin(2x) + C$

(b) $\int x^2e^{ax}\mathrm{d}x$, where a is a constant.

answer: $\frac{1}{a}x^2e^{ax} - \frac{2}{a^2}xe^{ax} + \frac{2}{a^3}e^{ax} + C$

(c) $\int x^2e^{-ax}\mathrm{d}x$, where a is a constant.

answer: $-\frac{1}{a}x^2e^{-ax} - \frac{2}{a^2}xe^{-ax} - \frac{2}{a^3}e^{-ax} + C$

(d) $\int x^2\frac{(e^{ax}+e^{-ax})^2}{4}\mathrm{d}x$, where a is a constant.

answer: $\frac{1}{8}\left(a^{-1}x^2e^{2ax} - a^{-1}x^2e^{-2ax} - a^{-2}xe^{2ax} - a^{-2}xe^{-2ax} + \frac{1}{2}a^{-3}e^{2ax} - \frac{1}{2}a^{-3}e^{-2ax} + \frac{2}{3}x^3\right) + C$

(e) $\int \frac{1}{\cos^2 x}\mathrm{d}x$. (Hint: This problem does not require integration by parts. What is the derivative of $\tan x$?)

answer: $\tan x + C$

(f) $\int(\tan^2 x)\mathrm{d}x$. (Hint: This problem does not require integration by parts. We can use $\tan^2 x = \frac{1}{\cos^2 x} - 1$ and the previous problem.)

(g) $\int x\tan^2 x\mathrm{d}x$. (Hint: $\tan^2 x\mathrm{d}x = \mathrm{d}(F(x))$, where $F(x)$ is the answer from the preceding problem).

answer: $-\frac{x^2}{2} + x\tan x + \ln|\cos x| + C$

(h) $\int e^{-\sqrt{x}}\mathrm{d}x$.

answer: $-2\sqrt{x}e^{-\sqrt{x}} - 2e^{-\sqrt{x}} + C$

(i) $\int \cos^2 x\ \mathrm{d}x$.

answer: $\frac{1}{4}\sin(2x) + \frac{x}{2} + C$

(j) $\int \frac{x}{1+x^2}\mathrm{d}x$ (Hint: use substitution rule, don't use integration by parts)

answer: $\frac{\ln\left(1+x^2\right)}{2} + C$

(k) $\int(\arctan x)\mathrm{d}x$.

answer: $x\arctan x - \frac{\ln\left(1+x^2\right)}{2} + C$

(l) $\int(\arcsin x)\mathrm{d}x$.

answer: $x\arcsin x + \sqrt{1-x^2} + C$

(m) $\int(\arcsin x)^2\mathrm{d}x$. (Hint: Try substituting $x = \sin y$.)

answer: $x(\arcsin x)^2 + 2\sqrt{1-x^2}\arcsin x - 2x + C$

(n) $\int \arctan\left(\frac{1}{x}\right)\mathrm{d}x$.

(o) $\int \sin xe^x\mathrm{d}x$

answer: $\frac{1}{2}\left(e^x\sin x - e^x\cos x\right) + C$

(p) $\int \cos xe^x\mathrm{d}x$

answer: $\frac{1}{2}\left(e^x\cos x + e^x\sin x\right) + C$

(q) $\int \sin(\ln(x))\mathrm{d}x$.

answer: $\frac{x}{2}\left(\sin(\ln x) - \cos(\ln x)\right) + C$

(r) $\int \cos(\ln(x))\mathrm{d}x$.

answer: $\frac{x}{2}\left(\cos(\ln x) + \sin(\ln x)\right) + C$

(s) $\int \ln x\mathrm{d}x$

answer: $x\ln|x| - x + C$

(t) $\int x\ln x\ \mathrm{d}x$.

answer: $\frac{1}{2}x^2\ln|x| - \frac{x^2}{4} + C$

(u) $\int \frac{\ln x}{\sqrt{x}}\mathrm{d}x$.

answer: $2\sqrt{x}(\ln x - 2) + C$

(v) $\int(\ln x)^2\mathrm{d}x$.

answer: $x(\ln x)^2 - 2x\ln x + 2x + C$

(w) $\int(\ln x)^3\mathrm{d}x$.

answer: $x(\ln x)^3 - 3x(\ln x)^2 + 6x\ln x - 6x + C$

(x) $\int x^2\cos^2 x\mathrm{d}x$. (This problem is related to Problem 7.d as $\cos x = \frac{e^{ix}+e^{-ix}}{2}$).

answer: $\frac{1}{4}x^2\sin\ (2x) + \frac{1}{4}x\cos\ (2x) - \frac{1}{8}\sin\ (2x) + \frac{1}{6}x^3$

Solution. 7.g.

$$
\begin{array}{rcll}
\displaystyle\int x\tan^2 x\mathrm{d}x &=& \displaystyle\int x\left(\sec^2 x-1\right)\mathrm{d}x & \left|\ \text{use } \sec^2 x-1=\tan^2 x\right. \\
&=& \displaystyle\int x\left(\sec^2 x-1\right)\mathrm{d}x & \\
&=& \displaystyle -\int x\mathrm{d}x+\int x\sec^2 x\mathrm{d}x & \left|\ \text{use } \mathrm{d}(\tan x)=\sec^2 x\mathrm{d}x\right. \\
&=& \displaystyle -\frac{x^2}{2}+\int x\mathrm{d}(\tan x) & \left|\ \text{integrate by parts}\right. \\
&=& \displaystyle -\frac{x^2}{2}+x\tan x-\int \tan x\mathrm{d}x & \\
&=& \displaystyle -\frac{x^2}{2}+x\tan x-\int \frac{\sin x}{\cos x}\mathrm{d}x & \left|\ \text{use } \sin x\mathrm{d}x=-\mathrm{d}(\cos x)\right. \\
&=& \displaystyle -\frac{x^2}{2}+x\tan x+\int \frac{\mathrm{d}(\cos x)}{\cos x} & \left|\ \text{Set } y=\cos x\right. \\
&=& \displaystyle -\frac{x^2}{2}+x\tan x+\int \frac{1}{y}\mathrm{d}y & \\
&=& \displaystyle -\frac{x^2}{2}+x\tan x+\ln|y|+C & \left|\ \text{Substitute back } y=\cos x\right. \\
&=& \displaystyle -\frac{x^2}{2}+x\tan x+\ln|\cos x|+C \quad . &
\end{array}
$$

Solution. 7.h.

$$
\begin{array}{rcll}
\displaystyle\int e^{-\sqrt{x}}\mathrm{d}x &=& \displaystyle\int 2ye^{-y}\mathrm{d}y & \left|\ \text{Subst.: } \begin{array}{rcl}\sqrt{x}&=&y\\ \frac{1}{2\sqrt{x}}\mathrm{d}x&=&\mathrm{d}y\\ \mathrm{d}x&=&2y\mathrm{d}y\end{array}\right. \\
&=& \displaystyle\int 2y\mathrm{d}\left(-e^{-y}\right) & \left|\ \text{int. by parts}\right. \\
&=& -2ye^{-y}+2\int e^{-y}\mathrm{d}y & \\
&=& -2ye^{-y}-2e^{-y}+C & \\
&=& -2\sqrt{x}e^{-\sqrt{x}}-2e^{-\sqrt{x}}+C \quad . &
\end{array}
$$

Solution. 7.i. Later, we shall study general methods for solving trigonometric integrals that will cover this example. Let us however show one way to solve this integral by integration by parts.

$$
\begin{array}{ll}
\displaystyle\int \cos^2 x\mathrm{d}x = x\cos^2 x-\int x\mathrm{d}(\cos^2 x) & \\
\displaystyle\qquad = x\cos^2 x-\int x2\cos x(-\sin x)\mathrm{d}x & \left|\ \sin(2x)=2\sin x\cos x\right. \\
\displaystyle\qquad = x\cos^2 x+\int x\sin(2x)\mathrm{d}x & \\
\displaystyle\qquad = x\cos^2 x+\int x\mathrm{d}\left(\frac{-\cos(2x)}{2}\right) & \\
\displaystyle\qquad = x\cos^2 x+x\left(\frac{-\cos(2x)}{2}\right)-\int\left(\frac{-\cos(2x)}{2}\right)\mathrm{d}x & \\
\displaystyle\qquad = \frac{x}{2}\left(2\cos^2 x-\cos(2x)\right)+\frac{\sin(2x)}{4}+C & \left|\ \cos(2x)=\cos^2 x-\sin^2 x\right. \\
\displaystyle\qquad = \frac{x}{2}\left(2\cos^2 x-(\cos^2 x-\sin^2 x)\right)+\frac{\sin(2x)}{4}+C & \left|\ \cos^2 x+\sin^2 x=1\right. \\
\displaystyle\qquad = \frac{x}{2}+\frac{\sin(2x)}{4}+C \quad . &
\end{array}
$$

Solution. 7.k

$$
\begin{array}{rcl}
\displaystyle\int \arctan x\mathrm{d}x &=& \displaystyle x\arctan x-\int x\mathrm{d}(\arctan x) \\
&=& \displaystyle x\arctan x-\int \frac{x}{x^2+1}\mathrm{d}x \\
&=& \displaystyle x\arctan x-\int \frac{\frac{1}{2}\mathrm{d}(x^2)}{x^2+1} \\
&=& \displaystyle x\arctan x-\int \frac{\frac{1}{2}\mathrm{d}(x^2+1)}{x^2+1} \\
&=& \displaystyle x\arctan x-\frac{1}{2}\ln(x^2+1)+C \quad .
\end{array}
$$

Solution. 7.m.

$$\begin{array}{rcll}
\displaystyle\int (\arcsin x)^2 \mathrm{d}x & = & \displaystyle\int \left(\arcsin(\sin y)\right)^2 \mathrm{d}(\sin y) & \Big| \text{ Set } x = \sin y \\
& = & \displaystyle\int y^2 \cos y \mathrm{d}y = \int y^2 \mathrm{d}(\sin y) & \Big| \text{ Integrate by parts} \\
& = & \displaystyle y^2 \sin y - \int 2y \sin y \mathrm{d}y & \\
& = & \displaystyle y^2 \sin y + \int 2y \mathrm{d}(\cos y) & \Big| \text{ Integrate by parts} \\
& = & \displaystyle y^2 \sin y + 2y \cos y - 2 \int \cos y \mathrm{d}y & \\
& = & y^2 \sin y + 2y \cos y - 2 \sin y + C & \Big| \text{ Substitute } y = \arcsin x \\
& = & x(\arcsin x)^2 & \\
& & +2\sqrt{1-x^2} \arcsin x - 2x + C \quad . &
\end{array}$$

Solution. 7.o

$$\begin{array}{rcll}
\displaystyle\int \sin x \underbrace{e^x \mathrm{d}x}_{=\mathrm{d}e^x} & = & \displaystyle\sin x e^x - \int e^x \mathrm{d}(\sin x) = \sin x e^x - \int \cos x \underbrace{e^x \mathrm{d}x}_{=\mathrm{d}e^x} & \\
& = & \displaystyle\sin x e^x - e^x \cos x + \int e^x \mathrm{d}(\cos x) & \\
& = & e^x \sin x - e^x \cos x - \int e^x \sin x \mathrm{d}x & \Big| \begin{array}{l}\text{add } \int e^x \sin x \mathrm{d}x \\ \text{to both sides}\end{array} \\
\displaystyle 2\int \sin x e^x \mathrm{d}x & = & \sin x e^x - e^x \cos x & \\
\displaystyle\int \sin x e^x \mathrm{d}x & = & \displaystyle\frac{1}{2}\left(\sin x e^x - e^x \cos x\right) \quad . &
\end{array}$$

Solution. 7.q.

$$\begin{array}{rll}
\displaystyle\int \sin(\ln x)\mathrm{d}x & \displaystyle = x\sin(\ln x) - \int x \mathrm{d}(\sin(\ln x)) & \Big| \text{ int. by parts} \\
& \displaystyle = x\sin(\ln x) - \int x\left(\cos(\ln x)\right)(\ln x)' \mathrm{d}x & \\
& \displaystyle = x\sin(\ln x) - \int \cos(\ln x)\mathrm{d}x & \Big| \text{ int. by parts} \\
& \displaystyle = x\sin(\ln x) - \left(x\cos(\ln x) - \int x\mathrm{d}(\cos(\ln x))\right) & \\
& \displaystyle = x\sin(\ln x) - x\cos(\ln x) + \int x(-\sin(\ln x))(\ln x)'\mathrm{d}x & \\
& \displaystyle = x\sin(\ln x) - x\cos(\ln x) - \int \sin(\ln x)\mathrm{d}x & \Big| \begin{array}{l}\text{add } \int \sin(\ln x)\mathrm{d}x \\ \text{to both sides}\end{array} \\
\displaystyle 2\int \sin(\ln x)\mathrm{d}x & = x\sin(\ln x) - x\cos(\ln x) & \\
\displaystyle\int \sin(\ln x)\mathrm{d}x & \displaystyle = \frac{x}{2}\left(\sin(\ln x) - \cos(\ln x)\right) \quad . &
\end{array}$$

Solution. 7.s

$$\int \ln x \mathrm{d}x = x\ln x - \int x\mathrm{d}(\ln x) = x\ln x - \int \frac{x}{x}\mathrm{d}x = x\ln x - x + C \quad .$$

Solution. 7.u

$$\begin{array}{rcll}
\displaystyle\int \frac{\ln x}{\sqrt{x}}\mathrm{d}x & = & \displaystyle\int (\ln x)2\mathrm{d}\left(\sqrt{x}\right) & \Big| \text{ integrate by parts} \\
& = & \displaystyle(\ln x)2\sqrt{x} - \int 2\sqrt{x}\mathrm{d}(\ln x) & \\
& = & \displaystyle 2\sqrt{x}\ln x - 2\int \frac{\sqrt{x}}{x}\mathrm{d}x & \\
& = & \displaystyle 2\sqrt{x}\ln x - 2\int x^{-\frac{1}{2}}\mathrm{d}x & \\
& = & 2\sqrt{x}\ln x - 4\sqrt{x} + C & \\
& = & 2\sqrt{x}(\ln x - 2) + C \quad . &
\end{array}$$

8. Compute $\int x^n e^x \mathrm{d}x$, where n is a non-negative integer.

Solution. 8

$$\begin{array}{rcl}
\displaystyle\int x^n e^x \mathrm{d}x & = & \displaystyle\int x^n \mathrm{d}e^x \\
& = & \displaystyle x^n e^x - \int e^x \mathrm{d}x^n \\
& = & \displaystyle x^n e^x - n\int x^{n-1}e^x \mathrm{d}x \\
& = & \displaystyle x^n e^x - n\left(\int x^{n-1}\mathrm{d}e^x\right) \\
& = & \displaystyle x^n e^x - n\left(x^{n-1}e^x - \int (n-1)x^{n-2}e^x \mathrm{d}x\right) \\
& = & \displaystyle x^n e^x - nx^{n-1}e^x + n(n-1)\int x^{n-2}e^x \mathrm{d}x \\
& = & \ldots \text{(continue above process)} \ldots \\
& = & x^n e^x - nx^{n-1}e^x + n(n-1)x^{n-2}e^x + \ldots \\
& & +(-1)^k n(n-1)(n-2)\ldots(n-k+1)x^{n-k}e^x \\
& & + \cdots + (-1)^n n! e^x + C \\
& = & \displaystyle C + \sum_{k=0}^{n} (-1)^n \frac{n!}{(n-k)!} x^{n-k}e^x \quad .
\end{array}$$