Calculus I
Homework
Definite integrals and areas between curves

1. (a) Find the area of the region bounded by the curves $y = 2x^2$ and $y = 4 + x^2$.

answer: $\frac{32}{3}$

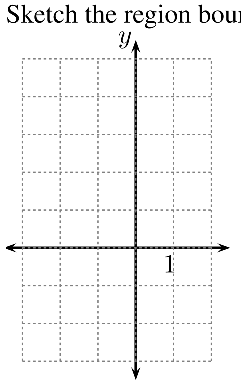
(b) Find the area of the region bounded by the curves $x = 4 - y^2$ and $y = 2 - x$.

answer: $\frac{9}{2}$

(c) Find the area of the region bounded by the curves $y = x^2$ and $y = 2x^2 + x - 2$.

answer: $\frac{9}{2}$

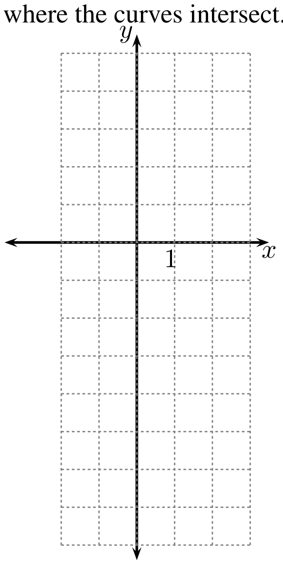
(d)
- Sketch the region bounded by the curves $y = x^2$ and $y = 2x^2 + x - 2$.



- Find the area of the region.

answer: $\frac{9}{2}$

(e)
- Sketch the region bounded by the curves $y = -x^2 + 2x - 1$ and $y = -2x^2 + 3x + 1$. Make sure to indicate the points where the curves intersect.



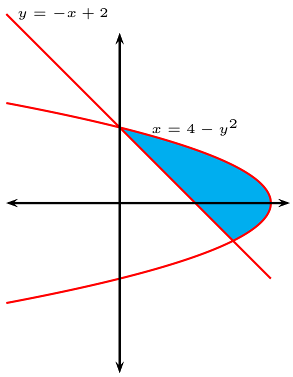
- Find the area of the region.

Solution. 1.b. $x = 4 - y^2$ is a parabola (here we consider x as a function of y). $y = -x + 2$ implies that $x = 2 - y$ and so the two curves intersect when

$$\begin{aligned} 4-y^2 &= 2-y \\ -y^2+y+2 &= 0 \\ -(y+1)(y-2) &= 0 \\ y &= -1 \text{ or } 2 \quad . \end{aligned}$$

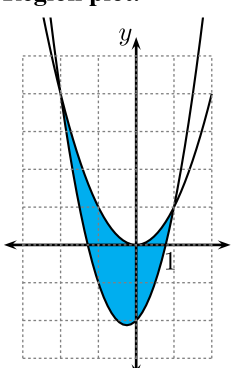
As $x = 2-y$, this implies that $x = 0$ when $y = 2$ and $x = 3$ when $y = -1$, or in other words the points of intersection are $(0, 2)$ and $(3, -1)$. Therefore we the region is the one plotted below. Integrating with respect to y, we get that the area is

$$\begin{aligned} A &= \int\limits_{-1}^{2} \left|4-x^2-(-x+2)\right| \mathrm{d}y = \int\limits_{-1}^{2} \left(-y^2+y+2\right) \mathrm{d}y \\ &= \left[-\frac{y^3}{3}+\frac{y^2}{2}+2y\right]_{-1}^{2} = -\frac{8}{3}+2+4-\left(-\frac{(-1)^3}{3}+\frac{(-1)^2}{2}-2\right) \\ &= \frac{9}{2} \quad . \end{aligned}$$



Solution. 1.d

Region plot.



The intersection between the two parabolas are found via

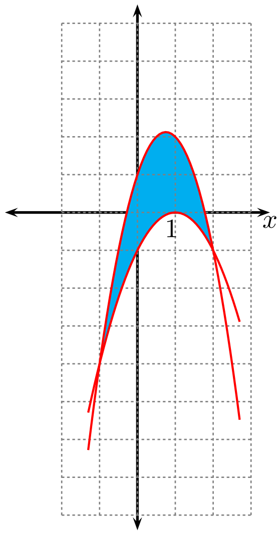
$$\begin{aligned} x^2 &= 2x^2+x-2 \\ x^2+x-2 &= 0 \\ (x-1)(x+2) &= 0 \\ x=1 & \quad x=-2 \\ y=1 & \quad y=4. \end{aligned}$$

Area of the region.

$$\begin{aligned} A &= \int_{-2}^{1} \left|x^2-(2x^2+x-2)\right| \mathrm{d}x && \Big| \; x^2 > (2x^2+x-2) \text{ for } x \in [-2,1] \text{ (from plot)} \\ &= \int_{-2}^{1} \left(x^2-(2x^2+x-2)\right) \mathrm{d}x \\ &= \left[-\frac{1}{3}x^3-\frac{1}{2}x^2+2x\right]_{-2}^{1} \\ &= \frac{9}{2}. \end{aligned}$$

Solution. 1.e

Region plot.



The intersections between the two parabolas are found via

$$\begin{array}{rcl} -2x^2+3x+1 & = & -x^2+2x-1 \\ -x^2+x+2 & = & 0 \\ -(x+1)(x-2) & = & 0 \\ x=-1 & \text{or} & x=2 \\ y=-4 & & y=-1. \end{array}$$

Area of the region.

$$\begin{aligned} A &= \int_{-1}^{2} \left|-2x^2+3x+1-(-x^2+2x-1)\right| \mathrm{d}x \qquad \left|\ \begin{array}{l} -2x^2+3x+1 > -x^2+2x-1 \\ \text{for } x \in [-1,2] \text{ (from plot)} \end{array}\right. \\ &= \int_{-1}^{2} \left(-2x^2+3x+1-(-x^2+2x-1)\right) \mathrm{d}x \\ &= \int_{-1}^{2} \left(-x^2+x+2\right) \mathrm{d}x \\ &= \left[-\frac{1}{3}x^3+\frac{1}{2}x^2+2x\right]_{-1}^{2} \\ &= \left(-\frac{1}{3}2^3+\frac{1}{2}2^2+2\cdot 2\right)-\left(-\frac{1}{3}(-1)^3+\frac{1}{2}(-1)^2+2(-1)\right) \\ &= \frac{9}{2}. \end{aligned}$$