Calculus II
Trigonometric integrals

Todor Milev

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Outline

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Integrals of the form $\int R(\cos\theta, \sin\theta)\mathrm{d}\theta$, R

Let R be an arbitrary rational function in two variables (quotient of polynomials in two variables).

Question

Can we integrate $\int R(\cos\theta, \sin\theta)\mathrm{d}\theta$?

- Yes. We will learn how in what follows.
- The algorithm for integration is roughly:
  - Apply the substitution $\theta = 2\arctan t$ to transform to integral of rational function.
  - Solve as previously studied.

The rationalizing substitution $\theta = 2\arctan t$

Let R- rational function in two variables. $\int R(\cos\theta, \sin\theta) d\theta$ can be integrated via the substitution $\theta = 2\arctan t$. How does this transform $\sin\theta$, $\cos\theta$? How does this transform $d\theta$? How is t expressed via θ?

$$\begin{aligned}
\sin\theta &= \sin(2\arctan t) = \frac{2\tan(\arctan t)}{1+\tan^2(\arctan t)} = \frac{2t}{1+t^2}\\
\cos\theta &= \cos(2\arctan t) = \frac{1-\tan^2(\arctan t)}{1+\tan^2(\arctan t)} = \frac{1-t^2}{1+t^2}
\end{aligned}$$

Recall the expression of $\sin(2z), \cos(2z)$ via $\tan z$:

$$\begin{aligned}
\sin(2z) &= 2\sin z\cos z = \frac{2\sin z\cos z\frac{1}{\cos^2 z}}{(\cos^2 z+\sin^2 z)\frac{1}{\cos^2 z}} = \frac{2\tan z}{1+\tan^2 z} \quad .\\
\cos(2z) &= \cos^2 z - \sin^2 z = \frac{(\cos^2 z-\sin^2 z)\frac{1}{\cos^2 z}}{(\cos^2 z+\sin^2 z)\frac{1}{\cos^2 z}} = \frac{1-\tan^2 z}{1+\tan^2 z} .
\end{aligned}$$

The rationalizing substitution $\theta = 2\arctan t$

Let R- rational function in two variables. $\int R(\cos\theta, \sin\theta) d\theta$ can be integrated via the substitution $\theta = 2\arctan t$. How does this transform $\sin\theta$, $\cos\theta$? How does this transform $d\theta$? How is t expressed via θ?

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\sin\theta &= \sin(2\arctan t) = \frac{2\tan(\arctan t)}{1+\tan^2(\arctan t)} = \frac{2t}{1+t^2}\\
\cos\theta &= \cos(2\arctan t) = \frac{1-\tan^2(\arctan t)}{1+\tan^2(\arctan t)} = \frac{1-t^2}{1+t^2}\\
d\theta &= 2d(\arctan t) = \frac{2}{1+t^2}dt\\
t &= \tan\left(\frac{\theta}{2}\right)
\end{aligned}$$

Theorem

The substitution given above transforms $\int R(\cos\theta, \sin\theta) d\theta$ to an integral of a rational function of t.

Example

Let $\theta = 2\arctan t$, $\cos\theta = \frac{1-t^2}{1+t^2}$, $\sin\theta = \frac{2t}{1+t^2}$, $z = \frac{3}{\sqrt{5}}\left(t+\frac{1}{3}\right)$.

$$
\begin{aligned}
\int \frac{\mathrm{d}\theta}{2\sin\theta - \cos\theta + 5} &= \int \frac{2\mathrm{d}t}{(1+t^2)\left(2\frac{2t}{t^2+1} - \frac{(1-t^2)}{1+t^2} + 5\right)}\\
&= \int \frac{2\mathrm{d}t}{6t^2+4t+4}\\
&= \int \frac{\mathrm{d}t}{3t^2+2t+2}\\
\text{(complete square)} \quad &= \int \frac{\mathrm{d}t}{3\left(t^2+2t\frac{1}{3}+\frac{1}{9}-\frac{1}{9}+\frac{2}{3}\right)}\\
&= \frac{1}{3}\int \frac{\mathrm{d}t}{\left(t+\frac{1}{3}\right)^2+\frac{5}{9}}\\
&= \frac{1}{3}\int \frac{\mathrm{d}t}{\frac{5}{9}\left(\frac{9}{5}\left(t+\frac{1}{3}\right)^2+1\right)}
\end{aligned}
$$

Example

Let $\theta = 2\arctan t$, $\cos\theta = \frac{1-t^2}{1+t^2}$, $\sin\theta = \frac{2t}{1+t^2}$, $z = \frac{3}{\sqrt{5}}\left(t+\frac{1}{3}\right)$.

$$
\begin{aligned}
\int \frac{\mathrm{d}\theta}{2\sin\theta - \cos\theta + 5} &= \frac{1}{3}\int \frac{\mathrm{d}t}{\frac{5}{9}\left(\frac{9}{5}\left(t+\frac{1}{3}\right)^2+1\right)} \\
&= \frac{3}{5}\int \frac{\frac{\sqrt{5}}{3}\mathrm{d}\left(\frac{3}{\sqrt{5}}\left(t+\frac{1}{3}\right)\right)}{\left(\left(\frac{3}{\sqrt{5}}\left(t+\frac{1}{3}\right)\right)^2+1\right)} \\
&= \frac{\sqrt{5}}{5}\int \frac{\mathrm{d}z}{z^2+1} \\
&= \frac{\sqrt{5}}{5}\arctan z + C \\
&= \frac{\sqrt{5}}{5}\arctan\left(\frac{3}{\sqrt{5}}\left(t+\frac{1}{3}\right)\right) + C \\
&= \frac{\sqrt{5}}{5}\arctan\left(\frac{3}{\sqrt{5}}\left(\tan\left(\frac{\theta}{2}\right)+\frac{1}{3}\right)\right) + C
\end{aligned}
$$

The integral $\int \sec\theta \mathrm{d}\theta$ appears often in practice. A quicker solution will be shown later, but first we show the standard method.

Example

Set $\theta = 2\arctan t$, $\cos\theta = \dfrac{1-\tan^2(\frac{\theta}{2})}{1+\tan^2\left(\frac{\theta}{2}\right)} = \dfrac{1-t^2}{1+t^2}$, $\mathrm{d}\theta = 2\dfrac{1}{1+t^2}\mathrm{d}t$.

$$
\begin{aligned}
\int \sec\theta \mathrm{d}\theta &= \int \frac{1}{\cos\theta}\mathrm{d}\theta = \int \frac{1}{\left(\frac{1-t^2}{\cancel{1+t^2}}\right)}\frac{2}{\cancel{(1+t^2)}}\mathrm{d}t \\
&= \int \frac{2}{1-t^2}\mathrm{d}t = \int \left(\frac{1}{1-t}+\frac{1}{1+t}\right)\mathrm{d}t \quad \Big| \text{ part. fractions} \\
&= -\ln|1-t| + \ln|1+t| + C \\
&= \ln\left|\frac{1+t}{1-t}\right| + C \\
&= \ln\left|\frac{1+\tan\left(\frac{\theta}{2}\right)}{1-\tan\left(\frac{\theta}{2}\right)}\right| + C
\end{aligned}
$$

The integral $\int \sec\theta d\theta$ appears often in practice. A quicker solution will be shown later, but first we show the standard method.

Example

Set $\theta = 2\arctan t$, $\cos\theta = \dfrac{1-\tan^2(\frac{\theta}{2})}{1+\tan^2\left(\frac{\theta}{2}\right)} = \dfrac{1-t^2}{1+t^2}$, $d\theta = 2\dfrac{1}{1+t^2}dt$.

$$\int \sec\theta d\theta \quad = \quad \ln\left|\frac{1+\tan\left(\frac{\theta}{2}\right)}{1-\tan\left(\frac{\theta}{2}\right)}\right| + C$$

This is a perfectly good answer, however there's a simplification:

$$\begin{aligned}
\tan\theta + \sec\theta &= \frac{\sin\theta + 1}{\cos\theta} = \frac{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) + \sin^2\left(\frac{\theta}{2}\right) + \cos^2\left(\frac{\theta}{2}\right)}{\cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)} \\
&= \frac{\left(\sin\left(\frac{\theta}{2}\right) + \cos\left(\frac{\theta}{2}\right)\right)^{\cancel{2}}}{\left(\cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)\right)\cancel{\left(\cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)\right)}} \\
&= \frac{\sin\left(\frac{\theta}{2}\right) + \cos\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)} = \frac{1+\tan\left(\frac{\theta}{2}\right)}{1-\tan\left(\frac{\theta}{2}\right)} \quad .
\end{aligned}$$

The integral $\int \sec\theta d\theta$ appears often in practice. A quicker solution will be shown later, but first we show the standard method.

Example

Set $\theta = 2\arctan t$, $\cos\theta = \dfrac{1-\tan^2(\frac{\theta}{2})}{1+\tan^2\left(\frac{\theta}{2}\right)} = \dfrac{1-t^2}{1+t^2}$, $d\theta = 2\dfrac{1}{1+t^2}dt$.

$$\int \sec\theta d\theta \quad = \quad \ln\left|\tan\theta + \sec\theta\right| + C$$

This is a perfectly good answer, however there's a simplification:

$$\begin{aligned}
\tan\theta + \sec\theta &= \frac{\sin\theta + 1}{\cos\theta} = \frac{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) + \sin^2\left(\frac{\theta}{2}\right) + \cos^2\left(\frac{\theta}{2}\right)}{\cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)} \\
&= \frac{\left(\sin\left(\frac{\theta}{2}\right) + \cos\left(\frac{\theta}{2}\right)\right)^{\cancel{2}}}{\left(\cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)\right)\cancel{\left(\cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)\right)}} \\
&= \frac{\sin\left(\frac{\theta}{2}\right) + \cos\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)} = \frac{1 + \tan\left(\frac{\theta}{2}\right)}{1 - \tan\left(\frac{\theta}{2}\right)} \quad .
\end{aligned}$$

Trigonometric Integrals - quick ad hoc techniques

- As we saw, every rational trigonometric expression can be integrated with the substitution $\theta = 2\arctan t$.
- This integration technique results in rather long computations.
- Particular integral types may be computable with quicker ad hoc techniques.
- We illustrate such techniques on examples.
- Examples to which our ad hoc techniques apply arise from integrals needed outside of the subject of Calculus II, so these techniques are important.
- The trigonometric integral we saw, $\int \frac{\mathrm{d}\theta}{2\sin\theta - \cos\theta + 5}$, will not work with any of following ad-hoc techniques, so the general method is important as well.

Example

$$\begin{aligned}
\int \sin^3 x \mathrm{d}x &= \int \sin^2 x \sin x \mathrm{d}x \\
&= \int \sin^2 x \mathrm{d}(-\cos x) \\
&= \int (-1)\left(1 - \cos^2 x\right) \mathrm{d}(\cos x) \\
&= \int \left(\cos^2 x - 1\right) \mathrm{d}(\cos x) \\
&= \int \left(u^2 - 1\right) \mathrm{d}u \\
&= \frac{u^3}{3} - u + C \\
&= \frac{1}{3}\cos^3 x - \cos x + C \quad .
\end{aligned}$$

Can we rewrite $\sin^2 x$ via $\cos x$?

Set $u = \cos x$

Example

$$\begin{aligned}
\int \cos^5 x \sin^2 x \mathrm{d}x &= \int \cos^4 x \sin^2 x \cos x \mathrm{d}x \\
&= \int \cos^4 x \sin^2 x \mathrm{d}(\sin x) && \text{Can we rewrite } \cos^4 x \text{ via } \sin x? \\
&= \int \left(\cos^2 x\right)^2 \sin^2 x \mathrm{d}(\sin x) \\
&= \int \left(1-\sin^2 x\right)^2 \sin^2 x \mathrm{d}(\sin x) && \text{Set } u = \sin x \\
&= \int \left(1-u^2\right)^2 u^2 \mathrm{d}u \\
&= \int \left(1-2u^2+u^4\right) u^2 \mathrm{d}u \\
&= \int \left(u^2-2u^4+u^6\right) \mathrm{d}u \\
&= \frac{u^3}{3} - 2\frac{u^5}{5} + \frac{u^7}{7} + C \\
&= \frac{\sin^3 x}{3} - 2\frac{\sin^5 x}{5} + \frac{\sin^7 x}{7} + C \quad .
\end{aligned}$$

$\displaystyle\int \sin^m x\cos^n x dx = \int \sin^m x \cos^{n-1} x d(\sin x)$	When $n-$ odd: $\cos x dx = d(\sin x)$
$\displaystyle= \int \sin^m x\left(1-\sin^2 x\right)^{\frac{n-1}{2}} d(\sin x)$	Express $\cos x$ via $\sin x$
$\displaystyle= \int u^m \left(1-u^2\right)^{\frac{n-1}{2}} du$	Set $\sin x = u$
$\displaystyle\int \sin^m x\cos^n x dx = \int \sin^{m-1} x \cos^n x d(-\cos x)$	When $m-$ odd: $\sin x dx = d(-\cos x)$
$\displaystyle= -\int \left(1-\cos^2 x\right)^{\frac{m-1}{2}} \cos^n x d(\cos x)$	Express $\cos x$ via $\sin x$
$\displaystyle= -\int \left(1-u^2\right)^{\frac{m-1}{2}} u^n du$	Set $\cos x = u$

If both m, n- even, use $\left|\begin{array}{rcl} \sin^2 x &=& \frac{1-\cos(2x)}{2} \\ \cos^2 x &=& \frac{\cos(2x)+1}{2}\end{array}\right.$ and substitute $s = 2x$ to lower trig powers. Repeat above considerations.

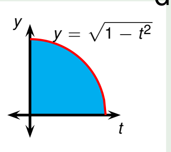
Example

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sin^2 x \mathrm{d}x &= \int_0^{\frac{\pi}{2}} \left(\frac{1-\cos(2x)}{2}\right) \mathrm{d}x \\ &= \left[\frac{x}{2} - \frac{\sin(2x)}{4}\right]_0^{\frac{\pi}{2}} \\ &= \left(\frac{\pi}{4} - \frac{\sin\pi}{4}\right) - \left(0 - \frac{\sin 0}{4}\right) = \frac{\pi}{4}.\end{aligned}$$

express $\sin^2 x$ via $\cos(2x)$

Example

Set $t = \cos x$, $x \in \left[0, \frac{\pi}{2}\right] \Rightarrow \sin x \geq 0$. Then $\mathrm{d}t = \mathrm{d}(\cos x) = -\sin x \mathrm{d}x$.



$$\begin{aligned}\int_{t=0}^{t=1} \sqrt{1-t^2} \mathrm{d}t &= -\int_{x=\frac{\pi}{2}}^{x=0} \sqrt{1-\cos^2 x} \sin x \mathrm{d}x \\ &= \int_{x=0}^{x=\frac{\pi}{2}} \sqrt{\sin^2 x} \sin x \mathrm{d}x \\ &= \int_0^{\frac{\pi}{2}} \sin^2 x \mathrm{d}x = \frac{\pi}{4} \quad .\end{aligned}$$

Example

$$\begin{aligned}
\int \tan^8 x \sec^4 x \mathrm{d}x &= \int \tan^8 x \sec^2 x \sec^2 x \mathrm{d}x \\
&= \int \tan^8 x \sec^2 x \mathrm{d}(\tan x) \\
&= \int \tan^8 x \left(1 + \tan^2 x\right) \mathrm{d}(\tan x) \\
&= \int u^8 \left(1 + u^2\right) \mathrm{d}u \\
&= \int \left(u^8 + u^{10}\right) \mathrm{d}u \\
&= \frac{u^9}{9} + \frac{u^{11}}{11} + C \\
&= \frac{\tan^9 x}{9} + \frac{\tan^{11} x}{11} + C \quad .
\end{aligned}$$

Can we rewrite $\sec^2 x$ via $\tan x$?

Set $u = \tan x$

Example

$$\begin{aligned}
\int \tan^5 x \sec^9 x \mathrm{d}x &= \int \tan^4 x \sec^8 x \tan x \sec x \mathrm{d}x \\
&= \int \tan^4 x \sec^8 x \mathrm{d}(\sec x) && \left| \begin{array}{l}\text{Can we rewrite} \\ \tan^4 x \text{ via } \sec x?\end{array}\right. \\
&= \int \left(\tan^2 x\right)^2 \sec^8 x \mathrm{d}(\sec x) \\
&= \int \left(\sec^2 x - 1\right)^2 \sec^8 x \mathrm{d}(\sec x) && \left| \text{Set } u = \sec x \right. \\
&= \int \left(1 - u^2\right)^2 u^8 \mathrm{d}u \\
&= \int \left(1 - 2u^2 + u^4\right) u^8 \mathrm{d}u \\
&= \int \left(u^8 - 2u^{10} + u^{12}\right) \mathrm{d}u \\
&= \frac{u^9}{9} - 2\frac{u^{11}}{11} + \frac{u^{13}}{13} + C \\
&= \frac{\sec^9 x}{9} - 2\frac{\sec^{11} x}{11} + \frac{\sec^{13} x}{13} + C \quad .
\end{aligned}$$

Partial strategy for fast evaluation of $\int \tan^m x \sec^n x dx$

$$\begin{aligned}\int \tan^m x \sec^n x dx &= \int \tan^m x \sec^{n-2} x d(\tan x)\\ &= \int \tan^m x \left(1+\tan^2 x\right)^{\frac{n-2}{2}} d(\tan x)\\ &= \int u^m \left(1+u^2\right)^{\frac{n-2}{2}} du\end{aligned}$$	n – even, $n \geq 2$ $\sec^2 x dx = d(\tan x)$ Express $\sec x$ via $\tan x$ Set $u = \tan x$
$$\begin{aligned}\int \tan^m x \sec^n x dx &= \int \tan^{m-1} x \sec^{n-1} x d(\sec x)\\ &= \int \left(\sec^2 x - 1\right)^{\frac{m-1}{2}} \sec^{n-1} x d(\sec x)\\ &= \int \left(u^2-1\right)^{\frac{m-1}{2}} u^n du\end{aligned}$$	m – odd, $n \geq 1$ $\tan x \sec x dx = d(\sec x)$ Express $\tan x$ via $\sec x$ Set $u = \sec x$

Outside of the above cases we either use more tricks or resort to the general method $x = 2\arctan t$.

Example

$$\begin{aligned}\int \tan x \mathrm{d}x &= \int \frac{\sin x}{\cos x}\mathrm{d}x = \int \frac{1}{\cos x}\mathrm{d}(-\cos x) && \Big| \text{ Set } u = \cos x\\ &= -\int \frac{\mathrm{d}u}{u} = -\ln|u| + C\\ &= -\ln|\cos x| + C = \ln|\sec x| + C\end{aligned}$$

The following can be/was computed via $x = 2\arctan t$. Alternatively:

Example

$$\begin{aligned}\int \sec x \mathrm{d}x &= \int \sec x \frac{(\sec x + \tan x)}{(\sec x + \tan x)}\mathrm{d}x\\ &= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\mathrm{d}x\\ &= \int \frac{\mathrm{d}(\tan x + \sec x)}{\sec x + \tan x} && \Big| \text{ Set } u = \sec x + \tan x\\ &= \int \frac{\mathrm{d}u}{u} = \ln|u| + C\\ &= \ln|\sec x + \tan x| + C.\end{aligned}$$

Example

$$\begin{aligned}
\int \tan^3 x dx &= \int \tan x \tan^2 x dx \\
&= \int \tan x \left(\sec^2 x - 1\right) dx \\
&= \int \tan x \sec^2 x dx - \int \tan x dx \\
&= \int \tan x d(\tan x) - \ln|\sec x| && \left| \text{ Set } u = \tan x \right. \\
&= \int u du + \ln\left|\frac{1}{\sec x}\right| \\
&= \frac{u^2}{2} + \ln|\cos x| + C \\
&= \frac{\tan^2 x}{2} + \ln|\cos x| + C
\end{aligned}$$

Example

$$
\begin{aligned}
\int \sec^3 x dx &= \int \sec x \sec^2 x dx \\
&= \int \sec x d(\tan x) \\
&= \sec x \tan x - \int \tan x d(\sec x) \\
&= \sec x \tan x - \int \tan^2 x \sec x dx \\
&= \sec x \tan x - \int (\sec^2 x - 1) \sec x dx \\
&= \sec x \tan x - \int \sec^3 x dx + \int \sec x dx \\
2\int \sec^3 x dx &= \sec x \tan x + \ln|\sec x + \tan x| + C \\
\int \sec^3 x dx &= \frac{1}{2}\left(\sec x \tan x + \ln|\sec x + \tan x|\right) + K.
\end{aligned}
$$

| Integrate by parts

To evaluate integrals of the form

1. $\int \sin(mx)\cos(nx)\mathrm{d}x$
2. $\int \sin(mx)\sin(nx)\mathrm{d}x$
3. $\int \cos(mx)\cos(nx)\mathrm{d}x$

use the corresponding identity:

1. $\sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$
2. $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$
3. $\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$

Example

$$
\begin{aligned}
\int \sin(4x)\cos(5x)\mathrm{d}x &= \int \frac{1}{2}[\sin(4x-5x)+\sin(4x+5x)]\mathrm{d}x \\
&= \frac{1}{2}\int (\sin(-x)+\sin(9x))\mathrm{d}x \\
&= \frac{1}{2}\int (-\sin x+\sin(9x))\mathrm{d}x \\
&= \frac{1}{2}(\cos x-\frac{1}{9}\cos(9x))+C
\end{aligned}
$$