Calculus II
Power series expansion of logarithms, part 1

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- To find C, plug in $x = 0$: $C = 0$.
- Therefore the theorem on integrating power series implies that

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  $$\ln(1-x) = -\sum_{n=1}^{\infty}\frac{x^n}{n}, \text{ for } |x|<1.$$
- By the same theorem, the radius of convergence remains $R=1$.