Calculus II
Basic and alternating series tests

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Alternating Series

Definition (Alternating Series)

An alternating series is a series whose terms are alternately positive and negative.

Examples

Here are two examples:

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$$

$$-\frac{1}{2}+\frac{2}{3}-\frac{3}{4}+\frac{4}{5}-\frac{5}{6}+\frac{6}{7}-\cdots=\sum_{n=1}^{\infty}(-1)^{n}\frac{n}{n+1}$$

The nth term of an alternating series has the form

$$a_n=(-1)^{n-1}b_n \qquad \text{or} \qquad a_n=(-1)^n b_n$$

where b_n is positive.

Theorem (The Alternating Series Test)

If the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1 - b_2 + b_3 - b_4 + b_5 - \cdots, \qquad b_n > 0$$

satisfies

1. $b_{n+1} \leq b_n$ *for all n and*
2. $\lim_{n\to\infty} b_n = 0$

then the series is convergent.

Example

The alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

satisfies

1. $b_{n+1} < b_n$ because $\frac{1}{n+1} < \frac{1}{n}$.
2. $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} = 0$.

Therefore the series is convergent by the Alternating Series Test.

Example

The series $\sum_{n=1}^{\infty}(-1)^n \frac{3n}{4n-1}$ is alternating, but

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{3n}{4n-1} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{3}{4-\frac{1}{n}} = \frac{3}{4}$$

Therefore the series is divergent by the basic Divergence Test.