Calculus I
Analytical graphing of functions in one variable

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Outline

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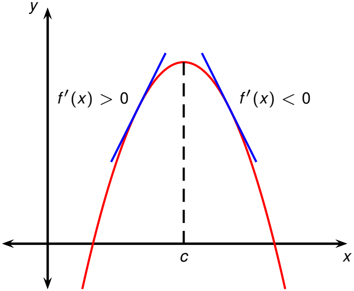
What Does f' Say About f?



- Consider the graph on the left.
- $f'(x) > 0$ to the left of c and $f'(x) < 0$ to the right of c.
- f is increasing to the left of c and decreasing to the right of c.
- This property holds more generally:

Increasing/Decreasing Test

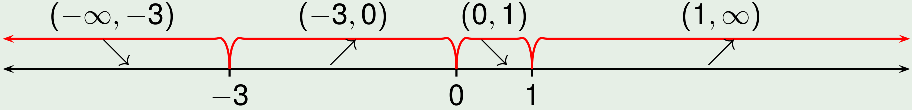
1. If $f'(x) > 0$ on an interval, then f is increasing on that interval.
2. If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

Example

Find where the function $f(x) = 3x^4 + 8x^3 - 18x^2 + 6$ is increasing and where it is decreasing.

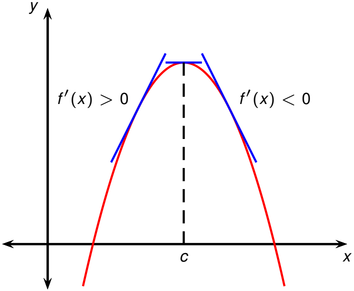
$$f'(x) = 12x^3 + 24x^2 - 36x = 12x(x^2 + 2x - 3) = 12x(x + 3)(x - 1)$$

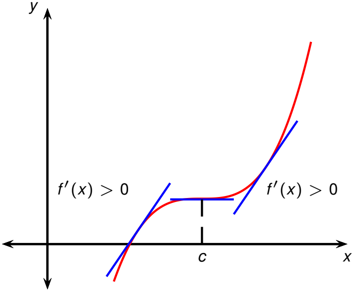
$f'(x)$ equals zero for $x = -3, 0, 1$. Therefore f' doesn't change sign in the intervals



Interval	$12x$	$x+3$	$x-1$	$f'(x)$	f
$(-\infty, -3)$	$-$	$-$	$-$	$-$	decreasing
$(-3, 0)$	$-$	$+$	$-$	$+$	increasing
$(0, 1)$	$+$	$+$	$-$	$-$	decreasing
$(1, \infty)$	$+$	$+$	$+$	$+$	increasing

- Recall: if f has a local max at c and $f'(c)$ exists, then $f'(c) = 0$. However if $f'(c) = 0$, it is not necessary that c be a local max.
- In the first picture, $f'(x) > 0$ to the left of c and $f'(x) < 0$ to the right of c.
- In other words, $f'(x)$ changes sign at c.
- In the second picture, $f'(x) > 0$ to the left of c and $f'(x) > 0$ to the right of c. $f'(x)$ doesn't change sign at c.
- In the first picture there's a local maximum, but not in the second.
- This suggests a way of testing for local maxima/minima.

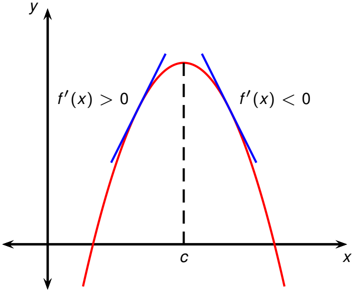


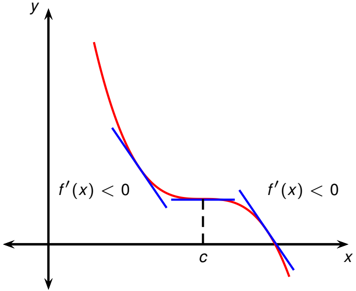


The First Derivative Test

Suppose $f'(c) = 0$ (i.e., f is differentiable at c and c is critical number for f).

1. If f' changes from positive to negative at c, then f has a local maximum at c.
2. If f' changes from negative to positive at c, then f has a local minimum at c.
3. If f' doesn't change signs at c, then f has no local maximum or minimum at c.





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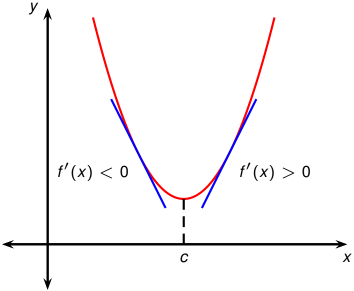

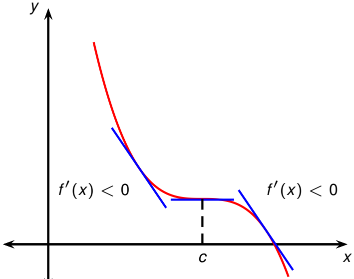

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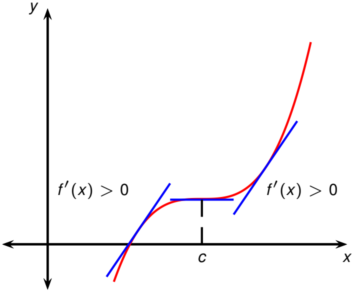
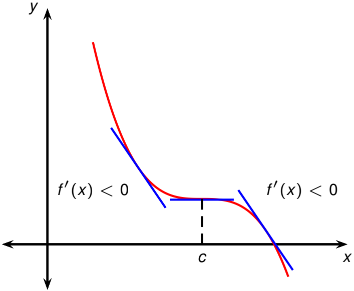
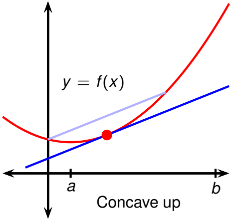
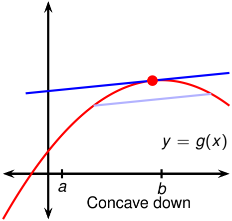




What Does f'' Say About f?

f and g are both increasing on (a, b), but “bend” in different directions.



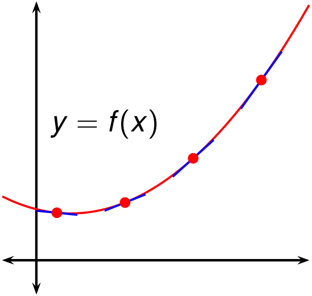


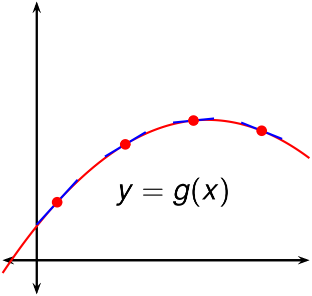
Definition (Concave Up/Concave Down, most general definition)

A function is called concave up/down if the line segment between any two points on its graph lies above/below the graph.

Theorem (Can be taken as a definition)

Let f be a differentiable function on an interval I. Then f is concave up (on I) if its graph lies above all of its tangents (on I), and f is concave down (on I) if its graph lies below all of its tangents (on I).





- In the graph of f the slopes of the tangent lines increase as we move from left to right.
- This means f' is an increasing function.
- This means f'' is positive on (a, b).
- Similarly g'' is negative on (a, b).

Concavity Test

1. If $f''(x) > 0$ for all x in I, then the graph of f is concave up on I.
2. If $f''(x) < 0$ for all x in I, then the graph of f is concave down on I.

Definition (Inflection Point)

A point $P = (x, f(x))$ on a curve $y = f(x)$ is called an inflection point if

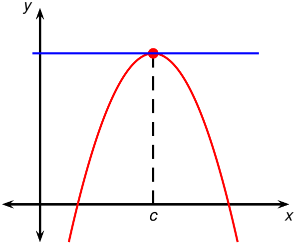
- $f''(x)$ exists
- the graph of f changes from concave up to concave down or from concave down to concave up at P.

In other words $P = (x, f(x))$ is an inflection point if f'' exists and changes signs at x.

This gives us a new way of checking if critical points are local maxima or local minima:

The Second Derivative Test
Suppose f'' is exists near c.

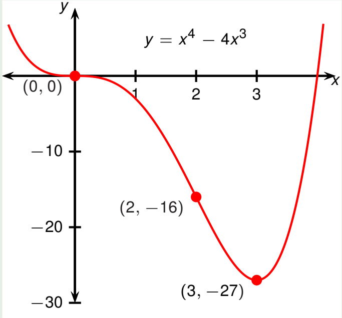
1. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.
2. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.



- $f'(c) = 0$, so f has a horizontal tangent at c.
- $f''(c) < 0$, so f is concave down near c.
- This means f lies below its horizontal tangent.
- This means $f(c)$ is a local maximum.

Example

Discuss the curve $y = f(x) = x^4 - 4x^3$ with respect to concavity, points of inflection, and local maxima and minima. Sketch the curve.

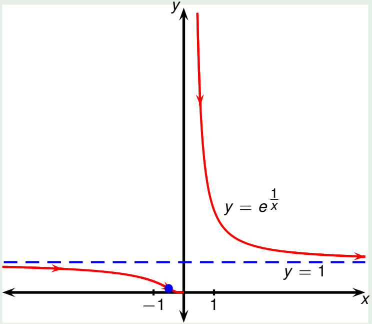


Interval	$f''(x)$	Concave
$(-\infty, 0)$	$+$	up
$(0, 2)$	$-$	down
$(2, \infty)$	$+$	up

- $f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3)$.
- $f''(x) = 12x^2 - 24x = 12x(x - 2)$.
- Critical numbers: 0 and 3.
- $f''(0) = 0$ and $f''(3) = 36 > 0$.
- Second Derivative Test:
- Localminimum at 3. $f(3) = -27$.
- No information about 0.
- First Derivative Test:
- f' is $-$ on $(-\infty, 0)$ and $-$ on $(0, 3)$.
- No local max or min at 0.
- Inflection points: $(0, 0)$ and $(2, -16)$.

Example

Draw the graph of $f(x) = e^{\frac{1}{x}}$.



- $f(x)$ is always positive.
- Domain: everything but 0.
- Check for vertical asymptote at 0.
- $t = 1/x$: $\lim_{x\to 0^+} e^{1/x} = \lim_{t\to\infty} e^t = \infty$.
- $t = 1/x$: $\lim_{x\to 0^-} e^{1/x} = \lim_{t\to -\infty} e^t = 0$.
- As $x \to \pm\infty$, $1/x \to 0$.
- Therefore $\lim_{x\to\pm\infty} e^{1/x} = 1$
- $y = 1$ is a horizontal asymptote.

$$f'(x) = e^{\frac{1}{x}}\left(\frac{1}{x}\right)' = e^{\frac{1}{x}}\left(-x^{-2}\right) = -\frac{e^{\frac{1}{x}}}{x^2}.$$

$$f''(x) = -\frac{\left(-\frac{e^{\frac{1}{x}}}{x^2}\right)x^2 - e^{\frac{1}{x}}(2x)}{x^4} = \frac{e^{\frac{1}{x}}(1+2x)}{x^4}.$$

Always decreasing. Inflection point: $(-1/2, e^{-2})$.

Guidelines for Sketching a Curve

The following items are to be considered when drawing a curve. Not every item is relevant to every function.

1. Determine the domain of the function.
2. Depending on availability, use computer software to plot.
3. Compute x, y intercepts.
4. Determine symmetries, periodicity.
5. Compute asymptotes - vertical, horizontal, optional - slanted.
6. Compute intervals of increase or decrease.
7. Compute local and global maxima and minima.
8. Compute concavity and points of inflection.

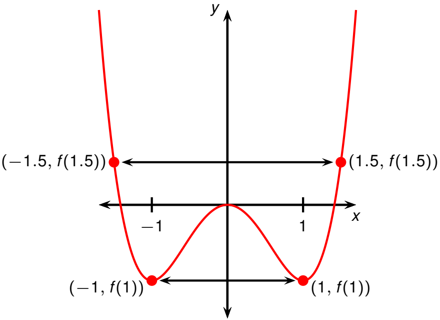
1. Domain
   - Find the domain of the function.
   - Remember the two restrictions: no dividing by 0, and no taking the even root of a negative number.
2. You can use computer software to plot your function.
   - Most computer software will ask you to specify the domain of the function explicitly.
   - Some software may be able to determine the (implied) domain of your function.
   - Software may not be always available (example: Calculus I exams).

3. Intercepts

- Find the intercepts of the function.
- $f(0)$ is the y-intercept.
- To find the x-intercepts, set $y = 0$ and solve for x.
- You can sometimes skip this step if the equation is too difficult to solve.

4. Symmetry, Periodicity

- If $f(-x) = f(x)$ for all x, then f is even.
- If $f(-x) = -f(x)$ for all x, then f is odd.
- If there is some number p such that $f(a+p) = f(a)$ for all a, then f is called periodic. The smallest such p is called its period.



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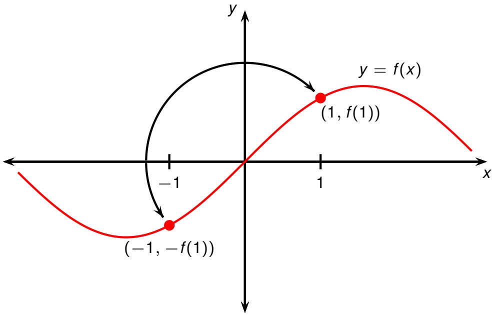

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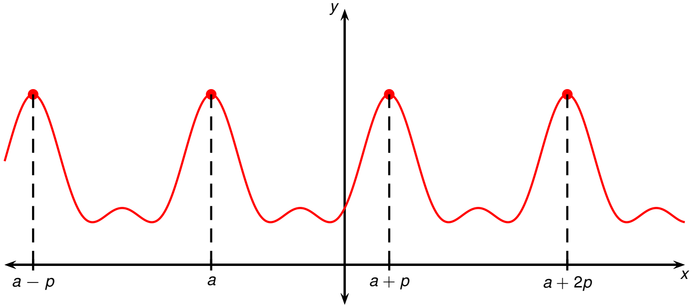


5. Asymptotes

- **Horizontal asymptotes** can be found by finding $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.
- If either of these equals a number L, then $y = L$ is a horizontal asymptote of f.
- If neither limit exists, there is no horizontal asymptote.
- The line $x = a$ is a **Vertical asymptote** of f if any of the following is true

$$\begin{array}{ll}\lim_{x\to a^+} f(x) = \infty & \lim_{x\to a^-} f(x) = \infty \\ \lim_{x\to a^+} f(x) = -\infty & \lim_{x\to a^-} f(x) = -\infty\end{array}$$

- We may discuss slant asymptotes in another lecture if time allows.

6. Intervals of increase or decrease

- To find intervals of increase or decrease, use the increasing/decreasing test.
- Compute f'.
- Find where f' is positive or negative.
- Where f' is positive, f is increasing.
- Where f' is negative, f is decreasing.

7. Local maxima and minima

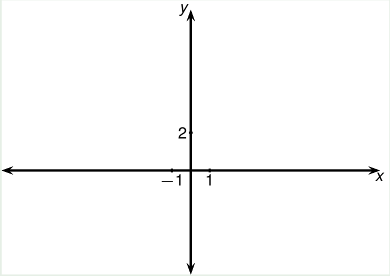
- Find the critical numbers of f (the numbers c where $f'(c)$ doesn't exist or $f'(c) = 0$).
- Use the First Derivative Test on each of these numbers:
- If f' changes from positive to negative at a critical number c, then c is a local maximum.
- If f' changes from negative to positive at a critical number c, then c is a local minimum.
- If f' doesn't change sign at a critical number c, then c is neither a local maximum nor a local minimum.

8. Concavity and points of inflection

- To find inflection points and intervals of concavity, use the concavity test.
- Compute f''.
- Find where f'' is positive or negative.
- Where f'' is positive, f is concave up.
- Where f'' is negative, f is concave down.
- Inflection points occur when f'' changes signs.

Example

Sketch the curve $y = \frac{2x^2}{x^2-1}$.

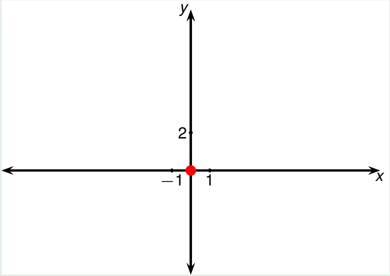


1. Domain

The domain of the function is $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$.

Example

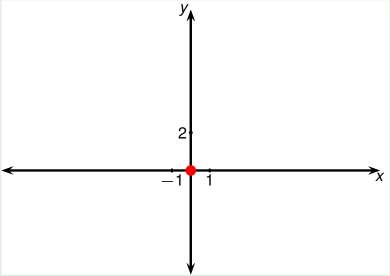
Sketch the curve $y = \frac{2x^2}{x^2-1}$.



3. Intercepts
   - y-intercept: $f(0) = 0$.
   - x-intercept: $f(x) = 0$ when $x = 0$.
   - The only intercept is $(0, 0)$.

Example

Sketch the curve $y = \frac{2x^2}{x^2-1}$.



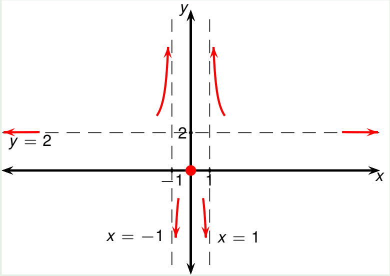
4. Symmetry

$$f(-x) = \frac{2(-x)^2}{(-x)^2 - 1} = \frac{2x^2}{x^2 - 1} = f(x)$$

Therefore f is even.

Example

Sketch the curve $y = \frac{2x^2}{x^2-1}$.



5. Asymptotes

$$\lim_{x\to\pm\infty} \frac{2x^2}{x^2-1} = \lim_{x\to\pm\infty} \frac{2}{1-1/x^2} = 2$$

$y = 2$ is a horizontal asymptote.

$$\lim_{x\to 1^+} \frac{2x^2}{x^2-1} = \infty$$

$$\lim_{x\to 1^-} \frac{2x^2}{x^2-1} = -\infty$$

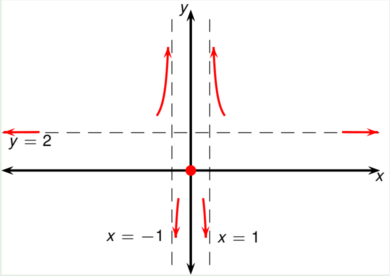
$$\lim_{x\to -1^+} \frac{2x^2}{x^2-1} = -\infty$$

$$\lim_{x\to -1^-} \frac{2x^2}{x^2-1} = \infty$$

$x = \pm 1$ are vertical asymptotes.

Example

Sketch the curve $y = \frac{2x^2}{x^2-1}$.



Interval	I/D	Concavity
$(-\infty, -1)$	I	
$(-1, 0)$	I	
$(0, 1)$	D	
$(1, \infty)$	D	

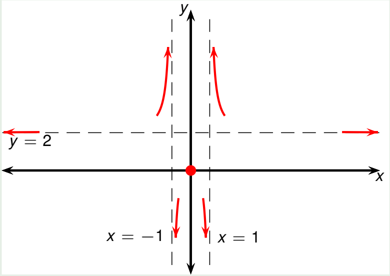
6. Intervals of increase or decrease

$$f'(x) = \frac{(x^2-1)(4x) - 2x^2(2x)}{(x^2-1)^2} = \frac{-4x}{(x^2-1)^2}$$

	$-4x$	$(x^2-1)^2$	f'
$(-\infty, -1)$	$+$	$+$	$+$
$(-1, 0)$	$+$	$+$	$+$
$(0, 1)$	$-$	$+$	$-$
$(1, \infty)$	$-$	$+$	$-$

Example

Sketch the curve $y = \frac{2x^2}{x^2-1}$.



Interval	I/D	Concavity
$(-\infty, -1)$	I	
$(-1, 0)$	I	
$(0, 1)$	D	
$(1, \infty)$	D	

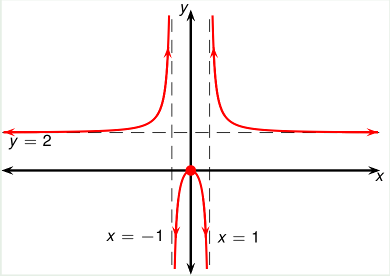
7. Local maxima and minima

	$-4x$	$(x^2-1)^2$	f'
$(-\infty, -1)$	$+$	$+$	$+$
$(-1, 0)$	$+$	$+$	$+$
$(0, 1)$	$-$	$+$	$-$
$(1, \infty)$	$-$	$+$	$-$

- f' changes sign from $+$ to $-$ at 0.
- Therefore $(0, 0)$ is a local maximum.

Example

Sketch the curve $y = \frac{2x^2}{x^2-1}$.



Interval	I/D	Concavity
$(-\infty, -1)$	I	up
$(-1, 0)$	I	down
$(0, 1)$	D	down
$(1, \infty)$	D	up

8. Concavity and points of inflection

$$f''(x) = \frac{-4(x^2-1)^2 + 4x \cdot 2(x^2-1)2x}{(x^2-1)^4} = \frac{12x^2+4}{(x^2-1)^3}$$

	$12x^2+4$	$(x^2-1)^3$	f''
$(-\infty, -1)$	$+$	$+$	$+$
$(-1, 1)$	$+$	$-$	$-$
$(1, \infty)$	$+$	$+$	$+$

No points of inflection because ± 1 are not in the domain of f.