

# Calculus II

## Homework

### Area locked by curve

1. Let  $x \in (0, 1)$ . Express the following using  $x$  and  $\sqrt{1 - x^2}$ .

- |                            |                            |
|----------------------------|----------------------------|
| (a) $\sin(\arcsin(x))$ .   | (e) $\sin(2 \arccos(x))$ . |
| (b) $\sin(2 \arcsin(x))$ . | (f) $\sin(3 \arccos(x))$ . |
| (c) $\sin(3 \arcsin(x))$ . | (g) $\cos(2 \arcsin(x))$ . |
| (d) $\sin(\arccos(x))$ .   | (h) $\cos(3 \arccos(x))$ . |

2. Express as the following as an algebraic expression of  $x$ . In other words, “get rid” of the trigonometric and inverse trigonometric expressions.

- |                                          |                                    |
|------------------------------------------|------------------------------------|
| (a) $\cos^2(\arctan x)$ .                | (c) $\frac{1}{\cos(\arcsin x)}$ .  |
| (b) $-\sin^2(\operatorname{arccot} x)$ . | (d) $-\frac{1}{\sin(\arccos x)}$ . |

3. Rewrite as a rational function of  $t$ . This problem will be later used to derive the Euler substitutions (an important technique for integrating).

- |                           |                                         |
|---------------------------|-----------------------------------------|
| (a) $\cos(2 \arctan t)$ . | (g) $\cos(2 \operatorname{arccot} t)$ . |
| (b) $\sin(2 \arctan t)$ . | (h) $\sin(2 \operatorname{arccot} t)$ . |
| (c) $\tan(2 \arctan t)$ . | (i) $\tan(2 \operatorname{arccot} t)$ . |
| (d) $\cot(2 \arctan t)$ . | (j) $\cot(2 \operatorname{arccot} t)$ . |
| (e) $\csc(2 \arctan t)$ . | (k) $\csc(2 \operatorname{arccot} t)$ . |
| (f) $\sec(2 \arctan t)$ . | (l) $\sec(2 \operatorname{arccot} t)$ . |

4. Compute the derivative (derive the formula).

- |                                    |                                                                                                                   |
|------------------------------------|-------------------------------------------------------------------------------------------------------------------|
| (a) $(\arctan x)'$ .               | (d) $(\arccos x)'$ .                                                                                              |
| (b) $(\operatorname{arccot} x)'$ . | (e) Let $\operatorname{arcsec}$ denote the inverse of the secant function. Compute $(\operatorname{arcsec} x)'$ . |
| (c) $(\arcsin x)'$ .               |                                                                                                                   |

5. (a) Let  $a + b \neq k\pi$ ,  $a \neq k\pi + \frac{\pi}{2}$  and  $b \neq k\pi + \frac{\pi}{2}$  for any  $k \in \mathbb{Z}$  (integers). Prove that

$$\frac{\tan a + \tan b}{1 - \tan a \tan b} = \tan(a + b) \quad .$$

(b) Let  $x$  and  $y$  be real. Prove that, for  $xy \neq 1$ , we have

$$\arctan x + \arctan y = \arctan \left( \frac{x + y}{1 - xy} \right)$$

if the left hand side lies between  $(-\frac{\pi}{2}, \frac{\pi}{2})$ .

6. Evaluate the indefinite integral. Illustrate the steps of your solutions.

$$(a) \int x \sin x dx.$$

$$(b) \int x e^{-x} dx.$$

$$(c) \int x^2 e^x dx.$$

$$(d) \int x \sin(-2x) dx.$$

$$(e) \int x^2 \cos(3x) dx.$$

$$(f) \int x^2 e^{-2x} dx.$$

$$(g) \int x \sin(2x) dx.$$

$$(h) \int x \cos(3x) dx.$$

$$(i) \int x^2 e^{2x} dx.$$

$$(j) \int x^3 e^x dx.$$

7. Evaluate the indefinite integral. Illustrate the steps of your solutions.

$$(a) \int x^2 \cos(2x) dx.$$

$$(b) \int x^2 e^{ax} dx, \text{ where } a \text{ is a constant.}$$

$$(c) \int x^2 e^{-ax} dx, \text{ where } a \text{ is a constant.}$$

$$(d) \int x^2 \frac{(e^{ax} + e^{-ax})^2}{4} dx, \text{ where } a \text{ is a constant.}$$

$$(e) \int \frac{1}{\cos^2 x} dx. \quad (\text{Hint: This problem does not require integration by parts. What is the derivative of } \tan x?)$$

$$(f) \int (\tan^2 x) dx. \quad (\text{Hint: This problem does not require integration by parts. We can use } \tan^2 x = \frac{1}{\cos^2 x} - 1 \text{ and the previous problem.})$$

$$(g) \int x \tan^2 x dx. \quad (\text{Hint: } \tan^2 x dx = d(F(x)), \text{ where } F(x) \text{ is the answer from the preceding problem}).$$

$$(h) \int e^{-\sqrt{x}} dx.$$

$$(i) \int \cos^2 x dx.$$

$$(j) \int \frac{x}{1+x^2} dx \quad (\text{Hint: use substitution rule, don't use integration by parts})$$

$$(k) \int (\arctan x) dx.$$

$$(l) \int (\arcsin x) dx.$$

$$(m) \int (\arcsin x)^2 dx. \quad (\text{Hint: Try substituting } x = \sin y.)$$

$$(n) \int \arctan\left(\frac{1}{x}\right) dx.$$

$$(o) \int \sin x e^x dx$$

$$(p) \int \cos x e^x dx$$

$$(q) \int \sin(\ln(x)) dx.$$

$$(r) \int \cos(\ln(x)) dx.$$

$$(s) \int \ln x dx$$

$$(t) \int x \ln x dx.$$

$$(u) \int \frac{\ln x}{\sqrt{x}} dx.$$

$$(v) \int (\ln x)^2 dx.$$

$$(w) \int (\ln x)^3 dx.$$

$$(x) \int x^2 \cos^2 x dx. \quad (\text{This problem is related to Problem 7.d as } \cos x = \frac{e^{ix} + e^{-ix}}{2}).$$

8. Compute  $\int x^n e^x dx$ , where  $n$  is a non-negative integer.

9. Integrate. Illustrate the steps of your solution.

$$(a) \int \frac{1}{x+1} dx$$

$$(b) \int \frac{x-1}{x+1} dx$$

$$(c) \int \frac{1}{(x+1)^2} dx$$

$$(d) \int \frac{x}{(x+1)^2} dx$$

$$(e) \int \frac{1}{(2x+3)^2} dx$$

$$(f) \int \frac{x}{2x^2+3} dx$$

$$(g) \int \frac{1}{2x^2+3} dx$$

$$(h) \int \frac{x}{2x^2+x+1} dx.$$

$$(i) \int \frac{x}{2x^2+x+3} dx$$

$$(j) \int \frac{x}{x^2-x+3} dx$$

$$(k) \int \frac{1}{(x^2 + 1)^2} dx$$

$$(m) \int \frac{1}{(x^2 + 1)^3} dx$$

$$(l) \int \frac{1}{(x^2 + x + 1)^2} dx$$

10. Let  $a, b, c, A, B$  be real numbers. Suppose in addition  $a \neq 0$  and  $b^2 - 4ac < 0$ . Integrate

$$\int \frac{Ax + B}{ax^2 + bx + c} dx \quad .$$

The purpose of this exercise is to produce a formula in form ready for implementation in a computer algebra system.

11. Let  $a, b, c, A, B$  be real numbers and let  $n > 1$  be an integer. Suppose in addition  $a \neq 0$  and  $b^2 - 4ac < 0$ . Let

$$J(n) = \int \frac{1}{(x^2 + \frac{b}{a}x + \frac{c}{a})^n} dx \quad .$$

(a) Express the integral

$$\int \frac{Ax + B}{(ax^2 + bx + c)^n} dx$$

via  $J(n)$ .

(b) Express  $J(n)$  recursively via  $J(n - 1)$

The purpose of this exercise is to produce a formula in form ready for implementation in a computer algebra system.

12. Integrate. Some of the examples require partial fraction decomposition and some do not. Illustrate the steps of your solution.

$$(a) \int \frac{1}{4x^2 + 4x + 1} dx$$

$$(h) \int \frac{x}{3x^2 + x - 2} dx$$

$$(b) \int \frac{1}{1 - x^2} dx$$

$$(i) \int \frac{x}{3x^2 + x + 2} dx$$

$$(c) \int \frac{1}{5 - x^2} dx$$

$$(j) \int \frac{x}{2x^2 + x + 1} dx$$

$$(d) \int \frac{x}{4x^2 + x + \frac{1}{16}} dx$$

$$(k) \int \frac{x}{2x^2 + x - 1} dx$$

$$(e) \int \frac{x + 1}{2x^2 + x} dx$$

$$(l) \int \frac{1}{x^2 + x + 1} dx$$

$$(f) \int \frac{x}{4x^2 + x + 5} dx$$

$$(m) \int \frac{1}{2x^2 + 5x + 1} dx$$

$$(g) \int \frac{x}{4x^2 + x - 5} dx$$

13. Evaluate the indefinite integral. Illustrate all steps of your solution.

$$(a) \int \frac{x^3 + 4}{x^2 + 4} dx$$

$$(h) \int \frac{15x^2 - 4x - 81}{(x - 3)(x + 4)(x - 1)} dx$$

$$(b) \int \frac{4x^2}{2x^2 - 1} dx$$

$$(i) \int \frac{x^4 + 10x^3 + 18x^2 + 2x - 13}{x^4 + 4x^3 + 3x^2 - 4x - 4} dx$$

$$(c) \int \frac{x^3}{x^2 + 2x - 3} dx$$

Check first that  $(x - 1)(x + 2)^2(x + 1) = x^4 + 4x^3 + 3x^2 - 4x - 4$ .

$$(d) \int \frac{x^3}{x^2 + 3x - 4} dx$$

$$(j) \int \frac{x^4}{(x^2 + 2)(x + 2)} dx$$

$$(e) \int \frac{x^3}{2x^2 + 3x - 5} dx$$

$$(k) \int \frac{x^5}{x^3 - 1} dx$$

$$(f) \int \frac{x^2 + 1}{(x - 3)(x - 2)^2} dx$$

$$(l) \int \frac{x^4}{(x^2 + 2)(x + 1)^2} dx$$

$$(g) \int \frac{x^4}{(x + 1)^2(x + 2)} dx$$

$$(m) \int \frac{3x^2 + 2x - 1}{(x-1)(x^2+1)} dx$$

$$(n) \int \frac{x^2 - 1}{x(x^2 + 1)^2} dx$$

14. Integrate

$$\int \frac{x^6 - x^5 + \frac{9}{2}x^4 - 4x^3 + \frac{13}{2}x^2 - \frac{7}{2}x + \frac{11}{4}}{x^5 - x^4 + 3x^3 - 3x^2 + \frac{9}{4}x - \frac{9}{4}} dx \quad .$$

15. Integrate.

$$(a) \int \frac{1}{3 + \cos x} dx.$$

$$(d) \int \frac{1}{2 + \tan x} dx. \text{ (Hint: this integral can be done simply with the substitution } x = \arctan t.)$$

$$(b) \int \frac{1}{4 + \cos x} dx.$$

$$(c) \int \frac{1}{3 + \sin x} dx.$$

$$(e) \int \frac{dx}{2 \sin x - \cos x + 5}.$$

16. Integrate. The answer key has not been proofread, use with caution.

$$(a) \int \sin(3x) \cos(2x) dx.$$

$$(b) \int \sin x \cos(5x) dx.$$

$$(c) \int \cos(3x) \sin(2x) dx.$$

$$(d) \int \sin(5x) \sin(3x) dx.$$

$$(e) \int \cos(x) \cos(3x) dx.$$

17. Integrate.

$$(a) \int \sin^2 x \cos x dx.$$

$$(c) \int \cos^3 x dx.$$

$$(b) \int \sin^2 x dx.$$

$$(d) \int \sin^3 x \cos^4 x dx.$$

18. Integrate.

$$(a) \int \sec x dx.$$

$$(b) \int \sec^3 x dx.$$

$$(c) \int \tan^3 x dx.$$

$$(d) \int \sec^2 x \tan^2 x dx.$$

19. Find a linear substitution (via completing the square) to transform the radical to a multiple of an expression of the form  $\sqrt{u^2 + 1}$ ,  $\sqrt{u^2 - 1}$  or  $\sqrt{1 - u^2}$ .

$$(a) \sqrt{x^2 + x + 1}.$$

$$(b) \sqrt{-2x^2 + x + 1}.$$

20. Compute the integral.

$$(a) \int \frac{\sqrt{1 + x^2}}{x^2} dx.$$

21. Compute the integral using a trigonometric substitution.

$$(a) \int \frac{\sqrt{9-x^2}}{x^2} dx \quad .$$

22. Compute the integral.

$$(a) \int \sqrt{x^2+1} dx$$

$$(b) \int \sqrt{x^2+2} dx$$

$$(c) \int \sqrt{x^2+x+1} dx$$

$$(d) \int \sqrt{(2x^2+2x+1)} dx$$

$$(e) \int \sqrt{(3x^2+2x+1)} dx$$

$$(f) \int \frac{\sqrt{x^2+1}}{x+1} dx$$

23. Let  $b^2 - 4ac < 0$  and  $a > 0$  be (real) numbers. Show that

$$\int \sqrt{(ax^2+bx+c)} dx = \frac{\sqrt{a}D}{2} \left( \ln \left( \sqrt{\left( \frac{2xa+b}{2\sqrt{Da}} \right)^2 + 1} + \frac{2xa+b}{2\sqrt{Da}} \right) + \frac{2xa+b}{2\sqrt{Da}} \sqrt{\left( \frac{2xa+b}{2\sqrt{Da}} \right)^2 + 1} \right) + C,$$

$$\text{where } D = \frac{4ac - b^2}{4a^2}.$$

24. Integrate

$$(a) \int \sqrt{1-x^2} dx$$

$$(b) \int \sqrt{2-x^2} dx$$

$$(c) \int \sqrt{-x^2+x+1} dx$$

$$(d) \int \sqrt{2-x-x^2} dx$$

$$(e) \int \frac{\sqrt{1-x^2}}{1+x} dx$$

$$(f) \int \frac{\sqrt{1-x^2}}{2+x} dx$$

25. Integrate

$$(a) \int \sqrt{x^2-1} dx$$

$$(b) \int \sqrt{x^2-2} dx$$

$$(c) \int \sqrt{2x^2+x-1} dx$$

$$(d) \int \sqrt{x^2+x-1} dx$$

26. (a) Express  $x$ ,  $dx$  and  $\sqrt{x^2+1}$  via  $\theta$  and  $d\theta$  for the trigonometric substitution  $x = \cot \theta$ ,  $\theta \in (0, \pi)$ .

(b) Express  $x$ ,  $dx$  and  $\sqrt{x^2+1}$  via  $t$  and  $dt$  for the Euler substitution  $x = \cot(2 \arctan t)$ ,  $t > 0$ . Express  $t$  via  $x$ .

27. Let the variables  $x$  and  $t$  be related via  $\sqrt{x^2+1} = x+t$ .

(a) Express  $x$  via  $t$ .

(b) Express  $\sqrt{x^2+1}$  via  $t$  alone.

(c) Express  $dx$  via  $t$  and  $dt$ .

28. (a) Express  $x$ ,  $dx$  and  $\sqrt{x^2 + 1}$  via  $\theta$  and  $d\theta$  for the trigonometric substitution  $x = \tan \theta$ ,  $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ .  
 (b) Express  $x$ ,  $dx$  and  $\sqrt{x^2 + 1}$  via  $t$  and  $dt$  for the Euler substitution  $x = \tan(2 \arctan t)$ ,  $t \in (-1, 1)$ . Express  $t$  via  $x$ .
29. Let the variables  $x$  and  $t$  be related via  $\sqrt{x^2 + 1} = \frac{x}{t} - 1$ .  
 (a) Express  $x$  via  $t$ .  
 (b) Express  $\sqrt{x^2 + 1}$  via  $t$  alone.  
 (c) Express  $dx$  via  $t$  and  $dt$ .
30. (a) Express  $x$ ,  $dx$  and  $\sqrt{1 - x^2}$  via  $\theta$  and  $d\theta$  for the trigonometric substitution  $x = \cos \theta$ ,  $\theta \in [0, \pi]$ .  
 (b) Express  $x$ ,  $dx$  and  $\sqrt{1 - x^2}$  via  $t$  and  $dt$  for the Euler substitution  $x = \cos(2 \arctan t)$ ,  $t \geq 0$ . Express  $t$  via  $x$ .
31. Let the variables  $x$  and  $t$  be related via  $\sqrt{-x^2 + 1} = (1 - x)t$ .  
 (a) Express  $x$  via  $t$ .  
 (b) Express  $\sqrt{-x^2 + 1}$  via  $t$  alone.  
 (c) Express  $dx$  via  $t$  and  $dt$ .
32. (a) Express  $x$ ,  $dx$  and  $\sqrt{1 - x^2}$  via  $\theta$  and  $d\theta$  for the trigonometric substitution  $x = \sin \theta$ ,  $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ .  
 (b) Express  $x$ ,  $dx$  and  $\sqrt{1 - x^2}$  via  $t$  and  $dt$  for the Euler substitution  $x = \sin(2 \arctan t)$ ,  $t \in [-1, 1]$ . Express  $t$  via  $x$ .
33. Let the variables  $x$  and  $t$  be related via  $\sqrt{-x^2 + 1} = 1 - xt$ .  
 (a) Express  $x$  via  $t$ .  
 (b) Express  $\sqrt{-x^2 + 1}$  via  $t$  alone.  
 (c) Express  $dx$  via  $t$  and  $dt$ .
34. (a) Express  $x$ ,  $dx$  and  $\sqrt{x^2 - 1}$  via  $\theta$  and  $d\theta$  for the trigonometric substitution  $x = \csc \theta$ ,  $\theta \in [0, \frac{\pi}{2}] \cup [\pi, \frac{3\pi}{2}]$ .  
 (b) Express  $x$ ,  $dx$  and  $\sqrt{x^2 - 1}$  via  $t$  and  $dt$  for the Euler substitution  $x = \sec(2 \arctan t)$ ,  $t \in (-\infty, -1) \cup [1, 0)$ . Express  $t$  via  $x$ .
35. Let the variables  $x$  and  $t$  be related via  $\sqrt{x^2 - 1} = (x + 1)t$ .  
 (a) Express  $x$  via  $t$ .  
 (b) Express  $\sqrt{x^2 - 1}$  via  $t$  alone.  
 (c) Express  $dx$  via  $t$  and  $dt$ .
36. (a) Express  $x$ ,  $dx$  and  $\sqrt{1 - x^2}$  via  $\theta$  and  $d\theta$  for the trigonometric substitution  $x = \csc \theta$ ,  $\theta \in [0, \frac{\pi}{2}] \cup [\pi, \frac{3\pi}{2}]$ .  
 (b) Express  $x$ ,  $dx$  and  $\sqrt{1 - x^2}$  via  $t$  and  $dt$  for the Euler substitution  $x = \csc(2 \arctan t)$ ,  $t \in (-\infty, -1) \cup [0, 1)$ . Express  $t$  via  $x$ .
37. Let the variables  $x$  and  $t$  be related via  $\sqrt{x^2 - 1} = \frac{1}{t} - x$ .  
 (a) Express  $x$  via  $t$ .  
 (b) Express  $\sqrt{x^2 - 1}$  via  $t$  alone.  
 (c) Express  $dx$  via  $t$  and  $dt$ .
38. Compute the limits. The answer key has not been fully proofread, use with caution.

(a)  $\lim_{x \rightarrow 0} \frac{\sin x}{x}$ .

(b)  $\lim_{x \rightarrow 0} \frac{x}{\ln(1 + x)}$ .

(c)  $\lim_{x \rightarrow 0} \frac{x^2}{x - \ln(1 + x)}$ .

(d)  $\lim_{x \rightarrow 0} \frac{x^2}{\sin x \ln(1 + x)}$ .

(e)  $\lim_{x \rightarrow 0} \frac{\sin^2 x}{(\ln(1 + x))^2}$ .

(f)  $\lim_{x \rightarrow 0} \frac{\cos x - 1}{\sin x \ln(1 + x)}$ .

(g)  $\lim_{x \rightarrow 0} \frac{\arctan x - x}{x^3}$ .

(h)  $\lim_{x \rightarrow 0} \frac{\arcsin x - x}{x^3}$ .

(i)  $\lim_{x \rightarrow 1} \frac{x}{x - 1} - \frac{1}{\ln x}$ .

(j)  $\lim_{x \rightarrow 0} \frac{\cos(nx) - \cos(mx)}{x^2}$ .

$$(k) \lim_{x \rightarrow 0} \frac{\arcsin x - x - \frac{1}{6}x^3}{\sin^5 x}.$$

$$(l) \lim_{x \rightarrow 1} \frac{\sin(\pi x) \ln x}{\cos(\pi x) + 1}.$$

$$(m) \lim_{x \rightarrow 0} \frac{\sin x - x}{\arcsin x - x}.$$

$$(n) \lim_{x \rightarrow 0} \frac{\sin x - x}{\arctan x - x}.$$

$$(o) \lim_{x \rightarrow \infty} x \sin\left(\frac{2}{x}\right).$$

39. Compute the limit.

$$(a) \lim_{x \rightarrow \infty} \left(\frac{x-2}{x}\right)^x.$$

$$(b) \lim_{x \rightarrow \infty} \left(\frac{x-2}{x}\right)^{2x}$$

$$(c) \lim_{x \rightarrow \infty} \left(\frac{x}{x+3}\right)^{2x}$$

40. Find the limit.

$$(a) \lim_{x \rightarrow \infty} \left(1 - \frac{2}{x}\right)^x.$$

$$(b) \lim_{x \rightarrow 0} (1-x)^{\frac{1}{x}}.$$

$$(c) \lim_{x \rightarrow \infty} \left(\frac{x}{x-5}\right)^x.$$

$$(d) \lim_{x \rightarrow \infty} \left(\frac{x}{x-2}\right)^{3x+2}.$$

41. Determine whether the integral is convergent or divergent. Motivate your answer.

$$(a) \int_2^{\infty} \frac{1}{(x-1)^{\frac{3}{2}}} dx.$$

$$(b) \int_{-1}^1 \frac{1}{\sqrt[5]{1+x}} dx.$$

$$(c) \int_1^{\infty} \frac{1}{\sqrt[5]{1+x}} dx.$$

$$(d) \int_{-1}^{\infty} \frac{1}{\sqrt[5]{1+x}} dx.$$

$$(e) \int_{-\infty}^0 \frac{1}{2-3x} dx.$$

$$(f) \int_{-\infty}^0 \frac{1}{(2-3x)^2} dx.$$

$$(g) \int_{-\infty}^0 \frac{1}{(2-3x)^{1.00000001}} dx.$$

$$(h) \int_{-2}^{\frac{1}{2}} \frac{1}{2x-1} dx.$$

$$(i) \int_{-1}^{\infty} e^{-3x} dx.$$

$$(j) \int_{-\infty}^5 2^x dx.$$

$$(k) \int_{-\infty}^{\infty} x^3 dx.$$

$$(l) \int_{-\infty}^{\infty} x e^{-x^2} dx.$$

$$(m) \int_0^{\infty} \sqrt{x} e^{-\sqrt{x}} dx.$$

$$(n) \int_0^{\infty} \sin^2 x dx.$$

$$(o) \int_0^5 \frac{1}{x^2 + x - 2} dx.$$

$$(p) \int_0^{\infty} \frac{1}{x^2 + x + 1} dx.$$

$$(q) \int_2^{\infty} \frac{1}{x^2 - x - 1} dx.$$

$$(r) \int_0^{\infty} \frac{1}{x^2 - x - 1} dx.$$

$$(s) \int_{-\infty}^{\infty} \frac{x^2}{x^4 + 2} dx.$$

$$(t) \int_{100}^{\infty} \frac{1}{x \ln x} dx.$$

$$(u) \int_{100}^{\infty} \frac{1}{x(\ln x)^2} dx.$$

$$(v) \int_0^1 \ln x dx.$$

$$(w) \int_0^1 \frac{\ln x}{\sqrt{x}} dx.$$

$$(x) \int_0^2 x^3 \ln x dx.$$

$$(y) \int_0^1 \frac{e^{\frac{1}{x}}}{x^2} dx.$$

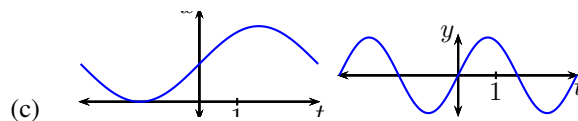
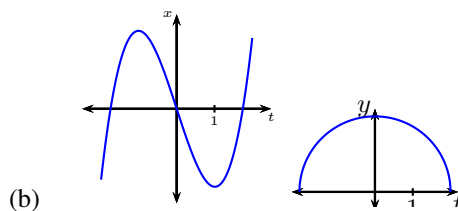
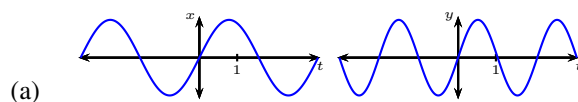
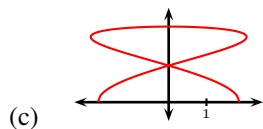
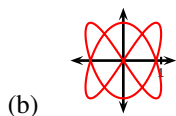
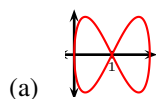
$$(z) \int_{-1}^0 \frac{e^{\frac{1}{x}}}{x^2} dx.$$

42. Determine whether the integral is convergent or divergent. Motivate your answer. The answer key has not been proofread, use with caution.

$$(a) \int_0^{\infty} \sin x^2 dx \text{ (This problem is more difficult and may re-}$$

quire knowledge of sequences to solve).

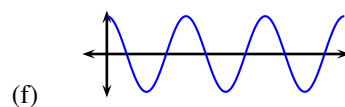
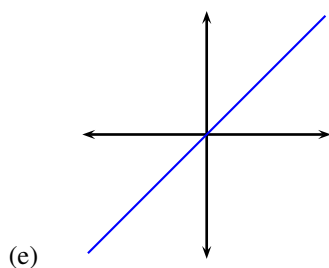
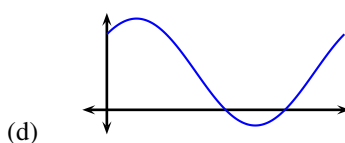
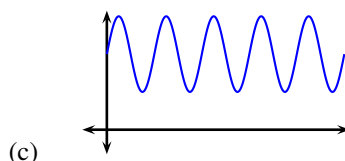
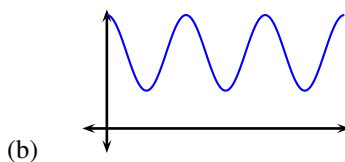
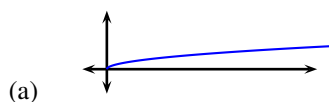
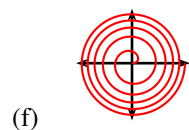
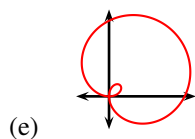
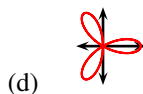
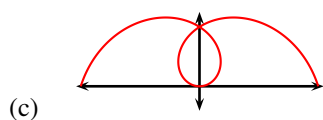
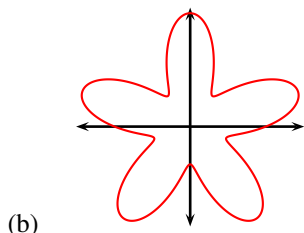
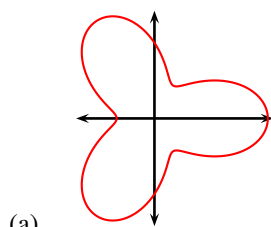
43. Match the graphs of the parametric equations  $x = f(t)$ ,  $y = g(t)$  with the graph of the parametric curve  $\gamma : \begin{cases} x = f(t) \\ y = g(t) \end{cases}$



44.



Match the graph of the curve to its graph in polar coordinates and to its polar parametric equations.



- (i)  $r = 1 + \sin(\theta) + \cos(\theta)$
- (ii)  $r = \theta, \theta \in [-\pi, \pi]$ .
- (iii)  $r = \cos(3\theta), \theta \in [0, 2\pi]$ .
- (iv)  $r = \frac{1}{4}\sqrt{\theta}, \theta \in [0, 10\pi]$ .
- (v)  $r = 2 + \sin(5\theta)$ .
- (vi)  $r = 2 + \cos(3\theta)$ .

45.

- (a) Sketch the curve given in polar coordinates by  $r = 2 \sin \theta$ . What kind of a figure is this curve? Find an equation satisfied by the curve in the  $(x, y)$ -coordinates.
- (b) Sketch the curve given in polar coordinates by  $r = 4 \cos \theta$ . What kind of a figure is this curve? Find an equation satisfied by the curve in the  $(x, y)$ -coordinates.
- (c) Sketch the curve given in polar coordinates by  $r = 2 \sec \theta$ . What kind of a figure is this curve? Find an equation satisfied by the curve in the  $(x, y)$ -coordinates.
- (d) Sketch the curve given in polar coordinates by  $r = 2 \csc \theta$ . What kind of a figure is this curve? Find an equation satisfied by the curve in the  $(x, y)$ -coordinates.
- (e) Sketch the curve given in polar coordinates by  $r = 2 \sec(\theta + \frac{\pi}{4})$ . What kind of a figure is this curve? Find an equation satisfied by the curve in the  $(x, y)$ -coordinates.
- (f) Sketch the curve given in polar coordinates by  $r = 2 \csc(\theta + \frac{\pi}{6})$ . What kind of a figure is this curve? Find an equation satisfied by the curve in the  $(x, y)$ -coordinates.

46. Find the values of the parameter  $t$  for which the curve has horizontal and vertical tangents.

- (a)  $x = t^2 - t + 1, y = t^2 + t - 1$
- (b)  $x = t^3 - t^2 - t + 1, y = t^2 - t - 1$ .
- (c)  $x = \cos(t), y = \sin(3t)$
- (d)  $x = \cos(t) + \sin(t), y = \sin(t)$ .

47. Show that the parametric curve has multiple tangents at the point and find their slopes.

- (a)  $x = \cos t, y = 2 \sin(2t)$ , two tangents at  $(x, y) = (0, 0)$ . of tangents.
- (b)  $x = \cos t \sin(3t), y = \sin(t) \sin(3t)$ , six tangents at  $(x, y) = (0, 0)$ .
- (c)  $x = \cos t, y = \sin(3t)$ , find the two points at which the curve has double tangent and find the slopes of both pairs
- (d)  $x = t^3 - t^2 - t + 1, y = t^2 - t - 1$ , find a point where the curve has double tangent and find the slopes of the tangents.

48. Find the length of the curve.

- (a)  $y = x^2, x \in [1, 2]$ .
- (b)  $y = \sqrt{x}, x \in [1, 2]$ .
- (c)  $x = \sqrt{t} - 2t$  and  $y = \frac{8}{3}t^{\frac{3}{4}}$  from  $t = 1$  to  $t = 4$ .
- (d)  $\gamma : \begin{cases} x(t) = \frac{1}{t} + \frac{t^3}{3} \\ y(t) = 2t \end{cases}, t \in [1, 2]$  .
- (e)  $\gamma : \begin{cases} x(t) = \frac{1}{t} + t \\ y(t) = 2 \ln t \end{cases}, t \in [1, 2]$  .
- (f) One arch of the cycloid

$$\gamma : \begin{cases} x(t) = t - \sin t \\ y(t) = 1 - \cos t \end{cases}, t \in [0, 2\pi]$$

- (g) The cardioid

$$\gamma : \begin{cases} x(t) = (1 + \sin t) \cos t \\ y(t) = (1 + \sin t) \sin t \end{cases}, t \in [0, 2\pi]$$

49. Set up an integral that expresses the length of the curve and find the length of the curve.

- (a)  $\begin{cases} x(t) = e^t + e^{-t} \\ y(t) = 5 - 2t \end{cases}, t \in [0, 3]$
- (b)  $\begin{cases} x(t) = \sin t + \cos t \\ y(t) = \sin t - \cos t \end{cases}, t \in [0, \pi]$

50. Give a geometric definition of the cycloid curve using a circle of radius 1. Using that definition, derive equations for the cycloid curve. Find area locked between one “arch” of the cycloid curve and the  $x$  axis.

51. (a) The curve given in polar coordinates by  $r = 1 + \sin 2\theta$  is plotted below by computer. Find the area lying outside of this curve and inside of the circle  $x^2 + y^2 = 1$ .
- (b) The curve given in polar coordinates by  $r = \cos(2\theta)$  is plotted below by computer. Find the area lying inside the curve and outside of the circle  $x^2 + y^2 = \frac{1}{4}$ .
- (c) Below is a computer generated plot of the curve  $r = \sin(2\theta)$ . Find the area locked inside one petal of the curve and outside of the circle  $x^2 + y^2 = \frac{1}{4}$ .

52. The answer key has not been proofread, use with caution.

- (a) Sketch the graph of the curve given in polar coordinates by  $r = 3 \sin(2\theta)$  and find the area of one petal.
- (b) Sketch the graph of the curve given in polar coordinates by  $r = 4 + 3 \sin \theta$  and find the area enclosed by the curve.