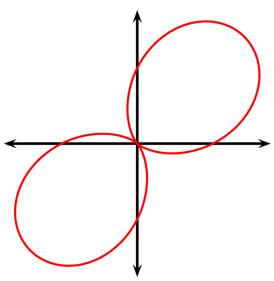
Calculus II
Homework
Area locked by curve

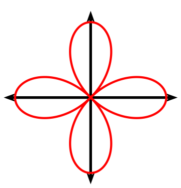
1. Give a geometric definition of the cycloid curve using a circle of radius 1. Using that definition, derive equations for the cycloid curve. Find area locked between one "arch" of the cycloid curve and the x axis.

2. (a) The curve given in polar coordinates by $r = 1 + \sin 2\theta$ is plotted below by computer. Find the area lying outside of this curve and inside of the circle $x^2 + y^2 = 1$.



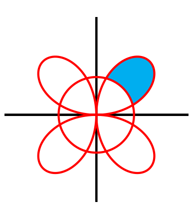
answer: $a = 2 - \frac{\pi}{4}$

(b) The curve given in polar coordinates by $r = \cos(2\theta)$ is plotted below by computer. Find the area lying inside the curve and outside of the circle $x^2 + y^2 = \frac{1}{4}$.



answer: $\frac{\pi}{6} + \frac{\sqrt{3}}{4}$

(c) Below is a computer generated plot of the curve $r = \sin(2\theta)$. Find the area locked inside one petal of the curve and outside of the circle $x^2 + y^2 = \dfrac{1}{4}$.



answer: $\frac{\pi}{24} + \frac{\sqrt{3}}{16}$

Solution. 2.a. A computer generated plot of the two curves is included below. The circle $x^2 + y^2 = 1$ has one-to-one polar representation given by $r = 1, \theta \in [0, 2\pi)$. Except the origin, which is traversed four times by the curve $r = 1 + \sin(2\theta)$, the second curve is in a one-to-one correspondence with points in the r, θ-plane given by the equation $r = 1 + \sin(2\theta), \theta \in [0, 2\pi)$. Since the two curves do not meet in the origin, we may conclude that the two curves may intersect only when their values for r and θ coincide. Therefore we have an intersection when

$$\begin{array}{rcll} 1 + \sin(2\theta) &=& 1 & \\ \sin(2\theta) &=& 0 & \\ \theta &=& 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2} & \big| \text{ because } \theta \in [0, 2\pi) \end{array}$$

Therefore the two curves meet in the points $(0,1)(-1,0)$ and $(0,-1),(1,0)$.

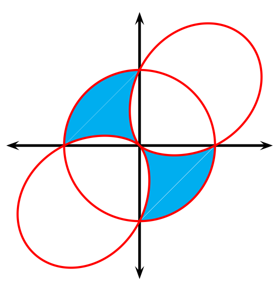
Denote the investigated region by A. From the computer-generated plot, it is clear that when a point has polar coordinates $\theta \in [\frac{\pi}{2},\pi] \cup [\frac{3\pi}{2}, 2\pi]$, $r \in [1+\sin(2\theta), 1]$ it lies in A. Furthermore, the points r, θ lying in the above intervals are in one-to-one correspondence with the points in A.

Suppose we have a curve $r = f(\theta), \theta \in [a,b]$ for which no two points lie on the same ray from the origin. Recall from theory that the area swept by that curve is given by

$$\int_a^b \frac{1}{2} f^2(\theta) \mathrm{d}\theta \quad .$$

Therefore the area a of A is computed via the integrals

$$\begin{array}{rcll} a & = & \displaystyle\int_{\frac{\pi}{2}}^{\pi} \frac{1}{2}\left(\underbrace{1}_{\text{outer curve}}{}^2 - \left(\underbrace{1+\sin(2\theta)}_{\text{inner curve}}\right)^2 \right) \mathrm{d}\theta + \int_{\frac{3\pi}{2}}^{2\pi} \frac{1}{2}\left(1^2 - (1+\sin(2\theta))^2\right) \mathrm{d}\theta & \text{use the symmetry of } A \\ & = & \displaystyle\int_{\frac{\pi}{2}}^{\pi} \left(1^2 - (1+\sin(2\theta))^2\right) \mathrm{d}\theta = \int_{\frac{\pi}{2}}^{\pi} \left(-2\sin(2\theta) - \sin^2(2\theta)\right) \mathrm{d}\theta & \text{use } \sin^2 z = \frac{1-\cos(2z)}{2} \\ & = & \displaystyle\int_{\frac{\pi}{2}}^{\pi} \left(-2\sin(2\theta) - \frac{1}{2} + \frac{1}{2}\cos(4\theta)\right) \mathrm{d}\theta = \left[\cos(2\theta) - \frac{1}{2}\theta - \frac{1}{8}\sin(4\theta)\right]_{\frac{\pi}{2}}^{\pi} & \\ & = & 2 - \frac{\pi}{4} \quad . & \end{array}$$



Solution. 2.b A computer generated plot of the figure is included below. The circle $x^2 + y^2 = \frac{1}{4}$ is centered at 0 and of radius $\frac{1}{2}$ and therefore can be parametrized in polar coordinates via $r = \frac{1}{2}, \theta \in [0, 2\pi]$.

Points with polar coordinates (r_1, θ_1) and (r_2, θ_2) coincide if one of the three holds:

- $r_1 = r_2 \neq 0$ and $\theta_1 = \theta_2 + 2k\pi, k \in \mathbb{Z}$,
- $r_1 = -r_2 \neq 0$ and $\theta_1 = \theta_2 + (2k+1)\pi, k \in \mathbb{Z}$,
- $r_1 = r_2 = 0$ and θ is arbitrary.

To find the intersection points of the two curves we have to explore each of the cases above. The third case is not possible as the circle does not pass through the origin. Suppose we are in the first case. Then the value of r (as a function of θ) is equal for the two curves. Thus the two curves intersect if

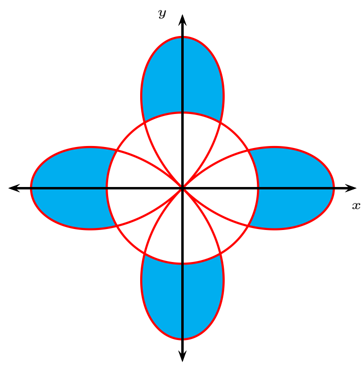
$$\begin{array}{rcll} r = \cos(2\theta) & = & \frac{1}{2} & \\ 2\theta & = & \pm\frac{\pi}{3} + 2k\pi & \text{where } k \in \mathbb{Z} \\ \theta & = & \pm\frac{\pi}{6} + k\pi & \text{where } k \in \mathbb{Z} \\ \theta & = & \frac{\pi}{6}, \frac{\pi}{6} + \pi, -\frac{\pi}{6} + \pi, -\frac{\pi}{6} + 2\pi & \text{all other values discarded as } \theta \in [0, 2\pi] \\ \theta & = & \frac{\pi}{6}, \frac{7\pi}{6}, \frac{5\pi}{6}, \frac{11\pi}{6} & \end{array}$$

This gives us only four intersection points, and the computer-generated plot shows eight. Therefore the second case must yield new intersection points: the two curves intersect also when

$$\begin{array}{rcll} r = \cos(2\theta) & = & -\frac{1}{2} & \\ 2\theta & = & \pm\frac{2\pi}{3} + 2k\pi & \text{where } k \in \mathbb{Z} \\ \theta & = & \pm\frac{\pi}{3} + k\pi & \text{where } k \in \mathbb{Z} \\ \theta & = & \frac{\pi}{3}, \frac{\pi}{3} + \pi, \frac{-\pi}{3} + \pi, \frac{-\pi}{3} + 2\pi & \text{all other values are discarded as } \theta \in [0, 2\pi] \\ \theta & = & \frac{\pi}{3}, \frac{4\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{3} \quad . & \end{array}$$

From the computer-generated plot below, we can see that the area we are looking for is 4 times the area locked between the two curves for $\theta \in \left[\frac{-\pi}{6}, \frac{\pi}{6}\right]$. Therefore the area we are looking for is given by

$$4\int\limits_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1}{2}\left(\cos^2(2\theta) - \left(\frac{1}{2}\right)^2\right) \mathrm{d}\theta \quad .$$



We leave the above integral to the reader.

Solution. 2.c. The circle $x^2 + y^2 = \frac{1}{4}$ is centered at 0 and of radius $\frac{1}{2}$ and therefore can be parametrized in polar coordinates via $r = \frac{1}{2}, \theta \in [0, 2\pi)$.

Points with polar coordinates (r_1, θ_1) and (r_2, θ_2) coincide if one of the three holds:

- $r_1 = r_2 \neq 0$ and $\theta_1 = \theta_2 + 2k\pi, k \in \mathbb{Z}$,
- $r_1 = -r_2 \neq 0$ and $\theta_1 = \theta_2 + (2k+1)\pi, k \in \mathbb{Z}$,
- $r_1 = r_2 = 0$ and θ is arbitrary.

To find the intersection points of the two curves we have to explore each of the cases above. The third case is not possible as the circle does not pass through the origin. Suppose we are in the first case. Then the value of r (as a function of θ) is equal for the two curves. Thus the two curves intersect if

$$\begin{array}{rcl|l} r = \sin(2\theta) & = & \frac{1}{2} & \\ 2\theta & = & \frac{\pi}{6} + 2k\pi \text{ or } \frac{5\pi}{6} + 2k\pi & \text{where } k \in \mathbb{Z} \\ \theta & = & \frac{\pi}{12} + k\pi \text{ or } \frac{5\pi}{12} & \text{where } k \in \mathbb{Z} \\ \theta & = & \frac{\pi}{12}, \frac{13\pi}{12}, \frac{5\pi}{12}, \frac{17\pi}{12} & \begin{array}{l}\text{other values discarded as} \\ \theta \in [0, 2\pi]\end{array} \end{array}$$

This gives us only four intersection points, and the computer-generated plot shows eight. Therefore the second case must yield 4 new intersection points. However, from the figure we see there are only two intersection points that participate in the boundary of our area, and both of those were found above. Therefore we shall not find the remaining 4 intersections.

Both the areas locked by the petal and the area locked by the section of the circle are found by the formula for the area locked by a polar curve. Subtracting the two we get that the area we are looking for is:

$$\begin{aligned} \text{Area} &= \int\limits_{\theta=\frac{\pi}{12}}^{\theta=\frac{5\pi}{12}} \frac{1}{2}\left(\sin^2(2\theta) - \left(\frac{1}{2}\right)^2\right) \mathrm{d}\theta \quad . \\ &= \frac{1}{2}\int\limits_{\theta=\frac{\pi}{12}}^{\theta=\frac{5\pi}{12}} \left(\frac{1-\cos(4\theta)}{2} - \frac{1}{4}\right) \mathrm{d}\theta \\ &= \frac{1}{2}\left[\frac{1}{4}\theta - \frac{\sin(4\theta)}{8}\right]_{\theta=\frac{\pi}{12}}^{\theta=\frac{5\pi}{12}} \\ &= \frac{\pi}{24} + \frac{\sqrt{3}}{16} \quad . \end{aligned}$$

3. The answer key has not been proofread, use with caution.

(a) Sketch the graph of the curve given in polar coordinates by $r = 3\sin(2\theta)$ and find the area of one petal.

answer: $\frac{9\pi}{8}$, curve sketch: 

(b) Sketch the graph of the curve given in polar coordinates by $r = 4 + 3\sin\theta$ and find the area enclosed by the curve.

answer: $\frac{41\pi}{2}$, curve sketch: