Precalculus
Homework
Logarithms basics

1. Use the definition of a logarithm to evaluate each of the following without using a calculator. The answer key has not been proofread, use with caution.

   (a) $\log_2 16$.

   answer: 4

   (b) $\log_3 \left(\frac{1}{9}\right)$.

   answer: -2

   (c) $\log_{10} 1000$.

   answer: 3

   (d) $\log_6 36^{-\frac{2}{3}}$.

   answer: $-\frac{4}{3}$

   (e) $\log_2(8\sqrt{2})$.

   answer: $\frac{7}{2}$

   (f) $\log_{\frac{1}{2}}(4)$.

   answer: -2

   (g) $\log_{\frac{1}{9}}(\sqrt{3})$.

   answer: $-\frac{1}{4}$

2. Find the exact value of each expression.

   (a) $\log_5 125$.

   answer: 3

   (b) $\log_3 \frac{1}{27}$.

   answer: -3

   (c) $\ln \left(\frac{1}{e}\right)$.

   answer: -1

   (d) $\log_{10} \sqrt{10}$.

   answer: $\frac{1}{2}$

   (e) $e^{\ln 4.5}$.

   answer: 4.5

   (f) $\log_{10} 0.0001$.

   answer: 4

   (g) $\log_{1.5} 2.25$.

   answer: 2

   (h) $\log_5 4 - \log_5 500$.

   answer: -3

   (i) $\log_2 6 - \log_2 15 + \log_2 20$.

   answer: 3

   (j) $\log_3 100 - \log_3 18 - \log_3 50$.

   answer: -2

   (k) $e^{-2\ln 5}$.

   answer: $\frac{1}{25}$

   (l) $\ln\left(\ln e^{e^{10}}\right)$.

   answer: 10

   (m) $\log_7 \left(\frac{49^x}{343^y}\right)$

   answer: $2x - 3y$

   Solution. 2.m.

$$\begin{aligned}
\log_7\left(\tfrac{49^x}{343^y}\right) &= \log_7 49^x - \log_7 343^y \\
&= x\log_7 49 - y\log_7 343 \\
\text{However } 49 = 7^2 \text{ and } 343 = 7^3, \text{ therefore } \log_7\left(\tfrac{49^x}{343^y}\right) &= 2x - 3y.
\end{aligned}$$

3. Using only the $\ln$ operation of your calculator compute the indicated logarithm. Confirm your computation numerically by exponentiation.

(a) $\log_5(13)$.

answer: $\frac{\ln 13}{\ln 5} \approx 1.59369$

(b) $\log_{12}(9)$.

answer: $\frac{\ln 9}{\ln 12} \approx 0.884228$

(c) $\log_{13}(101)$.

answer: $\frac{\ln 101}{\ln 13} \approx 1.79303$

(d) $\log_{10}(2015)$.

answer: $\frac{\ln 2015}{\ln 10} \approx 3.30428$

Solution.

$$\log_5(13) = \frac{\ln 13}{\ln 5} \approx \frac{2.564949357}{1.609437912} \approx 1.593693.$$

As a check of our computations, we compute by calculator: $13 = 5^{\log_5 13} \approx 5^{1.593693} \approx 13.000007508$, and our computations check out.

4. Express each of the following as a single logarithm. If possible, compute the logarithm without using a calculator. The answer key has not been proofread, use with caution.

(a) $\ln 4 + \ln 6 - \ln 5$.

answer: $\ln\left(\frac{24}{5}\right)$

(b) $2\ln 2 - 3\ln 3 + 4\ln 4$.

answer: $\ln\left(\frac{1024}{27}\right)$

(c) $\ln 36 - 2\ln 3 - 3\ln 2$.

answer: $-\ln 2 = \ln\left(\frac{1}{2}\right)$

(d) $\log_2(24) - \log_4 9$.

answer: 3

(e) $\log_7(24) + \log_{\frac{1}{7}} 3 - \log_{49}(64)$.

answer: 0

(f) $\log_3(24) + \log_3\left(\frac{3}{8}\right)$.

answer: 2

Solution. 4.b.

$$\begin{aligned} 2\ln 2 - 3\ln 3 + 4\ln 4 &= \ln 2^2 - \ln 3^3 + \ln 4^4 \\ &= \ln 4 - \ln 27 + \ln 256 \\ &= \ln\left(\frac{4}{27}\right) + \ln 256 \\ &= \ln\left(\frac{4\cdot 256}{27}\right) \\ &= \ln\left(\frac{1024}{27}\right). \end{aligned}$$

$\frac{1024}{27}$ is not a rational power of e, therefore $\ln\left(\frac{1024}{27}\right)$ is not a rational number and there are no further simplifications of the answer (except possibly a numerical approximation with a calculator or equivalent).

Solution. 4.e

$$
\begin{aligned}
\log_7(24) + \log_{\frac{1}{7}}(3) - \log_{49}(64) =& \log_7(24) + \frac{\log_7(3)}{\log_7\left(\frac{1}{7}\right)} - \frac{\log_7(64)}{\log_7(49)} && \text{common base} \\
=& \log_7(24) + \frac{\log_7(3)}{-1} - \frac{\log_7(64)}{2} && \text{simplify logarithms} \\
=& \log_7(24) - \log_7(3) - \frac{1}{2}\log_7(64) \\
=& \log_7\left(\frac{24}{3}\right) - \log_7\left(64^{\frac{1}{2}}\right) && \text{rule: } \log_a x - \log_a y = \log_a\left(\tfrac{x}{y}\right) \\
& && \text{rule: } \log_a x^r = r\log_a x \\
=& \log_7(8) - \log_7\left(\sqrt{64}\right) \\
=& \log_7 8 - \log_7 8 = 0 && \text{alternatively:} \\
=& \log_7\left(\frac{8}{8}\right) \\
=& \log_7(1) \\
=& 0.
\end{aligned}
$$

5. Demonstrate the identity(s).

   (a) $-\ln\left(\sqrt{1+x^2} - x\right) = \ln\left(x + \sqrt{1+x^2}\right)$