Precalculus
Exponent basics

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2019

Outline

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Properties of exponential expressions.

For integer x, y and bases a, b, we demonstrate the exponent rules by example.

1. $a^x a^y = a^{x+y}$
2. $\frac{a^x}{a^y} = a^{x-y}$
3. $(a^x)^y = a^{xy}$
4. $(ab)^x = a^x b^x$

These rules do continue to hold for all $a > 0$, $b > 0$ and arbitrary x and y. The rules do fail when $a < 0$, $b < 0$ and x, y are not integers.

$$\begin{aligned} 7^3 \cdot 7^2 &= (7 \cdot 7 \cdot 7)(7 \cdot 7) \\ &= 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \\ &= 7^5 \\ &= 7^{3+2}. \end{aligned}$$

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$$\begin{aligned} \frac{7^3}{7^2} &= \frac{\cancel{7} \cdot \cancel{7} \cdot 7}{\cancel{7} \cdot \cancel{7}} \\ &= 7 \\ &= 7^1 \\ &= 7^{3-2}. \end{aligned}$$

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$$\begin{aligned}\left(7^2\right)^4 &= 7^2 \cdot 7^2 \cdot 7^2 \cdot 7^2\\ &= (7\cdot 7)(7\cdot 7)(7\cdot 7)(7\cdot 7)\\ &= 7\cdot 7\cdot 7\cdot 7\cdot 7\cdot 7\cdot 7\cdot 7\\ &= 7^8\\ &= 7^{2\cdot 4}\end{aligned}$$

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$$\begin{aligned}(5 \cdot 7)^3 &= (5 \cdot 7)(5 \cdot 7)(5 \cdot 7)\\ &= 5 \cdot 7 \cdot 5 \cdot 7 \cdot 5 \cdot 7\\ &= 5 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdot 7\\ &= 5^3 \cdot 7^3\end{aligned}$$

Exponents overview

- For integer x, we know how to compute a^x as a function of a.
- How do we compute $f(x) = a^x$ when x is not an integer?
- We need to go back to the definition of a^x (for x non-integer).
- In what follows we give/recall an elementary way to define exponent.
- Then we give an alternative second definition.
- The second definition will be studied in sufficient depth only much later.
- The two definitions are equivalent: if we choose one definition the other becomes a theorem and the other way round.
- Choosing one definition makes some statements easier to prove and others more difficult.
- We shall discuss pros and cons of the two. In a nutshell:
  - the first elementary definition is easier to motivate;
  - the second alternative definition is easier to compute with.

Exponent definition using limits (approach I)

- For integer p we know to compute a^p.
- Therefore for integer q we know to compute $a^{\frac{1}{q}} = \sqrt[q]{a} = \max\{x | \text{ for which } x^q \leq a\}$.
- Therefore we know to compute $a^{\frac{p}{q}}$ for all rational $\frac{p}{q}$.
- We can then define
$$a^x = \lim_{\substack{y \to x \\ y\text{-rational}}} a^y$$
For example, a^π would be defined as the limit of the sequence $a^{3.14}, a^{3.141}, a^{3.1415}, \dots$.
- Cons: not computationally effective; not how computers compute.
- Pros: for non-integer x and y, it is very easy to prove that $a^{x+y} = a^x a^y$ - this follows from the definition of limit above.
- This is the definition assumed in many elementary courses.

Exponent definition using series (approach II)

- The following formula (studied much later) can be used as alternative definition.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots$$

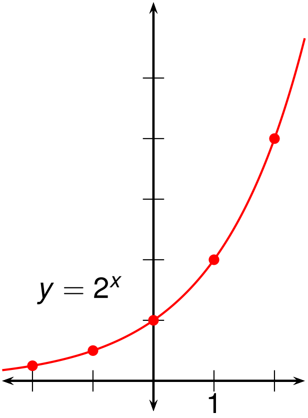
  Here $n! = 1 \cdot 2 \cdot 3 \cdot \dots \cdot (n-1) \cdot n$ and is read "n factorial".
- For $|x| < 1$ define

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + \frac{(-1)^{n+1} x^n}{n} + \dots$$

  Infinite sum studied much later.
- For arbitrary $a > 0$ define a^x as $a^x = e^{x \ln a}$.
- Cons: more difficult to prove $e^{x+y} = e^x e^y$ and $e^{\ln(1+x)} = 1 + x$, proof done later.
- Pros: this is how e^x and a^x are actually computed (by modern computers and by humans in the past).

Exponential Functions

The function $f(x) = 2^x$ is called an exponential function because the variable x is the exponent.

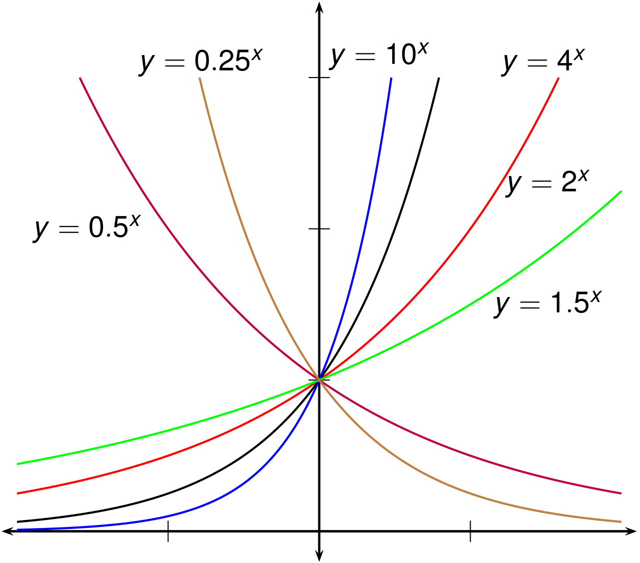


x	y
2	4
1	2
0	1
-1	$\frac{1}{2}$
-2	$\frac{1}{4}$

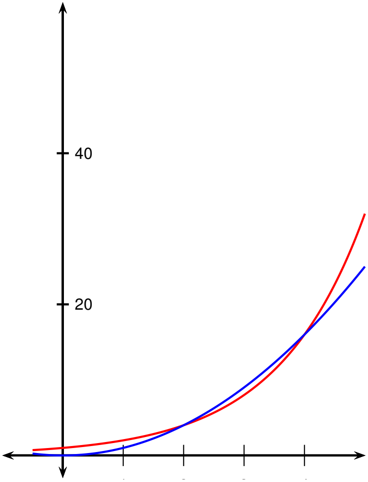
(Exponential Function Terminology)

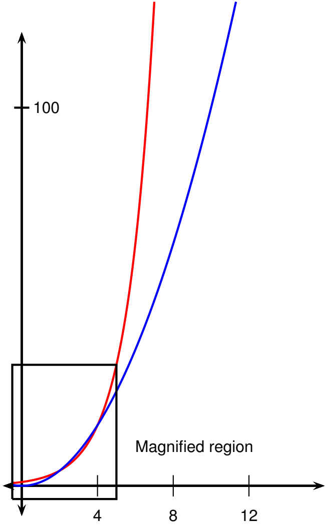
An exponential function is a function of the form $f(x) = a^x$, where a is a positive constant.

Graphs of various exponential functions.



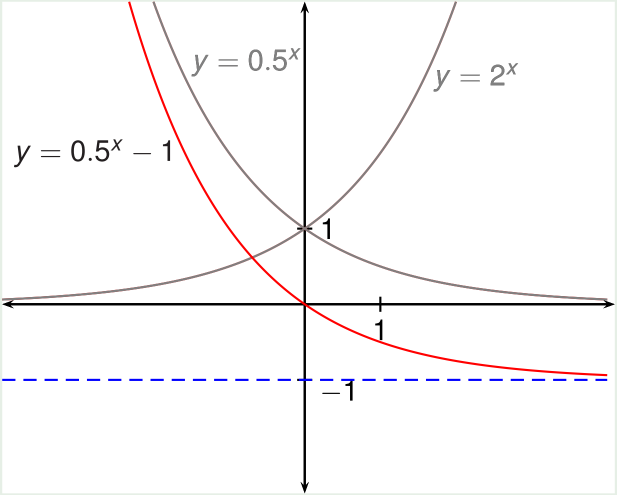
Graphical comparison of $y = 2^x$ with $y = x^2$. Axes have different scales.





Example

Draw the graph of the function $y = 2^{-x} - 1 = 0.5^x - 1 = \left(\frac{1}{2}\right)^x - 1$.
Assume the graph of $y = 2^x$ given.



- Plot of 2^x assumed given.
- Plot $f(-x)$ = reflect $f(x)$ across y axis.
- Plot $g(x) - 1$ = shift graph $g(x)$ 1 unit down.

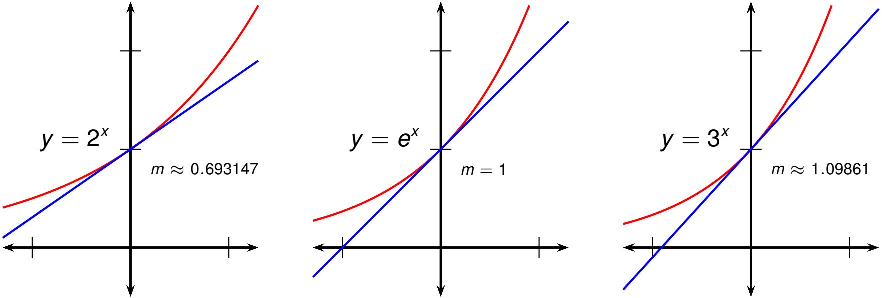
Proposition

Let $a > 0$, $a \neq 1$. *Let* x *and* y *be real numbers. Then* $a^x = a^y$ *if and only if* $x = y$.

- In other words, the exponent function a^x is one-to-one.

The Natural Exponential Function

- One base for an exponential function is especially useful.
- It has a special property: its tangent line at $x = 0$ has slope $m = 1$.
- We call this number e, known as Euler's number or Napier's constant.
- e is a number between 2 and 3.
- In fact, $e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots \approx 2.71828$.



Recall that $e = 1 + \frac{1}{1} + \frac{1}{2!} + \frac{1}{3!} + \cdots \approx 2.718281828$.

Theorem (The Number e as a Limit)

For large n we have that:

$$\begin{aligned} e &\approx \left(1 + \frac{1}{n}\right)^n \\ &\approx (1+n)^{\frac{1}{n}} \\ e^x &\approx \left(1 + \frac{x}{n}\right)^n \end{aligned}$$

All approximations become better as n increases.

- The approximation was discovered by Jacob Bernoulli (1655-1705) in order to apply to compound interest rate computations.

- In finance, compound interest is interest on a deposit which gets added automatically to the deposit so it earns additional interest from then on.
- The period in which this compounding process occurs is called compounding period.
- Annual compound interest rate of $k\%$ compounded once a year multiplies the current deposit by a factor of $\left(1+\frac{k}{100}\right)$.
- Therefore n years of annual compound interest rate of $k\%$ compounded once a year multiplies the original deposit by factor:

$$\underbrace{\underbrace{\underbrace{\left(1+\frac{k}{100}\right)}_{\text{after 1 year}}\cdot\left(1+\frac{k}{100}\right)}_{\text{after 2 years}}\cdot\dots\cdot\left(1+\frac{k}{100}\right)}_{\text{after } n \text{ years}}=\left(1+\frac{k}{100}\right)^n$$

Definition

The amount of money obtained from principal (original deposit) P after n years of annual compound interest rate of $k\%$, compounded once a year, is given by the formula

$$P\left(1+\frac{k}{100}\right)^n.$$

Example

You have 1000 USD kept at annual rate of 5%. The interest is compounded yearly. Approximate without using a calculator the amount of money you will have after 40 years. Check your approximation with a calculator.

Example

Decide, without using a calculator, which is more profitable: earning a yearly compound interest of 2% for 150 years or earning yearly simple interest of 11% for 150 years? Check your approximation with a calculator.

Example

When quickly computing interest rate “in the head”, financial advisors often use the following trick called the “rule of 72”. To find the time in years t needed for a sum to double under compound interest rate of $k\%$, financial advisors simply approximate $t \approx \frac{72}{k}$.
To illustrate the rule, under an interest rate of 2%, one needs approximately $\frac{72}{2} = 36$ years for the sum to double. Under interest rate of 6%, the sum doubles after only about $\frac{72}{6} = 12$ years. In 36 years an interest of 6% would double 3 times, in other words would increase by a factor of $2^3 = 8$.
Using the approximation $e \approx \left(1 + \frac{1}{n}\right)^n$ for large n, justify the rule of 72.