Calculus II

Series absolute convergence, the ratio and root tests

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2019

Outline

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Alternating Series

Definition (Alternating Series)

An alternating series is a series whose terms are alternately positive and negative.

Examples

Here are two examples:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$$

$$-\frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \frac{4}{5} - \frac{5}{6} + \frac{6}{7} - \cdots = \sum_{n=1}^{\infty} (-1)^{n} \frac{n}{n+1}$$

The nth term of an alternating series has the form

$$a_n = (-1)^{n-1} b_n \qquad \text{or} \qquad a_n = (-1)^n b_n$$

where b_n is positive.

Theorem (The Alternating Series Test)

If the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1 - b_2 + b_3 - b_4 + b_5 - \cdots, \qquad b_n > 0$$

satisfies

1. $b_{n+1} \leq b_n$ *for all* n *and*
2. $\lim_{n\to\infty} b_n = 0$

then the series is convergent.

Example

The alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

satisfies

1. $b_{n+1} < b_n$ because $\frac{1}{n+1} < \frac{1}{n}$.
2. $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} = 0$.

Therefore the series is convergent by the Alternating Series Test.

Example

The series $\sum_{n=1}^{\infty}(-1)^n\frac{3n}{4n-1}$ is alternating, but

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{3n}{4n-1}\cdot\frac{\frac{1}{n}}{\frac{1}{n}} = \lim_{n\to\infty}\frac{3}{4-\frac{1}{n}} = \frac{3}{4}$$

Therefore the series is divergent by the basic Divergence Test.

Estimating Sums

This theorem allows us to estimate the size of the remainder $R_n = s - s_n$ in an alternating series.

Theorem (Alternating Series Estimation Theorem)

Let $\sum (-1)^{n-1} b_n$ be the sum of an alternating series that satisfies

1. $0 \leq b_{n+1} \leq b_n$ *and*
2. $\lim\limits_{n\to\infty} b_n = 0.$

Then the size of the error is less than the first omitted term; that is,

$$|R_n| = |s - s_n| \leq b_{n+1}.$$

Example

Find the sum of $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ correct to three decimal places. ($0! = 1$.)

1. $b_{n+1} = \dfrac{1}{(n+1)!} = \dfrac{1}{n!(n+1)} < \dfrac{1}{n!} = b_n.$
2. $0 < \dfrac{1}{n!} < \dfrac{1}{n} \to 0$, so $b_n \to 0$ as $n \to \infty$.

- Therefore the series converges by the Alternating Series Test.

$$\begin{aligned} s &= \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \cdots \\ &= 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} - \frac{1}{5040} + \cdots \end{aligned}$$

- $|s - s_6| \leq b_7 = \frac{1}{5040} < 0.0002.$
- $s_6 = 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} \approx 0.368056.$
- The error of less than 0.0002 doesn't affect the third decimal place, so $s \approx s_6 \approx 0.368$.

Absolute Convergence and the Ratio and Root Tests

In this section, we start with any series $\sum a_n$ and consider the corresponding series

$$\sum |a_n| = |a_1| + |a_2| + |a_3| + \cdots$$

consisting of the absolute values of the terms of the original series.

Absolute Convergence

Definition (Absolutely Convergent)

A series $\sum a_n$ is called absolutely convergent if the series of absolute values $\sum |a_n|$ is convergent.

If $\sum a_n$ is a series with all positive terms, then $|a_n| = a_n$ and absolute convergence is the same thing as convergence in this case.

Example

The series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

is absolutely convergent because

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n^2} \right| = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

is a convergent p-series with $p = 2$.

Example

The alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

is convergent (by the alternating series test, as already demonstrated).

- Is it absolutely convergent?

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right| = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

- This is a p-series with $p = 1$.
- Therefore $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right|$ is divergent.
- Therefore $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is not absolutely convergent.

Definition (Conditionally Convergent)

A series $\sum a_n$ is called conditionally convergent if it is convergent but not absolutely convergent.

- The alternating harmonic series is conditionally convergent.
- Therefore it is possible for a series to be convergent but not absolutely convergent.
- Question: Is it possible for a series to be absolutely convergent but not convergent?
- Answer: No. This is the content of the next theorem.

Theorem (Absolute Convergence Implies Convergence)

If a series is absolutely convergent, then it is convergent.

Example

Determine whether

$$\sum_{n=1}^{\infty} \frac{\cos n}{n^2} = \frac{\cos 1}{1^2} + \frac{\cos 2}{2^2} + \frac{\cos 3}{3^2} + \frac{\cos 4}{4^2} + \cdots$$

is convergent or divergent.

- The series has positive and negative terms, but is not alternating.
- Use the Comparison Test:

$$\begin{array}{ccccc} 0 & \leq & \left|\cos n\right| & \leq & 1 \\ 0 & \leq & \dfrac{\left|\cos n\right|}{n^2} & \leq & \dfrac{1}{n^2} \end{array}$$

- $\sum \frac{1}{n^2}$ is a p-series with $p = 2$.
- Therefore $\sum \frac{1}{n^2}$ is convergent, and so by the Comparison Test, $\sum \frac{\left|\cos n\right|}{n^2}$ is also convergent.
- Therefore $\sum \frac{\cos n}{n^2}$ is absolutely convergent.
- Therefore by the previous theorem, $\sum \frac{\cos n}{n^2}$ is convergent.

The Ratio Test

Theorem (The Ratio Test)

1. *If* $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, *then the series* $\sum a_n$ *is absolutely convergent (and therefore convergent).*
2. *If* $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ *or* $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$, *then the series* $\sum a_n$ *is divergent.*
3. *If* $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L = 1$, *then the Ratio Test is inconclusive.*

The Ratio Test is inconclusive if $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = 1$.

Example

$$\sum_{n=1}^{\infty}\frac{1}{n^2}$$

- This is a p-series with $p = 2$.
- Therefore it is convergent.

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \frac{n^2}{(n+1)^2}\cdot\frac{\frac{1}{n^2}}{\frac{1}{n^2}} = \frac{1}{\left(1+\frac{1}{n}\right)^2} \to 1 \qquad \text{as } n\to\infty$$

Example

$$\sum_{n=1}^{\infty}\frac{1}{n}$$

- This is a p-series with $p = 1$.
- Therefore it is divergent.

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{1}{n+1}}{\frac{1}{n}} = \frac{n}{n+1}\cdot\frac{\frac{1}{n}}{\frac{1}{n}} = \frac{1}{1+\frac{1}{n}} \to 1 \qquad \text{as } n\to\infty$$

Example

Test the series $\sum_{n=1}^{\infty}(-1)^n \frac{n^3}{3^n}$ for absolute convergence.

$$
\begin{aligned}
\left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{(-1)^{n+1}\frac{(n+1)^3}{3^{n+1}}}{(-1)^n\frac{n^3}{3^n}}\right| \\
&= \frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3} \\
&= \frac{1}{3}\left(\frac{n+1}{n}\right)^3 \\
&= \frac{1}{3}\left(1+\frac{1}{n}\right)^3 \\
&\to \frac{1}{3} < 1
\end{aligned}
$$

Therefore the series is absolutely convergent by the Ratio Test.

Example

Test the convergence of the series $\sum\limits_{n=1}^{\infty} \frac{n^n}{3^n n!}$.

$$
\begin{aligned}
\left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{\frac{(n+1)^{n+1}}{3^{n+1}(n+1)!}}{\frac{n^n}{3^n n!}}\right| \\
&= \frac{(n+1)^{n+1}}{n^n} \cdot \frac{3^n n!}{3^{n+1}(n+1)!} \\
&= \frac{\cancel{(n+1)}(n+1)^n}{n^n} \cdot \frac{\cancel{3^n}\cancel{n!}}{3^{\cancel{n}+1}\cancel{(n+1)}\cancel{n!}} \\
&= \frac{1}{3}\left(\frac{n+1}{n}\right)^n = \frac{1}{3}\left(1+\frac{1}{n}\right)^n \\
&\to \frac{e}{3} < 1
\end{aligned}
$$

Therefore the series is convergent by the Ratio Test.

Example

Test the convergence of the series $\sum_{n=1}^{\infty} \frac{n^n}{n!}$.

$$\begin{aligned}
\left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{\frac{(n+1)^{n+1}}{(n+1)!}}{\frac{n^n}{n!}}\right| \\
&= \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} \\
&= \frac{(n+1)(n+1)^n}{(n+1)n!} \cdot \frac{n!}{n^n} \\
&= \left(\frac{n+1}{n}\right)^n = \left(1+\frac{1}{n}\right)^n \\
&\to e > 1
\end{aligned}$$

Therefore the series is divergent by the Ratio Test.

The Root Test

Theorem (The Root Test)

1. *If* $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, *then the series* $\sum a_n$ *is absolutely convergent (and therefore convergent).*
2. *If* $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ *or* $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, *then the series* $\sum a_n$ *is divergent.*
3. *If* $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L = 1$, *then the Root Test is inconclusive.*

If $L = 1$ in the Ratio Test, don't try the Root Test, because it will be inconclusive too.
If $L = 1$ in the Root Test, don't try the Ratio Test, because it will be inconclusive too.

Example

Test convergence of the series $\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$.

$$
\begin{aligned}
a_n &= \left(\frac{2n+3}{3n+2}\right)^n \\
\sqrt[n]{|a_n|} &= \frac{2n+3}{3n+2} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} \\
&= \frac{2+\frac{3}{n}}{3+\frac{2}{n}} \\
&\to \frac{2}{3} < 1
\end{aligned}
$$

Therefore the series is absolutely convergent by the Root Test.