Calculus I
Implicit derivatives, related rates

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2019

Outline

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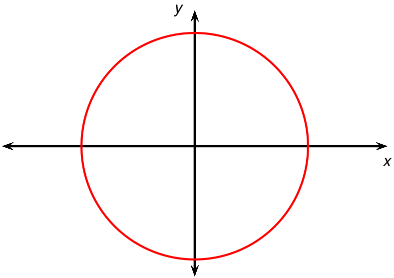
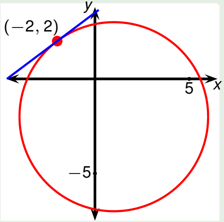
Implicit Differentiation

- So far, we have seen functions with formulas that express one varable explicitly in terms of the other.
- $y = \sqrt{x^3 + 1}$, $y = x \sin x$, etc.
- Some functions are given implicitly by a relation between x and y.
- $x^2 + y^2 = 1$ isn't the equation of any one function.
- Implicitly it gives two functions: $y = \sqrt{1 - x^2}$ and $y = -\sqrt{1 - x^2}$.
- How do we differentiate these functions?
- Differentiate both sides with respect to x, and then solve for y'.



Example

Find an equation of the tangent line to $(x-1)^2+(y+2)^2=25$ at $(-2,2)$.



Find $\frac{dy}{dx}$, given $(x-1)^2+(y+2)^2=25$:

$$\frac{d}{dx}\left((x-1)^2\right)+\frac{d}{dx}\left((y+2)^2\right)=\frac{d}{dx}(25)$$

$$2(x-1)\frac{d}{dx}(x-1)+2(y+2)\frac{d}{dx}(y+2)=0$$

$$2(x-1)(1)+2(y+2)\left(\frac{dy}{dx}\right)=0$$

$$2(y+2)\left(\frac{dy}{dx}\right)=2(1-x)$$

$$\frac{dy}{dx}=\frac{1-x}{y+2}$$

Plug in $(-2,2)$:

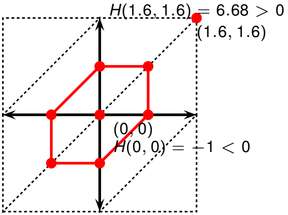
$$\frac{dy}{dx}=\frac{1-(-2)}{2+2}=\frac{3}{4}$$

Point-slope form:

$$y-2=\frac{3}{4}(x+2)$$

(Elementary Computer algorithm for sketching graphs)

Let H-continuous; is there simple algorithm to sketch $H(x, y) = 0$? Yes.



We illustrate the algorithm for:

$$x^2 + 2y^2 = 1$$
$$x^2 + 2y^2 - 1 = 0$$

Set $H(x, y) = x^2 + 2y^2 - 1$

- Elementary algorithm: fix large rectangle.
- Split the grid in triangular mesh. One strategy to do that is shown.
- For each triangle:
  - Fix two corners $P(x_P, y_P)$ and $Q(x_Q, y_Q)$.
  - If $H(x_P, y_P)$ and $H(x_Q, y_Q)$ have different sign then H must become zero somewhere on the segment between P and Q.
  - Select a point between P and Q and "guess" that H is zero there.
    - In our implementation, we select the midpoint (i.e., $\frac{1}{2}P + \frac{1}{2}Q$).
    - Connect the selected pts. for each triangle.
  - Repeat for ever finer grid.

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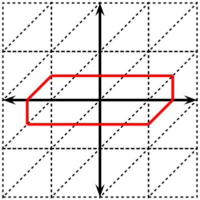

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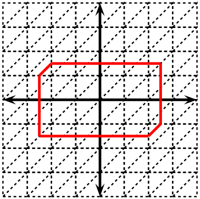

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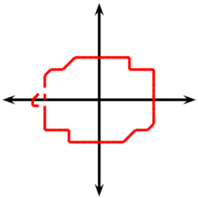

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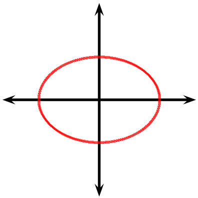

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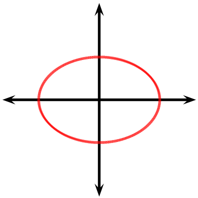

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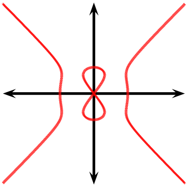

Illustrate the algorithm for:
$$y^2(y^2-3)=x^2(x^2-5)$$
$$H(x,y)=y^2(y^2-3)-x^2(x^2-5)$$

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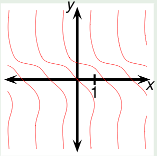
Example



Find y' as an expression of x and y.

$$
\begin{aligned}
\sin(2(x+y)) &= y^2\cos(2x).\\
\frac{d}{dx}(\sin(2(x+y))) &= \frac{d}{dx}(y^2\cos(2x))\\
\cos(2(x+y))\frac{d}{dx}(2(x+y)) &= \frac{d}{dx}(y^2)\cos(2x)+(y^2)\frac{d}{dx}(\cos(2x))\\
\cos(2(x+y))\left(2+2y'\right) &= 2yy'\cos(2x)+y^2(-\sin(2x))\frac{d}{dx}(2x)\\
2\cos(2(x+y))\left(1+y'\right) &= 2yy'\cos(2x)-y^2\sin(2x)2\\
\cos(2(x+y))+y'\cos(2(x+y)) &= yy'\cos(2x)-y^2\sin(2x)\\
y'\cos(2(x+y))-yy'\cos(2x) &= -\cos(2(x+y))-y^2\sin(2x)\\
y'(\cos(2(x+y))-y\cos(2x)) &= -\cos(2(x+y))-y^2\sin(2x)\\
y' &= \frac{-\cos(2(x+y))-y^2\sin(2x)}{\cos(2(x+y))-y\cos(2x)}.
\end{aligned}
$$

Example

Let $x^4 + y^4 = 16$. Find y''.

$$4x^3 + 4y^3 y' = 0$$

$$y' = -\frac{x^3}{y^3}.$$

$$y'' = \frac{\mathrm{d}}{\mathrm{d}x}\left(-\frac{x^3}{y^3}\right) = -\frac{\frac{\mathrm{d}}{\mathrm{d}x}\left(x^3\right)y^3 - x^3\frac{\mathrm{d}}{\mathrm{d}x}\left(y^3\right)}{\left(y^3\right)^2}$$

$$= -\frac{(3x^2)y^3 - x^3(3y^2y')}{y^6} = -\frac{3x^2y^3 - 3x^3y^2\left(-\frac{x^3}{y^3}\right)}{y^6}$$

$$= -\frac{3x^2(y^3 + \frac{x^4}{y})}{y^6} = -\frac{3x^2\left(\frac{y^4+x^4}{y}\right)}{y^6}$$

$$= -\frac{3x^2(y^4 + x^4)}{y^7} = -\frac{3x^2(16)}{y^7} = -48\frac{x^2}{y^7}.$$

Related Rates

- Suppose we are pumping a balloon with air.
- The balloon's volume is increasing.
- The balloon's radius is increasing.
- The rates of increase of these quantities are related to one another.
- It is easier to measure the rate of increase of volume.
- In a related rates problem, we compute the rate of change of one quantity in terms of the rate of change of another (which may be more easily measured).
- Procedure:
  1. Find an equation relating the two quantities.
  2. Use the Chain Rule to differentiate both sides with respect to time.

Example

Air is being pumped into a balloon such that its volume changes at a rate of 100 cm^3/s. How fast is the radius of the balloon increasing when the diameter is 50 cm?

- Let V denote the balloon's volume.
- Let r denote its radius.
- Given: $\frac{dV}{dt} = 100$ cm^3/s.
- Unknown: $\frac{dr}{dt}$ when $r = 25$ cm.

1. Find an equation relating the two quantities.
2. Use the Chain Rule to differentiate both sides.

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dt} = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right)$$

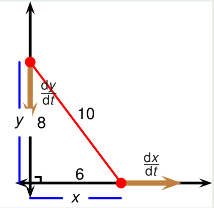
$$\frac{dV}{dt} = \frac{d}{dr}\left(\frac{4}{3}\pi r^3\right)\frac{dr}{dt}$$

$$\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{1}{4\pi r^2}\frac{dV}{dt}$$

$$\frac{dr}{dt} = \frac{1}{4\pi(25\text{cm})^2}100\frac{\text{cm}^3}{\text{s}} = \frac{1}{25\pi}\text{cm/s}$$

Example



10 ft ladder rests against a vertical wall. The bottom of the ladder slides away from the wall at a rate of 1 ft/s. How fast is the ladder top sliding down when the bottom is 6 ft from the wall?

- Let y= dist. from top to ground.
- Let x= dist. from bottom to wall.
- Given: $\frac{dx}{dt} = 1$ ft/s.
- Unknown: $\frac{dy}{dt}$ when $x = 6$ ft.
- Pythagorean Therem: $y = \sqrt{10^2 - 6^2} = 8$.
- Relationship b/n quantities.
- Differentiate (use Chain Rule).

$$x^2 + y^2 = 10^2 = 100$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}$$

$$\frac{dy}{dt} = -\frac{6\text{ ft}}{8\text{ ft}} \cdot 1\text{ ft/s}$$

$$= -3/4\text{ ft/s}.$$

Therefore the top of the ladder is falling at a rate of $3/4$ ft/s.