Arithmetics
Division and fractions
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Definition (Division, exact)

To divide a number p (**dividend**) by a number d (**divisor**) means to find a number q (**quotient**) so that

$$q \cdot d = p$$

Example

Divide 5 by 3.

$$\text{answer} = \frac{5}{3}$$

Observation

The quotient of two integers equals the fraction formed by putting the dividend as the numerator and the divisor as the denominator.

$$q = \frac{p}{d}$$

- Recall exact division: $$\begin{aligned} p' &= q' \cdot d' \\ q' &= \frac{p'}{d'} \end{aligned}$$
- Quotient may fail to reduce to an integer.
- What if we want an integer quotient?

Definition (Integer division with remainder)

To **divide** an integer $p > 0$ by an integer $d > 0$ **with remainder** $r \geq 0$ means to find the largest integer $q \geq 0$ and the smallest $0 \leq r$ so that:

$$p = q \cdot d + r$$

p is called the **dividend**, d is called the **divisor**, q is called the **quotient** and r is called the **remainder**.

Example

Divide 7 by 3 with remainder. $7 = 2 \cdot 3 + 1$.

- Differences between exact division integer division.
    - Integer division quotient is integer, exact division quotient is fraction.
    - Exact division: no notion of remainder.

- Recall integer division of p by d with remainder: $p = q \cdot d + r$.
- Integer division of p by d answers the following question:

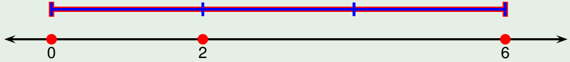
Question

How many times does the length d fit inside the length p?

- p may be divisible by d an exact number of times.
- p may fail to be integer-divisible by d, then we have remainder.

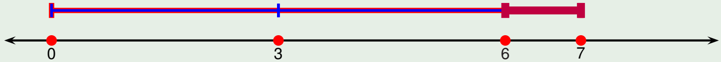
Example

Divide 6 by 2 with remainder. Solution: $6 = 3 \cdot 2 + 0$.



2 divides 6 exactly 3 times.

Divide 7 by 3. Solution: $7 = 2 \cdot 3 + 1$.



3 divides 7 non-exactly 2 times with remainder 1.

Example (Divisor 1-digit, quotient 1-digit)

Divide 7 by 2 with remainder. To solve, we need to answer: what is the largest integer which, when multiplied by 2, stays smaller than 7? Try:

$$\begin{array}{rcl} 0\cdot 2 &=& 0\\ 1\cdot 2 &=& 2\\ 2\cdot 2 &=& 4\\ 3\cdot 2 &=& 6\\ 4\cdot 2 &=& 8>7 \end{array}$$

$\Rightarrow 7=3\cdot 2+x$. Solve: $x=7-3\cdot 2=1$. Therefore $7=3\cdot 2+1$.

Observation (Question to answer when dividing with remainder)

What is the largest integer which, when multiplied by d, remains smaller than p?

- To answer this question, we guess quickly as shown above.
- Later on we learn to divide large numbers without guessing.
- However, we still need the guessing approach as a building block of the complete division algorithm.

Example (Integer division: 1-digit dividend, 1-digit divisor)

Divide with remainder:

$$
\begin{array}{rrcl}
5 \text{ by } 2: & 5 & = & 2\cdot 2+1\\
3 \text{ by } 5: & 5 & = & 0\cdot 5+3\\
8 \text{ by } 4: & 5 & = & 2\cdot 4\\
9 \text{ by } 8: & 5 & = & 1\cdot 8+1\\
3 \text{ by } 1: & 3 & = & 3\cdot 1\\
9 \text{ by } 2: & 6 & = & 4\cdot 2+1
\end{array}
$$

All quotients are known to be one-digit numbers.

Example (Integer division: 1-digit divisor, 1-digit quotient)

Divide with remainder:

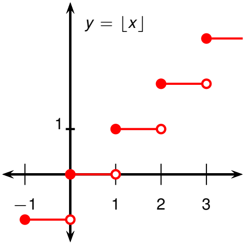
$$\begin{aligned}
12 \text{ by } 6: &\quad 12 &=& \quad 2\cdot 6\\
14 \text{ by } 5: &\quad 14 &=& \quad 2\cdot 5+4\\
37 \text{ by } 5: &\quad 37 &=& \quad 7\cdot 5+2\\
40 \text{ by } 9: &\quad 40 &=& \quad 4\cdot 9+4\\
49 \text{ by } 7: &\quad 49 &=& \quad 7\cdot 7\\
57 \text{ by } 8: &\quad 57 &=& \quad 7\cdot 8+1\\
67 \text{ by } 7: &\quad 67 &=& \quad 9\cdot 7+4\\
82 \text{ by } 9: &\quad 82 &=& \quad 9\cdot 9+1
\end{aligned}$$

All quotients are known to be one-digit numbers.

Definition (Greatest Integer Function)

The *greatest integer function* $\lfloor x \rfloor$ is defined as the largest integer that is less than or equal to x.

In computer science this function is called the *floor* function, also the *round-down* function.



$$\begin{aligned} \lfloor 4 \rfloor &= 4 \\ \lfloor 4.8 \rfloor &= 4 \\ \lfloor 1.5 \rfloor &= \lfloor 1 + 0.5 \rfloor = 1 \end{aligned}$$

$$\left\lfloor 1 + \frac{1}{2} \right\rfloor = 1$$

$$\left\lfloor \frac{3}{2} \right\rfloor = \left\lfloor \frac{2+1}{2} \right\rfloor = \left\lfloor \frac{2}{2} + \frac{1}{2} \right\rfloor = \left\lfloor 1 + \frac{1}{2} \right\rfloor = 1$$

$$\lfloor \pi \rfloor = \lfloor 3.1415\ldots \rfloor = 3$$

Example

Compute the floor (round-down) function.

$$
\begin{aligned}
\left\lfloor \frac{1}{3} \right\rfloor &= 0\\
\left\lfloor \frac{6}{7} \right\rfloor &= 0\\
\left\lfloor 2+\frac{3}{11} \right\rfloor &= 2+\left\lfloor \frac{3}{11} \right\rfloor = 2\\
\left\lfloor 10+\frac{1}{7} \right\rfloor &= 10+\left\lfloor \frac{1}{7} \right\rfloor = 10
\end{aligned}
$$

Observation

$$\left\lfloor \frac{p}{q} \right\rfloor = 0 \qquad \textit{whenever } \ 0 \leq p < q$$

Observation

$$\lfloor n+x \rfloor = n + \lfloor x \rfloor \qquad \textit{whenever n is integer}$$

Example

Compute the floor (round-down) of $\frac{8}{3}$.

$$\left\lfloor\frac{8}{3}\right\rfloor = \left\lfloor\frac{2\cdot 3+2}{3}\right\rfloor$$

Divide 8 by 3 with remainder. Try:

$$\begin{aligned} 0\cdot 3 &= 0\\ 1\cdot 3 &= 3\\ 2\cdot 3 &= 6\\ 3\cdot 3 &= 9>8 \end{aligned}$$

$$= \left\lfloor\frac{2\cdot \cancel{3}}{\cancel{3}}+\frac{2}{3}\right\rfloor$$

$$= \left\lfloor 2+\frac{2}{3}\right\rfloor = 2$$

because $2\leq 2+\frac{2}{3}<3$

Observation

The floor (round-down) of $\frac{p}{q}$ is computed as

$$\left\lfloor\frac{p}{d}\right\rfloor = q,$$

where q is the the quotient obtained by integer division of p by d.

Example

Compute the floor (round-down) function.

$$
\begin{aligned}
\left\lfloor \frac{4}{3} \right\rfloor &= \left\lfloor \frac{1\cdot 3+1}{3} \right\rfloor = \left\lfloor \frac{1\cdot 3}{3} + \frac{1}{3} \right\rfloor = \left\lfloor 1 + \frac{1}{3} \right\rfloor = 1\\
\left\lfloor \frac{15}{2} \right\rfloor &= \left\lfloor \frac{7\cdot 2+1}{2} \right\rfloor = \left\lfloor \frac{7\cdot 2}{2} + \frac{1}{2} \right\rfloor = \left\lfloor 7 + \frac{1}{2} \right\rfloor = 7\\
\left\lfloor \frac{24}{4} \right\rfloor &= \left\lfloor \frac{6\cdot 4}{4} \right\rfloor = \lfloor 6 \rfloor = 6\\
\left\lfloor \frac{43}{5} \right\rfloor &= \left\lfloor \frac{8\cdot 5+3}{5} \right\rfloor = \left\lfloor \frac{8\cdot 5}{5} + \frac{3}{5} \right\rfloor = \left\lfloor 8 + \frac{3}{5} \right\rfloor = 8\\
\left\lfloor \frac{56}{7} \right\rfloor &= \left\lfloor \frac{8\cdot 7}{7} \right\rfloor = \lfloor 8 \rfloor = 8\\
\left\lfloor \frac{79}{8} \right\rfloor &= \left\lfloor \frac{9\cdot 8+7}{8} \right\rfloor = \left\lfloor \frac{9\cdot 8}{8} + \frac{7}{8} \right\rfloor = \left\lfloor 9 + \frac{7}{8} \right\rfloor = 9\\
\left\lfloor \frac{80}{9} \right\rfloor &= \left\lfloor \frac{8\cdot 9+8}{9} \right\rfloor = \left\lfloor \frac{8\cdot 9}{9} + \frac{8}{9} \right\rfloor = \left\lfloor 8 + \frac{8}{9} \right\rfloor = 8
\end{aligned}
$$