Arithmetics
Fraction basics
calculator-algebra.org

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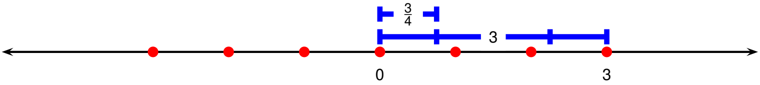
Division and the number line

- Numbers represent lengths by measuring distances.



- A segment can be divided into equal parts.
- The parts may no longer have lengths that are exact integers.
- Such lengths are represented by fractions.
- These lengths are called **fractions** and are denoted by:

$$\frac{a}{b} \qquad \text{or} \qquad a/b$$

- Number on top: **numerator**, represents length of divided segment.
- Number on bottom: **denominator**, represents the number of segments we are dividing into.

Observation

For any number a, the following equality holds.

$$\frac{a}{1} = a$$

Example

Simplify.

$$\begin{aligned} \frac{5}{1} &= 5 \\ \frac{-3}{1} &= -3 \end{aligned}$$

- Division by one corresponds to “division of a segment into one equal part”.
- “Division of a segment into one equal part” is interpreted as not dividing the segment at all.

Example (Reading fractions)

Read the fraction. Honor your English dialect naming convention, if different from the one given here.

- $\frac{1}{3}$ one third (can also say “a third”).
- $\frac{1}{2}$ one half.
- $\frac{1}{4}$ one quarter; one fourth.
- $\frac{3}{4}$ three quarters, three fourths.
- $\frac{6}{7}$ six sevenths.
- $\frac{11}{6}$ eleven sixths.

- Most frequently, a fraction $\frac{a}{b}$ is read along the template “*a* *b*-th(s)”.
- Most important exceptions: $\dfrac{1}{2}$ is read as “one half” (or just “half”). $\dfrac{1}{4}$ is read as both “one fourth” and as ”one quarter”.

Observation

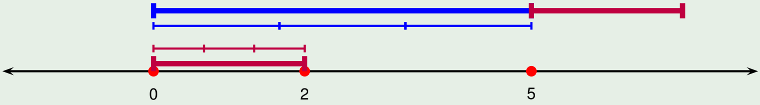
Fractions with same denominator are added by adding their numerators.

$$\frac{a}{c} \pm \frac{b}{c} = \frac{a \pm b}{c}$$

A similar rule holds for subtraction.

Example

$$\begin{aligned} \frac{2}{3} + \frac{5}{3} &= \frac{2+5}{3} = \frac{7}{3} \\ \frac{4}{3} - \frac{1}{3} &= \frac{4-1}{3} = \frac{3}{3} \end{aligned}$$



Example (Add numbers with same denominator)

$$
\begin{aligned}
\frac{1}{4}+\frac{2}{4} &= \frac{3}{4}\\
\frac{7}{6}+\frac{10}{6} &= \frac{17}{6}\\
\frac{302}{111}+\frac{24}{111} &= \frac{326}{111}\\
\frac{3}{6}-\frac{2}{6} &= \frac{1}{6}\\
\frac{103}{101}-\frac{4}{101} &= \frac{99}{101}\\
\frac{3}{8}-\frac{6}{8} &= \frac{-3}{8}=-\frac{3}{8}\\
\frac{11}{13}-\frac{20}{13} &= \frac{-9}{13}=-\frac{9}{13}
\end{aligned}
$$

Definition (Factor a number (properly))

To factor an integer a (properly) means to find integers $b > 1$ and $c > 1$ so that

$$a = \pm b \cdot c$$

The numbers b and c are called **factors.**

Example (Proper factorization)

$$\begin{array}{rcl} 4 &=& 2 \cdot 2 \\ 6 &=& 3 \cdot 2 = 2 \cdot 3 \\ -8 &=& -2 \cdot 4 = -4 \cdot 2 = -2 \cdot 2 \cdot 2 \end{array}$$

Example (Not a proper factorization)

$$\begin{array}{rcl} -3 &=& (-1) \cdot 3 \\ 4 &=& 1 \cdot 4 \\ 1 &=& 2 \cdot \frac{1}{2} \\ 1 &=& 0.25 \cdot 4 \end{array}$$

Definition

An integer greater than one is prime if it cannot be factored properly.

- 0 and 1 are not prime by definition.
- Whether negatives can be prime is usually left undefined.
  - Both options could be made to make sense.
  - To avoid confusion: avoid question of prime negative numbers.

Example

	Prime?	Full factorization		Prime?	Full factorization
1	no	-	9	no	$3 \cdot 3$
2	yes	2	10	no	$2 \cdot 5$
3	yes	3	11	yes	11
4	no	$2 \cdot 2$	12	no	$2 \cdot 2 \cdot 3$
5	yes	5	13	yes	13
6	no	$2 \cdot 3$	14	no	$2 \cdot 7$
7	yes	7	15	no	$3 \cdot 5$
8	no	$2 \cdot 2 \cdot 2$	16	no	$2 \cdot 2 \cdot 2 \cdot 2$

Definition (Factor positive integer completely)

To factor a positive integer completely means to write it as a product of prime factors.

$$x = p_1 \cdot p_2 \cdots p_n$$

- It is best practice to sort (order) the prime factors. Most frequently used order: smaller factors come first.

Lemma (Unique prime factorization)

Up to shuffling prime factors, there is only one way to factor a number.

- Consequence: two numbers are equal if and only if their sorted prime factorizations are equal.
- When factoring, we may or may not use exponent notation:

$$\begin{aligned} 16 &= 4 \cdot 4 = \underbrace{2 \cdot 2 \cdot 2 \cdot 2}_{\text{4 copies}} = 2^4 \\ 36 &= 4 \cdot 9 = 2 \cdot 2 \cdot 3 \cdot 3 = 2^2 \cdot 3^2 \end{aligned}$$

Example

Factor the number completely. If applicable, show two answers: with and without exponent notation (x^3 vs $x \cdot x \cdot x$).

$$
\begin{aligned}
4 &= 2 \cdot 2 = 2^2 \\
6 &= 2 \cdot 3 \\
7 &= \text{already factored (prime number)} \\
8 &= 2 \cdot 4 = 2 \cdot 2 \cdot 2 = 2^3 \\
9 &= 3 \cdot 3 = 3^2 \\
15 &= 3 \cdot 5 \\
24 &= 8 \cdot 3 = 2 \cdot 4 \cdot 3 = 2 \cdot 2 \cdot 2 \cdot 3 = 2^3 \cdot 3 \\
36 &= 6 \cdot 6 = 2 \cdot 3 \cdot 2 \cdot 3 = 2 \cdot 2 \cdot 3 \cdot 3 = 2^2 \cdot 3^2 \\
52 &= 4 \cdot 13 = 2 \cdot 2 \cdot 13 = 2^2 \cdot 13 \\
67 &= \text{already factored (prime number)} \\
91 &= 7 \cdot 13
\end{aligned}
$$

Definition (Factor a number)

To factor an integer a means to find integers $b > 1$ and $c > 1$ so that

$$a = \pm b \cdot c$$

We say that b, c are **factors** of a.

Definition (Prime number)

A number is prime if it cannot be factored.

Definition (Complete factorization)

To find a complete factorization of an integer a means to find prime numbers $p_1 > 1, p_2 > 1, \dots, p_n > 1$ with

$$a = \pm p_1 \cdot p_2 \cdots p_n$$

- Is the number 67414977753059 prime? No

$$\underbrace{67414977753059}_{\text{not prime}} = \underbrace{11494253}_{\text{prime}} \cdot \underbrace{5865103}_{\text{prime}}$$

- Even when x is large, there exist fast computer algorithms to check whether x is prime.
- In other words, there exist fast algorithms for knowing whether a proper factorization exists.
- However, even when we know x can be factored, as of 2019, there are no known fast computer algorithms for finding an actual factorization.
- In fact, the number above was generated by first making two large primes and then multiplying them.
- Each of the two known large primes above was in turn generated by trying large integers at random.

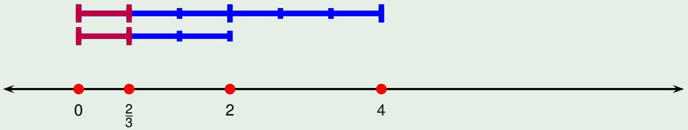
Observation

Fractions do not change when we multiply their numerator and denominator by the same number.

$$\frac{a}{b} = \frac{x \cdot a}{x \cdot b}$$

Example

$$\begin{aligned}\frac{2}{3} &= \frac{2\cdot 2}{2\cdot 3} = \frac{4}{6}\\ &= \frac{3\cdot 2}{3\cdot 3} = \frac{6}{9}\end{aligned}$$



Definition (Reduce a fraction)

A positive fraction $\frac{a}{b}$ is a reduction of a positive fraction $\frac{A}{B}$ when

$$\frac{A}{B} = \frac{a}{b}$$

and $\frac{a}{b}$ has smaller numerator and denominator, i.e., $A > a$ and $B > b$.

Recall that $\frac{x \cdot a}{x \cdot b} = \frac{a}{b}$.

Example

Reduce the fraction.

$$\begin{aligned}
\frac{4}{6} &= \frac{2\cdot 2}{2\cdot 3} = \frac{2}{3}\\
\frac{2}{4} &= \frac{2\cdot 1}{2\cdot 2} = \frac{1}{2}\\
\frac{3}{9} &= \frac{3\cdot 1}{3\cdot 3} = \frac{1}{3}
\end{aligned}$$

- To reduce a fraction, we use the rule:

$$\frac{x \cdot a}{x \cdot b} = \frac{\cancel{x} \cdot a}{\cancel{x} \cdot b} = \frac{a}{b}.$$

- To reduce excessive copying: use cancel notation.
- The uses of the cancel notation will become apparent in examples.
- Rules.
  - Use a single slanted line.
  - Unless circumstances dictate otherwise, slant your line from lower left corner to top right corner.
  - Do not use crosses, smudges, or any other notation that obscures the expression below the cancel line.

Example

Simplify (reduce) the fraction.

$$
\begin{aligned}
\frac{9}{12} &= \frac{\cancel{3}\cdot 3}{\cancel{3}\cdot 4} = \frac{3}{4}\\
\frac{14}{6} &= \frac{\cancel{2}\cdot 7}{\cancel{2}\cdot 3} = \frac{7}{3}\\
\frac{6}{2} &= \frac{\cancel{2}\cdot 3}{\cancel{2}\cdot 1} = \frac{3}{1} = 3\\
\frac{6}{5} &= \text{already simplified: } \frac{6}{5} = \frac{3\cdot 2}{5}, \text{ no primes to cancel}\\
\frac{10}{15} &= \frac{2\cdot \cancel{5}}{3\cdot \cancel{5}} = \frac{2}{3}\\
\frac{8}{6} &= \frac{\cancel{2}\cdot 4}{\cancel{2}\cdot 3} = \frac{4}{3}
\end{aligned}
$$

Lemma

If there are no common prime factors between the numerator and the denominator, the fraction is reduced.

- There exists an algorithm for simplifying every fraction.
- The algorithm is considerably different from what we exercised so far: it does not require us to factor the numerator and denominator.
- Instead, the algorithm involves the notion of a greatest common divisor (GCD).
- We will study greatest common divisors (GCD) and the fraction simplification GCD algorithm later.
- That fraction simplification GCD algorithm is well-suited for and very fast on a computer.
- This algorithm is also practical for use by hand.
- However on small examples the factorization-cancellation guess-work technique shown in examples is faster for a human.

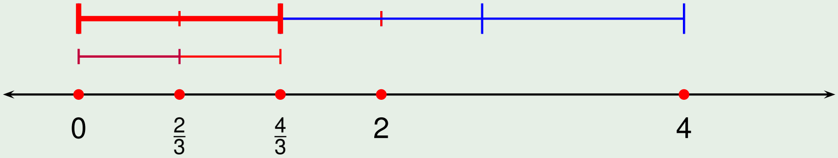
Observation

$$x \cdot \frac{a}{b} = \frac{x \cdot a}{b}$$

Multiplying a fraction by a number is equivalent to multiplying its numerator by that number.

Example

$$2 \cdot \frac{2}{3} = \frac{4}{3}$$



Example

Simplify the expression to a single reduced fraction.

$$
\begin{array}{rcl}
2 \cdot \frac{2}{3} & = & \frac{2\cdot 2}{3} = \frac{4}{3} \\
3 \cdot \frac{2}{15} & = & \frac{\cancel{3}\cdot 2}{\cancel{3}\cdot 5} = \frac{2}{5} \\
3 \cdot \frac{1}{3} & = & \frac{\cancel{3}}{\cancel{3}} = 1 \\
7 \cdot \frac{3}{21} & = & \frac{7\cdot 3}{21} = \frac{\cancel{21}}{\cancel{21}} = 1 \\
\text{[alternatively]} & = & 7 \cdot \frac{\cancel{3}}{\cancel{3}\cdot 7} = 7 \cdot \frac{1}{7} = \frac{\cancel{7}}{\cancel{7}} = 1 \\
6 \cdot \frac{2}{15} & = & \frac{6\cdot 2}{15} = \frac{2\cdot \cancel{3}\cdot 2}{\cancel{3}\cdot 5} = \frac{4}{5} \\
4 \cdot \frac{5}{18} & = & \frac{4\cdot 5}{3\cdot 6} = \frac{2\cdot \cancel{2}\cdot 5}{3\cdot \cancel{2}\cdot 3} = \frac{2\cdot 5}{3\cdot 3} = \frac{10}{9}
\end{array}
$$