Calculus II
Homework
Building block integrals

1. Integrate. Illustrate the steps of your solution.

(a) $\displaystyle\int \frac{1}{x+1}\mathrm{d}x$

answer: $\ln|x+1| + C$

(b) $\displaystyle\int \frac{x-1}{x+1}\mathrm{d}x$

answer: $x - 2\ln|x+1| + C$

(c) $\displaystyle\int \frac{1}{(x+1)^2}\mathrm{d}x$

answer: $-\frac{1}{x+1} + C$

(d) $\displaystyle\int \frac{x}{(x+1)^2}\mathrm{d}x$

answer: $\ln|x+1| + \frac{1}{x+1} + C$

(e) $\displaystyle\int \frac{1}{(2x+3)^2}\mathrm{d}x$

answer: $-\frac{1}{2(2x+3)} + C$

(f) $\displaystyle\int \frac{x}{2x^2+3}\mathrm{d}x$

answer: $\frac{1}{4}\ln\left(2x^2+3\right) + C$

(g) $\displaystyle\int \frac{1}{2x^2+3}\mathrm{d}x$

answer: $\frac{1}{\sqrt{6}}\arctan\left(\sqrt{\frac{2}{3}}x\right)$

(h) $\displaystyle\int \frac{x}{2x^2+x+1}\mathrm{d}x$.

answer: $\frac{1}{4}\ln\left(x^2+\frac{1}{2}x+\frac{1}{2}\right) - \frac{\sqrt{7}}{14}\arctan\left(\frac{4x+1}{\sqrt{7}}\right) + C$

(i) $\displaystyle\int \frac{x}{2x^2+x+3}\mathrm{d}x$

answer: $\frac{1}{4}\ln\left(2x^2+x+3\right) - \frac{1}{2\sqrt{23}}\arctan\left(\frac{4x+1}{\sqrt{23}}\right) + C$

(j) $\displaystyle\int \frac{x}{x^2-x+3}\mathrm{d}x$

answer: $\frac{1}{2}\ln\left|x^2-x+3\right| + \frac{1}{\sqrt{11}}\arctan\left(\frac{x-\frac{1}{2}}{\frac{\sqrt{11}}{2}}\right) + C$

(k) $\displaystyle\int \frac{1}{\left(x^2+1\right)^2}\mathrm{d}x$

answer: $\frac{1}{2}x\left(x^2+1\right)^{-1} + \frac{1}{2}\arctan(x) + C$

(l) $\displaystyle\int \frac{1}{\left(x^2+x+1\right)^2}\mathrm{d}x$

answer: $\frac{2}{3}x\left(x^2+x+1\right)^{-1} + \frac{1}{3}\left(x^2+x+1\right)^{-1} + \frac{4}{9}\sqrt{3}\arctan\left(\frac{2}{3}\sqrt{3}x + \frac{\sqrt{3}}{3}\right)$

(m) $\displaystyle\int \frac{1}{\left(x^2+1\right)^3}\mathrm{d}x$

answer: $\frac{3}{8}x\left(x^2+1\right)^{-1} + \frac{1}{4}x\left(x^2+1\right)^{-2} + \frac{3}{8}\arctan(x) + C$

Solution. 1.h.

$$
\begin{aligned}
\int \frac{x}{2x^2+x+1}\mathrm{d}x &= \int \frac{x}{2\left(x^2+2x\frac{1}{4}+\frac{1}{2}\right)}\mathrm{d}x \\
&= \int \frac{x}{2\left(x^2+2x\frac{1}{4}+\frac{1}{16}-\frac{1}{16}+\frac{1}{2}\right)}\mathrm{d}x && \text{complete square in denominator} \\
&= \frac{1}{2}\int \frac{x}{\left(x+\frac{1}{4}\right)^2+\frac{7}{16}}\mathrm{d}x \\
&= \frac{1}{2}\int \frac{x+\frac{1}{4}-\frac{1}{4}}{\left(x+\frac{1}{4}\right)^2+\frac{7}{16}}\mathrm{d}\left(x+\frac{1}{4}\right) && \text{Set } u = x+\tfrac{1}{4} \\
&= \frac{1}{2}\int \frac{u-\frac{1}{4}}{u^2+\frac{7}{16}}\mathrm{d}u \\
&= \frac{1}{2}\left(\int \frac{u}{u^2+\frac{7}{16}}\mathrm{d}u - \frac{1}{4}\int \frac{1}{u^2+\frac{7}{16}}\mathrm{d}u\right) \\
&= \frac{1}{2}\left(\frac{1}{2}\ln\left(u^2+\frac{7}{16}\right) - \frac{1}{4\sqrt{\frac{7}{16}}}\arctan\left(\frac{u}{\sqrt{\frac{7}{16}}}\right)\right) + K \\
&= \frac{1}{4}\ln\left(x^2+\frac{1}{2}x+\frac{1}{2}\right) - \frac{\sqrt{7}}{14}\arctan\left(\frac{4x+1}{\sqrt{7}}\right) + K \quad .
\end{aligned}
$$

Solution. 1.1

$$
\begin{aligned}
\int \frac{1}{\left(x^2+x+1\right)^2}\mathrm{d}x &= \int \frac{1}{\left(\left(x^2+2x\frac{1}{2}+\frac{1}{4}\right)-\frac{1}{4}+1\right)^2}\mathrm{d}x && \text{complete the square}\\
&= \int \frac{1}{\left(\left(x+\frac{1}{2}\right)^2+\frac{3}{4}\right)^2}\mathrm{d}\left(x+\frac{1}{2}\right) && \text{Set } w = x+\tfrac{1}{2}\\
&= \int \frac{1}{\left(w^2+\frac{3}{4}\right)^2}\mathrm{d}w\\
&= \int \frac{1}{\left(\frac{3}{4}\left(\left(\frac{2w}{\sqrt{3}}\right)^2+1\right)\right)^2}\frac{\sqrt{3}}{2}\mathrm{d}\left(\frac{2w}{\sqrt{3}}\right) && \text{Set } z = \tfrac{2w}{\sqrt{3}}\\
&= \frac{\frac{\sqrt{3}}{2}}{\left(\frac{3}{4}\right)^2}\int \frac{1}{\left(z^2+1\right)^2}\mathrm{d}z\\
&= \frac{8\sqrt{3}}{9}\int \frac{1}{\left(z^2+1\right)^2}\mathrm{d}z \quad .
\end{aligned}
$$

The integral $\int \frac{1}{(z^2+1)^2}\mathrm{d}z$ was already studied; it was also given as an exercise in Problem 1.k. We leave the rest of the problem to the reader.

2. Let a, b, c, A, B be real numbers. Suppose in addition $a \neq 0$ and $b^2 - 4ac < 0$. Integrate

$$\int \frac{Ax+B}{ax^2+bx+c}\mathrm{d}x \quad .$$

The purpose of this exercise is to produce a formula in form ready for implementation in a computer algebra system.

Solution. 2.

$$
\begin{aligned}
\int \frac{Ax+B}{ax^2+bx+c}\mathrm{d}x &= \int \frac{Ax+B}{a\left(x^2+2x\frac{b}{2a}+\frac{c}{a}\right)}\mathrm{d}x\\
&= \int \frac{Ax+B}{a\left(x^2+2x\frac{b}{2a}+\frac{b^2}{4a^2}-\frac{b^2}{4a^2}+\frac{c}{a}\right)}\mathrm{d}x && \text{complete square in denominator}\\
&= \frac{1}{a}\int \frac{Ax+B}{\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}}\mathrm{d}x && \text{Set } D = \tfrac{4ac-b^2}{4a^2}\\
&= \frac{1}{a}\int \frac{A\left(x+\frac{b}{2a}-\frac{b}{2a}\right)+B}{\left(x+\frac{b}{2a}\right)^2+D}\mathrm{d}\left(x+\frac{b}{2a}\right) && \text{Set } u = x+\tfrac{b}{2a}\\
&= \frac{1}{a}\int \frac{Au+B-\frac{Ab}{2a}}{u^2+D}\mathrm{d}u && \text{Set } C = B-\tfrac{Ab}{2a}\\
&= \frac{1}{a}\left(A\int \frac{u}{u^2+D}\mathrm{d}u + C\int \frac{1}{u^2+D}\mathrm{d}u\right)\\
&= \frac{1}{a}\left(\frac{A}{2}\ln(u^2+D)+\frac{C}{\sqrt{D}}\arctan\left(\frac{u}{\sqrt{D}}\right)\right)+K\\
&= \frac{1}{a}\left(\frac{A}{2}\ln\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)\right.\\
&\quad \left.+\frac{C}{\sqrt{D}}\arctan\left(\frac{x+\frac{b}{2a}}{\sqrt{D}}\right)\right)+K.
\end{aligned}
$$

The solution is complete. Question to the student: where do we use $b^2-4ac<0$?

3. Let a, b, c, A, B be real numbers and let $n > 1$ be an integer. Suppose in addition $a \neq 0$ and $b^2 - 4ac < 0$. Let

$$J(n) = \int \frac{1}{\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)^n}\mathrm{d}x \quad .$$

(a) Express the integral

$$\int \frac{Ax+B}{\left(ax^2+bx+c\right)^n}\mathrm{d}x$$

via $J(n)$.

(b) Express $J(n)$ recursively via $J(n-1)$

The purpose of this exercise is to produce a formula in form ready for implementation in a computer algebra system.

Solution. 3.a.

$$
\begin{aligned}
\int \frac{Ax+B}{(ax^2+bx+c)^n}\mathrm{d}x &= \int \frac{Ax+B}{a^n\left(x^2+2x\frac{b}{2a}+\frac{c}{a}\right)^n}\mathrm{d}x \\
&= \int \frac{Ax+B}{a^n\left(x^2+2x\frac{b}{2a}+\frac{b^2}{4a^2}-\frac{b^2}{4a^2}+\frac{c}{a}\right)^n}\mathrm{d}x && \text{complete square in denominator} \\
&= \frac{1}{a^n}\int \frac{Ax+B}{\left(\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}\right)^n}\mathrm{d}x && \text{Set } D=\frac{4ac-b^2}{4a^2} \\
&= \frac{1}{a^n}\int \frac{A\left(x+\frac{b}{2a}-\frac{b}{2a}\right)+B}{\left(\left(x+\frac{b}{2a}\right)^2+D\right)^n}\mathrm{d}\left(x+\frac{b}{2a}\right) && \text{Set } u=x+\frac{b}{2a} \\
&= \frac{1}{a^n}\int \frac{Au+B-\frac{Ab}{2a}}{(u^2+D)^n}\mathrm{d}u && \text{Set } C=B-\frac{Ab}{2a} \\
&= \frac{1}{a^n}\left(A\int \frac{u}{(u^2+D)^n}\mathrm{d}u + C\int \frac{1}{(u^2+D)^n}\mathrm{d}u\right) \\
&= \frac{1}{a^n}\left(\frac{A}{2(1-n)}\left(u^2+D\right)^{1-n}+CJ(n)\right) \\
&= \frac{1}{a^n}\left(\frac{A}{2(1-n)}\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)^{1-n}+CJ(n)\right)
\end{aligned}
$$

Solution. 3.b. We use all notation and computations from the previous part of the problem. According to theory, in order to solve that integral, we are supposed to integrate by parts the simpler integral

$$
\begin{aligned}
J(n-1) &= \int \frac{1}{\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)^{n-1}}\mathrm{d}x = \int \frac{1}{(u^2+D)^{n-1}}\mathrm{d}u && \text{int. by parts} \\
&= \frac{u}{(u^2+D)^{n-1}} - \int u\,\mathrm{d}\left(\frac{1}{(u^2+D)^{n-1}}\right) \\
&= \frac{u}{(u^2+D)^{n-1}} + 2(n-1)\int \frac{u^2}{(u^2+D)^n}\mathrm{d}u \\
&= \frac{u}{(u^2+D)^{n-1}} + 2(n-1)\int \frac{u^2+D-D}{(u^2+D)^n}\mathrm{d}u \\
&= \frac{u}{(u^2+D)^{n-1}} + 2(n-1)J(n-1) - 2D(n-1)\int \frac{1}{(u^2+D)^n}\mathrm{d}u \\
&= \frac{u}{(u^2+D)^{n-1}} + 2(n-1)J(n-1) - 2D(n-1)J(n)
\end{aligned}
$$

In the above equality, we rearrange terms to get that

$$
\begin{aligned}
2D(n-1)J(n) &= \frac{u}{(u^2+D)^{n-1}} + (2n-3)J(n-1) \\
J(n) &= \frac{1}{D}\left(\frac{u}{2(n-1)\left(u^2+D\right)^{n-1}} + \frac{2n-3}{2n-2}J(n-1)\right) \\
&= \frac{1}{D}\left(\frac{x+\frac{b}{2a}}{(2n-2)\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)^{n-1}} + \frac{2n-3}{2n-2}J(n-1)\right) \quad .
\end{aligned}
$$