Calculus I
Reference: the Chain Rule statement and notation

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- It would be nice if we could find the derivative of f in terms of the derivatives of y and u.
- It turns out that the derivative of the composition $g \circ h$ is the product of the derivative of g and the derivative of h.
- This important fact is called the Chain Rule.

The Chain Rule

Let g and h be functions. Recall that the composite function $f = g \circ h$ is defined via $f(x) = g(h(x))$.

Theorem

Let h be differentiable at x and let g be a differentiable at $h(x)$. Then the composite function $f = g \circ h$ is differentiable at x and f' is given by the product

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f'(x) & = & g'(h(x)) \cdot h'(x) & & \text{(notation 1)} \\
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f'(x) = (g(u))' & = & g'(u)u' & \text{where } u = h(x) & \text{(notation 2)}
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$$\begin{array}{rcll} f'(x) &=& g'(h(x)) \cdot h'(x) & \textit{(notation 1)} \\ & & & \textit{equivalently:} \\ f'(x) = (g(u))' &=& g'(u)u' \quad \textit{where } u = h(x) & \textit{(notation 2)} \\ \dfrac{dy}{dx} &=& \dfrac{dy}{du}\dfrac{du}{dx} \quad \textit{where } y = g(u) & \textit{(notation 3)} \end{array} .$$

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f'(x) & = & g'(h(x)) \cdot {\color{red} h'(x)} & \textit{(notation 1)} \\
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The last equality uses the Leibniz notation (due to G. Leibniz (1646-1716)).

Chain rule notations

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$$\begin{array}{rclll} (g(h(x)))' & = & g'(h(x)) \cdot h'(x) & & \text{(notation 1)} \\ (g(u))' & = & g'(u)u' & \text{where } u = h(x) & \text{(notation 2)} \\ \frac{\mathrm{d}y}{\mathrm{d}x} & = & \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} & \text{where } y = g(u) & \text{(notation 3)} \end{array} .$$

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- There are additional notations (not covered here) used in practice.
- Whenever in doubt about derivative notation, if possible, request clarification.