Calculus I
Maxima and minima over closed intervals

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Outline

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Fermat's Theorem suggests that we should look at three types of points to find local maxima and minima:

1. Points c for which $f'(c) = 0$.
2. Points c for which $f'(c)$ doesn't exist.
3. Points c at ends of intervals where f is defined. Here, we need also that f be defined at c.

Definition (Critical Number)

A critical number of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ doesn't exist.

Fermat's Theorem says that if f has a local maximum or minimum at c, and c is not an endpoint, then c is a critical number for f.

Example

Find the critical numbers of $f(x) = x^{\frac{1}{4}}\left(4 - x^2\right)$.

$$\begin{aligned} f(x) &= x^{\frac{1}{4}}\left(4 - x^2\right) \\ &= 4x^{\frac{1}{4}} - x^{\frac{9}{4}} \\ \text{so } f'(x) &= x^{-\frac{3}{4}} - \frac{9}{4}x^{\frac{5}{4}} \\ &= \frac{1}{x^{\frac{3}{4}}} - \frac{9}{4}x^{\frac{5}{4}} \\ &= \frac{4 - 9x^2}{4x^{\frac{3}{4}}} \end{aligned}$$

$$\begin{aligned} \text{Set } f'(x) &= 0 \\ \frac{4 - 9x^2}{4x^{\frac{3}{4}}} &= 0 \\ 4 - 9x^2 &= 0 \\ (2 - 3x)(2 + 3x) &= 0 \\ x &= \pm\frac{2}{3} \end{aligned}$$

- Critical numbers occur:
  1. Where $f'(x)$ isn't defined: 0.
  2. Where $f'(x) = 0$: $\frac{2}{3}$ and $-\frac{2}{3}$.
- f isn't defined at $-\frac{2}{3}$. Therefore the critical numbers are 0 and $\frac{2}{3}$.

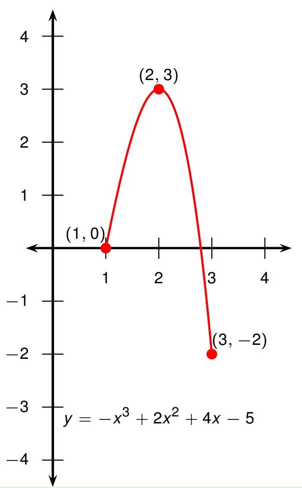
The Closed Interval Method

We know from the Extreme Value Theorem that a continuous function attains its maximum and minimum on a closed interval $[a, b]$. The maximum might occur at an endpoint. The minimum might occur at an endpoint.
To find the maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

1. Find the values of f at the critical numbers of f in $[a, b]$.
   - Find the values c with $f'(c) = 0$.
   - Find the values c where f' does not exist.
2. Find the values of f at the endpoints a and b.
3. The maximum of f is maximum of the preceding values; the minimum value is the minimum.

Example

Find the maximum and minimum values of the function $f(x) = -x^3 + 2x^2 + 4x - 5$ on the interval $[1, 3]$.



$$\begin{aligned} f'(x) &= -3x^2 + 4x + 4 \\ &= (-3x - 2)(x - 2) \end{aligned}$$

If $f'(x) = 0$, $x = -\frac{2}{3}$ or 2.

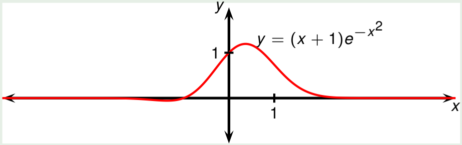
Need to check:

1. The critical numbers of f in $[a, b]$.
2. The endpoints a and b.

x	$f(x)$
1	0
2	3
3	−2

Maximum on $[1, 3]$: 3. Minimum on $[1, 3]$: -2.

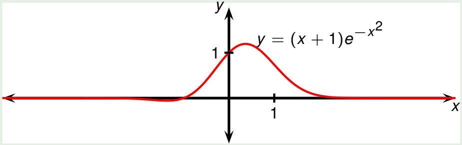
Example



Find the value of x for which
$$f(x) = (x+1)e^{-x^2}$$
attains its maximum in the interval $[-5, 5]$. Use the given plot.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left((x+1)e^{-x^2}\right) &= \frac{\mathrm{d}}{\mathrm{d}x}(x+1)e^{-x^2} + (x+1)\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{-x^2}\right)\\
&= 1\cdot e^{-x^2} + (x+1)\,e^{-x^2}\left(-x^2\right)'\\
&= 1\cdot e^{-x^2} + (x+1)e^{-x^2}(-2x)\\
&= (1+(x+1)(-2x))\,e^{-x^2} = \left(-2x^2-2x+1\right)e^{-x^2}
\end{aligned}$$

Example



Find the value of x for which
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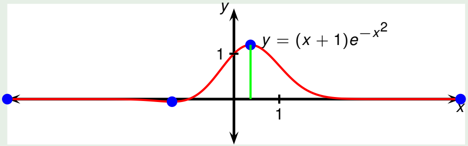
$$\frac{\mathrm{d}}{\mathrm{d}x}\left((x+1)e^{-x^2}\right) = (1+(x+1)(-2x))\,e^{-x^2} = \left(-2x^2-2x+1\right)e^{-x^2}$$

Find critical points: set $f'(x)=0$ and solve for x:

$$\begin{aligned} (-2x^2-2x+1)e^{-x^2} &= 0 && \Big| \text{Div. by } e^{-x^2}\neq 0\\ -2x^2-2x+1 &= 0\\ x_1, x_2 &= \frac{-(-2)\pm\sqrt{(-2)^2-4(-2)\cdot 1}}{2(-2)}\\ &= \frac{2\pm\sqrt{12}}{-4} = \frac{-2\mp 2\sqrt{3}}{4} = \frac{-1\pm\sqrt{3}}{2} \end{aligned}$$

Example



Find the value of x for which
$$f(x)=(x+1)e^{-x^2}$$
attains its maximum in the interval $[-5,5]$. Use the given plot.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left((x+1)e^{-x^2}\right)=(1+(x+1)(-2x))\,e^{-x^2}=\left(-2x^2-2x+1\right)e^{-x^2}$$

Find critical points: set $f'(x)=0$ and solve for x:

$$(-2x^2-2x+1)e^{-x^2}=0 \qquad \Big| \text{Div. by } e^{-x^2}\neq 0$$

$$x_1,x_2=\frac{2\pm\sqrt{12}}{-4}=\frac{-2\mp 2\sqrt{3}}{4}=\frac{-1\pm\sqrt{3}}{2}$$

Compare the values of f at the endpoints and the critical points:

	x	$f(x)$
	-5	close to 0 from plot
	$\frac{-1-\sqrt{3}}{2}$	negative, min from plot
Final answer:	$\frac{-1+\sqrt{3}}{2}$	positive, max from plot
	5	close to 0 from plot

One variable optimization Problems

The problem of finding minimum/maximum of a differentiable one-variable function often arises in practice.

Problem (One variable optimization problem statement)

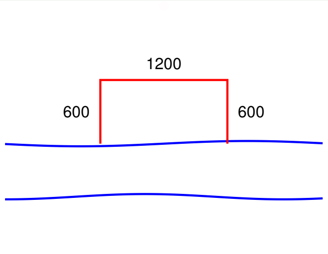
Given a function $f(x)$, find the maximum and/or the minimum of $f(x)$, and the values of x for which the minima/maxima are achieved.

Optimization problems are usually not formulated directly in the above form. Solving an optimization problem involves the following steps.

1. Draw a picture of the problem. Assign variable names to the involved quantities. Determine which quantity is being maximized/minimized.
2. Express all involved quantities in terms of only one of them. If you cannot do that then the problem is not in one variable (i.e., lies outside of the scope of Calculus I).
3. Use the closed interval method to find the maximum/minimum value of the desired quantity.

Example

A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. He doesn't need to put fencing along the river. What are the dimensions of the field with the largest area?



Let x and y denote the depth and width of the rectangle (in feet). Let A be its area.

$$\begin{aligned} 2x + y &= 2400 \\ y &= 2400 - 2x \\ A &= xy = x(2400 - 2x) \\ &= 2400x - 2x^2 \end{aligned}$$

Notice that $0 \leq x \leq 1200$.

Maximize the function $A(x)$:

$$A'(x) = 2400 - 4x$$

Critical number: $x = 600$.

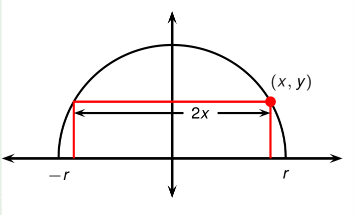
Area $= A = xy$

x	$A(x)$
0	0
600	720,000
1200	0

Therefore the maximum area occurs when $x = 600$ft and $y = 1200$ft.

Example

Find the largest possible area of a rectangle inscribed in a semicircle of radius r.



Let the semicircle have center at the origin. Let (x, y) -coord. of top right corner of rectangle. Let A be its area.

$$\begin{aligned} A &= \text{base} \cdot \text{height} \\ &= 2x \cdot y = 2x \cdot \sqrt{r^2 - x^2} \\ A' &= 2\sqrt{r^2 - x^2} - \frac{2x^2}{\sqrt{r^2 - x^2}} \\ &= \frac{2(r^2 - x^2)}{\sqrt{r^2 - x^2}} - \frac{2x^2}{\sqrt{r^2 - x^2}} = \frac{2(r^2 - 2x^2)}{\sqrt{r^2 - x^2}} \end{aligned}$$

Critical numbers: $x = \frac{r}{\sqrt{2}}$ and r.

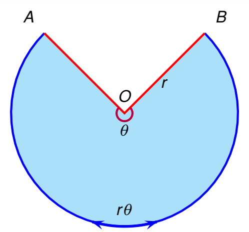
To eliminate y, use that (x, y) lies on the semicircle.

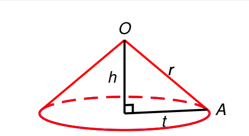
$$\begin{aligned} y^2 &= r^2 - x^2 \\ y &= \sqrt{r^2 - x^2} \end{aligned}$$

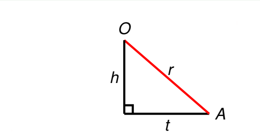
We have $0 \leq x \leq r$ and so the critical numbers together with the endpoints are $x = 0, \frac{r}{\sqrt{2}}, r$. Since $A(0) = 0 = A(r)$, the max is achieved at $x = y = \frac{r}{\sqrt{2}}$. The max area is $A(\frac{r}{\sqrt{2}}) = 2\frac{r}{\sqrt{2}}\sqrt{r^2 - \frac{r^2}{2}} = r^2$.

Example

A cone is folded from a wedge-shaped profile of radius r. Find the maximal possible volume V of such a cone.







Set h - cone height, t - cone radius. Then $V = \frac{1}{3}(\text{area cone base})h = \frac{1}{3}\pi t^2 h$. Let θ - angle of the wedge. Then arc$AB = r\theta$ = perimeter cone base = $2\pi t$. Therefore $t = \frac{r\theta}{2\pi}$. Then

$$h = \sqrt{r^2 - t^2} = \sqrt{r^2 - \left(\frac{r\theta}{2\pi}\right)^2} = \frac{r}{2\pi}\sqrt{4\pi^2 - \theta^2},$$

Example

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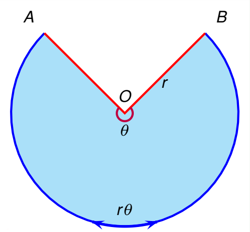

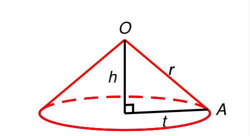

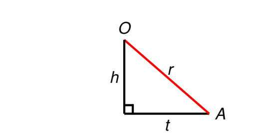

Example

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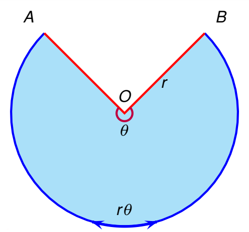
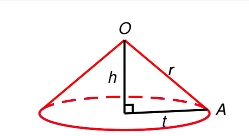
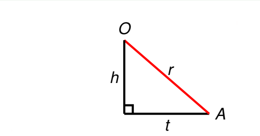






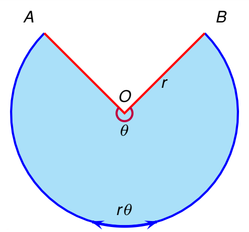
We reduced the problem to: find the maximum of

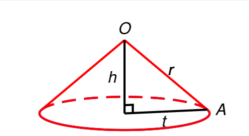
$$V = \frac{r^3}{24\pi^2}\theta^2\sqrt{4\pi^2-\theta^2}, \qquad 0 \leq \theta \leq 2\pi$$

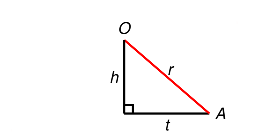
as function of θ (using the closed interval method). We need to find the critical points of V, i.e., the values of θ for which $\frac{\mathrm{d}V}{\mathrm{d}\theta} = 0$ and the values of θ for which $\frac{\mathrm{d}V}{\mathrm{d}\theta}$ is not defined.

Example

A cone is folded from a wedge-shaped profile of radius r. Find the maximal possible volume V of such a cone.



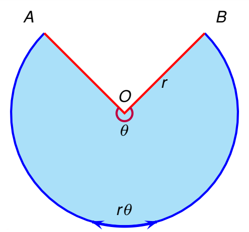


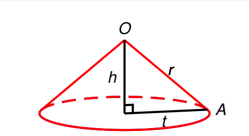


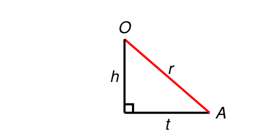
$$
\begin{aligned}
V &= \frac{r^3}{24\pi^2}\theta^2\sqrt{4\pi^2-\theta^2}, \qquad 0\leq\theta\leq 2\pi\\
\frac{\mathrm{d}V}{\mathrm{d}\theta} &= \left(\frac{r^3}{24\pi^2}\right)\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\theta^2\right)\sqrt{4\pi^2-\theta^2}\\
&\quad+\left(\frac{r^3}{24\pi^2}\right)\theta^2\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sqrt{4\pi^2-\theta^2}\right)\\
&= \left(\frac{r^3}{24\pi^2}\right)(2\theta)\sqrt{4\pi^2-\theta^2}\\
&\quad+\left(\frac{r^3}{24\pi^2}\right)\theta^2\left(\frac{1}{2}\frac{\frac{\mathrm{d}}{\mathrm{d}\theta}(-\theta^2)}{\sqrt{4\pi^2-\theta^2}}\right)\\
&= \left(\frac{r^3}{24\pi^2}\right)\frac{2\theta(4\pi^2-\theta^2)-\theta^3}{\sqrt{4\pi^2-\theta^2}}\\
&= \left(\frac{r^3}{24\pi^2}\right)\frac{8\theta\pi^2-3\theta^3}{\sqrt{4\pi^2-\theta^2}}
\end{aligned}
$$

Example

A cone is folded from a wedge-shaped profile of radius r. Find the maximal possible volume V of such a cone.







$$\begin{aligned} V &= \frac{r^3}{24\pi^2}\theta^2\sqrt{4\pi^2-\theta^2}, \qquad 0\leq\theta\leq 2\pi \\ \frac{\mathrm{d}V}{\mathrm{d}\theta} &= \left(\frac{r^3}{24\pi^2}\right)\frac{8\theta\pi^2-3\theta^3}{\sqrt{4\pi^2-\theta^2}} \end{aligned}$$

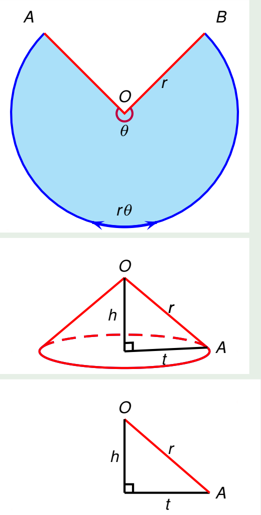
We have that $\frac{\mathrm{d}V}{\mathrm{d}\theta}=0$ when

$$\begin{aligned} 8\theta\pi^2-3\theta^3 &= 0 \\ \theta(8\pi^2-3\theta^2) &= 0 \\ -3\theta\left(\theta-\sqrt{\frac{8}{3}}\pi\right)\left(\theta+\sqrt{\frac{8}{3}}\pi\right) &= 0. \end{aligned}$$

Therefore θ is critical point for V if $\theta=0$, $\theta=\sqrt{\frac{8}{3}}\pi$, or $\theta=2\pi$ (note $\theta=-\sqrt{\frac{8}{3}}\pi$ is outside of the domain of V). For $\theta=0, 2\pi$ the volume V is 0, so the maximum volume is attained at $\theta=\sqrt{\frac{8}{3}}\pi$.

Example

A cone is folded from a wedge-shaped profile of radius r. Find the maximal possible volume V of such a cone.



$$V(\theta) = \frac{r^3}{24\pi^2}\theta^2\sqrt{4\pi^2 - \theta^2}$$

Finally, the answer to the problem is

$$\begin{aligned} V_{max} &= V\left(\sqrt{\frac{8}{3}}\pi\right) \\ &= \frac{r^3}{24\pi^2}\left(\sqrt{\frac{8}{3}}\pi\right)^2\sqrt{4\pi^2 - \left(\sqrt{\frac{8}{3}}\pi\right)^2} \\ &= \frac{r^3}{9}\pi\sqrt{4 - \frac{8}{3}} \\ &= \pi\frac{r^3}{9}\sqrt{\frac{4}{3}} = \frac{2\pi r^3}{9\sqrt{3}} \end{aligned}$$