Calculus II
Homework
Series absolute convergence, the ratio and root tests

1. Establish whether the series is convergent or divergent. Use the ratio or root tests. Show all your work. The answer key has not been proofread, use with caution.

   (a) $\displaystyle\sum_{n=0}^{\infty}(-1)^n n^2 3^{-n}$ — answer: convergent, straightforward with ratio test

   (b) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n+1}{4n}\right)^n$ — answer: convergent, straightforward with root test

   (c) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{4n+1}{n}\right)^n$ — answer: divergent, straightforward with root test

   (d) $\displaystyle\sum_{n=1}^{\infty}\frac{n^n}{4^n n!}$ — answer: convergent, use ratio test

   (e) $\displaystyle\sum_{n=1}^{\infty}\frac{(4n)^n}{n!}$ — answer: divergent, use ratio test

   Solution. 1.a We proceed with the ratio test; the alternating series test works too, however that approach is a lot less straightforward and we leave it to the reader.
   Let the n^{th} term of the series be $a_n = (-1)^n n^2 3^{-n}$. The ratio test states that if the limit $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ exists and is less than 1, then the series is convergent, and if the limit exists and is greater than 1, then then the series is divergent.

$$\begin{array}{rcl}\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &=& \displaystyle\lim_{n\to\infty}\left|\frac{(-1)^{n+1}3^{-n-1}(n+1)^2}{(-1)^n 3^{-n} n^2}\right| \\ &=& \displaystyle\lim_{n\to\infty}\left|\frac{1}{3}\left(1+\frac{1}{n}\right)^2\right| \\ &=& \frac{1}{3} < 1 \quad .\end{array}$$

   Therefore the series is convergent by the ratio test.

   Solution. 1.e The series can quickly be shown to be divergent by showing that $\lim\limits_{n\to\infty}\frac{(4n)^n}{n!} = \infty$. Nonetheless we will use the ratio test, as it provides insight to what happens when we replace the constant 4 with another constant. In order to establish the divergence of

$$\sum_{n=1}^{\infty}\frac{(4n)^n}{n!} \quad ,$$

   we shall use the ratio test. We recall that the ratio test states that if $\lim\limits_{n\to\infty}\dfrac{a_{n+1}}{a_n}$ exists and is equal to L, then if $L > 1$ the series is divergent and if $L < 1$ the series is convergent (if $L = 1$ the test is inconclusive).
   We compute:

$$\begin{array}{rcl}\displaystyle\lim_{n\to\infty}\frac{a_{n+1}}{a_n} &=& \displaystyle\lim_{n\to\infty}\left|\frac{(4n+4)^{n+1}n!}{(n+1)!(4n)^n}\right| \\ &=& \displaystyle\lim_{n\to\infty}\left|\frac{(4n+4)(4n+4)^n}{(n+1)(4n)^n}\right| \\ &=& \displaystyle\left(\lim_{n\to\infty}\frac{4n+4}{n+1}\right)\left(\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n\right) = 4e > 1 \quad ,\end{array}$$

   and therefore the series is divergent.

2. Except for $x = \pm e$, use the ratio test to determine all real values of x for which

$$\sum_{n=0}^{\infty} x^n \frac{n!}{n^n}$$

is convergent. You are expected to use in your solution the fact that

$$\lim_{x \to 0} \left(1 + \frac{x}{n}\right)^n = e^x \quad .$$