Precalculus
Graphs of trig functions; inverse trig

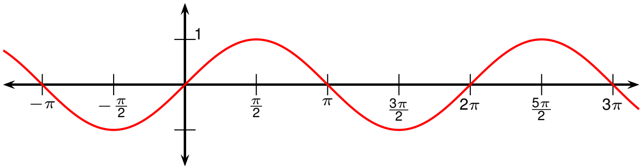
Todor Milev

2019

Outline

1. Inverse Trigonometric Functions
   - The arcsine function
   - The arccosine function
   - The arctangent and the remaining inverse trig functions
   - Trigonometric Functions with Inverse Trig Arguments

Inverse Trigonometric Functions



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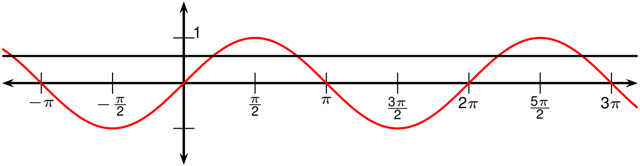

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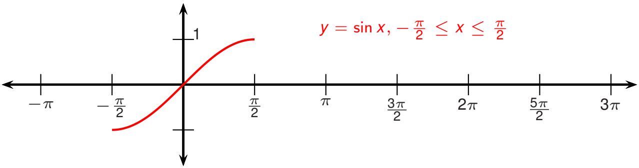

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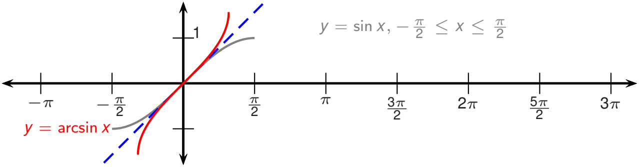

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- We call it $\arcsin$ or $\sin^{-1}$.

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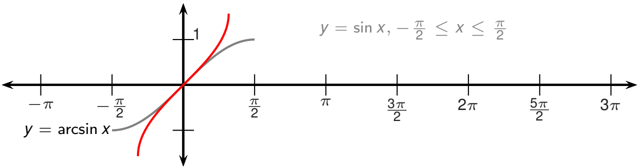

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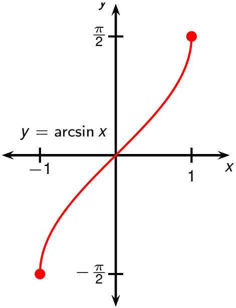

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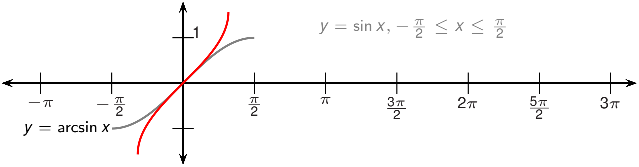

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- It is if we restrict the domain to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.
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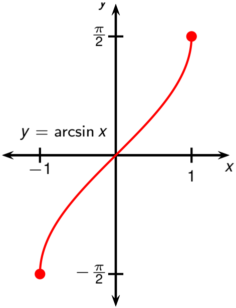
- We call it arcsin or $\sin^{-1}$.
- $\arcsin x = y \Leftrightarrow \sin y = x$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.



Example

$$\text{Find } \arcsin\left(\frac{1}{2}\right).$$

Observation

- $\arcsin y =$ *the appropriate angle whose sine equals y.*

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- $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$.

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- $\arcsin y =$ *the appropriate angle whose sine equals* y.
- *Important: the output angle must lie in the interval* $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

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$$\text{Find } \arcsin\left(\frac{1}{2}\right).$$

- $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$.
- $-\frac{\pi}{2} \leq \frac{\pi}{6} \leq \frac{\pi}{2}$.

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$$\text{Find } \arcsin\left(\frac{1}{2}\right).$$

- $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$.
- $-\frac{\pi}{2} \leq \frac{\pi}{6} \leq \frac{\pi}{2}$.
- Therefore $\arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$.

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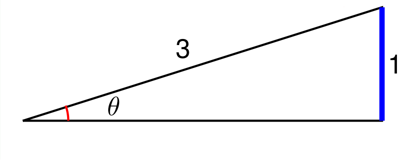

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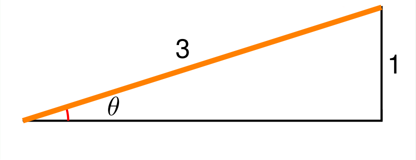

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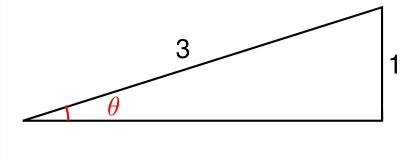

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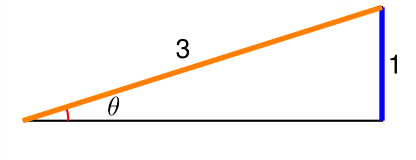

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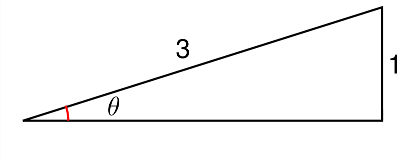

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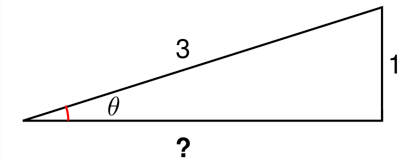
- Length of adjacent side $=$ **?**



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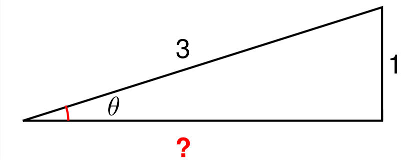
- Length of adjacent side $= \sqrt{3^2 - 1^2}$



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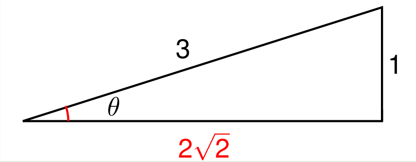
- Length of adjacent side $= \sqrt{3^2 - 1^2} = \sqrt{8} = 2\sqrt{2}$.



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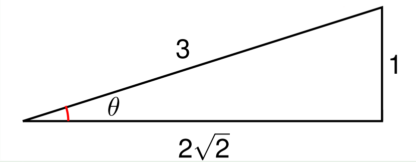

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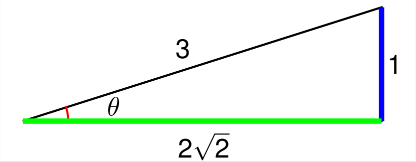

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- Therefore $\arcsin(\sin 1.5) = 1.5$.

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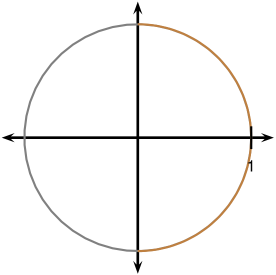
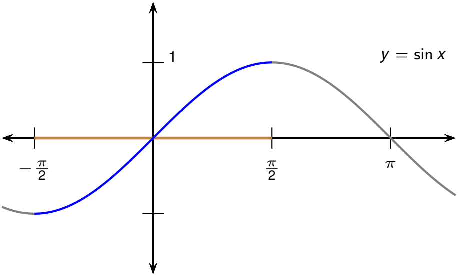
Find $\arcsin(\sin 2)$.





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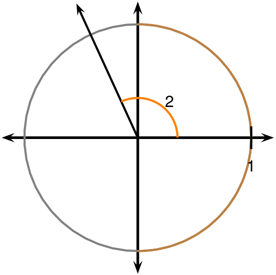

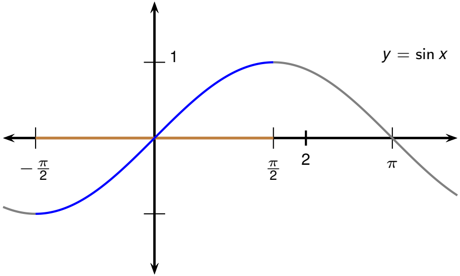

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- 2 is not between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.
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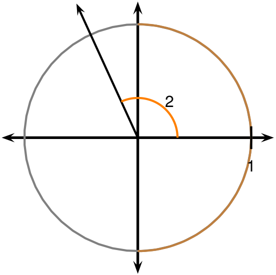

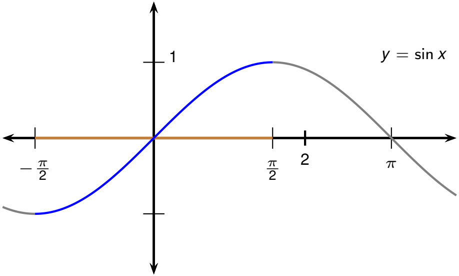

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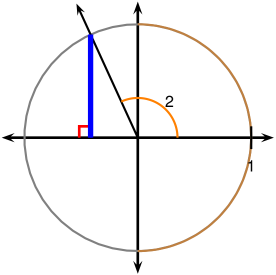

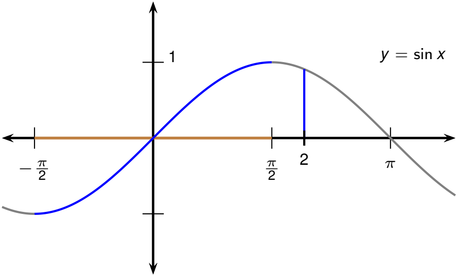

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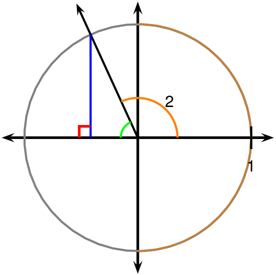

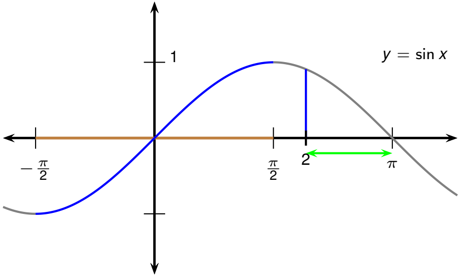

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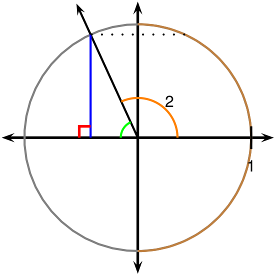

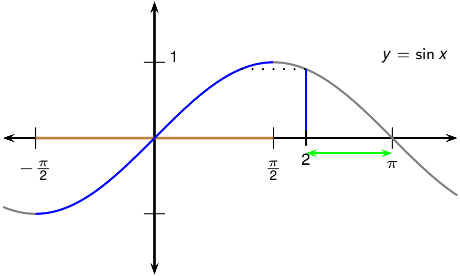

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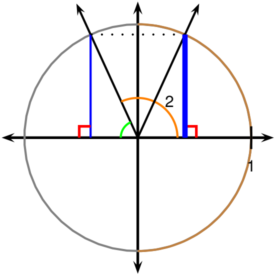

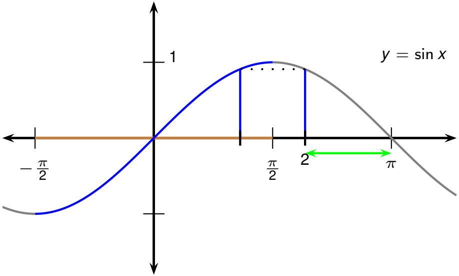

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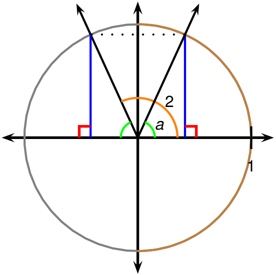

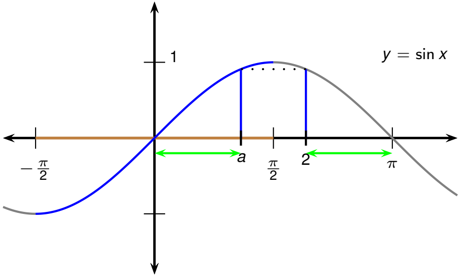

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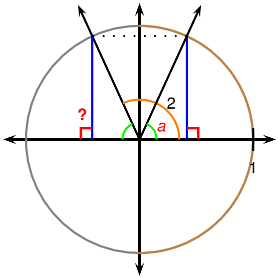
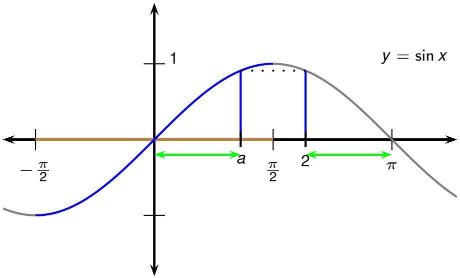
$$a = ?$$





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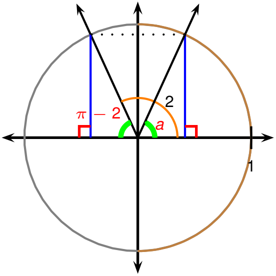
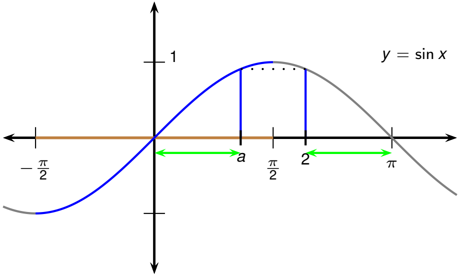
$$a = \pi - 2.$$





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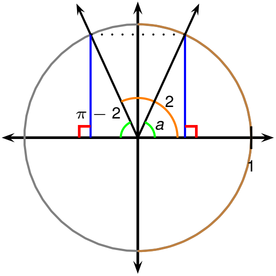

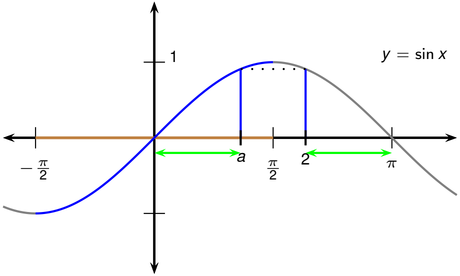

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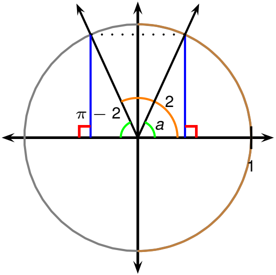

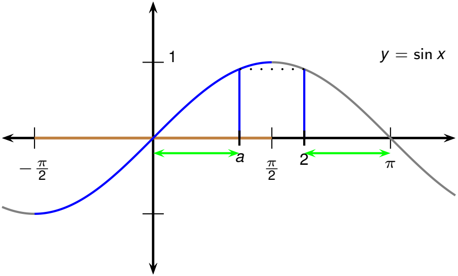

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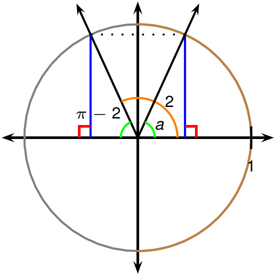
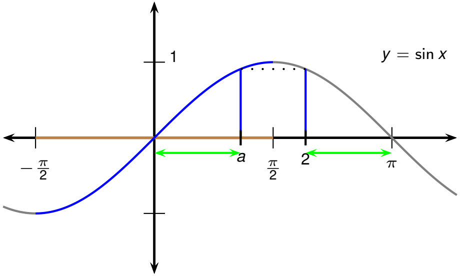
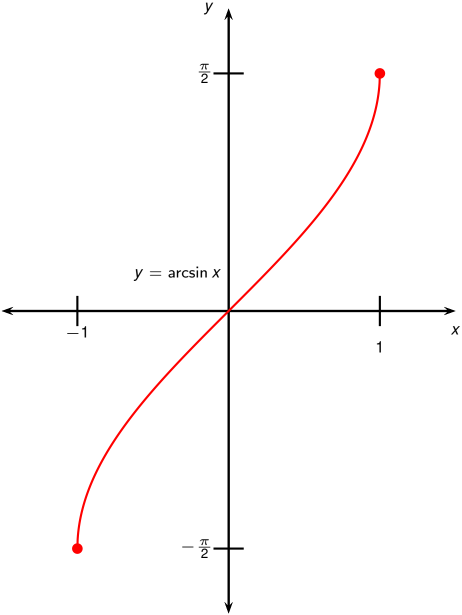
$$\begin{aligned}\arcsin(\sin 2) &= \arcsin(\sin a)\\ &= a = \pi - 2.\end{aligned}$$





Important facts about arcsin:



1. Domain: **?**
2. Range: **?**
3. $\arcsin x = y \Leftrightarrow \sin y = x$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$.
4. $\arcsin(\sin x) = x$ for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.
5. $\sin(\arcsin x) = x$ for $-1 \leq x \leq 1$.
6. $\frac{\mathrm{d}}{\mathrm{d}x}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$.

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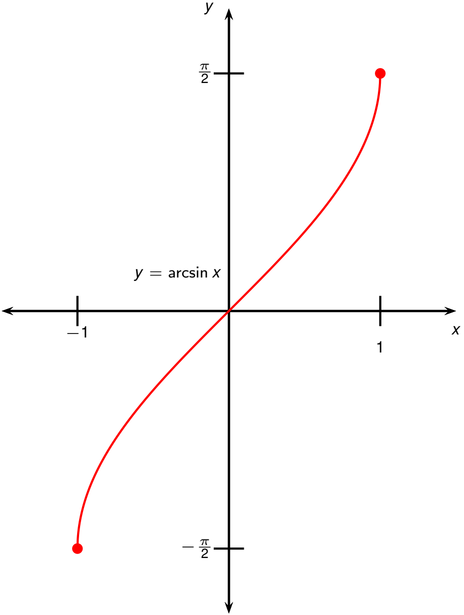

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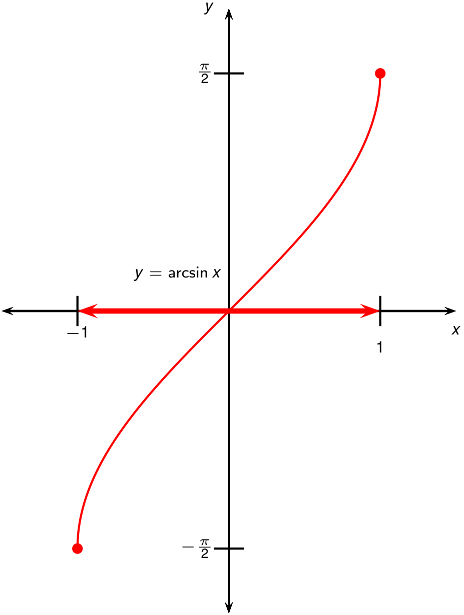

1. Domain: $[-1, 1]$.
2. Range: **?**
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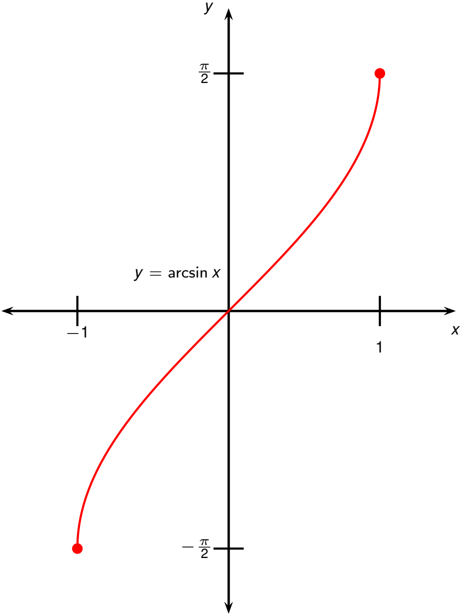

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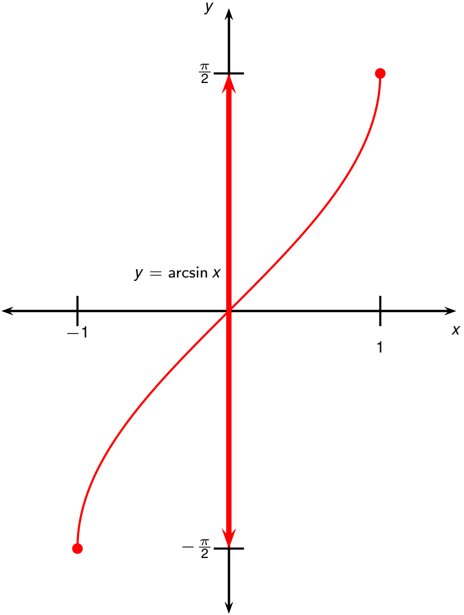

1. Domain: $[-1, 1]$.
2. Range: $[-\frac{\pi}{2}, \frac{\pi}{2}]$.
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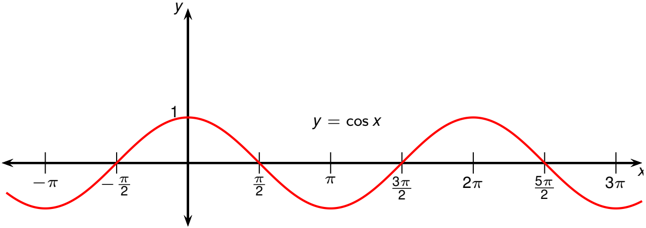
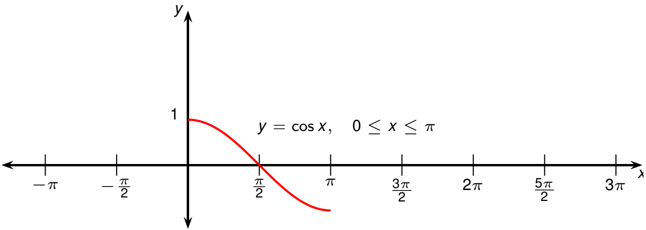
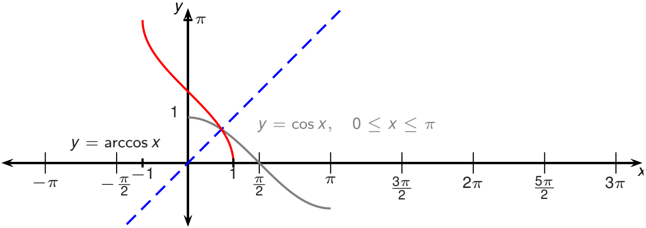


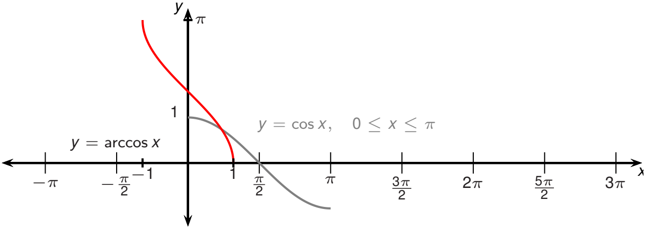
- Same for $\cos x$.

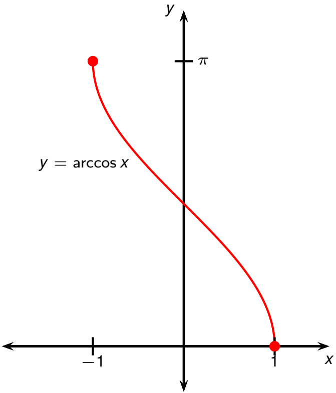


- Same for $\cos x$.
- Restrict the domain to $[0, \pi]$.



- Same for $\cos x$.
- Restrict the domain to $[0, \pi]$.
- The inverse is called arccos or $\cos^{-1}$.





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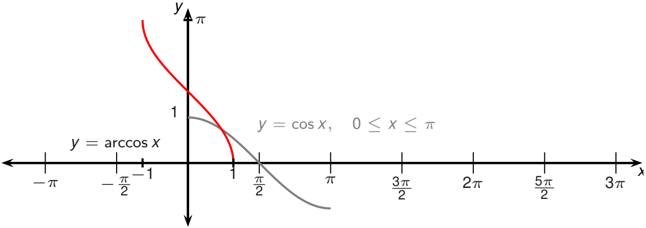
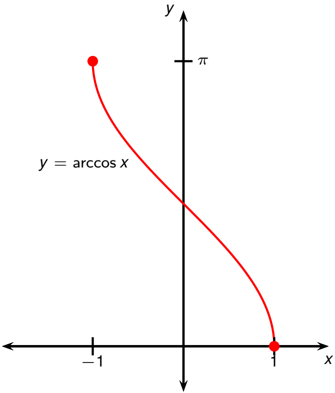
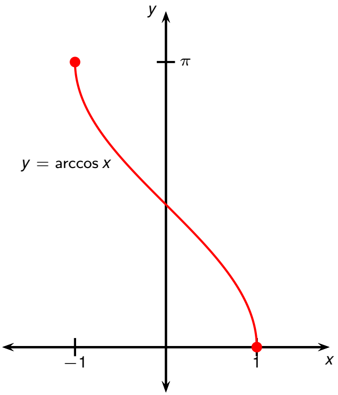


- Same for $\cos x$.
- Restrict the domain to $[0, \pi]$.
- The inverse is called arccos or $\cos^{-1}$.
- $\arccos(x) = y \Leftrightarrow \cos y = x$ and $0 \leq y \leq \pi$.



Important facts about arccos:



1. Domain:
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4. $\arccos(\cos x) = x$ for $0 \leq x \leq \pi$.
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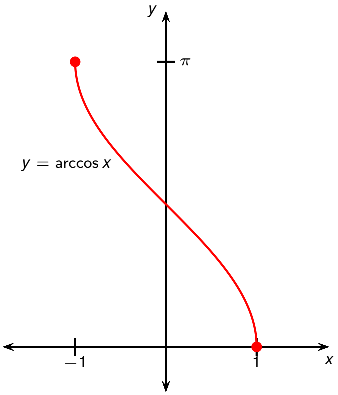

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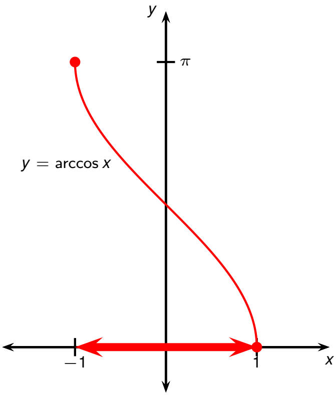

1. Domain: $[-1, 1]$.
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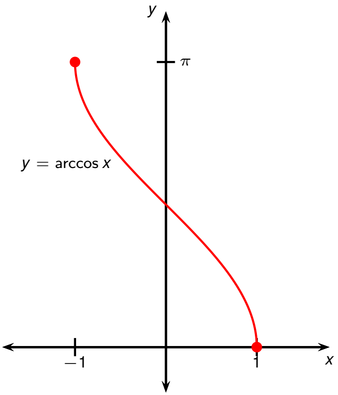

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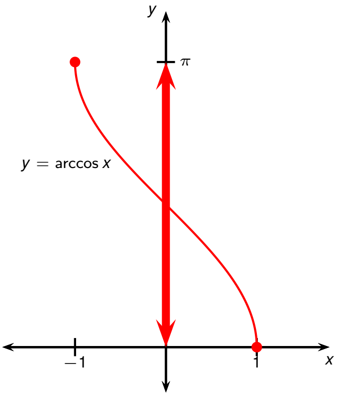

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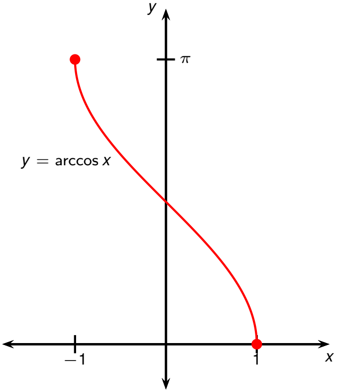

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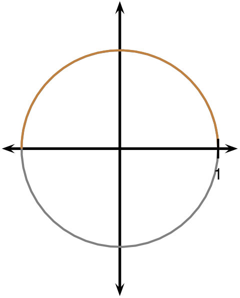
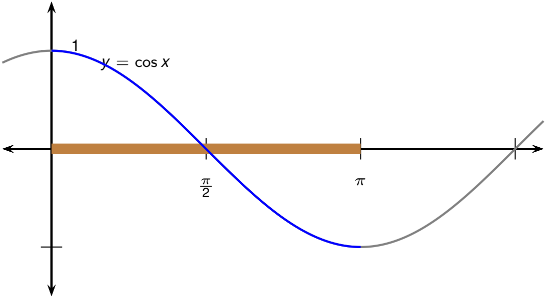
(The proof is similar to the proof of the formula for the derivative of $\arcsin x$.)

Example

Find $\arccos(\cos 4)$.

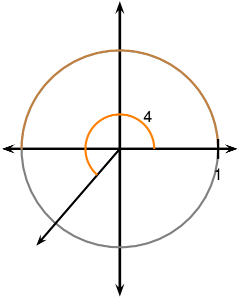


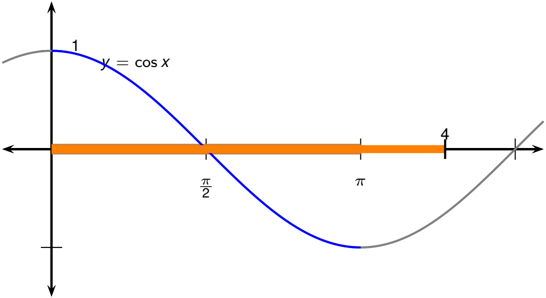


Example

Find $\arccos(\cos 4)$.

- 4 is not between 0 and π.

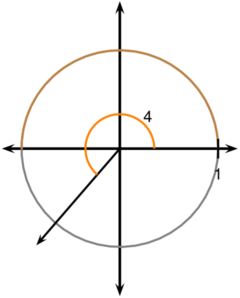


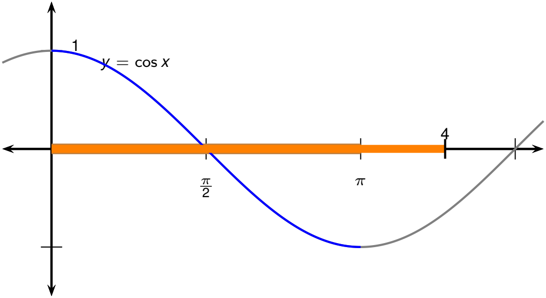


Example

Find $\arccos(\cos 4)$.

- 4 is not between 0 and π.
- We need the angle *a* between 0 and π for which $\cos 4 = \cos a$.





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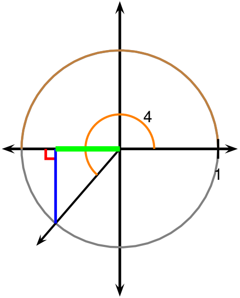

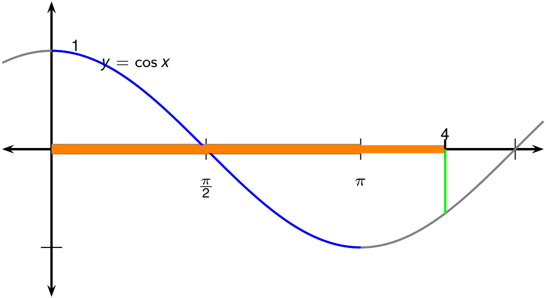

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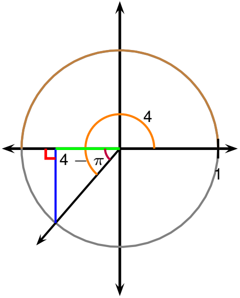

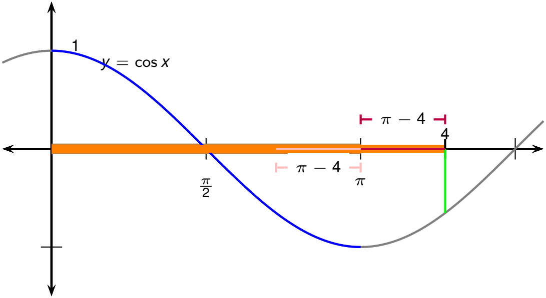

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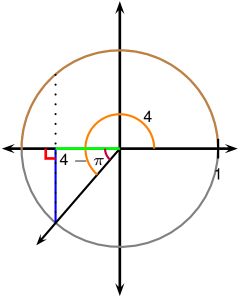

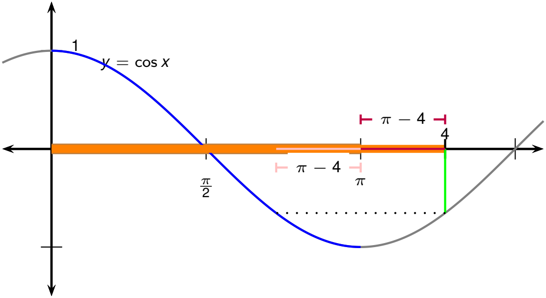

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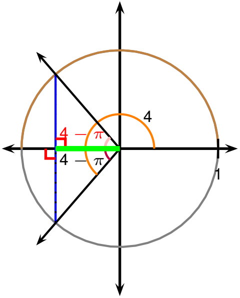

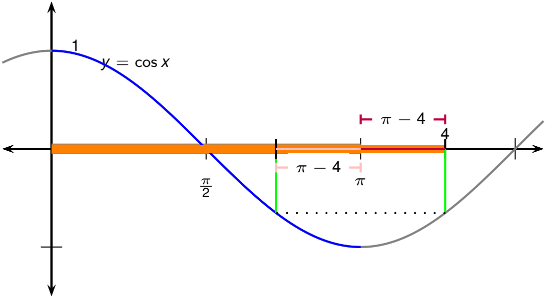

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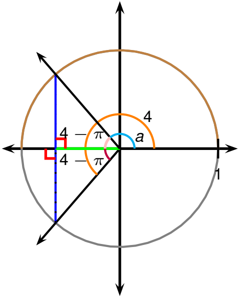

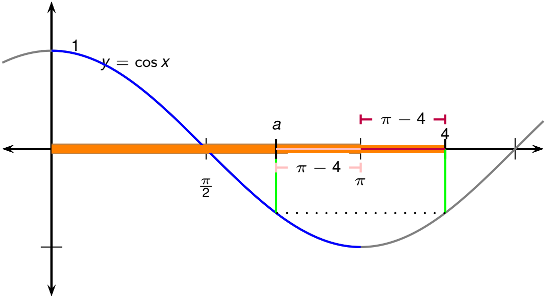

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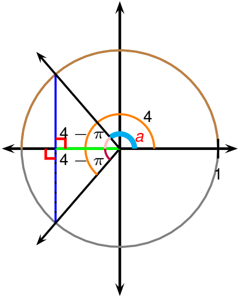
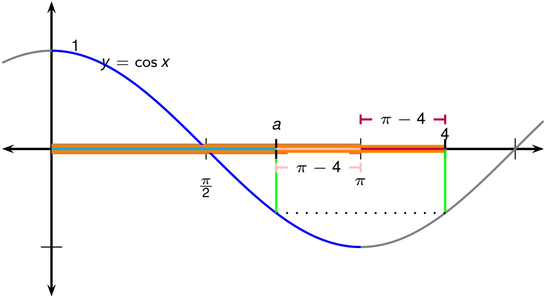
$$a = ?$$



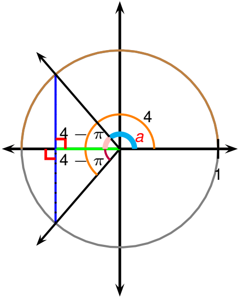


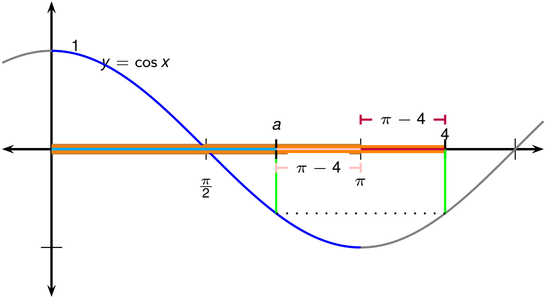
Example

Find $\arccos(\cos 4)$.

- 4 is not between 0 and π.
- We need the angle a between 0 and π for which $\cos 4 = \cos a$.

$$a = \pi - (4 - \pi)$$



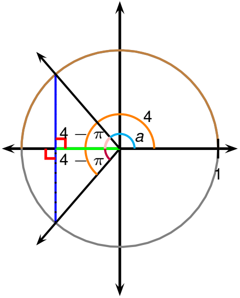


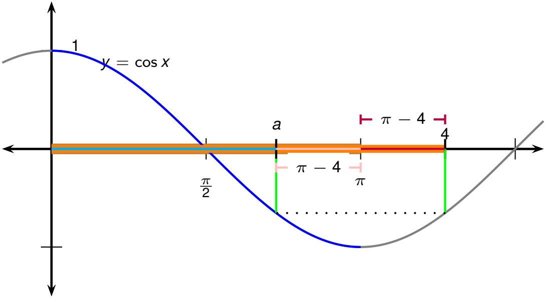
Example

Find $\arccos(\cos 4)$.

- 4 is not between 0 and π.
- We need the angle a between 0 and π for which $\cos 4 = \cos a$.

$$a = \pi - (4 - \pi) = 2\pi - 4$$



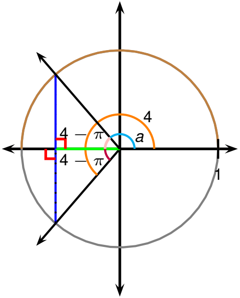


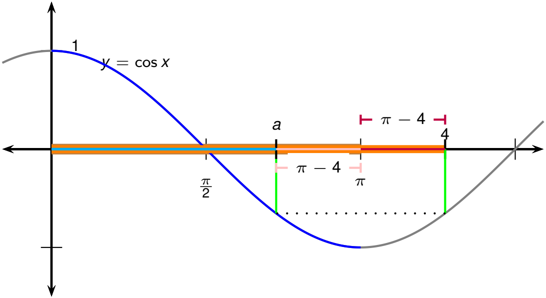
Example

Find $\arccos(\cos 4)$.

- 4 is not between 0 and π.
- We need the angle a between 0 and π for which $\cos 4 = \cos a$.

$$\begin{array}{rrcl} & a & = & \pi - (4-\pi) = 2\pi - 4 \\ \text{Therefore} & \arccos(\cos 4) & = & \arccos(\cos a) \end{array}$$





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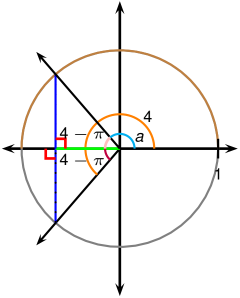

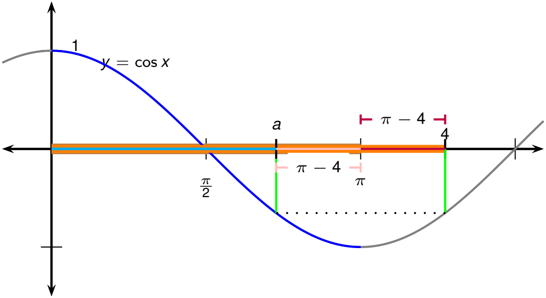

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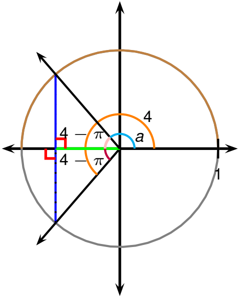
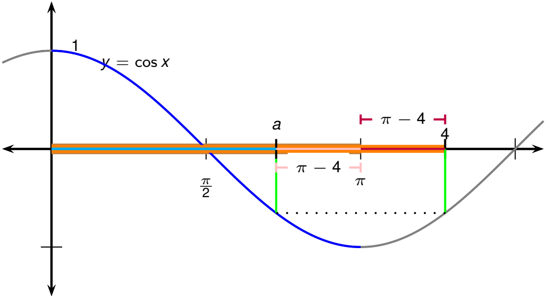
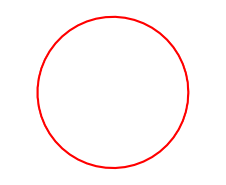




Example

The horizon at sea is viewed at from a ship height of 10m. How far does the ship need to travel to reach the horizon just observed?

Example



The horizon at sea is viewed at from a ship height of 10m. How far does the ship need to travel to reach the horizon just observed? Assume earth is round with radius 6371 km and that the ship sails along the shortest curved path.

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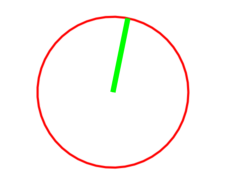

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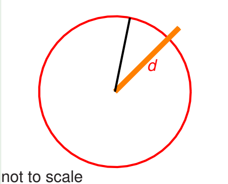
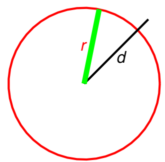
Example



The horizon at sea is viewed at from a ship height of 10m. How far does the ship need to travel to reach the horizon just observed? Assume earth is round with radius 6371 km and that the ship sails along the shortest curved path.

- Let d be the distance from eyes of seaman to the center of earth.

Example



not to scale

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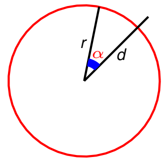
- Let d be the distance from eyes of seaman to the center of earth.
- Let r be the radius of earth.

Example



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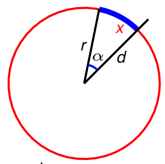
- Let d be the distance from eyes of seaman to the center of earth.
- Let r be the radius of earth. Let α be the indicated angle.

Example



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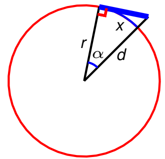
- Let r be the radius of earth. Let α be the indicated angle.
- Let the distance to the horizon be x.

Example



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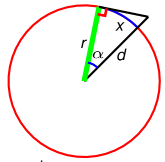
Example



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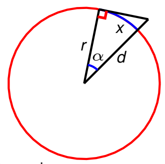
$r = 6371$km

Example



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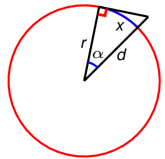
- Let *r* be the radius of earth. Let α be the indicated angle.
- Let the distance to the horizon be *x*.

$r = 6371$km
$d = \mathbf{?}$

Example



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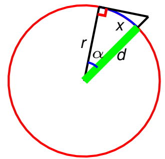
- Let the distance to the horizon be *x*.

$r = 6371\text{km}$

$d = 6371\text{km} + 0.01\text{km}$

Example



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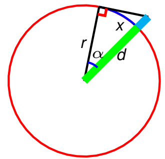
r=6371km

d=6371km + 0.01km

Example



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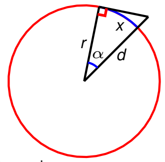
$$r = 6371\text{km}$$
$$d = 6371\text{km} + 0.01\text{km}$$

Example



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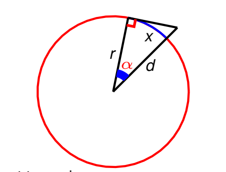
- Let the distance to the horizon be x.

r=6371km

d=6371km + 0.01km = 6371.01km

Example



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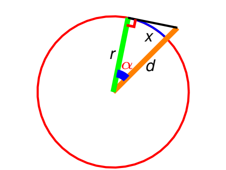
$r = 6371\text{km}$

$d = 6371\text{km} + 0.01\text{km} = 6371.01\text{km}$

$\cos\alpha = \textbf{?}$

Example



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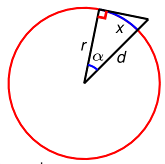
$r = 6371\text{km}$

$d = 6371\text{km} + 0.01\text{km} = 6371.01\text{km}$

$$\cos\alpha = \frac{r}{d}$$

Example



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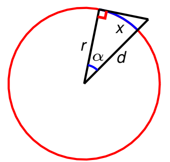
$$\begin{aligned} r &= 6371\text{km} \\ d &= 6371\text{km} + 0.01\text{km} = 6371.01\text{km} \\ \cos\alpha &= \frac{r}{d} \\ \alpha &= \arccos\left(\frac{r}{d}\right) \end{aligned}$$

Example



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\cos\alpha &= \frac{r}{d}\\
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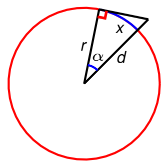
x &= \mathbf{?}
\end{aligned}
$$

Example



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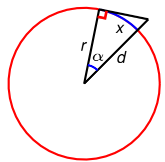
$$

Example



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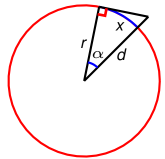
Example



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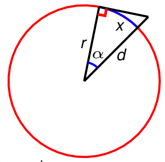
x &= r\alpha = r\arccos\left(\frac{r}{d}\right) = 6371\text{km}\arccos\left(\frac{6371\text{km}}{6371.01\text{km}}\right)
\end{aligned}
$$

Example



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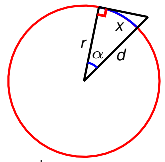
Example



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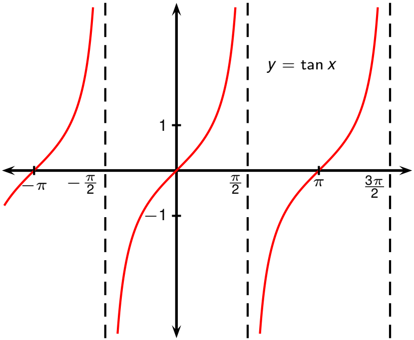
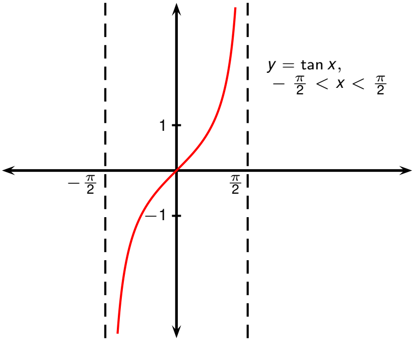
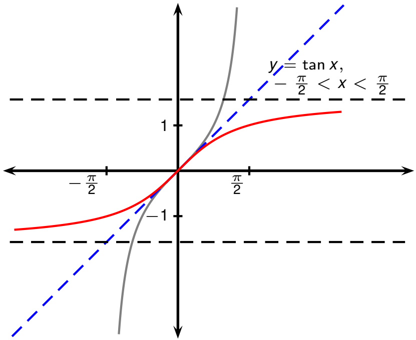
x &= r\alpha = r\arccos\left(\frac{r}{d}\right) = 6371\text{km}\arccos\left(\frac{6371\text{km}}{6371.01\text{km}}\right) \approx 11.29\text{km}
\end{aligned}
$$



- $\tan x$ isn't one-to-one.



- $\tan x$ isn't one-to-one.
- Restrict the domain to $(-\frac{\pi}{2}, \frac{\pi}{2})$.



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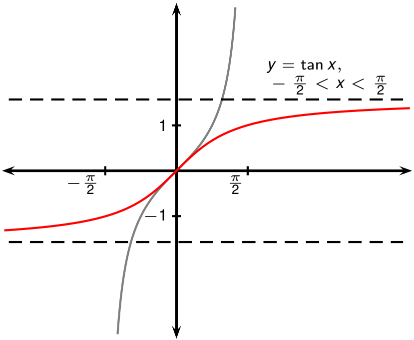

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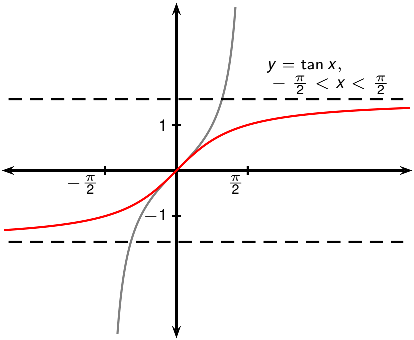
- $\arctan x = y \Leftrightarrow \tan y = x$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$.



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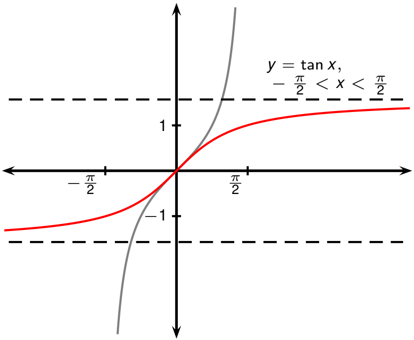
- Domain of $\arctan$: **?**
- Range of $\arctan$:



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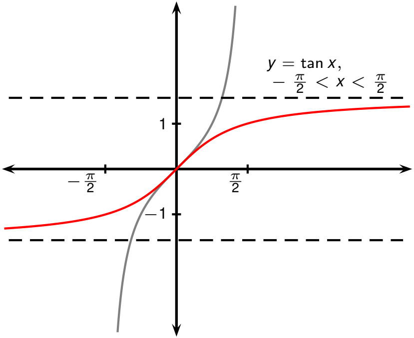
- $\arctan x = y \Leftrightarrow \tan y = x$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$.
- Domain of $\arctan$: $(-\infty, \infty)$.
- Range of $\arctan$:



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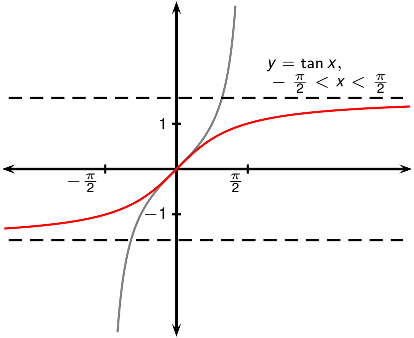

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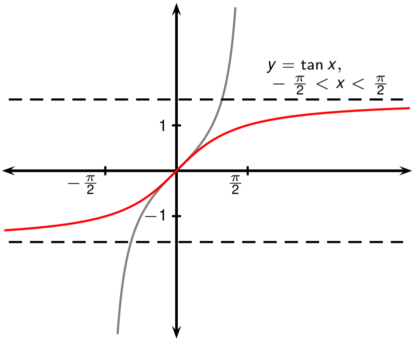

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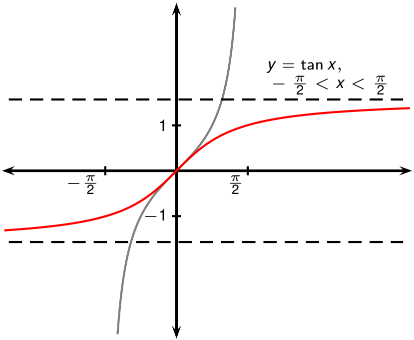
- $\lim_{x\to\infty} \arctan x =$ **?**
- $\lim_{x\to-\infty} \arctan x =$



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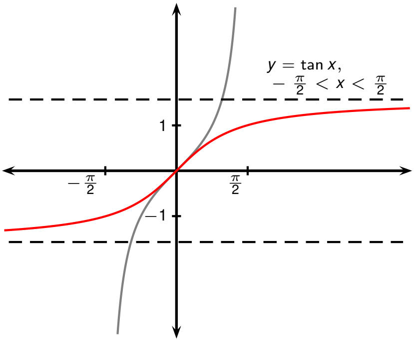
- $\lim_{x\to\infty} \arctan x = \frac{\pi}{2}$.
- $\lim_{x\to-\infty} \arctan x =$



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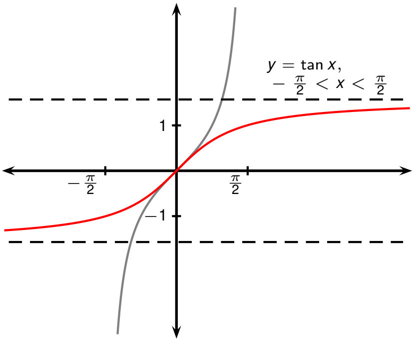
- $\lim\limits_{x\to\infty} \arctan x = \dfrac{\pi}{2}$.
- $\lim\limits_{x\to-\infty} \arctan x = \textbf{?}$



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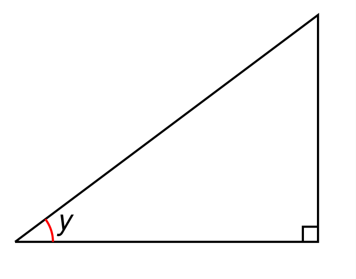
- Domain of $\arctan$: $(-\infty, \infty)$.
- Range of $\arctan$: $(-\frac{\pi}{2}, \frac{\pi}{2})$.
- $\lim\limits_{x\to\infty} \arctan x = \dfrac{\pi}{2}.$
- $\lim\limits_{x\to-\infty} \arctan x = -\dfrac{\pi}{2}.$

Example

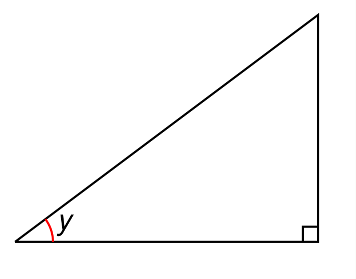
Simplify the expression $\cos(\arctan x)$.



Example

Simplify the expression $\cos(\arctan x)$.

- Let $y = \arctan x$, so $\tan y = x$.



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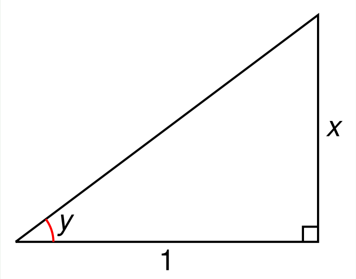
Simplify the expression $\cos(\arctan x)$.

- Let $y = \arctan x$, so $\tan y = x$.
- Draw a right triangle with opposite x and adjacent 1.



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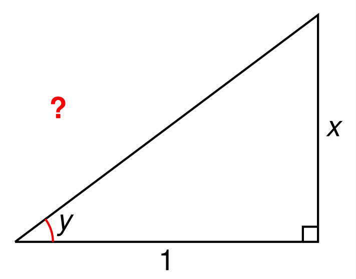
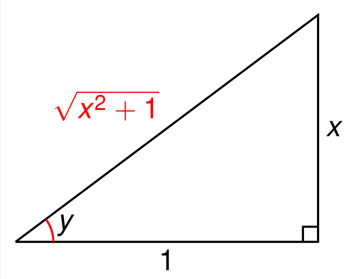


Example

Simplify the expression $\cos(\arctan x)$.

- Let $y = \arctan x$, so $\tan y = x$.
- Draw a right triangle with opposite x and adjacent 1.
- Length of hypotenuse $= \sqrt{1^2 + x^2}$.



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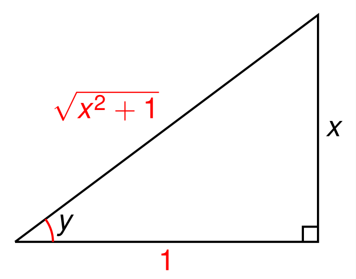

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- Let $y = \arctan x$, so $\tan y = x$.
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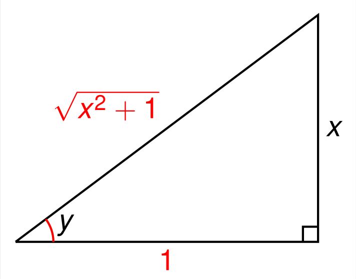

The remaining inverse trigonometric functions aren't used as often:

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y = \operatorname{arccsc} x & (|x| \geq 1) & \Leftrightarrow & \csc y = x & \text{and} & y \in \left(0, \frac{\pi}{2}\right] \cup \left(\pi, \frac{3\pi}{2}\right] \\
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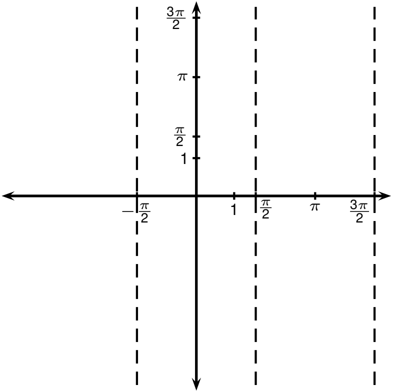
We will however make use of $\operatorname{arcsec} x$: we discuss in detail its domain.

$$y = \operatorname{arcsec} x \quad (|x| \geq 1) \quad \Leftrightarrow \quad \sec y = x \quad \text{and} \quad y \in \textbf{?}$$



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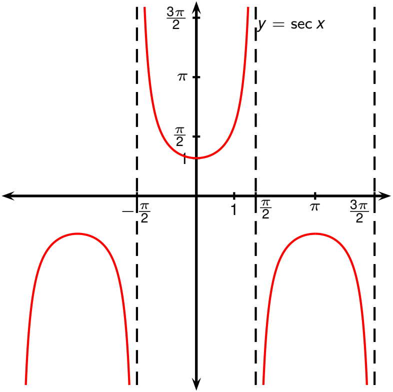

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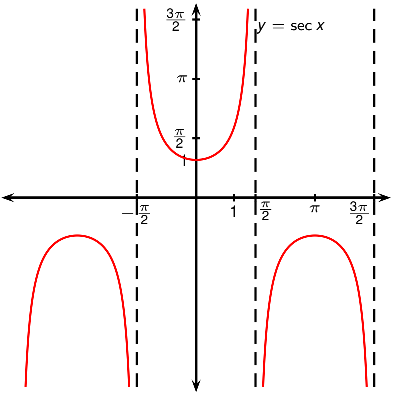

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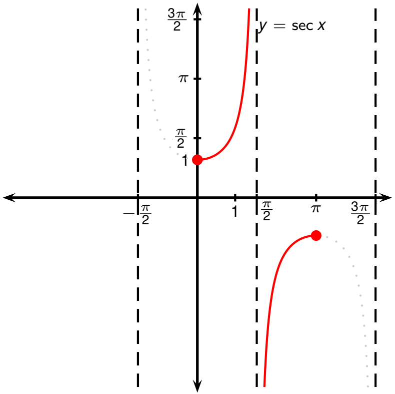

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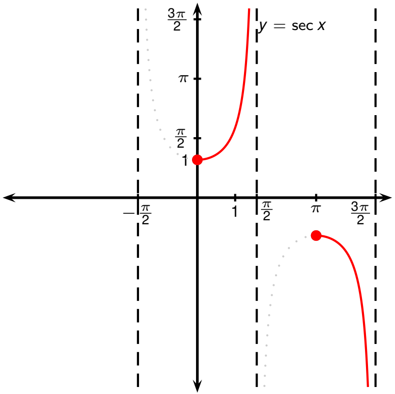

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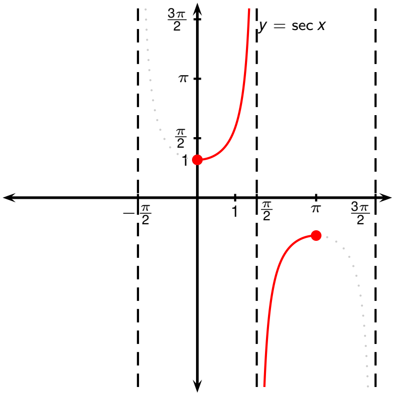

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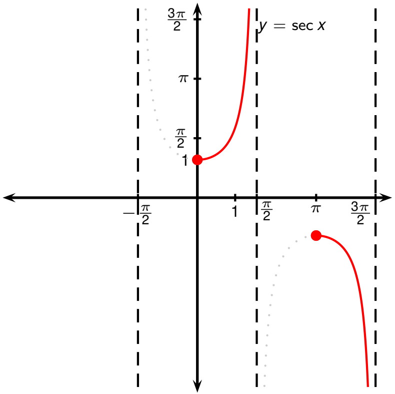

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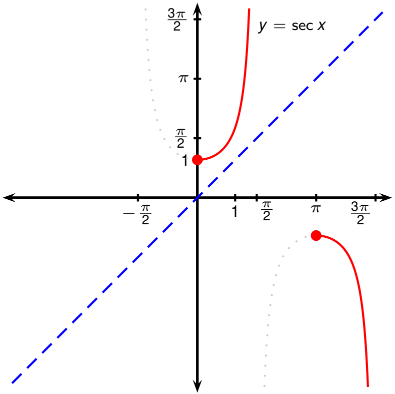

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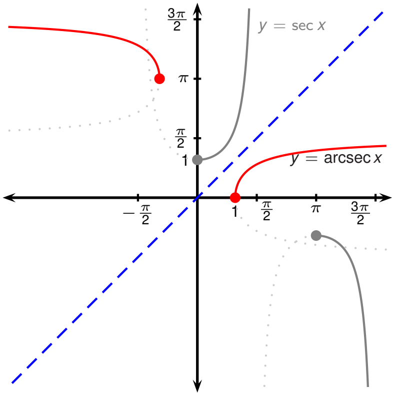

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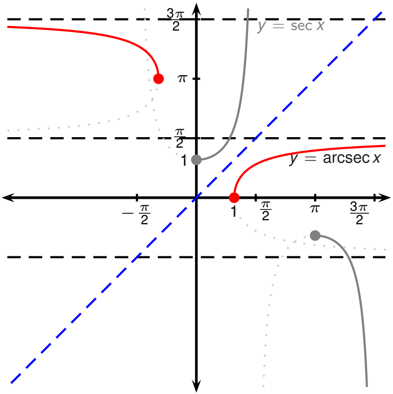

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Rewrite $\sin(2\arccos(x))$ as an algebraic expression of x and $\sqrt{1-x^2}$.

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$\sin(2\arccos(x)) = \sin(2y)$ | Set $y = \arccos x$

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Set $y = \arccos x$

Express via $\sin y, \cos y$

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$$\cos(3\arccos(x)) = \cos(3y) \qquad \left|\ y = \arccos x\right.$$

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&= (\textbf{?} \qquad\qquad\qquad)\cos y && \text{Express via} \\
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$= \cos(2y)\cos y - \sin(2y)\sin y$	Angle sum f-la
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\end{aligned}
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Example

Rewrite $\cos(3\arccos(x))$ as an algebraic expression of x and $\sqrt{1-x^2}$. To simplify $\arccos x$ we try to use $\cos(\arccos x) = x$. Therefore our aim is to rewrite the expression only using the $\cos$ function.

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&= 4\cos^3 y - 3\cos y \\
&= 4x^3 - 3x && \left|\; x = \cos y \right.
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