Calculus I
Derivative of $x^m e^x$

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Theorem (The Product Rule)

If f and g are both differentiable, then

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x).$$

Proof.

Let $F(x) = f(x)g(x)$. Then

$$\begin{aligned}
F'(x) &= \lim_{h\to 0} \frac{F(x+h)-F(x)}{h} = \lim_{h\to 0} \frac{f(x+h)g(x+h)-f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h)-f(x+h)g(x)+f(x+h)g(x)-f(x)g(x)}{h} \\
&= \lim_{h\to 0} \left(f(x+h)\frac{g(x+h)-g(x)}{h} + g(x)\frac{f(x+h)-f(x)}{h} \right) \\
&= \lim_{h\to 0} f(x+h) \cdot \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \\
&\quad + \lim_{h\to 0} g(x) \cdot \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = f(x)g'(x) + g(x)f'(x). \quad \square
\end{aligned}$$

Example (Product Rule, polynomial times the Natural Exponential Function)

$$\text{Differentiate} \quad f(x) = x^3 e^x.$$

$$\begin{aligned} \text{Product Rule:} \quad f'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(x^3\right)(e^x) + \left(x^3\right)\frac{\mathrm{d}}{\mathrm{d}x}(e^x) \\ &= \left(3x^2\right)(e^x) + \left(x^3\right)(e^x) \\ &= e^x\left(x^3 + 3x^2\right). \end{aligned}$$