Calculus II
Simplify $\sin(k \arcsin x)$, $\cos(k \arcsin x)$, $\sin(k \arccos x)$, $\cos(k \arccos x)$

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2019

Example

Rewrite $\sin(2\arccos(x))$ as an algebraic expression of x and $\sqrt{1-x^2}$.

$$\sin(2\arccos(x))$$

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To simplify $\arccos x$ we try to use $\cos(\arccos x) = x$.

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$$\sin(2\arccos(x)) = \sin(2y) \qquad \Big| \text{ Set } y = \arccos x$$

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$$\begin{aligned} \sin(2\arccos(x)) &= \sin(2y) \\ &= \textbf{?} \end{aligned}$$

Set $y = \arccos x$
Express via $\sin y, \cos y$

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$$\begin{aligned} \sin(2\arccos(x)) &= \sin(2y) \\ &= 2\cos y \sin y \end{aligned}$$

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&= 2\cos y \sin y && \text{Express via } \sin y, \cos y \\
&= 2\cos y \left(\pm\sqrt{1-\cos^2 y}\right) && \text{Express } \sin y \text{ via } \cos y
\end{aligned}
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&= 2\cos y\sqrt{1-\cos^2 y} && \sin y > 0 \text{ because}\\
& && 0 \leq y \leq \pi
\end{aligned}$$

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$= 2\cos y\left(\pm\sqrt{1-\cos^2 y}\right)$	Express $\sin y$ via $\cos y$
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$$\begin{aligned} \cos(3\arccos(x)) &= \cos(3y) = \cos(2y+y) && \Big|\ y = \arccos x \\ &= \cos(2y)\cos y - \sin(2y)\sin y && \Big|\ \text{Angle sum f-la} \end{aligned}$$

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$= (\textbf{?} \qquad\qquad)\cos y$	Express via
$- \textbf{?} \qquad\qquad \sin y$	$\sin y, \cos y$

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&\quad - 2\sin y \cos y \sin y && \Big| \; \sin y, \cos y \\
&= \cos^3 y - \sin^2 y \cos y - 2\sin^2 y \cos y \\
&= \cos^3 y - 3\sin^2 y \cos y && \Big| \; \text{Express } \sin y \text{ via } \cos y \\
&= \cos^3 y - 3(\textbf{?} \qquad\qquad)\cos y
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&= \cos^3 y - \sin^2 y\cos y - 2\sin^2 y\cos y \\
&= \cos^3 y - 3{\color{red}\sin^2 y}\cos y && \Big|\; {\color{red}\text{Express } \sin y \text{ via } \cos y} \\
&= \cos^3 y - 3({\color{red}1 - \cos^2 y})\cos y
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& = \cos^3 y - \sin^2 y\cos y - 2\sin^2 y\cos y & & \\
& = \cos^3 y - 3\sin^2 y\cos y & & \text{Express } \sin y \text{ via } \cos y \\
& = \cos^3 y - 3(1-\cos^2 y)\cos y & & \\
& = 4\cos^3 y - 3\cos y & &
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