

Precalculus

Trig cofunction identities and angle-sum formulas

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Outline

- 1 Cofunction identities
- 2 Trigonometric Functions of Sums of Angles
- 3 Double Angle Formulas

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Cofunction identities

Proposition (Cofunction identities)

$$\begin{array}{llll} \sin\left(\frac{\pi}{2} - \alpha\right) & = & \cos \alpha & \sin\left(\frac{\pi}{2} + \alpha\right) & = & \cos \alpha \\ \cos\left(\frac{\pi}{2} - \alpha\right) & = & \sin \alpha & \cos\left(\frac{\pi}{2} + \alpha\right) & = & -\sin \alpha \end{array}$$

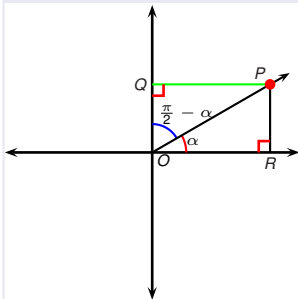
- The proof each formula is broken into 4 cases depending on which quadrant contains α .
- This makes a total of 4 formulas \times 4 cases per formula = 16 cases.
- We show only a few of the cases.
- The proof provides intuition why the formulas are true.
- The Quadrant I part of the proof serves as a visual aid for memorization.
- There is an algebraically simpler (but theoretically advanced) way to prove the above identities through the angle sum formulas, derived in turn from Euler's formula (studied later/in another course).

Cofunction identities

Proposition (Cofunction identities)

$$\begin{aligned}\sin\left(\frac{\pi}{2} - \alpha\right) &= \cos \alpha & \sin\left(\frac{\pi}{2} + \alpha\right) &= \cos \alpha \\ \cos\left(\frac{\pi}{2} - \alpha\right) &= \sin \alpha & \cos\left(\frac{\pi}{2} + \alpha\right) &= -\sin \alpha\end{aligned}$$

Part of Proof.



We are showing $\sin\left(\frac{\pi}{2} - \alpha\right) = \cos \alpha$ when α is in quadrant I.

$$\begin{aligned}\sin\left(\frac{\pi}{2} - \alpha\right) &= \frac{|PQ|}{|OP|} & \left| \square ORPQ \right. \\ &= \frac{|OR|}{|OP|} \\ &= \cos \alpha & \left. \text{as desired} \right.\end{aligned}$$

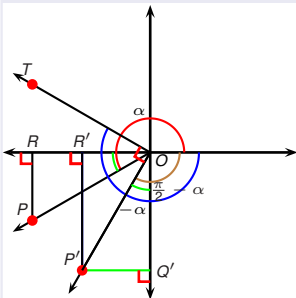


Cofunction identities

Proposition (Cofunction identities)

$$\begin{aligned}\sin\left(\frac{\pi}{2} - \alpha\right) &= \cos \alpha & \sin\left(\frac{\pi}{2} + \alpha\right) &= \cos \alpha \\ \cos\left(\frac{\pi}{2} - \alpha\right) &= \sin \alpha & \cos\left(\frac{\pi}{2} + \alpha\right) &= -\sin \alpha\end{aligned}$$

Part of Proof.



We are showing $\sin\left(\frac{\pi}{2} - \alpha\right) = \cos \alpha$ when α is in Quadrant III. It follows $\frac{\pi}{2} - \alpha$ is in Quadrant III.

$$\begin{aligned}\sin\left(\frac{\pi}{2} - \alpha\right) &= -\frac{|P'R'|}{|OP'|} = -\frac{|OQ'|}{|OP'|} \quad \square OR'P'Q' \\ &= -\frac{|OR|}{|OP|} \\ &= \cos \alpha\end{aligned} \quad \Big| \text{ as desired}$$

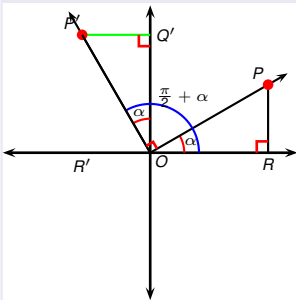


Cofunction identities

Proposition (Cofunction identities)

$$\begin{aligned}\sin\left(\frac{\pi}{2} - \alpha\right) &= \cos \alpha & \sin\left(\frac{\pi}{2} + \alpha\right) &= \cos \alpha \\ \cos\left(\frac{\pi}{2} - \alpha\right) &= \sin \alpha & \cos\left(\frac{\pi}{2} + \alpha\right) &= -\sin \alpha\end{aligned}$$

Part of Proof.



We show $\cos\left(\frac{\pi}{2} + \alpha\right) = -\sin \alpha$ when α is in Quadrant I. It follows $\frac{\pi}{2} + \alpha$ is in Quadrant II.

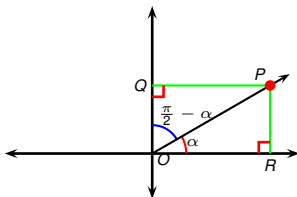
$$\begin{aligned}\cos\left(\frac{\pi}{2} + \alpha\right) &= -\frac{|OR'|}{|OP'|} && \square ORPQ \\ &= -\frac{|P'Q'|}{|OP'|} \\ &= -\frac{|PR|}{|OP|} \\ &= -\sin \alpha. && \text{as desired } \square\end{aligned}$$

Cofunction identities

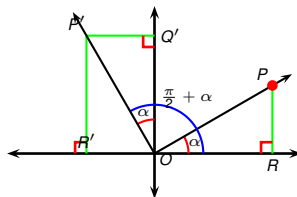
Proposition (Cofunction identities)

$$\begin{aligned}\sin\left(\frac{\pi}{2} - \alpha\right) &= \cos \alpha & \sin\left(\frac{\pi}{2} + \alpha\right) &= \cos \alpha \\ \cos\left(\frac{\pi}{2} - \alpha\right) &= \sin \alpha & \cos\left(\frac{\pi}{2} + \alpha\right) &= -\sin \alpha\end{aligned}$$

To memorize the cofunction identities it suffices to memorize the Quadrant I case via the two diagrams below.



$$\begin{aligned}\sin\left(\frac{\pi}{2} - \alpha\right) &= \frac{|PQ|}{|OP|} \\ \cos\left(\frac{\pi}{2} - \alpha\right) &= \frac{|OR|}{|OP|}\end{aligned}$$

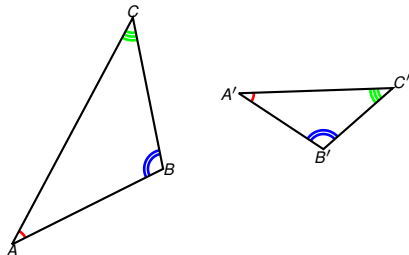


$$\begin{aligned}\sin\left(\frac{\pi}{2} + \alpha\right) &= \frac{|OQ'|}{|OP|} \\ \cos\left(\frac{\pi}{2} + \alpha\right) &= -\frac{|PQ'|}{|OP'|} = \frac{|PR|}{|OP|}\end{aligned}$$

Definition (Similar triangles)

We say that a triangle $\triangle ABC$ is similar to a triangle $\triangle A'B'C'$ if the two triangles have equal angles.

- The equal angles are assumed given in the same order for both triangles, that is, $\angle ABC = \angle A'B'C'$, $\angle BCA = \angle B'C'A'$, $\angle CAB = \angle C'A'B'$.

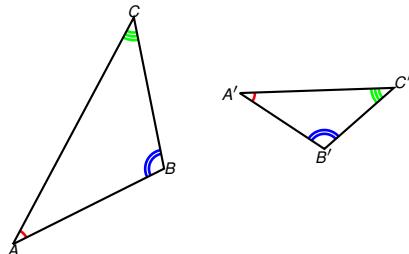


The following statement is proved in the subject of Euclidean (planar) geometry.

Theorem (Similar triangles have equal side ratios)

Let $\triangle ABC$ and $\triangle A'B'C'$ be two similar triangles. Then the ratios of the lengths of the sides of the two triangles are equal, that is

$$\frac{|AB|}{|BC|} = \frac{|A'B'|}{|B'C'|} \quad \frac{|BC|}{|CA|} = \frac{|B'C'|}{|C'A'|} \quad \frac{|CA|}{|AB|} = \frac{|C'A'|}{|A'B'|}$$



$$\begin{array}{l|l} |DA| = |QB| & \square DABQ \\ = \sin \alpha |CB| & \triangle CQB \\ = \sin \alpha \sin \beta |OC| & \triangle OBC \\ = \sin \alpha \sin \beta & \end{array}$$

Trig Functions of Sums and Differences of Angles

Theorem

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$$

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

- We gave a geometric proof of the sum formulas when the two angles are acute and their sum is less than $\pi = 90^\circ$.
- The theorem holds for all angles α, β without any restrictions.
- This can be shown by combining the preceding proof with identities such as $\cos\left(\frac{\pi}{2} - \alpha\right) = \sin \alpha$, $\cos\left(\frac{\pi}{2} + \alpha\right) = -\sin \alpha$.
- There is a theoretically more advanced (but algebraically simpler) proof using Euler's formula (to be studied later/in another course).
- The difference formulas are a consequence of the sum formulas and the fact that \sin is an odd function and \cos is even.

Trig Functions of Differences of Angles

Example

Prove the identities

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$$

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

from the (already demonstrated) identities

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$\sin(\alpha - \beta) = \sin(\alpha + (-\beta))$$

$$= \sin \alpha \cos(-\beta) + \cos \alpha \sin(-\beta) \quad \left| \begin{array}{l} \cos \text{ is even ,} \\ \sin \text{ is odd} \end{array} \right.$$

$$= \sin \alpha \cos \beta - \cos \alpha \sin \beta$$

$$\cos(\alpha - \beta) = \cos(\alpha + (-\beta))$$

$$= \cos \alpha \cos(-\beta) - \sin \alpha \sin(-\beta) \quad \left| \begin{array}{l} \cos \text{ is even ,} \\ \sin \text{ is odd} \end{array} \right.$$

$$= \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

Example

Find the exact value of the trigonometric function using radicals.

$$\cos(105^\circ) = \cos(45^\circ + 60^\circ)$$

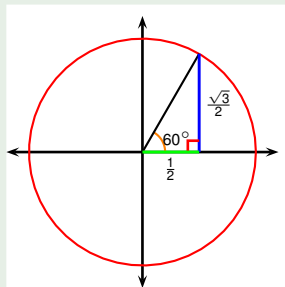
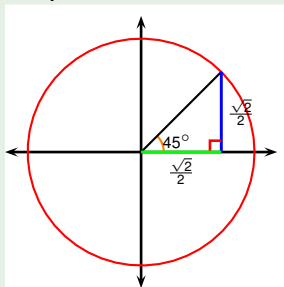
$$= \cos(45^\circ) \cos(60^\circ) - \sin(45^\circ) \sin(60^\circ)$$

$$= \frac{\sqrt{2}}{2} \cdot \frac{1}{2} - \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2}$$

$$= \frac{\sqrt{2} - \sqrt{6}}{4}$$

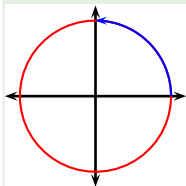
we know the trig
f-ns of 45° and 60°

Angle sum f-la



Example

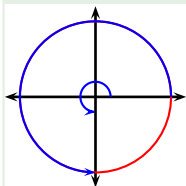
Use the angle sum/difference formulas to simplify.



$$\begin{aligned}\cos\left(\frac{\pi}{2} - x\right) &= \cos\left(\frac{\pi}{2}\right) \cos x + \sin\left(\frac{\pi}{2}\right) \sin x \\ &= 0 \cdot \cos(x) + 1 \cdot \sin x \\ &= \sin x\end{aligned}$$

Example

Use the angle sum/difference formulas to simplify.



$$\begin{aligned}
 \cot \left(\frac{3\pi}{2} + x \right) &= \frac{\cos \left(\frac{3\pi}{2} + x \right)}{\sin \left(\frac{3\pi}{2} + x \right)} \\
 &= \frac{\cos \left(\frac{3\pi}{2} \right) \cos x - \sin \left(\frac{3\pi}{2} \right) \sin x}{\sin \left(\frac{3\pi}{2} \right) \cos x + \cos \left(\frac{3\pi}{2} \right) \sin x} \\
 &= \frac{0 \cdot \cos x - (-1) \sin x}{(-1) \cos x + 0 \cdot \sin x} \\
 &= \frac{-\cos x}{-\sin x} = -\frac{\sin x}{\cos x} \\
 &= -\tan x
 \end{aligned}$$

Example

Show that $\tan(\pi + x) = \tan x$ using the angle sum formulas.

$$\begin{aligned}\tan(\pi + x) &= \frac{\sin(\pi + x)}{\cos(\pi + x)} \\&= \frac{\sin \pi \cos x + \cos \pi \sin x}{\cos \pi \cos x - \sin \pi \sin x} \\&= \frac{0 \cdot \cos x + (-1) \cdot \sin x}{(-1) \cdot \cos x - 0 \cdot \sin x} \\&= \frac{-\sin x}{-\cos x} \\&= \frac{\sin x}{\cos x} \\&= \tan x,\end{aligned}$$

as desired.

Proposition (\tan, \cot are π -periodic)

The tangent and cotangent functions are π -periodic, in other words,

$$\tan(\theta + \pi) = \tan \theta$$

$$\cot(\theta + \pi) = \cot \theta$$

Recall the angle sum formula $\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$.

Example

Show that the Pythagorean identity $\sin^2 \theta + \cos^2 \theta = 1$ follows from the angle difference formula.

$$\begin{aligned} 1 &= \cos 0 \\ &= \cos(\theta - \theta) \\ &= \cos \theta \cos \theta + \sin \theta \sin \theta \\ &= \cos^2 \theta + \sin^2 \theta, \end{aligned}$$

as desired.

Example

Prove the angle sum formula $\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$.

$$\begin{aligned}
 \tan(\alpha + \beta) &= \frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)} \\
 &= \frac{(\sin \alpha \cos \beta + \cos \alpha \sin \beta) \frac{1}{\cos \alpha \cos \beta}}{(\cos \alpha \cos \beta - \sin \alpha \sin \beta) \frac{1}{\cos \alpha \cos \beta}} \\
 &= \frac{\frac{\sin \alpha \cancel{\cos \beta}}{\cos \alpha \cancel{\cos \beta}} + \frac{\cancel{\cos \alpha} \sin \beta}{\cancel{\cos \alpha} \cos \beta}}{\frac{\cancel{\cos \alpha} \cos \beta}{\cancel{\cos \alpha} \cos \beta} - \frac{\sin \alpha \sin \beta}{\cos \alpha \cos \beta}} \\
 &= \frac{\frac{\sin \alpha}{\cos \alpha} + \frac{\sin \beta}{\cos \beta}}{1 - \frac{\sin \alpha}{\cos \alpha} \cdot \frac{\sin \beta}{\cos \beta}} \\
 &= \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}
 \end{aligned}$$

Double angle formulas

Proposition (Double angle formulas)

$$\begin{aligned}\sin(2\alpha) &= 2 \sin \alpha \cos \alpha \\ \cos(2\alpha) &= \cos^2 \alpha - \sin^2 \alpha \\ &= 2 \cos^2 \alpha - 1 \\ &= 1 - 2 \sin^2 \alpha\end{aligned}$$

- The double angle formulas play a special role in integration.

Example

Derive the double-angle formulas.

$$\begin{aligned}\sin(2\alpha) &= \sin(\alpha + \alpha) \\ &= \sin \alpha \cos \alpha + \cos \alpha \sin \alpha \\ &= 2 \sin \alpha \cos \alpha\end{aligned}$$

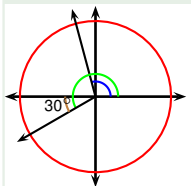
$$\begin{aligned}\cos(2\alpha) &= \cos(\alpha + \alpha) \\ &= \cos \alpha \cos \alpha - \sin \alpha \sin \alpha \\ &= \cos^2 \alpha - \sin^2 \alpha \\ &= \cos^2 \alpha - (1 - \cos^2 \alpha) \\ &= 2 \cos^2 \alpha - 1 \\ &= 1 - \sin^2 \alpha - \sin^2 \alpha \\ &= 1 - 2 \sin^2 \alpha\end{aligned}$$

Recall the half angle formula $\cos \alpha = \pm \sqrt{\frac{1 + \cos(2\alpha)}{2}}$.

Example

Using radicals, find the exact value of the trigonometric expression.

$$\begin{aligned}
 \cos 105^\circ &= \pm \sqrt{\frac{1 + \cos(2 \cdot 105^\circ)}{2}} && \left| \cos 105^\circ < 0 \right. \\
 &= -\sqrt{\frac{1 + \cos(210^\circ)}{2}} \\
 &= -\sqrt{\frac{1 - \cos(30^\circ)}{2}} \\
 &= -\sqrt{\frac{1 - \frac{\sqrt{3}}{2}}{2}} = -\sqrt{\frac{2 - \sqrt{3}}{2 \cdot 2}} \\
 &= -\frac{\sqrt{2 - \sqrt{3}}}{2}
 \end{aligned}$$



Proposition (Power-Reducing Formulas)

$$\sin^2 \alpha = \frac{1 - \cos(2\alpha)}{2} \quad \cos^2 \alpha = \frac{1 + \cos(2\alpha)}{2}$$

Proof.

$$\begin{aligned} \cos(2\alpha) &= 1 - 2\sin^2 \alpha & \cos(2\alpha) &= 2\cos^2 \alpha - 1 \\ 2\sin^2 \alpha &= 1 - \cos(2\alpha) & 2\cos^2 \alpha &= 1 + \cos(2\alpha) \\ \sin^2 \alpha &= \frac{1 - \cos(2\alpha)}{2} & \cos^2 \alpha &= \frac{1 + \cos(2\alpha)}{2} \end{aligned}$$



Corollary

$$\sin \alpha = \pm \sqrt{\frac{1 - \cos(2\alpha)}{2}} \quad \cos \alpha = \pm \sqrt{\frac{1 + \cos(2\alpha)}{2}}$$

Corollary (Half-Angle Formulas)

$$\sin\left(\frac{\beta}{2}\right) = \pm \sqrt{\frac{1 - \cos \beta}{2}} \quad \cos\left(\frac{\beta}{2}\right) = \pm \sqrt{\frac{1 + \cos \beta}{2}}$$

Proposition (Power-Reducing Formulas)

$$\sin^2 \alpha = \frac{1 - \cos(2\alpha)}{2} \quad \cos^2 \alpha = \frac{1 + \cos(2\alpha)}{2}$$

- The power reducing formulas are used to express $\sin^k \alpha$ and $\cos^k \alpha$ via lower powers of the sin and cos functions (applied to angles other than α).
- This technique will play a key role in integration (studied later/in another course).

Recall the formulas: $\sin^2 \beta = \frac{1 - \cos(2\beta)}{2}$, $\cos^2 \beta = \frac{\cos(2\beta) + 1}{2}$.

Example

Rewrite $\sin^4 \alpha$ in terms of first powers of the cosines and sines of multiples of the angle α .

$$\begin{aligned}\sin^4 \alpha &= (\sin^2 \alpha)^2 \\&= \left(\frac{1 - \cos(2\alpha)}{2} \right)^2 \\&= \frac{1}{4} (1 - 2\cos(2\alpha) + \cos^2(2\alpha)) \\&= \frac{1}{4} \left(1 - 2\cos(2\alpha) + \frac{\cos(2 \cdot 2\alpha) + 1}{2} \right) \\&= \frac{1}{4} \left(1 - 2\cos(2\alpha) + \frac{\cos(2 \cdot 2\alpha)}{2} + \frac{1}{2} \right) \\&= \frac{1}{4} \left(\frac{3}{2} - 2\cos(2\alpha) + \frac{\cos(4\alpha)}{2} \right) \\&= \frac{1}{8} (3 - 4\cos(2\alpha) + \cos(4\alpha))\end{aligned}$$