

Precalculus

Exponent basics

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Outline

- 1 Exponents
 - Two ways to define exponents
 - Basic properties
 - The Natural Exponential Function

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$$\begin{aligned} 1 \quad & a^x a^y = a^{x+y} \\ 2 \quad & \frac{a^x}{a^y} = a^{x-y} \\ 3 \quad & (a^x)^y = a^{xy} \end{aligned}$$

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 - the second alternative definition is easier to compute with.

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- This is the definition assumed in many elementary courses.

Exponent definition using series (approach II)

- The following formula (studied much later) can be used as alternative definition.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

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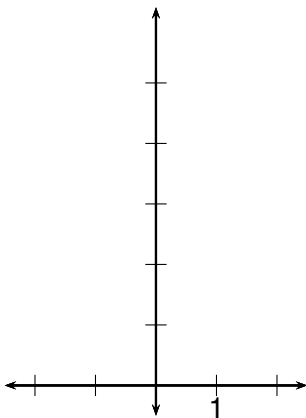
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- Pros: this is how e^x and a^x are actually computed (by modern computers and by humans in the past).

Exponential Functions

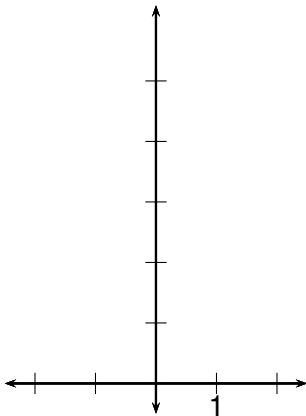
The function $f(x) = 2^x$ is called an exponential function because the variable x is the exponent.



x	y
2	
1	
0	
-1	
-2	

Exponential Functions

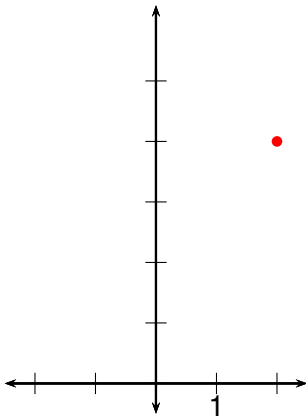
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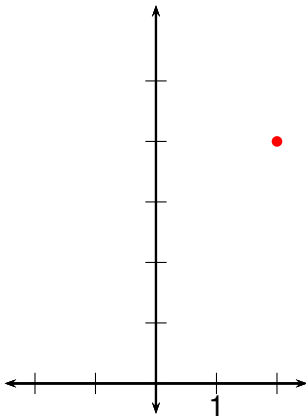
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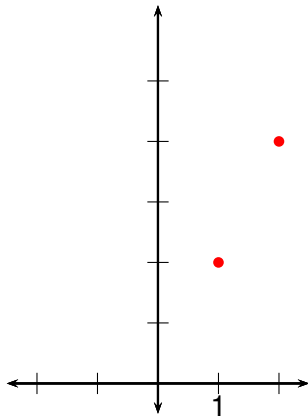
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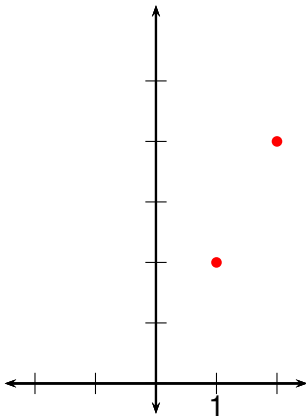
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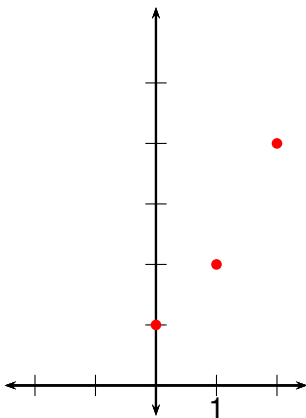
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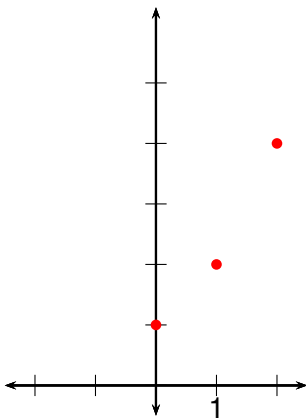
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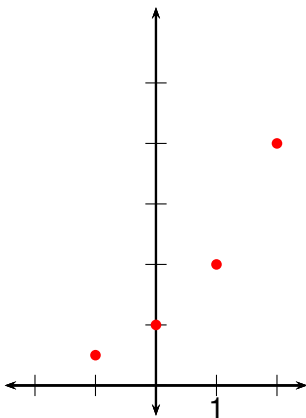
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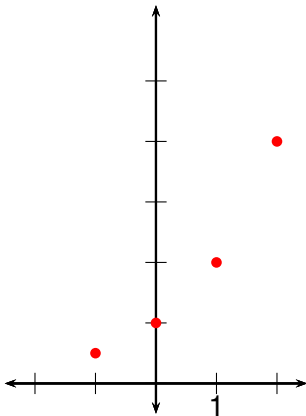
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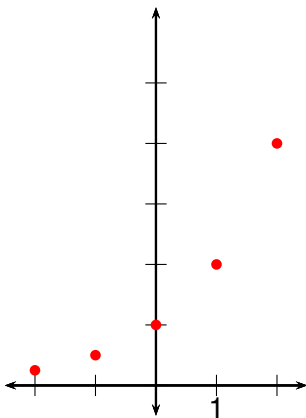
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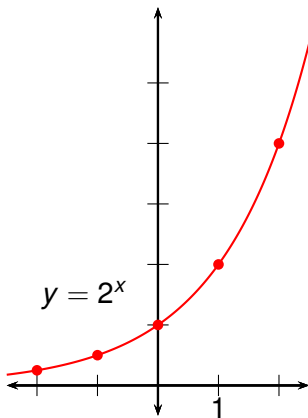
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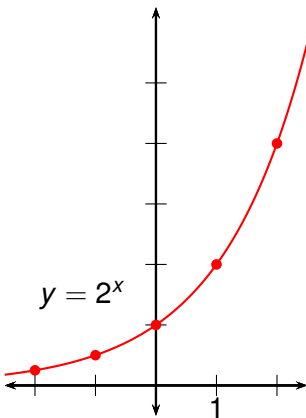
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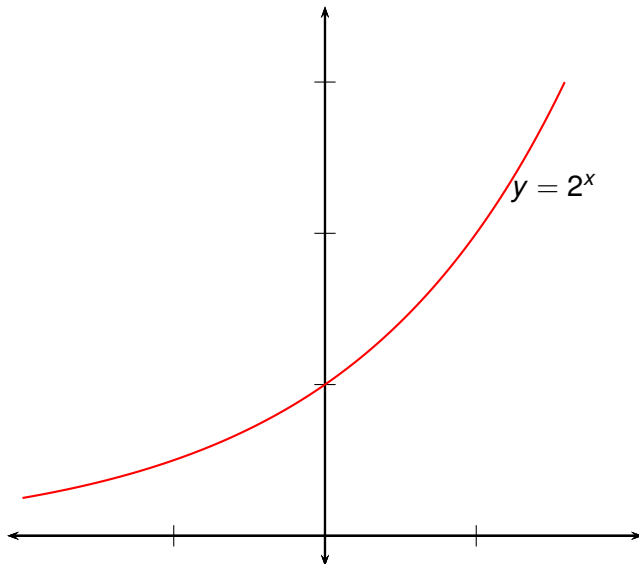


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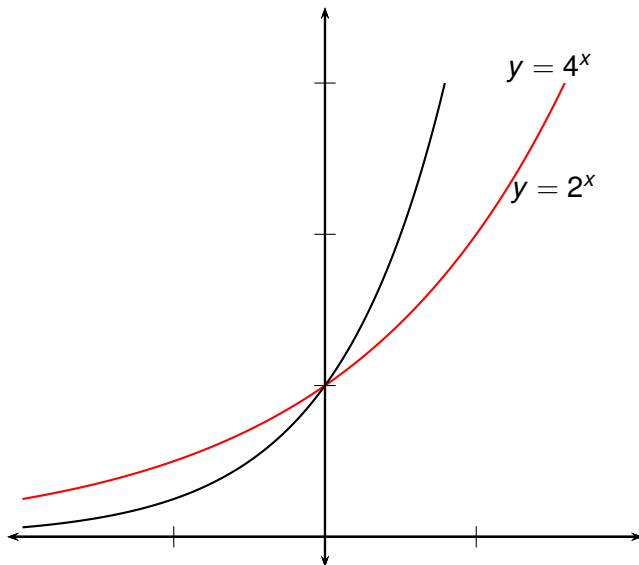
(Exponential Function Terminology)

An exponential function is a function of the form $f(x) = a^x$, where a is a positive constant.

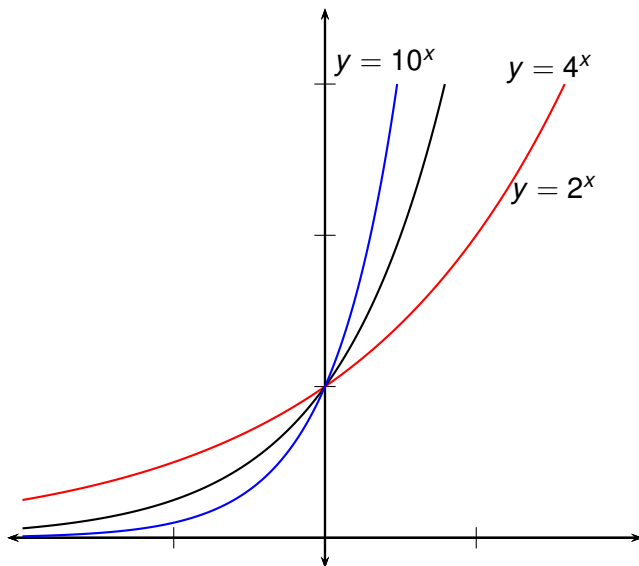
Graphs of various exponential functions.



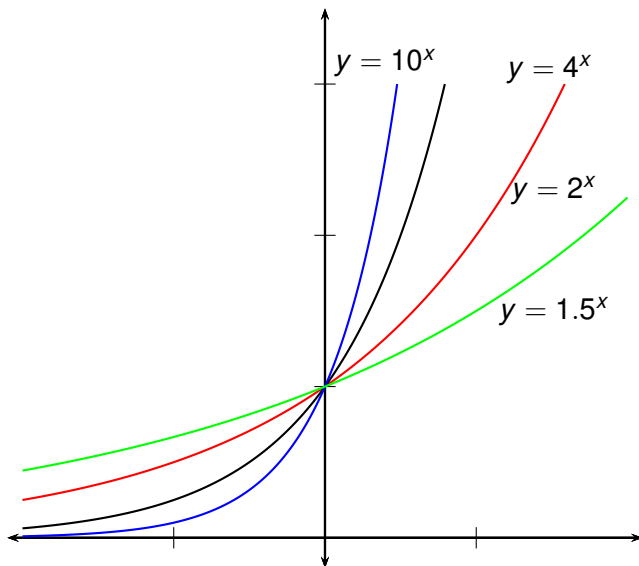
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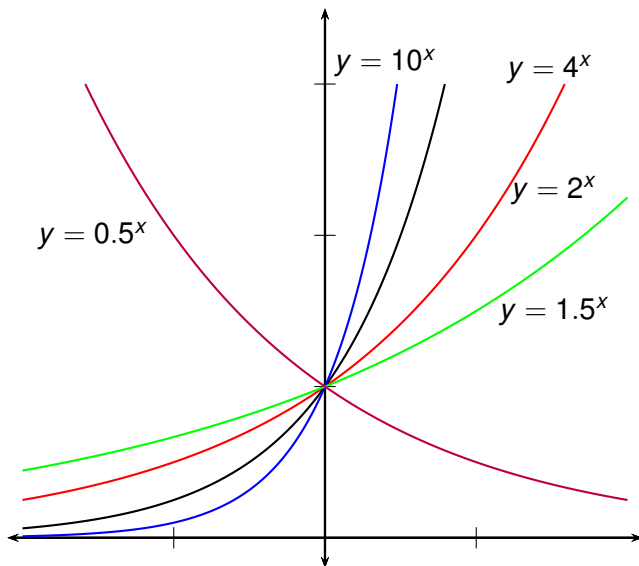
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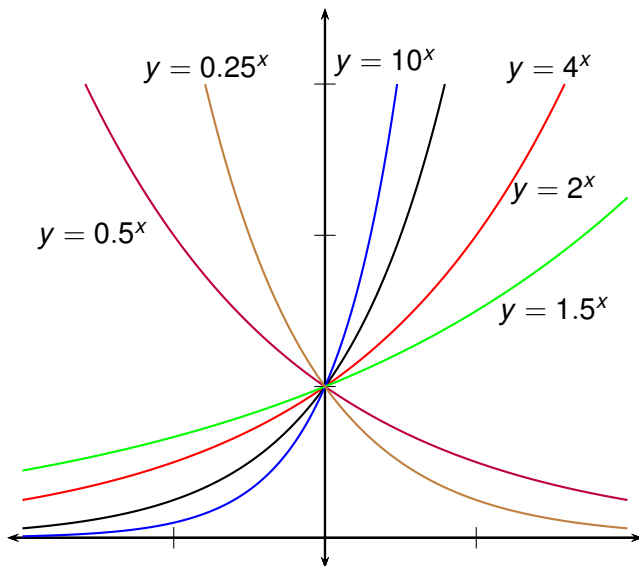
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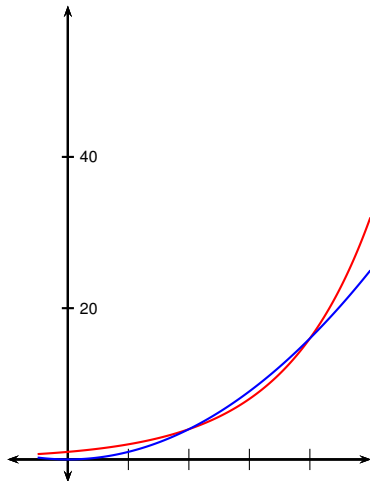
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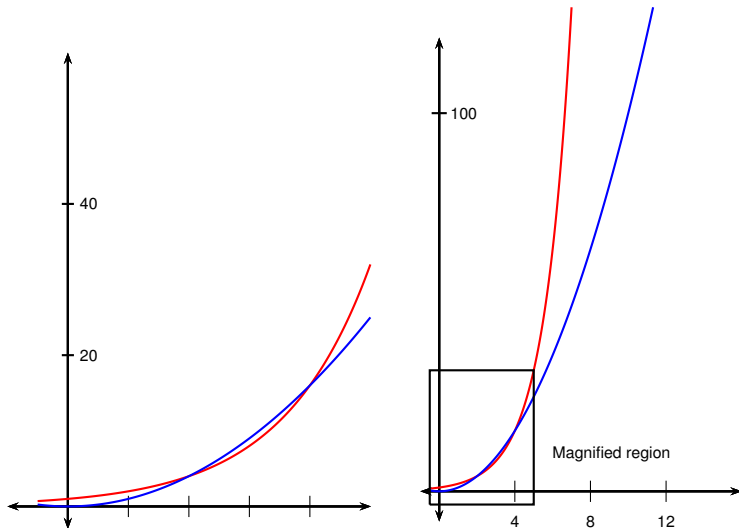
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Graphical comparison of $y = 2^x$ with $y = x^2$. Axes have different scales.

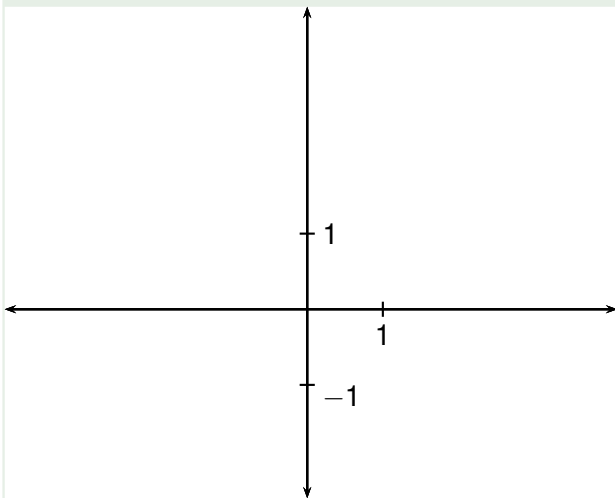


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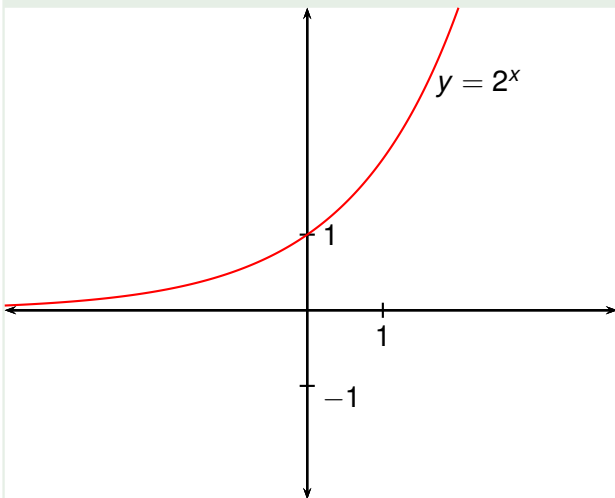
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Draw the graph of the function $y = 2^{-x} - 1 = 0.5^x - 1 = \left(\frac{1}{2}\right)^x - 1$.
Assume the graph of $y = 2^x$ given.



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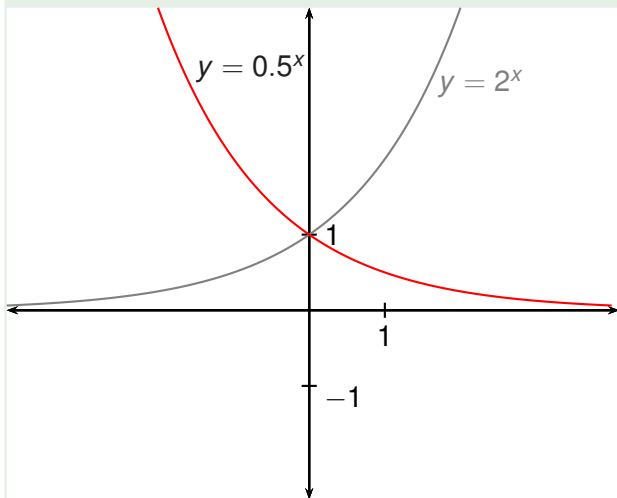
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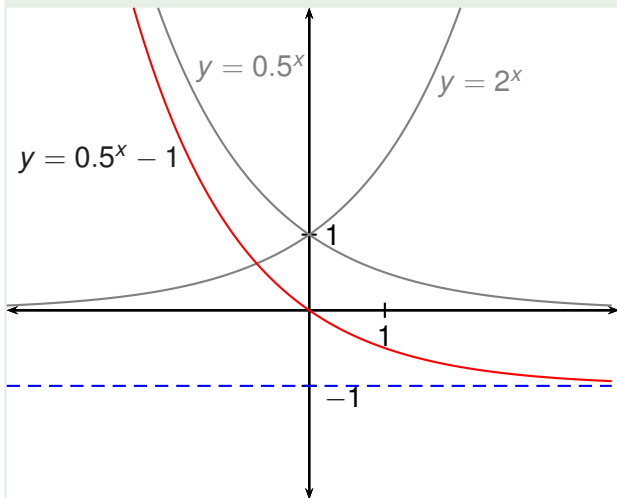
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- Plot $g(x) - 1 =$ shift graph $g(x)$ 1 unit down.

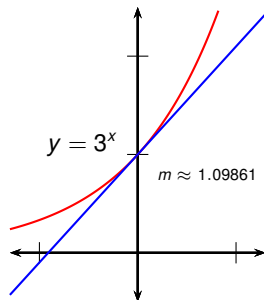
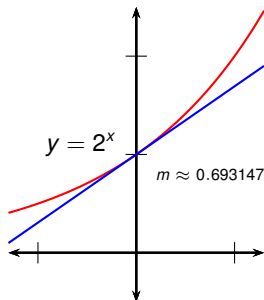
Proposition

Let $a > 0$, $a \neq 1$. Let x and y be real numbers. Then $a^x = a^y$ if and only if $x = y$.

- In other words, the exponent function a^x is one-to-one.

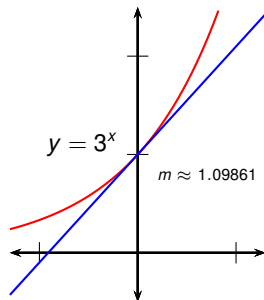
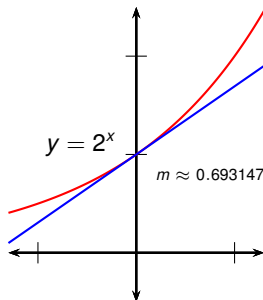
The Natural Exponential Function

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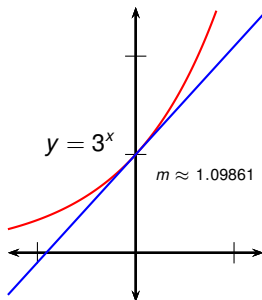
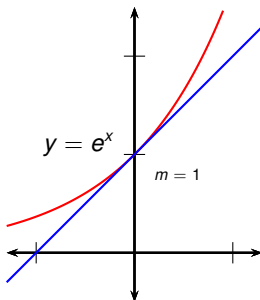
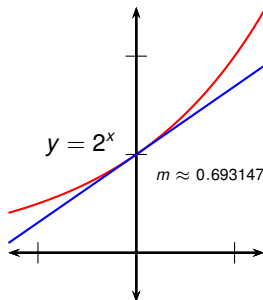
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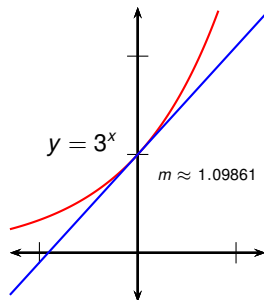
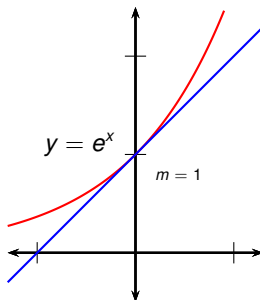
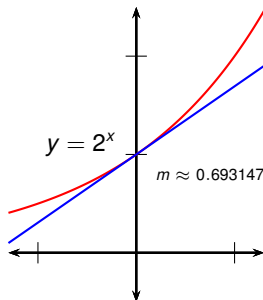
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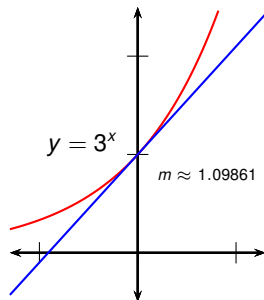
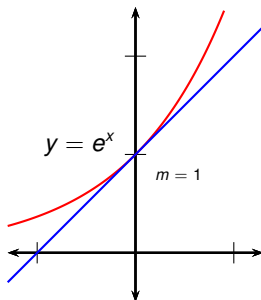
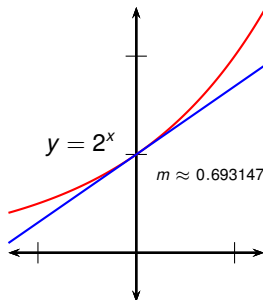
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- e is a number between 2 and 3.
- In fact, $e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots \approx 2.71828$.



Recall that $e = 1 + \frac{1}{1} + \frac{1}{2!} + \frac{1}{3!} + \cdots \approx 2.718281828$.

Theorem (The Number e as a Limit)

For large n we have that:

$$\begin{aligned} e &\approx \left(1 + \frac{1}{n}\right)^n \\ &\approx (1 + n)^{\frac{1}{n}} \\ e^x &\approx \left(1 + \frac{x}{n}\right)^n \end{aligned}$$

All approximations become better as n increases.

- The approximation was discovered by Jacob Bernoulli (1655-1705) in order to apply to compound interest rate computations.

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Definition

The amount of money obtained from principal (original deposit) P after n years of annual compound interest rate of $k\%$, compounded once a year, is given by the formula

$$P \left(1 + \frac{k}{100} \right)^n .$$

Example

You have 1000 USD kept at annual rate of 5%. The interest is compounded yearly. Approximate without using a calculator the amount of money you will have after 40 years. Check your approximation with a calculator.

Example

Decide, without using a calculator, which is more profitable: earning a yearly compound interest of 2% for 150 years or earning yearly simple interest of 11% for 150 years? Check your approximation with a calculator.

Example

When quickly computing interest rate “in the head”, financial advisors often use the following trick called the “rule of 72”. To find the time in years t needed for a sum to double under compound interest rate of $k\%$, financial advisors simply approximate $t \approx \frac{72}{k}$.

To illustrate the rule, under an interest rate of 2% , one needs approximately $\frac{72}{2} = 36$ years for the sum to double. Under interest rate of 6% , the sum doubles after only about $\frac{72}{6} = 12$ years. In 36 years an interest of 6% would double 3 times, in other words would increase by a factor of $2^3 = 8$.

Using the approximation $e \approx (1 + \frac{1}{n})^n$ for large n , justify the rule of 72.