Precalculus
Completing the square

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Example (Completing the square)

Complete the square.

$$
\begin{aligned}
3x^2 - 5x + 1 &= 3\left(x^2 - \frac{5}{3}x\right) + 1 \\
&= 3\left(x^2 - 2 \cdot \frac{5}{2 \cdot 3}x\right) + 1 \\
&= 3\left(x^2 - 2 \cdot \frac{5}{6}x + \left(\frac{5}{6}\right)^2 - \left(\frac{5}{6}\right)^2\right) + 1 \\
&= 3\left(\left(x - \frac{5}{6}\right)^2 - \frac{25}{36}\right) + 1 \\
&= 3\left(x - \frac{5}{6}\right)^2 - \frac{25}{12} + 1 \\
&= 3\left(x - \frac{5}{6}\right)^2 - \frac{13}{12}.
\end{aligned}
$$

Definition (Completing the square)

Let $a \neq 0$. To *complete the square* means to carry out the following algebraic manipulation.

$$
\begin{aligned}
ax^2 + bx + c &= a\left(x^2 + \frac{b}{a}x\right) + c \\
&= a\left(x^2 + 2 \cdot \frac{b}{2a}x\right) + c \\
&= a\left(x^2 + 2\frac{b}{2a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right) + c \quad \left| \begin{array}{l} \text{Add \& subtract} \\ \left(\frac{b}{2a}\right)^2 \\ \text{use} \\ (A+B)^2 = \\ A^2 + 2AB + B^2 \end{array} \right. \\
&= a\left(\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2}\right) + c \\
&= a\left(x + \frac{b}{2a}\right)^2 - \cancel{a} \cdot \frac{b^2}{4a^{\cancel{2}}} + c \\
&= a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}.
\end{aligned}
$$