# Calculus I
# The Fundamental Theorem of Calculus, Part I

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# Outline

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- Part 2 of the FTC roughly says "integration undoes differentiation."
- Part 2 of the FTC was already studied as the Evaluation Theorem. It allows us to compute integrals by finding antiderivatives, without writing limits of Riemann sums.
- Part 1 of the FTC roughly says "differentiation undoes integration."
- Part 1 of the FTC deals with functions of the form

$$g(x) = \int_a^x f(t)\mathrm{d}t$$

where $f$ is a continuous function on $[a, b]$ and $x$ varies between $a$ and $b$.

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- If we let $x$ vary, then $\int_a^x f(t)\mathrm{d}t$ varies.
- If $f$ is positive, then $g$ can be interpreted as the area under $f$ from $a$ to $x$.

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**Theorem (The Fundamental Theorem of Calculus, Part 1)**

*If $f$ is continuous on $[a,b]$, then the function $g$ defined by*

$$g(x) = \int_a^x f(t) \mathrm{d}t$$

*is continuous on $[a,b]$ and differentiable on $(a,b)$, and $g'(x) = f(x)$.*

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For each formula $g(x)$, find the derivative $g'(x)$.

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| $\displaystyle\int_{35}^x \frac{1+r^2+4r^3}{1-r^4}\mathrm{d}r$ | |
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$$\begin{aligned} \text{Differentiate} \quad y &= \int_0^{x^4} \sec t \, dt. \\ \text{Let} \quad u &= x^4. \\ \text{Then} \quad y &= \int_0^{u} \sec t \, dt. \end{aligned}$$

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**Theorem (The Fundamental Theorem of Calculus)**

*Suppose $f$ is continuous on $[a, b]$. Then*

1. *If* $G(x) = \int_a^x f(t)\mathrm{d}t$, *then* $G'(x) = f(x)$.
2. $\int_a^b f(x)\mathrm{d}x = F(b) - F(a)$, *where $F$ is any antiderivative of $f$.*

We already studied part 2 of the FTC as the Evaluation Theorem.

## Theorem

*Let $A, B$-numbers, $a(x), b(x)$ -differentiable functions with $A < a(x) < B$, $A < b(x) < B$. Let $f$ - continuous on $[A, B]$ and $G(x) = \int_{a(x)}^{b(x)} f(t)dt$. Then $G'(x) = f(b(x))b'(x) - f(a(x))a'(x)$.*

## Proof.

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$$G(x) = \int_{a(x)}^{b(x)} f(t)dt = \int_{c}^{b(x)} f(t)dt + \int_{a(x)}^{c} f(t)dt$$

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Then

$G'(x) = (h(b(x)) - h(a(x)))'$

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Then using the chain rule we get
$G'(x) = (h(b(x)) - h(a(x)))' = h'(b(x))b'(x) - h'(a(x))a'(x)$

□

## Theorem

*Let $A, B$-numbers, $a(x), b(x)$ -differentiable functions with $A < a(x) < B$, $A < b(x) < B$. Let $f$ - continuous on $[A, B]$ and $G(x) = \int_{a(x)}^{b(x)} f(t)dt$. Then $G'(x) = f(b(x))b'(x) - f(a(x))a'(x)$.*

## Proof.

Let $c \in (A, B)$. Set $h(u) = \int_c^u f(t)dt$. FTC part 1 states that $h'(u) = f(u)$.

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Problems similar to the following often appear on Calculus I exams.

**Example**

Let $G(x) = \int_{\sqrt{x}}^{x^2} \ln t dt$, $x > 0$. Find $G'(x)$.

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*Let $f$ be a function continuous on $[a, b]$ and let $G(x) = \int_a^x f(t)\mathrm{d}t$ for all $x \in [a, b]$. Then $G$ is differentiable and $G'(x) = f(x)$.*

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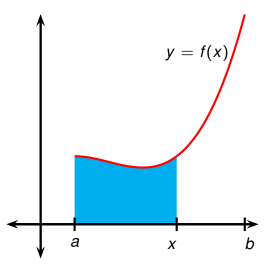

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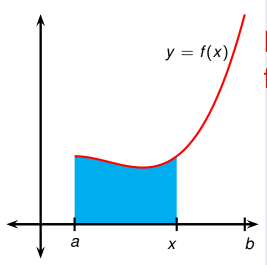


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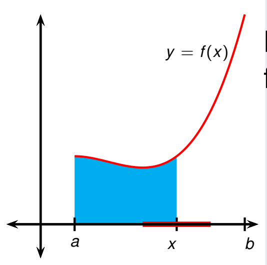


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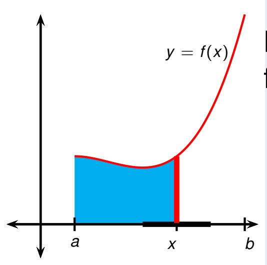


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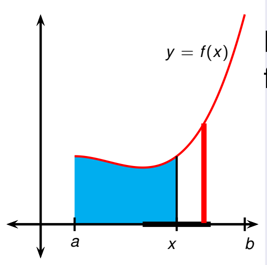


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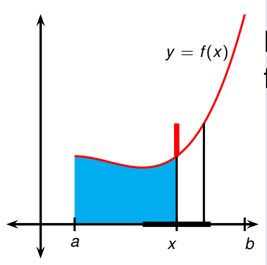


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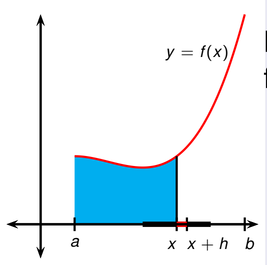


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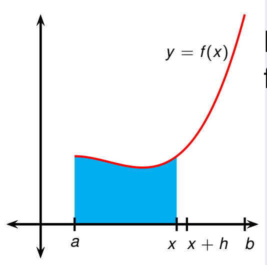


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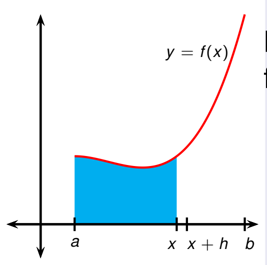


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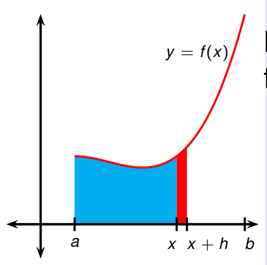


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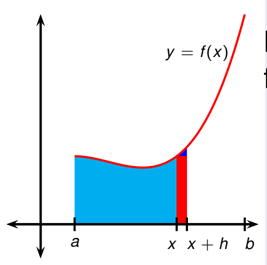


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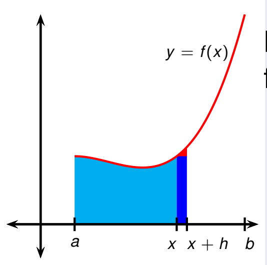


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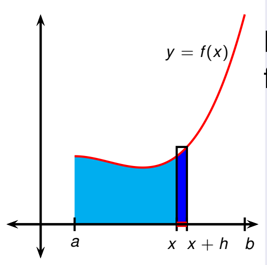


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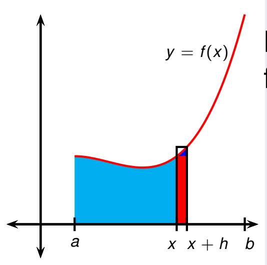


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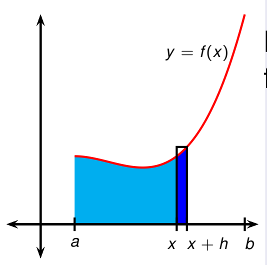


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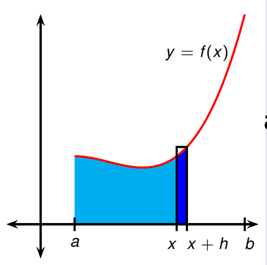


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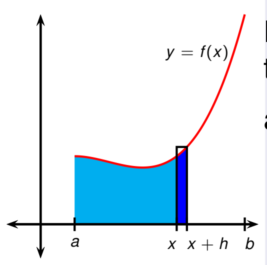


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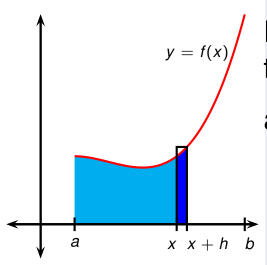


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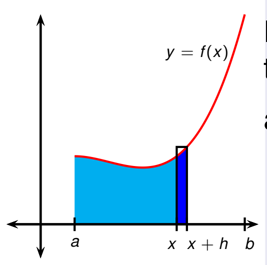


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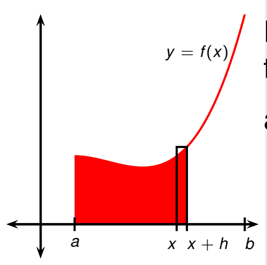


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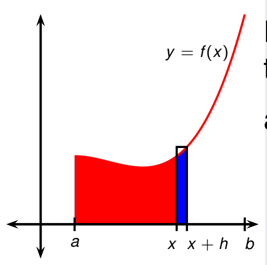


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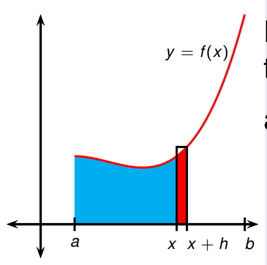


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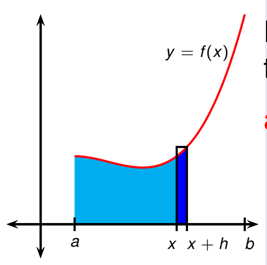


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- The Evaluation Theorem says that, if $f$ is continuous on $[a, b]$, then
$$\int_a^b f(x)\mathrm{d}x = F(b) - F(a),$$
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**Theorem (The Net Change Theorem)**

*The integral of the rate of change is the net change:*

$$\int_a^b F'(x)\mathrm{d}x = F(b) - F(a).$$

- If an object moves along a straight line with position function $s(t)$, then its velocity is $v(t) = s'(t)$.
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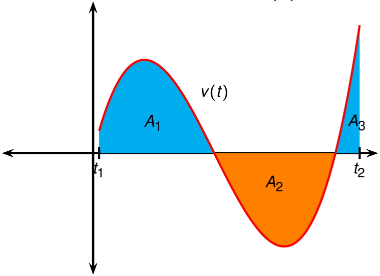


$$\begin{aligned} \text{displacement} &= \int_{t_1}^{t_2} v(t)\mathrm{d}t \\ &= A_1 - A_2 + A_3 \\ \text{distance} &= \int_{t_1}^{t_2} |v(t)|\mathrm{d}t \\ &= A_1 + A_2 + A_3 \end{aligned}$$

## Example

A particle moves along a line so that its velocity at time $t$ is $v(t) = t^2 - t - 6$ (measured in meters per second).

1. Find the displacement of the particle during the time period $1 \leq t \leq 4$.
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Therefore the particle moves 4.5m to the left.

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&= \frac{61}{6} \approx 10.17\mathrm{m}
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- Velocity is the antiderivative of acceleration.
- If we know the acceleration and the initial values $s(0)$ and $v(0)$ for position and velocity, then we can find $s(t)$ by antidifferentiating twice.

An object near the Earth is subject to a gravitational force that produces a downward acceleration of 32 ft/s$^2$ (or 9.8 m/s$^2$).

## Example

A ball is thrown upward with a speed of 48 ft/s from the edge of a cliff 432 ft above the ground. Find its height above the ground $t$ seconds later.

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A ball is thrown upward with a speed of 48 ft/s from the edge of a cliff 432 ft above the ground. Find its height above the ground $t$ seconds later.

$$\begin{aligned} v'(t) &= a(t) = -32 \\ v(t) &= -32t + C \\ &= -32t + 48 \end{aligned}$$

To find $C$, use the fact that $v(0) = 48$.

$$\begin{aligned} v(0) &= 48 \\ -32 \cdot 0 + C &= 48 \\ C &= 48 \end{aligned}$$

$$\begin{aligned} s'(t) &= -32t + 48 \\ s(t) &= -16t^2 + 48t + D \\ &= -16t^2 + 48t + 432 \end{aligned}$$

To find $D$, use the fact that $s(0) = 432$.

$$\begin{aligned} s(0) &= 432 \\ -16 \cdot 0^2 + 48 \cdot 0 + D &= 432 \\ D &= 432 \end{aligned}$$