Calculus II
Homework
Integrals of involving radicals of quadratics

1. Let $x \in (0, 1)$. Express the following using x and $\sqrt{1 - x^2}$.

   (a) $\sin(\arcsin(x))$.
   (b) $\sin(2 \arcsin(x))$.
   (c) $\sin(3 \arcsin(x))$.
   (d) $\sin(\arccos(x))$.
   (e) $\sin(2 \arccos(x))$.
   (f) $\sin(3 \arccos(x))$.
   (g) $\cos(2 \arcsin(x))$.
   (h) $\cos(3 \arccos(x))$.

2. Express as the following as an algebraic expression of x. In other words, "get rid" of the trigonometric and inverse trigonometric expressions.

   (a) $\cos^2(\arctan x)$.
   (b) $-\sin^2(\operatorname{arccot} x)$.
   (c) $\dfrac{1}{\cos(\arcsin x)}$.
   (d) $-\dfrac{1}{\sin(\arccos x)}$.

3. Rewrite as a rational function of t. This problem will be later used to derive the Euler substitutions (an important technique for integrating).

   (a) $\cos(2 \arctan t)$.
   (b) $\sin(2 \arctan t)$.
   (c) $\tan(2 \arctan t)$.
   (d) $\cot(2 \arctan t)$.
   (e) $\csc(2 \arctan t)$.
   (f) $\sec(2 \arctan t)$.
   (g) $\cos(2 \operatorname{arccot} t)$.
   (h) $\sin(2 \operatorname{arccot} t)$.
   (i) $\tan(2 \operatorname{arccot} t)$.
   (j) $\cot(2 \operatorname{arccot} t)$.
   (k) $\csc(2 \operatorname{arccot} t)$.
   (l) $\sec(2 \operatorname{arccot} t)$.

4. Compute the derivative (derive the formula).

   (a) $(\arctan x)'$.
   (b) $(\operatorname{arccot} x)'$.
   (c) $(\arcsin x)'$.
   (d) $(\arccos x)'$.
   (e) Let arcsec denote the inverse of the secant function. Compute $(\operatorname{arcsec} x)'$.

5. (a) Let $a + b \neq k\pi$, $a \neq k\pi + \frac{\pi}{2}$ and $b \neq k\pi + \frac{\pi}{2}$ for any $k \in \mathbb{Z}$ (integers). Prove that

   $$\frac{\tan a + \tan b}{1 - \tan a \tan b} = \tan(a + b) \quad .$$

   (b) Let x and y be real. Prove that, for $xy \neq 1$, we have

   $$\arctan x + \arctan y = \arctan\left(\frac{x + y}{1 - xy}\right)$$

   if the left hand side lies between $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

6. Evaluate the indefinite integral. Illustrate the steps of your solutions.

(a) $\int x \sin x \mathrm{d}x$.

(b) $\int x e^{-x} \mathrm{d}x$.

(c) $\int x^2 e^x \mathrm{d}x$.

(d) $\int x \sin(-2x) \mathrm{d}x$.

(e) $\int x^2 \cos(3x) \mathrm{d}x$.

(f) $\int x^2 e^{-2x} \mathrm{d}x$.

(g) $\int x \sin(2x) \mathrm{d}x$.

(h) $\int x \cos(3x) \mathrm{d}x$.

(i) $\int x^2 e^{2x} \mathrm{d}x$.

(j) $\int x^3 e^x \mathrm{d}x$.

7. Evaluate the indefinite integral. Illustrate the steps of your solutions.

(a) $\int x^2 \cos(2x) \mathrm{d}x$.

(b) $\int x^2 e^{ax} \mathrm{d}x$, where a is a constant.

(c) $\int x^2 e^{-ax} \mathrm{d}x$, where a is a constant.

(d) $\int x^2 \frac{(e^{ax} + e^{-ax})^2}{4} \mathrm{d}x$, where a is a constant.

(e) $\int \frac{1}{\cos^2 x} \mathrm{d}x$. (Hint: This problem does not require integration by parts. What is the derivative of $\tan x$?)

(f) $\int (\tan^2 x) \mathrm{d}x$. (Hint: This problem does not require integration by parts. We can use $\tan^2 x = \frac{1}{\cos^2 x} - 1$ and the previous problem.)

(g) $\int x \tan^2 x \mathrm{d}x$. (Hint: $\tan^2 x \mathrm{d}x = \mathrm{d}(F(x))$, where $F(x)$ is the answer from the preceding problem).

(h) $\int e^{-\sqrt{x}} \mathrm{d}x$.

(i) $\int \cos^2 x \, \mathrm{d}x$.

(j) $\int \frac{x}{1+x^2} \mathrm{d}x$ (Hint: use substitution rule, don't use integration by parts)

(k) $\int (\arctan x) \mathrm{d}x$.

(l) $\int (\arcsin x) \mathrm{d}x$.

(m) $\int (\arcsin x)^2 \mathrm{d}x$. (Hint: Try substituting $x = \sin y$.)

(n) $\int \arctan\left(\frac{1}{x}\right) \mathrm{d}x$.

(o) $\int \sin x e^x \mathrm{d}x$

(p) $\int \cos x e^x \mathrm{d}x$

(q) $\int \sin(\ln(x)) \mathrm{d}x$.

(r) $\int \cos(\ln(x)) \mathrm{d}x$.

(s) $\int \ln x \mathrm{d}x$

(t) $\int x \ln x \, \mathrm{d}x$.

(u) $\int \frac{\ln x}{\sqrt{x}} \mathrm{d}x$.

(v) $\int (\ln x)^2 \mathrm{d}x$.

(w) $\int (\ln x)^3 \mathrm{d}x$.

(x) $\int x^2 \cos^2 x \mathrm{d}x$. (This problem is related to Problem 7.d as $\cos x = \frac{e^{ix} + e^{-ix}}{2}$).

8. Compute $\int x^n e^x \mathrm{d}x$, where n is a non-negative integer.

9. Integrate. Illustrate the steps of your solution.

(a) $\int \frac{1}{x+1} \mathrm{d}x$

(b) $\int \frac{x-1}{x+1} \mathrm{d}x$

(c) $\int \frac{1}{(x+1)^2} \mathrm{d}x$

(d) $\int \frac{x}{(x+1)^2} \mathrm{d}x$

(e) $\int \frac{1}{(2x+3)^2} \mathrm{d}x$

(f) $\int \frac{x}{2x^2+3} \mathrm{d}x$

(g) $\int \frac{1}{2x^2+3} \mathrm{d}x$

(h) $\int \frac{x}{2x^2+x+1} \mathrm{d}x$.

(i) $\int \frac{x}{2x^2+x+3} \mathrm{d}x$

(j) $\int \frac{x}{x^2-x+3} \mathrm{d}x$

(k) $\displaystyle\int \frac{1}{(x^2+1)^2} \mathrm{d}x$

(l) $\displaystyle\int \frac{1}{(x^2+x+1)^2} \mathrm{d}x$

(m) $\displaystyle\int \frac{1}{(x^2+1)^3} \mathrm{d}x$

10. Let a, b, c, A, B be real numbers. Suppose in addition $a \neq 0$ and $b^2 - 4ac < 0$. Integrate

$$\int \frac{Ax+B}{ax^2+bx+c} \mathrm{d}x \quad .$$

The purpose of this exercise is to produce a formula in form ready for implementation in a computer algebra system.

11. Let a, b, c, A, B be real numbers and let $n > 1$ be an integer. Suppose in addition $a \neq 0$ and $b^2 - 4ac < 0$. Let

$$J(n) = \int \frac{1}{\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)^n} \mathrm{d}x \quad .$$

(a) Express the integral

$$\int \frac{Ax+B}{(ax^2+bx+c)^n} \mathrm{d}x$$

via $J(n)$.

(b) Express $J(n)$ recursively via $J(n-1)$

The purpose of this exercise is to produce a formula in form ready for implementation in a computer algebra system.

12. Integrate. Some of the examples require partial fraction decomposition and some do not. Illustrate the steps of your solution.

(a) $\displaystyle\int \frac{1}{4x^2+4x+1} \mathrm{d}x$

(b) $\displaystyle\int \frac{1}{1-x^2} \mathrm{d}x$

(c) $\displaystyle\int \frac{1}{5-x^2} \mathrm{d}x$

(d) $\displaystyle\int \frac{x}{4x^2+x+\frac{1}{16}} \mathrm{d}x$

(e) $\displaystyle\int \frac{x+1}{2x^2+x} \mathrm{d}x$

(f) $\displaystyle\int \frac{x}{4x^2+x+5} \mathrm{d}x$

(g) $\displaystyle\int \frac{x}{4x^2+x-5} \mathrm{d}x$

(h) $\displaystyle\int \frac{x}{3x^2+x-2} \mathrm{d}x$

(i) $\displaystyle\int \frac{x}{3x^2+x+2} \mathrm{d}x$

(j) $\displaystyle\int \frac{x}{2x^2+x+1} \mathrm{d}x$

(k) $\displaystyle\int \frac{x}{2x^2+x-1} \mathrm{d}x$

(l) $\displaystyle\int \frac{1}{x^2+x+1} \mathrm{d}x$

(m) $\displaystyle\int \frac{1}{2x^2+5x+1} \mathrm{d}x$

13. Evaluate the indefinite integral. Illustrate all steps of your solution.

(a) $\displaystyle\int \frac{x^3+4}{x^2+4} \mathrm{d}x$

(b) $\displaystyle\int \frac{4x^2}{2x^2-1} \mathrm{d}x$

(c) $\displaystyle\int \frac{x^3}{x^2+2x-3} \mathrm{d}x$

(d) $\displaystyle\int \frac{x^3}{x^2+3x-4} \mathrm{d}x$

(e) $\displaystyle\int \frac{x^3}{2x^2+3x-5} \mathrm{d}x$

(f) $\displaystyle\int \frac{x^2+1}{(x-3)(x-2)^2} \mathrm{d}x$

(g) $\displaystyle\int \frac{x^4}{(x+1)^2(x+2)} \mathrm{d}x$

(h) $\displaystyle\int \frac{15x^2-4x-81}{(x-3)(x+4)(x-1)} \mathrm{d}x$

(i) $\displaystyle\int \frac{x^4+10x^3+18x^2+2x-13}{x^4+4x^3+3x^2-4x-4} \mathrm{d}x$

Check first that $(x-1)(x+2)^2(x+1) = x^4+4x^3+3x^2-4x-4$.

(j) $\displaystyle\int \frac{x^4}{(x^2+2)(x+2)} \mathrm{d}x$

(k) $\displaystyle\int \frac{x^5}{x^3-1} \mathrm{d}x$

(l) $\displaystyle\int \frac{x^4}{(x^2+2)(x+1)^2} \mathrm{d}x$

(m) $\displaystyle\int \frac{3x^2+2x-1}{(x-1)(x^2+1)}\mathrm{d}x$

(n) $\displaystyle\int \frac{x^2-1}{x(x^2+1)^2}\mathrm{d}x$

14. Integrate

$$\int \frac{x^6 - x^5 + \frac{9}{2}x^4 - 4x^3 + \frac{13}{2}x^2 - \frac{7}{2}x + \frac{11}{4}}{x^5 - x^4 + 3x^3 - 3x^2 + \frac{9}{4}x - \frac{9}{4}}\mathrm{d}x \quad .$$

15. Integrate.

(a) $\displaystyle\int \frac{1}{3+\cos x}\mathrm{d}x.$

(b) $\displaystyle\int \frac{1}{4+\cos x}\mathrm{d}x.$

(c) $\displaystyle\int \frac{1}{3+\sin x}\mathrm{d}x.$

(d) $\displaystyle\int \frac{1}{2+\tan x}\mathrm{d}x.$ (Hint: this integral can be done simply with the substitution $x = \arctan t$.)

(e) $\displaystyle\int \frac{\mathrm{d}x}{2\sin x - \cos x + 5}.$

16. Integrate. The answer key has not been proofread, use with caution.

(a) $\displaystyle\int \sin(3x)\cos(2x)\mathrm{d}x.$

(b) $\displaystyle\int \sin x\cos(5x)\mathrm{d}x.$

(c) $\displaystyle\int \cos(3x)\sin(2x)\mathrm{d}x.$

(d) $\displaystyle\int \sin(5x)\sin(3x)\mathrm{d}x.$

(e) $\displaystyle\int \cos(x)\cos(3x)\mathrm{d}x.$

17. Integrate.

(a) $\displaystyle\int \sin^2 x\cos x\mathrm{d}x.$

(b) $\displaystyle\int \sin^2 x\mathrm{d}x.$

(c) $\displaystyle\int \cos^3 x\mathrm{d}x.$

(d) $\displaystyle\int \sin^3 x\cos^4 x\mathrm{d}x.$

18. Integrate.

(a) $\displaystyle\int \sec x\mathrm{d}x.$

(b) $\displaystyle\int \sec^3 x\mathrm{d}x.$

(c) $\displaystyle\int \tan^3 x\mathrm{d}x.$

(d) $\displaystyle\int \sec^2 x\tan^2 x\mathrm{d}x.$

19. Find a linear substitution (via completing the square) to transform the radical to a multiple of an expression of the form $\sqrt{u^2+1}$, $\sqrt{u^2-1}$ or $\sqrt{1-u^2}$.

(a) $\sqrt{x^2+x+1}.$

(b) $\sqrt{-2x^2+x+1}.$

20. Compute the integral.

(a) $\displaystyle\int \frac{\sqrt{1+x^2}}{x^2}\mathrm{d}x.$

21. Compute the integral using a trigonometric substitution.

(a) $\int \frac{\sqrt{9-x^2}}{x^2}\mathrm{d}x$.

22. Compute the integral.

    (a) $\int \sqrt{x^2+1}\mathrm{d}x$

    (b) $\int \sqrt{x^2+2}\mathrm{d}x$

    (c) $\int \sqrt{x^2+x+1}\mathrm{d}x$

    (d) $\int \sqrt{(2x^2+2x+1)}\mathrm{d}x$

    (e) $\int \sqrt{(3x^2+2x+1)}\mathrm{d}x$

    (f) $\int \frac{\sqrt{x^2+1}}{x+1}\mathrm{d}x$

23. Let $b^2-4ac<0$ and $a>0$ be (real) numbers. Show that

    $\int \sqrt{(ax^2+bx+c)}\mathrm{d}x = \frac{\sqrt{a}D}{2}\left(\ln\left(\sqrt{\left(\frac{2xa+b}{2\sqrt{D}a}\right)^2+1}+\frac{2xa+b}{2\sqrt{D}a}\right)+\frac{2xa+b}{2\sqrt{D}a}\sqrt{\left(\frac{2xa+b}{2\sqrt{D}a}\right)^2+1}\right)+C,$

    where $D=\frac{4ac-b^2}{4a^2}$.

24. Integrate

    (a) $\int \sqrt{1-x^2}\mathrm{d}x$

    (b) $\int \sqrt{2-x^2}\mathrm{d}x$

    (c) $\int \sqrt{-x^2+x+1}\mathrm{d}x$

    (d) $\int \sqrt{2-x-x^2}\mathrm{d}x$

    (e) $\int \frac{\sqrt{1-x^2}}{1+x}\mathrm{d}x$

    (f) $\int \frac{\sqrt{1-x^2}}{2+x}\mathrm{d}x$

25. Integrate

    (a) $\int \sqrt{x^2-1}\mathrm{d}x$

    (b) $\int \sqrt{x^2-2}\mathrm{d}x$

    (c) $\int \sqrt{2x^2+x-1}\mathrm{d}x$

    (d) $\int \sqrt{x^2+x-1}\mathrm{d}x$

26. (a) Express $x, \mathrm{d}x$ and $\sqrt{x^2+1}$ via θ and $\mathrm{d}\theta$ for the trigonometric substitution $x=\cot\theta$, $\theta\in(0,\pi)$.

    (b) Express $x, \mathrm{d}x$ and $\sqrt{x^2+1}$ via t and $\mathrm{d}t$ for the Euler substitution $x=\cot(2\arctan t)$, $t>0$. Express t via x.

27. Let the variables x and t be related via $\sqrt{x^2+1}=x+t$.

    (a) Express x via t.

    (b) Express $\sqrt{x^2+1}$ via t alone.

    (c) Express $\mathrm{d}x$ via t and $\mathrm{d}t$.

28. (a) Express $x, \mathrm{d}x$ and $\sqrt{x^2+1}$ via θ and $\mathrm{d}\theta$ for the trigonometric substitution $x = \tan\theta$, $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

    (b) Express $x, \mathrm{d}x$ and $\sqrt{x^2+1}$ via t and $\mathrm{d}t$ for the Euler substitution $x = \tan(2\arctan t)$, $t \in (-1, 1)$. Express t via x.

29. Let the variables x and t be related via $\sqrt{x^2+1} = \frac{x}{t} - 1$.

    (a) Express x via t.

    (b) Express $\sqrt{x^2+1}$ via t alone.

    (c) Express $\mathrm{d}x$ via t and $\mathrm{d}t$.

30. (a) Express $x, \mathrm{d}x$ and $\sqrt{1-x^2}$ via θ and $\mathrm{d}\theta$ for the trigonometric substitution $x = \cos\theta$, $\theta \in [0, \pi]$.

    (b) Express $x, \mathrm{d}x$ and $\sqrt{1-x^2}$ via t and $\mathrm{d}t$ for the Euler substitution $x = \cos(2\arctan t)$, $t \geq 0$. Express t via x.

31. Let the variables x and t be related via $\sqrt{-x^2+1} = (1-x)t$.

    (a) Express x via t.

    (b) Express $\sqrt{-x^2+1}$ via t alone.

    (c) Express $\mathrm{d}x$ via t and $\mathrm{d}t$.

32. (a) Express $x, \mathrm{d}x$ and $\sqrt{1-x^2}$ via θ and $\mathrm{d}\theta$ for the trigonometric substitution $x = \sin\theta$, $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

    (b) Express $x, \mathrm{d}x$ and $\sqrt{1-x^2}$ via t and $\mathrm{d}t$ for the Euler substitution $x = \sin(2\arctan t)$, $t \in [-1, 1]$. Express t via x.

33. Let the variables x and t be related via $\sqrt{-x^2+1} = 1 - xt$.

    (a) Express x via t.

    (b) Express $\sqrt{-x^2+1}$ via t alone.

    (c) Express $\mathrm{d}x$ via t and $\mathrm{d}t$.

34. (a) Express $x, \mathrm{d}x$ and $\sqrt{x^2-1}$ via θ and $\mathrm{d}\theta$ for the trigonometric substitution $x = \csc\theta$, $\theta \in \left[0, \frac{\pi}{2}\right] \cup \left[\pi, \frac{3\pi}{2}\right)$.

    (b) Express $x, \mathrm{d}x$ and $\sqrt{1-x^2}$ via t and $\mathrm{d}t$ for the Euler substitution $x = \sec(2\arctan t)$, $t \in (-\infty, -1) \cup [1, 0)$. Express t via x.

35. Let the variables x and t be related via $\sqrt{x^2-1} = (x+1)t$.

    (a) Express x via t.

    (b) Express $\sqrt{x^2-1}$ via t alone.

    (c) Express $\mathrm{d}x$ via t and $\mathrm{d}t$.

36. (a) Express $x, \mathrm{d}x$ and $\sqrt{1-x^2}$ via θ and $\mathrm{d}\theta$ for the trigonometric substitution $x = \csc\theta$, $\theta \in \left[0, \frac{\pi}{2}\right] \cup \left[\pi, \frac{3\pi}{2}\right)$.

    (b) Express $x, \mathrm{d}x$ and $\sqrt{1-x^2}$ via t and $\mathrm{d}t$ for the Euler substitution $x = \csc(2\arctan t)$, $t \in (-\infty, -1) \cup [0, 1)$. Express t via x.

37. Let the variables x and t be related via $\sqrt{x^2-1} = \frac{1}{t} - x$.

    (a) Express x via t.

    (b) Express $\sqrt{x^2-1}$ via t alone.

    (c) Express $\mathrm{d}x$ via t and $\mathrm{d}t$.