

# Calculus I

## Newton's Method

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# Outline

## 1 Newton's Method

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# Newton's Method

Find the roots of these equations:

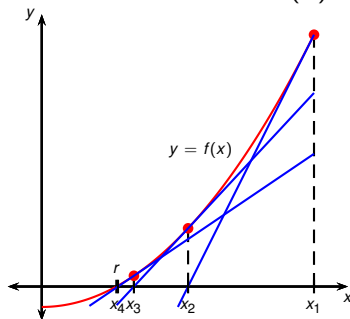
$$x^3 - 5x^2 - 6x = 0$$

$$x(x - 6)(x + 1) = 0$$

$$48x(1 + x)^{60} - (1 + x)^{60} + 1 = 0$$

- Roots:  $x = 0, -1$ , or  $6$ .
- No problem.
- Problem.
- Plug it into a computer algebra system. The non-zero root is about  $0.0076$ .
- How does the computer find the root?
- Probably using Newton's Method.

Goal: find a root  $r$  of  $f(x)$ .



- Pick a number  $x_1$ .
- Find the tangent to  $f$  at  $(x_1, f(x_1))$ .
- Call the  $x$ -intercept of this line  $x_2$ .
- Repeat the process using  $x_2$ .
- Find the tangent to  $f$  at  $(x_2, f(x_2))$ .
- Call the  $x$ -intercept of this line  $x_3$ , and so on.

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$\vdots$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Equation:  $y - f(x_n) = f'(x_n)(x - x_n)$   
 x-intercept:  $0 - f(x_n) = f'(x_n)(x_{n+1} - x_n)$   
 $f'(x_n)x_n - f(x_n) = f'(x_n)x_{n+1}$   
 $x_{n+1} = \frac{f'(x_n)x_n - f(x_n)}{f'(x_n)}$   
 $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

- Newton's Method gives us a sequence  $x_1, x_2, x_3, \dots$  of approximations to a root  $r$  of a function  $f(x)$ .
- If the  $n$ th approximation is  $x_n$  and  $f'(x_n) \neq 0$ , then the  $(n+1)$ st approximation is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

- If the numbers  $x_n$  become closer and closer to  $r$  as  $n$  becomes large, we say that the sequence converges to  $r$ .
- The sequence does not always converge.

### Example (Newton's Method, Example 1, p. 313)

Starting with  $x_1 = 2$ , find the third approximation  $x_3$  to the root of the equation  $x^3 - 2x - 5 = 0$ .

$$f(x) = x^3 - 2x - 5.$$

$$f'(x) = 3x^2 - 2.$$

$$\text{Newton's Method: } x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 2x_n - 5}{3x_n^2 - 2}$$

$$\begin{aligned} x_2 &= x_1 - \frac{x_1^3 - 2x_1 - 5}{3x_1^2 - 2} \\ &= (2) - \frac{(2)^3 - 2(2) - 5}{3(2)^2 - 2} \\ &= 2.1. \end{aligned}$$

$$\begin{aligned} x_3 &= x_2 - \frac{x_2^3 - 2x_2 - 5}{3x_2^2 - 2} \\ &= (2.1) - \frac{(2.1)^3 - 2(2.1) - 5}{3(2.1)^2 - 2} \\ &= 2.0946. \end{aligned}$$

## Example (Newton's Method)

Starting with  $x_1 = 5$ , use two steps of Newton's Method to approximate  $\sqrt{28}$ .

$$f(x) = x^2 - 28.$$

$$f'(x) = 2x.$$

$$\text{Newton's Method: } x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - 28}{2x_n}$$

$$\begin{aligned}x_2 &= x_1 - \frac{x_1^2 - 28}{2x_1} \\&= (5) - \frac{(5)^2 - 28}{2(5)} \\&= 5.3.\end{aligned}$$

$$\begin{aligned}x_3 &= x_2 - \frac{x_2^2 - 28}{2x_2} \\&= (5.3) - \frac{(5.3)^2 - 28}{2(5.3)} \\&= 5609/1060.\end{aligned}$$