Calculus I Linearization and differentials

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2019

Outline

Linear Approximations

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Differentials

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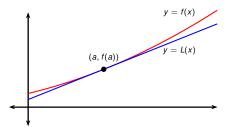
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Linear Approximations and Differentials

- Main idea: A curve is very close to its tangent line at the point of tangency.
- We can use the tangent line at (a, f(a)) as an approximation to the curve y = f(x).
- This approximation works well as long as x is near a.



Linear Approximations 5/15

Definition (Linearization of *f* at *a*)

The linear function whose graph is the tangent line at (a, f(a)) is called the linearization of f at a. Its equation is

$$L(x) = f(a) + f'(a)(x - a).$$

Definition (Linear Approximation of f(x) near a)

The approximation

$$f(x) \approx f(a) + f'(a)(x - a)$$

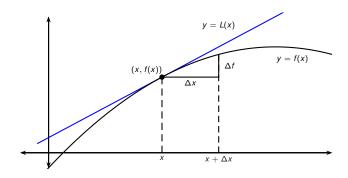
is called the linear approximation of f at a.

Let
$$y = f(x)$$
, $\Delta y := f(x) - f(a)$, and $\Delta x := x - a$.

Definition (Linear approx. y = f(x) near a, alternative notation)

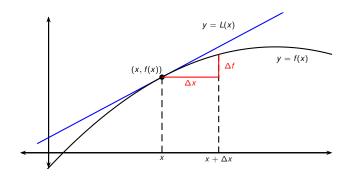
$$\Delta y \approx \frac{dy}{dx} \Delta x$$
.

Linear approximations



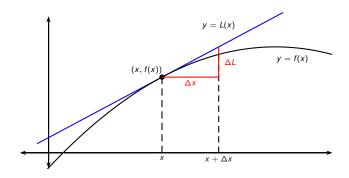
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Run	Δx	Δx
Rise	Δf	ΔL
Formula	$\Delta f = f(x + \Delta x) - f(x)$	$\Delta L = (\Delta x)f'(x)$

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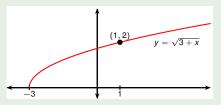
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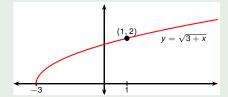
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Linear Approximations 7.

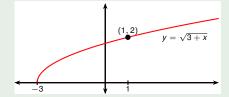
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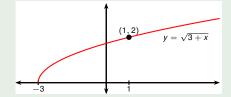
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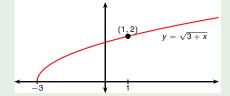


- $f'(x) = \frac{1}{2\sqrt{x+3}}.$
- f(1) = ?
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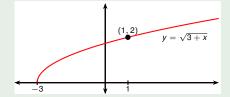
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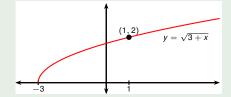
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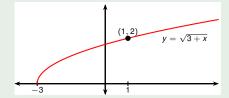


Find the linearization of the function $f(x) = \sqrt{x+3}$ at a=1 and use it to approximate the numbers $\sqrt{3.98}$ and $\sqrt{4.05}$. Are these approximations overestimates or underestimates?

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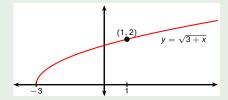
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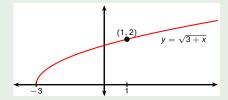
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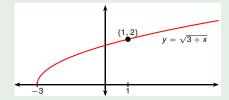
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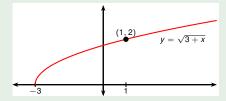
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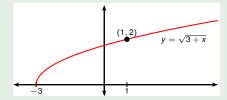
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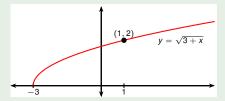
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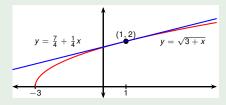
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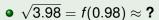
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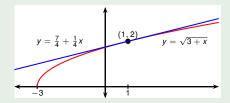
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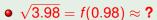
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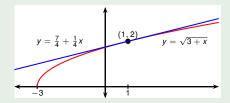
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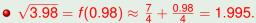
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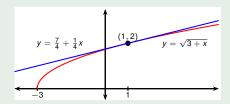
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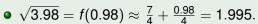
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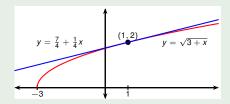
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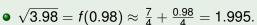
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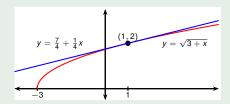
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$$\sqrt{4.05} = f(1.05) \approx \frac{7}{4} + \frac{1.05}{4} = 2.0125$$
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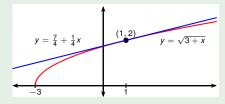
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Linearization:

$$L(x) = 2 + \frac{1}{4}(x - 1)$$
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The graph of the linearization is above the curve, so these are overestimates.

•
$$\sqrt{3.98} = f(0.98) \approx \frac{7}{4} + \frac{0.98}{4} = 1.995$$
.

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Linear Approximations 8/15

Example

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- When x = 2 and $\Delta x = 0.05$, we have:
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- Therefore $\Delta Ly = 0.7$, an approximation of $\Delta y = 0.717625$.

Differentials

Differentials

- If we substitute Δy by the formal expression dy and Δx by the formal expression dx, the expression dx appears to "cancel" to give a formal identity.

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- Define the differential d and the differential forms dx, d(f(x)) by requesting that d and dx satisfy the transformation law

$$d(f(x)) = f'(x)dx$$

for any differentiable function f(x). In abbreviated notation:

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Expressions containing expression of the form d(something) are called differential forms.

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- If we substitute Δy by the formal expression dy and Δx by the formal expression dx, the expression dx appears to "cancel" to give a formal identity.
- Define the *differential* d and the *differential forms* dx, d(f(x)) by requesting that d and dx satisfy the transformation law

$$d(f(x)) = f'(x)dx$$

for any differentiable function f(x). In abbreviated notation:

$$df = f' dx$$

Expressions containing expression of the form d(something) are called differential forms.

Differentials

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- Do not confuse differentials with derivatives. The correct equality is this.

$$df(x) = f'(x) dx$$

Example

Compute the differential (via dx).

$$d(x^2)$$

Example

Compute the differential (via dx).

$$d\left(x^2\right) = \left(x^2\right)' dx$$

Example

Compute the differential (via dx).

$$d(x^2) = (x^2)' dx = 2x dx .$$

Example

Compute the differential (via dx).

$$d(\sqrt{x})$$

Example

Compute the differential (via dx).

$$d(\sqrt{x}) = (\sqrt{x})' dx$$

Example

Compute the differential (via dx).

$$d(\sqrt{x}) = (\sqrt{x})' dx = \frac{1}{2\sqrt{x}} dx .$$

 All rules for computing with derivatives have analogues for computing with differential forms.

- All rules for computing with derivatives have analogues for computing with differential forms.
- The rules for computing differential forms are a direct consequence of the corresponding derivative rules and the transformation law d(f(x)) = f'(x)dx.

Rule name: product rule. Differential rule

Derivative rule
$$(fg)' = f'g + fg'$$

Rule name: product rule. Differential rule d(fg) = gdf + fdg

Derivative rule
$$(fg)' = f'g + fg'$$

Rule name: constant derivative rule.

Differential rule
$$d(fg) = gdf + fdg$$

$$(fg)' = f'g + fg'$$

 $(c)' = 0$

Rule name: constant derivative rule.

Differential rule

$$d(fg) = gdf + fdg$$
$$dc = 0 = 0dx$$

Derivative rule

(c)' = 0

$$(fg)' = f'g + fg'$$

Rule name: Differential rule d(fg) = gdf + fdgdc = 0 = 0dx

Derivative rule

$$(fg)' = f'g + fg'$$

 $(c)' = 0$
 $(cf)' = cf'$

Rule name:

Differential rule d(fg) = gdf + fdg dc = 0 = 0dx d(cf) = cdf

Derivative rule (fg)' = f'g + fg'

$$(c)' = 0$$

$$(cf)' = cf'$$

Rule name: sum rule.
Differential rule d(fg) = gdf + fdg

$$d(fg) = gdf + fdg$$
$$dc = 0 = 0dx$$
$$d(cf) = cdf$$

Derivative rule (fg)' = f'g + fg' (c)' = 0 (cf)' = cf'(f+q)' = f'+q'

Rule name: sum rule.

Differential rule d(fg) = gdf + fdg dc = 0 = 0dx d(cf) = cdf

d(f+g)=df+dg

Derivative rule

$$(fg)'=f'g+fg'$$

$$(c)' = 0$$

$$(cf)' = cf'$$

$$(G') = G'$$
$$(f+g)' = f'+g'$$

c-const.

Rule name: chain rule. Differential rule

Differential rule

$$d(fg) = gdf + fdg$$

$$dc = 0 = 0dx$$

$$d(cf) = cdf$$

$$d(f + g) = df + dg$$

Derivative rule

$$(fg)' = f'g + fg'$$

 $(c)' = 0$
 $(cf)' = cf'$
 $(f+g)' = f' + g'$

c-const.

$$(f(g(x)))' = f'(g(x))g'(x)$$

Rule name: chain rule.

Differential rule

$$d(fg) = gdf + fdg$$

$$dc = 0 = 0dx$$

$$d(cf) = cdf$$

$$d(f+g)=df+dg$$

$$df(g(x)) = f'(g(x))dg(x)$$

= $f'(g(x))g'(x)dx$

$$\mathsf{d} f(g) = f'(g) \mathsf{d} g$$

Derivative rule

$$(fg)' = f'g + fg'$$

$$(c)'=0$$

$$(cf)' = cf'$$

$$(f+g)'=f'+g'$$

$$= f'(g(x))g'(x)dx \quad (f(g(x)))' = f'(g(x))g'(x)$$

c-const.

Rule name: power rule.

Differential rule

$$d(fg) = gdf + fdg$$
$$dc = 0 = 0dx$$

$$d(cf) = cdf$$

$$d(f+g)=df+dg$$

$$df(g(x)) = f'(g(x))dg(x)$$

$$= f'(g(x))g'(x)c$$

$$= f'(a)da$$

df(g) = f'(g)dg

Derivative rule

$$(fq)' = f'q + fq'$$

$$(c)'=0$$

$$(cf)' = cf'$$

$$(f+g)'=f'+g'$$

$$= f'(g(x))g'(x)dx \quad (f(g(x)))' = f'(g(x))g'(x)$$

$$(x^n)' = nx^{n-1}$$

c-const.

Rule name: power rule.

Differential rule
$$d(fg) = gdf + fdg$$

$$dc = 0 = 0 dx$$

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$$= f'(g(x))g'(x)dx$$

Derivative rule

$$(fg)'=f'g+fg'$$

$$(c)'=0$$

$$(cf)'=cf'$$

$$(f+g)'=f'+g'$$

$$= f'(g(x))g'(x)dx \quad (f(g(x)))' = f'(g(x))g'(x)$$

$$\frac{\mathrm{d}f(g)}{\mathrm{d}(x^n) = nx^{n-1}\mathrm{d}x}$$

$$\overline{(x^n)'=nx^{n-1}}$$

c-const.

Rule name: exponent derivative rule.

Differential rule Derivative
$$d(fq) = qdf + fdq$$
 $(fq)' = f$

$$d(tg) = gdt + tdg \qquad (tg)' = t'g$$

$$dc = 0 = 0dx \qquad (c)' = 0$$

$$d(cf) = cdf \qquad (cf)' = cf'$$

$$d(f+g) = df + dg$$

$$df(g(x)) = f'(g(x))dg(x)$$

$$= f'(g(x))g'(x)dx \quad (f(g(x)))' = f'(g(x))g'(x)$$

$$\frac{\mathsf{d} f(g)}{\mathsf{d} f(g)} = f'(g) \mathsf{d} g$$

$$\frac{dr(g)}{d(x^n) = nx^{n-1}dx}$$

$$(fq)' = f'q + fq'$$

$$(c)'=0$$

$$(CI)' = CI'$$
$$(f+a)' = f' + a'$$

$$(x^n)' = nx^{n-1}$$
$$(e^x)' = e^x$$

c-const.

Rule name: exponent derivative rule.

Differential rule Derivative rule
$$d(fg) = gdf + fdg \qquad (fg)' = f'g + fg'$$

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$$d(f+g) = df + dg \qquad (f+g)' = f' + g'$$

$$df(g(x)) = f'(g(x))dg(x) \qquad = f'(g(x))g'(x)dx \qquad (f(g(x)))' = f'(g(x))g'(x)$$

$$df(g) = f'(g)dg$$

$$d(x^n) = nx^{n-1}dx \qquad (x^n)' = nx^{n-1}$$

$$d(e^x) = e^x dx \qquad (e^x)' = e^x$$

Differential rule
$$d(fg) = gdf + fdg$$
 $(fg)' = f'g + fg'$ $(c)' = 0$ c -const. $d(cf) = cdf$ $(cf)' = cf'$ c -const. $d(f+g) = df + dg$ $(f+g)' = f' + g'$ $df(g(x)) = f'(g(x))dg(x)$ $= f'(g(x))g'(x)dx$ $(f(g(x)))' = f'(g(x))g'(x)$ $df(g) = f'(g)dg$ $d(x^n) = nx^{n-1}dx$ $d(e^x) = e^xdx$ $(e^x)' = e^x$ $(sin x)' = cos x$

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$$df(g) = f'(g)dg$$

$$d(x^n) = nx^{n-1}dx \qquad (x^n)' = nx^{n-1}$$

$$d(e^x) = e^x dx \qquad (e^x)' = e^x$$

$$d(\sin x) = \cos x dx \qquad (\sin x)' = \cos x$$

$$(\cos x)' = -\sin x$$

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Differentials are especially efficient at "encoding" the chain rule.

Example

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$$\mathsf{d}\left(\mathsf{In}\left(1+\sqrt{1+x^2}\right)\right)$$

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Set
$$u = 1 + \sqrt{1 + x^2}$$
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$$d\left(\ln\left(1+\sqrt{1+x^2}\right)\right) = d\left(\ln u\right)$$

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Example

Compute the differential d $\left(\ln\left(1+\sqrt{1+x^2}\right)\right)$.

Set $u = 1 + \sqrt{1 + x^2}$.

$$d\left(\ln\left(1+\sqrt{1+x^2}\right)\right) = d\left(\ln u\right) = \frac{1}{u}du = \frac{1}{u}d\left(1+\sqrt{1+x^2}\right) =$$

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Example

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.
Set $u=1+\sqrt{1+x^2}$. Set $v=1+x^2$.

$$d\left(\ln\left(1+\sqrt{1+x^{2}}\right)\right) = d\left(\ln u\right) = \frac{1}{u}du = \frac{1}{u}d\left(1+\sqrt{1+x^{2}}\right) = \frac{1}{u}d\left(\sqrt{1+x^{2}}\right) = \frac{1}{u}d\left(\sqrt{1+x^{2}}\right) = \frac{1}{u}d\left(v^{\frac{1}{2}}\right) = \frac{1}{u}\frac{1}{2}v^{-\frac{1}{2}}dv$$

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