Precalculus
Euler's formula and trigonometric identities

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Outline

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Euler's Formula

Theorem (Euler's Formula)

$$e^{ix} = \cos x + i \sin x,$$

where $e \approx 2.71828$ *is Euler's/Napier's constant .*

Proof.

Recall $n! = 1 \cdot 2 \cdot 3 \cdot \dots \cdot (n-1) \cdot n$. Borrow from Calc II the f-las:

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$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + \frac{(-1)^n x^{2n}}{(2n)!} + \dots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots$$

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Rearrange.

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Rearrange. Plug-in $z = ix$. Use $i^2 = -1$. Multiply $\sin x$ by i. Add to get $e^{ix} = \cos x + i \sin x$. □

Trigonometric Identities Revisited

- $e^{ix} = \cos x + i \sin x$ (Euler's Formula).
- $e^{ix} e^{iy} = e^{ix+iy} = e^{i(x+y)}$ (exponentiation rule: valid for $\mathbb{C}$).
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All trigonometric formulas can be easily derived using the above formulas.

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$$\begin{array}{rcl} \sin(x+y) & = & \sin x \cos y + \sin y \cos x \\ \cos(x+y) & = & \cos x \cos y - \sin x \sin y \end{array} \quad .$$

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- By convention, when dealing with trigonometric identities we do not account for the domains of the involved expressions.
- For example, $\frac{\sin\theta}{\sin\theta} = 1$ is considered a valid trigonometric identity, although, when considered as a function, the left hand side is not defined for $\theta \neq 0$.

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- To prove a trigonometric identity means to show that the two sides of the equality sign are equivalent.

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- To prove a trigonometric identity means to show that the two sides of the equality sign are equivalent.
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- The discussion here also applies for trigonometric identities in more than one variables.

Types of identites

- In the present course we deal with two basic types of trigonometric identities.
- First, identities that involve operations on the arguments of the trigonometric functions.
  - Example: $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ (this is one of the angle sum identities); $\sin\theta + \sin(-\theta) = 0$.
  - Such identities can be proved using the angle sum formulas and the even/odd function properties of $\sin, \cos$.
- Second, identities that involve trigonometric functions of one variable.
  - Example: $\tan^2\theta + 1 = \sec^2\theta$.
  - Such identities can be proved only using the already demonstrated Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$.
- The Pythagorean identity follows from the angle sum formulas and the even/odd function properties of $\sin, \cos$, so all trigonometric identities follow from those properties alone.

Example

Demonstrate the trigonometric identity $\csc^2\theta - 1 = \cot^2\theta$.

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We transform the left hand side to the right one.

$$\begin{aligned}
\csc^2\theta - 1 &= \frac{1}{\sin^2\theta} - 1 \\
&= \frac{1-\sin^2\theta}{\sin^2\theta} \\
&= \frac{\cos^2\theta}{\sin^2\theta} \\
&= \cot^2\theta \qquad \Big| \text{ as desired.}
\end{aligned}$$

Example

Verify the trigonometric identity $2\csc^2\alpha = \dfrac{1}{1-\cos\alpha} + \dfrac{1}{1+\cos\alpha}$

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We transform the right hand side to the left.

$$\begin{aligned}
\frac{1}{1-\cos\alpha} + \frac{1}{1+\cos\alpha} &= \frac{(1+\cos\alpha)}{(1-\cos\alpha)(1+\cos\alpha)} + \frac{(1-\cos\alpha)}{(1-\cos\alpha)(1+\cos\alpha)}\\
&= \frac{1+\cos\alpha+1-\cos\alpha}{1-\cos^2\alpha}\\
&= \frac{2}{\sin^2\alpha}\\
&= 2\csc^2\alpha
\end{aligned}$$

as desired.

Example

Verify the identity $\ln(\sec\theta - 1) + \ln(\sec\theta + 1) - 2\ln(\sec\theta) = 2\ln(\sin\theta)$, where $0 < \theta < \frac{\pi}{2}$.

Example

Verify the identity $\ln(\sec\theta - 1) + \ln(\sec\theta + 1) - 2\ln(\sec\theta) = 2\ln(\sin\theta)$, where $0 < \theta < \frac{\pi}{2}$. We transform the left hand side to the right.

$$\begin{aligned}
& \ln(\sec\theta - 1) + \ln(\sec\theta + 1) - 2\ln(\sec\theta) \\
= \ & \ln\left((\sec\theta - 1)(\sec\theta + 1)\right) - \ln\left(\sec^2\theta\right) \\
= \ & \ln\left(\sec^2\theta - 1\right) - \ln\left(\sec^2\theta\right) \\
= \ & \ln\left(\frac{\sec^2\theta - 1}{\sec^2\theta}\right) \\
= \ & \ln\left(1 - \frac{1}{\sec^2\theta}\right) \\
= \ & \ln\left(1 - \cos^2\theta\right) \\
= \ & \ln\left(\sin^2\theta\right) \\
= \ & 2\ln\left(\sin\theta\right)
\end{aligned}$$

as desired.

Example

Verify the identity $\tan x + \cot x = \sec x \csc x$.

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$$\begin{aligned}
\tan x + \cot x &= \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} \\
&= \frac{\sin^2 x}{\sin x \cos x} + \frac{\cos^2 x}{\sin x \cos x} \\
&= \frac{\sin^2 x + \cos^2 x}{\sin x \cos x} \\
&= \frac{1}{\sin x \cos x} \\
&= \frac{1}{\sin x}\frac{1}{\cos x} \\
&= \csc x \sec x,
\end{aligned}$$

as desired.

Example

Prove the trigonometric identity.

$$(\sin\theta + \cos\theta)^2 = 1 + \sin(2\theta)$$

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$(\sin\theta + \cos\theta)^2 =$ **?**

$(A + B)^2 =$ **?**

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$$(\sin\theta + \cos\theta)^2 = \sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta \quad \Bigg| \quad \begin{array}{l}(A+B)^2 = \\ A^2 + 2AB + B^2\end{array}$$

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Here we explicitly permit the use of the Pythagorean identities and the double angle f-las:

$$\begin{aligned}
\cos^2\theta + \sin^2\theta &= 1\\
\sin(2\theta) &= 2\sin\theta\cos\theta\\
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$$\begin{aligned}(\sin\theta+\cos\theta)^2 &= \sin^2\theta+2\sin\theta\cos\theta+\cos^2\theta\\ &= 1+\sin(2\theta)\end{aligned}$$

$(A+B)^2 = A^2+2AB+B^2$

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Express $\sin(3x)$ and $\cos(3x)$ via $\cos x$ and $\sin x$.

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Recall the formulas
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The real parts of the starting and final expression must be equal; therefore:

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All angles here are multiples of $\frac{\theta}{2}$, so set $\varphi = \frac{\theta}{2}$, $\theta = 2\varphi$.

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$$A^2 + 2AB + B^2 = (A+B)^2$$

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$A^2+2AB+B^2$
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as desired.

Strategy for proving trigonometric identities

An expression is rational trigonometric if it is written using $\sin\theta, \cos\theta$ and the four arithmetic operations.

Question

Is there a general method for proving all rational trigonometric identities in one variable?

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- The method is rather cumbersome for a human and is best suited for computers.

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  - You may need to use trig functions of angles smaller than θ, for example $\sin\left(\frac{\theta}{2}\right), \cos\left(\frac{\theta}{2}\right)$.

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  - You may need to use trig functions of angles smaller than θ, for example $\sin\left(\frac{\theta}{2}\right)$, $\cos\left(\frac{\theta}{2}\right)$.
  - A fraction of θ such that all appearing angles are integer multiples of it will always work.

Proving the following identities is a good exercise.

1. $\sin\theta\cot\theta = \cos\theta$.
2. $(\sin\theta + \cos\theta)^2 = 1 + \sin(2\theta)$.
3. $\sec\theta - \cos\theta = \tan\theta\sin\theta$.
4. $\tan^2\theta - \sin^2\theta = \tan^2\theta\sin^2\theta$.
5. $\cot^2\theta + \sec^2\theta = \tan^2\theta + \csc^2\theta$.
6. $2\csc(2\theta) = \sec\theta\csc\theta$.
7. $\tan(2\theta) = \dfrac{2\tan\theta}{1-\tan^2\theta}$.
8. $\dfrac{1}{1-\sin\theta} + \dfrac{1}{1+\sin\theta} = 2\sec^2\theta$.
9. $\tan\alpha + \tan\beta = \dfrac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta}$.
10. $\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta}$.
11. $\sin(3\theta) + \sin\theta = 2\sin(2\theta)\cos\theta$.
12. $\cos(3\theta) = 4\cos^3\theta - 3\cos\theta$.
13. $1 + \tan^2\theta = \sec^2\theta$.
14. $1 + \csc^2\theta = \cot^2\theta$.
15. $2\cos^2(2x) =$
    $2\sin^4\theta + 2\cos^4\theta - \sin^2(2\theta)$.
16. $\dfrac{1+\tan\left(\frac{\theta}{2}\right)}{1-\tan\left(\frac{\theta}{2}\right)} = \tan\theta + \sec\theta$.