

Arithmetics
Division and fractions
calculator-algebra.org

Todor Milev

2019

Definition (Division, exact)

To divide a number p (**dividend**) by a number d (**divisor**) means to find a number q (**quotient**) so that

$$q \cdot d = p$$

Example

Divide 5 by 3.

$$\text{answer} = \frac{5}{3}$$

Observation

The quotient of two integers equals the fraction formed by putting the dividend as the numerator and the divisor as the denominator.

$$q = \frac{p}{d}$$

- Recall exact division:
$$\begin{aligned} p' &= q' \cdot d' \\ q' &= \frac{p'}{d'} \end{aligned}$$
- Quotient may fail to reduce to an integer.
- What if we want an integer quotient?

Definition (Integer division with remainder)

To **divide** an integer $p > 0$ by an integer $d > 0$ **with remainder** $r \geq 0$ means to find the largest integer $q \geq 0$ and the smallest $0 \leq r$ so that:

$$p = q \cdot d + r$$

p is called the **dividend**, d is called the **divisor**, q is called the **quotient** and r is called the **remainder**.

Example

Divide 7 by 3 with remainder. $7 = 2 \cdot 3 + 1$.

- Differences between exact division integer division.
 - Integer division quotient is integer, exact division quotient is fraction.
 - Exact division: no notion of remainder.

- Recall integer division of p by d with remainder: $p = q \cdot d + r$.
- Integer division of p by d answers the following question:

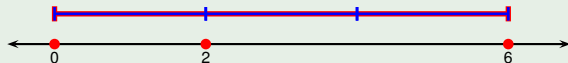
Question

How many times does the length d fit inside the length p ?

- p may be divisible by d an exact number of times.
- p may fail to be integer-divisible by d , then we have remainder.

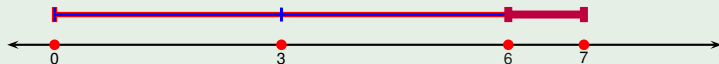
Example

Divide 6 by 2 with remainder. Solution: $6 = 3 \cdot 2 + 0$.



2 divides 6 exactly 3 times.

Divide 7 by 3. Solution: $7 = 2 \cdot 3 + 1$.



3 divides 7 non-exactly 2 times with remainder 1.

Example (Divisor 1-digit, quotient 1-digit)

Divide 7 by 2 with remainder. To solve, we need to answer: what is the largest integer which, when multiplied by 2, stays smaller than 7? Try:

$$0 \cdot 2 = 0$$

$$1 \cdot 2 = 2$$

$$2 \cdot 2 = 4$$

$$3 \cdot 2 = 6$$

$$4 \cdot 2 = 8 > 7$$

$\Rightarrow 7 = 3 \cdot 2 + x$. Solve: $x = 7 - 3 \cdot 2 = 1$. Therefore $7 = 3 \cdot 2 + 1$.

Observation (Question to answer when dividing with remainder)

What is the largest integer which, when multiplied by d , remains smaller than p ?

- To answer this question, we guess quickly as shown above.
- Later on we learn to divide large numbers without guessing.
- However, we still need the guessing approach as a building block of the complete division algorithm.

Example (Integer division: 1-digit dividend, 1-digit divisor)

Divide with remainder:

$$5 \text{ by } 2 : \quad 5 = 2 \cdot 2 + 1$$

$$3 \text{ by } 5 : \quad 5 = 0 \cdot 5 + 3$$

$$8 \text{ by } 4 : \quad 5 = 2 \cdot 4$$

$$9 \text{ by } 8 : \quad 5 = 1 \cdot 8 + 1$$

$$3 \text{ by } 1 : \quad 3 = 3 \cdot 1$$

$$9 \text{ by } 2 : \quad 6 = 4 \cdot 2 + 1$$

All quotients are known to be one-digit numbers.

Example (Integer division: 1-digit divisor, 1-digit quotient)

Divide with remainder:

$$12 \text{ by } 6 : \quad 12 = 2 \cdot 6$$

$$14 \text{ by } 5 : \quad 14 = 2 \cdot 5 + 4$$

$$37 \text{ by } 5 : \quad 37 = 7 \cdot 5 + 2$$

$$40 \text{ by } 9 : \quad 40 = 4 \cdot 9 + 4$$

$$49 \text{ by } 7 : \quad 49 = 7 \cdot 7$$

$$57 \text{ by } 8 : \quad 57 = 7 \cdot 8 + 1$$

$$67 \text{ by } 7 : \quad 67 = 9 \cdot 7 + 4$$

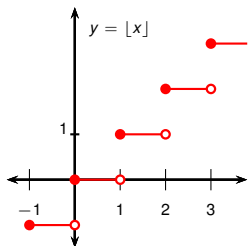
$$82 \text{ by } 9 : \quad 82 = 9 \cdot 9 + 1$$

All quotients are known to be one-digit numbers.

Definition (Greatest Integer Function)

The *greatest integer function* $\lfloor x \rfloor$ is defined as the largest integer that is less than or equal to x .

In computer science this function is called the *floor* function, also the *round-down* function.



$$\lfloor 4 \rfloor = 4$$

$$\lfloor 4.8 \rfloor = 4$$

$$\lfloor 1.5 \rfloor = \lfloor 1 + 0.5 \rfloor = 1$$

$$\left\lfloor 1 + \frac{1}{2} \right\rfloor = 1$$

$$\left\lfloor \frac{3}{2} \right\rfloor = \left\lfloor \frac{2+1}{2} \right\rfloor = \left\lfloor \frac{2}{2} + \frac{1}{2} \right\rfloor = \left\lfloor 1 + \frac{1}{2} \right\rfloor = 1$$

$$\lfloor \pi \rfloor = \lfloor 3.1415\dots \rfloor = 3$$

Example

Compute the floor (round-down) function.

$$\begin{aligned}\left\lfloor \frac{1}{3} \right\rfloor &= 0 \\ \left\lfloor \frac{6}{7} \right\rfloor &= 0 \\ \left\lfloor 2 + \frac{3}{11} \right\rfloor &= 2 + \left\lfloor \frac{3}{11} \right\rfloor = 2 \\ \left\lfloor 10 + \frac{1}{7} \right\rfloor &= 10 + \left\lfloor \frac{1}{7} \right\rfloor = 10\end{aligned}$$

Observation

$$\left\lfloor \frac{p}{q} \right\rfloor = 0 \quad \text{whenever } 0 \leq p < q$$

Observation

$$\lfloor n + x \rfloor = n + \lfloor x \rfloor \quad \text{whenever } n \text{ is integer}$$

Example

Compute the floor (round-down) of $\frac{8}{3}$.

$$\left\lfloor \frac{8}{3} \right\rfloor = \left\lfloor \frac{2 \cdot 3 + 2}{3} \right\rfloor$$

$$= \left\lfloor \frac{2 \cdot \cancel{3}}{\cancel{3}} + \frac{2}{3} \right\rfloor$$

$$= \left\lfloor 2 + \frac{2}{3} \right\rfloor = 2$$

Divide 8 by 3 with remainder. Try:

$$0 \cdot 3 = 0$$

$$1 \cdot 3 = 3$$

$$2 \cdot 3 = 6$$

$$3 \cdot 3 = 9 > 8$$

$$\text{because } 2 \leq 2 + \frac{2}{3} < 3$$

Observation

The floor (round-down) of $\frac{p}{q}$ is computed as

$$\left\lfloor \frac{p}{d} \right\rfloor = q,$$

where q is the the quotient obtained by integer division of p by d .

Example

Compute the floor (round-down) function.

$$\begin{aligned} \left\lfloor \frac{4}{3} \right\rfloor &= \left\lfloor \frac{1 \cdot 3 + 1}{3} \right\rfloor = \left\lfloor \frac{1 \cdot 3}{3} + \frac{1}{3} \right\rfloor = \left\lfloor 1 + \frac{1}{3} \right\rfloor = 1 \\ \left\lfloor \frac{15}{2} \right\rfloor &= \left\lfloor \frac{7 \cdot 2 + 1}{2} \right\rfloor = \left\lfloor \frac{7 \cdot 2}{2} + \frac{1}{2} \right\rfloor = \left\lfloor 7 + \frac{1}{2} \right\rfloor = 7 \\ \left\lfloor \frac{24}{4} \right\rfloor &= \left\lfloor \frac{6 \cdot 4}{4} \right\rfloor = \lfloor 6 \rfloor = 6 \\ \left\lfloor \frac{43}{5} \right\rfloor &= \left\lfloor \frac{8 \cdot 5 + 3}{5} \right\rfloor = \left\lfloor \frac{8 \cdot 5}{5} + \frac{3}{5} \right\rfloor = \left\lfloor 8 + \frac{3}{5} \right\rfloor = 8 \\ \left\lfloor \frac{56}{7} \right\rfloor &= \left\lfloor \frac{8 \cdot 7}{7} \right\rfloor = \lfloor 8 \rfloor = 8 \\ \left\lfloor \frac{79}{8} \right\rfloor &= \left\lfloor \frac{9 \cdot 8 + 7}{8} \right\rfloor = \left\lfloor \frac{9 \cdot 8}{8} + \frac{7}{8} \right\rfloor = \left\lfloor 9 + \frac{7}{8} \right\rfloor = 9 \\ \left\lfloor \frac{80}{9} \right\rfloor &= \left\lfloor \frac{8 \cdot 9 + 8}{9} \right\rfloor = \left\lfloor \frac{8 \cdot 9}{9} + \frac{8}{9} \right\rfloor = \left\lfloor 8 + \frac{8}{9} \right\rfloor = 8 \end{aligned}$$