

# Precalculus

## Graphs of trig functions; inverse trig

Todor Milev

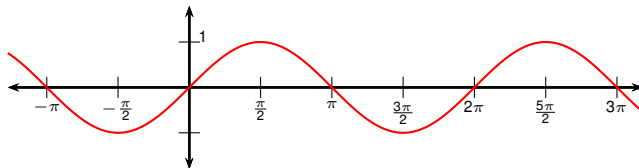
2019

# Outline

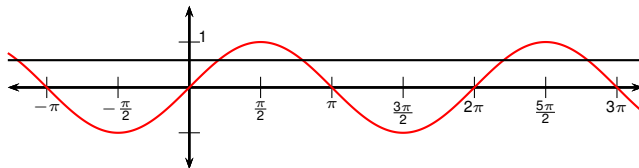
## 1 Inverse Trigonometric Functions

- The arcsine function
- The arccosine function
- The arctangent and the remaining inverse trig functions
- Trigonometric Functions with Inverse Trig Arguments

# Inverse Trigonometric Functions

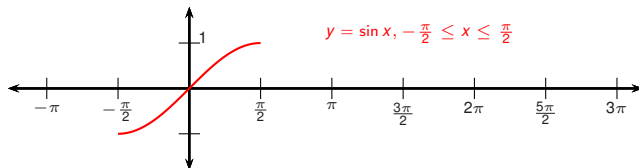


# Inverse Trigonometric Functions



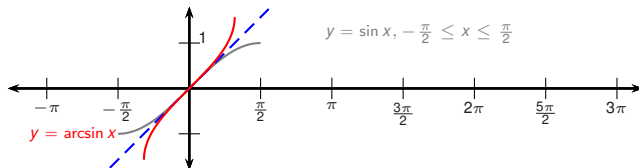
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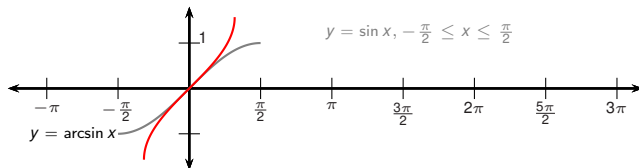
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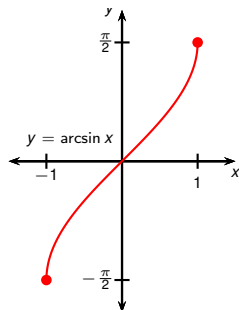


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- Then it has an inverse function.
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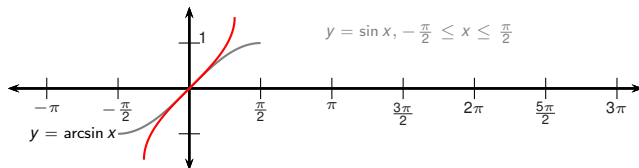
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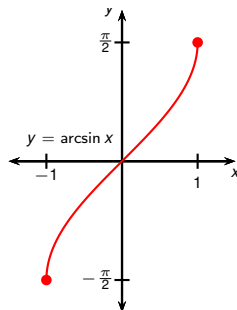
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- It is if we restrict the domain to  $[-\frac{\pi}{2}, \frac{\pi}{2}]$ .
- Then it has an inverse function.
- We call it arcsin or  $\sin^{-1}$ .
- $\arcsin x = y \Leftrightarrow \sin y = x$  and  $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ .





## Example

Find  $\arcsin\left(\frac{1}{2}\right)$ .

## Observation

- $\arcsin y =$  *the appropriate angle whose sine equals  $y$ .*

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- *Important: the output angle must lie in the interval  $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ .*

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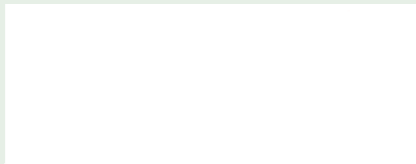
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Find  $\arcsin\left(\frac{1}{2}\right)$ .

- $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$ .
- $-\frac{\pi}{2} \leq \frac{\pi}{6} \leq \frac{\pi}{2}$ .
- Therefore  $\arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$ .

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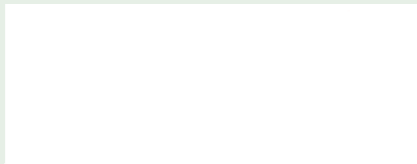
Find  $\tan \left( \arcsin \left( \frac{1}{3} \right) \right)$ .



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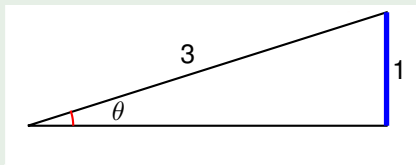
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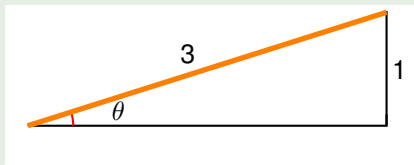
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- Draw a right triangle with **opposite side 1** and hypotenuse 3.



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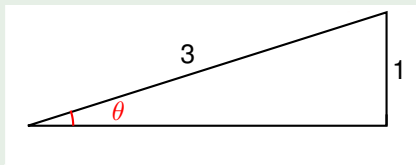
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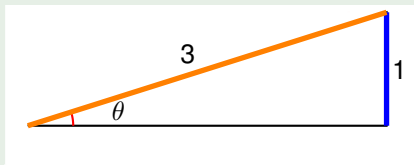
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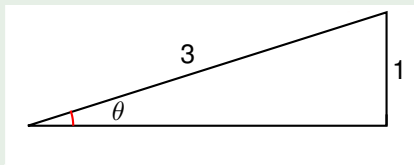
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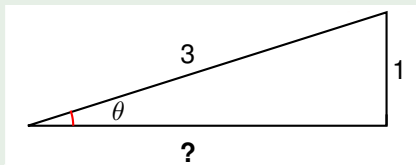




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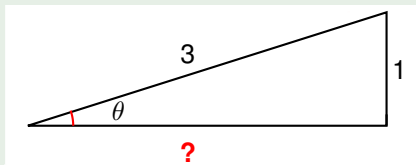
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- Length of adjacent side = ?



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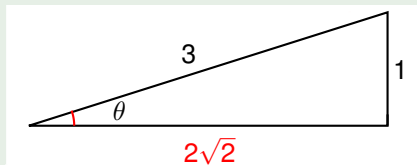
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- Length of adjacent side =  $\sqrt{3^2 - 1^2}$



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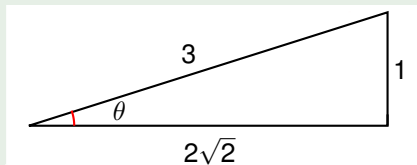
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- Length of adjacent side  $= \sqrt{3^2 - 1^2} = \sqrt{8} = 2\sqrt{2}$ .



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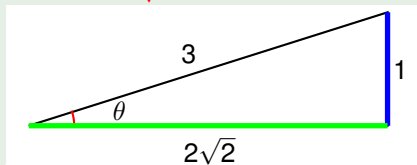
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- Then  $\tan \left( \arcsin \left( \frac{1}{3} \right) \right) = ?$



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- Length of adjacent side  $= \sqrt{3^2 - 1^2} = \sqrt{8} = 2\sqrt{2}$ .
- Then  $\tan \left( \arcsin \left( \frac{1}{3} \right) \right) = \frac{1}{2\sqrt{2}}$ .



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- Therefore  $-\frac{\pi}{2} \leq 1.5 \leq \frac{\pi}{2}$ .

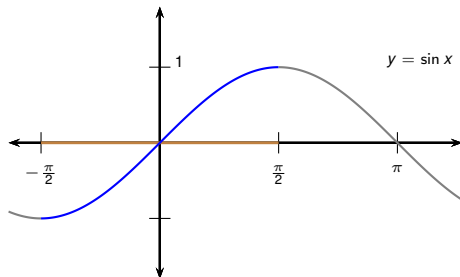
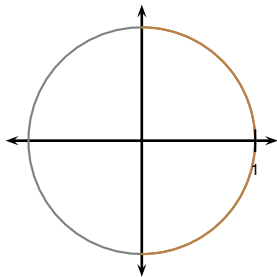
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- Therefore  $\arcsin(\sin 1.5) = 1.5$ .

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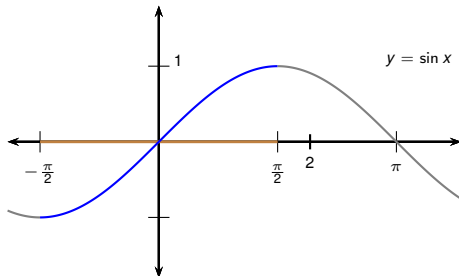
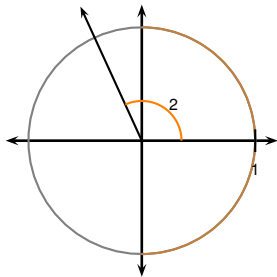
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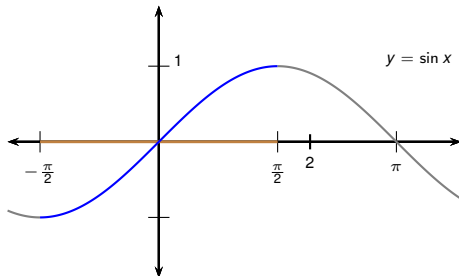
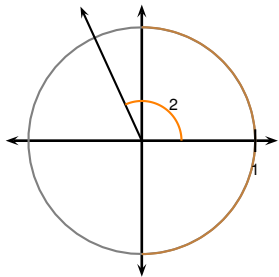
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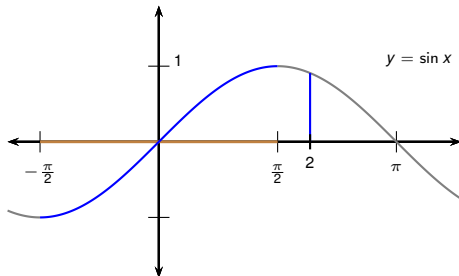
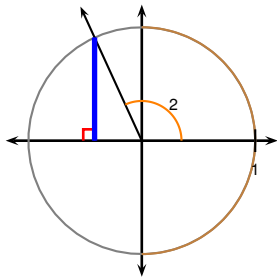
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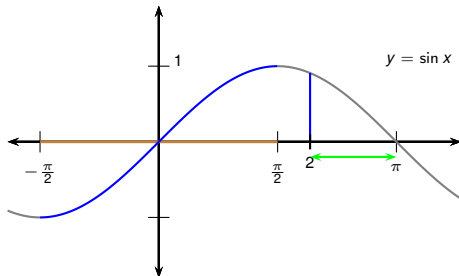
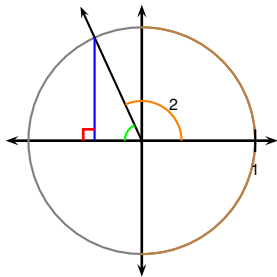
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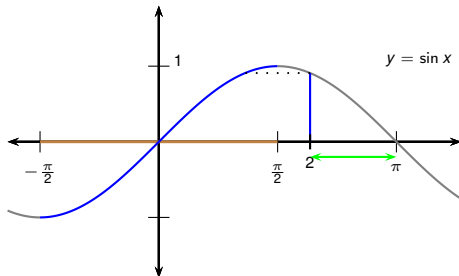
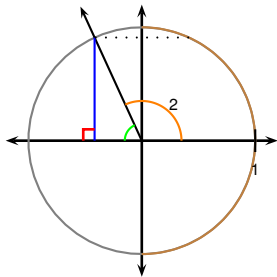
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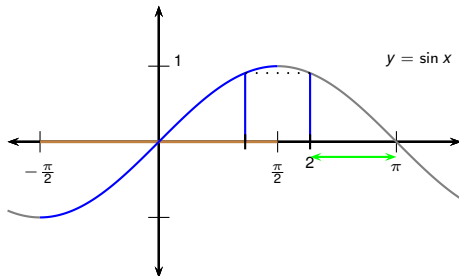
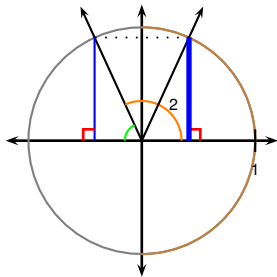




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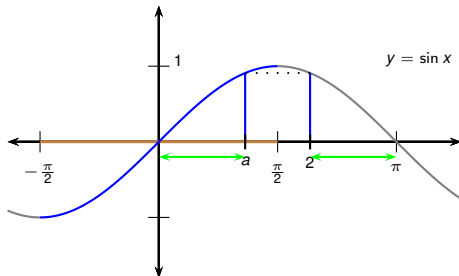
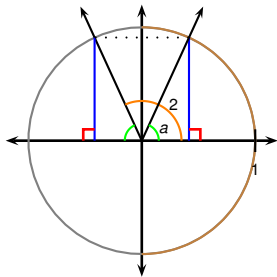
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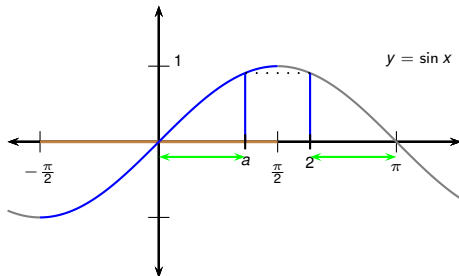
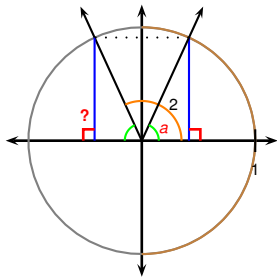


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$$a = ?$$

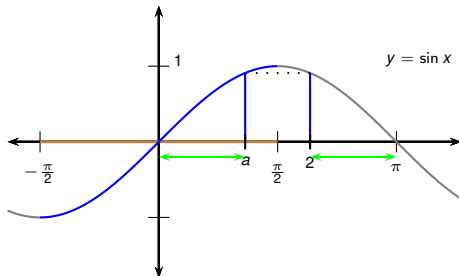
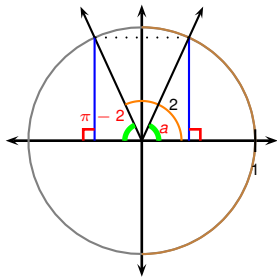


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$$a = \pi - 2.$$



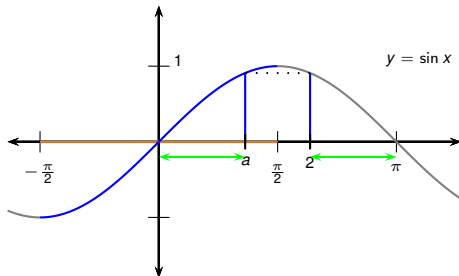
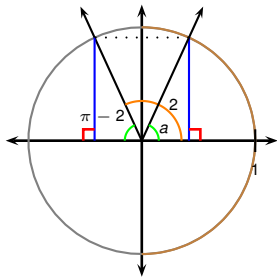
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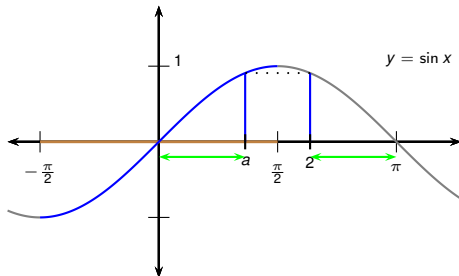
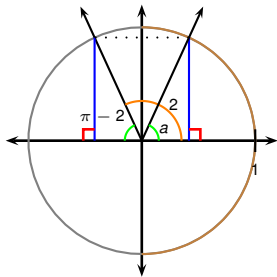
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$$\begin{aligned} \text{Therefore } \arcsin(\sin 2) &= \arcsin(\sin a) \\ &= a \end{aligned}$$



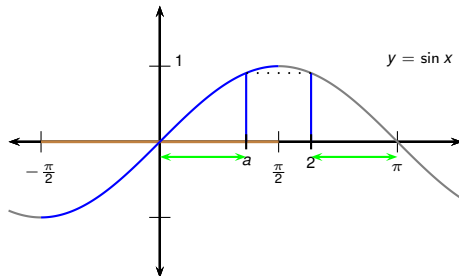
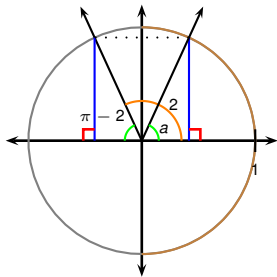
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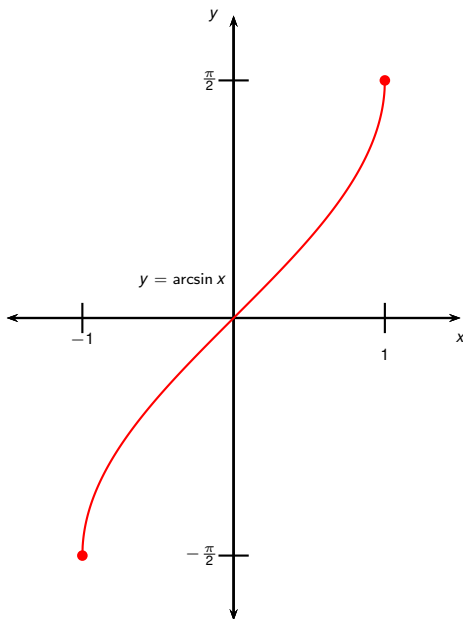
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$$\begin{aligned} \text{Therefore } \arcsin(\sin 2) &= \arcsin(\sin a) \\ &= a = \pi - 2. \end{aligned}$$



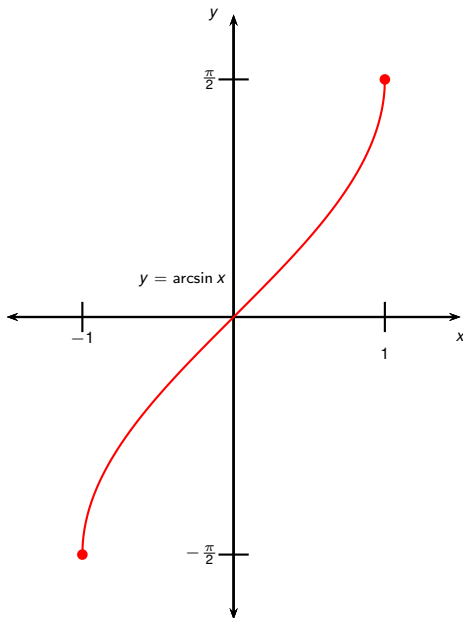
## Important facts about arcsin:



- 1 Domain: ?
- 2 Range: ?
- 3  $\arcsin x = y \Leftrightarrow \sin y = x$  and  $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ .
- 4  $\arcsin(\sin x) = x$  for  $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ .
- 5  $\sin(\arcsin x) = x$  for  $-1 \leq x \leq 1$ .
- 6  $\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$ .



## Important facts about arcsin:



1 Domain: ?

2 Range: ?

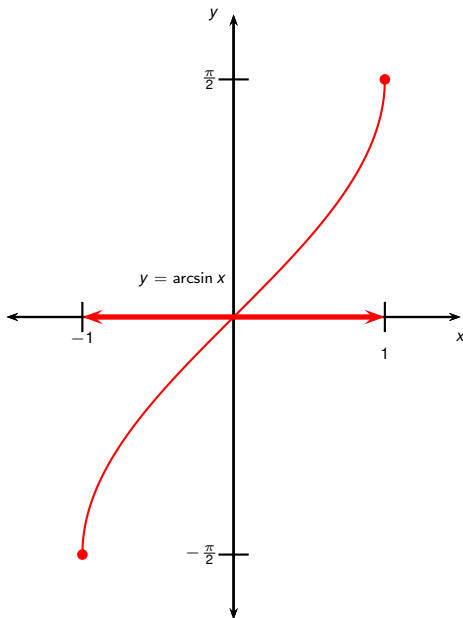
3  $\arcsin x = y \Leftrightarrow \sin y = x$  and  $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ .

4  $\arcsin(\sin x) = x$  for  $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ .

5  $\sin(\arcsin x) = x$  for  $-1 \leq x \leq 1$ .

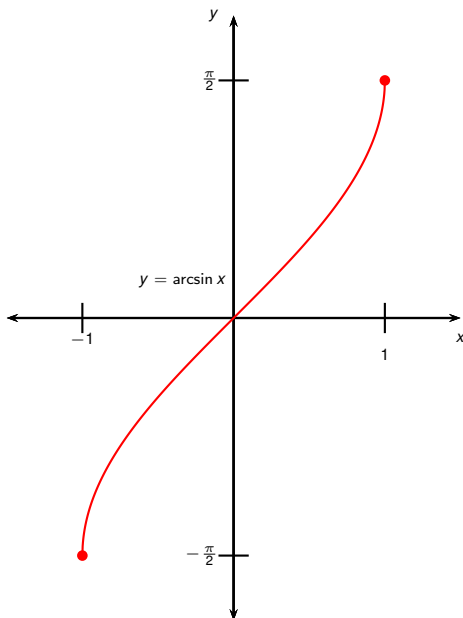
6  $\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$ .

## Important facts about arcsin:



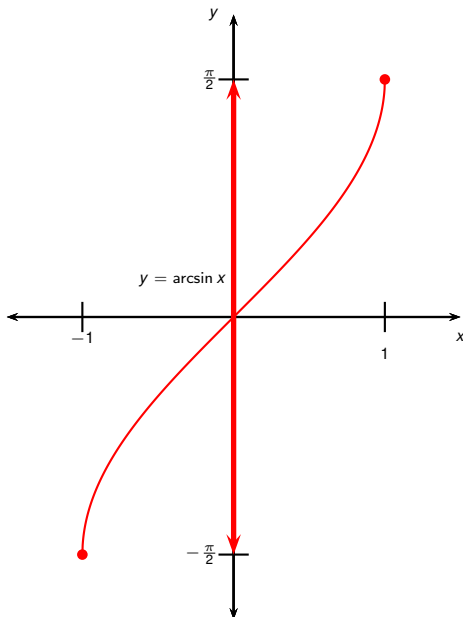
- ① Domain:  $[-1, 1]$ .
- ② Range: ?
- ③  $\arcsin x = y \Leftrightarrow \sin y = x$  and  $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ .
- ④  $\arcsin(\sin x) = x$  for  $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ .
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## Important facts about arcsin:

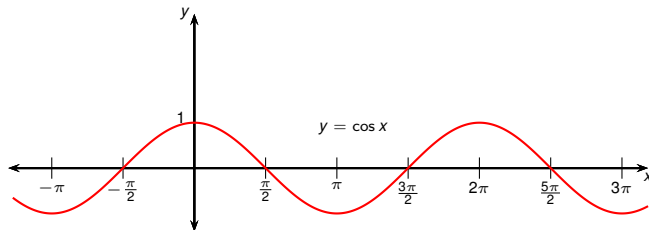


- 1 Domain:  $[-1, 1]$ .
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- 3  $\arcsin x = y \Leftrightarrow \sin y = x$  and  $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ .
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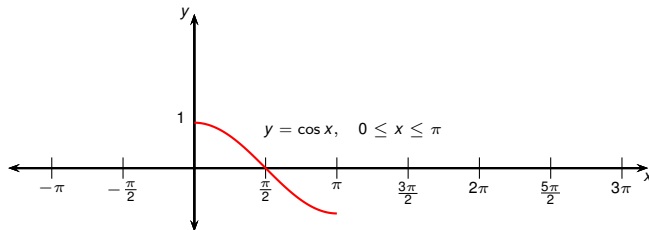
## Important facts about arcsin:



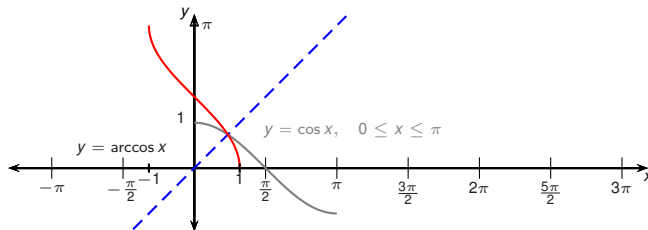
- 1 Domain:  $[-1, 1]$ .
- 2 Range:  $[-\frac{\pi}{2}, \frac{\pi}{2}]$ .
- 3  $\arcsin x = y \Leftrightarrow \sin y = x$  and  $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ .
- 4  $\arcsin(\sin x) = x$  for  $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ .
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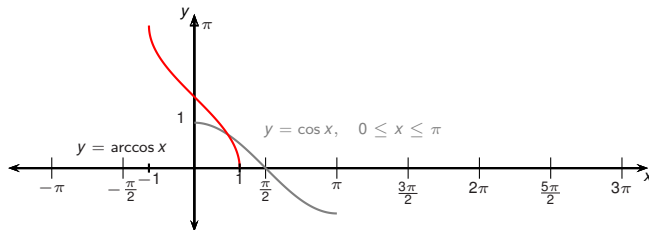
- Same for  $\cos x$ .



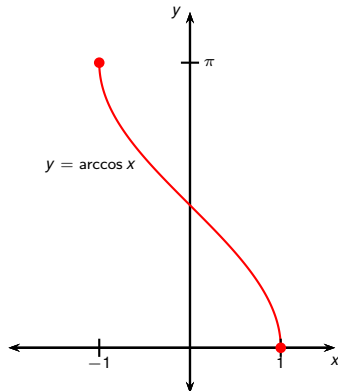
- Same for  $\cos x$ .
- Restrict the domain to  $[0, \pi]$ .



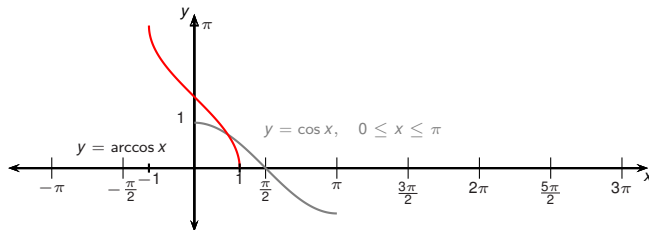
- Same for  $\cos x$ .
- Restrict the domain to  $[0, \pi]$ .
- The inverse is called arccos or  $\cos^{-1}$ .



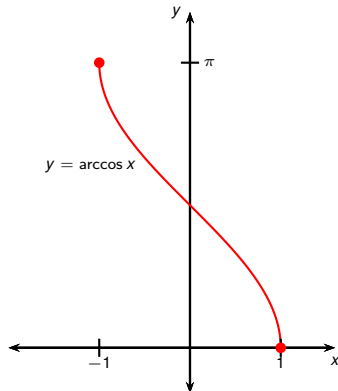
- Same for  $\cos x$ .
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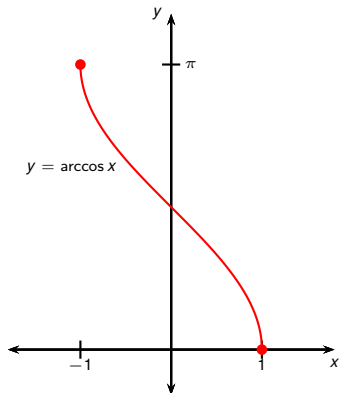




- Same for  $\cos x$ .
- Restrict the domain to  $[0, \pi]$ .
- The inverse is called arccos or  $\cos^{-1}$ .
- $\arccos(x) = y \Leftrightarrow \cos y = x$  and  $0 \leq y \leq \pi$ .

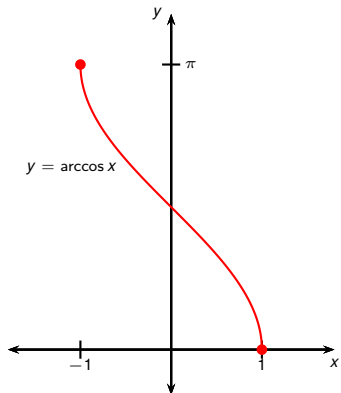


## Important facts about arccos:



- 1 Domain:
- 2 Range:
- 3  $\arccos x = y \Leftrightarrow \cos y = x$  and  $0 \leq y \leq \pi$ .
- 4  $\arccos(\cos x) = x$  for  $0 \leq x \leq \pi$ .
- 5  $\cos(\arccos x) = x$  for  $-1 \leq x \leq 1$ .
- 6  $\frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1-x^2}}$ .

## Important facts about arccos:



1 Domain: ?

2 Range:

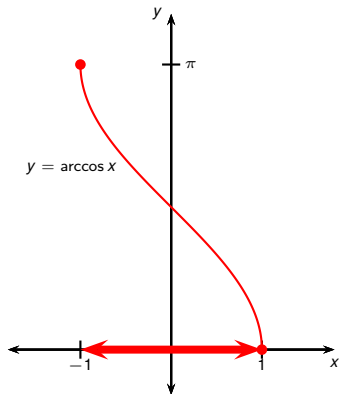
3  $\arccos x = y \Leftrightarrow \cos y = x$  and  $0 \leq y \leq \pi$ .

4  $\arccos(\cos x) = x$  for  $0 \leq x \leq \pi$ .

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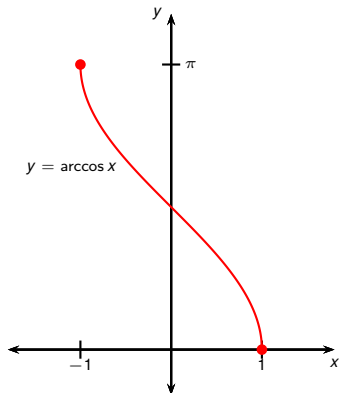
6  $\frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1-x^2}}$ .

## Important facts about arccos:



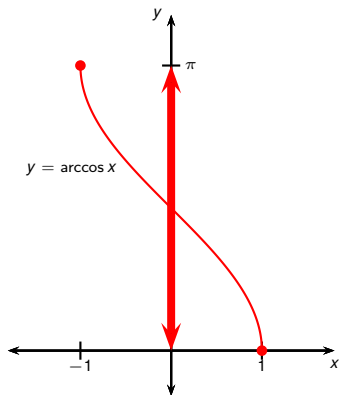
- 1 Domain:  $[-1, 1]$ .
- 2 Range:
- 3  $\arccos x = y \Leftrightarrow \cos y = x$  and  $0 \leq y \leq \pi$ .
- 4  $\arccos(\cos x) = x$  for  $0 \leq x \leq \pi$ .
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## Important facts about arccos:



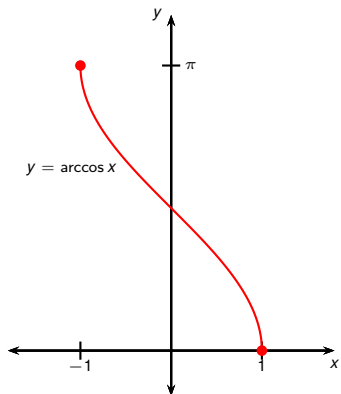
- 1 Domain:  $[-1, 1]$ .
- 2 **Range: ?**
- 3  $\arccos x = y \Leftrightarrow \cos y = x$  and  $0 \leq y \leq \pi$ .
- 4  $\arccos(\cos x) = x$  for  $0 \leq x \leq \pi$ .
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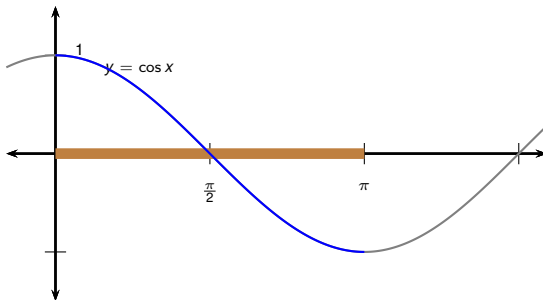
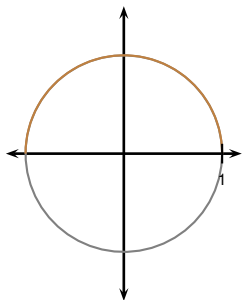
## Important facts about arccos:



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- 6  $\frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1-x^2}}$ .  
(The proof is similar to the proof of the formula for the derivative of  $\arcsin x$ .)

## Example

Find  $\arccos(\cos 4)$ .

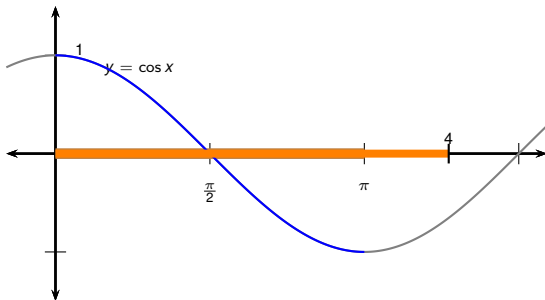
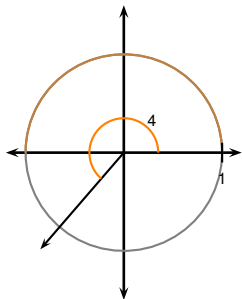




## Example

Find  $\arccos(\cos 4)$ .

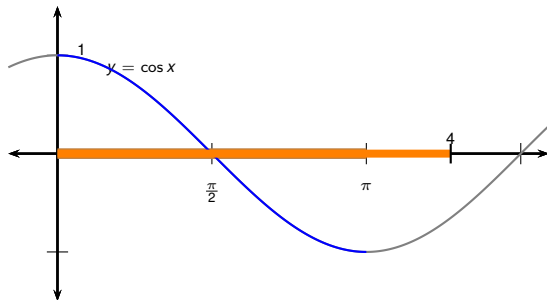
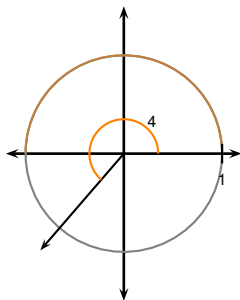
- 4 is not between 0 and  $\pi$ .



## Example

Find  $\arccos(\cos 4)$ .

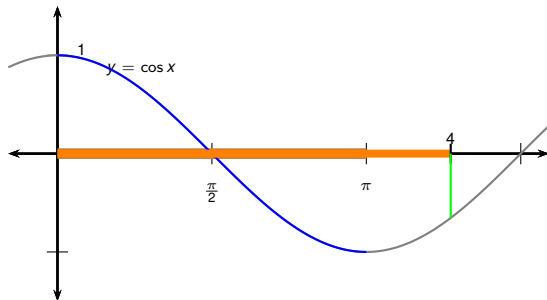
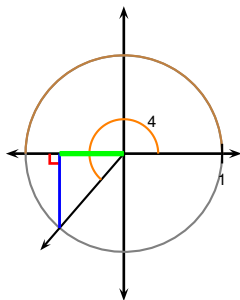
- 4 is not between 0 and  $\pi$ .
- We need the angle  $a$  between 0 and  $\pi$  for which  $\cos 4 = \cos a$ .



## Example

Find  $\arccos(\cos 4)$ .

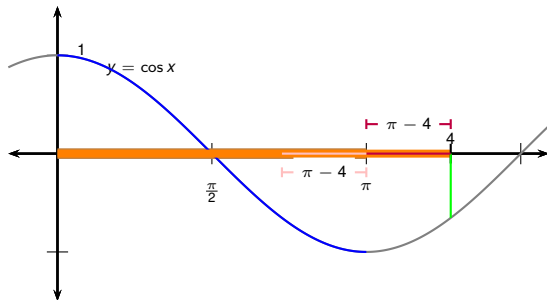
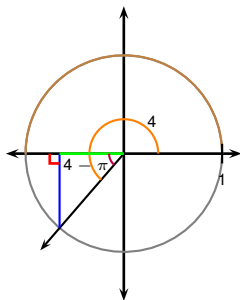
- 4 is not between 0 and  $\pi$ .
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## Example

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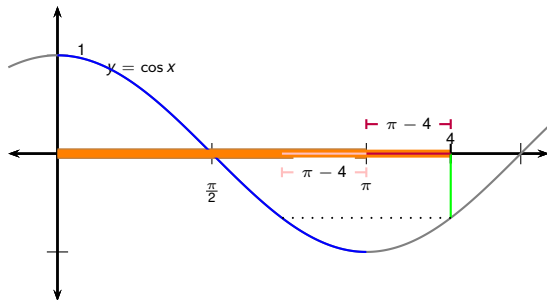
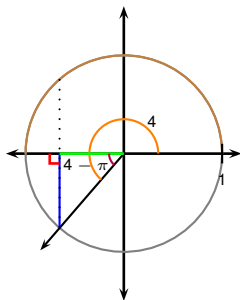
- 4 is not between 0 and  $\pi$ .
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## Example

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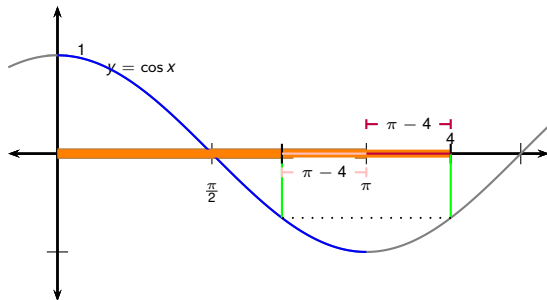
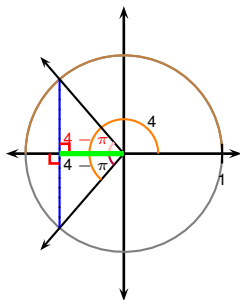
- 4 is not between 0 and  $\pi$ .
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## Example

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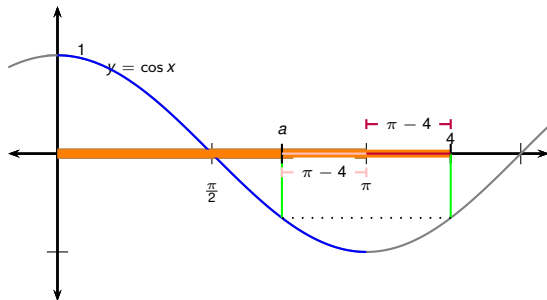
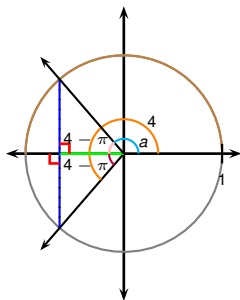
- 4 is not between 0 and  $\pi$ .
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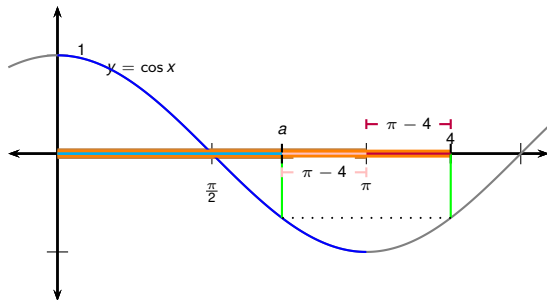
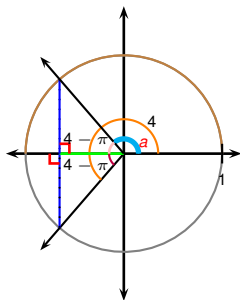


## Example

Find  $\arccos(\cos 4)$ .

- 4 is not between 0 and  $\pi$ .
- We need the angle  $a$  between 0 and  $\pi$  for which  $\cos 4 = \cos a$ .

$$a = ?$$



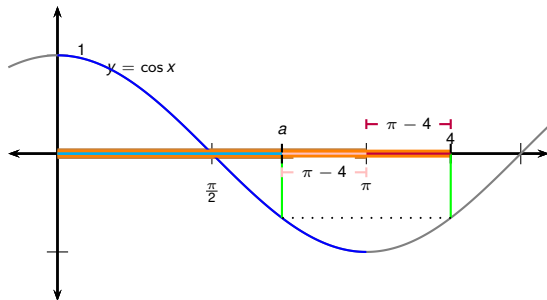
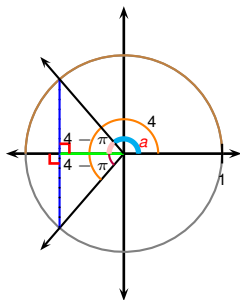


## Example

Find  $\arccos(\cos 4)$ .

- 4 is not between 0 and  $\pi$ .
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$$a = \pi - (4 - \pi)$$

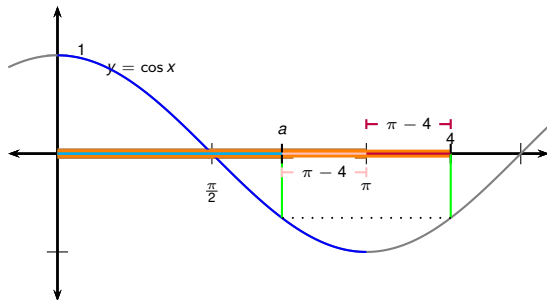
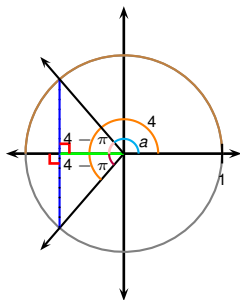


## Example

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- 4 is not between 0 and  $\pi$ .
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$$a = \pi - (4 - \pi) = 2\pi - 4$$



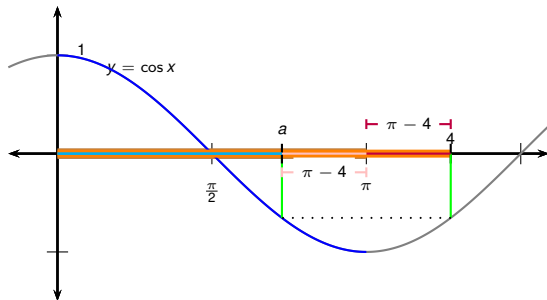
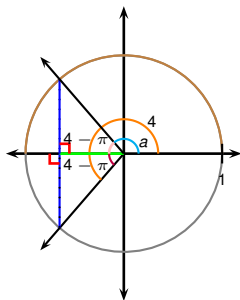
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$$\text{Therefore } \arccos(\cos 4) = \arccos(\cos a)$$



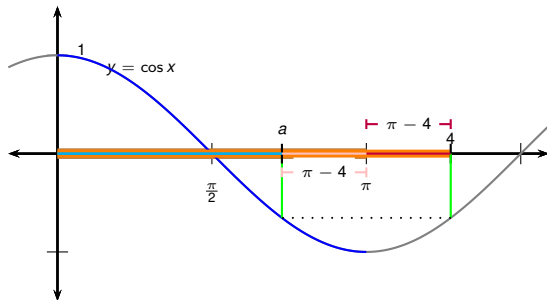
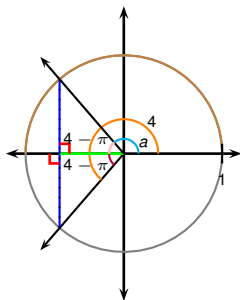
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$$a = \pi - (4 - \pi) = 2\pi - 4$$

$$\begin{aligned} \text{Therefore } \arccos(\cos 4) &= \arccos(\cos a) \\ &= a \end{aligned}$$



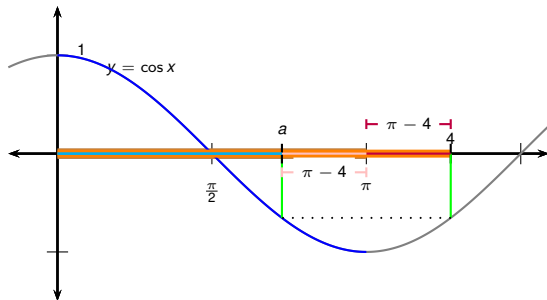
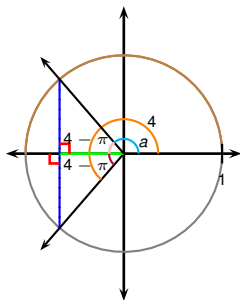
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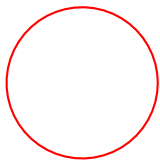
$$\begin{aligned} \text{Therefore } \arccos(\cos 4) &= \arccos(\cos a) \\ &= a = 2\pi - 4. \end{aligned}$$



## Example

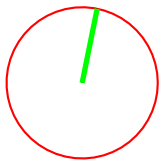
The horizon at sea is viewed at from a ship height of 10m. How far does the ship need to travel to reach the horizon just observed?

## Example



The horizon at sea is viewed at from a ship height of 10m. How far does the ship need to travel to reach the horizon just observed? Assume earth is round with radius 6371 km and that the ship sails along the shortest curved path.

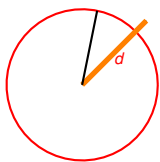
## Example



The horizon at sea is viewed at from a ship height of 10m. How far does the ship need to travel to reach the horizon just observed? Assume earth is round with **radius** 6371 km and that the ship sails along the shortest curved path.



## Example

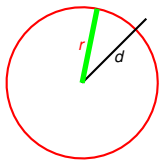


not to scale

The horizon at sea is viewed at from a ship height of 10m. How far does the ship need to travel to reach the horizon just observed? Assume earth is round with radius 6371 km and that the ship sails along the shortest curved path.

- Let  $d$  be the distance from eyes of seaman to the center of earth.

## Example

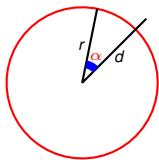


not to scale

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- Let  $d$  be the distance from eyes of seaman to the center of earth.
- Let  $r$  be the radius of earth.

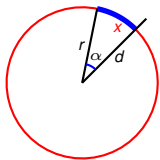
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- Let  $r$  be the radius of earth. Let  $\alpha$  be the indicated angle.

## Example

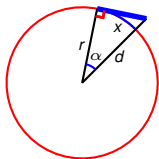


not to scale

The horizon at sea is viewed at from a ship height of 10m. How far does the ship need to travel to reach the horizon just observed? Assume earth is round with radius 6371 km and that **the ship sails along the shortest curved path**.

- Let  $d$  be the distance from eyes of seaman to the center of earth.
- Let  $r$  be the radius of earth. Let  $\alpha$  be the indicated angle.
- Let the distance to the horizon be  $x$ .

## Example

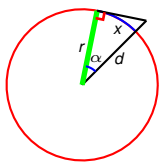


not to scale

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## Example

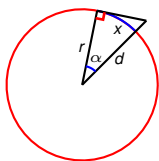


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$$r=6371\text{km}$$

## Example



not to scale

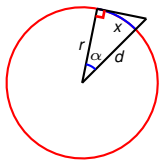
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$$d=?$$

## Example



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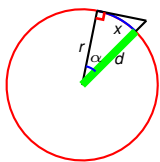
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$$r = 6371 \text{ km}$$

$$d = 6371 \text{ km} + 0.01 \text{ km}$$



## Example



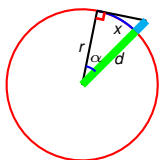
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## Example



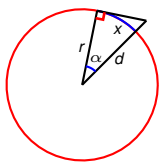
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## Example



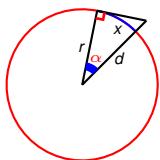
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## Example



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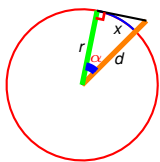
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## Example



not to scale

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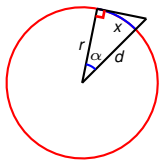
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## Example



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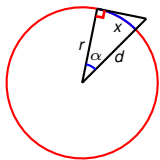
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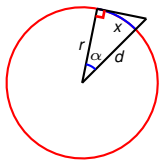
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not to scale

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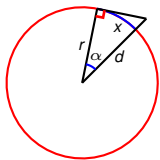
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## Example



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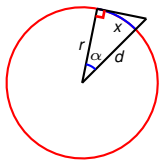
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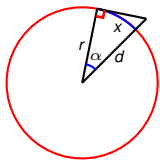
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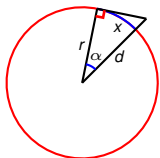
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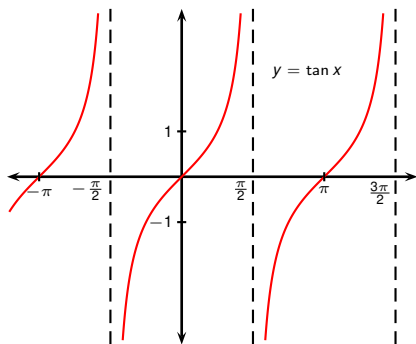
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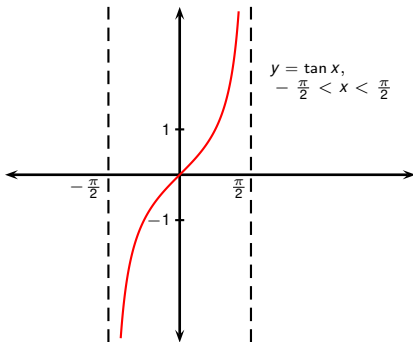
$$\alpha = \arccos\left(\frac{r}{d}\right)$$

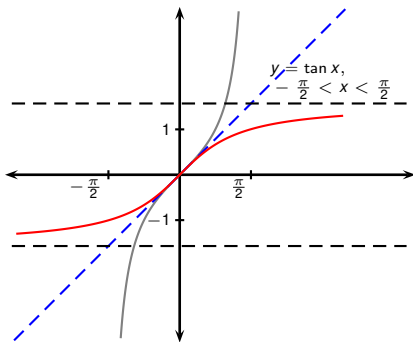
$$x = r\alpha = r \arccos\left(\frac{r}{d}\right) = 6371 \text{ km} \arccos\left(\frac{6371 \text{ km}}{6371.01 \text{ km}}\right) \approx 11.29 \text{ km}$$

- $\tan x$  isn't one-to-one.

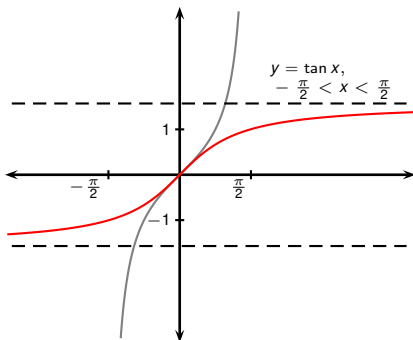


- $\tan x$  isn't one-to-one.
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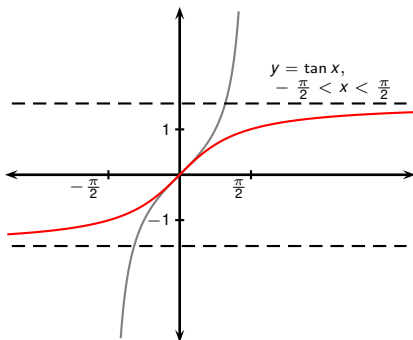


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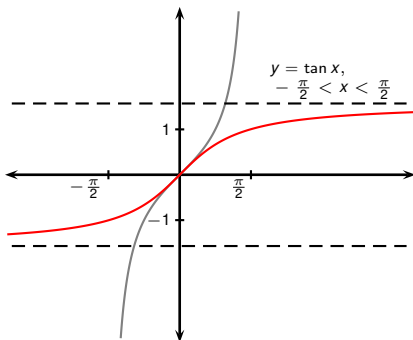


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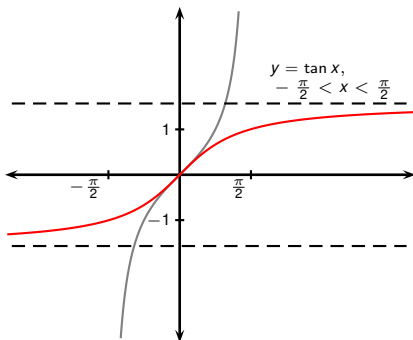




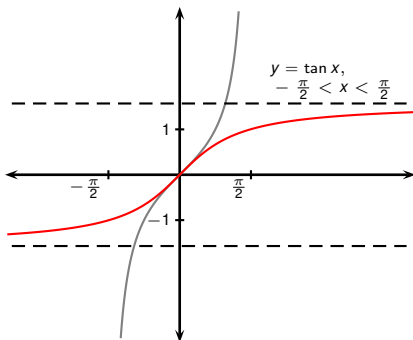
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- **Domain of  $\arctan$ : ?**
- Range of  $\arctan$ :



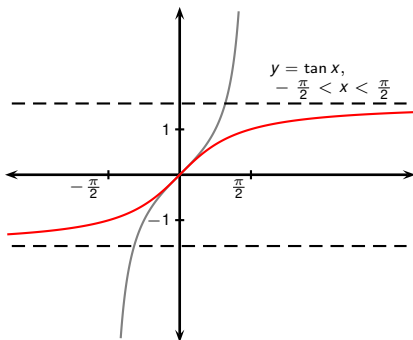
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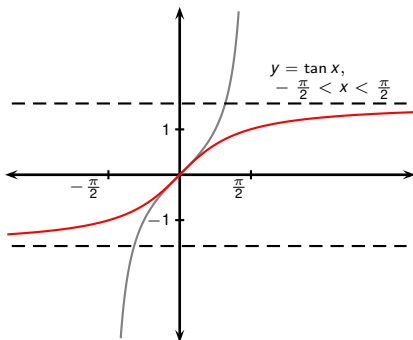
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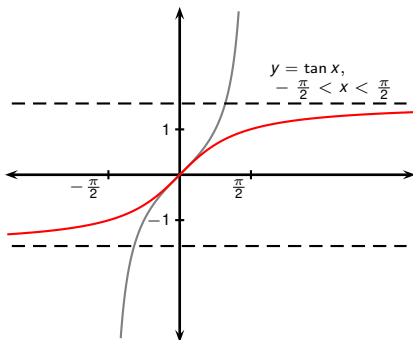
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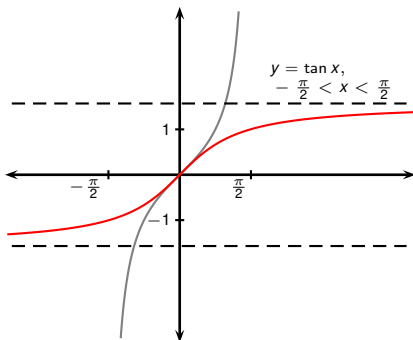
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- $\lim_{x \rightarrow \infty} \arctan x = ?$
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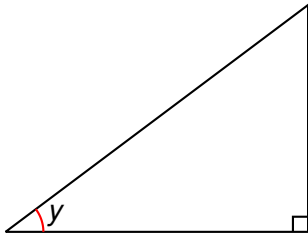


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## Example

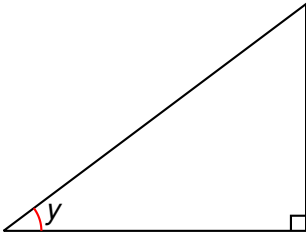
Simplify the expression  $\cos(\arctan x)$ .



## Example

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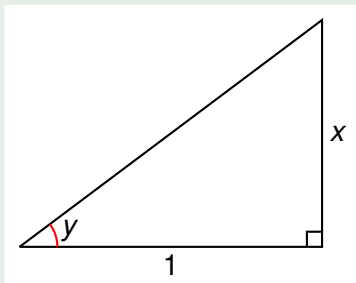
- Let  $y = \arctan x$ , so  $\tan y = x$ .



## Example

Simplify the expression  $\cos(\arctan x)$ .

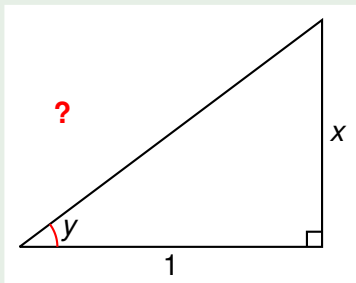
- Let  $y = \arctan x$ , so  $\tan y = x$ .
- Draw a right triangle with opposite  $x$  and adjacent  $1$ .



## Example

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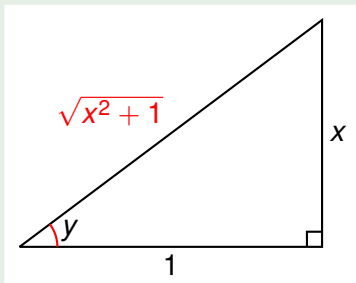
- Let  $y = \arctan x$ , so  $\tan y = x$ .
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- Length of hypotenuse = ?



## Example

Simplify the expression  $\cos(\arctan x)$ .

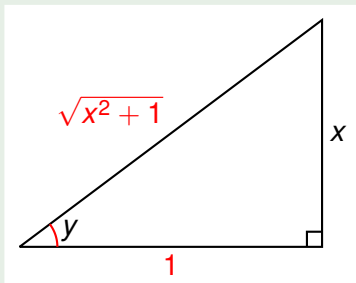
- Let  $y = \arctan x$ , so  $\tan y = x$ .
- Draw a right triangle with opposite  $x$  and adjacent 1.
- Length of hypotenuse =  $\sqrt{1^2 + x^2}$ .



## Example

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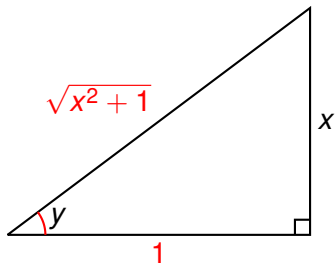
- Let  $y = \arctan x$ , so  $\tan y = x$ .
- Draw a right triangle with opposite  $x$  and adjacent  $1$ .
- Length of hypotenuse =  $\sqrt{1^2 + x^2}$ .
- Then  $\cos(\arctan x) = ?$



## Example

Simplify the expression  $\cos(\arctan x)$ .

- Let  $y = \arctan x$ , so  $\tan y = x$ .
- Draw a right triangle with opposite  $x$  and adjacent  $1$ .
- Length of hypotenuse =  $\sqrt{1^2 + x^2}$ .
- Then  $\cos(\arctan x) = \frac{1}{\sqrt{1+x^2}}$ .



The remaining inverse trigonometric functions aren't used as often:

$$\begin{aligned}y = \operatorname{arccsc} x \quad (|x| \geq 1) &\Leftrightarrow \csc y = x \quad \text{and} \quad y \in \left(0, \frac{\pi}{2}\right] \cup \left(\pi, \frac{3\pi}{2}\right] \\y = \operatorname{arcsec} x \quad (|x| \geq 1) &\Leftrightarrow \sec y = x \quad \text{and} \quad y \in \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right) \\y = \operatorname{arccot} x \quad (|x| \in \mathbb{R}) &\Leftrightarrow \cot y = x \quad \text{and} \quad y \in (0, \pi)\end{aligned}$$

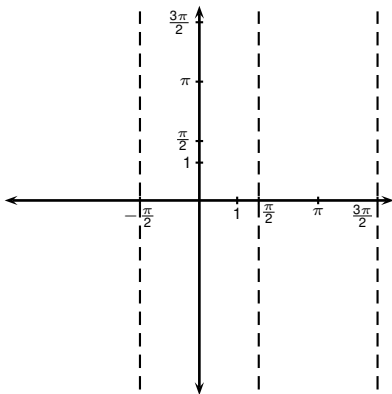


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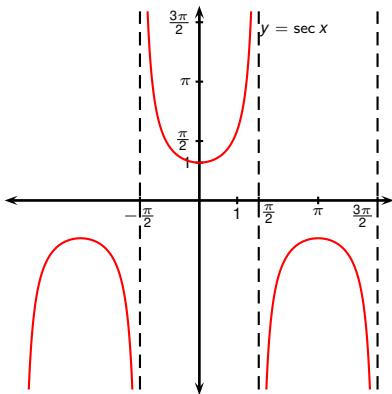
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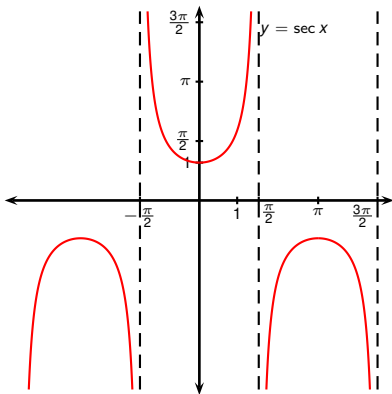
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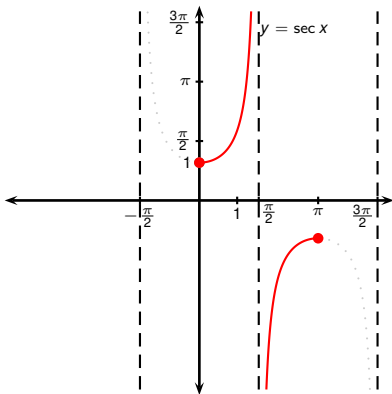
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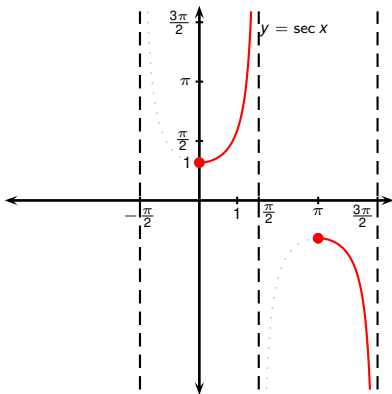
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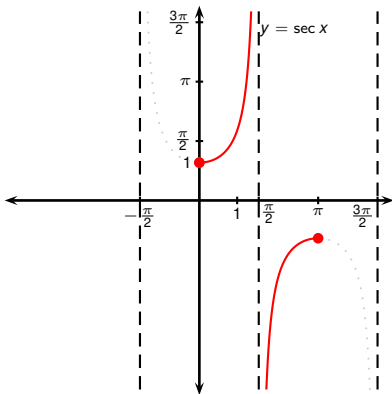
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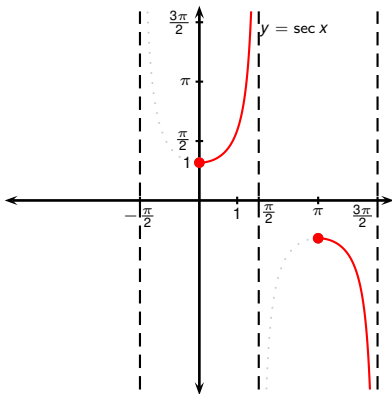
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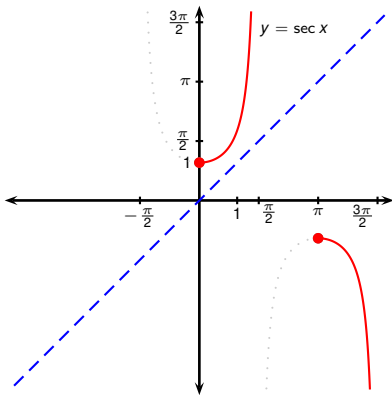


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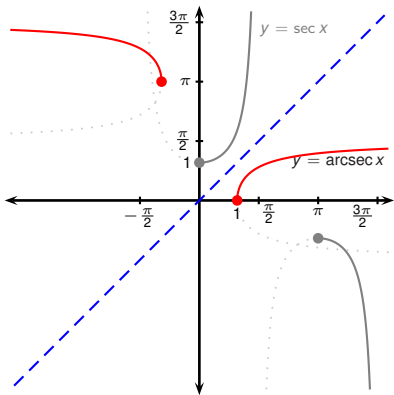
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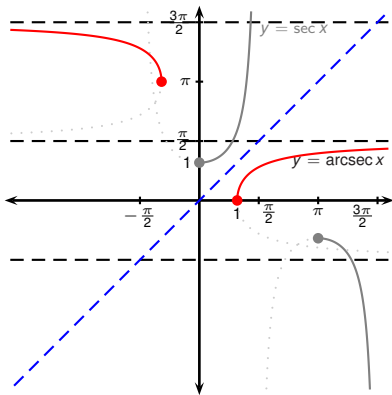
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Rewrite  $\sin(2 \arccos(x))$  as an algebraic expression of  $x$  and  $\sqrt{1 - x^2}$ .

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Set  $y = \arccos x$   
Express via  $\sin y, \cos y$



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$$\begin{aligned}\sin(2 \arccos(x)) &= \sin(2y) \\ &= 2 \cos y \sin y \\ &= 2 \cos y \left( \pm \sqrt{1 - \cos^2 y} \right)\end{aligned} \quad \left| \begin{array}{l} \text{Set } y = \arccos x \\ \text{Express via } \sin y, \cos y \\ \text{Express } \sin y \text{ via } \cos y \end{array} \right.$$

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 \cos(3 \arccos(x)) &= \cos(3y) = \cos(2y + y) \\
 &= \cos(2y) \cos y - \sin(2y) \sin y \\
 &= (\cos^2 y - \sin^2 y) \cos y \\
 &\quad - 2 \sin y \cos y \sin y \\
 &= \cos^3 y - \sin^2 y \cos y - 2 \sin^2 y \cos y \\
 &= \cos^3 y - 3 \sin^2 y \cos y \\
 &= \cos^3 y - 3(\text{?}) \cos y
 \end{aligned}$$

$y = \arccos x$   
 Angle sum f-la  
 Express via  
 $\sin y, \cos y$

Express  $\sin y$   
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## Example

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 &= \cos^3 y - \sin^2 y \cos y - 2 \sin^2 y \cos y \\
 &= \cos^3 y - 3 \sin^2 y \cos y & \text{Express } \sin y \\
 &= \cos^3 y - 3(1 - \cos^2 y) \cos y & \text{via } \cos y \\
 &= 4\cos^3 y - 3 \cos y
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 &= \cos^3 y - 3(1 - \cos^2 y) \cos y \\
 &= 4\cos^3 y - 3\cos y \\
 &= 4x^3 - 3x
 \end{aligned}$$

$y = \arccos x$   
 Angle sum f-la  
 Express via  
 $\sin y, \cos y$

Express  $\sin y$   
 via  $\cos y$

$x = \cos y$

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