

# Calculus II

## L'Hospital's rule

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# Outline

- 1 Indeterminate Forms and L'Hospital's Rule
  - Indeterminate Products
  - Indeterminate Differences
  - Indeterminate Powers

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## Example

Find  $\lim_{x \rightarrow 1} \frac{\ln x}{x-1}$ .

- $\lim_{x \rightarrow 1} \ln x = 0$ .
- $\lim_{x \rightarrow 1} (x - 1) = 0$ .
- We don't get any cancellation between top and bottom.
- We need new techniques.

## Theorem (L'Hospital's Rule)

*Suppose that  $f$  and  $g$  are differentiable and  $g'(x) \neq 0$  on an open interval that contains  $a$  (except possibly at  $a$ ). Suppose that*

$$\lim_{x \rightarrow a} f(x) = 0 \qquad \text{and} \qquad \lim_{x \rightarrow a} g(x) = 0$$

$$\text{or that} \qquad \lim_{x \rightarrow a} f(x) = \pm\infty \qquad \text{and} \qquad \lim_{x \rightarrow a} g(x) = \pm\infty$$

*(In other words, we have an indeterminate form of type  $0/0$  or  $\infty/\infty$ .)*  
*Then*

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}.$$

## Example

Find  $\lim_{x \rightarrow 1} \frac{\ln x}{x-1}$ .

- $\lim_{x \rightarrow 1} \ln x = 0$ .
- $\lim_{x \rightarrow 1} (x - 1) = 0$ .
- This is an indeterminate form of type  $0/0$ .
- Apply L'Hospital's rule:

$$\lim_{x \rightarrow 1} \frac{\ln x}{x-1} = \lim_{x \rightarrow 1} \frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(x-1)} = \lim_{x \rightarrow 1} \frac{1/x}{1} = \lim_{x \rightarrow 1} \frac{1}{x} = 1.$$

## Example

Find  $\lim_{x \rightarrow \infty} \frac{e^x}{x^2}$ .

- $\lim_{x \rightarrow \infty} e^x = \infty$ .
- $\lim_{x \rightarrow \infty} x^2 = \infty$ .
- This is an indeterminate form of type  $\infty/\infty$ .
- Apply L'Hospital's rule:

$$\lim_{x \rightarrow \infty} \frac{e^x}{x^2} = \lim_{x \rightarrow \infty} \frac{\frac{d}{dx}(e^x)}{\frac{d}{dx}(x^2)} = \lim_{x \rightarrow \infty} \frac{e^x}{2x}$$

- $\lim_{x \rightarrow \infty} 2x = \infty$ .
- This is an indeterminate form of type  $\infty/\infty$ .
- Apply L'Hospital's rule again:

$$\lim_{x \rightarrow \infty} \frac{e^x}{x^2} = \lim_{x \rightarrow \infty} \frac{e^x}{2x} = \lim_{x \rightarrow \infty} \frac{e^x}{2} = \infty.$$

# Indeterminate Products

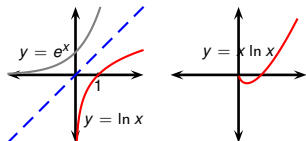
If  $\lim_{x \rightarrow a} f(x) = 0$  and  $\lim_{x \rightarrow a} g(x) = \pm\infty$ , then it isn't clear what  $\lim_{x \rightarrow a} (fg)(x)$  will be.

In such a case, write the product  $fg$  as a quotient:

$$fg = \frac{f}{1/g} \quad \text{or} \quad fg = \frac{g}{1/f}.$$

This converts the given limit into an indeterminate form of type  $0/0$  or  $\infty/\infty$ .





## Example

Evaluate  $\lim_{x \rightarrow 0^+} x \ln x$ .

- $\lim_{x \rightarrow 0^+} \ln x = -\infty$ .
- $\lim_{x \rightarrow 0^+} x = 0$ .
- This is an indeterminate form of type  $0(-\infty)$  (or  $-\infty/(1/0)$ ).
- Apply L'Hospital's rule:

$$\begin{aligned}
 \lim_{x \rightarrow 0^+} x \ln x &= \lim_{x \rightarrow 0^+} \frac{\ln x}{\frac{1}{x}} = \lim_{x \rightarrow 0^+} \frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}\left(\frac{1}{x}\right)} \\
 &= \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x \rightarrow 0^+} (-x) = 0.
 \end{aligned}$$

# Indeterminate Differences

If  $\lim_{x \rightarrow a} f(x) = \infty$  and  $\lim_{x \rightarrow a} g(x) = \infty$ , then the limit

$$\lim_{x \rightarrow a} [f(x) - g(x)]$$

is called an indeterminate form of type  $\infty - \infty$ .

To compute such a limit, try to convert it into a quotient (by using a common denominator, or by rationalizing, or by factoring out a common factor).

## Example

Evaluate  $\lim_{x \rightarrow (\pi/2)^-} (\sec x - \tan x)$ .

- $\lim_{x \rightarrow (\pi/2)^-} \sec x = \infty$ .
- $\lim_{x \rightarrow (\pi/2)^-} \tan x = \infty$ .
- This is an indeterminate form of type  $\infty - \infty$ .
- Apply L'Hospital's rule:

$$\begin{aligned}\lim_{x \rightarrow (\pi/2)^-} (\sec x - \tan x) &= \lim_{x \rightarrow (\pi/2)^-} \left( \frac{1}{\cos x} - \frac{\sin x}{\cos x} \right) \\&= \lim_{x \rightarrow (\pi/2)^-} \frac{1 - \sin x}{\cos x} \\&\quad \text{(indeterminate form of type } 0/0.) \\&= \lim_{x \rightarrow (\pi/2)^-} \frac{\frac{d}{dx}(1 - \sin x)}{\frac{d}{dx} \cos x} \\&= \lim_{x \rightarrow (\pi/2)^-} \frac{-\cos x}{-\sin x} = 0\end{aligned}$$

# Indeterminate Powers

Several indeterminate forms arise from the limit  $\lim_{x \rightarrow a} f(x)^{g(x)}$ .

$$\lim_{x \rightarrow a} f(x) = 0 \quad \text{and} \quad \lim_{x \rightarrow a} g(x) = 0 \quad \text{type } 0^0$$

$$\lim_{x \rightarrow a} f(x) = \infty \quad \text{and} \quad \lim_{x \rightarrow a} g(x) = 0 \quad \text{type } \infty^0$$

$$\lim_{x \rightarrow a} f(x) = 1 \quad \text{and} \quad \lim_{x \rightarrow a} g(x) = \pm\infty \quad \text{type } 1^\infty$$

These can all be solved either by taking the natural logarithm:

$$\text{let } y = [f(x)]^{g(x)}, \text{ then } \ln y = g(x) \ln f(x)$$

or by writing the function as an exponential:

$$[f(x)]^{g(x)} = e^{g(x) \ln f(x)}.$$

## Example

Find  $\lim_{x \rightarrow 0^+} x^x$ .

- $0^x = 0$  for any  $x > 0$ .
- $x^0 = 1$  for any  $x \neq 0$ .
- This is an indeterminate form of type  $0^0$ .
- Write as an exponential:
- $x^x = e^{x \ln x}$ .
- Recall that  $\lim_{x \rightarrow 0^+} x \ln x = 0$ .
- Therefore

$$\lim_{x \rightarrow 0^+} x^x = \lim_{x \rightarrow 0^+} e^{x \ln x} = e^0 = 1$$

## Example

$$\begin{aligned}
 \lim_{x \rightarrow \infty} \left(1 + \frac{k}{x}\right)^x &= \lim_{x \rightarrow \infty} e^{\ln\left(1 + \frac{k}{x}\right)^x} && \text{exponent= continuous f-n} \\
 &= e^{\lim_{x \rightarrow \infty} \ln\left(1 + \frac{k}{x}\right)^x} = e^k && \text{limit computed below} \\
 \lim_{x \rightarrow \infty} \ln\left(1 + \frac{k}{x}\right)^x &= \lim_{x \rightarrow \infty} x \ln\left(1 + \frac{k}{x}\right) \\
 &= \lim_{x \rightarrow \infty} \frac{\frac{d}{dx} (\ln(1 + \frac{k}{x}))}{\frac{d}{dx} (\frac{1}{x})} && \text{form "0/0", use L'Hospital} \\
 &= \lim_{x \rightarrow \infty} \frac{\frac{1}{1 + \frac{k}{x}} \left(1 + \frac{k}{x}\right)'}{-\frac{1}{x^2}} \\
 &= \lim_{x \rightarrow \infty} \frac{\frac{1}{1 + \frac{k}{x}} \left(-\frac{k}{x^2}\right)}{-\frac{1}{x^2}} \\
 &= \lim_{x \rightarrow \infty} \frac{k}{1 + \frac{k}{x}} = k
 \end{aligned}$$