Calculus II
Trigonometric integrals

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Outline

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Integrals of the form $\int R(\cos\theta, \sin\theta)\mathrm{d}\theta$, R

Let R be an arbitrary rational function in two variables (quotient of polynomials in two variables).

Question

Can we integrate $\int R(\cos\theta, \sin\theta)\mathrm{d}\theta$?

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- Yes. We will learn how in what follows.

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- Yes. We will learn how in what follows.
- The algorithm for integration is roughly:
  - Apply the substitution $\theta = 2\arctan t$ to transform to integral of rational function.
  - Solve as previously studied.

The rationalizing substitution $\theta = 2 \arctan t$

Recall the expression of $\sin(2z), \cos(2z)$ via $\tan z$:

$$\sin(2z) = \textbf{?}$$

$$\cos(2z)$$

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\sin\theta &= \sin(2\arctan t) = \frac{2\tan(\arctan t)}{1+\tan^2(\arctan t)} = \frac{2t}{1+t^2} \\
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t &= ?
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Theorem

The substitution given above transforms $\int R(\cos\theta, \sin\theta) \mathrm{d}\theta$ *to an integral of a rational function of* t.

Example

$$\int \frac{\mathrm{d}\theta}{2\sin\theta - \cos\theta + 5}$$

Example

Let $\theta = 2\arctan t$, $\cos\theta = \frac{1-t^2}{1+t^2}$, $\sin\theta = \frac{2t}{1+t^2}$

$$\int \frac{\mathrm{d}\theta}{2\sin\theta - \cos\theta + 5} = \int \frac{2\mathrm{d}t}{(1+t^2)\left(2\frac{2t}{t^2+1} - \frac{(1-t^2)}{1+t^2} + 5\right)}$$

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$$\begin{aligned}\int \frac{\mathrm{d}\theta}{2\sin\theta - \cos\theta + 5} &= \int \frac{2\mathrm{d}t}{(1+t^2)\left(2\frac{2t}{t^2+1} - \frac{(1-t^2)}{1+t^2} + 5\right)} \\ &= \int \frac{2\mathrm{d}t}{6t^2 + 4t + 4}\end{aligned}$$

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&= \int \frac{2\mathrm{d}t}{6t^2 + 4t + 4} \\
&= \int \frac{\mathrm{d}t}{3t^2 + 2t + 2}
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\text{(complete square)} \quad &= \int \frac{dt}{3\left(t^2 + 2t\frac{1}{3} + \frac{1}{9} - \frac{1}{9} + \frac{2}{3}\right)}
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&= \frac{1}{3}\int \frac{\mathrm{d}t}{\left(t+\frac{1}{3}\right)^2+\frac{5}{9}} \\
&= \frac{1}{3}\int \frac{\mathrm{d}t}{\frac{5}{9}\left(\frac{9}{5}\left(t+\frac{1}{3}\right)^2+1\right)}
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&= \frac{1}{3}\int \frac{\mathrm{d}t}{\frac{5}{9}\left(\frac{9}{5}\left(t+\frac{1}{3}\right)^2 + 1\right)}
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$$\int \frac{\mathrm{d}\theta}{2\sin\theta - \cos\theta + 5} = \frac{1}{3}\int \frac{\mathrm{d}t}{\frac{5}{9}\left(\frac{9}{5}\left(t+\frac{1}{3}\right)^2 + 1\right)}$$

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$$= \frac{3}{5}\int \frac{\frac{\sqrt{5}}{3} d\left(\frac{3}{\sqrt{5}}\left(t \qquad \right)\right)}{\left(\left(\frac{3}{\sqrt{5}}\left(t+\frac{1}{3}\right)\right)^2+1\right)}$$

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$$\begin{aligned}
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&= \frac{3}{5}\int \frac{\frac{\sqrt{5}}{3}\mathrm{d}\left(\frac{3}{\sqrt{5}}\left(t+\frac{1}{3}\right)\right)}{\left(\left(\frac{3}{\sqrt{5}}\left(t+\frac{1}{3}\right)\right)^2+1\right)}\\
&= \frac{\sqrt{5}}{5}\int \frac{\mathrm{d}z}{z^2+1}
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&= \frac{\sqrt{5}}{5}\int \frac{\mathrm{d}z}{z^2+1} \\
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&= \frac{\sqrt{5}}{5}\int \frac{\mathrm{d}z}{z^2+1}\\
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&= \frac{\sqrt{5}}{5}\arctan z + C \\
&= \frac{\sqrt{5}}{5}\arctan\left(\frac{3}{\sqrt{5}}\left(t+\frac{1}{3}\right)\right) + C \\
&= \frac{\sqrt{5}}{5}\arctan\left(\frac{3}{\sqrt{5}}\left(\tan\left(\frac{\theta}{2}\right)+\frac{1}{3}\right)\right) + C
\end{aligned}
$$

The integral $\int \sec\theta \mathrm{d}\theta$ appears often in practice. A quicker solution will be shown later, but first we show the standard method.

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Example

Set $\theta = 2\arctan t$, $\cos\theta = \frac{1-\tan^2(\frac{\theta}{2})}{1+\tan^2\left(\frac{\theta}{2}\right)} = \frac{1-t^2}{1+t^2}$, .

$$\int \sec\theta \mathrm{d}\theta = \int \frac{1}{\cos\theta}\mathrm{d}\theta = \int \frac{1}{\left(\frac{1-t^2}{1+t^2}\right)} \frac{2}{(1+t^2)}\mathrm{d}t$$

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$$\begin{aligned}\int \sec\theta \mathrm{d}\theta &= \int \frac{1}{\cos\theta}\mathrm{d}\theta = \int \frac{1}{\left(\frac{1-t^2}{\cancel{1+t^2}}\right)} \frac{2}{(\cancel{1+t^2})}\mathrm{d}t \\ &= \int \frac{2}{1-t^2}\mathrm{d}t\end{aligned}$$

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$$\begin{aligned}
\int \sec\theta \mathrm{d}\theta &= \int \frac{1}{\cos\theta}\mathrm{d}\theta = \int \frac{1}{\left(\frac{1-t^2}{\cancel{1+t^2}}\right)} \frac{2}{\cancel{(1+t^2)}}\mathrm{d}t \\
&= \int \frac{2}{1-t^2}\mathrm{d}t = \int \left(\frac{1}{1-t} + \frac{1}{1+t}\right)\mathrm{d}t \quad \Big| \text{ part. fractions} \\
&= -\ln|1-t| + \ln|1+t| + C
\end{aligned}$$

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&= -\ln|1-t| + \ln|1+t| + C \\
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$$\int \sec\theta \mathrm{d}\theta = \ln\left|\frac{1+\tan\left(\frac{\theta}{2}\right)}{1-\tan\left(\frac{\theta}{2}\right)}\right| + C$$

This is a perfectly good answer, however there's a simplification:

$$\tan\theta + \sec\theta =$$

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This is a perfectly good answer, however there's a simplification:

$$\tan\theta + \sec\theta \quad = \quad \frac{\sin\theta + 1}{\cos\theta} = \frac{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) + \sin^2\left(\frac{\theta}{2}\right) + \cos^2\left(\frac{\theta}{2}\right)}{\cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)}$$

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This is a perfectly good answer, however there's a simplification:

$$\tan\theta + \sec\theta \quad = \quad \frac{\sin\theta + 1}{\cos\theta} = \frac{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) + \sin^2\left(\frac{\theta}{2}\right) + \cos^2\left(\frac{\theta}{2}\right)}{\cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)}$$

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- Examples to which our ad hoc techniques apply arise from integrals needed outside of the subject of Calculus II, so these techniques are important.
- The trigonometric integral we saw, $\int \frac{\mathrm{d}\theta}{2\sin\theta - \cos\theta + 5}$, will not work with any of following ad-hoc techniques, so the general method is important as well.

Example

$$\int \sin^3 x \mathrm{d}x$$

Example

$$\int \sin^3 x \mathrm{d}x = \int \sin^2 x \sin x \mathrm{d}x$$

Example

$$\begin{aligned}\int \sin^3 x \mathrm{d}x &= \int \sin^2 x \sin x \mathrm{d}x \\ &= \int \sin^2 x \mathrm{d}(\textbf{?} \qquad)\end{aligned}$$

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$$\begin{aligned}\int \sin^3 x\mathrm{d}x &= \int \sin^2 x \sin x\mathrm{d}x\\ &= \int \sin^2 x\mathrm{d}(-\cos x)\\ &= \int (-1)\left(\textbf{?} \qquad\qquad \right)\mathrm{d}(\cos x)\end{aligned}$$

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Can we rewrite $\sin^2 x$ via $\cos x$?

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$$\begin{aligned}\int \sin^3 x \mathrm{d}x &= \int \sin^2 x \sin x \mathrm{d}x \\ &= \int \sin^2 x \mathrm{d}(-\cos x) \\ &= \int (-1)\left(1-\cos^2 x\right)\mathrm{d}(\cos x)\end{aligned}$$

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Set $u = \cos x$

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&= \int \left(u^2 - 1\right) \mathrm{d}u \\
&= \frac{u^3}{3} - u + C
\end{aligned}$$

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\begin{aligned}
\int \sin^3 x dx &= \int \sin^2 x \sin x dx \\
&= \int \sin^2 x d(-\cos x) \\
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&= \int \left(\cos^2 x - 1\right) d(\cos x) \\
&= \int \left(u^2 - 1\right) du \\
&= \frac{u^3}{3} - u + C \\
&= \frac{1}{3}\cos^3 x - \cos x + C \quad .
\end{aligned}$$

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$$\int \cos^5 x \sin^2 x \mathrm{d}x$$

Example

$$\int \cos^5 x \sin^2 x \mathrm{d}x = \int \cos^4 x \sin^2 x \cos x \mathrm{d}x$$

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$$\begin{aligned}\int \cos^5 x \sin^2 x \mathrm{d}x &= \int \cos^4 x \sin^2 x \cos x \mathrm{d}x\\ &= \int \cos^4 x \sin^2 x \mathrm{d}(\mathbf{?} \quad)\end{aligned}$$

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$$\begin{aligned}\int \cos^5 x \sin^2 x \mathrm{d}x &= \int \cos^4 x \sin^2 x \cos x \mathrm{d}x \\ &= \int \cos^4 x \sin^2 x \mathrm{d}(\sin x)\end{aligned}$$

Example

$$\begin{aligned}\int \cos^5 x \sin^2 x \mathrm{d}x &= \int \cos^4 x \sin^2 x \cos x \mathrm{d}x \\ &= \int {\color{red}\cos^4 x} \sin^2 x \mathrm{d}(\sin x)\end{aligned}$$

Can we rewrite $\cos^4 x$ via $\sin x$?

Example

$$
\begin{aligned}
\int \cos^5 x \sin^2 x \mathrm{d}x &= \int \cos^4 x \sin^2 x \cos x \mathrm{d}x \\
&= \int \cos^4 x \sin^2 x \mathrm{d}(\sin x) \\
&= \int \left(\cos^2 x\right)^2 \sin^2 x \mathrm{d}(\sin x)
\end{aligned}
$$

Can we rewrite $\cos^4 x$ via $\sin x$?

Example

$$\begin{aligned}\int \cos^5 x \sin^2 x \mathrm{d}x &= \int \cos^4 x \sin^2 x \cos x \mathrm{d}x \\ &= \int \cos^4 x \sin^2 x \mathrm{d}(\sin x) \\ &= \int \left(\cos^2 x\right)^2 \sin^2 x \mathrm{d}(\sin x) \\ &= \int \left(1 - \sin^2 x\right)^2 \sin^2 x \mathrm{d}(\sin x)\end{aligned}$$

Can we rewrite $\cos^4 x$ via $\sin x$?

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$$\begin{aligned}
\int \cos^5 x \sin^2 x \mathrm{d}x &= \int \cos^4 x \sin^2 x \cos x \mathrm{d}x \\
&= \int \cos^4 x \sin^2 x \mathrm{d}(\sin x) && \text{Can we rewrite } \cos^4 x \text{ via } \sin x\text{?} \\
&= \int \left(\cos^2 x\right)^2 \sin^2 x \mathrm{d}(\sin x) \\
&= \int \left(1 - \sin^2 x\right)^2 \sin^2 x \mathrm{d}(\sin x) && \text{Set } u = \sin x \\
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&= \frac{u^3}{3} - 2\frac{u^5}{5} + \frac{u^7}{7} + C
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&= \int \left(1 - 2u^2 + u^4\right) u^2 \mathrm{d}u \\
&= \int \left(u^2 - 2u^4 + u^6\right) \mathrm{d}u \\
&= \frac{u^3}{3} - 2\frac{u^5}{5} + \frac{u^7}{7} + C \\
&= \frac{\sin^3 x}{3} - 2\frac{\sin^5 x}{5} + \frac{\sin^7 x}{7} + C \quad .
\end{aligned}
$$

Can we rewrite $\cos^4 x$ via $\sin x$?

Set $u = \sin x$

$$\int \sin^m x \cos^n x \mathrm{d}x$$

When n – odd:

$$\int \sin^m x \cos^n x \mathrm{d}x$$

When m – odd:

$$\int \sin^m x \cos^n x dx = \int \sin^m x \cos^{n-1} x d(\sin x)$$

When n – odd:
$\cos x dx$
$= d(\sin x)$

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$$= \int \sin^m x \left(1 - \sin^2 x\right)^{\frac{n-1}{2}} d(\sin x)$$

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If both m, n- even,

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If both m, n- even, use $\left|\begin{array}{rcl} \sin^2 x & = & \frac{1-\cos(2x)}{2} \\ \cos^2 x & = & \frac{\cos(2x)+1}{2} \end{array}\right.$ and substitute $s = 2x$ to lower trig powers. Repeat above considerations.

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Example

$$\int_0^{\frac{\pi}{2}} \sin^2 x \mathrm{d}x$$

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$$\int_0^{\frac{\pi}{2}} \sin^2 x \mathrm{d}x = \int_0^{\frac{\pi}{2}} \left(\frac{1 - \cos(2x)}{2} \right) \mathrm{d}x$$

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$$\int_{t=0}^{t=1} \sqrt{1-t^2}\mathrm{d}t$$

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$$\int_0^{\frac{\pi}{2}} \sin^2 x dx = \int_0^{\frac{\pi}{2}} \left(\frac{1-\cos(2x)}{2} \right) dx$$

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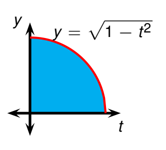

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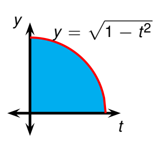
Set $t = \cos x$, $x \in \left[0, \frac{\pi}{2}\right]$.



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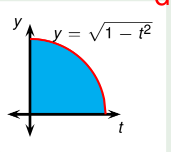

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Set $t = \cos x$, $x \in \left[0, \frac{\pi}{2}\right]$. Then

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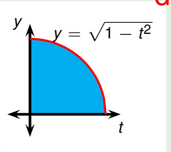

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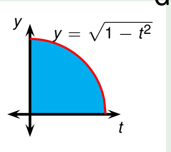

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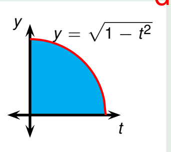

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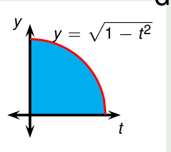

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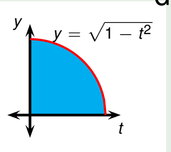

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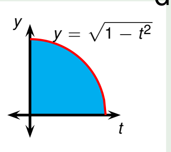

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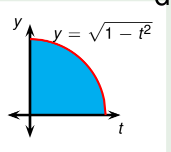

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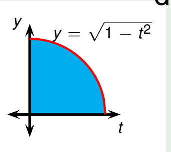
Example

Set $t = \cos x$, $x \in \left[0, \frac{\pi}{2}\right] \Rightarrow \sin x \geq 0$. Then

$\mathrm{d}t = \mathrm{d}(\cos x) = -\sin x \mathrm{d}x$.



$$\int_{t=0}^{t=1} \sqrt{1-t^2}\mathrm{d}t = -\int_{x=\frac{\pi}{2}}^{x=0} \sqrt{1-\cos^2 x} \sin x \mathrm{d}x$$

$$= \int_{x=0}^{x=\frac{\pi}{2}} \sqrt{\sin^2 x} \sin x \mathrm{d}x$$

$$= \int_0^{\frac{\pi}{2}} \sin^2 x \mathrm{d}x$$

Example

$$\int_0^{\frac{\pi}{2}} \sin^2 x dx = \int_0^{\frac{\pi}{2}} \left(\frac{1-\cos(2x)}{2}\right) dx$$

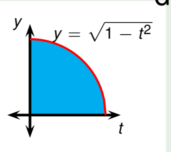
express $\sin^2 x$ via $\cos(2x)$

$$= \left[\frac{x}{2} - \frac{\sin(2x)}{4}\right]_0^{\frac{\pi}{2}}$$

$$= \left(\frac{\pi}{4} - \frac{\sin\pi}{4}\right) - \left(0 - \frac{\sin 0}{4}\right) = \frac{\pi}{4}.$$

Example

Set $t = \cos x$, $x \in \left[0, \frac{\pi}{2}\right] \Rightarrow \sin x \geq 0$. Then $dt = d(\cos x) = -\sin x dx$.



$$\int_{t=0}^{t=1} \sqrt{1-t^2} dt = -\int_{x=\frac{\pi}{2}}^{x=0} \sqrt{1-\cos^2 x} \sin x dx$$

$$= \int_{x=0}^{x=\frac{\pi}{2}} \sqrt{\sin^2 x} \sin x dx$$

$$= \int_0^{\frac{\pi}{2}} \sin^2 x dx = \frac{\pi}{4} \quad .$$

Example

$$\int \tan^8 x \sec^4 x \mathrm{d}x$$

Example

$$\int \tan^8 x \sec^4 x dx = \int \tan^8 x \sec^2 x \sec^2 x dx$$

Example

$$
\begin{aligned}
\int \tan^8 x \sec^4 x \mathrm{d}x &= \int \tan^8 x \sec^2 x \sec^2 x \mathrm{d}x \\
&= \int \tan^8 x \sec^2 x \mathrm{d}\left(\textbf{?} \qquad \right)
\end{aligned}
$$

Example

$$\begin{aligned}\int \tan^8 x \sec^4 x \mathrm{d}x &= \int \tan^8 x \sec^2 x \sec^2 x \mathrm{d}x \\ &= \int \tan^8 x \sec^2 x \mathrm{d}(\tan x)\end{aligned}$$

Example

$$\begin{aligned}\int \tan^8 x \sec^4 x \mathrm{d}x &= \int \tan^8 x \sec^2 x \sec^2 x \mathrm{d}x \\ &= \int \tan^8 x \sec^2 x \mathrm{d}(\tan x) \\ &= \int \tan^8 x \left(\textbf{?} \qquad\qquad \right) \mathrm{d}(\tan x)\end{aligned}$$

Can we rewrite $\sec^2 x$ via $\tan x$?

Example

$$\begin{aligned}\int \tan^8 x \sec^4 x dx &= \int \tan^8 x \sec^2 x \sec^2 x dx \\ &= \int \tan^8 x \sec^2 x d(\tan x) \\ &= \int \tan^8 x \left(1 + \tan^2 x\right) d(\tan x)\end{aligned}$$

Can we rewrite $\sec^2 x$ via $\tan x$?

Example

$$\begin{aligned}\int \tan^8 x \sec^4 x dx &= \int \tan^8 x \sec^2 x \sec^2 x dx \\ &= \int \tan^8 x \sec^2 x d(\tan x) \\ &= \int \tan^8 x \left(1 + \tan^2 x\right) d(\tan x) \\ &= \int u^8 \left(1 + u^2\right) du\end{aligned}$$

Can we rewrite $\sec^2 x$ via $\tan x$?

Set $u = \tan x$

Example

$$
\begin{aligned}
\int \tan^8 x \sec^4 x \mathrm{d}x &= \int \tan^8 x \sec^2 x \sec^2 x \mathrm{d}x \\
&= \int \tan^8 x \sec^2 x \mathrm{d}(\tan x) \\
&= \int \tan^8 x \left(1 + \tan^2 x\right) \mathrm{d}(\tan x) \\
&= \int u^8 \left(1+u^2\right) \mathrm{d}u \\
&= \int \left(u^8 + u^{10}\right) \mathrm{d}u
\end{aligned}
$$

Can we rewrite $\sec^2 x$ via $\tan x$?

Set $u = \tan x$

Example

$$\begin{aligned}
\int \tan^8 x \sec^4 x dx &= \int \tan^8 x \sec^2 x \sec^2 x dx \\
&= \int \tan^8 x \sec^2 x d(\tan x) \\
&= \int \tan^8 x \left(1 + \tan^2 x\right) d(\tan x) \\
&= \int u^8 \left(1+u^2\right) du \\
&= \int \left(u^8 + u^{10}\right) du \\
&= ?
\end{aligned}$$

Can we rewrite $\sec^2 x$ via $\tan x$?

Set $u = \tan x$

Example

$$\begin{aligned}
\int \tan^8 x \sec^4 x dx &= \int \tan^8 x \sec^2 x \sec^2 x dx \\
&= \int \tan^8 x \sec^2 x d\left(\tan x\right) \\
&= \int \tan^8 x \left(1 + \tan^2 x\right) d(\tan x) \\
&= \int u^8 \left(1 + u^2\right) du \\
&= \int \left(u^8 + u^{10}\right) du \\
&= \frac{u^9}{9} + \frac{u^{11}}{11} + C
\end{aligned}$$

Can we rewrite $\sec^2 x$ via $\tan x$?

Set $u = \tan x$

Example

$$\begin{aligned}
\int \tan^8 x \sec^4 x \mathrm{d}x &= \int \tan^8 x \sec^2 x \sec^2 x \mathrm{d}x \\
&= \int \tan^8 x \sec^2 x \mathrm{d}(\tan x) \\
&= \int \tan^8 x \left(1+\tan^2 x\right) \mathrm{d}(\tan x) \\
&= \int u^8 \left(1+u^2\right) \mathrm{d}u \\
&= \int \left(u^8 + u^{10}\right) \mathrm{d}u \\
&= \frac{u^9}{9} + \frac{u^{11}}{11} + C
\end{aligned}$$

Can we rewrite $\sec^2 x$ via $\tan x$?

Set $u = \tan x$

Example

$$
\begin{aligned}
\int \tan^8 x \sec^4 x dx &= \int \tan^8 x \sec^2 x \sec^2 x dx \\
&= \int \tan^8 x \sec^2 x d\left(\tan x\right) \\
&= \int \tan^8 x \left(1 + \tan^2 x\right) d(\tan x) \\
&= \int u^8 \left(1 + u^2\right) du \\
&= \int \left(u^8 + u^{10}\right) du \\
&= \frac{u^9}{9} + \frac{u^{11}}{11} + C
\end{aligned}
$$

Can we rewrite $\sec^2 x$ via $\tan x$?

Set $u = \tan x$

Example

$$\begin{aligned}
\int \tan^8 x \sec^4 x dx &= \int \tan^8 x \sec^2 x \sec^2 x dx \\
&= \int \tan^8 x \sec^2 x d(\tan x) \\
&= \int \tan^8 x \left(1+\tan^2 x\right) d(\tan x) \\
&= \int u^8 \left(1+u^2\right) du \\
&= \int \left(u^8+u^{10}\right) du \\
&= \frac{u^9}{9} + \frac{u^{11}}{11} + C \\
&= \frac{\tan^9 x}{9} + \frac{\tan^{11} x}{11} + C \quad .
\end{aligned}$$

Can we rewrite $\sec^2 x$ via $\tan x$?

Set $u = \tan x$

Example

$$\int \tan^5 x \sec^9 x \mathrm{d}x$$

Example

$$\int \tan^5 x \sec^9 x \mathrm{d}x = \int \tan^4 x \sec^8 x \tan x \sec x \mathrm{d}x$$

Example

$$\begin{aligned}\int \tan^5 x \sec^9 x \mathrm{d}x &= \int \tan^4 x \sec^8 x \tan x \sec x \mathrm{d}x \\ &= \int \tan^4 x \sec^8 x \mathrm{d}(\mathbf{?} \quad)\end{aligned}$$

Example

$$\begin{aligned}\int \tan^5 x \sec^9 x \mathrm{d}x &= \int \tan^4 x \sec^8 x \tan x \sec x \mathrm{d}x\\ &= \int \tan^4 x \sec^8 x \mathrm{d}(\sec x)\end{aligned}$$

Example

$$\begin{aligned}\int \tan^5 x \sec^9 x \mathrm{d}x &= \int \tan^4 x \sec^8 x \tan x \sec x \mathrm{d}x \\ &= \int \tan^4 x \sec^8 x \mathrm{d}(\sec x)\end{aligned}$$

Can we rewrite $\tan^4 x$ via $\sec x$?

Example

$$\begin{aligned}\int \tan^5 x \sec^9 x \mathrm{d}x &= \int \tan^4 x \sec^8 x \tan x \sec x \mathrm{d}x \\ &= \int \tan^4 x \sec^8 x \mathrm{d}(\sec x) \\ &= \int \left(\tan^2 x\right)^2 \sec^8 x \mathrm{d}(\sec x)\end{aligned}$$

Can we rewrite $\tan^4 x$ via $\sec x$?

Example

$$\begin{aligned}\int \tan^5 x \sec^9 x \mathrm{d}x &= \int \tan^4 x \sec^8 x \tan x \sec x \mathrm{d}x \\ &= \int \tan^4 x \sec^8 x \mathrm{d}(\sec x) \\ &= \int \left(\tan^2 x\right)^2 \sec^8 x \mathrm{d}(\sec x) \\ &= \int \left(\sec^2 x - 1\right)^2 \sec^8 x \mathrm{d}(\sec x)\end{aligned}$$

Can we rewrite $\tan^4 x$ via $\sec x$?

Example

$$\begin{aligned}\int \tan^5 x \sec^9 x \mathrm{d}x &= \int \tan^4 x \sec^8 x \tan x \sec x \mathrm{d}x \\ &= \int \tan^4 x \sec^8 x \mathrm{d}(\sec x) \\ &= \int \left(\tan^2 x\right)^2 \sec^8 x \mathrm{d}(\sec x) \\ &= \int \left(\sec^2 x - 1\right)^2 \sec^8 x \mathrm{d}(\sec x) \\ &= \int \left(1 - u^2\right)^2 u^8 \mathrm{d}u\end{aligned}$$

Can we rewrite $\tan^4 x$ via $\sec x$?

Set $u = \sec x$

Example

$$\begin{aligned}
\int \tan^5 x \sec^9 x \mathrm{d}x &= \int \tan^4 x \sec^8 x \tan x \sec x \mathrm{d}x \\
&= \int \tan^4 x \sec^8 x \mathrm{d}(\sec x) && \Bigg| \text{Can we rewrite } \tan^4 x \text{ via } \sec x? \\
&= \int \left(\tan^2 x\right)^2 \sec^8 x \mathrm{d}(\sec x) \\
&= \int \left(\sec^2 x - 1\right)^2 \sec^8 x \mathrm{d}(\sec x) && \Bigg| \text{Set } u = \sec x \\
&= \int \left(1 - u^2\right)^2 u^8 \mathrm{d}u \\
&= \int \left(1 - 2u^2 + u^4\right) u^8 \mathrm{d}u
\end{aligned}$$

Example

$$\begin{aligned}
\int \tan^5 x \sec^9 x \mathrm{d}x &= \int \tan^4 x \sec^8 x \tan x \sec x \mathrm{d}x \\
&= \int \tan^4 x \sec^8 x \mathrm{d}(\sec x) \\
&= \int \left(\tan^2 x\right)^2 \sec^8 x \mathrm{d}(\sec x) \\
&= \int \left(\sec^2 x - 1\right)^2 \sec^8 x \mathrm{d}(\sec x) \\
&= \int \left(1 - u^2\right)^2 u^8 \mathrm{d}u \\
&= \int \left(1 - 2u^2 + u^4\right) u^8 \mathrm{d}u \\
&= \int \left(u^8 - 2u^{10} + u^{12}\right) \mathrm{d}u
\end{aligned}$$

Can we rewrite $\tan^4 x$ via $\sec x$?

Set $u = \sec x$

Example

$$\begin{aligned}
\int \tan^5 x \sec^9 x \mathrm{d}x &= \int \tan^4 x \sec^8 x \tan x \sec x \mathrm{d}x \\
&= \int \tan^4 x \sec^8 x \mathrm{d}(\sec x) \\
&= \int \left(\tan^2 x\right)^2 \sec^8 x \mathrm{d}(\sec x) \\
&= \int \left(\sec^2 x - 1\right)^2 \sec^8 x \mathrm{d}(\sec x) \\
&= \int \left(1 - u^2\right)^2 u^8 \mathrm{d}u \\
&= \int \left(1 - 2u^2 + u^4\right) u^8 \mathrm{d}u \\
&= \int \left(u^8 - 2u^{10} + u^{12}\right) \mathrm{d}u \\
&= ?
\end{aligned}$$

Can we rewrite $\tan^4 x$ via $\sec x$?

Set $u = \sec x$

Example

$$
\begin{aligned}
\int \tan^5 x \sec^9 x \mathrm{d}x &= \int \tan^4 x \sec^8 x \tan x \sec x \mathrm{d}x \\
&= \int \tan^4 x \sec^8 x \mathrm{d}(\sec x) && \text{Can we rewrite } \tan^4 x \text{ via } \sec x? \\
&= \int \left(\tan^2 x\right)^2 \sec^8 x \mathrm{d}(\sec x) \\
&= \int \left(\sec^2 x - 1\right)^2 \sec^8 x \mathrm{d}(\sec x) && \text{Set } u = \sec x \\
&= \int \left(1 - u^2\right)^2 u^8 \mathrm{d}u \\
&= \int \left(1 - 2u^2 + u^4\right) u^8 \mathrm{d}u \\
&= \int \left(u^8 - 2u^{10} + u^{12}\right) \mathrm{d}u \\
&= \frac{u^9}{9} - 2\frac{u^{11}}{11} + \frac{u^{13}}{13} + C
\end{aligned}
$$

Example

$$\begin{aligned}\int \tan^5 x \sec^9 x dx &= \int \tan^4 x \sec^8 x \tan x \sec x dx \\ &= \int \tan^4 x \sec^8 x d(\sec x) \\ &= \int \left(\tan^2 x\right)^2 \sec^8 x d(\sec x) \\ &= \int \left(\sec^2 x - 1\right)^2 \sec^8 x d(\sec x) \\ &= \int \left(1-u^2\right)^2 u^8 du \\ &= \int \left(1-2u^2+u^4\right) u^8 du \\ &= \int \left(u^8 - 2u^{10} + u^{12}\right) du \\ &= \frac{u^9}{9} - 2\frac{u^{11}}{11} + \frac{u^{13}}{13} + C\end{aligned}$$

Can we rewrite $\tan^4 x$ via $\sec x$?

Set $u = \sec x$

Example

$$\begin{aligned}
\int \tan^5 x \sec^9 x \mathrm{d}x &= \int \tan^4 x \sec^8 x \tan x \sec x \mathrm{d}x \\
&= \int \tan^4 x \sec^8 x \mathrm{d}(\sec x) \\
&= \int \left(\tan^2 x\right)^2 \sec^8 x \mathrm{d}(\sec x) \\
&= \int \left(\sec^2 x - 1\right)^2 \sec^8 x \mathrm{d}(\sec x) \\
&= \int \left(1 - u^2\right)^2 u^8 \mathrm{d}u \\
&= \int \left(1 - 2u^2 + u^4\right) u^8 \mathrm{d}u \\
&= \int \left(u^8 - 2u^{10} + u^{12}\right) \mathrm{d}u \\
&= \frac{u^9}{9} - 2\frac{u^{11}}{11} + \frac{u^{13}}{13} + C
\end{aligned}$$

Can we rewrite $\tan^4 x$ via $\sec x$?

Set $u = \sec x$

Example

$$\begin{aligned}\int \tan^5 x \sec^9 x \mathrm{d}x &= \int \tan^4 x \sec^8 x \tan x \sec x \mathrm{d}x \\ &= \int \tan^4 x \sec^8 x \mathrm{d}(\sec x) \\ &= \int \left(\tan^2 x\right)^2 \sec^8 x \mathrm{d}(\sec x) \\ &= \int \left(\sec^2 x - 1\right)^2 \sec^8 x \mathrm{d}(\sec x) \\ &= \int \left(1 - u^2\right)^2 u^8 \mathrm{d}u \\ &= \int \left(1 - 2u^2 + u^4\right) u^8 \mathrm{d}u \\ &= \int \left(u^8 - 2u^{10} + u^{12}\right) \mathrm{d}u \\ &= \frac{u^9}{9} - 2\frac{u^{11}}{11} + \frac{u^{13}}{13} + C\end{aligned}$$

Can we rewrite $\tan^4 x$ via $\sec x$?

Set $u = \sec x$

Example

$$\begin{aligned}
\int \tan^5 x \sec^9 x \mathrm{d}x &= \int \tan^4 x \sec^8 x \tan x \sec x \mathrm{d}x \\
&= \int \tan^4 x \sec^8 x \mathrm{d}(\sec x) \\
&= \int \left(\tan^2 x\right)^2 \sec^8 x \mathrm{d}(\sec x) \\
&= \int \left(\sec^2 x - 1\right)^2 \sec^8 x \mathrm{d}(\sec x) \\
&= \int \left(1 - u^2\right)^2 u^8 \mathrm{d}u \\
&= \int \left(1 - 2u^2 + u^4\right) u^8 \mathrm{d}u \\
&= \int \left(u^8 - 2u^{10} + u^{12}\right) \mathrm{d}u \\
&= \frac{u^9}{9} - 2\frac{u^{11}}{11} + \frac{u^{13}}{13} + C \\
&= \frac{\sec^9 x}{9} - 2\frac{\sec^{11} x}{11} + \frac{\sec^{13} x}{13} + C \quad .
\end{aligned}$$

Can we rewrite $\tan^4 x$ via $\sec x$?

Set $u = \sec x$

Partial strategy for fast evaluation of $\int \tan^m x \sec^n x \mathrm{d}x$

$\displaystyle\int \tan^m x \sec^n x \mathrm{d}x$	n – even, $n \geq 2$
$\displaystyle\int \tan^m x \sec^n x \mathrm{d}x$	m – odd, $n \geq 1$

Partial strategy for fast evaluation of $\int \tan^m x \sec^n x \mathrm{d}x$

$$\int \tan^m x \sec^n x \mathrm{d}x = \int \tan^m x \sec^{n-2} x \mathrm{d}(\tan x)$$

n – even, $n \geq 2$

$\sec^2 x \mathrm{d}x = \mathrm{d}(\tan x)$

$$\int \tan^m x \sec^n x \mathrm{d}x$$

m – odd, $n \geq 1$

Partial strategy for fast evaluation of $\int \tan^m x \sec^n x \mathrm{d}x$

$$\int \tan^m x \sec^n x \mathrm{d}x = \int \tan^m x \sec^{n-2} x \mathrm{d}(\tan x)$$

$$= \int \tan^m x \left(1+\tan^2 x\right)^{\frac{n-2}{2}} \mathrm{d}(\tan x)$$

n – even, $n \geq 2$

$\sec^2 x \mathrm{d}x$

$= \mathrm{d}(\tan x)$

Express $\sec x$ via $\tan x$

$$\int \tan^m x \sec^n x \mathrm{d}x$$

m – odd, $n \geq 1$

Partial strategy for fast evaluation of $\int \tan^m x \sec^n x dx$

$$\int \tan^m x \sec^n x dx = \int \tan^m x \sec^{n-2} x d(\tan x)$$

$$= \int \tan^m x \left(1 + \tan^2 x\right)^{\frac{n-2}{2}} d(\tan x)$$

n – even, $n \geq 2$
$\sec^2 x dx$
$= d(\tan x)$
Express $\sec x$
via $\tan x$

$$\int \tan^m x \sec^n x dx$$

m – odd, $n \geq 1$

Partial strategy for fast evaluation of $\int \tan^m x \sec^n x \mathrm{d}x$

$$\begin{aligned}
\int \tan^m x \sec^n x \mathrm{d}x &= \int \tan^m x \sec^{n-2} x \mathrm{d}(\tan x) \\
&= \int \tan^m x \left(1 + \tan^2 x\right)^{\frac{n-2}{2}} \mathrm{d}(\tan x) \\
&= \int u^m \left(1 + u^2\right)^{\frac{n-2}{2}} \mathrm{d}u
\end{aligned}$$

n – even, $n \geq 2$

$\sec^2 x \mathrm{d}x = \mathrm{d}(\tan x)$

Express $\sec x$ via $\tan x$

Set $u = \tan x$

$$\int \tan^m x \sec^n x \mathrm{d}x$$

m – odd, $n \geq 1$

Partial strategy for fast evaluation of $\int \tan^m x \sec^n x dx$

$\int \tan^m x \sec^n x dx = \int \tan^m x \sec^{n-2} x d(\tan x)$	n – even, $n \geq 2$ $\sec^2 x dx$ $= d(\tan x)$
$= \int \tan^m x \left(1 + \tan^2 x\right)^{\frac{n-2}{2}} d(\tan x)$	Express $\sec x$ via $\tan x$
$= \int u^m \left(1 + u^2\right)^{\frac{n-2}{2}} du$	Set $u = \tan x$
$\int \tan^m x \sec^n x dx = \int \tan^{m-1} x \sec^{n-1} x d(\sec x)$	m – odd, $n \geq 1$ $\tan x \sec x dx$ $= d(\sec x)$

Partial strategy for fast evaluation of $\int \tan^m x \sec^n x dx$

$$\begin{aligned}\int \tan^m x \sec^n x dx &= \int \tan^m x \sec^{n-2} x d(\tan x)\\ &= \int \tan^m x \left(1+\tan^2 x\right)^{\frac{n-2}{2}} d(\tan x)\\ &= \int u^m \left(1+u^2\right)^{\frac{n-2}{2}} du\end{aligned}$$	n – even, $n \geq 2$; $\sec^2 x dx = d(\tan x)$; Express $\sec x$ via $\tan x$; Set $u = \tan x$
$$\begin{aligned}\int \tan^m x \sec^n x dx &= \int \tan^{m-1} x \sec^{n-1} x d(\sec x)\\ &= \int \left(\sec^2 x - 1\right)^{\frac{m-1}{2}} \sec^{n-1} x d(\sec x)\end{aligned}$$	m – odd, $n \geq 1$; $\tan x \sec x dx = d(\sec x)$; Express $\tan x$ via $\sec x$

Partial strategy for fast evaluation of $\int \tan^m x \sec^n x \mathrm{d}x$

$\int \tan^m x \sec^n x \mathrm{d}x = \int \tan^m x \sec^{n-2} x \mathrm{d}(\tan x)$	n – even, $n \geq 2$; $\sec^2 x\mathrm{d}x = \mathrm{d}(\tan x)$
$= \int \tan^m x \left(1+\tan^2 x\right)^{\frac{n-2}{2}} \mathrm{d}(\tan x)$	Express $\sec x$ via $\tan x$
$= \int u^m \left(1+u^2\right)^{\frac{n-2}{2}} \mathrm{d}u$	Set $u = \tan x$
$\int \tan^m x \sec^n x \mathrm{d}x = \int \tan^{m-1} x \sec^{n-1} x \mathrm{d}(\sec x)$	m – odd, $n \geq 1$; $\tan x \sec x\mathrm{d}x = \mathrm{d}(\sec x)$
$= \int \left(\sec^2 x - 1\right)^{\frac{m-1}{2}} \sec^{n-1} x \mathrm{d}(\sec x)$	Express $\tan x$ via $\sec x$

Partial strategy for fast evaluation of $\int \tan^m x \sec^n x dx$

$\displaystyle\int \tan^m x \sec^n x dx = \int \tan^m x \sec^{n-2} x d(\tan x)$	n – even, $n \geq 2$ $\sec^2 x dx$ $= d(\tan x)$
$\displaystyle= \int \tan^m x \left(1+\tan^2 x\right)^{\frac{n-2}{2}} d(\tan x)$	Express $\sec x$ via $\tan x$
$\displaystyle= \int u^m \left(1+u^2\right)^{\frac{n-2}{2}} du$	Set $u = \tan x$
$\displaystyle\int \tan^m x \sec^n x dx = \int \tan^{m-1} x \sec^{n-1} x d(\sec x)$	m – odd, $n \geq 1$ $\tan x \sec x dx$ $= d(\sec x)$
$\displaystyle= \int \left(\sec^2 x - 1\right)^{\frac{m-1}{2}} \sec^{n-1} x d(\sec x)$	Express $\tan x$ via $\sec x$

Partial strategy for fast evaluation of $\int \tan^m x \sec^n x dx$

$$\begin{aligned}\int \tan^m x \sec^n x dx &= \int \tan^m x \sec^{n-2} x d(\tan x)\\ &= \int \tan^m x \left(1+\tan^2 x\right)^{\frac{n-2}{2}} d(\tan x)\\ &= \int u^m \left(1+u^2\right)^{\frac{n-2}{2}} du\end{aligned}$$	n – even, $n \geq 2$ $\sec^2 x dx$ $= d(\tan x)$ Express $\sec x$ via $\tan x$ Set $u = \tan x$
$$\begin{aligned}\int \tan^m x \sec^n x dx &= \int \tan^{m-1} x \sec^{n-1} x d(\sec x)\\ &= \int \left(\sec^2 x - 1\right)^{\frac{m-1}{2}} \sec^{n-1} x d(\sec x)\\ &= \int \left(u^2-1\right)^{\frac{m-1}{2}} u^n du\end{aligned}$$	m – odd, $n \geq 1$ $\tan x \sec x dx$ $= d(\sec x)$ Express $\tan x$ via $\sec x$ Set $u = \sec x$

Partial strategy for fast evaluation of $\int \tan^m x \sec^n x dx$

$\int \tan^m x \sec^n x dx = \int \tan^m x \sec^{n-2} x d(\tan x)$	n – even, $n \geq 2$ $\sec^2 x dx$ $= d(\tan x)$
$= \int \tan^m x \left(1+\tan^2 x\right)^{\frac{n-2}{2}} d(\tan x)$	Express $\sec x$ via $\tan x$
$= \int u^m \left(1+u^2\right)^{\frac{n-2}{2}} du$	Set $u = \tan x$
$\int \tan^m x \sec^n x dx = \int \tan^{m-1} x \sec^{n-1} x d(\sec x)$	m – odd, $n \geq 1$ $\tan x \sec x dx$ $= d(\sec x)$
$= \int \left(\sec^2 x - 1\right)^{\frac{m-1}{2}} \sec^{n-1} x d(\sec x)$	Express $\tan x$ via $\sec x$
$= \int \left(u^2 - 1\right)^{\frac{m-1}{2}} u^n du$	Set $u = \sec x$

Outside of the above cases we either use more tricks or resort to the general method $x = 2 \arctan t$.

Partial strategy for fast evaluation of $\int \tan^m x \sec^n x dx$

$$\begin{aligned}\int \tan^m x \sec^n x dx &= \int \tan^m x \sec^{n-2} x d(\tan x) \\ &= \int \tan^m x \left(1+\tan^2 x\right)^{\frac{n-2}{2}} d(\tan x) \\ &= \int u^m \left(1+u^2\right)^{\frac{n-2}{2}} du\end{aligned}$$	n – even, $n \geq 2$ $\sec^2 x dx$ $= d(\tan x)$ Express $\sec x$ via $\tan x$ Set $u = \tan x$
$$\begin{aligned}\int \tan^m x \sec^n x dx &= \int \tan^{m-1} x \sec^{n-1} x d(\sec x) \\ &= \int \left(\sec^2 x - 1\right)^{\frac{m-1}{2}} \sec^{n-1} x d(\sec x) \\ &= \int \left(u^2 - 1\right)^{\frac{m-1}{2}} u^n du\end{aligned}$$	m – odd, $n \geq 1$ $\tan x \sec x dx$ $= d(\sec x)$ Express $\tan x$ via $\sec x$ Set $u = \sec x$

Outside of the above cases we either use more tricks or resort to the general method $x = 2 \arctan t$.

Example

$$\int \tan x \mathrm{d}x$$

Example

Example

$$\int \tan x \mathrm{d}x \quad = \quad \int \frac{\sin x}{\cos x} \mathrm{d}x$$

Example

Example

$$\int \tan x \mathrm{d}x = \int \frac{\sin x}{\cos x}\mathrm{d}x = \int \frac{1}{\cos x}\mathrm{d}(\textbf{?} \qquad)$$

Example

Example

$$\int \tan x \mathrm{d}x \quad = \quad \int \frac{\sin x}{\cos x}\mathrm{d}x = \int \frac{1}{\cos x}\mathrm{d}(-\cos x)$$

Example

Example

$$\begin{aligned}\int \tan x \mathrm{d}x &= \int \frac{\sin x}{\cos x}\mathrm{d}x = \int \frac{1}{\cos x}\mathrm{d}(-\cos x) \\ &= -\int \frac{\mathrm{d}u}{u}\end{aligned}$$

Set $u = \cos x$

Example

Example

$$\begin{aligned}\int \tan x \mathrm{d}x &= \int \frac{\sin x}{\cos x}\mathrm{d}x = \int \frac{1}{\cos x}\mathrm{d}(-\cos x) \\ &= -\int \frac{\mathrm{d}u}{u}\end{aligned}$$

Set $u = \cos x$

Example

Example

$$\begin{aligned}\int \tan x \mathrm{d}x &= \int \frac{\sin x}{\cos x}\mathrm{d}x = \int \frac{1}{\cos x}\mathrm{d}(-\cos x) \quad \Big| \text{ Set } u = \cos x\\ &= -\int \frac{\mathrm{d}u}{u} = -\ln|u| + C\end{aligned}$$

Example

Example

$$\begin{aligned}\int \tan x \mathrm{d}x &= \int \frac{\sin x}{\cos x}\mathrm{d}x = \int \frac{1}{\cos x}\mathrm{d}(-\cos x) \\ &= -\int \frac{\mathrm{d}u}{u} = -\ln|u| + C \\ &= -\ln|\cos x| + C\end{aligned}$$

Set $u = \cos x$

Example

Example

$$\begin{aligned}
\int \tan x \mathrm{d}x &= \int \frac{\sin x}{\cos x}\mathrm{d}x = \int \frac{1}{\cos x}\mathrm{d}(-\cos x) && \Big| \text{ Set } u = \cos x \\
&= -\int \frac{\mathrm{d}u}{u} = -\ln|u| + C \\
&= -\ln|\cos x| + C = \ln|\sec x| + C
\end{aligned}$$

Example

Example

$$\begin{aligned}\int \tan x \mathrm{d}x &= \int \frac{\sin x}{\cos x}\mathrm{d}x = \int \frac{1}{\cos x}\mathrm{d}(-\cos x) && \Big| \text{ Set } u = \cos x\\ &= -\int \frac{\mathrm{d}u}{u} = -\ln|u| + C\\ &= -\ln|\cos x| + C = \ln|\sec x| + C\end{aligned}$$

The following can be/was computed via $x = 2\arctan t$.

Example

$$\int \sec x \mathrm{d}x$$

Example

$$\begin{aligned}\int \tan x \mathrm{d}x &= \int \frac{\sin x}{\cos x}\mathrm{d}x = \int \frac{1}{\cos x}\mathrm{d}(-\cos x) \quad \Big| \text{ Set } u = \cos x\\ &= -\int \frac{\mathrm{d}u}{u} = -\ln|u| + C\\ &= -\ln|\cos x| + C = \ln|\sec x| + C\end{aligned}$$

The following can be/was computed via $x = 2\arctan t$. Alternatively:

Example

$$\int \sec x \mathrm{d}x = \int \sec x \frac{(\sec x + \tan x)}{(\sec x + \tan x)}\mathrm{d}x$$

Example

$$\begin{aligned}\int \tan x \mathrm{d}x &= \int \frac{\sin x}{\cos x}\mathrm{d}x = \int \frac{1}{\cos x}\mathrm{d}(-\cos x) \quad \Big| \text{ Set } u = \cos x\\ &= -\int \frac{\mathrm{d}u}{u} = -\ln|u| + C\\ &= -\ln|\cos x| + C = \ln|\sec x| + C\end{aligned}$$

The following can be/was computed via $x = 2\arctan t$. Alternatively:

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$$\begin{aligned}\int \sec x \mathrm{d}x &= \int \sec x \frac{(\sec x + \tan x)}{(\sec x + \tan x)}\mathrm{d}x\\ &= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\mathrm{d}x\end{aligned}$$

Example

$$\begin{aligned}\int \tan x \mathrm{d}x &= \int \frac{\sin x}{\cos x}\mathrm{d}x = \int \frac{1}{\cos x}\mathrm{d}(-\cos x) &&\Big|\ \text{Set } u = \cos x\\ &= -\int \frac{\mathrm{d}u}{u} = -\ln|u| + C\\ &= -\ln|\cos x| + C = \ln|\sec x| + C\end{aligned}$$

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Example

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The following can be/was computed via $x = 2\arctan t$. Alternatively:

Example

$$\begin{aligned}\int \sec x \mathrm{d}x &= \int \sec x \frac{(\sec x + \tan x)}{(\sec x + \tan x)}\mathrm{d}x\\ &= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\mathrm{d}x\\ &= \int \frac{\mathrm{d}(\tan x + \sec x)}{\sec x + \tan x}\end{aligned}$$

Example

$$\begin{aligned}\int \tan x \mathrm{d}x &= \int \frac{\sin x}{\cos x}\mathrm{d}x = \int \frac{1}{\cos x}\mathrm{d}(-\cos x) \qquad \Big| \text{ Set } u = \cos x\\ &= -\int \frac{\mathrm{d}u}{u} = -\ln|u| + C\\ &= -\ln|\cos x| + C = \ln|\sec x| + C\end{aligned}$$

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Example

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Example

$$\begin{aligned}\int \tan x \mathrm{d}x &= \int \frac{\sin x}{\cos x}\mathrm{d}x = \int \frac{1}{\cos x}\mathrm{d}(-\cos x) && \Big| \text{ Set } u = \cos x\\ &= -\int \frac{\mathrm{d}u}{u} = -\ln|u| + C\\ &= -\ln|\cos x| + C = \ln|\sec x| + C\end{aligned}$$

The following can be/was computed via $x = 2\arctan t$. Alternatively:

Example

$$\begin{aligned}\int \sec x \mathrm{d}x &= \int \sec x \frac{(\sec x + \tan x)}{(\sec x + \tan x)}\mathrm{d}x\\ &= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\mathrm{d}x\\ &= \int \frac{\mathrm{d}(\tan x + \sec x)}{\sec x + \tan x} && \Big| \text{ Set } u = \sec x + \tan x\\ &= \int \frac{\mathrm{d}u}{u}\end{aligned}$$

Example

$$\begin{aligned}\int \tan x \mathrm{d}x &= \int \frac{\sin x}{\cos x}\mathrm{d}x = \int \frac{1}{\cos x}\mathrm{d}(-\cos x) && \Big| \text{ Set } u = \cos x\\ &= -\int \frac{\mathrm{d}u}{u} = -\ln|u| + C\\ &= -\ln|\cos x| + C = \ln|\sec x| + C\end{aligned}$$

The following can be/was computed via $x = 2\arctan t$. Alternatively:

Example

$$\begin{aligned}\int \sec x \mathrm{d}x &= \int \sec x \frac{(\sec x + \tan x)}{(\sec x + \tan x)}\mathrm{d}x\\ &= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\mathrm{d}x\\ &= \int \frac{\mathrm{d}(\tan x + \sec x)}{\sec x + \tan x} && \Big| \text{ Set } u = \sec x + \tan x\\ &= \int \frac{\mathrm{d}u}{u} = \ln|u| + C\end{aligned}$$

Example

$$\begin{aligned}\int \tan x \mathrm{d}x &= \int \frac{\sin x}{\cos x}\mathrm{d}x = \int \frac{1}{\cos x}\mathrm{d}(-\cos x) && \Big|\ \text{Set } u = \cos x\\ &= -\int \frac{\mathrm{d}u}{u} = -\ln|u| + C \\ &= -\ln|\cos x| + C = \ln|\sec x| + C\end{aligned}$$

The following can be/was computed via $x = 2\arctan t$. Alternatively:

Example

$$\begin{aligned}\int \sec x \mathrm{d}x &= \int \sec x \frac{(\sec x + \tan x)}{(\sec x + \tan x)}\mathrm{d}x \\ &= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\mathrm{d}x \\ &= \int \frac{\mathrm{d}(\tan x + \sec x)}{\sec x + \tan x} && \Big|\ \text{Set } u = \sec x + \tan x \\ &= \int \frac{\mathrm{d}u}{u} = \ln|u| + C \\ &= \ln|\sec x + \tan x| + C.\end{aligned}$$

Example

$$\int \tan^3 x \mathrm{d}x$$

Example

$$\int \tan^3 x \mathrm{d}x = \int \tan x \tan^2 x \mathrm{d}x$$

Example

$$\begin{aligned}\int \tan^3 x \mathrm{d}x &= \int \tan x \tan^2 x \mathrm{d}x\\ &= \int \tan x \left(\sec^2 x - 1\right) \mathrm{d}x\end{aligned}$$

Example

$$
\begin{aligned}
\int \tan^3 x \mathrm{d}x &= \int \tan x \tan^2 x \mathrm{d}x \\
&= \int \tan x \left(\sec^2 x - 1 \right) \mathrm{d}x \\
&= \int \tan x \sec^2 x \mathrm{d}x - \int \tan x \mathrm{d}x
\end{aligned}
$$

Example

$$
\begin{aligned}
\int \tan^3 x \mathrm{d}x &= \int \tan x \tan^2 x \mathrm{d}x \\
&= \int \tan x \left(\sec^2 x - 1\right) \mathrm{d}x \\
&= \int \tan x \sec^2 x \mathrm{d}x - \int \tan x \mathrm{d}x
\end{aligned}
$$

Example

$$
\begin{aligned}
\int \tan^3 x \mathrm{d}x &= \int \tan x \tan^2 x \mathrm{d}x \\
&= \int \tan x \left(\sec^2 x - 1 \right) \mathrm{d}x \\
&= \int \tan x \sec^2 x \mathrm{d}x - \int \tan x \mathrm{d}x \\
&= \int \tan x \mathrm{d}(\mathbf{?} \qquad) - \mathbf{?}
\end{aligned}
$$

Example

$$\begin{aligned}
\int \tan^3 x dx &= \int \tan x \tan^2 x dx \\
&= \int \tan x \left(\sec^2 x - 1\right) dx \\
&= \int \tan x \sec^2 x dx - \int \tan x dx \\
&= \int \tan x d(\tan x) - \textbf{?}
\end{aligned}$$

Example

$$
\begin{aligned}
\int \tan^3 x dx &= \int \tan x \tan^2 x dx \\
&= \int \tan x \left(\sec^2 x - 1\right) dx \\
&= \int \tan x \sec^2 x dx - \int \tan x dx \\
&= \int \tan x d(\tan x) - \mathbf{?}
\end{aligned}
$$

Example

$$
\begin{aligned}
\int \tan^3 x \mathrm{d}x &= \int \tan x \tan^2 x \mathrm{d}x \\
&= \int \tan x \left(\sec^2 x - 1\right) \mathrm{d}x \\
&= \int \tan x \sec^2 x \mathrm{d}x - \int \tan x \mathrm{d}x \\
&= \int \tan x \mathrm{d}(\tan x) - \ln|\sec x|
\end{aligned}
$$

Example

$$
\begin{aligned}
\int \tan^3 x \mathrm{d}x &= \int \tan x \tan^2 x \mathrm{d}x \\
&= \int \tan x \left(\sec^2 x - 1\right) \mathrm{d}x \\
&= \int \tan x \sec^2 x \mathrm{d}x - \int \tan x \mathrm{d}x \\
&= \int \tan x \mathrm{d}(\tan x) - \ln|\sec x| & \Big| \text{ Set } u = \tan x \\
&= \int u \mathrm{d}u + \ln\left|\frac{1}{\sec x}\right|
\end{aligned}
$$

Example

$$\begin{aligned}
\int \tan^3 x \mathrm{d}x &= \int \tan x \tan^2 x \mathrm{d}x \\
&= \int \tan x \left(\sec^2 x - 1\right) \mathrm{d}x \\
&= \int \tan x \sec^2 x \mathrm{d}x - \int \tan x \mathrm{d}x \\
&= \int \tan x \mathrm{d}(\tan x) - \ln|\sec x| && \Big| \text{ Set } u = \tan x \\
&= \int u \mathrm{d}u + \ln\left|\frac{1}{\sec x}\right|
\end{aligned}$$

Example

$$\begin{aligned}
\int \tan^3 x \mathrm{d}x &= \int \tan x \tan^2 x \mathrm{d}x \\
&= \int \tan x \left(\sec^2 x - 1\right) \mathrm{d}x \\
&= \int \tan x \sec^2 x \mathrm{d}x - \int \tan x \mathrm{d}x \\
&= \int \tan x \mathrm{d}(\tan x) - \ln|\sec x| && \Bigg| \text{ Set } u = \tan x \\
&= \int u \mathrm{d}u + \ln\left|\frac{1}{\sec x}\right| \\
&= \frac{u^2}{2} + \ln|\cos x| + C
\end{aligned}$$

Example

$$\begin{aligned}
\int \tan^3 x dx &= \int \tan x \tan^2 x dx \\
&= \int \tan x \left(\sec^2 x - 1\right) dx \\
&= \int \tan x \sec^2 x dx - \int \tan x dx \\
&= \int \tan x d(\tan x) - \ln|\sec x| \qquad \Big| \text{ Set } u = \tan x \\
&= \int u du + \ln\left|\frac{1}{\sec x}\right| \\
&= \frac{u^2}{2} + \ln|\cos x| + C
\end{aligned}$$

Example

$$
\begin{aligned}
\int \tan^3 x dx &= \int \tan x \tan^2 x dx \\
&= \int \tan x \left(\sec^2 x - 1\right) dx \\
&= \int \tan x \sec^2 x dx - \int \tan x dx \\
&= \int \tan x d(\tan x) - \ln|\sec x| \qquad \Big| \text{ Set } u = \tan x \\
&= \int u du + \ln\left|\frac{1}{\sec x}\right| \\
&= \frac{u^2}{2} + \ln|\cos x| + C \\
&= \frac{\tan^2 x}{2} + \ln|\cos x| + C
\end{aligned}
$$

Example

$$\int \sec^3 x \mathrm{d}x$$

Example

$$\int \sec^3 x \mathrm{d}x = \int \sec x \sec^2 x \mathrm{d}x$$

Example

$$
\begin{aligned}
\int \sec^3 x \mathrm{d}x &= \int \sec x \sec^2 x \mathrm{d}x \\
&= \int \sec x \mathrm{d}(\textbf{?} \quad)
\end{aligned}
$$

Example

$$\begin{aligned} \int \sec^3 x \mathrm{d}x &= \int \sec x \sec^2 x \mathrm{d}x \\ &= \int \sec x \mathrm{d}(\tan x) \end{aligned}$$

Example

$$\begin{aligned}\int \sec^3 x \mathrm{d}x &= \int \sec x \sec^2 x \mathrm{d}x \\ &= \int \sec x \mathrm{d}(\tan x) \\ &= \sec x \tan x - \int \tan x \mathrm{d}(\sec x)\end{aligned}$$

Integrate by parts

Example

$$
\begin{aligned}
\int \sec^3 x dx &= \int \sec x \sec^2 x dx \\
&= \int \sec x d(\tan x) \\
&= \sec x \tan x - \int \tan x d(\sec x)
\end{aligned}
$$

Integrate by parts

Example

$$\begin{aligned}\int \sec^3 x \mathrm{d}x &= \int \sec x \sec^2 x \mathrm{d}x \\ &= \int \sec x \mathrm{d}(\tan x) \\ &= \sec x \tan x - \int \tan x \mathrm{d}(\sec x) \\ &= \sec x \tan x - \int \tan \quad x \textbf{?} \qquad \mathrm{d}x\end{aligned}$$

Integrate by parts

Example

$$\begin{aligned}\int \sec^3 x\,\mathrm{d}x &= \int \sec x \sec^2 x\,\mathrm{d}x \\ &= \int \sec x\,\mathrm{d}(\tan x) \\ &= \sec x \tan x - \int \tan x\,\mathrm{d}(\sec x) \\ &= \sec x \tan x - \int \tan\ x \sec x \tan x\,\mathrm{d}x\end{aligned}$$

| Integrate by parts

Example

$$\begin{aligned}\int \sec^3 x \mathrm{d}x &= \int \sec x \sec^2 x \mathrm{d}x \\ &= \int \sec x \mathrm{d}(\tan x) \\ &= \sec x \tan x - \int \tan x \mathrm{d}(\sec x) \\ &= \sec x \tan x - \int \tan\ \ x \sec x \tan x \mathrm{d}x\end{aligned}$$

Integrate by parts

Example

$$\begin{aligned}\int \sec^3 x\mathrm{d}x &= \int \sec x \sec^2 x\mathrm{d}x \\ &= \int \sec x\mathrm{d}(\tan x) \\ &= \sec x \tan x - \int \tan x\mathrm{d}(\sec x) \\ &= \sec x \tan x - \int \tan^2 x \sec x\mathrm{d}x\end{aligned}$$

Integrate by parts

Example

$$
\begin{aligned}
\int \sec^3 x \mathrm{d}x &= \int \sec x \sec^2 x \mathrm{d}x \\
&= \int \sec x \mathrm{d}(\tan x) \\
&= \sec x \tan x - \int \tan x \mathrm{d}(\sec x) \\
&= \sec x \tan x - \int \tan^2 x \sec x \mathrm{d}x \\
&= \sec x \tan x - \int (\sec^2 x - 1) \sec x \mathrm{d}x
\end{aligned}
$$

Integrate by parts

Example

$$
\begin{aligned}
\int \sec^3 x dx &= \int \sec x \sec^2 x dx \\
&= \int \sec x d(\tan x) \\
&= \sec x \tan x - \int \tan x d(\sec x) \\
&= \sec x \tan x - \int \tan^2 x \sec x dx \\
&= \sec x \tan x - \int (\sec^2 x - 1) \sec x dx \\
&= \sec x \tan x - \int \sec^3 x dx + \int \sec x dx
\end{aligned}
$$

Integrate by parts

Example

$$
\begin{aligned}
\int \sec^3 x \mathrm{d}x &= \int \sec x \sec^2 x \mathrm{d}x \\
&= \int \sec x \mathrm{d}(\tan x) \\
&= \sec x \tan x - \int \tan x \mathrm{d}(\sec x) \\
&= \sec x \tan x - \int \tan^2 x \sec x \mathrm{d}x \\
&= \sec x \tan x - \int (\sec^2 x - 1) \sec x \mathrm{d}x \\
&= \sec x \tan x - \int \sec^3 x \mathrm{d}x + \int \sec x \mathrm{d}x
\end{aligned}
$$

Integrate by parts

Example

$$
\begin{aligned}
\int \sec^3 x \mathrm{d}x &= \int \sec x \sec^2 x \mathrm{d}x \\
&= \int \sec x \mathrm{d}(\tan x) \\
&= \sec x \tan x - \int \tan x \mathrm{d}(\sec x) \\
&= \sec x \tan x - \int \tan^2 x \sec x \mathrm{d}x \\
&= \sec x \tan x - \int (\sec^2 x - 1) \sec x \mathrm{d}x \\
&= \sec x \tan x - \int \sec^3 x \mathrm{d}x + \int \sec x \mathrm{d}x
\end{aligned}
$$

Integrate by parts

Example

$$
\begin{aligned}
\int \sec^3 x dx &= \int \sec x \sec^2 x dx \\
&= \int \sec x d(\tan x) \\
&= \sec x \tan x - \int \tan x d(\sec x) \\
&= \sec x \tan x - \int \tan^2 x \sec x dx \\
&= \sec x \tan x - \int (\sec^2 x - 1) \sec x dx \\
&= \sec x \tan x - \int \sec^3 x dx + \int \sec x dx \\
2\int \sec^3 x dx &= \sec x \tan x + \textbf{?} \qquad + C
\end{aligned}
$$

Integrate by parts

Example

$$\begin{aligned}
\int \sec^3 x dx &= \int \sec x \sec^2 x dx \\
&= \int \sec x d(\tan x) \\
&= \sec x \tan x - \int \tan x d(\sec x) \\
&= \sec x \tan x - \int \tan^2 x \sec x dx \\
&= \sec x \tan x - \int (\sec^2 x - 1) \sec x dx \\
&= \sec x \tan x - \int \sec^3 x dx + \int \sec x dx \\
2 \int \sec^3 x dx &= \sec x \tan x + \textbf{?} \qquad + C
\end{aligned}$$

Integrate by parts

Example

$$
\begin{aligned}
\int \sec^3 x \mathrm{d}x &= \int \sec x \sec^2 x \mathrm{d}x \\
&= \int \sec x \mathrm{d}(\tan x) && \left| \begin{array}{l}\text{Integrate}\\ \text{by parts}\end{array}\right. \\
&= \sec x \tan x - \int \tan x \mathrm{d}(\sec x) \\
&= \sec x \tan x - \int \tan^2 x \sec x \mathrm{d}x \\
&= \sec x \tan x - \int (\sec^2 x - 1) \sec x \mathrm{d}x \\
&= \sec x \tan x - \int \sec^3 x \mathrm{d}x + {\color{red}\int \sec x \mathrm{d}x} \\
2\int \sec^3 x \mathrm{d}x &= \sec x \tan x + {\color{red}\ln\left|\sec x + \tan x\right|} + C
\end{aligned}
$$

Example

$$
\begin{aligned}
\int \sec^3 x dx &= \int \sec x \sec^2 x dx \\
&= \int \sec x d(\tan x) \\
&= \sec x \tan x - \int \tan x d(\sec x) \\
&= \sec x \tan x - \int \tan^2 x \sec x dx \\
&= \sec x \tan x - \int (\sec^2 x - 1) \sec x dx \\
&= \sec x \tan x - \int \sec^3 x dx + \int \sec x dx \\
2 \int \sec^3 x dx &= \sec x \tan x + \ln |\sec x + \tan x| + C \\
\int \sec^3 x dx &= \frac{1}{2} (\sec x \tan x + \ln |\sec x + \tan x|) + K.
\end{aligned}
$$

Integrate by parts

To evaluate integrals of the form

1. $\int \sin(mx)\cos(nx)\mathrm{d}x$
2. $\int \sin(mx)\sin(nx)\mathrm{d}x$
3. $\int \cos(mx)\cos(nx)\mathrm{d}x$

use the corresponding identity:

1. $\sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$
2. $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$
3. $\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$

Example

$$\int \sin(4x)\cos(5x)\mathrm{d}x$$

Example

$$\int \sin(4x)\cos(5x)\mathrm{d}x = \int \frac{1}{2}[\sin(4x-5x)+\sin(4x+5x)]\mathrm{d}x$$

Example

$$
\begin{aligned}
\int \sin(4x)\cos(5x)\mathrm{d}x &= \int \frac{1}{2}[\sin(4x-5x)+\sin(4x+5x)]\mathrm{d}x \\
&= \frac{1}{2}\int (\sin(-x)+\sin(9x))\mathrm{d}x
\end{aligned}
$$

Example

$$
\begin{aligned}
\int \sin(4x)\cos(5x)\mathrm{d}x &= \int \frac{1}{2}[\sin(4x-5x)+\sin(4x+5x)]\mathrm{d}x \\
&= \frac{1}{2}\int(\sin(-x)+\sin(9x))\mathrm{d}x \\
&= \frac{1}{2}\int(-\sin x+\sin(9x))\mathrm{d}x
\end{aligned}
$$

Example

$$\begin{aligned}
\int \sin(4x)\cos(5x)\mathrm{d}x &= \int \frac{1}{2}[\sin(4x-5x)+\sin(4x+5x)]\mathrm{d}x \\
&= \frac{1}{2}\int(\sin(-x)+\sin(9x))\mathrm{d}x \\
&= \frac{1}{2}\int(-\sin x+\sin(9x))\mathrm{d}x \\
&= \frac{1}{2}(\cos x-\frac{1}{9}\cos(9x))+C
\end{aligned}$$