

Calculus II

Integrals of the form $\int \arctan(mx) dx$

Todor Milev

2019

Integration by parts: $\int u dv = uv - \int v du.$

Example

$$\int_0^1 \arctan x dx =$$

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$$\int_0^1 \arctan x dx = ?$$

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$$\int_0^1 \arctan x dx = [(\arctan x)x]_{x=0}^{x=1} - \int_0^1 x d(\arctan x)$$

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$$\int_0^1 \arctan x dx = \left[(\arctan x) x \right]_{x=0}^{x=1} - \int_0^1 x d(\arctan x)$$

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Example

$$\begin{aligned} \int_0^1 \arctan x dx &= [(\arctan x)x]_{x=0}^{x=1} - \int_0^1 x d(\arctan x) \\ &= 1 \cdot \arctan 1 - 0 \cdot \arctan 0 - \int_{x=0}^{x=1} x? \end{aligned}$$

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 \int_0^1 \arctan x dx &= [(\arctan x)x]_{x=0}^{x=1} - \int_0^1 x d(\arctan x) \\
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 &= ? - \int_{x=0}^{x=1} \frac{1}{1+x^2} d(?)
 \end{aligned}$$

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 \int_0^1 \arctan x dx &= [(\arctan x)x]_{x=0}^{x=1} - \int_0^1 x d(\arctan x) \\
 &= 1 \cdot \arctan 1 - 0 \cdot \arctan 0 - \int_{x=0}^{x=1} x \frac{1}{1+x^2} dx \\
 &= \frac{\pi}{4} - \int_{x=0}^{x=1} \frac{1}{1+x^2} d(?)
 \end{aligned}$$

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$$\begin{aligned}
 \int_0^1 \arctan x dx &= [(\arctan x)x]_{x=0}^{x=1} - \int_0^1 x d(\arctan x) \\
 &= 1 \cdot \arctan 1 - 0 \cdot \arctan 0 - \int_{x=0}^{x=1} \textcolor{red}{x} \frac{1}{1+x^2} \textcolor{red}{dx} \\
 &= \frac{\pi}{4} - \int_{x=0}^{x=1} \frac{1}{1+x^2} d(\textcolor{red}{?})
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 &= \frac{\pi}{4} - \frac{1}{2} \int_{x=0}^{x=1} \frac{1}{1+x^2} d(\textcolor{red}{1} + x^2)
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Example

Set $w = 1 + x^2$.

$$\begin{aligned}
 \int_0^1 \arctan x dx &= [(\arctan x)x]_{x=0}^{x=1} - \int_0^1 x d(\arctan x) \\
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 &= \frac{\pi}{4} - \frac{1}{2} \int_{x=0}^{x=1} \frac{1}{1+x^2} d(1+x^2) \\
 &= \frac{\pi}{4} - \frac{1}{2} \int_{x=0}^{x=1} \frac{1}{w} dw
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 &= \frac{\pi}{4} - \frac{1}{2} \int_{x=0}^{x=1} \frac{1}{w} dw = \frac{\pi}{4} - \frac{1}{2} [\ln |w|]_{x=0}^{x=1}
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 &= \frac{\pi}{4} - \frac{1}{2} \left[\ln(1+x^2) \right]_{x=0}^{x=1}
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 &= \frac{\pi}{4} - \frac{1}{2} \left[\ln(1+x^2) \right]_{x=0}^{x=1} \\
 &= \frac{\pi}{4} - \frac{1}{2} (\ln 2 - \ln 1)
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 &= \frac{\pi}{4} - \frac{1}{2} (\ln 2 - \ln 1) = \frac{\pi}{4} - \frac{1}{2} \ln 2.
 \end{aligned}$$