Calculus II
Series basic facts

Todor Milev

2019

Outline

Outline

Outline

License to use and redistribute

Theorem

If the series $\sum_{n=1}^{\infty} a_n$ *is convergent, then* $\lim_{n\to\infty} a_n = 0$.

Proof.

Theorem

If the series $\sum_{n=1}^{\infty} a_n$ *is convergent, then* $\lim_{n\to\infty} a_n = 0$.

Proof.

- Let $s_n = a_1 + a_2 + \cdots + a_n$.

Theorem

If the series $\sum_{n=1}^{\infty} a_n$ *is convergent, then* $\lim_{n\to\infty} a_n = 0$.

Proof.

- Let $s_n = a_1 + a_2 + \cdots + a_n$.
- Then $a_n = s_n - s_{n-1}$.

Theorem

If the series $\sum_{n=1}^{\infty} a_n$ *is convergent, then* $\lim_{n\to\infty} a_n = 0$.

Proof.

- Let $s_n = a_1 + a_2 + \cdots + a_n$.
- Then $a_n = s_n - s_{n-1}$.
- Since $\sum_{n=1}^{\infty} a_n$ is convergent, the sequence $\{s_n\}$ is convergent.

Theorem

If the series $\sum_{n=1}^{\infty} a_n$ *is convergent, then* $\lim_{n\to\infty} a_n = 0$.

Proof.

- Let $s_n = a_1 + a_2 + \cdots + a_n$.
- Then $a_n = s_n - s_{n-1}$.
- Since $\sum_{n=1}^{\infty} a_n$ is convergent, the sequence $\{s_n\}$ is convergent.
- Let $\lim_{n\to\infty} s_n = s$.

Theorem

If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

Proof.

- Let $s_n = a_1 + a_2 + \cdots + a_n$.
- Then $a_n = s_n - s_{n-1}$.
- Since $\sum_{n=1}^{\infty} a_n$ is convergent, the sequence $\{s_n\}$ is convergent.
- Let $\lim_{n\to\infty} s_n = s$.
- Then $\lim_{n\to\infty} s_{n-1} =$

Theorem

If the series $\sum_{n=1}^{\infty} a_n$ *is convergent, then* $\lim_{n\to\infty} a_n = 0$.

Proof.

- Let $s_n = a_1 + a_2 + \cdots + a_n$.
- Then $a_n = s_n - s_{n-1}$.
- Since $\sum_{n=1}^{\infty} a_n$ is convergent, the sequence $\{s_n\}$ is convergent.
- Let $\lim_{n\to\infty} s_n = s$.
- Then $\lim_{n\to\infty} s_{n-1} = s$.

Theorem

If the series $\sum_{n=1}^{\infty} a_n$ *is convergent, then* $\lim_{n\to\infty} a_n = 0$.

Proof.

- Let $s_n = a_1 + a_2 + \cdots + a_n$.
- Then $a_n = s_n - s_{n-1}$.
- Since $\sum_{n=1}^{\infty} a_n$ is convergent, the sequence $\{s_n\}$ is convergent.
- Let $\lim_{n\to\infty} s_n = s$.
- Then $\lim_{n\to\infty} s_{n-1} = s$.
- Therefore

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (s_n - s_{n-1})$$

Theorem

If the series $\sum_{n=1}^{\infty} a_n$ *is convergent, then* $\lim_{n\to\infty} a_n = 0$.

Proof.

- Let $s_n = a_1 + a_2 + \cdots + a_n$.
- Then $a_n = s_n - s_{n-1}$.
- Since $\sum_{n=1}^{\infty} a_n$ is convergent, the sequence $\{s_n\}$ is convergent.
- Let $\lim_{n\to\infty} s_n = s$.
- Then $\lim_{n\to\infty} s_{n-1} = s$.
- Therefore

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (s_n - s_{n-1}) = s - s$$

Theorem

If the series $\sum_{n=1}^{\infty} a_n$ *is convergent, then* $\lim_{n\to\infty} a_n = 0$.

Proof.

- Let $s_n = a_1 + a_2 + \cdots + a_n$.
- Then $a_n = s_n - s_{n-1}$.
- Since $\sum_{n=1}^{\infty} a_n$ is convergent, the sequence $\{s_n\}$ is convergent.
- Let $\lim_{n\to\infty} s_n = s$.
- Then $\lim_{n\to\infty} s_{n-1} = s$.
- Therefore

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (s_n - s_{n-1}) = s - s = 0 \qquad \square$$

Theorem

If the series $\sum_{n=1}^{\infty} a_n$ *is convergent, then* $\lim_{n\to\infty} a_n = 0$.

This is just a restatement of the previous theorem:

Theorem (The Divergence Test)

If $\lim_{n\to\infty} a_n$ *doesn't exist or if* $\lim_{n\to\infty} a_n \neq 0$, *then the series* $\sum_{n=1}^{\infty} a_n$ *is divergent.*

Example

Show that the series $\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ diverges.

Example

Show that the series $\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ diverges.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{5n^2+4}$$

Example

Show that the series $\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ diverges.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{5n^2+4} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}}$$

Example

Show that the series $\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ diverges.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{5n^2+4} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{1}{5+\frac{4}{n^2}}$$

Example

Show that the series $\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ diverges.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{5n^2+4} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{1}{5+\frac{4}{n^2}} = \frac{1}{5} \neq 0$$

Example

Show that the series $\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ diverges.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{5n^2+4} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{1}{5+\frac{4}{n^2}} = \frac{1}{5} \neq 0$$

Therefore, by the Divergence Test, the series diverges.

The Integral Test and Estimates of Sums

- In general, it is not easy to find the sum of a series.
- We could do this for $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ because we found a simple formula for the nth partial sum s_n.
- In the next few sections, we'll learn techniques for showing whether a series is convergent or divergent without explicitly computing its sum.

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

- Use a computer to calculate partial sums.

n	$s_n = \sum_{i=1}^{n} \frac{1}{i^2}$
5	1.4636
10	1.5498
50	1.6251
100	1.6350
500	1.6429
1000	1.6439
5000	1.6447

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

- Use a computer to calculate partial sums.
- Appears to be converging.

n	$s_n = \sum_{i=1}^{n} \frac{1}{i^2}$
5	1.4636
10	1.5498
50	1.6251
100	1.6350
500	1.6429
1000	1.6439
5000	1.6447

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

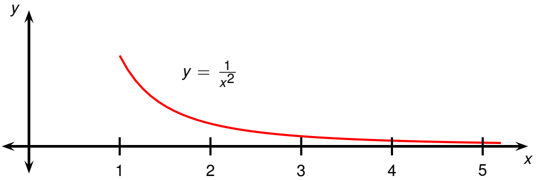
- Use a computer to calculate partial sums.
- Appears to be converging.
- How do we prove it?

n	$s_n = \sum_{i=1}^{n} \frac{1}{i^2}$
5	1.4636
10	1.5498
50	1.6251
100	1.6350
500	1.6429
1000	1.6439
5000	1.6447

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

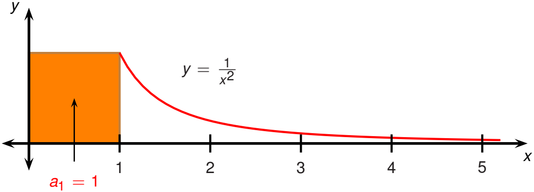
- Use a computer to calculate partial sums.
- Appears to be converging.
- How do we prove it?
- Use $f(x) = \frac{1}{x^2}$.

n	$s_n = \sum_{i=1}^{n} \frac{1}{i^2}$
5	1.4636
10	1.5498
50	1.6251
100	1.6350
500	1.6429
1000	1.6439
5000	1.6447



$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

- Use a computer to calculate partial sums.
- Appears to be converging.
- How do we prove it?
- Use $f(x) = \frac{1}{x^2}$.

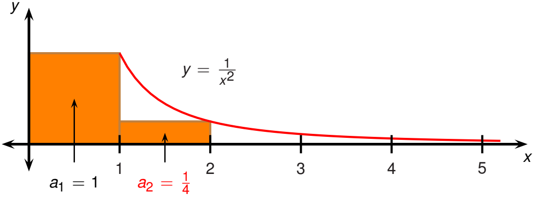


n	$s_n = \sum_{i=1}^{n} \frac{1}{i^2}$
5	1.4636
10	1.5498
50	1.6251
100	1.6350
500	1.6429
1000	1.6439
5000	1.6447

- $\frac{1}{1^2}$ is the area of a rectangle.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

- Use a computer to calculate partial sums.
- Appears to be converging.
- How do we prove it?
- Use $f(x) = \frac{1}{x^2}$.



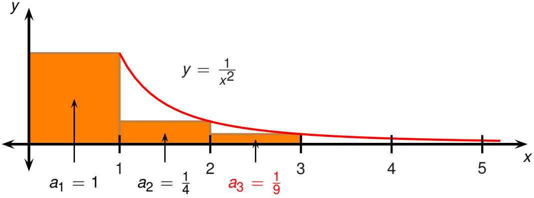
n	$s_n = \sum_{i=1}^{n} \frac{1}{i^2}$
5	1.4636
10	1.5498
50	1.6251
100	1.6350
500	1.6429
1000	1.6439
5000	1.6447

- $\frac{1}{1^2}$ is the area of a rectangle.
- So is $\frac{1}{2^2} = \frac{1}{4}$.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

n	$s_n = \sum_{i=1}^{n} \frac{1}{i^2}$
5	1.4636
10	1.5498
50	1.6251
100	1.6350
500	1.6429
1000	1.6439
5000	1.6447

- Use a computer to calculate partial sums.
- Appears to be converging.
- How do we prove it?
- Use $f(x) = \frac{1}{x^2}$.

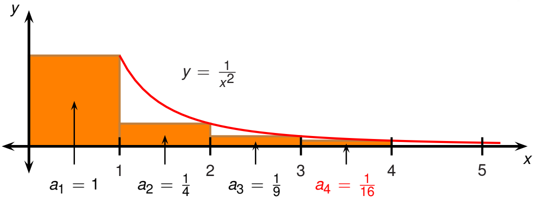


- $\frac{1}{1^2}$ is the area of a rectangle.
- So is $\frac{1}{2^2} = \frac{1}{4}$.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

n	$s_n = \sum_{i=1}^{n} \frac{1}{i^2}$
5	1.4636
10	1.5498
50	1.6251
100	1.6350
500	1.6429
1000	1.6439
5000	1.6447

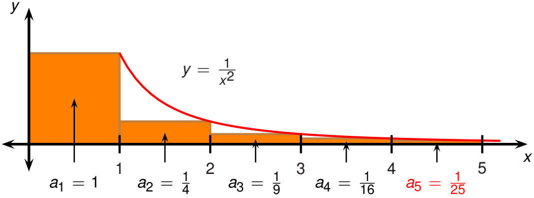
- Use a computer to calculate partial sums.
- Appears to be converging.
- How do we prove it?
- Use $f(x) = \frac{1}{x^2}$.



- $\frac{1}{1^2}$ is the area of a rectangle.
- So is $\frac{1}{2^2} = \frac{1}{4}$.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

- Use a computer to calculate partial sums.
- Appears to be converging.
- How do we prove it?
- Use $f(x) = \frac{1}{x^2}$.



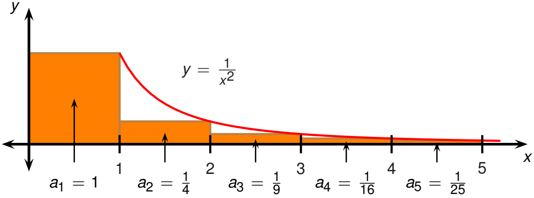
n	$s_n = \sum_{i=1}^{n} \frac{1}{i^2}$
5	1.4636
10	1.5498
50	1.6251
100	1.6350
500	1.6429
1000	1.6439
5000	1.6447

- $\frac{1}{1^2}$ is the area of a rectangle.
- So is $\frac{1}{2^2} = \frac{1}{4}$.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

n	$s_n = \sum_{i=1}^{n} \frac{1}{i^2}$
5	1.4636
10	1.5498
50	1.6251
100	1.6350
500	1.6429
1000	1.6439
5000	1.6447

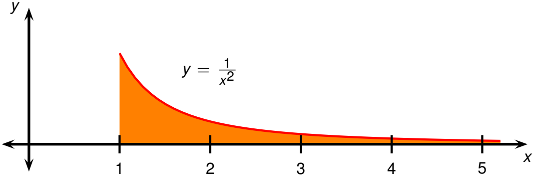
- Use a computer to calculate partial sums.
- Appears to be converging.
- How do we prove it?
- Use $f(x) = \frac{1}{x^2}$.



- $\frac{1}{1^2}$ is the area of a rectangle.
- So is $\frac{1}{2^2} = \frac{1}{4}$.
- The improper integral $\int_1^\infty \frac{1}{x^2} dx$ is

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

- Use a computer to calculate partial sums.
- Appears to be converging.
- How do we prove it?
- Use $f(x) = \frac{1}{x^2}$.



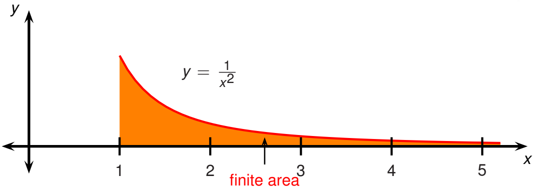
n	$s_n = \sum_{i=1}^{n} \frac{1}{i^2}$
5	1.4636
10	1.5498
50	1.6251
100	1.6350
500	1.6429
1000	1.6439
5000	1.6447

- $\frac{1}{1^2}$ is the area of a rectangle.
- So is $\frac{1}{2^2} = \frac{1}{4}$.
- The improper integral $\int_1^\infty \frac{1}{x^2} dx$ is **?**

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

n	$s_n = \sum_{i=1}^{n} \frac{1}{i^2}$
5	1.4636
10	1.5498
50	1.6251
100	1.6350
500	1.6429
1000	1.6439
5000	1.6447

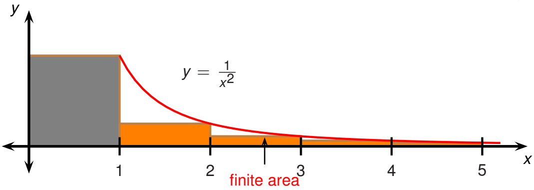
- Use a computer to calculate partial sums.
- Appears to be converging.
- How do we prove it?
- Use $f(x) = \frac{1}{x^2}$.



- $\frac{1}{1^2}$ is the area of a rectangle.
- So is $\frac{1}{2^2} = \frac{1}{4}$.
- The improper integral $\int_1^\infty \frac{1}{x^2} dx$ is convergent.

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots$$

- Use a computer to calculate partial sums.
- Appears to be converging.
- How do we prove it?
- Use $f(x)=\frac{1}{x^2}$.



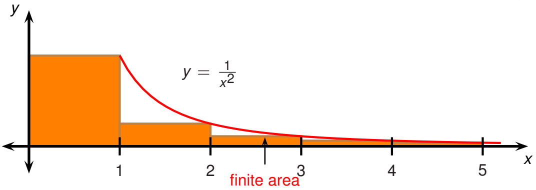
n	$s_n=\sum_{i=1}^{n}\frac{1}{i^2}$
5	1.4636
10	1.5498
50	1.6251
100	1.6350
500	1.6429
1000	1.6439
5000	1.6447

- $\frac{1}{1^2}$ is the area of a rectangle.
- So is $\frac{1}{2^2}=\frac{1}{4}$.
- The improper integral $\int_1^{\infty}\frac{1}{x^2}dx$ is convergent.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

n	$s_n = \sum_{i=1}^{n} \frac{1}{i^2}$
5	1.4636
10	1.5498
50	1.6251
100	1.6350
500	1.6429
1000	1.6439
5000	1.6447

- Use a computer to calculate partial sums.
- Appears to be converging.
- How do we prove it?
- Use $f(x) = \frac{1}{x^2}$.



- $\frac{1}{1^2}$ is the area of a rectangle.
- So is $\frac{1}{2^2} = \frac{1}{4}$.
- The improper integral $\int_1^{\infty} \frac{1}{x^2} dx$ is convergent.
- Therefore $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots$$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots$$

- Use a computer to calculate partial sums.

n	$s_n = \sum_{i=1}^{n} \frac{1}{\sqrt{i}}$
5	3.2317
10	5.0210
50	12.7524
100	18.5896
500	43.2834
1000	61.8010
5000	139.9681

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots$$

- Use a computer to calculate partial sums.
- Appears to be diverging.

n	$s_n = \sum_{i=1}^{n} \frac{1}{\sqrt{i}}$
5	3.2317
10	5.0210
50	12.7524
100	18.5896
500	43.2834
1000	61.8010
5000	139.9681

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots$$

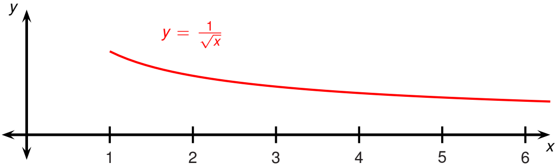
- Use a computer to calculate partial sums.
- Appears to be diverging.
- How do we prove it?

n	$s_n = \sum_{i=1}^{n} \frac{1}{\sqrt{i}}$
5	3.2317
10	5.0210
50	12.7524
100	18.5896
500	43.2834
1000	61.8010
5000	139.9681

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots$$

- Use a computer to calculate partial sums.
- Appears to be diverging.
- How do we prove it?
- Use $f(x) = \frac{1}{\sqrt{x}}$.

n	$s_n = \sum_{i=1}^{n} \frac{1}{\sqrt{i}}$
5	3.2317
10	5.0210
50	12.7524
100	18.5896
500	43.2834
1000	61.8010
5000	139.9681

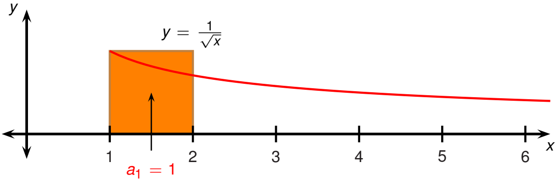


$$\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots$$

- Use a computer to calculate partial sums.
- Appears to be diverging.
- How do we prove it?
- Use $f(x) = \frac{1}{\sqrt{x}}$.

n	$s_n = \sum_{i=1}^{n} \frac{1}{\sqrt{i}}$
5	3.2317
10	5.0210
50	12.7524
100	18.5896
500	43.2834
1000	61.8010
5000	139.9681

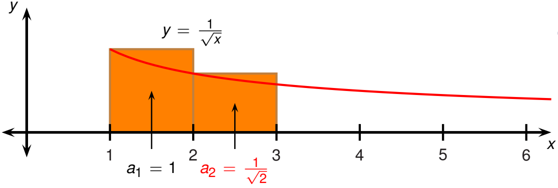
- $\frac{1}{\sqrt{1}}$ is the area of a rectangle.



$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots$$

n	$s_n = \sum_{i=1}^{n} \frac{1}{\sqrt{i}}$
5	3.2317
10	5.0210
50	12.7524
100	18.5896
500	43.2834
1000	61.8010
5000	139.9681

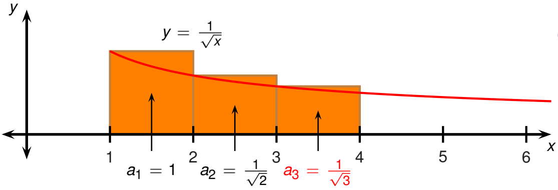
- Use a computer to calculate partial sums.
- Appears to be diverging.
- How do we prove it?
- Use $f(x) = \frac{1}{\sqrt{x}}$.



- $\frac{1}{\sqrt{1}}$ is the area of a rectangle.
- So is $\frac{1}{\sqrt{2}}$.

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots$$

- Use a computer to calculate partial sums.
- Appears to be diverging.
- How do we prove it?
- Use $f(x) = \frac{1}{\sqrt{x}}$.



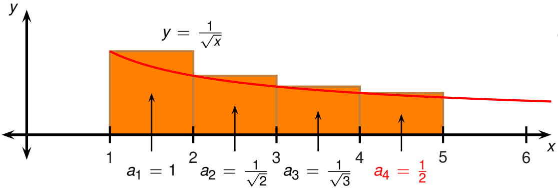
n	$s_n = \sum_{i=1}^{n} \frac{1}{\sqrt{i}}$
5	3.2317
10	5.0210
50	12.7524
100	18.5896
500	43.2834
1000	61.8010
5000	139.9681

- $\frac{1}{\sqrt{1}}$ is the area of a rectangle.
- So is $\frac{1}{\sqrt{2}}$.

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots$$

n	$s_n = \sum_{i=1}^{n} \frac{1}{\sqrt{i}}$
5	3.2317
10	5.0210
50	12.7524
100	18.5896
500	43.2834
1000	61.8010
5000	139.9681

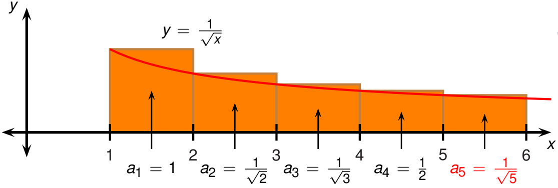
- Use a computer to calculate partial sums.
- Appears to be diverging.
- How do we prove it?
- Use $f(x) = \frac{1}{\sqrt{x}}$.



- $\frac{1}{\sqrt{1}}$ is the area of a rectangle.
- So is $\frac{1}{\sqrt{2}}$.

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots$$

- Use a computer to calculate partial sums.
- Appears to be diverging.
- How do we prove it?
- Use $f(x) = \frac{1}{\sqrt{x}}$.

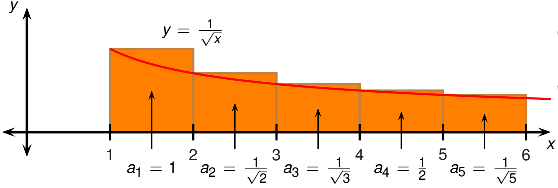


n	$s_n = \sum_{i=1}^{n} \frac{1}{\sqrt{i}}$
5	3.2317
10	5.0210
50	12.7524
100	18.5896
500	43.2834
1000	61.8010
5000	139.9681

- $\frac{1}{\sqrt{1}}$ is the area of a rectangle.
- So is $\frac{1}{\sqrt{2}}$.

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots$$

- Use a computer to calculate partial sums.
- Appears to be diverging.
- How do we prove it?
- Use $f(x) = \frac{1}{\sqrt{x}}$.



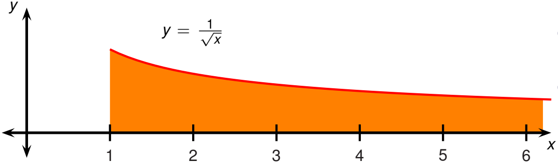
n	$s_n = \sum_{i=1}^{n} \frac{1}{\sqrt{i}}$
5	3.2317
10	5.0210
50	12.7524
100	18.5896
500	43.2834
1000	61.8010
5000	139.9681

- $\frac{1}{\sqrt{1}}$ is the area of a rectangle.
- So is $\frac{1}{\sqrt{2}}$.
- $\int_1^{\infty} \frac{1}{\sqrt{x}} dx$ is **?**

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots$$

n	$s_n = \sum_{i=1}^{n} \frac{1}{\sqrt{i}}$
5	3.2317
10	5.0210
50	12.7524
100	18.5896
500	43.2834
1000	61.8010
5000	139.9681

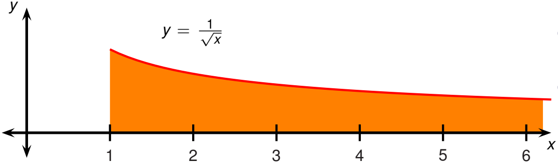
- Use a computer to calculate partial sums.
- Appears to be diverging.
- How do we prove it?
- Use $f(x) = \frac{1}{\sqrt{x}}$.



- $\frac{1}{\sqrt{1}}$ is the area of a rectangle.
- So is $\frac{1}{\sqrt{2}}$.
- $\int_1^{\infty} \frac{1}{\sqrt{x}} dx$ is **?**

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots$$

- Use a computer to calculate partial sums.
- Appears to be diverging.
- How do we prove it?
- Use $f(x) = \frac{1}{\sqrt{x}}$.

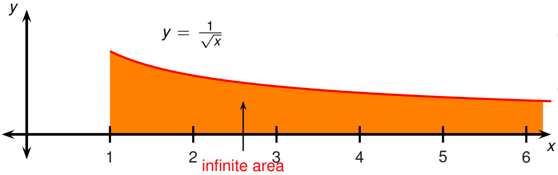


n	$s_n = \sum_{i=1}^{n} \frac{1}{\sqrt{i}}$
5	3.2317
10	5.0210
50	12.7524
100	18.5896
500	43.2834
1000	61.8010
5000	139.9681

- $\frac{1}{\sqrt{1}}$ is the area of a rectangle.
- So is $\frac{1}{\sqrt{2}}$.
- $\int_1^{\infty} \frac{1}{\sqrt{x}} dx$ is **?**

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots$$

- Use a computer to calculate partial sums.
- Appears to be diverging.
- How do we prove it?
- Use $f(x) = \frac{1}{\sqrt{x}}$.



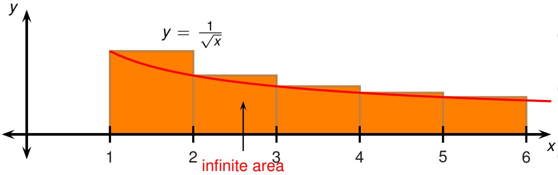
n	$s_n = \sum_{i=1}^{n} \frac{1}{\sqrt{i}}$
5	3.2317
10	5.0210
50	12.7524
100	18.5896
500	43.2834
1000	61.8010
5000	139.9681

- $\frac{1}{\sqrt{1}}$ is the area of a rectangle.
- So is $\frac{1}{\sqrt{2}}$.
- $\int_1^{\infty} \frac{1}{\sqrt{x}} dx$ is divergent.

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots$$

n	$s_n = \sum_{i=1}^{n} \frac{1}{\sqrt{i}}$
5	3.2317
10	5.0210
50	12.7524
100	18.5896
500	43.2834
1000	61.8010
5000	139.9681

- Use a computer to calculate partial sums.
- Appears to be diverging.
- How do we prove it?
- Use $f(x) = \frac{1}{\sqrt{x}}$.



- $\frac{1}{\sqrt{1}}$ is the area of a rectangle.
- So is $\frac{1}{\sqrt{2}}$.
- $\int_1^\infty \frac{1}{\sqrt{x}} dx$ is divergent.
- Therefore $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is divergent.

Theorem (The Integral Test)

Let f be a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\mathrm{d}x$ is convergent. In other words,

1. *If $\displaystyle\int_1^{\infty} f(x)\mathrm{d}x$ is convergent, then $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent.*
2. *If $\displaystyle\int_1^{\infty} f(x)\mathrm{d}x$ is divergent, then $\displaystyle\sum_{n=1}^{\infty} a_n$ is divergent.*

Theorem (The Integral Test)

Let f be a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\mathrm{d}x$ is convergent. In other words,

1. *If* $\displaystyle\int_1^{\infty} f(x)\mathrm{d}x$ *is convergent, then* $\displaystyle\sum_{n=1}^{\infty} a_n$ *is convergent.*
2. *If* $\displaystyle\int_1^{\infty} f(x)\mathrm{d}x$ *is divergent, then* $\displaystyle\sum_{n=1}^{\infty} a_n$ *is divergent.*

Note that it is not necessary to start the series or the integral at $n = 1$. For instance, to test the series

$$\sum_{n=4}^{\infty} \frac{1}{(n-3)^2}$$

we would use

$$\int_4^{\infty} \frac{1}{(x-3)^2}\mathrm{d}x$$

Example

Test the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ for convergence.

Example

Test the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ for convergence.

$f(x) = \frac{1}{x^2+1}$ is continuous, positive, and decreasing on $[1, \infty)$, so use the Integral Test.

Example

Test the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ for convergence.

$f(x) = \frac{1}{x^2+1}$ is continuous, positive, and decreasing on $[1, \infty)$, so use the Integral Test.

$$\int_1^{\infty} \frac{1}{x^2+1} \mathrm{d}x \quad = \quad \lim_{t \to \infty} \int_1^{t} \frac{1}{x^2+1} \mathrm{d}x$$

Example

Test the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ for convergence.

$f(x) = \frac{1}{x^2+1}$ is continuous, positive, and decreasing on $[1, \infty)$, so use the Integral Test.

$$\begin{aligned}\int_1^{\infty} \frac{1}{x^2+1} \mathrm{d}x &= \lim_{t\to\infty} \int_1^t \frac{1}{x^2+1} \mathrm{d}x \\ &= \lim_{t\to\infty} \left[\textbf{?} \qquad \right]_1^t\end{aligned}$$

Example

Test the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ for convergence.

$f(x) = \frac{1}{x^2+1}$ is continuous, positive, and decreasing on $[1, \infty)$, so use the Integral Test.

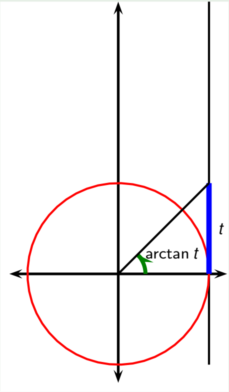
$$\begin{aligned}\int_1^{\infty} \frac{1}{x^2+1}\mathrm{d}x &= \lim_{t\to\infty} \int_1^{t} \frac{1}{x^2+1}\mathrm{d}x \\ &= \lim_{t\to\infty} \left[\arctan x\right]_1^t\end{aligned}$$

Example

Test the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ for convergence.

$f(x) = \frac{1}{x^2+1}$ is continuous, positive, and decreasing on $[1, \infty)$, so use the Integral Test.

$$\begin{aligned}
\int_1^{\infty} \frac{1}{x^2+1} \mathrm{d}x &= \lim_{t\to\infty} \int_1^t \frac{1}{x^2+1} \mathrm{d}x \\
&= \lim_{t\to\infty} \left[\arctan x\right]_1^t \\
&= \lim_{t\to\infty} \left(\arctan t - \mathbf{?}\right)
\end{aligned}$$

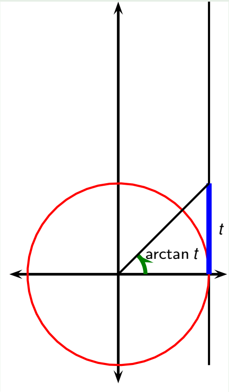


Example

Test the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ for convergence.

$f(x) = \frac{1}{x^2+1}$ is continuous, positive, and decreasing on $[1, \infty)$, so use the Integral Test.

$$\begin{aligned}\int_1^{\infty} \frac{1}{x^2+1} \mathrm{d}x &= \lim_{t\to\infty} \int_1^{t} \frac{1}{x^2+1} \mathrm{d}x \\ &= \lim_{t\to\infty} \left[\arctan x\right]_1^t \\ &= \lim_{t\to\infty} \left(\arctan t - \mathbf{?}\right)\end{aligned}$$

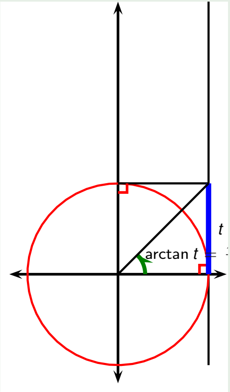


Example

Test the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ for convergence.

$f(x) = \frac{1}{x^2+1}$ is continuous, positive, and decreasing on $[1, \infty)$, so use the Integral Test.

$$\begin{aligned}
\int_1^{\infty} \frac{1}{x^2+1} \mathrm{d}x &= \lim_{t\to\infty} \int_1^{t} \frac{1}{x^2+1} \mathrm{d}x \\
&= \lim_{t\to\infty} \left[\arctan x\right]_1^t \\
&= \lim_{t\to\infty} \left(\arctan t - \frac{\pi}{4}\right)
\end{aligned}$$

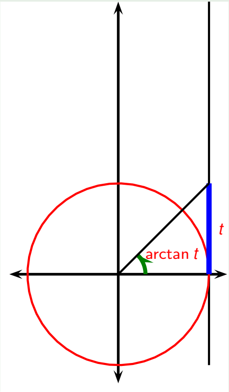


Example

Test the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ for convergence.

$f(x) = \frac{1}{x^2+1}$ is continuous, positive, and decreasing on $[1, \infty)$, so use the Integral Test.

$$\begin{aligned}
\int_1^{\infty} \frac{1}{x^2+1} dx &= \lim_{t\to\infty} \int_1^t \frac{1}{x^2+1} dx \\
&= \lim_{t\to\infty} \left[\arctan x\right]_1^t \\
&= \lim_{t\to\infty} \left(\arctan t - \frac{\pi}{4}\right) \\
&= \textbf{?} - \frac{\pi}{4}
\end{aligned}$$

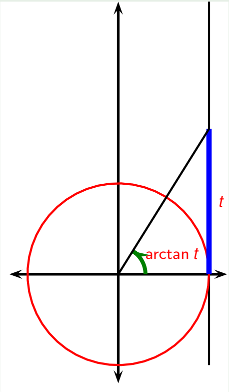


Example

Test the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ for convergence.

$f(x) = \frac{1}{x^2+1}$ is continuous, positive, and decreasing on $[1, \infty)$, so use the Integral Test.

$$\begin{aligned}
\int_1^{\infty} \frac{1}{x^2+1} \mathrm{d}x &= \lim_{t\to\infty} \int_1^{t} \frac{1}{x^2+1} \mathrm{d}x \\
&= \lim_{t\to\infty} \left[\arctan x\right]_1^t \\
&= \lim_{t\to\infty} \left(\arctan t - \frac{\pi}{4}\right) \\
&= \mathbf{?} - \frac{\pi}{4}
\end{aligned}$$

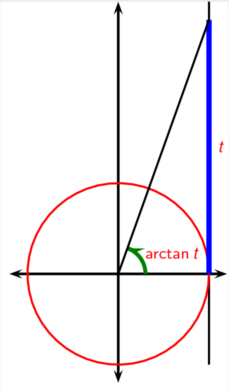


Example

Test the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ for convergence.

$f(x) = \frac{1}{x^2+1}$ is continuous, positive, and decreasing on $[1, \infty)$, so use the Integral Test.

$$\begin{aligned}
\int_1^{\infty} \frac{1}{x^2+1} \mathrm{d}x &= \lim_{t\to\infty} \int_1^{t} \frac{1}{x^2+1} \mathrm{d}x \\
&= \lim_{t\to\infty} \left[\arctan x\right]_1^t \\
&= \lim_{t\to\infty} \left(\arctan t - \frac{\pi}{4}\right) \\
&= \textbf{?} - \frac{\pi}{4}
\end{aligned}$$

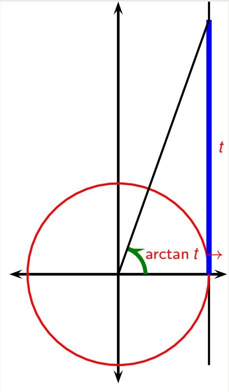


Example

Test the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ for convergence.

$f(x) = \frac{1}{x^2+1}$ is continuous, positive, and decreasing on $[1,\infty)$, so use the Integral Test.

$$\begin{aligned}
\int_1^{\infty} \frac{1}{x^2+1} dx &= \lim_{t\to\infty} \int_1^t \frac{1}{x^2+1} dx \\
&= \lim_{t\to\infty} \left[\arctan x\right]_1^t \\
&= \lim_{t\to\infty} \left(\arctan t - \frac{\pi}{4}\right) \\
&= \frac{\pi}{2} - \frac{\pi}{4}
\end{aligned}$$

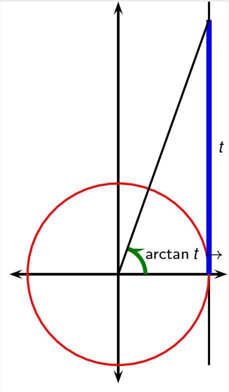


Example

Test the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ for convergence.

$f(x) = \frac{1}{x^2+1}$ is continuous, positive, and decreasing on $[1, \infty)$, so use the Integral Test.

$$\begin{aligned}
\int_1^{\infty} \frac{1}{x^2+1} dx &= \lim_{t\to\infty} \int_1^{t} \frac{1}{x^2+1} dx \\
&= \lim_{t\to\infty} \left[\arctan x\right]_1^t \\
&= \lim_{t\to\infty} \left(\arctan t - \frac{\pi}{4}\right) \\
&= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}
\end{aligned}$$

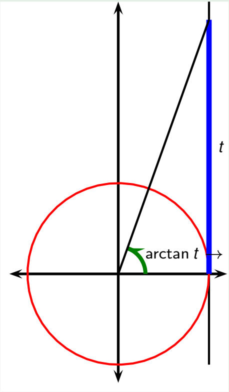


Example

Test the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ for convergence.

$f(x) = \frac{1}{x^2+1}$ is continuous, positive, and decreasing on $[1, \infty)$, so use the Integral Test.

$$\begin{aligned}
\int_1^{\infty} \frac{1}{x^2+1} \mathrm{d}x &= \lim_{t\to\infty} \int_1^{t} \frac{1}{x^2+1} \mathrm{d}x \\
&= \lim_{t\to\infty} \left[\arctan x\right]_1^t \\
&= \lim_{t\to\infty} \left(\arctan t - \frac{\pi}{4}\right) \\
&= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}
\end{aligned}$$



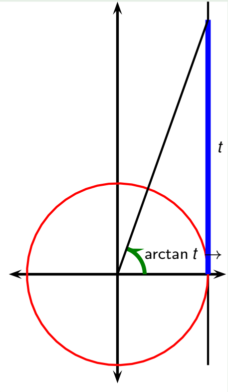
Therefore $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ is **?**

Example

Test the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ for convergence.

$f(x) = \frac{1}{x^2+1}$ is continuous, positive, and decreasing on $[1, \infty)$, so use the Integral Test.

$$\begin{aligned}
\int_1^{\infty} \frac{1}{x^2+1} \mathrm{d}x &= \lim_{t\to\infty} \int_1^{t} \frac{1}{x^2+1} \mathrm{d}x \\
&= \lim_{t\to\infty} \left[\arctan x\right]_1^t \\
&= \lim_{t\to\infty} \left(\arctan t - \frac{\pi}{4}\right) \\
&= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}
\end{aligned}$$



Therefore $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ is convergent.

Example

For which values of p is the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ convergent?

Example

For which values of p is the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ convergent?

- If $p < 0$, then $\lim\limits_{n\to\infty} \frac{1}{n^p} = \textbf{?}$

Example

For which values of p is the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ convergent?

- If $p < 0$, then $\lim_{n\to\infty} \frac{1}{n^p} = \infty$.

Example

For which values of p is the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ convergent?

- If $p < 0$, then $\lim_{n\to\infty} \frac{1}{n^p} = \infty$.
- If $p = 0$, then $\lim_{n\to\infty} \frac{1}{n^p} =$ **?**

Example

For which values of p is the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ convergent?

- If $p < 0$, then $\lim\limits_{n\to\infty} \frac{1}{n^p} = \infty$.
- If $p = 0$, then $\lim\limits_{n\to\infty} \frac{1}{n^p} = 1$.

Example

For which values of p is the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ convergent?

- If $p < 0$, then $\lim_{n\to\infty} \frac{1}{n^p} = \infty$.
- If $p = 0$, then $\lim_{n\to\infty} \frac{1}{n^p} = 1$.
- Therefore for $p \leq 0$ the series is **?**

Example

For which values of p is the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ convergent?

- If $p < 0$, then $\lim_{n\to\infty} \frac{1}{n^p} = \infty$.
- If $p = 0$, then $\lim_{n\to\infty} \frac{1}{n^p} = 1$.
- Therefore for $p \leq 0$ the series is divergent.

Example

For which values of p is the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ convergent?

- If $p < 0$, then $\lim_{n\to\infty} \frac{1}{n^p} = \infty$.
- If $p = 0$, then $\lim_{n\to\infty} \frac{1}{n^p} = 1$.
- Therefore for $p \leq 0$ the series is divergent.
- It remains to investigate the case $p > 0$. If $p > 0$, then $f(x) = \frac{1}{x^p}$ is continuous, positive, and decreasing on $[1, \infty)$, so we can use the Integral Test.

Example

For which values of p is the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ convergent?

- If $p < 0$, then $\lim_{n\to\infty} \frac{1}{n^p} = \infty$.
- If $p = 0$, then $\lim_{n\to\infty} \frac{1}{n^p} = 1$.
- Therefore for $p \leq 0$ the series is divergent.
- It remains to investigate the case $p > 0$. If $p > 0$, then $f(x) = \frac{1}{x^p}$ is continuous, positive, and decreasing on $[1, \infty)$, so we can use the Integral Test.

$$\int_1^{\infty} \frac{1}{x^p} dx = \begin{cases} \text{convergent} & \text{when} \quad \textbf{?} \\ \text{divergent} & \text{when} \quad \textbf{?} \end{cases}$$

Example

For which values of p is the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ convergent?

- If $p < 0$, then $\displaystyle\lim_{n\to\infty} \frac{1}{n^p} = \infty$.
- If $p = 0$, then $\displaystyle\lim_{n\to\infty} \frac{1}{n^p} = 1$.
- Therefore for $p \leq 0$ the series is divergent.
- It remains to investigate the case $p > 0$. If $p > 0$, then $f(x) = \frac{1}{x^p}$ is continuous, positive, and decreasing on $[1, \infty)$, so we can use the Integral Test.

$$\int_1^{\infty} \frac{1}{x^p} \mathrm{d}x = \begin{cases} \text{convergent} & \text{when} \quad p > 1 \\ \text{divergent} & \text{when} \quad \textbf{?} \end{cases}$$

Example

For which values of p is the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ convergent?

- If $p < 0$, then $\lim\limits_{n\to\infty} \frac{1}{n^p} = \infty$.
- If $p = 0$, then $\lim\limits_{n\to\infty} \frac{1}{n^p} = 1$.
- Therefore for $p \leq 0$ the series is divergent.
- It remains to investigate the case $p > 0$. If $p > 0$, then $f(x) = \frac{1}{x^p}$ is continuous, positive, and decreasing on $[1, \infty)$, so we can use the Integral Test.

$$\int_1^{\infty} \frac{1}{x^p} \mathrm{d}x = \begin{cases} \text{convergent} & \text{when} \quad p > 1 \\ \text{divergent} & \text{when} \quad ? \end{cases}$$

Example

For which values of p is the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ convergent?

- If $p < 0$, then $\lim\limits_{n\to\infty} \frac{1}{n^p} = \infty$.
- If $p = 0$, then $\lim\limits_{n\to\infty} \frac{1}{n^p} = 1$.
- Therefore for $p \leq 0$ the series is divergent.
- It remains to investigate the case $p > 0$. If $p > 0$, then $f(x) = \frac{1}{x^p}$ is continuous, positive, and decreasing on $[1, \infty)$, so we can use the Integral Test.

$$\int_1^{\infty} \frac{1}{x^p} \mathrm{d}x = \begin{cases} \text{convergent} & \text{when} \quad p > 1 \\ \text{divergent} & \text{when} \quad p \leq 1 \end{cases}$$

Example

For which values of p is the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ convergent?

- If $p < 0$, then $\lim\limits_{n\to\infty} \frac{1}{n^p} = \infty$.
- If $p = 0$, then $\lim\limits_{n\to\infty} \frac{1}{n^p} = 1$.
- Therefore for $p \leq 0$ the series is divergent.
- It remains to investigate the case $p > 0$. If $p > 0$, then $f(x) = \frac{1}{x^p}$ is continuous, positive, and decreasing on $[1, \infty)$, so we can use the Integral Test.

$$\int_1^{\infty} \frac{1}{x^p} \mathrm{d}x = \begin{cases} \text{convergent} & \text{when} \quad p > 1 \\ \text{divergent} & \text{when} \quad p \leq 1 \end{cases}$$

- $\Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent when $p > 1$ and divergent when $p \leq 1$.

Example

For which values of p is the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ convergent?

- If $p < 0$, then $\lim\limits_{n\to\infty} \frac{1}{n^p} = \infty$.
- If $p = 0$, then $\lim\limits_{n\to\infty} \frac{1}{n^p} = 1$.
- Therefore for $p \leq 0$ the series is divergent.
- It remains to investigate the case $p > 0$. If $p > 0$, then $f(x) = \frac{1}{x^p}$ is continuous, positive, and decreasing on $[1, \infty)$, so we can use the Integral Test.
$$\int_1^{\infty} \frac{1}{x^p} dx = \begin{cases} \text{convergent} & \text{when} \quad p > 1 \\ \text{divergent} & \text{when} \quad p \leq 1 \end{cases}$$
- $\Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent when $p > 1$ and divergent when $p \leq 1$.

This theorem summarizes the results of the previous example.

Theorem (p-series Convergence)

The p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ *is convergent if* $p > 1$ *and divergent if* $p \leq 1$.

Example

Test the series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$ for convergence.

Example

Test the series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$ for convergence.

- $f(x) = \frac{\ln x}{x}$ is continuous and positive ($x > 0$).

Example

Test the series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$ for convergence.

- $f(x) = \frac{\ln x}{x}$ is continuous and positive $(x > 0)$.
- To establish where $f(x)$ is decreasing, take the derivative.

Example

Test the series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$ for convergence.

- $f(x) = \frac{\ln x}{x}$ is continuous and positive ($x > 0$).
- To establish where $f(x)$ is decreasing, take the derivative.

$$f'(x) = \frac{\left(\frac{1}{x}\right)(x) - (\ln x)(1)}{x^2}$$

Example

Test the series $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ for convergence.

- $f(x) = \frac{\ln x}{x}$ is continuous and positive $(x > 0)$.
- To establish where $f(x)$ is decreasing, take the derivative.

$$f'(x) = \frac{\left(\frac{1}{x}\right)(x) - (\ln x)(1)}{x^2} = \frac{1 - \ln x}{x^2}$$

Example

Test the series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$ for convergence.

- $f(x) = \frac{\ln x}{x}$ is continuous and positive ($x > 0$).
- To establish where $f(x)$ is decreasing, take the derivative.

$$f'(x) = \frac{\left(\frac{1}{x}\right)(x) - (\ln x)(1)}{x^2} = \frac{1 - \ln x}{x^2}$$

- This is negative for all $x >$ **?**

Example

Test the series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$ for convergence.

- $f(x) = \frac{\ln x}{x}$ is continuous and positive ($x > 0$).
- To establish where $f(x)$ is decreasing, take the derivative.
$$f'(x) = \frac{\left(\frac{1}{x}\right)(x) - (\ln x)(1)}{x^2} = \frac{1 - \ln x}{x^2}$$
- This is negative for all $x > e$.

Example

Test the series $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ for convergence.

- $f(x) = \frac{\ln x}{x}$ is continuous and positive ($x > 0$).
- To establish where $f(x)$ is decreasing, take the derivative.

$$f'(x) = \frac{\left(\frac{1}{x}\right)(x) - (\ln x)(1)}{x^2} = \frac{1 - \ln x}{x^2}$$

- This is negative for all $x > e$.
- Therefore f is decreasing for all $x > e$.

Example

Test the series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$ for convergence.

- $f(x) = \frac{\ln x}{x}$ is continuous and positive ($x > 0$).
- To establish where $f(x)$ is decreasing, take the derivative.
$$f'(x) = \frac{\left(\frac{1}{x}\right)(x) - (\ln x)(1)}{x^2} = \frac{1 - \ln x}{x^2}$$
- This is negative for all $x > e$.
- Therefore f is decreasing for all $x > e$.

$$\int_1^{\infty} \frac{\ln x}{x} \mathrm{d}x \quad = \quad \lim_{t \to \infty} \int_1^{t} \frac{\ln x}{x} \mathrm{d}x$$

Example

Test the series $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ for convergence.

- $f(x) = \frac{\ln x}{x}$ is continuous and positive ($x > 0$).
- To establish where $f(x)$ is decreasing, take the derivative.

$$f'(x) = \frac{\left(\frac{1}{x}\right)(x) - (\ln x)(1)}{x^2} = \frac{1 - \ln x}{x^2}$$

- This is negative for all $x > e$.
- Therefore f is decreasing for all $x > e$.

$$\int_1^{\infty} \frac{\ln x}{x} \mathrm{d}x = \lim_{t\to\infty} \int_1^t \frac{\ln x}{x} \mathrm{d}x = \lim_{t\to\infty} \left[\textbf{?} \qquad \right]_1^t$$

Example

Test the series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$ for convergence.

- $f(x) = \frac{\ln x}{x}$ is continuous and positive ($x > 0$).
- To establish where $f(x)$ is decreasing, take the derivative.
$$f'(x) = \frac{\left(\frac{1}{x}\right)(x) - (\ln x)(1)}{x^2} = \frac{1 - \ln x}{x^2}$$
- This is negative for all $x > e$.
- Therefore f is decreasing for all $x > e$.

$$\int_1^{\infty} \frac{\ln x}{x} \mathrm{d}x = \lim_{t\to\infty} \int_1^t \frac{\ln x}{x} \mathrm{d}x = \lim_{t\to\infty} \left[\frac{(\ln x)^2}{2} \right]_1^t$$

Example

Test the series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$ for convergence.

- $f(x) = \frac{\ln x}{x}$ is continuous and positive ($x > 0$).
- To establish where $f(x)$ is decreasing, take the derivative.
$$f'(x) = \frac{\left(\frac{1}{x}\right)(x) - (\ln x)(1)}{x^2} = \frac{1 - \ln x}{x^2}$$
- This is negative for all $x > e$.
- Therefore f is decreasing for all $x > e$.

$$\begin{aligned}\int_1^\infty \frac{\ln x}{x} \mathrm{d}x &= \lim_{t\to\infty} \int_1^t \frac{\ln x}{x} \mathrm{d}x = \lim_{t\to\infty} \left[\frac{(\ln x)^2}{2} \right]_1^t \\ &= \lim_{t\to\infty} \left(\frac{1}{2} (\ln t)^2 - \textbf{?} \right)\end{aligned}$$

Example

Test the series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$ for convergence.

- $f(x) = \frac{\ln x}{x}$ is continuous and positive ($x > 0$).
- To establish where $f(x)$ is decreasing, take the derivative.
$$f'(x) = \frac{\left(\frac{1}{x}\right)(x) - (\ln x)(1)}{x^2} = \frac{1 - \ln x}{x^2}$$
- This is negative for all $x > e$.
- Therefore f is decreasing for all $x > e$.

$$\begin{aligned}\int_1^{\infty} \frac{\ln x}{x} \mathrm{d}x &= \lim_{t\to\infty} \int_1^t \frac{\ln x}{x} \mathrm{d}x = \lim_{t\to\infty} \left[\frac{(\ln x)^2}{2} \right]_1^t \\ &= \lim_{t\to\infty} \left(\frac{1}{2} (\ln t)^2 - \textbf{?} \right)\end{aligned}$$

Example

Test the series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$ for convergence.

- $f(x) = \frac{\ln x}{x}$ is continuous and positive ($x > 0$).
- To establish where $f(x)$ is decreasing, take the derivative.

$$f'(x) = \frac{\left(\frac{1}{x}\right)(x) - (\ln x)(1)}{x^2} = \frac{1 - \ln x}{x^2}$$

- This is negative for all $x > e$.
- Therefore f is decreasing for all $x > e$.

$$\begin{aligned}\int_1^{\infty} \frac{\ln x}{x} \mathrm{d}x &= \lim_{t\to\infty} \int_1^t \frac{\ln x}{x} \mathrm{d}x = \lim_{t\to\infty} \left[\frac{(\ln x)^2}{2} \right]_1^t \\ &= \lim_{t\to\infty} \left(\frac{1}{2}(\ln t)^2 - 0 \right)\end{aligned}$$

Example

Test the series $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ for convergence.

- $f(x) = \frac{\ln x}{x}$ is continuous and positive ($x > 0$).
- To establish where $f(x)$ is decreasing, take the derivative.

$$f'(x) = \frac{\left(\frac{1}{x}\right)(x) - (\ln x)(1)}{x^2} = \frac{1 - \ln x}{x^2}$$

- This is negative for all $x > e$.
- Therefore f is decreasing for all $x > e$.

$$\begin{aligned}\int_1^\infty \frac{\ln x}{x} dx &= \lim_{t\to\infty} \int_1^t \frac{\ln x}{x} dx = \lim_{t\to\infty} \left[\frac{(\ln x)^2}{2}\right]_1^t \\ &= \lim_{t\to\infty} \left(\frac{1}{2}(\ln t)^2 - 0\right) = \textbf{?}\end{aligned}$$

Example

Test the series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$ for convergence.

- $f(x) = \frac{\ln x}{x}$ is continuous and positive ($x > 0$).
- To establish where $f(x)$ is decreasing, take the derivative.
$$f'(x) = \frac{\left(\frac{1}{x}\right)(x) - (\ln x)(1)}{x^2} = \frac{1 - \ln x}{x^2}$$
- This is negative for all $x > e$.
- Therefore f is decreasing for all $x > e$.

$$\begin{aligned}\int_1^{\infty} \frac{\ln x}{x} \mathrm{d}x &= \lim_{t\to\infty} \int_1^t \frac{\ln x}{x} \mathrm{d}x = \lim_{t\to\infty} \left[\frac{(\ln x)^2}{2}\right]_1^t \\ &= \lim_{t\to\infty} \left(\frac{1}{2}(\ln t)^2 - 0\right) = \infty\end{aligned}$$

Example

Test the series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$ for convergence.

- $f(x) = \frac{\ln x}{x}$ is continuous and positive ($x > 0$).
- To establish where $f(x)$ is decreasing, take the derivative.

$$f'(x) = \frac{\left(\frac{1}{x}\right)(x) - (\ln x)(1)}{x^2} = \frac{1 - \ln x}{x^2}$$

- This is negative for all $x > e$.
- Therefore f is decreasing for all $x > e$.

$$\begin{aligned}\int_1^{\infty} \frac{\ln x}{x} dx &= \lim_{t\to\infty} \int_1^{t} \frac{\ln x}{x} dx = \lim_{t\to\infty} \left[\frac{(\ln x)^2}{2}\right]_1^t \\ &= \lim_{t\to\infty} \left(\frac{1}{2}(\ln t)^2 - 0\right) = \infty\end{aligned}$$

Therefore $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ is **?**

Example

Test the series $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ for convergence.

- $f(x) = \frac{\ln x}{x}$ is continuous and positive ($x > 0$).
- To establish where $f(x)$ is decreasing, take the derivative.

$$f'(x) = \frac{\left(\frac{1}{x}\right)(x) - (\ln x)(1)}{x^2} = \frac{1 - \ln x}{x^2}$$

- This is negative for all $x > e$.
- Therefore f is decreasing for all $x > e$.

$$\begin{aligned} \int_1^{\infty} \frac{\ln x}{x} dx &= \lim_{t\to\infty} \int_1^t \frac{\ln x}{x} dx = \lim_{t\to\infty} \left[\frac{(\ln x)^2}{2} \right]_1^t \\ &= \lim_{t\to\infty} \left(\frac{1}{2} (\ln t)^2 - 0 \right) = \infty \end{aligned}$$

Therefore $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ is divergent.

Estimating the Sum of a Series

- Suppose we have already used the Integral Test to show that $\sum a_n$ converges.
- Now we want to find an approximation to the sum of the series.

Estimating the Sum of a Series

- Suppose we have already used the Integral Test to show that $\sum a_n$ converges.
- Now we want to find an approximation to the sum of the series.
- Any partial sum s_n is an approximation. But how good?

Estimating the Sum of a Series

- Suppose we have already used the Integral Test to show that $\sum a_n$ converges.
- Now we want to find an approximation to the sum of the series.
- Any partial sum s_n is an approximation. But how good?
- Estimate the size of the remainder $R_n = s - s_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots$.

Estimating the Sum of a Series

- Suppose we have already used the Integral Test to show that $\sum a_n$ converges.
- Now we want to find an approximation to the sum of the series.
- Any partial sum s_n is an approximation. But how good?
- Estimate the size of the remainder $R_n = s - s_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots$.
- Suppose $f(n) = a_n$. Draw rectangles with heights $a_{n+1}, a_{n+2}, \ldots$.

Estimating the Sum of a Series

- Suppose we have already used the Integral Test to show that $\sum a_n$ converges.
- Now we want to find an approximation to the sum of the series.
- Any partial sum s_n is an approximation. But how good?
- Estimate the size of the remainder $R_n = s - s_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots$.
- Suppose $f(n) = a_n$. Draw rectangles with heights $a_{n+1}, a_{n+2}, \ldots$.
- Use the right endpoints to find the height: then the rectangles are under the curve $y = f(x)$.
- Use the left endpoints to find the height: then the rectangles are above the curve $y = f(x)$.

Estimating the Sum of a Series

- Suppose we have already used the Integral Test to show that $\sum a_n$ converges.
- Now we want to find an approximation to the sum of the series.
- Any partial sum s_n is an approximation. But how good?
- Estimate the size of the remainder $R_n = s - s_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots$.
- Suppose $f(n) = a_n$. Draw rectangles with heights $a_{n+1}, a_{n+2}, \ldots$.
- Use the right endpoints to find the height: then the rectangles are under the curve $y = f(x)$.
- $R_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots \leq \int_n^\infty f(x)\mathrm{d}x$.
- Use the left endpoints to find the height: then the rectangles are above the curve $y = f(x)$.

Estimating the Sum of a Series

- Suppose we have already used the Integral Test to show that $\sum a_n$ converges.
- Now we want to find an approximation to the sum of the series.
- Any partial sum s_n is an approximation. But how good?
- Estimate the size of the remainder $R_n = s - s_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots$.
- Suppose $f(n) = a_n$. Draw rectangles with heights $a_{n+1}, a_{n+2}, \ldots$.
- Use the right endpoints to find the height: then the rectangles are under the curve $y = f(x)$.
- $R_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots \leq \int_n^\infty f(x)\mathrm{d}x$.
- Use the left endpoints to find the height: then the rectangles are above the curve $y = f(x)$.
- $R_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots \geq \int_{n+1}^\infty f(x)\mathrm{d}x$.

Remainder Estimate for the Integral Test

Suppose $f(k) = a_k$, where f is continuous, positive, and decreasing for $x \geq n$, and $\sum a_k$ is convergent with sum s. If $R_n = s - s_n$, then

$$\int_{n+1}^{\infty} f(x)\mathrm{d}x \leq R_n \leq \int_{n}^{\infty} f(x)\mathrm{d}x$$

Example (Example 5, p. 737)

Approximate the sum of $\sum \frac{1}{n^3}$ using the first 10 terms. Estimate the error involved in this approximation. How many terms are required to get an accuracy of 0.0005 or better?

Example (Example 5, p. 737)

Approximate the sum of $\sum \frac{1}{n^3}$ using the first 10 terms. Estimate the error involved in this approximation. How many terms are required to get an accuracy of 0.0005 or better?

$$\int_n^\infty \frac{1}{x^3} \mathrm{d}x =$$

Example (Example 5, p. 737)

Approximate the sum of $\sum \frac{1}{n^3}$ using the first 10 terms. Estimate the error involved in this approximation. How many terms are required to get an accuracy of 0.0005 or better?

$$\int_n^\infty \frac{1}{x^3} \mathrm{d}x = \lim_{t\to\infty} \left[-\frac{1}{2x^2} \right]_n^t$$

Example (Example 5, p. 737)

Approximate the sum of $\sum \frac{1}{n^3}$ using the first 10 terms. Estimate the error involved in this approximation. How many terms are required to get an accuracy of 0.0005 or better?

$$\int_n^\infty \frac{1}{x^3}\mathrm{d}x = \lim_{t\to\infty}\left[-\frac{1}{2x^2}\right]_n^t = \lim_{t\to\infty}\left(-\frac{1}{2t^2}+\frac{1}{2n^2}\right) =$$

Example (Example 5, p. 737)

Approximate the sum of $\sum \frac{1}{n^3}$ using the first 10 terms. Estimate the error involved in this approximation. How many terms are required to get an accuracy of 0.0005 or better?

$$\int_n^\infty \frac{1}{x^3} \mathrm{d}x = \lim_{t\to\infty} \left[-\frac{1}{2x^2} \right]_n^t = \lim_{t\to\infty} \left(-\frac{1}{2t^2} + \frac{1}{2n^2} \right) = \frac{1}{2n^2}$$

Example (Example 5, p. 737)

Approximate the sum of $\sum \frac{1}{n^3}$ using the first 10 terms. Estimate the error involved in this approximation. How many terms are required to get an accuracy of 0.0005 or better?

$$\int_n^{\infty} \frac{1}{x^3} \mathrm{d}x = \lim_{t\to\infty} \left[-\frac{1}{2x^2} \right]_n^t = \lim_{t\to\infty} \left(-\frac{1}{2t^2} + \frac{1}{2n^2} \right) = \frac{1}{2n^2}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^3} \approx s_{10} = \frac{1}{1^3} + \frac{1}{2^3} + \cdots + \frac{1}{10^3} \approx 1.975$$

Example (Example 5, p. 737)

Approximate the sum of $\sum \frac{1}{n^3}$ using the first 10 terms. Estimate the error involved in this approximation. How many terms are required to get an accuracy of 0.0005 or better?

$$\int_n^\infty \frac{1}{x^3}\mathrm{d}x = \lim_{t\to\infty}\left[-\frac{1}{2x^2}\right]_n^t = \lim_{t\to\infty}\left(-\frac{1}{2t^2}+\frac{1}{2n^2}\right) = \frac{1}{2n^2}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^3} \approx s_{10} = \frac{1}{1^3}+\frac{1}{2^3}+\cdots+\frac{1}{10^3} \approx 1.975$$

$$R_{10} \leq \int_{10}^\infty \frac{1}{x^3}\mathrm{d}x =$$

Example (Example 5, p. 737)

Approximate the sum of $\sum \frac{1}{n^3}$ using the first 10 terms. Estimate the error involved in this approximation. How many terms are required to get an accuracy of 0.0005 or better?

$$\int_n^\infty \frac{1}{x^3}dx = \lim_{t\to\infty}\left[-\frac{1}{2x^2}\right]_n^t = \lim_{t\to\infty}\left(-\frac{1}{2t^2}+\frac{1}{2n^2}\right) = \frac{1}{2n^2}$$

$$\sum_{n=1}^\infty \frac{1}{n^3} \approx s_{10} = \frac{1}{1^3}+\frac{1}{2^3}+\cdots+\frac{1}{10^3} \approx 1.975$$

$$R_{10} \leq \int_{10}^\infty \frac{1}{x^3}dx = \frac{1}{2(10)^2}$$

Example (Example 5, p. 737)

Approximate the sum of $\sum \frac{1}{n^3}$ using the first 10 terms. Estimate the error involved in this approximation. How many terms are required to get an accuracy of 0.0005 or better?

$$\int_n^\infty \frac{1}{x^3}\mathrm{d}x = \lim_{t\to\infty}\left[-\frac{1}{2x^2}\right]_n^t = \lim_{t\to\infty}\left(-\frac{1}{2t^2}+\frac{1}{2n^2}\right) = \frac{1}{2n^2}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^3} \approx s_{10} = \frac{1}{1^3}+\frac{1}{2^3}+\cdots+\frac{1}{10^3} \approx 1.975$$

$$R_{10} \leq \int_{10}^\infty \frac{1}{x^3}\mathrm{d}x = \frac{1}{2(10)^2} = \frac{1}{200}$$

Example (Example 5, p. 737)

Approximate the sum of $\sum \frac{1}{n^3}$ using the first 10 terms. Estimate the error involved in this approximation. How many terms are required to get an accuracy of 0.0005 or better?

$$\int_n^\infty \frac{1}{x^3}\mathrm{d}x = \lim_{t\to\infty}\left[-\frac{1}{2x^2}\right]_n^t = \lim_{t\to\infty}\left(-\frac{1}{2t^2}+\frac{1}{2n^2}\right) = \frac{1}{2n^2}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^3} \approx s_{10} = \frac{1}{1^3}+\frac{1}{2^3}+\dots+\frac{1}{10^3} \approx 1.975$$

$$R_{10} \leq \int_{10}^\infty \frac{1}{x^3}\mathrm{d}x = \frac{1}{2(10)^2} = \frac{1}{200}$$

Therefore the error is at most 0.005.

Example (Example 5, p. 737)

Approximate the sum of $\sum \frac{1}{n^3}$ using the first 10 terms. Estimate the error involved in this approximation. How many terms are required to get an accuracy of 0.0005 or better?

$$\int_n^\infty \frac{1}{x^3}\mathrm{d}x = \lim_{t\to\infty}\left[-\frac{1}{2x^2}\right]_n^t = \lim_{t\to\infty}\left(-\frac{1}{2t^2}+\frac{1}{2n^2}\right) = \frac{1}{2n^2}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^3} \approx s_{10} = \frac{1}{1^3}+\frac{1}{2^3}+\cdots+\frac{1}{10^3} \approx 1.975$$

$$R_{10} \leq \int_{10}^\infty \frac{1}{x^3}\mathrm{d}x = \frac{1}{2(10)^2} = \frac{1}{200}$$

Therefore the error is at most 0.005.

To get an accuracy of 0.0005 or better, we want $R_n \leq 0.0005$. Since $R_n \leq \frac{1}{2n^2}$, we want

$$\frac{1}{2n^2} \leq 0.0005, \qquad \text{or} \qquad n \geq \sqrt{1000} \approx 31.6$$

$$\int_{n+1}^{\infty} f(x)\mathrm{d}x \quad \leq \quad R_n \quad \leq \quad \int_{n}^{\infty} f(x)\mathrm{d}x$$

$$\begin{array}{rcccc} \int_{n+1}^{\infty} f(x)\mathrm{d}x & \leq & R_n & \leq & \int_n^{\infty} f(x)\mathrm{d}x \\ s_n + \int_{n+1}^{\infty} f(x)\mathrm{d}x & \leq & s_n + R_n & \leq & s_n + \int_n^{\infty} f(x)\mathrm{d}x \end{array}$$

- Add s_n to both sides of both inequalities.

$$
\begin{array}{rcccl}
\int_{n+1}^{\infty} f(x)\mathrm{d}x & \leq & R_n & \leq & \int_{n}^{\infty} f(x)\mathrm{d}x \\
s_n + \int_{n+1}^{\infty} f(x)\mathrm{d}x & \leq & s_n + R_n & \leq & s_n + \int_{n}^{\infty} f(x)\mathrm{d}x \\
s_n + \int_{n+1}^{\infty} f(x)\mathrm{d}x & \leq & & \leq & s_n + \int_{n}^{\infty} f(x)\mathrm{d}x
\end{array}
$$

- Add s_n to both sides of both inequalities.

$$\begin{array}{rcccl} \int_{n+1}^{\infty} f(x)\mathrm{d}x & \leq & R_n & \leq & \int_{n}^{\infty} f(x)\mathrm{d}x \\ s_n + \int_{n+1}^{\infty} f(x)\mathrm{d}x & \leq & s_n + R_n & \leq & s_n + \int_{n}^{\infty} f(x)\mathrm{d}x \\ s_n + \int_{n+1}^{\infty} f(x)\mathrm{d}x & \leq & s & \leq & s_n + \int_{n}^{\infty} f(x)\mathrm{d}x \end{array}$$

- Add s_n to both sides of both inequalities.

$$\begin{array}{rcccl} \int_{n+1}^{\infty} f(x)\mathrm{d}x & \leq & R_n & \leq & \int_{n}^{\infty} f(x)\mathrm{d}x \\ s_n + \int_{n+1}^{\infty} f(x)\mathrm{d}x & \leq & s_n + R_n & \leq & s_n + \int_{n}^{\infty} f(x)\mathrm{d}x \\ s_n + \int_{n+1}^{\infty} f(x)\mathrm{d}x & \leq & s & \leq & s_n + \int_{n}^{\infty} f(x)\mathrm{d}x \end{array}$$

- Add s_n to both sides of both inequalities.
- This gives upper and lower bounds for s.

$$
\begin{array}{rcccl}
\int_{n+1}^{\infty} f(x)\mathrm{d}x & \leq & R_n & \leq & \int_{n}^{\infty} f(x)\mathrm{d}x \\
s_n + \int_{n+1}^{\infty} f(x)\mathrm{d}x & \leq & s_n + R_n & \leq & s_n + \int_{n}^{\infty} f(x)\mathrm{d}x \\
s_n + \int_{n+1}^{\infty} f(x)\mathrm{d}x & \leq & s & \leq & s_n + \int_{n}^{\infty} f(x)\mathrm{d}x
\end{array}
$$

- Add s_n to both sides of both inequalities.
- This gives upper and lower bounds for s.
- This is a better approximation than just using s_n.

The Comparison Tests

- In the Comparison Tests, the idea is to compare a given series with another series that is known to be convergent or divergent.
- Consider the series $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$.
- This reminds us of the series

The Comparison Tests

- In the Comparison Tests, the idea is to compare a given series with another series that is known to be convergent or divergent.
- Consider the series $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$.
- This reminds us of the series

The Comparison Tests

- In the Comparison Tests, the idea is to compare a given series with another series that is known to be convergent or divergent.
- Consider the series $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$.
- This reminds us of the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$.

The Comparison Tests

- In the Comparison Tests, the idea is to compare a given series with another series that is known to be convergent or divergent.
- Consider the series $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$.
- This reminds us of the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$.
- $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a geometric series with $a =$ and $r =$

The Comparison Tests

- In the Comparison Tests, the idea is to compare a given series with another series that is known to be convergent or divergent.
- Consider the series $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$.
- This reminds us of the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$.
- $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a geometric series with $a = \frac{1}{2}$ and $r =$

The Comparison Tests

- In the Comparison Tests, the idea is to compare a given series with another series that is known to be convergent or divergent.
- Consider the series $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$.
- This reminds us of the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$.
- $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a geometric series with $a = \frac{1}{2}$ and $r =$

The Comparison Tests

- In the Comparison Tests, the idea is to compare a given series with another series that is known to be convergent or divergent.
- Consider the series $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$.
- This reminds us of the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$.
- $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a geometric series with $a = \frac{1}{2}$ and $r = \frac{1}{2}$.

The Comparison Tests

- In the Comparison Tests, the idea is to compare a given series with another series that is known to be convergent or divergent.
- Consider the series $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$.
- This reminds us of the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$.
- $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a geometric series with $a = \frac{1}{2}$ and $r = \frac{1}{2}$.
- Therefore $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is

The Comparison Tests

- In the Comparison Tests, the idea is to compare a given series with another series that is known to be convergent or divergent.
- Consider the series $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$.
- This reminds us of the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$.
- $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a geometric series with $a = \frac{1}{2}$ and $r = \frac{1}{2}$.
- Therefore $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is convergent.

The Comparison Tests

- In the Comparison Tests, the idea is to compare a given series with another series that is known to be convergent or divergent.
- Consider the series $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$.
- This reminds us of the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$.
- $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a geometric series with $a = \frac{1}{2}$ and $r = \frac{1}{2}$.
- Therefore $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is convergent.

$$\frac{1}{2^i + 1} \qquad \frac{1}{2^i}$$

The Comparison Tests

- In the Comparison Tests, the idea is to compare a given series with another series that is known to be convergent or divergent.
- Consider the series $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$.
- This reminds us of the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$.
- $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a geometric series with $a = \frac{1}{2}$ and $r = \frac{1}{2}$.
- Therefore $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is convergent.

$$\frac{1}{2^i + 1} < \frac{1}{2^i}$$

The Comparison Tests

- In the Comparison Tests, the idea is to compare a given series with another series that is known to be convergent or divergent.
- Consider the series $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$.
- This reminds us of the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$.
- $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a geometric series with $a = \frac{1}{2}$ and $r = \frac{1}{2}$.
- Therefore $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is convergent.

$$\frac{1}{2^i+1} < \frac{1}{2^i}$$

$$\sum_{i=1}^{n} \frac{1}{2^i+1} < \sum_{i=1}^{n} \frac{1}{2^i}$$

The Comparison Tests

- In the Comparison Tests, the idea is to compare a given series with another series that is known to be convergent or divergent.
- Consider the series $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$.
- This reminds us of the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$.
- $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a geometric series with $a = \frac{1}{2}$ and $r = \frac{1}{2}$.
- Therefore $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is convergent.

$$\begin{aligned} \frac{1}{2^i+1} &< \frac{1}{2^i} \\ \sum_{i=1}^{n} \frac{1}{2^i+1} &< \sum_{i=1}^{n} \frac{1}{2^i} < \sum_{i=1}^{\infty} \frac{1}{2^i} \end{aligned}$$

The Comparison Tests

- In the Comparison Tests, the idea is to compare a given series with another series that is known to be convergent or divergent.
- Consider the series $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$.
- This reminds us of the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$.
- $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a geometric series with $a = \frac{1}{2}$ and $r = \frac{1}{2}$.
- Therefore $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is convergent.

$$\begin{aligned} \frac{1}{2^i+1} &< \frac{1}{2^i} \\ \sum_{i=1}^{n} \frac{1}{2^i+1} &< \sum_{i=1}^{n} \frac{1}{2^i} < \sum_{i=1}^{\infty} \frac{1}{2^i} = \end{aligned}$$

The Comparison Tests

- In the Comparison Tests, the idea is to compare a given series with another series that is known to be convergent or divergent.
- Consider the series $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$.
- This reminds us of the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$.
- $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a geometric series with $a = \frac{1}{2}$ and $r = \frac{1}{2}$.
- Therefore $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is convergent.

$$
\begin{aligned}
\frac{1}{2^i+1} &< \frac{1}{2^i} \\
\sum_{i=1}^{n} \frac{1}{2^i+1} &< \sum_{i=1}^{n} \frac{1}{2^i} < \sum_{i=1}^{\infty} \frac{1}{2^i} = 1
\end{aligned}
$$

The Comparison Tests

- In the Comparison Tests, the idea is to compare a given series with another series that is known to be convergent or divergent.
- Consider the series $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$.
- This reminds us of the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$.
- $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a geometric series with $a = \frac{1}{2}$ and $r = \frac{1}{2}$.
- Therefore $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is convergent.

$$\begin{aligned} \frac{1}{2^i+1} &< \frac{1}{2^i} \\ \sum_{i=1}^{n} \frac{1}{2^i+1} &< \sum_{i=1}^{n} \frac{1}{2^i} < \sum_{i=1}^{\infty} \frac{1}{2^i} = 1 \end{aligned}$$

- The partial sums of $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$ are increasing and are bounded above by 1.

The Comparison Tests

- In the Comparison Tests, the idea is to compare a given series with another series that is known to be convergent or divergent.
- Consider the series $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$.
- This reminds us of the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$.
- $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a geometric series with $a = \frac{1}{2}$ and $r = \frac{1}{2}$.
- Therefore $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is convergent.

$$\begin{aligned} \frac{1}{2^i+1} &< \frac{1}{2^i} \\ \sum_{i=1}^{n} \frac{1}{2^i+1} &< \sum_{i=1}^{n} \frac{1}{2^i} < \sum_{i=1}^{\infty} \frac{1}{2^i} = 1 \end{aligned}$$

- The partial sums of $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$ are increasing and are bounded above by 1.
- Therefore $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$ is

The Comparison Tests

- In the Comparison Tests, the idea is to compare a given series with another series that is known to be convergent or divergent.
- Consider the series $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$.
- This reminds us of the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$.
- $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a geometric series with $a = \frac{1}{2}$ and $r = \frac{1}{2}$.
- Therefore $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is convergent.

$$\begin{aligned} \frac{1}{2^i+1} &< \frac{1}{2^i} \\ \sum_{i=1}^{n} \frac{1}{2^i+1} &< \sum_{i=1}^{n} \frac{1}{2^i} < \sum_{i=1}^{\infty} \frac{1}{2^i} = 1 \end{aligned}$$

- The partial sums of $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$ are increasing and are bounded above by 1.
- Therefore $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$ is convergent.

Theorem (The Comparison Test)

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

1. *If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is also convergent.*
2. *If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum a_n$ is also divergent.*

Theorem (The Comparison Test)

Suppose that $\sum a_n$ *and* $\sum b_n$ *are series with positive terms.*

1. *If* $\sum b_n$ *is convergent and* $a_n \leq b_n$ *for all* n*, then* $\sum a_n$ *is also convergent.*
2. *If* $\sum b_n$ *is divergent and* $a_n \geq b_n$ *for all* n*, then* $\sum a_n$ *is also divergent.*

When we use the Comparison Test, we need to have some series $\sum b_n$ that we know in order to make a comparison. Usually $\sum b_n$ is one of

- A p-series
- A geometric series

Theorem (The Comparison Test)

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

1. *If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is also convergent.*
2. *If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum a_n$ is also divergent.*

When we use the Comparison Test, we need to have some series $\sum b_n$ that we know in order to make a comparison. Usually $\sum b_n$ is one of

- A p-series ($\sum \frac{1}{n^p}$ converges if and diverges if)
- A geometric series

Theorem (The Comparison Test)

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

1. *If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is also convergent.*
2. *If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum a_n$ is also divergent.*

When we use the Comparison Test, we need to have some series $\sum b_n$ that we know in order to make a comparison. Usually $\sum b_n$ is one of

- A p-series ($\sum \frac{1}{n^p}$ converges if $p > 1$ and diverges if)
- A geometric series

Theorem (The Comparison Test)

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

1. *If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is also convergent.*
2. *If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum a_n$ is also divergent.*

When we use the Comparison Test, we need to have some series $\sum b_n$ that we know in order to make a comparison. Usually $\sum b_n$ is one of

- A p-series ($\sum \frac{1}{n^p}$ converges if $p > 1$ and diverges if)
- A geometric series

Theorem (The Comparison Test)

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

1. *If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is also convergent.*
2. *If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum a_n$ is also divergent.*

When we use the Comparison Test, we need to have some series $\sum b_n$ that we know in order to make a comparison. Usually $\sum b_n$ is one of

- A p-series ($\sum \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$)
- A geometric series

Theorem (The Comparison Test)

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

1. *If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is also convergent.*
2. *If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum a_n$ is also divergent.*

When we use the Comparison Test, we need to have some series $\sum b_n$ that we know in order to make a comparison. Usually $\sum b_n$ is one of

- A p-series ($\sum \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$)
- A geometric series ($\sum ar^{n-1}$ converges if and diverges if)

Theorem (The Comparison Test)

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

1. *If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is also convergent.*
2. *If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum a_n$ is also divergent.*

When we use the Comparison Test, we need to have some series $\sum b_n$ that we know in order to make a comparison. Usually $\sum b_n$ is one of

- A p-series ($\sum \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$)
- A geometric series ($\sum ar^{n-1}$ converges if $|r| < 1$ and diverges if)

Theorem (The Comparison Test)

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

1. *If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is also convergent.*
2. *If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum a_n$ is also divergent.*

When we use the Comparison Test, we need to have some series $\sum b_n$ that we know in order to make a comparison. Usually $\sum b_n$ is one of

- A p-series ($\sum \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$)
- A geometric series ($\sum ar^{n-1}$ converges if $|r| < 1$ and diverges if)

Theorem (The Comparison Test)

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

1. *If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is also convergent.*
2. *If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum a_n$ is also divergent.*

When we use the Comparison Test, we need to have some series $\sum b_n$ that we know in order to make a comparison. Usually $\sum b_n$ is one of

- A p-series ($\sum \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$)
- A geometric series ($\sum ar^{n-1}$ converges if $|r| < 1$ and diverges if $|r| \geq 1$)

Example

Determine if $\sum_{n=1}^{\infty} \frac{5}{2n^2+7n+3}$ converges or diverges.

Example

Determine if $\sum_{n=1}^{\infty} \frac{5}{2n^2+7n+3}$ converges or diverges.

- As $n \to \infty$, the dominant term in the denominator is $2n^2$, so compare with $\frac{5}{2n^2}$.

$$\frac{5}{2n^2+7n+3} \qquad \frac{5}{2n^2}$$

Example

Determine if $\sum_{n=1}^{\infty} \frac{5}{2n^2+7n+3}$ converges or diverges.

- As $n \to \infty$, the dominant term in the denominator is $2n^2$, so compare with $\frac{5}{2n^2}$.

$$\frac{5}{2n^2+7n+3} < \frac{5}{2n^2}$$

Example

Determine if $\sum_{n=1}^{\infty} \frac{5}{2n^2+7n+3}$ converges or diverges.

- As $n \to \infty$, the dominant term in the denominator is $2n^2$, so compare with $\frac{5}{2n^2}$.

$$\frac{5}{2n^2+7n+3} < \frac{5}{2n^2}$$

$$\sum_{n=1}^{\infty} \frac{5}{2n^2} = \frac{5}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Example

Determine if $\sum_{n=1}^{\infty} \frac{5}{2n^2+7n+3}$ converges or diverges.

- As $n \to \infty$, the dominant term in the denominator is $2n^2$, so compare with $\frac{5}{2n^2}$.

$$\frac{5}{2n^2+7n+3} < \frac{5}{2n^2}$$

$$\sum_{n=1}^{\infty} \frac{5}{2n^2} = \frac{5}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

- This is a constant times a p-series with $p =$

Example

Determine if $\sum_{n=1}^{\infty} \frac{5}{2n^2+7n+3}$ converges or diverges.

- As $n \to \infty$, the dominant term in the denominator is $2n^2$, so compare with $\frac{5}{2n^2}$.

$$\frac{5}{2n^2+7n+3} < \frac{5}{2n^2}$$

$$\sum_{n=1}^{\infty} \frac{5}{2n^2} = \frac{5}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

- This is a constant times a p-series with $p = 2 > 1$.

Example

Determine if $\sum_{n=1}^{\infty} \frac{5}{2n^2+7n+3}$ converges or diverges.

- As $n \to \infty$, the dominant term in the denominator is $2n^2$, so compare with $\frac{5}{2n^2}$.

$$\frac{5}{2n^2+7n+3} < \frac{5}{2n^2}$$

$$\sum_{n=1}^{\infty} \frac{5}{2n^2} = \frac{5}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

- This is a constant times a p-series with $p = 2 > 1$.
- Therefore $\sum_{n=1}^{\infty} \frac{5}{2n^2}$ is

Example

Determine if $\sum_{n=1}^{\infty} \frac{5}{2n^2+7n+3}$ converges or diverges.

- As $n \to \infty$, the dominant term in the denominator is $2n^2$, so compare with $\frac{5}{2n^2}$.

$$\frac{5}{2n^2+7n+3} < \frac{5}{2n^2}$$

$$\sum_{n=1}^{\infty} \frac{5}{2n^2} = \frac{5}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

- This is a constant times a p-series with $p = 2 > 1$.
- Therefore $\sum_{n=1}^{\infty} \frac{5}{2n^2}$ is convergent.

Example

Determine if $\sum_{n=1}^{\infty} \frac{5}{2n^2+7n+3}$ converges or diverges.

- As $n \to \infty$, the dominant term in the denominator is $2n^2$, so compare with $\frac{5}{2n^2}$.

$$\frac{5}{2n^2+7n+3} < \frac{5}{2n^2}$$

$$\sum_{n=1}^{\infty} \frac{5}{2n^2} = \frac{5}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

- This is a constant times a p-series with $p = 2 > 1$.
- Therefore $\sum_{n=1}^{\infty} \frac{5}{2n^2}$ is convergent.
- Therefore $\sum_{n=1}^{\infty} \frac{5}{2n^2+7n+3}$ is by the Comparison Test.

Example

Determine if $\sum_{n=1}^{\infty} \frac{5}{2n^2+7n+3}$ converges or diverges.

- As $n \to \infty$, the dominant term in the denominator is $2n^2$, so compare with $\frac{5}{2n^2}$.

$$\frac{5}{2n^2+7n+3} < \frac{5}{2n^2}$$

$$\sum_{n=1}^{\infty} \frac{5}{2n^2} = \frac{5}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

- This is a constant times a p-series with $p = 2 > 1$.
- Therefore $\sum_{n=1}^{\infty} \frac{5}{2n^2}$ is convergent.
- Therefore $\sum_{n=1}^{\infty} \frac{5}{2n^2+7n+3}$ is convergent by the Comparison Test.

Example

Determine if $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ converges or diverges.

Example

Determine if $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ converges or diverges.

- We could use the Integral Test to find this.

Example

Determine if $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ converges or diverges.

- We could use the Integral Test to find this.
- The Comparison Test is even easier.

Example

Determine if $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ converges or diverges.

- We could use the Integral Test to find this.
- The Comparison Test is even easier.

$$\frac{\ln n}{n} \qquad \frac{1}{n} \qquad \text{if } n \geq 3$$

Example

Determine if $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ converges or diverges.

- We could use the Integral Test to find this.
- The Comparison Test is even easier.

$$\frac{\ln n}{n} > \frac{1}{n} \qquad \text{if } n \geq 3$$

Example

Determine if $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ converges or diverges.

- We could use the Integral Test to find this.
- The Comparison Test is even easier.

$$\frac{\ln n}{n} > \frac{1}{n} \qquad \text{if } n \geq 3$$

- $\sum_{n=1}^{\infty} \frac{1}{n}$ is a p-series with $p =$

Example

Determine if $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ converges or diverges.

- We could use the Integral Test to find this.
- The Comparison Test is even easier.

$$\frac{\ln n}{n} > \frac{1}{n} \qquad \text{if } n \geq 3$$

- $\sum_{n=1}^{\infty} \frac{1}{n}$ is a p-series with $p = 1$.

Example

Determine if $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ converges or diverges.

- We could use the Integral Test to find this.
- The Comparison Test is even easier.

$$\frac{\ln n}{n} > \frac{1}{n} \qquad \text{if } n \geq 3$$

- $\sum_{n=1}^{\infty} \frac{1}{n}$ is a p-series with $p = 1$.
- Therefore $\sum_{n=1}^{\infty} \frac{1}{n}$ is

Example

Determine if $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ converges or diverges.

- We could use the Integral Test to find this.
- The Comparison Test is even easier.

$$\frac{\ln n}{n} > \frac{1}{n} \qquad \text{if } n \geq 3$$

- $\sum_{n=1}^{\infty} \frac{1}{n}$ is a p-series with $p = 1$.
- Therefore $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Example

Determine if $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ converges or diverges.

- We could use the Integral Test to find this.
- The Comparison Test is even easier.

$$\frac{\ln n}{n} > \frac{1}{n} \qquad \text{if } n \geq 3$$

- $\sum_{n=1}^{\infty} \frac{1}{n}$ is a p-series with $p = 1$.
- Therefore $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.
- Therefore $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ is by the Comparison Test.

Example

Determine if $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ converges or diverges.

- We could use the Integral Test to find this.
- The Comparison Test is even easier.

$$\frac{\ln n}{n} > \frac{1}{n} \qquad \text{if } n \geq 3$$

- $\sum_{n=1}^{\infty} \frac{1}{n}$ is a p-series with $p = 1$.
- Therefore $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.
- Therefore $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ is divergent by the Comparison Test.

In order to use the comparison test to see if $\sum a_n$ is convergent or divergent, we need the terms a_n to be

1. smaller than the terms of a convergent series, or
2. bigger than the terms of a divergent series.

In order to use the comparison test to see if $\sum a_n$ is convergent or divergent, we need the terms a_n to be

1. smaller than the terms of a convergent series, or
2. bigger than the terms of a divergent series.

If the terms a_n are

1. bigger than the terms of a convergent series, or
2. smaller than the terms of a divergent series,

then the Comparison Test gives no information.

In order to use the comparison test to see if $\sum a_n$ is convergent or divergent, we need the terms a_n to be

1. smaller than the terms of a convergent series, or
2. bigger than the terms of a divergent series.

If the terms a_n are

1. bigger than the terms of a convergent series, or
2. smaller than the terms of a divergent series,

then the Comparison Test gives no information.

- Consider the series $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$.

In order to use the comparison test to see if $\sum a_n$ is convergent or divergent, we need the terms a_n to be

1. smaller than the terms of a convergent series, or
2. bigger than the terms of a divergent series.

If the terms a_n are

1. bigger than the terms of a convergent series, or
2. smaller than the terms of a divergent series,

then the Comparison Test gives no information.

- Consider the series $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$.

$$\frac{1}{2^n-1} \qquad \frac{1}{2^n}$$

In order to use the comparison test to see if $\sum a_n$ is convergent or divergent, we need the terms a_n to be

1. smaller than the terms of a convergent series, or
2. bigger than the terms of a divergent series.

If the terms a_n are

1. bigger than the terms of a convergent series, or
2. smaller than the terms of a divergent series,

then the Comparison Test gives no information.

- Consider the series $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$.

$$\frac{1}{2^n - 1} > \frac{1}{2^n}$$

In order to use the comparison test to see if $\sum a_n$ is convergent or divergent, we need the terms a_n to be

1. smaller than the terms of a convergent series, or
2. bigger than the terms of a divergent series.

If the terms a_n are

1. bigger than the terms of a convergent series, or
2. smaller than the terms of a divergent series,

then the Comparison Test gives no information.

- Consider the series $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$.

$$\frac{1}{2^n - 1} > \frac{1}{2^n}$$

- The Comparison Test tells us nothing here.

In order to use the comparison test to see if $\sum a_n$ is convergent or divergent, we need the terms a_n to be

1. smaller than the terms of a convergent series, or
2. bigger than the terms of a divergent series.

If the terms a_n are

1. bigger than the terms of a convergent series, or
2. smaller than the terms of a divergent series,

then the Comparison Test gives no information.

- Consider the series $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$.

$$\frac{1}{2^n-1} > \frac{1}{2^n}$$

- The Comparison Test tells us nothing here.
- Nevertheless, we think $\sum \frac{1}{2^n-1}$ should converge, because it's so close to $\sum \frac{1}{2^n}$.

Theorem (The Limit Comparison Test)

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both series diverge.

Theorem (The Limit Comparison Test)

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both series diverge.

The main thing to check is that c is finite and non-zero.

Example

Test the series $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ for convergence or divergence.

Example

Test the series $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ for convergence or divergence.
Use the Limit Comparison Test with

$$a_n = \frac{1}{2^n - 1}, \qquad b_n = \frac{1}{2^n}$$

Example

Test the series $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ for convergence or divergence.
Use the Limit Comparison Test with

$$a_n = \frac{1}{2^n - 1}, \qquad b_n = \frac{1}{2^n}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} \quad = \quad \lim_{n\to\infty} \frac{\frac{1}{2^n-1}}{\frac{1}{2^n}}$$

Example

Test the series $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ for convergence or divergence.
Use the Limit Comparison Test with

$$a_n = \frac{1}{2^n - 1}, \qquad b_n = \frac{1}{2^n}$$

$$\begin{aligned}\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{\frac{1}{2^n-1}}{\frac{1}{2^n}} \\ &= \lim_{n\to\infty} \frac{2^n}{2^n - 1}\end{aligned}$$

Example

Test the series $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ for convergence or divergence.
Use the Limit Comparison Test with

$$a_n = \frac{1}{2^n-1}, \qquad b_n = \frac{1}{2^n}$$

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{\frac{1}{2^n-1}}{\frac{1}{2^n}} \\
&= \lim_{n\to\infty} \frac{2^n}{2^n-1} \cdot \frac{\frac{1}{2^n}}{\frac{1}{2^n}}
\end{aligned}$$

Example

Test the series $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ for convergence or divergence.
Use the Limit Comparison Test with

$$a_n = \frac{1}{2^n - 1}, \qquad b_n = \frac{1}{2^n}$$

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{\frac{1}{2^n-1}}{\frac{1}{2^n}} \\
&= \lim_{n\to\infty} \frac{2^n}{2^n - 1} \cdot \frac{\frac{1}{2^n}}{\frac{1}{2^n}} \\
&= \lim_{n\to\infty} \frac{1}{1 - \frac{1}{2^n}}
\end{aligned}$$

Example

Test the series $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ for convergence or divergence.
Use the Limit Comparison Test with

$$a_n = \frac{1}{2^n - 1}, \qquad b_n = \frac{1}{2^n}$$

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{\frac{1}{2^n-1}}{\frac{1}{2^n}} \\
&= \lim_{n\to\infty} \frac{2^n}{2^n - 1} \cdot \frac{\frac{1}{2^n}}{\frac{1}{2^n}} \\
&= \lim_{n\to\infty} \frac{1}{1 - \frac{1}{2^n}} = 1 > 0
\end{aligned}$$

Example

Test the series $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ for convergence or divergence.
Use the Limit Comparison Test with

$$a_n = \frac{1}{2^n - 1}, \qquad b_n = \frac{1}{2^n}$$

$$\begin{aligned} \lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{\frac{1}{2^n-1}}{\frac{1}{2^n}} \\ &= \lim_{n\to\infty} \frac{2^n}{2^n-1} \cdot \frac{\frac{1}{2^n}}{\frac{1}{2^n}} \\ &= \lim_{n\to\infty} \frac{1}{1-\frac{1}{2^n}} = 1 > 0 \end{aligned}$$

- $\sum \frac{1}{2^n}$ is a geometric series.

Example

Test the series $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ for convergence or divergence.
Use the Limit Comparison Test with

$$a_n = \frac{1}{2^n - 1}, \qquad b_n = \frac{1}{2^n}$$

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{\frac{1}{2^n-1}}{\frac{1}{2^n}} \\
&= \lim_{n\to\infty} \frac{2^n}{2^n - 1} \cdot \frac{\frac{1}{2^n}}{\frac{1}{2^n}} \\
&= \lim_{n\to\infty} \frac{1}{1 - \frac{1}{2^n}} = 1 > 0
\end{aligned}$$

- $\sum \frac{1}{2^n}$ is a convergent geometric series.

Example

Test the series $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ for convergence or divergence.
Use the Limit Comparison Test with

$$a_n = \frac{1}{2^n - 1}, \qquad b_n = \frac{1}{2^n}$$

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{\frac{1}{2^n-1}}{\frac{1}{2^n}} \\
&= \lim_{n\to\infty} \frac{2^n}{2^n-1} \cdot \frac{\frac{1}{2^n}}{\frac{1}{2^n}} \\
&= \lim_{n\to\infty} \frac{1}{1-\frac{1}{2^n}} = 1 > 0
\end{aligned}$$

- $\sum \frac{1}{2^n}$ is a convergent geometric series.
- By the Limit Comparison Test $\sum \frac{1}{2^n-1}$ is convergent too.

Example

Test the series $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{7+n^5}}$ for convergence or divergence.

Example

Test the series $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{7+n^5}}$ for convergence or divergence.

- The dominant part of the numerator is and the dominant part of the denominator is

$$a_n = \frac{2n^2+3n}{\sqrt{7+n^5}}, \qquad b_n = \frac{\quad}{\quad}$$

Example

Test the series $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{7+n^5}}$ for convergence or divergence.

- The dominant part of the numerator is $2n^2$ and the dominant part of the denominator is

$$a_n = \frac{2n^2+3n}{\sqrt{7+n^5}}, \qquad b_n = \frac{2n^2}{}$$

Example

Test the series $\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{7+n^5}}$ for convergence or divergence.

- The dominant part of the numerator is $2n^2$ and the dominant part of the denominator is

$$a_n = \frac{2n^2+3n}{\sqrt{7+n^5}}, \qquad b_n = \frac{2n^2}{}$$

Example

Test the series $\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{7+n^5}}$ for convergence or divergence.

- The dominant part of the numerator is $2n^2$ and the dominant part of the denominator is $\sqrt{n^5} = n^{5/2}$.

$$a_n = \frac{2n^2+3n}{\sqrt{7+n^5}}, \qquad b_n = \frac{2n^2}{n^{5/2}}$$

Example

Test the series $\sum_{n=1}^{\infty} \frac{2n^2 + 3n}{\sqrt{7 + n^5}}$ for convergence or divergence.

- The dominant part of the numerator is $2n^2$ and the dominant part of the denominator is $\sqrt{n^5} = n^{5/2}$.

$$a_n = \frac{2n^2 + 3n}{\sqrt{7 + n^5}}, \qquad b_n = \frac{2n^2}{n^{5/2}} = \frac{2}{n^{1/2}}$$

Example

Test the series $\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{7+n^5}}$ for convergence or divergence.

- The dominant part of the numerator is $2n^2$ and the dominant part of the denominator is $\sqrt{n^5} = n^{5/2}$.

$$\begin{aligned} a_n &= \frac{2n^2+3n}{\sqrt{7+n^5}}, \qquad b_n = \frac{2n^2}{n^{5/2}} = \frac{2}{n^{1/2}} \\ \lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{2n^2+3n}{\sqrt{7+n^5}} \cdot \frac{n^{1/2}}{2} \end{aligned}$$

Example

Test the series $\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{7+n^5}}$ for convergence or divergence.

- The dominant part of the numerator is $2n^2$ and the dominant part of the denominator is $\sqrt{n^5} = n^{5/2}$.

$$\begin{aligned} a_n &= \frac{2n^2+3n}{\sqrt{7+n^5}}, \qquad b_n = \frac{2n^2}{n^{5/2}} = \frac{2}{n^{1/2}} \\ \lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{2n^2+3n}{\sqrt{7+n^5}} \cdot \frac{n^{1/2}}{2} = \lim_{n\to\infty} \frac{2n^{5/2}+3n^{3/2}}{2\sqrt{7+n^5}} \end{aligned}$$

Example

Test the series $\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{7+n^5}}$ for convergence or divergence.

- The dominant part of the numerator is $2n^2$ and the dominant part of the denominator is $\sqrt{n^5} = n^{5/2}$.

$$\begin{aligned} a_n &= \frac{2n^2+3n}{\sqrt{7+n^5}}, \qquad b_n = \frac{2n^2}{n^{5/2}} = \frac{2}{n^{1/2}} \\ \lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{2n^2+3n}{\sqrt{7+n^5}} \cdot \frac{n^{1/2}}{2} = \lim_{n\to\infty} \frac{2n^{5/2}+3n^{3/2}}{2\sqrt{7+n^5}} \frac{\frac{1}{n^{5/2}}}{\frac{1}{n^{5/2}}} \end{aligned}$$

Example

Test the series $\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{7+n^5}}$ for convergence or divergence.

- The dominant part of the numerator is $2n^2$ and the dominant part of the denominator is $\sqrt{n^5} = n^{5/2}$.

$$
\begin{aligned}
a_n &= \frac{2n^2+3n}{\sqrt{7+n^5}}, \qquad b_n = \frac{2n^2}{n^{5/2}} = \frac{2}{n^{1/2}} \\
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{2n^2+3n}{\sqrt{7+n^5}} \cdot \frac{n^{1/2}}{2} = \lim_{n\to\infty} \frac{2n^{5/2}+3n^{3/2}}{2\sqrt{7+n^5}} \frac{\frac{1}{n^{5/2}}}{\frac{1}{n^{5/2}}} \\
&= \lim_{n\to\infty} \frac{2+\frac{3}{n}}{2\sqrt{\frac{7}{n^5}+1}}
\end{aligned}
$$

Example

Test the series $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{7+n^5}}$ for convergence or divergence.

- The dominant part of the numerator is $2n^2$ and the dominant part of the denominator is $\sqrt{n^5} = n^{5/2}$.

$$
\begin{aligned}
a_n &= \frac{2n^2+3n}{\sqrt{7+n^5}}, \qquad b_n = \frac{2n^2}{n^{5/2}} = \frac{2}{n^{1/2}} \\
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{2n^2+3n}{\sqrt{7+n^5}} \cdot \frac{n^{1/2}}{2} = \lim_{n\to\infty} \frac{2n^{5/2}+3n^{3/2}}{2\sqrt{7+n^5}} \frac{\frac{1}{n^{5/2}}}{\frac{1}{n^{5/2}}} \\
&= \lim_{n\to\infty} \frac{2+\frac{3}{n}}{2\sqrt{\frac{7}{n^5}+1}} = 1 > 0
\end{aligned}
$$

Example

Test the series $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{7+n^5}}$ for convergence or divergence.

- The dominant part of the numerator is $2n^2$ and the dominant part of the denominator is $\sqrt{n^5} = n^{5/2}$.

$$\begin{aligned}
a_n &= \frac{2n^2+3n}{\sqrt{7+n^5}}, \qquad b_n = \frac{2n^2}{n^{5/2}} = \frac{2}{n^{1/2}} \\
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{2n^2+3n}{\sqrt{7+n^5}} \cdot \frac{n^{1/2}}{2} = \lim_{n\to\infty} \frac{2n^{5/2}+3n^{3/2}}{2\sqrt{7+n^5}} \frac{\frac{1}{n^{5/2}}}{\frac{1}{n^{5/2}}} \\
&= \lim_{n\to\infty} \frac{2+\frac{3}{n}}{2\sqrt{\frac{7}{n^5}+1}} = 1 > 0
\end{aligned}$$

- $\sum \frac{2}{n^{\frac{1}{2}}}$ is a constant multiple of a p-series with $p =$

Example

Test the series $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{7+n^5}}$ for convergence or divergence.

- The dominant part of the numerator is $2n^2$ and the dominant part of the denominator is $\sqrt{n^5} = n^{5/2}$.

$$\begin{aligned}
a_n &= \frac{2n^2+3n}{\sqrt{7+n^5}}, \qquad b_n = \frac{2n^2}{n^{5/2}} = \frac{2}{n^{1/2}} \\
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{2n^2+3n}{\sqrt{7+n^5}} \cdot \frac{n^{1/2}}{2} = \lim_{n\to\infty} \frac{2n^{5/2}+3n^{3/2}}{2\sqrt{7+n^5}} \frac{\frac{1}{n^{5/2}}}{\frac{1}{n^{5/2}}} \\
&= \lim_{n\to\infty} \frac{2+\frac{3}{n}}{2\sqrt{\frac{7}{n^5}+1}} = 1 > 0
\end{aligned}$$

- $\sum \frac{2}{n^{\frac{1}{2}}}$ is a constant multiple of a p-series with $p = \frac{1}{2}$.

Example

Test the series $\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{7+n^5}}$ for convergence or divergence.

- The dominant part of the numerator is $2n^2$ and the dominant part of the denominator is $\sqrt{n^5} = n^{5/2}$.

$$
\begin{aligned}
a_n &= \frac{2n^2+3n}{\sqrt{7+n^5}}, \qquad b_n = \frac{2n^2}{n^{5/2}} = \frac{2}{n^{1/2}} \\
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{2n^2+3n}{\sqrt{7+n^5}} \cdot \frac{n^{1/2}}{2} = \lim_{n\to\infty} \frac{2n^{5/2}+3n^{3/2}}{2\sqrt{7+n^5}} \frac{\frac{1}{n^{5/2}}}{\frac{1}{n^{5/2}}} \\
&= \lim_{n\to\infty} \frac{2+\frac{3}{n}}{2\sqrt{\frac{7}{n^5}+1}} = 1 > 0
\end{aligned}
$$

- $\sum \frac{2}{n^{\frac{1}{2}}}$ is a constant multiple of a p-series with $p = \frac{1}{2}$.
- Therefore $\sum \frac{2}{n^{\frac{1}{2}}}$ is

Example

Test the series $\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{7+n^5}}$ for convergence or divergence.

- The dominant part of the numerator is $2n^2$ and the dominant part of the denominator is $\sqrt{n^5} = n^{5/2}$.

$$\begin{aligned} a_n &= \frac{2n^2+3n}{\sqrt{7+n^5}}, \qquad b_n = \frac{2n^2}{n^{5/2}} = \frac{2}{n^{1/2}} \\ \lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{2n^2+3n}{\sqrt{7+n^5}} \cdot \frac{n^{1/2}}{2} = \lim_{n\to\infty} \frac{2n^{5/2}+3n^{3/2}}{2\sqrt{7+n^5}} \frac{\frac{1}{n^{5/2}}}{\frac{1}{n^{5/2}}} \\ &= \lim_{n\to\infty} \frac{2+\frac{3}{n}}{2\sqrt{\frac{7}{n^5}+1}} = 1 > 0 \end{aligned}$$

- $\sum \frac{2}{n^{\frac{1}{2}}}$ is a constant multiple of a p-series with $p = \frac{1}{2}$.
- Therefore $\sum \frac{2}{n^{\frac{1}{2}}}$ is divergent

Example

Test the series $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{7+n^5}}$ for convergence or divergence.

- The dominant part of the numerator is $2n^2$ and the dominant part of the denominator is $\sqrt{n^5} = n^{5/2}$.

$$
\begin{aligned}
a_n &= \frac{2n^2+3n}{\sqrt{7+n^5}}, \qquad b_n = \frac{2n^2}{n^{5/2}} = \frac{2}{n^{1/2}} \\
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{2n^2+3n}{\sqrt{7+n^5}} \cdot \frac{n^{1/2}}{2} = \lim_{n\to\infty} \frac{2n^{5/2}+3n^{3/2}}{2\sqrt{7+n^5}} \frac{\frac{1}{n^{5/2}}}{\frac{1}{n^{5/2}}} \\
&= \lim_{n\to\infty} \frac{2+\frac{3}{n}}{2\sqrt{\frac{7}{n^5}+1}} = 1 > 0
\end{aligned}
$$

- $\sum \frac{2}{n^{\frac{1}{2}}}$ is a constant multiple of a p-series with $p = \frac{1}{2}$.
- Therefore $\sum \frac{2}{n^{\frac{1}{2}}}$ is divergent, and so is $\sum \frac{2n^2+3n}{\sqrt{7+n^5}}$.