Calculus II
Integrals of the form $\displaystyle\int \frac{Ax+B}{ax^2+bx+c}\mathrm{d}x$, denominator has no real roots

Todor Milev

2019

Linear substitutions leading to blocks IIa and IIIa

Building block IIa: $\int \frac{x}{1+x^2} \mathrm{d}x = \frac{1}{2}\ln(1+x^2) + C$.
Building block IIIa: $\int \frac{1}{1+x^2} \mathrm{d}x = \arctan x + C$.

- Let $ax^2 + bx + c$ have no real roots.
- We can find p, q so that the linear substitution $u = px + q$ transforms the quadratic to:
$$ax^2 + bx + c = r(u^2 + 1)$$
(where r is some number to be determined).
- To find p, q, we complete the square.
- In this way, integrals of the form $\displaystyle\int \frac{Ax + B}{ax^2 + bx + c} \mathrm{d}x$ are transformed to combinations of building blocks IIa and IIIa.
- We show examples; the general case is analogous and we leave it to the student.

Linear substitutions leading to blocks IIa and IIIa

Building block IIa: $\int \frac{x}{1+x^2} \mathrm{d}x = \frac{1}{2}\ln(1+x^2) + C$.
Building block IIIa: $\int \frac{1}{1+x^2} \mathrm{d}x = \arctan x + C$.

Example

No real roots $\Rightarrow$ complete the square. Let $u = x + \frac{1}{2}$, let $z = \frac{2u}{\sqrt{3}}$.

$$\begin{aligned}
\int \frac{x}{x^2+x+1}\mathrm{d}x &= \int \frac{x}{x^2 + 2\cdot\frac{1}{2}x + \frac{1}{4} - \frac{1}{4} + 1}\mathrm{d}x \\
&= \int \frac{x + \frac{1}{2} - \frac{1}{2}}{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}}\mathrm{d}\left(x + \frac{1}{2}\right) \\
&= \int \frac{u - \frac{1}{2}}{u^2 + \frac{3}{4}}\mathrm{d}u \\
&= \int \frac{u}{u^2+\frac{3}{4}}\mathrm{d}u - \frac{1}{2}\int \frac{1}{u^2+\frac{3}{4}}\mathrm{d}u
\end{aligned}$$

Linear substitutions leading to blocks IIa and IIIa

Building block IIa: $\int \frac{x}{1+x^2} \mathrm{d}x = \frac{1}{2}\ln(1+x^2) + C$.
Building block IIIa: $\int \frac{1}{1+x^2} \mathrm{d}x = \arctan x + C$.

Example

No real roots $\Rightarrow$ complete the square. Let $u = x + \frac{1}{2}$, let $z = \frac{2u}{\sqrt{3}}$.

$$\int \frac{x}{x^2+x+1}\mathrm{d}x = \int \frac{u}{u^2+\frac{3}{4}}\mathrm{d}u - \frac{1}{2}\int \frac{1}{u^2+\frac{3}{4}}\mathrm{d}u$$

$$\begin{aligned}\int \frac{1}{u^2+\frac{3}{4}}\mathrm{d}u &= \int \frac{1}{\frac{3}{4}\left(\frac{4}{3}u^2+1\right)}\mathrm{d}u \\ &= \int \frac{1}{\frac{3}{4}\left(\left(\frac{2u}{\sqrt{3}}\right)^2+1\right)}\frac{\sqrt{3}}{2}\mathrm{d}\left(\frac{2u}{\sqrt{3}}\right) \\ &= \frac{2\sqrt{3}}{3}\int \frac{1}{z^2+1}\mathrm{d}z = \frac{2\sqrt{3}}{3}\arctan z + C\end{aligned}$$

Linear substitutions leading to blocks IIa and IIIa

Building block IIa: $\int \frac{x}{1+x^2}\mathrm{d}x = \frac{1}{2}\ln(1+x^2) + C$.
Building block IIIa: $\int \frac{1}{1+x^2}\mathrm{d}x = \arctan x + C$.

Example

No real roots $\Rightarrow$ complete the square. Let $u = x + \frac{1}{2}$, let $z = \frac{2u}{\sqrt{3}}$.

$$\begin{aligned}\int \frac{x}{x^2+x+1}\mathrm{d}x &= \int \frac{u}{u^2+\frac{3}{4}}\mathrm{d}u - \frac{1}{2}\int \frac{1}{u^2+\frac{3}{4}}\mathrm{d}u\\ &= \int \frac{u}{u^2+\frac{3}{4}}\mathrm{d}u - \frac{1}{2}\frac{2\sqrt{3}}{3}\arctan z + C\end{aligned}$$

$$\begin{aligned}\int \frac{u}{u^2+\frac{3}{4}}\mathrm{d}u &= \int \frac{1}{u^2+\frac{3}{4}}\mathrm{d}\left(\frac{u^2}{2}\right)\\ &= \frac{1}{2}\int \frac{1}{u^2+\frac{3}{4}}\mathrm{d}\left(u^2+\frac{3}{4}\right) = \frac{1}{2}\ln\left(u^2+\frac{3}{4}\right) + C\end{aligned}$$

Linear substitutions leading to blocks IIa and IIIa

Building block IIa: $\int \frac{x}{1+x^2}\mathrm{d}x = \frac{1}{2}\ln(1+x^2) + C$.
Building block IIIa: $\int \frac{1}{1+x^2}\mathrm{d}x = \arctan x + C$.

Example

No real roots $\Rightarrow$ complete the square. Let $u = x + \frac{1}{2}$, let $z = \frac{2u}{\sqrt{3}}$.

$$\begin{aligned}
\int \frac{x}{x^2+x+1}\mathrm{d}x &= \int \frac{u}{u^2+\frac{3}{4}}\mathrm{d}u - \frac{1}{2}\int \frac{1}{u^2+\frac{3}{4}}\mathrm{d}u\\
&= \frac{1}{2}\ln\left(u^2+\frac{3}{4}\right) - \frac{1}{2}\frac{2\sqrt{3}}{3}\arctan z + C\\
&= \frac{1}{2}\ln\left(\left(x+\frac{1}{2}\right)^2+\frac{3}{4}\right) - \frac{\sqrt{3}}{3}\arctan\left(\frac{2u}{\sqrt{3}}\right) + C\\
&= \frac{1}{2}\ln\left(x^2+x+1\right) - \frac{\sqrt{3}}{3}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + C
\end{aligned}$$