Calculus I
Limits involving infinity

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Outline

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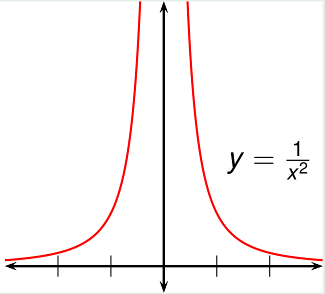
Infinite Limits

Example

Find $\lim_{x \to 0} \frac{1}{x^2}$ if it exists.



x	$\frac{1}{x^2}$
± 1	1
± 0.5	4
± 0.2	25
± 0.1	100
± 0.05	400
± 0.01	10,000
± 0.001	1,000,000

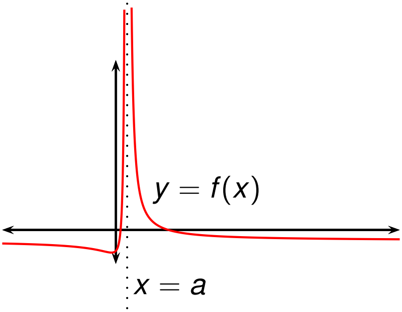
- As x gets close to 0, so does x^2, so $\frac{1}{x^2}$ gets large.
- $\frac{1}{x^2}$ can be made arbitrarily large by taking x close enough to 0.
- $f(x)$ doesn't approach a number, so $\lim_{x \to 0} \frac{1}{x^2}$ doesn't exist.

Definition (Infinite Limit)

Let f be a function defined on both sides of a, except perhaps at a. Then

$$\lim_{x \to a} f(x) = \infty$$

means the values of $f(x)$ can be made arbitrarily large by taking x sufficiently close to a, but not equal to a.



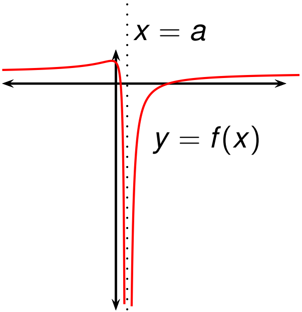
- Other notation: $f(x) \to \infty$ as $x \to a$.
- In such cases, the limit does not exist.
- ∞ is not a number. The notation $\lim_{x \to a} f(x) = \infty$ expresses the particular way in which the limit doesn't exist.

Definition (Infinite Limit)

Let f be a function defined on both sides of a, except perhaps at a. Then

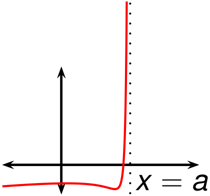
$$\lim_{x \to a} f(x) = -\infty$$

means the values of $f(x)$ can be made arbitrarily negative by taking x sufficiently close to a, but not equal to a.

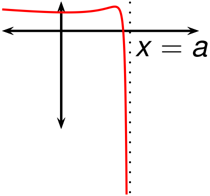


- Here, by "arbitrarily negative" we mean the number is negative with large absolute value.
- In such cases, the limit does not exist.
- $-\infty$ is not a number. The notation $\lim_{x \to a} f(x) = -\infty$ expresses the particular way in which the limit doesn't exist.

There are similar definitions for one-sided limits:

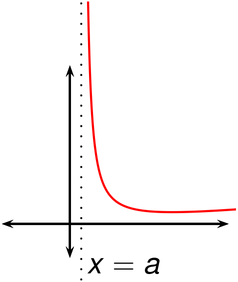


$$\lim_{x \to a^-} f(x) = \infty$$

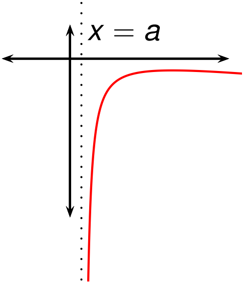


$$\lim_{x \to a^-} f(x) = -\infty$$

$x \to a^-$ means we only consider $x < a$.



$$\lim_{x \to a^+} f(x) = \infty$$



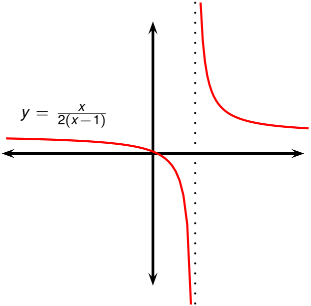
$$\lim_{x \to a^+} f(x) = -\infty$$

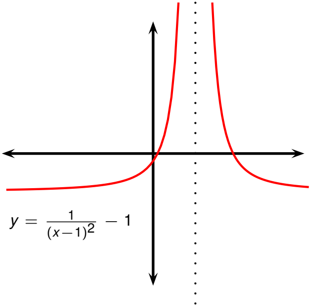
$x \to a^+$ means we only consider $x > a$.

Definition (Vertical Asymptote)

The line $x = a$ is called a vertical asymptote of the curve $y = f(x)$ if at least one of the following statements is true:

$$\lim_{x\to a} f(x) = \infty \qquad \lim_{x\to a^-} f(x) = \infty \qquad \lim_{x\to a^+} f(x) = \infty$$
$$\lim_{x\to a} f(x) = -\infty \qquad \lim_{x\to a^-} f(x) = -\infty \qquad \lim_{x\to a^+} f(x) = -\infty$$



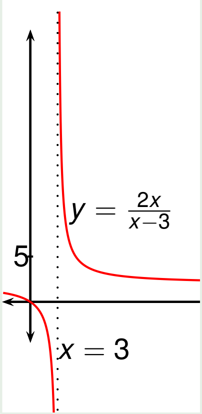


Example

Find $\lim\limits_{x\to 3^+} \frac{2x}{x-3}$ and $\lim\limits_{x\to 3^-} \frac{2x}{x-3}$.

$\lim\limits_{x\to 3^+} \frac{2x}{x-3} = \infty$.

$\lim\limits_{x\to 3^-} \frac{2x}{x-3} = -\infty$.



- If x is near 3 but larger than 3, the denominator $x - 3$ is a small positive number and $2x$ is close to 6.
- So the quotient $\frac{2x}{x-3}$ is a large positive number.
- If x is near 3 but smaller than 3, the denominator $x - 3$ is a negative number with small absolute value and $2x$ is close to 6.
- So $\frac{2x}{x-3}$ is a negative number with large absolute value.
- $x = 3$ is a vertical asymptote for $f(x) = \frac{2x}{x-3}$.

$$\lim_{x \to a} f(x)$$

If we plug in *a* and get

$$f(a) = \frac{\text{something different from } 0}{0},$$

then the limit will be DNE, ∞, or $-\infty$.
To determine what the answer is, this is what we do:

1. Factor.
2. Determine if each factor is positive or negative.
3. An odd number of negative factors means the limit is $-\infty$.
4. An even number of negative factors means the limit is ∞.
5. For a two-sided limit, the answer is DNE unless the left limit and the right limit are either both ∞ or both $-\infty$.

Example (Infinite Limit)

$$\text{Find} \quad \lim_{x\to 1^+} \frac{x^2-3x}{x^2-3x+2}$$

$$\text{Plug in 1:} \quad \frac{(1)^2-3(1)}{(1)^2-3(1)+2} = \frac{-2}{0}$$

The numerator is non-zero and the denominator is zero. Therefore the answer is DNE, ∞, or $-\infty$.

$$\begin{aligned}\text{Factor:} \quad \lim_{x\to 1^+} \frac{x^2-3x}{x^2-3x+2} &= \lim_{x\to 1^+} \frac{x(x-3)}{(x-2)(x-1)}\\ &\to \frac{(\,+\,)(\,-\,)}{(\,-\,)(\,+\,)}\\ &= (+)\end{aligned}$$

$$\text{Therefore} \quad \lim_{x\to 1^+} \frac{x^2-3x}{x^2-3x+2} =$$

Example (Infinite Limit)

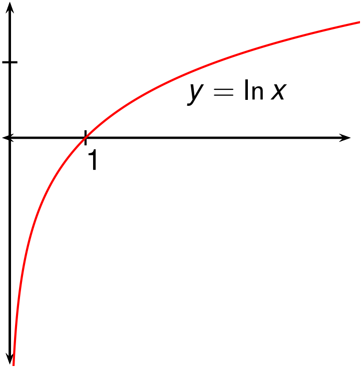
$$\text{Find} \quad \lim_{x\to -1} \frac{x^2+5x+6}{x^3+2x^2+x}$$

$$\text{Plug in } -1\text{:} \quad \frac{(-1)^2+5(-1)+6}{(-1)^3+2(-1)^2+(-1)} = \frac{2}{0}$$

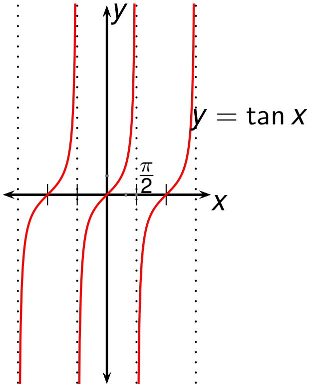
The numerator is non-zero and the denominator is zero. Therefore the answer is DNE, ∞, or $-\infty$.

$$\begin{aligned}
\text{Factor:} \quad \lim_{x\to -1} \frac{x^2+5x+6}{x^3+2x^2+x} &= \lim_{x\to -1} \frac{(x+2)(x+3)}{x(x+1)^2} \\
&\to \frac{(\ +\ )(\ +\ )}{(\ -\ )(\ +\ )} \\
&= (-)
\end{aligned}$$

$$\text{Therefore} \quad \lim_{x\to -1} \frac{x^2+5x+6}{x^3+2x^2+x} =$$



$$\lim_{x \to 0^+} \ln x = -\infty$$

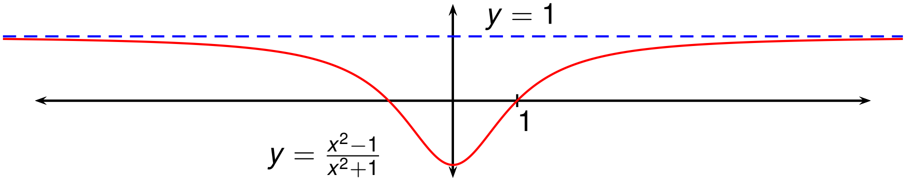


$$\lim_{x \to \frac{\pi}{2}^+} \tan x = -\infty$$

$$\lim_{x \to \frac{\pi}{2}^-} \tan x = \infty$$

$$\lim_{x \to \frac{\pi}{2}} \tan x = \text{DNE}$$

Limits at Infinity; Horizontal Asymptotes



x	$f(x)$
0	-1
± 1	0
± 2	0.600000
± 3	0.800000
± 4	0.882353
± 5	0.923077
± 10	0.980198

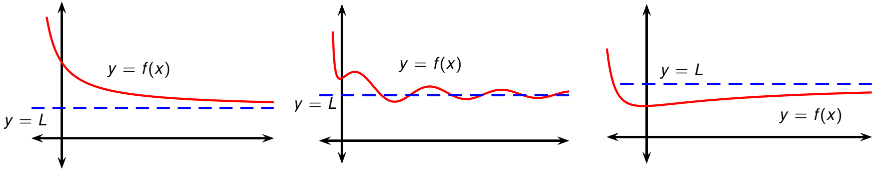
- Consider $f(x) = \frac{x^2-1}{x^2+1}$ as x becomes large.
- The values of $f(x)$ get closer and closer to 1.
- We express this by writing $\lim\limits_{x\to\infty} f(x) = 1$.
- When x is very negative, $f(x)$ is also near 1.
- We express this by writing $\lim\limits_{x\to-\infty} f(x) = 1$.

Definition (Limit at Infinity)

Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x\to\infty} f(x) = L$$

means that the values of f can be made arbitrarily close to L by taking x sufficiently large.



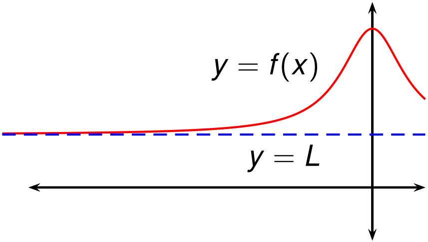
- There are many ways that this can happen.
- Other notation: $f(x) \to L$ as $x \to \infty$.
- ∞ is not a number.

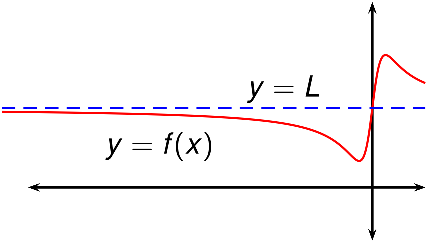
Definition (Limit at Minus Infinity)

Let f be a function defined on some interval $(-\infty, b)$. Then

$$\lim_{x \to -\infty} f(x) = L$$

means that the values of f can be made arbitrarily close to L by taking x sufficiently negative.



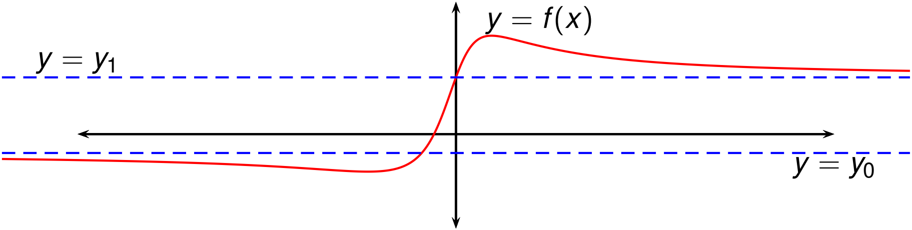


Definition (Horizontal Asymptote)

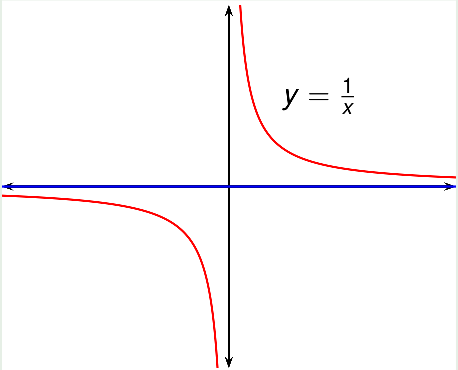
The line $y = L$ is called a horizontal asymptote of f if either

$$\lim_{x\to\infty} f(x) = L \qquad \text{or} \qquad \lim_{x\to-\infty} f(x) = L.$$

- For example, $y = 1$ is a horizontal asymptote for $f(x) = \frac{x^2-1}{x^2+1}$.
- Can a function have two horizontal asymptotes? Yes.



Example



$$\frac{1}{100} = 0.01, \qquad \frac{1}{10,000} = 0.0001$$
$$\frac{1}{1,000,000} = 0.000001$$

Find $\lim_{x\to\infty} \frac{1}{x}$ and $\lim_{x\to-\infty} \frac{1}{x}$.

- When x is large, $\frac{1}{x}$ is small.
- By taking x large enough, we can make $\frac{1}{x}$ as small as we like.
- Therefore $\lim_{x\to\infty} \frac{1}{x} = 0$.
- Similarly, $\lim_{x\to-\infty} \frac{1}{x} = 0$.
- $y = 0$ (the x-axis) is a horizontal asymptote for the curve $y = \frac{1}{x}$.

We can generalize the previous example to other powers of x:

Theorem (Infinite Limits of $\frac{1}{x^r}$)

If $r > 0$ is a rational number, then

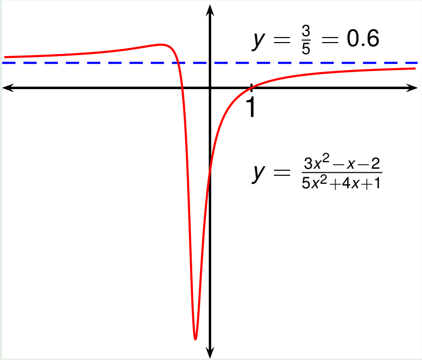
$$\lim_{x \to \infty} \frac{1}{x^r} = 0.$$

If $r > 0$ is a rational number such that x^r is defined for all x, then

$$\lim_{x \to -\infty} \frac{1}{x^r} = 0.$$

Example

Evaluate $\lim_{x\to\infty} \frac{3x^2-x-2}{5x^2+4x+1}$.



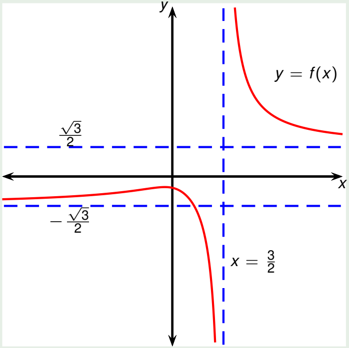
A similar calculation shows that the limit as $x \to -\infty$ is also $\frac{3}{5}$.

Standard approach: divide top and bottom by the highest power of x in the denominator.

$$
\begin{aligned}
& \lim_{x\to\infty} \frac{(3x^2-x-2)}{(5x^2+4x+1)} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} \\
= & \lim_{x\to\infty} \frac{3-\frac{1}{x}-\frac{2}{x^2}}{5+\frac{4}{x}+\frac{1}{x^2}} \\
= & \frac{\lim\limits_{x\to\infty} 3 - \lim\limits_{x\to\infty}\frac{1}{x} - 2\lim\limits_{x\to\infty}\frac{1}{x^2}}{\lim\limits_{x\to\infty} 5 + 4\lim\limits_{x\to\infty}\frac{1}{x} + \lim\limits_{x\to\infty}\frac{1}{x^2}} \\
= & \frac{3-0-0}{5+0+0} = \frac{3}{5}
\end{aligned}
$$

Example

Find the horizontal and vertical asymptotes of $f(x)=\frac{\sqrt{3x^2+1}}{2x-3}$.



If $x>0$ then $x=\sqrt{x^2}$.
If $x<0$ then $x=-\sqrt{x^2}$.
Vertical Asymptote:
$x=\frac{3}{2}$.

$$\lim_{x\to\infty}\frac{\sqrt{3x^2+1}}{2x-3}\cdot\frac{\frac{1}{x}}{\frac{1}{x}}=\lim_{x\to\infty}\frac{\sqrt{3x^2+1}}{2x-3}\cdot\frac{\frac{1}{\sqrt{x^2}}}{\frac{1}{x}}$$

$$=\lim_{x\to\infty}\frac{\sqrt{3+\frac{1}{x^2}}}{2-\frac{3}{x}}=\frac{\sqrt{\lim\limits_{x\to\infty}3+\lim\limits_{x\to\infty}\frac{1}{x^2}}}{\lim\limits_{x\to\infty}2-3\lim\limits_{x\to\infty}\frac{1}{x}}$$

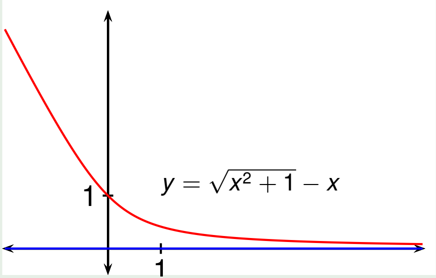
$$=\frac{\sqrt{3+0}}{2-0}=\frac{\sqrt{3}}{2}$$

$$\lim_{x\to-\infty}\frac{\sqrt{3x^2+1}}{2x-3}\cdot\frac{\frac{1}{x}}{\frac{1}{x}}=\lim_{x\to-\infty}\frac{\sqrt{3x^2+1}}{2x-3}\cdot\frac{\frac{-1}{\sqrt{x^2}}}{\frac{1}{x}}$$

$$=\lim_{x\to-\infty}-\frac{\sqrt{3+\frac{1}{x^2}}}{2-\frac{3}{x}}=-\frac{\sqrt{3}}{2}$$

Example

Evaluate $\lim\limits_{x\to\infty} \sqrt{x^2+1} - x$.



- $\sqrt{x^2+1} \to \infty$ and $x \to \infty$ as $x \to \infty$.
- It isn't clear what happens to the difference.
- Divide top & bottom by x.

- Standard approach: multiply top and bottom by $\pm$conjugate radical.

$$\lim_{x\to\infty} \left(\sqrt{x^2+1} - x\right) \frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x}$$
$$= \lim_{x\to\infty} \frac{(\cancel{x^2}+1) - \cancel{x^2}}{\sqrt{x^2+1}+x}$$
$$= \lim_{x\to\infty} \frac{1}{\left(\sqrt{x^2+1}+x\right)} \cdot \frac{\frac{1}{x}}{\frac{1}{x}}$$
$$= \lim_{x\to\infty} \frac{\frac{1}{x}}{\sqrt{1+\frac{1}{x^2}}+1}$$
$$= \frac{0}{\sqrt{1+0}+1} = 0$$

Infinite Limits at Infinity

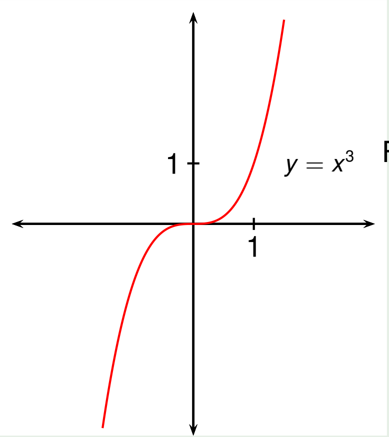
We write

$$\lim_{x \to \infty} f(x) = \infty$$

to mean that $f(x)$ becomes large as x becomes large. We attach similar meaning to

$$\lim_{x \to \infty} f(x) = -\infty, \qquad \lim_{x \to -\infty} f(x) = \infty, \qquad \lim_{x \to -\infty} = -\infty$$

Example



Find $\lim_{x\to\infty} x^3$ and $\lim_{x\to-\infty} x^3$.

- When x is large, so is x^3.
- By taking x large enough, we can make x^3 arbitrarily large.
- Therefore $\lim_{x\to\infty} x^3 = \infty$.
- Similarly, $\lim_{x\to-\infty} x^3 = -\infty$.

$10^3 = 1000, \qquad 100^3 = 1,000,000,$
$1000^3 = 1,000,000,000$

Example

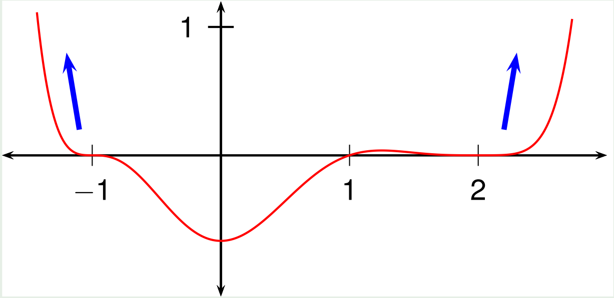
Find $\lim_{x\to\infty}(x^2 - x)$.

- WRONG: $\lim_{x\to\infty}(x^2 - x) = \lim_{x\to\infty} x^2 - \lim_{x\to\infty} x = \infty - \infty = 0$.
- The limit laws don't apply here as the limits on the right don't exist (recall: limits equal to ∞ don't exist).
- Furthermore arithmetics with ∞ is not allowed: ∞ isn't a number.
- Instead: $\lim_{x\to\infty}(x^2 - x) = \lim_{x\to\infty} x(x-1) = \infty$.
- This is because x and $x - 1$ both become arbitrarily large as $x \to \infty$.

Example

Find the limits as $x \to \infty$ and $x \to -\infty$ of

$$y = \frac{1}{24}(x-2)^4(x+1)^3(x-1).$$



$$\lim_{x\to\infty} \frac{1}{24}\underset{(+)}{(x-2)^4}\,\underset{(+)}{(x+1)^3}\,\underset{(+)}{(x-1)} = \infty$$

$$\lim_{x\to-\infty} \frac{1}{24}\underset{(+)}{(x-2)^4}\,\underset{(-)}{(x+1)^3}\,\underset{(-)}{(x-1)} = \infty$$