Calculus I
Derivative of $(a(x))^{b(x)}$

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Example

Differentiate $x^{\tan x}$, where $x > 0$.

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{\tan x}\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\left(e^{\ln x}\right)^{\tan x}\right) && \text{Convert base to } e^{?} \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{(\ln x)\tan x}\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{u}\right) && \text{Set } (\ln x)\tan x = u \\
&= \frac{\mathrm{d}}{\mathrm{d}u}\left(e^{u}\right)\frac{\mathrm{d}u}{\mathrm{d}x} && \text{Chain rule} \\
&= e^{u}\frac{\mathrm{d}}{\mathrm{d}x}\left((\ln x)\tan x\right) \\
&= e^{(\ln x)\tan x}\left((\ln x)'\tan x + (\ln x)(\tan x)'\right) && \text{Prod. rule} \\
&= x^{\tan x}\left(\frac{1}{x}\tan x + (\ln x)\sec^2 x\right)
\end{aligned}
$$

Example

Differentiate $(3x+1)^{\ln x}$, where $3x+1>0$.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left((3x+1)^{\ln x}\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\left(e^{\ln(3x+1)}\right)^{\ln x}\right) && \Big|\ \text{Convert base to } e^{?}\\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{\ln(3x+1)\ln x}\right)\\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{u}\right) = \frac{\mathrm{d}}{\mathrm{d}u}\left(e^{u}\right)\frac{\mathrm{d}u}{\mathrm{d}x} && \Big|\ \text{Set } \ln(3x+1)\ln x = u\\
&= e^{u}\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln(3x+1)\ln x\right)\\
&= e^{\ln(3x+1)\ln x}\left((\ln(3x+1))'\ln x+\ln(3x+1)(\ln x)'\right)\\
&= (3x+1)^{\ln x}\left(\frac{(3x+1)'}{3x+1}\ln x+\ln(3x+1)\frac{1}{x}\right)\\
&= (3x+1)^{\ln x}\left(\frac{3\ln x}{3x+1}+\ln(3x+1)\frac{1}{x}\right)
\end{aligned}$$

Example

Differentiate $(3x+1)^{\ln x}$, where $3x+1>0$.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left((3x+1)^{\ln x}\right) = (3x+1)^{\ln x}\left(\frac{3\ln x}{3x+1} + \ln(3x+1)\frac{1}{x}\right)$$

Theorem

$$\frac{\mathrm{d}}{\mathrm{d}x}\left((a(x))^{b(x)}\right) = (a(x))^{b(x)}\left(\frac{a'(x)}{a(x)}b(x) + \ln(a(x))b'(x)\right), \quad a(x)>0$$