

Calculus II

Integral of rational function with cubic denominator, part 1

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2019

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- We show how to find A_1, A_2, \dots, A_k on examples.