

Precalculus

Express $\sin(kx)$, $\cos(kx)$ via $\sin x$, $\cos x$ using Euler's formula

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2019

Example

Express $\sin(3x)$ and $\cos(3x)$ via $\cos x$ and $\sin x$.

- Recall Euler's formula: $e^{i\alpha} = \cos \alpha + i \sin \alpha$.

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$$\begin{aligned} & \cos(3x) + i \sin(3x) \\ &= e^{3ix} \end{aligned}$$

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- Recall Euler's formula: $e^{i\alpha} = \cos \alpha + i \sin \alpha$.
- Recall the formula: $(a + b)^3 = ?$

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The real parts of the starting and final expression must be equal;
therefore:

$$\cos(3x) = \cos^3 x - 3\cos x \sin^2 x$$

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The real parts of the starting and final expression must be equal; likewise the imaginary parts must be equal; therefore:

$$\begin{aligned}
 \cos(3x) &= \cos^3 x - 3\cos x \sin^2 x \\
 \sin(3x) &= 3\cos^2 x \sin x - \sin^3 x
 \end{aligned}$$