Calculus I
Trigonometric derivatives

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2019

Outline

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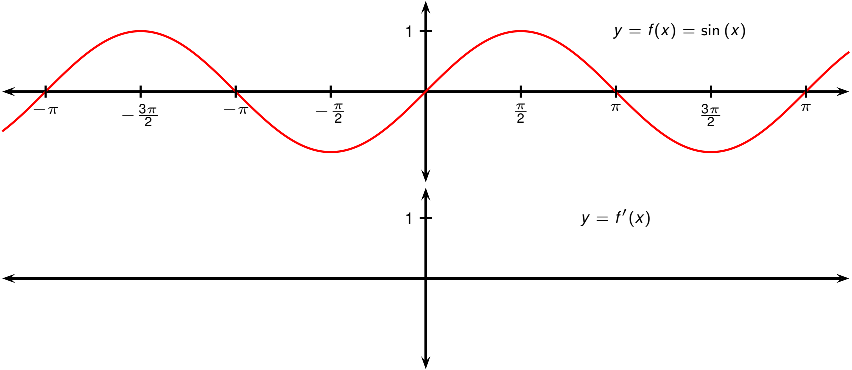
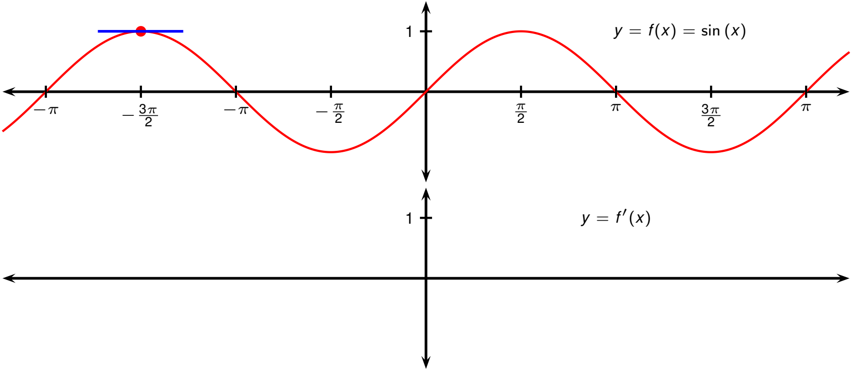
Derivatives of Trigonometric Functions



What is the derivative of $f(x) = \sin x$?

Derivatives of Trigonometric Functions



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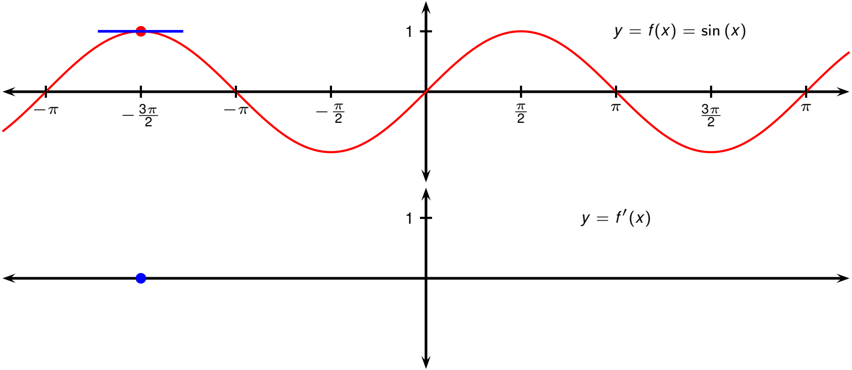

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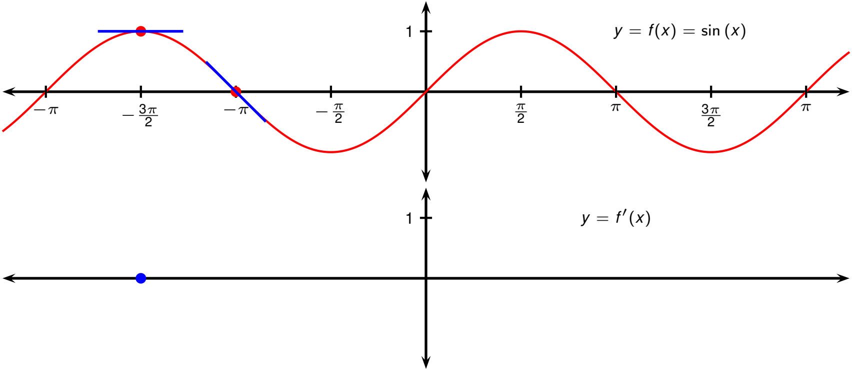

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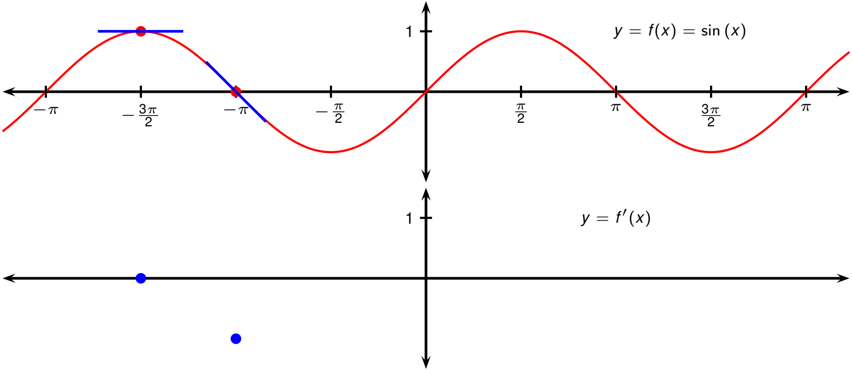

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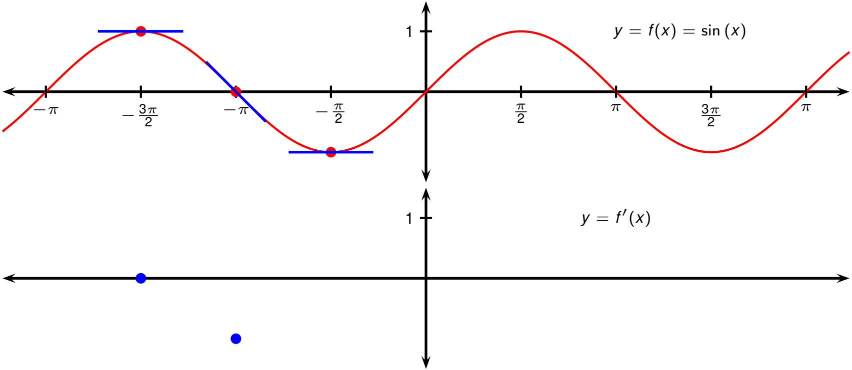

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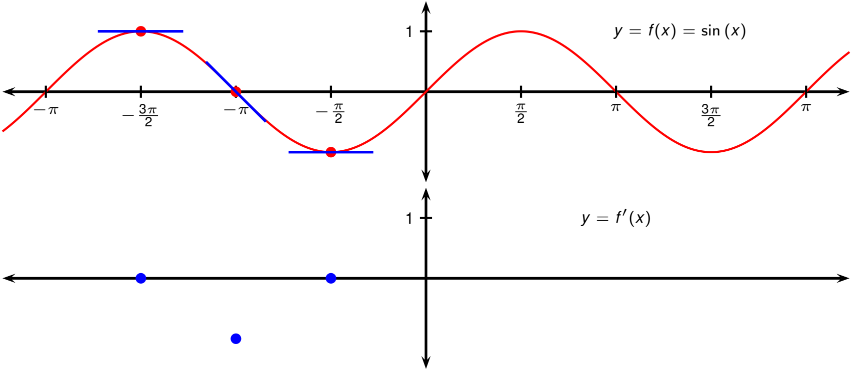

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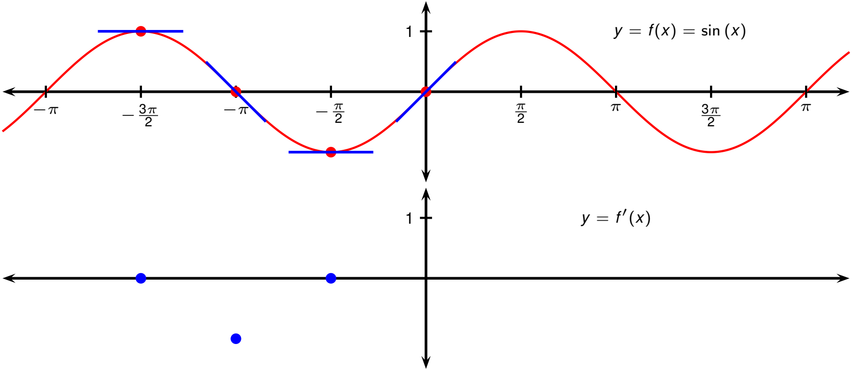

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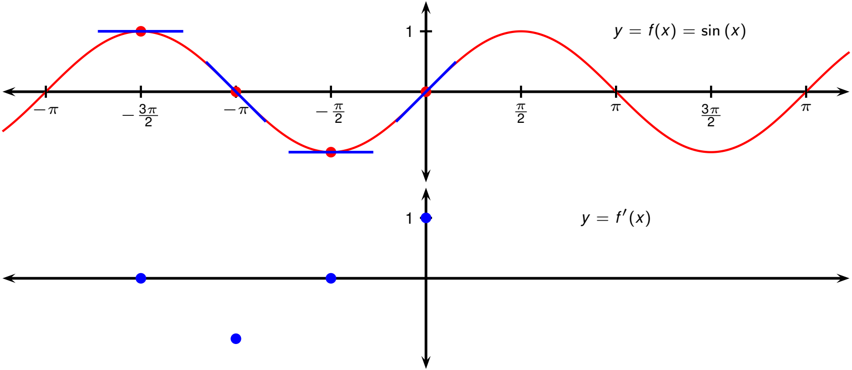

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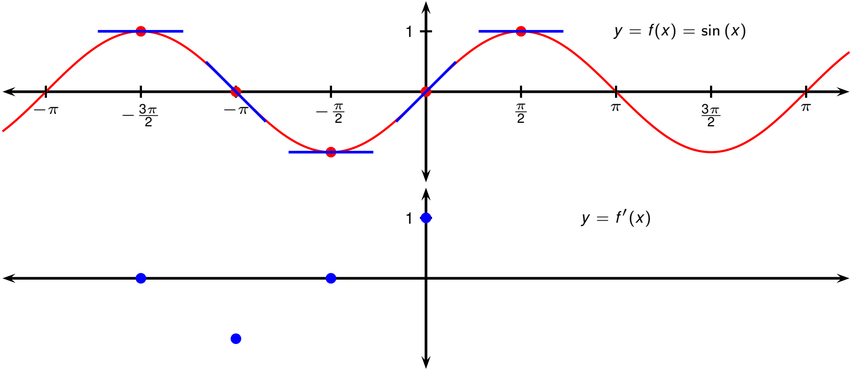

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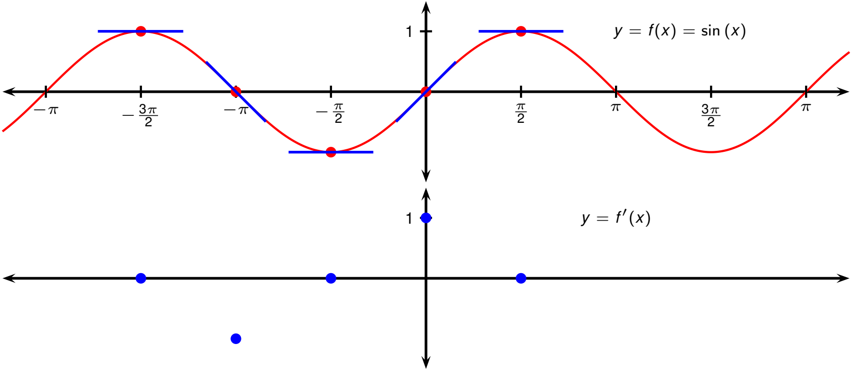

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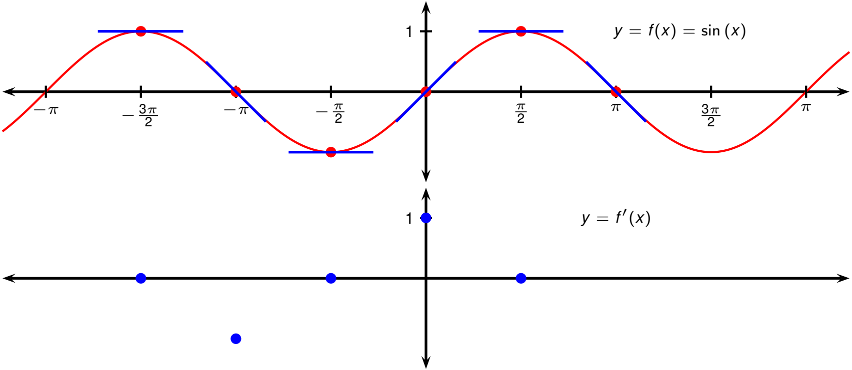

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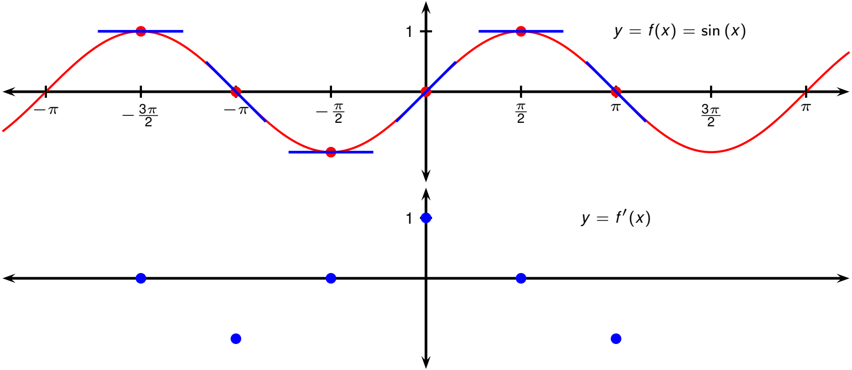

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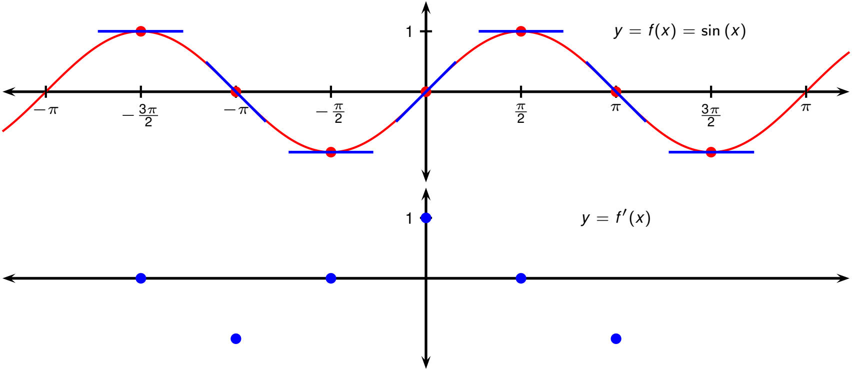

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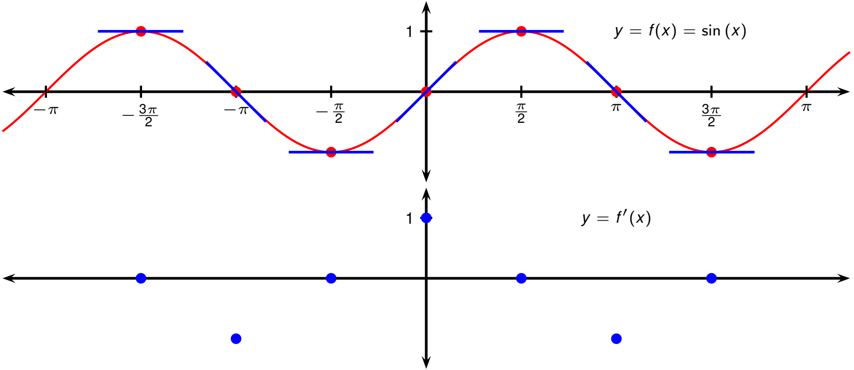

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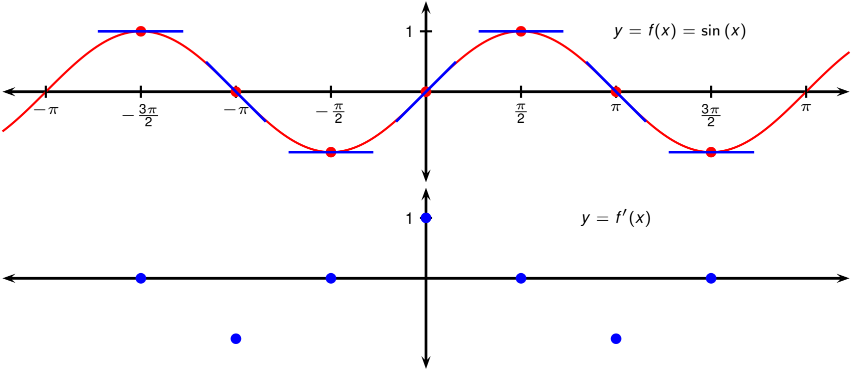

What is the derivative of $f(x) = \sin x$? It looks like $\cos x$.

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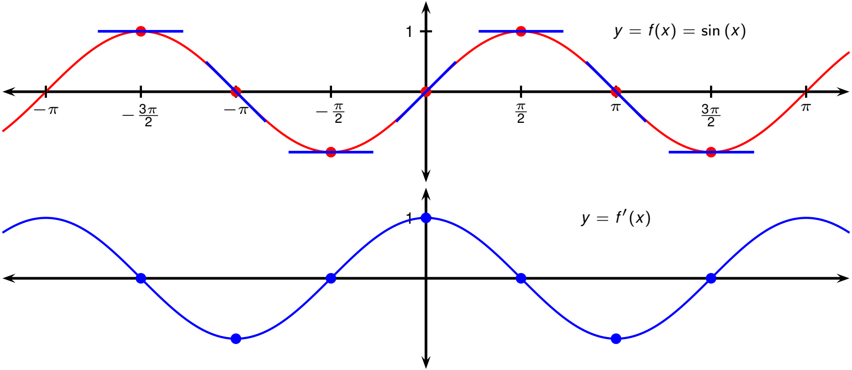

What is the derivative of $f(x) = \sin x$? It looks like $\cos x$.

Let $\quad f(x) = \sin x.$

$$\text{Let} \quad f(x) = \sin x.$$

$$\text{Then} \quad f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Let $f(x) = \sin x$.

Then $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h\to 0} \dfrac{\sin(x+h) - \sin x}{h}$

Let $f(x) = \sin x$.

Then $$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$$
$$= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

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$$= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$= \lim_{h\to 0} \left(\frac{\sin x \cos h - \sin x}{h} + \frac{\cos x \sin h}{h} \right)$$

Let $f(x) = \sin x$.

Then
$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}$$
$$= \lim_{h\to 0}\frac{\sin x\cos h + \cos x \sin h - \sin x}{h}$$
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$$= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$= \lim_{h\to 0} \left(\frac{\sin x \cos h - \sin x}{h} + \frac{\cos x \sin h}{h} \right)$$

$$= \lim_{h\to 0} \left(\sin x \left(\frac{\cos h - 1}{h} \right) + \cos x \left(\frac{\sin h}{h} \right) \right)$$

$$\text{Let}\quad f(x) = \sin x.$$

$$\begin{aligned}
\text{Then}\quad f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sin(x+h)-\sin x}{h}\\
&= \lim_{h\to 0} \frac{\sin x\cos h + \cos x \sin h - \sin x}{h}\\
&= \lim_{h\to 0} \left(\frac{\sin x\cos h - \sin x}{h} + \frac{\cos x\sin h}{h}\right)\\
&= \lim_{h\to 0} \left(\sin x\left(\frac{\cos h - 1}{h}\right) + \cos x\left(\frac{\sin h}{h}\right)\right)\\
&= \lim_{h\to 0} \sin x \cdot \lim_{h\to 0}\left(\frac{\cos h - 1}{h}\right) + \lim_{h\to 0}\cos x \cdot \lim_{h\to 0}\left(\frac{\sin h}{h}\right)
\end{aligned}$$

Let $f(x) = \sin x$.

Then
$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} \\
&= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\
&= \lim_{h\to 0} \left(\frac{\sin x \cos h - \sin x}{h} + \frac{\cos x \sin h}{h} \right) \\
&= \lim_{h\to 0} \left(\sin x \left(\frac{\cos h - 1}{h} \right) + \cos x \left(\frac{\sin h}{h} \right) \right) \\
&= \lim_{h\to 0} \sin x \cdot \lim_{h\to 0} \left(\frac{\cos h - 1}{h} \right) + \lim_{h\to 0} \cos x \cdot \lim_{h\to 0} \left(\frac{\sin h}{h} \right) \\
&= \mathbf{?} \quad \cdot \lim_{h\to 0} \left(\frac{\cos h - 1}{h} \right) + \mathbf{?} \quad \cdot \lim_{h\to 0} \left(\frac{\sin h}{h} \right)
\end{aligned}$$

Let $f(x) = \sin x.$

Then
$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sin(x+h)-\sin x}{h}\\
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&= \lim_{h\to 0} \sin x \cdot \lim_{h\to 0} \left(\frac{\cos h - 1}{h} \right) + \lim_{h\to 0} \cos x \cdot \lim_{h\to 0} \left(\frac{\sin h}{h} \right)\\
&= \sin x \cdot \lim_{h\to 0} \left(\frac{\cos h - 1}{h} \right) + \mathbf{?} \quad \cdot \lim_{h\to 0} \left(\frac{\sin h}{h} \right)
\end{aligned}
$$

Let $\quad f(x) = \sin x.$

Then $\quad f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \dfrac{\sin(x+h)-\sin x}{h}$

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$$= \lim_{h\to 0} \sin x \cdot \lim_{h\to 0} \left(\frac{\cos h - 1}{h} \right) + \lim_{h\to 0} \cos x \cdot \lim_{h\to 0} \left(\frac{\sin h}{h} \right)$$

$$= \sin x \cdot \lim_{h\to 0} \left(\frac{\cos h - 1}{h} \right) + \mathbf{?} \quad \cdot \lim_{h\to 0} \left(\frac{\sin h}{h} \right)$$

Let $f(x) = \sin x$.

Then
$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}\\
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\end{aligned}
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$$\text{Let} \quad f(x) = \sin x.$$

$$\begin{aligned}
\text{Then} \quad f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sin(x+h)-\sin x}{h} \\
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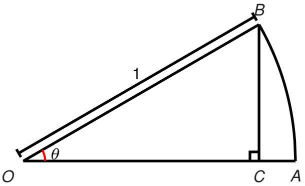
&= \sin x \cdot \lim_{h\to 0} \left(\frac{\cos h - 1}{h} \right) + \cos x \cdot \lim_{h\to 0} \left(\frac{\sin h}{h} \right)
\end{aligned}$$

We need to do more work to find the other two limits.

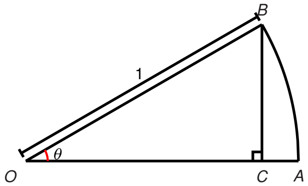
Claim: $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$

Suppose $0 < \theta < \frac{\pi}{2}$.



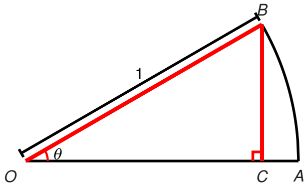
Claim: $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$

Suppose $0 < \theta < \frac{\pi}{2}$. Write $\sin\theta$ using ratios of side lengths of a triangle.



$$\sin\theta = \textbf{?}$$

Claim: $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$

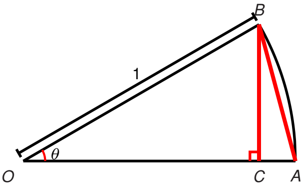


Suppose $0 < \theta < \frac{\pi}{2}$. Write $\sin\theta$ using ratios of side lengths of a triangle.

$$\sin\theta = \frac{|BC|}{|OB|} = |BC|$$

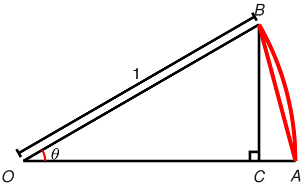
Claim: $\displaystyle\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$

Suppose $0 < \theta < \frac{\pi}{2}$. Write $\sin\theta$ using ratios of side lengths of a triangle.



$$\sin\theta = \frac{|BC|}{|OB|} = |BC| < |AB|$$

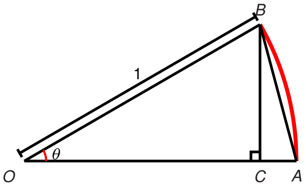
Claim: $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$



Suppose $0 < \theta < \frac{\pi}{2}$. Write $\sin\theta$ using ratios of side lengths of a triangle.

$$\sin\theta = \frac{|BC|}{|OB|} = |BC| < |AB| < \operatorname{arc}AB$$

Claim: $\displaystyle\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$

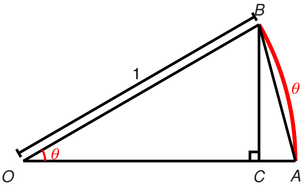


Suppose $0<\theta<\frac{\pi}{2}$. Write $\sin\theta$ using ratios of side lengths of a triangle.

$$\sin\theta=\frac{|BC|}{|OB|}=|BC|<|AB|<\mathrm{arc}AB=\textbf{?}$$

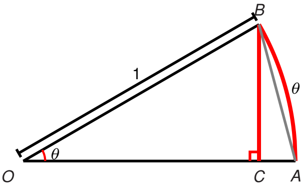
Claim: $$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$$

Suppose $0 < \theta < \frac{\pi}{2}$. Write $\sin \theta$ using ratios of side lengths of a triangle.



$$\sin \theta = \frac{|BC|}{|OB|} = |BC| < |AB| < \text{arc}AB = \theta$$

Claim: $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$



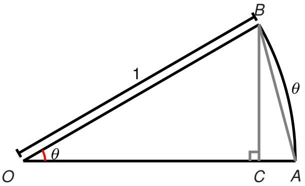
Suppose $0 < \theta < \frac{\pi}{2}$. Write $\sin\theta$ using ratios of side lengths of a triangle.

$$\sin\theta = \frac{|BC|}{|OB|} = |BC| < |AB| < \mathrm{arc}AB = \theta$$

Therefore $\sin\theta < \theta$

Claim: $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$

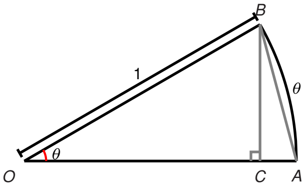
Suppose $0 < \theta < \frac{\pi}{2}$. Write $\sin\theta$ using ratios of side lengths of a triangle.



$$\sin\theta = \frac{|BC|}{|OB|} = |BC| < |AB| < \mathrm{arc}AB = \theta$$

Therefore $\sin\theta < \theta$ and therefore $\frac{\sin\theta}{\theta} < 1$.

Claim: $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$



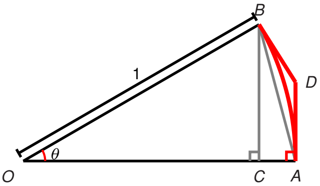
Suppose $0 < \theta < \frac{\pi}{2}$. Write $\sin \theta$ using ratios of side lengths of a triangle.

$$\sin \theta = \frac{|BC|}{|OB|} = |BC| < |AB| < \text{arc}AB = \theta$$

Therefore $\sin \theta < \theta$ and therefore $\frac{\sin \theta}{\theta} < 1$.

$\theta = \text{arc}AB$

Claim: $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$



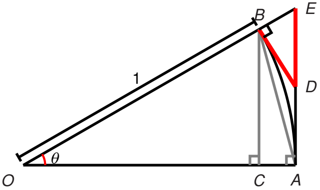
Suppose $0 < \theta < \frac{\pi}{2}$. Write $\sin\theta$ using ratios of side lengths of a triangle.

$$\sin\theta = \frac{|BC|}{|OB|} = |BC| < |AB| < \mathrm{arc}AB = \theta$$

Therefore $\sin\theta < \theta$ and therefore $\frac{\sin\theta}{\theta} < 1$.

$$\theta = \mathrm{arc}AB < |AD| + |DB|$$

Claim: $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$



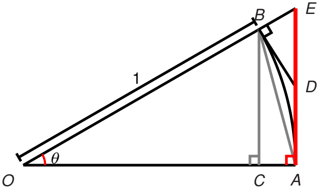
Suppose $0 < \theta < \frac{\pi}{2}$. Write $\sin \theta$ using ratios of side lengths of a triangle.

$$\sin \theta = \frac{|BC|}{|OB|} = |BC| < |AB| < \mathrm{arc}AB = \theta$$

Therefore $\sin \theta < \theta$ and therefore $\frac{\sin \theta}{\theta} < 1$.

$$\theta = \mathrm{arc}AB < |AD| + |DB| < |AD| + |DE|$$

Claim: $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$



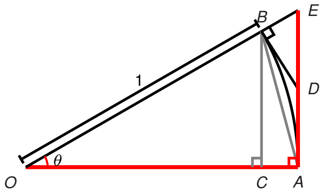
Suppose $0 < \theta < \frac{\pi}{2}$. Write $\sin\theta$ using ratios of side lengths of a triangle.

$$\sin\theta = \frac{|BC|}{|OB|} = |BC| < |AB| < \mathrm{arc}AB = \theta$$

Therefore $\sin\theta < \theta$ and therefore $\frac{\sin\theta}{\theta} < 1$.

$$\theta = \mathrm{arc}AB < |AD| + |DB| < |AD| + |DE|$$
$$= |AE|$$

Claim: $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$



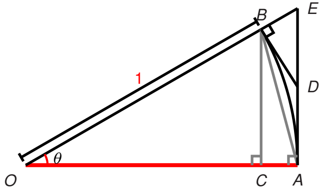
Suppose $0 < \theta < \frac{\pi}{2}$. Write $\sin\theta$ using ratios of side lengths of a triangle.

$$\sin\theta = \frac{|BC|}{|OB|} = |BC| < |AB| < \text{arc}AB = \theta$$

Therefore $\sin\theta < \theta$ and therefore $\frac{\sin\theta}{\theta} < 1$.

$$\theta = \text{arc}AB < |AD| + |DB| < |AD| + |DE|$$
$$= |AE| = |OA|\tan\theta$$

Claim: $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$



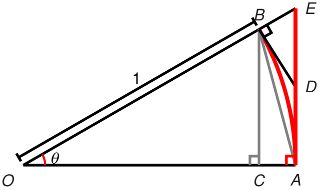
Suppose $0 < \theta < \frac{\pi}{2}$. Write $\sin\theta$ using ratios of side lengths of a triangle.

$$\sin\theta = \frac{|BC|}{|OB|} = |BC| < |AB| < \mathrm{arc}AB = \theta$$

Therefore $\sin\theta < \theta$ and therefore $\frac{\sin\theta}{\theta} < 1$.

$$\begin{aligned}\theta = \mathrm{arc}AB < |AD| + |DB| &< |AD| + |DE| \\ = |AE| = |OA| \tan\theta &= \tan\theta\end{aligned}$$

Claim: $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$



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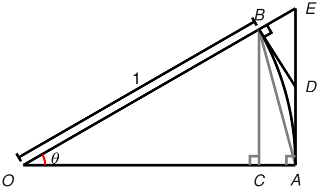

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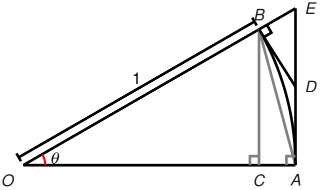

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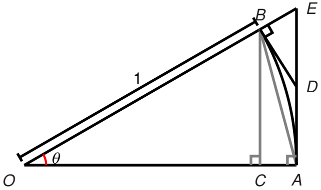

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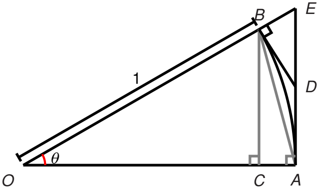

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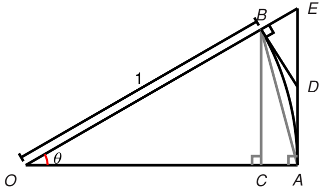

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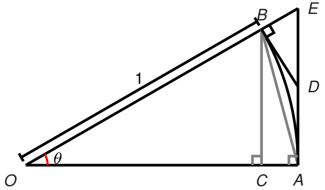

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$\lim_{\theta\to 0} \cos\theta =$ **?**

Claim: $\lim\limits_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$

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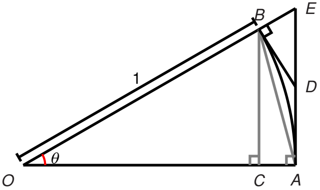

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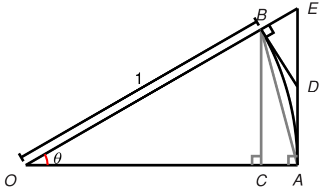
$\lim\limits_{\theta\to 0}\cos\theta = 1$

Claim: $\lim\limits_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$



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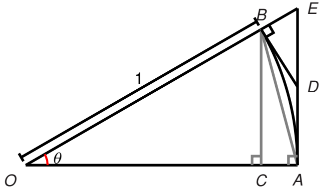
Claim: $\displaystyle\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$



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Therefore $\theta<\tan\theta=\frac{\sin\theta}{\cos\theta}$, so $\cos\theta<\frac{\sin\theta}{\theta}$.

$$\cos\theta<\frac{\sin\theta}{\theta}<1$$

$\displaystyle\lim_{\theta\to 0}\cos\theta=1$ and $\displaystyle\lim_{\theta\to 0}1=1$, so by the Squeeze Theorem $\displaystyle\lim_{\theta\to 0^+}\frac{\sin\theta}{\theta}=1$.

Claim: $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$

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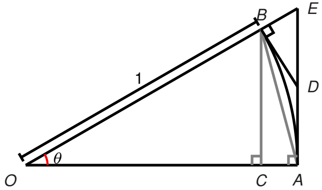

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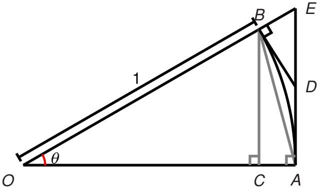
Claim: $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$



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Let $\quad f(x) = \sin x.$

Then $\quad f'(x) = \lim_{h\to 0} \sin x \cdot \lim_{h\to 0} \left(\frac{\cos h - 1}{h} \right) + \lim_{h\to 0} \cos x \cdot \lim_{h\to 0} \left(\frac{\sin h}{h} \right)$

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$= \mathbf{?} \cdot \mathbf{?} + \mathbf{?} \cdot \mathbf{?}$

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$$= \sin x \cdot \lim_{h\to 0}\left(\frac{\cos h - 1}{h}\right) + \cos x \cdot \mathbf{?}$$

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&= \sin x \cdot \lim_{h\to 0} \left(\frac{\cos h - 1}{h} \right) + \cos x \cdot \textbf{?}
\end{aligned}
$$

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$$= \sin x \cdot \lim_{h\to 0}\left(\frac{\cos h - 1}{h}\right) + \cos x \cdot 1$$

We need to find

$$\lim_{h\to 0} \frac{\cos h - 1}{h}$$

Let $f(x) = \sin x.$

Then
$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \sin x \cdot \lim_{h\to 0} \left(\frac{\cos h - 1}{h}\right) + \lim_{h\to 0} \cos x \cdot \lim_{h\to 0}\left(\frac{\sin h}{h}\right)\\
&= \sin x \cdot \lim_{h\to 0}\left(\frac{\cos h - 1}{h}\right) + \cos x \cdot 1
\end{aligned}
$$

We need to find

$$
\lim_{h\to 0} \frac{\cos h - 1}{h} = \lim_{h\to 0} \frac{(\cos h - 1)}{h} \cdot \frac{({\color{red}\cos h + 1})}{({\color{red}\cos h + 1})}
$$

Let $\quad f(x) = \sin x.$

$$\begin{aligned} \text{Then} \quad f'(x) &= \lim_{h\to 0} \sin x \cdot \lim_{h\to 0}\left(\frac{\cos h - 1}{h}\right) + \lim_{h\to 0} \cos x \cdot \lim_{h\to 0}\left(\frac{\sin h}{h}\right) \\ &= \sin x \cdot \lim_{h\to 0}\left(\frac{\cos h - 1}{h}\right) + \cos x \cdot 1 \end{aligned}$$

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$$= \lim_{h\to 0} \frac{-\sin^2 h}{h(\cos h + 1)}$$

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$$= -\lim_{h\to 0}\frac{\sin h}{h}\cdot\lim_{h\to 0}\frac{\sin h}{\cos h + 1} = -1\cdot\left(\frac{0}{1+1}\right)$$

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We need to find

$$\begin{aligned} \lim_{h\to 0}\frac{\cos h - 1}{h} &= \lim_{h\to 0}\frac{(\cos h - 1)}{h}\cdot\frac{(\cos h + 1)}{(\cos h + 1)} = \lim_{h\to 0}\frac{\cos^2 h - 1}{h(\cos h + 1)} \\ &= \lim_{h\to 0}\frac{-\sin^2 h}{h(\cos h + 1)} = -\lim_{h\to 0}\left(\frac{\sin h}{h}\cdot\frac{\sin h}{\cos h + 1}\right) \\ &= -\lim_{h\to 0}\frac{\sin h}{h}\cdot\lim_{h\to 0}\frac{\sin h}{\cos h + 1} = -1\cdot\left(\frac{0}{1+1}\right) = 0 \end{aligned}$$

Let $\quad f(x) = \sin x.$

Then $\quad f'(x) = \lim_{h\to 0} \sin x \cdot \lim_{h\to 0}\left(\frac{\cos h - 1}{h}\right) + \lim_{h\to 0} \cos x \cdot \lim_{h\to 0}\left(\frac{\sin h}{h}\right)$

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Differentiate $y = \dfrac{e^x}{2 + \sin x}$.

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&= \frac{\left(e^x\right)\left(2+\sin x\right) - \left(e^x\right)\left(\cos x\right)}{\left(2+\sin x\right)^2} \\
&= \frac{2e^x + e^x \sin x - e^x \cos x}{(2+\sin x)^2} \\
&= \frac{e^x(2+\sin x - \cos x)}{(2+\sin x)^2}.
\end{aligned}$$

Example (Trigonometric limit)

Find $\displaystyle \lim_{x\to 0} \frac{2x}{\sin(9x)}$

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Let $\theta = 9x$.

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Theorem (The Derivative of $\cos x$)

$$\frac{d}{dx}(\cos x) = -\sin x$$

- This can be proved in a similar fashion as the formula for $\sin x$.
- Alternatively, this can be proved using the derivative of $\sin x$ and (the not yet studied) Implicit Differentiation and Chain Rule.

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Theorem (The Derivative of Tangent)

$$\frac{\mathrm{d}}{\mathrm{d}x}(\tan x) = \sec^2 x.$$

Proof.

$\square$

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Let $y = \tan x = \textbf{?}$

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&= \sec^2 x.
\end{aligned}$$

□

Derivatives of Trigonometric Functions

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sin x) = \cos x \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\csc x) = -\csc x \cot x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\cos x) = -\sin x \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\sec x) = \sec x \tan x$$

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Example (Quotient Rule, Trig)

Differentiate $y = \dfrac{\sec x}{1 + \tan x}$.

Quotient Rule:

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\frac{\mathrm{d}}{\mathrm{d}x}(\sec x)(1+\tan x) - (\sec x)\frac{\mathrm{d}}{\mathrm{d}x}(1+\tan x)}{(1+\tan x)^2} \\ &= \frac{(\sec x \tan x)\,(1+\tan x) - (\sec x)\,(\sec^2 x)}{(1+\tan x)^2}\end{aligned}$$

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Example (Using the Product Rule twice)

Differentiate:

$$y = \theta e^{\theta}(\tan\theta + \sec\theta).$$

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Example

Find the 27th derivative of $f(x) = \cos x$.

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$$
\begin{aligned}
f'(x) &= \\
f''(x) &= \\
f'''(x) &= \\
f^{(4)}(x) &= \\
f^{(5)}(x) &=
\end{aligned}
$$

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Find the 27th derivative of $f(x) = \cos x$.

$$f'(x) = \textbf{?}$$

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Example

Find the 27th derivative of $f(x) = \cos x$.

$$f'(x) = -\sin x$$

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- The derivatives repeat in a cycle of length 4.

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- The derivatives repeat in a cycle of length 4.
- $f^{(24)}(x) = $ **?** .

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Find the 27th derivative of $f(x) = \cos x$.

$$\begin{aligned} f'(x) &= -\sin x \\ f''(x) &= -\cos x \\ f'''(x) &= \sin x \\ f^{(4)}(x) &= \cos x \\ f^{(5)}(x) &= -\sin x \end{aligned}$$

- The derivatives repeat in a cycle of length 4.
- $f^{(24)}(x) = \cos x$.

Example

Find the 27th derivative of $f(x) = \cos x$.

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f'(x) &= -\sin x \\
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- The derivatives repeat in a cycle of length 4.
- $f^{(24)}(x) = \cos x$.
- Differentiate three more times: $f^{(27)}(x) =$ **?** .

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Find the 27th derivative of $f(x) = \cos x$.

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- The derivatives repeat in a cycle of length 4.
- $f^{(24)}(x) = \cos x$.
- Differentiate three more times: $f^{(27)}(x) = \sin x$.