

Calculus I

Homework

Inverse functions

1. Find the inverse function. You are asked to do the algebra only; you are not asked to determine the domain or range of the function or its inverse.

(a) $f(x) = 3x^2 + 4x - 7$, where $x \geq -\frac{2}{3}$.

$$\frac{5}{22} - \frac{2}{x} = x \quad , \quad \frac{5}{x^2 + 3x + 2} + \frac{2}{x} = (x)_{-1}^{-1} \text{ : ANSWER}$$

(b) $f(x) = 2x^2 + 3x - 5$, where $x \geq -\frac{3}{4}$.

$$\frac{8}{64} - \frac{2}{x} = x \quad , \quad \frac{7}{x^2 + 8x + 16} + \frac{7}{x} = (x)_{-1}^{-1} \text{ : ANSWER}$$

(c) $f(x) = \frac{2x+5}{x-4}$, where $x \neq 4$.

$$\frac{2}{3} \neq x \quad , \quad \frac{2-x}{5+x\frac{2}{3}} = (x)_{-1}^{-1} \text{ : ANSWER}$$

(d) $f(x) = \frac{3x+5}{2x-4}$, where $x \neq 2$.

$$\frac{2}{3} \neq x \quad , \quad \frac{3-x}{5+x\frac{2}{3}} = (x)_{-1}^{-1} \text{ : ANSWER}$$

(e) $f(x) = \frac{5x+6}{4x+5}$, where $x \neq -\frac{5}{4}$.

$$\frac{7}{5} \neq x \quad , \quad \frac{5-x\frac{7}{5}}{9+x\frac{5}{9}} = (x)_{-1}^{-1} \text{ : ANSWER}$$

(f) $f(x) = \frac{2x-3}{-3x+4}$, where $x \neq \frac{4}{3}$.

$$\frac{8}{2} \neq x \quad , \quad \frac{2-x\frac{8}{2}}{3+x\frac{2}{3}} = (x)_{-1}^{-1} \text{ : ANSWER}$$

Solution. 1.d This is a concise solution written in form suitable for test taking.

$$\begin{aligned} y &= \frac{3x+5}{2x-4} \\ y(2x-4) &= 3x+5 \\ 2xy-4y &= 3x+5 \\ 2xy-3x &= 4y+5 \\ x(2y-3) &= 4y+5 \\ x &= \frac{4y+5}{2y-3} \\ \text{Therefore } f^{-1}(y) &= \frac{4y+5}{2y-3} \\ f^{-1}(x) &= \frac{4x+5}{2x-3} \end{aligned}$$

Solution. 1.e. Set $f(x) = y$. Then

$$\begin{aligned} y &= \frac{5x+6}{4x+5} \\ y(4x+5) &= 5x+6 \\ x(4y-5) &= -5y+6 \\ x &= \frac{-5y+6}{4y-5} \end{aligned}$$

Therefore the function $x = g(y) = \frac{-5y+6}{4y-5}$ is the inverse of $f(x)$. We write $g = f^{-1}$. The function $g = f^{-1}$ is defined for $y \neq \frac{5}{4}$. For our final answer we relabel the argument of g to x :

$$g(x) = f^{-1}(x) = \frac{-5x+6}{4x-5}.$$

Let us check our work. In order for f and g to be inverses, we need that $g(f(x))$ be equal to x .

$$g(f(x)) = \frac{-5f(x)+6}{4f(x)-5} = \frac{-5\frac{(5x+6)}{4x+5}+6}{4\frac{(5x+6)}{4x+5}-5} = \frac{-5(5x+6)+6(4x+5)}{4(5x+6)-5(4x+5)} = \frac{-x}{-1} = x,$$

as expected.

2. Find the inverse function and its domain.

(a) $y = \ln(x+3).$

(b) $y = 4 \ln(x-3) - 4.$

(c) $y = 2 \ln(-2x+4) + 1$

(d) $f(x) = e^{x^3}.$

(e) $y = (\ln x)^2, x \geq 1.$

(f) $y = \frac{e^x}{1+2e^x}.$

(g) $f(x) = 2^{2x} + 2^x - 2.$

Solution. 2.a

$$y = \ln(x+3)$$

$$e^y = e^{\ln(x+3)}$$

$$e^y = x+3$$

$$e^y - 3 = x$$

$$\text{Therefore } f^{-1}(y) = e^y - 3.$$

The domain of e^y is all real numbers, so the domain of f^{-1} is all real numbers.

Solution. 2.b

$$4 \ln(x-3) - 4 = y$$

$$4 \ln(x-3) = y+4$$

$$\ln(x-3) = \frac{y+4}{4} \quad \left| \text{exponentiate} \right.$$

$$e^{\ln(x-3)} = e^{\frac{y+4}{4}}$$

$$x-3 = e^{\frac{y+4}{4}}$$

$$f^{-1}(y) = x = e^{\frac{y+4}{4}} + 3$$

$$f^{-1}(x) = e^{\frac{x+4}{4}} + 3 \quad \left| \text{relabel.} \right.$$

The domain of f^{-1} is all real numbers (no restrictions on the domain).

Solution. 2.c

$$\begin{aligned} y &= (\ln x)^2 & \left| \text{take } \sqrt{} \text{ on both sides, } y \geq 0 \right. \\ \sqrt{y} &= \ln x & \left| \text{exponentiate} \right. \\ e^{\sqrt{y}} &= e^{\ln x} = x \\ f^{-1}(y) &= e^{\sqrt{y}} \\ f^{-1}(x) &= e^{\sqrt{x}} \end{aligned}$$

Solution. 2.f

$$\begin{aligned}y &= \frac{e^x}{1 + 2e^x} \\y(1 + 2e^x) &= e^x \\y &= e^x(1 - 2y) \\\frac{y}{1 - 2y} &= e^x \\\ln \frac{y}{1 - 2y} &= \ln e^x \\\ln \frac{y}{1 - 2y} &= x\end{aligned}$$

$$\text{Therefore } f^{-1}(y) = \ln \frac{y}{1 - 2y}.$$

The natural logarithm function is only defined for positive input values. Therefore the domain is the set of all y for which

$$\frac{y}{1 - 2y} > 0.$$

This inequality holds if the numerator and denominator are both positive or both negative. This happens if either

- (a) $y > 0$ and $y < \frac{1}{2}$, or
- (b) $y < 0$ and $y > \frac{1}{2}$.

The latter option is impossible, so the domain is $\{y \in \mathbb{R} \mid 0 < y < \frac{1}{2}\}$.