Precalculus
Exponential and logarithtmic models

Todor Milev

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Outline

1 Models Involving Logarithms and Exponents

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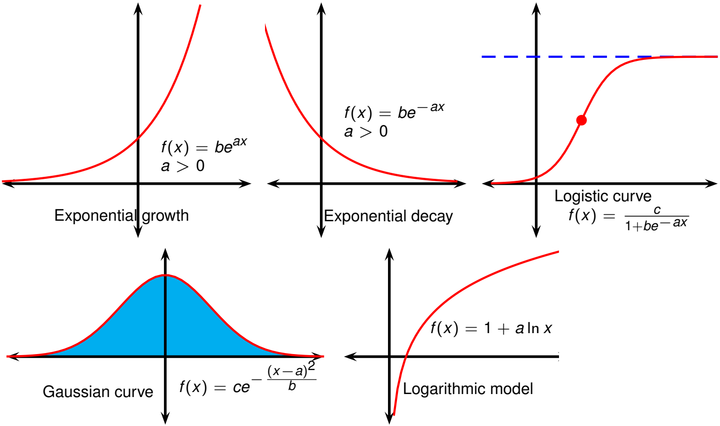
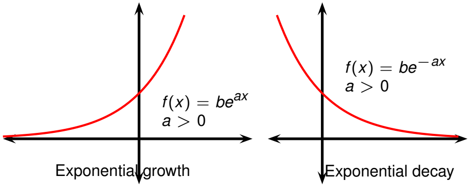
A few commonly used mathematical models

The following functions are commonly used as mathematical models to approximate solutions to certain problems arising from practice.



Exponential growth/decay model

- Exponential growth models can be applied to model (the initial stage of) growth in biological systems where the environment limitations are negligible.
- Exponential decay models use similar formulas but have negative coefficients in the exponent. Exponential decay models are used (among other applications) for carbon dating.



Example

1 day after the start of hypothetical experiment a population of fruit flies was measured to have 100 individuals. 2 days after the start there were 150 flies. Write down an exponential growth law that fits this data. According to the model, how may fruit were there at the start of the experiment? After 5 days?

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$$\begin{aligned} be^{1\cdot a} &= 100\\ be^{a} &= 100\\ b &= 100e^{-a} \end{aligned}$$

$$\begin{aligned} be^{2a} &= 150\\ 100e^{-a}e^{2a} &= 150\\ e^{a} &= \frac{3}{2}\\ a &= \ln\left(\frac{3}{2}\right) \approx 0.405\\ b &= 100e^{-a} = \frac{100}{e^a} = \frac{100}{\frac{3}{2}} = \frac{200}{3} \approx 67. \end{aligned}$$

In the start of the hypothetical experiment there must have been $be^{a\cdot 0} = b \approx 67$ flies. Under the exponential growth model, after 5 one would expect $be^{5a} = 67(e^a)^5 = 67\left(\frac{3}{2}\right)^5 = 67\cdot\frac{243}{32} \approx 509$ flies.

Example

Most of the carbon in a living organism is the stable isotope ^{12}C but there is a near constant ratio of the radioactive isotope ^{14}C. After death, the radioactive carbon gradually decays, leaving a smaller ratio of radioactive carbon compared to living organisms. An approximate law measuring the ratio of ^{14}C to ^{12}C (by mass) is given by the formula $10^{-12}e^{-\frac{t}{8223}}$, where t is the number of years after death. A fossil has ratio of ^{14}C to ^{12}C equal to $0.5 \cdot 10^{-13}$. Estimate the age of the fossil.

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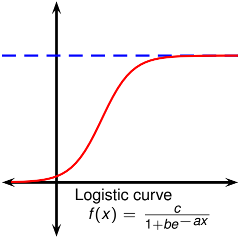
$$\begin{aligned}
10^{-12}e^{-\frac{t}{8223}} &= \frac{1}{2} \cdot 10^{-13} \\
e^{-\frac{t}{8223}} &= \frac{1}{2} \cdot 10^{-1} = \frac{1}{20} \\
-\frac{t}{8223} &= \ln\left(\frac{1}{20}\right) \\
t &= -8223 \ln\left(\frac{1}{20}\right) = 8223 \ln(20) \\
t &\approx 24633.
\end{aligned}$$

We estimate the fossil's age to be between 24000 and 25000 years.

Logistic growth model

- The Logistic growth model is the simplest mathematical model which satisfies the following assumptions.



Logistic curve
$f(x) = \frac{c}{1+be^{-ax}}$

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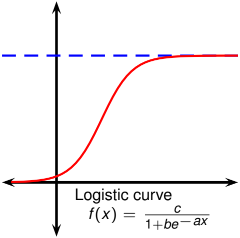
  - We assume near exponential growth for small populations (relative to the environment's resources).



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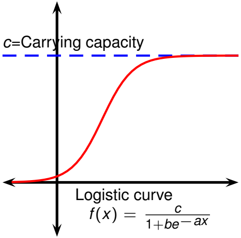
  - We assume negative growth when the population exceeds a threshold number called the environment carrying capacity.



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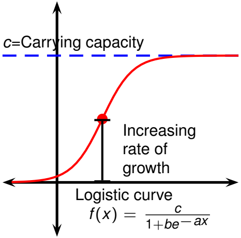

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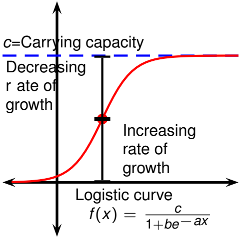

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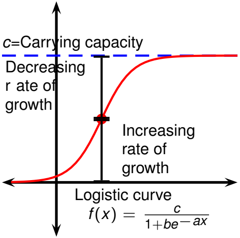
- The Logistic growth model is the simplest mathematical model which satisfies the following assumptions.
  - We assume near exponential growth for small populations (relative to the environment's resources).
  - We assume negative growth when the population exceeds a threshold number called the environment carrying capacity.
- Although it's one of the simplest mathematical models, the logistic growth model is used in practice.



Example

In a hypothetical experiment, the number of E. Coli bacteria cells is modeled with a logistic curve $E(t) = \dfrac{2.6 \times 10^{11}}{1 + (3.94 \times 10^{9})e^{-1.387t}}$, where t measures time in hours since the start of the experiment.

- According to the model, approximately how many cells were there at the start of the experiment?

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- According to the model, approximately how many cells were there at the start of the experiment?

According to the model, the number of cells in the beginning was

$$
\begin{aligned}
E(0) &= \frac{2.6 \times 10^{11}}{1 + (3.94 \times 10^9)e^{-1.387 \cdot 0}} = \frac{2.6 \times 10^{11}}{1 + (3.94 \times 10^9)e^{0}} \\
&= \frac{2.6 \times 10^{11}}{1 + (3.94 \times 10^9)} \approx \frac{2.6 \times 10^{11}}{3.94 \times 10^9} \\
&= \frac{2.6 \times 10^{2}}{3.94} \approx \frac{260}{3.94} \approx 65.99 \approx 66 \text{ cells.} \qquad \Big| \text{ calculator}
\end{aligned}
$$

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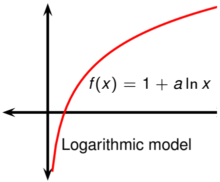
- According to the model, how many hours are needed for the number of cells to be approximately 10^{11}?

To find the hours t needed to get 10^{11} cells, we solve $E(t) = 10^{11}$.

$$\begin{aligned}
10^{11} &= \frac{2.6 \times 10^{11}}{1 + (3.94 \times 10^9)e^{-1.387t}} \\
1 + 3.94 \times 10^9 e^{-1.387t} &= 2.6 \\
e^{-1.387x} &= \frac{1.6}{3.94} \times 10^{-9} && \Big| \text{ take } \ln \\
-1.387t &= \ln\left(\frac{1.6}{3.94} \times 10^{-9}\right) \\
t &= \frac{\ln(1.6) - \ln(3.94) - 9\ln(10)}{-1.387} \\
&\approx 15.6 \text{ hours} && \Big| \text{ calculator}
\end{aligned}$$

Logarithmic models

- Logarithmic functions are commonly used as mathematical models.
- Examples include, among others, the Richter earthquake scale and chemical reaction kinetics (speed of chemical reactions).



Example

The Richter magnitude M_L of an earthquake is determined from the logarithm of the amplitude A of waves recorded by seismographs (with adjustment to compensate for the distance between the measuring station and the estimated epicenter of the earthquake). The formula is

$$M_L = \log_{10} A - J_0(\delta),$$

where $J_0(\delta)$ depends on the distance δ from the epicenter. Compare the amplitudes A_1 and A_2 of the seism. waves of two hypothetical earthquakes of magnitudes 4 and 7.5 with the same epicenter.

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$$\begin{array}{rrcl} & \log_{10} A_2 - J_0(\delta) & = & 7.5 \\ - & \log_{10} A_1 - J_0(\delta) & = & 4 \\ & \log_{10} A_2 - \log_{10} A_1 & = & 3.5 \\ & \log_{10}\left(\dfrac{A_2}{A_1}\right) & = & 3.5 \\ & \dfrac{A_2}{A_1} & = & 10^{3.5} \\ & A_2 & = & 10^{3.5} A_1 \approx 3162 A_1 . \end{array}$$

The stronger earthquake has about 3160 times larger wave amplitude.