Precalculus
Angle sum formulas memorization

Todor Milev

2019

$\sin(\alpha+\beta), \cos(\alpha+\beta)$ via $\sin\alpha, \sin\beta, \cos\alpha, \cos\beta$

$\sin(\alpha+\beta) =$ **?**

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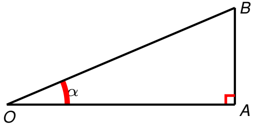

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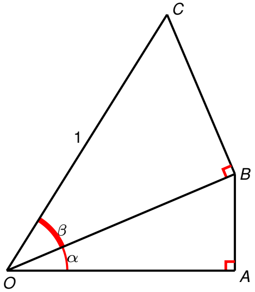

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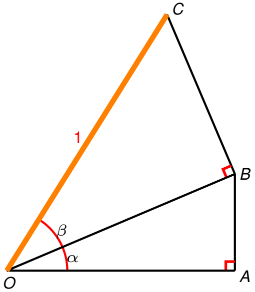

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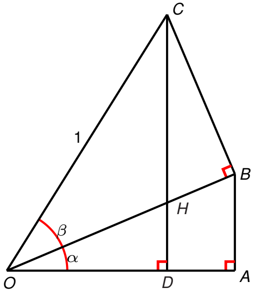

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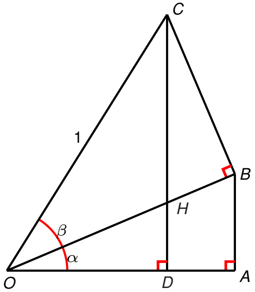

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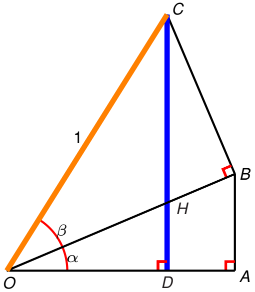

$$\sin(\alpha+\beta) = \frac{|CD|}{|OC|}$$

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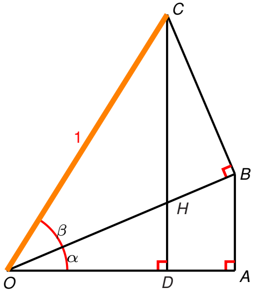

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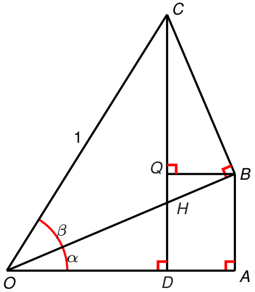

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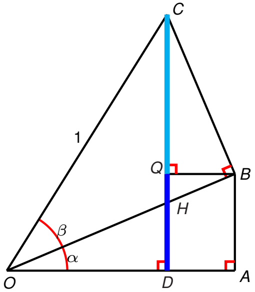

$$\sin(\alpha+\beta) = \frac{|CD|}{|OC|} = |CD|$$
$$= |QD| + |CQ|$$

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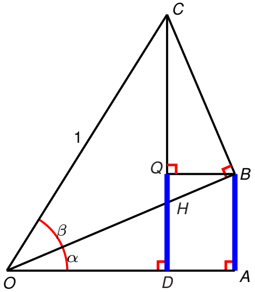

$$|QD| = |BA| \qquad \Big| \; \square DABQ$$

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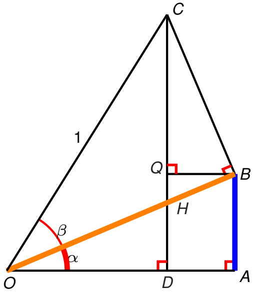

$$\sin(\alpha+\beta) = \frac{|CD|}{|OC|} = |CD|$$
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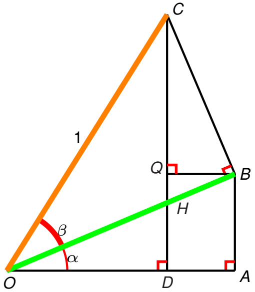
$$\cos(\alpha+\beta) = \textbf{?}$$

$$|QD| = |BA|$$
$$= \sin\alpha |OB|$$

$\square DABQ$

$\triangle OAB$

$\sin(\alpha+\beta)$, $\cos(\alpha+\beta)$ via $\sin\alpha$, $\sin\beta$, $\cos\alpha$, $\cos\beta$



$$\begin{aligned}|QD| &= |BA| \\ &= \sin\alpha|OB| \\ &= \sin\alpha\cos\beta|OC|\end{aligned} \quad \left|\begin{array}{l}\Box DABQ \\ \triangle OAB \\ \triangle OBC\end{array}\right.$$

$$\begin{aligned}\sin(\alpha+\beta) &= \frac{|CD|}{|OC|} = |CD| \\ &= |QD| + |CQ|\end{aligned}$$

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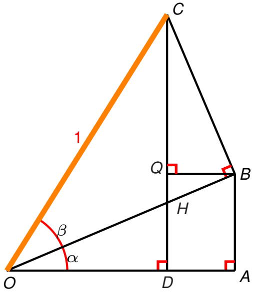

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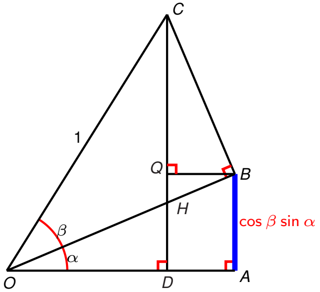

$$\begin{aligned} \sin(\alpha+\beta) &= \frac{|CD|}{|OC|} = |CD| \\ &= |QD| + |CQ| \\ &= \sin\alpha\cos\beta + \textbf{?} \end{aligned}$$

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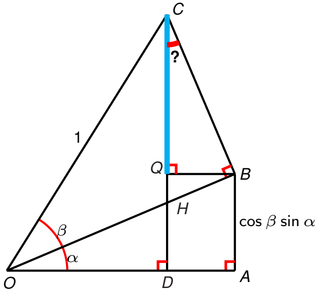

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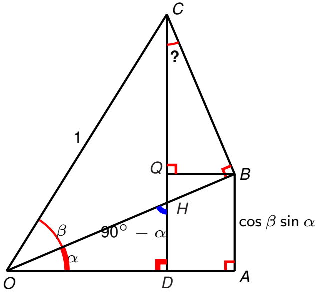

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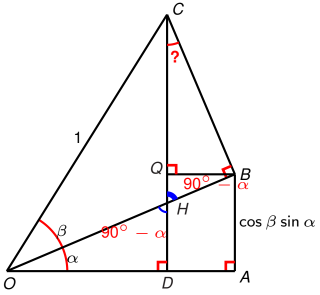

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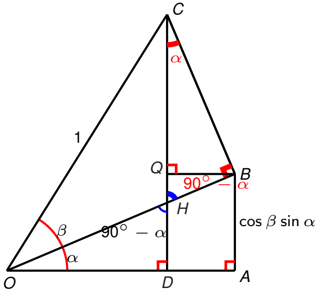

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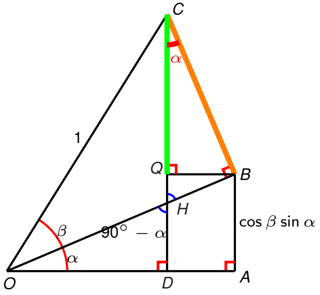

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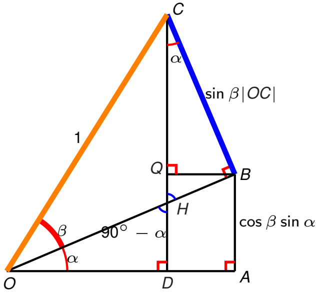

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\begin{aligned}
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&= \sin\alpha\cos\beta \\
|CQ| &= \cos\alpha {\color{red}|CB|} && \triangle CQB \\
&= \cos\alpha {\color{red}\sin\beta |OC|} && \triangle OBC
\end{aligned}
$$

$$
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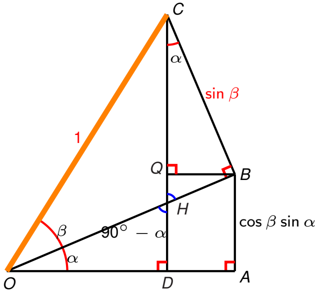

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$$

$$
\begin{aligned}
|CQ| &= \cos\alpha |CB| && \triangle CQB\\
&= \cos\alpha \sin\beta |OC| && \triangle OBC\\
&= \cos\alpha \sin\beta
\end{aligned}
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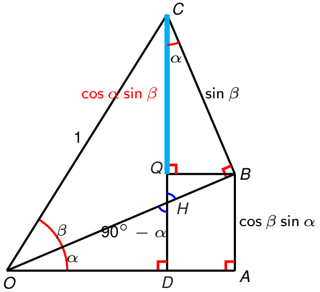

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$$

$$
\begin{aligned}
{\color{red}|CQ|} &= \cos\alpha|CB| && \triangle CQB\\
&= \cos\alpha\sin\beta|OC| && \triangle OBC\\
&{\color{red}= \cos\alpha\sin\beta}
\end{aligned}
$$

$$
\begin{aligned}
\sin(\alpha+\beta) &= \frac{|CD|}{|OC|} = |CD|\\
&= |QD| + {\color{red}|CQ|}\\
&= \sin\alpha\cos\beta + {\color{red}\cos\alpha\sin\beta}
\end{aligned}
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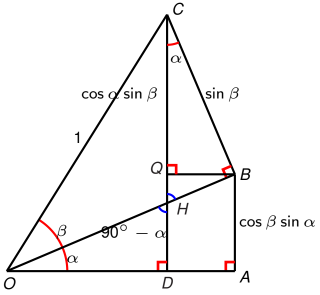

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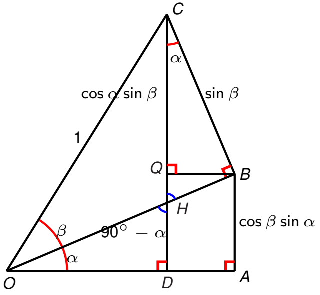

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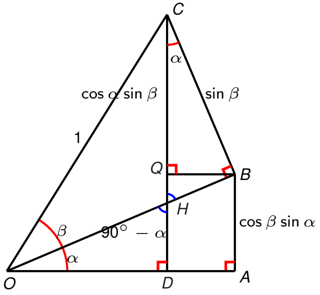
$$\begin{aligned}\cos(\alpha+\beta) &= \frac{|OD|}{|OC|} = |OD| \\ &= |OA| - |DA| \\ &= \cos\alpha\cos\beta - \sin\alpha\sin\beta\end{aligned}$$

$$\begin{aligned}|QD| &= |BA| && \square DABQ \\ &= \sin\alpha |OB| && \triangle OAB \\ &= \sin\alpha\cos\beta |OC| && \triangle OBC \\ &= \sin\alpha\cos\beta \\ |CQ| &= \cos\alpha |CB| && \triangle CQB \\ &= \cos\alpha\sin\beta|OC| && \triangle OBC \\ &= \cos\alpha\sin\beta \\ |OA| &= \cos\alpha|OB| && \triangle OAB \\ &= \cos\alpha\cos\beta|OC| && \triangle OBC \\ &= \cos\alpha\cos\beta \\ |DA| &= |QB| && \square DABQ \\ &= \sin\alpha|CB| && \triangle CQB \\ &= \sin\alpha\sin\beta|OC| && \triangle OBC \\ &= \sin\alpha\sin\beta\end{aligned}$$

$\sin(\alpha+\beta), \cos(\alpha+\beta)$ via $\sin\alpha, \sin\beta, \cos\alpha, \cos\beta$



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Trig Functions of Sums and Differences of Angles

Theorem

$$\begin{aligned} \sin(\alpha+\beta) &= \sin\alpha\cos\beta+\cos\alpha\sin\beta \\ \cos(\alpha+\beta) &= \cos\alpha\cos\beta-\sin\alpha\sin\beta \end{aligned}$$

Trig Functions of Sums and Differences of Angles

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- We gave a geometric proof of the sum formulas when the two angles are acute and their sum is less than $\pi = 90^\circ$.

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- We gave a geometric proof of the sum formulas when the two angles are acute and their sum is less than $\pi = 90^\circ$.
- The theorem holds for all angles α, β without any restrictions.

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- We gave a geometric proof of the sum formulas when the two angles are acute and their sum is less than $\pi = 90^\circ$.
- The theorem holds for all angles α, β without any restrictions.
- This can be shown by combining the preceding proof with identities such as $\cos\left(\frac{\pi}{2}-\alpha\right)=\sin\alpha$, $\cos\left(\frac{\pi}{2}+\alpha\right)=-\sin\alpha$.

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\sin(\alpha+\beta) &= \sin\alpha\cos\beta+\cos\alpha\sin\beta \\
\cos(\alpha+\beta) &= \cos\alpha\cos\beta-\sin\alpha\sin\beta
\end{aligned}
$$

- We gave a geometric proof of the sum formulas when the two angles are acute and their sum is less than $\pi = 90^\circ$.
- The theorem holds for all angles α, β without any restrictions.
- This can be shown by combining the preceding proof with identities such as $\cos\left(\frac{\pi}{2}-\alpha\right)=\sin\alpha$, $\cos\left(\frac{\pi}{2}+\alpha\right)=-\sin\alpha$.
- There is a theoretically more advanced (but algebraically simpler) proof using Euler's formula (to be studied later/in another course).

Trig Functions of Sums and Differences of Angles

Theorem

$$\begin{aligned}
\sin(\alpha+\beta) &= \sin\alpha\cos\beta+\cos\alpha\sin\beta\\
\sin(\alpha-\beta) &= \sin\alpha\cos\beta-\cos\alpha\sin\beta\\
\cos(\alpha+\beta) &= \cos\alpha\cos\beta-\sin\alpha\sin\beta\\
\cos(\alpha-\beta) &= \cos\alpha\cos\beta+\sin\alpha\sin\beta
\end{aligned}$$

- We gave a geometric proof of the sum formulas when the two angles are acute and their sum is less than $\pi = 90^\circ$.
- The theorem holds for all angles α, β without any restrictions.
- This can be shown by combining the preceding proof with identities such as $\cos\left(\frac{\pi}{2}-\alpha\right) = \sin\alpha$, $\cos\left(\frac{\pi}{2}+\alpha\right) = -\sin\alpha$.
- There is a theoretically more advanced (but algebraically simpler) proof using Euler's formula (to be studied later/in another course).
- The difference formulas are a consequence of the sum formulas and the fact that $\sin$ is an odd function and $\cos$ is even.

Trig Functions of Differences of Angles

Example

Prove the identities

$$\begin{aligned}\sin(\alpha-\beta) &= \sin\alpha\cos\beta-\cos\alpha\sin\beta\\ \cos(\alpha-\beta) &= \cos\alpha\cos\beta+\sin\alpha\sin\beta\end{aligned}$$

from the (already demonstrated) identities

$$\begin{aligned}\sin(\alpha+\beta) &= \sin\alpha\cos\beta+\cos\alpha\sin\beta\\ \cos(\alpha+\beta) &= \cos\alpha\cos\beta-\sin\alpha\sin\beta\end{aligned}$$

$$\begin{aligned}\sin(\alpha-\beta) &= \sin(\alpha+(-\beta))\\ &= \sin\alpha\cos(-\beta)+\cos\alpha\sin(-\beta) && \left|\begin{array}{l}\cos \text{ is even },\\ \sin \text{ is odd}\end{array}\right.\\ &= \sin\alpha\cos\beta-\cos\alpha\sin\beta\\ \cos(\alpha-\beta) &= \cos(\alpha+(-\beta))\\ &= \cos\alpha\cos(-\beta)-\sin\alpha\sin(-\beta) && \left|\begin{array}{l}\cos \text{ is even },\\ \sin \text{ is odd}\end{array}\right.\\ &= \cos\alpha\cos\beta+\cos\alpha\sin\beta\end{aligned}$$