Precalculus
Polynomial division and factorization of cubics with rational root

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Outline

Example (Polynomial long division)

Divide with quotient and remainder $x^3 + 2x^2 + 1$ by $x - 1$.

$$
\begin{array}{rr|l}
\textbf{Quotient:} & & x^2 + 3x + 3 \\
\hline
x - 1 & & x^3 + 2x^2 \qquad + 1 \\
- & & x^3 - x^2 \\
\hline
 & & \quad 3x^2 \qquad + 1 \\
- & & \quad 3x^2 - 3x \\
\hline
 & & \qquad\quad 3x + 1 \\
- & & \qquad\quad 3x - 3 \\
\hline
\textbf{Remainder:} & & \qquad\qquad 4
\end{array}
$$

$$
\begin{array}{rcl}
\textbf{(Dividend)} & = & \textbf{(Quotient)} \cdot \textbf{(Divisor)} + \textbf{(Remainder)} \\
(x^3 + 2x^2 + 1) & = & (x^2 + 3x + 3) \cdot (x - 1) + 4
\end{array}
$$

Example

Demonstrate that $6x^3 - 19x^2 + 17x - 3$ is divisible by $2x - 3$ using polynomial long division. Use your work to factor the cubic. Solve the equation $6x^3 - 19x^2 + 17x - 3 = 0$.

$$
\begin{array}{rr}
\textbf{Quotient:} & 3x^2 - 5x + 1 \\
2x - 3 & \overline{\big)\, 6x^3 - 19x^2 + 17x - 3} \\
- & 6x^3 - 9x^2 \\ \hline
 & -10x^2 + 17x - 3 \\
- & -10x^2 + 15x \\ \hline
 & 2x - 3 \\
- & 2x - 3 \\ \hline
\textbf{Remainder:} & 0
\end{array}
$$

$$\textbf{(Dividend)} = \textbf{(Quotient)} \cdot \textbf{(Divisor)} + \textbf{(Remainder)}$$

$$(6x^3 - 19x^2 + 17x - 3) = (3x^2 - 5x + 1) \cdot (2x - 3)$$

Example

Demonstrate that $6x^3 - 19x^2 + 17x - 3$ is divisible by $2x - 3$ using polynomial long division. Use your work to factor the cubic. Solve the equation $6x^3 - 19x^2 + 17x - 3 = 0$.

$$\begin{aligned}(6x^3 - 19x^2 + 17x - 3) &= (3x^2 - 5x + 1) \cdot (2x - 3)\\ &= 3\left(x - \left(\tfrac{5+\sqrt{13}}{6}\right)\right)\left(x - \left(\tfrac{5-\sqrt{13}}{6}\right)\right)(2x - 3)\end{aligned}$$

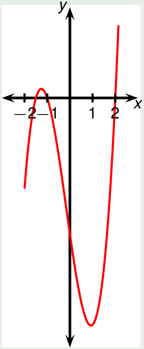
No easy factorization of quadratic, so use formula:

$x_1, x_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-5) \pm \sqrt{(-5)^2 - 4\cdot 3\cdot 1}}{2\cdot 3} = \frac{5 \pm \sqrt{13}}{6}$ We are ready to solve the equation.

$$\begin{aligned}6x^3 - 19x^2 + 17x - 3 &= 0\\ 3\left(x - \left(\tfrac{5+\sqrt{13}}{6}\right)\right)\left(x - \left(\tfrac{5-\sqrt{13}}{6}\right)\right)(2x - 3) &= 0\end{aligned}$$

$$\begin{aligned}2x - 3 &= 0 \quad \textbf{or} \quad x = \left(\tfrac{5+\sqrt{13}}{6}\right) \quad \textbf{or} \quad x = \left(\tfrac{5-\sqrt{13}}{6}\right)\\ x &= \tfrac{3}{2}\end{aligned}$$

Example



Plot the left hand side of the equation with a graphing calculator. Solve the equation.

$$\begin{aligned} 2x^3 + x^2 - 7x - 6 &= 0 \\ (2x+3)(x+1)(x-2) &= 0 \end{aligned}$$

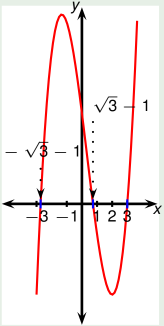
$$x = -\frac{3}{2} \quad \textbf{or} \quad x = -1 \quad \textbf{or} \quad x = 2$$

Make sure to practice with the graphing calculator you will on your exam(s). The graph appears to intersect the x axis at: $-1.5, -1, 2$. The left hand side should factor as:

$$2(x - (-1.5))(x - (-1))(x - 2) = (2x+3)(x+1)(x-2)$$
$$= (2x^2 + 5x + 3)(x - 2) = (2x^3 + 5x^2 + 3x) - (4x^2 + 10x + 6)$$
$$= 2x^3 + x^2 - 7x - 6$$

Check work to make sure we guessed the roots correctly.

Example



Plot the left hand side of the equation with a graphing calculator. Solve the equation.

$$\begin{aligned} x^3 - x^2 - 8x + 6 &= 0 \\ (x-3)(x^2+2x-2) &= 0 \end{aligned}$$

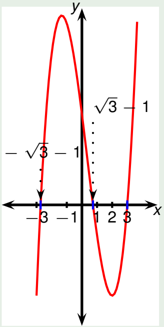
The graph appears to intersect the x axis at: $-\sqrt{3}-1$, $\sqrt{3}-1, 3$. What are the two roots besides 3?

Quotient: $x^2 + 2x - 2$

$$\begin{array}{rl} & x^2 + 2x - 2 \\ \hline x - 3 \; \big) & x^3 - x^2 - 8x + 6 \\ - & x^3 - 3x^2 \\ \hline & 2x^2 - 8x + 6 \\ - & 2x^2 - 6x \\ \hline & -2x + 6 \\ - & -2x + 6 \\ \hline & 0 \end{array}$$

Remainder: 0

Example



Plot the left hand side of the equation with a graphing calculator. Solve the equation.

$$\begin{aligned} x^3 - x^2 - 8x + 6 &= 0 \\ (x-3)(x^2+2x-2) &= 0 \end{aligned}$$

$$x - 3 = 0 \quad \textbf{or} \quad x = \frac{-2 \pm \sqrt{(2)^2 - 4\cdot 1\cdot(-2)}}{2\cdot 1}$$

$$x = 3 \qquad x = \frac{-2\pm\sqrt{12}}{2}$$

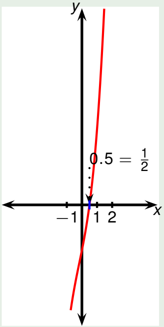
$$x = \frac{-2\pm 2\sqrt{3}}{2} = -1\pm\sqrt{3}.$$

The graph appears to intersect the x axis at: $-\sqrt{3}-1, \sqrt{3}-1, 3$. What are the two roots besides 3?

Final answer:

$x = 3$ **or** $x = -1-\sqrt{3}$ **or** $x = -1+\sqrt{3}$.

Example



Plot the left hand side of the equation with a graphing calculator. Find all real solutions of the equation.

$$\begin{aligned} 2x^3+x^2+5x-3 &= 0\\ \left(x-\tfrac{1}{2}\right)\left(2x^2+2x+6\right)+0 &= 0 \end{aligned}$$

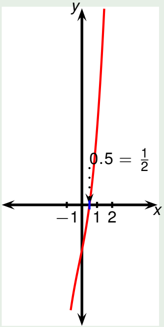
We see only one root, $x = 0.5 = \frac{1}{2}$. Is our guess correct? Is there another root (far away from 0)? Factor:

Quotient: $2x^2 + 2x + 6$

$$\begin{array}{r|l} x-\frac{1}{2} & 2x^3+x^2+5x-3 \\ - & 2x^3-x^2 \\ \hline & \phantom{2x^3+{}}2x^2+5x-3 \\ - & \phantom{2x^3+{}}2x^2-x \\ \hline & \phantom{2x^3+x^2+{}}6x-3 \\ - & \phantom{2x^3+x^2+{}}6x-3 \\ \hline & \phantom{2x^3+x^2+6x-{}}0 \end{array}$$

Remainder: 0

Example



Plot the left hand side of the equation with a graphing calculator. Find all real solutions of the equation.

$$\begin{aligned} 2x^3 + x^2 + 5x - 3 &= 0 \\ \left(x - \tfrac{1}{2}\right)\left(2x^2 + 2x + 6\right) + 0 &= 0 \end{aligned}$$

$$x - \frac{1}{2} = 0 \quad \textbf{or} \quad x = \frac{-2 \pm \sqrt{2^2 - 4\cdot 2\cdot 6}}{2\cdot 2}$$

$$x = \frac{1}{2} \qquad\qquad x = \frac{-2 \pm \sqrt{-44}}{2\cdot 2}$$

no real solution

We see only one root, $x = 0.5 = \frac{1}{2}$. Is our guess correct? Is there another root (far away from 0)?