Calculus I
Homework
Limits involving infinity

1. Show the following limits do not exist and compute whether they evaluate to ∞, $-\infty$, or neither.

(a) $\displaystyle\lim_{x\to 3^+} \frac{x^2+x-1}{x^2-2x-3}$. answer: ∞

(b) $\displaystyle\lim_{x\to 3^-} \frac{x^2+x-1}{x^2-2x-3}$. answer: $-\infty$

(c) $\displaystyle\lim_{x\to 1^+} \frac{x^2+1}{\sqrt{x^2+3}-2}$. answer: ∞

(d) $\displaystyle\lim_{x\to 1^-} \frac{x^2+1}{\sqrt{x^2+3}-2}$. answer: $-\infty$

(e) $\displaystyle\lim_{x\to 2^+} \frac{\sqrt{x^3-8}}{-x^2+x+2}$. answer: $-\infty$

(f) $\displaystyle\lim_{x\to -1^+} \frac{\sqrt[3]{x^2+2x+1}}{x^2-2x-3}$. answer: $-\infty$

2. Find the limit or show that it does not exist. If the limit does not exist, indicate whether it is $\pm\infty$, or neither. The answer key has not been proofread, use with caution.

(a) $\displaystyle\lim_{x\to\infty} \frac{x-2}{2x+1}$. answer: $\frac{1}{2}$

(b) $\displaystyle\lim_{x\to\infty} \frac{1-x^2}{x^3-x-1}$. answer: 0

(c) $\displaystyle\lim_{x\to-\infty} \frac{x-2}{x^2+5}$. answer: 0

(d) $\displaystyle\lim_{x\to-\infty} \frac{3x^3+2}{2x^3-4x+5}$. answer: $\frac{3}{2}$

(e) $\displaystyle\lim_{x\to\infty} \frac{\sqrt{x}+x^2}{\sqrt{x}-x^2}$. answer: -1

(f) $\displaystyle\lim_{x\to\infty} \frac{3-x\sqrt{x}}{2x^{\frac{3}{2}}-2}$. answer: $-\frac{1}{2}$

(g) $\displaystyle\lim_{x\to\infty} \frac{(2x^2+3)^2}{(x-1)^2(x^2+1)}$. answer: 4

(h) $\displaystyle\lim_{x\to\infty} \frac{x^2-3}{\sqrt{x^4+3}}$. answer: 1

(i) $\displaystyle\lim_{x\to-\infty} \frac{\sqrt{x^2+1}}{x+1}$. answer: -1

(j) $\displaystyle\lim_{x\to\infty} \frac{\sqrt{16x^6-3x}}{x^3+2}$. answer: 4

(k) $\displaystyle\lim_{x\to-\infty} \frac{\sqrt{16x^6-3x}}{x^3+2}$. answer: -4

(l) $\displaystyle\lim_{x\to\infty} \frac{\sqrt{3x^2+2x+1}}{x+1}$. answer: $\sqrt{3}$

(m) $\displaystyle\lim_{x\to\infty} \sqrt{4x^2+x}-2x$. answer: $\frac{1}{4}$

(n) $\displaystyle\lim_{x\to-\infty} x+\sqrt{x^2+3x}$. answer: $-\frac{3}{2}$

(o) $\displaystyle\lim_{x\to\infty} \sqrt{x^2+2x}-\sqrt{x^2-2x}$. answer: 2

(p) $\displaystyle\lim_{x\to-\infty} \sqrt{x^2+x}-\sqrt{x^2-x}$. answer: -1

(q) $\displaystyle\lim_{x\to\infty} \sqrt{x^2+ax}-\sqrt{x^2+bx}$. answer: $\frac{a-b}{2}$

(r) $\displaystyle\lim_{x\to\infty} \cos x$. answer: DNE

(s) $\displaystyle\lim_{x\to\infty} \frac{x^4+x}{x^3-x+2}$. answer: ∞

(t) $\displaystyle\lim_{x\to\infty} \sqrt{x^2+1}$. answer: ∞

(u) $\displaystyle\lim_{x\to-\infty} (x^4+x^5)$. answer: $-\infty$

(v) $\displaystyle\lim_{x\to-\infty} \frac{\sqrt{1+x^6}}{1+x^2}$. answer: ∞

(w) $\displaystyle\lim_{x\to\infty} (x-\sqrt{x})$. answer: ∞

(x) $\displaystyle\lim_{x\to\infty} (x^2-x^3)$. answer: $-\infty$

(y) $\displaystyle\lim_{x\to\infty} x\sin x$. answer: DNE

(z) $\displaystyle\lim_{x\to\infty} \sqrt{x}\sin x$. answer: DNE

Solution. 2.d.

$$
\begin{aligned}
\lim_{x\to-\infty} \frac{3x^3+2}{2x^3-4x+5} &= \lim_{x\to-\infty} \frac{(3x^3+2)\frac{1}{x^3}}{(2x^3-4x+5)\frac{1}{x^3}} \quad &&\text{Divide top and bottom}\\
&= \lim_{x\to-\infty} \frac{3+\frac{2}{x^3}}{2-\frac{4}{x^2}+\frac{5}{x^3}} \quad &&\text{by highest term in denominator}\\
&= \frac{3+0}{2-0+0} = \frac{3}{2}.
\end{aligned}
$$

Solution. 2.i

$$\lim_{x \to -\infty} \frac{\sqrt{x^2+1}}{x+1} = \lim_{x \to -\infty} \frac{\frac{1}{x}\sqrt{x^2+1}}{\frac{1}{x}(x+1)} = \lim_{x \to -\infty} \frac{-\frac{1}{\sqrt{x^2}}\sqrt{x^2+1}}{\frac{1}{x}(x+1)} \qquad \Big| \; x = -\sqrt{x^2}, \text{ whenever } x < 0$$

$$= \lim_{x \to -\infty} \frac{-\sqrt{\frac{x^2+1}{x^2}}}{1+\frac{1}{x}} = \lim_{x \to -\infty} \frac{-\sqrt{1+\underbrace{\frac{1}{x^2}}_{\to 0}}}{1+\underbrace{\frac{1}{x}}_{x \to 0}}$$

$$= 1.$$

Solution. 2.k.

$$\lim_{x \to -\infty} \frac{\sqrt{16x^6 - 3x}}{x^3+2} = \lim_{x \to -\infty} \frac{\sqrt{x^6\left(16-\frac{3}{x^5}\right)}}{x^3+2}$$

$$= \lim_{x \to -\infty} \frac{\sqrt{x^6}\sqrt{\left(16-\frac{3}{x^5}\right)}}{x^3+2} \qquad \Big| \; \sqrt{x^6} = -x^3 \text{ because } x < 0 \text{ as } x \to -\infty$$

$$= \lim_{x \to -\infty} \frac{-x^3\sqrt{\left(16-\frac{3}{x^5}\right)}}{x^3+2} \qquad \Big| \; \text{Divide by highest order term in denominator}$$

$$= \lim_{x \to -\infty} \frac{-x^3\sqrt{\left(16-\frac{3}{x^5}\right)}}{x^3+2}$$

$$= \lim_{x \to -\infty} \frac{-\cancel{x^3}\sqrt{\left(16-\frac{3}{x^5}\right)}\frac{1}{\cancel{x^3}}}{(x^3+2)\frac{1}{x^3}}$$

$$= \lim_{x \to -\infty} \frac{-\sqrt{\left(16-\underbrace{\frac{3}{x^5}}_{\to 0}\right)}}{1+\underbrace{\frac{2}{x^3}}_{\to 0}}$$

$$= \frac{-\sqrt{16}}{1} = -4.$$

Solution. 2.l

$$\lim_{x \to \infty} \frac{\sqrt{3x^2+2x+1}}{x+1} = \lim_{x \to \infty} \frac{\frac{1}{x}\sqrt{3x^2+2x+1}}{\frac{1}{x}(x+1)}$$

$$= \lim_{x \to \infty} \frac{\sqrt{\frac{3x^2+2x+1}{x^2}}}{\left(1+\frac{1}{x}\right)}$$

$$= \lim_{x \to \infty} \frac{\sqrt{3+\frac{2}{x}+\frac{1}{x^2}}}{\left(1+\frac{1}{x}\right)}$$

$$= \frac{\sqrt{3+0+0}}{1+0}$$

$$= \sqrt{3}.$$

Solution. 2.p.

$$\lim_{x \to -\infty} \sqrt{x^2+x} - \sqrt{x^2-x} = \lim_{x \to -\infty} \left(\sqrt{x^2+x}-\sqrt{x^2-x}\right)\frac{\left(\sqrt{x^2+x}+\sqrt{x^2-x}\right)}{\left(\sqrt{x^2+x}+\sqrt{x^2-x}\right)}$$

$$= \lim_{x \to -\infty} \frac{x^2+x-(x^2-x)}{\sqrt{x^2+x}+\sqrt{x^2-x}} = \lim_{x \to -\infty} \frac{2x\frac{1}{x}}{\left(\sqrt{x^2+x}+\sqrt{x^2-x}\right)\frac{1}{x}}$$

$$= \lim_{x \to -\infty} \frac{2}{\frac{\sqrt{x^2+x}}{x}+\frac{\sqrt{x^2-x}}{x}} = \lim_{x \to -\infty} \frac{2}{-\sqrt{\frac{x^2+x}{x^2}}-\sqrt{\frac{x^2-x}{x^2}}}$$

$$= \lim_{x \to -\infty} \frac{2}{-\sqrt{1+\frac{1}{x}}-\sqrt{1-\frac{1}{x}}} = \frac{2}{-\sqrt{1+0}-\sqrt{1-0}} = -1.$$

The sign highlighted in red arises from the fact that, for negative x, we have that $x = -\sqrt{x^2}$.

3. Find the horizontal and vertical asymptotes of the graph of the function. If a graphing device is available, check your work by plotting the function.

(a) $y = \dfrac{2x}{\sqrt{x^2 + x + 3} - 3}$.

answer: vertical: $x = -1$, $x = 3$, horizontal: $y = -5$

(b) $y = \dfrac{3x^2}{\sqrt{x^2 + 2x + 10} - 5}$.

answer Vertical: $x = 2$, $x = -3$, horizontal: $y = 2$, $y = -2$

(c) $y = \dfrac{3x + 1}{x - 2}$.

answer Vertical: $x = 3$, $x = -5$, horizontal: none.

(d) $y = \dfrac{x^2 - 1}{2x^2 - 3x - 2}$.

answer vertical: $x = 2$, horizontal: $y = 3$

(e) $y = \dfrac{2x^2 - 2x - 1}{x^2 + x - 2}$.

answer vertical: $x = 2$, $x = -\frac{1}{2}$, horizontal: $y = \frac{1}{2}$

(f) $f(x) = \dfrac{-5x^2 - 3x + 5}{x^2 - 2x - 3}$.

answer vertical: $x = 1$, $x = -2$, horizontal: $y = 2$

(g) $y = \dfrac{1 + x^4}{x^2 - x^4}$.

answer vertical: $x = 0$, $x = 1$, $x = -1$, horizontal: $y = -1$

(h) $y = \dfrac{x^3 - x}{x^2 - 7x + 6}$.

answer vertical: $x = 6$, no horizontal asymptote

(i) $y = \dfrac{x - 9}{\sqrt{4x^2 + 3x + 3}}$.

answer: no vertical asymptote, horizontal: $y = \pm \frac{1}{2}$

(j) $y = \dfrac{\sqrt{x^2 + 1} - x}{x}$.

answer vertical: $x = 0$, horizontal: $y = 0$, $y = -2$

(k) $f(x) = \dfrac{x}{\sqrt{x^2 + 3} - 2x}$

answer vertical: $x = 1$, horizontal: $y = \frac{1}{3}$, $y = -1$

Solution. 3.a **Vertical asymptotes.** A function $f(x)$ has a vertical asymptote at $x = a$ if $\lim_{x \to a} f(x) = \pm\infty$.

The function is algebraic, and therefore has a finite limit at every point it is defined (i.e., no asymptote). Therefore the function can have vertical asymptotes only for those x for which $f(x)$ is not defined. The function is not defined for $\sqrt{x^2 + x + 3} - 3 = 0$, which has two solutions, $x = 2$ and $x = -3$. These are precisely the vertical asymptotes: indeed,

$$\lim_{x \to 2^+} \frac{2x}{\sqrt{x^2 + x + 3} - 3} = \infty \qquad \lim_{x \to 2^-} \frac{2x}{\sqrt{x^2 + x + 3} - 3} = -\infty$$

and

$$\lim_{x \to -3^+} \frac{2x}{\sqrt{x^2 + x + 3} - 3} = \infty \qquad \lim_{x \to -3^-} \frac{2x}{\sqrt{x^2 + x + 3} - 3} = -\infty$$

Horizontal asymptotes. A function $f(x)$ has a horizontal asymptote if $\lim_{x \to \pm\infty} f(x)$ exists. If that limit exists, and is some number, say, N, then $y = N$ is the equation of the corresponding asymptote.
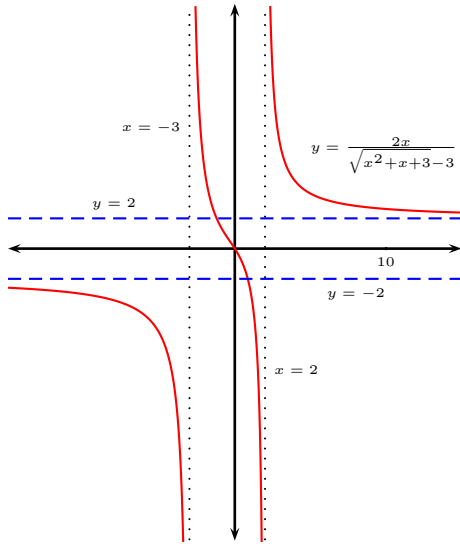
Consider the limit $x \to -\infty$. We have that

$$
\begin{aligned}
\lim_{x \to -\infty} \frac{2x}{\sqrt{x^2 + 3x + 3} - 3} &= \lim_{x \to -\infty} \frac{2}{\frac{\sqrt{x^2 + x + 3}}{x} - \frac{3}{x}} \\
&= \lim_{x \to -\infty} \frac{2}{-\sqrt{\frac{x^2 + 3x + 3}{x^2}} - \frac{3}{x}} \\
&= \lim_{x \to -\infty} \frac{2}{-\sqrt{1 + \frac{3}{x} + \frac{3}{x^2}} - \frac{3}{x}} \\
&= \frac{\lim_{x \to -\infty} 2}{-\sqrt{\lim_{x \to -\infty} 1 + \lim_{x \to -\infty} \frac{3}{x} + \lim_{x \to -\infty} \frac{3}{x^2}} - \lim_{x \to -\infty} \frac{3}{x}} \\
&= \frac{2}{-\sqrt{1 + 0 + 0} - 0} \\
&= -2 \quad .
\end{aligned}
$$

$\left| \quad \frac{1}{x} = -\sqrt{\frac{1}{x^2}} \text{ when } x < 0 \right.$

Therefore $y = -2$ is a horizontal asymptote.

The case $x \to \infty$, is handled similarly and yields that $y = 2$ is a horizontal asymptote.

A computer generated graph confirms our computations.

Solution. 3.d

Vertical asymptotes. A function $f(x)$ has a vertical asymptote at $x = a$ if $\lim\limits_{x \to a} f(x) = \pm\infty$.

The function is algebraic, and therefore has a finite limit at every point it is defined (i.e., no asymptote). Therefore the function can have vertical asymptotes only for those x for which $f(x)$ is not defined. The function is not defined for $2x^2 - 3x - 2 = 0$, which has two solutions, $x = 2$ and $x = -\frac{1}{2}$. These are precisely the vertical asymptotes: indeed,

$$\lim_{x \to 2^+} \frac{x^2 - 1}{2x^2 - 3x - 2} = \lim_{x \to 2^+} \frac{x^2 - 1}{2(x-2)\left(x + \frac{1}{2}\right)} = \infty \qquad \left| \text{ Limit of form } \frac{(+)}{(+)(+)} \right.$$

$$\lim_{x \to 2^-} \frac{x^2 - 1}{2x^2 - 3x - 2} = \lim_{x \to 2^-} \frac{x^2 - 1}{2(x-2)\left(x + \frac{1}{2}\right)} = -\infty \qquad \left| \text{ Limit of form } \frac{(+)}{(-)(+)} \right.$$

and

$$\lim_{x \to -\frac{1}{2}^+} \frac{x^2 - 1}{2x^2 - 3x - 2} = \lim_{x \to -\frac{1}{2}^+} \frac{x^2 - 1}{2(x-2)\left(x + \frac{1}{2}\right)} = \infty \qquad \left| \text{ Limit of form } \frac{(-)}{(+)(-)} \right.$$

$$\lim_{x \to -\frac{1}{2}^-} \frac{x^2 - 1}{2x^2 - 3x - 2} = \lim_{x \to -\frac{1}{2}^-} \frac{x^2 - 1}{2(x-2)\left(x + \frac{1}{2}\right)} = -\infty \qquad \left| \text{ Limit of form } \frac{(-)}{(-)(-)} \right.$$

Horizontal asymptotes. A function $f(x)$ has a horizontal asymptote if $\lim\limits_{x \to \pm\infty} f(x)$ exists. If that limit exists, and is some number, say, N, then $y = N$ is the equation of the corresponding asymptote.
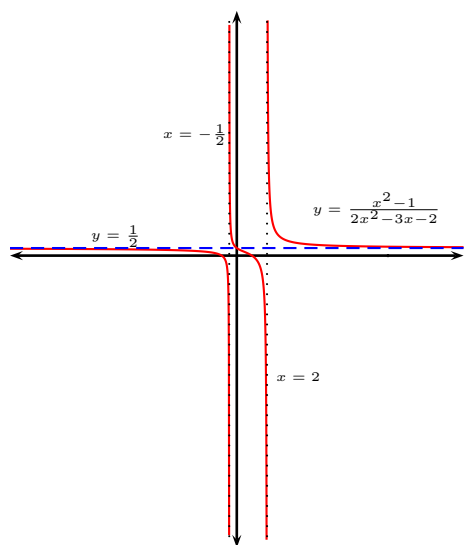
We have that

$$\lim_{x \to \infty} \frac{x^2 - 1}{2x^2 - 3x - 2} = \lim_{x \to \infty} \frac{\left(x^2 - 1\right)\frac{1}{x^2}}{\left(2x^2 - 3x - 2\right)\frac{1}{x^2}} \qquad \left| \text{ Divide by highest term in den.} \right.$$

$$= \lim_{x \to \infty} \frac{1 - \frac{1}{x^2}}{2 - \frac{3}{x} - \frac{2}{x^2}}$$

$$= \frac{\lim\limits_{x \to \infty} 1 - \lim\limits_{x \to \infty} \frac{1}{x^2}}{\lim\limits_{x \to \infty} 2 - \lim\limits_{x \to \infty} \frac{3}{x} - \lim\limits_{x \to \infty} \frac{2}{x^2}} \qquad \left| \text{ Step may be skipped} \right.$$

$$= \frac{1 - 0}{2 - 0 - 0}$$

$$= \frac{1}{2}$$

A similar computation shows that

$$\lim_{x \to -\infty} \frac{x^2 - 1}{2x^2 - 3x - 2} = \frac{1}{2}$$

Therefore $y = \frac{1}{2}$ is the only horizontal asymptote, valid in both directions ($x \to \pm\infty$).

A computer generated graph confirms our computations.

Solution. 3.f

Vertical asymptotes. The function is rational, and therefore has a finite limit (and therefore no vertical asymptote) at every point it its domain. The function is not defined for $x^2 - 2x - 3 = 0$, which has two solutions, $x = -1$ and $x = 3$. These are precisely the vertical asymptotes: indeed,

$$
\begin{aligned}
\lim_{x \to -1^+} \frac{-5x^2 - 3x + 5}{x^2 - 2x - 3} &= \lim_{x \to -1^+} \frac{-5x^2 - 3x + 5}{(x+1)(x-3)} = -\infty & \left| \text{Limit of form } \frac{(+)}{(+)(-)} \right. \\
\lim_{x \to -1^-} \frac{-5x^2 - 3x + 5}{x^2 - 2x - 3} &= \lim_{x \to -1^-} \frac{-5x^2 - 3x + 5}{(x+1)(x-3)} = \infty & \left| \text{Limit of form } \frac{(+)}{(-)(-)} \right.
\end{aligned}
$$

and

$$
\begin{aligned}
\lim_{x \to 3^+} \frac{-5x^2 - 3x + 5}{x^2 - 2x - 3} &= \lim_{x \to 3^+} \frac{-5x^2 - 3x + 5}{(x+1)(x-3)} = -\infty & \left| \text{Limit of form } \frac{(-)}{(+)(+)} \right. \\
\lim_{x \to 3^-} \frac{-5x^2 - 3x + 5}{x^2 - 2x - 3} &= \lim_{x \to 3^-} \frac{-5x^2 - 3x + 5}{(x+1)(x-3)} = \infty & \left| \text{Limit of form } \frac{(-)}{(+)(-)} \right.
\end{aligned}
$$

Horizontal asymptotes.

$$
\begin{aligned}
\lim_{x \to \pm\infty} \frac{-5x^2 - 3x + 5}{x^2 - 2x - 3} &= \lim_{x \to \pm\infty} \frac{\left(-5x^2 - 3x + 5\right)\frac{1}{x^2}}{\left(x^2 - 2x - 3\right)\frac{1}{x^2}} & \left| \text{Divide by highest term in den.} \right. \\
&= \lim_{x \to \pm\infty} \frac{-5 - \frac{3}{x} + \frac{5}{x^2}}{1 - \frac{2}{x} - \frac{3}{x^2}} & \\
&= \frac{-\lim_{x \to \pm\infty} 5 - \lim_{x \to \pm\infty} \frac{3}{x} + \lim_{x \to \pm\infty} \frac{5}{x^2}}{\lim_{x \to \pm\infty} 1 - \lim_{x \to \pm\infty} \frac{2}{x} - \lim_{x \to \pm\infty} \frac{3}{x^2}} & \left| \text{Step may be skipped} \right. \\
&= \frac{-5 - 0 + 0}{1 - 0 - 0} & \\
&= -5.
\end{aligned}
$$

Therefore $y = -5$ is the only horizontal asymptote, valid in both directions ($x \to \pm\infty$).

A computer generated graph confirms our computations.

$$x = -1$$

$$y = \frac{-5x^2 - 3x + 5}{x^2 - 2x - 3}$$

$$y = -5$$

$$x = 3$$
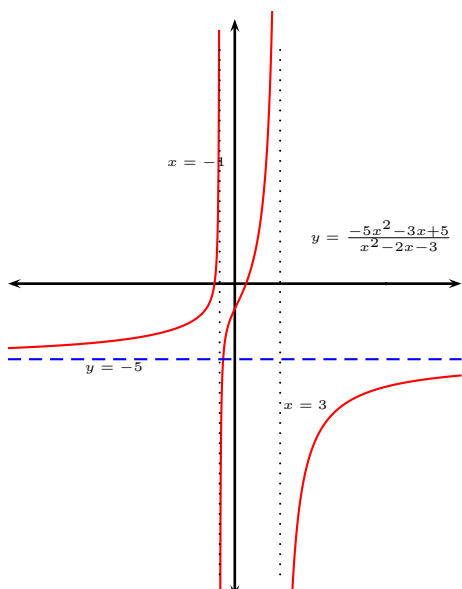
Solution. 3.k

Vertical asymptotes. A function $f(x)$ has a vertical asymptote at $x = a$ if $\lim\limits_{x \to a} f(x) = \pm\infty$.

The function is algebraic, and therefore has a finite limit at every point it is defined (i.e., no asymptote). Therefore the function can have vertical asymptotes only for those x for which $f(x)$ is not defined. The function is not defined for

$$
\begin{aligned}
\sqrt{x^2 + 3} - 2x &= 0 \\
\sqrt{x^2 + 3} &= 2x \\
x^2 + 3 &= 4x^2 \\
3x^2 - 3 &= 0 \\
3(x - 1)(x + 1) &= 0 \\
x = 1 \quad \text{or} \quad x &= -1
\end{aligned}
$$

square both sides
may introduce extraneous solutions

$x = -1$ is extraneous:
$$\sqrt{(-1)^2 + 3} - (-1)2 = 4 \neq 0$$

$x = -1$ is indeed a vertical asymptote:

$$\lim_{x \to 1^+} \frac{x}{\sqrt{x^2 + 3} - 2x} = \infty \qquad \lim_{x \to 1^-} \frac{x}{\sqrt{x^2 + 3} - 2x} = -\infty.$$

Horizontal asymptotes.

$$
\begin{aligned}
\lim_{x \to -\infty} \frac{x}{\sqrt{x^2 + 3} - 2x}
&= \lim_{x \to -\infty} \frac{1}{\frac{\sqrt{x^2+3}}{x} - 2} \\
&= \lim_{x \to -\infty} \frac{1}{-\sqrt{\frac{x^2+3}{x^2}} - 2} \qquad\qquad \left| \ \frac{1}{x} = -\sqrt{\frac{1}{x^2}} \text{ when } x < 0 \right. \\
&= \lim_{x \to -\infty} \frac{1}{-\sqrt{1 + \frac{3}{x^2}} - 2} \\
&= \frac{1}{-\sqrt{1 + 0} - 2} \\
&= -\frac{1}{3}. \\[4pt]
\lim_{x \to -\infty} \frac{x}{\sqrt{x^2 + 3} - 2x}
&= \lim_{x \to \infty} \frac{1}{\frac{\sqrt{x^2+3}}{x} - 2} \\
&= \lim_{x \to \infty} \frac{1}{\sqrt{\frac{x^2+3}{x^2}} - 2} \qquad\qquad \left| \ \frac{1}{x} = \sqrt{\frac{1}{x^2}} \text{ when } x > 0 \right. \\
&= \lim_{x \to \infty} \frac{1}{\sqrt{1 + \frac{3}{x^2}} - 2} \\
&= \frac{1}{\sqrt{1 + 0} - 2} \\
&= -1.
\end{aligned}
$$

Therefore $y = -\frac{1}{3}$ and $y = -1$ are the two horizontal asymptotes.

A computer generated graph confirms our computations.