Precalculus
Completing the square

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Example (Completing the square)

Complete the square.

$$
\begin{aligned}
3x^2 - 5x + 1 &= 3\left(x^2 - \frac{5}{3}x\right) + 1 \\
&= 3\left(x^2 - 2 \cdot \frac{5}{2 \cdot 3}x\right) + 1 \\
&= 3\left(x^2 - 2 \cdot \frac{5}{6}x + \left(\frac{5}{6}\right)^2 - \left(\frac{5}{6}\right)^2\right) + 1 \\
&= 3\left(\left(x - \frac{5}{6}\right)^2 - \frac{25}{36}\right) + 1 \\
&= 3\left(x - \frac{5}{6}\right)^2 - \frac{25}{12} + 1 \\
&= 3\left(x - \frac{5}{6}\right)^2 - \frac{13}{12}.
\end{aligned}
$$

Definition (Completing the square)

Let $a \neq 0$. To *complete the square* means to carry out the following algebraic manipulation.

$$
\begin{aligned}
ax^2+bx+c &= a\left(x^2+\frac{b}{a}x\right)+c \\
&= a\left(x^2+2\cdot\frac{b}{2a}x\right)+c \\
&= a\left(x^2+2\frac{b}{2a}x+\left(\frac{b}{2a}\right)^2-\left(\frac{b}{2a}\right)^2\right)+c \\
&= a\left(\left(x+\frac{b}{2a}\right)^2-\frac{b^2}{4a^2}\right)+c \\
&= a\left(x+\frac{b}{2a}\right)^2-\cancel{a}\cdot\frac{b^2}{4\cancel{a}^2}+c \\
&= a\left(x+\frac{b}{2a}\right)^2+c-\frac{b^2}{4a}.
\end{aligned}
$$

Add & subtract $\left(\frac{b}{2a}\right)^2$

use $(A+B)^2 = A^2+2AB+B^2$