

Precalculus

Homework

Inverse functions

1. Convert from degrees to radians.

(a) 15° .

ANSWER: $\frac{12}{\pi} \approx 0.261799388$

(b) 30° .

ANSWER: $\frac{6}{\pi} \approx 0.523598776$

(c) 36° .

ANSWER: $\frac{5}{\pi} \approx 0.628318531$

(d) 45° .

ANSWER: $\frac{4}{\pi} \approx 0.785398163$

(e) 60° .

ANSWER: $\frac{3}{2\pi} \approx 1.047197551$

(f) 75° .

ANSWER: $\frac{12}{5\pi} \approx 1.308997$

(g) 90° .

ANSWER: $\frac{2}{\pi}$

(h) 120° .

(i) 135° .

(j) 150° .

(k) 180° .

(l) 225° .

(m) 270° .

(n) 305° .

ANSWER: $\frac{3}{2\pi}$

(o) 360° .

ANSWER: $\frac{4}{3\pi}$

(p) 405° .

ANSWER: $\frac{6}{5\pi}$

(q) 1200° .

ANSWER: π

(r) -900° .

ANSWER: $-\frac{4}{5\pi}$

(s) -2014° .

ANSWER: $\frac{7}{3\pi}$

ANSWER: $-\frac{1007}{90}\pi \approx -35.150931$

ANSWER: $\left(\frac{\pi}{900}\right)^\circ \approx 286^\circ$

ANSWER: -362520°

ANSWER: 362520°

2. Convert from radians to degrees. The answer key has not been proofread, use with caution.

(a) 4π .

ANSWER: 720°

(b) $-\frac{7}{6}\pi$.

ANSWER: -210°

(c) $\frac{7}{12}\pi$.

ANSWER: 105°

(d) $\frac{4}{3}\pi$.

(e) $-\frac{3}{8}\pi$.

(f) 2014π .

ANSWER: 240°

ANSWER: -67.5°

(g) 5 .

(h) -2014 .

3. Find the indicated circle arc-length. The answer key has not been proofread, use with caution.

(a) Circle of radius 3, arc of measure 36° .

ANSWER: $\frac{5}{3}\pi \approx 1.884956$

(b) Circle of radius $\frac{1}{2}$, arc of measure 100° .

ANSWER: $\frac{18}{5}\pi \approx 0.872665$

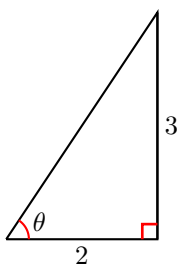
(c) Circle of radius 1, arc of measure 3 (radians).

ANSWER: 3

(d) Circle of radius 3, arc of measure 300° .

ANSWER: $5\pi \approx 15.707963$

4. Find the 6 trigonometric functions of the indicated angle in the indicated right triangle.



(a)

$$\frac{3}{\sqrt{13}} = \theta \csc \theta, \frac{2}{\sqrt{13}} = \theta \sec \theta, \frac{3}{2} = \theta \cot \theta, \frac{2}{3} = \theta \tan \theta, \frac{1}{\sqrt{13}} = \theta \cos \theta, \frac{3}{\sqrt{13}} = \theta \sin \theta$$



(b)

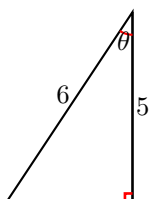
$$\sqrt{5} = \theta \csc \theta, \frac{2}{\sqrt{5}} = \theta \sec \theta, \frac{1}{2} = \theta \cot \theta, \frac{2}{1} = \theta \tan \theta, \frac{2}{\sqrt{5}} = \theta \cos \theta, \frac{1}{\sqrt{5}} = \theta \sin \theta$$



(c)

$$\frac{2}{\sqrt{29}} = \theta \csc \theta, \frac{5}{\sqrt{29}} = \theta \sec \theta, \frac{2}{5} = \theta \cot \theta, \frac{5}{2} = \theta \tan \theta, \frac{6\sqrt{29}}{2} = \theta \cos \theta, \frac{6\sqrt{29}}{5} = \theta \sin \theta$$

(d)



$$\frac{11}{\sqrt{13}} = \theta \csc \theta, \frac{11}{9} = \theta \sec \theta, \frac{11}{5} = \theta \cot \theta, \frac{5}{11} = \theta \tan \theta, \frac{9}{\sqrt{13}} = \theta \cos \theta, \frac{9}{11} = \theta \sin \theta$$

5. Find the exact value of the trigonometric function (using radicals).

(a) $\cos 135^\circ$.

ANSWER:

(b) $\sin 225^\circ$.

ANSWER:

(c) $\cos 495^\circ$.

ANSWER:

(d) $\sin 560^\circ$.

ANSWER:

(e) $\sin \left(\frac{3\pi}{2} \right)$.

ANSWER:

(f) $\cos \left(\frac{11\pi}{6} \right)$.

ANSWER:

$$(g) \sin\left(\frac{2015\pi}{3}\right).$$

$$(h) \cos\left(\frac{17\pi}{3}\right).$$

6. Find all solutions of the equation in the interval $[0, 2\pi)$. The answer key has not been proofread, use with caution.

$$(a) \sin x = -\frac{\sqrt{2}}{2}.$$

$$(b) \cos x = \frac{\sqrt{3}}{2}.$$

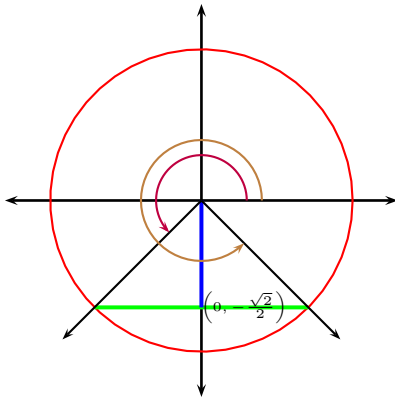
$$(c) \sin(3x) = \frac{1}{2}.$$

$$(d) \cos(7x) = 0.$$

$$(e) \cos\left(3x + \frac{\pi}{2}\right) = 0.$$

$$(f) \sin\left(5x - \frac{\pi}{3}\right) = 0.$$

Solution. 6.a



$$\sin x = -\frac{\sqrt{2}}{2}$$

Since $\sin x$ is negative it must be either in Quadrant III or IV. Therefore the angle x is coterminal either with $225^\circ = \frac{5\pi}{4}$ (Quadrant III) or $315^\circ = \frac{7\pi}{4}$ (Quadrant IV).

Case 1. x is coterminal with $225^\circ = \frac{5\pi}{4}$. We can compute

$$\begin{aligned} x &= \frac{5\pi}{4} + 2k\pi & \left| \begin{array}{l} k \text{ is any integer} \end{array} \right. \\ x &= \frac{5\pi}{4} + \frac{8k\pi}{4} \\ x &= \frac{5\pi + 8k\pi}{4} \\ x &= \frac{\pi(5 + 8k)}{4} \end{aligned}$$

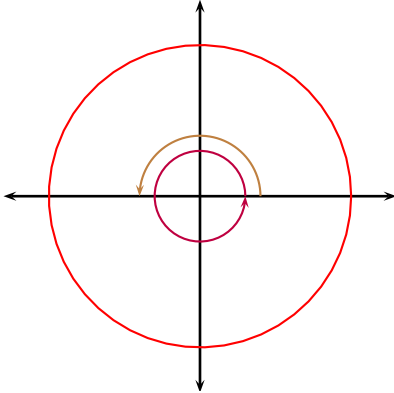
We are looking for solutions in the interval $[0, 2\pi)$ and so we must discard those values of the integer k for which $\frac{\pi(5+8k)}{4}$ is negative or is greater than or equal to 2π . Therefore the only solution in this case is $x = \frac{5\pi}{4}$.

Case 2.

$$\begin{aligned}
 x &= \frac{7\pi}{4} + 2k\pi \\
 x &= \frac{7\pi}{4} + \frac{8k\pi}{4} \\
 x &= \frac{7\pi + 8k\pi}{4} \\
 x &= \frac{\pi(7 + 8k)}{4}
 \end{aligned}$$

We are looking for solutions in the interval $[0, 2\pi)$ and so we must discard those values of the integer k for which $\frac{\pi(7+8k)}{4}$ is negative or is greater than or equal to 2π . Therefore the only solution in this case is $x = \frac{7\pi}{4}$.

Solution. 6.f



$$\sin\left(5x - \frac{\pi}{3}\right) = 0$$

Since $\sin 0 = 0$ and $\sin 180^\circ = \sin \pi = 0$, the angle $5x - \frac{\pi}{3}$ must be coterminal with 0 or π .

Case 1. $5x - \frac{\pi}{3}$ is coterminal with 0. We compute

$$\begin{aligned}
 5x - \frac{\pi}{3} &= 0 + 2k\pi \\
 5x &= \frac{\pi}{3} + 2k\pi \\
 x &= \frac{\frac{\pi}{3} + 2k\pi}{5} \\
 x &= \frac{\frac{\pi}{3} + \frac{6k\pi}{3}}{5} \\
 x &= \frac{\frac{\pi + 6k\pi}{3}}{5} \\
 x &= \frac{\pi + 6k\pi}{15} \\
 x &= \frac{\pi(1 + 6k)}{15}
 \end{aligned}$$

$$x = \cancel{\frac{\pi}{15}}, \frac{\pi[1 + 6(0)]}{15}, \frac{\pi[1 + 6(1)]}{15}, \frac{\pi[1 + 6(2)]}{15}, \frac{\pi(1 + 12)}{15}, \frac{\pi[1 + 6(3)]}{15}, \frac{\pi[1 + 6(4)]}{15}, \cancel{\frac{\pi(1 + 24)}{15}}$$

$$x = \frac{\pi}{15}, \frac{7\pi}{15}, \frac{13\pi}{15}, \frac{19\pi}{15}, \frac{25\pi}{15}.$$

Discard other values of k as they yield angles outside of $[0, 2\pi)$

Case 2.

$$\begin{aligned}
5x - \frac{\pi}{3} &= \pi + 2k\pi \\
5x &= \pi + \frac{\pi}{3} + 2k\pi \\
5x &= \frac{4\pi}{3} + 2k\pi \\
x &= \frac{\frac{4\pi}{3} + 2k\pi}{5} \\
x &= \frac{\frac{4\pi}{3} + \frac{6k\pi}{3}}{5} \\
x &= \frac{\frac{4\pi + 6k\pi}{3}}{5} \\
x &= \frac{4\pi + 6k\pi}{15} \\
x &= \frac{2\pi(2 + 3k)}{15}
\end{aligned}$$

$$x = \cancel{\frac{2\pi}{15}}, \frac{2\pi[2 + 3(0)]}{15}, \frac{2\pi[2 + 3(1)]}{15}, \frac{2\pi[2 + 3(2)]}{15}, \frac{2\pi[2 + 3(3)]}{15}, \frac{2\pi[2 + 3(4)]}{15}, \dots$$

$$x = \frac{4\pi}{15}, \frac{10\pi}{15}, \frac{16\pi}{15}, \frac{22\pi}{15}, \frac{28\pi}{15}.$$

Discard other values of k as they yield angles outside of $[0, 2\pi)$

Our final answer (combined from the two cases) is $x = \frac{\pi}{15}, \frac{4\pi}{15}, \frac{7\pi}{15}, \frac{2\pi}{3}, \frac{13\pi}{15}, \frac{16\pi}{15}, \frac{19\pi}{15}, \frac{22\pi}{15}, \frac{5\pi}{3}$ or $\frac{28\pi}{15}$.

7. Use the known values of $\sin 30^\circ$, $\cos 30^\circ$, $\sin 45^\circ$, $\cos 45^\circ$, $\sin 60^\circ$, $\cos 60^\circ$, \dots , the angle sum formulas and the cofunction identities to find an exact value (using radicals) for the trigonometric function.

(a) The six trigonometric functions of $105^\circ = 45^\circ + 60^\circ$:

$$\bullet \sin(105^\circ).$$

$$\bullet \cos(105^\circ). \text{ Should your answer be a positive or a negative number?}$$

$$\bullet \tan(105^\circ).$$

$$\bullet \cot(105^\circ).$$

$$\bullet \sec(105^\circ).$$

$$\bullet \csc(105^\circ).$$

(b) The six trigonometric functions of $\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}$:

$$\bullet \sin\left(\frac{\pi}{12}\right).$$

$$\bullet \cos\left(\frac{\pi}{12}\right). \text{ Should } \sin\left(\frac{\pi}{12}\right) \text{ be larger or smaller than } \cos\left(\frac{\pi}{12}\right)?$$

$$\bullet \tan\left(\frac{\pi}{12}\right).$$

$$\bullet \cot\left(\frac{\pi}{12}\right).$$

$$\bullet \sec\left(\frac{\pi}{12}\right).$$

$$\bullet \csc\left(\frac{\pi}{12}\right).$$

8. Simplify to a trigonometric function of the angle θ . The answer key has not been proofread, use with caution.

$$(a) \sin\left(\frac{\pi}{2} - \theta\right).$$

$$(b) \cos\left(\frac{13\pi}{2} - \theta\right).$$

$$(c) \tan(\pi - \theta)$$

$$(d) \cot\left(\frac{3\pi}{2} - \theta\right)$$

$$(e) \csc\left(\frac{3\pi}{2} + \theta\right)$$

9. Using the power-reducing formulas, rewrite the expression in terms of first powers of the cosines and sines of multiples of the angle θ .

(a) $\sin^4 \theta$.

ANSWER: $\frac{8}{1} \cos(\theta) - \frac{7}{1} \cos(2\theta) + \frac{8}{3}$

(b) $\cos^4 \theta$.

ANSWER: $\frac{8}{1} \cos(\theta) + \frac{7}{1} \cos(2\theta) + \frac{8}{3}$

(c) $\sin^6 \theta$.

ANSWER: $\sin^6 \theta = -\frac{1}{3} \cos(6\theta) + \frac{6}{3} \cos(4\theta) - \frac{15}{3} \cos(2\theta) + \frac{10}{3}$

(d) $\cos^6 \theta$.

ANSWER: $\cos^6 \theta = \frac{1}{3} \cos(6\theta) + \frac{6}{3} \cos(4\theta) + \frac{15}{3} \cos(2\theta) + \frac{10}{3}$

10. Use the sum-to-product formulas to find all solutions of the trigonometric equation in the interval $[0, 2\pi)$.

Please note that typing a query such as “solve(sin(x)+sin(3x)=0)” at www.wolframalpha.com will provide you with a correct answer and a function plot.

(a) $\sin(x) + \sin(3x) = 0$.

ANSWER: $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$

(b) $\cos(x) + \cos(-3x) = 0$.

ANSWER: $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}, \frac{11\pi}{2}, \frac{13\pi}{2}, \frac{15\pi}{2}$

(c) $\sin(x) - \sin(3x) = 0$.

ANSWER: $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}, \frac{11\pi}{2}, \frac{13\pi}{2}, \frac{15\pi}{2}$

(d) $\cos(2x) - \cos(3x) = 0$.

ANSWER: $x = 0, \frac{\pi}{8}, \frac{5\pi}{8}, \frac{3\pi}{4}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{5\pi}{4}, \frac{11\pi}{8}, \frac{13\pi}{8}, \frac{7\pi}{4}, \frac{15\pi}{8}, \frac{17\pi}{8}, \frac{9\pi}{4}$

11. Find the inverse function. You are asked to do the algebra only; you are not asked to determine the domain or range of the function or its inverse.

(a) $f(x) = 3x^2 + 4x - 7$, where $x \geq -\frac{2}{3}$.

ANSWER: $f^{-1}(x) = \frac{\sqrt{12x+25} - 2}{6}$

(b) $f(x) = 2x^2 + 3x - 5$, where $x \geq -\frac{3}{4}$.

ANSWER: $f^{-1}(x) = \frac{\sqrt{4x+25} + 3}{4}$

(c) $f(x) = \frac{2x+5}{x-4}$, where $x \neq 4$.

ANSWER: $f^{-1}(x) = \frac{2-x}{x+5}$

(d) $f(x) = \frac{3x+5}{2x-4}$, where $x \neq 2$.

ANSWER: $f^{-1}(x) = \frac{2x-3}{x+5}$

(e) $f(x) = \frac{5x+6}{4x+5}$, where $x \neq -\frac{5}{4}$.

ANSWER: $f^{-1}(x) = \frac{5-x}{x+6}$

(f) $f(x) = \frac{2x-3}{-3x+4}$, where $x \neq \frac{4}{3}$.

ANSWER: $f^{-1}(x) = \frac{3-x}{x+4}$

Solution. 11.d This is a concise solution written in form suitable for test taking.

$$\begin{aligned} y &= \frac{3x+5}{2x-4} \\ y(2x-4) &= 3x+5 \\ 2xy-4y &= 3x+5 \\ 2xy-3x &= 4y+5 \\ x(2y-3) &= 4y+5 \\ x &= \frac{4y+5}{2y-3} \\ \text{Therefore } f^{-1}(y) &= \frac{4y+5}{2y-3} \\ f^{-1}(x) &= \frac{4x+5}{2x-3} \end{aligned}$$

Solution. 11.e. Set $f(x) = y$. Then

$$\begin{aligned} y &= \frac{5x+6}{4x+5} \\ y(4x+5) &= 5x+6 \\ x(4y-5) &= -5y+6 \\ x &= \frac{-5y+6}{4y-5}. \end{aligned}$$

Therefore the function $x = g(y) = \frac{-5y+6}{4y-5}$ is the inverse of $f(x)$. We write $g = f^{-1}$. The function $g = f^{-1}$ is defined for $y \neq \frac{5}{4}$. For our final answer we relabel the argument of g to x :

$$g(x) = f^{-1}(x) = \frac{-5x+6}{4x-5}.$$

Let us check our work. In order for f and g to be inverses, we need that $g(f(x))$ be equal to x .

$$g(f(x)) = \frac{-5f(x)+6}{4f(x)-5} = \frac{-5\frac{(5x+6)}{4x+5}+6}{4\frac{(5x+6)}{4x+5}-5} = \frac{-5(5x+6)+6(4x+5)}{4(5x+6)-5(4x+5)} = \frac{-x}{-1} = x,$$

as expected.

12. Find the inverse function and its domain.

(a) $y = \ln(x+3)$.

(b) $y = 4 \ln(x-3) - 4$.

(c) $y = 2 \ln(-2x+4) + 1$

(d) $f(x) = e^{x^3}$.

(e) $y = (\ln x)^2, x \geq 1$.

(f) $y = \frac{e^x}{1+2e^x}$.

(g) $f(x) = 2^{2x} + 2^x - 2$.

Solution. 12.a

$$\begin{aligned} y &= \ln(x+3) \\ e^y &= e^{\ln(x+3)} \\ e^y &= x+3 \\ e^y - 3 &= x \end{aligned}$$

Therefore $f^{-1}(y) = e^y - 3$.

The domain of e^y is all real numbers, so the domain of f^{-1} is all real numbers.

Solution. 12.b

$$\begin{aligned} 4 \ln(x-3) - 4 &= y \\ 4 \ln(x-3) &= y+4 \\ \ln(x-3) &= \frac{y+4}{4} && \left| \text{exponentiate} \right. \\ e^{\ln(x-3)} &= e^{\frac{y+4}{4}} \\ x-3 &= e^{\frac{y+4}{4}} \\ f^{-1}(y) = x &= e^{\frac{y+4}{4}} + 3 \\ f^{-1}(x) &= e^{\frac{x+4}{4}} + 3 && \left| \text{relabel.} \right. \end{aligned}$$

The domain of f^{-1} is all real numbers (no restrictions on the domain).

Solution. 12.e

$$\begin{array}{rcl} y & = & (\ln x)^2 \\ \sqrt{y} & = & \ln x \\ e^{\sqrt{y}} & = & e^{\ln x} = x \end{array} \quad \left| \begin{array}{l} \text{take } \sqrt{} \text{ on both sides, } y \geq 0 \\ \text{exponentiate} \end{array} \right.$$
$$\begin{array}{rcl} f^{-1}(y) & = & e^{\sqrt{y}} \\ f^{-1}(x) & = & e^{\sqrt{x}} \end{array}$$

Solution. 12.f

$$\begin{aligned} y &= \frac{e^x}{1 + 2e^x} \\ y(1 + 2e^x) &= e^x \\ y &= e^x(1 - 2y) \\ \frac{y}{1 - 2y} &= e^x \\ \ln \frac{y}{1 - 2y} &= \ln e^x \\ \ln \frac{y}{1 - 2y} &= x \\ \text{Therefore } f^{-1}(y) &= \ln \frac{y}{1 - 2y}. \end{aligned}$$

The natural logarithm function is only defined for positive input values. Therefore the domain is the set of all y for which

$$\frac{y}{1 - 2y} > 0.$$

This inequality holds if the numerator and denominator are both positive or both negative. This happens if either

- (a) $y > 0$ and $y < \frac{1}{2}$, or
- (b) $y < 0$ and $y > \frac{1}{2}$.

The latter option is impossible, so the domain is $\{y \in \mathbb{R} \mid 0 < y < \frac{1}{2}\}$.