Precalculus
Angle sum formulas memorization

Todor Milev

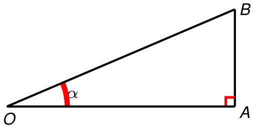
2019

$\sin(\alpha+\beta), \cos(\alpha+\beta)$ via $\sin\alpha, \sin\beta, \cos\alpha, \cos\beta$

$\sin(\alpha+\beta) =$ **?**

$\cos(\alpha+\beta) =$ **?**

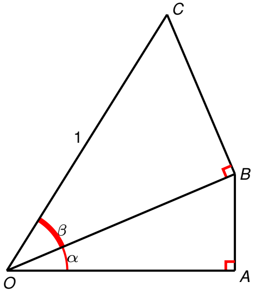
$\sin(\alpha+\beta), \cos(\alpha+\beta)$ via $\sin\alpha, \sin\beta, \cos\alpha, \cos\beta$



$\sin(\alpha+\beta) =$ **?**

$\cos(\alpha+\beta) =$ **?**

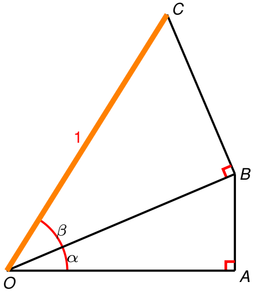
$\sin(\alpha+\beta)$, $\cos(\alpha+\beta)$ via $\sin\alpha$, $\sin\beta$, $\cos\alpha$, $\cos\beta$



$\sin(\alpha+\beta) =$ **?**

$\cos(\alpha+\beta) =$ **?**

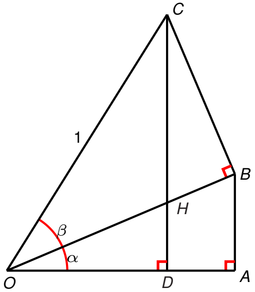
$\sin(\alpha+\beta)$, $\cos(\alpha+\beta)$ via $\sin\alpha$, $\sin\beta$, $\cos\alpha$, $\cos\beta$



$\sin(\alpha+\beta) =$ **?**

$\cos(\alpha+\beta) =$ **?**

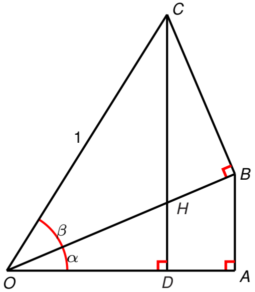
$\sin(\alpha+\beta), \cos(\alpha+\beta)$ via $\sin\alpha, \sin\beta, \cos\alpha, \cos\beta$



$\sin(\alpha+\beta) =$ **?**

$\cos(\alpha+\beta) =$ **?**

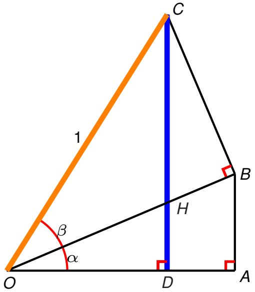
$\sin(\alpha+\beta), \cos(\alpha+\beta)$ via $\sin\alpha, \sin\beta, \cos\alpha, \cos\beta$



$\sin(\alpha+\beta) =$ **?**

$\cos(\alpha+\beta) =$ **?**

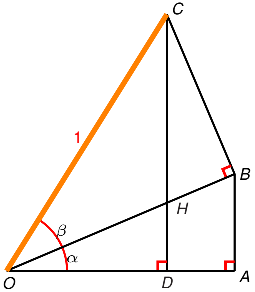
$\sin(\alpha+\beta)$, $\cos(\alpha+\beta)$ via $\sin\alpha$, $\sin\beta$, $\cos\alpha$, $\cos\beta$



$$\sin(\alpha+\beta) = \frac{|CD|}{|OC|}$$

$$\cos(\alpha+\beta) = \textbf{?}$$

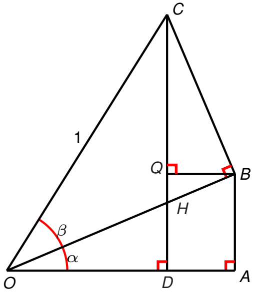
$\sin(\alpha+\beta), \cos(\alpha+\beta)$ via $\sin\alpha, \sin\beta, \cos\alpha, \cos\beta$



$$\sin(\alpha+\beta) = \frac{|CD|}{|OC|} = |CD|$$

$$\cos(\alpha+\beta) = \textbf{?}$$

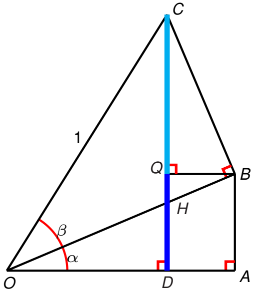
$\sin(\alpha+\beta), \cos(\alpha+\beta)$ via $\sin\alpha, \sin\beta, \cos\alpha, \cos\beta$



$$\sin(\alpha+\beta) = \frac{|CD|}{|OC|} = |CD|$$

$$\cos(\alpha+\beta) = \textbf{?}$$

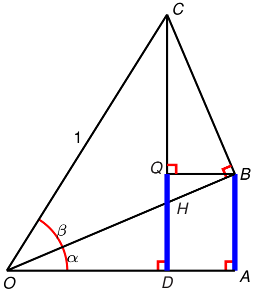
$\sin(\alpha+\beta)$, $\cos(\alpha+\beta)$ via $\sin\alpha$, $\sin\beta$, $\cos\alpha$, $\cos\beta$



$$\sin(\alpha+\beta) = \frac{|CD|}{|OC|} = |CD|$$
$$= |QD| + |CQ|$$

$$\cos(\alpha+\beta) = \textbf{?}$$

$\sin(\alpha+\beta), \cos(\alpha+\beta)$ via $\sin\alpha, \sin\beta, \cos\alpha, \cos\beta$

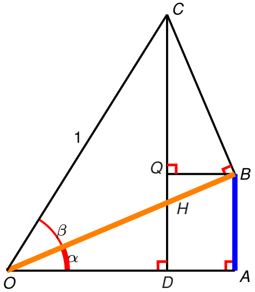


$|QD| = |BA|$ $\quad\Big|\ \square DABQ$

$$\sin(\alpha+\beta) = \frac{|CD|}{|OC|} = |CD|$$
$$= |QD| + |CQ|$$

$$\cos(\alpha+\beta) = \textbf{?}$$

$\sin(\alpha+\beta)$, $\cos(\alpha+\beta)$ via $\sin\alpha$, $\sin\beta$, $\cos\alpha$, $\cos\beta$

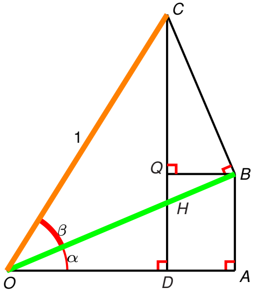


$$\sin(\alpha+\beta) = \frac{|CD|}{|OC|} = |CD| = |QD| + |CQ|$$

$$\cos(\alpha+\beta) = \textbf{?}$$

$$\begin{aligned} |QD| &= |BA| \\ &= \sin\alpha |OB| \end{aligned} \quad \left| \begin{array}{l} \square DABQ \\ \triangle OAB \end{array} \right.$$

$\sin(\alpha+\beta), \cos(\alpha+\beta)$ via $\sin\alpha, \sin\beta, \cos\alpha, \cos\beta$

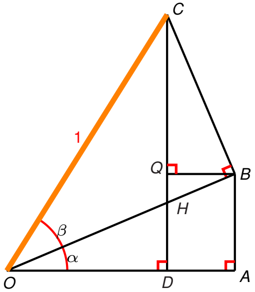


$$
\begin{aligned}
|QD| &= |BA| && \square DABQ\\
&= \sin\alpha\, {\color{red}|OB|} && \triangle OAB\\
&= \sin\alpha\, {\color{red}\cos\beta\, |OC|} && \triangle OBC
\end{aligned}
$$

$$
\begin{aligned}
\sin(\alpha+\beta) &= \frac{|CD|}{|OC|} = |CD|\\
&= |QD| + |CQ|
\end{aligned}
$$

$$\cos(\alpha+\beta) = \textbf{?}$$

$\sin(\alpha+\beta)$, $\cos(\alpha+\beta)$ via $\sin\alpha$, $\sin\beta$, $\cos\alpha$, $\cos\beta$

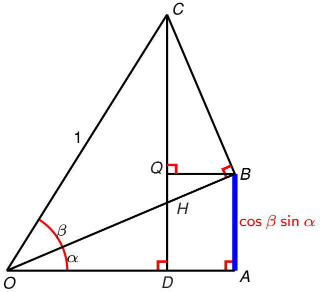


$$\begin{array}{rl|l} |QD| &= |BA| & \square DABQ \\ &= \sin\alpha |OB| & \triangle OAB \\ &= \sin\alpha\cos\beta {\color{red}|OC|} & \triangle OBC \\ &= \sin\alpha\cos\beta & \end{array}$$

$$\begin{aligned} \sin(\alpha+\beta) &= \frac{|CD|}{|OC|} = |CD| \\ &= |QD| + |CQ| \end{aligned}$$

$$\cos(\alpha+\beta) = \textbf{?}$$

$\sin(\alpha+\beta), \cos(\alpha+\beta)$ via $\sin\alpha, \sin\beta, \cos\alpha, \cos\beta$

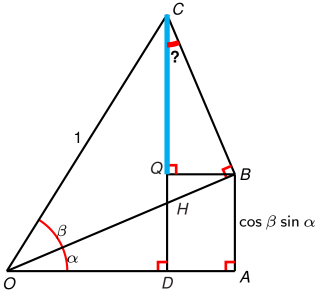


$$\begin{aligned}|QD| &= |BA| && \square DABQ\\ &= \sin\alpha|OB| && \triangle OAB\\ &= \sin\alpha\cos\beta|OC| && \triangle OBC\\ &= \sin\alpha\cos\beta\end{aligned}$$

$$\begin{aligned}\sin(\alpha+\beta) &= \frac{|CD|}{|OC|} = |CD|\\ &= |QD| + |CQ|\\ &= \sin\alpha\cos\beta + \textbf{?}\end{aligned}$$

$$\cos(\alpha+\beta) = \textbf{?}$$

$\sin(\alpha+\beta), \cos(\alpha+\beta)$ via $\sin\alpha, \sin\beta, \cos\alpha, \cos\beta$

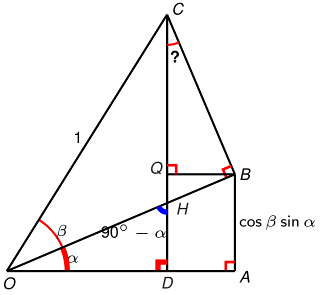


$$\begin{aligned} |QD| &= |BA| && \square DABQ \\ &= \sin\alpha |OB| && \triangle OAB \\ &= \sin\alpha\cos\beta |OC| && \triangle OBC \\ &= \sin\alpha\cos\beta \\ {\color{red}|CQ|} &= \end{aligned}$$

$$\begin{aligned} \sin(\alpha+\beta) &= \frac{|CD|}{|OC|} = |CD| \\ &= |QD| + {\color{red}|CQ|} \\ &= \sin\alpha\cos\beta + \mathbf{?} \end{aligned}$$

$$\cos(\alpha+\beta) = \mathbf{?}$$

$\sin(\alpha+\beta)$, $\cos(\alpha+\beta)$ via $\sin\alpha$, $\sin\beta$, $\cos\alpha$, $\cos\beta$

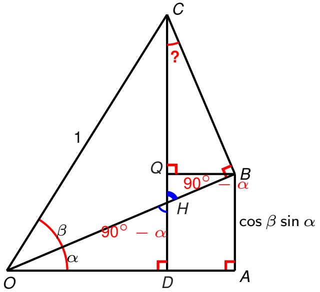


$$\begin{aligned}\sin(\alpha+\beta) &= \frac{|CD|}{|OC|} = |CD| \\ &= |QD| + |CQ| \\ &= \sin\alpha\cos\beta + \textbf{?}\end{aligned}$$

$$\cos(\alpha+\beta) = \textbf{?}$$

$$\begin{aligned}|QD| &= |BA| && \square DABQ \\ &= \sin\alpha|OB| && \triangle OAB \\ &= \sin\alpha\cos\beta|OC| && \triangle OBC \\ &= \sin\alpha\cos\beta \\ |CQ| &=\end{aligned}$$

$\sin(\alpha+\beta)$, $\cos(\alpha+\beta)$ via $\sin\alpha$, $\sin\beta$, $\cos\alpha$, $\cos\beta$

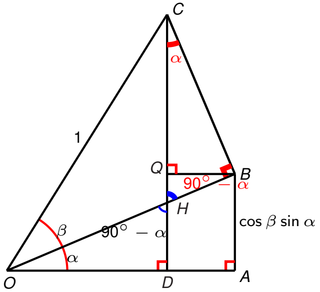


$$\sin(\alpha+\beta) = \frac{|CD|}{|OC|} = |CD|$$
$$= |QD| + |CQ|$$
$$= \sin\alpha\cos\beta + \textbf{?}$$

$$\cos(\alpha+\beta) = \textbf{?}$$

$\|QD\| = \|BA\|$		$\square DABQ$
$= \sin\alpha\|OB\|$		$\triangle OAB$
$= \sin\alpha\cos\beta\|OC\|$		$\triangle OBC$
$= \sin\alpha\cos\beta$		
$\|CQ\| =$		

$\sin(\alpha+\beta)$, $\cos(\alpha+\beta)$ via $\sin\alpha$, $\sin\beta$, $\cos\alpha$, $\cos\beta$

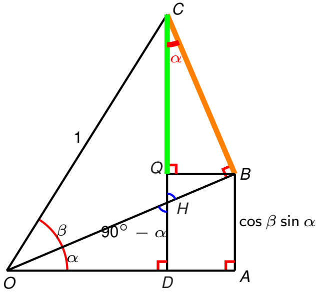


$$\sin(\alpha+\beta) = \frac{|CD|}{|OC|} = |CD|$$
$$= |QD| + |CQ|$$
$$= \sin\alpha\cos\beta + \textbf{?}$$

$$\cos(\alpha+\beta) = \textbf{?}$$

$$\begin{aligned} |QD| &= |BA| &&\Big| \; \square DABQ \\ &= \sin\alpha |OB| &&\Big| \; \triangle OAB \\ &= \sin\alpha\cos\beta |OC| &&\Big| \; \triangle OBC \\ &= \sin\alpha\cos\beta \\ |CQ| &= \end{aligned}$$

$\sin(\alpha+\beta)$, $\cos(\alpha+\beta)$ via $\sin\alpha$, $\sin\beta$, $\cos\alpha$, $\cos\beta$

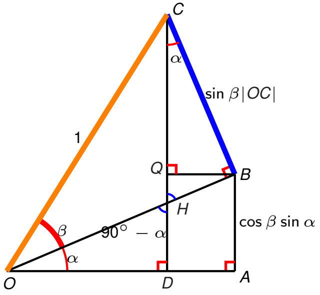


$$\sin(\alpha+\beta) = \frac{|CD|}{|OC|} = |CD|$$
$$= |QD| + |CQ|$$
$$= \sin\alpha\cos\beta + \textbf{?}$$

$$\cos(\alpha+\beta) = \textbf{?}$$

$$\begin{aligned} |QD| &= |BA| && \square DABQ \\ &= \sin\alpha |OB| && \triangle OAB \\ &= \sin\alpha\cos\beta |OC| && \triangle OBC \\ &= \sin\alpha\cos\beta && \\ |CQ| &= \cos\alpha |CB| && \triangle CQB \end{aligned}$$

$\sin(\alpha+\beta), \cos(\alpha+\beta)$ via $\sin\alpha, \sin\beta, \cos\alpha, \cos\beta$

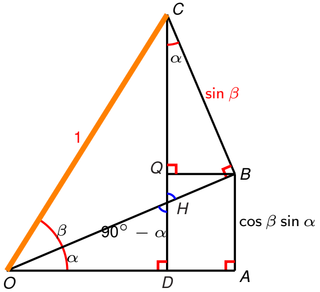


$$\begin{array}{rll|l} |QD| &= |BA| & & \square DABQ \\ &= \sin\alpha |OB| & & \triangle OAB \\ &= \sin\alpha\cos\beta |OC| & & \triangle OBC \\ &= \sin\alpha\cos\beta & & \\ |CQ| &= \cos\alpha \color{red}{|CB|} & & \triangle CQB \\ &= \cos\alpha \color{red}{\sin\beta |OC|} & & \triangle OBC \end{array}$$

$$\begin{aligned} \sin(\alpha+\beta) &= \frac{|CD|}{|OC|} = |CD| \\ &= |QD| + |CQ| \\ &= \sin\alpha\cos\beta + \textbf{?} \end{aligned}$$

$$\cos(\alpha+\beta) = \textbf{?}$$

$\sin(\alpha+\beta), \cos(\alpha+\beta)$ via $\sin\alpha, \sin\beta, \cos\alpha, \cos\beta$

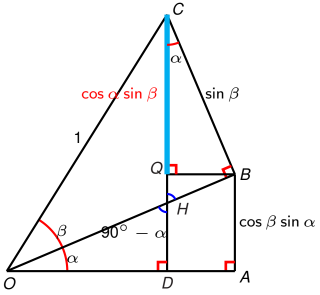


$$
\begin{aligned}
|QD| &= |BA| && \square DABQ \\
&= \sin\alpha |OB| && \triangle OAB \\
&= \sin\alpha\cos\beta |OC| && \triangle OBC \\
&= \sin\alpha\cos\beta \\
|CQ| &= \cos\alpha |CB| && \triangle CQB \\
&= \cos\alpha {\color{red}\sin\beta |OC|} && \triangle OBC \\
&= \cos\alpha {\color{red}\sin\beta}
\end{aligned}
$$

$$
\begin{aligned}
\sin(\alpha+\beta) &= \frac{|CD|}{|OC|} = |CD| \\
&= |QD| + |CQ| \\
&= \sin\alpha\cos\beta + {\color{red}\textbf{?}}
\end{aligned}
$$

$$\cos(\alpha+\beta) = \textbf{?}$$

$\sin(\alpha+\beta)$, $\cos(\alpha+\beta)$ via $\sin\alpha$, $\sin\beta$, $\cos\alpha$, $\cos\beta$

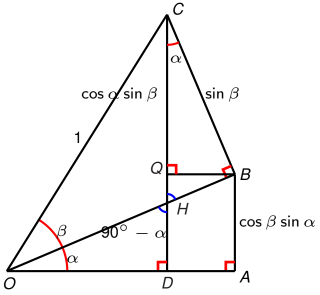


$$\begin{aligned} |QD| &= |BA| && \square DABQ \\ &= \sin\alpha|OB| && \triangle OAB \\ &= \sin\alpha\cos\beta|OC| && \triangle OBC \\ &= \sin\alpha\cos\beta \\ |CQ| &= \cos\alpha|CB| && \triangle CQB \\ &= \cos\alpha\sin\beta|OC| && \triangle OBC \\ &= \cos\alpha\sin\beta \end{aligned}$$

$$\begin{aligned} \sin(\alpha+\beta) &= \frac{|CD|}{|OC|} = |CD| \\ &= |QD| + |CQ| \\ &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \end{aligned}$$

$\cos(\alpha+\beta) =$ **?**

$\sin(\alpha+\beta), \cos(\alpha+\beta)$ via $\sin\alpha, \sin\beta, \cos\alpha, \cos\beta$

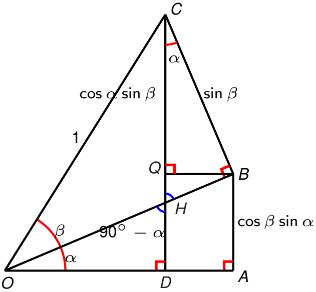


$$
\begin{aligned}
|QD| &= |BA| && \square DABQ \\
&= \sin\alpha |OB| && \triangle OAB \\
&= \sin\alpha \cos\beta |OC| && \triangle OBC \\
&= \sin\alpha \cos\beta \\
|CQ| &= \cos\alpha |CB| && \triangle CQB \\
&= \cos\alpha \sin\beta |OC| && \triangle OBC \\
&= \cos\alpha \sin\beta
\end{aligned}
$$

$$
\begin{aligned}
\sin(\alpha+\beta) &= \frac{|CD|}{|OC|} = |CD| \\
&= |QD| + |CQ| \\
&= \sin\alpha\cos\beta + \cos\alpha\sin\beta
\end{aligned}
$$

$\cos(\alpha+\beta) =$ **?**

$\sin(\alpha+\beta), \cos(\alpha+\beta)$ via $\sin\alpha, \sin\beta, \cos\alpha, \cos\beta$

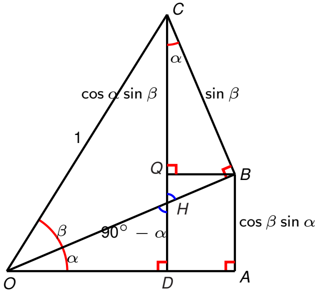


$$\begin{aligned}
\sin(\alpha+\beta) &= \frac{|CD|}{|OC|} = |CD| \\
&= |QD| + |CQ| \\
&= \sin\alpha\cos\beta + \cos\alpha\sin\beta
\end{aligned}$$

$$\begin{aligned}
\cos(\alpha+\beta) &= \frac{|OD|}{|OC|} = |OD| \\
&= |OA| - |DA| \\
&= \cos\alpha\cos\beta - \sin\alpha\sin\beta
\end{aligned}$$

$$\begin{array}{rl|l}
|QD| & = |BA| & \square DABQ \\
& = \sin\alpha|OB| & \triangle OAB \\
& = \sin\alpha\cos\beta|OC| & \triangle OBC \\
& = \sin\alpha\cos\beta & \\
|CQ| & = \cos\alpha|CB| & \triangle CQB \\
& = \cos\alpha\sin\beta|OC| & \triangle OBC \\
& = \cos\alpha\sin\beta & \\
|OA| & = \cos\alpha|OB| & \triangle OAB \\
& = \cos\alpha\cos\beta|OC| & \triangle OBC \\
& = \cos\alpha\cos\beta & \\
|DA| & = |QB| & \square DABQ \\
& = \sin\alpha|CB| & \triangle CQB \\
& = \sin\alpha\sin\beta|OC| & \triangle OBC \\
& = \sin\alpha\sin\beta &
\end{array}$$

$\sin(\alpha+\beta), \cos(\alpha+\beta)$ via $\sin\alpha, \sin\beta, \cos\alpha, \cos\beta$



$$\sin(\alpha+\beta) = \frac{|CD|}{|OC|} = |CD| = |QD| + |CQ| = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$\cos(\alpha+\beta) = \frac{|OD|}{|OC|} = |OD| = |OA| - |DA| = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\begin{array}{rll}
|QD| &= |BA| & \square DABQ \\
&= \sin\alpha|OB| & \triangle OAB \\
&= \sin\alpha\cos\beta|OC| & \triangle OBC \\
&= \sin\alpha\cos\beta & \\
|CQ| &= \cos\alpha|CB| & \triangle CQB \\
&= \cos\alpha\sin\beta|OC| & \triangle OBC \\
&= \cos\alpha\sin\beta & \\
|OA| &= \cos\alpha|OB| & \triangle OAB \\
&= \cos\alpha\cos\beta|OC| & \triangle OBC \\
&= \cos\alpha\cos\beta & \\
|DA| &= |QB| & \square DABQ \\
&= \sin\alpha|CB| & \triangle CQB \\
&= \sin\alpha\sin\beta|OC| & \triangle OBC \\
&= \sin\alpha\sin\beta &
\end{array}$$

Trig Functions of Sums and Differences of Angles

Theorem

$$\begin{aligned}\sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta\\ \cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta\end{aligned}$$

Trig Functions of Sums and Differences of Angles

Theorem

$$\begin{aligned}\sin(\alpha+\beta) &= \sin\alpha\cos\beta+\cos\alpha\sin\beta\\ \cos(\alpha+\beta) &= \cos\alpha\cos\beta-\sin\alpha\sin\beta\end{aligned}$$

- We gave a geometric proof of the sum formulas when the two angles are acute and their sum is less than $\pi = 90^\circ$.

Trig Functions of Sums and Differences of Angles

Theorem

$$\begin{aligned} \sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \\ \cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \end{aligned}$$

- We gave a geometric proof of the sum formulas when the two angles are acute and their sum is less than $\pi = 90^\circ$.
- The theorem holds for all angles α, β without any restrictions.

Trig Functions of Sums and Differences of Angles

Theorem

$$\begin{aligned}\sin(\alpha+\beta) &= \sin\alpha\cos\beta+\cos\alpha\sin\beta\\ \cos(\alpha+\beta) &= \cos\alpha\cos\beta-\sin\alpha\sin\beta\end{aligned}$$

- We gave a geometric proof of the sum formulas when the two angles are acute and their sum is less than $\pi = 90^\circ$.
- The theorem holds for all angles α, β without any restrictions.
- This can be shown by combining the preceding proof with identities such as $\cos\left(\frac{\pi}{2}-\alpha\right) = \sin\alpha$, $\cos\left(\frac{\pi}{2}+\alpha\right) = -\sin\alpha$.

Trig Functions of Sums and Differences of Angles

Theorem

$$\begin{aligned}\sin(\alpha+\beta) &= \sin\alpha\cos\beta+\cos\alpha\sin\beta\\ \cos(\alpha+\beta) &= \cos\alpha\cos\beta-\sin\alpha\sin\beta\end{aligned}$$

- We gave a geometric proof of the sum formulas when the two angles are acute and their sum is less than $\pi = 90^\circ$.
- The theorem holds for all angles α, β without any restrictions.
- This can be shown by combining the preceding proof with identities such as $\cos\left(\frac{\pi}{2}-\alpha\right) = \sin\alpha$, $\cos\left(\frac{\pi}{2}+\alpha\right) = -\sin\alpha$.
- There is a theoretically more advanced (but algebraically simpler) proof using Euler's formula (to be studied later/in another course).

Trig Functions of Sums and Differences of Angles

Theorem

$$\begin{aligned}
\sin(\alpha+\beta) &= \sin\alpha\cos\beta+\cos\alpha\sin\beta\\
\sin(\alpha-\beta) &= \sin\alpha\cos\beta-\cos\alpha\sin\beta\\
\cos(\alpha+\beta) &= \cos\alpha\cos\beta-\sin\alpha\sin\beta\\
\cos(\alpha-\beta) &= \cos\alpha\cos\beta+\sin\alpha\sin\beta
\end{aligned}$$

- We gave a geometric proof of the sum formulas when the two angles are acute and their sum is less than $\pi = 90^\circ$.
- The theorem holds for all angles α, β without any restrictions.
- This can be shown by combining the preceding proof with identities such as $\cos\left(\frac{\pi}{2}-\alpha\right) = \sin\alpha$, $\cos\left(\frac{\pi}{2}+\alpha\right) = -\sin\alpha$.
- There is a theoretically more advanced (but algebraically simpler) proof using Euler's formula (to be studied later/in another course).
- The difference formulas are a consequence of the sum formulas and the fact that $\sin$ is an odd function and $\cos$ is even.

Trig Functions of Differences of Angles

Example

Prove the identities

$$\begin{aligned}\sin(\alpha-\beta) &= \sin\alpha\cos\beta-\cos\alpha\sin\beta\\ \cos(\alpha-\beta) &= \cos\alpha\cos\beta+\sin\alpha\sin\beta\end{aligned}$$

from the (already demonstrated) identities

$$\begin{aligned}\sin(\alpha+\beta) &= \sin\alpha\cos\beta+\cos\alpha\sin\beta\\ \cos(\alpha+\beta) &= \cos\alpha\cos\beta-\sin\alpha\sin\beta\end{aligned}$$

$$\begin{aligned}\sin(\alpha-\beta) &= \sin(\alpha+(-\beta))\\ &= \sin\alpha\cos(-\beta)+\cos\alpha\sin(-\beta) && \left|\ \begin{array}{l}\cos \text{ is even },\\ \sin \text{ is odd}\end{array}\right.\\ &= \sin\alpha\cos\beta-\cos\alpha\sin\beta\\ \cos(\alpha-\beta) &= \cos(\alpha+(-\beta))\\ &= \cos\alpha\cos(-\beta)-\sin\alpha\sin(-\beta) && \left|\ \begin{array}{l}\cos \text{ is even },\\ \sin \text{ is odd}\end{array}\right.\\ &= \cos\alpha\cos\beta+\cos\alpha\sin\beta\end{aligned}$$