

Precalculus

Homework

Graphs of trig functions; inverse trig

1. Find each of the following values. Express your answers precisely, not as decimals.

(a) $\arcsin(\sin 4)$.

ANSWER: $\pi - 4$

(b) $\arcsin(\sin 0.5)$.

ANSWER: 0.5

(c) $\arcsin(\cos 120^\circ)$.

ANSWER: $-\frac{\pi}{6}$

(d) $\arccos(\cos(3))$.

ANSWER: 3

(e) $\arccos(\cos(-2))$.

ANSWER: 2

(f) $\arccos(\sin(-4))$.

ANSWER: $\frac{3\pi}{2} - 4 \approx 0.712389$

(g) $\arctan(\tan 5)$.

ANSWER: $5 - 2\pi$

Solution. 1.g $\frac{3\pi}{2} \approx 4.71$ and $2\pi \approx 6.28$, so

$$\begin{aligned} \frac{3\pi}{2} &< 5 < 2\pi \\ \text{Therefore } -\frac{\pi}{2} &< 5 - 2\pi < 0 < \frac{\pi}{2}. \end{aligned}$$

Therefore $5 - 2\pi$ is in the restricted domain of the tangent function. Moreover, the tangent function is π -periodic, so $\tan 5 = \tan(5 - 2\pi)$. Therefore $\arctan(\tan 5) = 5 - 2\pi$.

2. Express as the following as an algebraic expression of x . In other words, “get rid” of the trigonometric and inverse trigonometric expressions.

(a) $\cos^2(\arctan x)$.

ANSWER: $\frac{1}{1+x^2}$

(b) $-\sin^2(\operatorname{arccot} x)$.

ANSWER: $\frac{x^2+1}{x^2}$

(d) $-\frac{1}{\sin(\arccos x)}$.

ANSWER: $-\frac{x^2+1}{x}$

(c) $\frac{1}{\cos(\arcsin x)}$.

ANSWER: $-\frac{x^2-1}{1}$

Solution. 2.b. We follow the strategy outlined in the end of the solution of Problem 3.c. We set $y = \operatorname{arccot} x$. Then we need to express $-\sin^2 y$ via $\cot y$. That is a matter of algebra:

$$\begin{aligned}
-\sin^2(\operatorname{arccot} x) &= -\sin^2 y && \left| \begin{array}{l} \text{Set } y = \operatorname{arccot} x \\ \text{use } \sin^2 y + \cos^2 y = 1 \end{array} \right. \\
&= -\frac{\sin^2 y}{\sin^2 y + \cos^2 y} \\
&= -\frac{\sin^2 y}{1} \\
&= -\frac{\sin^2 y + \cos^2 y}{1} \\
&= -\frac{1}{1 + \cot^2 y} && \left| \begin{array}{l} \text{Substitute back } \cot y = x \end{array} \right. \\
&= -\frac{1}{1 + x^2} .
\end{aligned}$$

3. Let $x \in (0, 1)$. Express the following using x and $\sqrt{1 - x^2}$.

(a) $\sin(\arcsin(x))$.

(e) $\sin(2 \arccos(x))$.

(b) $\sin(2 \arcsin(x))$.

(f) $\sin(3 \arccos(x))$.

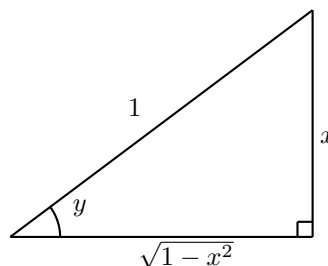
(c) $\sin(3 \arcsin(x))$.

(g) $\cos(2 \arcsin(x))$.

(d) $\sin(\arccos(x))$.

(h) $\cos(3 \arccos(x))$.

Solution. 3.b. Let $y = \arcsin x$. Then $\sin y = x$, and we can draw a right triangle with opposite side length x and hypotenuse length 1 to find the other trigonometric ratios of y .



Then $\cos y = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$. Now we use the double angle formula to find $\sin(2 \arcsin x)$.

$$\begin{aligned}
\sin(2 \arcsin x) &= \sin(2y) \\
&= 2 \sin y \cos y \\
&= 2x \sqrt{1 - x^2}.
\end{aligned}$$

Solution. 3.c. Use the result of Problem 3.b. This also requires the addition formula for sine:

$$\sin(A + B) = \sin A \cos B + \sin B \cos A,$$

and the double angle formula for cosine:

$$\cos(2y) = \cos^2 y - \sin^2 y.$$

$$\begin{aligned}
\sin(3 \arcsin x) &= \sin(3y) \\
&= \sin(2y + y) \\
&= \sin(2y) \cos y + \sin y \cos(2y) && \left| \begin{array}{l} \text{Use addition formula} \\ \text{Use double angle formulas} \end{array} \right. \\
&= (2 \sin y \cos y) \cos y + \sin y (\cos^2 y - \sin^2 y) \\
&= 2 \sin y \cos^2 y + \sin y \cos^2 y - \sin^3 y \\
&= 3 \sin y \cos^2 y - \sin^3 y \\
&= 3 \sin y (1 - \sin^2 y) - \sin^3 y \\
&= 3x(1 - x^2) - x^3 \\
&= 3x - 4x^3.
\end{aligned}$$

The solution is complete. A careful look at the solution above reveals a strategy useful for problems similar to this one.

- (a) Identify the inverse trigonometric expression- $\arcsin x, \arccos x, \arctan x, \dots$. In the present problem that was $y = \arcsin x$.
- (b) The problem is therefore a trigonometric function of y .
- (c) Using trig identities and algebra, rewrite the problem as a trigonometric expression involving only the trig function that transforms y to x . In the present problem we rewrote everything using $\sin y$.
- (d) Use the fact that $\sin(\arcsin x) = x, \cos(\arccos x) = x, \dots$, etc. to simplify.

Solution. 3.f We use the same strategy outlined in the end of the solution of Problem 3.c. Set $y = \arccos x$ and so $\cos(y) = x$. Therefore:

$$\begin{aligned}
\sin(3y) &= \sin(2y + y) \\
&= \sin(2y) \cos y + \sin y \cos(2y) \\
&= 2 \sin y \cos y \cos y + \sin y (2 \cos^2 y - 1) \\
&= 2 \sin y \cos^2 y + \sin y (2 \cos^2 y - 1) \\
&= \sin y (4 \cos^2 y - 1) && \left| \begin{array}{l} \text{use } \cos y = x \\ \sin y = \sqrt{1 - x^2} \end{array} \right. \\
&= \sqrt{1 - x^2} (4x^2 - 1) \quad .
\end{aligned}$$