Precalculus
Additional trigonometric identity exercises

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Proving the following identities is a good exercise.

1. $\sin\theta\cot\theta = \cos\theta$.
2. $(\sin\theta + \cos\theta)^2 = 1 + \sin(2\theta)$.
3. $\sec\theta - \cos\theta = \tan\theta\sin\theta$.
4. $\tan^2\theta - \sin^2\theta = \tan^2\theta\sin^2\theta$.
5. $\cot^2\theta + \sec^2\theta = \tan^2\theta + \csc^2\theta$.
6. $2\csc(2\theta) = \sec\theta\csc\theta$.
7. $\tan(2\theta) = \dfrac{2\tan\theta}{1-\tan^2\theta}$.
8. $\dfrac{1}{1-\sin\theta} + \dfrac{1}{1+\sin\theta} = 2\sec^2\theta$.
9. $\tan\alpha + \tan\beta = \dfrac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta}$.
10. $\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta}$.
11. $\sin(3\theta) + \sin\theta = 2\sin(2\theta)\cos\theta$.
12. $\cos(3\theta) = 4\cos^3\theta - 3\cos\theta$.
13. $1 + \tan^2\theta = \sec^2\theta$.
14. $1 + \csc^2\theta = \cot^2\theta$.
15. $2\cos^2(2x) =$ $2\sin^4\theta + 2\cos^4\theta - \sin^2(2\theta)$.
16. $\dfrac{1+\tan\left(\frac{\theta}{2}\right)}{1-\tan\left(\frac{\theta}{2}\right)} = \tan\theta + \sec\theta$.

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$(A + B)^2 =$
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$$(\sin\theta + \cos\theta)^2 = \sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta \quad \Bigg| \quad \begin{array}{l}(A+B)^2 = \\ A^2 + 2AB + B^2\end{array}$$

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Here we explicitly permit the use of the Pythagorean identities and the double angle f-las:

$$\begin{aligned}\cos^2\theta+\sin^2\theta &= 1\\ \sin(2\theta) &= 2\sin\theta\cos\theta\\ \cos(2\theta) &= \cos^2\theta-\sin^2\theta\end{aligned}$$

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All angles here are multiples of $\frac{\theta}{2}$, so set $\varphi = \frac{\theta}{2}$

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Prove the identity $\tan\theta + \sec\theta = \dfrac{1+\tan\left(\frac{\theta}{2}\right)}{1-\tan\left(\frac{\theta}{2}\right)}$

All angles here are multiples of $\frac{\theta}{2}$, so set $\varphi = \frac{\theta}{2}$, $\theta =$ **?** .

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Prove the identity $\tan\theta + \sec\theta = \dfrac{1 + \tan\left(\frac{\theta}{2}\right)}{1 - \tan\left(\frac{\theta}{2}\right)}$

All angles here are multiples of $\frac{\theta}{2}$, so set $\varphi = \frac{\theta}{2}$, $\theta = 2\varphi$.

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$A^2 + 2AB + B^2 = (A+B)^2$

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