Calculus II
Power series expansions related to exponents, part 2

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- Therefore by the Ratio Test the series converges for all x.

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- Just like the Maclaurin series, this series also represents e^x.

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Recall that $e^y = \sum_{n=0}^{\infty} \frac{y^n}{n!}$

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Set $y = x - 3$

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The radius of convergence was already computed to be $R = \infty$.