

Precalculus

Compute trigonometric function of a complementary angle, part 1

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Cofunction identities

Proposition (Cofunction identities)

$$\begin{aligned}\sin\left(\frac{\pi}{2} - \alpha\right) &= \cos \alpha & \sin\left(\frac{\pi}{2} + \alpha\right) &= \cos \alpha \\ \cos\left(\frac{\pi}{2} - \alpha\right) &= \sin \alpha & \cos\left(\frac{\pi}{2} + \alpha\right) &= -\sin \alpha\end{aligned}$$

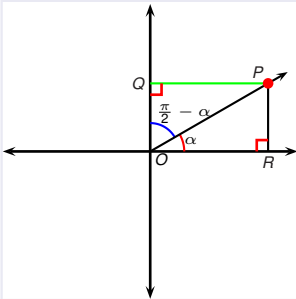
- The proof each formula is broken into 4 cases depending on which quadrant contains α .
- This makes a total of 4 formulas \times 4 cases per formula = 16 cases.
- We show only a few of the cases.
- The proof provides intuition why the formulas are true.
- The Quadrant I part of the proof serves as a visual aid for memorization.
- There is an algebraically simpler (but theoretically advanced) way to prove the above identities through the angle sum formulas, derived in turn from Euler's formula (studied later/in another course).

Cofunction identities

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Part of Proof.



We are showing $\sin\left(\frac{\pi}{2} - \alpha\right) = \cos \alpha$ when α is in quadrant I.

$$\begin{aligned}\sin\left(\frac{\pi}{2} - \alpha\right) &= \frac{|PQ|}{|OP|} & \left| \square ORPQ \right. \\ &= \frac{|OR|}{|OP|} \\ &= \cos \alpha & \left| \text{as desired} \right.\end{aligned}$$

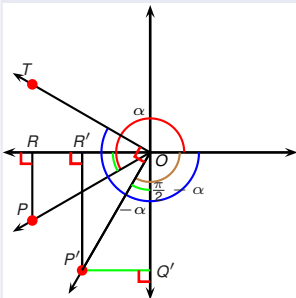


Cofunction identities

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$$\begin{aligned}\sin\left(\frac{\pi}{2} - \alpha\right) &= \cos \alpha & \sin\left(\frac{\pi}{2} + \alpha\right) &= \cos \alpha \\ \cos\left(\frac{\pi}{2} - \alpha\right) &= \sin \alpha & \cos\left(\frac{\pi}{2} + \alpha\right) &= -\sin \alpha\end{aligned}$$

Part of Proof.



We are showing $\sin\left(\frac{\pi}{2} - \alpha\right) = \cos \alpha$ when α is in Quadrant III. It follows $\frac{\pi}{2} - \alpha$ is in Quadrant III.

$$\begin{aligned}\sin\left(\frac{\pi}{2} - \alpha\right) &= -\frac{|P'R'|}{|OP'|} = -\frac{|OQ'|}{|OP'|} \quad \square OR'P'Q' \\ &= -\frac{|OR|}{|OP|} \\ &= \cos \alpha\end{aligned} \quad \Big| \text{ as desired}$$

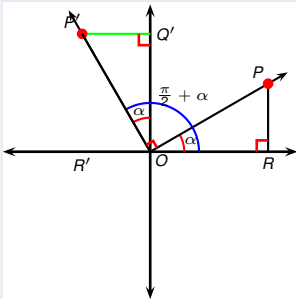


Cofunction identities

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Part of Proof.



We show $\cos\left(\frac{\pi}{2} + \alpha\right) = -\sin \alpha$ when α is in Quadrant I. It follows $\frac{\pi}{2} + \alpha$ is in Quadrant II.

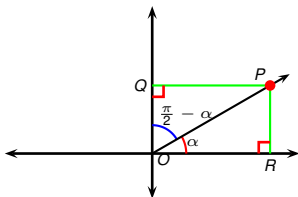
$$\begin{aligned}\cos\left(\frac{\pi}{2} + \alpha\right) &= -\frac{|OR'|}{|OP'|} && \square ORPQ \\ &= -\frac{|P'Q'|}{|OP'|} \\ &= -\frac{|PR|}{|OP|} \\ &= -\sin \alpha. && \text{as desired } \square\end{aligned}$$

Cofunction identities

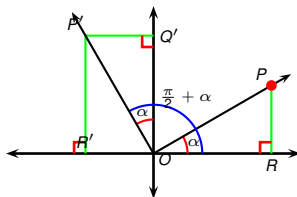
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To memorize the cofunction identities it suffices to memorize the Quadrant I case via the two diagrams below.



$$\begin{aligned}\sin\left(\frac{\pi}{2} - \alpha\right) &= \frac{|PQ|}{|OP|} \\ \cos\left(\frac{\pi}{2} - \alpha\right) &= \frac{|OQ|}{|OP|}\end{aligned}$$



$$\begin{aligned}\sin\left(\frac{\pi}{2} + \alpha\right) &= \frac{|OQ'|}{|OP|} \\ \cos\left(\frac{\pi}{2} + \alpha\right) &= -\frac{|PQ'|}{|OP'|} = \frac{|PR|}{|OP|}\end{aligned}$$