Precalculus
Euler's formula and trigonometric identities

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Outline

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Euler's Formula

Theorem (Euler's Formula)

$$e^{ix} = \cos x + i \sin x,$$

where $e \approx 2.71828$ is Euler's/Napier's constant .

Proof.

Recall $n! = 1 \cdot 2 \cdot 3 \cdot \dots \cdot (n-1) \cdot n$. Borrow from Calc II the f-las:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + \frac{(-1)^n x^{2n}}{(2n)!} + \dots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots$$

Euler's Formula

Theorem (Euler's Formula)

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Proof.

Recall $n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot (n-1) \cdot n$. Borrow from Calc II the f-las:

$$\begin{array}{rcccccccc}
i \sin x & = & & ix & & -i\frac{x^3}{3!} & & +i\frac{x^5}{5!} & - \dots \\
\cos x & = & 1 & & -\frac{x^2}{2!} & & +\frac{x^4}{4!} & & + \dots \\
\hline
e^{ix} & = & 1 & +ix & -\frac{x^2}{2!} & -i\frac{x^3}{3!} & +\frac{x^4}{4!} & +i\frac{x^5}{5!} & - \dots
\end{array}$$

Rearrange. Plug-in $z = ix$. Use $i^2 = -1$. Multiply $\sin x$ by i. Add to get $e^{ix} = \cos x + i \sin x$. $\square$

Trigonometric Identities Revisited

- $e^{ix} = \cos x + i \sin x$ (Euler's Formula).
- $e^{ix} e^{iy} = e^{ix+iy} = e^{i(x+y)}$ (exponentiation rule: valid for $\mathbb{C}$).
- $e^0 = 1$ (exponentiation rule).
- $\sin(-x) = -\sin x$, $\cos(-x) = \cos x$ (easy to remember).

Example

$$\begin{aligned} \sin(x+y) &= \sin x \cos y + \sin y \cos x \\ \cos(x+y) &= \cos x \cos y - \sin x \sin y \quad . \end{aligned}$$

Proof.

$$\begin{aligned} e^{i(x+y)} &= \cos(x+y) + i \sin(x+y) \\ e^{ix} e^{iy} &= \cos(x+y) + i \sin(x+y) \\ (\cos x + i \sin x)(\cos y + i \sin y) &= \cos(x+y) + i \sin(x+y) \\ \cos x \cos y - \sin x \sin y + i(\sin x \cos y + \sin y \cos x) &= \cos(x+y) + i \sin(x+y) \end{aligned}$$

Compare coefficient in front of i and remaining terms to get the desired equalities.

□

Trigonometric Identities Revisited

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Example

$$\sin^2 x + \cos^2 x = 1$$

Proof.

$$\begin{aligned} 1 &= e^0 \\ &= e^{ix-ix} = e^{ix}e^{-ix} = (\cos x + i \sin x)(\cos(-x) + i\sin(-x)) \\ &= (\cos x + i \sin x)(\cos x - i \sin x) = \cos^2 x - i^2 \sin^2 x \\ &= \cos^2 x + \sin^2 x \quad . \end{aligned}$$

□

Trigonometric Identities Revisited

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- $e^0 = 1$ (exponentiation rule).
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Example

$$\begin{aligned} \sin(2x) &= 2\sin x \cos x \\ \cos(2x) &= \cos^2 x - \sin^2 x \end{aligned} \quad .$$

Proof.

$$\begin{aligned} e^{i(2x)} &= \cos(2x) + i\sin(2x) \\ e^{ix}e^{ix} &= \cos(2x) + i\sin(2x) \\ (\cos x + i\sin x)^2 = (\cos x + i\sin x)(\cos x + i\sin x) &= \cos(2x) + i\sin(2x) \\ \cos^2 x - \sin^2 x + i(2\sin x\cos x) &= \cos(2x) + i\sin(2x) \end{aligned}$$

Compare coefficient in front of i and remaining terms to get the desired equalities. □

Trigonometric Identities

Definition (Trigonometric Identity)

A trigonometric identity is an equality between the trigonometric functions in one or more variables that holds for all values of the involved variables in the domains of all of the expressions.

- By convention, when dealing with trigonometric identities we do not account for the domains of the involved expressions.
- For example, $\frac{\sin\theta}{\sin\theta} = 1$ is considered a valid trigonometric identity, although, when considered as a function, the left hand side is not defined for $\theta \neq 0$.

Proof of a Trigonometric Identity

Let F and G be expressions that give a trigonometric identity: $F(\sin\theta, \cos\theta) = G(\sin\theta, \cos\theta)$.

- To prove a trigonometric identity means to show that the two sides of the equality sign are equivalent.
- There are two ways to do this (in the present course the first way will be preferred).
- First method: transform the left and right hand sides to an equal expression. In particular:
  - we can choose to transform the left hand side to the right;
  - we can choose to transform the right hand side to the left;
  - we can choose to transform both sides to a third equivalent expression.
- Second method: start with an already known identity and transform it, by a series of equivalent transformations, to the identity we desire to prove.
- The discussion here also applies for trigonometric identities in more than one variables.

Types of identites

- In the present course we deal with two basic types of trigonometric identities.
- First, identities that involve operations on the arguments of the trigonometric functions.
  - Example: $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ (this is one of the angle sum identities); $\sin\theta + \sin(-\theta) = 0$.
  - Such identities can be proved using the angle sum formulas and the even/odd function properties of $\sin, \cos$.
- Second, identities that involve trigonometric functions of one variable.
  - Example: $\tan^2\theta + 1 = \sec^2\theta$.
  - Such identities can be proved only using the already demonstrated Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$.
- The Pythagorean identity follows from the angle sum formulas and the even/odd function properties of $\sin, \cos$, so all trigonometric identities follow from those properties alone.

Example

Demonstrate the trigonometric identity $\csc^2\theta - 1 = \cot^2\theta$.
We transform the left hand side to the right one.

$$\begin{aligned}
\csc^2\theta - 1 &= \frac{1}{\sin^2\theta} - 1 \\
&= \frac{1-\sin^2\theta}{\sin^2\theta} \\
&= \frac{\cos^2\theta}{\sin^2\theta} \\
&= \cot^2\theta && \Big| \text{ as desired.}
\end{aligned}$$

Example

Verify the trigonometric identity $2\csc^2\alpha = \dfrac{1}{1-\cos\alpha} + \dfrac{1}{1+\cos\alpha}$

We transform the right hand side to the left.

$$
\begin{aligned}
\frac{1}{1-\cos\alpha} + \frac{1}{1+\cos\alpha} &= \frac{(1+\cos\alpha)}{(1-\cos\alpha)(1+\cos\alpha)} + \frac{(1-\cos\alpha)}{(1-\cos\alpha)(1+\cos\alpha)} \\
&= \frac{1+\cos\alpha+1-\cos\alpha}{1-\cos^2\alpha} \\
&= \frac{2}{\sin^2\alpha} \\
&= 2\csc^2\alpha
\end{aligned}
$$

as desired.

Example

Verify the identity $\ln(\sec\theta - 1) + \ln(\sec\theta + 1) - 2\ln(\sec\theta) = 2\ln(\sin\theta)$, where $0 < \theta < \frac{\pi}{2}$. We transform the left hand side to the right.

$$
\begin{aligned}
& \ln(\sec\theta - 1) + \ln(\sec\theta + 1) - 2\ln(\sec\theta) \\
= & \ln\left((\sec\theta - 1)(\sec\theta + 1)\right) - \ln\left(\sec^2\theta\right) \\
= & \ln\left(\sec^2\theta - 1\right) - \ln\left(\sec^2\theta\right) \\
= & \ln\left(\frac{\sec^2\theta - 1}{\sec^2\theta}\right) \\
= & \ln\left(1 - \frac{1}{\sec^2\theta}\right) \\
= & \ln\left(1 - \cos^2\theta\right) \\
= & \ln\left(\sin^2\theta\right) \\
= & 2\ln\left(\sin\theta\right)
\end{aligned}
$$

as desired.

Example

Verify the identity $\tan x + \cot x = \sec x \csc x$.

$$\begin{aligned}
\tan x + \cot x &= \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} \\
&= \frac{\sin^2 x}{\sin x \cos x} + \frac{\cos^2 x}{\sin x \cos x} \\
&= \frac{\sin^2 x + \cos^2 x}{\sin x \cos x} \\
&= \frac{1}{\sin x \cos x} \\
&= \frac{1}{\sin x}\frac{1}{\cos x} \\
&= \csc x \sec x,
\end{aligned}$$

as desired.

Here we explicitly permit the use of the Pythagorean identities and the double angle f-las:

$$\begin{aligned}\cos^2\theta+\sin^2\theta &= 1\\ \sin(2\theta) &= 2\sin\theta\cos\theta\\ \cos(2\theta) &= \cos^2\theta-\sin^2\theta\end{aligned}$$

Example

Prove the trigonometric identity.

$$(\sin\theta+\cos\theta)^2 = 1+\sin(2\theta)$$

We need to transform both sides to the same expression. In this case, we choose to transform the left hand side to the right:

$$\begin{aligned}(\sin\theta+\cos\theta)^2 &= \sin^2\theta+2\sin\theta\cos\theta+\cos^2\theta\\ &= 1+\sin(2\theta)\end{aligned}$$

$$\left|\ \begin{array}{l}(A+B)^2=\\ A^2+2AB+B^2\end{array}\right.$$

Recall the formulas
$$\begin{aligned}\sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta\\ \cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta.\end{aligned}$$

Example

Express $\sin(3x)$ and $\cos(3x)$ via $\cos x$ and $\sin x$.

$$\begin{aligned}
\sin(3x) &= \sin(x+2x)\\
&= \sin x\cos(2x) + \cos x\sin(2x)\\
&= \sin x\left(\cos^2 x - \sin^2 x\right) + \cos x(2\sin x\cos x)\\
&= \sin x\cos^2 x - \sin^3 x + 2\sin x\cos^2 x\\
&= 3\sin x\cos^2 x - \sin^3 x\\
\cos(3x) &= \cos(x+2x)\\
&= \cos x\cos(2x) - \sin x\sin(2x)\\
&= \cos x(\cos^2 x - \sin^2 x) - \sin x(2\sin x\cos x)\\
&= \cos^3 x - \cos x\sin^2 x - 2\cos x\sin^2 x\\
&= \cos^3 x - 3\cos x\sin^2 x
\end{aligned}$$

- Recall Euler's formula: $e^{i\alpha} = \cos\alpha + i\sin\alpha$.
- Recall the formula: $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.

Example

Express $\sin(3x)$ and $\cos(3x)$ via $\cos x$ and $\sin x$.

$$
\begin{array}{ll}
\ \cos(3x) + i\sin(3x) & \Big|\ \text{Euler's f-la} \\
= e^{3ix} & \\
= \left(e^{ix}\right)^3 = (\cos x + i\sin x)^3 & \Big|\ \text{Euler's f-la} \\
= \cos^3 x + 3\cos^2 x(i\sin x) + 3\cos x(i\sin x)^2 + (i\sin x)^3 & \\
= \cos^3 x + 3i\cos^2 x\sin x + 3i^2\cos x\sin^2 x + i^3\sin^3 x & \\
= \cos^3 x + 3i\cos^2 x\sin x - 3\cos x\sin^2 x - i\sin^3 x & \Big|\ \text{Use } i^2 = -1 \\
= (\cos^3 x - 3\cos x\sin^2 x) + i(3\cos^2 x\sin x - \sin^3 x) &
\end{array}
$$

The real parts of the starting and final expression must be equal; likewise the imaginary parts must be equal; therefore:

$$
\begin{array}{rcl}
\cos(3x) & = & \cos^3 x - 3\cos x\sin^2 x \\
\sin(3x) & = & 3\cos^2 x\sin x - \sin^3 x
\end{array}
$$

Example

Prove the identity $\tan\theta + \sec\theta = \dfrac{1+\tan\left(\frac{\theta}{2}\right)}{1-\tan\left(\frac{\theta}{2}\right)}$

All angles here are multiples of $\frac{\theta}{2}$, so set $\varphi = \frac{\theta}{2}$, $\theta = 2\varphi$.

$$
\begin{aligned}
\tan(2\varphi)+\sec(2\varphi) &= \frac{\sin(2\varphi)}{\cos(2\varphi)} + \frac{1}{\cos(2\varphi)} \\
&= \frac{\sin(2\varphi)+1}{\cos(2\varphi)} \\
&= \frac{2\sin\varphi\cos\varphi + \sin^2\varphi + \cos^2\varphi}{\cos^2\varphi - \sin^2\varphi} && \left|\ \begin{aligned} &A^2+2AB+B^2 \\ &=(A+B)^2 \end{aligned}\right. \\
&= \frac{(\cos\varphi+\sin\varphi)^{\cancel{2}}}{(\cos\varphi-\sin\varphi)\cancel{(\cos\varphi+\sin\varphi)}} && \left|\ \begin{aligned} &A^2-B^2= \\ &(A-B)(A+B) \end{aligned}\right. \\
&= \frac{(\cos\varphi+\sin\varphi)\frac{1}{\cos\varphi}}{(\cos\varphi-\sin\varphi)\frac{1}{\cos\varphi}} = \frac{1+\frac{\sin\varphi}{\cos\varphi}}{1-\frac{\sin\varphi}{\cos\varphi}} \\
&= \frac{1+\tan\varphi}{1-\tan\varphi} && \left|\ \text{as desired.}\right.
\end{aligned}
$$

Strategy for proving trigonometric identities

An expression is rational trigonometric if it is written using $\sin\theta, \cos\theta$ and the four arithmetic operations.

Question

Is there a general method for proving all rational trigonometric identities in one variable?

- Given a number of variables and relations between them, there is an algorithm to check whether (rational) expressions in those variables are equal under the given relations.
- Thus, if we pick two variables s and c, and a single relation
$$s^2 + c^2 = 1$$
there is a standard method to verify whether two (rational) expressions in s and c are equal.
- The method is rather cumbersome for a human and is best suited for computers.

Strategy for proving trigonometric identities

An expression is rational trigonometric if it is written using $\sin\theta, \cos\theta$ and the four arithmetic operations.

Question

Is there a general method for proving all rational trigonometric identities in one variable?

- Yes.
- For expressions that depend only on $\sin\theta$ and $\cos\theta$, algebra tells us when two expressions in those are equal.
- Problems depending on $\cos\theta, \sin\theta$ alone will always be doable via easy ad-hoc tricks using
$$\sin^2\theta + \cos^2\theta = 1.$$
- The full method will not be needed in this course.
  - The full method: set $s = \sin\theta, c = \cos\theta$.
  - Check whether the two expressions in s, c are equal under the relation $s^2 + c^2 = 1$. (The method lies outside of present scope).

Strategy for proving trigonometric identities

An expression is rational trigonometric if it is written using $\sin\theta, \cos\theta$ and the four arithmetic operations.

Question

Is there a general method for proving all rational trigonometric identities in one variable?

- To prove a general trigonometric identity:
  - Use angle sum/double angle sum formulas to convert all formulas to trig expression depending only on $\sin\theta$, $\cos\theta$.
  - Use $\sin^2\theta + \cos^2\theta = 1$ to show the two formulas are equal (usage: ad-hoc).
  - You may need to use trig functions of angles smaller than θ, for example $\sin\left(\frac{\theta}{2}\right), \cos\left(\frac{\theta}{2}\right)$.
  - A fraction of θ such that all appearing angles are integer multiples of it will always work.

Proving the following identities is a good exercise.

1. $\sin\theta\cot\theta = \cos\theta$.
2. $(\sin\theta + \cos\theta)^2 = 1 + \sin(2\theta)$.
3. $\sec\theta - \cos\theta = \tan\theta\sin\theta$.
4. $\tan^2\theta - \sin^2\theta = \tan^2\theta\sin^2\theta$.
5. $\cot^2\theta + \sec^2\theta = \tan^2\theta + \csc^2\theta$.
6. $2\csc(2\theta) = \sec\theta\csc\theta$.
7. $\tan(2\theta) = \dfrac{2\tan\theta}{1-\tan^2\theta}$.
8. $\dfrac{1}{1-\sin\theta} + \dfrac{1}{1+\sin\theta} = 2\sec^2\theta$.
9. $\tan\alpha + \tan\beta = \dfrac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta}$.
10. $\tan(\alpha+\beta) = \dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$.
11. $\sin(3\theta) + \sin\theta = 2\sin(2\theta)\cos\theta$.
12. $\cos(3\theta) = 4\cos^3\theta - 3\cos\theta$.
13. $1 + \tan^2\theta = \sec^2\theta$.
14. $1 + \csc^2\theta = \cot^2\theta$.
15. $2\cos^2(2x) =$ $2\sin^4\theta + 2\cos^4\theta - \sin^2(2\theta)$.
16. $\dfrac{1+\tan\left(\frac{\theta}{2}\right)}{1-\tan\left(\frac{\theta}{2}\right)} = \tan\theta + \sec\theta$.