

Calculus II

Building block integrals

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Outline

- 1 Integration of Rational Functions
 - Building block integrals

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 - We use linear substitutions to transform each piece to one of 3 pairs of basic building block integrals.
 - We solve each building block integral and collect the terms.
- We study the algorithm “from the ground up”: we start with the building blocks.

The building blocks

Let n be a positive integer.

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- The case $n = 1$ is special for each of the building blocks:

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- The case $n = 1$ we call respectively building block Ia, IIa and IIIa.
The case $n > 1$ we call respectively building block Ib, IIb and IIIb.
This “building block” terminology is for our convenience, and is not a part of standard mathematical terminology.

Building block Ia

Building block Ia: $\int \frac{1}{x} dx$.

Example

Integrate building block Ia

$$\int \frac{1}{x} dx$$

Building block Ia

Building block Ia: $\int \frac{1}{x} dx$.

Example

Integrate building block Ia

$$\int \frac{1}{x} dx = ?$$

Building block Ia

Building block Ia: $\int \frac{1}{x} dx$.

Example

Integrate building block Ia

$$\int \frac{1}{x} dx = \ln |x| + C$$

Linear substitutions leading to building block Ia

Building block Ia: $\int \frac{1}{x} dx = \ln |x| + C.$

Example

Integrate

$$\int \frac{1}{-4x + 5} dx$$

Linear substitutions leading to building block Ia

Building block Ia: $\int \frac{1}{x} dx = \ln |x| + C.$

Example

Integrate

$$\int \frac{1}{-4x+5} dx = \int \frac{1}{(-4x+5)} \frac{d(-4x)}{(-4)}$$

Linear substitutions leading to building block Ia

Building block Ia: $\int \frac{1}{x} dx = \ln |x| + C.$

Example

Integrate

$$\begin{aligned}\int \frac{1}{-4x+5} dx &= \int \frac{1}{(-4x+5)} \frac{\overset{\text{red}}{d(-4x)}}{(-4)} \\ &= \int \frac{1}{(-4x+5)} \frac{\overset{\text{red}}{d(-4x+5)}}{(-4)}\end{aligned}$$

Linear substitutions leading to building block Ia

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Integrate

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Set $u = -4x + 5$

Linear substitutions leading to building block Ia

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$$\begin{aligned}\int \frac{1}{-4x+5} dx &= \int \frac{1}{(-4x+5)} \frac{d(-4x)}{(-4)} \\ &= \int \frac{1}{(-4x+5)} \frac{d(-4x+5)}{(-4)} && \left| \text{Set } u = -4x+5 \right. \\ &= \int \frac{1}{u} \frac{du}{(-4)} \\ &= -\frac{1}{4} \int u^{-1} du\end{aligned}$$

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Lin. subst. leading to building block 1a: general case

Building block 1a: $\int \frac{1}{x} dx = \ln |x| + C.$

Example

Integrate

$$\begin{aligned} \int \frac{1}{ax+b} dx &= \int \frac{1}{(ax+b)} \frac{d(ax)}{a} \\ &= \int \frac{1}{(ax+b)} \frac{d(ax+b)}{a} && \left| \text{Set } u = ax+b \right. \\ &= \int \frac{1}{u} \frac{du}{a} \\ &= \frac{1}{a} \int u^{-1} du = \frac{1}{a} \ln |u| + C \\ &= \frac{1}{a} \ln |ax+b| + C. \end{aligned}$$

Building block Ib

Building block Ib: $\int \frac{1}{x^n} dx = \int x^{-n} dx, n \neq 1.$

Example (Block Ib)

$$\int \frac{1}{x^n} dx$$

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Example (Block Ib)

$$\int \frac{1}{x^n} dx = \int x^{-n} dx = ?$$

Building block Ib

Building block Ib: $\int \frac{1}{x^n} dx = \int x^{-n} dx, n \neq 1.$

Example (Block Ib)

$$\int \frac{1}{x^n} dx = \int x^{-n} dx = \frac{x^{-n+1}}{-n+1} + C$$

Linear substitutions leading to building block Ib

Building block Ib: $\int \frac{1}{x^n} dx = \int x^{-n} dx = \frac{x^{-n+1}}{-n+1} + C, n \neq 1.$

Example

Integrate

$$\int \frac{1}{(3x+5)^3} dx$$

Linear substitutions leading to building block Ib

Building block Ib: $\int \frac{1}{x^n} dx = \int x^{-n} dx = \frac{x^{-n+1}}{-n+1} + C, n \neq 1.$

Example

Integrate

$$\int \frac{1}{(3x+5)^3} dx = \int \frac{1}{(3x+5)^3} \frac{d(\textcolor{red}{3}x)}{\textcolor{red}{3}}$$

Linear substitutions leading to building block Ib

Building block Ib: $\int \frac{1}{x^n} dx = \int x^{-n} dx = \frac{x^{-n+1}}{-n+1} + C, n \neq 1.$

Example

Integrate

$$\begin{aligned} \int \frac{1}{(3x+5)^3} dx &= \int \frac{1}{(3x+5)^3} \frac{\textcolor{red}{d(3x)}}{3} \\ &= \int \frac{1}{(3x+5)^3} \frac{\textcolor{red}{d(3x+5)}}{3} \end{aligned}$$

Linear substitutions leading to building block Ib

Building block Ib: $\int \frac{1}{x^n} dx = \int x^{-n} dx = \frac{x^{-n+1}}{-n+1} + C, n \neq 1.$

Example

Integrate

$$\begin{aligned} \int \frac{1}{(3x+5)^3} dx &= \int \frac{1}{(3x+5)^3} \frac{d(3x)}{3} \\ &= \int \frac{1}{(\textcolor{red}{3x+5})^3} \frac{d(\textcolor{red}{3x+5})}{3} && \left| \text{Set } \textcolor{red}{u} = \textcolor{red}{3x+5} \right. \\ &= \int \frac{1}{\textcolor{red}{u}^3} \frac{d\textcolor{red}{u}}{3} \end{aligned}$$

Linear substitutions leading to building block Ib

Building block Ib: $\int \frac{1}{x^n} dx = \int x^{-n} dx = \frac{x^{-n+1}}{-n+1} + C, n \neq 1.$

Example

Integrate

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Linear substitutions leading to building block Ib

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Linear substitutions leading to building block Ib

Building block Ib: $\int \frac{1}{x^n} dx = \int x^{-n} dx = \frac{x^{-n+1}}{-n+1} + C, n \neq 1.$

Example

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Lin. subst. leading to building block Ib: general case

Building block Ib: $\int \frac{1}{x^n} dx = \int x^{-n} dx = \frac{x^{-n+1}}{-n+1} + C, n \neq 1.$

Example

Let $n \neq 1$. Integrate

$$\begin{aligned}
 \int \frac{1}{(ax+b)^n} dx &= \int \frac{1}{(ax+b)^n} \frac{d(ax)}{a} \\
 &= \int \frac{1}{(ax+b)^n} \frac{d(ax+b)}{a} && \left| \text{Set } u = ax+b \right. \\
 &= \int \frac{1}{u^n} \frac{du}{a} \\
 &= \frac{1}{a} \int u^{-n} du = -\frac{1}{a} \frac{u^{-n+1}}{n-1} + C \\
 &= -\frac{1}{a(n-1)(ax+b)^{n-1}} + C.
 \end{aligned}$$

Building blocks IIa and IIIa

Building block IIa: $\int \frac{x}{1+x^2} dx$. Building block IIIa: $\int \frac{1}{1+x^2} dx$.

Example (Block IIa)

$$\int \frac{x}{1+x^2} dx$$

Building blocks IIa and IIIa

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Example (Block IIa)

$$\int \frac{x}{1+x^2} dx = \int \frac{1}{(1+x^2)} \frac{d(x^2)}{2}$$

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Example (Block IIa)

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Example (Block IIa)

$$\begin{aligned}\int \frac{x}{1+x^2} dx &= \int \frac{1}{(1+x^2)} \frac{d(x^2)}{2} \\ &= \int \frac{1}{\overset{1+x^2}{1+x^2}} \frac{\overset{d(1+x^2)}{d(1+x^2)}}{2} \\ &= \int \frac{1}{\overset{u}{u}} \frac{\overset{du}{du}}{2}\end{aligned}$$

Set $u = 1 + x^2$

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Set $u = 1 + x^2$

Building blocks IIa and IIIa

Building block IIa: $\int \frac{x}{1+x^2} dx$. Building block IIIa: $\int \frac{1}{1+x^2} dx$.

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Example (Block IIIa)

$$\int \frac{1}{1+x^2} dx$$

Building blocks IIa and IIIa

Building block IIa: $\int \frac{x}{1+x^2} dx$. Building block IIIa: $\int \frac{1}{1+x^2} dx$.

Example (Block IIa)

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Example (Block IIIa)

$$\int \frac{1}{1+x^2} dx = ?$$

Building blocks IIa and IIIa

Building block IIa: $\int \frac{x}{1+x^2} dx$. Building block IIIa: $\int \frac{1}{1+x^2} dx$.

Example (Block IIa)

$$\begin{aligned}\int \frac{x}{1+x^2} dx &= \int \frac{1}{(1+x^2)} \frac{d(x^2)}{2} \\ &= \int \frac{1}{1+x^2} \frac{d(1+x^2)}{2} \\ &= \int \frac{1}{u} \frac{du}{2} \\ &= \frac{1}{2} \ln |u| + C = \frac{1}{2} \ln (1+x^2) + C \quad .\end{aligned} \quad \left| \quad \text{Set } u = 1+x^2 \right.$$

Example (Block IIIa)

$$\int \frac{1}{1+x^2} dx = \arctan x + C$$

Linear substitutions leading to block IIa

Building block IIa: $\int \frac{x}{1+x^2} dx = \frac{1}{2} \ln(1+x^2) + C.$

Example

$$\int \frac{x}{2x^2+3} dx$$

Linear substitutions leading to block IIa

Building block IIa: $\int \frac{x}{1+x^2} dx = \frac{1}{2} \ln(1+x^2) + C$.

“Theoretical way” to solve example below: transform to IIa; this is slow.

Example

$$\int \frac{x}{2x^2+3} dx$$

Linear substitutions leading to block IIa

Building block IIa: $\int \frac{x}{1+x^2} dx = \frac{1}{2} \ln(1+x^2) + C$.

“Theoretical way” to solve example below: transform to IIa; this is slow.

Feel free to skip slide, we will redo in next slide with a shortcut.

Example

$$\int \frac{x}{2x^2+3} dx$$

Linear substitutions leading to block IIa

Building block IIa: $\int \frac{x}{1+x^2} dx = \frac{1}{2} \ln(1+x^2) + C$.

“Theoretical way” to solve example below: transform to IIa; this is slow.

Feel free to skip slide, we will redo in next slide with a shortcut.

Example

$$\int \frac{x}{2x^2+3} dx = \int \frac{x}{3\left(\frac{2}{3}x^2+1\right)} dx = \int \frac{x}{3\left(\left(\sqrt{\frac{2}{3}}x\right)^2+1\right)} dx$$

$$= \frac{3}{2} \int \frac{\sqrt{\frac{2}{3}}x}{3\left(\left(\sqrt{\frac{2}{3}}x\right)^2+1\right)} d\left(\sqrt{\frac{2}{3}}x\right)$$

$$\left| \text{Set } u = \sqrt{\frac{2}{3}}x \right.$$

$$= \frac{1}{2} \int \frac{u}{u^2+1} du = \frac{1}{4} \ln(1+u^2) + C$$

$$= \frac{1}{4} \ln\left(\frac{1}{3}(2x^2+3)\right) + C$$

$$= \frac{1}{4} \ln(2x^2+3) + \frac{\ln\left(\frac{1}{3}\right)}{4} + C$$

$$= \frac{1}{4} \ln(2x^2+3) + K.$$

Linear substitutions leading to blocks IIa

Building block IIa: $\int \frac{x}{1+x^2} dx = \frac{1}{2} \ln(1+x^2) + C$.

The example below can be done directly, without transforming to block IIa.

Example

$$\int \frac{x}{2x^2 + 3} dx$$

Linear substitutions leading to blocks IIa

Building block IIa: $\int \frac{x}{1+x^2} dx = \frac{1}{2} \ln(1+x^2) + C$.

The example below can be done directly, without transforming to block IIa.

Example

$$\int \frac{x}{2x^2+3} dx = \int \frac{1}{2x^2+3} d\left(\frac{x^2}{2}\right)$$

Linear substitutions leading to blocks IIa

Building block IIa: $\int \frac{x}{1+x^2} dx = \frac{1}{2} \ln(1+x^2) + C$.

The example below can be done directly, without transforming to block IIa.

Example

$$\begin{aligned} \int \frac{x}{2x^2+3} dx &= \int \frac{1}{2x^2+3} d\left(\frac{x^2}{2}\right) \\ &= \int \frac{1}{2x^2+3} d\left(\frac{2x^2}{4}\right) \end{aligned}$$

Linear substitutions leading to blocks IIa

Building block IIa: $\int \frac{x}{1+x^2} dx = \frac{1}{2} \ln(1+x^2) + C$.

The example below can be done directly, without transforming to block IIa.

Example

$$\begin{aligned} \int \frac{x}{2x^2+3} dx &= \int \frac{1}{2x^2+3} d\left(\frac{x^2}{2}\right) \\ &= \int \frac{1}{2x^2+3} d\left(\frac{2x^2+3}{4}\right) \end{aligned}$$

Linear substitutions leading to blocks IIa

Building block IIa: $\int \frac{x}{1+x^2} dx = \frac{1}{2} \ln(1+x^2) + C$.

The example below can be done directly, without transforming to block IIa.

Example

$$\begin{aligned} \int \frac{x}{2x^2+3} dx &= \int \frac{1}{2x^2+3} d\left(\frac{x^2}{2}\right) \\ &= \int \frac{1}{\color{red}{2x^2+3}} d\left(\frac{\color{red}{2x^2+3}}{4}\right) \quad \Bigg| \quad \text{Set } \color{red}{u} = \color{red}{2x^2+3} \\ &= \frac{1}{4} \int \frac{1}{\color{red}{u}} d\color{red}{u} \end{aligned}$$

Linear substitutions leading to blocks IIa

Building block IIa: $\int \frac{x}{1+x^2} dx = \frac{1}{2} \ln(1+x^2) + C$.

The example below can be done directly, without transforming to block IIa.

Example

$$\begin{aligned} \int \frac{x}{2x^2+3} dx &= \int \frac{1}{2x^2+3} d\left(\frac{x^2}{2}\right) \\ &= \int \frac{1}{2x^2+3} d\left(\frac{2x^2+3}{4}\right) && \left| \text{Set } u = 2x^2+3 \right. \\ &= \frac{1}{4} \int \frac{1}{u} du \\ &= \frac{1}{4} \ln |u| + C \end{aligned}$$

Linear substitutions leading to blocks IIa

Building block IIa: $\int \frac{x}{1+x^2} dx = \frac{1}{2} \ln(1+x^2) + C$.

The example below can be done directly, without transforming to block IIa.

Example

$$\begin{aligned} \int \frac{x}{2x^2+3} dx &= \int \frac{1}{2x^2+3} d\left(\frac{x^2}{2}\right) \\ &= \int \frac{1}{2x^2+3} d\left(\frac{2x^2+3}{4}\right) && \left| \text{Set } u = 2x^2+3 \right. \\ &= \frac{1}{4} \int \frac{1}{u} du \\ &= \frac{1}{4} \ln |u| + C \\ &= \frac{1}{4} \ln(2x^2+3) + C \end{aligned}$$

Linear substitutions leading to blocks IIa and IIIa

Building block IIIa: $\int \frac{1}{u^2+1} du = ? + C.$

Example

$$\int \frac{1}{x^2 + 2} dx$$

Linear substitutions leading to blocks IIa and IIIa

Building block IIIa: $\int \frac{1}{u^2+1} du = ? + C.$

Example

$$\int \frac{1}{x^2 + 2} dx$$

Linear substitutions leading to blocks IIa and IIIa

Building block IIIa: $\int \frac{1}{u^2 + 1} du = ? + C.$

Example

$$\int \frac{1}{x^2 + 2} dx$$

Linear substitutions leading to blocks IIa and IIIa

Building block IIIa: $\int \frac{1}{u^2+1} du = ? + C.$

Example

$$\int \frac{1}{x^2 + 2} dx = \int \frac{1}{2 \left(\frac{1}{2}x^2 + 1 \right)} dx$$

Linear substitutions leading to blocks IIa and IIIa

Building block IIIa: $\int \frac{1}{u^2+1} du = ? + C.$

Example

$$\int \frac{1}{x^2 + 2} dx = \int \frac{1}{2 \left(\frac{1}{2} x^2 + 1 \right)} dx$$

Linear substitutions leading to blocks IIa and IIIa

Building block IIIa: $\int \frac{1}{u^2+1} du = ? + C.$

Example

$$\int \frac{1}{x^2+2} dx = \int \frac{1}{2(\frac{1}{2}x^2+1)} dx$$

Linear substitutions leading to blocks IIa and IIIa

Building block IIIa: $\int \frac{1}{u^2+1} du = ? + C.$

Example

$$\begin{aligned}\int \frac{1}{x^2+2} dx &= \int \frac{1}{2\left(\frac{1}{2}x^2+1\right)} dx \\ &= \int \frac{1}{2\left(\left(\frac{x}{\sqrt{2}}\right)^2+1\right)} d\left(\frac{x}{\sqrt{2}}\right)\end{aligned}$$

Linear substitutions leading to blocks IIa and IIIa

Building block IIIa: $\int \frac{1}{u^2+1} du = ? + C.$

Example

$$\begin{aligned} \int \frac{1}{x^2+2} dx &= \int \frac{1}{2\left(\frac{1}{2}x^2+1\right)} dx && \left| \text{Use } 2 = (\sqrt{2})^2 \right. \\ &= \int \frac{1}{2\left(\left(\frac{x}{\sqrt{2}}\right)^2+1\right)} d\left(\frac{x}{\sqrt{2}}\right) \end{aligned}$$

Linear substitutions leading to blocks IIa and IIIa

Building block IIIa: $\int \frac{1}{u^2+1} du = ? + C.$

Example

$$\begin{aligned}
 \int \frac{1}{x^2+2} dx &= \int \frac{1}{2\left(\frac{1}{2}x^2+1\right)} dx && \left| \begin{array}{l} \text{Use } 2 = (\sqrt{2})^2 \\ \text{Set } \frac{x}{\sqrt{2}} = u \end{array} \right. \\
 &= \int \frac{1}{2\left(\left(\frac{x}{\sqrt{2}}\right)^2+1\right)} d\left(\frac{x}{\sqrt{2}}\right)
 \end{aligned}$$

Linear substitutions leading to blocks IIa and IIIa

Building block IIIa: $\int \frac{1}{u^2+1} du = ? + C.$

Example

$$\begin{aligned} \int \frac{1}{x^2+2} dx &= \int \frac{1}{2\left(\frac{1}{2}x^2+1\right)} dx \\ &= \int \frac{1}{2\left(\left(\frac{x}{\sqrt{2}}\right)^2+1\right)} \sqrt{2} d\left(\frac{x}{\sqrt{2}}\right) \end{aligned} \quad \left| \begin{array}{l} \text{Use } 2 = (\sqrt{2})^2 \\ \text{Set } \frac{x}{\sqrt{2}} = u \end{array} \right.$$

Linear substitutions leading to blocks IIa and IIIa

Building block IIIa: $\int \frac{1}{u^2+1} du = ? + C.$

Example

$$\begin{aligned}\int \frac{1}{x^2+2} dx &= \int \frac{1}{2\left(\frac{1}{2}x^2+1\right)} dx && \left| \begin{array}{l} \text{Use } 2 = (\sqrt{2})^2 \\ \text{Set } \frac{x}{\sqrt{2}} = u \end{array} \right. \\ &= \int \frac{1}{2\left(\left(\frac{x}{\sqrt{2}}\right)^2+1\right)} \sqrt{2} d\left(\frac{x}{\sqrt{2}}\right) \\ &= \frac{1}{\sqrt{2}} \int \frac{1}{u^2+1} du\end{aligned}$$

Linear substitutions leading to blocks IIa and IIIa

Building block IIIa: $\int \frac{1}{u^2+1} du = ? + C.$

Example

$$\begin{aligned}
 \int \frac{1}{x^2+2} dx &= \int \frac{1}{2\left(\frac{1}{2}x^2+1\right)} dx && \left| \begin{array}{l} \text{Use } 2 = (\sqrt{2})^2 \\ \text{Set } \frac{x}{\sqrt{2}} = u \end{array} \right. \\
 &= \int \frac{1}{2\left(\left(\frac{x}{\sqrt{2}}\right)^2+1\right)} \sqrt{2} d\left(\frac{x}{\sqrt{2}}\right) \\
 &= \frac{1}{\sqrt{2}} \int \frac{1}{u^2+1} du
 \end{aligned}$$

Linear substitutions leading to blocks IIa and IIIa

Building block IIIa: $\int \frac{1}{u^2+1} du = ? + C.$

Example

$$\begin{aligned}
 \int \frac{1}{x^2+2} dx &= \int \frac{1}{2\left(\frac{1}{2}x^2+1\right)} dx && \left| \begin{array}{l} \text{Use } 2 = (\sqrt{2})^2 \\ \text{Set } \frac{x}{\sqrt{2}} = u \end{array} \right. \\
 &= \int \frac{1}{2\left(\left(\frac{x}{\sqrt{2}}\right)^2+1\right)} \sqrt{2} d\left(\frac{x}{\sqrt{2}}\right) \\
 &= \frac{1}{\sqrt{2}} \int \frac{1}{u^2+1} du \\
 &= \frac{1}{\sqrt{2}} ? + C
 \end{aligned}$$

Linear substitutions leading to blocks IIa and IIIa

Building block IIIa: $\int \frac{1}{u^2+1} du = \arctan u + C$.

Example

$$\begin{aligned} \int \frac{1}{x^2+2} dx &= \int \frac{1}{2\left(\frac{1}{2}x^2+1\right)} dx && \left| \begin{array}{l} \text{Use } 2 = (\sqrt{2})^2 \\ \text{Set } \frac{x}{\sqrt{2}} = u \end{array} \right. \\ &= \int \frac{1}{2\left(\left(\frac{x}{\sqrt{2}}\right)^2+1\right)} \sqrt{2} d\left(\frac{x}{\sqrt{2}}\right) \\ &= \frac{1}{\sqrt{2}} \int \frac{1}{u^2+1} du \\ &= \frac{1}{\sqrt{2}} \arctan(u) + C \end{aligned}$$

Linear substitutions leading to blocks IIa and IIIa

Building block IIIa: $\int \frac{1}{u^2+1} du = \arctan u + C$.

Example

$$\begin{aligned}\int \frac{1}{x^2+2} dx &= \int \frac{1}{2\left(\frac{1}{2}x^2+1\right)} dx \\ &= \int \frac{1}{2\left(\left(\frac{x}{\sqrt{2}}\right)^2+1\right)} \sqrt{2} d\left(\frac{x}{\sqrt{2}}\right) \\ &= \frac{1}{\sqrt{2}} \int \frac{1}{u^2+1} du \\ &= \frac{1}{\sqrt{2}} \arctan(\textcolor{red}{u}) + C \\ &= \frac{1}{\sqrt{2}} \arctan\left(\frac{\textcolor{red}{x}}{\sqrt{2}}\right) + C\end{aligned}$$

Use $2 = (\sqrt{2})^2$

Set $\frac{\textcolor{red}{x}}{\sqrt{2}} = \textcolor{red}{u}$

Linear substitutions leading to blocks IIa and IIIa

Building block IIIa: $\int \frac{1}{u^2+1} du = \arctan u + C$.

Example

$$\begin{aligned} \int \frac{1}{x^2+2} dx &= \int \frac{1}{2\left(\frac{1}{2}x^2+1\right)} dx && \left| \begin{array}{l} \text{Use } 2 = (\sqrt{2})^2 \\ \text{Set } \frac{x}{\sqrt{2}} = u \end{array} \right. \\ &= \int \frac{1}{2\left(\left(\frac{x}{\sqrt{2}}\right)^2+1\right)} \sqrt{2} d\left(\frac{x}{\sqrt{2}}\right) \\ &= \frac{1}{\sqrt{2}} \int \frac{1}{u^2+1} du \\ &= \frac{1}{\sqrt{2}} \arctan(u) + C \\ &= \frac{1}{\sqrt{2}} \arctan\left(\frac{x}{\sqrt{2}}\right) + C \end{aligned}$$

Linear substitutions leading to blocks IIa and IIIa

Building block IIIa: $\int \frac{1}{u^2+1} du = \arctan u + C.$

Example

$$\begin{aligned}
 \int \frac{1}{x^2 + 2} dx &= \int \frac{1}{2 \left(\frac{1}{2} x^2 + 1 \right)} dx && \left| \begin{array}{l} \text{Use } 2 = (\sqrt{2})^2 \\ \text{Set } \frac{x}{\sqrt{2}} = u \end{array} \right. \\
 &= \int \frac{1}{2 \left(\left(\frac{x}{\sqrt{2}} \right)^2 + 1 \right)} \sqrt{2} d \left(\frac{x}{\sqrt{2}} \right) \\
 &= \frac{1}{\sqrt{2}} \int \frac{1}{u^2 + 1} du \\
 &= \frac{1}{\sqrt{2}} \arctan(u) + C \\
 &= \frac{1}{\sqrt{2}} \arctan \left(\frac{x}{\sqrt{2}} \right) + C
 \end{aligned}$$

Linear substitutions leading to blocks IIa and IIIa

Building block IIIa: $\int \frac{1}{x^2+1} dx = \arctan x + C$. **Let $a > 0$.**

Example

$$\begin{aligned}\int \frac{1}{x^2 + a} dx &= \int \frac{1}{a \left(\frac{1}{a} x^2 + 1 \right)} dx \\&= \int \frac{1}{a \left(\left(\frac{x}{\sqrt{a}} \right)^2 + 1 \right)} \sqrt{a} d \left(\frac{x}{\sqrt{a}} \right) \\&= \frac{1}{\sqrt{a}} \int \frac{1}{u^2 + 1} du \\&= \frac{1}{\sqrt{a}} \arctan(u) + C \\&= \frac{1}{\sqrt{a}} \arctan \left(\frac{x}{\sqrt{a}} \right) + C\end{aligned}$$

Use $a = (\sqrt{a})^2$

Set $u = \frac{x}{\sqrt{a}}$

Linear substitutions leading to blocks IIa and IIIa

Building block IIIa: $\int \frac{1}{x^2+1} dx = \arctan x + C$. **Let $a > 0$.**

Example

$$\begin{aligned}
 \int \frac{1}{x^2 + a} dx &= \int \frac{1}{a \left(\frac{1}{a} x^2 + 1 \right)} dx && \left| \begin{array}{l} \text{Use } a = (\sqrt{a})^2 \\ \text{Set } u = \frac{x}{\sqrt{a}} \end{array} \right. \\
 &= \int \frac{1}{a \left(\left(\frac{x}{\sqrt{a}} \right)^2 + 1 \right)} \sqrt{a} du \left(\frac{x}{\sqrt{a}} \right) \\
 &= \frac{1}{\sqrt{a}} \int \frac{1}{u^2 + 1} du \\
 &= \frac{1}{\sqrt{a}} \arctan(u) + C \\
 &= \frac{1}{\sqrt{a}} \arctan \left(\frac{x}{\sqrt{a}} \right) + C
 \end{aligned}$$

Linear substitutions leading to blocks IIa and IIIa

Building block IIIa: $\int \frac{1}{x^2+1} dx = \arctan x + C$. **Let** $a > 0$.

Example

$$\begin{aligned} \int \frac{1}{x^2+a} dx &= \int \frac{1}{a\left(\frac{1}{a}x^2+1\right)} dx && \left| \begin{array}{l} \text{Use } a = (\sqrt{a})^2 \\ \text{Set } u = \frac{x}{\sqrt{a}} \end{array} \right. \\ &= \int \frac{1}{a\left(\left(\frac{x}{\sqrt{a}}\right)^2+1\right)} \sqrt{a} du \\ &= \frac{1}{\sqrt{a}} \int \frac{1}{u^2+1} du \\ &= \frac{1}{\sqrt{a}} \arctan(u) + C \\ &= \frac{1}{\sqrt{a}} \arctan\left(\frac{x}{\sqrt{a}}\right) + C \end{aligned}$$

Linear substitutions leading to blocks IIa and IIIa

Building block IIIa: $\int \frac{1}{x^2+1} dx = \arctan x + C$. **Let $a > 0$.**

Example

$$\begin{aligned}
 \int \frac{1}{x^2 + a} dx &= \int \frac{1}{a\left(\frac{1}{a}x^2 + 1\right)} dx && \left| \begin{array}{l} \text{Use } a = (\sqrt{a})^2 \\ \text{Set } u = \frac{x}{\sqrt{a}} \end{array} \right. \\
 &= \int \frac{1}{a\left(\left(\frac{x}{\sqrt{a}}\right)^2 + 1\right)} \sqrt{a} du \\
 &= \frac{1}{\sqrt{a}} \int \frac{1}{u^2 + 1} du \\
 &= \frac{1}{\sqrt{a}} \arctan(u) + C \\
 &= \frac{1}{\sqrt{a}} \arctan\left(\frac{x}{\sqrt{a}}\right) + C
 \end{aligned}$$

Linear substitutions leading to blocks IIa and IIIa

Building block IIa: $\int \frac{x}{1+x^2} dx = \frac{1}{2} \ln(1+x^2) + C.$

Building block IIIa: $\int \frac{1}{1+x^2} dx = \arctan x + C.$

- Let $ax^2 + bx + c$ have no real roots.

Linear substitutions leading to blocks IIa and IIIa

Building block IIa: $\int \frac{x}{1+x^2} dx = \frac{1}{2} \ln(1+x^2) + C.$

Building block IIIa: $\int \frac{1}{1+x^2} dx = \arctan x + C.$

- Let $ax^2 + bx + c$ have no real roots.
- We can find p, q so that the linear substitution $u = px + q$ transforms the quadratic to:

$$ax^2 + bx + c = r(u^2 + 1)$$

(where r is some number to be determined).

Linear substitutions leading to blocks IIa and IIIa

Building block IIa: $\int \frac{x}{1+x^2} dx = \frac{1}{2} \ln(1+x^2) + C.$

Building block IIIa: $\int \frac{1}{1+x^2} dx = \arctan x + C.$

- Let $ax^2 + bx + c$ have no real roots.
- We can find p, q so that the linear substitution $u = px + q$ transforms the quadratic to:

$$ax^2 + bx + c = r(u^2 + 1)$$

(where r is some number to be determined).

- To find p, q , we **complete the square**.

Linear substitutions leading to blocks IIa and IIIa

Building block IIa: $\int \frac{x}{1+x^2} dx = \frac{1}{2} \ln(1+x^2) + C.$

Building block IIIa: $\int \frac{1}{1+x^2} dx = \arctan x + C.$

- Let $ax^2 + bx + c$ have no real roots.
- We can find p, q so that the linear substitution $u = px + q$ transforms the quadratic to:

$$ax^2 + bx + c = r(u^2 + 1)$$

(where r is some number to be determined).

- To find p, q , we complete the square.
- In this way, integrals of the form $\int \frac{Ax + B}{ax^2 + bx + c} dx$ are transformed to combinations of building blocks IIa and IIIa.

Linear substitutions leading to blocks IIa and IIIa

Building block IIa: $\int \frac{x}{1+x^2} dx = \frac{1}{2} \ln(1+x^2) + C.$

Building block IIIa: $\int \frac{1}{1+x^2} dx = \arctan x + C.$

- Let $ax^2 + bx + c$ have no real roots.
- We can find p, q so that the linear substitution $u = px + q$ transforms the quadratic to:

$$ax^2 + bx + c = r(u^2 + 1)$$

(where r is some number to be determined).

- To find p, q , we complete the square.
- In this way, integrals of the form $\int \frac{Ax + B}{ax^2 + bx + c} dx$ are transformed to combinations of building blocks IIa and IIIa.
- We show examples; the general case is analogous and we leave it to the student.

Linear substitutions leading to blocks IIa and IIIa

Building block IIa: $\int \frac{x}{1+x^2} dx = \frac{1}{2} \ln(1+x^2) + C.$

Building block IIIa: $\int \frac{1}{1+x^2} dx = \arctan x + C.$

Example

$$\int \frac{x}{x^2 + x + 1} dx =$$

Linear substitutions leading to blocks IIa and IIIa

Building block IIa: $\int \frac{x}{1+x^2} dx = \frac{1}{2} \ln(1+x^2) + C.$

Building block IIIa: $\int \frac{1}{1+x^2} dx = \arctan x + C.$

Example

No real roots \Rightarrow complete the square.

$$\int \frac{x}{x^2 + x + 1} dx = \int \frac{x}{x^2 + 2 \cdot \frac{1}{2}x + \frac{1}{4} - \frac{1}{4} + 1} dx$$

Linear substitutions leading to blocks IIa and IIIa

Building block IIa: $\int \frac{x}{1+x^2} dx = \frac{1}{2} \ln(1+x^2) + C.$

Building block IIIa: $\int \frac{1}{1+x^2} dx = \arctan x + C.$

Example

No real roots \Rightarrow complete the square.

$$\int \frac{x}{x^2 + x + 1} dx = \int \frac{x}{x^2 + 2 \cdot \frac{1}{2}x + \frac{1}{4} - \frac{1}{4} + 1} dx$$

Linear substitutions leading to blocks IIa and IIIa

Building block IIa: $\int \frac{x}{1+x^2} dx = \frac{1}{2} \ln(1+x^2) + C.$

Building block IIIa: $\int \frac{1}{1+x^2} dx = \arctan x + C.$

Example

No real roots \Rightarrow complete the square.

$$\begin{aligned} \int \frac{x}{x^2 + x + 1} dx &= \int \frac{x}{x^2 + 2 \cdot \frac{1}{2}x + \frac{1}{4} - \frac{1}{4} + 1} dx \\ &= \int \frac{x + \frac{1}{2} - \frac{1}{2}}{\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}} d\left(x + \frac{1}{2}\right) \end{aligned}$$

Linear substitutions leading to blocks IIa and IIIa

Building block IIa: $\int \frac{x}{1+x^2} dx = \frac{1}{2} \ln(1+x^2) + C.$

Building block IIIa: $\int \frac{1}{1+x^2} dx = \arctan x + C.$

Example

No real roots \Rightarrow complete the square.

$$\begin{aligned} \int \frac{x}{x^2 + x + 1} dx &= \int \frac{x}{x^2 + 2 \cdot \frac{1}{2}x + \frac{1}{4} - \frac{1}{4} + 1} dx \\ &= \int \frac{x + \frac{1}{2} - \frac{1}{2}}{\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}} d\left(x + \frac{1}{2}\right) \end{aligned}$$

Linear substitutions leading to blocks IIa and IIIa

Building block IIa: $\int \frac{x}{1+x^2} dx = \frac{1}{2} \ln(1+x^2) + C.$

Building block IIIa: $\int \frac{1}{1+x^2} dx = \arctan x + C.$

Example

No real roots \Rightarrow complete the square.

$$\begin{aligned} \int \frac{x}{x^2 + x + 1} dx &= \int \frac{x}{x^2 + 2 \cdot \frac{1}{2}x + \frac{1}{4} - \frac{1}{4} + 1} dx \\ &= \int \frac{x + \frac{1}{2} - \frac{1}{2}}{\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}} d\left(x + \frac{1}{2}\right) \end{aligned}$$

Linear substitutions leading to blocks IIa and IIIa

Building block IIa: $\int \frac{x}{1+x^2} dx = \frac{1}{2} \ln(1+x^2) + C.$

Building block IIIa: $\int \frac{1}{1+x^2} dx = \arctan x + C.$

Example

No real roots \Rightarrow complete the square. Let $u = x + \frac{1}{2}$

$$\begin{aligned} \int \frac{x}{x^2 + x + 1} dx &= \int \frac{x}{x^2 + 2 \cdot \frac{1}{2}x + \frac{1}{4} - \frac{1}{4} + 1} dx \\ &= \int \frac{\color{red}{x} + \frac{\color{red}{1}}{\color{red}{2}} - \frac{1}{2}}{\left(\color{red}{x} + \frac{\color{red}{1}}{\color{red}{2}}\right)^2 + \frac{3}{4}} d\left(\color{red}{x} + \frac{\color{red}{1}}{\color{red}{2}}\right) \\ &= \int \frac{\color{red}{u} - \frac{1}{2}}{\color{red}{u}^2 + \frac{3}{4}} d\color{red}{u} \end{aligned}$$

Linear substitutions leading to blocks IIa and IIIa

Building block IIa: $\int \frac{x}{1+x^2} dx = \frac{1}{2} \ln(1+x^2) + C.$

Building block IIIa: $\int \frac{1}{1+x^2} dx = \arctan x + C.$

Example

No real roots \Rightarrow complete the square. Let $u = x + \frac{1}{2}$

$$\begin{aligned}
 \int \frac{x}{x^2 + x + 1} dx &= \int \frac{x}{x^2 + 2 \cdot \frac{1}{2}x + \frac{1}{4} - \frac{1}{4} + 1} dx \\
 &= \int \frac{x + \frac{1}{2} - \frac{1}{2}}{\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}} d\left(x + \frac{1}{2}\right) \\
 &= \int \frac{\textcolor{red}{u} - \frac{1}{2}}{u^2 + \frac{3}{4}} du \\
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Linear substitutions leading to blocks IIa and IIIa

Building block IIa: $\int \frac{x}{1+x^2} dx = \frac{1}{2} \ln(1+x^2) + C.$

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Example

No real roots \Rightarrow complete the square. Let $u = x + \frac{1}{2}$

$$\begin{aligned}\int \frac{x}{x^2 + x + 1} dx &= \int \frac{x}{x^2 + 2 \cdot \frac{1}{2}x + \frac{1}{4} - \frac{1}{4} + 1} dx \\&= \int \frac{x + \frac{1}{2} - \frac{1}{2}}{\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}} d\left(x + \frac{1}{2}\right) \\&= \int \frac{u - \frac{1}{2}}{u^2 + \frac{3}{4}} du \\&= \int \frac{u}{u^2 + \frac{3}{4}} du - \frac{1}{2} \int \frac{1}{u^2 + \frac{3}{4}} du\end{aligned}$$

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Building blocks IIa and IIb

We solve building block IIb. For completeness, we solve block IIa again as well.

Example

$$\int \frac{x}{(x^2 + 1)^n} dx$$

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$$\int \frac{x}{(x^2 + 1)^n} dx = \int \frac{1}{(x^2 + 1)^n} \frac{d(x^2 + 1)}{2}$$

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$$\begin{aligned}\int \frac{x}{(x^2 + 1)^n} dx &= \int \frac{1}{(x^2 + 1)^n} \frac{d(x^2 + 1)}{2} \\ &= \frac{1}{2} \int u^{-n} du\end{aligned}$$

where we used the substitution $u = x^2 + 1$.

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Building block IIIb: example illustrating main idea

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Integrate $\int \frac{dx}{(x^2+1)^2}$. We start with an already known integral:

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Integrate $\int \frac{dx}{(x^2+1)^2}$. We start with an already known integral:

$$\arctan x + C = \int \frac{1}{x^2 + 1} dx$$

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Building block IIIb: example illustrating main idea

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Rearrange terms

$$2 \int \frac{dx}{(1+x^2)^2} = \left(\frac{x}{x^2+1} + \arctan x \right) + L \quad .$$

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Rearrange terms and divide by 2 to get the desired integral:

$$\int \frac{dx}{(1+x^2)^2} = \frac{1}{2} \left(\frac{x}{x^2+1} + \arctan x \right) + K.$$

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$$\int \frac{1}{(x^2 + 1)} dx = \arctan x + C \quad .$$

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- In this way we end up expressing $J(n)$ via $J(n-1)$.
- We work our way from $J(n)$ to $J(n-1)$, from $J(n-1)$ to $J(n-2)$, and so on, until we get to $J(1)$.

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and so on.

Example

Recall that $J(n) = \int \frac{1}{(x^2+1)^n} dx$. We have that:

$$J(n-1) = \frac{x}{(x^2+1)^{n-1}} + 2(n-1)J(n-1) - 2(n-1)J(n) \quad .$$

Rearrange to get:

$$\begin{aligned} 2(n-1)J(n) &= \frac{x}{(x^2+1)^{n-1}} + (2n-3)J(n-1) \\ J(n) &= \frac{x}{(2n-2)(x^2+1)^{n-1}} + \frac{2n-3}{2n-2}J(n-1) \quad . \end{aligned}$$

In this way we expressed $J(n)$ using $J(n-1)$. We apply the above formula consecutively:

$$J(n) = \frac{x}{(2n-2)(x^2+1)^{n-1}} + \frac{2n-3}{2n-2} \left(\frac{x}{(2n-4)(x^2+1)^{n-2}} + \frac{2n-5}{2n-4} J(n-2) \right) = \dots$$

and so on. The above can be used to write a formula for the final result, but that is as complicated as the process above.

Building block integral summary

| Type | a | b | Type a, lin. sub. | Type b, lin. sub |
|------|---------------------------|-------------------------------|---|---|
| I | $\int \frac{1}{x} dx$ | $\int \frac{1}{x^n} dx$ | $\int \frac{A}{ax+b} dx$ | $\int \frac{A}{(ax+b)^n} dx$ |
| II | $\int \frac{x}{x^2+1} dx$ | $\int \frac{x}{(x^2+1)^n} dx$ | $\int \frac{A(x+\frac{b}{2a})}{ax^2+bx+c} dx$ | $\int \frac{A(x+\frac{b}{2a})}{(ax^2+bx+c)^n} dx$ |
| III | $\int \frac{1}{x^2+1} dx$ | $\int \frac{1}{(x^2+1)^n} dx$ | $\int \frac{B}{ax^2+bx+c} dx$ | $\int \frac{B}{(ax^2+bx+c)^n} dx$ |

where A, B are arbitrary constants and a, b, c are constants with $b^2 - 4ac < 0$. The quadratics in the denominators have no real roots.

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- We solved building blocks I, II and III

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where A, B are arbitrary constants and a, b, c are constants with $b^2 - 4ac < 0$. The quadratics in the denominators have no real roots.

- We solved building blocks I, II and III in almost complete detail.

Building block integral summary

| Type | a | b | Type a, lin. sub. | Type b, lin. sub |
|------|---------------------------|-------------------------------|---|---|
| I | $\int \frac{1}{x} dx$ | $\int \frac{1}{x^n} dx$ | $\int \frac{A}{ax+b} dx$ | $\int \frac{A}{(ax+b)^n} dx$ |
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where A, B are arbitrary constants and a, b, c are constants with $b^2 - 4ac < 0$. The quadratics in the denominators have no real roots.

- We solved building blocks I, II and III in almost complete detail.
- The types in the remaining columns can be transformed to building block ones:

Building block integral summary

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- We solved building blocks I, II and III in almost complete detail.
- The types in the remaining columns can be transformed to building block ones:
 - Block I, linear substitutions: done in full detail.

Building block integral summary

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where A, B are arbitrary constants and a, b, c are constants with $b^2 - 4ac < 0$. The quadratics in the denominators have no real roots.

- We solved building blocks I, II and III in almost complete detail.
- The types in the remaining columns can be transformed to building block ones:
 - Block I, linear substitutions: done in full detail.
 - Block IIa, IIIa, linear substitutions: done in full detail,

Building block integral summary

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| II | $\int \frac{x}{x^2+1} dx$ | $\int \frac{x}{(x^2+1)^n} dx$ | $\int \frac{A(x+\frac{b}{2a})}{ax^2+bx+c} dx$ | $\int \frac{A(x+\frac{b}{2a})}{(ax^2+bx+c)^n} dx$ |
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where A, B are arbitrary constants and a, b, c are constants with $b^2 - 4ac < 0$. The quadratics in the denominators have no real roots.

- We solved building blocks I, II and III in almost complete detail.
- The types in the remaining columns can be transformed to building block ones:
 - Block I, linear substitutions: done in full detail.
 - Block IIa, IIIa, linear substitutions: done in full detail, **by means of completing the square.**

Building block integral summary

| Type | a | b | Type a, lin. sub. | Type b, lin. sub |
|------|---------------------------|-------------------------------|---|---|
| I | $\int \frac{1}{x} dx$ | $\int \frac{1}{x^n} dx$ | $\int \frac{A}{ax+b} dx$ | $\int \frac{A}{(ax+b)^n} dx$ |
| II | $\int \frac{x}{x^2+1} dx$ | $\int \frac{x}{(x^2+1)^n} dx$ | $\int \frac{A(x+\frac{b}{2a})}{ax^2+bx+c} dx$ | $\int \frac{A(x+\frac{b}{2a})}{(ax^2+bx+c)^n} dx$ |
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where A, B are arbitrary constants and a, b, c are constants with $b^2 - 4ac < 0$. The quadratics in the denominators have no real roots.

- We solved building blocks I, II and III in almost complete detail.
- The types in the remaining columns can be transformed to building block ones:
 - Block I, linear substitutions: done in full detail.
 - Block IIa, IIIa, linear substitutions: done in full detail, by means of completing the square.
 - Block IIb, IIIb, linear substitutions: **done by means of completing the square;**

Building block integral summary

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- We solved building blocks I, II and III in almost complete detail.
- The types in the remaining columns can be transformed to building block ones:
 - Block I, linear substitutions: done in full detail.
 - Block IIa, IIIa, linear substitutions: done in full detail, by means of completing the square.
 - Block IIb, IIIb, linear substitutions: done by means of completing the square; **computations are analogous and we leave them for exercise.**