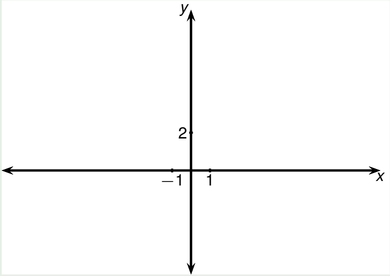
Calculus I
Intervals of increase and concavity, part 2

Todor Milev

2019

Example

Sketch the curve $y = \frac{2x^2}{x^2-1}$.

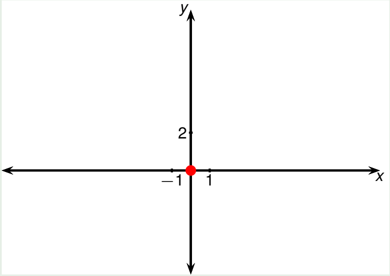


1. Domain

The domain of the function is $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$.

Example

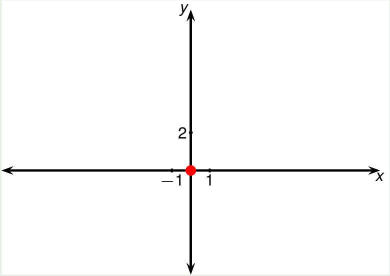
Sketch the curve $y = \frac{2x^2}{x^2-1}$.



3. Intercepts
   - y-intercept: $f(0) = 0$.
   - x-intercept: $f(x) = 0$ when $x = 0$.
   - The only intercept is $(0, 0)$.

Example

Sketch the curve $y = \frac{2x^2}{x^2-1}$.



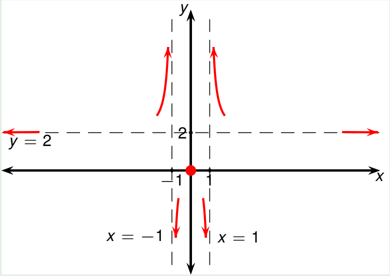
4. Symmetry

$$f(-x) = \frac{2(-x)^2}{(-x)^2 - 1} = \frac{2x^2}{x^2-1} = f(x)$$

Therefore f is even.

Example

Sketch the curve $y = \frac{2x^2}{x^2-1}$.



5. Asymptotes

$$\lim_{x\to\pm\infty} \frac{2x^2}{x^2-1} = \lim_{x\to\pm\infty} \frac{2}{1-1/x^2} = 2$$

$y = 2$ is a horizontal asymptote.

$$\lim_{x\to 1^+} \frac{2x^2}{x^2-1} = \infty$$

$$\lim_{x\to 1^-} \frac{2x^2}{x^2-1} = -\infty$$

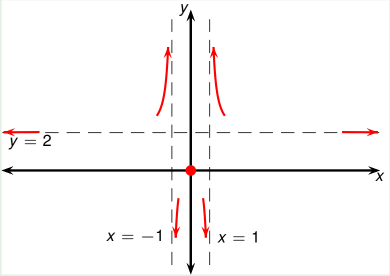
$$\lim_{x\to -1^+} \frac{2x^2}{x^2-1} = -\infty$$

$$\lim_{x\to -1^-} \frac{2x^2}{x^2-1} = \infty$$

$x = \pm 1$ are vertical asymptotes.

Example

Sketch the curve $y = \frac{2x^2}{x^2-1}$.



Interval	I/D	Concavity
$(-\infty, -1)$	I	
$(-1, 0)$	I	
$(0, 1)$	D	
$(1, \infty)$	D	

6. Intervals of increase or decrease

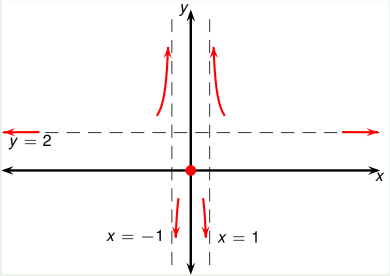
$$f'(x) = \frac{(x^2-1)(4x) - 2x^2(2x)}{(x^2-1)^2}$$

$$= \frac{-4x}{(x^2-1)^2}$$

	$-4x$	$(x^2-1)^2$	f'
$(-\infty, -1)$	$+$	$+$	$+$
$(-1, 0)$	$+$	$+$	$+$
$(0, 1)$	$-$	$+$	$-$
$(1, \infty)$	$-$	$+$	$-$

Example

Sketch the curve $y = \frac{2x^2}{x^2-1}$.



Interval	I/D	Concavity
$(-\infty, -1)$	I	
$(-1, 0)$	I	
$(0, 1)$	D	
$(1, \infty)$	D	

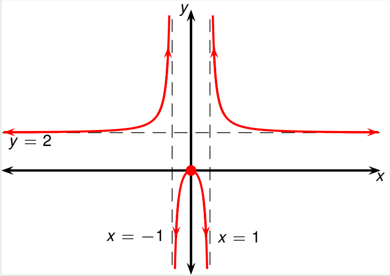
7. Local maxima and minima

	$-4x$	$(x^2-1)^2$	f'
$(-\infty, -1)$	$+$	$+$	$+$
$(-1, 0)$	$+$	$+$	$+$
$(0, 1)$	$-$	$+$	$-$
$(1, \infty)$	$-$	$+$	$-$

- f' changes sign from $+$ to $-$ at 0.
- Therefore $(0, 0)$ is a local maximum.

Example

Sketch the curve $y = \frac{2x^2}{x^2-1}$.



Interval	I/D	Concavity
$(-\infty, -1)$	I	up
$(-1, 0)$	I	down
$(0, 1)$	D	down
$(1, \infty)$	D	up

8. Concavity and points of inflection

$$f''(x) = \frac{-4(x^2-1)^2 + 4x \cdot 2(x^2-1)2x}{(x^2-1)^4} = \frac{12x^2+4}{(x^2-1)^3}$$

	$12x^2+4$	$(x^2-1)^3$	f''
$(-\infty, -1)$	$+$	$+$	$+$
$(-1, 1)$	$+$	$-$	$-$
$(1, \infty)$	$+$	$+$	$+$

No points of inflection because ± 1 are not in the domain of f.