Calculus II
Series basic facts

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Outline

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Theorem

If the series $\sum_{n=1}^{\infty} a_n$ *is convergent, then* $\lim_{n\to\infty} a_n = 0$.

Proof.

- Let $s_n = a_1 + a_2 + \cdots + a_n$.
- Then $a_n = s_n - s_{n-1}$.
- Since $\sum_{n=1}^{\infty} a_n$ is convergent, the sequence $\{s_n\}$ is convergent.
- Let $\lim_{n\to\infty} s_n = s$.
- Then $\lim_{n\to\infty} s_{n-1} = s$.
- Therefore

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (s_n - s_{n-1}) = s - s = 0 \qquad \square$$

Theorem

If the series $\sum_{n=1}^{\infty} a_n$ *is convergent, then* $\lim_{n\to\infty} a_n = 0$.

This is just a restatement of the previous theorem:

Theorem (The Divergence Test)

If $\lim_{n\to\infty} a_n$ *doesn't exist or if* $\lim_{n\to\infty} a_n \neq 0$*, then the series* $\sum_{n=1}^{\infty} a_n$ *is divergent.*

Example

Show that the series $\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ diverges.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{5n^2+4} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{1}{5+\frac{4}{n^2}} = \frac{1}{5} \neq 0$$

Therefore, by the Divergence Test, the series diverges.

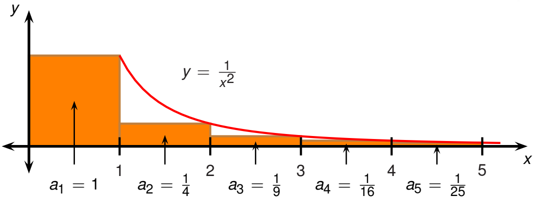
The Integral Test and Estimates of Sums

- In general, it is not easy to find the sum of a series.
- We could do this for $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ because we found a simple formula for the nth partial sum s_n.
- In the next few sections, we'll learn techniques for showing whether a series is convergent or divergent without explicitly computing its sum.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

n	$s_n = \sum_{i=1}^{n} \frac{1}{i^2}$
5	1.4636
10	1.5498
50	1.6251
100	1.6350
500	1.6429
1000	1.6439
5000	1.6447

- Use a computer to calculate partial sums.
- Appears to be converging.
- How do we prove it?
- Use $f(x) = \frac{1}{x^2}$.



- $\frac{1}{1^2}$ is the area of a rectangle.
- So is $\frac{1}{2^2} = \frac{1}{4}$.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

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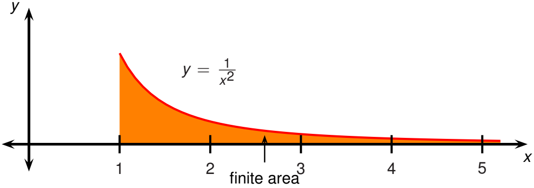

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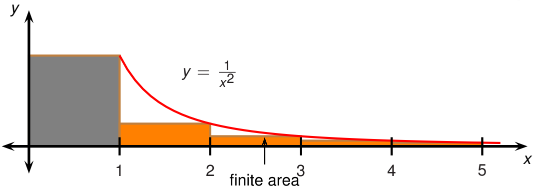

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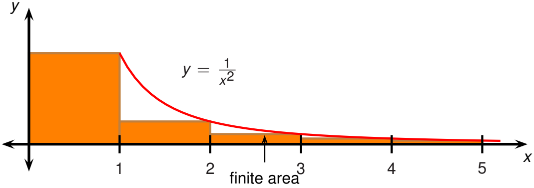

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- So is $\frac{1}{2^2} = \frac{1}{4}$.
- The improper integral $\int_1^{\infty} \frac{1}{x^2} dx$ is convergent.
- Therefore $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots$$

- Use a computer to calculate partial sums.
- Appears to be diverging.
- How do we prove it?
- Use $f(x) = \frac{1}{\sqrt{x}}$.

n	$s_n = \sum_{i=1}^{n} \frac{1}{\sqrt{i}}$
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10	5.0210
50	12.7524
100	18.5896
500	43.2834
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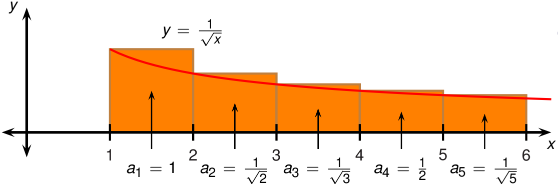

- $\frac{1}{\sqrt{1}}$ is the area of a rectangle.
- So is $\frac{1}{\sqrt{2}}$.

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots$$

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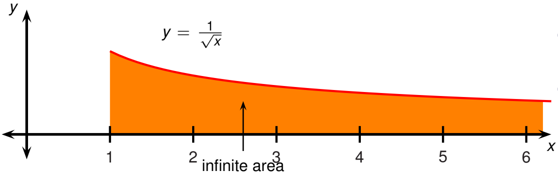

- $\frac{1}{\sqrt{1}}$ is the area of a rectangle.
- So is $\frac{1}{\sqrt{2}}$.
- $\int_1^\infty \frac{1}{\sqrt{x}} dx$ is divergent.

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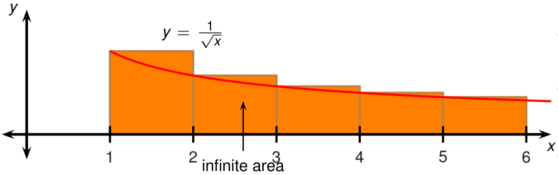


- $\frac{1}{\sqrt{1}}$ is the area of a rectangle.
- So is $\frac{1}{\sqrt{2}}$.
- $\int_1^{\infty} \frac{1}{\sqrt{x}} dx$ is divergent.
- Therefore $\sum\limits_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is divergent.

Theorem (The Integral Test)

Let f be a continuous, positive, decreasing function on $[1,\infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\mathrm{d}x$ is convergent. In other words,

1. *If* $\displaystyle\int_1^{\infty} f(x)\mathrm{d}x$ *is convergent, then* $\displaystyle\sum_{n=1}^{\infty} a_n$ *is convergent.*
2. *If* $\displaystyle\int_1^{\infty} f(x)\mathrm{d}x$ *is divergent, then* $\displaystyle\sum_{n=1}^{\infty} a_n$ *is divergent.*

Note that it is not necessary to start the series or the integral at $n = 1$. For instance, to test the series

$$\sum_{n=4}^{\infty} \frac{1}{(n-3)^2}$$

we would use

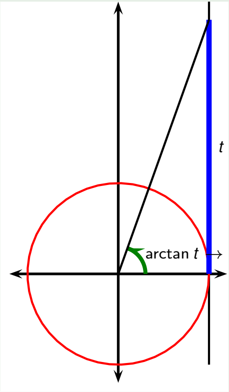
$$\int_4^{\infty} \frac{1}{(x-3)^2}\mathrm{d}x$$

Example

Test the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ for convergence.

$f(x) = \frac{1}{x^2+1}$ is continuous, positive, and decreasing on $[1, \infty)$, so use the Integral Test.

$$\begin{aligned}
\int_1^{\infty} \frac{1}{x^2+1} \mathrm{d}x &= \lim_{t\to\infty} \int_1^{t} \frac{1}{x^2+1} \mathrm{d}x \\
&= \lim_{t\to\infty} \left[\arctan x\right]_1^t \\
&= \lim_{t\to\infty} \left(\arctan t - \frac{\pi}{4}\right) \\
&= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}
\end{aligned}$$



Therefore $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ is convergent.

Example

For which values of p is the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ convergent?

- If $p < 0$, then $\lim\limits_{n\to\infty} \frac{1}{n^p} = \infty$.
- If $p = 0$, then $\lim\limits_{n\to\infty} \frac{1}{n^p} = 1$.
- Therefore for $p \leq 0$ the series is divergent.
- It remains to investigate the case $p > 0$. If $p > 0$, then $f(x) = \frac{1}{x^p}$ is continuous, positive, and decreasing on $[1, \infty)$, so we can use the Integral Test.
$$\int_1^{\infty} \frac{1}{x^p} \mathrm{d}x = \begin{cases} \text{convergent} & \text{when} \quad p > 1 \\ \text{divergent} & \text{when} \quad p \leq 1 \end{cases}$$
- $\Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent when $p > 1$ and divergent when $p \leq 1$.

This theorem summarizes the results of the previous example.

Theorem (p-series Convergence)

The p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ *is convergent if* $p > 1$ *and divergent if* $p \leq 1$.

Example

Test the series $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ for convergence.

- $f(x) = \frac{\ln x}{x}$ is continuous and positive ($x > 0$).
- To establish where $f(x)$ is decreasing, take the derivative.

$$f'(x) = \frac{\left(\frac{1}{x}\right)(x) - (\ln x)(1)}{x^2} = \frac{1 - \ln x}{x^2}$$

- This is negative for all $x > e$.
- Therefore f is decreasing for all $x > e$.

$$\begin{aligned} \int_1^{\infty} \frac{\ln x}{x} \mathrm{d}x &= \lim_{t\to\infty} \int_1^t \frac{\ln x}{x} \mathrm{d}x = \lim_{t\to\infty} \left[\frac{(\ln x)^2}{2} \right]_1^t \\ &= \lim_{t\to\infty} \left(\frac{1}{2}(\ln t)^2 - 0 \right) = \infty \end{aligned}$$

Therefore $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ is divergent.

Estimating the Sum of a Series

- Suppose we have already used the Integral Test to show that $\sum a_n$ converges.
- Now we want to find an approximation to the sum of the series.
- Any partial sum s_n is an approximation. But how good?
- Estimate the size of the remainder $R_n = s - s_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots$.
- Suppose $f(n) = a_n$. Draw rectangles with heights $a_{n+1}, a_{n+2}, \ldots$.
- Use the right endpoints to find the height: then the rectangles are under the curve $y = f(x)$.
- $R_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots \leq \int_n^\infty f(x)\mathrm{d}x$.
- Use the left endpoints to find the height: then the rectangles are above the curve $y = f(x)$.
- $R_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots \geq \int_{n+1}^\infty f(x)\mathrm{d}x$.

Remainder Estimate for the Integral Test

Suppose $f(k) = a_k$, where f is continuous, positive, and decreasing for $x \geq n$, and $\sum a_k$ is convergent with sum s. If $R_n = s - s_n$, then

$$\int_{n+1}^{\infty} f(x)\mathrm{d}x \leq R_n \leq \int_{n}^{\infty} f(x)\mathrm{d}x$$

Example (Example 5, p. 737)

Approximate the sum of $\sum \frac{1}{n^3}$ using the first 10 terms. Estimate the error involved in this approximation. How many terms are required to get an accuracy of 0.0005 or better?

$$\int_n^\infty \frac{1}{x^3}dx = \lim_{t\to\infty}\left[-\frac{1}{2x^2}\right]_n^t = \lim_{t\to\infty}\left(-\frac{1}{2t^2}+\frac{1}{2n^2}\right) = \frac{1}{2n^2}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^3} \approx s_{10} = \frac{1}{1^3}+\frac{1}{2^3}+\cdots+\frac{1}{10^3} \approx 1.975$$

$$R_{10} \leq \int_{10}^\infty \frac{1}{x^3}dx = \frac{1}{2(10)^2} = \frac{1}{200}$$

Therefore the error is at most 0.005.
To get an accuracy of 0.0005 or better, we want $R_n \leq 0.0005$. Since $R_n \leq \frac{1}{2n^2}$, we want

$$\frac{1}{2n^2} \leq 0.0005, \qquad \text{or} \qquad n \geq \sqrt{1000} \approx 31.6$$

$$
\begin{array}{rcccl}
\int_{n+1}^{\infty} f(x)\mathrm{d}x & \leq & R_n & \leq & \int_{n}^{\infty} f(x)\mathrm{d}x \\
s_n + \int_{n+1}^{\infty} f(x)\mathrm{d}x & \leq & s_n + R_n & \leq & s_n + \int_{n}^{\infty} f(x)\mathrm{d}x \\
s_n + \int_{n+1}^{\infty} f(x)\mathrm{d}x & \leq & s & \leq & s_n + \int_{n}^{\infty} f(x)\mathrm{d}x
\end{array}
$$

- Add s_n to both sides of both inequalities.
- This gives upper and lower bounds for s.
- This is a better approximation than just using s_n.

The Comparison Tests

- In the Comparison Tests, the idea is to compare a given series with another series that is known to be convergent or divergent.
- Consider the series $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$.
- This reminds us of the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$.
- $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a geometric series with $a = \frac{1}{2}$ and $r = \frac{1}{2}$.
- Therefore $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is convergent.

$$
\begin{aligned}
\frac{1}{2^i+1} &< \frac{1}{2^i} \\
\sum_{i=1}^{n} \frac{1}{2^i+1} &< \sum_{i=1}^{n} \frac{1}{2^i} < \sum_{i=1}^{\infty} \frac{1}{2^i} = 1
\end{aligned}
$$

- The partial sums of $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$ are increasing and are bounded above by 1.
- Therefore $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$ is convergent.

Theorem (The Comparison Test)

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

1. *If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is also convergent.*
2. *If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum a_n$ is also divergent.*

When we use the Comparison Test, we need to have some series $\sum b_n$ that we know in order to make a comparison. Usually $\sum b_n$ is one of

- A p-series ($\sum \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$)
- A geometric series ($\sum ar^{n-1}$ converges if $|r| < 1$ and diverges if $|r| \geq 1$)

Example

Determine if $\sum_{n=1}^{\infty} \frac{5}{2n^2+7n+3}$ converges or diverges.

- As $n \to \infty$, the dominant term in the denominator is $2n^2$, so compare with $\frac{5}{2n^2}$.

$$\frac{5}{2n^2+7n+3} < \frac{5}{2n^2}$$

$$\sum_{n=1}^{\infty} \frac{5}{2n^2} = \frac{5}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

- This is a constant times a p-series with $p = 2 > 1$.
- Therefore $\sum_{n=1}^{\infty} \frac{5}{2n^2}$ is convergent.
- Therefore $\sum_{n=1}^{\infty} \frac{5}{2n^2+7n+3}$ is convergent by the Comparison Test.

Example

Determine if $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ converges or diverges.

- We could use the Integral Test to find this.
- The Comparison Test is even easier.

$$\frac{\ln n}{n} > \frac{1}{n} \qquad \text{if } n \geq 3$$

- $\sum_{n=1}^{\infty} \frac{1}{n}$ is a p-series with $p = 1$.
- Therefore $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.
- Therefore $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ is divergent by the Comparison Test.

In order to use the comparison test to see if $\sum a_n$ is convergent or divergent, we need the terms a_n to be

1. smaller than the terms of a convergent series, or
2. bigger than the terms of a divergent series.

If the terms a_n are

1. bigger than the terms of a convergent series, or
2. smaller than the terms of a divergent series,

then the Comparison Test gives no information.

- Consider the series $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$.

$$\frac{1}{2^n-1} > \frac{1}{2^n}$$

- The Comparison Test tells us nothing here.
- Nevertheless, we think $\sum \frac{1}{2^n-1}$ should converge, because it's so close to $\sum \frac{1}{2^n}$.

Theorem (The Limit Comparison Test)

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both series diverge.

The main thing to check is that c is finite and non-zero.

Example

Test the series $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ for convergence or divergence.
Use the Limit Comparison Test with

$$a_n = \frac{1}{2^n - 1}, \qquad b_n = \frac{1}{2^n}$$

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{\frac{1}{2^n-1}}{\frac{1}{2^n}} \\
&= \lim_{n\to\infty} \frac{2^n}{2^n - 1} \cdot \frac{\frac{1}{2^n}}{\frac{1}{2^n}} \\
&= \lim_{n\to\infty} \frac{1}{1 - \frac{1}{2^n}} = 1 > 0
\end{aligned}$$

- $\sum \frac{1}{2^n}$ is a convergent geometric series.
- By the Limit Comparison Test $\sum \frac{1}{2^n-1}$ is convergent too.

Example

Test the series $\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{7+n^5}}$ for convergence or divergence.

- The dominant part of the numerator is $2n^2$ and the dominant part of the denominator is $\sqrt{n^5} = n^{5/2}$.

$$
\begin{aligned}
a_n &= \frac{2n^2+3n}{\sqrt{7+n^5}}, \qquad b_n = \frac{2n^2}{n^{5/2}} = \frac{2}{n^{1/2}} \\
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{2n^2+3n}{\sqrt{7+n^5}} \cdot \frac{n^{1/2}}{2} = \lim_{n\to\infty} \frac{2n^{5/2}+3n^{3/2}}{2\sqrt{7+n^5}} \frac{\frac{1}{n^{5/2}}}{\frac{1}{n^{5/2}}} \\
&= \lim_{n\to\infty} \frac{2+\frac{3}{n}}{2\sqrt{\frac{7}{n^5}+1}} = 1 > 0
\end{aligned}
$$

- $\sum \frac{2}{n^{\frac{1}{2}}}$ is a constant multiple of a p-series with $p = \frac{1}{2}$.
- Therefore $\sum \frac{2}{n^{\frac{1}{2}}}$ is divergent, and so is $\sum \frac{2n^2+3n}{\sqrt{7+n^5}}$.