Calculus I
Trigonometric derivatives

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Outline

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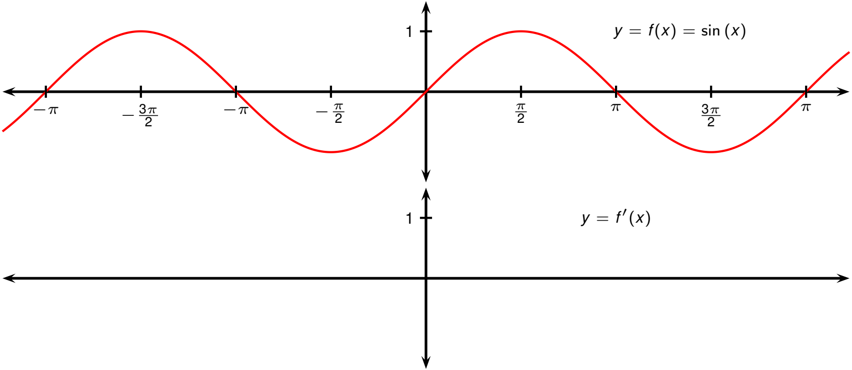
Derivatives of Trigonometric Functions



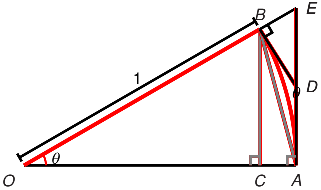
What is the derivative of $f(x) = \sin x$? It looks like $\cos x$.

$$
\begin{aligned}
\text{Let}\quad f(x) &= \sin x. \\
\text{Then}\quad f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} \\
&= \lim_{h\to 0}\frac{\sin x\cos h+\cos x\sin h-\sin x}{h} \\
&= \lim_{h\to 0}\left(\frac{\sin x\cos h-\sin x}{h}+\frac{\cos x\sin h}{h}\right) \\
&= \lim_{h\to 0}\left(\sin x\left(\frac{\cos h-1}{h}\right)+\cos x\left(\frac{\sin h}{h}\right)\right) \\
&= \lim_{h\to 0}\sin x\cdot\lim_{h\to 0}\left(\frac{\cos h-1}{h}\right)+\lim_{h\to 0}\cos x\cdot\lim_{h\to 0}\left(\frac{\sin h}{h}\right) \\
&= \sin x\cdot\lim_{h\to 0}\left(\frac{\cos h-1}{h}\right)+\ \cos x\cdot\lim_{h\to 0}\left(\frac{\sin h}{h}\right)
\end{aligned}
$$

We need to do more work to find the other two limits.

Claim: $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$

Suppose $0 < \theta < \frac{\pi}{2}$. Write $\sin\theta$ using ratios of side lengths of a triangle.



$$\sin\theta = \frac{|BC|}{|OB|} = |BC| < |AB| < \text{arc}AB = \theta$$

Therefore $\sin\theta < \theta$ and therefore $\frac{\sin\theta}{\theta} < 1$.

$$\theta = \text{arc}AB < |AD| + |DB| < |AD| + |DE|$$
$$= |AE| = |OA| \tan\theta = \tan\theta$$

Therefore $\theta < \tan\theta = \frac{\sin\theta}{\cos\theta}$, so $\cos\theta < \frac{\sin\theta}{\theta}$.

$$\cos\theta < \frac{\sin\theta}{\theta} < 1$$

$\lim_{\theta \to 0} \cos\theta = 1$ and $\lim_{\theta \to 0} 1 = 1$, so by the Squeeze Theorem $\lim_{\theta \to 0^+} \frac{\sin\theta}{\theta} = 1$. $\frac{\sin\theta}{\theta}$ is even, so the left limit is also 1.

$$\text{Let} \quad f(x) = \sin x.$$

$$\text{Then} \quad f'(x) = \lim_{h\to 0} \sin x \cdot \lim_{h\to 0}\left(\frac{\cos h - 1}{h}\right) + \lim_{h\to 0} \cos x \cdot \lim_{h\to 0}\left(\frac{\sin h}{h}\right)$$

$$= \sin x \cdot \lim_{h\to 0}\left(\frac{\cos h - 1}{h}\right) + \cos x \cdot 1$$

We need to find

$$\lim_{h\to 0}\frac{\cos h - 1}{h} = \lim_{h\to 0}\frac{(\cos h - 1)}{h}\cdot\frac{(\cos h + 1)}{(\cos h + 1)} = \lim_{h\to 0}\frac{\cos^2 h - 1}{h(\cos h + 1)}$$

$$= \lim_{h\to 0}\frac{-\sin^2 h}{h(\cos h + 1)} = -\lim_{h\to 0}\left(\frac{\sin h}{h}\cdot\frac{\sin h}{\cos h + 1}\right)$$

$$= -\lim_{h\to 0}\frac{\sin h}{h}\cdot\lim_{h\to 0}\frac{\sin h}{\cos h + 1} = -1\cdot\left(\frac{0}{1+1}\right) = 0$$

Theorem (The Derivative of $\sin x$)

$$\frac{d}{dx}(\sin x) = \cos x$$

Example (Product Rule, Product Rule with Sine)

Differentiate $f(x) = x \sin x.$

$$\text{Product Rule:} \quad \begin{aligned} f'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}(x)(\sin x) + (x)\frac{\mathrm{d}}{\mathrm{d}x}(\sin x) \\ &= (1)\ (\sin x) + (x)(\cos x) \\ &= x\cos x + \sin x. \end{aligned}$$

Example (Quotient Rule, Natural Exponential Function and Sine)

Differentiate $y = \dfrac{e^x}{2+\sin x}$.

Quotient Rule:

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\frac{\mathrm{d}}{\mathrm{d}x}\left(e^x\right)\left(2+\sin x\right) - \left(e^x\right)\frac{\mathrm{d}}{\mathrm{d}x}\left(2+\sin x\right)}{\left(2+\sin x\right)^2}\\
&= \frac{\left(e^x\right)\left(2+\sin x\right) - \left(e^x\right)\left(\cos x\right)}{\left(2+\sin x\right)^2}\\
&= \frac{2e^x + e^x\sin x - e^x\cos x}{(2+\sin x)^2}\\
&= \frac{e^x(2+\sin x - \cos x)}{(2+\sin x)^2}.
\end{aligned}$$

Example (Trigonometric limit)

$$\text{Find} \quad \lim_{x\to 0}\frac{2x}{\sin(9x)} = \lim_{x\to 0}\frac{2x}{\sin(9x)}\cdot\frac{9}{9}$$
$$= \lim_{x\to 0}\frac{2}{9}\cdot\frac{9x}{\sin(9x)}$$
$$= \lim_{x\to 0}\frac{2}{9}\cdot\frac{1}{\frac{\sin(9x)}{9x}} = \lim_{\theta\to 0}\frac{2}{9}\cdot\frac{1}{\frac{\sin\theta}{\theta}}.$$

$$\text{Let} \quad \theta = 9x.$$

$$\text{As} \quad x\to 0, \quad \theta\to 0.$$

$$\text{Then} \quad \lim_{x\to 0}\frac{2x}{\sin(9x)} = \frac{2}{9}\cdot\frac{1}{\lim\limits_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right)}$$
$$= \frac{2}{9}\cdot\frac{1}{1} = \frac{2}{9}.$$

Theorem (The Derivative of $\cos x$)

$$\frac{d}{dx}(\cos x) = -\sin x$$

- This can be proved in a similar fashion as the formula for $\sin x$.
- Alternatively, this can be proved using the derivative of $\sin x$ and (the not yet studied) Implicit Differentiation and Chain Rule.

Example (Product Rule, with Cosine)

$$
\begin{aligned}
\text{Differentiate} \quad f(x) &= x\cos x. \\
\text{Product Rule:} \quad f'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}(x)(\cos x) + (x)\frac{\mathrm{d}}{\mathrm{d}x}(\cos x) \\
&= (1)\,(\cos x) + (x)(-\sin x) \\
&= -x\sin x + \cos x.
\end{aligned}
$$

Theorem (The Derivative of Tangent)

$$\frac{d}{dx}(\tan x) = \sec^2 x.$$

Proof.

$$\text{Let} \quad y = \tan x = \frac{\sin x}{\cos x}.$$

Quotient Rule:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{\frac{d}{dx}(\sin x)(\cos x) - (\sin x)\frac{d}{dx}(\cos x)}{(\cos x)^2} \\
&= \frac{(\cos x)\,(\cos x) - (\sin x)(\,-\sin x)}{\cos^2 x} \\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} \\
&= \sec^2 x.
\end{aligned}$$

$\square$

Derivatives of Trigonometric Functions

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sin x) = \cos x \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\csc x) = -\csc x \cot x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\cos x) = -\sin x \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\sec x) = \sec x \tan x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\tan x) = \sec^2 x \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\cot x) = -\csc^2 x$$

Example (Quotient Rule, Trig)

Differentiate $y = \frac{\sec x}{1 + \tan x}$.

Quotient Rule:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{\frac{d}{dx}(\sec x)(1+\tan x) - (\sec x)\frac{d}{dx}(1+\tan x)}{(1+\tan x)^2} \\
&= \frac{(\sec x \tan x)\ (1+\tan x) - (\sec x)\left(\sec^2 x\right)}{(1+\tan x)^2} \\
&= \frac{\sec x(\tan x + \tan^2 x - \sec^2 x)}{(1+\tan x)^2} \\
&= \frac{\sec x(\tan x + (-1))}{(1+\tan x)^2} = \frac{\sec x(\tan x - 1)}{(1+\tan x)^2}.
\end{aligned}$$

Example (Using the Product Rule twice)

Differentiate:

$$y = \theta e^{\theta}(\tan\theta + \sec\theta).$$

Product Rule:

$$y' = \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\theta e^{\theta}\right)(\tan\theta + \sec\theta) + \theta e^{\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}(\tan\theta + \sec\theta)$$

Product Rule:

$$= \left(\theta\frac{\mathrm{d}}{\mathrm{d}\theta}\left(e^{\theta}\right) + \frac{\mathrm{d}}{\mathrm{d}\theta}(\theta)e^{\theta}\right)(\tan\theta + \sec\theta) + \theta e^{\theta}\left(\sec^2\theta + \tan\theta\sec\theta\right)$$

$$= \left(\theta(e^{\theta}) + (1)e^{\theta}\right)(\tan\theta + \sec\theta) + \theta e^{\theta}(\sec^2\theta + \tan\theta\sec\theta)$$

$$= \theta e^{\theta}\sec\theta(\sec\theta + \tan\theta) + e^{\theta}(\theta + 1)(\tan\theta + \sec\theta)$$

$$= (\theta\sec\theta + \theta + 1)e^{\theta}(\tan\theta + \sec\theta).$$

Example

Find the 27th derivative of $f(x) = \cos x$.

$$f'(x) = -\sin x$$
$$f''(x) = -\cos x$$
$$f'''(x) = \sin x$$
$$f^{(4)}(x) = \cos x$$
$$f^{(5)}(x) = -\sin x$$

- The derivatives repeat in a cycle of length 4.
- $f^{(24)}(x) = \cos x$.
- Differentiate three more times: $f^{(27)}(x) = \sin x$.