Calculus I
Homework
Trigonometry review

1. Evaluate the difference quotient and simplify your answer.

   (a) $\dfrac{f(2+h)-f(2)}{h}$, where $f(x) = x^2 - x - 1$.

   answer: $h + 3$

   (b) $\dfrac{f(a+h)-f(a)}{h}$, where $f(x) = x^2$.

   answer: $h + 2a$

   (c) $\dfrac{f(a+h)-f(a)}{h}$, where $f(x) = x^3$.

   answer: $h^2 + 3a^2 + 3ah$

   (d) $\dfrac{f(a+h)-f(a)}{h}$, where $f(x) = x^4$.

   answer: $6a^2h + 4ah^2 + h^3 + 4a^3$

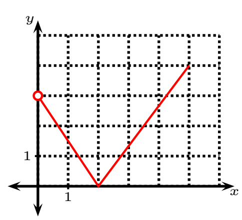
   (e) $\dfrac{f(x)-f(a)}{x-a}$, where $f(x) = \frac{1}{x}$.

   answer: $-\frac{1}{ax}$.

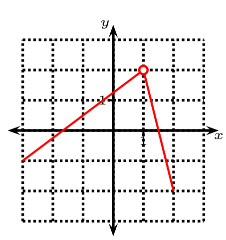
   (f) $\dfrac{f(x)-f(1)}{x-1}$, where $f(x) = \frac{x-1}{x+1}$.

   answer: $\frac{1}{x+1}$.

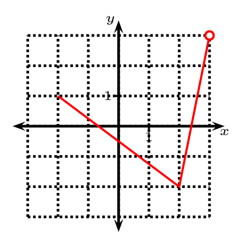
2. Write down a formula for a function whose graphs is given below. The graphs are up to scale. Please note that there is more than one way to write down a correct answer.

   (a) 

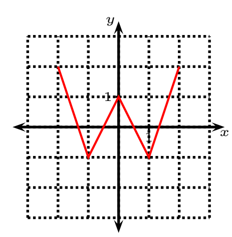
   answer: $y = \begin{cases} -\frac{3}{2}x + 3 & \text{if } 0 < x \leq 2 \\ \frac{4}{3}x - \frac{8}{3} & \text{if } 2 < x \leq 6 \end{cases}$

   (b) 

   answer: $y = \begin{cases} \frac{3}{4}x + \frac{5}{4} & \text{if } -3 \leq x < 1 \\ -4x + 6 & \text{if } 1 < x \leq 2 \end{cases}$

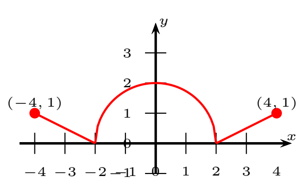
   (c) 

   answer: $y = \begin{cases} -\frac{3}{4}x - \frac{1}{2} & \text{if } -2 \leq x < 2 \\ 5x - 12 & \text{if } 2 \leq x < 3 \end{cases}$

   (d) 

   answer: $y = \begin{cases} -3x - 4 & \text{if } -2 \leq x < -1 \\ 2x + 1 & \text{if } -1 \leq x < 0 \\ -2x + 1 & \text{if } 0 \leq x < 1 \\ 3x - 4 & \text{if } 1 \leq x \leq 2 \end{cases}$

3. Write down formulas for function whose graphs are as follows. The graphs are up to scale. All arcs are parts of circles.

   (a) 

4. Evaluate the difference quotient and simplify your answer.

(a) $\dfrac{f(2+h)-f(2)}{h}$, where $f(x) = x^2 - x - 1$.

answer: $h+3$

(b) $\dfrac{f(a+h)-f(a)}{h}$, where $f(x) = x^2$.

answer: $h+2a$

(c) $\dfrac{f(a+h)-f(a)}{h}$, where $f(x) = x^3$.

answer: h^2+3a^2+3ah

(d) $\dfrac{f(a+h)-f(a)}{h}$, where $f(x) = x^4$.

answer: $6a^2h+4ah^2+h^3+4a^3$

(e) $\dfrac{f(x)-f(a)}{x-a}$, where $f(x) = \frac{1}{x}$.

answer: $-\frac{1}{ax}$.

(f) $\dfrac{f(x)-f(1)}{x-1}$, where $f(x) = \frac{x-1}{x+1}$.

answer: $\frac{1}{x+1}$.

5. Find the implied domain of the function.

(a) $f(x) = \dfrac{x+4}{x^2-4}$.

answer: $x \neq \pm 2$, alternatively: $x \in (-\infty,-2) \cup (-2,2) \cup (2,\infty)$

(b) $f(x) = \dfrac{2x^3-5}{x^2+5x+6}$.

answer: $x \neq -2,-3$, alternatively: $x \in (-\infty,-3) \cup (-3,-2) \cup (-2,\infty)$

(c) $f(t) = \sqrt[3]{3t-1}$.

answer: $x \in \mathbb{R}$ (the domain is all real numbers)

(d) $g(t) = \sqrt{5-t} - \sqrt{1+t}$.

answer: $x \in [-1,5]$.

(e) $h(x) = \dfrac{1}{\sqrt[6]{x^2-7x}}$.

answer: $x \in (-\infty,0) \cup (7,\infty)$.

(f) $f(u) = \dfrac{u+1}{1+\frac{1}{u+1}}$.

answer: $u \neq -1,-2$ or $u \in (-\infty,-2) \cup (-2,-1) \cup (-1,\infty)$..

(g) $F(x) = \sqrt{10-\sqrt{x}}$.

answer: $x \in [0,100]$

6. Find the implied domain of the function.

(a) $f(x) = \dfrac{x+4}{x^2-4}$.

answer: $x \neq \pm 2$, alternatively: $x \in (-\infty,-2) \cup (-2,2) \cup (2,\infty)$

(b) $f(x) = \dfrac{2x^3-5}{x^2+5x+6}$.

answer: $x \neq -2,-3$, alternatively: $x \in (-\infty,-3) \cup (-3,-2) \cup (-2,\infty)$

(c) $f(t) = \sqrt[3]{3t-1}$.

answer: $x \in \mathbb{R}$ (the domain is all real numbers)

(d) $g(t) = \sqrt{5-t} - \sqrt{1+t}$.

answer: $x \in [-1,5]$.

(e) $h(x) = \dfrac{1}{\sqrt[6]{x^2-7x}}$.

answer: $x \in (-\infty,0) \cup (7,\infty)$.

(f) $f(u) = \dfrac{u+1}{1+\frac{1}{u+1}}$.

answer: $u \neq -1,-2$ or $u \in (-\infty,-2) \cup (-2,-1) \cup (-1,\infty)$..

(g) $F(x) = \sqrt{10-\sqrt{x}}$.

answer: $x \in [0,100]$

7. Compute the composite functions $(f \circ g)(x)$, $(g \circ f)(x)$. Simplify your answer to a single fraction. Find the domain of the composite function.

(a) $f(x) = \dfrac{x+2}{x-2}, g(x) = \dfrac{x-1}{x+2}$.

answer: $(f \circ g)(x) = \frac{3+3x}{-5-x}$ $x \neq -2,-5$
$(g \circ f)(x) = \frac{4}{-2+3x}$ $x \neq 2, \frac{2}{3}$

(b) $f(x) = \dfrac{x+1}{3x-2}, g(x) = \dfrac{x-2}{x-1}$.

answer: $(f \circ g)(x) = \frac{-3+2x}{-4+x}$ $x \neq 4,1$
$(g \circ f)(x) = \frac{5-5x}{3-2x}$ $x \neq \frac{3}{2}, \frac{2}{3}$

(c) $f(x) = \dfrac{2x+1}{3x-1}, g(x) = \dfrac{x-2}{2x-1}$.

answer: $(f \circ g)(x) = \frac{-5+4x}{-5+x}$ $x \neq 5, \frac{1}{2}$
$(g \circ f)(x) = \frac{3-4x}{3+x}$ $x \neq -3, \frac{1}{3}$

(d) $f(x) = \dfrac{x+1}{x-2}, g(x) = \dfrac{x+2}{2x-1}$.

answer: $(f \circ g)(x) = \frac{1+3x}{4-3x}$ $x \neq \frac{4}{3}, \frac{1}{2}$
$(g \circ f)(x) = \frac{-3+3x}{4+x}$ $x \neq -4, 2$

(e) $f(x) = \dfrac{5x+1}{4x-1}, g(x) = \dfrac{4x-1}{3x+1}$.

answer: $(f \circ g)(x) = \frac{-4+23x}{-5+13x}$ $x \neq \frac{5}{13}, -\frac{1}{3}$
$(g \circ f)(x) = \frac{5+16x}{2+19x}$ $x \neq -\frac{2}{19}, \frac{1}{4}$

(f) $f(x) = \dfrac{3x-5}{x-2}$, $g(x) = \dfrac{x-2}{x-4}$.

answer: $(f\circ g)(x) = \frac{-2x+14}{-x+6}$ $\quad x \neq 6, 4$
$(g\circ f)(x) = \frac{x-1}{-x+3}$ $\quad x \neq 3, 2$

(g) $f(x) = \dfrac{x-3}{x+2}$, $g(y) = \dfrac{y+3}{y-4}$.

answer: $(f\circ g)(x) = \frac{-2x+15}{3x-5}$ $\quad x \neq \frac{5}{3}, 4$
$(g\circ f)(x) = \frac{4x+3}{-3x-11}$ $\quad x \neq -\frac{11}{3}, -2$

8. Find the functions $f\circ g$, $g\circ f$, $f\circ f$ and $g\circ g$ and their implied domains. The answer key has not been proofread, use with caution.

(a) $f(x) = x^2+1$, $g(x) = x+1$.

answer: Domain, all 4 cases: $x \in \mathbb{R}$ (all reals)
in some order: $(1+x)^2+1$, $(x)^2+2$, $((x)^2+1)^2+1$, $2+x$

(b) $f(x) = \sqrt{x+1}$, $g(x) = x+1$.

answer: Domain of $f\circ g$ is $x \geq -2$. Domain of $g\circ f$ is $x\geq -1$
Domain of $f\circ f$ is $x \geq -1$. Domain of $g \circ g$ is all reals ($x\in\mathbb{R}$).
in some order: $\sqrt{2+x}$, $1+\sqrt{1+x}$, $\sqrt{1+\sqrt{1+x}}$, $2+x$

(c) $f(x) = 2x$, $g(x) = \tan x$.

In this subproblem, you are not required to find the domain.

answer: Domain $f\circ f$: all reals ($x\in\mathbb{R}$). Domain $g\circ f$: $x \neq (2k+1)\frac{\pi}{2}$ for all $k\in\mathbb{Z}$
Domain $f\circ g$: $x\neq (4k+1)\frac{\pi}{4}$, $x \neq (4k+3)\frac{\pi}{4}$ for all $k \in \mathbb{Z}$
Domain $g\circ g$: $x \neq (2k+1)\frac{\pi}{2}$ and $x \neq k\pi + \arctan\left(\frac{\pi}{2}\right)$ for all $k\in\mathbb{Z}$
in some order: $2\tan x$, $\tan(2x)$, $4x$, $\tan(\tan x)$

(d) $f(x) = \dfrac{x+1}{x-1}$, $g(x) = \dfrac{x-1}{x+1}$.

answer: Domain $f\circ f$: $x\neq 1$. Domain $g\circ g$: $x \neq 0, x\neq -1$
Domain $f\circ g$: $x \neq -1$. Domain $g \circ f$: $x\neq 0, x \neq 1$
in some order: $-x$, $\frac{1}{x}$, x, $-\frac{1}{x}$

9. Convert from degrees to radians.

(a) 15°. answer: $\frac{\pi}{12} \approx 0.261799388$

(b) 30°. answer: $\frac{\pi}{6} \approx 0.523598776$

(c) 36°. answer: $\frac{\pi}{5} \approx 0.628318531$

(d) 45°. answer: $\frac{\pi}{4} \approx 0.785398163$

(e) 60°. answer: $\frac{\pi}{3} \approx 1.047197551$

(f) 75°. answer: $\frac{5\pi}{12} \approx 1.308997$

(g) 90°. answer: $\frac{\pi}{2}$

(h) 120°. answer: $\frac{2\pi}{3}$

(i) 135°. answer: $\frac{3\pi}{4}$

(j) 150°. answer: $\frac{5\pi}{6}$

(k) 180°. answer: π

(l) 225°. answer: $\frac{5\pi}{4}$

(m) 270°. answer: $\frac{3\pi}{2}$

(n) 305°. answer: $\frac{61\pi}{36} \approx 5.323254$

(o) 360°. answer: 2π

(p) 405°. answer: $\frac{9\pi}{4}$

(q) 1200°. answer: $\frac{20\pi}{3}$

(r) -900°. answer: -5π

(s) -2014°. answer: $-\frac{1007}{90}\pi \approx -35.150931$

10. Convert from radians to degrees. The answer key has not been proofread, use with caution.

(a) 4π. answer: 720°

(b) $-\frac{7}{6}\pi$. answer: -210°

(c) $\frac{7}{12}\pi$. answer: 105°

(d) $\frac{4}{3}\pi$. answer: 240°

(e) $-\frac{3}{8}\pi$. answer: -67.5°

(f) 2014π. answer: 362520°

(g) 5. answer: $\left(\frac{900}{\pi}\right)^\circ \approx 286^\circ$

(h) -2014. answer: -362520°

11. Prove the trigonometry identities.

(a) $\sin\theta\cot\theta = \cos\theta$.
(b) $(\sin\theta+\cos\theta)^2 = 1+\sin(2\theta)$.
(c) $\sec\theta - \cos\theta = \tan\theta\sin\theta$.
(d) $\tan^2\theta - \sin^2\theta = \tan^2\theta\sin^2\theta$.
(e) $\cot^2\theta + \sec^2\theta = \tan^2\theta + \csc^2\theta$.
(f) $2\csc(2\theta) = \sec\theta\csc\theta$.

(g) $\tan(2\theta) = \dfrac{2\tan\theta}{1-\tan^2\theta}$.

(h) $\dfrac{1}{1-\sin\theta} + \dfrac{1}{1+\sin\theta} = 2\sec^2\theta$.

(i) $\tan\alpha + \tan\beta = \dfrac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta}$.

(j) $\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$.

(k) $\sin(3\theta) + \sin\theta = 2\sin(2\theta)\cos\theta$.

(l) $\cos(3\theta) = 4\cos^3\theta - 3\cos\theta$.

(m) $1 + \tan^2\theta = \sec^2\theta$.

(n) $1 + \csc^2\theta = \cot^2\theta$.

(o) $2\cos^2(2x) = 2\sin^4\theta + 2\cos^4\theta - \sin^2(2\theta)$.

(p) $\dfrac{1+\tan\left(\frac{\theta}{2}\right)}{1-\tan\left(\frac{\theta}{2}\right)} = \tan\theta + \sec\theta$.

12. Find all values of x in the interval $[0, 2\pi]$ that satisfy the equation.

(a) $2\cos x - 1 = 0$.

answer: $x = \frac{\pi}{3}$ or $x = \frac{5\pi}{3}$

(b) $\sin(2x) = \cos x$.

answer: $x = \frac{\pi}{2}$, $x = \frac{3\pi}{2}$, $x = \frac{\pi}{6}$, or $x = \frac{5\pi}{6}$

(c) $\sqrt{3}\sin x = \sin(2x)$.

answer: $x = \frac{\pi}{6}$, $\frac{11\pi}{6}$, 0, π, 2π.

(d) $2\sin^2 x = 1$.

answer: $x = \frac{\pi}{4}$, $x = \frac{3\pi}{4}$, $x = \frac{5\pi}{4}$, or $x = \frac{7\pi}{4}$

(e) $2 + \cos(2x) = 3\cos x$.

answer: $x = 0$, $x = 2\pi$, $x = \frac{\pi}{3}$, or $x = \frac{5\pi}{3}$

(f) $2\cos x + \sin(2x) = 0$.

answer: $x = \frac{\pi}{2}$, $x = \frac{3\pi}{2}$

(g) $2\cos^2 x - (1+\sqrt{2})\cos x + \dfrac{\sqrt{2}}{2} = 0$.

answer: $x = \frac{\pi}{4}$, $\frac{\pi}{3}$, $\frac{5\pi}{3}$, $\frac{7\pi}{4}$.

(h) $|\tan x| = 1$.

answer: $x = \frac{\pi}{4}$, $x = \frac{3\pi}{4}$, $x = \frac{5\pi}{4}$, or $x = \frac{7\pi}{4}$

(i) $3\cot^2 x = 1$.

answer: $x = \frac{\pi}{3}$, $x = \frac{2\pi}{3}$, $x = \frac{4\pi}{3}$, or $x = \frac{5\pi}{3}$

(j) $\sin x = \tan x$.

answer: $x = 0$, $x = \pi$, or $x = 2\pi$

Solution. 12.g Set $\cos x = u$. Then

$$2\cos^2 x - (1+\sqrt{2})\cos x + \frac{\sqrt{2}}{2} = 0$$

becomes

$$2u^2 - (1+\sqrt{2})u + \frac{\sqrt{2}}{2} = 0.$$

This is a quadratic equation in u and therefore has solutions

$$\begin{aligned} u_1, u_2 &= \frac{1+\sqrt{2}\pm\sqrt{(1+\sqrt{2})^2 - 4\sqrt{2}}}{4} \\ &= \frac{1+\sqrt{2}\pm\sqrt{1-2\sqrt{2}+2}}{4} \\ &= \frac{1+\sqrt{2}\pm\sqrt{(1-\sqrt{2})^2}}{4} \\ &= \frac{1+\sqrt{2}\pm(1-\sqrt{2})}{4} = \begin{cases} \frac{1}{2} & \text{or} \\ \frac{\sqrt{2}}{2} \end{cases} \end{aligned}$$

Therefore $u = \cos x = \frac{1}{2}$ or $u = \cos x = \frac{\sqrt{2}}{2}$, and, as x is in the interval $[0, 2\pi]$, we get $x = \frac{\pi}{3}, \frac{5\pi}{3}$ (for $\cos x = \frac{1}{2}$) or $x = \frac{\pi}{4}, \frac{7\pi}{4}$ (for $\cos x = \frac{\sqrt{2}}{2}$).