Calculus II
Express $\sin(kx), \cos(kx)$ via $\sin x, \cos x$ using Euler's formula.

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- Recall Euler's formula: $e^{i\alpha} = \cos\alpha + i\sin\alpha$.
- Recall the formula: $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.

Example

Express $\sin(3x)$ and $\cos(3x)$ via $\cos x$ and $\sin x$.

$$\begin{array}{ll}
\cos(3x) + i\sin(3x) & \Big|\ \text{Euler's f-la} \\
= e^{3ix} & \\
= \left(e^{ix}\right)^3 = (\cos x + i\sin x)^3 & \Big|\ \text{Euler's f-la} \\
= \cos^3 x + 3\cos^2 x(i\sin x) + 3\cos x(i\sin x)^2 + (i\sin x)^3 & \\
= \cos^3 x + 3i\cos^2 x\sin x + 3i^2\cos x\sin^2 x + i^3\sin^3 x & \\
= \cos^3 x + 3i\cos^2 x\sin x - 3\cos x\sin^2 x - i\sin^3 x & \Big|\ \text{Use } i^2 = -1 \\
= (\cos^3 x - 3\cos x\sin^2 x) + i(3\cos^2 x\sin x - \sin^3 x) &
\end{array}$$

The real parts of the starting and final expression must be equal; likewise the imaginary parts must be equal; therefore:

$$\begin{aligned}
\cos(3x) &= \cos^3 x - 3\cos x\sin^2 x \\
\sin(3x) &= 3\cos^2 x\sin x - \sin^3 x
\end{aligned}$$