Calculus I
Exponents and logarithms review

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Outline

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Properties of exponential expressions.

For integer x, y and bases a, b, we demonstrate the exponent rules by example.

1. $a^x a^y = a^{x+y}$
2. $\frac{a^x}{a^y} = a^{x-y}$
3. $(a^x)^y = a^{xy}$
4. $(ab)^x = a^x b^x$

These rules do continue to hold for all $a > 0$, $b > 0$ and arbitrary x and y. The rules do fail when $a < 0$, $b < 0$ and x, y are not integers.

$$\begin{aligned}7^3 \cdot 7^2 &= (7 \cdot 7 \cdot 7)(7 \cdot 7)\\ &= 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7\\ &= 7^5\\ &= 7^{3+2}.\end{aligned}$$

Properties of exponential expressions.

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$$\begin{aligned}\frac{7^3}{7^2} &= \frac{\cancel{7}\cdot\cancel{7}\cdot 7}{\cancel{7}\cdot\cancel{7}}\\ &= 7\\ &= 7^1\\ &= 7^{3-2}.\end{aligned}$$

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$$\begin{aligned}\left(7^2\right)^4 &= 7^2 \cdot 7^2 \cdot 7^2 \cdot 7^2\\ &= (7\cdot 7)(7\cdot 7)(7\cdot 7)(7\cdot 7)\\ &= 7\cdot 7\cdot 7\cdot 7\cdot 7\cdot 7\cdot 7\cdot 7\\ &= 7^8\\ &= 7^{2\cdot 4}\end{aligned}$$

Properties of exponential expressions.

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$$\begin{aligned}(5\cdot 7)^3 &= (5\cdot 7)(5\cdot 7)(5\cdot 7)\\ &= 5\cdot 7\cdot 5\cdot 7\cdot 5\cdot 7\\ &= 5\cdot 5\cdot 5\cdot 7\cdot 7\cdot 7\\ &= 5^3\cdot 7^3\end{aligned}$$

Exponents overview

- For integer x, we know how to compute a^x as a function of a.
- How do we compute $f(x) = a^x$ when x is not an integer?
- We need to go back to the definition of a^x (for x non-integer).
- In what follows we give/recall an elementary way to define exponent.
- Then we give an alternative second definition.
- The second definition will be studied in sufficient depth only much later.
- The two definitions are equivalent: if we choose one definition the other becomes a theorem and the other way round.
- Choosing one definition makes some statements easier to prove and others more difficult.
- We shall discuss pros and cons of the two. In a nutshell:
    - the first elementary definition is easier to motivate;
    - the second alternative definition is easier to compute with.

Exponent definition using limits (approach I)

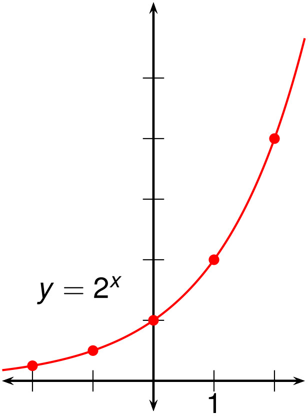
- For integer p we know to compute a^p.
- Therefore for integer q we know to compute $a^{\frac{1}{q}} = \sqrt[q]{a} = \max\{x |$ for which $x^q \leq a\}$.
- Therefore we know to compute $a^{\frac{p}{q}}$ for all rational $\frac{p}{q}$.
- We can then define
$$a^x = \lim_{\substack{y \to x \\ y\text{-rational}}} a^y$$
For example, a^π would be defined as the limit of the sequence $a^{3.14}, a^{3.141}, a^{3.1415}, \dots$.
- Cons: not computationally effective; not how computers compute.
- Pros: for non-integer x and y, it is very easy to prove that $a^{x+y} = a^x a^y$ - this follows from the definition of limit above.
- This is the definition assumed in many elementary courses.

Exponent definition using series (approach II)

- The following formula (studied much later) can be used as alternative definition.
$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots$$
Here $n! = 1 \cdot 2 \cdot 3 \cdot \dots \cdot (n-1) \cdot n$ and is read "n factorial".
- For $|x| < 1$ define
$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + \frac{(-1)^{n+1} x^n}{n} + \dots$$
Infinite sum studied much later.
- For arbitrary $a > 0$ define a^x as $a^x = e^{x \ln a}$.
- Cons: more difficult to prove $e^{x+y} = e^x e^y$ and $e^{\ln(1+x)} = 1 + x$, proof done later.
- Pros: this is how e^x and a^x are actually computed (by modern computers and by humans in the past).

Exponential Functions

The function $f(x) = 2^x$ is called an exponential function because the variable x is the exponent.

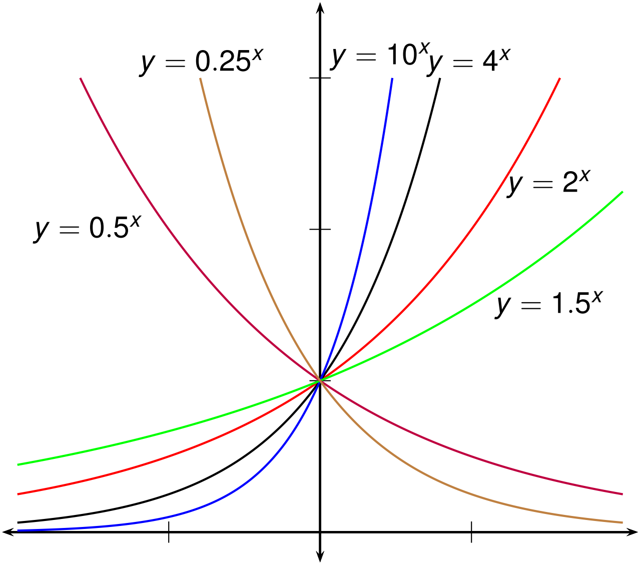


x	y
2	4
1	2
0	1
-1	$\frac{1}{2}$
-2	$\frac{1}{4}$

(Exponential Function Terminology)

An exponential function is a function of the form $f(x) = a^x$, where a is a positive constant.

Graphs of various exponential functions.



Observations

- a^x is always positive.
- $a^0 = 1$ for all a.

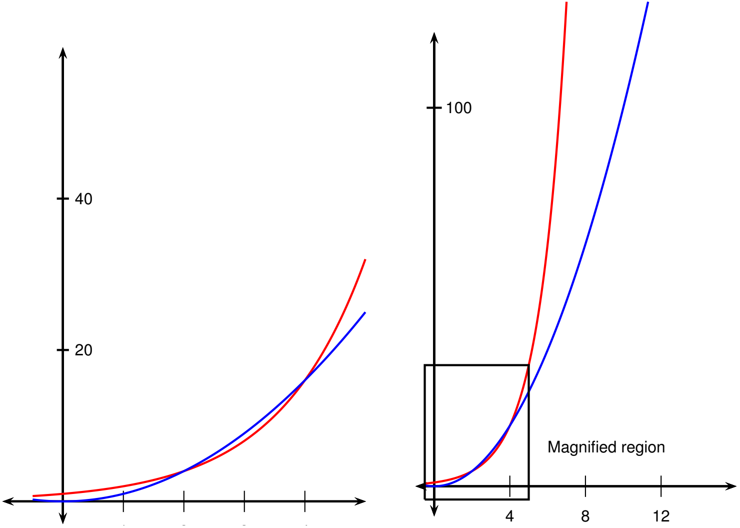
For $a > 1$:

- $\lim_{x\to\infty} a^x = \infty.$
- $\lim_{x\to-\infty} a^x = 0.$

$a < 1$:

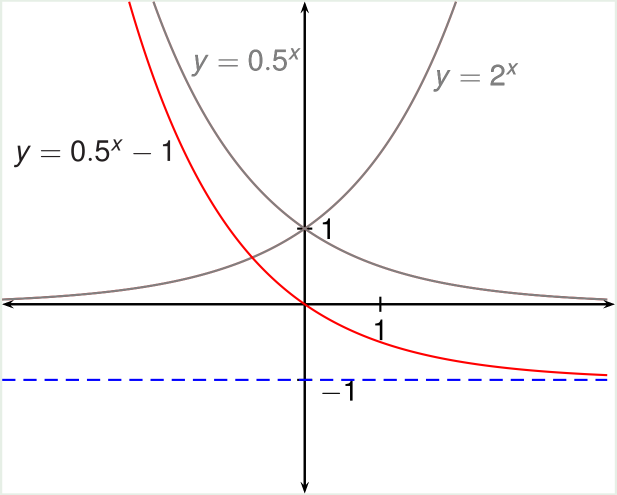
- $\lim_{x\to\infty} a^x = 0.$
- $\lim_{x\to-\infty} a^x = \infty.$

Graphical comparison of $y = 2^x$ with $y = x^2$. Axes have different scales.



Example

Draw the graph of the function $y = 2^{-x} - 1 = 0.5^x - 1 = \left(\frac{1}{2}\right)^x - 1$. Assume the graph of $y = 2^x$ given.



- Plot of 2^x assumed given.
- Plot $f(-x)$ = reflect $f(x)$ across y axis.
- Plot $g(x) - 1$ = shift graph $g(x)$ 1 unit down.

Example (Solve exponential equation without logarithms)

Solve for t.

Find a common base:

$$\begin{aligned}
16^{4t} &= 8^{t-2}\\
\left(2^4\right)^{4t} &= \left(2^3\right)^{t-2}\\
2^{16t} &= 2^{3t-6}\\
16t &= 3t-6\\
13t &= -6\\
t &= -\frac{6}{13}.
\end{aligned}$$

Example (Solving a quadratic exponential equation)

Solve for x.

$$\begin{aligned} 9^x &= 2\cdot 3^x + 63 \\ 9^x - 2\cdot 3^x - 63 &= 0 && \Big| \text{ Substitute } u = 3^x \\ u^2 - 2u - 63 &= 0 \\ (u-9)(u+7) &= 0 \end{aligned}$$

$$\begin{array}{rclcrcl} u &=& 9 & \text{or} & u &=& -7 \\ 3^x &=& 9 & \text{or} & 3^x &=& -7 \\ x &=& 2 & & & & \text{no real solution} \end{array}$$

Example (Solving an exponential word problem)

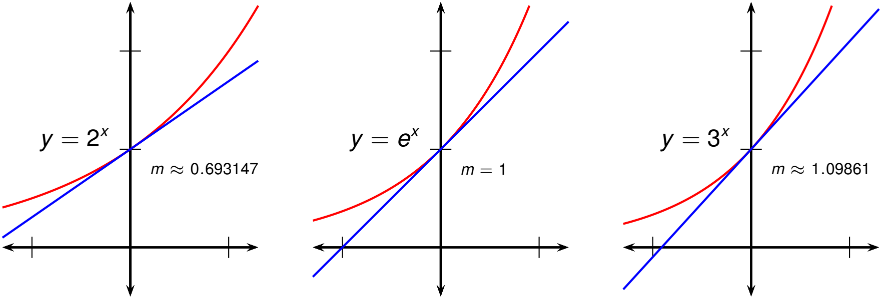
A farmer buys 48 chickens and 6 rabbits. The chicken population doubles each year, and the rabbit population doubles every six months. When does the farmer have the same number of chickens as rabbits?

Let $c(t)$ denote the number of chickens after t years, and let $r(t)$ denote the number of rabbits after t years.

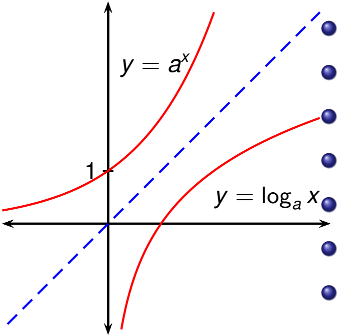
$$\begin{aligned} \text{Solve for } t: \quad &= \\ &= \\ &= \\ &= \\ &= \\ &= \\ t &= \end{aligned}$$

The Natural Exponential Function

- One base for an exponential function is especially useful.
- It has a special property: its tangent line at $x = 0$ has slope $m = 1$.
- We call this number e, known as Euler's number or Napier's constant.
- e is a number between 2 and 3.
- In fact, $e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots \approx 2.71828$.



Logarithmic Functions



- Suppose $a > 0$, $a \neq 1$.
- Let $f(x) = a^x$.
- Then f is either increasing or decreasing.
- Therefore f is one-to-one.
- Therefore f has an inverse function, f^{-1}.
- The graph shows $y = a^x$ for $a > 1$.
- The graph of $y = \log_a x$ is the reflection of this in the line $y = x$.

Definition ($\log_a x$)

The inverse function of $f(x) = a^x$ is called the logarithmic function with base a, and is written $\log_a x$. It is defined by the formula

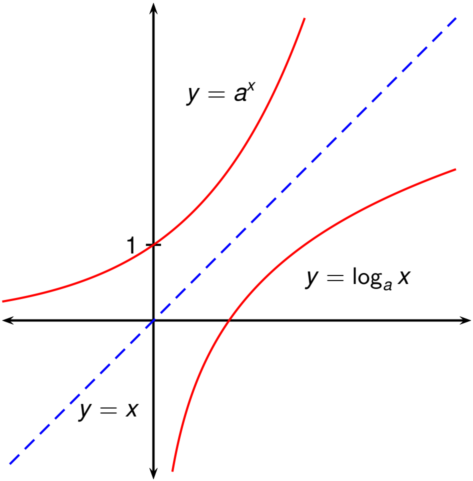
$$\log_a x = y \qquad \Leftrightarrow \qquad a^y = x.$$

If $x > 0$, then $\log_a x$ is the exponent to which the base a must be raised to give x.

Example

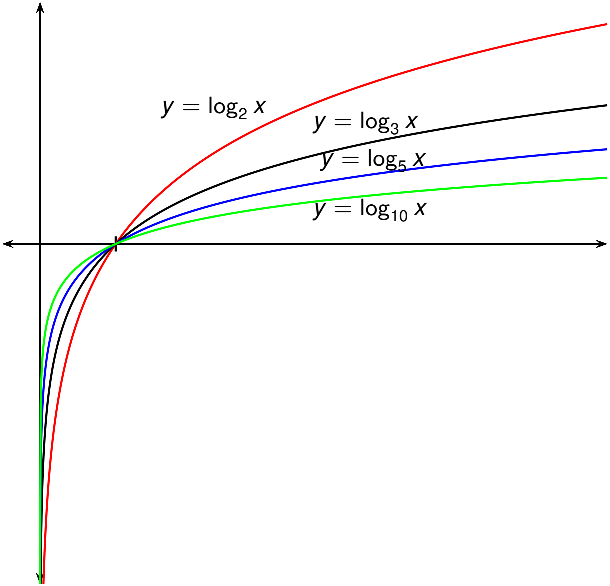
Evaluate:

1. $\log_3 81 = 4$ because $3^4 = 81$.
2. $\log_{25} 5 = \frac{1}{2}$ because $25^{\frac{1}{2}} = \sqrt{25} = 5$.
3. $\log_{10} 0.001 = -3$ because $10^{-3} = 0.001$.



- Suppose $a > 1$.
- Domain of a^x: $\mathbb{R}$.
- Range of a^x: $(0, \infty)$.
- Domain of $\log_a x$: $(0, \infty)$.
- Range of $\log_a x$: $\mathbb{R}$.
- $\log_a(a^x) = x$ for $x \in \mathbb{R}$.
- $a^{\log_a x} = x$ for $x > 0$.

Graphs of various logarithmic functions with $a > 1$



Theorem (Properties of Logarithmic Functions)

If $a > 1$, the function $f(x) = \log_a x$ is a one-to-one, continuous, increasing function with domain $(0, \infty)$ and range $\mathbb{R}$. If $x, y, a, b > 0$ and r is any real number, then

1. $\log_a(xy) = \log_a x + \log_a y$.
2. $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$.
3. $\log_a(x^r) = r \log_a x$.
4. $\log_a(x) = \log_b x \log_a b = \frac{\log_b x}{\log_b a} = \frac{\ln x}{\ln a}$.

Use the properties of logarithms to evaluate the following.

Example

$$\begin{aligned} \log_4 2 + \log_4 32 &= \log_4(2 \cdot 32) \\ &= \log_4(64) \\ &= 3 \\ & \text{(because } 4^3 = 64.\text{)} \end{aligned}$$

Example

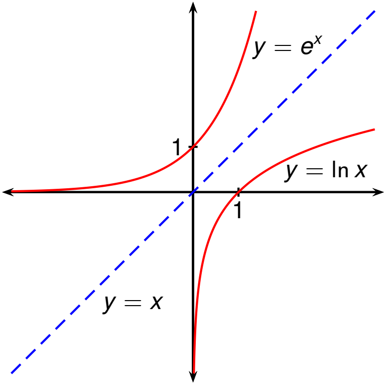
$$\begin{aligned} \log_2 80 - \log_2 5 &= \log_2 \left(\frac{80}{5}\right) \\ &= \log_2(16) \\ &= 4 \\ & \text{(because } 2^4 = 16.\text{)} \end{aligned}$$

Natural Logarithms

Definition ($\ln x$)

The logarithm with base e is called the natural logarithm, and has a special notation:

$$\log_e x = \ln x.$$



- $\ln x = y \quad \Leftrightarrow \quad e^y = x$.
- $\ln(e^x) = x$ for $x \in \mathbb{R}$.
- $e^{\ln x} = x$ for $x > 0$.

Summary of logarithm notation conventions

	Name	ISO notation	Other notation	Used in
$\log_2(x)$	binary logarithm	$\mathrm{lb}(x)$	$\mathrm{ld}(x)$, $\log(x)$, $\lg(x)$	computer science, information theory, music theory, photography
$\log_e(x)$	natural logarithm	$\ln(x)$	$\log(x)$	mathematics, physics, chemistry, statistics, economics, information theory, and engineering
$\log_{10}(x)$	common logarithm	$\lg(x)$	$\log(x)$	various engineering, logarithm tables, handheld calculators, spectroscopy

Table source: Wikipedia

- Standardized in ISO_31-11 (International Standards Organization).

Example

Solve the equation.

$$\begin{aligned} e^{5-3x} &= 10 \\ \ln(e^{5-3x}) &= \ln 10 \\ 5-3x &= \ln 10 \\ 3x &= 5-\ln 10 \\ x &= \frac{5-\ln 10}{3} \\ \text{Calculator:}\quad x &\approx 0.8991. \end{aligned}$$

Example

Solve the equation

$$e^{2x} - 3e^{x} - 4 = 0$$

Set $e^x = u$. Then $e^{2x} = u^2$.

$$\begin{aligned} u^2 - 3u - 4 &= 0 \\ (u-4)(u+1) &= 0 \end{aligned}$$

$$\begin{array}{rcr} u = 4 & \text{or} & u = -1 \\ e^x = 4 & \text{or} & e^x = -1 \\ x = \ln 4 & \text{or} & \text{no real solution} \\ x \approx 1.3863 & & \end{array}$$

Example

Solve the equation

$$4^{x+1} - 2^{x+2} - 3 = 0$$

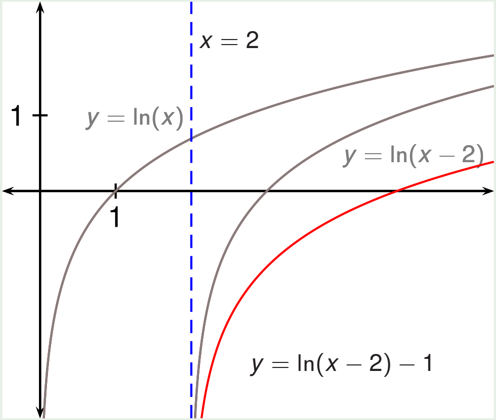
Set $u = 2^x$. Then $4^{x+1} = 4u^2, 2^{x+2} = 4u$.

$$\begin{aligned} 4u^2 - 4u - 3 &= 0 \\ (2u-3)(2u+1) &= 0 \end{aligned}$$

$$\begin{aligned} 2u - 3 = 0 &\quad \text{or} \quad 2u + 1 = 0 \\ u = \frac{3}{2} &\quad \text{or} \quad u = -\frac{1}{2} \\ 2^x = \frac{3}{2} &\quad \text{or} \quad 2^x = -\frac{1}{2} \\ x = \log_2\left(\frac{3}{2}\right) = \frac{\ln\left(\frac{3}{2}\right)}{\ln 2} \approx 0.58496 &\quad \text{or} \quad \text{no real solution} \end{aligned}$$

Example

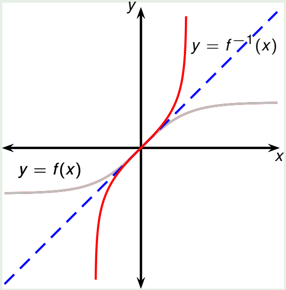
Draw the graph of $y = \ln(x-2) - 1$.



- Graph $y = \ln(x)$ assumed known.
- $f(x-2)$ shifts graph 2 units to the right.
- $g(x) - 1$ shifts graph 1 unit down.

Example

Find $f^{-1}(x)$ for
$$f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$
$f = \tanh$ = hyperbolic tangent function.



Final answer, relabeled:
$$f^{-1}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

$$\begin{aligned}
\frac{e^x - e^{-x}}{e^x + e^{-x}} &= y \\
\frac{\left(u - \frac{1}{u}\right)u}{\left(u + \frac{1}{u}\right)u} &= y \\
\frac{u^2 - 1}{u^2 + 1} &= y \\
u^2 - 1 &= y(u^2 + 1) \\
u^2(1 - y) &= 1 + y \\
u^2 &= \frac{1+y}{1-y} \\
(e^x)^2 &= \frac{1+y}{1-y} \\
e^{2x} &= \frac{1+y}{1-y} \\
x &= \frac{1}{2}\ln\left(\frac{1+y}{1-y}\right)
\end{aligned}$$

Set $u = e^x$
$e^{-x} = \frac{1}{e^x} = \frac{1}{u}$

Take ln