Calculus II
Integral of rational function with cubic denominator, part 3

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2019

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- The for each quadratic factor we need to add a partial fraction of the form
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