

Precalculus

Inverse functions

Todor Milev

2019

Outline

1

Inverse Functions

- One-to-one Functions
- The Definition of the Inverse of f

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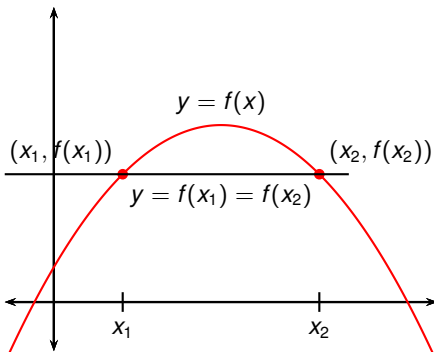
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One-to-one Functions

Definition (One-to-one Function)

A function f is a one-to-one function if it never takes on the same value twice; that is,

$$f(x_1) \neq f(x_2) \text{ whenever } x_1 \neq x_2.$$



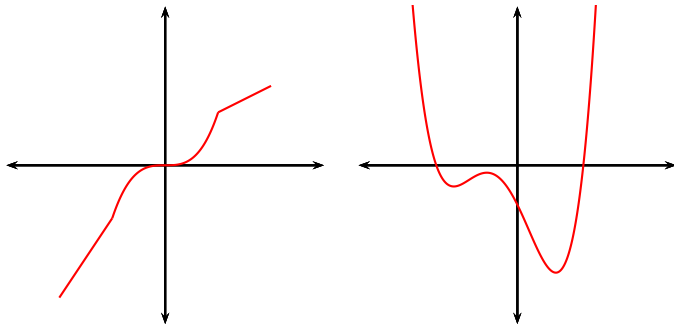
← This function is not one-to-one.

Question: How can we tell from the graph of a function whether it is one-to-one or not?

Answer: Use the horizontal line test.

The Horizontal Line Test.

A function is one-to-one if and only if no horizontal line intersects it more than once.

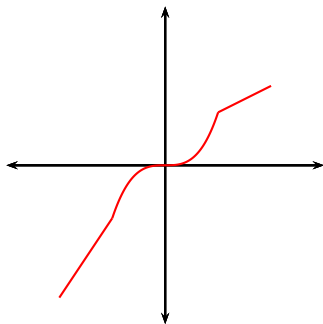


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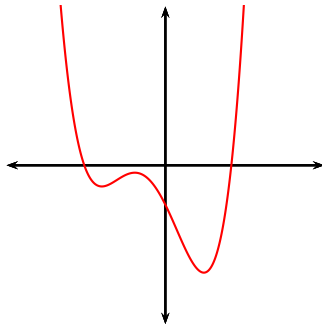
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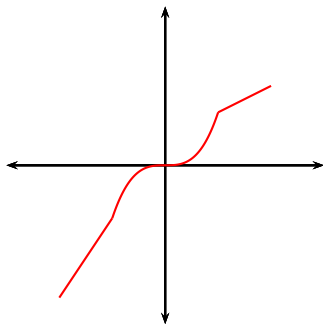


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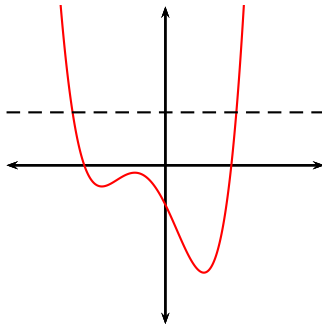
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One-to-one



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The Definition of the Inverse of f

Definition (f^{-1})

Let f be a one-to-one function with domain A and range B . Then the inverse of f is the function f^{-1} that has domain B and range A and is defined by

$$f^{-1}(y) = x \quad \Leftrightarrow \quad f(x) = y$$

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Example ($f(x) = x^3$)

The inverse of $f(x) = x^3$ is $f^{-1}(x) = \sqrt[3]{x}$. This is because if $y = x^3$, then

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No one blamed English language of being logical.

-Bjarne Stroustrup, creator of the programming language C++

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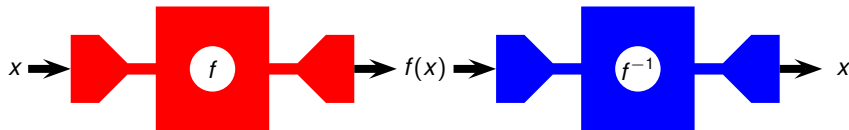
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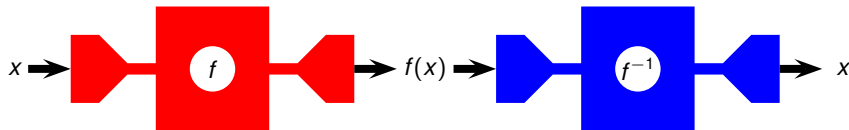
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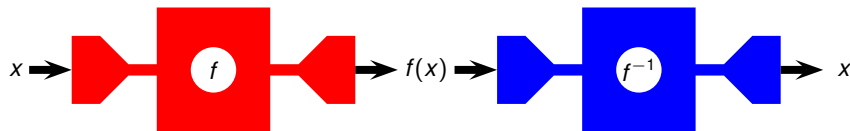
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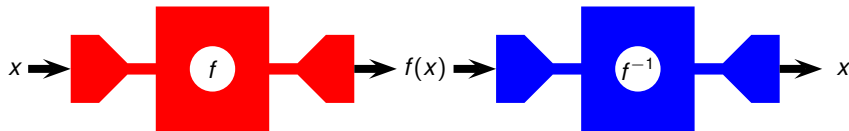
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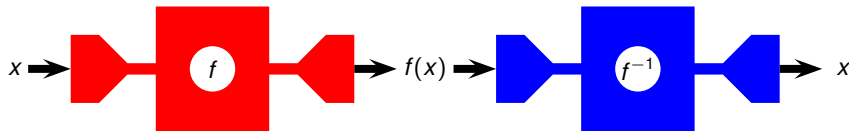
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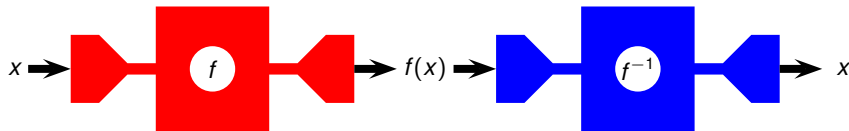
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If $f(x) = 2x + \sin 2x + e^{\frac{x}{2}}$, find $f^{-1}(1)$. You do not need to show that f has an inverse.

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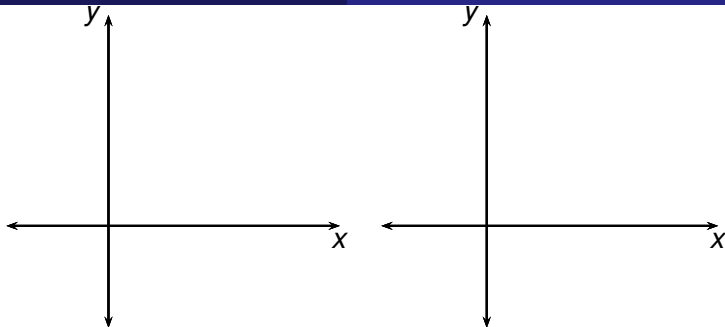
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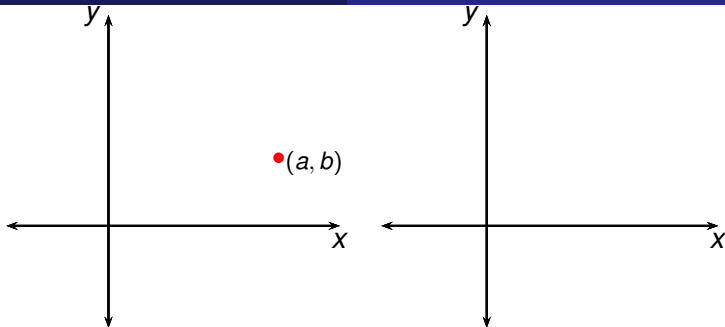
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Therefore $f^{-1}(1) = 0$.

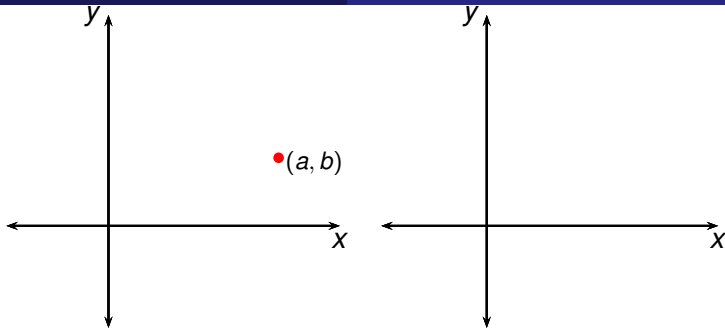


Interchanging x and y suggests relation between the graphs of f^{-1} and f :



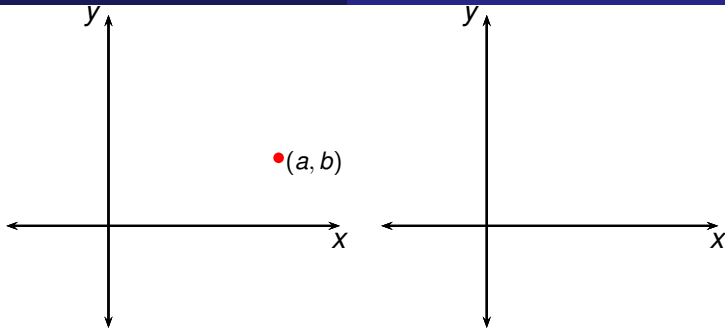
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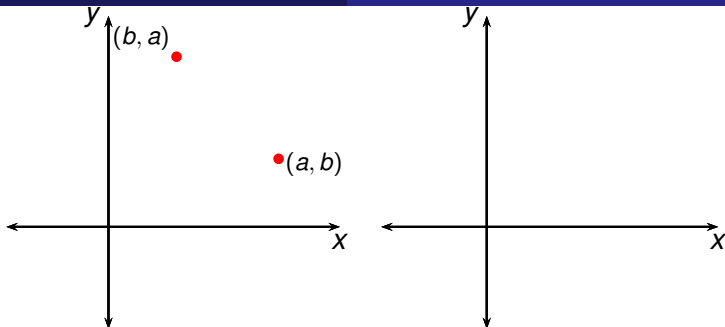
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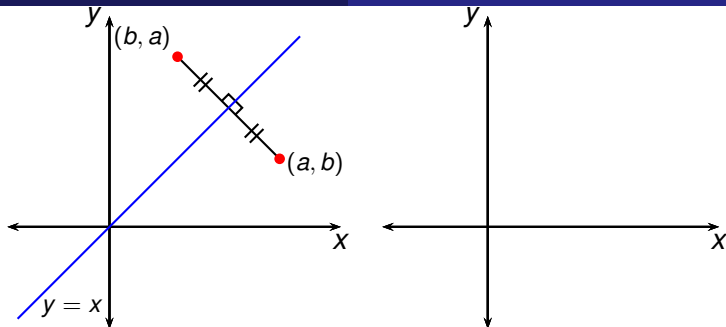
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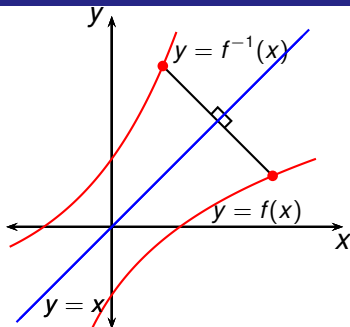
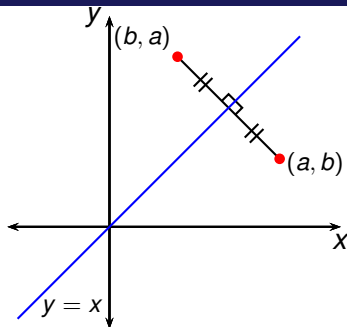
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- Then (b, a) is on the graph of f^{-1} .



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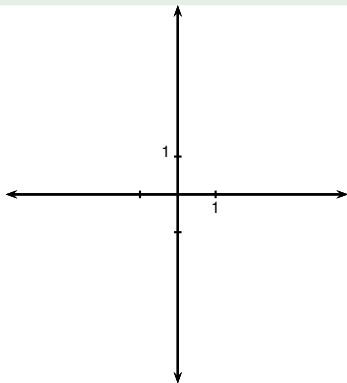
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- Then (b, a) is on the graph of f^{-1} .
- (b, a) is the reflection of (a, b) in the line $y = x$.



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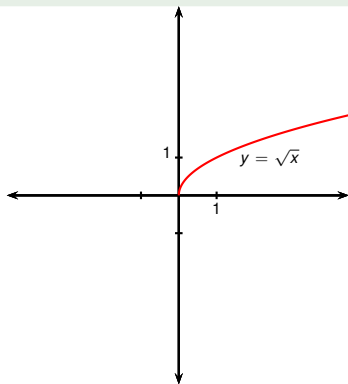
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- Then (b, a) is on the graph of f^{-1} .
- (b, a) is the reflection of (a, b) in the line $y = x$.
- Thus the graph of f^{-1} is obtained by reflecting the graph of f in the line $y = x$.

Example



Sketch the graph of $f(x) = \sqrt{-x - 1}$ and its inverse function.

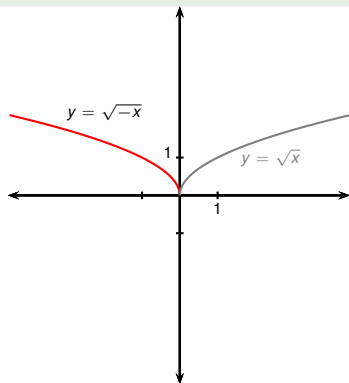
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Sketch the graph of $f(x) = \sqrt{-x - 1}$ and its inverse function.

- Draw the graph of $y = \sqrt{x}$.

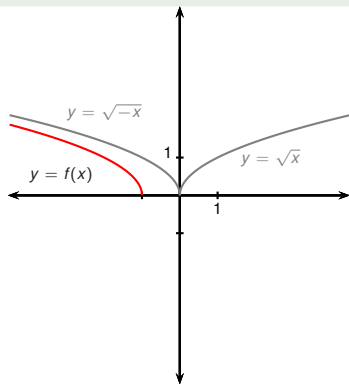
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Sketch the graph of $f(x) = \sqrt{-x - 1}$ and its inverse function.

- Draw the graph of $y = \sqrt{x}$.
- $y = \sqrt{-x}$ is the reflection of $y = \sqrt{x}$ in the y -axis.

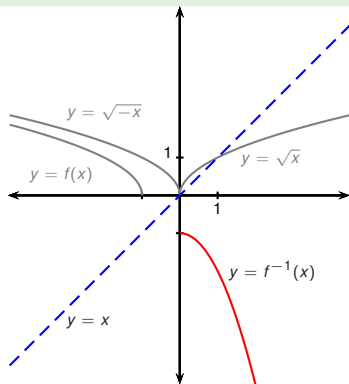
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- Draw the graph of $y = \sqrt{x}$.
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- $y = f(x) = \sqrt{-(x+1)} = \sqrt{-x-1}$ is the shift of $y = \sqrt{-x}$ **one unit to the left**.

Example

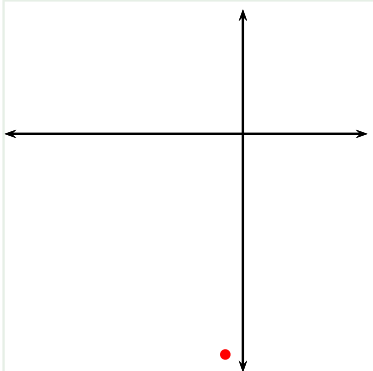


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- $y = f^{-1}(x)$ is the reflection of $y = f(x)$ across the line $y = x$.

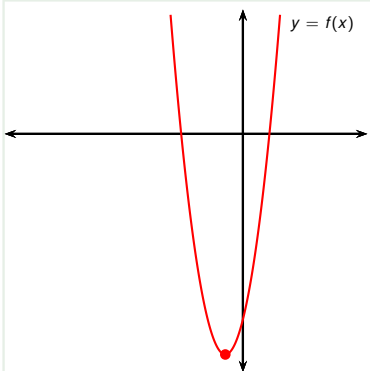
Example ()

Given: $f(x) = 3x^2 + 4x - 7$ with domain $x \geq -\frac{2}{3}$. Find $f^{-1}(x)$.



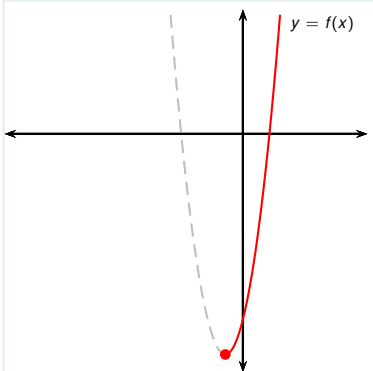
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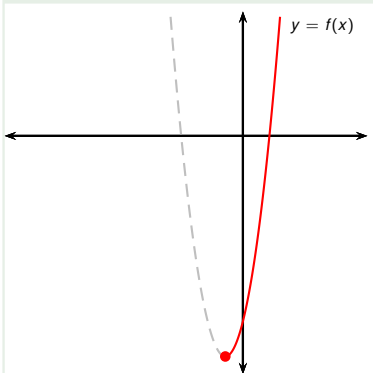
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Example ()

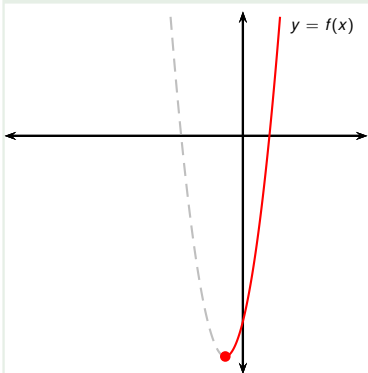
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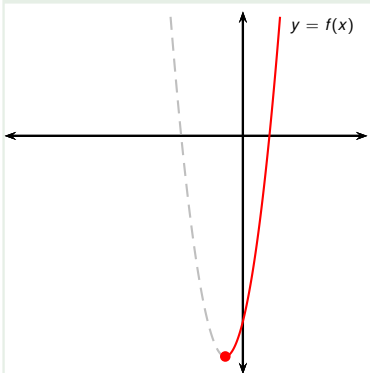
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That's a quadratic equation in x . Solve:

$$\frac{-4 \pm \sqrt{4^2 - 4 \cdot 3 \cdot (-y - 7)}}{2 \cdot 3}$$

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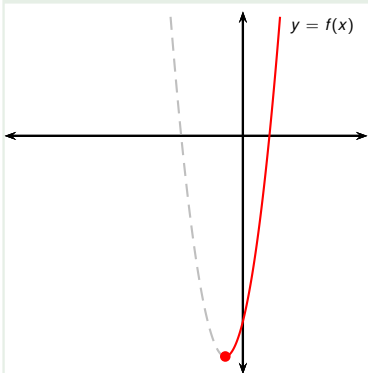
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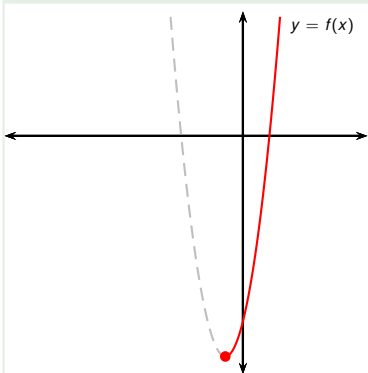
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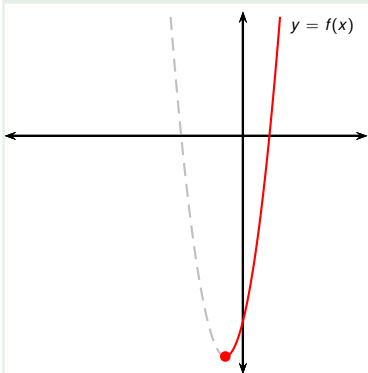
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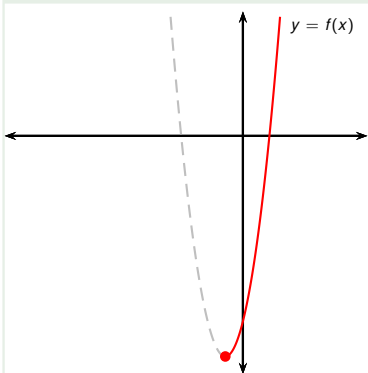
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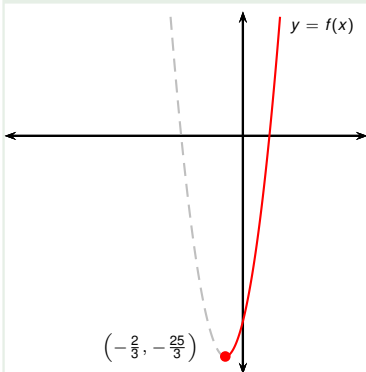
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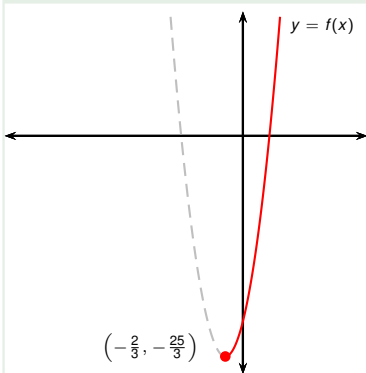
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Given: $f(x) = 3x^2 + 4x - 7$ with domain $x \geq -\frac{2}{3}$. Find $f^{-1}(x)$.



answer

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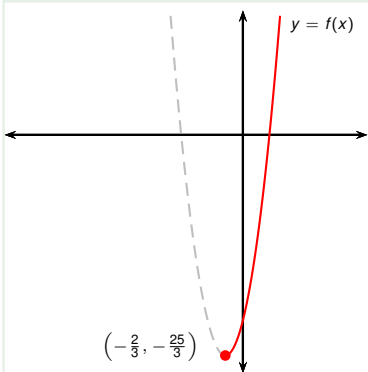
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Final answer, **relabelled**:

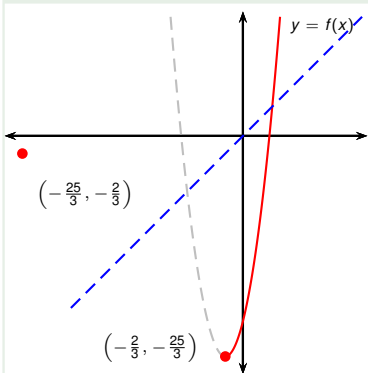
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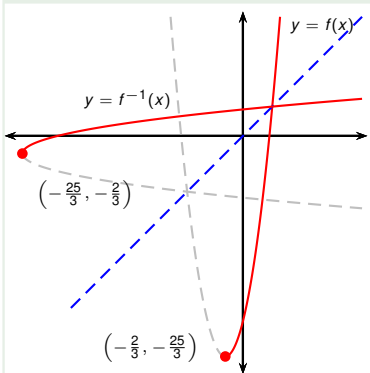
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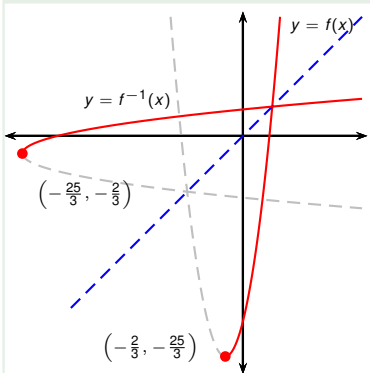
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Given: $f(x) = 3x^2 + 4x - 7$ with domain $x \geq -\frac{2}{3}$. Find $f^{-1}(x)$.



Final answer, relabelled:

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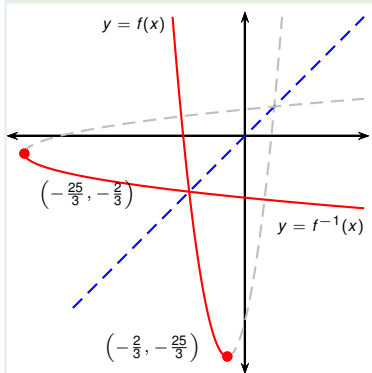
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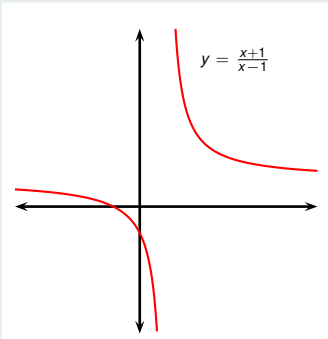
$$f^{-1}(x) = -\frac{2}{3} - \frac{\sqrt{25 + 3x}}{3}$$

We are given $x \leq -\frac{2}{3}$, therefore

$$x = -\frac{2}{3} - \frac{\sqrt{25 + 3y}}{3} = f^{-1}(y).$$

Example

Find $f^{-1}(x)$ where $f(x) = \frac{x+1}{x-1}$.

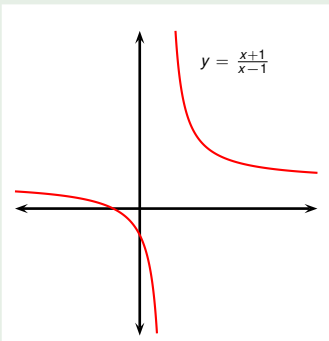


Example

Find $f^{-1}(x)$ where $f(x) = \frac{x+1}{x-1}$.

We deal with domains and ranges later:

$$y = \frac{x+1}{x-1}$$

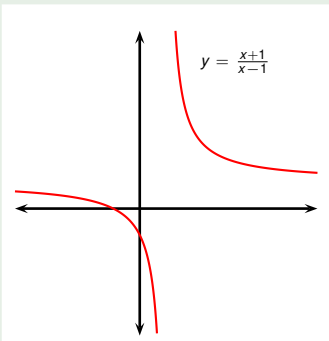


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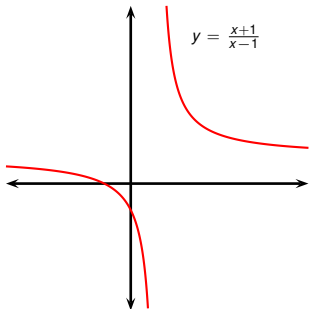
$$\begin{array}{lcl} y & = & \frac{x+1}{x-1} \\ y(x-1) & = & x+1 \end{array} \quad \left| \begin{array}{l} \text{mult. by } (x-1) \end{array} \right.$$



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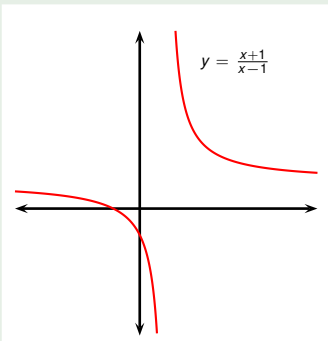


$$\begin{array}{rcl|l} y & = & \frac{x+1}{x-1} & \text{mult. by } (x-1) \\ y(x-1) & = & x+1 & \\ x(y-1) & = & y+1 & \end{array}$$

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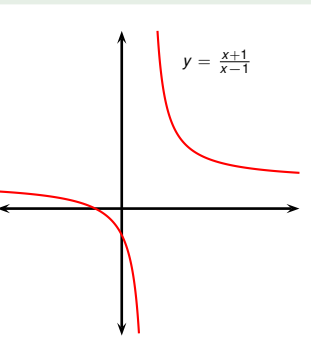


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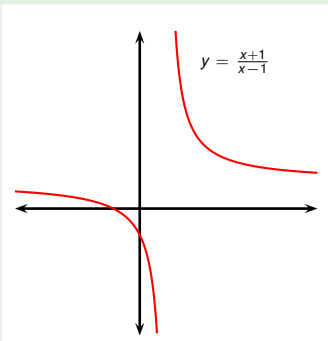


$$\begin{array}{rcl|l} y & = & \frac{x+1}{x-1} & \text{mult. by } (x-1) \\ \textcolor{red}{y}(x-1) & = & x+1 & \\ x(y-1) & = & \textcolor{red}{y}+1 & \end{array}$$

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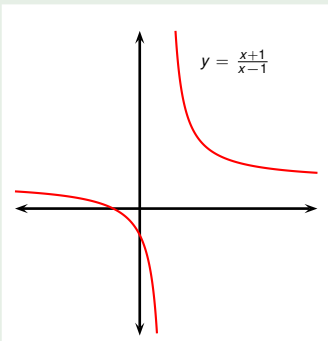


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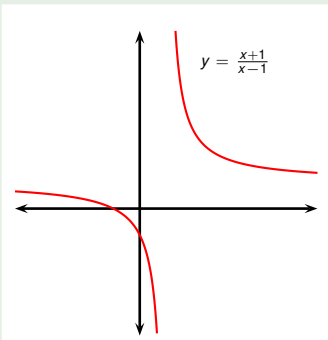


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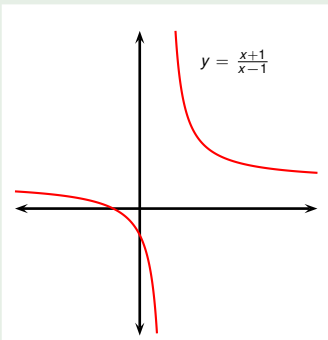


$$\begin{aligned} y &= \frac{x+1}{x-1} && \left| \begin{array}{l} \text{mult. by } (x-1) \end{array} \right. \\ y(x-1) &= x+1 \\ x(y-1) &= y+1 && \left| \begin{array}{l} \text{div. by } (y-1) \end{array} \right. \\ f^{-1}(y) = x &= \frac{y+1}{y-1} \end{aligned}$$

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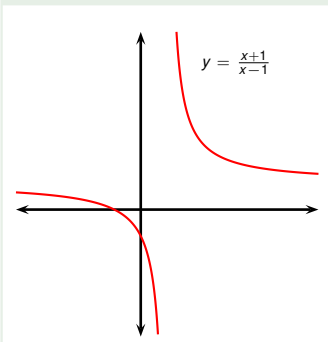


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 \end{array}$$

Example

Find $f^{-1}(x)$ where $f(x) = \frac{x+1}{x-1}$.

We deal with domains and ranges later:



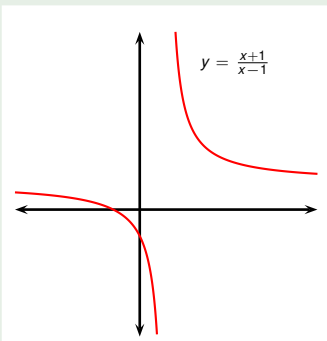
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 \end{array}$$

Answer: $f^{-1}(x) = \frac{x+1}{x-1}$

Example

Find $f^{-1}(x)$ where $f(x) = \frac{x+1}{x-1}$.

We deal with domains and ranges later:



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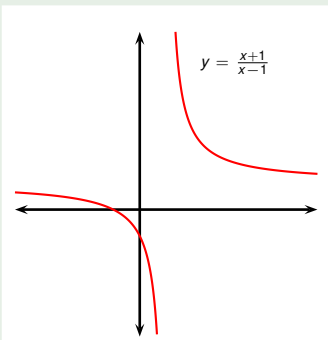
We divided by $y-1$ so $y \neq 1$.

Answer: $f^{-1}(x) = \frac{x+1}{x-1}$

Example

Find $f^{-1}(x)$ where $f(x) = \frac{x+1}{x-1}$.

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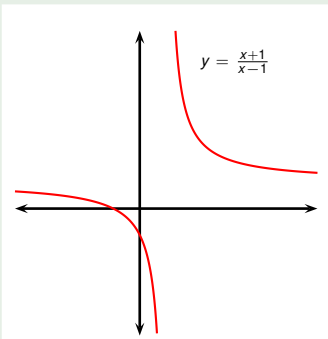
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Answer: $f^{-1}(x) = \frac{x+1}{x-1}$,
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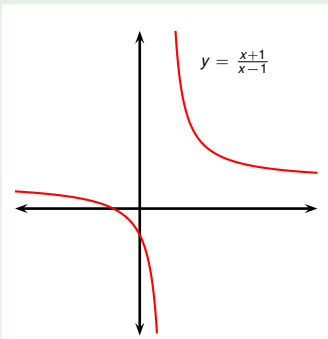
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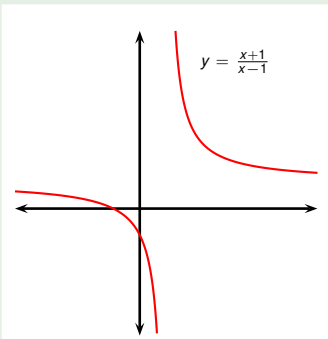
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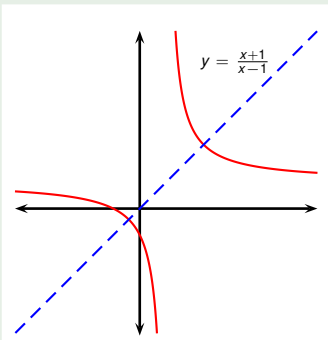
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