Calculus II
Integrals of the form $\int \sqrt{ax^2+c}\,\mathrm{d}x$, $a, c > 0$

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Euler substitution: $x = \frac{1}{2}\left(\frac{1}{t} - t\right)$, $\sqrt{x^2+1} = \frac{1}{2}\left(\frac{1}{t} + t\right)$,
$t = \sqrt{x^2+1} - x$, $\mathrm{d}x = -\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t$. Recall $t > 0$.

Example

$$\begin{aligned}
\int \sqrt{x^2+1}\,\mathrm{d}x &= -\int \frac{1}{2}\left(\frac{1}{t} + t\right)\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t \\
&= -\frac{1}{4}\int \left(\frac{1}{t^3} + 2\frac{1}{t} + t\right)\mathrm{d}t \\
&= -\frac{1}{4}\left(-\frac{t^{-2}}{2} + 2\ln|t| + \frac{t^2}{2}\right) + C \\
&= \frac{1}{2}\left(\frac{1}{2}\left(t^{-1} - t\right)\frac{1}{2}\left(t^{-1} + t\right)\right) - \frac{1}{2}\ln t + C \\
&= \frac{1}{2}x\sqrt{x^2+1} - \frac{1}{2}\ln\left(\sqrt{x^2+1} - x\right) + C \\
&= \frac{1}{2}x\sqrt{x^2+1} + \frac{1}{2}\ln\frac{\sqrt{x^2+1} + x}{\left(\sqrt{x^2+1} - x\right)\left(\sqrt{x^2+1} + x\right)} + C \\
&= \frac{1}{2}x\sqrt{x^2+1} + \frac{1}{2}\ln\left(\sqrt{x^2+1} + x\right) + C
\end{aligned}$$

Example

$$
\begin{aligned}
\int \frac{1}{x^2\sqrt{x^2+9}}\mathrm{d}x &= \int \frac{1}{x^2 3\sqrt{\left(\frac{x}{3}\right)^2+1}}\mathrm{d}x \\
&= \int \frac{1}{(3\cot\theta)^2 3\sqrt{\cot^2\theta+1}}\mathrm{d}(3\cot\theta) \\
&= \int \frac{1}{27\cot^2\theta\sqrt{\csc^2\theta}}\left(-3\csc^2\theta\right)\mathrm{d}\theta \\
&= \frac{1}{9}\int \frac{-\csc^2\theta}{\cot^2\theta\csc\theta}\mathrm{d}\theta \\
&= \frac{1}{9}\int \frac{-\sin\theta}{\cos^2\theta}\mathrm{d}\theta = \frac{1}{9}\int \frac{1}{\cos^2\theta}\mathrm{d}(\cos\theta) \\
&= \frac{1}{9}\int \frac{\mathrm{d}u}{u^2} = -\frac{1}{9u}+C = -\frac{\sec\theta}{9}+C \\
&= -\frac{\sqrt{x^2+9}}{9x}+C
\end{aligned}
$$

Set
$\frac{x}{3} = \cot\theta$
$x = 3\cot\theta$

$\theta \in (0,\pi)$

$\theta \in (0,\pi) \Rightarrow$
$\csc\theta > 0$

Set $u = \cos\theta$