

# Precalculus

## Completing the square

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## Example (Completing the square)

Complete the square.

$$3x^2 - 5x + 1$$

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$$3x^2 - 5x + 1 = 3 \left( x^2 - ? x \right) + 1$$

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$$3x^2 - 5x + 1 = 3 \left( x^2 - \frac{5}{3}x \right) + 1$$

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$$\begin{aligned} 3x^2 - 5x + 1 &= 3 \left( x^2 - \frac{5}{3}x \right) + 1 \\ &= 3 \left( x^2 - \color{red}{2} \cdot \frac{5}{\color{red}{2} \cdot 3} x \right) + 1 \end{aligned}$$

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 3x^2 - 5x + 1 &= 3 \left( x^2 - \frac{5}{3}x \right) + 1 \\
 &= 3 \left( x^2 - 2 \cdot \frac{5}{\textcolor{red}{2} \cdot \textcolor{red}{3}}x \right) + 1 \\
 &= 3 \left( x^2 - 2 \cdot \frac{5}{\textcolor{red}{6}}x + \textcolor{red}{?} \quad - \textcolor{red}{?} \right) + 1
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Let  $a \neq 0$ . To *complete the square* means to carry out the following algebraic manipulation.

$$ax^2 + bx + c$$

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 &= a \left( \left( x + \frac{b}{2a} \right)^2 - \frac{b^2}{4a^2} \right) + c
 \end{aligned}
 \left| \begin{array}{l} \text{Add \& subtract} \\ \left( \frac{b}{2a} \right)^2 \\ \text{use} \\ (A+B)^2 = \\ A^2 + 2AB + B^2 \end{array} \right.$$



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 &= a \left( \left( x + \frac{b}{2a} \right)^2 - \frac{b^2}{4a^2} \right) + c \\
 &= a \left( x + \frac{b}{2a} \right)^2 - \cancel{a} \cdot \frac{b^2}{4\cancel{a}^2} + c \\
 &= a \left( x + \frac{b}{2a} \right)^2 + c - \frac{b^2}{4a}.
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 &= a \left( \left( x + \frac{b}{2a} \right)^2 - \frac{b^2}{4a^2} \right) + c \\
 &= a \left( x + \frac{b}{2a} \right)^2 - a \cdot \frac{b^2}{4a^2} + c \\
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