

Precalculus

Angles

Todor Milev

2019

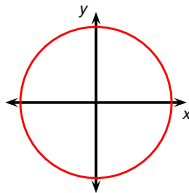
Outline

1 Angles

- The Unit circle
- Three Meanings of Angle
- Two Meanings of Rotation
- Angles and the Coordinate System
- Radians and Degrees
- Area cut off by an angle

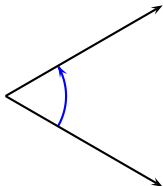
Definition

The *unit circle* is the circle with radius 1 and center at the center of the coordinate system.



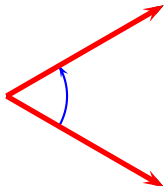
Three Meanings of the Term Angle

- The term “angle” is used to denote three distinct mathematical objects:



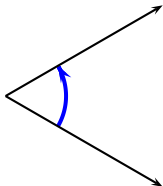
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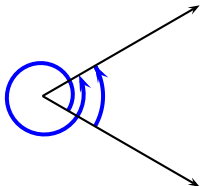
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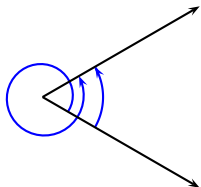
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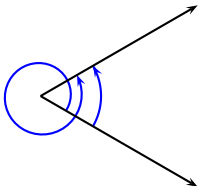
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 - the (geometric) angle formed by two rays,
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- **All three are referred to as “angle”**: use context to decide whether “angle” means “angle formed by two rays”, “angle measure” or “angle-measure of a rotation”.

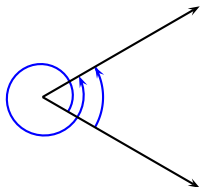


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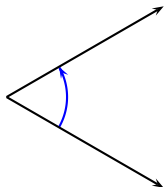


- The term “angle” is used to denote three distinct mathematical objects:
 - the (geometric) angle formed by two rays,
 - the angle-measure of such a geometric angle
 - the angle-measure of a rotation.
- All three are referred to as “angle”: use context to decide whether “angle” means “angle formed by two rays”, “angle measure” or “angle-measure of a rotation”.
- Except for a few introductory slides, we take full advantage of this convention.

Geometric angle definition

Definition (Geometric angle)

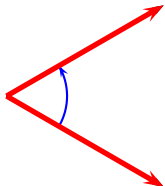
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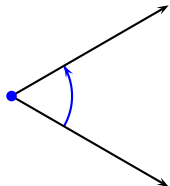
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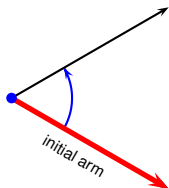


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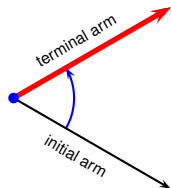
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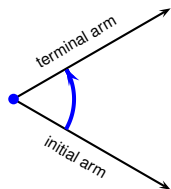


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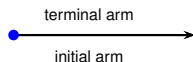


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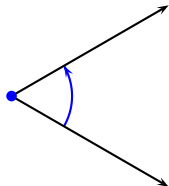
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- Angle measures are depicted as arcs pointing from the initial arm towards the terminal arm.
- By convention, the rays are allowed to coincide; the resulting angle is then called the *zero angle*.

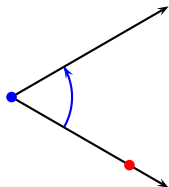
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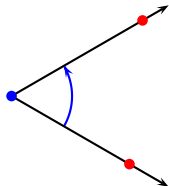
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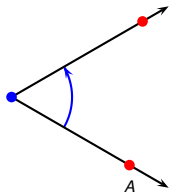


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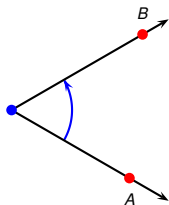
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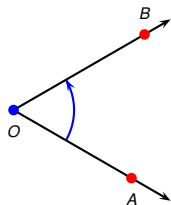
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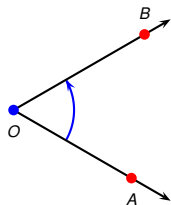


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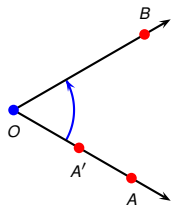
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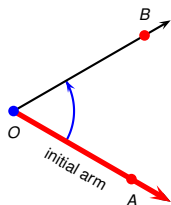
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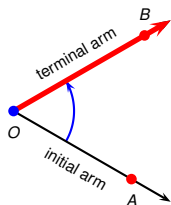
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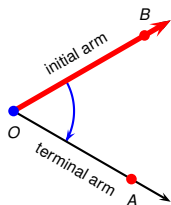
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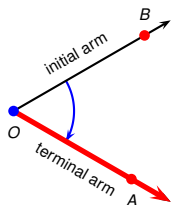
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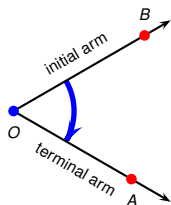
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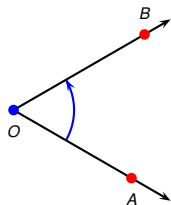
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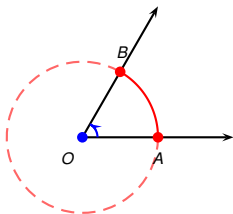


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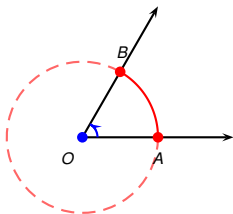


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- In this way $\angle AOB \neq \angle BOA$.



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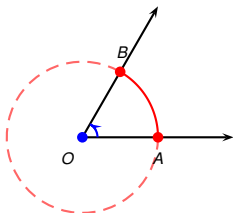
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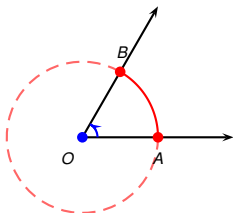
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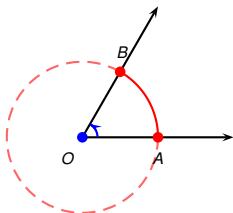
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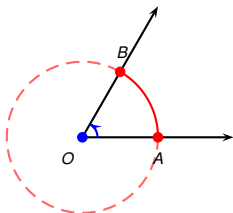
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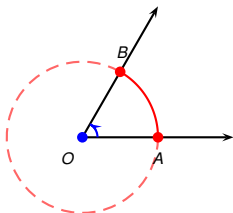
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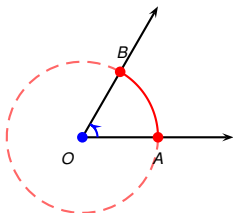
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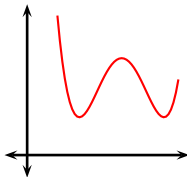
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- Angle measures are frequently denoted by greek letters such as $\alpha, \beta, \gamma, \theta, \dots$

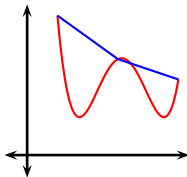
Arc-length of a circle arc

- There is a definition of arc-length of arbitrary smooth curve.



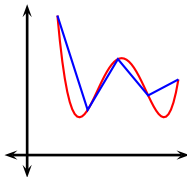
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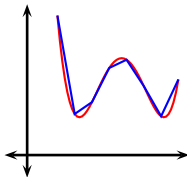
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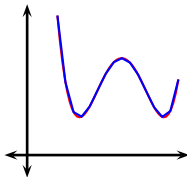
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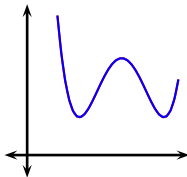
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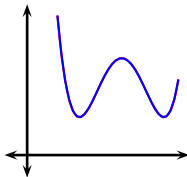
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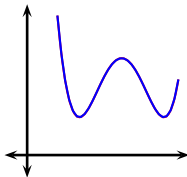
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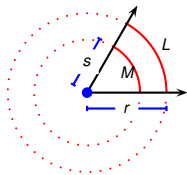


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- Until then we ask the reader to think of arc-length of a curve as the quantity obtained by “aligning a rope along the curve” and measuring the “length of this rope”.



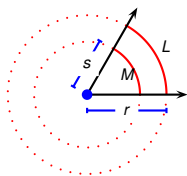
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Proposition

Let two circles have common center and radii s and r . Suppose an arbitrary geometric angle with vertex at the common center of the circles cuts off short arcs of length M and L . Then $\frac{s}{r} = \frac{M}{L}$.

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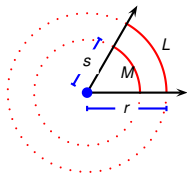


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Arc-length of a circle arc



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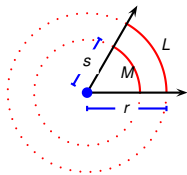
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Choose $s = 1$, relabel $M = \alpha$

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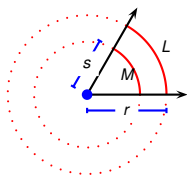
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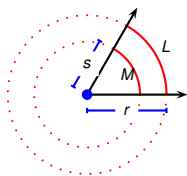
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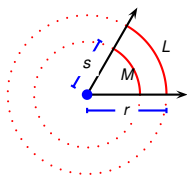
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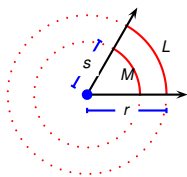
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The angle-measure of a geometric angle is the arc-length cut off from a radius 1 circle, therefore we get the following.

Arc-length of a circle arc



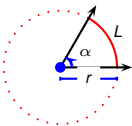
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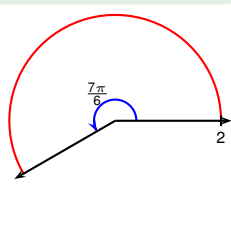
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Corollary

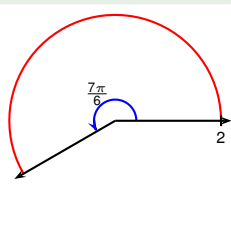
The arc-length cut off by an angle with measure α from a circle of radius r equals αr .

Example



Find the length of an arc of a circle of radius 2 cut off by an angle of measure $\frac{7\pi}{6}$ ($= 210^\circ$).

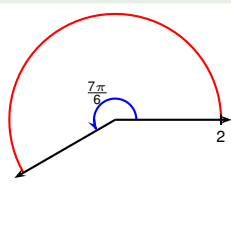
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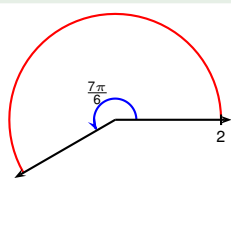
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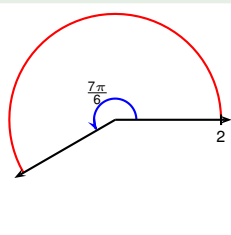
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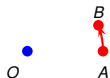
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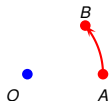
- The term rotation refers to two distinct objects:



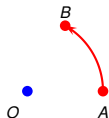
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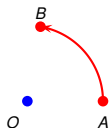
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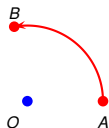
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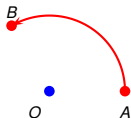
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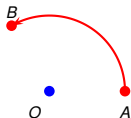
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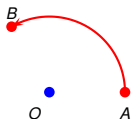
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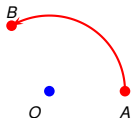
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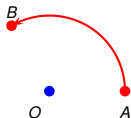
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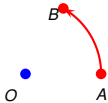


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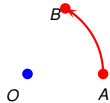
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- Whether the term rotation refers to continuous rotation or to “instantaneous” rotation should be inferred from context.





Definition (Continuous rotation)

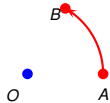
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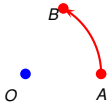
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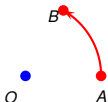
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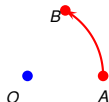
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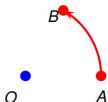
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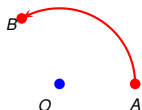
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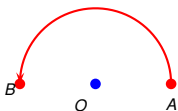
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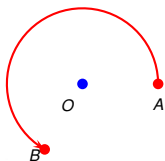
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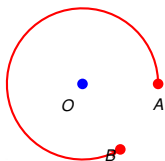
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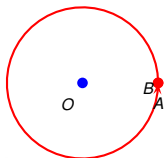
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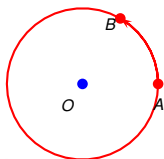
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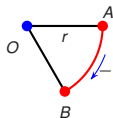
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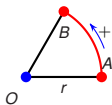
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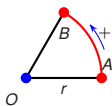
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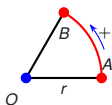
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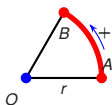
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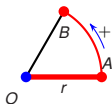
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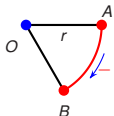


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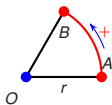
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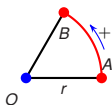
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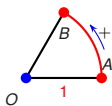
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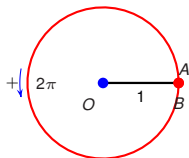
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- The radian measure of rotation equals the signed arc-length traveled by point at **distance 1 from the center**.

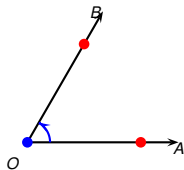


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Definition (Radian measure of proper continuous rotation)

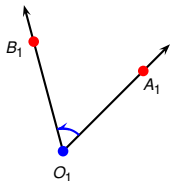
- The radian measure of rotation is a number whose magnitude equals the length of the arc traversed by a point divided by the distance of that point from the center of rotation.
- The sign of the radian measure is taken to be negative if the rotation is clockwise, else it is taken to be positive.
- The radian measure (radians for short) does not change when we change the point whose path length we are measuring.
- The radian measure of rotation equals the signed arc-length traveled by point at distance 1 from the center.
- A circle of radius 1 has circumference 2π , therefore a full counter-clockwise turn is measured by 2π radians.

Equivalence of angles

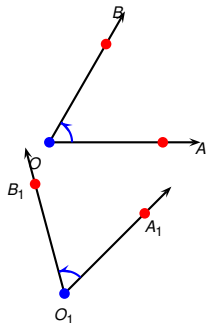


Definition (Congruent angles)

Two geometric angles are congruent (equivalent) if they can be transformed onto each other with a sequence of translations and rotations.



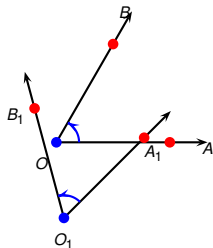
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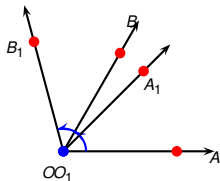
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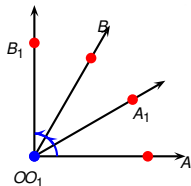
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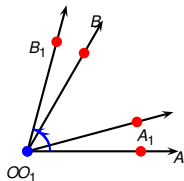
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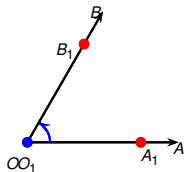
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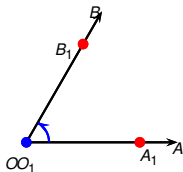
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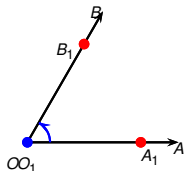
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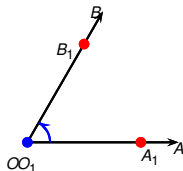
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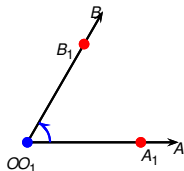
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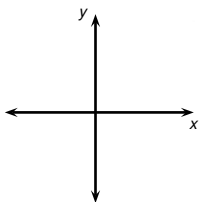
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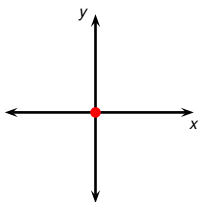
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Angles and the coordinate system



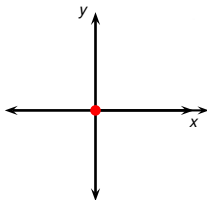
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Angles and the coordinate system



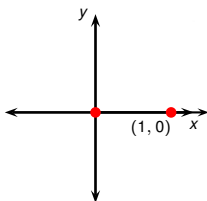
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Angles and the coordinate system



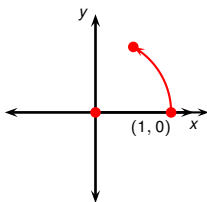
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Angles and the coordinate system



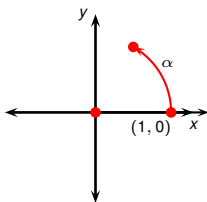
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Angles and the coordinate system



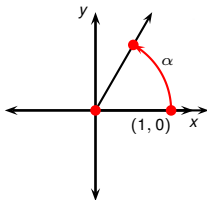
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Angles and the coordinate system



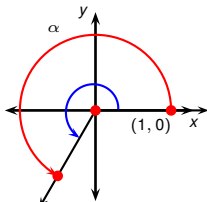
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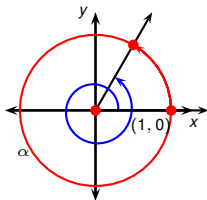
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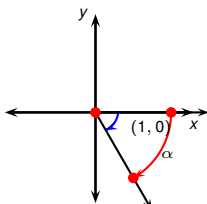
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Angles and the coordinate system



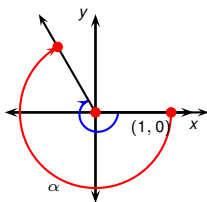
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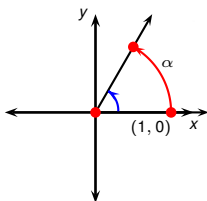
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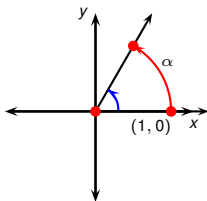
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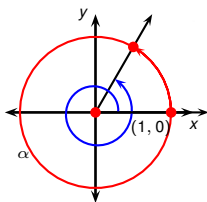
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$$t^\circ = \frac{t}{180}\pi \text{ (radians).}$$

Example

Convert from degrees to radians.

Deg.	45°	36°	-20°	360°	-720°	-225°	2015°
Rad.							

$$x = \frac{x}{\pi} 180^\circ.$$

Example

Convert from radians to degrees.

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Example

Convert from degrees to radians.

Deg.	45°	36°	-20°	360°	-720°	-225°	2015°
Rad.	$\frac{\pi}{4}$	$\frac{\pi}{5}$	$-\frac{\pi}{9}$	2π	-4π	$-\frac{5\pi}{4}$	

$$x = \frac{x}{\pi} 180^\circ.$$

Example

Convert from radians to degrees.

Rad.	$\frac{\pi}{3}$	$\frac{\pi}{10}$	$\frac{11\pi}{6}$	$\frac{7\pi}{4}$	$\frac{\pi}{7}$	$\frac{13\pi}{6}$	$-\frac{5\pi}{4}$	2
Deg.								

$$t^\circ = \frac{t}{180}\pi \text{ (radians).}$$

Example

Convert from degrees to radians.

Deg.	45°	36°	-20°	360°	-720°	-225°	2015°
Rad.	$\frac{\pi}{4}$	$\frac{\pi}{5}$	$-\frac{\pi}{9}$	2π	-4π	$-\frac{5\pi}{4}$?

$$x = \frac{x}{\pi} 180^\circ.$$

Example

Convert from radians to degrees.

Rad.	$\frac{\pi}{3}$	$\frac{\pi}{10}$	$\frac{11\pi}{6}$	$\frac{7\pi}{4}$	$\frac{\pi}{7}$	$\frac{13\pi}{6}$	$-\frac{5\pi}{4}$	2
Deg.								

$$t^\circ = \frac{t}{180}\pi \text{ (radians).}$$

Example

Convert from degrees to radians.

Deg.	45°	36°	-20°	360°	-720°	-225°	2015°
Rad.	$\frac{\pi}{4}$	$\frac{\pi}{5}$	$-\frac{\pi}{9}$	2π	-4π	$-\frac{5\pi}{4}$	$\frac{403}{36}\pi$

$$x = \frac{x}{\pi} 180^\circ.$$

Example

Convert from radians to degrees.

Rad.	$\frac{\pi}{3}$	$\frac{\pi}{10}$	$\frac{11\pi}{6}$	$\frac{7\pi}{4}$	$\frac{\pi}{7}$	$\frac{13\pi}{6}$	$-\frac{5\pi}{4}$	2
Deg.								

$$t^\circ = \frac{t}{180}\pi \text{ (radians).}$$

Example

Convert from degrees to radians.

Deg.	45°	36°	-20°	360°	-720°	-225°	2015°
Rad.	$\frac{\pi}{4}$	$\frac{\pi}{5}$	$-\frac{\pi}{9}$	2π	-4π	$-\frac{5\pi}{4}$	$\frac{403}{36}\pi$

$$x = \frac{x}{\pi} 180^\circ.$$

Example

Convert from radians to degrees.

Rad.	$\frac{\pi}{3}$	$\frac{\pi}{10}$	$\frac{11\pi}{6}$	$\frac{7\pi}{4}$	$\frac{\pi}{7}$	$\frac{13\pi}{6}$	$-\frac{5\pi}{4}$	2
Deg.	?							

$$t^{\circ} = \frac{t}{180}\pi \text{ (radians).}$$

Example

Convert from degrees to radians.

Deg.	45°	36°	-20°	360°	-720°	-225°	2015°
Rad.	$\frac{\pi}{4}$	$\frac{\pi}{5}$	$-\frac{\pi}{9}$	2π	-4π	$-\frac{5\pi}{4}$	$\frac{403}{36}\pi$

$$x = \frac{x}{\pi}180^{\circ}.$$

Example

Convert from radians to degrees.

Rad.	$\frac{\pi}{3}$	$\frac{\pi}{10}$	$\frac{11\pi}{6}$	$\frac{7\pi}{4}$	$\frac{\pi}{7}$	$\frac{13\pi}{6}$	$-\frac{5\pi}{4}$	2
Deg.	60°							

$$t^\circ = \frac{t}{180}\pi \text{ (radians).}$$

Example

Convert from degrees to radians.

Deg.	45°	36°	-20°	360°	-720°	-225°	2015°
Rad.	$\frac{\pi}{4}$	$\frac{\pi}{5}$	$-\frac{\pi}{9}$	2π	-4π	$-\frac{5\pi}{4}$	$\frac{403}{36}\pi$

$$x = \frac{x}{\pi} 180^\circ.$$

Example

Convert from radians to degrees.

Rad.	$\frac{\pi}{3}$	$\frac{\pi}{10}$	$\frac{11\pi}{6}$	$\frac{7\pi}{4}$	$\frac{\pi}{7}$	$\frac{13\pi}{6}$	$-\frac{5\pi}{4}$	2
Deg.	60°	?						

$$t^{\circ} = \frac{t}{180}\pi \text{ (radians).}$$

Example

Convert from degrees to radians.

Deg.	45°	36°	-20°	360°	-720°	-225°	2015°
Rad.	$\frac{\pi}{4}$	$\frac{\pi}{5}$	$-\frac{\pi}{9}$	2π	-4π	$-\frac{5\pi}{4}$	$\frac{403}{36}\pi$

$$x = \frac{x}{\pi} 180^{\circ}.$$

Example

Convert from radians to degrees.

Rad.	$\frac{\pi}{3}$	$\frac{\pi}{10}$	$\frac{11\pi}{6}$	$\frac{7\pi}{4}$	$\frac{\pi}{7}$	$\frac{13\pi}{6}$	$-\frac{5\pi}{4}$	2
Deg.	60°	18°						

$$t^\circ = \frac{t}{180}\pi \text{ (radians).}$$

Example

Convert from degrees to radians.

Deg.	45°	36°	-20°	360°	-720°	-225°	2015°
Rad.	$\frac{\pi}{4}$	$\frac{\pi}{5}$	$-\frac{\pi}{9}$	2π	-4π	$-\frac{5\pi}{4}$	$\frac{403}{36}\pi$

$$x = \frac{x}{\pi} 180^\circ.$$

Example

Convert from radians to degrees.

Rad.	$\frac{\pi}{3}$	$\frac{\pi}{10}$	$\frac{11\pi}{6}$	$\frac{7\pi}{4}$	$\frac{\pi}{7}$	$\frac{13\pi}{6}$	$-\frac{5\pi}{4}$	2
Deg.	60°	18°	?					

$$t^{\circ} = \frac{t}{180}\pi \text{ (radians).}$$

Example

Convert from degrees to radians.

Deg.	45°	36°	-20°	360°	-720°	-225°	2015°
Rad.	$\frac{\pi}{4}$	$\frac{\pi}{5}$	$-\frac{\pi}{9}$	2π	-4π	$-\frac{5\pi}{4}$	$\frac{403}{36}\pi$

$$x = \frac{x}{\pi}180^{\circ}.$$

Example

Convert from radians to degrees.

Rad.	$\frac{\pi}{3}$	$\frac{\pi}{10}$	$\frac{11\pi}{6}$	$\frac{7\pi}{4}$	$\frac{\pi}{7}$	$\frac{13\pi}{6}$	$-\frac{5\pi}{4}$	2
Deg.	60°	18°	330°					

$$t^{\circ} = \frac{t}{180}\pi \text{ (radians).}$$

Example

Convert from degrees to radians.

Deg.	45°	36°	-20°	360°	-720°	-225°	2015°
Rad.	$\frac{\pi}{4}$	$\frac{\pi}{5}$	$-\frac{\pi}{9}$	2π	-4π	$-\frac{5\pi}{4}$	$\frac{403}{36}\pi$

$$x = \frac{x}{\pi}180^{\circ}.$$

Example

Convert from radians to degrees.

Rad.	$\frac{\pi}{3}$	$\frac{\pi}{10}$	$\frac{11\pi}{6}$	$\frac{7\pi}{4}$	$\frac{\pi}{7}$	$\frac{13\pi}{6}$	$-\frac{5\pi}{4}$	2
Deg.	60°	18°	330°	?				

$$t^{\circ} = \frac{t}{180}\pi \text{ (radians).}$$

Example

Convert from degrees to radians.

Deg.	45°	36°	-20°	360°	-720°	-225°	2015°
Rad.	$\frac{\pi}{4}$	$\frac{\pi}{5}$	$-\frac{\pi}{9}$	2π	-4π	$-\frac{5\pi}{4}$	$\frac{403}{36}\pi$

$$x = \frac{x}{\pi}180^{\circ}.$$

Example

Convert from radians to degrees.

Rad.	$\frac{\pi}{3}$	$\frac{\pi}{10}$	$\frac{11\pi}{6}$	$\frac{7\pi}{4}$	$\frac{\pi}{7}$	$\frac{13\pi}{6}$	$-\frac{5\pi}{4}$	2
Deg.	60°	18°	330°	315°				

$$t^{\circ} = \frac{t}{180}\pi \text{ (radians).}$$

Example

Convert from degrees to radians.

Deg.	45°	36°	-20°	360°	-720°	-225°	2015°
Rad.	$\frac{\pi}{4}$	$\frac{\pi}{5}$	$-\frac{\pi}{9}$	2π	-4π	$-\frac{5\pi}{4}$	$\frac{403}{36}\pi$

$$x = \frac{x}{\pi}180^{\circ}.$$

Example

Convert from radians to degrees.

Rad.	$\frac{\pi}{3}$	$\frac{\pi}{10}$	$\frac{11\pi}{6}$	$\frac{7\pi}{4}$	$\frac{\pi}{7}$	$\frac{13\pi}{6}$	$-\frac{5\pi}{4}$	2
Deg.	60°	18°	330°	315°	?			

$$t^{\circ} = \frac{t}{180}\pi \text{ (radians).}$$

Example

Convert from degrees to radians.

Deg.	45°	36°	-20°	360°	-720°	-225°	2015°
Rad.	$\frac{\pi}{4}$	$\frac{\pi}{5}$	$-\frac{\pi}{9}$	2π	-4π	$-\frac{5\pi}{4}$	$\frac{403}{36}\pi$

$$x = \frac{x}{\pi}180^{\circ}.$$

Example

Convert from radians to degrees.

Rad.	$\frac{\pi}{3}$	$\frac{\pi}{10}$	$\frac{11\pi}{6}$	$\frac{7\pi}{4}$	$\frac{\pi}{7}$	$\frac{13\pi}{6}$	$-\frac{5\pi}{4}$	2
Deg.	60°	18°	330°	315°	$\frac{180^{\circ}}{7} \approx 25.7^{\circ}$			

$$t^{\circ} = \frac{t}{180} \pi \text{ (radians).}$$

Example

Convert from degrees to radians.

Deg.	45°	36°	-20°	360°	-720°	-225°	2015°
Rad.	$\frac{\pi}{4}$	$\frac{\pi}{5}$	$-\frac{\pi}{9}$	2π	-4π	$-\frac{5\pi}{4}$	$\frac{403}{36}\pi$

$$x = \frac{x}{\pi} 180^{\circ}.$$

Example

Convert from radians to degrees.

Rad.	$\frac{\pi}{3}$	$\frac{\pi}{10}$	$\frac{11\pi}{6}$	$\frac{7\pi}{4}$	$\frac{\pi}{7}$	$\frac{13\pi}{6}$	$-\frac{5\pi}{4}$	2
Deg.	60°	18°	330°	315°	$\frac{180^{\circ}}{7} \approx 25.7^{\circ}$?		

$$t^\circ = \frac{t}{180}\pi \text{ (radians).}$$

Example

Convert from degrees to radians.

Deg.	45°	36°	-20°	360°	-720°	-225°	2015°
Rad.	$\frac{\pi}{4}$	$\frac{\pi}{5}$	$-\frac{\pi}{9}$	2π	-4π	$-\frac{5\pi}{4}$	$\frac{403}{36}\pi$

$$x = \frac{x}{\pi} 180^\circ.$$

Example

Convert from radians to degrees.

Rad.	$\frac{\pi}{3}$	$\frac{\pi}{10}$	$\frac{11\pi}{6}$	$\frac{7\pi}{4}$	$\frac{\pi}{7}$	$\frac{13\pi}{6}$	$-\frac{5\pi}{4}$	2
Deg.	60°	18°	330°	315°	$\frac{180^\circ}{7} \approx 25.7^\circ$	390°		

$$t^\circ = \frac{t}{180}\pi \text{ (radians).}$$

Example

Convert from degrees to radians.

Deg.	45°	36°	-20°	360°	-720°	-225°	2015°
Rad.	$\frac{\pi}{4}$	$\frac{\pi}{5}$	$-\frac{\pi}{9}$	2π	-4π	$-\frac{5\pi}{4}$	$\frac{403}{36}\pi$

$$x = \frac{x}{\pi} 180^\circ.$$

Example

Convert from radians to degrees.

Rad.	$\frac{\pi}{3}$	$\frac{\pi}{10}$	$\frac{11\pi}{6}$	$\frac{7\pi}{4}$	$\frac{\pi}{7}$	$\frac{13\pi}{6}$	$-\frac{5\pi}{4}$	2
Deg.	60°	18°	330°	315°	$\frac{180^\circ}{7} \approx 25.7^\circ$	390°	?	

$$t^\circ = \frac{t}{180}\pi \text{ (radians).}$$

Example

Convert from degrees to radians.

Deg.	45°	36°	-20°	360°	-720°	-225°	2015°
Rad.	$\frac{\pi}{4}$	$\frac{\pi}{5}$	$-\frac{\pi}{9}$	2π	-4π	$-\frac{5\pi}{4}$	$\frac{403}{36}\pi$

$$x = \frac{x}{\pi} 180^\circ.$$

Example

Convert from radians to degrees.

Rad.	$\frac{\pi}{3}$	$\frac{\pi}{10}$	$\frac{11\pi}{6}$	$\frac{7\pi}{4}$	$\frac{\pi}{7}$	$\frac{13\pi}{6}$	$-\frac{5\pi}{4}$	2
Deg.	60°	18°	330°	315°	$\frac{180^\circ}{7} \approx 25.7^\circ$	390°	-225°	

$$t^{\circ} = \frac{t}{180} \pi \text{ (radians).}$$

Example

Convert from degrees to radians.

Deg.	45°	36°	-20°	360°	-720°	-225°	2015°
Rad.	$\frac{\pi}{4}$	$\frac{\pi}{5}$	$-\frac{\pi}{9}$	2π	-4π	$-\frac{5\pi}{4}$	$\frac{403}{36}\pi$

$$x = \frac{x}{\pi} 180^{\circ}.$$

Example

Convert from radians to degrees.

Rad.	$\frac{\pi}{3}$	$\frac{\pi}{10}$	$\frac{11\pi}{6}$	$\frac{7\pi}{4}$	$\frac{\pi}{7}$	$\frac{13\pi}{6}$	$-\frac{5\pi}{4}$	2
Deg.	60°	18°	330°	315°	$\frac{180^{\circ}}{7} \approx 25.7^{\circ}$	390°	-225°	?

$$t^{\circ} = \frac{t}{180}\pi \text{ (radians).}$$

Example

Convert from degrees to radians.

Deg.	45°	36°	-20°	360°	-720°	-225°	2015°
Rad.	$\frac{\pi}{4}$	$\frac{\pi}{5}$	$-\frac{\pi}{9}$	2π	-4π	$-\frac{5\pi}{4}$	$\frac{403}{36}\pi$

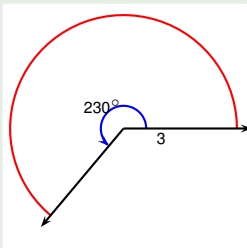
$$x = \frac{x}{\pi}180^{\circ}.$$

Example

Convert from radians to degrees.

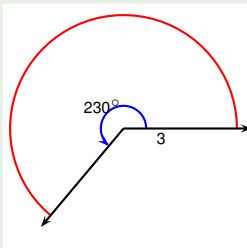
Rad.	$\frac{\pi}{3}$	$\frac{\pi}{10}$	$\frac{11\pi}{6}$	$\frac{7\pi}{4}$	$\frac{\pi}{7}$	$\frac{13\pi}{6}$	$-\frac{5\pi}{4}$	2
Deg.	60°	18°	330°	315°	$\frac{180^{\circ}}{7} \approx 25.7^{\circ}$	390°	-225°	$\frac{2}{\pi}180^{\circ} \approx 114.6^{\circ}$

Example



Find the length of an arc of a circle of radius 3 cut off by an angle of measure 230° .

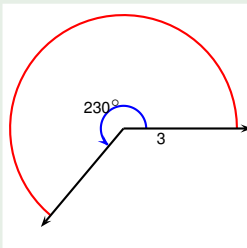
Example



Find the length of an arc of a circle of radius 3 cut off by an angle of measure 230° .

$$\text{arc-length} = ar$$

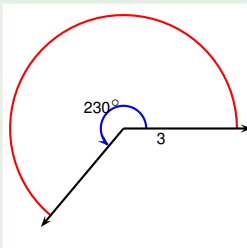
Example



Find the length of an arc of a circle of **radius 3** cut off by an angle of measure 230° .

$$\text{arc-length} = \alpha r = ? \cdot 3$$

Example

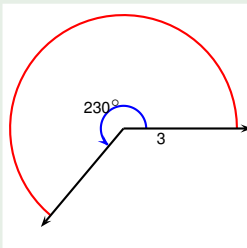


Find the length of an arc of a circle of radius 3 cut off by an **angle of measure 230°** .

$$\alpha = 230^\circ$$

$$\text{arc-length} = \alpha r = ? \cdot 3$$

Example



Find the length of an arc of a circle of radius 3 cut off by an angle of measure 230° .

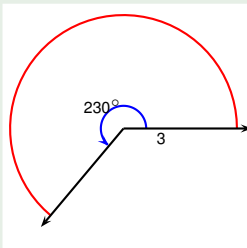
$$\alpha = 230^\circ$$

$$= ?$$

Convert to radians

$$\text{arc-length} = \alpha r = ? \cdot 3$$

Example



Find the length of an arc of a circle of radius 3 cut off by an angle of measure 230° .

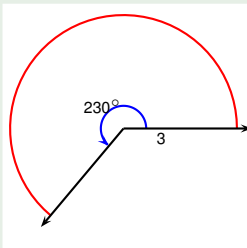
$$\alpha = 230^\circ$$

$$= 230^\circ \frac{\pi \text{ rad}}{180^\circ}$$

Convert to radians

$$\text{arc-length} = \alpha r = ? \cdot 3$$

Example



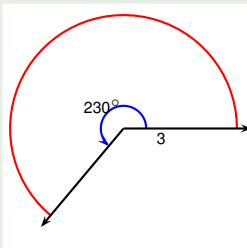
Find the length of an arc of a circle of radius 3 cut off by an angle of measure 230° .

$$\begin{aligned}\alpha &= 230^\circ \\ &= 230^\circ \frac{\pi \text{ rad}}{180^\circ} = \frac{23}{18} \pi \text{ rad}\end{aligned}$$

Convert to radians

$$\text{arc-length} = \alpha r = ? \cdot 3$$

Example



Find the length of an arc of a circle of radius 3 cut off by an angle of measure 230° .

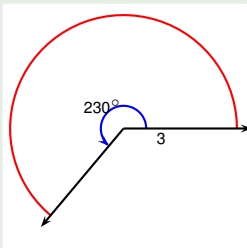
$$\alpha = 230^\circ$$

$$= 230^\circ \frac{\pi \text{ rad}}{180^\circ} = \frac{23}{18} \pi \text{ rad}$$

Convert to radians

$$\text{arc-length} = \alpha r = \frac{23\pi}{18} \cdot 3$$

Example



Find the length of an arc of a circle of radius 3 cut off by an angle of measure 230° .

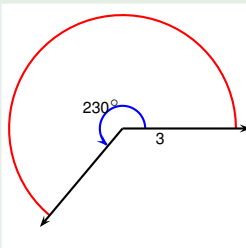
$$\alpha = 230^\circ$$

$$= 230^\circ \frac{\pi \text{ rad}}{180^\circ} = \frac{23}{18} \pi \text{ rad}$$

$$\text{arc-length} = \alpha r = \frac{23\pi}{18} \cdot 3 = \frac{23\pi}{6}$$

Convert to radians

Example



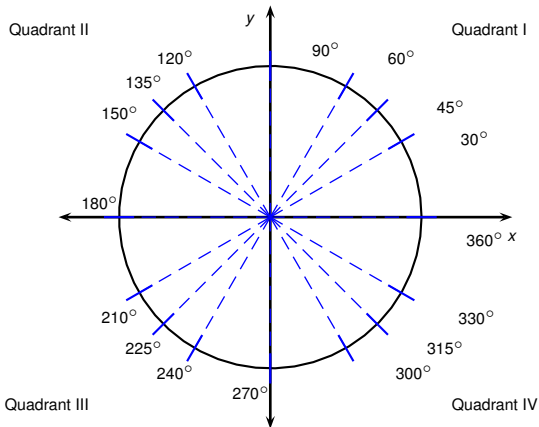
Find the length of an arc of a circle of radius 3 cut off by an angle of measure 230° .

$$\alpha = 230^\circ$$

$$= 230^\circ \frac{\pi \text{ rad}}{180^\circ} = \frac{23}{18} \pi \text{ rad}$$

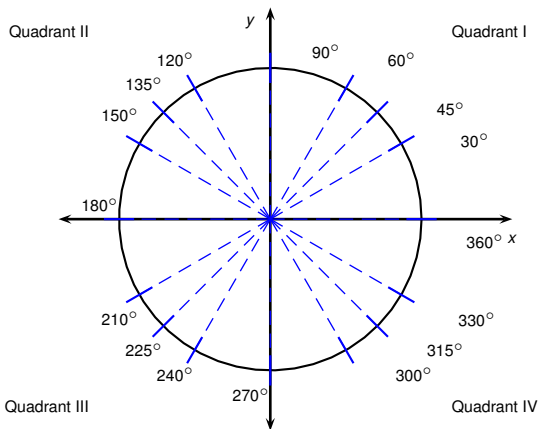
Convert to radians

$$\text{arc-length} = \alpha r = \frac{23\pi}{18} \cdot 3 = \frac{23\pi}{6} \approx 12.043$$



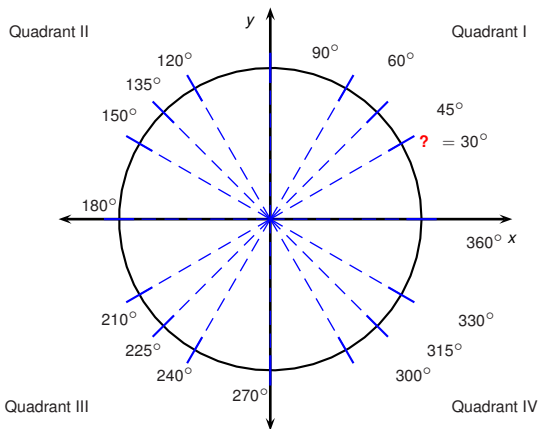
The most frequently encountered angles are given in the table below.

Deg.	0°	30°	45°	60°	90°	120°	135°	150°	180°	270°	360°
Rad.	?										



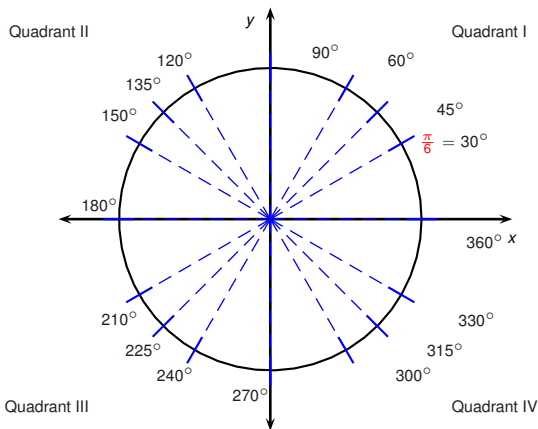
The most frequently encountered angles are given in the table below.

Deg.	0°	30°	45°	60°	90°	120°	135°	150°	180°	270°	360°
Rad.	0										



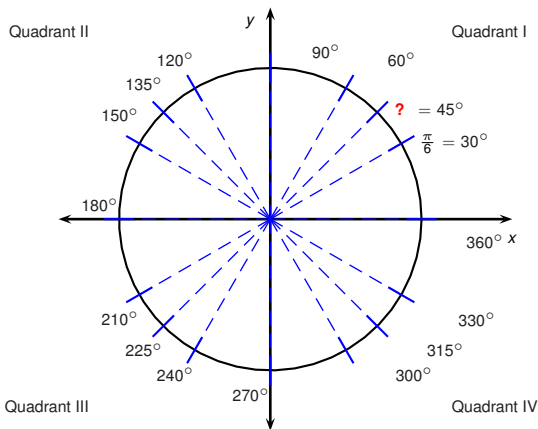
The most frequently encountered angles are given in the table below.

Deg.	0°	30°	45°	60°	90°	120°	135°	150°	180°	270°	360°
Rad.	0	?									



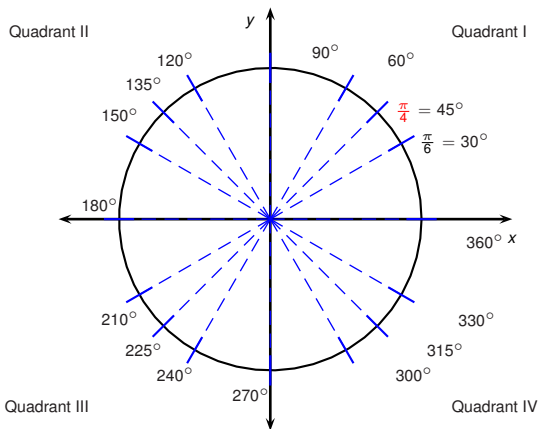
The most frequently encountered angles are given in the table below.

Deg.	0°	30°	45°	60°	90°	120°	135°	150°	180°	270°	360°
Rad.	0	$\frac{\pi}{6}$									



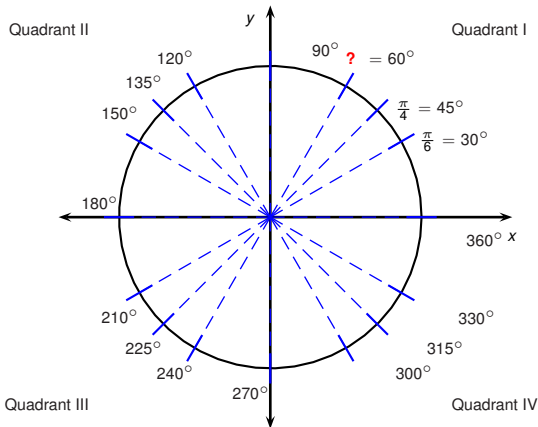
The most frequently encountered angles are given in the table below.

Deg.	0°	30°	45°	60°	90°	120°	135°	150°	180°	270°	360°
Rad.	0	$\frac{\pi}{6}$?								



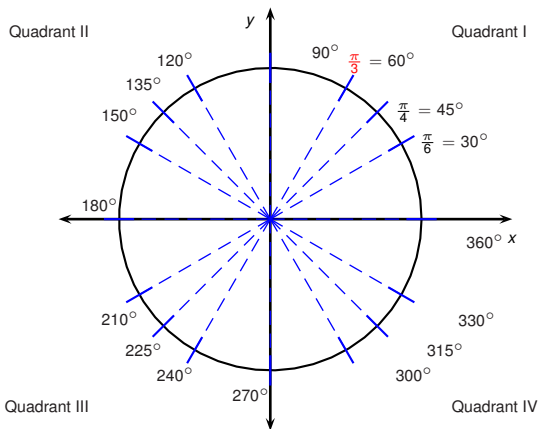
The most frequently encountered angles are given in the table below.

Deg.	0°	30°	45°	60°	90°	120°	135°	150°	180°	270°	360°
Rad.	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$								



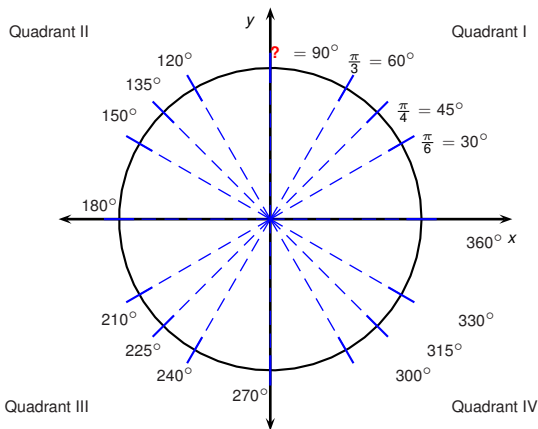
The most frequently encountered angles are given in the table below.

Deg.	0°	30°	45°	60°	90°	120°	135°	150°	180°	270°	360°
Rad.	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$?							



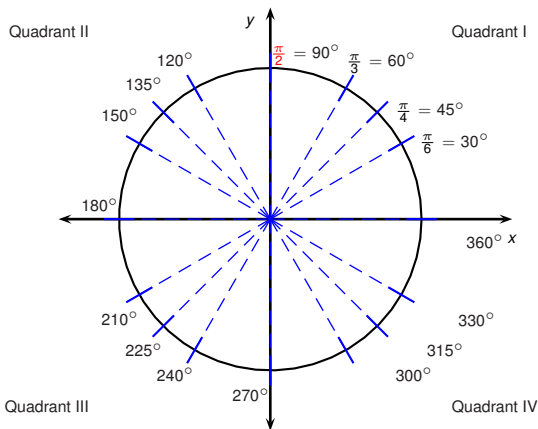
The most frequently encountered angles are given in the table below.

Deg.	0°	30°	45°	60°	90°	120°	135°	150°	180°	270°	360°
Rad.	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$							



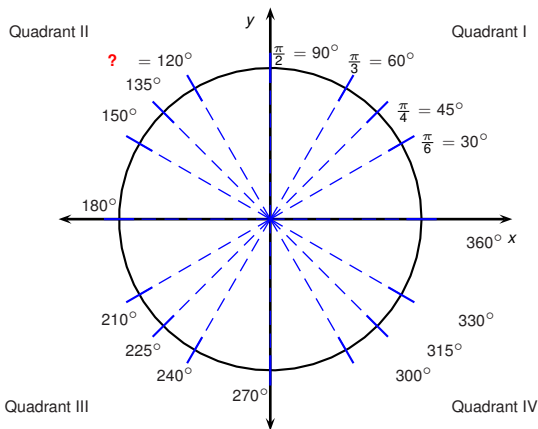
The most frequently encountered angles are given in the table below.

Deg.	0°	30°	45°	60°	90°	120°	135°	150°	180°	270°	360°
Rad.	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$?						



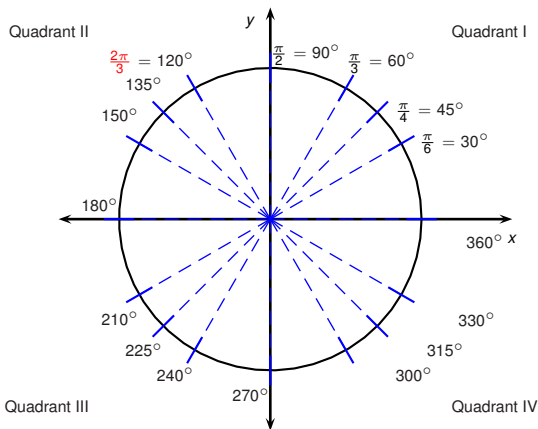
The most frequently encountered angles are given in the table below.

Deg.	0°	30°	45°	60°	90°	120°	135°	150°	180°	270°	360°
Rad.	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$						



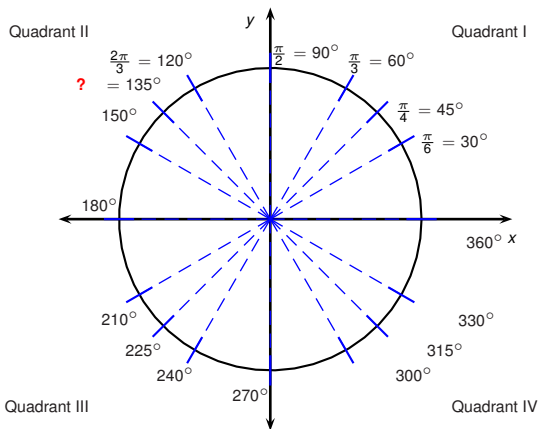
The most frequently encountered angles are given in the table below.

Deg.	0°	30°	45°	60°	90°	120°	135°	150°	180°	270°	360°
Rad.	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$?					



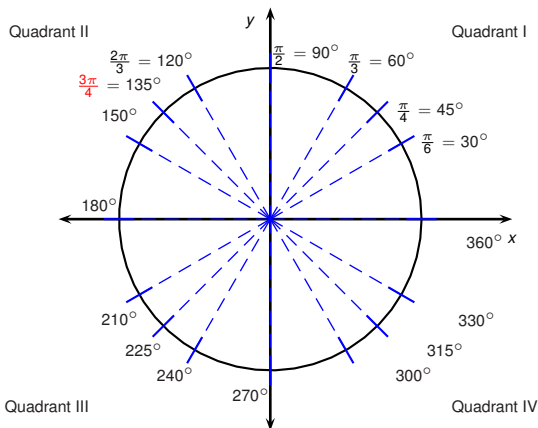
The most frequently encountered angles are given in the table below.

Deg.	0°	30°	45°	60°	90°	120°	135°	150°	180°	270°	360°
Rad.	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$					



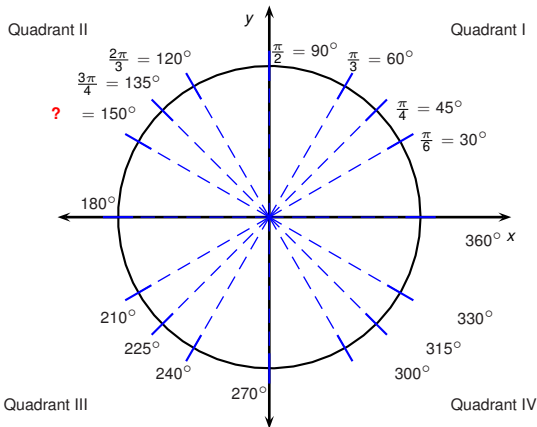
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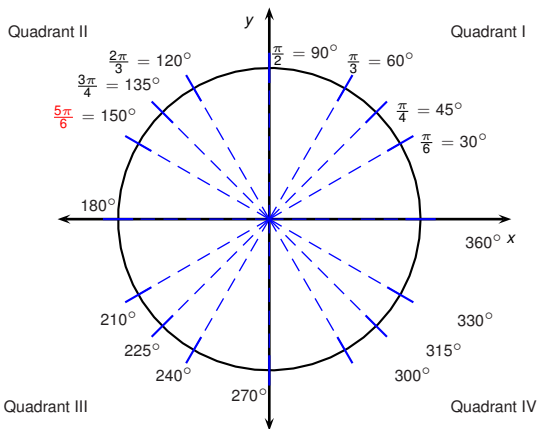
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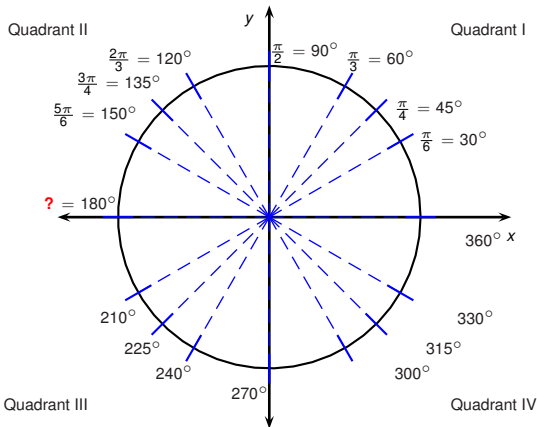
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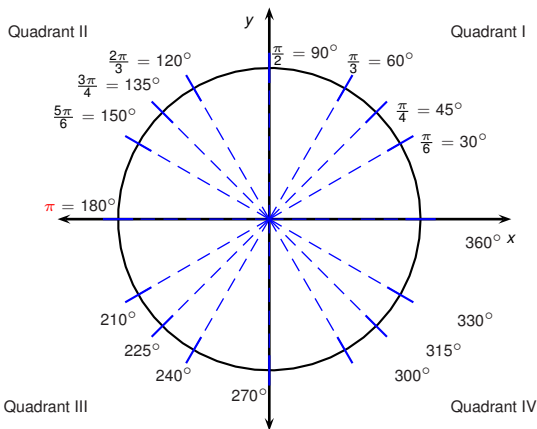
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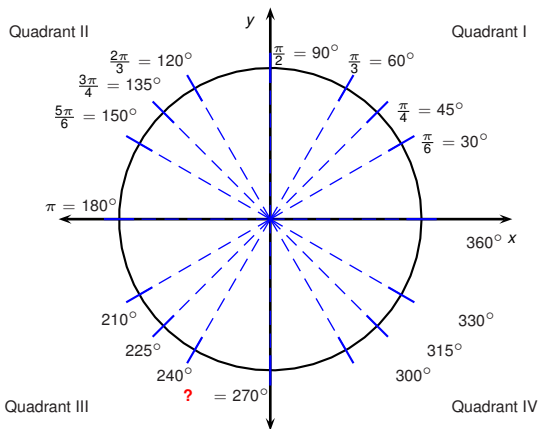
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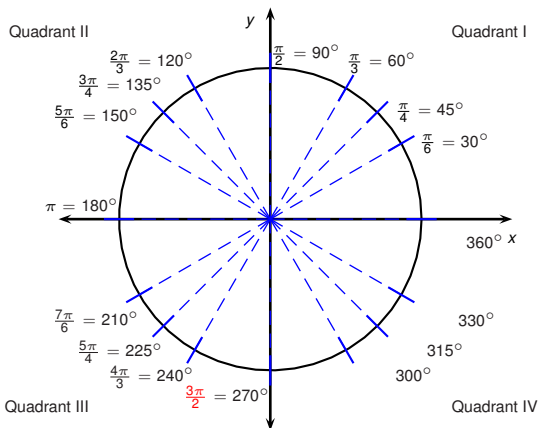
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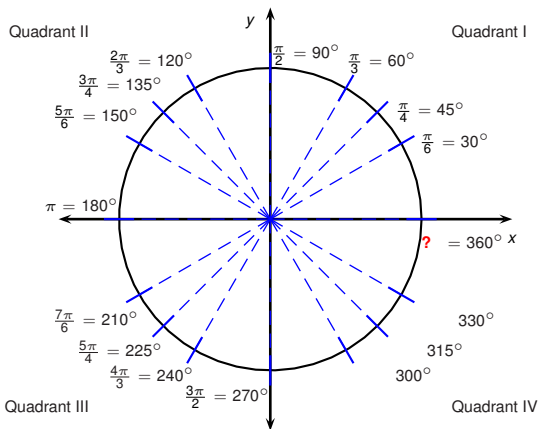
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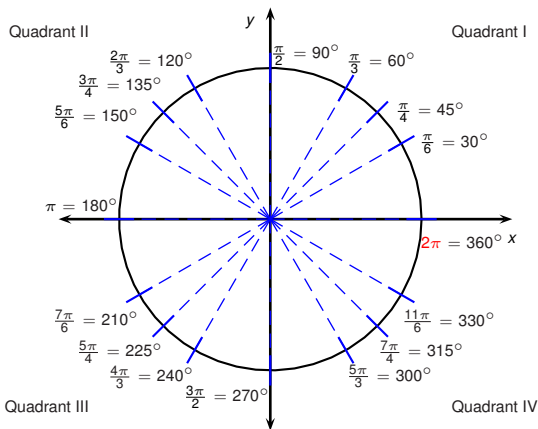
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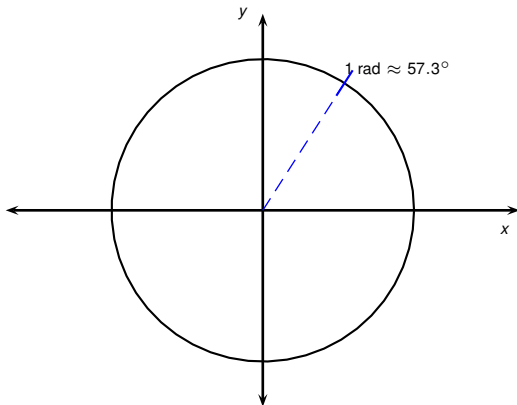
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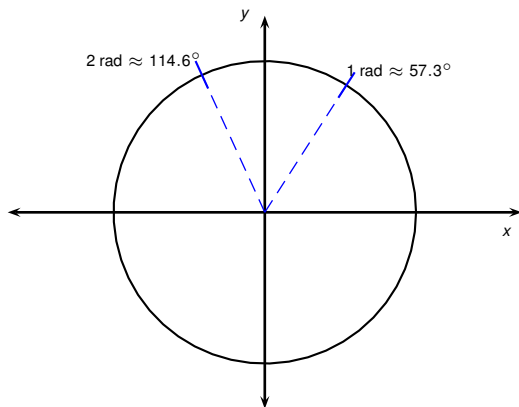


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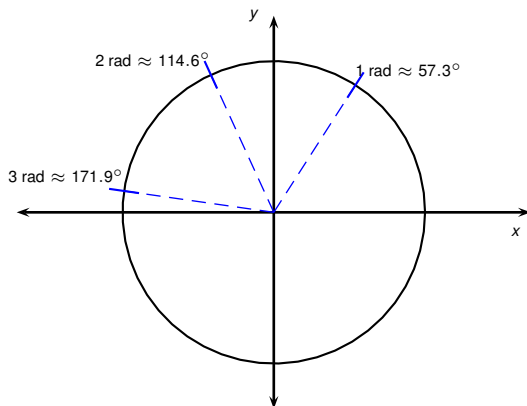
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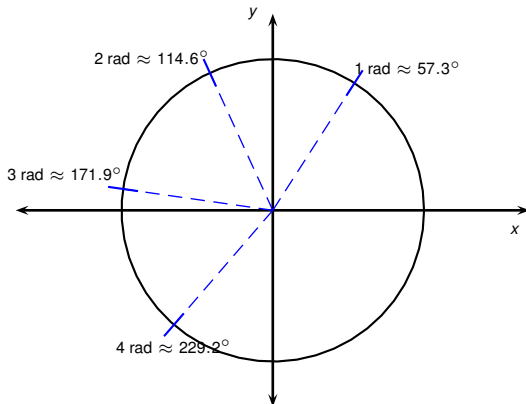
- Integer quantities of radians are not rational multiples of (the measure of) a half-turn and are not easy to compute with.



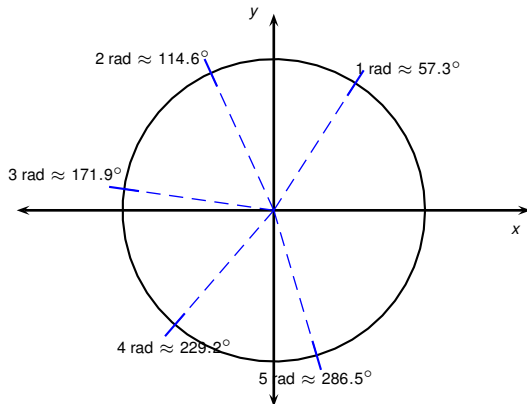
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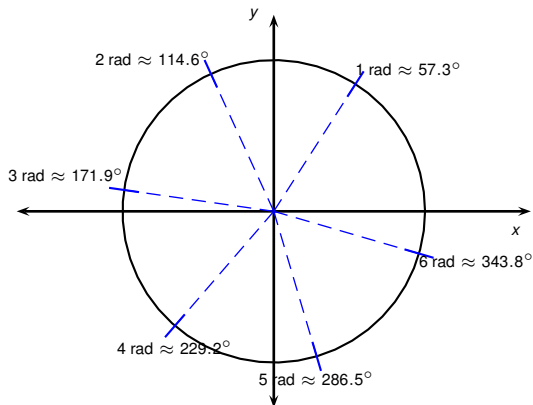
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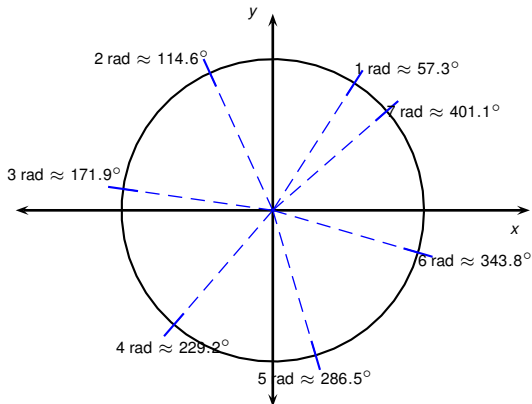
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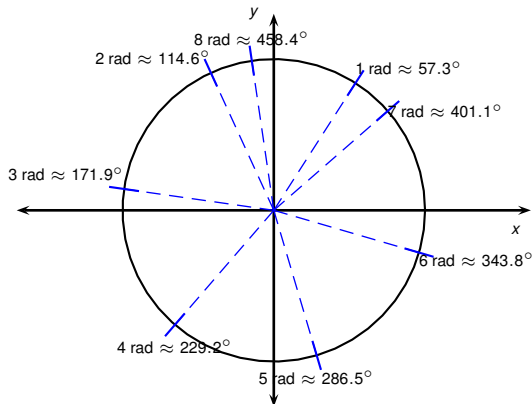
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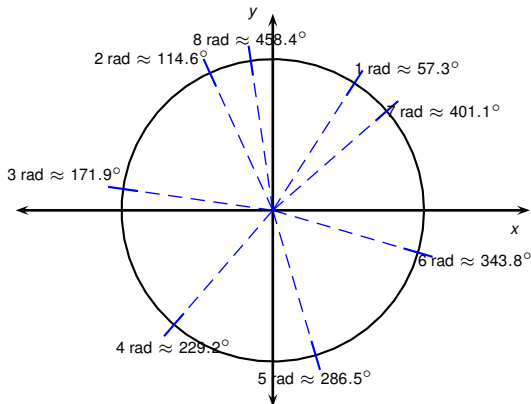
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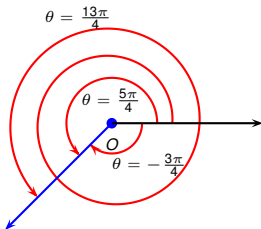
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- Integer quantities of radians are not rational multiples of (the measure of) a half-turn and are not easy to compute with.
- For example to determine in which quadrant is an angle of k radians located one needs to know the numerical value of $\frac{k}{\pi}$, which requires knowledge of π with great numerical accuracy.

Definition (Coterminal Angles)

Two angles (angle measures) are called coterminal if the corresponding geometric angles have the same initial and terminal sides.



Observation

The set of angles coterminal with α consists of the angles $\alpha + 2k\pi$, where k runs over the set of integers. In other words, the angles coterminal with α are the angles:

$$\dots, \alpha - 6\pi, \alpha - 4\pi, \alpha - 2\pi, \alpha, \alpha + 2\pi, \alpha + 4\pi, \alpha + 6\pi, \dots$$

Example

- Find all angles that are coterminal to $\frac{\pi}{4}$.
- Find all angles in the interval $[-2\pi, \pi]$ that are coterminal to $\frac{\pi}{4}$.

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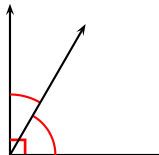
$$\cancel{\dots}, \cancel{\frac{\pi}{4} - 4\pi}, \frac{\pi}{4} - 2\pi, \frac{\pi}{4}, \cancel{\frac{\pi}{4} + 2\pi}, \cancel{\frac{\pi}{4} + 4\pi}, \dots$$

Our final answer is $-\frac{7\pi}{4}, \frac{\pi}{4}$

Complementary angles

Definition

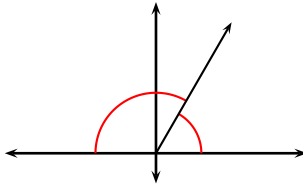
Two positive angles are called complementary when they sum to a right angle, i.e., an angle of measure $\frac{\pi}{2} = 90^\circ$.



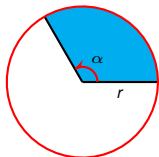
Supplementary angles

Definition

Two positive angles are called supplementary when they sum to $\pi = 180^\circ$.



A sector of a circle is the region cut off from a circle by an angle whose vertex is at the center of the circle.



Proposition (Area of a circle sector)

The area of a circle sector equals

$$\frac{1}{2}\alpha r^2,$$

where α is the angle of the sector.