Calculus II
Partial fractions

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Outline

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From building blocks to all rational functions: example

- We know how to solve $\int \frac{2}{x-1}\mathrm{d}x$ and $\int \frac{1}{x+2}\mathrm{d}x$.
- Consider the difference

$$\frac{2}{x-1} - \frac{1}{x+2} = \frac{2(x+2)-(x-1)}{(x-1)(x+2)} = \frac{x+5}{x^2+x-2} \quad .$$

- We can now solve the following integral:

$$\int \frac{x+5}{x^2+x-2}\mathrm{d}x = \int \left(\frac{2}{x-1} - \frac{1}{x+2}\right)\mathrm{d}x = 2\ln|x-1| - \ln|x+2| + C$$

- From (linear substitutions of) basic building blocks we constructed a larger example, which we can therefore solve.
- We now learn how to do the reverse procedure: given a rational function, split it into “partial fractions”.

Partial fractions definition

Definition

A partial fraction is rational function of one of the 2 forms below.

- $\frac{A}{(ax+b)^n}$, $n \geq 1$.
- $\frac{Ax+B}{(ax^2+bx+c)^n}$, where $b^2 - 4ac < 0$ and $n \geq 1$.

Theorem

Every rational function can be written as a sum of a polynomial and partial fractions.

- We already learned know how to integrate all partial fractions (using linear substitutions and building blocks I, II and III).
- Thus, if we can produce the partial fractions whose existence is promised by the theorem, we can integrate all rational functions.

Review of polynomial notation

- Recall that a rational function is a function of the form

$$f(x) = \frac{P(x)}{Q(x)}$$

  where P and $Q \neq 0$ are polynomials.
- Recall that the degree of P is the highest power of x in P that has a non-zero coefficient.

Ensure denominator degree > numerator degree

- To decompose $\frac{P(x)}{Q(x)}$ in partial fractions we ensure first the degree of the numerator is smaller than the degree of the denominator.
- We recall that to divide the dividend $P(x)$ by the divisor $Q(x)$ to get quotient $S(x)$ with remainder $R(x)$ means to find polynomials $S(x), R(x)$ such that $\deg R < \deg Q$ and

$$
\begin{array}{rcll}
P(x) & = & S(x)Q(x) + R(x) & \left| \text{ divide by } Q(x) \right. \\
\frac{P(x)}{Q(x)} & = & \frac{S(x)\cancel{Q(x)}}{\cancel{Q(x)}} + \frac{R(x)}{Q(x)} & \\
\frac{P(x)}{Q(x)} & = & S(x) + \frac{R(x)}{Q(x)} &
\end{array}
$$

- The above transforms $\frac{P(x)}{Q(x)}$ to a polynomial plus a fraction in which the numerator has degree smaller than the denominator.
- The polynomials $Q(x)$ and $S(x)$ are computed via polynomial long division. We recall the procedure through examples.

Example

Find $\int \frac{x^3+x}{x-1} \mathrm{d}x$.

$$
\begin{array}{r|rrrr}
 & x^2 & + \; x & + \; 2 & \\
\hline
x-1 & x^3 & & + \; x & \\
 & x^3 - x^2 & & & \\
\hline
 & & x^2 & + \; x & \\
 & & x^2 & - \; x & \\
\hline
 & & & 2x & \\
 & & & 2x & - \; 2 \\
\hline
 & & & & 2
\end{array}
$$

$$
\begin{aligned}
& \int \frac{x^3+x}{x-1} \mathrm{d}x \\
= \quad & \int \left(x^2 + x + 2 + \frac{2}{x-1} \right) \mathrm{d}x \\
= \quad & \frac{x^3}{3} + \frac{x^2}{2} + 2x \\
& + 2 \ln |x-1| + C
\end{aligned}
$$

- The next step in producing a partial fraction decomposition is to factor the denominator $Q(x)$.
- Factoring of $Q(x)$ can always be done in quadratic and linear terms as asserted in the following.

Corollary (Corollary to the Fundamental Theorem of Algebra)

Let $Q(x)$ be a polynomial (with real coefficients). Then $Q(x)$ can be factored as a product of terms of the form $(ax+b)^n$ (powers of linear terms) and product of terms of the form $(ax^2+bx+c)^n$ with $b^2-4ac<0$ (powers of quadratic terms).

- The above result is a corollary to the Fundamental Theorem of Algebra. We state the Fundamental Theorem of algebra without proving it.

Theorem (The Fundamental Theorem of Algebra)

Every polynomial has at least one complex root.

- Let $\frac{R(x)}{Q(x)}$ be a rational function with $\deg Q > \deg R$.
- Suppose $Q(x)$ factors into factors of the form

$$(ax+b)^N \qquad \text{and} \qquad (ax^2+bx+c)^M.$$

- Then we can split $\frac{R(x)}{Q(x)}$ into sum of partial fractions of the form

$$\frac{A_i}{(ax+b)^i}, \text{ with } i \leq N \qquad \text{or} \qquad \frac{B_jx+C_j}{(ax^2+bx+c)^j}, \text{ with } j \leq M,$$

  where the A_i's are constants - one for each power $1 \leq i \leq N$ and the B_j and C_j's are constants - one pair for each power $1 \leq j \leq M$.
- We use N different constants for each new linear factor of the form $(ax+b)^N$ and $2 \times M$ different constants for each factor of the form $(ax^2+bx+c)^N$.
- Thus the total number of constants used equals the degree of Q.
- The difficulty of finding the constants A_i, B_j, C_j increases as the number of distinct factors increases, as well as when the exponents of those factors increase.

$Q(x)$ has distinct linear factors

- Suppose $Q(x)$ is a product of distinct linear factors:

$$Q(x) = (a_1x + b_1)(a_2x + b_2)\cdots(a_kx + b_k)$$

  where no factor is repeated and no factor is a constant multiple of another.

- Then there exist constants $A_1, A_2, \dots, A_k$ such that

$$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \dots + \frac{A_k}{a_kx + b_k}$$

- We show how to find $A_1, A_2, \cdots, A_k$ on examples.

Example

Find $\int \frac{x^2+2x-1}{2x^3+3x^2-2x} \mathrm{d}x$.

- $\deg(x^2+2x-1) < \deg(2x^3+3x^2-2x)$: don't divide.
- Factor denominator: $2x^3+3x^2-2x = x(2x-1)(x+2)$.

$$\begin{aligned}
\frac{x^2+2x-1}{x(2x-1)(x+2)} &= \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2} \\
x^2+2x-1 &= A(2x-1)(x+2) + Bx(x+2) + Cx(2x-1) \\
x^2+2x-1 &= (2A+B+2C)x^2 + (3A+2B-C)x - 2A
\end{aligned}$$

$$\begin{aligned}
2A + \quad B + 2C &= \quad 1 \\
3A + 2B - \quad C &= \quad 2 \\
-2A \qquad\qquad &= -1
\end{aligned}$$

Solution:
$A = \frac{1}{2}, B = \frac{1}{5}, C = -\frac{1}{10}$.

$$\begin{aligned}
&\int \frac{x^2+2x-1}{2x^3+3x^2-2x} \mathrm{d}x \\
&= \int \left(\frac{1}{2}\ \frac{1}{x} + \frac{1}{5}\ \frac{1}{2x-1} - \frac{1}{10}\ \frac{1}{x+2} \right) \mathrm{d}x \\
&= \frac{1}{2} \ln|x| + \frac{1}{10} \ln|2x-1| \\
&\qquad - \frac{1}{10} \ln|x+2| + K
\end{aligned}$$

NOTE: There is a quick trick to find A, B, and C.

$$x^2+2x-1 = A(2x-1)(x+2)+Bx(x+2)+Cx(2x-1)$$

To find A, set $x=0$; to find B, set $x=\frac{1}{2}$; to find C, set $x=-2$.

$$\begin{aligned}
0^2+2\cdot 0-1 &= A(2\cdot 0-1)(0+2)\\
-1 &= -2A\\
A &= \frac{1}{2}
\end{aligned}$$

$$\begin{aligned}
\left(\frac{1}{2}\right)^2+2\cdot\frac{1}{2}-1 &= B\left(\frac{1}{2}\right)\left(\frac{1}{2}+2\right)\\
\frac{1}{4} &= \frac{5}{4}B\\
B &= \frac{1}{5}
\end{aligned}$$

$$\begin{aligned}
(-2)^2+2(-2)-1 &= C(-2)(2(-2)-1)\\
-1 &= 10C\\
C &= -\frac{1}{10}
\end{aligned}$$

$Q(x)$ has linear factors with higher multiplicity

- Suppose $Q(x)$ is a product of linear factors, some of which appear with power greater than 1.
- For example suppose the first linear factor has power r, that is, $(a_1x + b_1)^r$ occurs in the factorization of $Q(x)$.
- Then instead of a single term $\frac{A}{a_1x+b_1}$ we use

$$\frac{A_1}{a_1x + b_1} + \frac{A_2}{(a_1x + b_1)^2} + \cdots + \frac{A_r}{(a_1x + b_1)^r}$$

- In a similar fashion we add more partial fractions to account for all other terms of the form $(a_sx + b_s)^t$.

Example

$$\int \frac{x^4+x^3-4x^2+4x}{x^3-x^2-x+1}dx = \int \left(x+2+\frac{1}{x-1}+\frac{1}{(x-1)^2}-\frac{2}{x+1}\right) dx$$
$$= \frac{x^2}{2}+2x+\ln|x-1|-\frac{1}{x-1}-2\ln|x+1|+K$$

- Divide: $\frac{x^4+x^3-4x^2+4x}{x^3-x^2-x+1} = x+2+\frac{-x^2+5x-2}{x^3-x^2-x+1} = x+2+\frac{-x^2+5x-2}{(x-1)^2(x+1)}$.
- Factor denominator: $x^3-x^2-x+1 = (x-1)^2(x+1)$.
- Set up the partial fraction decomposition:
$$\begin{aligned}\frac{-x^2+5x-2}{(x-1)^2(x+1)} &= \frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x+1}\\ -x^2+5x-2 &= A(x-1)(x+1)+B(x+1)+C(x-1)^2\end{aligned}$$
- Plug-in $x=-1$: $-(-1)^2+5(-1)-2 = C(-1-1)^2 \Rightarrow C=-2$.
- Plug-in $x=1$: $-(1)^2+1\cdot 5-2 = B(1+1) \Rightarrow B=1$.
- Plug-in $x=0$:
$-2 = A(0-1)(0+1)+1\cdot\ (0+1)+(-2)\ \ (0-1)^2 \Rightarrow A=1$.

$Q(x)$ contains quadratic factors, multiplicity 1

- Suppose $Q(x)$ contains quadratic factors $ax^2 + bx + c$ with where $b^2 - 4ac < 0$ (i.e., the factor is irreducible).
- Suppose none of the quadratic factors is repeated.
- The for each quadratic factor we need to add a partial fraction of the form
$$\frac{Ax + B}{ax^2 + bx + c}.$$

Example

Find $\int \frac{2x^2-x+4}{x^3+4x}\mathrm{d}x$.

- $\deg(2x^2 - x + 4) < \deg(x^3 + 4x)$: don't divide.
- Factor denominator: $x^3 + 4x = x(x^2 + 4)$.

$$\begin{aligned}
\frac{2x^2 - x + 4}{x(x^2+4)} &= \frac{A}{x} + \frac{Bx + C}{(x^2+4)} \\
2x^2 - x + 4 &= A(x^2+4) + (Bx + C)x \\
2x^2 - x + 4 &= (A+B)x^2 + Cx + 4A
\end{aligned}$$

$$A = 1 \qquad C = -1 \qquad A + B = 2, \text{ therefore } B = 1$$

$$\begin{aligned}
\int \frac{2x^2 - x + 4}{x(x^2+4)}\mathrm{d}x &= \int \left(\frac{1}{x} + \frac{x-1}{x^2+4}\right)\mathrm{d}x \\
&= \int \frac{1}{x}\mathrm{d}x + \int \frac{x}{x^2+4}\mathrm{d}x - \int \frac{1}{x^2+4}\mathrm{d}x \\
&= \ln|x| + \frac{1}{2}\ln(x^2+4) - \frac{1}{2}\arctan\left(\frac{x}{2}\right) + K
\end{aligned}$$

$Q(x)$ has quadratic factors with multiplicity > 1

- Suppose $Q(x)$ has the factor $(ax^2 + bx + c)^r$, where $b^2 - 4ac < 0$ and $r > 1$.
- Then the partial fraction decomposition should include summands of the form

$$\frac{A_1 x + B_1}{ax^2 + bx + c} + \frac{A_2 x + B_2}{(ax^2 + bx + c)^2} + \dots + \frac{A_r x + B_r}{(ax^2 + bx + c)^r}$$

Example

Write out the form of the partial fraction decomposition of

$$\frac{x^3+x^2+1}{x(x-1)(x^2+x+1)(x^2+1)^3}$$

$$= \frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{x^2+x+1} + \frac{Ex+F}{x^2+1} + \frac{Gx+H}{(x^2+1)^2} + \frac{Ix+J}{(x^2+1)^3}.$$

For example of this size it makes sense to use a computer algebra system; one such system easily produces the decomposition:

$$= \frac{-1}{x} + \frac{\frac{1}{8}}{x-1} + \frac{-x-1}{(x^2+x+1)} + \frac{\frac{15}{8}x-\frac{1}{8}}{(x^2+1)} + \frac{\frac{3}{4}x+\frac{3}{4}}{(x^2+1)^2} + \frac{-\frac{x}{2}+\frac{1}{2}}{(x^2+1)^3}.$$