Calculus II
Power series expansion of rational functions with linear denominator, part 2

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Example

Find a power series representation for $\frac{1}{x+2}$.

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To find interval of convergence:

$$\begin{aligned}\left|-\frac{x}{2}\right| &< 1 \\ |x| &< 2\end{aligned}$$

Therefore the interval of convergence is $x \in (-2, 2)$.

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- The interval of convergence is again $x \in (-2, 2)$.