Calculus II
Integration of rational functions: plan for algorithm

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Integrating arbitrary rational functions

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  - We solve each building block integral and collect the terms.
- We study the algorithm "from the ground up": we start with the building blocks.

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- The case $n = 1$ is special for each of the building blocks:
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This "building block" terminology is for our convenience, and is not a part of standard mathematical terminology.