

Precalculus

Complex numbers definition; overview of numbers

Todor Milev

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Outline

1 Complex numbers definition

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- 1 Complex numbers definition
- 2 Complex numbers multiplication and addition

Definition (Complex numbers)

The set of complex numbers \mathbb{C} is defined as the set

$$\{a + bi \mid a, b - \text{real numbers}\},$$

where the number i is a number for which

$$i^2 = -1 \quad .$$

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$$\begin{aligned}(a + bi)(c + di) &= ac + adi + bci + bdi^2 = \textcolor{red}{ac} + adi + bci - \textcolor{red}{bd} \\ &= (\textcolor{red}{ac} - \textcolor{red}{bd}) + i(ad + bc)\end{aligned}$$

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Let $u = 2 + 3i$, $v = 5 - 7i$.

Example (Addition)

$$u + v =$$

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Example (Subtraction)

$$u - v =$$

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Example (Multiplication)

$$u \cdot v =$$

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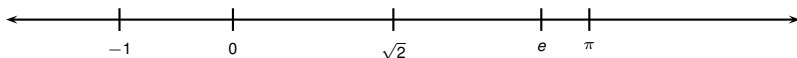
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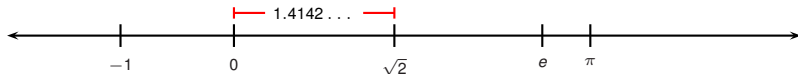
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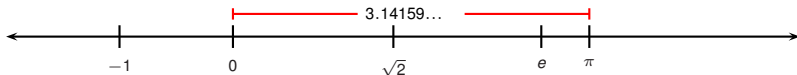
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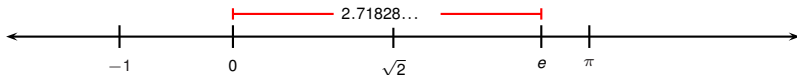
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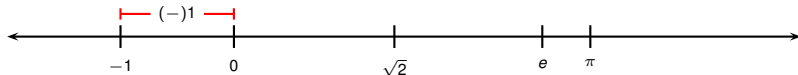
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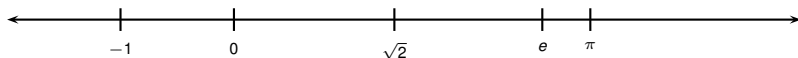
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$$-i,$$

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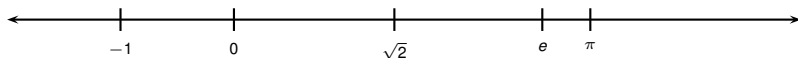
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- Geometric interpretation of complex numbers: beyond our scope.