Calculus II
Integration by parts

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Outline

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Integration by Parts

Every differentiation rule corresponds to a differential form rule which in turn corresponds to an integration rule.

$$
\begin{array}{rcll}
(uv)' &=& u'v + uv' & \text{Product Rule} \\
\mathrm{d}(uv) &=& v\mathrm{d}u + u\mathrm{d}v & \text{Differential Prod. Rule} \\
\int \mathrm{d}(uv) &=& \int v\mathrm{d}u + \int u\mathrm{d}v & \text{integration of the above} \\
uv &=& \int v\mathrm{d}u + \int u\mathrm{d}v & \text{rearrange} \\
\int u\mathrm{d}v &=& uv - \int v\mathrm{d}u &
\end{array}
$$

We just proved the following.

Proposition ((Rule of) Integration by Parts)

$$\int u\mathrm{d}v = uv - \int v\mathrm{d}u$$

Integration by parts: strategy for applying

Integration by parts:

$$\int u \mathrm{d}v = uv - \int v \mathrm{d}u.$$

Generally: Choose u in this order: **LIPET**

Logs, Inverse trig, Polynomial, Exponential, Trig

Integration by parts: $\int u\mathrm{d}v = uv - \int v\mathrm{d}u.$

Example

$$\int x\sin x\mathrm{d}x = \int x\mathrm{d}(-\cos x) \qquad \Big| \quad \sin x\mathrm{d}x = \mathrm{d}(-\cos x)$$

$$\begin{aligned} &= x(-\cos x) - \int(-\cos x)\mathrm{d}x \\ &= -x\cos x + \int \cos x\mathrm{d}x \\ &= -x\cos x + \sin x + C \end{aligned}$$

Integration by parts: $\displaystyle\int u\mathrm{d}v = uv - \int v\mathrm{d}u.$

Example

$$\begin{aligned}
\int \ln x\mathrm{d}x &= (\ln x)x - \int x\mathrm{d}(\ln x) && \Big|\ \text{integrate by parts}\\
&= x\ln x - \int x(\ln x)'\mathrm{d}x\\
&= x\ln x - \int x\frac{1}{x}\mathrm{d}x\\
&= x\ln x - \int \mathrm{d}x\\
&= x\ln x - x + C \quad .
\end{aligned}$$

Integration by parts: $\displaystyle\int u\mathrm{d}v = uv - \int v\mathrm{d}u.$

Example

$$
\begin{array}{rcll}
\displaystyle\int t^2 e^t \mathrm{d}t &=& \displaystyle\int t^2 \mathrm{d}\left(e^t\right) & \Big| \text{ integrate by parts} \\
&=& \displaystyle t^2 e^t - \int e^t \mathrm{d}\left(t^2\right) & \\
&=& \displaystyle t^2 e^t - \int e^t 2t\mathrm{d}t & \\
&=& \displaystyle t^2 e^t - 2\int t\mathrm{d}(e^t) & \Big| \text{ integrate by parts} \\
&=& \displaystyle t^2 e^t - 2\left(te^t - \int e^t \mathrm{d}t\right) & \\
&=& t^2 e^t - 2te^t + 2e^t + C &
\end{array}
$$

Integration by parts: $\int u \mathrm{d}v = uv - \int v \mathrm{d}u.$

Example

$$
\begin{aligned}
\int e^x \sin x \mathrm{d}x &= \int \sin x \,\mathrm{d}\left(e^x\right) \\
&= (\sin x) e^x - \int e^x \mathrm{d}(\sin x) \\
&= e^x \sin x - \int e^x \cos x \mathrm{d}x \\
&= e^x \sin x - \int \cos x \mathrm{d}\left(e^x\right) \\
&= e^x \sin x - \left((\cos x) e^x - \int e^x \mathrm{d}(\cos x) \right) \\
&= e^x \sin x - \cos x e^x + \int e^x (-\sin x) \mathrm{d}x \\
&= e^x \sin x - \cos x e^x - \int e^x \sin x \mathrm{d}x
\end{aligned}
$$

Integration by parts: $\int u\mathrm{d}v = uv - \int v\mathrm{d}u.$

Example

$$
\begin{aligned}
\int e^x \sin x \mathrm{d}x &= e^x \sin x - \cos x e^x - \int e^x \sin x \mathrm{d}x \\
2\int e^x \sin x \mathrm{d}x &= e^x \sin x - \cos x e^x \\
\int e^x \sin x \mathrm{d}x &= \frac{1}{2}\left(e^x \sin x - \cos x e^x\right) + C
\end{aligned}
$$

Integration by parts: $\int u \mathrm{d}v = uv - \int v \mathrm{d}u$.

Example

Set $w = 1 + x^2$.

$$
\begin{aligned}
\int_0^1 \arctan x \mathrm{d}x &= \left[(\arctan x)x\right]_{x=0}^{x=1} - \int_0^1 x \mathrm{d}\left(\arctan x\right) \\
&= 1 \cdot \arctan 1 - 0 \cdot \arctan 0 - \int_{x=0}^{x=1} x \frac{1}{1+x^2} \mathrm{d}x \\
&= \frac{\pi}{4} - \int_{x=0}^{x=1} \frac{1}{1+x^2} \mathrm{d}\left(\frac{x^2}{2}\right) \\
&= \frac{\pi}{4} - \frac{1}{2} \int_{x=0}^{x=1} \frac{1}{1+x^2} \mathrm{d}(1+x^2) \\
&= \frac{\pi}{4} - \frac{1}{2} \int_{x=0}^{x=1} \frac{1}{w} \mathrm{d}w = \frac{\pi}{4} - \frac{1}{2} \left[\ln|w|\right]_{x=0}^{x=1} \\
&= \frac{\pi}{4} - \frac{1}{2} \left[\ln\left(1+x^2\right)\right]_{x=0}^{x=1} \\
&= \frac{\pi}{4} - \frac{1}{2} \left(\ln 2 - \ln 1\right) = \frac{\pi}{4} - \frac{1}{2} \ln 2 \quad .
\end{aligned}
$$