

Precalculus

Equations formed by setting trigonometric sum equal to 0

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Proposition (Product to sum formulas)

$$\begin{aligned}\sin \alpha \sin \beta &= \frac{1}{2} (\cos(\alpha - \beta) - \cos(\alpha + \beta)) \\ \cos \alpha \cos \beta &= \frac{1}{2} (\cos(\alpha - \beta) + \cos(\alpha + \beta)) \\ \sin \alpha \cos \beta &= \frac{1}{2} (\sin(\alpha + \beta) + \sin(\alpha - \beta))\end{aligned}$$

Proof.

$$\begin{array}{rcll} + & \cos \alpha \cos \beta + \cancel{\sin \alpha \sin \beta} & = & \cos(\alpha - \beta) \\ & \cos \alpha \cos \beta - \cancel{\sin \alpha \sin \beta} & = & \cos(\alpha + \beta) \\ & \hline & 2 \cos \alpha \cos \beta & = & \cos(\alpha - \beta) + \cos(\alpha + \beta) \\ - & \cos \alpha \cos \beta + \sin \alpha \sin \beta & = & \cos(\alpha - \beta) \\ & \cos \alpha \cos \beta - \sin \alpha \sin \beta & = & \cos(\alpha + \beta) \\ & \hline & 2 \sin \alpha \sin \beta & = & \cos(\alpha - \beta) - \cos(\alpha + \beta) \\ + & \sin \alpha \cos \beta + \cos \alpha \sin \beta & = & \sin(\alpha + \beta) \\ & \sin \alpha \cos \beta - \cos \alpha \sin \beta & = & \sin(\alpha - \beta) \\ & \hline & 2 \sin \alpha \cos \beta & = & \sin(\alpha + \beta) + \sin(\alpha - \beta) \end{array}$$



Proposition (Product to sum formulas)

$$\begin{aligned}\sin \alpha \sin \beta &= \frac{1}{2} (\cos(\alpha - \beta) - \cos(\alpha + \beta)) \\ \cos \alpha \cos \beta &= \frac{1}{2} (\cos(\alpha - \beta) + \cos(\alpha + \beta)) \\ \sin \alpha \cos \beta &= \frac{1}{2} (\sin(\alpha + \beta) + \sin(\alpha - \beta))\end{aligned}$$

- Product to sum formulas are used when integrating (a topic to be studied later/in another course).

Proposition (Sum to product formulas)

$$\sin \alpha + \sin \beta = 2 \sin \left(\frac{\alpha + \beta}{2} \right) \cos \left(\frac{\alpha - \beta}{2} \right)$$

$$\sin \alpha - \sin \beta = 2 \sin \left(\frac{\alpha - \beta}{2} \right) \cos \left(\frac{\alpha + \beta}{2} \right)$$

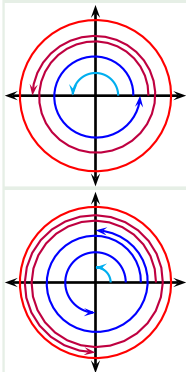
$$\cos \alpha + \cos \beta = 2 \cos \left(\frac{\alpha + \beta}{2} \right) \cos \left(\frac{\alpha - \beta}{2} \right)$$

$$\cos \alpha - \cos \beta = 2 \sin \left(\frac{\alpha + \beta}{2} \right) \sin \left(\frac{\alpha - \beta}{2} \right)$$

Recall the formula $\sin \alpha + \sin \beta = 2 \sin \left(\frac{\alpha + \beta}{2} \right) \cos \left(\frac{\alpha - \beta}{2} \right)$

Example

Find all solutions in the interval $[0, 2\pi)$.

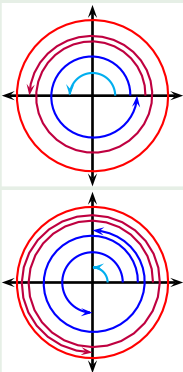


$$\begin{aligned} \sin(2x) + \sin(5x) &= 0 \quad | \text{ use f-l-a} \\ 2 \sin \left(\frac{2x + 5x}{2} \right) \cos \left(\frac{2x - 5x}{2} \right) &= 0 \\ 2 \sin \left(\frac{7}{2}x \right) \cos \left(-\frac{3}{2}x \right) &= 0 \quad | \begin{array}{l} \cos \\ \text{is even} \end{array} \\ 2 \sin \left(\frac{7}{2}x \right) \cos \left(\frac{3}{2}x \right) &= 0 \end{aligned}$$

Recall the formula $\sin \alpha + \sin \beta = 2 \sin \left(\frac{\alpha + \beta}{2} \right) \cos \left(\frac{\alpha - \beta}{2} \right)$

Example

Find all solutions in the interval $[0, 2\pi)$.



$$\sin(2x) + \sin(5x) = 0 \quad | \text{ use formula}$$

$$2 \sin \left(\frac{7}{2}x \right) \cos \left(\frac{3}{2}x \right) = 0$$

$$\sin \left(\frac{7}{2}x \right) = 0$$

$$\frac{7}{2}x = k\pi$$

$$x = \frac{2k\pi}{7}$$

k – integer

$$x = \cancel{\cdot}, \cancel{\frac{2\pi}{7}}, 0, \frac{2\pi}{7}, \frac{4\pi}{7}, \frac{6\pi}{7}, \frac{8\pi}{7}, \frac{10\pi}{7}, \frac{12\pi}{7}, \frac{14\pi}{7}, \cancel{\cdot}$$

or

$$\cos \left(\frac{3}{2}x \right) = 0$$

$$\frac{3}{2}x = \frac{\pi}{2} + k\pi = \frac{(2k+1)\pi}{2}$$

k – integer

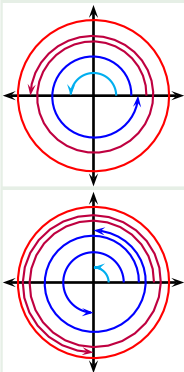
$$x = \frac{(2k+1)\pi}{3}$$

$$x = \cancel{\cdot}, \cancel{\frac{\pi}{3}}, \frac{\pi}{3}, \frac{3\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \cancel{\cdot}$$

Recall the formula $\sin \alpha + \sin \beta = 2 \sin \left(\frac{\alpha + \beta}{2} \right) \cos \left(\frac{\alpha - \beta}{2} \right)$

Example

Find all solutions in the interval $[0, 2\pi)$.



$$\sin(2x) + \sin(5x) = 0 \quad | \text{ use formula}$$

$$2 \sin \left(\frac{7}{2}x \right) \cos \left(\frac{3}{2}x \right) = 0$$

$$x = \cancel{\cdot}, \cancel{-\frac{2\pi}{7}}, 0, \frac{2\pi}{7}, \frac{4\pi}{7}, \frac{6\pi}{7}, \frac{8\pi}{7}, \frac{10\pi}{7}, \frac{12\pi}{7}, \frac{14\pi}{7}, \cancel{\cdot}$$

or

$$x = \cancel{\cdot}, \cancel{-\frac{\pi}{3}}, \frac{\pi}{3}, \frac{3\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \cancel{\cdot}$$

