

# Precalculus

## Trigonometry and triangles

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# Outline

- 1 Law of sines
- 2 Law of cosines

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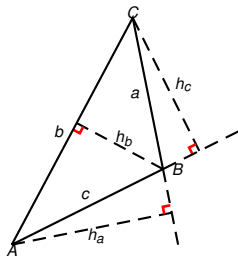
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# Triangle area = $\frac{1}{2}$ base $\cdot$ height

Let  $\triangle ABC$  have side lengths  $a, b, c$  and height lengths  $h_a, h_b, h_c$ , as indicated - side  $a$  is opposite to vertex  $A$  and  $h_a$  starts at  $A$ , and so on.

## Proposition (Triangle area)

$$\text{Area}(\triangle ABC) = \frac{1}{2} \text{height} \cdot \text{base} = \frac{1}{2} h_a a = \frac{1}{2} h_b b = \frac{1}{2} h_c c.$$



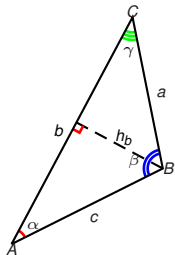
# Triangle area from two sides and angle between them

Let  $\triangle ABC$  have sides lengths  $a, b, c$  angles  $\alpha, \beta, \gamma$ , as indicated:  $\alpha$  is opposite to  $a$ ,  $\beta$  is opposite to  $b$ ,  $\gamma$  is opposite to  $c$ .

## Proposition ( $\triangle$ area from two sides and angle between them)

*The area of a triangle is half the product of the lengths of two of its sides times the sine of the angle between them. In other words,*

$$\text{Area}(\triangle ABC) = \frac{ab \sin \gamma}{2} = \frac{bc \sin \alpha}{2} = \frac{ca \sin \beta}{2}$$



## Proof.

$$\begin{aligned} \text{Area}(\triangle ABC) &= \frac{\text{base} \cdot \text{height}}{2} = \frac{bh_b}{2} \\ &= \frac{ba \sin \gamma}{2}. \end{aligned}$$

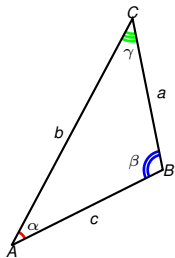
The proof of the other two cases is similar. □

# Law of sines

Let  $\triangle ABC$  have sides lengths  $a, b, c$  angles  $\alpha, \beta, \gamma$ , as indicated:  $\alpha$  is opposite to  $a$ ,  $\beta$  is opposite to  $b$ ,  $\gamma$  is opposite to  $c$ .

## Proposition (Law of Sines)

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma}.$$

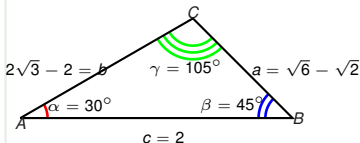


## Proof.

$$\begin{aligned} \text{Area}(\triangle ABC) &= \frac{ab \sin \gamma}{2} = \frac{bc \sin \alpha}{2} & \left| \text{Div. by } \frac{b}{2} \right. \\ a \sin \gamma &= c \sin \alpha \\ \frac{a}{\sin \alpha} &= \frac{c}{\sin \gamma}. \end{aligned}$$

The remaining cases are similar. □

## Example

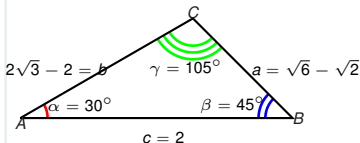


A triangle has a side of length  $2\text{cm}$ ; the two angles adjacent to it are  $30^\circ$  and  $45^\circ$ .

- Find the other two sides of the triangle.
- Find the area of the triangle.

- Let the known side be  $c = 2\text{cm}$ .
- Let the known angles  $30^\circ$ ,  $45^\circ$  be arranged as in the figure, and let the third angle be  $\gamma = 180^\circ - 30^\circ - 45^\circ = 180^\circ - 75^\circ = 105^\circ$ .
- Label the unknown sides  $a$ ,  $b$  as indicated.

## Example



A triangle has a side of length  $2\text{cm}$ ; the two angles adjacent to it are  $30^\circ$  and  $45^\circ$ .

- Find the other two sides of the triangle.
- Find the area of the triangle.

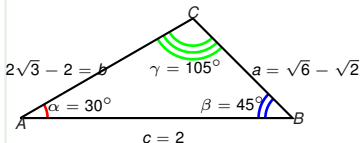
$$\begin{aligned}\sin 105^\circ &= \sin(60^\circ + 45^\circ) = \sin 60^\circ \cos 45^\circ + \cos 60^\circ \sin 45^\circ \\ &= \frac{\sqrt{3}}{2} \frac{\sqrt{2}}{2} + \frac{1}{2} \frac{\sqrt{2}}{2} = \frac{\sqrt{6} + \sqrt{2}}{4}\end{aligned}$$

$$\frac{a}{\sin \alpha} = \frac{c}{\sin \gamma} \quad \text{Law of sines}$$

$$\begin{aligned}a &= \frac{c \sin \alpha}{\sin \gamma} = \frac{2 \sin 30^\circ}{\sin 105^\circ} = \frac{2 \cdot \frac{1}{2}}{\frac{\sqrt{6} + \sqrt{2}}{4}} = \frac{4(\sqrt{6} - \sqrt{2})}{(\sqrt{6} + \sqrt{2})(\sqrt{6} - \sqrt{2})} \\ &= \frac{4(\sqrt{6} - \sqrt{2})}{6 - 2} = \sqrt{6} - \sqrt{2}\end{aligned}$$



## Example



A triangle has a side of length  $2\text{cm}$ ; the two angles adjacent to it are  $30^\circ$  and  $45^\circ$ .

- Find the other two sides of the triangle.
- Find the area of the triangle.

| Law of sines

$$\frac{b}{\sin \beta} = \frac{c}{\sin \gamma}$$

$$b = \frac{c \sin \beta}{\sin \gamma} = \frac{2 \sin 45^\circ}{\sin 105^\circ} = \frac{\cancel{2} \frac{\sqrt{2}}{2}}{\frac{\sqrt{6} + \sqrt{2}}{4}} = \frac{4\sqrt{2}(\sqrt{6} - \sqrt{2})}{(\sqrt{6} + \sqrt{2})(\sqrt{6} - \sqrt{2})}$$

$$= \frac{\cancel{4}\sqrt{2}(\sqrt{6} - \sqrt{2})}{\cancel{4}} = 2\sqrt{3} - 2$$

$$\text{Area} = \frac{bc \sin \alpha}{2} = \frac{(2\sqrt{3} - 2)\cancel{2} \frac{1}{2}}{2} = \sqrt{3} - 1 \quad \text{cm}^2$$

Let  $\triangle ABC$  have sides lengths  $a, b, c$  angles  $\alpha, \beta, \gamma$ , as indicated.

## Proposition (Law of Cosines)

$$c^2 = a^2 + b^2 - 2ab \cos \gamma$$

$$a^2 = b^2 + c^2 - 2bc \cos \alpha$$

$$b^2 = c^2 + a^2 - 2ca \cos \beta$$

Proof if  $\gamma < 90^\circ$ .

Drop a perpendicular  $h$  from  $B$  to  $AC$ .

$$|CD| = a \cos \gamma$$

$$h = a \sin \gamma$$

$$|AD| = b - |CD| = b - a \cos \gamma$$

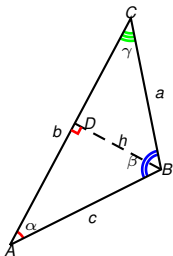
$$c^2 = |AD|^2 + h^2$$

$$= (b - a \cos \gamma)^2 + (a \sin \gamma)^2$$

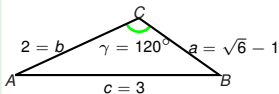
$$= b^2 - 2ab \cos \gamma + a^2 \cos^2 \gamma + a^2 \sin^2 \gamma$$

$$= b^2 - 2ab \cos \gamma + a^2.$$

Pyth. thm.  
 $\triangle BDA$



## Example



The longest side of a triangle has length 3 and the angle opposite to it is  $120^\circ$ . Another side of that triangle has length 2.

- Find the length of the third side.
- Find the area of the triangle.

$$a^2 + b^2 - 2ab \cos \gamma = c^2$$

$$a^2 + 2^2 - 2a \cdot 2 \cdot \cos 120^\circ = 3^2$$

$$a^2 - 4a \left( -\frac{1}{2} \right) - 5 = 0$$

$$a^2 + 2a - 5 = 0$$

$$a = \frac{-2 \pm \sqrt{2^2 - 4 \cdot (-5) \cdot 1}}{2 \cdot 1}$$

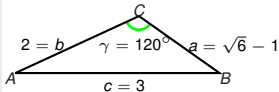
$$= \frac{-2 \pm \sqrt{24}}{2} = \frac{-2 + 2\sqrt{6}}{2}$$

$$= -1 + \sqrt{6}$$

Law of cosines  
Solve for  $a$  :

$$a > 0$$

## Example



The longest side of a triangle has length 3 and the angle opposite to it is  $120^\circ$ . Another side of that triangle has length 2.

- Find the length of the third side.
- Find the area of the triangle.

$$a^2 + b^2 - 2ab \cos \gamma = c^2$$

$$a^2 + 2^2 - 2a \cdot 2 \cdot \cos 120^\circ = 3^2$$

Law of cosines  
Solve for  $a$  :

$$a = \frac{-2 \pm \sqrt{2^2 - 4 \cdot (-5) \cdot 1}}{2 \cdot 1}$$

$$= -1 + \sqrt{6}$$

$$\text{Area} = \frac{ab \sin \gamma}{2} = \frac{(\sqrt{6} - 1) \cancel{2} \sqrt{3}}{\cancel{2} \cdot 2}$$

$$= \frac{3\sqrt{2} - \sqrt{3}}{2}$$