Calculus I
Logarithmic derivatives

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Outline

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1. Derivatives of Logarithmic Functions

2. Derivative of $a(x)^{b(x)}$
   - Logarithmic Differentiation
   - The Number e as a Limit

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Derivatives of Logarithmic Functions

Theorem (The Derivative of $\log_a x$)

$$\frac{\mathrm{d}}{\mathrm{d}x}(\log_a x) = \frac{1}{x \ln a}.$$

Proof.

$\square$

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Proof.

Let $y = \log_a x$.

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$$\text{Let } y = \log_a x.$$
$$\text{Then } a^y = x.$$

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$$\begin{aligned} \text{Let } y &= \log_a x. \\ \text{Then } a^y &= x. \\ \text{Differentiate implicitly:} \quad ? \qquad &= ? \end{aligned}$$

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Example (Chain Rule)

Differentiate $\quad f(x) = \log_3(5^x + 1).$

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$$\begin{aligned} \text{Differentiate} \quad & f(x) = \log_3(5^x + 1). \\ \text{Let} \quad & {\color{red} h(x) = \textbf{?}} \\ \text{Let} \quad & g(x) = \textbf{?} \\ \text{Then} \quad & f(x) = g(h(x)). \end{aligned}$$

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& = \left(\frac{1}{h(x)\ln 3}\right)(5^x \ln 5)
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\text{Chain Rule:} \quad f'(x) &= g'(h(x))h'(x) \\
&= \left(\frac{1}{h(x)\ln 3}\right)\left(5^x \ln 5\right) \\
&= \frac{5^x \ln 5}{(5^x + 1)\ln 3}.
\end{aligned}
$$

Theorem (The Derivative of $\log_a x$)

$$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}.$$

$\ln x = \log_e x$. Therefore when we set $a = e$ we get the derivative of $\ln x$:

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Example

Compute the indicated derivative.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(2\ln(3x-1)\right)$$

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$$\frac{d}{dx}\left(2\ln(3x-1)\right) = 2\cdot\frac{d}{dx}\left(\ln(3x-1)\right)$$

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$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(2\ln(3x-1)\right) &= 2\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln(3x-1)\right)\\ &= 2\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln u\right) && \Big|\ \text{Set } 3x-1=u\end{aligned}$$

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$$\begin{aligned}\frac{d}{dx}\left(2\ln(3x-1)\right) &= 2\cdot\frac{d}{dx}\left(\ln(3x-1)\right) \\ &= 2\cdot\frac{d}{dx}\left(\ln u\right) && \Bigg| \text{ Set } 3x-1=u \\ &= 2\cdot\frac{d}{du}(\ln u)\cdot\frac{du}{dx}\end{aligned}$$

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$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(2\ln(3x-1)\right) &= 2\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln(3x-1)\right) \\
&= 2\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln u\right) && \Big| \text{ Set } 3x-1=u \\
&= 2\cdot\frac{\mathrm{d}}{\mathrm{d}u}(\ln u)\cdot\frac{\mathrm{d}u}{\mathrm{d}x} \\
&= 2\cdot\frac{1}{u}\cdot\frac{\mathrm{d}}{\mathrm{d}x}(3x-1)
\end{aligned}$$

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\frac{d}{dx}\left(2\ln(3x-1)\right) &= 2\cdot\frac{d}{dx}\left(\ln(3x-1)\right) \\
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&= 2\cdot\frac{1}{u}\cdot\frac{\mathrm{d}}{\mathrm{d}x}(3x-1) \\
&= 2\cdot\frac{1}{3x-1}\cdot 3
\end{aligned}$$

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&= 2\cdot\frac{d}{du}(\ln u)\cdot\frac{du}{dx} \\
&= 2\cdot\frac{1}{u}\cdot\frac{d}{dx}(3x-1) \\
&= 2\cdot\frac{1}{3x-1}\cdot 3 \\
&= \frac{6}{3x-1}
\end{aligned}
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Example

Compute the given derivative.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln \sqrt[3]{4x-1}\right)$$

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$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln\sqrt[3]{4x-1}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\ln(4x-1)^{\frac{1}{3}}\right)$$

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$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln\sqrt[3]{4x-1}\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\ln(4x-1)^{\frac{1}{3}}\right)\\ &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{3}\ln(4x-1)\right)\end{aligned}$$

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&= \frac{1}{3}\frac{d}{dx}\left(\ln(4x-1)\right)\\
&= \frac{1}{3}\,\mathbf{?}
\end{aligned}$$

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\frac{d}{dx}\left(\ln \sqrt[3]{4x-1}\right) &= \frac{d}{dx}\left(\ln(4x-1)^{\frac{1}{3}}\right)\\
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&= \frac{1}{3}\frac{d}{dx}\left(\ln(4x-1)\right)\\
&= \frac{1}{3}\frac{(4x-1)'}{4x-1}
\end{aligned}$$

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&= \frac{4}{3(4x-1)}
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Example (Chain Rule, Natural Logarithm)

Differentiate $y = \ln(e^x \sec x)$.

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Differentiate $y = \ln(e^x \sec x)$.

$$\begin{aligned} y &= \ln e^x + \ln(\sec x) \\ &= \textbf{?} + \ln(\sec x). \end{aligned}$$

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Differentiate $y = \ln(e^x \sec x)$.

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\begin{aligned}
y &= \ln e^x + \ln(\sec x)\\
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\end{aligned}
$$

Example (Chain Rule, Natural Logarithm)

Differentiate $y = \ln(e^x \sec x)$.

$$y = \ln e^x + \ln(\sec x)$$

$$= x + \ln(\sec x).$$

$$\frac{dy}{dx} = \mathbf{?} + \frac{d}{dx}(\ln(\sec x))$$

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Differentiate $y = \ln(e^x \sec x)$.

$$\begin{aligned} y &= \ln e^x + \ln(\sec x) \\ &= x + \ln(\sec x). \\ \frac{dy}{dx} &= 1 + \frac{d}{dx}(\ln(\sec x)) \end{aligned}$$

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Let $u = \textbf{?}$

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Let $u = \sec x$.

Then $\ln(\sec x) = \ln u$.

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Differentiate $y = \ln(e^x \sec x)$.

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\text{Let } u &= \sec x.\\
\text{Then } \ln(\sec x) &= \ln u.\\
\text{Chain Rule: } \frac{\mathrm{d}y}{\mathrm{d}x} &= 1 + \frac{\mathrm{d}}{\mathrm{d}u}(\ln u)\frac{\mathrm{d}u}{\mathrm{d}x}
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Chain Rule: $\frac{dy}{dx} = 1 + \frac{d}{du}(\ln u)\frac{du}{dx}$

$$= 1 + \left(\frac{1}{u}\right)(\textbf{?} \qquad\qquad)$$

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$$\text{Differentiate} \quad y = \ln(e^x \sec x).$$

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\text{Chain Rule:} \quad \frac{dy}{dx} &= 1 + \frac{d}{du}(\ln u)\frac{du}{dx} \\
&= 1 + \left(\frac{1}{u}\right)(\sec x \tan x) \\
&= 1 + \frac{1}{\cancel{\sec x}}\cancel{\sec x}\tan x \\
&= 1 + \tan x.
\end{aligned}$$

Example

Find $f'(x)$ if $f(x) = \ln|x|$.

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$$= \frac{1}{x} \text{ if } x \neq 0.$$

Example

Differentiate $x^{\tan x}$, where $x > 0$.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{\tan x}\right)$$

Example

Differentiate $x^{\tan x}$, where $x > 0$.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{\tan x}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\left(e^{\mathbf{?}}\quad\right)^{\tan x}\right)$$

Convert base to $e^{?}$

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Convert base to e?

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Differentiate $x^{\tan x}$, where $x > 0$.

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\left(x^{\tan x}\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\left(e^{\ln x}\right)^{\tan x}\right) && \Big| \text{ Convert base to } e^? \\ &= \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{(\ln x)\tan x}\right) \\ &= \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{u}\right) && \Big| \text{ Set } (\ln x)\tan x = u \end{aligned}$$

Example

Differentiate $x^{\tan x}$, where $x > 0$.

$$\begin{aligned}
\frac{d}{dx}\left(x^{\tan x}\right) &= \frac{d}{dx}\left(\left(e^{\ln x}\right)^{\tan x}\right) && \text{Convert base to } e^{?}\\
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\end{aligned}$$

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Convert base to e?

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&= {\color{red}\frac{\mathrm{d}}{\mathrm{d}u}\left(e^{u}\right)}\frac{\mathrm{d}u}{\mathrm{d}x} && \text{Chain rule} \\
&= {\color{red}\textbf{?}}\ \frac{\mathrm{d}}{\mathrm{d}x}\left((\ln x)\tan x\right)
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&= e^{u}\frac{\mathrm{d}}{\mathrm{d}x}\left((\ln x){\color{red}\tan x}\right) \\
&= e^{(\ln x)\tan x}\left((\ln x)'{\color{red}\tan x} + (\ln x)\left({\color{red}\tan x}\right)'\right) && {\color{red}\text{Prod. rule}}
\end{aligned}$$

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Differentiate $x^{\tan x}$, where $x > 0$.

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&= e^{(\ln x)\tan x}\left((\ln x)'\tan x + (\ln x)\,{\color{red}(\tan x)'}\right) && \text{Prod. rule}\\
&= x^{\tan x}\left(\frac{1}{x}\tan x + (\ln x)\,{\color{red}\sec^2 x}\right)
\end{aligned}
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Example

Differentiate $(3x+1)^{\ln x}$, where $3x+1>0$.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left((3x+1)^{\ln x}\right)$$

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$$\frac{\mathrm{d}}{\mathrm{d}x}\left((3x+1)^{\ln x}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\left(e^{\mathbf{?}} \qquad \right)^{\ln x}\right) \qquad \Bigg| \text{ Convert base to } e^{?}$$

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&= \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{\ln(3x+1)\ln x}\right) \\
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Steps in Logarithmic Differentiation

1. Take natural logarithms of both sides of an equation $y = f(x)$.
2. Use the properties of logarithms to simplify.
3. Differentiate implicitly with respect to x.
4. Solve the resulting equation for y'.

Note: If $f(x) < 0$, then we use $\ln |f(x)|$ instead as $\ln(f(x))$ is not defined. We computed the derivative of $\ln |f(x)|$ in the previous lecture.

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Theorem (The Number e as a Limit)

$$e = \lim_{x\to 0}(1+x)^{\frac{1}{x}} = \lim_{y\to\infty}\left(1+\frac{1}{y}\right)^{y}.$$

Proof.

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