Calculus II
Integral of rational function with cubic denominator, part 2

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Example

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- In a similar fashion we add more partial fractions to account for all other terms of the form $(a_sx + b_s)^t$.