Calculus II
Power series, full lecture

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Outline

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Power Series

Definition (Power Series)

A power series is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

where x is a variable and the c_n's are constants called the coefficients of the series.

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  whose domain is the set of all x for which the series converges.
- f resembles a polynomial, except it has infinitely many terms.

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A series of the form
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- We use the convention that $(x-a)^0 = 1$, even if $x = a$.
- If $x = a$, then all terms are 0 for $n \geq 1$, so the series always converges when $x = a$.

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- Therefore by the Ratio Test the series diverges for all $x \neq 0$.
- Therefore the series only converges for $x = 0$.

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- Therefore by the Ratio Test the series converges for all x.
- Therefore the domain of the function is $(-\infty, \infty)$, or $\mathbb{R}$.

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- The series converges if $2\leq x<4$ and diverges otherwise.

Theorem (Convergence of Power Series)

For a power series $\sum c_n(x-a)^n$, there are three possibilities:

1. *The series converges only when $x = a$.*
2. *The series converges for all x.*
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3. In the third case, the inequality $|x-a| < R$ can be rewritten $a - R < x < a + R$.

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- In general, the Ratio Test (or Root Test) should be used to find the radius of convergence R.
- The Ratio and Root Tests will always fail when x is an endpoint $a - R$ or $a + R$, so the endpoints must be checked with another test.

Example

Find the radius of convergence and interval of convergence of the series $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$.

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- Ratio Test: it converges if $3|x| < 1$ and diverges if $3|x| > 1$.
- So it converges if $|x| < \frac{1}{3}$ and diverges if $|x| > \frac{1}{3}$.

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- Therefore $R = \frac{1}{3}$.
- If we use $x = \frac{1}{3}$, we get $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$, which is
- If we use $x = -\frac{1}{3}$, we get $\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}}$, which is

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$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(-3)^{n+1}x^{n+1}}{\sqrt{n+2}}\cdot\frac{\sqrt{n+1}}{(-3)^n x^n}\right| \\ &= \lim_{n\to\infty} 3|x|\sqrt{\frac{n+1}{n+2}\cdot\frac{\frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{n}}}} = \lim_{n\to\infty} 3|x|\sqrt{\frac{1+\frac{1}{n}}{1+\frac{2}{n}}} = 3|x|\end{aligned}$$

- Ratio Test: it converges if $3|x| < 1$ and diverges if $3|x| > 1$.
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- Therefore $R = \frac{1}{3}$.
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- In this way $g(x) = \sum_{n=0}^{\infty} x^n$ is a new way to compute/expresses the function $f(x) = \frac{1}{1-x}$ for $|x| < 1$.
- Except for their domains, the functions $g(x)$ and $f(x)$ coincide.

Recall the geometric series formula:

$$\frac{1}{1-y} = \sum_{n=0}^{\infty} y^n = 1 + y + y^2 + y^3 + \dots \quad \text{if \& only if } |y| < 1 \quad .$$

Example

Write $\frac{1}{1+x^2}$ as a power series and find the interval of convergence.

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$$\frac{1}{1+x^2} = \frac{1}{1-\left(-x^2\right)} = \sum_{n=0}^{\infty} \left(-x^2\right)^n \qquad \Bigg| \quad \text{if \& only if } |-x^2| < 1$$

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\frac{1}{2+x} &= \frac{1}{2\left(1+\frac{x}{2}\right)} \\
&= \frac{1}{2}\cdot\frac{1}{\left(1-\left(-\frac{x}{2}\right)\right)} = \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n \qquad \Bigg| \quad \text{if \& only if } \left|-\tfrac{x}{2}\right| < 1 \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^n \\
&= \frac{1}{2}-\frac{x}{4}+\frac{x^2}{8}-\frac{x^3}{16}+\dots
\end{aligned}$$

To find interval of convergence:

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\left|-\frac{x}{2}\right| &< 1 \\
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Therefore the interval of convergence is $x \in (-2, 2)$.

Example

Find a power series representation for $\frac{x^3}{x+2}$.

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&= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} \mathbf{?}
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if & only if $|x| < 2$

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- Another way to write this is $\frac{x^3}{x+2} = \sum_{n=3}^{\infty} \frac{(-1)^{n-1}}{2^{n-2}} x^n$.

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- The interval of convergence is again $x \in (-2, 2)$.

Differentiation and Integration of Power Series

Theorem (Differentiation and Integration of Power Series)

If a power series $\sum c_n(x-a)^n$ has radius of convergence $R > 0$, then the function f defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ and

1. $$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}.$$
2. $$\int f(x)\,\mathrm{d}x = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots$$ $$= C + \sum_{n=0}^{\infty} c_n\frac{(x-a)^{n+1}}{n+1}.$$

- This is called term-by-term differentiation and integration.
- Another way of saying it is

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\sum_{n=0}^{\infty} c_n(x-a)^n\right] = \sum_{n=0}^{\infty} \frac{\mathrm{d}}{\mathrm{d}x}\left[c_n(x-a)^n\right]$$

$$\int\left[\sum_{n=0}^{\infty} c_n(x-a)^n\right]\mathrm{d}x = \sum_{n=0}^{\infty} \int \left[c_n(x-a)^n\right]\mathrm{d}x$$

- We can treat power series like polynomials with infinitely many terms.

Example

Find the derivative of the Bessel function

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

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Find the derivative of the Bessel function

$$\begin{aligned} J_0(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2} \\ J_0'(x) &= \sum_{n=0}^{\infty} \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{(-1)^n x^{2n}}{2^{2n}(n!)^2} \right) \end{aligned}$$

Example

Find the derivative of the Bessel function

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J_0(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2} \\
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&= \sum_{n=0}^{\infty} \frac{(-1)^n 2n x^{2n-1}}{2^{2n}(n!)^2}
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- $J_0(x)$ is defined

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- $J_0(x)$ is defined everywhere.

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- $J_0(x)$ is defined everywhere.
- Therefore its derivative $J_0'(x)$ is also defined everywhere.

Example

Find a power series for $\ln(1-x)$ and state its radius of convergence.

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$$\ln(1-x) \quad = \quad \int d(\ln(1-x)) \qquad \Big| \text{ up to const.}$$

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$$\begin{aligned}\ln(1-x) &= \int \mathrm{d}(\ln(1-x)) = \int (\ln(1-x))' \mathrm{d}x \quad \Big| \text{ up to const.} \\ &= \int \textbf{?} \qquad \mathrm{d}x\end{aligned}$$

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$$\begin{aligned}\ln(1-x) &= \int \mathrm{d}(\ln(1-x)) = \int (\ln(1-x))' \mathrm{d}x & \Big| \text{ up to const.}\\ &= \int \left(-\frac{1}{1-x}\right)\mathrm{d}x\end{aligned}$$

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$$\begin{array}{rcll} \ln(1-x) &=& \int \mathrm{d}(\ln(1-x)) = \int (\ln(1-x))'\mathrm{d}x & \Big| \text{ up to const.} \\ &=& \int \left(-\frac{1}{1-x}\right)\mathrm{d}x & \\ &=& -\int \Big(\mathbf{?} \qquad\qquad\qquad\qquad \Big)\mathrm{d}x & \Big| \text{ for } |x|<1 \end{array}$$

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$$\begin{array}{rcll} \ln(1-x) &=& \int d(\ln(1-x)) = \int (\ln(1-x))' dx & \Big| \text{ up to const.} \\ &=& \int \left(-\frac{1}{1-x}\right) dx & \\ &=& -\int \left(1 + x + x^2 + x^3 + \cdots\right) dx & \Big| \text{ for } |x| < 1 \\ &=& -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots\right) & \end{array}$$

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- To find C, plug in $x = 0$: $C =$ **?**

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&=& \displaystyle C - \sum_{n=1}^{\infty} \frac{x^n}{n} &
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- To find C, plug in $x = 0$: $C = 0$.
- Therefore the theorem on integrating power series implies that $\ln(1-x) = -\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}$, for $|x| < 1$.

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- To find C, plug in $x=0$: $C=0$.
- Therefore the theorem on integrating power series implies that
  $$\ln(1-x) = -\sum_{n=1}^{\infty}\frac{x^n}{n}, \text{ for } |x|<1.$$
- By the same theorem, the radius of convergence remains $R=1$.

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Find a power series for $\arctan x$ and state its radius of convergence.

$\arctan(x)$

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$$\begin{aligned} \arctan(x) &= \int d(\arctan x) = \int (\arctan x)' dx && \Big| \text{ up to const.} \\ &= \int \left(\frac{1}{1+x^2} \right) dx \end{aligned}$$

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Find a power series for $\arctan x$ and state its radius of convergence.

$$\begin{array}{rcl|l}
\arctan(x) & = & \displaystyle\int d(\arctan x) = \int (\arctan x)' dx & \text{up to const.} \\
& = & \displaystyle\int \left(\frac{1}{1+x^2}\right) dx = \int \left(\frac{1}{1-(-x^2)}\right) dx & \\
& = & \displaystyle\int \left(\mathbf{?} \qquad\qquad\qquad \right) dx & \text{for } |x| < 1
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&= \int \left(1 - x^2 + x^4 - x^6 + \cdots \right) \mathrm{d}x && \Big| \text{ for } |x| < 1
\end{aligned}$$

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\arctan(x) &= \int d(\arctan x) = \int (\arctan x)' dx && \text{up to const.}\\
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&= \int \left(1 - x^2 + x^4 - x^6 + \cdots\right) dx && \text{for } |x| < 1\\
&= \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\right)
\end{aligned}$$

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Find a power series for $\arctan x$ and state its radius of convergence.

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- By the same theorem, the radius of convergence remains $R = 1$.

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- Therefore $c_n = \frac{f^{(n)}(a)}{n!}$.

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If f has a power series representation at a, that is, if

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Definition (Taylor Series)

This series is called the Taylor series of f.

The case when $a = 0$ is special enough to have its own name:

Definition (Maclaurin Series)

The Maclaurin series of f is the Taylor series of f centered at $a = 0$. In other words, it is the series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \cdots$$

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$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$
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- Therefore $R = \infty$.

Example

Find the sum of the series

$$\sum_{n=0}^{\infty}(-1)^n\frac{1}{2^n n!} = 1-\frac{1}{2\cdot 1!}+\frac{1}{4\cdot 2!}-\frac{1}{8\cdot 3!}+\cdots$$

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- Just like the Maclaurin series, this series also represents e^x.

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Recall that $e^y = \sum_{n=0}^{\infty} \frac{y^n}{n!}$

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Set $y = x - 3$

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The radius of convergence was already computed to be $R = \infty$.

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Find the Maclaurin series of $f(x) = \sin x$ and its radius of convergence.

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Therefore $R =$.

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Therefore $R = \infty$. It can be shown that this series sums to $\sin x$.

Example

Find the sum of the series

$$\sum_{n=0}^{\infty}(-1)^n\frac{\pi^{2n+1}}{2^{2n+1}(2n+1)!}=\frac{\pi}{2}-\frac{\pi^3}{8\cdot 3!}+\frac{\pi^5}{32\cdot 5!}-\frac{\pi^7}{128\cdot 7!}+\cdots$$

Example

Find the sum of the series

$$\sum_{n=0}^{\infty}(-1)^n\frac{\pi^{2n+1}}{2^{2n+1}(2n+1)!} = \frac{\pi}{2} - \frac{\pi^3}{8\cdot 3!} + \frac{\pi^5}{32\cdot 5!} - \frac{\pi^7}{128\cdot 7!} + \cdots$$

$$\sin x = \sum_{n=0}^{\infty}$$

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$$\sum_{n=0}^{\infty}(-1)^n\frac{\pi^{2n+1}}{2^{2n+1}(2n+1)!}=\frac{\pi}{2}-\frac{\pi^3}{8\cdot 3!}+\frac{\pi^5}{32\cdot 5!}-\frac{\pi^7}{128\cdot 7!}+\cdots$$

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$$\sum_{n=0}^{\infty}(-1)^n\frac{\pi^{2n+1}}{2^{2n+1}(2n+1)!} = \sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)!}\left(\frac{\pi}{2}\right)^{2n+1}$$

Example

Find the sum of the series

$$\sum_{n=0}^{\infty}(-1)^n\frac{\pi^{2n+1}}{2^{2n+1}(2n+1)!}=\frac{\pi}{2}-\frac{\pi^3}{8\cdot 3!}+\frac{\pi^5}{32\cdot 5!}-\frac{\pi^7}{128\cdot 7!}+\cdots$$

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$$\begin{aligned}\sum_{n=0}^{\infty}(-1)^n\frac{\pi^{2n+1}}{2^{2n+1}(2n+1)!} &= \sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)!}\left(\frac{\pi}{2}\right)^{2n+1}\\ &= \sin\frac{\pi}{2}\end{aligned}$$

Example

Find the sum of the series

$$\sum_{n=0}^{\infty}(-1)^n\frac{\pi^{2n+1}}{2^{2n+1}(2n+1)!}=\frac{\pi}{2}-\frac{\pi^3}{8\cdot 3!}+\frac{\pi^5}{32\cdot 5!}-\frac{\pi^7}{128\cdot 7!}+\cdots$$

$$\begin{aligned}
\sin x &= \sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)!}x^{2n+1}\\
\sum_{n=0}^{\infty}(-1)^n\frac{\pi^{2n+1}}{2^{2n+1}(2n+1)!} &= \sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)!}\left(\frac{\pi}{2}\right)^{2n+1}\\
&= \sin\frac{\pi}{2}\\
&= 1
\end{aligned}$$

Example

Find the Maclaurin series for $\cos x$.

Example

Find the Maclaurin series for $\cos x$.

$$\cos x = \frac{\mathrm{d}}{\mathrm{d}x}\left(\quad \right)$$

Example

Find the Maclaurin series for $\cos x$.

$$\cos x = \frac{d}{dx}(\sin x)$$

Example

Find the Maclaurin series for $\cos x$.

$$\begin{aligned} \cos x &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\sin x\right) \\ &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\qquad\qquad\qquad \right) \end{aligned}$$

Example

Find the Maclaurin series for $\cos x$.

$$\begin{aligned}\cos x &= \frac{d}{dx}\left(\sin x\right)\\ &= \frac{d}{dx}\left(\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}\right)\end{aligned}$$

Example

Find the Maclaurin series for $\cos x$.

$$\begin{aligned}
\cos x &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\sin x\right)\\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}\right)\\
&= \sum_{n=0}^{\infty}\frac{\mathrm{d}}{\mathrm{d}x}\left((-1)^n\frac{x^{2n+1}}{(2n+1)!}\right)
\end{aligned}$$

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Find the Maclaurin series for $\cos x$.

$$\begin{aligned}
\cos x &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\sin x\right)\\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}\right)\\
&= \sum_{n=0}^{\infty}\frac{\mathrm{d}}{\mathrm{d}x}\left((-1)^n\frac{x^{2n+1}}{(2n+1)!}\right)\\
&= \sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)x^{2n}}{(2n+1)!}
\end{aligned}$$

Example

Find the Maclaurin series for $\cos x$.

$$
\begin{aligned}
\cos x &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\sin x\right)\\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}\right)\\
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&= \sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)x^{2n}}{(2n+1)!}\\
&= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{}
\end{aligned}
$$

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Find the Maclaurin series for $\cos x$.

$$\begin{aligned}
\cos x &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\sin x\right)\\
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&= \sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)x^{2n}}{(2n+1)!}\\
&= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}
\end{aligned}$$

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Find the Maclaurin series for $\cos x$.

$$
\begin{aligned}
\cos x &= \frac{\mathrm{d}}{\mathrm{d}x}(\sin x) \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}\right) \\
&= \sum_{n=0}^{\infty} \frac{\mathrm{d}}{\mathrm{d}x}\left((-1)^n \frac{x^{2n+1}}{(2n+1)!}\right) \\
&= \sum_{n=0}^{\infty}(-1)^n \frac{(2n+1)x^{2n}}{(2n+1)!} \\
&= \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!} \\
&= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots
\end{aligned}
$$

Example

Find the Maclaurin series for $\cos x$.

$$
\begin{aligned}
\cos x &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\sin x\right)\\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}\right)\\
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&= 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots
\end{aligned}
$$

The series for $\sin x$ converges everywhere, so the series for $\cos x$ does too.

Example

Find the Maclaurin series for $x \cos x$.

Example

Find the Maclaurin series for $x \cos x$.

$$x \cos x = x$$

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Find the Maclaurin series for $x \cos x$.

$$x \cos x = x \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

Example

Find the Maclaurin series for $x \cos x$.

$$\begin{aligned} x\cos x &= x\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} \\ &= \sum_{n=0}^{\infty}(-1)^n\frac{\phantom{x^{2n+1}}}{(2n)!} \end{aligned}$$

Example

Find the Maclaurin series for $x \cos x$.

$$\begin{aligned} x \cos x &= x \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n)!} \end{aligned}$$

Example

Find the Maclaurin series for $x \cos x$.

$$
\begin{aligned}
x \cos x &= x \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n)!} \\
&= x - \frac{x^3}{2!} + \frac{x^5}{4!} - \frac{x^7}{6!} + \cdots
\end{aligned}
$$

Here is a table of some important Maclaurin series we have learned:

Function	Series	R
$\dfrac{1}{1-x}$	$= \displaystyle\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$	1
$\arctan x$	$= \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$	1
e^x	$= \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$	∞
$\sin x$	$= \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$	∞
$\cos x$	$= \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$	∞

Example

Use a power series to find $\lim_{x \to 0} \dfrac{e^x - 1 - x}{x^2}$.

Example

Use a power series to find $\lim\limits_{x\to 0} \dfrac{e^x - 1 - x}{x^2}$.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Example

Use a power series to find $\lim_{x \to 0} \frac{e^x - 1 - x}{x^2}$.

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \\ e^x - 1 - x &= \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \end{aligned}$$

Example

Use a power series to find $\lim\limits_{x \to 0} \dfrac{e^x - 1 - x}{x^2}$.

$$\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \\
e^x - 1 - x &= \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \\
\frac{e^x - 1 - x}{x^2} &= \frac{1}{2!} + \frac{x}{3!} + \frac{x^2}{4!} + \cdots
\end{aligned}$$

Example

Use a power series to find $\lim\limits_{x\to 0} \dfrac{e^x - 1 - x}{x^2}$.

$$
\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \\
e^x - 1 - x &= \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \\
\frac{e^x - 1 - x}{x^2} &= \frac{1}{2!} + \frac{x}{3!} + \frac{x^2}{4!} + \cdots \\
\lim_{x\to 0} \frac{e^x - 1 - x}{x^2} &= \lim_{x\to 0} \left(\frac{1}{2} + \frac{x}{3!} + \frac{x^2}{4!} + \cdots \right) \qquad .
\end{aligned}
$$

Example

Use a power series to find $\lim\limits_{x\to 0}\dfrac{e^x-1-x}{x^2}$.

$$
\begin{aligned}
e^x &= 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\\
e^x-1-x &= \frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots\\
\frac{e^x-1-x}{x^2} &= \frac{1}{2!}+\frac{x}{3!}+\frac{x^2}{4!}+\cdots\\
\lim_{x\to 0}\frac{e^x-1-x}{x^2} &= \lim_{x\to 0}\left(\frac{1}{2}+\frac{x}{3!}+\frac{x^2}{4!}+\cdots\right) = \qquad .
\end{aligned}
$$

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Use a power series to find $\lim_{x\to 0} \frac{e^x - 1 - x}{x^2}$.

$$\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \\
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\frac{e^x - 1 - x}{x^2} &= \frac{1}{2!} + \frac{x}{3!} + \frac{x^2}{4!} + \cdots \\
\lim_{x\to 0} \frac{e^x - 1 - x}{x^2} &= \lim_{x\to 0}\left(\frac{1}{2} + \frac{x}{3!} + \frac{x^2}{4!} + \cdots\right) = \frac{1}{2} \quad .
\end{aligned}$$

Example

Use a power series to find $\lim_{x\to 0} \frac{x - \sin x}{x^3}$.

Example

Use a power series to find $\lim_{x \to 0} \frac{x - \sin x}{x^3}$.

$$\sin x \quad = \quad x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

Example

Use a power series to find $\lim_{x\to 0} \frac{x - \sin x}{x^3}$.

$$\begin{aligned} \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\ -\sin x &= -x + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \cdots \end{aligned}$$

Example

Use a power series to find $\lim\limits_{x\to 0} \dfrac{x-\sin x}{x^3}$.

$$\begin{aligned}
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\
-\sin x &= -x + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \cdots \\
x - \sin x &= \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \cdots
\end{aligned}$$

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Use a power series to find $\lim_{x\to 0} \frac{x - \sin x}{x^3}$.

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x - \sin x &= \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \cdots \\
\frac{x - \sin x}{x^3} &= \frac{1}{3!} - \frac{x^2}{5!} + \frac{x^4}{7!} - \cdots
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Use a power series to find $\lim_{x\to 0} \frac{x - \sin x}{x^3}$.

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\frac{x - \sin x}{x^3} &= \frac{1}{3!} - \frac{x^2}{5!} + \frac{x^4}{7!} - \cdots \\
\lim_{x\to 0} \frac{x - \sin x}{x^3} &= \lim_{x\to 0} \left(\frac{1}{6} - \frac{x^2}{5!} + \frac{x^4}{7!} - \cdots \right)
\end{aligned}$$

Example

Use a power series to find $\lim\limits_{x\to 0} \dfrac{x-\sin x}{x^3}$.

$$
\begin{aligned}
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\
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\lim_{x\to 0} \frac{x-\sin x}{x^3} &= \lim_{x\to 0} \left(\frac{1}{6} - \frac{x^2}{5!} + \frac{x^4}{7!} - \cdots \right) =
\end{aligned}
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\frac{x - \sin x}{x^3} &= \frac{1}{3!} - \frac{x^2}{5!} + \frac{x^4}{7!} - \cdots \\
\lim_{x\to 0} \frac{x - \sin x}{x^3} &= \lim_{x\to 0} \left(\frac{1}{6} - \frac{x^2}{5!} + \frac{x^4}{7!} - \cdots \right) = \frac{1}{6}
\end{aligned}$$