

# Precalculus

## Complex numbers definition; overview of numbers

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# Outline

## 1 Complex numbers definition

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- 2 Complex numbers multiplication and addition

## Definition (Complex numbers)

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where the number  $i$  is a number for which

$$i^2 = -1 \quad .$$

The number  $i$  is called the imaginary unit.

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$$\begin{aligned}(a + bi)(c + di) &= ac + adi + bci + bdi^2 = \textcolor{red}{ac} + adi + bci - \textcolor{red}{bd} \\ &= (\textcolor{red}{ac} - \textcolor{red}{bd}) + i(ad + bc)\end{aligned}$$



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Let  $u = 2 + 3i$ ,  $v = 5 - 7i$ .

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$$u + v =$$

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### Example (Subtraction)

$$u - v =$$

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### Example (Multiplication)

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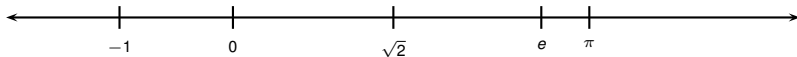
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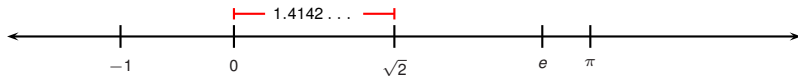
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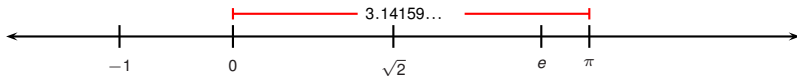
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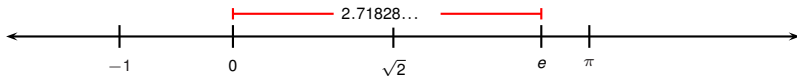
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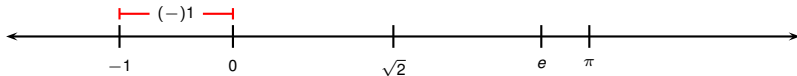
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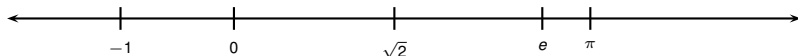
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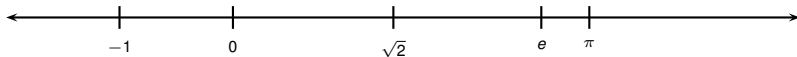
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- Geometric interpretation of complex numbers: beyond our scope.