Calculus II
Sequences

Todor Milev

2019

Outline

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- We are interested to study sequences such as:

$$\begin{array}{l} (a_1, a_2, a_3 \dots) \\ (1, 2, 3, \dots) \\ (1, 3, 5, 7, \dots) \\ (1, -1, 1, -1, \dots) \quad . \end{array}$$

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A sequence is an ordered collection of objects in which repetitions are allowed.

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- We start by a few examples.

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where a_n denotes the nth term.

$$\begin{aligned} a_1 &= 2 \cdot 1 = 2 \\ a_2 &= 2 \cdot 2 = 4 \\ a_3 &= 2 \cdot 3 = 6 \\ a_4 &= 2 \cdot 4 = 8 \\ &\vdots \end{aligned}$$

Example

The sequence

$$(-1, 1, -1, 1, -1, 1, \dots)$$

can be written as $b_n = (-1)^n$.

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The sequence

$$\left(\frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, -\frac{1}{16}, \dots\right)$$

can be written as $d_n = -\left(-\frac{1}{2}\right)^n$.

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A sequence is a list of numbers
written in a definite order

Definition (Sequence indexed by the integers)

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- The use of $\{\}$ versus $()$ differs between authors and instructors.

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$$\begin{array}{rcl} & & (1, 2, 3, 4, 5, 6, 7, 8, 9, 10) \\ (a_n)_{n=1}^{5} & \text{for} & (a_1, a_2, a_3, a_4, a_5) \end{array}$$

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    - by specifying a formula for the n^{th} term;
    - by recursion;
    - by specifying a property of integers and constructing a sequence of all integers with that property.

Sequences via formulas

- Sequences can be defined by presenting a formula to obtain the n^{th} term a_n as a function of the index n.

Example

$$a_n = \frac{n}{n+1} \qquad \left(\frac{n}{n+1}\right)_{n=1}^{\infty}$$

$$a_n = \frac{(-1)^n(n+1)}{3^n} \qquad \left(\frac{(-1)^n(n+1)}{3^n}\right)_{n=1}^{\infty}$$

$$a_n = \sqrt{n-3}, n \geq 3 \qquad \left(\sqrt{n-3}\right)_{n=3}^{\infty}$$

$$a_n = \cos\left(\frac{n\pi}{6}\right), n \geq 0 \qquad \left(\cos\frac{n\pi}{6}\right)_{n=0}^{\infty}$$

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$a_n = \sqrt{n-3}, n \geq 3$	$\left(\sqrt{n-3}\right)_{n=3}^{\infty}$	$\left(0, 1, \sqrt{2}, \sqrt{3}, \dots\right)$
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Example (Sequences via formulas: find sequence terms)

Find the first five terms of each of the following sequences.

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$$2, 5, 10, 17, 26, \dots$$

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Find a formula for the general term a_n of the sequence

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- The n^{th} term has numerator **?**

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Find a formula for the nth term of each of the following sequences.

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$$\left(2, \frac{1}{2}, \frac{1}{8}, \frac{1}{32}, \frac{1}{128}, \dots\right)$$

2. $b_n =$

$$-1, 4, -9, 16, -25, \dots$$

3. $c_n =$

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- We found the sequence $\left(0, \frac{1}{4}, -\frac{2}{8}, \frac{3}{16}, -\frac{4}{32}, \frac{5}{64}, \dots\right)$ can be given by: $a_n = (-1)^n \frac{n-1}{2^n}$

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Define recursively the Fibonacci sequence $(f_n)_{n=1}^{\infty}$ by requesting that

$$f_1 = 1 \qquad f_2 = 1 \qquad f_n = f_{n-1} + f_{n-2}, \quad n \geq 3.$$

The first few terms are

$$1, 1, 2, 3, 5, 8, \mathbf{?}$$

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- In fact the Fibonacci sequence can be described by a formula, but it is not very simple: $a_n = \frac{\sqrt{5}}{5}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$.

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- For example, a crude test for whether a number is prime is to check whether it is divisible by all positive numbers smaller than it.
- Our sequence is well defined; we could generate it, say, by computer.
- However, we have given no closed or even recursive formula to generate the entire sequence.

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- We note that in addition to the illustrated ways to define sequences, we are also free to use for the task any well-posed statement.

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1. Let a_n be the n^{th} digit in the decimal expansion of the number e.

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1. Let a_n be the n^{th} digit in the decimal expansion of the number e. The first few terms of (a_n):

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2. Consider the sequence (p_n), where p_n is the population of the world as of January 1 of year n.

Definition (Arithmetic sequence)

An arithmetic sequence is one in which successive terms differ by a constant number. This constant is called the difference of the arithmetic sequence.

Example (Which are arithmetic?)

$$\begin{array}{rrrrrll}
1, & 2, & 3, & 4, & 5, & \dots & \text{is arithmetic with difference } 1. \\
23, & 16, & 9, & 2, & -5, & \dots & \text{is arithmetic with difference } -7. \\
8, & 9, & 12, & 17, & 24, & \dots & \text{is not arithmetic.} \\
 & & & & & & (9-8=1 \text{ but } 12-9=3.)
\end{array}$$

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Sequence	Arithmetic?	Difference	First term	nth term
$1, -1, 1, -1, \ldots$				
$\frac{1}{6}, \frac{1}{2}, \frac{5}{6}, \frac{7}{6}, \frac{3}{2}, \ldots$				
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$2, 2, 2, 2, \dots$	yes	0	2	

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$\frac{1}{6}, \frac{1}{2}, \frac{5}{6}, \frac{7}{6}, \frac{3}{2}, \dots$	yes	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6} + \frac{1}{3}(n-1)$
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$2, 2, 2, 2, \dots$	yes	0	2	$2 + 0(n-1)$

If an arithmetic sequence has difference d, then the nth term has formula

$$a_n = a_1 + d(n-1),$$

where a_1 is the first term.

Definition (Geometric sequence)

A geometric sequence is one in which each term is obtained by multiplying the previous one by the same constant. This constant is called the ratio of the geometric sequence.

Example (Which are geometric?)

$$\begin{array}{rrrrrll} 2, & 4, & 8, & 16, & 32, & \dots & \text{is geometric with ratio } 2. \\ 1, & -3, & 9, & -27, & 81, & \dots & \text{is geometric with ratio } -3. \\ -42, & -14, & -21, & 31, & -22, & \dots & \text{is not geometric.} \\ & & & & & & \left(\frac{-14}{-42} = \frac{1}{3} \text{ but } \frac{-21}{-14} = \frac{3}{2}.\right) \end{array}$$

Example (Arithmetic and geometric)

Sequence	Arithmetic/ geometric	Diff.	Ratio	a_1	a_n
$\frac{2}{3}, \frac{4}{9}, \frac{8}{27}, \frac{16}{81}, \dots$					
$7, 3, -1, -5, \dots$					
$4, 4, 4, 4, \dots$					
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$7, 3, -1, -5, \dots$	arithmetic	-4	—	7	
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$7, 3, -1, -5, \dots$	arithmetic	-4	—	7	$7-4(n-1)$
$4, 4, 4, 4, \dots$					
$\pi, -\pi^2, \pi^3, -\pi^4, \dots$					
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Example (Arithmetic and geometric)

Sequence	Arithmetic/ geometric	Diff.	Ratio	a_1	a_n
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$4, 4, 4, 4, \dots$	both	0			
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$4, 4, 4, 4, \dots$	both	0	1	4	
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$1, 1, 2, 2, 3, 3, \dots$	neither	—	—	1	$\left\lceil \frac{n}{2} \right\rceil$

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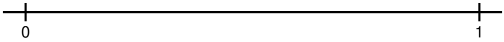
$7, 3, -1, -5, \dots$	arithmetic	-4	—	7	$7-4(n-1)$
$4, 4, 4, 4, \dots$	both	0	1	4	$4 = 4(1)^{n-1}$
$\pi, -\pi^2, \pi^3, -\pi^4, \dots$	geometric	—	$-\pi$	π	$\pi(-\pi)^{n-1}$
$1, 1, 2, 2, 3, 3, \dots$	neither	—	—	1	$\lceil \frac{n}{2} \rceil$

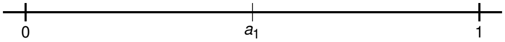
If a geometric sequence has ratio r, then the nth term has formula

$$a_n = a_1 r^{n-1}.$$

where a_1 is the first term.



- The sequence $a_n = \frac{n}{n+1}$ can be plotted on a number line or using Cartesian coordinates.



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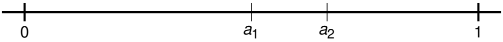

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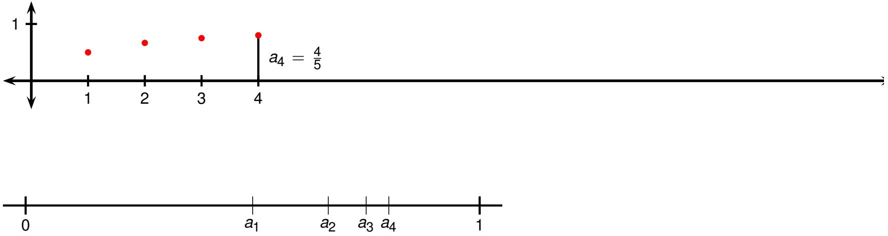

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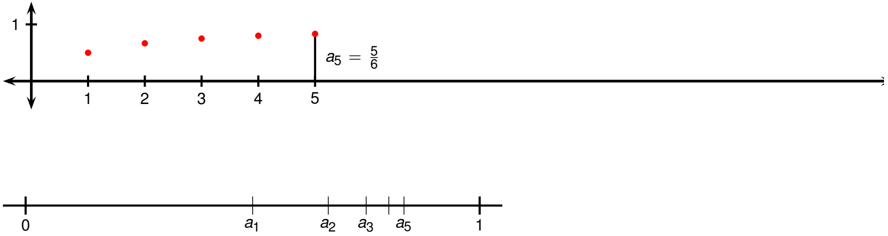

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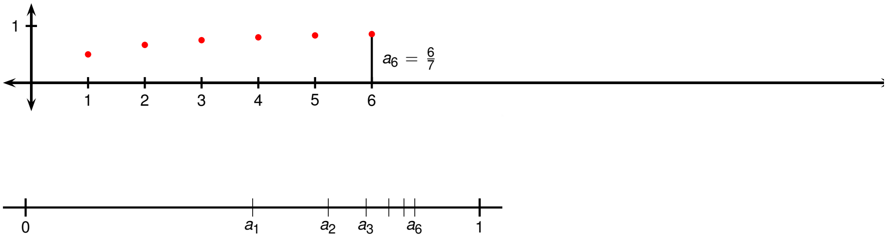

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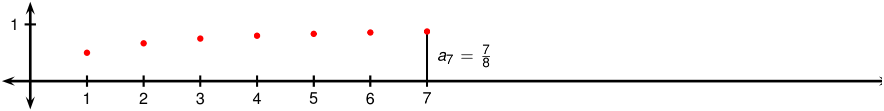

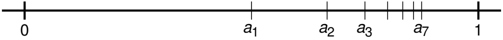

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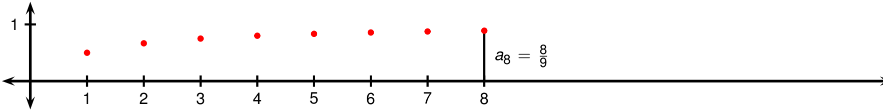

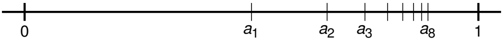

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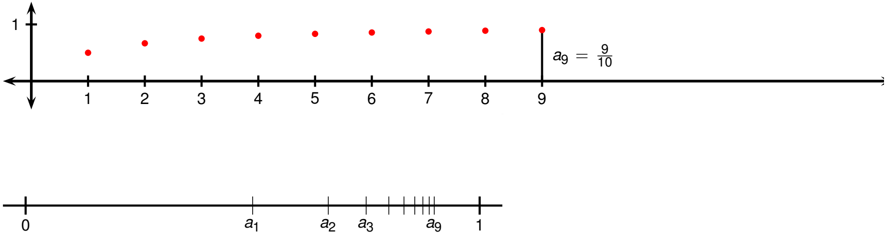

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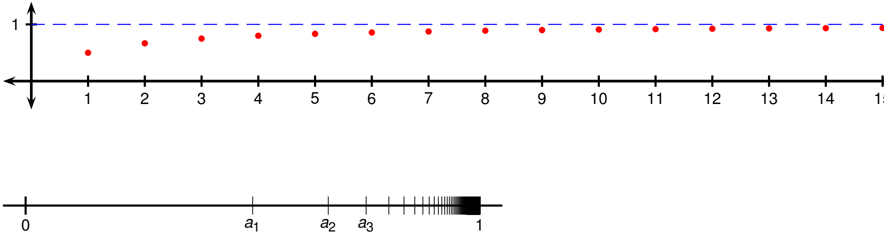

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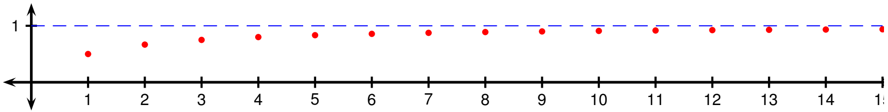
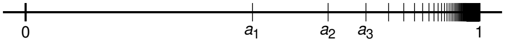
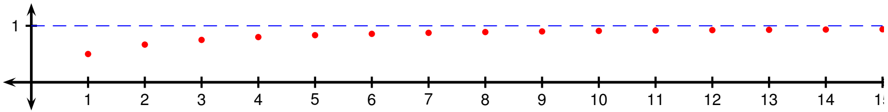
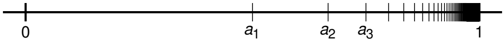




- The sequence $a_n = \frac{n}{n+1}$ can be plotted on a number line or using Cartesian coordinates.
- From the pictures, the terms in the sequence appear to approach 1 as n gets larger.





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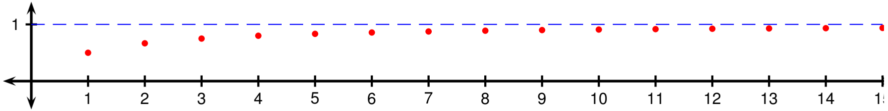

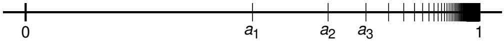

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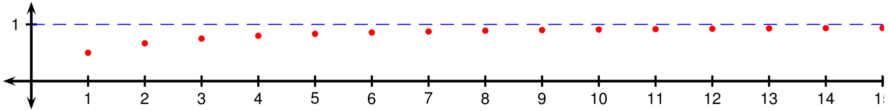

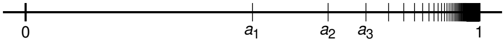

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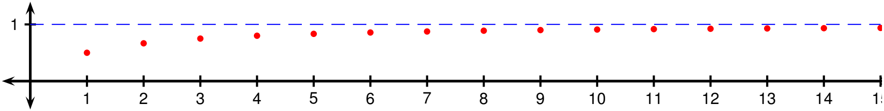

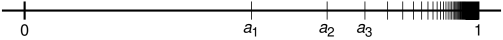

- The sequence $a_n = \frac{n}{n+1}$ can be plotted on a number line or using Cartesian coordinates.
- From the pictures, the terms in the sequence appear to approach 1 as n gets larger.
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- This can be made arbitrarily small by choosing n large enough.
- We express this by writing $\lim\limits_{n\to\infty} \frac{n}{n+1} = 1$.

Definition (Limit of a Sequence)

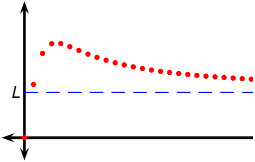
A sequence $\{a_n\}$ has the limit L, and we write

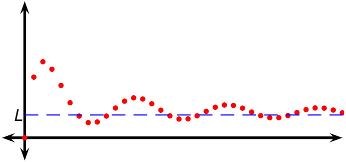
$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L \text{ as } n \to \infty$$

if we can make a_n as close to L as we like by taking n large enough.

Definition (Convergent)

A sequence that has a limit is called convergent. A sequence that has no limit is called divergent.

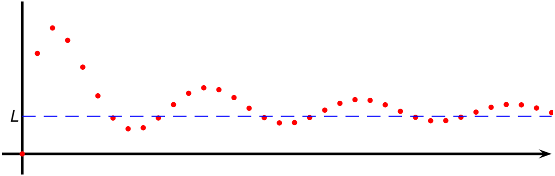




If you compare the definition of the limit of a sequence with the definition of the infinite limit of a function, you'll see that the only difference between

$$\lim_{n\to\infty} a_n = L \qquad \text{and} \qquad \lim_{x\to\infty} f(x) = L$$

is that n is required to be an integer.



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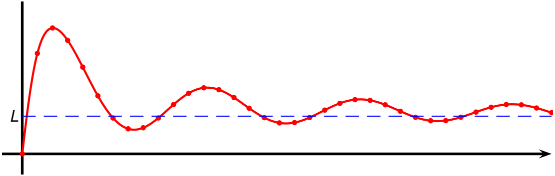

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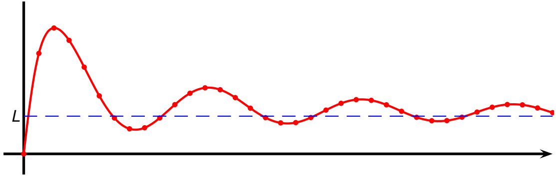

Theorem

If $\lim_{x\to\infty} f(x) = L$ *and* $f(n) = a_n$ *for all integers n, then* $\lim_{n\to\infty} a_n = L$*.*

Example

Find $\lim_{n\to\infty} \frac{n}{n+1}$.

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You can probably guess what $\lim_{n\to\infty} a_n = -\infty$ means.

The Limit Laws for continuous functions also hold for sequences:
If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then

1. $\lim\limits_{n\to\infty}(a_n + b_n) = \lim\limits_{n\to\infty} a_n + \lim\limits_{n\to\infty} b_n$
2. $\lim\limits_{n\to\infty}(a_n - b_n) = \lim\limits_{n\to\infty} a_n - \lim\limits_{n\to\infty} b_n$
3. $\lim\limits_{n\to\infty} ca_n = c \lim\limits_{n\to\infty} a_n$
4. $\lim\limits_{n\to\infty}(a_n b_n) = \lim\limits_{n\to\infty} a_n \cdot \lim\limits_{n\to\infty} b_n$
5. $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}$ if $\lim_{n\to\infty} b_n \neq 0$
6. $\lim\limits_{n\to\infty} a_n^p = \left[\lim\limits_{n\to\infty} a_n\right]^p$ if $p > 0$ and $a_n > 0$.

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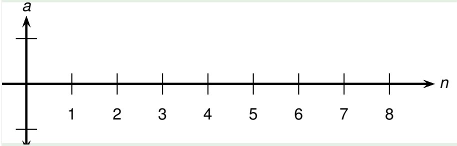
- Therefore

$$\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{x\to\infty} f(x) = 0$$

Example

Is the sequence $a_n = (-1)^n$ convergent or divergent?



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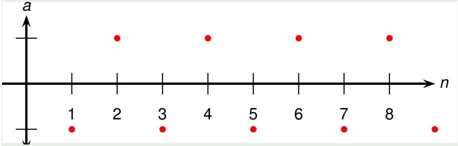

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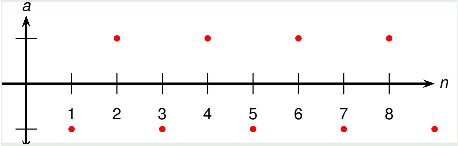

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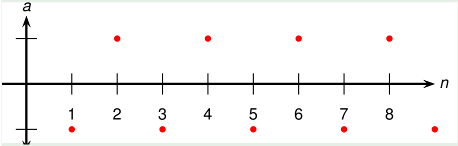

- The terms oscillate between -1 and 1 infinitely many times.
- Therefore the sequence doesn't approach any number.
- $\{a_n\}$ is divergent.

Theorem (The Squeeze Theorem for Sequences)

If $a_n \leq b_n \leq c_n$ *for* $n \geq n_0$ *and* $\lim\limits_{n\to\infty} a_n = L = \lim\limits_{n\to\infty} c_n$, *then* $\lim\limits_{n\to\infty} b_n = L$.

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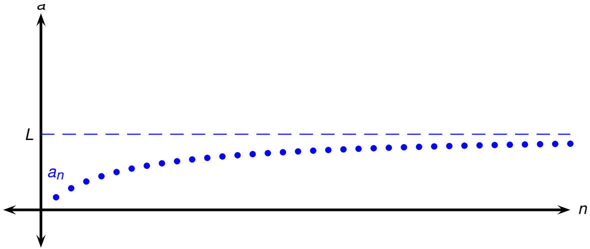

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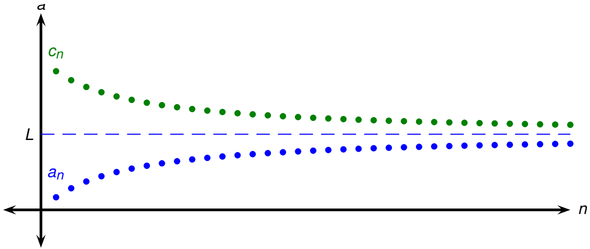

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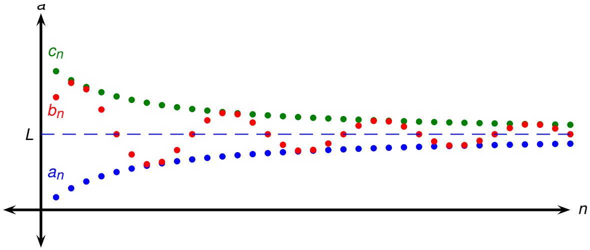

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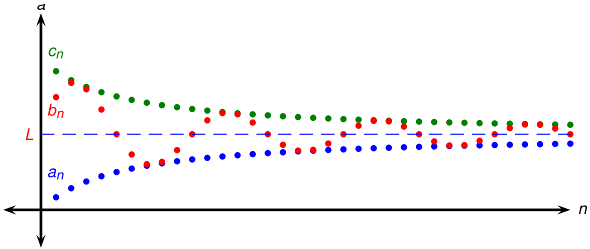

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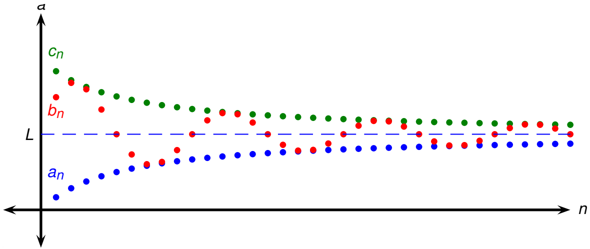

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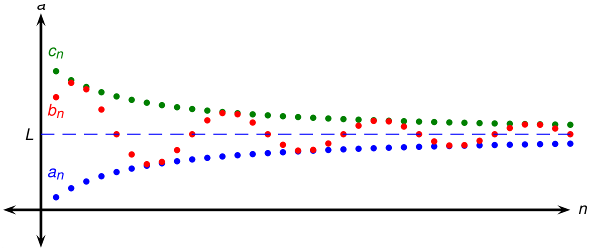

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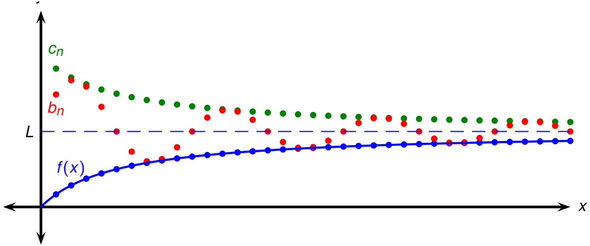

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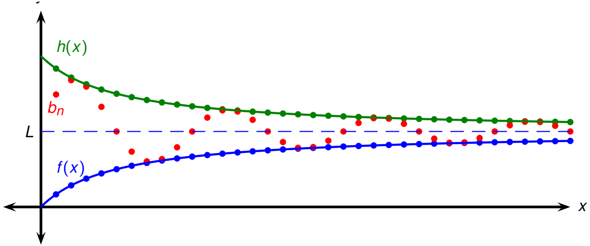

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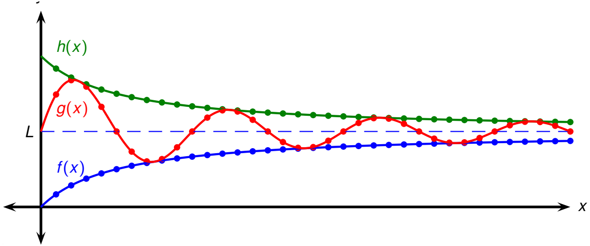

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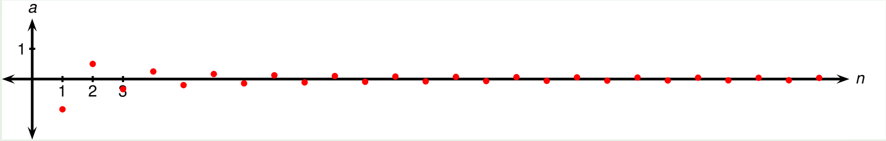

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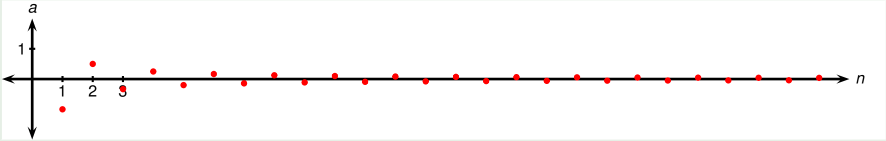

Theorem

If $\lim_{n\to\infty} a_n = L$ *and the function* f *is continuous at* L*, then*

$$\lim_{n\to\infty} f(a_n) = f(L)$$

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Sine is continuous at 0.

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- Since $\frac{1}{n} \to 0$ as $n \to \infty$, by the Squeeze Theorem $a_n \to 0$ as $n \to \infty$.

Example

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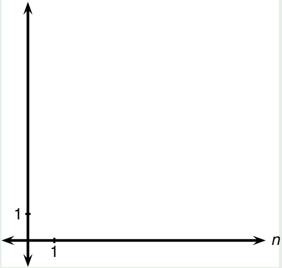

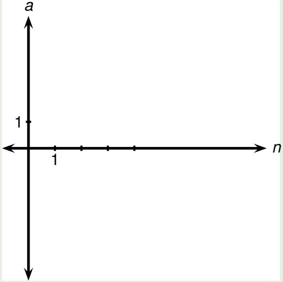

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Consider the exponential function $y = r^x$.

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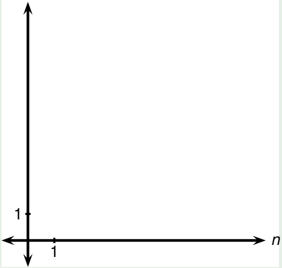

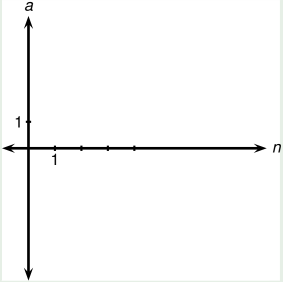

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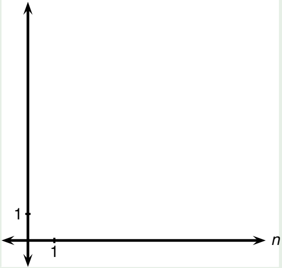

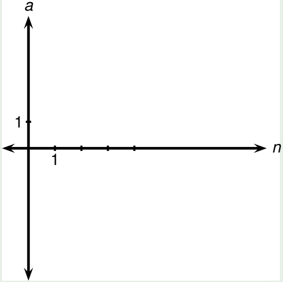

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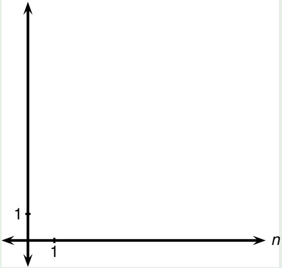

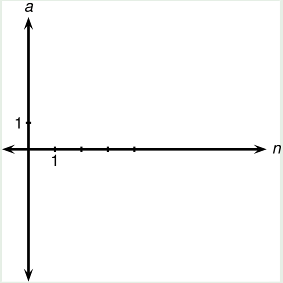

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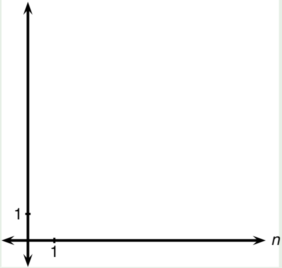

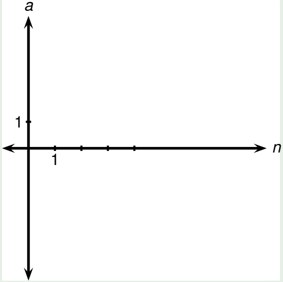

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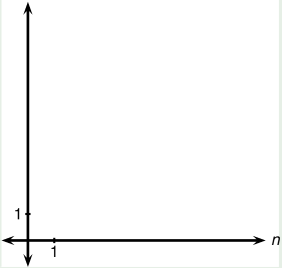

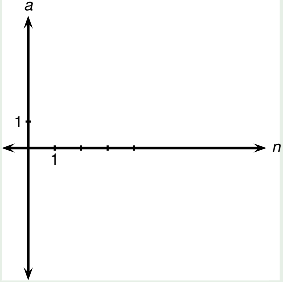

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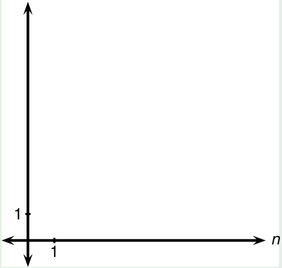

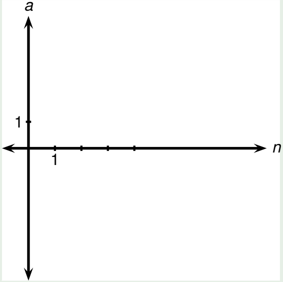

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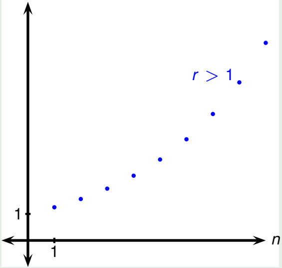

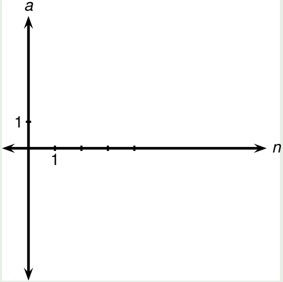

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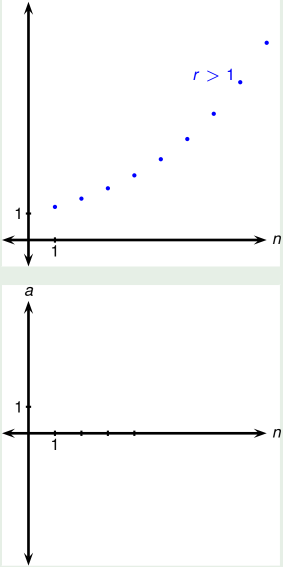

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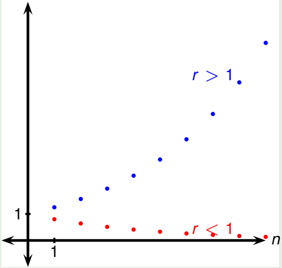

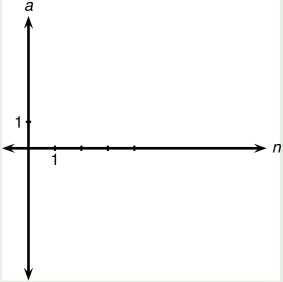

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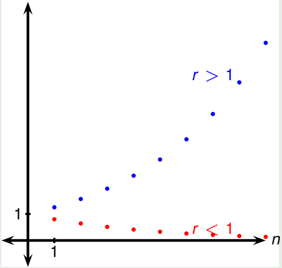

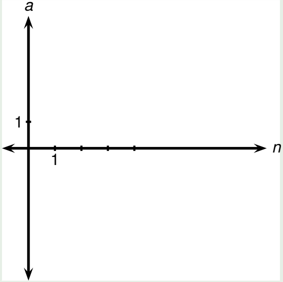

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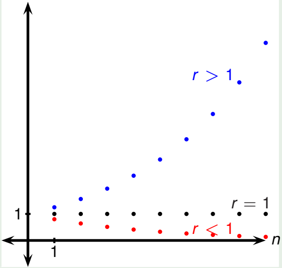

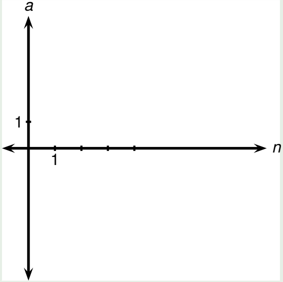

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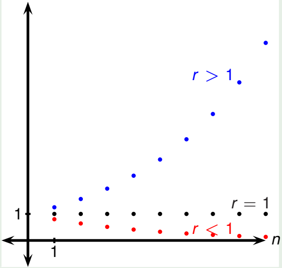

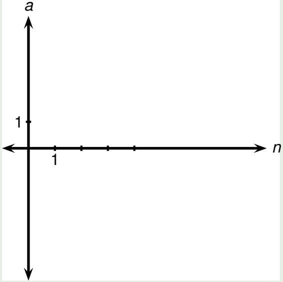

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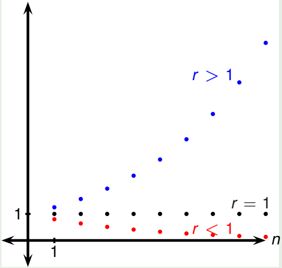

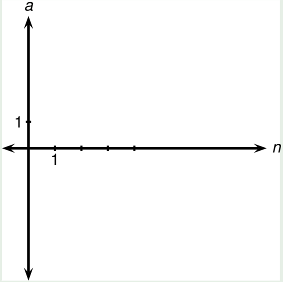

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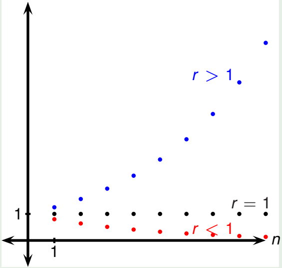
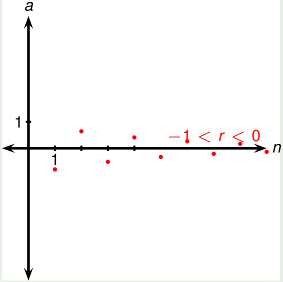
If $-1 < r < 0$, then $0 < |r| < 1$, and

$$\lim_{n\to\infty} |r^n| = \lim_{n\to\infty} |r|^n = 0$$

Therefore $\lim\limits_{n\to\infty} r^n = 0$.





Example

For what values of r is the sequence $\{r^n\}$ convergent?

Consider the exponential function $y = r^x$.

$$\lim_{x\to\infty} r^x = \begin{cases} \infty & \text{if} \quad r > 1 \\ 0 & \text{if} \quad 0 < r < 1 \end{cases}$$

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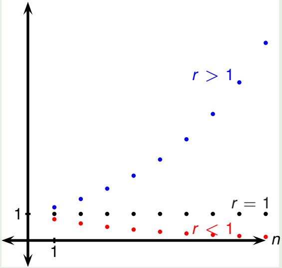
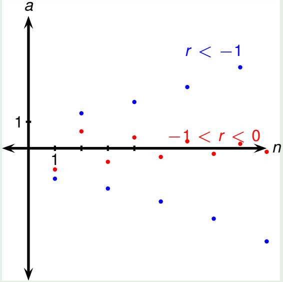
If $r \leq -1$, then r^n diverges.





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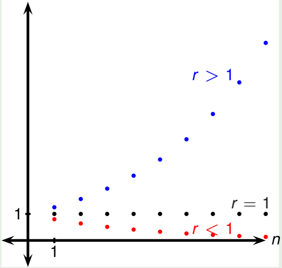
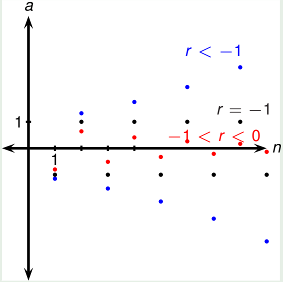
If $r \le -1$, then r^n diverges. In particular, $(-1)^n$ diverges.





This theorem summarizes the results of the previous example.

Theorem (Convergence of Geometric Sequences)

The sequence $\{r^n\}$ is convergent if $-1 < r \leq 1$ and divergent otherwise.

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \textit{if} \quad -1 < r < 1 \\ 1 & \textit{if} \quad r = 1 \end{cases}$$

Definition (Increasing and Decreasing)

A sequence $\{a_n\}$ is called increasing if $a_n < a_{n+1}$ for all $n \geq 1$. In other words, $\{a_n\}$ is increasing if $a_1 < a_2 < a_3 < \cdots$.
A sequence $\{a_n\}$ is called decreasing if $a_n > a_{n+1}$ for all $n \geq 1$. In other words, $\{a_n\}$ is decreasing if $a_1 > a_2 > a_3 > \cdots$.
A sequence is called monotonic if it is either increasing or decreasing.

Example

The sequence $\left\{\frac{1}{2n+1}\right\}$ is decreasing because

$$a_n = \frac{1}{2n+1} \qquad a_{n+1} = \frac{1}{2(n+1)+1} = \frac{1}{2n+3}$$

and

$$\frac{1}{2n+1} > \frac{1}{2n+3}$$

because the denominator of the latter is bigger.

Definition (Bounded Sequence)

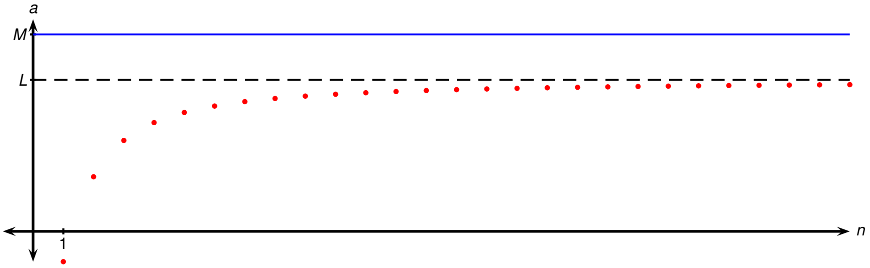
A sequence $\{a_n\}$ is called bounded above if there exists a number M such that

$$a_n < M \qquad \text{for all} \qquad n \geq 1.$$

It is called bounded below if there exists a number M such that

$$a_n > M \qquad \text{for all} \qquad n \geq 1.$$

A bounded sequence is a sequence that is bounded below and above.



Theorem (Monotonic Sequence Theorem)

Every bounded, monotonic sequence is convergent.