# Precalculus
# Euler's formula memorization

Todor Milev

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# Euler's Formula

## Theorem (Euler's Formula)

$$e^{ix} = \cos x + i \sin x,$$

*where* $e \approx 2.71828$ *is Euler's/Napier's constant .*

## Proof.

Recall $n! = 1 \cdot 2 \cdot 3 \cdot \dots \cdot (n-1) \cdot n$. Borrow from Calc II the f-las:

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$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + \frac{(-1)^n x^{2n}}{(2n)!} + \dots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots$$

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e^z & = & 1 & +z & +\frac{z^2}{2!} & +\frac{z^3}{3!} & +\frac{z^4}{4!} & +\frac{z^5}{5!} & +\dots
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Rearrange. Plug-in $z = ix$. Use $i^2 = -1$.

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Rearrange. Plug-in $z = ix$. Use $i^2 = -1$. Multiply $\sin x$ by $i$. Add to get $e^{ix} = \cos x + i \sin x$. $\square$

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- $e^{ix} = \cos x + i \sin x$ (Euler's Formula).
- $e^{ix} e^{iy} = e^{ix+iy} = e^{i(x+y)}$ (exponentiation rule: valid for $\mathbb{C}$).
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All trigonometric formulas can be easily derived using the above formulas.

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Compare coefficient in front of $i$ and remaining terms to get the desired equalities. □

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