Calculus II
Homework
Improper integrals

1. Determine whether the integral is convergent or divergent. Motivate your answer.

(a) $\displaystyle\int_{2}^{\infty} \frac{1}{(x-1)^{\frac{3}{2}}}\mathrm{d}x.$

answer: convergent

(b) $\displaystyle\int_{-1}^{1} \frac{1}{\sqrt[5]{1+x}}\mathrm{d}x.$

answer: convergent

(c) $\displaystyle\int_{1}^{\infty} \frac{1}{\sqrt[5]{1+x}}\mathrm{d}x.$

answer: divergent

(d) $\displaystyle\int_{-1}^{\infty} \frac{1}{\sqrt[5]{1+x}}\mathrm{d}x.$

answer: divergent

(e) $\displaystyle\int_{-\infty}^{0} \frac{1}{2-3x}\mathrm{d}x.$

answer: divergent

(f) $\displaystyle\int_{-\infty}^{0} \frac{1}{(2-3x)^2}\mathrm{d}x.$

answer: convergent

(g) $\displaystyle\int_{-\infty}^{0} \frac{1}{(2-3x)^{1.00000001}}\mathrm{d}x.$

answer: convergent

(h) $\displaystyle\int_{-2}^{\frac{1}{2}} \frac{1}{2x-1}\mathrm{d}x.$

answer: divergent

(i) $\displaystyle\int_{-1}^{\infty} e^{-3x}\mathrm{d}x.$

answer: convergent, equals $\frac{e^3}{3}$

(j) $\displaystyle\int_{-\infty}^{5} 2^x\mathrm{d}x.$

answer: convergent

(k) $\displaystyle\int_{-\infty}^{\infty} x^3\mathrm{d}x.$

answer: divergent

(l) $\displaystyle\int_{-\infty}^{\infty} xe^{-x^2}\mathrm{d}x.$

answer: convergent, equals 0

(m) $\displaystyle\int_{0}^{\infty} \sqrt{x}e^{-\sqrt{x}}\mathrm{d}x.$

answer: convergent, equals 4

(n) $\displaystyle\int_{0}^{\infty} \sin^2 x\mathrm{d}x.$

answer: divergent

(o) $\displaystyle\int_{0}^{5} \frac{1}{x^2+x-2}\mathrm{d}x.$

answer: divergent

(p) $\displaystyle\int_{0}^{\infty} \frac{1}{x^2+x+1}\mathrm{d}x.$

answer: convergent

(q) $\displaystyle\int_{2}^{\infty} \frac{1}{x^2-x-1}\mathrm{d}x.$

answer: convergent

(r) $\displaystyle\int_{0}^{\infty} \frac{1}{x^2-x-1}\mathrm{d}x.$

answer: divergent

(s) $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{x^4+2}\mathrm{d}x.$

answer: convergent

(t) $\displaystyle\int_{100}^{\infty} \frac{1}{x\ln x}\mathrm{d}x.$

answer: divergent

(u) $\displaystyle\int_{100}^{\infty} \frac{1}{x(\ln x)^2}\mathrm{d}x.$

answer: convergent

(v) $\displaystyle\int_{0}^{1} \ln x\mathrm{d}x.$

answer: convergent

(w) $\displaystyle\int_{0}^{1} \frac{\ln x}{\sqrt{x}}\mathrm{d}x.$

answer: convergent

(x) $\displaystyle\int\limits_0^2 x^3 \ln x \mathrm{d}x.$

answer: divergent

answer: convergent, equals $-1 + 4\ln 2$

(y) $\displaystyle\int\limits_0^1 \frac{e^{\frac{1}{x}}}{x^2} \mathrm{d}x.$

(z) $\displaystyle\int\limits_{-1}^0 \frac{e^{\frac{1}{x}}}{x^2} \mathrm{d}x.$

answer: convergent

Solution. 1.m It is possible to show that this integral is convergent by using the comparison theorem. However, we shall use direct integration instead. First, we solve the indefinite integral:

$$\begin{array}{rcll}
\displaystyle\int \sqrt{x} e^{-\sqrt{x}} \mathrm{d}x & = & \displaystyle\int \sqrt{x} e^{-\sqrt{x}} \frac{2\sqrt{x}\mathrm{d}x}{2\sqrt{x}} & \Big| \text{ use } \mathrm{d}\sqrt{x} = \frac{\mathrm{d}x}{2\sqrt{x}} \\
& = & \displaystyle\int \sqrt{x} e^{-\sqrt{x}} \left(2\sqrt{x}\mathrm{d}\sqrt{x}\right) & \Big| \text{ Set } \sqrt{x} = u \\
& = & \displaystyle 2\int u^2 e^{-u} \mathrm{d}u & \\
& = & \displaystyle 2\left(-\int u^2 \mathrm{d}\left(e^{-u}\right)\right) & \Big| \text{ integrate by parts} \\
& = & \displaystyle 2\left(-u^2 e^{-u} + \int e^{-u} \mathrm{d}\left(u^2\right)\right) & \\
& = & \displaystyle 2\left(-u^2 e^{-u} + \int 2u e^{-u} \mathrm{d}u\right) & \\
& = & \displaystyle 2\left(-u^2 e^{-u} - \int 2u \mathrm{d}e^{-u}\right) & \Big| \text{ integrate by parts again} \\
& = & \displaystyle 2\left(-u^2 e^{-u} - 2u e^{-u} + \int 2 e^{-u} \mathrm{d}u\right) & \\
& = & 2\left(-u^2 e^{-u} - 2u e^{-u} - 2e^{-u}\right) + C & \\
& = & 2\left(-x e^{-\sqrt{x}} - 2\sqrt{x} e^{-\sqrt{x}} - 2e^{-\sqrt{x}}\right) + C &
\end{array}$$

Therefore

$$\begin{array}{rcll}
\displaystyle\int\limits_0^\infty \sqrt{x} e^{-\sqrt{x}} \mathrm{d}x & = & \displaystyle\lim_{t\to\infty} 2\left[-x e^{-\sqrt{x}} - 2\sqrt{x} e^{-\sqrt{x}} - 2e^{-\sqrt{x}}\right]_0^\infty & \\
& = & \displaystyle 4 + \lim_{t\to\infty} 4\left(-t e^{-\sqrt{t}} - \sqrt{t} e^{-\sqrt{t}} - e^{-\sqrt{t}}\right) & \Big| \text{ Set } u = \sqrt{t} \\
& = & \displaystyle 4 - 4\lim_{u\to\infty}\left(u^2 e^{-u} + u e^{-u} + e^{-u}\right) & \\
& = & \displaystyle 4 - 4\lim_{u\to\infty} \frac{u^2 + u + 1}{e^u} & \Big| \text{ use L'Hospital's rule for limit, see below} \\
& = & 4 \quad , &
\end{array}$$

and the integral converges to 4. In the above computation we used the following limit computation

$$\begin{array}{rcll}
\displaystyle\lim_{u\to\infty} \frac{u^2 + u + 1}{e^u} & = & \displaystyle\lim_{u\to\infty} \frac{2u + 1}{e^u} & \Big| \text{ Apply L'Hospital's rule} \\
& = & \displaystyle\lim_{u\to\infty} \frac{2}{e^u} & \\
& = & 0 \quad . &
\end{array}$$

Solution. 1.s The integrand is a rational function and therefore we can solve this problem by finding the indefinite integral and then computing the limit. We would need to start by factoring $x^4 + 2$ into irreducible quadratic factors - that is already quite laborious:

$$x^4 + 2 = \left(x^2 + \sqrt[4]{8}x + \sqrt{2}\right)\left(x^2 - \sqrt[4]{8}x + \sqrt{2}\right) \quad .$$

The problem asks us only to establish the convergence of the integral; it does not ask us to compute its actual numerical value. Therefore we can give a much simpler solution. The function is even and therefore it suffices to establish whether $\displaystyle\int\limits_0^\infty \frac{x^2}{x^4+2}\mathrm{d}x$ is convergent.

We have that

$$\int_0^\infty \frac{x^2}{x^4+2}\mathrm{d}x = \int_0^1 \frac{x^2}{x^4+2}\mathrm{d}x + \int_1^\infty \frac{x^2}{x^4+2}\mathrm{d}x \quad .$$

The function $\frac{x^2}{x^4+2}$ is continuous so $\int_0^1 \frac{x^2}{x^4+2}\mathrm{d}x$ integrates to a number, which does not affect the convergence of the above expression. Therefore the convergence of our integral is governed by the convergence of $\int_1^\infty \frac{x^2}{x^4+2}\mathrm{d}x$. To establish that that integral is convergent, we use the comparison theorem as follows.

$$\begin{array}{rcl|l}
\displaystyle\int_1^\infty \frac{x^2}{x^4+2}\mathrm{d}x & \leq & \displaystyle\int_1^\infty \frac{x^2}{x^4}\mathrm{d}x & \begin{array}{l}\text{we have that } x^4+2 > x^4 \\ \text{and therefore } \dfrac{x^2}{x^4+2} \leq \dfrac{x^2}{x^4}\end{array} \\
& = & \displaystyle\int_1^\infty x^{-2}\mathrm{d}x & \\
& = & \displaystyle\lim_{t\to\infty} \left[-\frac{1}{x}\right]_1^t & \\
& = & \displaystyle\lim_{t\to\infty} 1 - \frac{1}{t} & \\
& = & 1 \quad . &
\end{array}$$

In this way we showed $\displaystyle\int_1^\infty \frac{x^2}{x^4+2}\mathrm{d}x \leq 1$. Therefore, as $\dfrac{x^2}{x^4+2} \geq 0$ is positive, we can apply the comparison theorem to get that $\displaystyle\int_1^\infty \frac{x^2}{x^4+2}\mathrm{d}x$ is convergent.

2. Determine whether the integral is convergent or divergent. Motivate your answer. The answer key has not been proofread, use with caution.

   (a) $\displaystyle\int_0^\infty \sin x^2 \mathrm{d}x$ (This problem is more difficult and may require knowledge of sequences to solve).

   answer: convergent