Calculus II

Integrals of the form $\int \sqrt{ax^2+bx+c}dx$, quadratic has no real roots.

Todor Milev

2019

Linear substitutions to simplify radicals $\sqrt{ay^2 + by + c}$

- Using linear substitutions, radicals of form $\sqrt{ay^2 + by + c}$, $a \neq 0$, $b^2 - 4ac \neq 0$ can be transformed to (multiple of):
  - $\sqrt{x^2 + 1}$
  - $\sqrt{-x^2 + 1}$
  - $\sqrt{x^2 - 1}$.
- We already studied how to do that using completing the square when dealing with rational functions.

Recall: linear substitution is subst. of the form $u = px + q$.

Example

Use linear substitution to transform $\sqrt{x^2+x+1}$ to multiple of $\sqrt{u^2+1}$.

$$
\begin{aligned}
\sqrt{x^2+x+1} &= \sqrt{x^2 + 2\cdot\frac{1}{2}x + \frac{1}{4} - \frac{1}{4} + 1} \\
&= \sqrt{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}} \\
&= \sqrt{\frac{3}{4}\left(\frac{4}{3}\left(x+\frac{1}{2}\right)^2 + 1\right)} \\
&= \frac{\sqrt{3}}{2}\sqrt{\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right)^2 + 1} \\
&= \frac{\sqrt{3}}{2}\sqrt{u^2+1},
\end{aligned}
$$

where $u = \frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right) = \frac{2\sqrt{3}}{3}x + \frac{\sqrt{3}}{3}$.