

# Precalculus

## Degree lowering formulas

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## Proposition (Power-Reducing Formulas)

$$\sin^2 \alpha = \frac{1 - \cos(2\alpha)}{2} \quad \cos^2 \alpha = \frac{1 + \cos(2\alpha)}{2}$$

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$$\begin{aligned}\cos(2\alpha) &= 1 - 2\sin^2 \alpha \\ \color{red}{2}\sin^2 \alpha &= 1 - \cos(2\alpha) \\ \sin^2 \alpha &= \frac{1 - \cos(2\alpha)}{\color{red}{2}}\end{aligned}$$



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$$\sin \alpha = \pm \sqrt{\frac{1 - \cos(2\alpha)}{2}} \quad \cos \alpha = \pm \sqrt{\frac{1 + \cos(2\alpha)}{2}}$$

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$$\sin \alpha = \pm \sqrt{\frac{1 - \cos(2\alpha)}{2}} \quad \cos \alpha = \pm \sqrt{\frac{1 + \cos(2\alpha)}{2}}$$

### Corollary (Half-Angle Formulas)

$$\sin \left( \frac{\beta}{2} \right) = \pm \sqrt{\frac{1 - \cos \beta}{2}} \quad \cos \left( \frac{\beta}{2} \right) = \pm \sqrt{\frac{1 + \cos \beta}{2}}$$



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- The power reducing formulas are used to express  $\sin^k \alpha$  and  $\cos^k \alpha$  via lower powers of the sin and cos functions (applied to angles other than  $\alpha$ ).

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- The power reducing formulas are used to express  $\sin^k \alpha$  and  $\cos^k \alpha$  via lower powers of the sin and cos functions (applied to angles other than  $\alpha$ ).
- This technique will play a key role in integration (studied later/in another course).

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Rewrite  $\sin^4 \alpha$  in terms of first powers of the cosines and sines of multiples of the angle  $\alpha$ .

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Recall the formulas:  $\sin^2 \beta = ?$  ,  $\cos^2 \beta = ?$  .

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$$\begin{aligned} \sin^4 \alpha &= (\sin^2 \alpha)^2 \\ &= \left( \frac{1 - \cos(2\alpha)}{2} \right)^2 \end{aligned}$$

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$$\begin{aligned} \sin^4 \alpha &= (\sin^2 \alpha)^2 \\ &= \left( \frac{1 - \cos(2\alpha)}{2} \right)^2 \\ &= \frac{1}{4} (1 - 2\cos(2\alpha) + \cos^2(2\alpha)) \end{aligned}$$

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 &= \frac{1}{8} (3 - 4\cos(2\alpha) + \cos(4\alpha))
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