Precalculus
Compute trigonometric function of a complementary angle, part 1

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Cofunction identities

Proposition (Cofunction identities)

$$\begin{array}{rclrcl}\sin\left(\frac{\pi}{2}-\alpha\right) & = & \cos\alpha & \sin\left(\frac{\pi}{2}+\alpha\right) & = & \cos\alpha\\ \cos\left(\frac{\pi}{2}-\alpha\right) & = & \sin\alpha & \cos\left(\frac{\pi}{2}+\alpha\right) & = & -\sin\alpha\end{array}$$

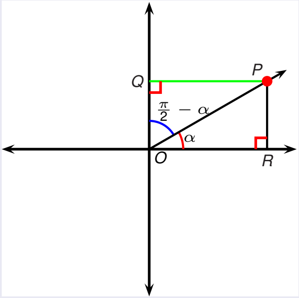
- The proof each formula is broken into 4 cases depending on which quadrant contains α.
- This makes a total of 4 formulas $\times 4$ cases per formula = 16 cases.
- We show only a few of the cases.
- The proof provides intuition why the formulas are true.
- The Quadrant I part of the proof serves as a visual aid for memorization.
- There is an algebraically simpler (but theoretically advanced) way to prove the above identities through the angle sum f-las, derived in turn from Euler's formula (studied later/in another course).

Cofunction identities

Proposition (Cofunction identities)

$$
\begin{aligned}
\sin\left(\frac{\pi}{2}-\alpha\right) &= \cos\alpha & \sin\left(\frac{\pi}{2}+\alpha\right) &= \cos\alpha\\
\cos\left(\frac{\pi}{2}-\alpha\right) &= \sin\alpha & \cos\left(\frac{\pi}{2}+\alpha\right) &= -\sin\alpha
\end{aligned}
$$

Part of Proof.



We are showing $\sin\left(\frac{\pi}{2}-\alpha\right)=\cos\alpha$ when α is in quadrant I.

$$
\begin{aligned}
\sin\left(\frac{\pi}{2}-\alpha\right) &= \frac{|PQ|}{|OP|} && \Big| \ \square ORPQ\\
&= \frac{|OR|}{|OP|}\\
&= \cos\alpha && \Big| \ \text{as desired}
\end{aligned}
$$

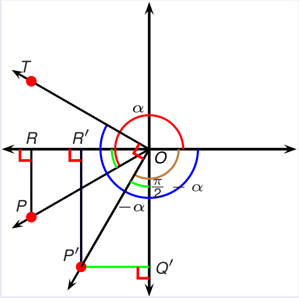
$\square$

Cofunction identities

Proposition (Cofunction identities)

$$\begin{aligned} \sin\left(\frac{\pi}{2}-\alpha\right) &= \cos\alpha & \sin\left(\frac{\pi}{2}+\alpha\right) &= \cos\alpha \\ \cos\left(\frac{\pi}{2}-\alpha\right) &= \sin\alpha & \cos\left(\frac{\pi}{2}+\alpha\right) &= -\sin\alpha \end{aligned}$$

Part of Proof.



We are showing $\sin\left(\frac{\pi}{2}-\alpha\right)=\cos\alpha$ when α is in Quadrant III. It follows $\frac{\pi}{2}-\alpha$ is in Quadrant III.

$$\begin{aligned} \sin\left(\frac{\pi}{2}-\alpha\right) &= -\frac{|P'R'|}{|OP'|} = -\frac{|OQ'|}{|OP'|} && \Big|\ \square OR'P'Q' \\ &= -\frac{|OR|}{|OP|} \\ &= \cos\alpha && \Big|\ \text{as desired} \end{aligned}$$

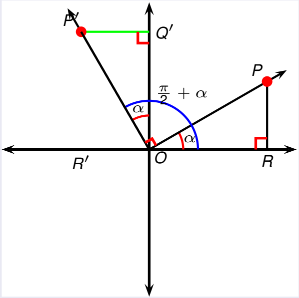
□

Cofunction identities

Proposition (Cofunction identities)

$$
\begin{aligned}
\sin\left(\frac{\pi}{2}-\alpha\right) &= \cos\alpha & \sin\left(\frac{\pi}{2}+\alpha\right) &= \cos\alpha \\
\cos\left(\frac{\pi}{2}-\alpha\right) &= \sin\alpha & \cos\left(\frac{\pi}{2}+\alpha\right) &= -\sin\alpha
\end{aligned}
$$

Part of Proof.



We show $\cos\left(\frac{\pi}{2}+\alpha\right) = -\sin\alpha$ when α is in Quadrant I. It follows $\frac{\pi}{2}+\alpha$ is in Quadrant II.

$$
\begin{aligned}
\cos\left(\frac{\pi}{2}+\alpha\right) &= -\frac{|OR'|}{|OP'|} && \Big|\ \square ORPQ \\
&= -\frac{|P'Q'|}{|OP'|} \\
&= -\frac{|PR|}{|OP|} \\
&= -\sin\alpha. && \Big|\ \text{as desired}
\end{aligned}
$$

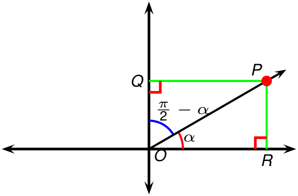
□

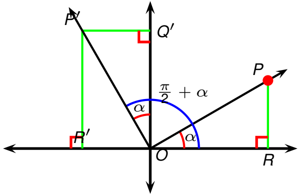
Cofunction identities

Proposition (Cofunction identities)

$$
\begin{array}{rclrcl}
\sin\left(\frac{\pi}{2}-\alpha\right) &=& \cos\alpha & \sin\left(\frac{\pi}{2}+\alpha\right) &=& \cos\alpha \\
\cos\left(\frac{\pi}{2}-\alpha\right) &=& \sin\alpha & \cos\left(\frac{\pi}{2}+\alpha\right) &=& -\sin\alpha
\end{array}
$$

To memorize the cofunction identities it suffices to memorize the Quadrant I case via the two diagrams below.





$$
\begin{array}{rcl}
\sin\left(\frac{\pi}{2}-\alpha\right) &=& \frac{|PQ|}{|OP|} \\
\cos\left(\frac{\pi}{2}-\alpha\right) &=& \frac{|OQ|}{|OP|}
\end{array}
$$

$$
\begin{array}{rcl}
\sin\left(\frac{\pi}{2}+\alpha\right) &=& \frac{|OQ'|}{|OP|} \\
\cos\left(\frac{\pi}{2}+\alpha\right) &=& -\frac{|PQ'|}{|OP'|} = \frac{|PR|}{|OP|}
\end{array}
$$