

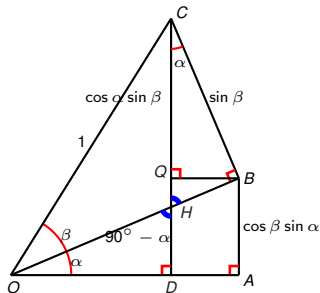
Precalculus

Angle sum formulas memorization

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$\sin(\alpha + \beta), \cos(\alpha + \beta)$ via $\sin \alpha, \sin \beta, \cos \alpha, \cos \beta$



$$\begin{aligned}
 \sin(\alpha + \beta) &= \frac{|CD|}{|OC|} = |CD| \\
 &= |QD| + |CQ| \\
 &= \sin \alpha \cos \beta + \cos \alpha \sin \beta \\
 \cos(\alpha + \beta) &= \frac{|OD|}{|OC|} = |OD| \\
 &= |OA| - |DA| \\
 &= \cos \alpha \cos \beta - \sin \alpha \sin \beta
 \end{aligned}$$

$$\begin{aligned}
 |QD| &= |BA| & \square DABQ \\
 &= \sin \alpha |OB| & \triangle OAB \\
 &= \sin \alpha \cos \beta |OC| & \triangle OBC \\
 &= \sin \alpha \cos \beta \\
 |CQ| &= \cos \alpha |CB| & \triangle CQB \\
 &= \cos \alpha \sin \beta |OC| & \triangle OBC \\
 &= \cos \alpha \sin \beta \\
 |OA| &= \cos \alpha |OB| & \triangle OAB \\
 &= \cos \alpha \cos \beta |OC| & \triangle OBC \\
 &= \cos \alpha \cos \beta \\
 |DA| &= |QB| & \square DABQ \\
 &= \sin \alpha |CB| & \triangle CQB \\
 &= \sin \alpha \sin \beta |OC| & \triangle OBC \\
 &= \sin \alpha \sin \beta
 \end{aligned}$$

Trig Functions of Sums and Differences of Angles

Theorem

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$$

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

- We gave a geometric proof of the sum formulas when the two angles are acute and their sum is less than $\pi = 90^\circ$.
- The theorem holds for all angles α, β without any restrictions.
- This can be shown by combining the preceding proof with identities such as $\cos\left(\frac{\pi}{2} - \alpha\right) = \sin \alpha$, $\cos\left(\frac{\pi}{2} + \alpha\right) = -\sin \alpha$.
- There is a theoretically more advanced (but algebraically simpler) proof using Euler's formula (to be studied later/in another course).
- The difference formulas are a consequence of the sum formulas and the fact that \sin is an odd function and \cos is even.

Trig Functions of Differences of Angles

Example

Prove the identities

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$$

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

from the (already demonstrated) identities

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$\sin(\alpha - \beta) = \sin(\alpha + (-\beta))$$

$$= \sin \alpha \cos(-\beta) + \cos \alpha \sin(-\beta) \quad \left| \begin{array}{l} \cos \text{ is even ,} \\ \sin \text{ is odd} \end{array} \right.$$

$$= \sin \alpha \cos \beta - \cos \alpha \sin \beta$$

$$\cos(\alpha - \beta) = \cos(\alpha + (-\beta))$$

$$= \cos \alpha \cos(-\beta) - \sin \alpha \sin(-\beta) \quad \left| \begin{array}{l} \cos \text{ is even ,} \\ \sin \text{ is odd} \end{array} \right.$$

$$= \cos \alpha \cos \beta + \sin \alpha \sin \beta$$