

Calculus II

Integrals of the form $\int \sin^n x \cos^m x dx$, at least
one power is odd

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2019

Example

$$\int \sin^3 x dx$$

Example

$$\int \sin^3 x dx = \int \sin^2 x \sin x dx$$

Example

$$\begin{aligned}\int \sin^3 x dx &= \int \sin^2 x \sin x dx \\ &= \int \sin^2 x d(\quad)\end{aligned}$$

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$$\begin{aligned}\int \sin^3 x dx &= \int \sin^2 x \sin x dx \\ &= \int \sin^2 x d(-\cos x)\end{aligned}$$

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 \int \sin^3 x dx &= \int \sin^2 x \sin x dx \\
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 \end{aligned}$$

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Can we rewrite
 $\sin^2 x$ via $\cos x$?

Example

$$\begin{aligned}\int \sin^3 x dx &= \int \sin^2 x \sin x dx \\ &= \int \sin^2 x d(-\cos x) \\ &= \int (-1) (1 - \cos^2 x) d(\cos x)\end{aligned}$$

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Set $u = \cos x$

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 &= \frac{u^3}{3} - u + C
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 &= \int (u^2 - 1) du \\
 &= \frac{u^3}{3} - u + C \\
 &= \frac{1}{3} \cos^3 x - \cos x + C .
 \end{aligned}$$

Can we rewrite
 $\sin^2 x$ via $\cos x$?

Set $u = \cos x$

Example

$$\int \cos^5 x \sin^2 x dx$$

Example

$$\int \cos^5 x \sin^2 x dx = \int \cos^4 x \sin^2 x \cos x dx$$

Example

$$\begin{aligned}\int \cos^5 x \sin^2 x dx &= \int \cos^4 x \sin^2 x \cos x dx \\ &= \int \cos^4 x \sin^2 x d(?)\end{aligned}$$

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 \int \cos^5 x \sin^2 x dx &= \int \cos^4 x \sin^2 x \cos x dx \\
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 &= \int (1 - \sin^2 x)^2 \sin^2 x d(\sin x)
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 $\cos^4 x$ via $\sin x$?

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 &= \int (\cos^2 x)^2 \sin^2 x d(\sin x) \\
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 &= \int (u^2 - 2u^4 + u^6) du
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 &= \frac{u^3}{3} - 2\frac{u^5}{5} + \frac{u^7}{7} + C
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 &= \frac{u^3}{3} - 2\frac{u^5}{5} + \frac{u^7}{7} + C \\
 &= \frac{\sin^3 x}{3} - 2\frac{\sin^5 x}{5} + \frac{\sin^7 x}{7} + C .
 \end{aligned}$$