Calculus II
Integration of rational functions: plan for algorithm

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Integrating arbitrary rational functions

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Question

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  - We use linear substitutions to transform each piece to one of 3 pairs of basic building block integrals.
  - We solve each building block integral and collect the terms.
- We study the algorithm “from the ground up”: we start with the building blocks.

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- The case $n = 1$ is special for each of the building blocks:
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- The case $n = 1$ we call respectively building block Ia, IIa and IIIa. The case $n > 1$ we call respectively building block Ib, IIb and IIIb. This “building block” terminology is for our convenience, and is not a part of standard mathematical terminology.