Calculus I
Reference: the Chain Rule statement and notation

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The Chain Rule

- What is the derivative of $f(x) = \sqrt{x^2 + 1}$?
- The Power Rule doesn't tell us how to find the derivative.
- f is a composite function $g \circ h$:
- $y = g(u) = \sqrt{u}$.
- $u = h(x) = x^2 + 1$.
- Then $y = f(x) = g(h(x)) = g(x^2 + 1) = \sqrt{x^2 + 1}$.
- We know the derivatives of g and h:
- $g'(u) = \frac{1}{2}u^{-\frac{1}{2}}$.
- $h'(x) = 2x$.
- It would be nice if we could find the derivative of f in terms of the derivatives of y and u.
- It turns out that the derivative of the composition $g \circ h$ is the product of the derivative of g and the derivative of h.
- This important fact is called the Chain Rule.

The Chain Rule

Let g and h be functions. Recall that the composite function $f = g \circ h$ is defined via $f(x) = g(h(x))$.

Theorem

Let h be differentiable at x and let g be a differentiable at $h(x)$. Then the composite function $f = g \circ h$ is differentiable at x and f' is given by the product

$$
\begin{array}{rcllr}
f'(x) & = & g'(h(x)) \cdot h'(x) & & \textit{(notation 1)} \\
 & & & & \textit{equivalently:} \\
f'(x) = (g(u))' & = & g'(u)u' & \textit{where } u = h(x) & \textit{(notation 2)} \\
\frac{dy}{dx} & = & \frac{dy}{du}\frac{du}{dx} & \textit{where } y = g(u) & \textit{(notation 3)}
\end{array} .
$$

The last equality uses the Leibniz notation (due to G. Leibniz (1646-1716)).

Chain rule notations

- As we saw, the chain rule can be written using a number of notations:

$$
\begin{array}{rclrl}
(g(h(x)))' & = & g'(h(x)) \cdot h'(x) & & \text{(notation 1)}\\
(g(u))' & = & g'(u)u' & \text{where } u = h(x) & \text{(notation 2)}\\
\frac{\mathrm{d}y}{\mathrm{d}x} & = & \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} & \text{where } y = g(u) & \text{(notation 3)}
\end{array} .
$$

- The three notations are all accepted and can be used interchangeably.
- Most authors tend to prefer one of these notations over the others.
- In order to exercise ourselves we shall use all three notations throughout our course.
- There are additional notations (not covered here) used in practice.
- Whenever in doubt about derivative notation, if possible, request clarification.