Calculus I
The Fundamental Theorem of Calculus, Part I

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Outline

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The Fundamental Theorem of Calculus

- The Fundamental Theorem of Calculus has two parts.
- Part 2 of the FTC roughly says "integration undoes differentiation."
- Part 2 of the FTC was already studied as the Evaluation Theorem. It allows us to compute integrals by finding antiderivatives, without writing limits of Riemann sums.
- Part 1 of the FTC roughly says "differentiation undoes integration."
- Part 1 of the FTC deals with functions of the form

$$g(x) = \int_a^x f(t)\mathrm{d}t$$

  where f is a continuous function on $[a, b]$ and x varies between a and b.

$$g(x) = \int_a^x f(t)\mathrm{d}t$$

- g depends only on x.
- If x is a fixed number, then $\int_a^x f(t)\mathrm{d}t$ is a fixed number.
- If we let x vary, then $\int_a^x f(t)\mathrm{d}t$ varies.
- If f is positive, then g can be interpreted as the area under f from a to x.

Example (FTC Part 1)

If $g(x) = \int_1^x (e^t + 2t)\mathrm{d}t,$ find $g'(x)$.

$$\begin{aligned} g(x) &= \left[e^t + t^2\right]_1^x \\ &= (e^x + x^2) - (e^1 + 1^2) \\ &= e^x + x^2 - e - 1. \end{aligned}$$

$$\begin{aligned} g'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}(e^x + x^2 - e - 1) \\ &= e^x + 2x - 0 - 0 \\ &= e^x + 2x. \end{aligned}$$

Theorem (The Fundamental Theorem of Calculus, Part 1)

If f is continuous on $[a, b]$, then the function g defined by

$$g(x) = \int_a^x f(t)\mathrm{d}t$$

is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$.

Example

Find the derivative of $g(x) = \int_0^x \sqrt{1+t^2}\mathrm{d}t$.

- $f(t) = \sqrt{1+t^2}$ is continuous.
- By the FTC, Part 1,

$$g'(x) = \sqrt{1+x^2}$$

Example (FTC, Part 1)

For each formula $g(x)$, find the derivative $g'(x)$.

$g(x)$	$g'(x)$
$\displaystyle\int_0^x \sin(t^2+1)\cos(t^3+2)\mathrm{d}t$	$\sin(x^2+1)\cos(x^3+2)$
$\displaystyle\int_{35}^x \frac{1+r^2+4r^3}{1-r^4}\mathrm{d}r$	$\displaystyle\frac{1+x^2+4x^3}{1-x^4}$
$\displaystyle\int_{-1}^x \frac{\cos 2\theta+1}{1+\sin^2\theta}\mathrm{d}\theta$	$\displaystyle\frac{\cos 2x+1}{1+\sin^2 x}$

Example (Chain Rule, FTC Part 1)

$$\text{Differentiate} \quad y = \int_0^{x^4} \sec t \, dt.$$

$$\text{Let} \quad u = x^4.$$

$$\text{Then} \quad y = \int_0^{u} \sec t \, dt.$$

$$\text{Chain Rule:} \quad \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

$$= (\sec u)\left(4x^3\right)$$

$$= 4x^3 \sec(x^4).$$

Theorem (The Fundamental Theorem of Calculus)

Suppose f is continuous on $[a, b]$. Then

1. *If $G(x) = \int_a^x f(t)\mathrm{d}t$, then $G'(x) = f(x)$.*
2. $\int_a^b f(x)\mathrm{d}x = F(b) - F(a)$, *where F is any antiderivative of f.*

We already studied part 2 of the FTC as the Evaluation Theorem.

Theorem

Let A, B-numbers, $a(x), b(x)$ -differentiable functions with $A < a(x) < B$, $A < b(x) < B$. Let f - continuous on $[A, B]$ and $G(x) = \int_{a(x)}^{b(x)} f(t)dt$. Then $G'(x) = f(b(x))b'(x) - f(a(x))a'(x)$.

Proof.

Let $c \in (A, B)$. Set $h(u) = \int_c^u f(t)dt$. FTC part 1 states that $h'(u) = f(u)$.

$$\begin{aligned} G(x) &= \int_{a(x)}^{b(x)} f(t)dt = \int_c^{b(x)} f(t)dt + \int_{a(x)}^c f(t)dt \\ &= \int_c^{b(x)} f(t)dt - \int_c^{a(x)} f(t)dt = h(b(x)) - h(a(x)) \end{aligned} .$$

Then using the chain rule we get $G'(x) = (h(b(x)) - h(a(x)))' = h'(b(x))b'(x) - h'(a(x))a'(x) = f(b(x))b'(x) - f(a(x))a'(x)$, as desired. □

Problems similar to the following often appear on Calculus I exams.

Example

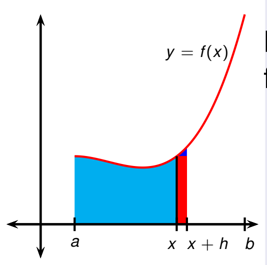
Let $G(x) = \int_{\sqrt{x}}^{x^2} \ln t dt$, $x > 0$. Find $G'(x)$.

$$G'(x) = (\ln x^2)(x^2)' - (\ln \sqrt{x})(\sqrt{x})' = \left(4x - \frac{1}{4}x^{-\frac{1}{2}}\right) \ln x.$$

Theorem (The Fundamental Theorem of Calculus part 1)

Let f be a function continuous on $[a, b]$ and let $G(x) = \int_a^x f(t)\mathrm{d}t$ for all $x \in [a, b]$. Then G is differentiable and $G'(x) = f(x)$.

Proof.



Let $\varepsilon > 0$. There exists δ such that $|f(t) - f(x)| < \varepsilon$ for all t for which $|x - t| < \delta$. Then for all $0 < h < \delta$:

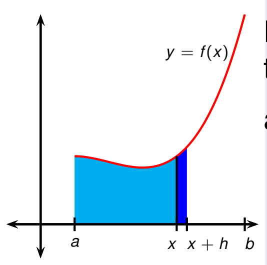
$$\begin{array}{rcl|l}
\varepsilon > & f(t) - f(x) & > -\varepsilon & \text{integrate} \\
h\varepsilon > & \int_x^{x+h}(f(t) - f(x))\mathrm{d}t & > -h\varepsilon & \text{divide by } h \\
\varepsilon > & \frac{\int_x^{x+h}(f(t)-f(x))\mathrm{d}t}{h} & > -\varepsilon & \\
\varepsilon > & \frac{\int_x^{x+h} f(t)\mathrm{d}t}{h} - \frac{hf(x)}{h} & > -\varepsilon & \\
\varepsilon > & \left| \frac{\int_x^{x+h} f(t)\mathrm{d}t}{h} - f(x) \right| & &
\end{array}$$

□

Theorem (The Fundamental Theorem of Calculus part 1)

Let f be a function continuous on $[a, b]$ and let $G(x) = \int_a^x f(t)dt$ for all $x \in [a, b]$. Then G is differentiable and $G'(x) = f(x)$.

Proof.



In analogous fashion we can handle the case $h < 0$, to prove: for any $\varepsilon > 0$ there exists $\delta > 0$ so that for all $0 < h$ $|h| < \delta$ we have $\left| \frac{\int_x^{x+h} f(t)dt}{h} - f(x) \right| < \varepsilon$.

$$
\begin{aligned}
G'(x) &= \lim_{h\to 0} \frac{G(x+h) - G(x)}{h} \\
&= \lim_{h\to 0} \frac{\int_a^{x+h} f(t)dt - \int_a^x f(t)dt}{h} \\
&= \lim_{h\to 0} \frac{\int_x^{x+h} f(t)dt}{h} = f(x)
\end{aligned}
$$

- The Evaluation Theorem says that, if f is continuous on $[a, b]$, then
$$\int_a^b f(x)\mathrm{d}x = F(b) - F(a),$$
where $F(x)$ is an antiderivative of $f(x)$.
- This means $F' = f$, so
$$\int_a^b F'(x)\mathrm{d}x = F(b) - F(a),$$
- $F'(x)$ is the rate of change of $y = F(x)$ with respect to x.
- $F(b) - F(a)$ is the net change in y as x changes from a to b.

Theorem (The Net Change Theorem)

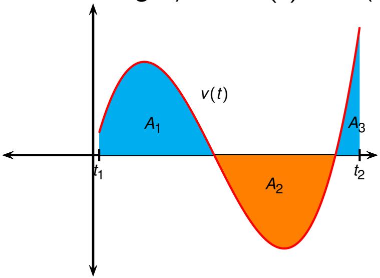
The integral of the rate of change is the net change:

$$\int_a^b F'(x)\mathrm{d}x = F(b) - F(a).$$

- If an object moves along a straight line with position function $s(t)$, then its velocity is $v(t) = s'(t)$.
- In this case, the Net Change Theorem says

$$\int_{t_1}^{t_2} v(t)\mathrm{d}t = s(t_2) - s(t_1).$$

- This is the displacement, or net change of position.
- If we want to calculate the distance the object travels, we have to consider separately the intervals where $v(t) \geq 0$ (object moves to the right) and $v(t) \leq 0$ (object moves to the left).



$$\begin{aligned} \text{displacement} &= \int_{t_1}^{t_2} v(t)\mathrm{d}t \\ &= A_1 - A_2 + A_3 \\ \text{distance} &= \int_{t_1}^{t_2} |v(t)|\mathrm{d}t \\ &= A_1 + A_2 + A_3 \end{aligned}$$

Example

A particle moves along a line so that its velocity at time t is $v(t) = t^2 - t - 6$ (measured in meters per second).

1. Find the displacement of the particle during the time period $1 \leq t \leq 4$.
2. Find the distance traveled during this time period.

Example

A particle moves along a line so that its velocity at time t is $v(t) = t^2 - t - 6$ (measured in meters per second).

1. Find the displacement of the particle during the time period $1 \leq t \leq 4$.

The displacement is

$$
\begin{aligned}
s(4) - s(1) &= \int_1^4 v(t)\mathrm{d}t = \int_1^4 (t^2 - t - 6)\mathrm{d}t \\
&= \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_1^4 \\
&= \left(\frac{4^3}{3} - \frac{4^2}{2} - 6\cdot 4\right) - \left(\frac{1^3}{3} - \frac{1^2}{2} - 6\cdot 1\right) \\
&= -\frac{9}{2}.
\end{aligned}
$$

Therefore the particle moves 4.5m to the left.

Example

A particle moves along a line so that its velocity at time t is $v(t) = t^2 - t - 6$ (measured in meters per second).

2. Find the distance traveled during this time period.

$v(t) = t^2 - t - 6 = (t-3)(t+2)$ and so $v(t) \leq 0$ on the interval $[1,3]$ and $v(t) \geq 0$ on the interval $[3,4]$.
The distance is

$$\begin{aligned}
\int_1^4 |v(t)| \mathrm{d}t &= \int_1^3 [-v(t)] \mathrm{d}t + \int_3^4 v(t) \mathrm{d}t \\
&= \int_1^3 (-t^2 + t + 6) \mathrm{d}t + \int_3^4 (t^2 - t - 6) \mathrm{d}t \\
&= \left[-\frac{t^3}{3} + \frac{t^2}{2} + 6t \right]_1^3 + \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t \right]_3^4 \\
&= \frac{61}{6} \approx 10.17 \mathrm{m}
\end{aligned}$$

Rectilinear Motion

- Suppose a particle is moving in a straight line, with position function $s(t)$.
- Its velocity is $v(t) = s'(t)$.
- Its acceleration is $a(t) = v'(t)$.
- Position is the antiderivative of velocity.
- Velocity is the antiderivative of acceleration.
- If we know the acceleration and the initial values $s(0)$ and $v(0)$ for position and velocity, then we can find $s(t)$ by antidifferentiating twice.

An object near the Earth is subject to a gravitational force that produces a downward acceleration of 32 ft/s^2 (or 9.8 m/s^2).

Example

A ball is thrown upward with a speed of 48 ft/s from the edge of a cliff 432 ft above the ground. Find its height above the ground t seconds later.

$$\begin{aligned} v'(t) &= a(t) = -32 \\ v(t) &= -32t + C \\ &= -32t + 48 \end{aligned}$$

To find C, use the fact that $v(0) = 48$.

$$\begin{aligned} v(0) &= 48 \\ -32 \cdot 0 + C &= 48 \\ C &= 48 \end{aligned}$$

$$\begin{aligned} s'(t) &= -32t + 48 \\ s(t) &= -16t^2 + 48t + D \\ &= -16t^2 + 48t + 432 \end{aligned}$$

To find D, use the fact that $s(0) = 432$.

$$\begin{aligned} s(0) &= 432 \\ -16 \cdot 0^2 + 48 \cdot 0 + D &= 432 \\ D &= 432 \end{aligned}$$