

Precalculus

Express $\sin(kx)$, $\cos(kx)$ via $\sin x$, $\cos x$ using Euler's formula

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- Recall Euler's formula: $e^{i\alpha} = \cos \alpha + i \sin \alpha$.
- Recall the formula: $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.

Example

Express $\sin(3x)$ and $\cos(3x)$ via $\cos x$ and $\sin x$.

$$\begin{aligned}
 & \cos(3x) + i \sin(3x) && | \text{ Euler's f-la} \\
 &= e^{3ix} \\
 &= (e^{ix})^3 = (\cos x + i \sin x)^3 && | \text{ Euler's f-la} \\
 &= \cos^3 x + 3\cos^2 x(i \sin x) + 3\cos x(i \sin x)^2 + (i \sin x)^3 \\
 &= \cos^3 x + 3i \cos^2 x \sin x + 3i^2 \cos x \sin^2 x + i^3 \sin^3 x \\
 &= \cos^3 x + 3i \cos^2 x \sin x - 3\cos x \sin^2 x - i \sin^3 x && | \text{ Use } i^2 = -1 \\
 &= (\cos^3 x - 3\cos x \sin^2 x) + i(3\cos^2 x \sin x - \sin^3 x)
 \end{aligned}$$

The real parts of the starting and final expression must be equal; likewise the imaginary parts must be equal; therefore:

$$\begin{aligned}
 \cos(3x) &= \cos^3 x - 3\cos x \sin^2 x \\
 \sin(3x) &= 3\cos^2 x \sin x - \sin^3 x
 \end{aligned}$$