Calculus II
Ratio test related to the exponent as a limit

Todor Milev

2019

Example

Test the convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{3^n n!}$.

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$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(n+1)^{n+1}}{3^{n+1}(n+1)!}}{\frac{n^n}{3^n n!}}\right|$$

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\left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{\frac{(n+1)^{n+1}}{3^{n+1}(n+1)!}}{\frac{n^n}{3^n n!}}\right| \\
&= \frac{(n+1)^{n+1}}{n^n} \cdot \frac{3^n n!}{3^{n+1}(n+1)!}
\end{aligned}
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&= \frac{(n+1)(n+1)^n}{n^n} \cdot \frac{3^n n!}{3^{n+1}(n+1)n!}
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&= \frac{(n+1)^{n+1}}{n^n} \cdot \frac{3^n n!}{3^{n+1}{\color{red}(n+1)!}} \\
&= \frac{(n+1)(n+1)^n}{n^n} \cdot \frac{3^n n!}{3^{n+1}{\color{red}(n+1)n!}}
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&= \frac{1}{3}\left(\frac{n+1}{n}\right)^n = \frac{1}{3}\left(1+\frac{1}{n}\right)^n
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&\to \frac{e}{3}
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Therefore the series is **?** by the Ratio Test.

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&= \frac{1}{3}\left(\frac{n+1}{n}\right)^n = \frac{1}{3}\left(1+\frac{1}{n}\right)^n \\
&\to \frac{e}{3} < 1
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&= \frac{(n+1)(n+1)^n}{(n+1)n!} \cdot \frac{n!}{n^n} \\
&= \left(\frac{n+1}{n}\right)^n = \left(1+\frac{1}{n}\right)^n
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&\to e
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&\to e
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Therefore the series is by the Ratio Test.

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&= \frac{(n+1)(n+1)^n}{(n+1)n!} \cdot \frac{n!}{n^n} \\
&= \left(\frac{n+1}{n}\right)^n = \left(1 + \frac{1}{n}\right)^n \\
&\to e > 1
\end{aligned}$$

Therefore the series is divergent by the Ratio Test.