Precalculus
Trigonometric equations and inequalities

Todor Milev

2019

Outline

License to use and redistribute

Trigonometric equations

- Some problems will not ask you to prove a trigonometric identity, but rather to solve a trigonometric equation.
- Consider the problem of finding all values of x for which $\sin x = \sin(2x) = 2\sin x \cos x$.
- This is not a trigonometric identity - the two sides are different.
- However, there are values for x which the above equality holds.

Example

Find all solutions and then find those that lie between -360° and 360°.

$$\sin\theta = \frac{\sqrt{3}}{2}$$

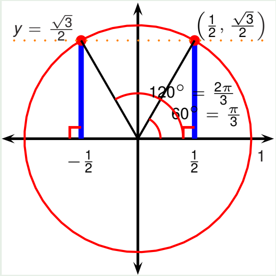
$$\theta = 60^\circ + k\cdot 360^\circ = \underset{\cdots}{\ldots}\ \underset{k=-2}{-660^\circ},\ \underset{k=-1}{-300^\circ},\ \underset{k=0}{60^\circ},\ \underset{k=1}{420^\circ},\ \underset{\cdots}{\ldots}$$

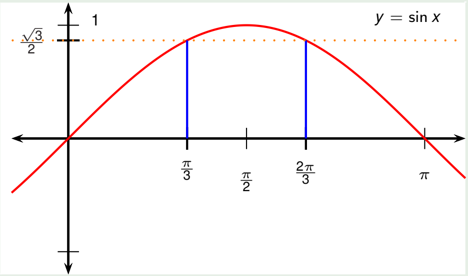
or

$$120^\circ + k\cdot 360^\circ = \ldots -600^\circ,\ -240^\circ,\ 120^\circ,\ 480^\circ,\ \ldots$$

$$\theta = \cancel{\ldots}\ \cancel{-660^\circ},\ -300^\circ,\ 60^\circ,\ \cancel{420^\circ},\ \cancel{\ldots}$$

$$\cancel{\ldots}\ \cancel{-600^\circ},\ -240^\circ,\ 120^\circ,\ \cancel{480^\circ},\ \cancel{\ldots}$$





Example

Find all solutions and then find those that lie between -180° and 180°.

$$\cos\theta = \frac{\sqrt{2}}{2}$$

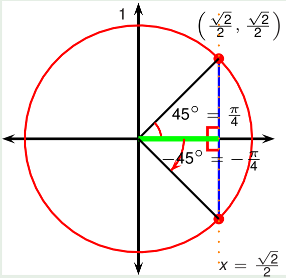
$$\theta = 45^\circ + k\cdot 360^\circ = \quad \dots \quad -675^\circ,\ -315^\circ,\ 45^\circ,\ 405^\circ,\ \dots$$

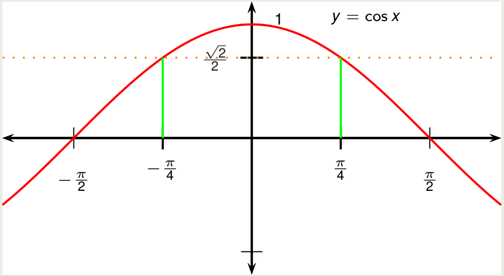
or $\qquad \cdots \quad k=-2 \quad k=-1 \quad k=0 \quad k=1 \quad \cdots$

$$-45^\circ + k\cdot 360^\circ = \dots \ -765^\circ,\ -405^\circ,\ -45^\circ,\ 315^\circ,\ \dots$$

$$\theta = \quad \cancel{\dots}\ \cancel{-675^\circ},\ \cancel{-315^\circ},\ 45^\circ,\ \cancel{405^\circ},\ \cancel{\dots}$$

$$\cancel{\dots}\ \cancel{-765^\circ},\ \cancel{-405^\circ},\ -45^\circ,\ \cancel{315^\circ},\ \cancel{\dots}$$





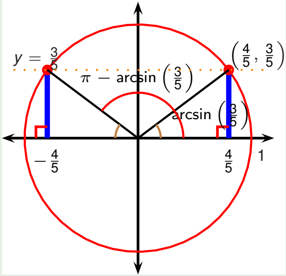
Example

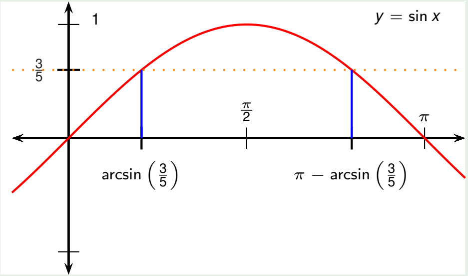
Find all solutions of the equation.

$$\begin{aligned} \sin\theta &= \frac{3}{5} \\ \theta &= \arcsin\left(\frac{3}{5}\right) + k\cdot(2\pi) \qquad \Big|\ \text{arcsin implies radians} \end{aligned}$$

or

$$\pi - \arcsin\left(\frac{3}{5}\right) + k\cdot(2\pi)$$





Example

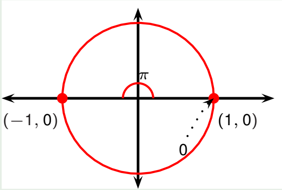
Find all values of θ in the interval $[0, 2\pi]$ such that $\sin\theta = \sin(2\theta)$.

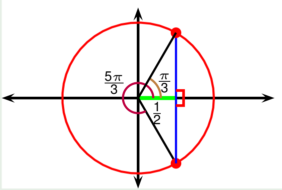
$$
\begin{aligned}
\sin\theta &= \sin(2\theta)\\
\sin\theta &= 2\sin\theta\cos\theta\\
0 &= 2\sin\theta\cos\theta - \sin\theta\\
0 &= \sin\theta(2\cos\theta - 1)
\end{aligned}
$$

$$
\begin{aligned}
\sin\theta &= 0\\
\theta &= 0 + 2k\pi\\
&\quad \text{or } \pi + 2k\pi\\
\theta &= 0 \text{ or } 2\pi \text{ or } \pi
\end{aligned}
$$

or

$$
\begin{aligned}
2\cos\theta - 1 &= 0\\
\cos\theta &= \tfrac{1}{2}\\
\theta &= \tfrac{\pi}{3} + 2k\pi \text{ or } \tfrac{5\pi}{3} + 2k\pi\\
\theta &= \tfrac{\pi}{3} \text{ or } \tfrac{5\pi}{3}
\end{aligned}
$$





Example

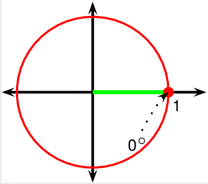
Find all values of θ in the interval $\theta \in [0, 2\pi]$ for which

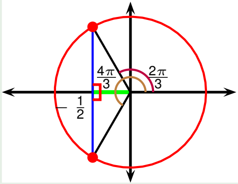
$$\begin{aligned}
\cos(2\theta) &= \cos\theta \\
\cos^2\theta - \sin^2\theta - \cos\theta &= 0 && \Big| \text{ Express via } \cos\theta \\
\cos^2\theta - (1-\cos^2\theta) - \cos\theta &= 0 \\
2\cos^2\theta - \cos\theta - 1 &= 0 && \Big| \text{ Set } \cos\theta = u \\
2u^2 - u - 1 &= 0 \\
(u-1)(2u+1) &= 0
\end{aligned}$$

$$\begin{aligned}
u - 1 &= 0 \\
\cos\theta &= 1 \\
\theta &= 0 + 2k\pi \\
\theta &= 0 \text{ or } 2\pi
\end{aligned}$$

or

$$\begin{aligned}
2u + 1 &= 0 \\
\cos\theta &= -\frac{1}{2} \\
\theta &= \frac{2\pi}{3} + 2k\pi \text{ or } \frac{4\pi}{3} + 2k\pi \\
\theta &= \frac{2\pi}{3} \text{ or } \frac{4\pi}{3}
\end{aligned}$$





Strategy for solving trigonometric equations

- Suppose we want to solve an algebraic trigonometric equation.
- More precisely, the equation should be an algebraic expressions of the trigonometric functions of a single variable.
- Here is a general strategy for solving such a problem:
    - Using trig identities, rewrite in terms of $\sin x$ and $\cos x$ only.
    - Suppose $x \in [2n\pi, (2n+1)\pi]$.
        - Set $\sin x = \sqrt{1-\cos^2 x}$ (allowed due to restrictions on x).
        - Set $\cos x = u$. Solve the resulting algebraic equation for u.
        - For the found solutions for u, solve $\cos x = u$.
        - Check whether your solutions satisfy $x \in [2n\pi, (2n+1)\pi]$.
    - Suppose $x \in [(2n-1)\pi, 2n\pi]$.
        - Set $\sin x = -\sqrt{1-\cos^2 x}$ (allowed due to restrictions on x).
        - Set $\cos x = u$. Solve the resulting algebraic equation for u.
        - For the found solutions for u, solve $\cos x = u$.
        - Check whether your solutions satisfy $x \in [(2n-1)\pi, 2n\pi]$.
- A similar strategy exists for $u = \sin x$ instead of $u = \cos x$.
- Problems requiring full algorithm may be too hard for Calc exams.

Proposition (Product to sum formulas)

$$
\begin{aligned}
\sin\alpha\sin\beta &= \frac{1}{2}\left(\cos(\alpha-\beta)-\cos(\alpha+\beta)\right)\\
\cos\alpha\cos\beta &= \frac{1}{2}\left(\cos(\alpha-\beta)+\cos(\alpha+\beta)\right)\\
\sin\alpha\cos\beta &= \frac{1}{2}\left(\sin(\alpha+\beta)+\sin(\alpha-\beta)\right)
\end{aligned}
$$

Proof.

$$
\begin{array}{crcl}
+ & \cos\alpha\cos\beta + \cancel{\sin\alpha\sin\beta} &=& \cos(\alpha-\beta)\\
 & \cos\alpha\cos\beta - \cancel{\sin\alpha\sin\beta} &=& \cos(\alpha+\beta)\\
 & 2\cos\alpha\cos\beta &=& \cos(\alpha-\beta)+\cos(\alpha+\beta)\\
\hline
- & \cos\alpha\cos\beta + \sin\alpha\sin\beta &=& \cos(\alpha-\beta)\\
 & \cos\alpha\cos\beta - \sin\alpha\sin\beta &=& \cos(\alpha+\beta)\\
 & 2\sin\alpha\sin\beta &=& \cos(\alpha-\beta)-\cos(\alpha+\beta)\\
\hline
+ & \sin\alpha\cos\beta + \cos\alpha\sin\beta &=& \sin(\alpha+\beta)\\
 & \sin\alpha\cos\beta - \cos\alpha\sin\beta &=& \sin(\alpha-\beta)\\
 & 2\sin\alpha\cos\beta &=& \sin(\alpha+\beta)+\sin(\alpha+\beta)
\end{array}
$$

□

Proposition (Product to sum formulas)

$$
\begin{aligned}
\sin\alpha\sin\beta &= \frac{1}{2}\left(\cos(\alpha-\beta)-\cos(\alpha+\beta)\right)\\
\cos\alpha\cos\beta &= \frac{1}{2}\left(\cos(\alpha-\beta)+\cos(\alpha+\beta)\right)\\
\sin\alpha\cos\beta &= \frac{1}{2}\left(\sin(\alpha+\beta)+\sin(\alpha-\beta)\right)
\end{aligned}
$$

- Product to sum formulas are used when integrating (a topic to be studied later/in another course).

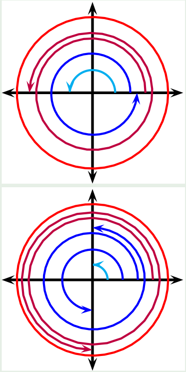
Proposition (Sum to product formulas)

$$\begin{aligned}
\sin\alpha + \sin\beta &= 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)\\
\sin\alpha - \sin\beta &= 2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)\\
\cos\alpha + \cos\beta &= 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)\\
\cos\alpha - \cos\beta &= 2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)
\end{aligned}$$

Recall the formula $\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

Example

Find all solutions in the interval $[0, 2\pi)$.



$$\sin(2x) + \sin(5x) = 0 \quad \Big| \text{ use f-la}$$

$$2\sin\left(\frac{2x+5x}{2}\right)\cos\left(\frac{2x-5x}{2}\right) = 0$$

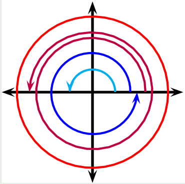
$$2\sin\left(\frac{7}{2}x\right)\cos\left(-\frac{3}{2}x\right) = 0 \quad \Big| \begin{array}{l}\cos\\ \text{is even}\end{array}$$

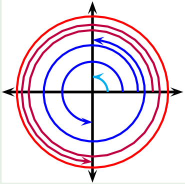
$$2\sin\left(\frac{7}{2}x\right)\cos\left(\frac{3}{2}x\right) = 0$$

Recall the formula $\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

Example

Find all solutions in the interval $[0, 2\pi)$.





$$\sin(2x) + \sin(5x) = 0 \quad \Big| \text{ use f-la}$$

$$2\sin\left(\frac{7}{2}x\right)\cos\left(\frac{3}{2}x\right) = 0$$

$$\sin\left(\tfrac{7}{2}x\right) = 0$$

$$\tfrac{7}{2}x = k\pi \qquad k - \text{integer}$$

$$x = \frac{2k\pi}{7}$$

$$x = \cancel{\dots}, \cancel{\frac{-2\pi}{7}}, 0, \frac{2\pi}{7}, \frac{4\pi}{7}, \frac{6\pi}{7}, \frac{8\pi}{7}, \frac{10\pi}{7}, \frac{12\pi}{7}, \cancel{\frac{14\pi}{7}}, \cancel{\dots}$$

or

$$\cos\left(\tfrac{3}{2}x\right) = 0$$

$$\tfrac{3}{2}x = \tfrac{\pi}{2} + k\pi = \frac{(2k+1)\pi}{2} \qquad k - \text{integer}$$

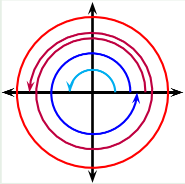
$$x = \frac{(2k+1)\pi}{3}$$

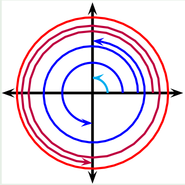
$$x = \cancel{\dots}, \cancel{-\frac{\pi}{3}}, \frac{\pi}{3}, \frac{3\pi}{3}, \frac{5\pi}{3}, \cancel{\frac{7\pi}{3}}, \cancel{\dots}$$

Recall the formula $\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

Example

Find all solutions in the interval $[0, 2\pi)$.





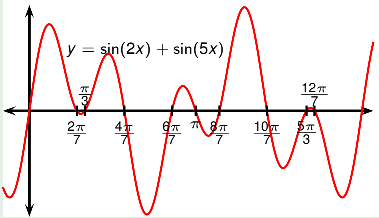
$$\sin(2x) + \sin(5x) = 0 \quad \Big| \text{ use f-la}$$

$$2\sin\left(\frac{7}{2}x\right)\cos\left(\frac{3}{2}x\right) = 0$$

$$x = \cancel{\ldots}, \cancel{\frac{-2\pi}{7}}, 0, \frac{2\pi}{7}, \frac{4\pi}{7}, \frac{6\pi}{7}, \frac{8\pi}{7}, \frac{10\pi}{7}, \frac{12\pi}{7}, \cancel{\frac{14\pi}{7}}, \cancel{\ldots}$$

or

$$x = \cancel{\ldots}, \cancel{-\frac{\pi}{3}}, \frac{\pi}{3}, \frac{3\pi}{3}, \frac{5\pi}{3}, \cancel{\frac{7\pi}{3}}, \cancel{\ldots}$$



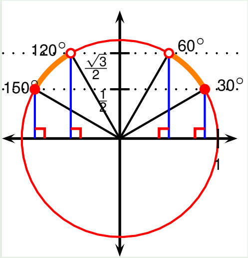
Example

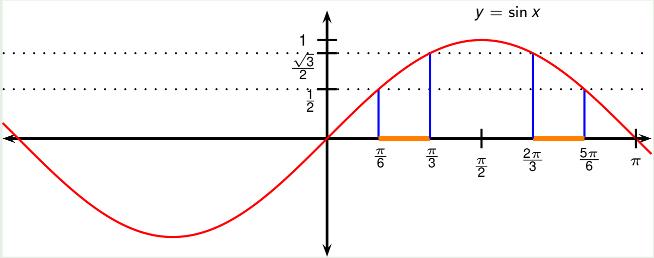
Solve. Among your solutions, find those between -360° and 450°.

$\frac{1}{2} \leq \sin\theta < \frac{\sqrt{3}}{2}$

$x \in [30^\circ + k360^\circ, 60^\circ + k360^\circ) \cup (120^\circ + k360^\circ, 150^\circ + k360^\circ]$

$$x \in \begin{array}{rl|l} & \cancel{\cdots} & \\ & \cup\, \cancel{[-690^\circ, -660^\circ)} \cup \cancel{(-600^\circ, -570^\circ]} & k = -2 \\ & \cup\, [-330^\circ, -300^\circ) \cup (-240^\circ, -210^\circ] & k = -1 \\ & \cup\, [30^\circ, 60^\circ) \cup (120^\circ, 150^\circ] & k = 0 \\ & \cup\, [390^\circ, 420^\circ) \cup \cancel{(480^\circ, 510^\circ]} & k = 1 \\ & \cancel{\cdots} & \end{array}$$





Example

Solve. Among your solutions, find those between -360° and 450°.

$\frac{1}{2} \leq \sin\theta < \frac{\sqrt{3}}{2}$

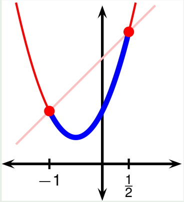
$x \in [30^\circ + k360^\circ, 60^\circ + k360^\circ) \cup (120^\circ + k360^\circ, 150^\circ + k360^\circ]$

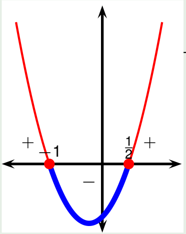
$$
x \in \begin{array}{rl|l}
& \cancel{\cdots} & \\
\cup\, \cancel{[-690^\circ, -660^\circ)} & \cup\, \cancel{(-600^\circ, -570^\circ]} & k = -2 \\
\cup\, [-330^\circ, -300^\circ) & \cup\, (-240^\circ, -210^\circ] & k = -1 \\
\cup\, [30^\circ, 60^\circ) & \cup\, (120^\circ, 150^\circ] & k = 0 \\
\cup\, [390^\circ, 420^\circ) & \cup\, \cancel{(480^\circ, 510^\circ]} & k = 1 \\
& \cancel{\cdots} &
\end{array}
$$

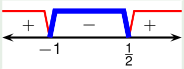
In radians:

$x \in \left[-\frac{11\pi}{6}, -\frac{5\pi}{3}\right) \cup \left[-\frac{4\pi}{3}, -\frac{7\pi}{6}\right) \cup \left[\frac{\pi}{6}, \frac{\pi}{3}\right) \cup \left[\frac{2\pi}{3}, \frac{5\pi}{6}\right) \cup \left[\frac{13\pi}{6}, \frac{7\pi}{3}\right)$

Example





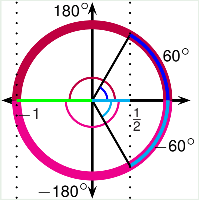


- Solve the inequality $2u^2+2u+1 \leq u+2$.
- Find all solutions of $2\cos^2\theta + 2\cos\theta + 1 \leq \cos\theta + 2$ lying in $[-360^\circ, 360^\circ]$.

$$\begin{array}{rcl}
2u^2+2u+1 &\leq& u+2\\
2u^2+u-1 &\leq& 0\\
2\left(u-\frac{1}{2}\right)(u+1) &\leq& 0\\
u &\in& \left[-1,\frac{1}{2}\right]
\end{array}$$

$$\begin{array}{rcll}
2\cos^2\theta+2\cos\theta+1 &\leq& \cos\theta+2 & \text{Set } \cos\theta = u\\
2u^2+2u+1 &\leq& u+2 &\\
u &\in& \left[-1,\frac{1}{2}\right] & \text{(solved above)}\\
\cos\theta &\in& \left[-1,\frac{1}{2}\right] &\\
-1\leq\cos\theta &\leq& \frac{1}{2} &
\end{array}$$

Example



- Solve the inequality $2u^2 + 2u + 1 \leq u + 2$.
- Find all solutions of $2\cos^2\theta + 2\cos\theta + 1 \leq \cos\theta + 2$ lying in $[-360°, 360°]$.

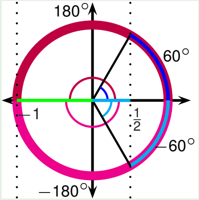
$$\cos\theta \in \left[-1, \tfrac{1}{2}\right]$$
$$-1 \leq \cos\theta \leq \frac{1}{2}$$

$$\theta \in [-180° + k360°, -60° + k360°] \cup [60° + k360°, 180° + k360°]$$

$$\theta \in \begin{array}{ll} \cdots \cup \cancel{[-540°, -420°]} \cup [-300°, -180°] & k=-1 \\ \cup [-180°, -60°] \cup [60°, 180°] & k=0 \\ \cup [180°, 300°] \cup \cancel{[420°, 540°]} \cdots & k=1 \end{array}$$

$$\theta \in [-300°, -60°] \cup [60°, 300°]$$

Example



- Solve the inequality $2u^2 + 2u + 1 \leq u + 2$.
- Find all solutions of $2\cos^2\theta + 2\cos\theta + 1 \leq \cos\theta + 2$ lying in $[-360°, 360°]$.

$\theta \in [-300°, -60°] \cup [60°, 300°]$