

# Precalculus

## Equations involving logarithms and exponents

Todor Milev

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# Outline

- 1 Equations involving logarithms
- 2 Equations involving exponents
- 3 Inverse function problems and exponents
- 4 Basic exponential inequalities

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## Example

Solve the equation.

$$\begin{aligned}\log_3(2x^2 + 1) &= 2 && | \text{Exponentiate base 3} \\ 3^{\log_3(2x^2 + 1)} &= 3^2 \\ 2x^2 + 1 &= 9 \\ 2x^2 &= 8 \\ x^2 &= \frac{8}{2} = 4 \\ x &= \pm\sqrt{4} = \pm 2 \\ x = 2 \text{ or } x = -2 && | \text{final answer}\end{aligned}$$

The logarithmic property  $\log_a(xy) = \log_a x + \log_a y$  holds only for positive  $x, y$ . Failure to check the positivity of  $x, y$  can result in extraneous (fake) solutions to logarithmic equations.

### Example

Solve the equation.

$$\log_2(x+2) + \log_2(x-1) = 2$$

$$\log_2((x+2)(x-1)) = 2$$

$$(x+2)(x-1) = 2^2$$

$$x^2 + x - 2 = 4$$

$$x^2 + x - 6 = 0$$

$$(x-2)(x+3) = 0$$

$$x = 2 \quad \text{or} \quad x = -3$$

$x = -3$  not a solution (outside of domain)

Domain:  $x > 1$

Exponentiate base 2

## Example (Solve exponential equation without logarithms)

Solve for  $t$ .

Find a common base:

$$\begin{aligned}16^{4t} &= 8^{t-2} \\ (2^4)^{4t} &= (2^3)^{t-2} \\ 2^{16t} &= 2^{3t-6} \\ 16t &= 3t - 6 \\ 13t &= -6 \\ t &= -\frac{6}{13}.\end{aligned}$$

## Example

Solve the equation.

$$\begin{aligned}2^{1-5x} &= 12 && | \text{ apply } \log_2 \\ \log_2(2^{1-5x}) &= \log_2 12 \\ 1 - 5x &= \log_2 12 = \log_2(4 \cdot 3) \\ 1 - 5x &= \log_2 4 + \log_2 3 \\ 1 - 5x &= 2 + \log_2 3 \\ 5x &= 1 - (2 + \log_2 3) \\ &= -1 - \log_2 3 \\ x &= \frac{-1 - \log_2 3}{5} \\ \text{Calculator: } x &\approx -0.516993.\end{aligned}$$

## Example

Solve the equation.

$$e^{x-3} = 2e^{2x-1}$$

Divide by  $e^{2x-1}$

$$\frac{e^{x-3}}{e^{2x-1}} = 2$$

$$e^{x-3-(2x-1)} = 2$$

$$e^{-x-2} = 2$$

Apply  $\ln$

$$-x - 2 = \ln 2$$

$$-x = \ln 2 + 2$$

$$x = -(\ln 2 + 2)$$

$$x = -\ln 2 - 2$$

Final answer

$$x \approx -2.693$$

Calculator



## Example

Solve.

$$3^{2x+5} = 5 \cdot 2^{-x+1}$$

Common base

$$(2^{\log_2 3})^{2x+5} = 5 \cdot 2^{-x+1}$$

$$a = b^{\log_b a}$$

$$2^{(\log_2 3)(2x+5)} = 5 \cdot 2^{-x+1}$$

$$\frac{2^{(\log_2 3)(2x+5)}}{2^{-x+1}} = 5$$

$$2^{(\log_2 3)(2x+5)-(-x+1)} = 5$$

Apply  $\log_2$

$$(\log_2 3)(2x + 5) + x - 1 = \log_2 5$$

$$x(2 \log_2 3 + 1) + 5 \log_2 3 = \log_2 5 + 1$$

$$x = \frac{\log_2 5 + 1 - 5 \log_2 3}{2 \log_2 3 + 1}$$

$$x \approx -1.1038$$

Calculator

## Example

Solve the equation.

$$e^{5-3x} = 10 \quad \text{apply } \ln$$

$$=$$

$$=$$

$$x =$$

Calculator:  $x \approx 0.8991.$

### Example (Solving an exponential word problem)

A farmer buys 48 chickens and 6 rabbits. The chicken population doubles each year, and the rabbit population doubles every six months. When does the farmer have the same number of chickens as rabbits?

Let  $c(t)$  denote the number of chickens after  $t$  years, and let  $r(t)$  denote the number of rabbits after  $t$  years.

Solve for  $t$ :  $=$

 $=$  $=$  $=$  $=$  $=$  $t =$

## Example (Solving a quadratic exponential equation)

Solve for  $x$ .

$$\begin{aligned}9^x &= 2 \cdot 3^x + 63 \\9^x - 2 \cdot 3^x - 63 &= 0 & \left| \text{Substitute } u = 3^x \right. \\u^2 - 2u - 63 &= 0 \\(u - 9)(u + 7) &= 0\end{aligned}$$

$$\begin{aligned}u &= 9 \quad \text{or} \quad u = -7 \\3^x &= 9 \quad \text{or} \quad 3^x = -7 \\x &= 2 & \quad \text{no real solution}\end{aligned}$$

## Example

Solve the equation

$$e^{2x} - 3e^x - 4 = 0$$

Set  $e^x = u$ . Then  $e^{2x} = u^2$ .

$$u^2 - 3u - 4 = 0$$

$$(u - 4)(u + 1) = 0$$

$$u = 4$$

or

$$u = -1$$

$$e^x = 4$$

or

$$e^x = -1$$

$$x = \ln 4$$

or

no real solution

$$x \approx 1.3863$$

## Example

Solve the equation

$$4^{x+1} - 2^{x+2} - 3 = 0$$

Set  $u = 2^x$ . Then  $4^{x+1} = 4u^2$ ,  $2^{x+2} = 4u$ .

$$4u^2 - 4u - 3 = 0$$

$$(2u - 3)(2u + 1) = 0$$

$$2u - 3 = 0 \quad \text{or} \quad 2u + 1 = 0$$

$$u = \frac{3}{2} \quad \text{or} \quad u = -\frac{1}{2}$$

$$2^x = \frac{3}{2} \quad \text{or} \quad 2^x = -\frac{1}{2}$$

$$x = \log_2 \left( \frac{3}{2} \right) = \frac{\ln \left( \frac{3}{2} \right)}{\ln 2} \approx 0.58496 \quad \text{or} \quad \text{no real solution}$$

## Example (Exponential equation that reduces to quadratic)

Solve the equation.

$$3^{2x} = 2 + 63 \cdot 3^{-2x}$$

$$3^{2x} - 2 - 63 \cdot 3^{-2x} = 0$$

$$u - 2 - 63u^{-1} = 0$$

$$u^2 - 2u - 63 = 0$$

$$(u - 9)(u + 7) = 0$$

$$u - 9 = 0 \quad \text{or} \quad u + 7 = 0$$

$$u = 9 \quad \text{or} \quad u = -7$$

$$3^{2x} = 9 \quad \text{or} \quad \text{no real solution}$$

$$2x = \log_3 9$$

$$2x = 2$$

$$x = 1$$

> 2 terms  $\Rightarrow$

transfer one side

$$3^{2x} = u$$

$$3^{-2x} = (3^{2x})^{-1} = u^{-1}$$

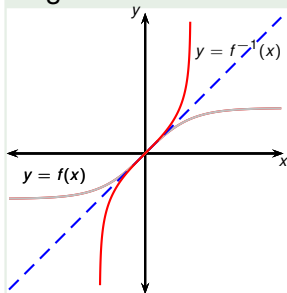
Multiply  $\cdot u$

## Example

Find  $f^{-1}(x)$  for

$$f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

$f = \tanh =$  hyperbolic tangent function.



Final answer, relabeled:

$$f^{-1}(x) = \frac{1}{2} \ln \left( \frac{1+x}{1-x} \right)$$

$$\frac{e^x - e^{-x}}{e^x + e^{-x}} = y$$

$$\frac{(u - \frac{1}{u})u}{(u + \frac{1}{u})u} = y$$

$$\frac{u^2 - 1}{u^2 + 1} = y$$

$$u^2 - 1 = y(u^2 + 1)$$

$$u^2(1 - y) = 1 + y$$

$$u^2 = \frac{1 + y}{1 - y}$$

$$(e^x)^2 = \frac{1 + y}{1 - y}$$

$$e^{2x} = \frac{1 + y}{1 - y}$$

$$x = \frac{1}{2} \ln \left( \frac{1 + y}{1 - y} \right)$$

$$\begin{aligned} \text{Set } u &= e^x \\ e^{-x} &= \frac{1}{e^x} = \frac{1}{u} \end{aligned}$$

Take  $\ln$



## Example

Solve the inequality.

$$2^{-3x-5} < 7$$

$$\log_2 2^{-3x-5} < \log_2 7$$

$$-3x - 5 < \log_2 7$$

$$-3x < \log_2 7 + 5$$

$$x > -\frac{\log_2 7 + 5}{3}$$

$$x \in \left( -\frac{5 + \log_2 7}{3}, \infty \right)$$

Logarithms preserve inequalities: apply  $\log_2$

Division by negative number flips inequalities

