Calculus II
Power series expansion of arctangent, part 1

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Example

Find a power series for $\arctan x$ and state its radius of convergence.

$\arctan(x)$

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- By the same theorem, the radius of convergence remains $R = 1$.