Calculus II
Homework
L'Hospital's rule

1. Compute the limits. The answer key has not been fully proofread, use with caution.

(a) $\displaystyle\lim_{x\to 0}\frac{\sin x}{x}$.

answer: 1

(b) $\displaystyle\lim_{x\to 0}\frac{x}{\ln(1+x)}$.

answer: 1

(c) $\displaystyle\lim_{x\to 0}\frac{x^2}{x-\ln(1+x)}$.

answer: 2

(d) $\displaystyle\lim_{x\to 0}\frac{x^2}{\sin x\ln(1+x)}$.

answer: 1

(e) $\displaystyle\lim_{x\to 0}\frac{\sin^2 x}{(\ln(1+x))^2}$.

answer: 1

(f) $\displaystyle\lim_{x\to 0}\frac{\cos x-1}{\sin x\ln(1+x)}$.

answer: $-\frac{1}{2}$

(g) $\displaystyle\lim_{x\to 0}\frac{\arctan x-x}{x^3}$.

answer: $-\frac{1}{3}$

(h) $\displaystyle\lim_{x\to 0}\frac{\arcsin x-x}{x^3}$.

answer: $\frac{1}{6}$

(i) $\displaystyle\lim_{x\to 1}\frac{x}{x-1}-\frac{1}{\ln x}$.

answer: $\frac{1}{2}$

(j) $\displaystyle\lim_{x\to 0}\frac{\cos(nx)-\cos(mx)}{x^2}$.

answer: $\frac{m^2-n^2}{2}$

(k) $\displaystyle\lim_{x\to 0}\frac{\arcsin x-x-\frac{1}{6}x^3}{\sin^5 x}$.

answer: $\frac{3}{40}$

(l) $\displaystyle\lim_{x\to 1}\frac{\sin(\pi x)\ln x}{\cos(\pi x)+1}$.

answer: $-\frac{2}{\pi}$

(m) $\displaystyle\lim_{x\to 0}\frac{\sin x-x}{\arcsin x-x}$.

answer: -1

(n) $\displaystyle\lim_{x\to 0}\frac{\sin x-x}{\arctan x-x}$.

answer: $\frac{1}{2}$

(o) $\displaystyle\lim_{x\to\infty}x\sin\left(\frac{2}{x}\right)$.

answer: 2

Solution. 1l The limit is of the form "$\frac{0}{0}$" so we are allowed to use L'Hospital's rule.

$$
\begin{array}{rcll}
\displaystyle\lim_{x\to 1}\frac{\sin(\pi x)\ln x}{\cos(\pi x)+1} &=& \displaystyle\lim_{x\to 1}\frac{(\sin(\pi x)\ln x)'}{(\cos(\pi x)+1)'} & \\
&=& \displaystyle\lim_{x\to 1}\frac{\left(\pi\cos(\pi x)\ln x+\sin(\pi x)\frac{1}{x}\right)}{(-\pi\sin(\pi x))} & \Big|\ \text{type “}\tfrac{0}{0}\text{”, L'Hospital's rule} \\
&=& \displaystyle\lim_{x\to 1}\frac{\left(\pi\cos(\pi x)\ln x+\sin(\pi x)\frac{1}{x}\right)'}{(-\pi\sin(\pi x))'} & \\
&=& \displaystyle\lim_{x\to 1}\frac{-\pi^2\sin(\pi x)\ln(x)+2\pi\cos(\pi x)x^{-1}-\sin(\pi x)x^{-2}}{(-\pi^2\cos(\pi x))} & \\
&=& \displaystyle\frac{-\pi^2\sin(\pi)\ln(1)+2\pi\cos(\pi)-\sin(\pi)}{(-\pi^2\cos(\pi))} & \\
&=& \displaystyle -\frac{2}{\pi}\quad . &
\end{array}
$$

Solution. 1n **Solution I.**

$$\begin{array}{rcll}
\displaystyle\lim_{x\to 0}\frac{\sin x - x}{\arctan x - x} & = & \displaystyle\lim_{x\to 0}\frac{\cos x - 1}{\frac{1}{1+x^2}-1} & \text{L'Hospital rule} \\
 & = & \displaystyle\lim_{x\to 0}\frac{-\sin x}{\frac{-2x}{(1+x^2)^2}} & \text{L'Hospital rule again} \\
 & = & \displaystyle\lim_{x\to 0}\frac{(1+x^2)^2}{2}\frac{\sin x}{x} & \\
 & = & \displaystyle\lim_{x\to 0}\frac{(1+x^2)^2}{2}\lim_{x\to 0}\frac{\sin x}{x} & \\
 & = & \displaystyle\frac{1}{2}\quad . &
\end{array}$$

Solution II.

$$\begin{array}{rcll}
\displaystyle\lim_{x\to 0}\frac{\sin x - x}{\arctan x - x} & = & \displaystyle\lim_{x\to 0}\frac{\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\dots\right)-x}{\left(x-\frac{x^3}{3}+\frac{x^5}{5}-\dots\right)-x} & \text{use the Maclaurin series of } \sin, \arctan \\
 & = & \displaystyle\lim_{x\to 0}\frac{-\frac{x^3}{6}+x^5\left(\frac{1}{5!}-\dots\right)}{-\frac{x^3}{3}+x^5\left(\frac{1}{5}-\dots\right)} & \text{The expressions in parenthesis are continous functions in x} \\
 & = & \displaystyle\lim_{x\to 0}\frac{-\frac{1}{6}+x^2\left(\frac{1}{5!}-\dots\right)}{-\frac{1}{3}+x^2\left(\frac{1}{5}-\dots\right)} & \\
 & = & \displaystyle\frac{-\frac{1}{6}+0}{\frac{1}{3}+0} & \\
 & = & \displaystyle\frac{1}{2}\quad . &
\end{array}$$

Solution. 1o.

$$\begin{array}{rcll}
\displaystyle\lim_{x\to\infty} x\sin\left(\frac{2}{x}\right) & = & \displaystyle\lim_{x\to\infty}\frac{\sin\left(\frac{2}{x}\right)}{\frac{1}{x}} & \text{indeterminate form} \\
 & & & \text{Use L'Hospital's rule} \\
 & = & \displaystyle\lim_{x\to\infty}\frac{\cos\left(\frac{2}{x}\right)\left(-\frac{2}{x^2}\right)}{-\frac{1}{x^2}} & \\
 & = & \displaystyle\lim_{x\to\infty} 2\cos\left(\frac{x}{2}\right) & \\
 & = & 2\quad . &
\end{array}$$

2. Compute the limit.

   (a) $\displaystyle\lim_{x\to\infty}\left(\frac{x-2}{x}\right)^x$.

   answer: e^{-2}

   (b) $\displaystyle\lim_{x\to\infty}\left(\frac{x-2}{x}\right)^{2x}$

   answer: e^{-4}.

   (c) $\displaystyle\lim_{x\to\infty}\left(\frac{x}{x+3}\right)^{2x}$

   answer: e^{-6}

Solution. 2.a.

Variant I.

$$\begin{array}{rcll}
\displaystyle\lim_{x\to\infty}\left(\frac{x-2}{x}\right)^x & = & \displaystyle\lim_{x\to\infty}\left(1-\frac{2}{x}\right)^x & \text{use } \displaystyle\lim_{x\to\infty}\left(1+\frac{k}{x}\right)^x = e^k \\
 & = & e^{-2}\quad . &
\end{array}$$

Variant II.

$$
\begin{aligned}
\lim_{x\to\infty}\left(\frac{x-2}{x}\right)^x &= \lim_{x\to\infty} e^{\ln\left(\left(\frac{x-2}{x}\right)^x\right)} \\
\lim_{x\to\infty}\ln\left(\left(\frac{x-2}{x}\right)^x\right) &= \lim_{x\to\infty} x\left(\ln(x-2)-\ln(x)\right) \\
&= \lim_{x\to\infty}\frac{\ln(x-2)-\ln(x)}{\frac{1}{x}} && \Big|\ \text{L'Hospital rule} \\
&= \lim_{x\to\infty}\frac{\frac{1}{x-2}-\frac{1}{x}}{-\frac{1}{x^2}} \\
&= \lim_{x\to\infty}\frac{-2x^2}{x^2-2x} = -2 && \Big|\ \text{Therefore} \\
\lim_{x\to\infty}\left(\frac{x-2}{x}\right)^x &= \lim_{x\to\infty} e^{\ln\left(\left(\frac{x-2}{x}\right)^x\right)} \\
&= e^{\lim\limits_{x\to\infty}\ln\left(\left(\frac{x-2}{x}\right)^x\right)} \\
&= e^{-2} \quad .
\end{aligned}
$$

3. Find the limit.

(a) $\displaystyle\lim_{x\to\infty}\left(1-\frac{2}{x}\right)^x$. answer: e^{-2}

(b) $\displaystyle\lim_{x\to 0}(1-x)^{\frac{1}{x}}$. answer: e^{-1}

(c) $\displaystyle\lim_{x\to\infty}\left(\frac{x}{x-5}\right)^x$. answer: e^{5}

(d) $\displaystyle\lim_{x\to\infty}\left(\frac{x}{x-2}\right)^{3x+2}$. answer: e^{6}