Precalculus
Solve triangle from side and two angles

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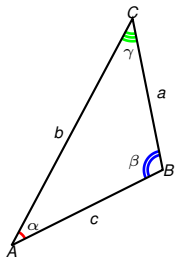
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Law of sines

Let $\triangle ABC$ have sides lengths a, b, c angles α, β, γ, as indicated: α is opposite to a, β is opposite to b, γ is opposite to c.

Proposition (Law of Sines)

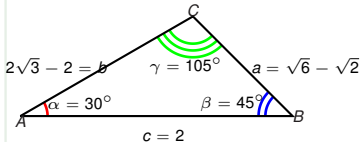$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma}.$$

Proof.

$$\begin{aligned}
\text{Area}(\triangle ABC) = \frac{ab \sin \gamma}{2} &= \frac{bc \sin \alpha}{2} \quad \Big| \text{ Div. by } \frac{b}{2} \\
a \sin \gamma &= c \sin \alpha \\
\frac{a}{\sin \alpha} &= \frac{c}{\sin \gamma}.
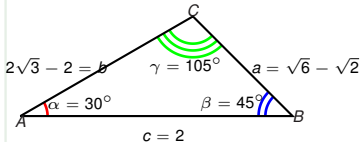\end{aligned}$$

The remaining cases are similar. □

Example

A triangle has a side of length 2*cm*; the two angles adjacent to it are $30°$ and $45°$.

- Find the other two sides of the triangle.
- Find the area of the triangle.

- Let the known side be $c = 2cm$.
- Let the known angles $30°$, $45°$ be arranged as in the figure, and let the third angle be $\gamma = 180° - 30° - 45° = 180° - 75° = 105°$.
- Label the unknown sides a, b as indicated.

Example

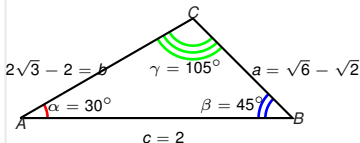A triangle has a side of length 2*cm*; the two angles adjacent to it are $30°$ and $45°$.

- Find the other two sides of the triangle.
- Find the area of the triangle.

$2\sqrt{3} - 2 = b$ $\gamma = 105°$ $a = \sqrt{6} - \sqrt{2}$

$\alpha = 30°$ $\beta = 45°$

$c = 2$

$$\sin 105° = \sin(60° + 45°) = \sin 60° \cos 45° + \cos 60° \sin 45°$$

$$= \frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2} + \frac{1}{2}\frac{\sqrt{2}}{2} = \frac{\sqrt{6} + \sqrt{2}}{4}$$

$$\frac{a}{\sin \alpha} = \frac{c}{\sin \gamma} \qquad |\text{Law of sines}$$

$$a = \frac{c \sin \alpha}{\sin \gamma} = \frac{2 \sin 30°}{\sin 105°} = \frac{2 \cdot \frac{1}{2}}{\frac{\sqrt{6}+\sqrt{2}}{4}} = \frac{4(\sqrt{6} - \sqrt{2})}{(\sqrt{6} + \sqrt{2})(\sqrt{6} - \sqrt{2})}$$

$$= \frac{4(\sqrt{6} - \sqrt{2})}{6 - 2} = \sqrt{6} - \sqrt{2}$$

Example

A triangle has a side of length 2*cm*; the two angles adjacent to it are 30° and 45°.

- Find the other two sides of the triangle.
- Find the area of the triangle.

$$\frac{b}{\sin \beta} = \frac{c}{\sin \gamma} \qquad |\text{Law of sines}$$

$$b = \frac{c \sin \beta}{\sin \gamma} = \frac{2 \sin 45°}{\sin 105°} = \frac{2 \frac{\sqrt{2}}{2}}{\frac{\sqrt{6}+\sqrt{2}}{4}} = \frac{4\sqrt{2}\left(\sqrt{6}-\sqrt{2}\right)}{\left(\sqrt{6}+\sqrt{2}\right)\left(\sqrt{6}-\sqrt{2}\right)}$$

$$= \frac{4\sqrt{2}(\sqrt{6}-\sqrt{2})}{4} = 2\sqrt{3} - 2$$

$$\text{Area} = \frac{bc \sin \alpha}{2} = \frac{(2\sqrt{3}-2)2\frac{1}{2}}{2} = \sqrt{3} - 1 \quad \text{cm}^2$$