Calculus II
Power series expansion of rational functions with linear denominator, part 2

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Example

Find a power series representation for $\frac{1}{x+2}$.

$$\begin{aligned}
\frac{1}{2+x} &= \frac{1}{2\left(1+\frac{x}{2}\right)} \\
&= \frac{1}{2}\cdot\frac{1}{\left(1-\left(-\frac{x}{2}\right)\right)} = \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n && \Big| \text{ if \& only if } \left|-\tfrac{x}{2}\right| < 1 \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^n \\
&= \frac{1}{2}-\frac{x}{4}+\frac{x^2}{8}-\frac{x^3}{16}+\dots
\end{aligned}$$

To find interval of convergence:

$$\begin{aligned}
\left|-\frac{x}{2}\right| &< 1 \\
|x| &< 2
\end{aligned}$$

Therefore the interval of convergence is $x \in (-2, 2)$.

Example

Find a power series representation for $\frac{x^3}{x+2}$.

$$\begin{aligned}
\frac{x^3}{x+2} &= x^3 \cdot \frac{1}{x+2} \\
&= x^3 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^n && \Bigg| \text{ if \& only if } |x| < 2 \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^{n+3} \\
&= \frac{x^3}{2} - \frac{x^4}{4} + \frac{x^5}{8} - \frac{x^6}{16} + \cdots
\end{aligned}$$

- Another way to write this is $\displaystyle \frac{x^3}{x+2} = \sum_{n=3}^{\infty} \frac{(-1)^{n-1}}{2^{n-2}} x^n$.
- The interval of convergence is again $x \in (-2, 2)$.