Calculus II

Integrals of the form $\displaystyle\int \frac{a}{bx^2+c}\mathrm{d}x,\ c>0$

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Linear substitutions leading to blocks IIa and IIIa

Building block IIIa: $\int \frac{1}{u^2+1}\mathrm{d}u =$ **?** $+ C$.

Example

$$\int \frac{1}{x^2+2}\mathrm{d}x$$

Linear substitutions leading to blocks IIa and IIIa

Building block IIIa: $\int \frac{1}{u^2+1} \mathrm{d}u = \mathbf{?} \qquad + C$.

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Example

$$\int \frac{1}{x^2 + 2} \mathrm{d}x = \int \frac{1}{2\left(\frac{1}{2}x^2 + 1\right)} \mathrm{d}x$$

Linear substitutions leading to blocks IIa and IIIa

Building block IIIa: $\int \frac{1}{u^2+1}\mathrm{d}u = \textbf{?} \qquad\qquad + C.$

Example

$$\int \frac{1}{x^2+2}\mathrm{d}x \quad = \quad \int \frac{1}{2\left(\frac{1}{2}x^2+1\right)}\mathrm{d}x$$

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Building block IIIa: $\int \frac{1}{u^2+1} \mathrm{d}u = \textbf{?} \qquad\qquad + C.$

Example

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Linear substitutions leading to blocks IIa and IIIa

Building block IIIa: $\int \frac{1}{u^2+1} \mathrm{d}u = \textbf{?} \qquad + C.$

Example

$$\begin{aligned}\int \frac{1}{x^2+2}\mathrm{d}x &= \int \frac{1}{2\left(\frac{1}{2}x^2+1\right)}\mathrm{d}x \\ &= \int \frac{1}{2\left(\left(\frac{x}{\sqrt{2}}\right)^2+1\right)} \quad \mathrm{d}\left(\; x \; \right)\end{aligned}$$

Linear substitutions leading to blocks IIa and IIIa

Building block IIIa: $\int \frac{1}{u^2+1} \mathrm{d}u = \textbf{?} \qquad\qquad + C.$

Example

$$\begin{aligned}\int \frac{1}{x^2+2}\mathrm{d}x &= \int \frac{1}{2\left(\frac{1}{2}x^2+1\right)}\mathrm{d}x \\ &= \int \frac{1}{2\left(\left(\frac{x}{\sqrt{2}}\right)^2+1\right)} \quad \mathrm{d}\left(\ x\ \right)\end{aligned}$$

Use $2 = \left(\sqrt{2}\right)^2$

Linear substitutions leading to blocks IIa and IIIa

Building block IIIa: $\int \frac{1}{u^2+1} \mathrm{d}u =$ **?** $\qquad + C$.

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$$\begin{aligned}\int \frac{1}{x^2+2}\mathrm{d}x &= \int \frac{1}{2\left(\frac{1}{2}x^2+1\right)}\mathrm{d}x \\ &= \int \frac{1}{2\left(\left(\frac{x}{\sqrt{2}}\right)^2+1\right)} \quad \mathrm{d}\left(\ x \ \right)\end{aligned}$$

Use $2 = \left(\sqrt{2}\right)^2$

Set $\frac{x}{\sqrt{2}} = u$

Linear substitutions leading to blocks IIa and IIIa

Building block IIIa: $\int \frac{1}{u^2+1} \mathrm{d}u =$ **?** $\qquad + C$.

Example

$$\begin{aligned}\int \frac{1}{x^2+2}\mathrm{d}x &= \int \frac{1}{2\left(\frac{1}{2}x^2+1\right)}\mathrm{d}x \\ &= \int \frac{1}{2\left(\left(\frac{x}{\sqrt{2}}\right)^2+1\right)} \sqrt{2}\mathrm{d}\left(\frac{x}{\sqrt{2}}\right)\end{aligned}$$

Use $2 = \left(\sqrt{2}\right)^2$

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Linear substitutions leading to blocks IIa and IIIa

Building block IIIa: $\int \frac{1}{u^2+1}\mathrm{d}u =$ **?** $\quad + C$.

Example

$$
\begin{aligned}
\int \frac{1}{x^2+2}\mathrm{d}x &= \int \frac{1}{2\left(\frac{1}{2}x^2+1\right)}\mathrm{d}x \\
&= \int \frac{1}{2\left(\left(\frac{x}{\sqrt{2}}\right)^2+1\right)}\sqrt{2}\mathrm{d}\left(\frac{x}{\sqrt{2}}\right) \\
&= \frac{1}{\sqrt{2}}\int \frac{1}{u^2+1}\mathrm{d}u
\end{aligned}
$$

Use $2 = \left(\sqrt{2}\right)^2$

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Building block IIIa: $\int \frac{1}{u^2+1} \mathrm{d}u =$ **?** $+ C$.

Example

$$\begin{aligned}\int \frac{1}{x^2+2}\mathrm{d}x &= \int \frac{1}{2\left(\frac{1}{2}x^2+1\right)}\mathrm{d}x \\ &= \int \frac{1}{2\left(\left(\frac{x}{\sqrt{2}}\right)^2+1\right)}\sqrt{2}\mathrm{d}\left(\frac{x}{\sqrt{2}}\right) \\ &= \frac{1}{\sqrt{2}}\int \frac{1}{u^2+1}\mathrm{d}u\end{aligned}$$

Use $2 = \left(\sqrt{2}\right)^2$

Set $\frac{x}{\sqrt{2}} = u$

Linear substitutions leading to blocks IIa and IIIa

Building block IIIa: $\int \frac{1}{u^2+1}du = \mathbf{?} \qquad + C$.

Example

$$
\begin{aligned}
\int \frac{1}{x^2+2}dx &= \int \frac{1}{2\left(\frac{1}{2}x^2+1\right)}dx \\
&= \int \frac{1}{2\left(\left(\frac{x}{\sqrt{2}}\right)^2+1\right)}\sqrt{2}d\left(\frac{x}{\sqrt{2}}\right) \\
&= \frac{1}{\sqrt{2}}\int \frac{1}{u^2+1}du \\
&= \frac{1}{\sqrt{2}}\mathbf{?} \qquad\qquad + C
\end{aligned}
$$

Use $2 = \left(\sqrt{2}\right)^2$

Set $\frac{x}{\sqrt{2}} = u$

Linear substitutions leading to blocks IIa and IIIa

Building block IIIa: $\int \frac{1}{u^2+1} \mathrm{d}u = \arctan u + C$.

Example

$$\begin{aligned}
\int \frac{1}{x^2+2}\mathrm{d}x &= \int \frac{1}{2\left(\frac{1}{2}x^2+1\right)}\mathrm{d}x \\
&= \int \frac{1}{2\left(\left(\frac{x}{\sqrt{2}}\right)^2+1\right)}\sqrt{2}\mathrm{d}\left(\frac{x}{\sqrt{2}}\right) \\
&= \frac{1}{\sqrt{2}}\int \frac{1}{u^2+1}\mathrm{d}u \\
&= \frac{1}{\sqrt{2}}\arctan(u) + C
\end{aligned}$$

Use $2 = \left(\sqrt{2}\right)^2$

Set $\frac{x}{\sqrt{2}} = u$

Linear substitutions leading to blocks IIa and IIIa

Building block IIIa: $\int \frac{1}{u^2+1} \mathrm{d}u = \arctan u + C$.

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&= \int \frac{1}{2\left(\left(\frac{x}{\sqrt{2}}\right)^2+1\right)}\sqrt{2}\mathrm{d}\left(\frac{x}{\sqrt{2}}\right) \\
&= \frac{1}{\sqrt{2}}\int \frac{1}{u^2+1}\mathrm{d}u \\
&= \frac{1}{\sqrt{2}}\arctan(u) + C \\
&= \frac{1}{\sqrt{2}}\arctan\left(\frac{x}{\sqrt{2}}\right) + C
\end{aligned}$$

Use $2 = \left(\sqrt{2}\right)^2$

Set $\frac{x}{\sqrt{2}} = u$

Linear substitutions leading to blocks IIa and IIIa

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&= \frac{1}{\sqrt{2}}\arctan(u) + C \\
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\end{aligned}$$

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Set $\frac{x}{\sqrt{2}} = u$

Linear substitutions leading to blocks IIa and IIIa

Building block IIIa: $\int \frac{1}{u^2+1}\mathrm{d}u = \arctan u + C$.

Example

$$\begin{aligned}
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&= \frac{1}{\sqrt{2}}\arctan(u)+C \\
&= \frac{1}{\sqrt{2}}\arctan\left(\frac{x}{\sqrt{2}}\right)+C
\end{aligned}$$

Use $2=\left(\sqrt{2}\right)^2$

Set $\frac{x}{\sqrt{2}}=u$

Linear substitutions leading to blocks IIa and IIIa

Building block IIIa: $\int \frac{1}{x^2+1}\mathrm{d}x = \arctan x + C$. **Let** $a > 0$**.**

Example

$$
\begin{aligned}
\int \frac{1}{x^2+a}\mathrm{d}x &= \int \frac{1}{a\left(\frac{1}{a}x^2+1\right)}\mathrm{d}x \\
&= \int \frac{1}{a\left(\left(\frac{x}{\sqrt{a}}\right)^2+1\right)}\sqrt{a}\,\mathrm{d}\left(\frac{x}{\sqrt{a}}\right) \\
&= \frac{1}{\sqrt{a}}\int \frac{1}{u^2+1}\mathrm{d}u \\
&= \frac{1}{\sqrt{a}}\arctan(u) + C \\
&= \frac{1}{\sqrt{a}}\arctan\left(\frac{x}{\sqrt{a}}\right) + C
\end{aligned}
$$

Use $a = \left(\sqrt{a}\right)^2$

Set $u = \frac{x}{\sqrt{a}}$

Linear substitutions leading to blocks IIa and IIIa

Building block IIIa: $\int \frac{1}{x^2+1}\mathrm{d}x = \arctan x + C$. **Let $a > 0$.**

Example

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&= \frac{1}{\sqrt{a}}\arctan(u) + C \\
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\end{aligned}$$

Use $a = \left(\sqrt{a}\right)^2$

Set $u = \frac{x}{\sqrt{a}}$

Linear substitutions leading to blocks IIa and IIIa

Building block IIIa: $\int \frac{1}{x^2+1} \mathrm{d}x = \arctan x + C$. **Let** $a > 0$.

Example

$$\begin{aligned}
\color{red}{\int \frac{1}{x^2+a} \mathrm{d}x} &= \int \frac{1}{a\left(\frac{1}{a}x^2+1\right)} \mathrm{d}x \\
&= \int \frac{1}{a\left(\left(\frac{x}{\sqrt{a}}\right)^2+1\right)} \sqrt{a}\,\mathrm{d}\left(\frac{x}{\sqrt{a}}\right) \\
&= \frac{1}{\sqrt{a}} \int \frac{1}{u^2+1} \mathrm{d}u \\
&= \frac{1}{\sqrt{a}} \arctan(u) + C \\
&= \color{red}{\frac{1}{\sqrt{a}} \arctan\left(\frac{x}{\sqrt{a}}\right) + C}
\end{aligned}$$

Use $a = \left(\sqrt{a}\right)^2$

Set $u = \frac{x}{\sqrt{a}}$

Linear substitutions leading to blocks IIa and IIIa

Building block IIIa: $\int \frac{1}{x^2+1} \mathrm{d}x = \arctan x + C$. **Let** $a > 0$**.**

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