Calculus II
Integrals of involving radicals of quadratics

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Outline

1. Integrals of form $\int R(x, \sqrt{ax^2+bx+c})\mathrm{d}x$, R - rational function
   - Transforming to the forms $\sqrt{x^2+1}, \sqrt{-x^2+1}, \sqrt{x^2-1}$
   - Table of Euler and trig substitutions
   - The case $\sqrt{x^2+1}$
   - The case $\sqrt{-x^2+1}$
   - The case $\sqrt{x^2-1}$

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Integrals of form $\int R(x, \sqrt{ax^2+bx+c})\mathrm{d}x$, R - rational function

Let $R(x, y)$ be an arbitrary rational expression in two variables (quotient of polynomials in two variables).

Question

Can we integrate $\displaystyle\int R\left(x, \sqrt{ax^2+bx+c}\right)\mathrm{d}x$*?*

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  - Use trigonometric substitution or Euler substitution to transform to trigonometric or rational function integral (no radicals).

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- We motivate why we need such integrals by examples such as computing the area of an ellipse.

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$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2\sin^2\theta} = \sqrt{a^2(1 - \sin^2\theta)} =$$

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- With $u = a^2 - x^2$, the new variable is a function of the old one.
- With $x = a \sin\theta$, the old variable is a function of the new one.

Linear substitutions to simplify radicals $\sqrt{ay^2+by+c}$

- Using linear substitutions, radicals of form $\sqrt{ay^2+by+c}$, $a\neq 0$, $b^2-4ac\neq 0$ can be transformed to (multiple of):
  - $\sqrt{x^2+1}$
  - $\sqrt{-x^2+1}$
  - $\sqrt{x^2-1}$.
- We already studied how to do that using completing the square when dealing with rational functions.

Recall: linear substitution is subst. of the form $u = px + q$.

Example

Use linear substitution to transform $\sqrt{x^2 + x + 1}$ to multiple of $\sqrt{u^2 + 1}$.

$$\sqrt{x^2 + x + 1} \quad =$$

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Example

Use linear substitution to transform $\sqrt{x^2 + x + 1}$ to multiple of $\sqrt{u^2 + 1}$.

$$\sqrt{x^2 + x + 1} \quad = \quad \sqrt{x^2 + 2 \cdot \frac{1}{2} x + \mathbf{?} - \mathbf{?} + 1}$$

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Example

Use linear substitution to transform $\sqrt{x^2+x+1}$ to multiple of $\sqrt{u^2+1}$.

$$\sqrt{x^2+x+1} = \sqrt{x^2+2\cdot\frac{1}{2}x+\frac{1}{4}-\frac{1}{4}+1}$$

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Use linear substitution to transform $\sqrt{x^2+x+1}$ to multiple of $\sqrt{u^2+1}$.

$$\begin{aligned}\sqrt{x^2+x+1} &= \sqrt{x^2+2\cdot\frac{1}{2}x+\frac{1}{4}-\frac{1}{4}+1}\\ &= \sqrt{\left(x+\textbf{?}\right)^2+\textbf{?}}\end{aligned}$$

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&= \sqrt{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}} \\
&= \sqrt{\frac{3}{4}\left(\frac{4}{3}\left(x+\frac{1}{2}\right)^2+1\right)}
\end{aligned}$$

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&= \sqrt{\frac{3}{4}\left(\frac{4}{3}\left(x+\frac{1}{2}\right)^2+1\right)} \\
&= \frac{\sqrt{3}}{2}\sqrt{\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right)^2+1}
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&= \frac{\sqrt{3}}{2}\sqrt{u^2+1},
\end{aligned}$$

where $u = \frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right) = \frac{2\sqrt{3}}{3}x + \frac{\sqrt{3}}{3}$.

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Example

Use linear subst. to transform $\sqrt{-2x^2 + x + 1}$ to multiple of $\sqrt{-u^2 + 1}$.

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Use linear subst. to transform $\sqrt{-2x^2 + x + 1}$ to multiple of $\sqrt{-u^2 + 1}$.

$$\sqrt{-2x^2 + x + 1} \quad = \quad \sqrt{-2\left(x^2 - \frac{1}{2}x - \frac{1}{2}\right)}$$

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&= \sqrt{-2\left(\left(x - \frac{1}{4}\right)^2 - \frac{9}{16}\right)} \\
&= \sqrt{\frac{9}{8}\left(-\frac{16}{9}\left(x - \frac{1}{4}\right)^2 + 1\right)}
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- Each of the above integrals can be transformed to a rational trigonometric integral using 3 pairs of substitutions:
  $x = \tan\theta$, $x = \cot\theta$; $x = \sin\theta$, $x = \cos\theta$; $x = \csc\theta$, $x = \sec\theta$.

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- The Euler substitutions directly transform the integral to a rational function integral.
- We will demonstrate that the Euler substitutions are rational.

Trigonometric substitution and Euler substitution

Expression	Substitution	Variable range	Relevant identity
$\sqrt{x^2+1}$	$x=\tan\theta$	$\theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$	$1+\tan^2\theta=\sec^2\theta$
	$x=\cot\theta$	$\theta\in(0,\pi)$	$1+\cot^2\theta=\csc^2\theta$
$\sqrt{-x^2+1}$	$x=\sin\theta$	$\theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$	$1-\sin^2\theta=\cos^2\theta$
	$x=\cos\theta$	$\theta\in(0,\pi)$	$1-\cos^2\theta=\cos^2\theta$
$\sqrt{x^2-1}$	$x=\csc\theta$	$\theta\in\left[0,\frac{\pi}{2}\right)\cup\left[\pi,\frac{3\pi}{2}\right)$	$\csc^2\theta-1=\cot^2\theta$
	$x=\sec\theta$	$\theta\in\left[0,\frac{\pi}{2}\right)\cup\left[\pi,\frac{3\pi}{2}\right)$	$\sec^2\theta-1=\tan^2\theta$
Euler substitution by applying in addition $\theta=2\arctan t$			
$\sqrt{x^2+1}$	$x=\frac{2t}{1-t^2}$	$-1<t<1$	(?)
	$x=\frac{1}{2}\left(\frac{1}{t}-t\right)$	$0<t$	(?)
$\sqrt{-x^2+1}$	$x=\frac{2t}{1+t^2}$	$-1\leq t\leq 1$	(?)
	$x=\frac{1-t^2}{1+t^2}$	$0<t$	(?)
$\sqrt{x^2-1}$	$x=\frac{1}{2}\left(\frac{1}{t}+t\right)$	$t\in(-\infty,-1)\cup[0,1)$	(?)
	$x=\frac{1+t^2}{1-t^2}$	$t\in(-\infty,-1)\cup[0,1)$	(?)

Trigonometric substitution $x = \cot\theta$ for $\sqrt{x^2+1}$

The trigonometric substitution $x = \cot\theta$, $\theta \in (0, \pi)$ for $\sqrt{x^2+1}$:

$$\sqrt{x^2+1} \quad =$$

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$$\sqrt{x^2+1} = \sqrt{\cot^2\theta + 1}$$

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when $\theta \in (0, \pi)$ we have $\sin\theta \geq 0$ and so $\sqrt{\sin^2\theta} = \sin\theta$

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Example

$$\int \frac{1}{x^2\sqrt{x^2+9}}\mathrm{d}x$$

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&= \int \frac{1}{(3\cot\theta)^2 3\sqrt{\cot^2\theta+1}}\mathrm{d}(3\cot\theta) \\
&= \int \frac{1}{27\cot^2\theta\sqrt{\csc^2\theta}}\left(-3\csc^2\theta\right)\mathrm{d}\theta \\
&= \frac{1}{9}\int \frac{-\csc^2\theta}{\cot^2\theta\csc\theta}\mathrm{d}\theta \\
&= \frac{1}{9}\int \frac{-\sin\theta}{\cos^2\theta}\mathrm{d}\theta = \frac{1}{9}\int\frac{1}{\cos^2\theta}\mathrm{d}(\textbf{?}\quad)
\end{aligned}
$$

Set
$\frac{x}{3} = \cot\theta$
$x = 3\cot\theta$

$\theta \in (0,\pi)$

$\theta \in (0,\pi) \Rightarrow$
$\csc\theta > 0$

Example

$$\begin{aligned}
\int \frac{1}{x^2\sqrt{x^2+9}}\mathrm{d}x &= \int \frac{1}{x^2 3\sqrt{\left(\frac{x}{3}\right)^2+1}}\mathrm{d}x \\
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\end{aligned}$$

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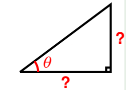

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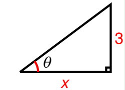

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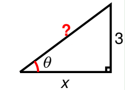

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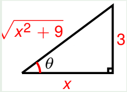

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&= \frac{1}{9}\int \frac{\mathrm{d}u}{u^2} = -\frac{1}{9u}+C = -\frac{\sec\theta}{9}+C \\
&= -\frac{\sqrt{x^2+9}}{9x}+C
\end{aligned}$$

Set
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$\theta \in (0, \pi)$

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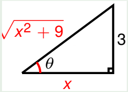

Euler subst. for $\sqrt{x^2+1}$ corresponding to $x = \cot\theta$

- $x = \cot\theta$ transforms $\mathrm{d}x, x, \sqrt{x^2+1}$ to trig form.
- $\theta = 2\arctan t$, transforms $\mathrm{d}\theta, \cos\theta, \sin\theta$ to rational form.

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- $x = \cot\theta$ transforms $\mathrm{d}x, x, \sqrt{x^2+1}$ to trig form.
- $\theta = 2\arctan t,\ \ t > 0$ transforms $\mathrm{d}\theta, \cos\theta, \sin\theta$ to rational form.

What if we compose the above? We get the Euler substitution:

$$x =$$

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What if we compose the above? We get the Euler substitution:

$$
\begin{aligned}
x &= \cot\theta \\
&= \cot(2\arctan t)
\end{aligned}
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What if we compose the above? We get the Euler substitution:

$$\begin{aligned} x &= \cot\theta \\ &= \cot(2\arctan t) && \Big|\text{Recall: } \cot(2z) = \frac{\cos(2z)}{\sin(2z)} \end{aligned}$$

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- $x = \cot\theta$ transforms $\mathrm{d}x, x, \sqrt{x^2+1}$ to trig form.
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What if we compose the above? We get the Euler substitution:

$$\begin{aligned} x &= \cot\theta \\ &= \cot\left(2\arctan t\right) && \middle|\text{Recall: } \cot(2z) = \frac{\cos(2z)}{\sin(2z)} = \frac{1-\tan^2 z}{2\tan z} \end{aligned}$$

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Euler subst. for $\sqrt{x^2+1}$ corresponding to $x = \cot\theta$

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Euler subst. for $\sqrt{x^2+1}$ corresponding to $x = \cot\theta$

- $x = \cot\theta$ transforms $\mathrm{d}x, x, \sqrt{x^2+1}$ to trig form.
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Euler subst. for $\sqrt{x^2+1}$ corresponding to $x = \cot\theta$

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Euler subst. for $\sqrt{x^2+1}$ corresponding to $x = \cot\theta$

- $x = \cot\theta$ transforms dx, x, $\sqrt{x^2+1}$ to trig form.
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We can furthermore compute

$$\sqrt{x^2+1} =$$

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- $x = \cot\theta$ transforms $\mathrm{d}x, x, \sqrt{x^2+1}$ to trig form.
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We can furthermore compute

$$\sqrt{x^2+1} = \sqrt{\frac{1}{4}\left(\frac{1}{t} - t\right)^2 + 1}$$

Euler subst. for $\sqrt{x^2+1}$ corresponding to $x = \cot\theta$

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We can furthermore compute

$$\begin{aligned}\sqrt{x^2+1} &= \sqrt{\frac{1}{4}\left(\frac{1}{t}-t\right)^2 + 1}\\ &= \frac{1}{2}\sqrt{\left(\frac{1}{t}-t\right)^2 + 4}\end{aligned}$$

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- $x = \cot\theta$ transforms $\mathrm{d}x, x, \sqrt{x^2+1}$ to trig form.
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We can furthermore compute

$$\begin{aligned} \sqrt{x^2+1} &= \sqrt{\frac{1}{4}\left(\frac{1}{t}-t\right)^2+1} \\ &= \frac{1}{2}\sqrt{\left(\frac{1}{t}-t\right)^2+4} \qquad \left| \; \left(\tfrac{1}{t}-t\right)^2+4 = \left(\tfrac{1}{t}+t\right)^2 \right. \end{aligned}$$

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- $\theta = 2\arctan t, \quad t > 0$ transforms $\mathrm{d}\theta, \cos\theta, \sin\theta$ to rational form.

What if we compose the above? We get the Euler substitution:

$$x = \frac{1}{2}\left(\frac{1}{t} - t\right) .$$

We can furthermore compute

$$\begin{aligned} \sqrt{x^2+1} &= \sqrt{\frac{1}{4}\left(\frac{1}{t}-t\right)^2 + 1} \\ &= \frac{1}{2}\sqrt{\left(\frac{1}{t}-t\right)^2 + 4} \qquad \left| \; \left(\frac{1}{t}-t\right)^2 + 4 = \left(\frac{1}{t}+t\right)^2 \right. \end{aligned}$$

Euler subst. for $\sqrt{x^2+1}$ corresponding to $x = \cot\theta$

- $x = \cot\theta$ transforms $\mathrm{d}x, x, \sqrt{x^2+1}$ to trig form.
- $\theta = 2\arctan t,\ \ t > 0$ transforms $\mathrm{d}\theta, \cos\theta, \sin\theta$ to rational form.

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Euler subst. for $\sqrt{x^2+1}$ corresponding to $x=\cot\theta$

- $x=\cot\theta$ transforms $\mathrm{d}x, x, \sqrt{x^2+1}$ to trig form.
- $\theta = 2\arctan t,\ \ t>0$ transforms $\mathrm{d}\theta, \cos\theta, \sin\theta$ to rational form.

What if we compose the above? We get the Euler substitution:

$$x = \frac{1}{2}\left(\frac{1}{t}-t\right) \quad .$$

We can furthermore compute

$$\begin{aligned}\sqrt{x^2+1} &= \sqrt{\frac{1}{4}\left(\frac{1}{t}-t\right)^2+1}\\ &= \frac{1}{2}\sqrt{\left(\frac{1}{t}+t\right)^2} \qquad \left|\ \begin{array}{l}\sqrt{\left(\frac{1}{t}+t\right)^2}=\frac{1}{t}+t\\ \text{because } t>0\end{array}\right.\end{aligned}$$

Euler subst. for $\sqrt{x^2+1}$ corresponding to $x = \cot\theta$

- $x = \cot\theta$ transforms $\mathrm{d}x, x, \sqrt{x^2+1}$ to trig form.
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&= \frac{1}{2}\sqrt{\left(\frac{1}{t}+t\right)^2} \\
&= \frac{1}{2}\left(\frac{1}{t}+t\right) .
\end{aligned}$$

$\sqrt{\left(\frac{1}{t}+t\right)^2} = \frac{1}{t}+t$
because $t > 0$

Euler subst. for $\sqrt{x^2+1}$ corresponding to $x = \cot\theta$

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- $x = \cot\theta$ transforms $\mathrm{d}x, x, \sqrt{x^2+1}$ to trig form.
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What if we compose the above? We get the Euler substitution:

$$x = \frac{1}{2}\left(\frac{1}{t} - t\right) \quad .$$

We can furthermore compute

$$\sqrt{x^2+1} = \frac{1}{2}\left(\frac{1}{t} + t\right) \quad .$$

Finally compute

$$\mathrm{d}x =$$

Euler subst. for $\sqrt{x^2+1}$ corresponding to $x = \cot\theta$

- $x = \cot\theta$ transforms $\mathrm{d}x, x, \sqrt{x^2+1}$ to trig form.
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What if we compose the above? We get the Euler substitution:

$$x = \frac{1}{2}\left(\frac{1}{t} - t\right) .$$

We can furthermore compute

$$\sqrt{x^2+1} = \frac{1}{2}\left(\frac{1}{t} + t\right) .$$

Finally compute

$$\mathrm{d}x = \mathrm{d}\left(\frac{1}{2}\left(\frac{1}{t} - t\right)\right)$$

Euler subst. for $\sqrt{x^2+1}$ corresponding to $x = \cot\theta$

- $x = \cot\theta$ transforms $\mathrm{d}x, x, \sqrt{x^2+1}$ to trig form.
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We can furthermore compute

$$\sqrt{x^2+1} = \frac{1}{2}\left(\frac{1}{t} + t\right) \quad .$$

Finally compute

$$\mathrm{d}x = \mathrm{d}\left(\frac{1}{2}\left(\frac{1}{t} - t\right)\right) =$$

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Finally compute

$$\mathrm{d}x = \mathrm{d}\left(\frac{1}{2}\left(\frac{1}{t} - t\right)\right) = -\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t$$

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Finally compute

$$\begin{aligned} \mathrm{d}x &= \mathrm{d}\left(\frac{1}{2}\left(\frac{1}{t} - t\right)\right) = -\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t \\ t &= \frac{1}{2}\left(\frac{1}{t} + t\right) - \frac{1}{2}\left(\frac{1}{t} - t\right) . \end{aligned}$$

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Finally compute

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Euler subst. for $\sqrt{x^2+1}$ corresponding to $x = \cot\theta$

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What if we compose the above? We get the Euler substitution:

$$x = \frac{1}{2}\left(\frac{1}{t} - t\right) \quad .$$

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Finally compute

$$\begin{aligned} \mathrm{d}x &= \mathrm{d}\left(\frac{1}{2}\left(\frac{1}{t} - t\right)\right) = -\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t \\ t &= \frac{1}{2}\left(\frac{1}{t} + t\right) - \frac{1}{2}\left(\frac{1}{t} - t\right) = \sqrt{x^2+1} - x \quad . \end{aligned}$$

Euler subst. for $\sqrt{x^2+1}$ corresponding to $x = \cot\theta$

- $x = \cot\theta$ transforms $\mathrm{d}x, x, \sqrt{x^2+1}$ to trig form.
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What if we compose the above? We get the Euler substitution:

$$x = \frac{1}{2}\left(\frac{1}{t} - t\right) \quad .$$

We can furthermore compute

$$\sqrt{x^2+1} = \frac{1}{2}\left(\frac{1}{t} + t\right) \quad .$$

Finally compute

$$\begin{aligned} \mathrm{d}x &= \mathrm{d}\left(\frac{1}{2}\left(\frac{1}{t} - t\right)\right) = -\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t \\ t &= \frac{1}{2}\left(\frac{1}{t} + t\right) - \frac{1}{2}\left(\frac{1}{t} - t\right) = \sqrt{x^2+1} - x \quad . \end{aligned}$$

Euler subst. for $\sqrt{x^2+1}$ corresponding to $x=\cot\theta$

- $x=\cot\theta$ transforms $\mathrm{d}x, x, \sqrt{x^2+1}$ to trig form.
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What if we compose the above? We get the Euler substitution:

$$x = \frac{1}{2}\left(\frac{1}{t}-t\right) \quad .$$

We can furthermore compute

$$\sqrt{x^2+1} = \frac{1}{2}\left(\frac{1}{t}+t\right) \quad .$$

Finally compute

$$\begin{aligned} \mathrm{d}x &= \mathrm{d}\left(\frac{1}{2}\left(\frac{1}{t}-t\right)\right) = -\frac{1}{2}\left(\frac{1}{t^2}+1\right)\mathrm{d}t \\ t &= \frac{1}{2}\left(\frac{1}{t}+t\right)-\frac{1}{2}\left(\frac{1}{t}-t\right) = \sqrt{x^2+1}-x \quad . \end{aligned}$$

Euler subst. for $\sqrt{x^2+1}$ corresponding to $x = \cot\theta$

- $x = \cot\theta$ transforms $\mathrm{d}x, x, \sqrt{x^2+1}$ to trig form.
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What if we compose the above? We get the Euler substitution:

Definition

The Euler substitution for $\sqrt{x^2+1}$ corresponding to $x = \cot\theta$ is given by:

$$\begin{aligned} x &= \frac{1}{2}\left(\frac{1}{t} - t\right), \qquad t > 0 \\ \sqrt{x^2+1} &= \frac{1}{2}\left(\frac{1}{t} + t\right) \\ \mathrm{d}x &= -\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t \\ t &= \sqrt{x^2+1} - x \quad . \end{aligned}$$

Euler substitution: $x = \frac{1}{2}\left(\frac{1}{t} - t\right)$, $\sqrt{x^2+1} = \frac{1}{2}\left(\frac{1}{t} + t\right)$,
$t = \sqrt{x^2+1} - x$, $\mathrm{d}x = -\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t$.

Example

$$\int \sqrt{x^2+1}\,\mathrm{d}x =$$

Euler substitution: $x = \frac{1}{2}\left(\frac{1}{t} - t\right)$, $\sqrt{x^2+1} = \frac{1}{2}\left(\frac{1}{t} + t\right)$,
$t = \sqrt{x^2+1} - x$, $\mathrm{d}x = -\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t$.

Example

$$\int \sqrt{x^2+1}\,\mathrm{d}x = -\int \frac{1}{2}\left(\frac{1}{t} + t\right)\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t$$

Euler substitution: $x = \frac{1}{2}\left(\frac{1}{t} - t\right)$, $\sqrt{x^2+1} = \frac{1}{2}\left(\frac{1}{t} + t\right)$,
$t = \sqrt{x^2+1} - x$, $\mathrm{d}x = -\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t$.

Example

$$\int \sqrt{x^2+1}\,\mathrm{d}x = -\int \frac{1}{2}\left(\frac{1}{t} + t\right)\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t$$

Euler substitution: $x = \frac{1}{2}\left(\frac{1}{t} - t\right)$, $\sqrt{x^2+1} = \frac{1}{2}\left(\frac{1}{t} + t\right)$,
$t = \sqrt{x^2+1} - x$, $\mathrm{d}x = -\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t$.

Example

$$\begin{aligned}\int \sqrt{x^2+1}\,\mathrm{d}x &= -\int \frac{1}{2}\left(\frac{1}{t} + t\right)\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t\\ &= -\frac{1}{4}\int\left(\frac{1}{t^3} + 2\frac{1}{t} + t\right)\mathrm{d}t\end{aligned}$$

Euler substitution: $x = \frac{1}{2}\left(\frac{1}{t} - t\right)$, $\sqrt{x^2+1} = \frac{1}{2}\left(\frac{1}{t} + t\right)$,
$t = \sqrt{x^2+1} - x$, $\mathrm{d}x = -\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t$.

Example

$$\begin{aligned}\int \sqrt{x^2+1}\,\mathrm{d}x &= -\int \frac{1}{2}\left(\frac{1}{t} + t\right)\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t \\ &= -\frac{1}{4}\int\left(\frac{1}{t^3} + 2\frac{1}{t} + t\right)\mathrm{d}t\end{aligned}$$

Euler substitution: $x = \frac{1}{2}\left(\frac{1}{t} - t\right)$, $\sqrt{x^2+1} = \frac{1}{2}\left(\frac{1}{t} + t\right)$,
$t = \sqrt{x^2+1} - x$, $\mathrm{d}x = -\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t$.

Example

$$\begin{aligned}\int \sqrt{x^2+1}\,\mathrm{d}x &= -\int \frac{1}{2}\left(\frac{1}{t} + t\right)\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t \\ &= -\frac{1}{4}\int\left(\frac{1}{t^3} + 2\frac{1}{t} + t\right)\mathrm{d}t\end{aligned}$$

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Example

$$\begin{aligned}\int \sqrt{x^2+1}\,\mathrm{d}x &= -\int \frac{1}{2}\left(\frac{1}{t} + t\right)\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t\\ &= -\frac{1}{4}\int\left(\frac{1}{t^3} + 2\frac{1}{t} + t\right)\mathrm{d}t\end{aligned}$$

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Example

$$\begin{aligned}\int \sqrt{x^2+1}\,\mathrm{d}x &= -\int \frac{1}{2}\left(\frac{1}{t} + t\right)\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t \\ &= -\frac{1}{4}\int\left(\frac{1}{t^3} + 2\frac{1}{t} + t\right)\mathrm{d}t\end{aligned}$$

Euler substitution: $x = \frac{1}{2}\left(\frac{1}{t} - t\right)$, $\sqrt{x^2+1} = \frac{1}{2}\left(\frac{1}{t} + t\right)$,
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Example

$$\begin{aligned}
\int \sqrt{x^2+1}\,\mathrm{d}x &= -\int \frac{1}{2}\left(\frac{1}{t} + t\right)\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t \\
&= -\frac{1}{4}\int\left(\frac{1}{t^3} + 2\frac{1}{t} + t\right)\mathrm{d}t \\
&= -\frac{1}{4}\left(-\frac{t^{-2}}{2} + 2\ln|t| + \frac{t^2}{2}\right) + C
\end{aligned}$$

Euler substitution: $x = \frac{1}{2}\left(\frac{1}{t} - t\right)$, $\sqrt{x^2+1} = \frac{1}{2}\left(\frac{1}{t} + t\right)$,
$t = \sqrt{x^2+1} - x$, $\mathrm{d}x = -\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t$.

Example

$$\begin{aligned}\int \sqrt{x^2+1}\,\mathrm{d}x &= -\int \frac{1}{2}\left(\frac{1}{t} + t\right)\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t\\ &= -\frac{1}{4}\int\left(\frac{1}{t^3} + 2\frac{1}{t} + t\right)\mathrm{d}t\\ &= -\frac{1}{4}\left(-\frac{t^{-2}}{2} + 2\ln|t| + \frac{t^2}{2}\right) + C\end{aligned}$$

Euler substitution: $x = \frac{1}{2}\left(\frac{1}{t} - t\right)$, $\sqrt{x^2+1} = \frac{1}{2}\left(\frac{1}{t} + t\right)$,
$t = \sqrt{x^2+1} - x$, $\mathrm{d}x = -\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t$.

Example

$$\begin{aligned}\int \sqrt{x^2+1}\,\mathrm{d}x &= -\int \frac{1}{2}\left(\frac{1}{t} + t\right)\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t\\ &= -\frac{1}{4}\int \left(\frac{1}{t^3} + 2\frac{1}{t} + t\right)\mathrm{d}t\\ &= -\frac{1}{4}\left(-\frac{t^{-2}}{2} + 2\ln|t| + \frac{t^2}{2}\right) + C\end{aligned}$$

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Example

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\int \sqrt{x^2+1}\,\mathrm{d}x &= -\int \frac{1}{2}\left(\frac{1}{t} + t\right)\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t \\
&= -\frac{1}{4}\int\left(\frac{1}{t^3} + 2\frac{1}{t} + t\right)\mathrm{d}t \\
&= -\frac{1}{4}\left(-\frac{t^{-2}}{2} + 2\ln|t| + \frac{t^2}{2}\right) + C \\
&= \frac{1}{2}\left(\frac{1}{2}\left(t^{-1} - t\right)\frac{1}{2}\left(t^{-1} + t\right)\right) - \frac{1}{2}\ln t + C
\end{aligned}$$

Euler substitution: $x = \frac{1}{2}\left(\frac{1}{t} - t\right)$, $\sqrt{x^2+1} = \frac{1}{2}\left(\frac{1}{t} + t\right)$,
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Example

$$
\begin{aligned}
\int \sqrt{x^2+1}\,\mathrm{d}x &= -\int \frac{1}{2}\left(\frac{1}{t} + t\right)\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t \\
&= -\frac{1}{4}\int \left(\frac{1}{t^3} + 2\frac{1}{t} + t\right)\mathrm{d}t \\
&= -\frac{1}{4}\left(-\frac{t^{-2}}{2} + 2\ln|t| + \frac{t^2}{2}\right) + C \\
&= \frac{1}{2}\left(\frac{1}{2}\left(t^{-1} - t\right)\frac{1}{2}\left(t^{-1} + t\right)\right) - \frac{1}{2}\ln t + C
\end{aligned}
$$

Euler substitution: $x = \frac{1}{2}\left(\frac{1}{t} - t\right)$, $\sqrt{x^2+1} = \frac{1}{2}\left(\frac{1}{t} + t\right)$,
$t = \sqrt{x^2+1} - x$, $\mathrm{d}x = -\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t$.

Example

$$
\begin{aligned}
\int \sqrt{x^2+1}\,\mathrm{d}x &= -\int \frac{1}{2}\left(\frac{1}{t} + t\right)\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t \\
&= -\frac{1}{4}\int \left(\frac{1}{t^3} + 2\frac{1}{t} + t\right)\mathrm{d}t \\
&= -\frac{1}{4}\left(-\frac{t^{-2}}{2} + 2\ln|t| + \frac{t^2}{2}\right) + C \\
&= \frac{1}{2}\left(\frac{1}{2}\left(t^{-1} - t\right)\frac{1}{2}\left(t^{-1} + t\right)\right) - \frac{1}{2}\ln t + C
\end{aligned}
$$

Euler substitution: $x = \frac{1}{2}\left(\frac{1}{t} - t\right)$, $\sqrt{x^2+1} = \frac{1}{2}\left(\frac{1}{t} + t\right)$,
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Example

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\int \sqrt{x^2+1}\,\mathrm{d}x &= -\int \frac{1}{2}\left(\frac{1}{t} + t\right)\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t \\
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&= -\frac{1}{4}\left(-\frac{t^{-2}}{2} + 2\ln|t| + \frac{t^2}{2}\right) + C \\
&= \frac{1}{2}\left(\frac{1}{2}\left(t^{-1} - t\right)\frac{1}{2}\left(t^{-1} + t\right)\right) - \frac{1}{2}\ln t + C
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Euler substitution: $x = \frac{1}{2}\left(\frac{1}{t} - t\right)$, $\sqrt{x^2+1} = \frac{1}{2}\left(\frac{1}{t} + t\right)$,
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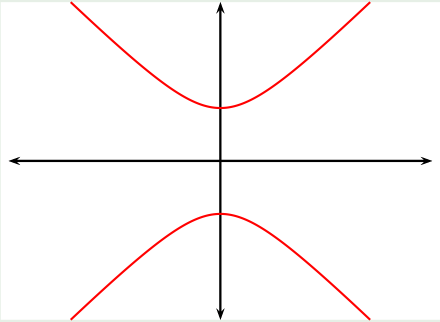
Example

Find the area locked b-n the hyperbolas $y = \pm\sqrt{x^2+1}$ and $x = \pm 2\sqrt{2}$.



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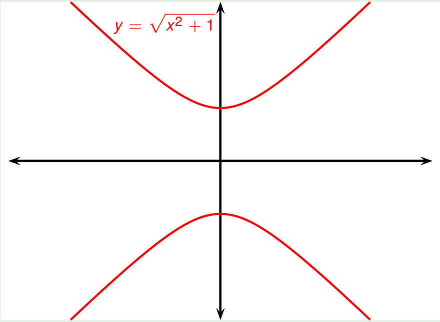

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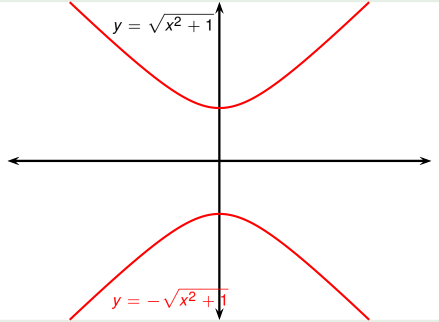

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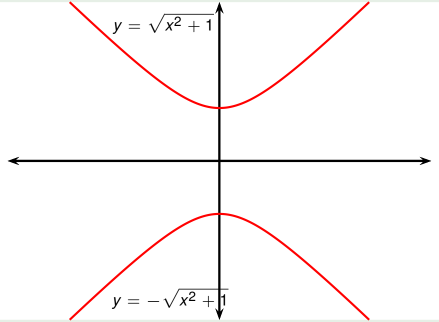

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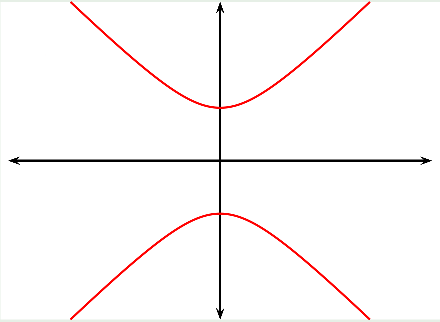

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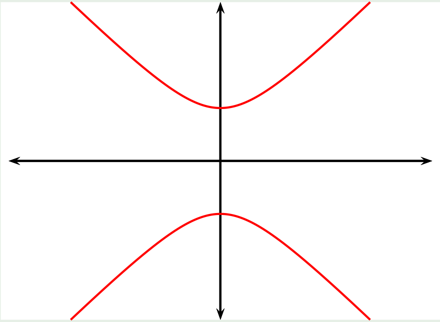

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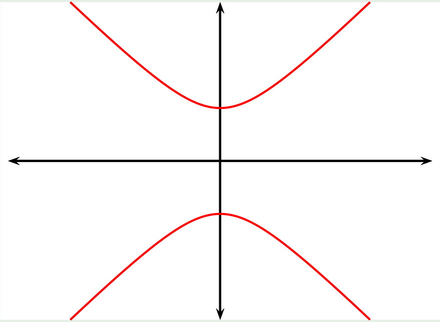

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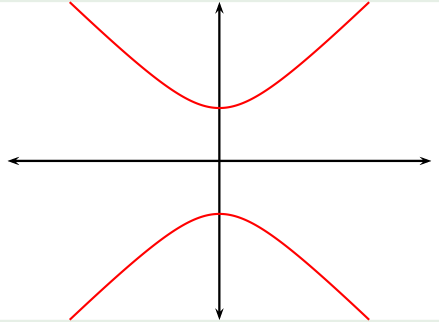

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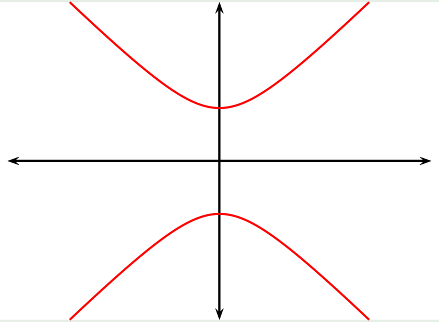

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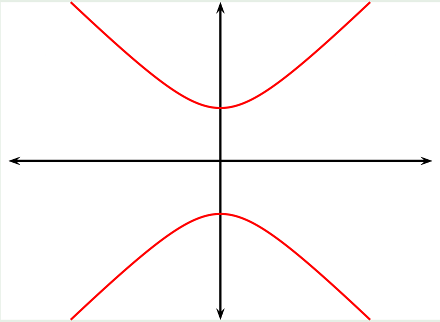

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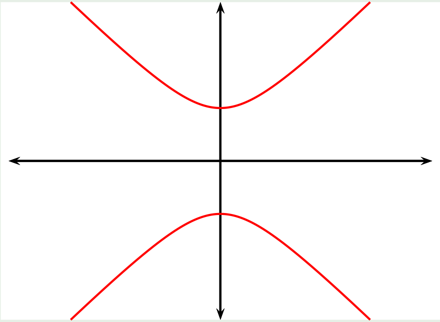

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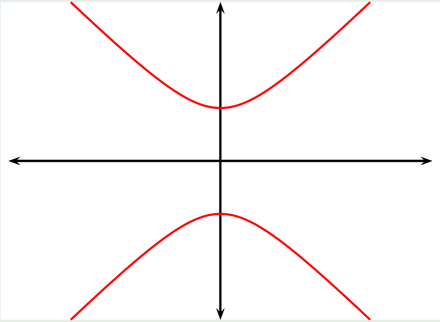

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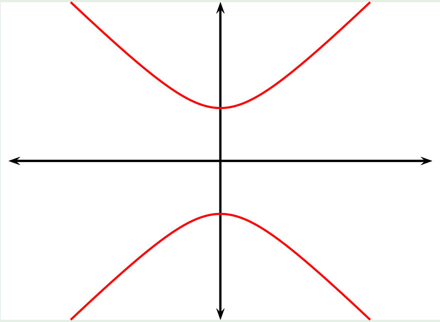

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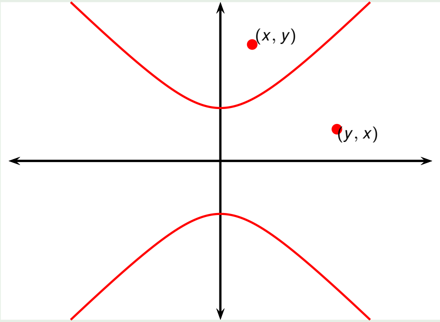

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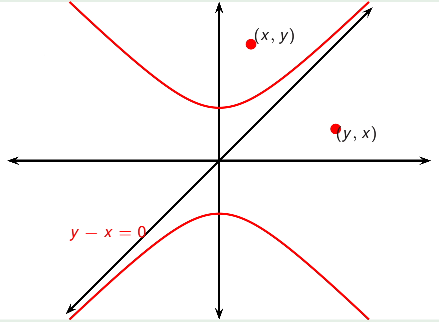

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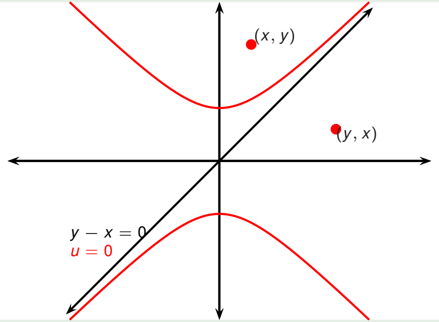

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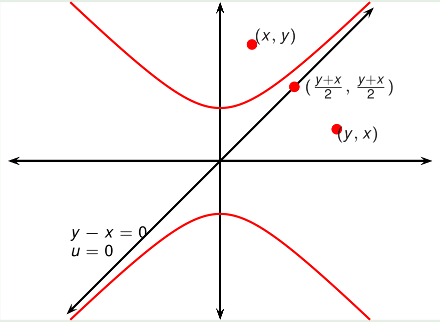

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x^2+1 &= y^2\\
y^2 - x^2 &= 1\\
\frac{\sqrt{2}}{2}(y-x)\frac{\sqrt{2}}{2}(y+x) &= \frac{1}{2}\\
uv &= \frac{1}{2}\\
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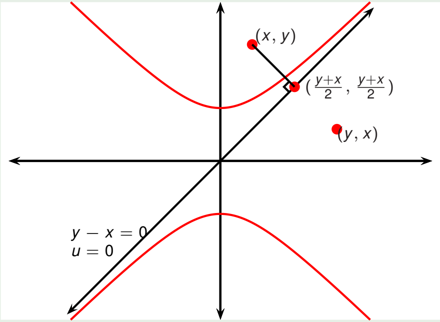
\end{aligned}$$

where $\left| \begin{array}{l} u = \frac{\sqrt{2}}{2}(y-x) \\ v = \frac{\sqrt{2}}{2}(y+x) \end{array} \right.$. Consider an arbitrary point (x, y).

Example

Find the area locked b-n the hyperbolas $y = \pm\sqrt{x^2+1}$ and $x = \pm 2\sqrt{2}$.



distance b-n (x, y) and line $u = 0$ equals

We studied $v = \frac{\frac{1}{2}}{u}$ is called a hyperbola: why do we call $y = \sqrt{x^2+1}$ hyperbola? Compute:

$$\begin{array}{rcl}
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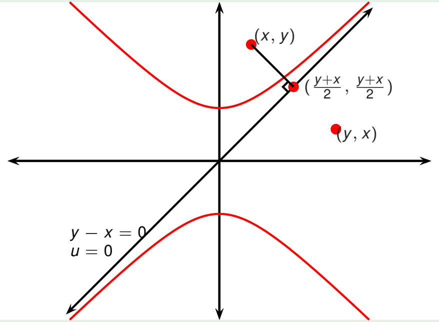

distance b-n (x, y) and line $u = 0$ equals

$$\sqrt{\left(x - \frac{(x+y)}{2}\right)^2 + \left(y - \frac{(x+y)}{2}\right)^2}$$

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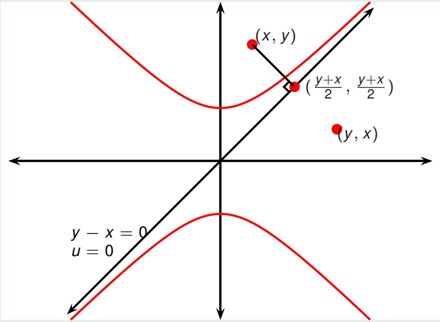

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$$\sqrt{\left(x - \frac{(x+y)}{2}\right)^2 + \left(y - \frac{(x+y)}{2}\right)^2}$$
$$= \sqrt{\frac{1}{2}(y-x)^2}$$

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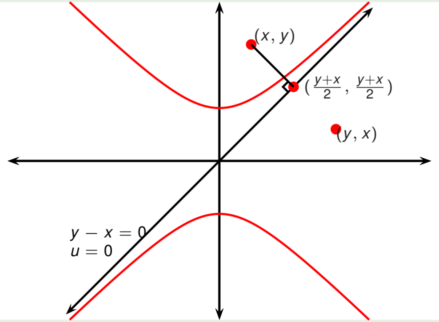

distance b-n (x, y) and line $u = 0$ equals

$$\sqrt{\left(x - \frac{(x+y)}{2}\right)^2 + \left(y - \frac{(x+y)}{2}\right)^2}$$

$$= \sqrt{\frac{1}{2}(y-x)^2} = \pm\frac{\sqrt{2}}{2}(y-x)$$

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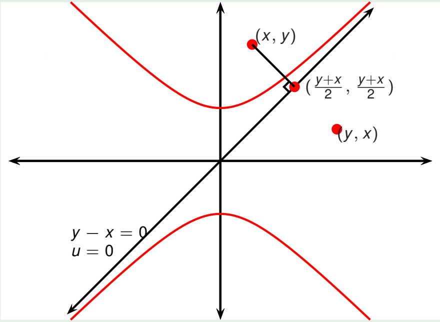

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Signed distance b-n (x, y) and line $u = 0$ equals

$$\pm\sqrt{\left(x - \frac{(x+y)}{2}\right)^2 + \left(y - \frac{(x+y)}{2}\right)^2}$$
$$= \pm\sqrt{\frac{1}{2}(y-x)^2} = \frac{\sqrt{2}}{2}(y-x)$$

Example

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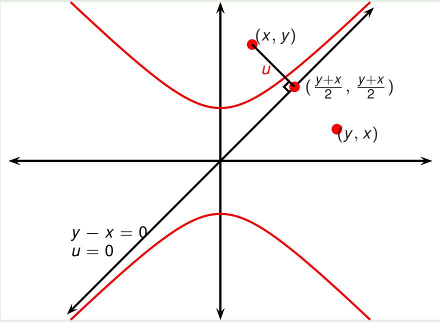

Signed distance b-n (x, y) and line $u = 0$ equals

$$\pm\sqrt{\left(x - \frac{(x+y)}{2}\right)^2 + \left(y - \frac{(x+y)}{2}\right)^2} = \pm\sqrt{\frac{1}{2}(y-x)^2} = \frac{\sqrt{2}}{2}(y-x) = u.$$

We studied $v = \frac{\frac{1}{2}}{u}$ is called a hyperbola: why do we call $y = \sqrt{x^2+1}$ hyperbola? Compute:

$$\begin{aligned}
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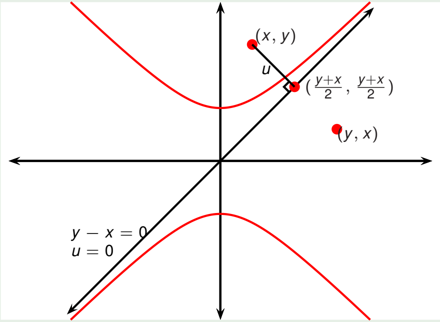

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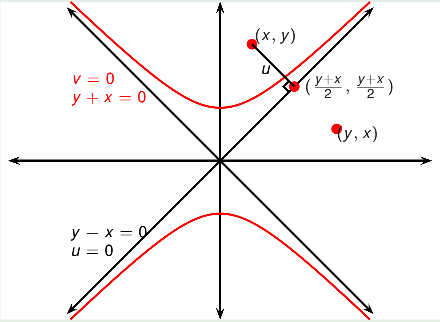
Example

Find the area locked b-n the hyperbolas $y=\pm\sqrt{x^2+1}$ and $x=\pm 2\sqrt{2}$.



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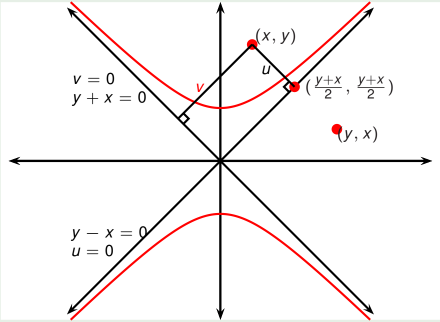

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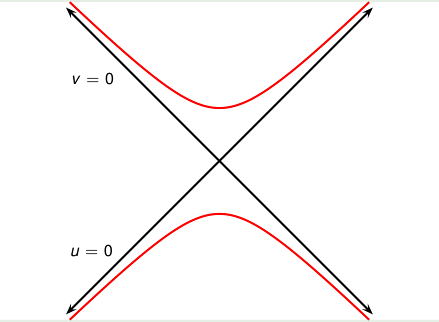

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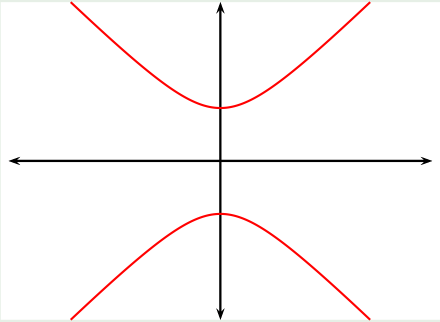
Signed distance b-n (x, y) and line $u = 0$ equals u. Similarly compute that signed distance b-n (x, y) and the line $v = 0$ equals v. $\Rightarrow y^2 - x^2 = 1$ is the hyperbola $v = \frac{1/2}{u}$ in the (u, v)-plane.

Example

Find the area locked b-n the hyperbolas $y = \pm\sqrt{x^2+1}$ and $x = \pm 2\sqrt{2}$.

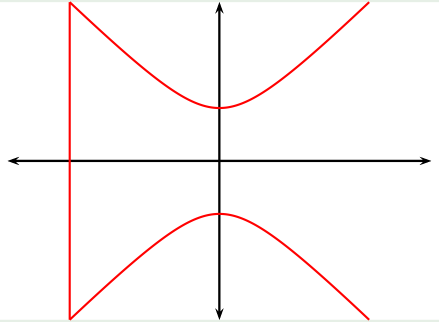


The area in question is:

$$\int_{?}^{?} 2\sqrt{x^2+1}\mathrm{d}x$$

Example

Find the area locked b-n the hyperbolas $y = \pm\sqrt{x^2+1}$ and $x = \pm 2\sqrt{2}$.

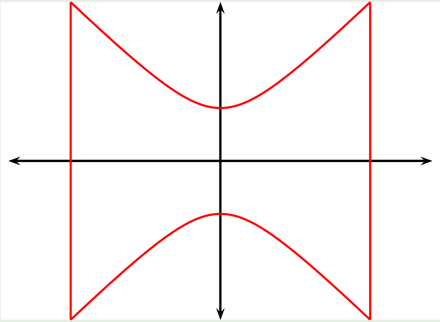


The area in question is:

$$\int_{-2\sqrt{2}}^{?} 2\sqrt{x^2+1}\,\mathrm{d}x$$

Example

Find the area locked b-n the hyperbolas $y = \pm\sqrt{x^2+1}$ and $x = \pm 2\sqrt{2}$.

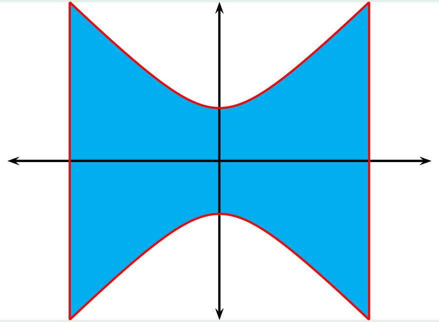


The area in question is:

$$\int_{-2\sqrt{2}}^{2\sqrt{2}} 2\sqrt{x^2+1}\,\mathrm{d}x$$

Example

Find the area locked b-n the hyperbolas $y = \pm\sqrt{x^2+1}$ and $x = \pm 2\sqrt{2}$.



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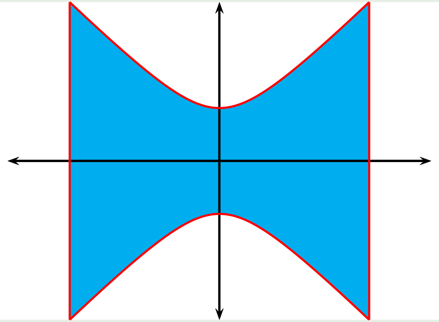


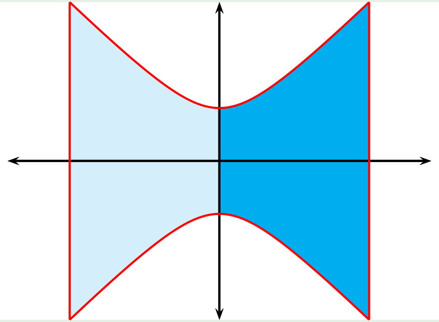
The area in question is:

$$\int_{-2\sqrt{2}}^{2\sqrt{2}} 2\sqrt{x^2+1}\,\mathrm{d}x$$

$$= \left[x\sqrt{x^2+1} + \ln\left(\sqrt{x^2+1}+x\right)\right]_{-2\sqrt{2}}^{2\sqrt{2}}$$

Example

Find the area locked b-n the hyperbolas $y = \pm\sqrt{x^2+1}$ and $x = \pm 2\sqrt{2}$.



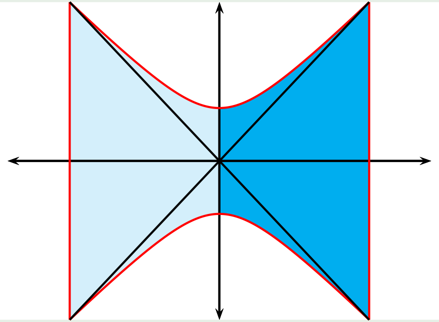
The area in question is:

$$\int\limits_{-2\sqrt{2}}^{2\sqrt{2}} 2\sqrt{x^2+1}\mathrm{d}x$$

$$= 2\left[x\sqrt{x^2+1} + \ln\left(\sqrt{x^2+1}+x\right)\right]_0^{2\sqrt{2}}$$

Example

Find the area locked b-n the hyperbolas $y = \pm\sqrt{x^2+1}$ and $x = \pm 2\sqrt{2}$.



The area in question is:

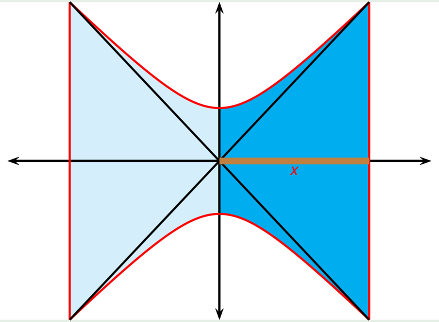
$$\int_{-2\sqrt{2}}^{2\sqrt{2}} 2\sqrt{x^2+1}\mathrm{d}x$$

$$= 2\left[x\sqrt{x^2+1}\right.$$

$$\left. + \ln\left(\sqrt{x^2+1}+x\right)\right]_0^{2\sqrt{2}}$$

Example

Find the area locked b-n the hyperbolas $y = \pm\sqrt{x^2+1}$ and $x = \pm 2\sqrt{2}$.



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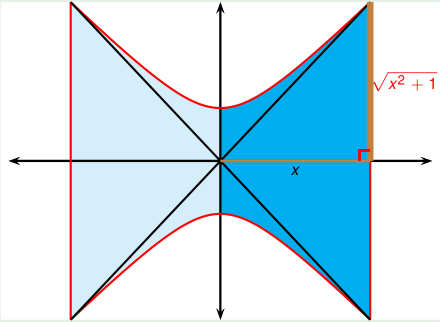

The area in question is:

$$\int\limits_{-2\sqrt{2}}^{2\sqrt{2}} 2\sqrt{x^2+1}\mathrm{d}x$$

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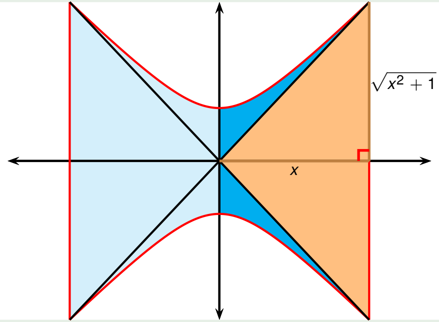

The area in question is:

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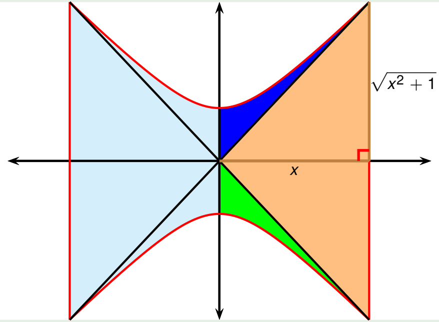
$$= 2\left[x\sqrt{x^2+1} + \ln\left(\sqrt{x^2+1} + x\right)\right]_0^{2\sqrt{2}}$$

Example

Find the area locked b-n the hyperbolas $y = \pm\sqrt{x^2+1}$ and $x = \pm 2\sqrt{2}$.



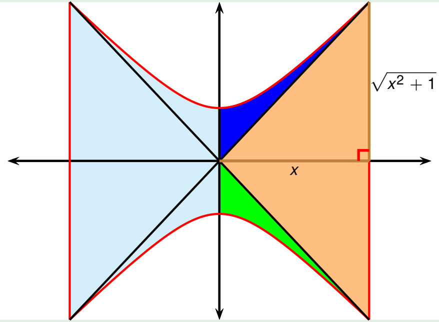
The area in question is:

$$\int_{-2\sqrt{2}}^{2\sqrt{2}} 2\sqrt{x^2+1}\mathrm{d}x$$

$$= 2\left[x\sqrt{x^2+1} + \ln\left(\sqrt{x^2+1}+x\right)\right]_0^{2\sqrt{2}}$$

Example

Find the area locked b-n the hyperbolas $y = \pm\sqrt{x^2+1}$ and $x = \pm 2\sqrt{2}$.



The area in question is:

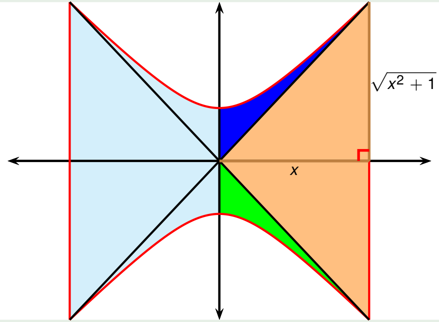
$$\int_{-2\sqrt{2}}^{2\sqrt{2}} 2\sqrt{x^2+1}\mathrm{d}x$$

$$= 2\left[x\sqrt{x^2+1} + \ln\left(\sqrt{x^2+1}+x\right)\right]_0^{2\sqrt{2}}$$

$$= 2\left(2\sqrt{2}\sqrt{(2\sqrt{2})^2+1}\right.$$

Example

Find the area locked b-n the hyperbolas $y = \pm\sqrt{x^2+1}$ and $x = \pm 2\sqrt{2}$.



The area in question is:

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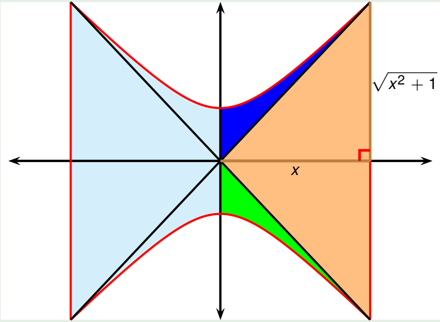
$$= 2\left[x\sqrt{x^2+1} + \ln\left(\sqrt{x^2+1}+x\right)\right]_0^{2\sqrt{2}}$$

$$= 2\left(2\sqrt{2}\sqrt{(2\sqrt{2})^2+1} + \ln\left(\sqrt{(2\sqrt{2})^2+1}+2\sqrt{2}\right)\right)$$

Example

Find the area locked b-n the hyperbolas $y = \pm\sqrt{x^2+1}$ and $x = \pm 2\sqrt{2}$.



The area in question is:

$$\int_{-2\sqrt{2}}^{2\sqrt{2}} 2\sqrt{x^2+1}\mathrm{d}x$$

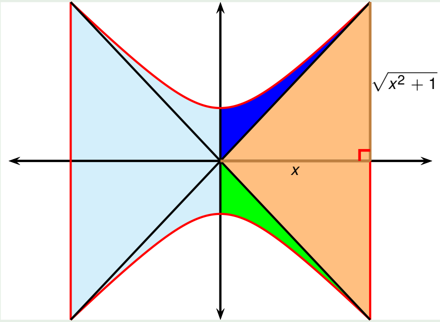
$$= 2\left[x\sqrt{x^2+1} + \ln\left(\sqrt{x^2+1}+x\right)\right]_0^{2\sqrt{2}}$$

$$= 2\left(2\sqrt{2}\sqrt{(2\sqrt{2})^2+1} + \ln\left(\sqrt{(2\sqrt{2})^2+1}+2\sqrt{2}\right)\right)$$

$$= 12\sqrt{2} + 2\ln\left(3+2\sqrt{2}\right)$$

Example

Find the area locked b-n the hyperbolas $y = \pm\sqrt{x^2+1}$ and $x = \pm 2\sqrt{2}$.



The area in question is:

$$\int_{-2\sqrt{2}}^{2\sqrt{2}} 2\sqrt{x^2+1}\mathrm{d}x$$

$$= 2\left[x\sqrt{x^2+1} + \ln\left(\sqrt{x^2+1}+x\right)\right]_0^{2\sqrt{2}}$$

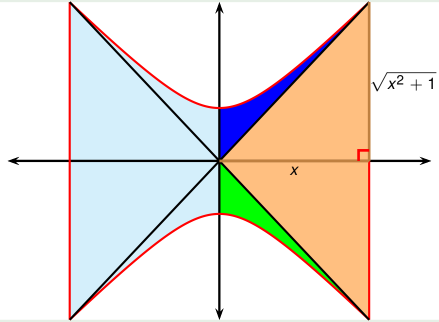
$$= 2\left(2\sqrt{2}\sqrt{(2\sqrt{2})^2+1} + \ln\left(\sqrt{(2\sqrt{2})^2+1}+2\sqrt{2}\right)\right)$$

$$= 12\sqrt{2} + 2\ln\left(3+2\sqrt{2}\right)$$

$$\approx 20.496$$

Example

Find the area locked b-n the hyperbolas $y=\pm\sqrt{x^2+1}$ and $x=\pm 2\sqrt{2}$.



- Recall: integral can be solved via $x=\tan\theta$.

The area in question is:

$$\int_{-2\sqrt{2}}^{2\sqrt{2}} 2\sqrt{x^2+1}\mathrm{d}x$$
$$= 2\left[x\sqrt{x^2+1}\right.$$
$$\left.+\ln\left(\sqrt{x^2+1}+x\right)\right]_0^{2\sqrt{2}}$$
$$= 2\left(2\sqrt{2}\sqrt{(2\sqrt{2})^2+1}\right.$$
$$\left.+\ln\left(\sqrt{(2\sqrt{2})^2+1}+2\sqrt{2}\right)\right)$$
$$= 12\sqrt{2}+2\ln\left(3+2\sqrt{2}\right)$$
$$\approx 20.496$$

Example

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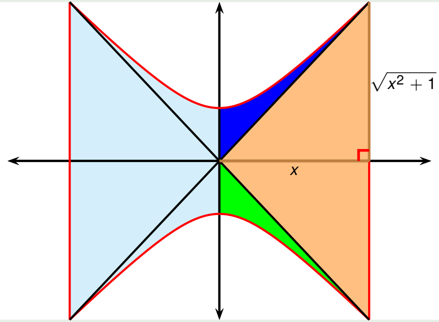

- Recall: integral can be solved via $x = \tan\theta$.
- Geometric interpretation of θ?

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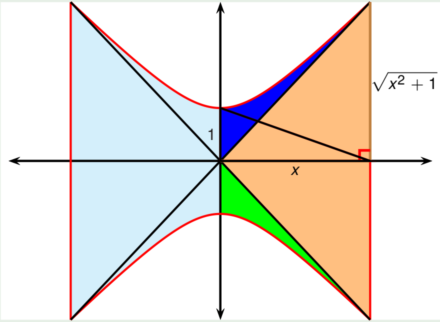

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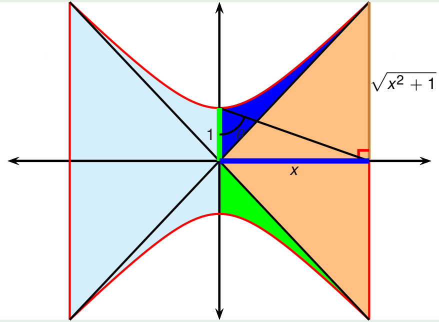

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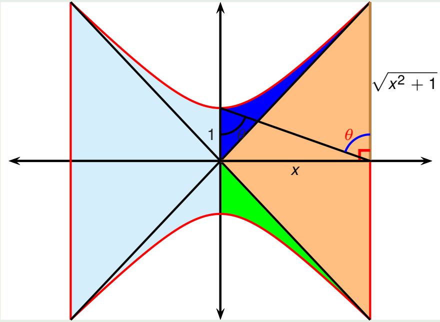

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Trigonometric substitution $x = \cos\theta$ for $\sqrt{-x^2+1}$

The trigonometric substitution $x = \cos\theta$, $\theta \in [0, \pi]$ for $\sqrt{-x^2+1}$:

$$\sqrt{-x^2+1} \quad =$$

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The trigonometric substitution $x = \cos\theta$, $\theta \in [0, \pi]$ for $\sqrt{-x^2+1}$:

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\sqrt{-x^2+1} &= \sqrt{1-\cos^2\theta} \\
&= \sqrt{\sin^2\theta} \\
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\end{aligned}
$$

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The trigonometric substitution $x = \cos\theta$, $\theta \in [0, \pi]$ for $\sqrt{-x^2+1}$:

$$\begin{array}{rcl} \sqrt{-x^2+1} &=& \sqrt{1-\cos^2\theta} \\ &=& \sqrt{\sin^2\theta} \\ &=& \sin\theta \quad . \end{array}$$

when $\theta \in [0, \pi]$ we have
$\sin\theta \geq 0$ and so $\sqrt{\sin^2\theta} = \sin\theta$

To summarize:

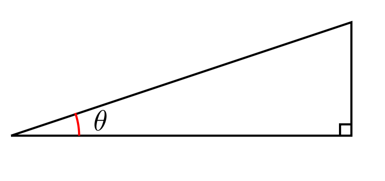
Definition

The trigonometric substitution $x = \cos\theta$, $\theta \in [0, \pi]$ for $\sqrt{-x^2+1}$ is given by:

$$\begin{array}{rcl} x &=& \cos\theta \\ \sqrt{-x^2+1} &=& \sin\theta \\ \mathrm{d}x &=& -\sin\theta \mathrm{d}\theta \\ \theta &=& \arccos x \quad . \end{array}$$

Example

Evaluate $\int \frac{\sqrt{9-x^2}}{x^2} \mathrm{d}x$.



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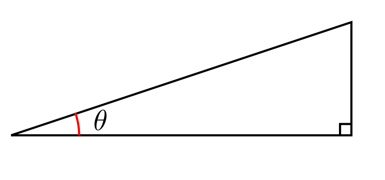

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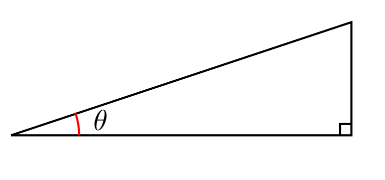

Example

Evaluate $\int \frac{\sqrt{9-x^2}}{x^2}\mathrm{d}x$.

- Let $x = 3\sin\theta$, where $-\pi/2 \leq \theta \leq \pi/2$.
- Then $\mathrm{d}x =$

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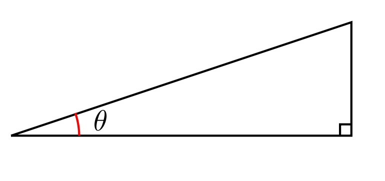

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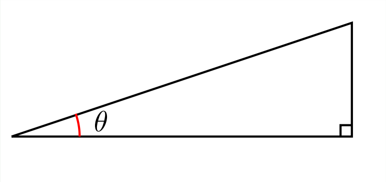

Example

Evaluate $\int \frac{\sqrt{9-x^2}}{x^2} dx$.

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- Then $dx = 3\cos\theta d\theta$.

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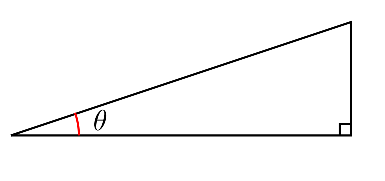

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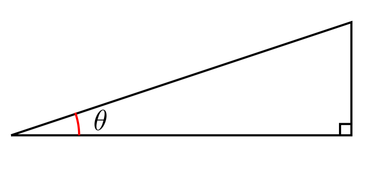

$$\sqrt{9 - x^2} = \sqrt{9 - 9\sin^2\theta} =$$

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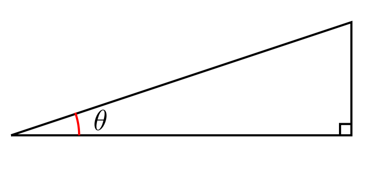

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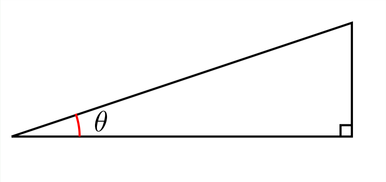

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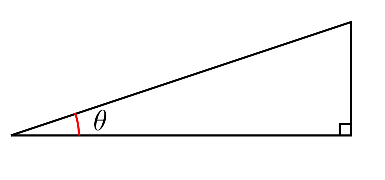

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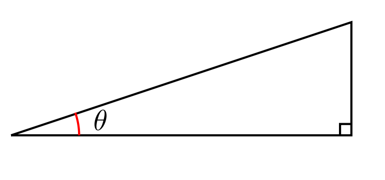

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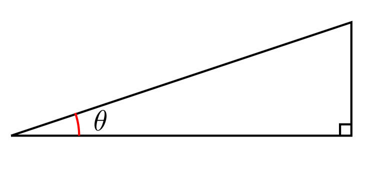

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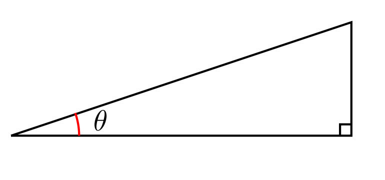

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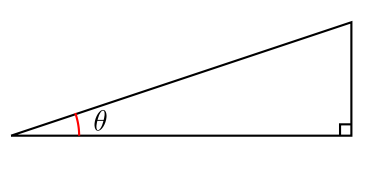

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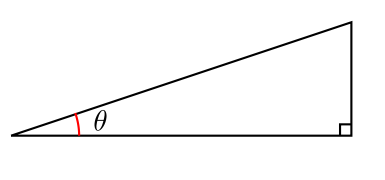

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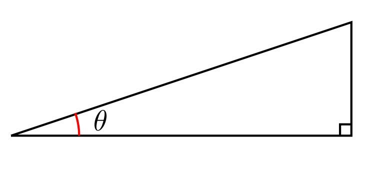

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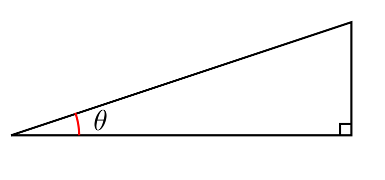

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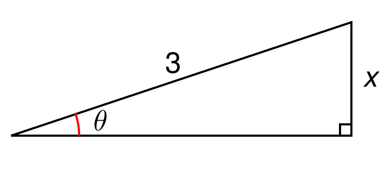

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- Let $x = 3\sin\theta$, where $-\pi/2 \leq \theta \leq \pi/2$.
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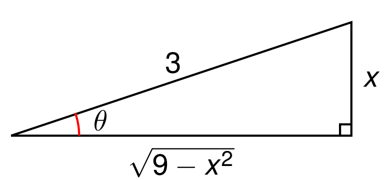


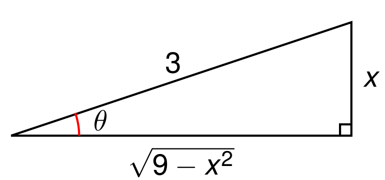
$$\sqrt{9-x^2} = \sqrt{9-9\sin^2\theta} = \sqrt{9\cos^2\theta} = 3|\cos\theta| = 3\cos\theta$$

$$\begin{aligned}\int \frac{\sqrt{9-x^2}}{x^2}\mathrm{d}x &= \int \frac{3\cos\theta}{9\sin^2\theta} 3\cos\theta \mathrm{d}\theta = \int \cot^2\theta \mathrm{d}\theta \\ &= \int (\csc^2\theta - 1)\mathrm{d}\theta = -\cot\theta - \theta + C\end{aligned}$$

Example

Evaluate $\int \frac{\sqrt{9-x^2}}{x^2}\mathrm{d}x$.

- Let $x = 3\sin\theta$, where $-\pi/2 \leq \theta \leq \pi/2$.
- Then $\mathrm{d}x = 3\cos\theta \mathrm{d}\theta$.



$$\sqrt{9-x^2} = \sqrt{9-9\sin^2\theta} = \sqrt{9\cos^2\theta} = 3|\cos\theta| = 3\cos\theta$$

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Example

Evaluate $\int \frac{\sqrt{9-x^2}}{x^2} \mathrm{d}x$.

- Let $x = 3\sin\theta$, where $-\pi/2 \leq \theta \leq \pi/2$.
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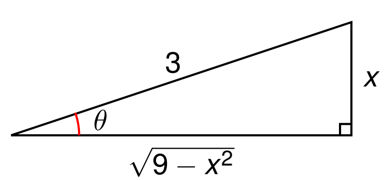
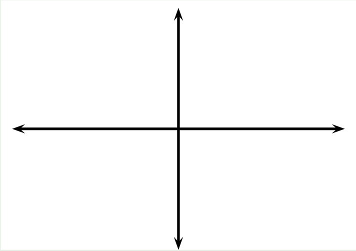


$$\sqrt{9-x^2} = \sqrt{9-9\sin^2\theta} = \sqrt{9\cos^2\theta} = 3|\cos\theta| = 3\cos\theta$$

$$\begin{aligned}
\int \frac{\sqrt{9-x^2}}{x^2}\mathrm{d}x &= \int \frac{3\cos\theta}{9\sin^2\theta} 3\cos\theta \mathrm{d}\theta = \int \cot^2\theta \mathrm{d}\theta \\
&= \int (\csc^2\theta - 1)\mathrm{d}\theta = -\cot\theta - \theta + C \\
&= -\frac{\sqrt{9-x^2}}{x} - \arcsin\left(\frac{x}{3}\right) + C
\end{aligned}$$

Example

Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a, b > 0$.



Example

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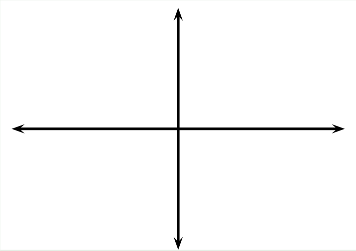
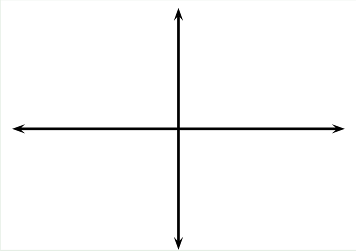


Express y via x:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Example

Find the area enclosed by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $a, b > 0$.



Express y via x:

$$\begin{aligned}\frac{x^2}{a^2}+\frac{y^2}{b^2} &= 1\\ \frac{y^2}{b^2} &= 1-\frac{x^2}{a^2}\end{aligned}$$

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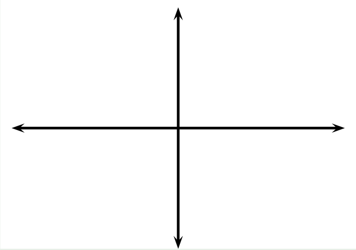
Find the area enclosed by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $a, b>0$.

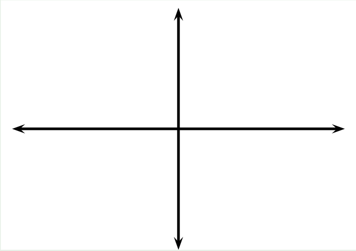


Express y via x:

$$\begin{aligned}\frac{x^2}{a^2}+\frac{y^2}{b^2} &= 1\\ \frac{y^2}{b^2} &= 1-\frac{x^2}{a^2}\\ y^2 &= b^2\left(1-\frac{x^2}{a^2}\right)\end{aligned}$$

Example

Find the area enclosed by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $a, b > 0$.



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\frac{x^2}{a^2}+\frac{y^2}{b^2} &= 1\\
\frac{y^2}{b^2} &= 1-\frac{x^2}{a^2}\\
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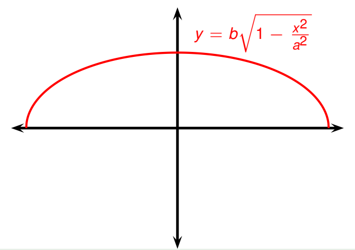
y &= \pm b\sqrt{1-\frac{x^2}{a^2}}
\end{aligned}$$

Example

Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a, b > 0$.



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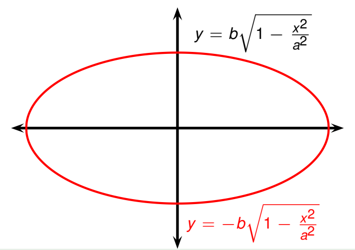

Express y via x:

$$\begin{aligned}\frac{x^2}{a^2}+\frac{y^2}{b^2} &= 1\\ \frac{y^2}{b^2} &= 1-\frac{x^2}{a^2}\\ y^2 &= b^2\left(1-\frac{x^2}{a^2}\right)\\ y &= \pm b\sqrt{1-\frac{x^2}{a^2}}\end{aligned}$$

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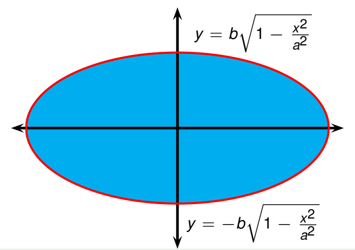

The area in question is

Express y via x:

$$\begin{aligned}
\frac{x^2}{a^2}+\frac{y^2}{b^2} &= 1\\
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y &= \pm b\sqrt{1-\frac{x^2}{a^2}}
\end{aligned}$$

Example

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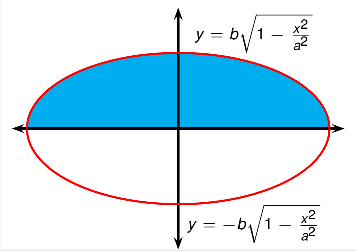

The area in question is

$$\int_{?}^{?} {\color{red}2} b\sqrt{1 - \frac{x^2}{a^2}}\mathrm{d}x$$

Express y via x:

$$\begin{aligned} \frac{x^2}{a^2} + \frac{y^2}{b^2} &= 1 \\ \frac{y^2}{b^2} &= 1 - \frac{x^2}{a^2} \\ y^2 &= b^2\left(1 - \frac{x^2}{a^2}\right) \\ y &= \pm b\sqrt{1 - \frac{x^2}{a^2}} \end{aligned}$$

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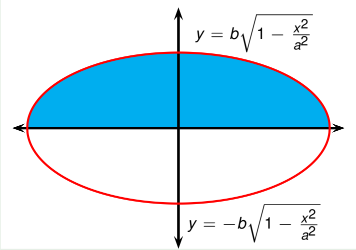
Find the area enclosed by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $a,b>0$.



The area in question is

$$\int_{?}^{?} 2b\sqrt{1-\frac{x^2}{a^2}}\mathrm{d}x$$

Express y via x:

$$\begin{aligned}\frac{x^2}{a^2}+\frac{y^2}{b^2} &= 1\\ \frac{y^2}{b^2} &= 1-\frac{x^2}{a^2}\\ y^2 &= b^2\left(1-\frac{x^2}{a^2}\right)\\ y &= \pm b\sqrt{1-\frac{x^2}{a^2}}\end{aligned}$$

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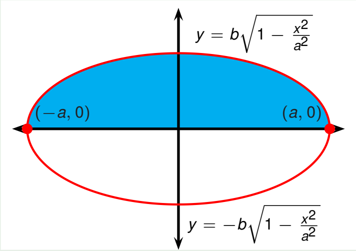


The area in question is

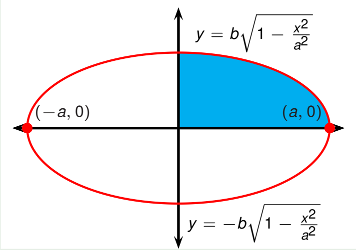
$$\int_{-a}^{a} 2b\sqrt{1-\frac{x^2}{a^2}}\mathrm{d}x$$

Express y via x:

$$\begin{aligned}
\frac{x^2}{a^2}+\frac{y^2}{b^2} &= 1\\
\frac{y^2}{b^2} &= 1-\frac{x^2}{a^2}\\
y^2 &= b^2\left(1-\frac{x^2}{a^2}\right)\\
y &= \pm b\sqrt{1-\frac{x^2}{a^2}}
\end{aligned}$$

Example

Find the area enclosed by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $a, b > 0$.



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$$
\begin{aligned}
\frac{x^2}{a^2}+\frac{y^2}{b^2} &= 1\\
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y &= \pm b\sqrt{1-\frac{x^2}{a^2}}
\end{aligned}
$$

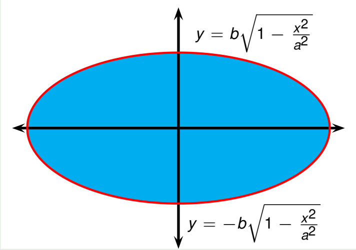
The area in question is

$$
\int_{-a}^{a} 2b\sqrt{1-\frac{x^2}{a^2}}\mathrm{d}x
$$

$$
= 4\int_{0}^{a} b\sqrt{1-\frac{x^2}{a^2}}\mathrm{d}x.
$$

Example

Find the area enclosed by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $a, b>0$.



$$\int_0^a \sqrt{1-\frac{x^2}{a^2}}\mathrm{d}x$$

The area in question is

$$\int\limits_{-a}^{a} 2b\sqrt{1-\frac{x^2}{a^2}}\mathrm{d}x$$

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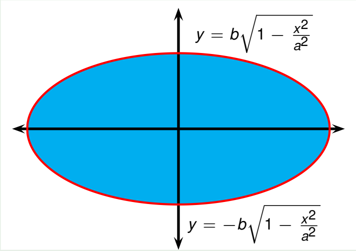

Trig subst.: set $x=a\sin\theta$, $\theta\in\left(0,\frac{\pi}{2}\right)$.

$$\int_0^a\sqrt{1-\frac{x^2}{a^2}}\mathrm{d}x$$

The area in question is

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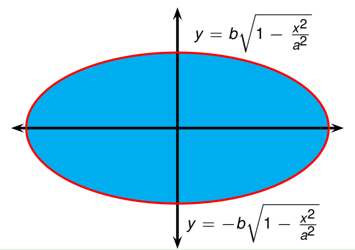

The area in question is

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Trig subst.: set $x = a\sin\theta$, $\theta \in \left(0, \frac{\pi}{2}\right)$.

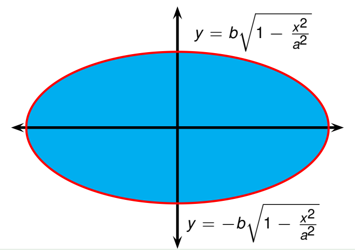
Compute: $\sqrt{1 - \frac{x^2}{a^2}} = \sqrt{1 - \frac{a^2\sin^2\theta}{a^2}}$

.

$$\int_0^a \sqrt{1 - \frac{x^2}{a^2}}\mathrm{d}x$$

Example

Find the area enclosed by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $a,b>0$.



Trig subst.: set $x=a\sin\theta$, $\theta\in\left(0,\frac{\pi}{2}\right)$.

Compute: $\sqrt{1-\frac{x^2}{a^2}}=\sqrt{1-\frac{a^2\sin^2\theta}{a^2}}=\sqrt{1-\sin^2\theta}$.

$$\int_0^a\sqrt{1-\frac{x^2}{a^2}}dx$$

The area in question is

$$\int_{-a}^{a}2b\sqrt{1-\frac{x^2}{a^2}}dx$$

$$=4\int_0^a b\sqrt{1-\frac{x^2}{a^2}}dx$$

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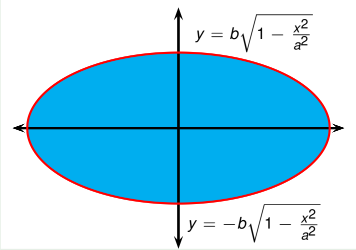

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Compute: $\sqrt{1-\frac{x^2}{a^2}}=\sqrt{1-\frac{a^2\sin^2\theta}{a^2}}=\sqrt{1-\sin^2\theta}=\cos\theta$.

$$\int_0^a \sqrt{1-\frac{x^2}{a^2}}\mathrm{d}x$$

The area in question is

$$\int_{-a}^{a} 2b\sqrt{1-\frac{x^2}{a^2}}\mathrm{d}x$$

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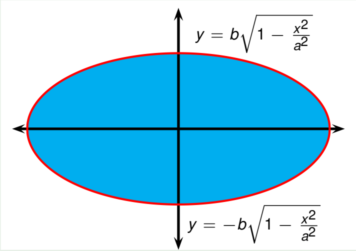


The area in question is

$$\int_{-a}^{a} 2b\sqrt{1-\frac{x^2}{a^2}}\mathrm{d}x$$

$$= 4\int_{0}^{a} b\sqrt{1-\frac{x^2}{a^2}}\mathrm{d}x$$

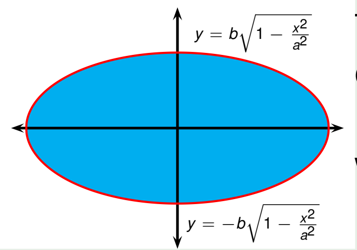
Trig subst.: set $x = a\sin\theta$, $\theta\in\left(0,\frac{\pi}{2}\right)$.

Compute: $\sqrt{1-\frac{x^2}{a^2}} = \sqrt{1-\frac{a^2\sin^2\theta}{a^2}} = \sqrt{1-\sin^2\theta} = \cos\theta$.

$$\int_0^a \sqrt{1-\frac{x^2}{a^2}}\mathrm{d}x = \int \cos\theta\, \mathrm{d}\left(a\sin\theta\right)$$

Example

Find the area enclosed by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $a, b>0$.



Trig subst.: set $x=a\sin\theta$, $\theta\in\left(0,\frac{\pi}{2}\right)$.
Compute: $\sqrt{1-\frac{x^2}{a^2}}=\sqrt{1-\frac{a^2\sin^2\theta}{a^2}}=\sqrt{1-\sin^2\theta}=\cos\theta$. When $x=0$, $\theta=0$ and when $x=a$, $\theta=\frac{\pi}{2}$.

$$\int_0^a\sqrt{1-\frac{x^2}{a^2}}\mathrm{d}x \quad = \quad \int_{0}^{\frac{\pi}{2}}\cos\theta\,\mathrm{d}\left(a\sin\theta\right)$$

The area in question is

$$\int\limits_{-a}^{a}2b\sqrt{1-\frac{x^2}{a^2}}\mathrm{d}x$$

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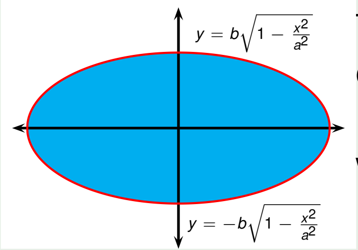
Example

Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a, b > 0$.



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Trig subst.: set $x = a\sin\theta$, $\theta \in \left(0, \frac{\pi}{2}\right)$.
Compute: $\sqrt{1 - \frac{x^2}{a^2}} = \sqrt{1 - \frac{a^2\sin^2\theta}{a^2}} = \sqrt{1 - \sin^2\theta} = \cos\theta$. When $x = 0$, $\theta = 0$ and when $x = a$, $\theta = \frac{\pi}{2}$.

$$\begin{aligned}\int_0^a \sqrt{1 - \frac{x^2}{a^2}}\mathrm{d}x &= \int_0^{\frac{\pi}{2}} \cos\theta\,\mathrm{d}\left(a\sin\theta\right)\\ &= a\int_0^{\frac{\pi}{2}} \cos^2\theta\mathrm{d}\theta\end{aligned}$$

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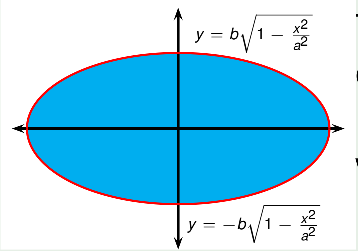

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Compute: $\sqrt{1 - \frac{x^2}{a^2}} = \sqrt{1 - \frac{a^2\sin^2\theta}{a^2}} = \sqrt{1 - \sin^2\theta} = \cos\theta$. When $x = 0$, $\theta = 0$ and when $x = a$, $\theta = \frac{\pi}{2}$.

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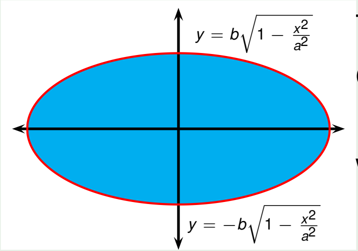

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Compute: $\sqrt{1 - \frac{x^2}{a^2}} = \sqrt{1 - \frac{a^2\sin^2\theta}{a^2}} = \sqrt{1 - \sin^2\theta} = \cos\theta$. When $x = 0$, $\theta = 0$ and when $x = a$, $\theta = \frac{\pi}{2}$.

$$\int_0^a \sqrt{1 - \frac{x^2}{a^2}}\mathrm{d}x = \int_0^{\frac{\pi}{2}} \cos\theta \,\mathrm{d}\left(a\sin\theta\right)$$
$$= a\int_0^{\frac{\pi}{2}} \cos^2\theta \mathrm{d}\theta$$
$$= a\int_0^{\frac{\pi}{2}} \frac{\cos(2\theta)+1}{2}\mathrm{d}\theta$$

The area in question is

$$\int_{-a}^{a} 2b\sqrt{1 - \frac{x^2}{a^2}}\mathrm{d}x$$
$$= 4\int_0^a b\sqrt{1 - \frac{x^2}{a^2}}\mathrm{d}x$$

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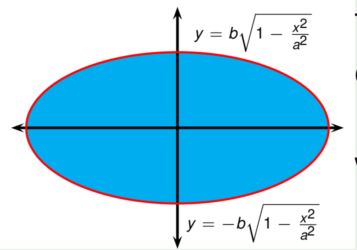

The area in question is

$$\int_{-a}^{a} 2b\sqrt{1-\frac{x^2}{a^2}}\mathrm{d}x$$

$$=4\int_{0}^{a} b\sqrt{1-\frac{x^2}{a^2}}\mathrm{d}x$$

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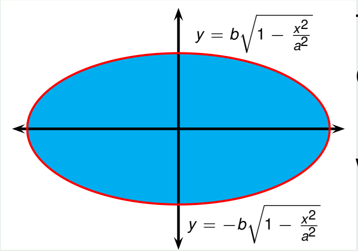
$$\begin{aligned}\int_0^a\sqrt{1-\frac{x^2}{a^2}}\mathrm{d}x &= \int_0^{\frac{\pi}{2}}\cos\theta\,\mathrm{d}\left(a\sin\theta\right)\\ &= a\int_0^{\frac{\pi}{2}}\cos^2\theta\mathrm{d}\theta\\ &= a\int_0^{\frac{\pi}{2}}\frac{\cos(2\theta)+1}{2}\mathrm{d}\theta\\ &= a\left[\frac{\sin(2\theta)}{4}+\frac{\theta}{2}\right]_{\theta=0}^{\theta=\frac{\pi}{2}}\end{aligned}$$

Example

Find the area enclosed by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $a, b>0$.



The area in question is

$$\int_{-a}^{a} 2b\sqrt{1-\frac{x^2}{a^2}}\mathrm{d}x$$

$$=4\int_{0}^{a} b\sqrt{1-\frac{x^2}{a^2}}\mathrm{d}x$$

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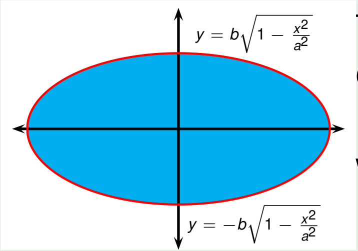

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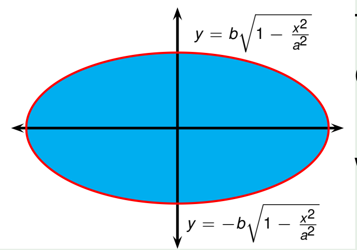
&= a\left[\frac{\sin(2\theta)}{4}+\frac{\theta}{2}\right]_{\theta=0}^{\theta=\frac{\pi}{2}}\\
&= a\left(0+\frac{\pi}{4}-(0+0)\right)
\end{aligned}$$

Example

Find the area enclosed by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $a, b > 0$.



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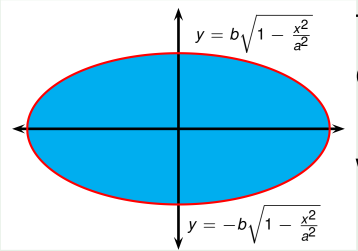

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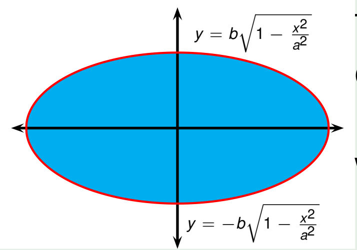

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- $\int \frac{x}{\sqrt{3-2x-x^2}}\mathrm{d}x = \int \frac{x}{\sqrt{4-(x+1)^2}}\mathrm{d}x = \int \frac{u-1}{\sqrt{4-u^2}}\mathrm{d}u$
- Let $u = 2\sin\theta$, where $-\pi/2 \leq \theta \leq \pi/2$. Then $\mathrm{d}u = 2\cos\theta\mathrm{d}\theta$.
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Euler subst. for $\sqrt{-x^2+1}$ corresponding to $x = \cos\theta$

- $x = \cos\theta$ transforms $\mathrm{d}x, x, \sqrt{-x^2+1}$ to trig form.
- $\theta = 2\arctan t$, transforms $\mathrm{d}\theta, \cos\theta, \sin\theta$ to rational form.

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- $x = \cos\theta$ transforms $\mathrm{d}x, x, \sqrt{-x^2+1}$ to trig form.
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What if we compose the above? We get the Euler substitution:

$$x =$$

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What if we compose the above? We get the Euler substitution:

$$\begin{aligned} x &= \cos\theta \\ &= \cos(2\arctan t) \end{aligned}$$

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$$\left|\ \cos(2z) = \right.$$

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$$\left| \cos(2z) = \frac{1-\tan^2 z}{1+\tan^2 z} \right.$$

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What if we compose the above? We get the Euler substitution:

$$\begin{aligned} x &= \cos\theta \\ &= \cos(2\arctan t) \\ &= \frac{1-\tan^2(\arctan t)}{1+\tan^2(\arctan t)} \end{aligned}$$

$$\cos(2z) = \frac{1-\tan^2 z}{1+\tan^2 z}$$

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$$\cos(2z) = \frac{1-\tan^2 z}{1+\tan^2 z}$$

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What if we compose the above? We get the Euler substitution:

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$$
\begin{array}{rcl}
x &=& \dfrac{1-t^2}{1+t^2} \\
\hline
\sqrt{-x^2+1} &=& \sqrt{1-\left(\dfrac{1-t^2}{1+t^2}\right)^2} \\
&=& \sqrt{\dfrac{(1+t^2)^2-(1-t^2)^2}{(1+t^2)^2}}
\end{array}
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\begin{aligned}
x &= \frac{1-t^2}{1+t^2}\\
\sqrt{-x^2+1} &= \sqrt{1-\left(\frac{1-t^2}{1+t^2}\right)^2}\\
&= \sqrt{\frac{(1+t^2)^2-(1-t^2)^2}{(1+t^2)^2}} \qquad \Big|\ (1+t^2)^2-(1-t^2)^2 = \textbf{?}
\end{aligned}
$$

Euler subst. for $\sqrt{-x^2+1}$ corresponding to $x=\cos\theta$

- $x=\cos\theta$ transforms $\mathrm{d}x, x, \sqrt{-x^2+1}$ to trig form.
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& = & \sqrt{\dfrac{(1+t^2)^2-(1-t^2)^2}{(1+t^2)^2}} \qquad \left| \; (1+t^2)^2-(1-t^2)^2 = 4t^2 \right.
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&= \sqrt{\frac{(1+t^2)^2-(1-t^2)^2}{(1+t^2)^2}} && \left|\ (1+t^2)^2-(1-t^2)^2 = 4t^2\right.\\
&= \sqrt{\frac{4t^2}{(1+t^2)^2}} && \left|\ \sqrt{4t^2} = 2t \text{ because } t>0\right.
\end{aligned}
$$

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$$\begin{aligned}
\sqrt{-x^2+1} &= \sqrt{1-\left(\frac{1-t^2}{1+t^2}\right)^2} \\
&= \sqrt{\frac{(1+t^2)^2-(1-t^2)^2}{(1+t^2)^2}} && \Big| \; (1+t^2)^2-(1-t^2)^2 = 4t^2 \\
&= \sqrt{\frac{4t^2}{(1+t^2)^2}} && \Big| \; \sqrt{4t^2} = 2t \text{ because } t > 0 \\
&= \frac{2t}{1+t^2}
\end{aligned}$$

Euler subst. for $\sqrt{-x^2+1}$ corresponding to $x = \cos\theta$

- $x = \cos\theta$ transforms $\mathrm{d}x, x, \sqrt{-x^2+1}$ to trig form.
- $\theta = 2\arctan t, \quad t > 0$ transforms $\mathrm{d}\theta, \cos\theta, \sin\theta$ to rational form.

What if we compose the above? We get the Euler substitution:

$$
\begin{array}{rcll}
x & = & \dfrac{1-t^2}{1+t^2} & \\
\hline
\sqrt{-x^2+1} & = & \sqrt{1-\left(\dfrac{1-t^2}{1+t^2}\right)^2} & \\
& = & \sqrt{\dfrac{(1+t^2)^2-(1-t^2)^2}{(1+t^2)^2}} & \left| \ (1+t^2)^2-(1-t^2)^2 = 4t^2 \right. \\
& = & \sqrt{\dfrac{4t^2}{(1+t^2)^2}} & \left| \ \sqrt{4t^2} = 2t \text{ because } t > 0 \right. \\
& = & \color{red}{\dfrac{2t}{1+t^2}} &
\end{array}
$$

Euler subst. for $\sqrt{-x^2+1}$ corresponding to $x = \cos\theta$

- $x = \cos\theta$ transforms $\mathrm{d}x, x, \sqrt{-x^2+1}$ to trig form.
- $\theta = 2\arctan t, \quad t > 0$ transforms $\mathrm{d}\theta, \cos\theta, \sin\theta$ to rational form.

What if we compose the above? We get the Euler substitution:

$$\begin{aligned} x &= \frac{1-t^2}{1+t^2} \\ \sqrt{-x^2+1} &= \frac{2t}{1+t^2} \end{aligned}$$

Euler subst. for $\sqrt{-x^2+1}$ corresponding to $x = \cos\theta$

- $x = \cos\theta$ transforms $\mathrm{d}x, x, \sqrt{-x^2+1}$ to trig form.
- $\theta = 2\arctan t, \quad t > 0$ transforms $\mathrm{d}\theta, \cos\theta, \sin\theta$ to rational form.

What if we compose the above? We get the Euler substitution:

$$\frac{x = \frac{1-t^2}{1+t^2}}{\sqrt{-x^2+1} = \frac{2t}{1+t^2}}$$

$$(1+t^2)x = 1-t^2$$

Euler subst. for $\sqrt{-x^2+1}$ corresponding to $x=\cos\theta$

- $x=\cos\theta$ transforms dx, x, $\sqrt{-x^2+1}$ to trig form.
- $\theta = 2\arctan t,\ \ t>0$ transforms dθ, $\cos\theta$, $\sin\theta$ to rational form.

What if we compose the above? We get the Euler substitution:

$$x = \frac{1-t^2}{1+t^2}$$

$$\sqrt{-x^2+1} = \frac{2t}{1+t^2}$$

$$(1+t^2)x = 1-t^2$$

Euler subst. for $\sqrt{-x^2+1}$ corresponding to $x = \cos\theta$

- $x = \cos\theta$ transforms $\mathrm{d}x, x, \sqrt{-x^2+1}$ to trig form.
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What if we compose the above? We get the Euler substitution:

$$
\begin{array}{rcl}
x & = & \dfrac{1-t^2}{1+t^2} \\
\hline
\sqrt{-x^2+1} & = & \dfrac{2t}{1+t^2} \\
\hline
(1+t^2)x & = & 1-t^2 \\
t^2(x+1) & = & 1-x
\end{array}
$$

Euler subst. for $\sqrt{-x^2+1}$ corresponding to $x = \cos\theta$

- $x = \cos\theta$ transforms $\mathrm{d}x, x, \sqrt{-x^2+1}$ to trig form.
- $\theta = 2\arctan t, \quad t > 0$ transforms $\mathrm{d}\theta, \cos\theta, \sin\theta$ to rational form.

What if we compose the above? We get the Euler substitution:

$$\begin{array}{rcl} x &=& \dfrac{1-t^2}{1+t^2} \\ \hline \sqrt{-x^2+1} &=& \dfrac{2t}{1+t^2} \\ \hline (1+t^2)x &=& 1-t^2 \\ t^2(x+1) &=& 1-x \end{array}$$

Euler subst. for $\sqrt{-x^2+1}$ corresponding to $x=\cos\theta$

- $x = \cos\theta$ transforms $\mathrm{d}x, x, \sqrt{-x^2+1}$ to trig form.
- $\theta = 2\arctan t,\quad t > 0$ transforms $\mathrm{d}\theta, \cos\theta, \sin\theta$ to rational form.

What if we compose the above? We get the Euler substitution:

$$x = \frac{1-t^2}{1+t^2}$$

$$\sqrt{-x^2+1} = \frac{2t}{1+t^2}$$

$$\begin{aligned}(1+t^2)x &= 1-t^2\\ t^2(x+1) &= 1-x\\ t^2 &= \frac{1-x}{1+x}\end{aligned}$$

Euler subst. for $\sqrt{-x^2+1}$ corresponding to $x = \cos\theta$

- $x = \cos\theta$ transforms dx, x, $\sqrt{-x^2+1}$ to trig form.
- $\theta = 2\arctan t$, $t > 0$ transforms dθ, $\cos\theta$, $\sin\theta$ to rational form.

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$$x = \frac{1-t^2}{1+t^2}$$

$$\sqrt{-x^2+1} = \frac{2t}{1+t^2}$$

$$\begin{aligned}
(1+t^2)x &= 1-t^2\\
t^2(x+1) &= 1-x\\
t^2 &= \frac{1-x}{1+x}\\
t &= \frac{\sqrt{1-x}}{\sqrt{1+x}} && \text{we use } t>0
\end{aligned}$$

Euler subst. for $\sqrt{-x^2+1}$ corresponding to $x = \cos\theta$

- $x = \cos\theta$ transforms dx, x, $\sqrt{-x^2+1}$ to trig form.
- $\theta = 2\arctan t$, $t > 0$ transforms dθ, $\cos\theta$, $\sin\theta$ to rational form.

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$$x = \frac{1-t^2}{1+t^2}$$

$$\sqrt{-x^2+1} = \frac{2t}{1+t^2}$$

$$\begin{aligned}
(1+t^2)x &= 1-t^2\\
t^2(x+1) &= 1-x\\
t^2 &= \frac{1-x}{1+x}\\
t &= \frac{\sqrt{1-x}}{\sqrt{1+x}}\frac{\sqrt{1+x}}{\sqrt{1+x}} && \text{we use } t > 0
\end{aligned}$$

Euler subst. for $\sqrt{-x^2+1}$ corresponding to $x = \cos\theta$

- $x = \cos\theta$ transforms $\mathrm{d}x, x, \sqrt{-x^2+1}$ to trig form.
- $\theta = 2\arctan t,\quad t > 0$ transforms $\mathrm{d}\theta, \cos\theta, \sin\theta$ to rational form.

What if we compose the above? We get the Euler substitution:

$$x = \frac{1-t^2}{1+t^2}$$

$$\sqrt{-x^2+1} = \frac{2t}{1+t^2}$$

$$\begin{aligned}
(1+t^2)x &= 1-t^2\\
t^2(x+1) &= 1-x\\
t^2 &= \frac{1-x}{1+x}\\
t &= \frac{\color{red}{\sqrt{1-x}}}{\sqrt{1+x}}\frac{\color{red}{\sqrt{1+x}}}{\sqrt{1+x}} = \frac{\color{red}{\sqrt{-x^2+1}}}{x+1} \qquad \text{we use } t > 0
\end{aligned}$$

Euler subst. for $\sqrt{-x^2+1}$ corresponding to $x=\cos\theta$

- $x=\cos\theta$ transforms $\mathrm{d}x, x, \sqrt{-x^2+1}$ to trig form.
- $\theta = 2\arctan t,\quad t>0$ transforms $\mathrm{d}\theta, \cos\theta, \sin\theta$ to rational form.

What if we compose the above? We get the Euler substitution:

$$x = \frac{1-t^2}{1+t^2}$$

$$\sqrt{-x^2+1} = \frac{2t}{1+t^2}$$

$$\begin{aligned}
(1+t^2)x &= 1-t^2\\
t^2(x+1) &= 1-x\\
t^2 &= \frac{1-x}{1+x}\\
t &= \frac{\sqrt{1-x}}{\sqrt{1+x}}\frac{\sqrt{1+x}}{\sqrt{1+x}} = \frac{\sqrt{-x^2+1}}{x+1} \qquad \text{we use } t>0
\end{aligned}$$

Euler subst. for $\sqrt{-x^2+1}$ corresponding to $x = \cos\theta$

- $x = \cos\theta$ transforms $\mathrm{d}x, x, \sqrt{-x^2+1}$ to trig form.
- $\theta = 2\arctan t,\quad t > 0$ transforms $\mathrm{d}\theta, \cos\theta, \sin\theta$ to rational form.

What if we compose the above? We get the Euler substitution:

$$x = \frac{1-t^2}{1+t^2}$$

$$\sqrt{-x^2+1} = \frac{2t}{1+t^2}$$

$$t = \frac{\sqrt{-x^2+1}}{x+1}$$

Euler subst. for $\sqrt{-x^2+1}$ corresponding to $x = \cos\theta$

- $x = \cos\theta$ transforms $\mathrm{d}x, x, \sqrt{-x^2+1}$ to trig form.
- $\theta = 2\arctan t, \quad t > 0$ transforms $\mathrm{d}\theta, \cos\theta, \sin\theta$ to rational form.

What if we compose the above? We get the Euler substitution:

$$x = \frac{1-t^2}{1+t^2}$$

$$\sqrt{-x^2+1} = \frac{2t}{1+t^2}$$

$$t = \frac{\sqrt{-x^2+1}}{x+1}$$

$$\mathrm{d}x$$

Euler subst. for $\sqrt{-x^2+1}$ corresponding to $x = \cos\theta$

- $x = \cos\theta$ transforms $\mathrm{d}x, x, \sqrt{-x^2+1}$ to trig form.
- $\theta = 2\arctan t,\quad t > 0$ transforms $\mathrm{d}\theta, \cos\theta, \sin\theta$ to rational form.

What if we compose the above? We get the Euler substitution:

$$x = \frac{1-t^2}{1+t^2}$$

$$\sqrt{-x^2+1} = \frac{2t}{1+t^2}$$

$$t = \frac{\sqrt{-x^2+1}}{x+1}$$

$$\mathrm{d}x = \mathrm{d}\left(\frac{1-t^2}{1+t^2}\right)$$

Euler subst. for $\sqrt{-x^2+1}$ corresponding to $x = \cos\theta$

- $x = \cos\theta$ transforms $\mathrm{d}x, x, \sqrt{-x^2+1}$ to trig form.
- $\theta = 2\arctan t,\ \ t > 0$ transforms $\mathrm{d}\theta, \cos\theta, \sin\theta$ to rational form.

What if we compose the above? We get the Euler substitution:

$$x = \frac{1-t^2}{1+t^2}$$

$$\sqrt{-x^2+1} = \frac{2t}{1+t^2}$$

$$t = \frac{\sqrt{-x^2+1}}{x+1}$$

$$\mathrm{d}x = \mathrm{d}\left(\frac{1-t^2}{1+t^2}\right) = \mathrm{d}\left(\frac{2-(1+t^2)}{1+t^2}\right)$$

Euler subst. for $\sqrt{-x^2+1}$ corresponding to $x = \cos\theta$

- $x = \cos\theta$ transforms $\mathrm{d}x, x, \sqrt{-x^2+1}$ to trig form.
- $\theta = 2\arctan t, \quad t > 0$ transforms $\mathrm{d}\theta, \cos\theta, \sin\theta$ to rational form.

What if we compose the above? We get the Euler substitution:

$$x = \frac{1-t^2}{1+t^2}$$

$$\sqrt{-x^2+1} = \frac{2t}{1+t^2}$$

$$t = \frac{\sqrt{-x^2+1}}{x+1}$$

$$\begin{aligned}\mathrm{d}x &= \mathrm{d}\left(\frac{1-t^2}{1+t^2}\right) = \mathrm{d}\left(\frac{2-(1+t^2)}{1+t^2}\right)\\ &= \mathrm{d}\left(\frac{2}{1+t^2} - 1\right)\end{aligned}$$

Euler subst. for $\sqrt{-x^2+1}$ corresponding to $x = \cos\theta$

- $x = \cos\theta$ transforms $\mathrm{d}x, x, \sqrt{-x^2+1}$ to trig form.
- $\theta = 2\arctan t, \quad t > 0$ transforms $\mathrm{d}\theta, \cos\theta, \sin\theta$ to rational form.

What if we compose the above? We get the Euler substitution:

$$
\begin{aligned}
x &= \frac{1-t^2}{1+t^2}\\
\sqrt{-x^2+1} &= \frac{2t}{1+t^2}\\
t &= \frac{\sqrt{-x^2+1}}{x+1}\\
\mathrm{d}x &= \mathrm{d}\left(\frac{1-t^2}{1+t^2}\right) = \mathrm{d}\left(\frac{2-(1+t^2)}{1+t^2}\right)\\
&= \mathrm{d}\left(\frac{2}{1+t^2} - 1\right) = -\frac{4t}{(1+t^2)^2}\mathrm{d}t
\end{aligned}
$$

Euler subst. for $\sqrt{-x^2+1}$ corresponding to $x = \cos\theta$

- $x = \cos\theta$ transforms $\mathrm{d}x, x, \sqrt{-x^2+1}$ to trig form.
- $\theta = 2\arctan t,\quad t > 0$ transforms $\mathrm{d}\theta, \cos\theta, \sin\theta$ to rational form.

What if we compose the above? We get the Euler substitution:

$$x = \frac{1-t^2}{1+t^2}$$

$$\sqrt{-x^2+1} = \frac{2t}{1+t^2}$$

$$t = \frac{\sqrt{-x^2+1}}{x+1}$$

$$\begin{aligned}\mathrm{d}x &= \mathrm{d}\left(\frac{1-t^2}{1+t^2}\right) = \mathrm{d}\left(\frac{2-(1+t^2)}{1+t^2}\right)\\ &= \mathrm{d}\left(\frac{2}{1+t^2} - 1\right) = -\frac{4t}{(1+t^2)^2}\mathrm{d}t\end{aligned}$$

Euler subst. for $\sqrt{-x^2+1}$ corresponding to $x = \cos\theta$

- $x = \cos\theta$ transforms $\mathrm{d}x, x, \sqrt{-x^2+1}$ to trig form.
- $\theta = 2\arctan t,\ \ t > 0$ transforms $\mathrm{d}\theta, \cos\theta, \sin\theta$ to rational form.

What if we compose the above? We get the Euler substitution:

Definition

The Euler substitution for $\sqrt{-x^2+1}$ corresponding to $x = \cos\theta$ is given by:

$$\begin{aligned} x &= \frac{1-t^2}{1+t^2}, \qquad t > 0 \\ \sqrt{-x^2+1} &= \frac{2t}{1+t^2} \\ \mathrm{d}x &= -\frac{4t}{(t^2+1)^2}\mathrm{d}t \\ t &= \frac{\sqrt{-x^2+1}}{x+1} \quad . \end{aligned}$$

Trigonometric substitution $x = \sec\theta$ for $\sqrt{x^2-1}$

The trigonometric substitution $x = \sec\theta$, $\theta \in \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$:

$$\sqrt{x^2-1} \quad =$$

Trigonometric substitution $x = \sec\theta$ for $\sqrt{x^2-1}$

The trigonometric substitution $x = \sec\theta$, $\theta \in \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$:

$$\sqrt{x^2-1} \quad = \quad \sqrt{\sec^2\theta - 1}$$

Trigonometric substitution $x = \sec\theta$ for $\sqrt{x^2-1}$

The trigonometric substitution $x = \sec\theta$, $\theta \in \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$:

$$\begin{aligned}\sqrt{x^2-1} &= \sqrt{\sec^2\theta - 1}\\ &= \sqrt{\frac{1}{\cos^2\theta} - 1}\end{aligned}$$

Trigonometric substitution $x = \sec\theta$ for $\sqrt{x^2-1}$

The trigonometric substitution $x = \sec\theta$, $\theta \in \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$:

$$\begin{aligned}
\sqrt{x^2-1} &= \sqrt{\sec^2\theta - 1} \\
&= \sqrt{\frac{1}{\cos^2\theta} - 1} \\
&= \sqrt{\frac{\sin^2\theta}{\cos^2\theta}}
\end{aligned}$$

Trigonometric substitution $x = \sec\theta$ for $\sqrt{x^2-1}$

The trigonometric substitution $x = \sec\theta$, $\theta \in \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$:

$$
\begin{aligned}
\sqrt{x^2-1} &= \sqrt{\sec^2\theta - 1}\\
&= \sqrt{\frac{1}{\cos^2\theta} - 1}\\
&= \sqrt{\frac{\sin^2\theta}{\cos^2\theta}}\\
&= \sqrt{\tan^2\theta}
\end{aligned}
$$

Trigonometric substitution $x = \sec\theta$ for $\sqrt{x^2-1}$

The trigonometric substitution $x = \sec\theta$, $\theta \in \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$:

$$\begin{aligned}
\sqrt{x^2-1} &= \sqrt{\sec^2\theta - 1} \\
&= \sqrt{\frac{1}{\cos^2\theta} - 1} \\
&= \sqrt{\frac{\sin^2\theta}{\cos^2\theta}} \\
&= \sqrt{\tan^2\theta}
\end{aligned}$$

when $\theta \in \theta \in \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$ we have $\tan\theta \geq 0$ and so $\sqrt{\tan^2\theta} = \tan\theta$

Trigonometric substitution $x = \sec\theta$ for $\sqrt{x^2-1}$

The trigonometric substitution $x = \sec\theta$, $\theta \in \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$:

$$\begin{aligned}
\sqrt{x^2-1} &= \sqrt{\sec^2\theta - 1} \\
&= \sqrt{\frac{1}{\cos^2\theta} - 1} \\
&= \sqrt{\frac{\sin^2\theta}{\cos^2\theta}} \\
&= \sqrt{\tan^2\theta} \\
&= \tan\theta \quad .
\end{aligned}$$

when $\theta \in \theta \in \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$ we have $\tan\theta \geq 0$ and so $\sqrt{\tan^2\theta} = \tan\theta$

Trigonometric substitution $x = \sec\theta$ for $\sqrt{x^2-1}$

The trigonometric substitution $x = \sec\theta$, $\theta \in \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$:

$$\begin{aligned}\sqrt{x^2-1} &= \sqrt{\sec^2\theta - 1}\\ &= \sqrt{\frac{1}{\cos^2\theta} - 1}\\ &= \sqrt{\frac{\sin^2\theta}{\cos^2\theta}}\\ &= \sqrt{\tan^2\theta}\\ &= \tan\theta \quad .\end{aligned}$$

when $\theta \in \theta \in \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$ we have $\tan\theta \geq 0$ and so $\sqrt{\tan^2\theta} = \tan\theta$

Trigonometric substitution $x = \sec\theta$ for $\sqrt{x^2-1}$

The trigonometric substitution $x = \sec\theta$, $\theta \in \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$:

$$\sqrt{x^2-1} \quad = \quad \tan\theta \quad .$$

Trigonometric substitution $x = \sec\theta$ for $\sqrt{x^2-1}$

The trigonometric substitution $x = \sec\theta$, $\theta \in \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$:

$$\sqrt{x^2-1} \quad = \quad \tan\theta \quad .$$

Definition

The trigonometric substitution $x = \sec\theta$, $\theta \in (0, \pi)$ for $\sqrt{x^2+1}$ is given by:

$$\begin{aligned} x &= \sec\theta = \frac{1}{\cos\theta} & \theta \in \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right) \\ \sqrt{x^2-1} &= \tan\theta & \\ \mathrm{d}x &= \textbf{?} & \mathrm{d}\theta \\ \theta &= \operatorname{arcsec} x \quad . & \end{aligned}$$

Trigonometric substitution $x = \sec\theta$ for $\sqrt{x^2-1}$

The trigonometric substitution $x = \sec\theta$, $\theta \in \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$:

$$\sqrt{x^2-1} = \tan\theta \quad .$$

Definition

The trigonometric substitution $x = \sec\theta$, $\theta \in (0,\pi)$ for $\sqrt{x^2+1}$ is given by:

$$
\begin{aligned}
x &= \sec\theta = \frac{1}{\cos\theta} \qquad \theta \in \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right) \\
\sqrt{x^2-1} &= \tan\theta \\
\mathrm{d}x &= \textbf{?} \qquad\qquad \mathrm{d}\theta \\
\theta &= \operatorname{arcsec} x \quad .
\end{aligned}
$$

Trigonometric substitution $x = \sec\theta$ for $\sqrt{x^2-1}$

The trigonometric substitution $x = \sec\theta$, $\theta \in \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$:

$$\sqrt{x^2-1} \quad = \quad \tan\theta \quad .$$

Definition

The trigonometric substitution $x = \sec\theta$, $\theta \in (0, \pi)$ for $\sqrt{x^2+1}$ is given by:

$$\begin{aligned} x &= \sec\theta = \frac{1}{\cos\theta} \qquad \theta \in \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right) \\ \sqrt{x^2-1} &= \tan\theta \\ \mathrm{d}x &= \textbf{?} \qquad\qquad\qquad \mathrm{d}\theta \\ \theta &= \operatorname{arcsec} x \quad . \end{aligned}$$

Trigonometric substitution $x = \sec\theta$ for $\sqrt{x^2-1}$

The trigonometric substitution $x = \sec\theta$, $\theta \in \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$:

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Definition

The trigonometric substitution $x = \sec\theta$, $\theta \in (0,\pi)$ for $\sqrt{x^2+1}$ is given by:

$$\begin{aligned} x &= \sec\theta = \frac{1}{\cos\theta} \qquad \theta \in \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right) \\ \sqrt{x^2-1} &= \tan\theta \\ \mathrm{d}x &= \textbf{?} \qquad\qquad\qquad \mathrm{d}\theta \\ \theta &= \operatorname{arcsec} x \quad . \end{aligned}$$

Trigonometric substitution $x = \sec\theta$ for $\sqrt{x^2-1}$

The trigonometric substitution $x = \sec\theta$, $\theta \in \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$:

$$\sqrt{x^2-1} = \tan\theta \quad .$$

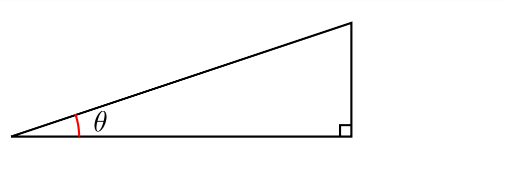
Definition

The trigonometric substitution $x = \sec\theta$, $\theta \in (0, \pi)$ for $\sqrt{x^2+1}$ is given by:

$$\begin{aligned} x &= \sec\theta = \frac{1}{\cos\theta} \qquad \theta \in \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right) \\ \sqrt{x^2-1} &= \tan\theta \\ \mathrm{d}x &= \frac{\sin\theta}{\cos^2\theta}\mathrm{d}\theta = \sec\theta\tan\theta\mathrm{d}\theta \\ \theta &= \operatorname{arcsec} x \quad . \end{aligned}$$

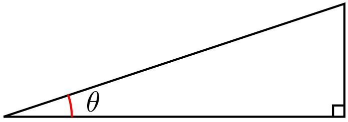
Example

Find $\int \frac{\mathrm{d}x}{\sqrt{x^2-a^2}}$, $a > 0$.



Example

Find $\int \frac{\mathrm{d}x}{\sqrt{x^2-a^2}}$, $a > 0$.

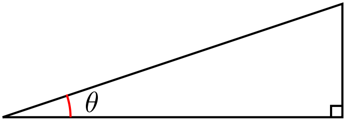


- $x =$

- $\mathrm{d}x =$
  $\sqrt{x^2 - a^2} =$

Example

Find $\int \frac{\mathrm{d}x}{\sqrt{x^2-a^2}}$, $a > 0$.



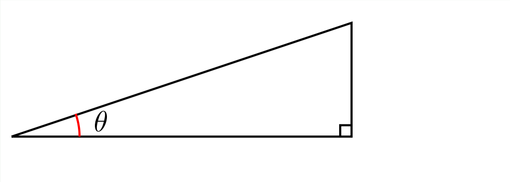
- $x =$

- $\mathrm{d}x =$
  $\sqrt{x^2-a^2} =$

Example

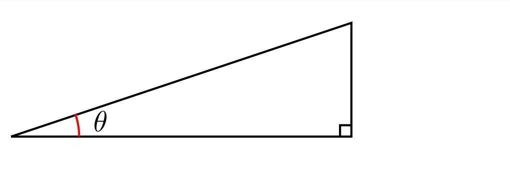
Find $\int \frac{\mathrm{d}x}{\sqrt{x^2-a^2}}$, $a > 0$.

- $x = a\sec\theta$,
  $0 < \theta < \pi/2$ or
  $\pi < \theta < 3\pi/2$.
- $\mathrm{d}x =$
  $\sqrt{x^2 - a^2} =$



Example

Find $\int \frac{\mathrm{d}x}{\sqrt{x^2-a^2}}$, $a>0$.



- $x = a\sec\theta$,
  $0<\theta<\pi/2$ or
  $\pi<\theta<3\pi/2$.
- $\mathrm{d}x =$
  $\sqrt{x^2-a^2} =$

Example

Find $\int \frac{\mathrm{d}x}{\sqrt{x^2-a^2}}$, $a > 0$.

- $x = a\sec\theta$, $0 < \theta < \pi/2$ or $\pi < \theta < 3\pi/2$.
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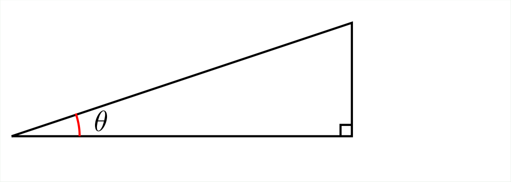
  $\sqrt{x^2-a^2} =$



Example

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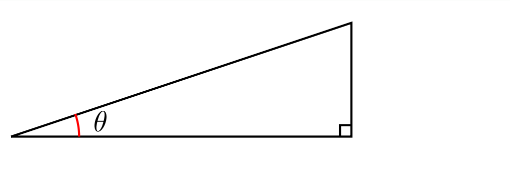
  $\sqrt{x^2 - a^2} = \sqrt{a^2\sec^2\theta - a^2} =$



Example

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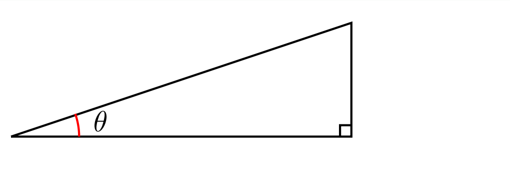

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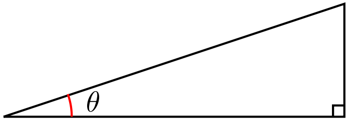
- $\mathrm{d}x = a\sec\theta\tan\theta\mathrm{d}\theta$.
  $\sqrt{x^2 - a^2} = \sqrt{a^2\sec^2\theta - a^2} = \sqrt{a^2\tan^2\theta} =$

Example

Find $\int \frac{\mathrm{d}x}{\sqrt{x^2-a^2}}$, $a > 0$.



- $x = a \sec\theta$,
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  $\sqrt{x^2-a^2} = \sqrt{a^2\sec^2\theta - a^2} = \sqrt{a^2\tan^2\theta} = a|\tan\theta| =$

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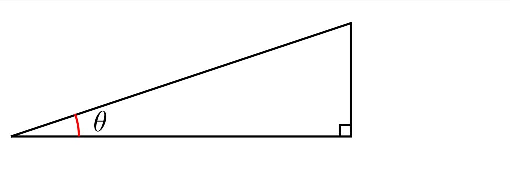

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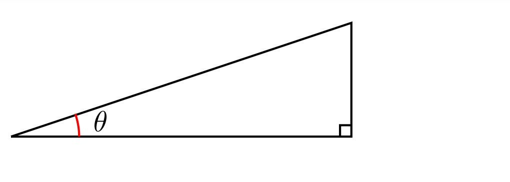

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$$\int \frac{\mathrm{d}x}{\sqrt{x^2-a^2}} = \int \frac{a\sec\theta\tan\theta\mathrm{d}\theta}{a\tan\theta}$$

Example

Find $\int \frac{\mathrm{d}x}{\sqrt{x^2-a^2}}$, $a>0$.

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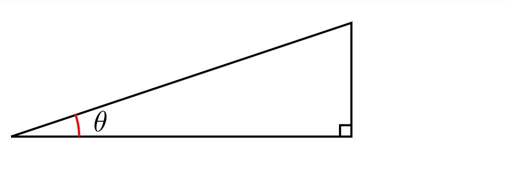

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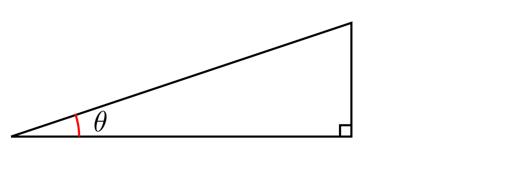
Example

Find $\int \frac{\mathrm{d}x}{\sqrt{x^2-a^2}}$, $a > 0$.



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$$\sqrt{x^2-a^2} = \sqrt{a^2\sec^2\theta - a^2} = \sqrt{a^2\tan^2\theta} = a|\tan\theta| = a\tan\theta$$

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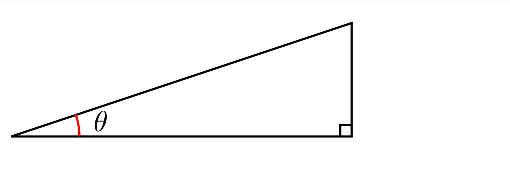

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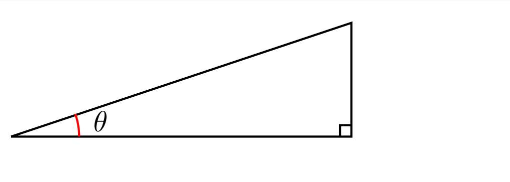

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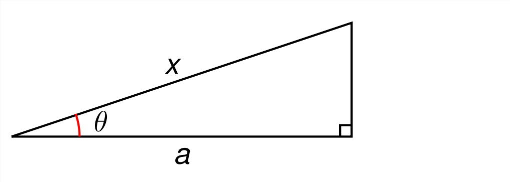
$$\begin{aligned}\int \frac{\mathrm{d}x}{\sqrt{x^2-a^2}} &= \int \frac{a\sec\theta\tan\theta\mathrm{d}\theta}{a\tan\theta} = \int \sec\theta\mathrm{d}\theta \\ &= \ln|\sec\theta + \tan\theta| + C\end{aligned}$$

Example

Find $\int \frac{dx}{\sqrt{x^2-a^2}}$, $a > 0$.



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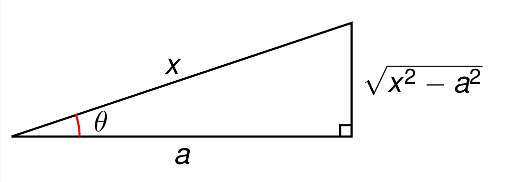

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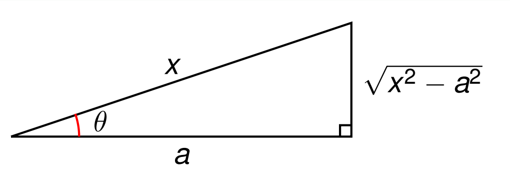

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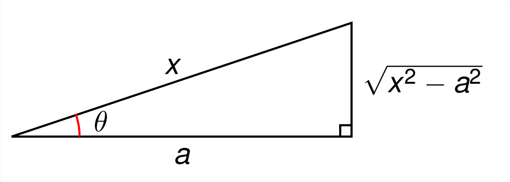
Find $\int \frac{\mathrm{d}x}{\sqrt{x^2-a^2}}$, $a > 0$.



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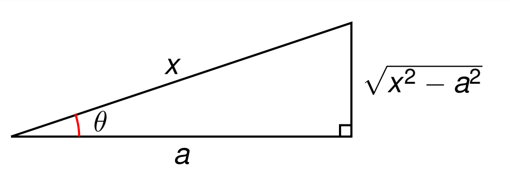
Example

Find $\int \frac{dx}{\sqrt{x^2-a^2}}$, $a > 0$.



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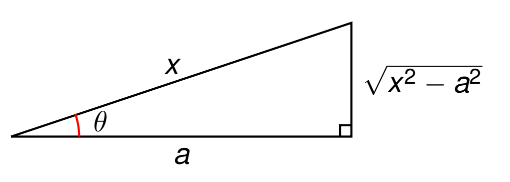
Find $\int \frac{\mathrm{d}x}{\sqrt{x^2-a^2}}$, $a > 0$.



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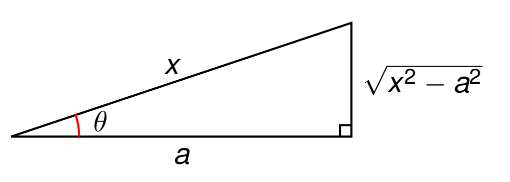
$$\begin{aligned}
\int \frac{\mathrm{d}x}{\sqrt{x^2-a^2}} &= \int \frac{a\sec\theta\tan\theta \mathrm{d}\theta}{a\tan\theta} = \int \sec\theta \mathrm{d}\theta \\
&= \ln|\sec\theta + \tan\theta| + C = \ln\left|\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}\right| + C \\
&= \ln\left|x + \sqrt{x^2-a^2}\right| - \ln a + C
\end{aligned}$$

Example

Find $\int \frac{\mathrm{d}x}{\sqrt{x^2-a^2}}$, $a > 0$.



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$$
\begin{aligned}
\int \frac{\mathrm{d}x}{\sqrt{x^2-a^2}} &= \int \frac{a\sec\theta\tan\theta\mathrm{d}\theta}{a\tan\theta} = \int \sec\theta\mathrm{d}\theta \\
&= \ln|\sec\theta + \tan\theta| + C = \ln\left|\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}\right| + C \\
&= \ln\left|x + \sqrt{x^2-a^2}\right| - \ln a + C
\end{aligned}
$$

Example

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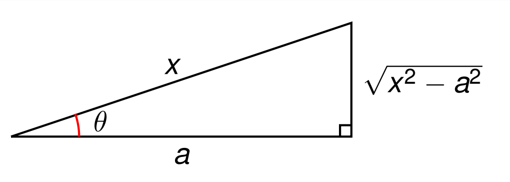

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$$\begin{aligned}
\int \frac{\mathrm{d}x}{\sqrt{x^2-a^2}} &= \int \frac{a\sec\theta\tan\theta\mathrm{d}\theta}{a\tan\theta} = \int \sec\theta\mathrm{d}\theta \\
&= \ln|\sec\theta+\tan\theta| + C = \ln\left|\frac{x}{a}+\frac{\sqrt{x^2-a^2}}{a}\right| + C \\
&= \ln\left|x+\sqrt{x^2-a^2}\right| + C_1
\end{aligned}$$

Euler substitution $x = \sec\theta$, $\theta = 2\arctan t$

- $x = \sec\theta$ transforms $\mathrm{d}x, x, \sqrt{x^2-1}$ to trig form.
- $\theta = 2\arctan t$, rationalizes $\mathrm{d}\theta, \cos\theta, \sin\theta$.

Euler substitution $x = \sec\theta$, $\theta = 2\arctan t$

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What if we compose the above?

Euler substitution $x = \sec\theta$, $\theta = 2\arctan t$

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What if we compose the above? We get the Euler substitution:

$$x \quad =$$

Euler substitution $x = \sec\theta$, $\theta = 2\arctan t$

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What if we compose the above? We get the Euler substitution:

$$x = \sec\theta = \frac{1}{\cos\theta}$$

Euler substitution $x = \sec\theta$, $\theta = 2\arctan t$

- $x = \sec\theta$ transforms $\mathrm{d}x, x, \sqrt{x^2-1}$ to trig form.
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What if we compose the above? We get the Euler substitution:

$$\begin{aligned} x &= \sec\theta = \frac{1}{\cos\theta} \\ &= \frac{1}{\cos(2\arctan t)} \end{aligned}$$

Euler substitution $x = \sec\theta$, $\theta = 2\arctan t$

- $x = \sec\theta$ transforms $\mathrm{d}x, x, \sqrt{x^2-1}$ to trig form.
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$$\begin{aligned} x &= \sec\theta = \frac{1}{\cos\theta} \\ &= \frac{1}{\cos(2\arctan t)} \end{aligned}$$

$\cos(2z) =$

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$$\begin{aligned} x &= \sec\theta = \frac{1}{\cos\theta} \\ &= \frac{1}{\cos(2\arctan t)} \end{aligned}$$

$$\cos(2z) = \frac{1-\tan^2 z}{1+\tan^2 z}$$

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What if we compose the above? We get the Euler substitution:

$$
\begin{aligned}
x &= \sec\theta = \frac{1}{\cos\theta} \\
&= \frac{1}{\cos(2\arctan t)} \\
&= \frac{1+\tan^2(\arctan t)}{1-\tan^2(\arctan t)}
\end{aligned}
$$

$$\left| \cos(2z) = \frac{1-\tan^2 z}{1+\tan^2 z} \right.$$

Euler substitution $x = \sec\theta$, $\theta = 2\arctan t$

- $x = \sec\theta$ transforms $\mathrm{d}x, x, \sqrt{x^2-1}$ to trig form.
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What if we compose the above? We get the Euler substitution:

$$\begin{aligned} x &= \sec\theta = \frac{1}{\cos\theta} \\ &= \frac{1}{\cos(2\arctan t)} \\ &= \frac{1+\tan^2(\arctan t)}{1-\tan^2(\arctan t)} \\ &= \frac{1+t^2}{1-t^2} \end{aligned}$$

$$\cos(2z) = \frac{1-\tan^2 z}{1+\tan^2 z}$$

Euler substitution $x = \sec\theta$, $\theta = 2\arctan t$

- $x = \sec\theta$ transforms $\mathrm{d}x, x, \sqrt{x^2-1}$ to trig form.
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What if we compose the above? We get the Euler substitution:

$$\begin{aligned} x &= \sec\theta = \frac{1}{\cos\theta} \\ &= \frac{1}{\cos(2\arctan t)} \\ &= \frac{1+\tan^2(\arctan t)}{1-\tan^2(\arctan t)} \\ &= \frac{1+t^2}{1-t^2} = \frac{2-(1-t^2)}{1-t^2} \end{aligned}$$

$$\left| \cos(2z) = \frac{1-\tan^2 z}{1+\tan^2 z} \right.$$

Euler substitution $x = \sec\theta$, $\theta = 2\arctan t$

- $x = \sec\theta$ transforms $\mathrm{d}x, x, \sqrt{x^2-1}$ to trig form.
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$$\begin{aligned} x &= \sec\theta = \frac{1}{\cos\theta} \\ &= \frac{1}{\cos(2\arctan t)} \\ &= \frac{1+\tan^2(\arctan t)}{1-\tan^2(\arctan t)} \\ &= \frac{1+t^2}{1-t^2} = \frac{2-(1-t^2)}{1-t^2} \\ &= -1 + \frac{2}{1-t^2} \end{aligned}$$

$$\cos(2z) = \frac{1-\tan^2 z}{1+\tan^2 z}$$

Euler substitution $x = \sec\theta$, $\theta = 2\arctan t$

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- $\theta = 2\arctan t,\ \ t \in (-\infty, -1) \cup [0, 1)$ rationalizes $\mathrm{d}\theta, \cos\theta, \sin\theta$.

What if we compose the above? We get the Euler substitution:

$$\begin{aligned} x &= \sec\theta = \frac{1}{\cos\theta} \\ &= \frac{1}{\cos(2\arctan t)} \\ &= \frac{1+\tan^2(\arctan t)}{1-\tan^2(\arctan t)} \\ &= \frac{1+t^2}{1-t^2} = \frac{2-(1-t^2)}{1-t^2} \\ &= -1 + \frac{2}{1-t^2} \end{aligned}$$

$$\cos(2z) = \frac{1-\tan^2 z}{1+\tan^2 z}$$

Euler substitution $x = \sec\theta$, $\theta = 2\arctan t$

- $x = \sec\theta$ transforms $\mathrm{d}x, x, \sqrt{x^2-1}$ to trig form.
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What if we compose the above? We get the Euler substitution:

$$x = -1 + \frac{2}{1-t^2}$$

$$\sqrt{x^2-1} =$$

Euler substitution $x = \sec\theta$, $\theta = 2\arctan t$

- $x = \sec\theta$ transforms $\mathrm{d}x, x, \sqrt{x^2-1}$ to trig form.
- $\theta = 2\arctan t,\ \ t \in (-\infty, -1) \cup [0, 1)$ rationalizes $\mathrm{d}\theta, \cos\theta, \sin\theta$.

What if we compose the above? We get the Euler substitution:

$$
\begin{aligned}
x &= -1 + \frac{2}{1-t^2} \\
\sqrt{x^2-1} &= \sqrt{\left(\frac{1+t^2}{1-t^2}\right)^2 - 1}
\end{aligned}
$$

Euler substitution $x = \sec\theta$, $\theta = 2\arctan t$

- $x = \sec\theta$ transforms $\mathrm{d}x, x, \sqrt{x^2-1}$ to trig form.
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$$x = -1 + \frac{2}{1-t^2}$$

$$\begin{aligned}
\sqrt{x^2-1} &= \sqrt{\left(\frac{1+t^2}{1-t^2}\right)^2 - 1} \\
&= \sqrt{\frac{(1+t^2)^2 - (1-t^2)^2}{(1-t^2)^2}}
\end{aligned}$$

Euler substitution $x = \sec\theta$, $\theta = 2\arctan t$

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$$\begin{aligned}
\sqrt{x^2-1} &= \sqrt{\left(\frac{1+t^2}{1-t^2}\right)^2 - 1} \\
&= \sqrt{\frac{(1+t^2)^2-(1-t^2)^2}{(1-t^2)^2}} \qquad \left| \; (1+t^2)^2-(1-t^2)^2 = \textbf{?} \right.
\end{aligned}$$

Euler substitution $x = \sec\theta$, $\theta = 2\arctan t$

- $x = \sec\theta$ transforms $\mathrm{d}x, x, \sqrt{x^2-1}$ to trig form.
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$$x = -1 + \frac{2}{1-t^2}$$

$$\begin{aligned}\sqrt{x^2-1} &= \sqrt{\left(\frac{1+t^2}{1-t^2}\right)^2 - 1}\\ &= \sqrt{\frac{(1+t^2)^2-(1-t^2)^2}{(1-t^2)^2}} && \left|\ (1+t^2)^2-(1-t^2)^2 = 4t^2\right.\end{aligned}$$

Euler substitution $x = \sec\theta$, $\theta = 2\arctan t$

- $x = \sec\theta$ transforms $\mathrm{d}x, x, \sqrt{x^2-1}$ to trig form.
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&= \sqrt{\frac{4t^2}{(1-t^2)^2}}
\end{aligned}$$

Euler substitution $x = \sec\theta$, $\theta = 2\arctan t$

- $x = \sec\theta$ transforms $\mathrm{d}x, x, \sqrt{x^2-1}$ to trig form.
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\begin{aligned}
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\hline
\sqrt{x^2-1} &= \sqrt{\left(\frac{1+t^2}{1-t^2}\right)^2 - 1} \\
&= \sqrt{\frac{(1+t^2)^2-(1-t^2)^2}{(1-t^2)^2}} && \Big| \; (1+t^2)^2-(1-t^2)^2 = 4t^2 \\
&= \sqrt{\frac{4t^2}{(1-t^2)^2}} && \Big| \; t, 1-t^2 \text{ have same sign when } t \in (-\infty,-1)\cup[0,1) \\
&= \frac{2t}{1-t^2}
\end{aligned}
$$

Euler substitution $x = \sec\theta$, $\theta = 2\arctan t$

- $x = \sec\theta$ transforms $\mathrm{d}x, x, \sqrt{x^2-1}$ to trig form.
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What if we compose the above? We get the Euler substitution:

$$x = -1 + \frac{2}{1-t^2}$$

$$\sqrt{x^2-1} = \frac{2t}{1-t^2}$$

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$$x = \frac{1+t^2}{1-t^2}$$

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- $x = \sec\theta$ transforms $\mathrm{d}x, x, \sqrt{x^2-1}$ to trig form.
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What if we compose the above? We get the Euler substitution:

$$
\begin{aligned}
x &= -1 + \frac{2}{1-t^2}\\
\sqrt{x^2-1} &= \frac{2t}{1-t^2}\\
x &= \frac{1+t^2}{\color{red}{1-t^2}}\\
(\color{red}{1-t^2})x &= 1+t^2
\end{aligned}
$$

Euler substitution $x = \sec\theta$, $\theta = 2\arctan t$

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$$x = -1 + \frac{2}{1-t^2}$$

$$\sqrt{x^2-1} = \frac{2t}{1-t^2}$$

$$\begin{aligned} x &= \frac{1+t^2}{1-t^2} \\ (1-t^2)x &= 1+t^2 \\ (1+x)t^2 &= x-1 \end{aligned}$$

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$$\sqrt{x^2-1} = \frac{2t}{1-t^2}$$

$$\begin{aligned} x &= \frac{1+t^2}{1-t^2} \\ (1-t^2)x &= 1+t^2 \\ {\color{red}(1+x)}t^2 &= x-1 \\ t^2 &= \frac{x-1}{\color{red}x+1} \end{aligned}$$

Euler substitution $x = \sec\theta$, $\theta = 2\arctan t$

- $x = \sec\theta$ transforms $\mathrm{d}x, x, \sqrt{x^2-1}$ to trig form.
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$$\begin{aligned}
x &= \frac{1+t^2}{1-t^2}\\
(1-t^2)x &= 1+t^2\\
(1+x)t^2 &= x-1\\
t^2 &= \frac{x-1}{x+1}\\
t &= \pm\sqrt{\frac{x-1}{x+1}}
\end{aligned}$$

Euler substitution $x = \sec\theta$, $\theta = 2\arctan t$

- $x = \sec\theta$ transforms $\mathrm{d}x, x, \sqrt{x^2-1}$ to trig form.
- $\theta = 2\arctan t,\ \ t \in (-\infty, -1) \cup [0, 1)$ rationalizes $\mathrm{d}\theta, \cos\theta, \sin\theta$.

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$$\mathrm{d}x =$$

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$$\sqrt{x^2-1} = \frac{2t}{1-t^2}$$

$$t = \pm\sqrt{\frac{x-1}{x+1}}$$

$$\mathrm{d}x = \mathrm{d}\left(-1 + \frac{2}{1-t^2}\right)$$

Euler substitution $x = \sec\theta$, $\theta = 2\arctan t$

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What if we compose the above? We get the Euler substitution:

$$
\begin{aligned}
x &= -1 + \frac{2}{1-t^2} \\
\sqrt{x^2-1} &= \frac{2t}{1-t^2} \\
t &= \pm\sqrt{\frac{x-1}{x+1}} \\
\mathrm{d}x &= \mathrm{d}\left(-1 + \frac{2}{1-t^2}\right) \\
&= \textbf{?}
\end{aligned}
$$

Euler substitution $x = \sec\theta$, $\theta = 2\arctan t$

- $x = \sec\theta$ transforms $\mathrm{d}x, x, \sqrt{x^2-1}$ to trig form.
- $\theta = 2\arctan t$, $t \in (-\infty, -1) \cup [0, 1)$ rationalizes $\mathrm{d}\theta, \cos\theta, \sin\theta$.

What if we compose the above? We get the Euler substitution:

$$x = -1 + \frac{2}{1-t^2}$$

$$\sqrt{x^2-1} = \frac{2t}{1-t^2}$$

$$t = \pm\sqrt{\frac{x-1}{x+1}}$$

$$\mathrm{d}x = \mathrm{d}\left(-1 + \frac{2}{1-t^2}\right) = \frac{4t}{(1-t^2)^2}\mathrm{d}t$$

Euler substitution $x = \sec\theta$, $\theta = 2\arctan t$

- $x = \sec\theta$ transforms $\mathrm{d}x, x, \sqrt{x^2-1}$ to trig form.
- $\theta = 2\arctan t$, $t \in (-\infty, -1) \cup [0, 1)$ rationalizes $\mathrm{d}\theta, \cos\theta, \sin\theta$.

What if we compose the above? We get the Euler substitution:

$$x = -1 + \frac{2}{1-t^2}$$

$$\sqrt{x^2-1} = \frac{2t}{1-t^2}$$

$$t = \pm\sqrt{\frac{x-1}{x+1}}$$

$$\begin{aligned} \mathrm{d}x &= \mathrm{d}\left(-1 + \frac{2}{1-t^2}\right) \\ &= \frac{4t}{(1-t^2)^2}\mathrm{d}t \end{aligned}$$

Euler substitution $x = \sec\theta$, $\theta = 2\arctan t$

- $x = \sec\theta$ transforms $\mathrm{d}x, x, \sqrt{x^2-1}$ to trig form.
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What if we compose the above? We get the Euler substitution:

Definition

The Euler substitution for $\sqrt{x^2-1}$ corresponding to $x = \sec\theta$ is given by:

$$\begin{aligned} x &= \frac{1+t^2}{1-t^2}, \qquad t \in (-\infty, -1) \cup [0, 1) \\ \sqrt{x^2-1} &= \frac{2t}{1-t^2} \\ \mathrm{d}x &= \frac{4t}{(1-t^2)^2}\mathrm{d}t \\ t &= \pm\frac{\sqrt{x^2-1}}{x+1} \quad . \end{aligned}$$

Rationalizing Substitutions

Some non-rational fractions can be changed into rational fractions by means of appropriate substitutions. In particular, when an integrand contains an expression of the form $\sqrt[n]{g(x)}$, the substitution $u = \sqrt[n]{g(x)}$ may be effective.

Example

$$\int \frac{\sqrt{x+4}}{x} \mathrm{d}x$$

Example

Let $u = \sqrt{x+4}$. Then $u^2 = x + 4$, so $x = \mathbf{?}$ and $\mathrm{d}x = \mathbf{?}$.

$$\int \frac{\sqrt{x+4}}{x} \mathrm{d}x = \int \frac{\mathbf{?}}{\mathbf{?}} \mathbf{?}$$

Example

Let $u = \sqrt{x+4}$. Then $u^2 = x + 4$, so $x = \textbf{?}$ and $\mathrm{d}x = \textbf{?}$.

$$\int \frac{\sqrt{x+4}}{x}\mathrm{d}x = \int \frac{u}{\textbf{?}}\textbf{?}$$

Example

Let $u = \sqrt{x+4}$. Then $u^2 = x + 4$, so $x = \textbf{?}$ and $\mathrm{d}x = \textbf{?}$.

$$\int \frac{\sqrt{x+4}}{x}\mathrm{d}x = \int \frac{u}{\textbf{?}}\textbf{?}$$

Example

Let $u = \sqrt{x+4}$. Then $u^2 = x + 4$, so $x = u^2 - 4$ and $\mathrm{d}x = \textbf{?}$.

$$\int \frac{\sqrt{x+4}}{x}\mathrm{d}x = \int \frac{u}{u^2-4}\textbf{?}$$

Example

Let $u = \sqrt{x+4}$. Then $u^2 = x + 4$, so $x = u^2 - 4$ and $dx = ?$.

$$\int \frac{\sqrt{x+4}}{x} dx = \int \frac{u}{u^2-4} ?$$

Example

Let $u = \sqrt{x+4}$. Then $u^2 = x + 4$, so $x = u^2 - 4$ and $\mathrm{d}x = 2u\mathrm{d}u$.

$$\int \frac{\sqrt{x+4}}{x}\mathrm{d}x = \int \frac{u}{u^2-4} 2u\mathrm{d}u$$

Example

Let $u=\sqrt{x+4}$. Then $u^2=x+4$, so $x=u^2-4$ and $\mathrm{d}x=2u\mathrm{d}u$.

$$\begin{aligned}\int\frac{\sqrt{x+4}}{x}\mathrm{d}x &= \int\frac{u}{u^2-4}2u\mathrm{d}u\\ &= 2\int\frac{u^2}{u^2-4}\mathrm{d}u\end{aligned}$$

Example

Let $u = \sqrt{x+4}$. Then $u^2 = x + 4$, so $x = u^2 - 4$ and $\mathrm{d}x = 2u\mathrm{d}u$.

$$\begin{aligned}
\int \frac{\sqrt{x+4}}{x}\mathrm{d}x &= \int \frac{u}{u^2-4} 2u\mathrm{d}u \\
&= 2\int \frac{u^2}{u^2-4}\mathrm{d}u \\
&= 2\int \left(1 + \frac{4}{u^2-4}\right)\mathrm{d}u && \Big| \text{ long division}
\end{aligned}$$

Example

Let $u = \sqrt{x+4}$. Then $u^2 = x + 4$, so $x = u^2 - 4$ and $\mathrm{d}x = 2u\mathrm{d}u$.

$$\begin{aligned}
\int \frac{\sqrt{x+4}}{x}\mathrm{d}x &= \int \frac{u}{u^2-4} 2u\mathrm{d}u \\
&= 2\int \frac{u^2}{u^2-4}\mathrm{d}u \\
&= 2\int \left(1 + \frac{4}{u^2-4}\right)\mathrm{d}u && \Big| \text{ long division} \\
&= 2\int \mathrm{d}u + 8\int \frac{\mathrm{d}u}{u^2-4}
\end{aligned}$$

Example

Let $u = \sqrt{x+4}$. Then $u^2 = x+4$, so $x = u^2 - 4$ and $\mathrm{d}x = 2u\mathrm{d}u$.

$$
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\int \frac{\sqrt{x+4}}{x}\mathrm{d}x &= \int \frac{u}{u^2-4} 2u\mathrm{d}u \\
&= 2\int \frac{u^2}{u^2-4}\mathrm{d}u \\
&= 2\int \left(1 + \frac{4}{u^2-4}\right)\mathrm{d}u && \Big|\ \text{long division} \\
&= 2\int \mathrm{d}u + 8\int \frac{\mathrm{d}u}{u^2-4} \\
&= 2\int \mathrm{d}u + 8\int \left(\frac{\frac{1}{4}}{u-2} - \frac{\frac{1}{4}}{u+2}\right)\mathrm{d}u && \Big|\ \text{partial fractions}
\end{aligned}
$$

Example

Let $u = \sqrt{x+4}$. Then $u^2 = x + 4$, so $x = u^2 - 4$ and $\mathrm{d}x = 2u\mathrm{d}u$.

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&= 2\int \frac{u^2}{u^2-4}\mathrm{d}u \\
&= 2\int \left(1 + \frac{4}{u^2-4}\right)\mathrm{d}u && \Big| \text{ long division} \\
&= 2\int \mathrm{d}u + 8\int \frac{\mathrm{d}u}{u^2-4} \\
&= 2\int \mathrm{d}u + 8\int \left(\frac{\frac{1}{4}}{u-2} - \frac{\frac{1}{4}}{u+2}\right)\mathrm{d}u && \Big| \text{ partial fractions} \\
&= 2u + 2(\ln|u-2| - \ln|u+2|) + C
\end{aligned}$$

Example

Let $u = \sqrt{x+4}$. Then $u^2 = x+4$, so $x = u^2 - 4$ and $\mathrm{d}x = 2u\mathrm{d}u$.

$$
\begin{aligned}
\int \frac{\sqrt{x+4}}{x}\mathrm{d}x &= \int \frac{u}{u^2-4} 2u\mathrm{d}u \\
&= 2\int \frac{u^2}{u^2-4}\mathrm{d}u \\
&= 2\int \left(1 + \frac{4}{u^2-4}\right)\mathrm{d}u && \Big|\ \text{long division} \\
&= 2\int \mathrm{d}u + 8\int \frac{\mathrm{d}u}{u^2-4} \\
&= 2\int \mathrm{d}u + 8\int \left(\frac{\frac{1}{4}}{u-2} - \frac{\frac{1}{4}}{u+2}\right)\mathrm{d}u && \Big|\ \text{partial fractions} \\
&= 2u + 2(\ln|u-2| - \ln|u+2|) + C \\
&= 2\sqrt{x+4} + 2\ln\left|\frac{\sqrt{x+4}-2}{\sqrt{x+4}+2}\right| + C
\end{aligned}
$$