Calculus I
Derivative of rational function, part 2

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Theorem (The Quotient Rule)

If f and g are differentiable and $g(x) \neq 0$, then

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f(x)}{g(x)}\right) = \frac{\frac{\mathrm{d}}{\mathrm{d}x}\left(f(x)\right)g(x) - f(x)\frac{\mathrm{d}}{\mathrm{d}x}\left(g(x)\right)}{\left(g(x)\right)^2}$$

(Leibniz notation)

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\end{aligned}$$

- The proof of the Quotient Rule is similar to the proof of the Product Rule.
- There is an alternative algebraic proof via the Product Rule, the Power Rule and the (not yet studied) Chain Rule.

Example (Quotient Rule, rational function)

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$$= \frac{\left(\textbf{?}\qquad\right)\left(-x^6 + 2\right) - \left(x^5 + 2x\right)\left(\textbf{?}\qquad\right)}{\left(-x^6 + 2\right)^2}$$

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$$= \frac{\left(5x^4 + 2\right)\left(-x^6 + 2\right) - \left(x^5 + 2x\right)\left(\textbf{?} \qquad\right)}{\left(-x^6 + 2\right)^2}$$

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$$= \frac{(-5x^{10} - 2x^6 + 10x^4 + 4) - (-6x^{10} - 12x^6)}{(-x^6+2)^2}$$

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&= \frac{\left(5x^4 + 2\right)\left(-x^6 + 2\right) - \left(x^5 + 2x\right)\left(-6x^5\right)}{\left(-x^6 + 2\right)^2} \\
&= \frac{(-5x^{10} - 2x^6 + 10x^4 + 4) - (-6x^{10} - 12x^6)}{(-x^6 + 2)^2} \\
&= \frac{x^{10} + 10x^6 + 10x^4 + 4}{(-x^6 + 2)^2}.
\end{aligned}$$