Precalculus
, Factorization of polynomials: overview

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2019

Outline

Recall that $i^2 = -1$, $\sqrt{-1} = i$.

Example (Polynomial factorizations)

$$2x^2 + 3x - 5 \quad =$$

$$x^2 + 1 \quad =$$

$$x^4 - 1 \quad =$$

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Every polynomial can be factored into product of linear terms

$$p(x) = a_0x^n + a_1x^{n-1} + \dots + a_n = a_0(x - x_1)\dots(x - x_n),$$

where $x_1, \dots, x_n$ *are the (not necessarily different) roots of* $p(x)$.

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Corollary

Every real polynomial can be factored into a product of real linear terms and real quadratic terms with no real roots, i.e., factors of form

- $(x - r)$, *where r is real and*
- $ax^2 + bx + c$ *with $b^2 - 4ac < 0$ where a, b, c are real.*

$$a_0x^n + a_1x^{n-1} + \dots + a_n = a_0(x - x_1)\dots(x - x_n)$$
$$= \text{prod. real quadratics no roots \& lin. terms.}$$

Example

$$2x^2 + 3x - 5 = (2x+5)(x-1) = 2\left(x - \left(-\tfrac{5}{2}\right)\right)(x-1)$$

$$x^2 + 1 = x^2 - (-1) = x^2 - i^2 = (x - (-i))(x - i)$$

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$=$prod. real quadratics no roots & lin. terms.

Example

$$2x^2+3x-5=(2x+5)(x-1)=2\left(x-\left(-\tfrac{5}{2}\right)\right)(x-1)$$ real roots

$$x^2+1=x^2-(-1)=x^2-i^2=(x-(-i))(x-i)$$ complex roots

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- In theory every polynomial can be factored.

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- Yes, with extra operations. Difficult: google Galois Theory to get started.

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These poly’s are equal	Type of factorization
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- We study those for cubics with the aid of scientific calculator.