Calculus I
Inverse functions

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Outline

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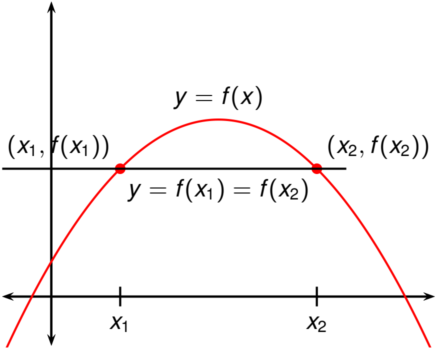
One-to-one Functions

Definition (One-to-one Function)

A function f is a one-to-one function if it never takes on the same value twice; that is,

$$f(x_1) \neq f(x_2) \text{ whenever } x_1 \neq x_2.$$

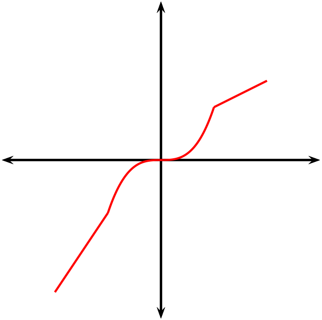


$\leftarrow$ This function is not one-to-one.

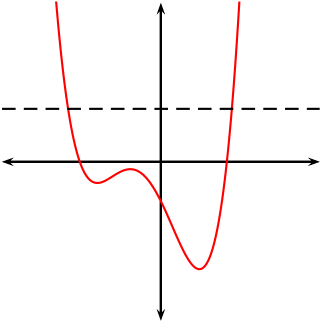
Question: How can we tell from the graph of a function whether it is one-to-one or not?
Answer: Use the horizontal line test.

The Horizontal Line Test.

A function is one-to-one if and only if no horizontal line intersects it more than once. □



One-to-one



Not one-to-one

The Definition of the Inverse of f

Definition (f^{-1})

Let f be a one-to-one function with domain A and range B. Then the inverse of f is the function f^{-1} that has domain B and range A and is defined by

$$f^{-1}(y) = x \qquad \Leftrightarrow \qquad f(x) = y$$

for all y in B.

Note:

- Only one-to-one functions have inverses.
- f^{-1} reverses the effect of f.
- domain of f^{-1} = range of f.
- range of f^{-1} = domain of f.

Example ($f(x) = x^3$)

The inverse of $f(x) = x^3$ is $f^{-1}(x) = \sqrt[3]{x}$. This is because if $y = x^3$, then

$$f^{-1}(y) = \sqrt[3]{y} = \sqrt[3]{x^3} = x.$$

The inverse of f is denoted as f^{-1}. This notation is one of the most frequent causes of student confusion. **WARNING:**

$$f^{-1}(x) \text{ does not mean } (f(x))^{-1} = \frac{1}{f(x)}.$$

The notations are different: the superscript -1 has different positions.

- f^{-1} is the compositional inverse of f.
- $\frac{1}{f(x)}$ is the multiplicative inverse of $f(x)$.
- $f^2(x)$ is an abbreviation for $(f(x))^2$, $f^3(x)$ is an abbreviation of $(f(x))^3$, and so on.
- However, $f^{-1}(x)$ is not the abbreviation of $(f(x))^{-1}$ and does not follow this pattern.

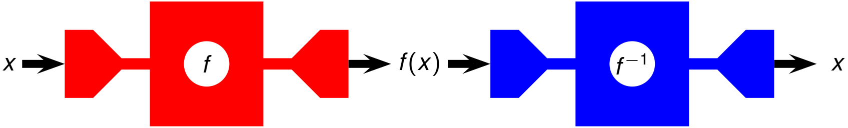
$$f^n(x) = \begin{cases} \text{stands for } (f(x))^n & \text{when } n = 1, 2, 3, \dots \\ \text{stands for inverse of } f \text{ applied to } x & \text{when } n = -1 \\ \text{should be avoided} & \text{when } n \neq -1, 1, 2, 3, \dots. \end{cases}$$

To reduce confusion, if possible, use $\frac{1}{f(x)}$ instead of $(f(x))^{-1}$.

$$f^{-1}(y) = x \qquad \Leftrightarrow \qquad f(x) = y.$$

Therefore

$$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = f^{-1}(y) = x.$$



Switch the roles of x and y:

$$f^{-1}(x) = y \qquad \Leftrightarrow \qquad f(y) = x.$$

Therefore

$$(f \circ f^{-1})(x) = f(f^{-1}(x)) = f(y) = x.$$

How to Find the Inverse of a One-to-one Function

1. Write $y = f(x)$.
2. Solve this equation for x in terms of y (if possible).

Example

If $f(x) = x^3 + 2$, find a formula for $f^{-1}(y)$.

$$\begin{aligned} y &= x^3 + 2 \\ x^3 &= y - 2 \\ x &= \sqrt[3]{y-2} \end{aligned}$$

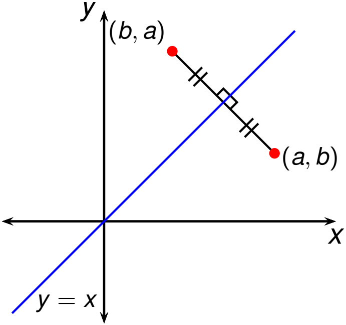
Therefore $x = f^{-1}(y) = \sqrt[3]{y-2}$. Sometimes we relabel x and y and write $f^{-1}(x) = \sqrt[3]{x-2}$. Whenever in doubt, do not relabel anything.

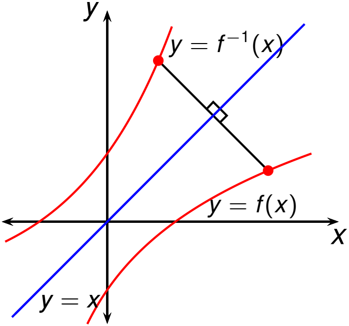
Example (Guess and Check)

If $f(x) = 2x + \sin 2x + e^{\frac{x}{2}}$, find $f^{-1}(1)$. You do not need to show that f has an inverse.

$$\begin{aligned} f(\quad) &= 2(\quad) + \sin 2(\quad) + e^{\frac{(\ )}{2}} \\ &= \\ &= 1. \end{aligned}$$

Therefore $\quad f^{-1}(1) =$

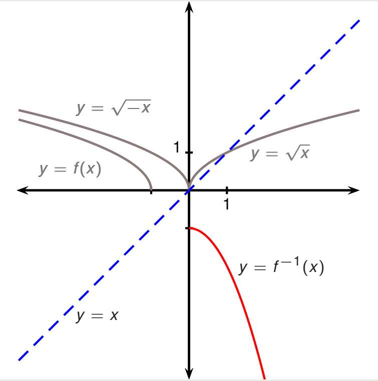




Interchanging x and y suggests relation between the graphs of f^{-1} and f:

- Suppose (a, b) is on the graph of f.
- Then $f(a) = b$.
- Then $f^{-1}(b) = a$.
- Then (b, a) is on the graph of f^{-1}.
- (b, a) is the reflection of (a, b) in the line $y = x$.
- Thus the graph of f^{-1} is obtained by reflecting the graph of f in the line $y = x$.

Example

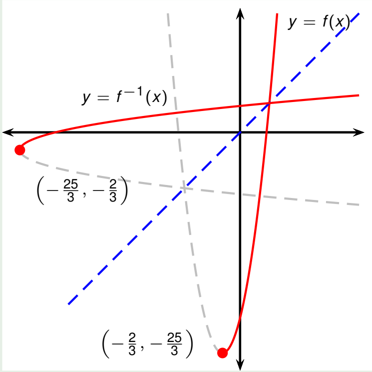


Sketch the graph of $f(x) = \sqrt{-x-1}$ and its inverse function.

- Draw the graph of $y = \sqrt{x}$.
- $y = \sqrt{-x}$ is the reflection of $y = \sqrt{x}$ in the y-axis.
- $y = f(x) = \sqrt{-(x+1)} = \sqrt{-x-1}$ is the shift of $y = \sqrt{-x}$ one unit to the left.
- $y = f^{-1}(x)$ is the reflection of $y = f(x)$ across the line $y = x$.

Example ( )

Given: $f(x) = 3x^2 + 4x - 7$ with domain $x \geq -\frac{2}{3}$. Find $f^{-1}(x)$.



$$\begin{aligned} 3x^2 + 4x - 7 &= y \\ 3x^2 + 4x + (-7 - y) &= 0 \end{aligned}$$

That's a quadratic equation in x. Solve:

$$\frac{-4 \pm \sqrt{4^2 - 4 \cdot 3 \cdot (-y - 7)}}{2 \cdot 3}$$

$$= -\frac{2 \pm \sqrt{25 + 3y}}{3} = -\frac{2}{3} \pm \frac{\sqrt{25 + 3y}}{3}$$

We are given $x \geq -\frac{2}{3}$, therefore

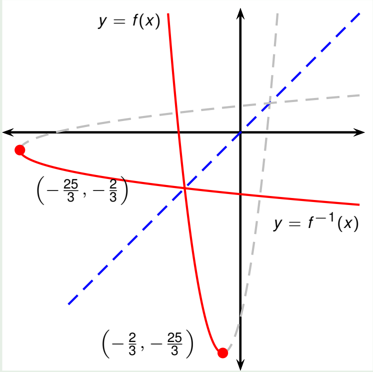
$$x = -\frac{2}{3} + \frac{\sqrt{25+3y}}{3} = f^{-1}(y).$$

Final answer, relabelled:

$$f^{-1}(x) = -\frac{2}{3} + \frac{\sqrt{25 + 3x}}{3}$$

Example (What if we change the problem to $x \leq -\frac{2}{3}$?)

Given: $f(x) = 3x^2 + 4x - 7$ with domain $x \leq -\frac{2}{3}$. Find $f^{-1}(x)$.



$$\begin{array}{rcl} 3x^2 + 4x - 7 &=& y \\ 3x^2 + 4x + (-7 - y) &=& 0 \end{array}$$

That's a quadratic equation in x. Solve:

$$\frac{-4 \pm \sqrt{4^2 - 4 \cdot 3 \cdot (-y - 7)}}{2 \cdot 3}$$

$$= -\frac{2 \pm \sqrt{25 + 3y}}{3} = -\frac{2}{3} \pm \frac{\sqrt{25 + 3y}}{3}$$

We are given $x \leq -\frac{2}{3}$, therefore

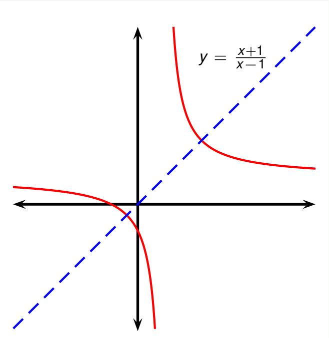
$$x = -\frac{2}{3} - \frac{\sqrt{25+3y}}{3} = f^{-1}(y).$$

Final answer, relabelled:

$$f^{-1}(x) = -\frac{2}{3} - \frac{\sqrt{25 + 3x}}{3}$$

Example

Find $f^{-1}(x)$ where $f(x) = \frac{x+1}{x-1}$.



Answer: $f^{-1}(x) = \frac{x+1}{x-1}$, $x \neq 1$.

We deal with domains and ranges later:

$$\begin{array}{rcl|l} y & = & \dfrac{x+1}{x-1} & \text{mult. by } (x-1) \\ y(x-1) & = & x+1 & \\ x(y-1) & = & y+1 & \text{div. by } (y-1) \\ f^{-1}(y) = x & = & \dfrac{y+1}{y-1} & \text{relabel } x, y \\ f^{-1}(x) & = & \dfrac{x+1}{x-1} & \end{array}$$

We divided by $y - 1$ so $y \neq 1$. Therefore the domain of f^{-1} is all real numbers except 1.

Can a non-identity function be its own inverse? Yes, f is.
What does it mean for f to be its own inverse?
Graph of f is symmetric across $y = x$.