Precalculus
Homework
Inverse functions

1. Find the inverse function. You are asked to do the algebra only; you are not asked to determine the domain or range of the function or its inverse.

   (a) $f(x) = 3x^2 + 4x - 7$, where $x \geq -\frac{2}{3}$.

   answer: $f^{-1}(x) = -\frac{2}{3} + \frac{\sqrt{25+3x}}{3}$, $x \geq -\frac{25}{3}$

   (b) $f(x) = 2x^2 + 3x - 5$, where $x \geq -\frac{3}{4}$.

   answer: $f^{-1}(x) = -\frac{3}{4} + \frac{\sqrt{49+8x}}{4}$, $x \geq -\frac{49}{8}$

   (c) $f(x) = \dfrac{2x+5}{x-4}$, where $x \neq 4$.

   answer: $f^{-1}(x) = \frac{4x+5}{x-2}$, $x \neq 2$

   (d) $f(x) = \dfrac{3x+5}{2x-4}$, where $x \neq 2$.

   answer: $f^{-1}(x) = \dfrac{4x+5}{2x-3}$, $x \neq \dfrac{3}{2}$

   (e) $f(x) = \dfrac{5x+6}{4x+5}$, where $x \neq -\frac{5}{4}$.

   answer: $f^{-1}(x) = \frac{-5x+6}{4x-5}$, $x \neq \frac{5}{4}$

   (f) $f(x) = \dfrac{2x-3}{-3x+4}$, where $x \neq \frac{4}{3}$..

   answer: $f^{-1}(x) = \frac{4x+3}{3x+2}$, $x \neq -\frac{2}{3}$

Solution. 1.d This is a concise solution written in form suitable for test taking.

$$\begin{array}{rrcl}
 & y & = & \dfrac{3x+5}{2x-4} \\
 & y(2x-4) & = & 3x+5 \\
 & 2xy-4y & = & 3x+5 \\
 & 2xy-3x & = & 4y+5 \\
 & x(2y-3) & = & 4y+5 \\
 & x & = & \dfrac{4y+5}{2y-3} \\
\text{Therefore} & f^{-1}(y) & = & \dfrac{5+4y}{2y-3} \\
 & f^{-1}(x) & = & \dfrac{5+4x}{2x-3}.
\end{array}$$

Solution. 1.e. Set $f(x) = y$. Then

$$\begin{array}{rcl}
y & = & \dfrac{5x+6}{4x+5} \\
y(4x+5) & = & 5x+6 \\
x(4y-5) & = & -5y+6 \\
x & = & \dfrac{-5y+6}{4y-5}.
\end{array}$$

Therefore the function $x = g(y) = \dfrac{-5y+6}{4y-5}$ is the inverse of $f(x)$. We write $g = f^{-1}$. The function $g = f^{-1}$ is defined for $y \neq \dfrac{5}{4}$. For our final answer we relabel the argument of g to x:

$$g(x) = f^{-1}(x) = \frac{-5x+6}{4x-5} \quad .$$

Let us check our work. In order for f and g to be inverses, we need that $g(f(x))$ be equal to x.

$$g(f(x)) = \frac{-5f(x)+6}{4f(x)-5} = \frac{-5\frac{(5x+6)}{4x+5}+6}{4\frac{(5x+6)}{4x+5}-5} = \frac{-5(5x+6)+6(4x+5)}{4(5x+6)-5(4x+5)} = \frac{-x}{-1} = x \quad ,$$

as expected.

2. Find the inverse function and its domain.

(a) $y = \ln(x+3)$.

answer: $f^{-1}(x) = e^x - 3$

(b) $y = 4\ln\,(x-3) - 4$.

answer: $f^{-1}(x) = e^{\frac{x+4}{4}x} + 3$

(c) $y = 2\ln\,(-2x+4) + 1$

answer: $f^{-1}(x) = -\frac{1}{2}e^{\frac{1}{2}x-\frac{1}{2}} + 2$

(d) $f(x) = e^{x^3}$.

answer: $f^{-1}(x) = \sqrt[3]{\ln x}, \quad x > 0$

(e) $y = (\ln x)^2$, $x \geq 1$.

answer: $f^{-1}(x) = e^{\sqrt{x}}, \quad x \geq 0$

(f) $y = \dfrac{e^x}{1+2e^x}$.

answer: $f^{-1}(x) = \ln\left(\frac{x}{1-2x}\right), \quad x \in \left(0, \frac{1}{2}\right)$

(g) $f(x) = 2^{2x} + 2^x - 2$.

answer: $f^{-1}(x) = \log_2 \frac{-1+\sqrt{9+4x}}{2}, \quad x \geq -2$

Solution. 2.a

$$\begin{aligned} y &= \ln(x+3) \\ e^y &= e^{\ln(x+3)} \\ e^y &= x+3 \\ e^y - 3 &= x \\ \text{Therefore} \quad f^{-1}(y) &= e^y - 3. \end{aligned}$$

The domain of e^y is all real numbers, so the domain of f^{-1} is all real numbers.

Solution. 2.b

$$\begin{array}{rcll} 4\ln(x-3)-4 &=& y & \\ 4\ln(x-3) &=& y+4 & \\ \ln(x-3) &=& \dfrac{y+4}{4} & \Big| \text{ exponentiate} \\ e^{\ln(x-3)} &=& e^{\frac{y+4}{4}} & \\ x-3 &=& e^{\frac{y+4}{4}} & \\ f^{-1}(y) = x &=& e^{\frac{y+4}{4}} + 3 & \\ f^{-1}(x) &=& e^{\frac{x+4}{4}} + 3 & \Big| \text{ relabel.} \end{array}$$

The domain of f^{-1} is all real numbers (no restrictions on the domain).

Solution. 2.e

$$\begin{array}{rcll} y &=& (\ln x)^2 & \Big| \text{ take } \sqrt{\ } \text{ on both sides, } y \geq 0 \\ \sqrt{y} &=& \ln x & \Big| \text{ exponentiate} \\ e^{\sqrt{y}} &=& e^{\ln x} = x & \\ f^{-1}(y) &=& e^{\sqrt{y}} & \\ f^{-1}(x) &=& e^{\sqrt{x}} & \end{array}$$

Solution. 2.f

$$
\begin{aligned}
y &= \frac{e^x}{1+2e^x} \\
y(1+2e^x) &= e^x \\
y &= e^x(1-2y) \\
\frac{y}{1-2y} &= e^x \\
\ln\frac{y}{1-2y} &= \ln e^x \\
\ln\frac{y}{1-2y} &= x \\
\text{Therefore} \quad f^{-1}(y) &= \ln\frac{y}{1-2y}.
\end{aligned}
$$

The natural logarithm function is only defined for positive input values. Therefore the domain is the set of all y for which

$$\frac{y}{1-2y} > 0.$$

This inequality holds if the numerator and denominator are both positive or both negative. This happens if either

(a) $y > 0$ and $y < \frac{1}{2}$, or

(b) $y < 0$ and $y > \frac{1}{2}$.

The latter option is impossible, so the domain is $\{y \in \mathbb{R} \mid 0 < y < \frac{1}{2}\}$.