Calculus II
Integrals of the form $\int \sqrt{ax^2 + bx + c}\,\mathbf{d}x$**,**
quadratic has no real roots.

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Linear substitutions to simplify radicals $\sqrt{ay^2 + by + c}$

- Using linear substitutions, radicals of form $\sqrt{ay^2 + by + c}$, $a \neq 0$,
  $b^2 - 4ac \neq 0$ can be transformed to (multiple of):
  - $\sqrt{x^2 + 1}$
  - $\sqrt{-x^2 + 1}$
  - $\sqrt{x^2 - 1}$.
- We already studied how to do that using completing the square when dealing with rational functions.

Recall: linear substitution is subst. of the form $u = px + q$.

Example

Use linear substitution to transform $\sqrt{x^2 + x + 1}$ to multiple of $\sqrt{u^2 + 1}$.

$$\sqrt{x^2 + x + 1} \;=\;$$

Recall: linear substitution is subst. of the form $u = px + q$.

Example

Use linear substitution to transform $\sqrt{x^2 + x + 1}$ to multiple of $\sqrt{u^2 + 1}$.

$$\sqrt{x^2 + x + 1} \;=\; \sqrt{x^2 + 2 \cdot \frac{1}{2}x + \textbf{?} - \textbf{?} + 1}$$

Recall: linear substitution is subst. of the form $u = px + q$.

Example

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$$\sqrt{x^2 + x + 1} \;=\; \sqrt{x^2 + 2 \cdot \frac{1}{2}x + \frac{1}{4} - \frac{1}{4} + 1}$$

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Example

Use linear substitution to transform $\sqrt{x^2 + x + 1}$ to multiple of $\sqrt{u^2 + 1}$.

$$
\begin{aligned}
\sqrt{x^2 + x + 1} &= \sqrt{x^2 + 2 \cdot \frac{1}{2}x + \frac{1}{4} - \frac{1}{4} + 1} \\
&= \sqrt{\left(x + \textbf{?}\right)^2 + \textbf{?}}
\end{aligned}
$$

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&= \sqrt{\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}} \\
&= \sqrt{\frac{3}{4}\left(\frac{4}{3}\left(x + \frac{1}{2}\right)^2 + 1\right)}
\end{aligned}
$$

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&= \sqrt{\frac{3}{4}\left(\frac{4}{3}\left(x + \frac{1}{2}\right)^2 + 1\right)} \\
&= \frac{\sqrt{3}}{2}\sqrt{\left(\frac{2}{\sqrt{3}}\left(x + \frac{1}{2}\right)\right)^2 + 1}
\end{aligned}
$$

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&= \frac{\sqrt{3}}{2}\sqrt{\left(\frac{2}{\sqrt{3}}\left(x + \frac{1}{2}\right)\right)^2 + 1} \\
&= \frac{\sqrt{3}}{2}\sqrt{u^2 + 1},
\end{aligned}
$$

where $u = \dfrac{2}{\sqrt{3}}\left(x + \dfrac{1}{2}\right) = \dfrac{2\sqrt{3}}{3}x + \dfrac{\sqrt{3}}{3}$.