Calculus II
Ratio test related to the exponent as a limit

Todor Milev

2019

Example

Test the convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{3^n n!}$.

$$
\begin{aligned}
\left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{\frac{(n+1)^{n+1}}{3^{n+1}(n+1)!}}{\frac{n^n}{3^n n!}}\right| \\
&= \frac{(n+1)^{n+1}}{n^n} \cdot \frac{3^n n!}{3^{n+1}(n+1)!} \\
&= \frac{\cancel{(n+1)}(n+1)^n}{n^n} \cdot \frac{\cancel{3^n}\cancel{n!}}{3^{\cancel{n}+1}\cancel{(n+1)}\cancel{n!}} \\
&= \frac{1}{3}\left(\frac{n+1}{n}\right)^n = \frac{1}{3}\left(1+\frac{1}{n}\right)^n \\
&\to \frac{e}{3} < 1
\end{aligned}
$$

Therefore the series is convergent by the Ratio Test.

Example

Test the convergence of the series $\sum_{n=1}^{\infty} \frac{n^n}{n!}$.

$$
\begin{aligned}
\left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{\frac{(n+1)^{n+1}}{(n+1)!}}{\frac{n^n}{n!}}\right| \\
&= \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} \\
&= \frac{(n+1)(n+1)^n}{(n+1)n!} \cdot \frac{n!}{n^n} \\
&= \left(\frac{n+1}{n}\right)^n = \left(1+\frac{1}{n}\right)^n \\
&\to e > 1
\end{aligned}
$$

Therefore the series is divergent by the Ratio Test.