Calculus II
Weierstrass substitution theory

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Integrals of the form $\int R(\cos\theta, \sin\theta)\mathrm{d}\theta$, R

Let R be an arbitrary rational function in two variables (quotient of polynomials in two variables).

Question

Can we integrate $\int R(\cos\theta, \sin\theta)\mathrm{d}\theta$?

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- Yes. We will learn how in what follows.

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  - Apply the substitution $\theta = 2 \arctan t$ to transform to integral of rational function.

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Question

Can we integrate $\int R(\cos\theta, \sin\theta)\mathrm{d}\theta$?

- Yes. We will learn how in what follows.
- The algorithm for integration is roughly:
  - Apply the substitution $\theta = 2\arctan t$ to transform to integral of rational function.
  - Solve as previously studied.

The rationalizing substitution $\theta = 2\arctan t$

Recall the expression of $\sin(2z), \cos(2z)$ via $\tan z$:

$$\sin(2z) = \textbf{?}$$

$$\cos(2z)$$

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\sin\theta &= \sin(2\arctan t) = \frac{2\tan(\arctan t)}{1+\tan^2(\arctan t)} = \frac{2t}{1+t^2} \\
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\mathrm{d}\theta &= 2\mathrm{d}(\arctan t) = \mathbf{?} \qquad \mathrm{d}t
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t &= \textbf{?}
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Theorem

The substitution given above transforms $\int R(\cos\theta, \sin\theta)\mathrm{d}\theta$ to an integral of a rational function of t.

The integral $\int \sec\theta \mathrm{d}\theta$ appears often in practice. A quicker solution will be shown later, but first we show the standard method.

Example

$$\int \sec\theta \mathrm{d}\theta$$

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$$\int \sec\theta \mathrm{d}\theta = \int \frac{1}{\cos\theta}\mathrm{d}\theta$$

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Set $\theta = 2\arctan t$, $\cos\theta = \frac{1-\tan^2(\frac{\theta}{2})}{1+\tan^2\left(\frac{\theta}{2}\right)} = \frac{1-t^2}{1+t^2}$, .

$$\int \sec\theta \mathrm{d}\theta \quad = \quad \int \frac{1}{\cos\theta}\mathrm{d}\theta = \int \frac{1}{\left(\frac{1-t^2}{1+t^2}\right)} \frac{2}{(1+t^2)}\mathrm{d}t$$

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Set $\theta = 2\arctan t$, $\cos\theta = \frac{1-\tan^2(\frac{\theta}{2})}{1+\tan^2\left(\frac{\theta}{2}\right)} = \frac{1-t^2}{1+t^2}$, $\mathrm{d}\theta = 2\frac{1}{1+t^2}\mathrm{d}t$.

$$\int \sec\theta \mathrm{d}\theta \quad = \quad \int \frac{1}{\cos\theta}\mathrm{d}\theta = \int \frac{1}{\left(\frac{1-t^2}{1+t^2}\right)} \frac{2}{(1+t^2)}\mathrm{d}t$$

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Set $\theta = 2\arctan t$, $\cos\theta = \dfrac{1-\tan^2(\frac{\theta}{2})}{1+\tan^2\left(\frac{\theta}{2}\right)} = \dfrac{1-t^2}{1+t^2}$, $\mathrm{d}\theta = 2\dfrac{1}{1+t^2}\mathrm{d}t$.

$$\begin{aligned}\int \sec\theta \mathrm{d}\theta &= \int \frac{1}{\cos\theta}\mathrm{d}\theta = \int \frac{1}{\left(\frac{1-t^2}{\cancel{1+t^2}}\right)}\frac{2}{(\cancel{1+t^2})}\mathrm{d}t \\ &= \int \frac{2}{1-t^2}\mathrm{d}t\end{aligned}$$

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Set $\theta = 2\arctan t$, $\cos\theta = \frac{1-\tan^2(\frac{\theta}{2})}{1+\tan^2\left(\frac{\theta}{2}\right)} = \frac{1-t^2}{1+t^2}$, $d\theta = 2\frac{1}{1+t^2}dt$.

$$\begin{aligned}\int \sec\theta d\theta &= \int \frac{1}{\cos\theta} d\theta = \int \frac{1}{\left(\frac{1-t^2}{\cancel{1+t^2}}\right)} \frac{2}{\cancel{(1+t^2)}} dt \\ &= \int \frac{2}{1-t^2} dt = \int \left(\frac{1}{1-t} + \frac{1}{1+t}\right) dt \quad \Big| \text{ part. fractions}\end{aligned}$$

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Set $\theta = 2\arctan t$, $\cos\theta = \frac{1-\tan^2(\frac{\theta}{2})}{1+\tan^2\left(\frac{\theta}{2}\right)} = \frac{1-t^2}{1+t^2}$, $d\theta = 2\frac{1}{1+t^2}dt$.

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