

Calculus I

Limits

Todor Milev

2019

Outline

- 1 The Limit of a Function
 - One-sided Limits

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- 1 The Limit of a Function
 - One-sided Limits
- 2 Calculating Limits Using Limit Laws

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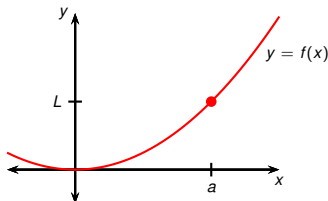
The Limit of a Function

Definition (The Limit of a Function)

We write

$$\lim_{x \rightarrow a} f(x) = L$$

and say “the limit of $f(x)$, as x approaches a , equals L ,” if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a (on either side of a) but not equal to a .



The Limit of a Function

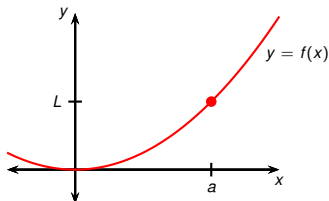
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Equivalent formulation. $\lim_{x \rightarrow a} f(x) = L$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - L| < \varepsilon$ for all x with $0 < |x - a| < \delta$.



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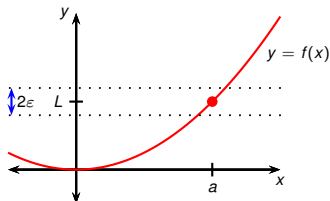
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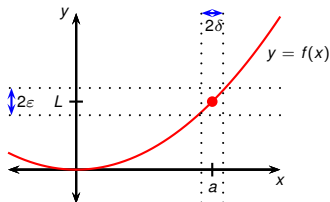
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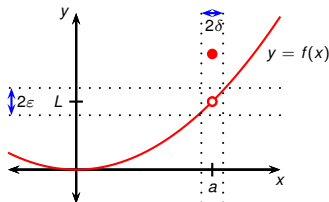
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Example

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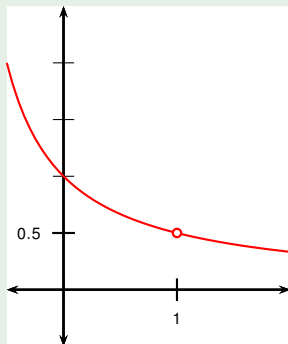
x	$f(x)$	x	$f(x)$
0.5	0.666667	1.5	0.400000
0.9	0.526316	1.1	0.476190
0.99	0.502513	1.01	0.497512
0.999	0.500250	1.001	0.499750
0.9999	0.500025	1.0001	0.499975

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- Notice that $\frac{x-1}{x^2-1}$ is not defined at 1.
- It is defined for values of x near 1.
- We guess that the limit is 0.5.

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- In this case, our guess is correct.

x	$f(x)$	x	$f(x)$
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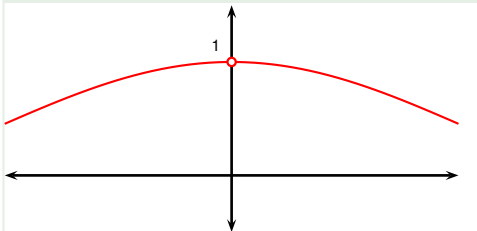
x	$f(x)$	x	$f(x)$
± 1.0	0.841471	± 0.1	0.998334
± 0.5	0.958851	± 0.05	0.999583
± 0.4	0.973546	± 0.01	0.999983
± 0.3	0.985067	± 0.005	0.999995
± 0.2	0.993347	± 0.001	0.999999

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x	$f(x)$	x	$f(x)$
1	$\sin \pi = 0$	$\frac{1}{2}$	$\sin(2\pi) = 0$
$\frac{1}{3}$	$\sin(3\pi) = 0$	$\frac{1}{4}$	$\sin(4\pi) = 0$
0.1	$\sin(10\pi) = 0$	0.01	$\sin(100\pi) = 0$

Example

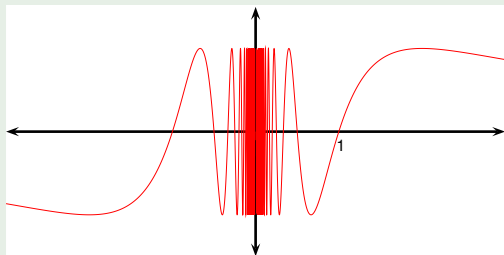
- Guess the value of $\lim_{x \rightarrow 0} \sin\left(\frac{\pi}{x}\right)$.
- Notice that $\sin\left(\frac{\pi}{x}\right)$ is not defined at 0.
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- We may guess that the limit is 0.

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- It is defined for values of x near 0.
- We may guess that the limit is 0.
- Such a guess would be **wrong**.

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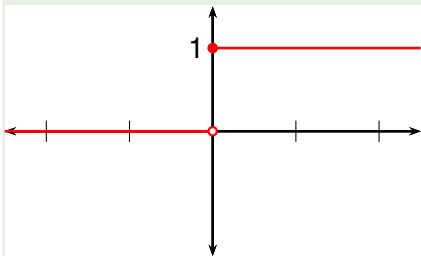


One-sided Limits

Example

The Heaviside function H is defined by

$$H(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}.$$



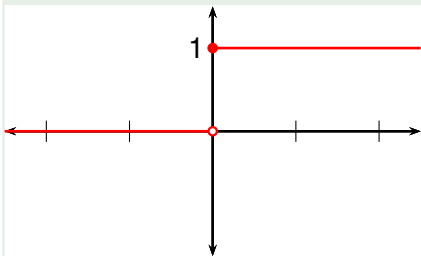
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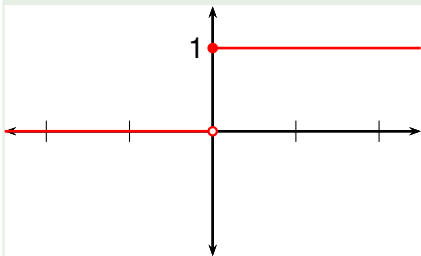


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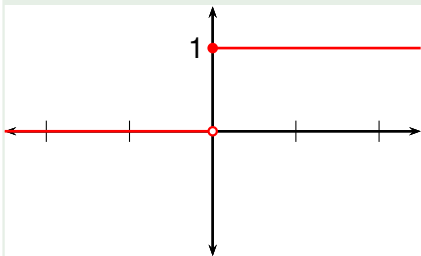
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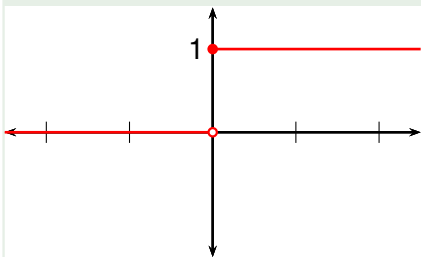
- As x approaches 0 from the left, $H(x)$ approaches 0.
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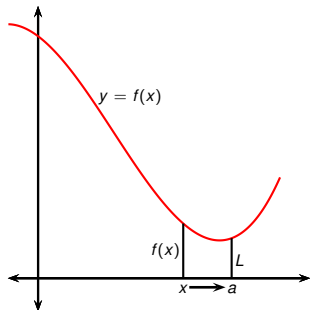
- As x approaches 0 from the left, $H(x)$ approaches 0.
- As x approaches 0 from the right, $H(x)$ approaches 1.
- There is no single number that $H(x)$ approaches as x approaches 0.
- Therefore $\lim_{x \rightarrow 0} H(x)$ doesn't exist.

Definition (Left-hand Limit)

We write

$$\lim_{x \rightarrow a^-} f(x) = L \quad \text{or} \quad \lim_{\substack{x \rightarrow a \\ x < a}} f(x) = L$$

and say the left-hand limit of $f(x)$ as x approaches a is equal to L if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to and less than a .

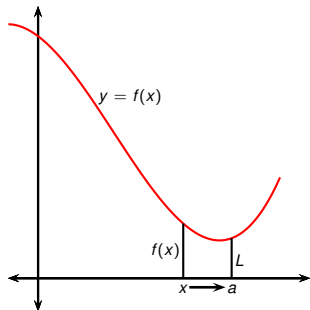


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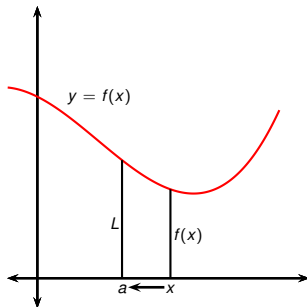
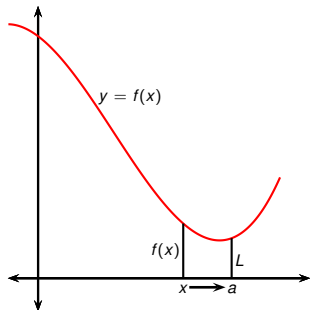
We can define a right-hand limit similarly.

Definition (**Right**-hand Limit)

We write

$$\lim_{x \rightarrow a^+} f(x) = L \quad \text{or} \quad \lim_{\substack{x \rightarrow a \\ x > a}} f(x) = L$$

and say the **right**-hand limit of $f(x)$ as x approaches a is equal to L if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to and **greater** than a .



We can define a **right**-hand limit similarly.

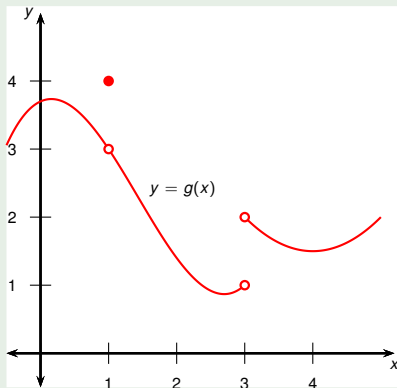
By comparing definitions, we can see that

$$\lim_{x \rightarrow a} f(x) = L \text{ if and only if } \lim_{x \rightarrow a^-} f(x) = L \text{ and } \lim_{x \rightarrow a^+} f(x) = L.$$

Example

The graph of a function g is shown to the right. Use it to state the values (if they exist) of the following:

$$\begin{array}{l|l} \lim_{x \rightarrow 1^-} g(x) = & \lim_{x \rightarrow 3^-} g(x) = \\ \lim_{x \rightarrow 1^+} g(x) = & \lim_{x \rightarrow 3^+} g(x) = \\ \lim_{x \rightarrow 1} g(x) = & \lim_{x \rightarrow 3} g(x) = \end{array}$$



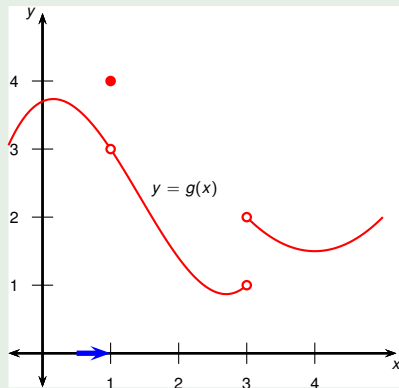
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The graph of a function g is shown to the right. Use it to state the values (if they exist) of the following:

$$\begin{array}{l|l} \lim_{x \rightarrow 1^-} g(x) = ? & \lim_{x \rightarrow 3^-} g(x) = \\ \lim_{x \rightarrow 1^+} g(x) = & \lim_{x \rightarrow 3^+} g(x) = \\ \lim_{x \rightarrow 1} g(x) = & \lim_{x \rightarrow 3} g(x) = \end{array}$$



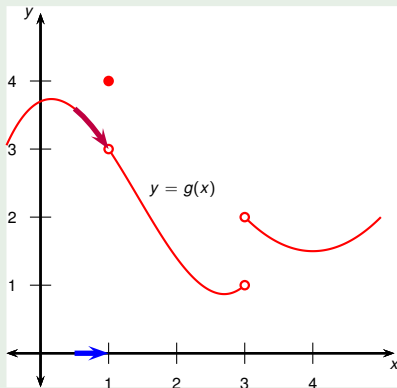
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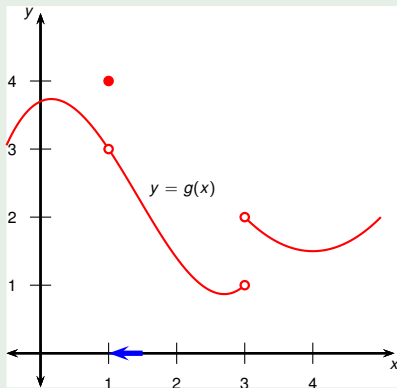
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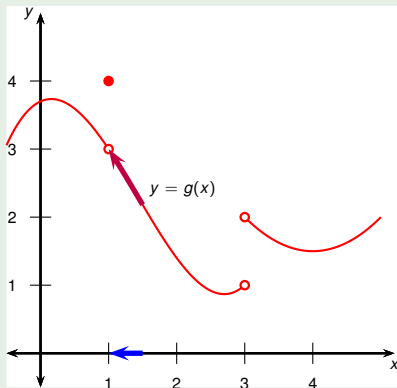
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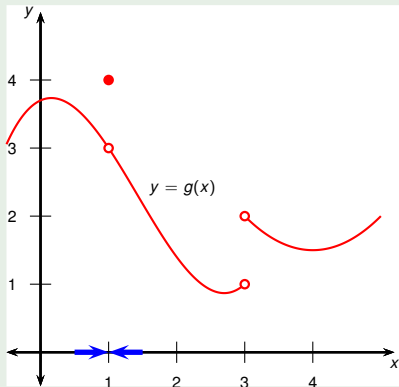
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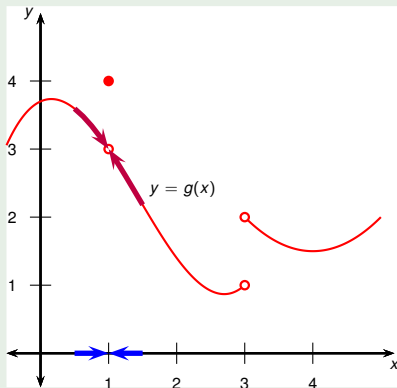
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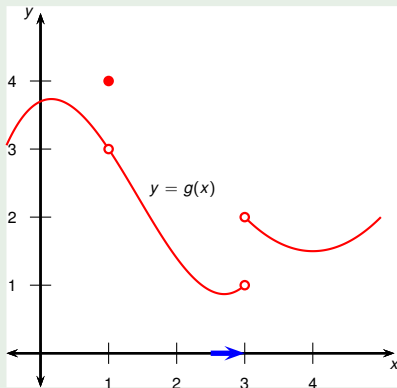
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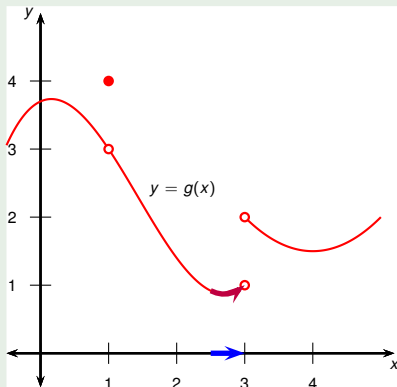
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The graph of a function g is shown to the right. Use it to state the values (if they exist) of the following:

$$\begin{array}{l|l} \lim_{x \rightarrow 1^-} g(x) = 3 & \lim_{x \rightarrow 3^-} g(x) = 1 \\ \lim_{x \rightarrow 1^+} g(x) = 3 & \lim_{x \rightarrow 3^+} g(x) = \\ \lim_{x \rightarrow 1} g(x) = 3 & \lim_{x \rightarrow 3} g(x) = \end{array}$$



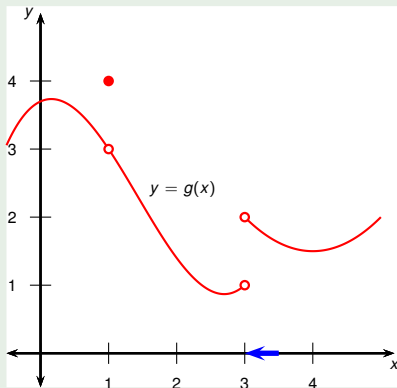
By comparing definitions, we can see that

$$\lim_{x \rightarrow a} f(x) = L \text{ if and only if } \lim_{x \rightarrow a^-} f(x) = L \text{ and } \lim_{x \rightarrow a^+} f(x) = L.$$

Example

The graph of a function g is shown to the right. Use it to state the values (if they exist) of the following:

$$\begin{array}{l|l} \lim_{x \rightarrow 1^-} g(x) = 3 & \lim_{x \rightarrow 3^-} g(x) = 1 \\ \lim_{x \rightarrow 1^+} g(x) = 3 & \lim_{x \rightarrow 3^+} g(x) = ? \\ \lim_{x \rightarrow 1} g(x) = 3 & \lim_{x \rightarrow 3} g(x) = \end{array}$$



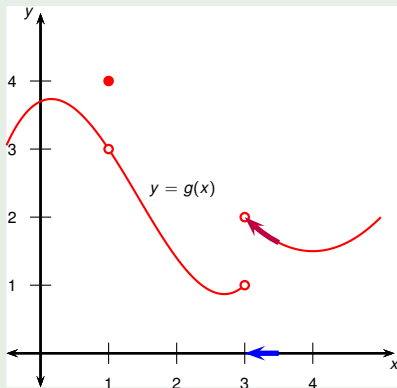
By comparing definitions, we can see that

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Example

The graph of a function g is shown to the right. Use it to state the values (if they exist) of the following:

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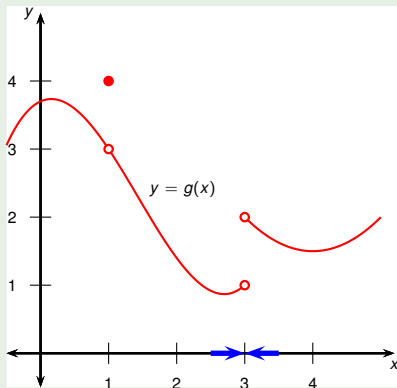
By comparing definitions, we can see that

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Example

The graph of a function g is shown to the right. Use it to state the values (if they exist) of the following:

$$\begin{array}{l|l} \lim_{x \rightarrow 1^-} g(x) = 3 & \lim_{x \rightarrow 3^-} g(x) = 1 \\ \lim_{x \rightarrow 1^+} g(x) = 3 & \lim_{x \rightarrow 3^+} g(x) = 2 \\ \lim_{x \rightarrow 1} g(x) = 3 & \lim_{x \rightarrow 3} g(x) = ? \end{array}$$



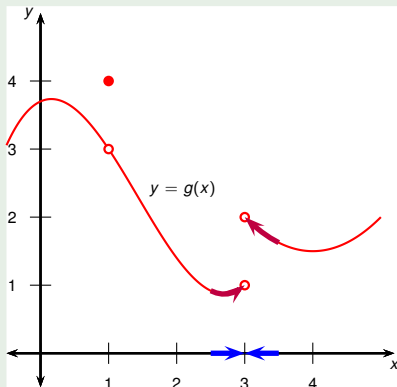
By comparing definitions, we can see that

$$\lim_{x \rightarrow a} f(x) = L \text{ if and only if } \lim_{x \rightarrow a^-} f(x) = L \text{ and } \lim_{x \rightarrow a^+} f(x) = L.$$

Example

The graph of a function g is shown to the right. Use it to state the values (if they exist) of the following:

$$\begin{array}{l|l} \lim_{x \rightarrow 1^-} g(x) = 3 & \lim_{x \rightarrow 3^-} g(x) = 1 \\ \lim_{x \rightarrow 1^+} g(x) = 3 & \lim_{x \rightarrow 3^+} g(x) = 2 \\ \lim_{x \rightarrow 1} g(x) = 3 & \lim_{x \rightarrow 3} g(x) = \text{DNE} \end{array}$$



Calculating Limits Using Limit Laws

Theorem (Limit Laws)

Suppose that c is a constant and that the limits $\lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow a} g(x)$ exist ($\pm\infty$ **not allowed**). Then

$$\textcircled{1} \quad \lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x).$$

$$\textcircled{2} \quad \lim_{x \rightarrow a} [f(x) - g(x)] = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x).$$

$$\textcircled{3} \quad \lim_{x \rightarrow a} [cf(x)] = c \lim_{x \rightarrow a} f(x).$$

$$\textcircled{4} \quad \lim_{x \rightarrow a} [f(x)g(x)] = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x).$$

$$\textcircled{5} \quad \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} \quad \text{if } \lim_{x \rightarrow a} g(x) \neq 0.$$

Calculating Limits Using Limit Laws

Theorem (Limit Laws)

Suppose that c is a constant and that the limits $\lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow a} g(x)$ exist ($\pm\infty$ **not allowed**). Then

$$\textcircled{1} \quad \lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x).$$

$$\textcircled{2} \quad \lim_{x \rightarrow a} [f(x) - g(x)] = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x).$$

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$$\textcircled{5} \quad \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} \quad \text{if } \lim_{x \rightarrow a} g(x) \neq 0.$$

Sum Law

Calculating Limits Using Limit Laws

Theorem (Limit Laws)

Suppose that c is a constant and that the limits $\lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow a} g(x)$ exist ($\pm\infty$ **not allowed**). Then

$$① \quad \lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x).$$

$$② \quad \lim_{x \rightarrow a} [f(x) - g(x)] = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x).$$

$$③ \quad \lim_{x \rightarrow a} [cf(x)] = c \lim_{x \rightarrow a} f(x).$$

$$④ \quad \lim_{x \rightarrow a} [f(x)g(x)] = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x).$$

$$⑤ \quad \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} \quad \text{if } \lim_{x \rightarrow a} g(x) \neq 0.$$

Difference Law

Calculating Limits Using Limit Laws

Theorem (Limit Laws)

Suppose that c is a constant and that the limits $\lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow a} g(x)$ exist ($\pm\infty$ **not allowed**). Then

$$1 \quad \lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x).$$

$$2 \quad \lim_{x \rightarrow a} [f(x) - g(x)] = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x).$$

$$3 \quad \lim_{x \rightarrow a} [cf(x)] = c \lim_{x \rightarrow a} f(x).$$

$$4 \quad \lim_{x \rightarrow a} [f(x)g(x)] = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x).$$

$$5 \quad \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} \quad \text{if } \lim_{x \rightarrow a} g(x) \neq 0.$$

Constant Multiple Law

Calculating Limits Using Limit Laws

Theorem (Limit Laws)

Suppose that c is a constant and that the limits $\lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow a} g(x)$ exist ($\pm\infty$ **not allowed**). Then

$$① \quad \lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x).$$

$$② \quad \lim_{x \rightarrow a} [f(x) - g(x)] = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x).$$

$$③ \quad \lim_{x \rightarrow a} [cf(x)] = c \lim_{x \rightarrow a} f(x).$$

$$④ \quad \lim_{x \rightarrow a} [f(x)g(x)] = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x).$$

$$⑤ \quad \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} \quad \text{if } \lim_{x \rightarrow a} g(x) \neq 0.$$

Product Law

Calculating Limits Using Limit Laws

Theorem (Limit Laws)

Suppose that c is a constant and that the limits $\lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow a} g(x)$ exist ($\pm\infty$ **not allowed**). Then

$$① \quad \lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x).$$

$$② \quad \lim_{x \rightarrow a} [f(x) - g(x)] = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x).$$

$$③ \quad \lim_{x \rightarrow a} [cf(x)] = c \lim_{x \rightarrow a} f(x).$$

$$④ \quad \lim_{x \rightarrow a} [f(x)g(x)] = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x).$$

$$⑤ \quad \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} \quad \text{if } \lim_{x \rightarrow a} g(x) \neq 0.$$

Quotient Law

Here are some other useful limit laws:

$$\textcircled{6} \quad \lim_{x \rightarrow a} [f(x)]^n = [\lim_{x \rightarrow a} f(x)]^n$$

$$\textcircled{7} \quad \lim_{x \rightarrow a} c = c.$$

$$\textcircled{8} \quad \lim_{x \rightarrow a} x = a.$$

$$\textcircled{9} \quad \lim_{x \rightarrow a} x^n = a^n.$$

$$\textcircled{10} \quad \lim_{x \rightarrow a} \sqrt[n]{x} = \sqrt[n]{a}, \text{ if } a > 0.$$

$$\textcircled{11} \quad \lim_{x \rightarrow a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow a} f(x)}, \text{ if } \lim_{x \rightarrow a} f(x) > 0.$$

Here are some other useful limit laws:

$$\textcircled{6} \quad \lim_{x \rightarrow a} [f(x)]^n = [\lim_{x \rightarrow a} f(x)]^n$$

$$\textcircled{7} \quad \lim_{x \rightarrow a} c = c.$$

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$$\textcircled{11} \quad \lim_{x \rightarrow a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow a} f(x)}, \text{ if } \lim_{x \rightarrow a} f(x) > 0.$$

Power Law

Here are some other useful limit laws:

$$\textcircled{6} \quad \lim_{x \rightarrow a} [f(x)]^n = [\lim_{x \rightarrow a} f(x)]^n$$

$$\textcircled{7} \quad \lim_{x \rightarrow a} c = c.$$

$$\textcircled{8} \quad \lim_{x \rightarrow a} x = a.$$

$$\textcircled{9} \quad \lim_{x \rightarrow a} x^n = a^n.$$

$$\textcircled{10} \quad \lim_{x \rightarrow a} \sqrt[n]{x} = \sqrt[n]{a}, \text{ if } a > 0.$$

$$\textcircled{11} \quad \lim_{x \rightarrow a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow a} f(x)}, \text{ if } \lim_{x \rightarrow a} f(x) > 0.$$

Root Law

Here are some other useful limit laws:

$$\textcircled{6} \quad \lim_{x \rightarrow a} [f(x)]^n = [\lim_{x \rightarrow a} f(x)]^n$$

$$\textcircled{7} \quad \lim_{x \rightarrow a} c = c.$$

$$\textcircled{8} \quad \lim_{x \rightarrow a} x = a.$$

$$\textcircled{9} \quad \lim_{x \rightarrow a} x^n = a^n.$$

$$\textcircled{10} \quad \lim_{x \rightarrow a} \sqrt[n]{x} = \sqrt[n]{a}, \text{ if } a > 0.$$

$$\textcircled{11} \quad \lim_{x \rightarrow a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow a} f(x)}, \text{ if } \lim_{x \rightarrow a} f(x) > 0.$$

Direct Substitution

Example

Evaluate the limit and justify each step:

$$\lim_{x \rightarrow 5} (2x^2 - 3x + 4)$$

Example

Evaluate the limit and justify each step:

$$\begin{aligned} & \lim_{x \rightarrow 5} (2x^2 - 3x + 4) \\ &= \lim_{x \rightarrow 5} (2x^2 - 3x) + \lim_{x \rightarrow 5} 4 \end{aligned} \quad \text{Law}$$

Example

Evaluate the limit and justify each step:

$$\begin{aligned} & \lim_{x \rightarrow 5} (2x^2 - 3x + 4) \\ &= \lim_{x \rightarrow 5} (2x^2 - 3x) + \lim_{x \rightarrow 5} 4 \end{aligned}$$

Law 1

Example

Evaluate the limit and justify each step:

$$\begin{aligned} & \lim_{x \rightarrow 5} (2x^2 - 3x + 4) \\ &= \lim_{x \rightarrow 5} (2x^2 - 3x) + \lim_{x \rightarrow 5} 4 && \text{Law 1} \\ &= \lim_{x \rightarrow 5} (2x^2) - \lim_{x \rightarrow 5} (3x) + \lim_{x \rightarrow 5} 4 && \text{Law} \end{aligned}$$

Example

Evaluate the limit and justify each step:

$$\begin{aligned} & \lim_{x \rightarrow 5} (2x^2 - 3x + 4) \\ &= \lim_{x \rightarrow 5} (2x^2 - 3x) + \lim_{x \rightarrow 5} 4 && \text{Law 1} \\ &= \lim_{x \rightarrow 5} (2x^2) - \lim_{x \rightarrow 5} (3x) + \lim_{x \rightarrow 5} 4 && \text{Law 2} \end{aligned}$$

Example

Evaluate the limit and justify each step:

$$\begin{aligned} & \lim_{x \rightarrow 5} (2x^2 - 3x + 4) \\ &= \lim_{x \rightarrow 5} (2x^2 - 3x) + \lim_{x \rightarrow 5} 4 && \text{Law 1} \\ &= \lim_{x \rightarrow 5} (2x^2) - \lim_{x \rightarrow 5} (3x) + \lim_{x \rightarrow 5} 4 && \text{Law 2} \\ &= 2 \lim_{x \rightarrow 5} x^2 - 3 \lim_{x \rightarrow 5} x + \lim_{x \rightarrow 5} 4 && \text{Law} \end{aligned}$$

Example

Evaluate the limit and justify each step:

$$\begin{aligned} & \lim_{x \rightarrow 5} (2x^2 - 3x + 4) \\ &= \lim_{x \rightarrow 5} (2x^2 - 3x) + \lim_{x \rightarrow 5} 4 && \text{Law 1} \\ &= \lim_{x \rightarrow 5} (2x^2) - \lim_{x \rightarrow 5} (3x) + \lim_{x \rightarrow 5} 4 && \text{Law 2} \\ &= 2 \lim_{x \rightarrow 5} x^2 - 3 \lim_{x \rightarrow 5} x + \lim_{x \rightarrow 5} 4 && \text{Law 3} \end{aligned}$$

Example

Evaluate the limit and justify each step:

$$\begin{aligned} & \lim_{x \rightarrow 5} (2x^2 - 3x + 4) \\ &= \lim_{x \rightarrow 5} (2x^2 - 3x) + \lim_{x \rightarrow 5} 4 && \text{Law 1} \\ &= \lim_{x \rightarrow 5} (2x^2) - \lim_{x \rightarrow 5} (3x) + \lim_{x \rightarrow 5} 4 && \text{Law 2} \\ &= 2 \lim_{x \rightarrow 5} x^2 - 3 \lim_{x \rightarrow 5} x + \lim_{x \rightarrow 5} 4 && \text{Law 3} \\ &= 2 \cdot 5^2 - 3 \cdot 5 + 4 && \text{Laws} \end{aligned}$$

Example

Evaluate the limit and justify each step:

$$\begin{aligned} & \lim_{x \rightarrow 5} (2x^2 - 3x + 4) \\ &= \lim_{x \rightarrow 5} (2x^2 - 3x) + \lim_{x \rightarrow 5} 4 && \text{Law 1} \\ &= \lim_{x \rightarrow 5} (2x^2) - \lim_{x \rightarrow 5} (3x) + \lim_{x \rightarrow 5} 4 && \text{Law 2} \\ &= 2 \lim_{x \rightarrow 5} x^2 - 3 \lim_{x \rightarrow 5} x + \lim_{x \rightarrow 5} 4 && \text{Law 3} \\ &= 2 \cdot 5^2 - 3 \cdot 5 + 4 && \text{Laws 7} \end{aligned}$$

Example

Evaluate the limit and justify each step:

$$\begin{aligned} & \lim_{x \rightarrow 5} (2x^2 - 3x + 4) \\ &= \lim_{x \rightarrow 5} (2x^2 - 3x) + \lim_{x \rightarrow 5} 4 && \text{Law 1} \\ &= \lim_{x \rightarrow 5} (2x^2) - \lim_{x \rightarrow 5} (3x) + \lim_{x \rightarrow 5} 4 && \text{Law 2} \\ &= 2 \lim_{x \rightarrow 5} x^2 - 3 \lim_{x \rightarrow 5} x + \lim_{x \rightarrow 5} 4 && \text{Law 3} \\ &= 2 \cdot 5^2 - 3 \cdot 5 + 4 && \text{Laws 7} \end{aligned}$$

Example

Evaluate the limit and justify each step:

$$\begin{aligned} & \lim_{x \rightarrow 5} (2x^2 - 3x + 4) \\ &= \lim_{x \rightarrow 5} (2x^2 - 3x) + \lim_{x \rightarrow 5} 4 && \text{Law 1} \\ &= \lim_{x \rightarrow 5} (2x^2) - \lim_{x \rightarrow 5} (3x) + \lim_{x \rightarrow 5} 4 && \text{Law 2} \\ &= 2 \lim_{x \rightarrow 5} x^2 - 3 \lim_{x \rightarrow 5} x + \lim_{x \rightarrow 5} 4 && \text{Law 3} \\ &= 2 \cdot 5^2 - 3 \cdot 5 + 4 && \text{Laws 7, 8} \end{aligned}$$

Example

Evaluate the limit and justify each step:

$$\begin{aligned} & \lim_{x \rightarrow 5} (2x^2 - 3x + 4) \\ &= \lim_{x \rightarrow 5} (2x^2 - 3x) + \lim_{x \rightarrow 5} 4 && \text{Law 1} \\ &= \lim_{x \rightarrow 5} (2x^2) - \lim_{x \rightarrow 5} (3x) + \lim_{x \rightarrow 5} 4 && \text{Law 2} \\ &= 2 \lim_{x \rightarrow 5} x^2 - 3 \lim_{x \rightarrow 5} x + \lim_{x \rightarrow 5} 4 && \text{Law 3} \\ &= 2 \cdot 5^2 - 3 \cdot 5 + 4 && \text{Laws 7, 8} \end{aligned}$$

Example

Evaluate the limit and justify each step:

$$\begin{aligned} & \lim_{x \rightarrow 5} (2x^2 - 3x + 4) \\ &= \lim_{x \rightarrow 5} (2x^2 - 3x) + \lim_{x \rightarrow 5} 4 && \text{Law 1} \\ &= \lim_{x \rightarrow 5} (2x^2) - \lim_{x \rightarrow 5} (3x) + \lim_{x \rightarrow 5} 4 && \text{Law 2} \\ &= 2 \lim_{x \rightarrow 5} x^2 - 3 \lim_{x \rightarrow 5} x + \lim_{x \rightarrow 5} 4 && \text{Law 3} \\ &= 2 \cdot 5^2 - 3 \cdot 5 + 4 && \text{Laws 7, 8, and 9} \end{aligned}$$

Example

Evaluate the limit and justify each step:

$$\begin{aligned} & \lim_{x \rightarrow 5} (2x^2 - 3x + 4) \\ &= \lim_{x \rightarrow 5} (2x^2 - 3x) + \lim_{x \rightarrow 5} 4 && \text{Law 1} \\ &= \lim_{x \rightarrow 5} (2x^2) - \lim_{x \rightarrow 5} (3x) + \lim_{x \rightarrow 5} 4 && \text{Law 2} \\ &= 2 \lim_{x \rightarrow 5} x^2 - 3 \lim_{x \rightarrow 5} x + \lim_{x \rightarrow 5} 4 && \text{Law 3} \\ &= 2 \cdot 5^2 - 3 \cdot 5 + 4 && \text{Laws 7, 8, and 9} \\ &= 39. \end{aligned}$$

Example (Limit Laws)

Evaluate the limit and justify each step:

$$\lim_{x \rightarrow 3} \frac{x + 2}{\sqrt{x - 1}(x + 1)^2}$$

Example (Limit Laws)

Evaluate the limit and justify each step:

$$\begin{aligned} & \lim_{x \rightarrow 3} \frac{x + 2}{\sqrt{x - 1}(x + 1)^2} \\ &= \frac{\lim_{x \rightarrow 3} (x + 2)}{\lim_{x \rightarrow 3} (\sqrt{x - 1}(x + 1)^2)} \end{aligned}$$

Law

Example (Limit Laws)

Evaluate the limit and justify each step:

$$\begin{aligned} & \lim_{x \rightarrow 3} \frac{x + 2}{\sqrt{x - 1}(x + 1)^2} \\ &= \frac{\lim_{x \rightarrow 3} (x + 2)}{\lim_{x \rightarrow 3} (\sqrt{x - 1}(x + 1)^2)} \end{aligned}$$

Law 5

Example (Limit Laws)

Evaluate the limit and justify each step:

$$\begin{aligned} & \lim_{x \rightarrow 3} \frac{x + 2}{\sqrt{x - 1}(x + 1)^2} \\ &= \frac{\lim_{x \rightarrow 3} (x + 2)}{\lim_{x \rightarrow 3} (\sqrt{x - 1}(x + 1)^2)} \quad \text{Law 5} \\ &= \frac{\lim_{x \rightarrow 3} (x + 2)}{\lim_{x \rightarrow 3} \sqrt{x - 1} \cdot \lim_{x \rightarrow 3} ((x + 1)^2)} \quad \text{Law} \end{aligned}$$

Example (Limit Laws)

Evaluate the limit and justify each step:

$$\begin{aligned} & \lim_{x \rightarrow 3} \frac{x + 2}{\sqrt{x - 1}(x + 1)^2} \\ &= \frac{\lim_{x \rightarrow 3} (x + 2)}{\lim_{x \rightarrow 3} (\sqrt{x - 1}(x + 1)^2)} \quad \text{Law 5} \\ &= \frac{\lim_{x \rightarrow 3} (x + 2)}{\lim_{x \rightarrow 3} \sqrt{x - 1} \cdot \lim_{x \rightarrow 3} ((x + 1)^2)} \quad \text{Law 4} \end{aligned}$$

Example (Limit Laws)

Evaluate the limit and justify each step:

$$\begin{aligned}
 & \lim_{x \rightarrow 3} \frac{x + 2}{\sqrt{x - 1}(x + 1)^2} \\
 = & \frac{\lim_{x \rightarrow 3} (x + 2)}{\lim_{x \rightarrow 3} (\sqrt{x - 1}(x + 1)^2)} && \text{Law 5} \\
 = & \frac{\lim_{x \rightarrow 3} (x + 2)}{\lim_{x \rightarrow 3} \sqrt{x - 1} \cdot \lim_{x \rightarrow 3} ((x + 1)^2)} && \text{Law 4} \\
 = & \frac{\lim_{x \rightarrow 3} (x + 2)}{\sqrt{\lim_{x \rightarrow 3} (x - 1)} (\lim_{x \rightarrow 3} (x + 1))^2} && \text{Laws}
 \end{aligned}$$

Example (Limit Laws)

Evaluate the limit and justify each step:

$$\begin{aligned}
 & \lim_{x \rightarrow 3} \frac{x + 2}{\sqrt{x - 1}(x + 1)^2} \\
 = & \frac{\lim_{x \rightarrow 3} (x + 2)}{\lim_{x \rightarrow 3} (\sqrt{x - 1}(x + 1)^2)} && \text{Law 5} \\
 = & \frac{\lim_{x \rightarrow 3} (x + 2)}{\lim_{x \rightarrow 3} \sqrt{x - 1} \cdot \lim_{x \rightarrow 3} ((x + 1)^2)} && \text{Law 4} \\
 = & \frac{\lim_{x \rightarrow 3} (x + 2)}{\sqrt{\lim_{x \rightarrow 3} (x - 1)} (\lim_{x \rightarrow 3} (x + 1))^2} && \text{Laws 11}
 \end{aligned}$$

Example (Limit Laws)

Evaluate the limit and justify each step:

$$\begin{aligned}
 & \lim_{x \rightarrow 3} \frac{x + 2}{\sqrt{x - 1}(x + 1)^2} \\
 = & \frac{\lim_{x \rightarrow 3} (x + 2)}{\lim_{x \rightarrow 3} (\sqrt{x - 1}(x + 1)^2)} && \text{Law 5} \\
 = & \frac{\lim_{x \rightarrow 3} (x + 2)}{\lim_{x \rightarrow 3} \sqrt{x - 1} \cdot \lim_{x \rightarrow 3} ((x + 1)^2)} && \text{Law 4} \\
 = & \frac{\lim_{x \rightarrow 3} (x + 2)}{\sqrt{\lim_{x \rightarrow 3} (x - 1)} (\lim_{x \rightarrow 3} (x + 1))^2} && \text{Laws 11}
 \end{aligned}$$

Example (Limit Laws)

Evaluate the limit and justify each step:

$$\begin{aligned} & \lim_{x \rightarrow 3} \frac{x+2}{\sqrt{x-1}(x+1)^2} \\ &= \frac{\lim_{x \rightarrow 3} (x+2)}{\lim_{x \rightarrow 3} (\sqrt{x-1}(x+1)^2)} && \text{Law 5} \\ &= \frac{\lim_{x \rightarrow 3} (x+2)}{\lim_{x \rightarrow 3} \sqrt{x-1} \cdot \lim_{x \rightarrow 3} ((x+1)^2)} && \text{Law 4} \\ &= \frac{\lim_{x \rightarrow 3} (x+2)}{\sqrt{\lim_{x \rightarrow 3} (x-1)} (\lim_{x \rightarrow 3} (x+1))^2} && \text{Laws 11 and 6} \end{aligned}$$

Example (Limit Laws)

Evaluate the limit and justify each step:

$$\begin{aligned}
 & \lim_{x \rightarrow 3} \frac{x + 2}{\sqrt{x - 1}(x + 1)^2} \\
 = & \frac{\lim_{x \rightarrow 3} (x + 2)}{\lim_{x \rightarrow 3} (\sqrt{x - 1}(x + 1)^2)} && \text{Law 5} \\
 = & \frac{\lim_{x \rightarrow 3} (x + 2)}{\lim_{x \rightarrow 3} \sqrt{x - 1} \cdot \lim_{x \rightarrow 3} ((x + 1)^2)} && \text{Law 4} \\
 = & \frac{\lim_{x \rightarrow 3} (x + 2)}{\sqrt{\lim_{x \rightarrow 3} (x - 1)} (\lim_{x \rightarrow 3} (x + 1))^2} && \text{Laws 11 and 6} \\
 = & \frac{\lim_{x \rightarrow 3} x + \lim_{x \rightarrow 3} 2}{\sqrt{\lim_{x \rightarrow 3} x - \lim_{x \rightarrow 3} 1} (\lim_{x \rightarrow 3} x + \lim_{x \rightarrow 3} 1)^2} && \text{Laws}
 \end{aligned}$$

Example (Limit Laws)

Evaluate the limit and justify each step:

$$\begin{aligned}
 & \lim_{x \rightarrow 3} \frac{x + 2}{\sqrt{x - 1}(x + 1)^2} \\
 = & \frac{\lim_{x \rightarrow 3} (x + 2)}{\lim_{x \rightarrow 3} (\sqrt{x - 1}(x + 1)^2)} && \text{Law 5} \\
 = & \frac{\lim_{x \rightarrow 3} (x + 2)}{\lim_{x \rightarrow 3} \sqrt{x - 1} \cdot \lim_{x \rightarrow 3} ((x + 1)^2)} && \text{Law 4} \\
 = & \frac{\lim_{x \rightarrow 3} (x + 2)}{\sqrt{\lim_{x \rightarrow 3} (x - 1)} (\lim_{x \rightarrow 3} (x + 1))^2} && \text{Laws 11 and 6} \\
 = & \frac{\lim_{x \rightarrow 3} x + \lim_{x \rightarrow 3} 2}{\sqrt{\lim_{x \rightarrow 3} x - \lim_{x \rightarrow 3} 1} (\lim_{x \rightarrow 3} x + \lim_{x \rightarrow 3} 1)^2} && \text{Laws 1}
 \end{aligned}$$

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 = & \frac{\lim_{x \rightarrow 3} x + \lim_{x \rightarrow 3} 2}{\sqrt{\lim_{x \rightarrow 3} x - \lim_{x \rightarrow 3} 1} (\lim_{x \rightarrow 3} x + \lim_{x \rightarrow 3} 1)^2} && \text{Laws 1 and 2} \\
 = & \frac{3 + 2}{\sqrt{3 - 1} (3 + 1)^2} && \text{Laws 8}
 \end{aligned}$$

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 = & \frac{\lim_{x \rightarrow 3} (x + 2)}{\sqrt{\lim_{x \rightarrow 3} (x - 1)} (\lim_{x \rightarrow 3} (x + 1))^2} && \text{Laws 11 and 6} \\
 = & \frac{\lim_{x \rightarrow 3} x + \textcolor{red}{\lim_{x \rightarrow 3} 2}}{\sqrt{\lim_{x \rightarrow 3} x - \textcolor{red}{\lim_{x \rightarrow 3} 1}} (\lim_{x \rightarrow 3} x + \textcolor{red}{\lim_{x \rightarrow 3} 1})^2} && \text{Laws 1 and 2} \\
 = & \frac{3 + \textcolor{red}{2}}{\sqrt{3 - \textcolor{red}{1}} (3 + \textcolor{red}{1})^2} && \textcolor{red}{\text{Laws 8}}
 \end{aligned}$$

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 = & \frac{\lim_{x \rightarrow 3} (x + 2)}{\sqrt{\lim_{x \rightarrow 3} (x - 1)} (\lim_{x \rightarrow 3} (x + 1))^2} && \text{Laws 11 and 6} \\
 = & \frac{\lim_{x \rightarrow 3} x + \textcolor{red}{\lim_{x \rightarrow 3} 2}}{\sqrt{\lim_{x \rightarrow 3} x - \textcolor{red}{\lim_{x \rightarrow 3} 1}} (\lim_{x \rightarrow 3} x + \textcolor{red}{\lim_{x \rightarrow 3} 1})^2} && \text{Laws 1 and 2} \\
 = & \frac{3 + \textcolor{red}{2}}{\sqrt{3 - \textcolor{red}{1}} (3 + \textcolor{red}{1})^2} && \text{Laws 8 and 7}
 \end{aligned}$$

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 = & \frac{3 + 2}{\sqrt{3 - 1} (3 + 1)^2} = \frac{5}{16\sqrt{2}}. && \text{Laws 8 and 7}
 \end{aligned}$$

Recall that every function which can be using the four arithmetic operations $(+, -, *, /)$ and radicals $\sqrt[n]{}$ is an algebraic function.

Theorem (Direct Substitution)

Let f be an algebraic function. Let the point a be in its domain (i.e., $f(a)$ is defined). Then $\lim_{x \rightarrow a} f(x) = f(a)$.

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This theorem is a partial case of the following theorem.

Theorem (Can be taken as definition)

Let f be a continuous function. Let the point a be in its domain (i.e., $f(a)$ is defined). Then $\lim_{x \rightarrow a} f(x) = f(a)$.

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Theorem (Can be taken as definition)

*Let f be a **continuous function**. Let the point a be in its domain (i.e., $f(a)$ is defined). Then $\lim_{x \rightarrow a} f(x) = f(a)$.*

Continuous functions will be defined later in this lecture.

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Therefore $\lim_{x \rightarrow 3} \frac{x + 2}{\sqrt{x - 1}(x + 1)^2} = \frac{5}{16\sqrt{2}}.$

Example (Limit in Which Direct Substitution Doesn't Work)

Find $\lim_{x \rightarrow 3} \frac{x^3 - 3x^2 + x - 3}{x^2 - 7x + 12}$

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Zero over zero is undefined, so we can't use direct substitution.

When computing a limit as x approaches a , we don't care what happens when $x = a$. This gives the following **useful fact**:

$$\text{If } f(x) = g(x)$$

when $x \neq a$,

$$\text{then } \lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x),$$

provided the limit exists.

We can use this fact to find $\lim_{x \rightarrow a} f(x)$ when $f(a)$ has the form $\frac{0}{0}$. In such a case, we use algebra to find a function $g(x)$ that agrees with $f(x)$ at all points except $x = a$. Here are some common techniques.

- 1 Factoring.
- 2 Using a conjugate radical.
- 3 Finding a common denominator.

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- 1 Factoring.
- 2 Using a conjugate radical.
- 3 Finding a common denominator.
- 4 **Using Taylor/Maclaurin series expansion. Studied in Calc II.**

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Factor: $\lim_{x \rightarrow 3} \frac{x^3 - 3x^2 + x - 3}{x^2 - 7x + 12} = \lim_{x \rightarrow 3} \frac{(x^2 + 1)(x - 3)}{?}$

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Factor: $\lim_{x \rightarrow 3} \frac{x^3 - 3x^2 + x - 3}{x^2 - 7x + 12} = \lim_{x \rightarrow 3} \frac{(x^2 + 1)(x - 3)}{(x - 4)(x - 3)}$

Example (Limit with Factoring)

$$\text{Find } \lim_{x \rightarrow 3} \frac{x^3 - 3x^2 + x - 3}{x^2 - 7x + 12}$$

$$\text{Plug in 3: } \frac{(3)^3 - 3(3)^2 + (3) - 3}{(3)^2 - 7(3) + 12} = \frac{0}{0}$$

Zero over zero is undefined, so we can't use direct substitution.

$$\begin{aligned} \text{Factor: } \lim_{x \rightarrow 3} \frac{x^3 - 3x^2 + x - 3}{x^2 - 7x + 12} &= \lim_{x \rightarrow 3} \frac{(x^2 + 1)\cancel{(x - 3)}}{(x - 4)\cancel{(x - 3)}} \\ &= \lim_{x \rightarrow 3} \frac{x^2 + 1}{x - 4} \end{aligned}$$

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Factor: $\lim_{x \rightarrow 3} \frac{x^3 - 3x^2 + x - 3}{x^2 - 7x + 12} = \lim_{x \rightarrow 3} \frac{(x^2 + 1)(\cancel{x - 3})}{(x - 4)(\cancel{x - 3})}$

$$= \lim_{x \rightarrow 3} \frac{x^2 + 1}{x - 4}$$

Plug in 3: $\lim_{x \rightarrow 3} \frac{x^3 - 3x^2 + x - 3}{x^2 - 7x + 12} = \frac{(3)^2 + 1}{(3) - 4}$

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$$= \lim_{x \rightarrow 3} \frac{x^2 + 1}{x - 4}$$

Plug in 3: $\lim_{x \rightarrow 3} \frac{x^3 - 3x^2 + x - 3}{x^2 - 7x + 12} = \frac{(3)^2 + 1}{(3) - 4}$

$$= \frac{10}{-1}$$

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$$= \lim_{x \rightarrow 3} \frac{x^2 + 1}{x - 4}$$

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$$= \frac{10}{-1}$$

$$= -10.$$

Example

Find $\lim_{t \rightarrow 0} \frac{\sqrt{t^2 + 9} - 3}{t^2}$

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Example

Find $\lim_{t \rightarrow 0} \frac{\sqrt{t^2 + 9} - 3}{t^2}$

Plug in 0: $\frac{\sqrt{(0)^2 + 9} - 3}{(0)^2} = \frac{0}{0}$

Zero over zero is undefined, so we can't use direct substitution.
Multiply top & bottom by (minus) the conjugate radical:

$$\lim_{t \rightarrow 0} \frac{\sqrt{t^2 + 9} - 3}{t^2} = \lim_{t \rightarrow 0} \frac{\sqrt{t^2 + 9} - 3}{t^2} \cdot \frac{?}{?}$$

Example

Find $\lim_{t \rightarrow 0} \frac{\sqrt{t^2 + 9} - 3}{t^2}$

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Example

Find $\lim_{t \rightarrow 0} \frac{\sqrt{t^2 + 9} - 3}{t^2}$

Plug in 0: $\frac{\sqrt{(0)^2 + 9} - 3}{(0)^2} = \frac{0}{0}$

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$$\begin{aligned} \lim_{t \rightarrow 0} \frac{\sqrt{t^2 + 9} - 3}{t^2} &= \lim_{t \rightarrow 0} \frac{\sqrt{t^2 + 9} - 3}{t^2} \cdot \frac{\sqrt{t^2 + 9} + 3}{\sqrt{t^2 + 9} + 3} \\ &= \lim_{t \rightarrow 0} \frac{?}{t^2 (\sqrt{t^2 + 9} + 3)} \end{aligned}$$

Example

$$\text{Find } \lim_{t \rightarrow 0} \frac{\sqrt{t^2 + 9} - 3}{t^2}$$

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Multiply top & bottom by (minus) the conjugate radical:

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Example

$$\text{Find } \lim_{t \rightarrow 0} \frac{\sqrt{t^2 + 9} - 3}{t^2}$$

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Example

Find $\lim_{t \rightarrow 0} \frac{\sqrt{t^2 + 9} - 3}{t^2}$

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Plug in 0: $= \frac{1}{\sqrt{(0)^2 + 9} + 3} = \frac{1}{6}.$

Example

Find $\lim_{x \rightarrow 1} g(x)$, where

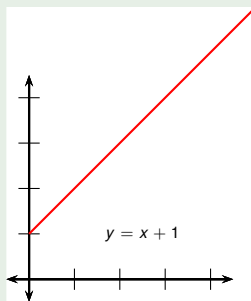
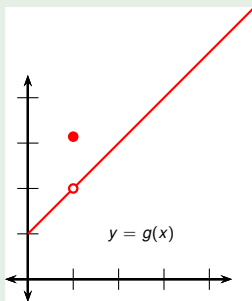
$$g(x) = \begin{cases} x + 1 & \text{if } x \neq 1 \\ \pi & \text{if } x = 1 \end{cases}$$

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g agrees with the function $f(x) = x + 1$ at every point except for $x = 1$.



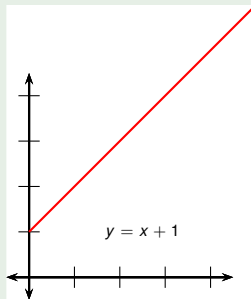
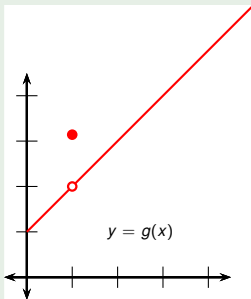
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$$\lim_{x \rightarrow 1} g(x) = \lim_{x \rightarrow 1} (x + 1) = ?$$



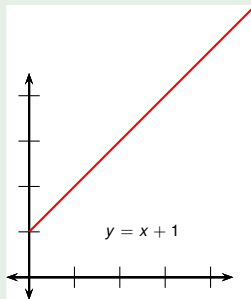
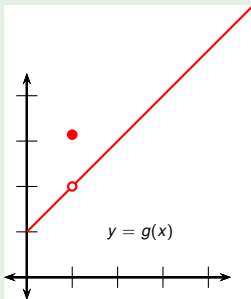
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$$\lim_{x \rightarrow 1} g(x) = \lim_{x \rightarrow 1} (x + 1) = 2.$$



Example (Limit with Factoring)

Find $\lim_{h \rightarrow 0} \frac{(3 + h)^2 - 9}{h}$

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Factor: $= \lim_{h \rightarrow 0} \frac{?}{h}$

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$$\text{Factor: } = \lim_{h \rightarrow 0} \frac{h(6 + h)}{h}$$

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$$\text{Plug in 0: } = (6 + (\textcolor{red}{0}))$$

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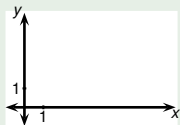
$$\text{Plug in 0: } = (6 + (0)) = 6.$$

Recall:

$$\lim_{x \rightarrow a} f(x) = L \quad \text{if and only if} \quad \lim_{x \rightarrow a^-} f(x) = L = \lim_{x \rightarrow a^+} f(x).$$

We can use this to find the limit of a piecewise defined function, or show that it doesn't exist.

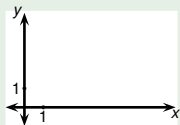
Example



$$f(x) = \begin{cases} \sqrt{x-4} & \text{if } x > 4 \\ 8-2x & \text{if } x < 4 \end{cases}$$

Determine whether $\lim_{x \rightarrow 4} f(x)$ exists.

Example

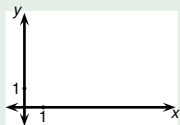


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$$\lim_{x \rightarrow 4^+} f(x)$$

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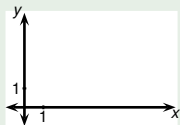


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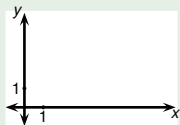


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Example



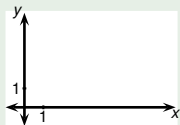
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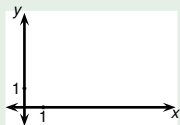
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Example



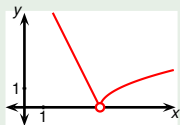
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The left and right hand limits are equal. Therefore the limit exists and

$$\lim_{x \rightarrow 4} f(x) = 0.$$

Theorem

If $f(x) \leq g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as x approaches a , then

$$\lim_{x \rightarrow a} f(x) \leq \lim_{x \rightarrow a} g(x).$$

Theorem (The Squeeze Theorem)

Suppose $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at a) and

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L$$

Then

$$\lim_{x \rightarrow a} g(x) = L.$$

Example

Show that $\lim_{x \rightarrow 0} x^2 \sin\left(\frac{8}{x}\right) = 0$.

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Doesn't work because $\lim_{x \rightarrow 0} \sin\left(\frac{8}{x}\right)$ doesn't exist.

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$$-1 \leq \sin\left(\frac{8}{x}\right) \leq 1.$$

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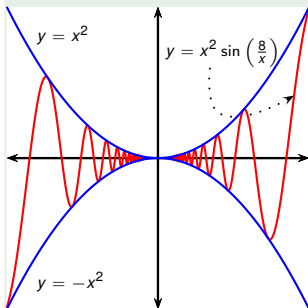
$$\lim_{x \rightarrow 0} x^2 = 0 \quad \text{and} \quad \lim_{x \rightarrow 0} (-x^2) = 0.$$

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Therefore by the Squeeze Theorem

$$\lim_{x \rightarrow 0} x^2 \sin\left(\frac{8}{x}\right) = 0.$$