

# Precalculus

## Trigonometric identities theory

Todor Milev

2019

# Proof of a Trigonometric Identity

Let  $F$  and  $G$  be expressions that give a trigonometric identity:

$$F(\sin \theta, \cos \theta) = G(\sin \theta, \cos \theta).$$

- To prove a trigonometric identity means to show that the two sides of the equality sign are equivalent.
- There are two ways to do this (in the present course the first way will be preferred).
- First method: transform the left and right hand sides to an equal expression. In particular:
  - we can choose to transform the left hand side to the right;
  - we can choose to transform the right hand side to the left;
  - we can choose to transform both sides to a third equivalent expression.
- Second method: start with an already known identity and transform it, by a series of equivalent transformations, to the identity we desire to prove.
- The discussion here also applies for trigonometric identities in more than one variables.

# Types of identities

- In the present course we deal with two basic types of trigonometric identities.
- First, identities that involve operations on the arguments of the trigonometric functions.
  - Example:  $\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$  (this is one of the angle sum identities);  $\sin \theta + \sin(-\theta) = 0$ .
  - Such identities can be proved using the angle sum formulas and the even/odd function properties of  $\sin$ ,  $\cos$ .
- Second, identities that involve trigonometric functions of one variable.
  - Example:  $\tan^2 \theta + 1 = \sec^2 \theta$ .
  - Such identities can be proved only using the already demonstrated Pythagorean identity  $\sin^2 \theta + \cos^2 \theta = 1$ .
- The Pythagorean identity follows from the angle sum formulas and the even/odd function properties of  $\sin$ ,  $\cos$ , so all trigonometric identities follow from those properties alone.

# Strategy for proving trigonometric identities

An expression is rational trigonometric if it is written using  $\sin \theta$ ,  $\cos \theta$  and the four arithmetic operations.

## Question

*Is there a general method for proving all rational trigonometric identities in one variable?*

- Given a number of variables and relations between them, there is an algorithm to check whether (rational) expressions in those variables are equal under the given relations.
- Thus, if we pick two variables  $s$  and  $c$ , and a single relation
$$s^2 + c^2 = 1$$
there is a standard method to verify whether two (rational) expressions in  $s$  and  $c$  are equal.
- The method is rather cumbersome for a human and is best suited for computers.

# Strategy for proving trigonometric identities

An expression is rational trigonometric if it is written using  $\sin \theta$ ,  $\cos \theta$  and the four arithmetic operations.

## Question

*Is there a general method for proving all rational trigonometric identities in one variable?*

- Yes.
- For expressions that depend only on  $\sin \theta$  and  $\cos \theta$ , algebra tells us when two expressions in those are equal.
- Problems depending on  $\cos \theta$ ,  $\sin \theta$  alone will always be doable via easy ad-hoc tricks using

$$\sin^2 \theta + \cos^2 \theta = 1.$$

- The full method will not be needed in this course.
  - The full method: set  $s = \sin \theta$ ,  $c = \cos \theta$ .
  - Check whether the two expressions in  $s, c$  are equal under the relation  $s^2 + c^2 = 1$ . (The method lies outside of present scope).

# Strategy for proving trigonometric identities

An expression is rational trigonometric if it is written using  $\sin \theta$ ,  $\cos \theta$  and the four arithmetic operations.

## Question

*Is there a general method for proving all rational trigonometric identities in one variable?*

- To prove a general trigonometric identity:
  - Use angle sum/double angle sum formulas to convert all formulas to trig expression depending only on  $\sin \theta$ ,  $\cos \theta$ .
  - Use  $\sin^2 \theta + \cos^2 \theta = 1$  to show the two formulas are equal (usage: ad-hoc).
  - You may need to use trig functions of angles smaller than  $\theta$ , for example  $\sin \left(\frac{\theta}{2}\right)$ ,  $\cos \left(\frac{\theta}{2}\right)$ .
  - A fraction of  $\theta$  such that all appearing angles are integer multiples of it will always work.