Calculus II
Polar coordinates

Todor Milev

2019

Outline

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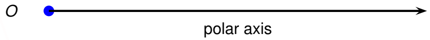

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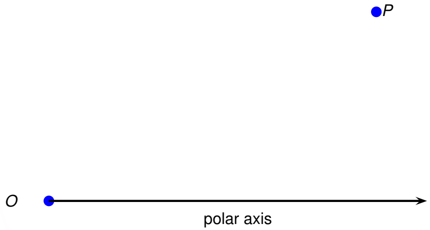

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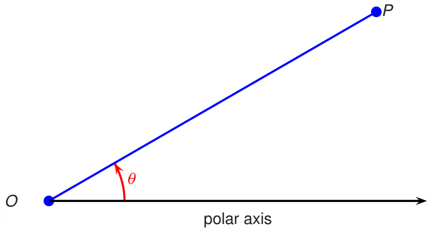

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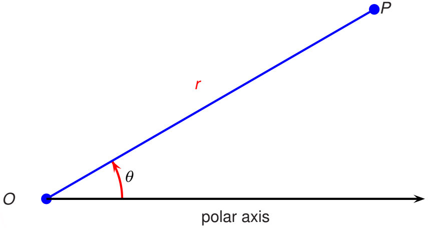

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- Let r denote the length of the segment OP.

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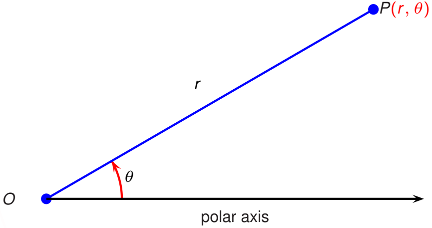

- Let P be a point in the plane.
- Let θ denote the angle between the polar axis and the line OP.
- Let r denote the length of the segment OP.
- Then P is represented by the ordered pair (r, θ).

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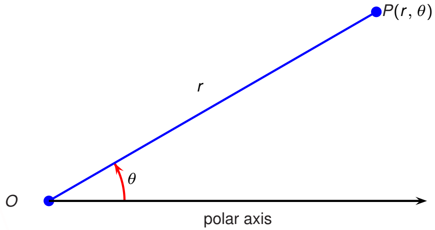

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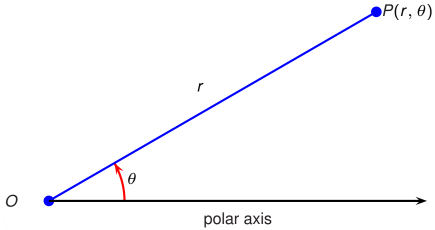

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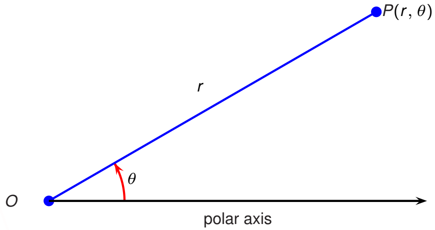

- The letters (x, y) imply Cartesian coordinates and the letters (r, θ)- polar. When we use other letters, it should be clear from context whether we mean Cartesian or polar coordinates. If not, one must request clarification.

1. What if θ is negative?
2. What if r is negative?
3. What if r is 0?

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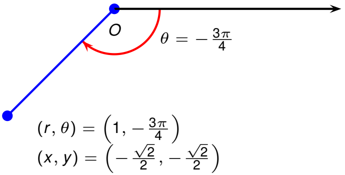

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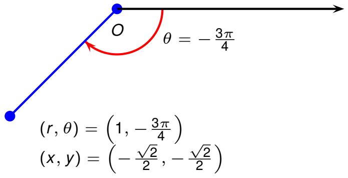

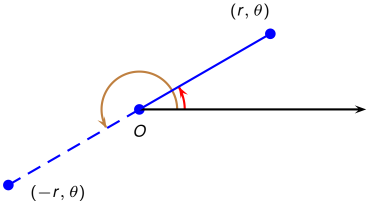

1. Positive angles θ are measured in the counterclockwise direction from O. Negative angles are measured in the clockwise direction.
2. Points with polar coordinates $(-r,\theta)$ and (r,θ) lie on the same line through O and at the same distance from O, but on opposite sides.

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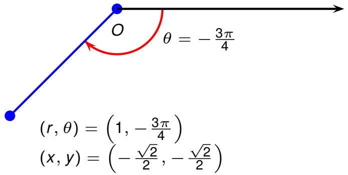
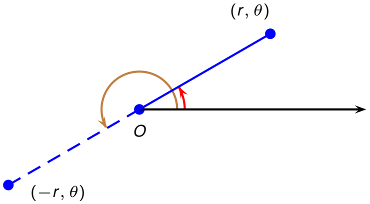
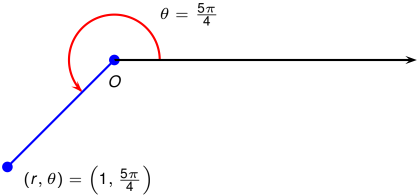
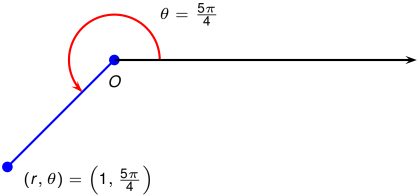


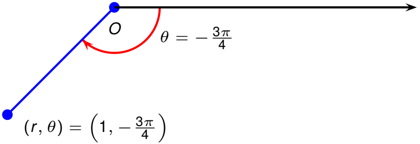


1. Positive angles θ are measured in the counterclockwise direction from O. Negative angles are measured in the clockwise direction.
2. Points with polar coordinates $(-r, \theta)$ and (r, θ) lie on the same line through O and at the same distance from O, but on opposite sides.
3. If $r = 0$, then $(0, \theta)$ represents O for all values of θ.

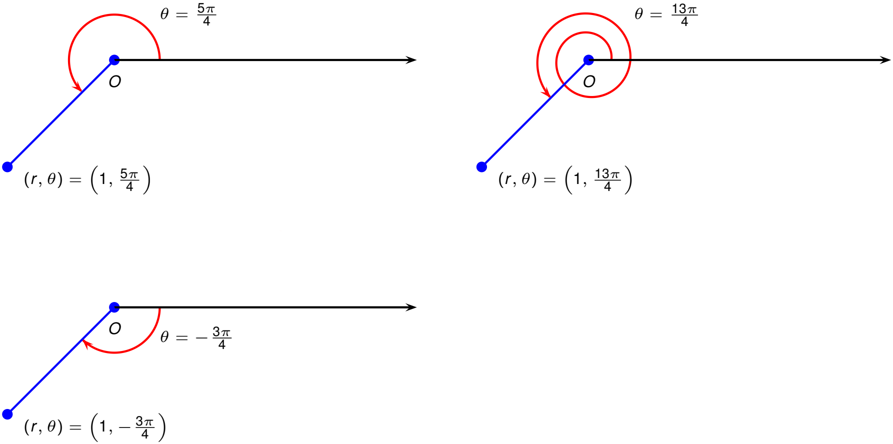


- There are many ways to represent the same point.

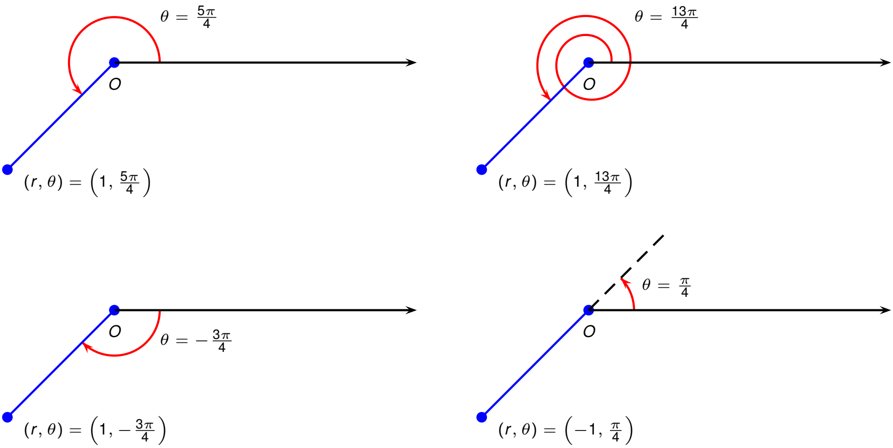




- There are many ways to represent the same point.
- We could use a negative θ.



- There are many ways to represent the same point.
- We could use a negative θ.
- We could go around more than once.



- There are many ways to represent the same point.
- We could use a negative θ.
- We could go around more than once.
- We could use a negative r.

- Let P_1 be point with polar coordinates (r_1, θ_1).
- Let P_2 be point with polar coordinates (r_2, θ_2).

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Observation

P_1 coincides with P_2 if one of the three mutually exclusive possibilities holds:

- $r_1 = r_2 \neq 0$ *and* $\theta_2 = \theta_1 + 2k\pi, k \in \mathbb{Z}$,
- $r_1 = -r_2 \neq 0$ *and* $\theta_2 = \theta_1 + (2k+1)\pi, k \in \mathbb{Z}$,
- $r_1 = r_2 = 0$ *and* θ *is arbitrary.*

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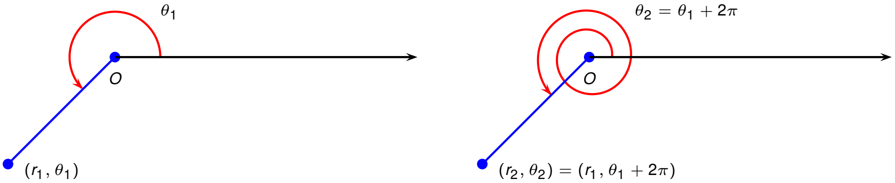

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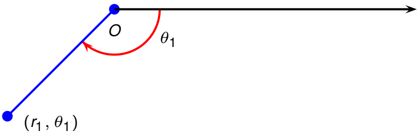
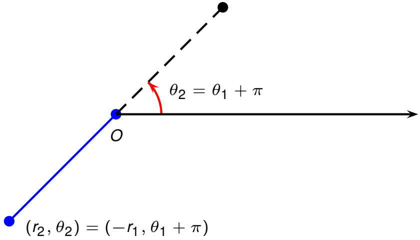
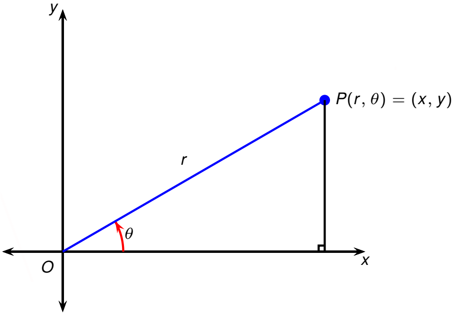




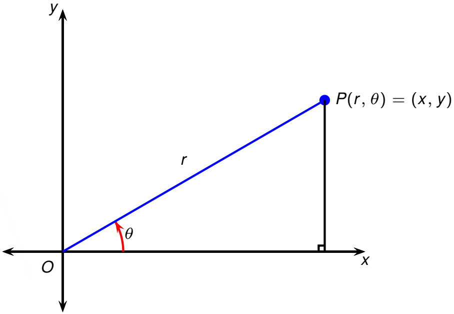
- How do we go from polar coordinates to Cartesian coordinates?



$$x =$$
$$y =$$

$$r =$$
$$\theta =$$

- How do we go from polar coordinates to Cartesian coordinates?
- Suppose a point has polar coordinates (r, θ) and Cartesian coordinates (x, y).



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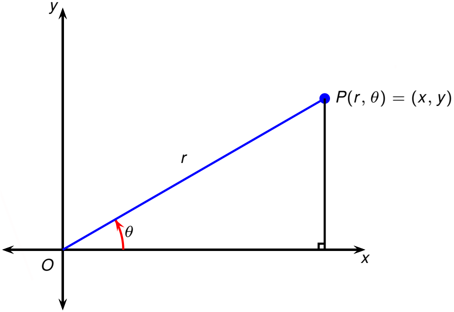

$$x =$$
$$y =$$

$$\cos\theta =$$
$$\sin\theta =$$

$$r =$$
$$\theta =$$

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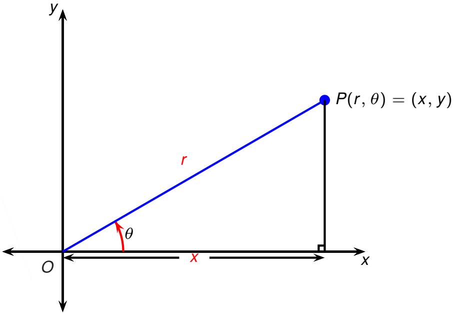

$$x =$$

$$y =$$

$$\cos\theta = \frac{x}{r}$$

$$\sin\theta =$$

$$r =$$

$$\theta =$$

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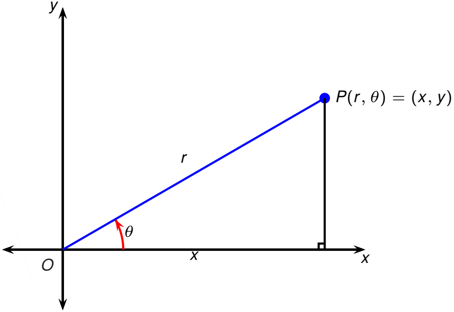

$$
\begin{aligned}
x &= \\
y &= \\
\hline
\cos\theta &= \frac{x}{r} \\
\sin\theta &= \\
\hline
r &= \\
\theta &=
\end{aligned}
$$

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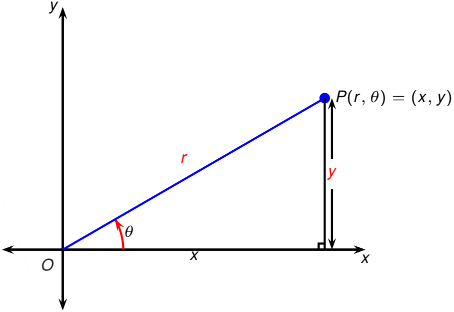

$$
\begin{aligned}
x &= \\
y &= \\
\hline
\cos\theta &= \frac{x}{r} \\
{\color{red}\sin\theta} &{\color{red}=} {\color{red}\frac{y}{r}} \\
\hline
r &= \\
\theta &=
\end{aligned}
$$

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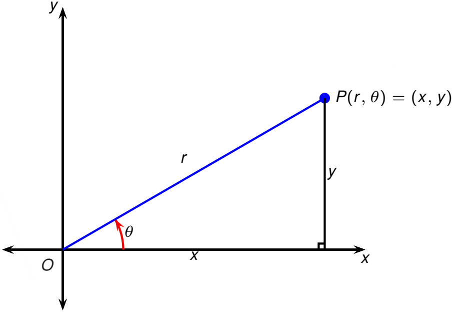

$$
\begin{aligned}
x &= r\cos\theta \\
y &= r\sin\theta \\
\hline
\cos\theta &= \frac{x}{r} \\
\sin\theta &= \frac{y}{r} \\
\hline
r &= \\
\theta &=
\end{aligned}
$$

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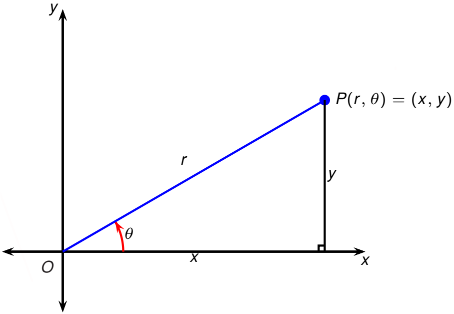
- Suppose a point has polar coordinates (r, θ) and Cartesian coordinates (x, y).
- How do we go from Cartesian coordinates to polar coordinates?



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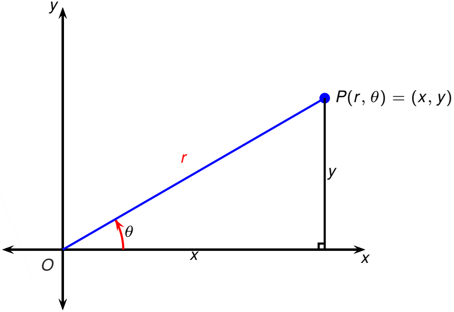

$$
\begin{aligned}
x &= r\cos\theta \\
y &= r\sin\theta \\
\hline
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{\color{red}r^2} &{\color{red}=} \\
\hline
{\color{red}r} &{\color{red}=} \\
\theta &=
\end{aligned}
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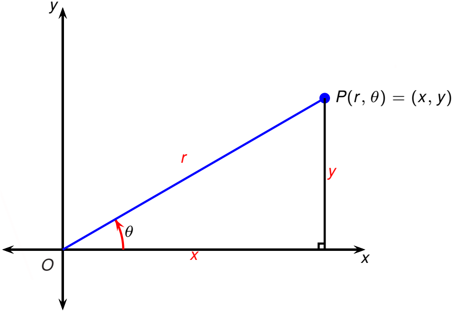

$$
\begin{aligned}
x &= r\cos\theta \\
y &= r\sin\theta \\
\hline
\cos\theta &= \frac{x}{r} \\
\sin\theta &= \frac{y}{r} \\
r^2 &= x^2 + y^2 \\
\hline
r &= \\
\theta &=
\end{aligned}
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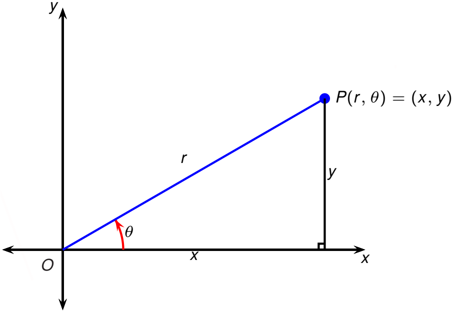

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r^2 &= x^2 + y^2 \\
\hline
r &= \sqrt{x^2 + y^2} \\
\theta &=
\end{aligned}
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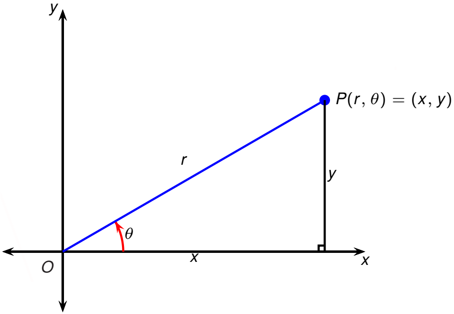

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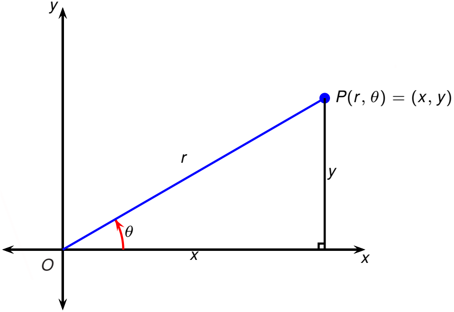

$$
\begin{array}{rcll}
x &=& r\cos\theta & \\
y &=& r\sin\theta & \\
\hline
\cos\theta &=& \frac{x}{r} & \\
\sin\theta &=& \frac{y}{r} & \\
r^2 &=& x^2 + y^2 & \\
\hline
r &=& \sqrt{x^2+y^2} & \\
\theta &=& \arcsin\left(\frac{y}{r}\right) & \text{if } x > 0 \\
&=& \arccos\left(\frac{x}{r}\right) & \text{if } y > 0 \\
&=& \arctan\left(\frac{y}{x}\right) & \text{if } x > 0
\end{array}
$$

Example

Convert the point $(2, \frac{\pi}{3})$ from polar to Cartesian coordinates.

Example

Convert the point $(2, \frac{\pi}{3})$ from polar to Cartesian coordinates.

$$x = r\cos\theta =$$

$$y = r\sin\theta =$$

Example

Convert the point $(2, \frac{\pi}{3})$ from polar to Cartesian coordinates.

$$x = r\cos\theta = \quad \cos$$

$$y = r\sin\theta =$$

Example

Convert the point $(2, \frac{\pi}{3})$ from polar to Cartesian coordinates.

$$x = r\cos\theta = 2\cos$$

$$y = r\sin\theta =$$

Example

Convert the point $(2, \frac{\pi}{3})$ from polar to Cartesian coordinates.

$$x = r\cos\theta = 2\cos$$

$$y = r\sin\theta =$$

Example

Convert the point $(2, \frac{\pi}{3})$ from polar to Cartesian coordinates.

$$x = r\cos\theta = 2\cos\frac{\pi}{3}$$

$$y = r\sin\theta =$$

Example

Convert the point $(2, \frac{\pi}{3})$ from polar to Cartesian coordinates.

$$x = r\cos\theta = 2\cos\frac{\pi}{3} = 2\left(\quad\right)$$

$$y = r\sin\theta =$$

Example

Convert the point $(2, \frac{\pi}{3})$ from polar to Cartesian coordinates.

$$x = r\cos\theta = 2\cos\frac{\pi}{3} = 2\left(\frac{1}{2}\right)$$

$$y = r\sin\theta =$$

Example

Convert the point $(2, \frac{\pi}{3})$ from polar to Cartesian coordinates.

$$x = r\cos\theta = 2\cos\frac{\pi}{3} = 2\left(\frac{1}{2}\right) = 1$$

$$y = r\sin\theta =$$

Example

Convert the point $(2, \frac{\pi}{3})$ from polar to Cartesian coordinates.

$$x = r\cos\theta = 2\cos\frac{\pi}{3} = 2\left(\frac{1}{2}\right) = 1$$

$$y = r\sin\theta = 2\sin\frac{\pi}{3}$$

Example

Convert the point $(2, \frac{\pi}{3})$ from polar to Cartesian coordinates.

$$x = r\cos\theta = 2\cos\frac{\pi}{3} = 2\left(\frac{1}{2}\right) = 1$$

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Example

Convert the point $(2, \frac{\pi}{3})$ from polar to Cartesian coordinates.

$$x = r\cos\theta = 2\cos\frac{\pi}{3} = 2\left(\frac{1}{2}\right) = 1$$

$$y = r\sin\theta = 2\sin\frac{\pi}{3} = 2\left(\frac{\sqrt{3}}{2}\right)$$

Example

Convert the point $(2, \frac{\pi}{3})$ from polar to Cartesian coordinates.

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$$y = r\sin\theta = 2\sin\frac{\pi}{3} = 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}$$

Example

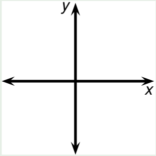
Convert the point $(2, \frac{\pi}{3})$ from polar to Cartesian coordinates.

$$x = r\cos\theta = 2\cos\frac{\pi}{3} = 2\left(\frac{1}{2}\right) = 1$$

$$y = r\sin\theta = 2\sin\frac{\pi}{3} = 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}$$

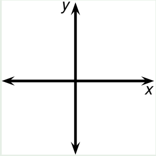
Therefore the point with polar coordinates $(2, \frac{\pi}{3})$ has Cartesian coordinates $(1, \sqrt{3})$.

Example



Represent the point with Cartesian coordinates $(1, -1)$ in terms of polar coordinates.

Example

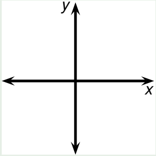


Represent the point with Cartesian coordinates $(1, -1)$ in terms of polar coordinates.

$$r = \pm\sqrt{x^2 + y^2}$$

$$\tan\theta = \frac{y}{x}$$

Example



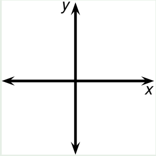
Represent the point with Cartesian coordinates $(1, -1)$ in terms of polar coordinates.

- Suppose r is positive.

$$r = \pm\sqrt{x^2 + y^2}$$

$$\tan\theta = \frac{y}{x}$$

Example



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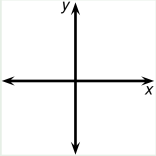
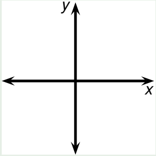


Represent the point with Cartesian coordinates $(1, -1)$ in terms of polar coordinates.

- Suppose r is positive.

$$
\begin{aligned}
r &= \sqrt{x^2 + y^2} \\
&= \sqrt{1^2 + (-1)^2} \\
&= \sqrt{2} \\
\tan\theta &= \frac{y}{x}
\end{aligned}
$$

Example



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\begin{aligned}
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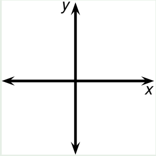
&= -1
\end{aligned}
$$

Example



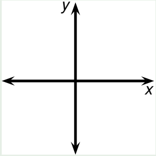
Represent the point with Cartesian coordinates $(1, -1)$ in terms of polar coordinates.

- Suppose r is positive.
- $\tan\theta = -1$ for $\theta = \frac{3\pi}{4}, \frac{7\pi}{4}$, and many other angles.

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{1^2 + (-1)^2} \\ &= \sqrt{2} \end{aligned}$$

$$\begin{aligned} \tan\theta &= \frac{y}{x} \\ &= -1 \end{aligned}$$

Example



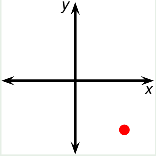
Represent the point with Cartesian coordinates $(1, -1)$ in terms of polar coordinates.

- Suppose r is positive.
- $\tan\theta = -1$ for $\theta = \frac{3\pi}{4}, \frac{7\pi}{4}$, and many other angles.
- $(1, -1)$ is in the quadrant.

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{1^2 + (-1)^2} \\ &= \sqrt{2} \end{aligned}$$

$$\begin{aligned} \tan\theta &= \frac{y}{x} \\ &= -1 \end{aligned}$$

Example



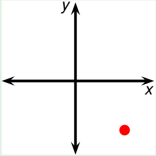
Represent the point with Cartesian coordinates $(1, -1)$ in terms of polar coordinates.

- Suppose r is positive.
- $\tan\theta = -1$ for $\theta = \frac{3\pi}{4}, \frac{7\pi}{4}$, and many other angles.
- $(1, -1)$ is in the fourth quadrant.

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{1^2 + (-1)^2} \\ &= \sqrt{2} \end{aligned}$$

$$\begin{aligned} \tan\theta &= \frac{y}{x} \\ &= -1 \end{aligned}$$

Example

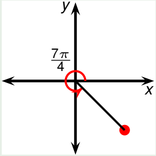


Represent the point with Cartesian coordinates $(1, -1)$ in terms of polar coordinates.

- Suppose r is positive.
- $\tan\theta = -1$ for $\theta = \frac{3\pi}{4}, \frac{7\pi}{4}$, and many other angles.
- $(1, -1)$ is in the fourth quadrant.
- Of the two values above, only $\theta =$ gives a point in the fourth quadrant.

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{1^2 + (-1)^2} \\ &= \sqrt{2} \\ \tan\theta &= \frac{y}{x} \\ &= -1 \end{aligned}$$

Example



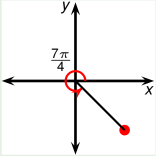
Represent the point with Cartesian coordinates $(1, -1)$ in terms of polar coordinates.

- Suppose r is positive.
- $\tan\theta = -1$ for $\theta = \frac{3\pi}{4}, \frac{7\pi}{4}$, and many other angles.
- $(1, -1)$ is in the fourth quadrant.
- Of the two values above, only $\theta = \frac{7\pi}{4}$ gives a point in the fourth quadrant.

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{1^2 + (-1)^2} \\ &= \sqrt{2} \end{aligned}$$

$$\begin{aligned} \tan\theta &= \frac{y}{x} \\ &= -1 \end{aligned}$$

Example

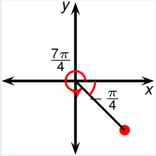


Represent the point with Cartesian coordinates $(1, -1)$ in terms of polar coordinates.

- Suppose r is positive.
- $\tan\theta = -1$ for $\theta = \frac{3\pi}{4}, \frac{7\pi}{4}$, and many other angles.
- $(1, -1)$ is in the fourth quadrant.
- Of the two values above, only $\theta = \frac{7\pi}{4}$ gives a point in the fourth quadrant.
- $\Rightarrow$ one representation of $(1, -1)$ in polar coordinates is $\left(\sqrt{2}, \frac{7\pi}{4}\right)$.

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{1^2 + (-1)^2} \\ &= \sqrt{2} \\ \tan\theta &= \frac{y}{x} \\ &= -1 \end{aligned}$$

Example



Represent the point with Cartesian coordinates $(1, -1)$ in terms of polar coordinates.

- Suppose r is positive.
- $\tan\theta = -1$ for $\theta = \frac{3\pi}{4}, \frac{7\pi}{4}$, and many other angles.
- $(1, -1)$ is in the fourth quadrant.
- Of the two values above, only $\theta = \frac{7\pi}{4}$ gives a point in the fourth quadrant.
- $\Rightarrow$ one representation of $(1, -1)$ in polar coordinates is $\left(\sqrt{2}, \frac{7\pi}{4}\right)$.
- $\left(\sqrt{2}, -\frac{\pi}{4}\right)$ is another.

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{1^2 + (-1)^2} \\ &= \sqrt{2} \\ \tan\theta &= \frac{y}{x} \\ &= -1 \end{aligned}$$