Calculus II
Homework
Partial fractions

1. Let $x \in (0,1)$. Express the following using x and $\sqrt{1-x^2}$.

   (a) $\sin(\arcsin(x))$.
   (b) $\sin(2\arcsin(x))$.
   (c) $\sin(3\arcsin(x))$.
   (d) $\sin(\arccos(x))$.
   (e) $\sin(2\arccos(x))$.
   (f) $\sin(3\arccos(x))$.
   (g) $\cos(2\arcsin(x))$.
   (h) $\cos(3\arccos(x))$.

2. Express as the following as an algebraic expression of x. In other words, "get rid" of the trigonometric and inverse trigonometric expressions.

   (a) $\cos^2(\arctan x)$.
   (b) $-\sin^2(\operatorname{arccot} x)$.
   (c) $\dfrac{1}{\cos(\arcsin x)}$.
   (d) $-\dfrac{1}{\sin(\arccos x)}$.

3. Rewrite as a rational function of t. This problem will be later used to derive the Euler substitutions (an important technique for integrating).

   (a) $\cos\left(2\arctan t\right)$.
   (b) $\sin\left(2\arctan t\right)$.
   (c) $\tan\left(2\arctan t\right)$.
   (d) $\cot\left(2\arctan t\right)$.
   (e) $\csc\left(2\arctan t\right)$.
   (f) $\sec\left(2\arctan t\right)$.
   (g) $\cos\left(2\operatorname{arccot} t\right)$.
   (h) $\sin\left(2\operatorname{arccot} t\right)$.
   (i) $\tan\left(2\operatorname{arccot} t\right)$.
   (j) $\cot\left(2\operatorname{arccot} t\right)$.
   (k) $\csc\left(2\operatorname{arccot} t\right)$.
   (l) $\sec\left(2\operatorname{arccot} t\right)$.

4. Compute the derivative (derive the formula).

   (a) $(\arctan x)'$.
   (b) $(\operatorname{arccot} x)'$.
   (c) $(\arcsin x)'$.
   (d) $(\arccos x)'$.
   (e) Let arcsec denote the inverse of the secant function. Compute $(\operatorname{arcsec} x)'$.

5. (a) Let $a+b \neq k\pi$, $a \neq k\pi + \frac{\pi}{2}$ and $b \neq k\pi + \frac{\pi}{2}$ for any $k \in \mathbb{Z}$ (integers). Prove that

   $$\frac{\tan a + \tan b}{1 - \tan a \tan b} = \tan(a+b) \quad .$$

   (b) Let x and y be real. Prove that, for $xy \neq 1$, we have

   $$\arctan x + \arctan y = \arctan\left(\frac{x+y}{1-xy}\right)$$

   if the left hand side lies between $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

6. Evaluate the indefinite integral. Illustrate the steps of your solutions.

(a) $\int x\sin x\mathrm{d}x$.

(b) $\int xe^{-x}\mathrm{d}x$.

(c) $\int x^2e^x\mathrm{d}x$.

(d) $\int x\sin(-2x)\mathrm{d}x$.

(e) $\int x^2\cos(3x)\mathrm{d}x$.

(f) $\int x^2e^{-2x}\mathrm{d}x$.

(g) $\int x\sin(2x)\mathrm{d}x$.

(h) $\int x\cos(3x)\mathrm{d}x$.

(i) $\int x^2e^{2x}\mathrm{d}x$.

(j) $\int x^3e^x\mathrm{d}x$.

7. Evaluate the indefinite integral. Illustrate the steps of your solutions.

(a) $\int x^2\cos(2x)\mathrm{d}x$.

(b) $\int x^2e^{ax}\mathrm{d}x$, where a is a constant.

(c) $\int x^2e^{-ax}\mathrm{d}x$, where a is a constant.

(d) $\int x^2\frac{(e^{ax}+e^{-ax})^2}{4}\mathrm{d}x$, where a is a constant.

(e) $\int \frac{1}{\cos^2 x}\mathrm{d}x$. (Hint: This problem does not require integration by parts. What is the derivative of $\tan x$?)

(f) $\int(\tan^2 x)\mathrm{d}x$. (Hint: This problem does not require integration by parts. We can use $\tan^2 x = \frac{1}{\cos^2 x} - 1$ and the previous problem.)

(g) $\int x\tan^2 x\mathrm{d}x$. (Hint: $\tan^2 x\mathrm{d}x = \mathrm{d}(F(x))$, where $F(x)$ is the answer from the preceding problem).

(h) $\int e^{-\sqrt{x}}\mathrm{d}x$.

(i) $\int \cos^2 x\ \mathrm{d}x$.

(j) $\int \frac{x}{1+x^2}\mathrm{d}x$ (Hint: use substitution rule, don't use integration by parts)

(k) $\int(\arctan x)\mathrm{d}x$.

(l) $\int(\arcsin x)\mathrm{d}x$.

(m) $\int(\arcsin x)^2\mathrm{d}x$. (Hint: Try substituting $x=\sin y$.)

(n) $\int \arctan\left(\frac{1}{x}\right)\mathrm{d}x$.

(o) $\int \sin xe^x\mathrm{d}x$

(p) $\int \cos xe^x\mathrm{d}x$

(q) $\int \sin(\ln(x))\mathrm{d}x$.

(r) $\int \cos(\ln(x))\mathrm{d}x$.

(s) $\int \ln x\mathrm{d}x$

(t) $\int x\ln x\ \mathrm{d}x$.

(u) $\int \frac{\ln x}{\sqrt{x}}\mathrm{d}x$.

(v) $\int(\ln x)^2\mathrm{d}x$.

(w) $\int(\ln x)^3\mathrm{d}x$.

(x) $\int x^2\cos^2 x\mathrm{d}x$. (This problem is related to Problem 7.d as $\cos x = \frac{e^{ix}+e^{-ix}}{2}$).

8. Compute $\int x^ne^x\mathrm{d}x$, where n is a non-negative integer.

9. Integrate. Illustrate the steps of your solution.

(a) $\int \frac{1}{x+1}\mathrm{d}x$

(b) $\int \frac{x-1}{x+1}\mathrm{d}x$

(c) $\int \frac{1}{(x+1)^2}\mathrm{d}x$

(d) $\int \frac{x}{(x+1)^2}\mathrm{d}x$

(e) $\int \frac{1}{(2x+3)^2}\mathrm{d}x$

(f) $\int \frac{x}{2x^2+3}\mathrm{d}x$

(g) $\int \frac{1}{2x^2+3}\mathrm{d}x$

(h) $\int \frac{x}{2x^2+x+1}\mathrm{d}x$.

(i) $\int \frac{x}{2x^2+x+3}\mathrm{d}x$

(j) $\int \frac{x}{x^2-x+3}\mathrm{d}x$

(k) $\int \frac{1}{(x^2+1)^2} \mathrm{d}x$

(l) $\int \frac{1}{(x^2+x+1)^2} \mathrm{d}x$

(m) $\int \frac{1}{(x^2+1)^3} \mathrm{d}x$

10. Let a, b, c, A, B be real numbers. Suppose in addition $a \neq 0$ and $b^2 - 4ac < 0$. Integrate

$$\int \frac{Ax+B}{ax^2+bx+c} \mathrm{d}x \quad .$$

The purpose of this exercise is to produce a formula in form ready for implementation in a computer algebra system.

11. Let a, b, c, A, B be real numbers and let $n > 1$ be an integer. Suppose in addition $a \neq 0$ and $b^2 - 4ac < 0$. Let

$$J(n) = \int \frac{1}{\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)^n} \mathrm{d}x \quad .$$

(a) Express the integral

$$\int \frac{Ax+B}{(ax^2+bx+c)^n} \mathrm{d}x$$

via $J(n)$.

(b) Express $J(n)$ recursively via $J(n-1)$

The purpose of this exercise is to produce a formula in form ready for implementation in a computer algebra system.

12. Integrate. Some of the examples require partial fraction decomposition and some do not. Illustrate the steps of your solution.

(a) $\int \frac{1}{4x^2+4x+1} \mathrm{d}x$

(b) $\int \frac{1}{1-x^2} \mathrm{d}x$

(c) $\int \frac{1}{5-x^2} \mathrm{d}x$

(d) $\int \frac{x}{4x^2+x+\frac{1}{16}} \mathrm{d}x$

(e) $\int \frac{x+1}{2x^2+x} \mathrm{d}x$

(f) $\int \frac{x}{4x^2+x+5} \mathrm{d}x$

(g) $\int \frac{x}{4x^2+x-5} \mathrm{d}x$

(h) $\int \frac{x}{3x^2+x-2} \mathrm{d}x$

(i) $\int \frac{x}{3x^2+x+2} \mathrm{d}x$

(j) $\int \frac{x}{2x^2+x+1} \mathrm{d}x$

(k) $\int \frac{x}{2x^2+x-1} \mathrm{d}x$

(l) $\int \frac{1}{x^2+x+1} \mathrm{d}x$

(m) $\int \frac{1}{2x^2+5x+1} \mathrm{d}x$

13. Evaluate the indefinite integral. Illustrate all steps of your solution.

(a) $\int \frac{x^3+4}{x^2+4} \mathrm{d}x$

(b) $\int \frac{4x^2}{2x^2-1} \mathrm{d}x$

(c) $\int \frac{x^3}{x^2+2x-3} \mathrm{d}x$

(d) $\int \frac{x^3}{x^2+3x-4} \mathrm{d}x$

(e) $\int \frac{x^3}{2x^2+3x-5} \mathrm{d}x$

(f) $\int \frac{x^2+1}{(x-3)(x-2)^2} \mathrm{d}x$

(g) $\int \frac{x^4}{(x+1)^2(x+2)} \mathrm{d}x$

(h) $\int \frac{15x^2-4x-81}{(x-3)(x+4)(x-1)} \mathrm{d}x$

(i) $\int \frac{x^4+10x^3+18x^2+2x-13}{x^4+4x^3+3x^2-4x-4} \mathrm{d}x$

Check first that $(x-1)(x+2)^2(x+1) = x^4+4x^3+3x^2-4x-4$.

(j) $\int \frac{x^4}{(x^2+2)(x+2)} \mathrm{d}x$

(k) $\int \frac{x^5}{x^3-1} \mathrm{d}x$

(l) $\int \frac{x^4}{(x^2+2)(x+1)^2} \mathrm{d}x$

(m) $\displaystyle\int \frac{3x^2+2x-1}{(x-1)(x^2+1)}\mathrm{d}x$

(n) $\displaystyle\int \frac{x^2-1}{x(x^2+1)^2}\mathrm{d}x$

14. Integrate

$$\int \frac{x^6 - x^5 + \frac{9}{2}x^4 - 4x^3 + \frac{13}{2}x^2 - \frac{7}{2}x + \frac{11}{4}}{x^5 - x^4 + 3x^3 - 3x^2 + \frac{9}{4}x - \frac{9}{4}}\mathrm{d}x \quad .$$