Calculus II
Integrals of the form $\displaystyle\int \frac{Ax+B}{(ax^2+bx+c)^n}\mathrm{d}x$, denominator has no real roots

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Building block IIIb: example illustrating main idea

Example

Integrate $\int \frac{dx}{(x^2+1)^2}$. We start with an already known integral:

$$\begin{aligned}
\arctan x + C &= \int \frac{1}{x^2+1} dx \\
&= \frac{1}{x^2+1} x - \int x d\left(\frac{1}{x^2+1}\right) \\
&= \frac{x}{x^2+1} - \int x\left(-\frac{2x}{(x^2+1)^2}\right) dx \\
&= \frac{x}{x^2+1} + 2\int \frac{-1+x^2+1}{(x^2+1)^2} dx \\
&= \frac{x}{x^2+1} + 2\int \frac{1}{x^2+1} dx - 2\int \frac{1}{(x^2+1)^2} dx \\
&= \frac{x}{x^2+1} + 2\arctan x - 2\int \frac{dx}{(x^2+1)^2}
\end{aligned}$$

Building block IIIb: example illustrating main idea

Example

Integrate $\int \frac{\mathrm{d}x}{(x^2+1)^2}$. We start with an already known integral:

$$\arctan x + C = \frac{x}{x^2+1} + 2\arctan x - 2\int \frac{\mathrm{d}x}{(x^2+1)^2}$$

Rearrange terms and divide by 2 to get the desired integral:

$$\int \frac{\mathrm{d}x}{(1+x^2)^2} = \frac{1}{2}\left(\frac{x}{x^2+1} + \arctan x\right) + K .$$

Building block IIIb

- Building block IIIa:

$$J(1) = \int \frac{1}{(x^2+1)} \mathrm{d}x = \arctan x + C \quad .$$

- Block IIIb:

$$J(n) = \int \frac{1}{(x^2+1)^n} \mathrm{d}x$$

- Unlike other cases, IIIb is much harder than IIIa.
- Set $J(n) = \int \frac{1}{(x^2+1)^n} \mathrm{d}x$. We are looking for a formula for $J(n)$. We know $J(1) = \arctan x + C$ (this is block IIIa).
- We start by $J(n-1) = \int \frac{1}{(x^2+1)^{n-1}} \mathrm{d}x$ and integrate by parts.
- In this way we end up expressing $J(n)$ via $J(n-1)$.
- We work our way from $J(n)$ to $J(n-1)$, from $J(n-1)$ to $J(n-2)$, and so on, until we get to $J(1)$.

Example

Recall that $J(n) = \int \frac{1}{(x^2+1)^n} \mathrm{d}x$. We have that:

$$
\begin{aligned}
J(n-1) &= \int \frac{1}{(x^2+1)^{n-1}} \mathrm{d}x \\
&= \frac{1}{(x^2+1)^{n-1}} x - \int x \mathrm{d} \left(\frac{1}{(1+x^2)^{n-1}} \right) \\
&= \frac{x}{(x^2+1)^{n-1}} - \int x \left(\frac{(-n+1)2x}{(1+x^2)^n} \right) \mathrm{d}x \\
&= \frac{x}{(x^2+1)^{n-1}} + 2(n-1) \int \frac{1+x^2-1}{(1+x^2)^n} \mathrm{d}x \\
&= \frac{x}{(x^2+1)^{n-1}} + 2(n-1) \int \frac{1}{(1+x^2)^{n-1}} \mathrm{d}x \\
&\quad -2(n-1) \int \frac{1}{(1+x^2)^n} \mathrm{d}x \\
&= \frac{x}{(x^2+1)^{n-1}} + 2(n-1)J(n-1) - 2(n-1)J(n) \quad .
\end{aligned}
$$

Example

Recall that $J(n) = \int \frac{1}{(x^2+1)^n} \mathrm{d}x$. We have that:

$$J(n-1) = \frac{x}{(x^2+1)^{n-1}} + 2(n-1)J(n-1) - 2(n-1)J(n) \quad .$$

Rearrange to get:

$$\begin{aligned} 2(n-1)J(n) &= \frac{x}{(x^2+1)^{n-1}} + (2n-3)J(n-1) \\ J(n) &= \frac{x}{(2n-2)(x^2+1)^{n-1}} + \frac{2n-3}{2n-2}J(n-1) \quad . \end{aligned}$$

In this way we expressed $J(n)$ using $J(n-1)$. We apply the above formula consecutively:
$J(n) = \frac{x}{(2n-2)(x^2+1)^{n-1}} + \frac{2n-3}{2n-2}\left(\frac{x}{(2n-4)(x^2+1)^{n-2}} + \frac{2n-5}{2n-4}J(n-2)\right) = \dots$
and so on. The above can be used to write a formula for the final result, but that is as complicated as the process above.