Calculus I
The Chain Rule

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Outline

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The Chain Rule

- What is the derivative of $f(x) = \sqrt{x^2+1}$?
- The Power Rule doesn't tell us how to find the derivative.
- f is a composite function $g \circ h$:
- $y = g(u) = \sqrt{u}$.
- $u = h(x) = x^2 + 1$.
- Then $y = f(x) = g(h(x)) = g(x^2+1) = \sqrt{x^2+1}$.
- We know the derivatives of g and h:
- $g'(u) = \frac{1}{2}u^{-\frac{1}{2}}$.
- $h'(x) = 2x$.
- It would be nice if we could find the derivative of f in terms of the derivatives of y and u.
- It turns out that the derivative of the composition $g \circ h$ is the product of the derivative of g and the derivative of h.
- This important fact is called the Chain Rule.

The Chain Rule

Let g and h be functions. Recall that the composite function $f = g \circ h$ is defined via $f(x) = g(h(x))$.

Theorem

Let h be differentiable at x and let g be a differentiable at $h(x)$. Then the composite function $f = g \circ h$ is differentiable at x and f' is given by the product

$$
\begin{array}{rcll}
f'(x) &=& g'(h(x)) \cdot h'(x) & \textit{(notation 1)} \\
&&& \textit{equivalently:} \\
f'(x) = (g(u))' &=& g'(u)u' \qquad \textit{where } u = h(x) & \textit{(notation 2)} \\
\dfrac{\mathrm{d}y}{\mathrm{d}x} &=& \dfrac{\mathrm{d}y}{\mathrm{d}u}\dfrac{\mathrm{d}u}{\mathrm{d}x} \qquad \textit{where } y = g(u) & \textit{(notation 3)} \quad .
\end{array}
$$

The last equality uses the Leibniz notation (due to G. Leibniz (1646-1716)).

Chain rule notations

- As we saw, the chain rule can be written using a number of notations:

$$
\begin{array}{rclrr}
(g(h(x)))' &=& g'(h(x)) \cdot h'(x) & & \text{(notation 1)}\\
(g(u))' &=& g'(u)u' & \text{where } u = h(x) & \text{(notation 2)}\\
\dfrac{\mathrm{d}y}{\mathrm{d}x} &=& \dfrac{\mathrm{d}y}{\mathrm{d}u}\dfrac{\mathrm{d}u}{\mathrm{d}x} & \text{where } y = g(u) & \text{(notation 3)}
\end{array} .
$$

- The three notations are all accepted and can be used interchangeably.
- Most authors tend to prefer one of these notations over the others.
- In order to exercise ourselves we shall use all three notations throughout our course.
- There are additional notations (not covered here) used in practice.
- Whenever in doubt about derivative notation, if possible, request clarification.

$$
\begin{array}{rcll}
(g(h(x)))' & = & g'(h(x)) \cdot h'(x) & & \text{(notation 1)} \\
(g(u))' & = & g'(u)u' & \text{where } u = h(x) & \text{(notation 2)} \\
\frac{dy}{dx} & = & \frac{dy}{du}\frac{du}{dx} & \text{where } y = g(u) & \text{(notation 3)}
\end{array} .
$$

Example (Chain Rule, Notation 1)

$$
\begin{array}{rrcl}
\text{Differentiate} & f(x) & = & \sqrt{x^2+1}. \\
\text{Let} & h(x) & & \\
\text{Let} & g(u) & = & \\
& & & \\
\text{Chain Rule:} & f'(x) & = & g'(h(x))h'(x) \\
& & = & \left(\qquad \right) (\quad)
\end{array}
$$

$$
\begin{aligned}
(g(h(x)))' &= g'(h(x)) \cdot h'(x) & & \text{(notation 1)}\\
(g(u))' &= g'(u)u' & \text{where } u = h(x) & \text{ (notation 2)}\\
\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} & \text{where } y = g(u) & \text{ (notation 3)} \quad .
\end{aligned}
$$

Example (Chain Rule, Notation 2)

$$
\begin{array}{rrcl}
\text{Differentiate} & f(x) & = & \sqrt{x^2+1}.\\
\text{Let} & u & = & \\
\text{Let} & g(u) & = & \\
\text{Then} & f(x) & = & g(u).\\
\text{Chain Rule:} & f'(x) & = & g'(u)u'
\end{array}
$$

$$
\begin{array}{rcll}
(g(h(x)))' &=& g'(h(x)) \cdot h'(x) & & \text{(notation 1)}\\
(g(u))' &=& g'(u)u' & \text{where } u = h(x) & \text{(notation 2)}\\
\frac{\mathrm{d}y}{\mathrm{d}x} &=& \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} & \text{where } y = g(u) & \text{(notation 3)}
\end{array} .
$$

Example (Chain Rule, Notation 3)

$$
\begin{array}{rrcl}
\text{Differentiate} & y &=& \sqrt{x^2+1}.\\
\text{Let} & u &=& \\
\text{Then} & y &=& \\
\text{Chain Rule:} & \frac{\mathrm{d}y}{\mathrm{d}x} &=& \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}\\
& &=& \left(\qquad\right)(\quad)\\
& &=&
\end{array}
$$

$$
\begin{array}{rcll}
(g(h(x)))' & = & g'(h(x)) \cdot h'(x) & & \text{(notation 1)} \\
(g(u))' & = & g'(u)u' & \text{where } u = h(x) & \text{(notation 2)} \\
\dfrac{dy}{dx} & = & \dfrac{dy}{du}\dfrac{du}{dx} & \text{where } y = g(u) & \text{(notation 3)}
\end{array} .
$$

Example (Chain Rule, Notation 1, square root of a trigonometric function)

$$
\begin{array}{rrcl}
\text{Differentiate} & f(x) & = & \sqrt{\sin x + 2}. \\
\text{Let} & h(x) & & \\
\text{Let} & g(u) & = & \\
& & & \\
\text{Chain Rule:} & f'(x) & = & g'(h(x))h'(x) \\
& & = & \left(\qquad \right) (\qquad)
\end{array}
$$

$$\begin{array}{rcll} (g(h(x)))' & = & g'(h(x)) \cdot h'(x) & & \text{(notation 1)} \\ (g(u))' & = & g'(u)u' & \text{where } u = h(x) & \text{(notation 2)} \\ \dfrac{\mathrm{d}y}{\mathrm{d}x} & = & \dfrac{\mathrm{d}y}{\mathrm{d}u}\dfrac{\mathrm{d}u}{\mathrm{d}x} & \text{where } y = g(u) & \text{(notation 3)} \end{array} .$$

Example (Chain Rule, Notation 2)

$$\begin{array}{rrcl} \text{Differentiate} & f(x) & = & \cos\left(x^3\right). \\ \text{Let} & u & = & \\ \text{Let} & g(u) & = & \\ \text{Then} & f(x) & = & g(u). \\ \text{Chain Rule:} & f'(x) & = & g'(u)u' \end{array}$$

$$
\begin{aligned}
(g(h(x)))' &= g'(h(x)) \cdot h'(x) & & \text{(notation 1)} \\
(g(u))' &= g'(u)u' & \text{where } u = h(x) & \text{(notation 2)} \\
\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} & \text{where } y = g(u) & \text{(notation 3)} \quad .
\end{aligned}
$$

Example (Chain Rule, Notation 2)

$$
\begin{aligned}
\text{Differentiate} \quad f(x) &= \cos^3 x. \\
\text{Let} \quad u &= \\
\text{Let} \quad g(u) &= \\
\text{Then} \quad f(x) &= g(u). \\
\text{Chain Rule:} \quad f'(x) &= g'(u)u'
\end{aligned}
$$

- In the example $y = \cos^3 x$, the outer function was a power function: $y = u^3$.
- The derivative was $\frac{dy}{dx} = 3u^2 \frac{du}{dx} = (3\cos^2 x)(-\sin x)$.
- We can generalize this:

Observation (The Power Rule Combined with the Chain Rule)

If n is any real number and $u = h(x)$ is differentiable, then

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$$

Alternatively,

$$\frac{d}{dx}[h(x)]^n = n[h(x)]^{n-1} \cdot h'(x)$$

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$$

$$\begin{aligned}
(g(h(x)))' &= g'(h(x))\cdot h'(x) && && \text{(notation 1)}\\
(g(u))' &= g'(u)u' && \text{where } u = h(x) && \text{(notation 2)}\\
\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} && \text{where } y = g(u) && \text{(notation 3)}
\end{aligned} .$$

Example (Chain Rule, Notation 3, Power Rule)

$$\begin{aligned}
\text{Differentiate} \quad y &= (x^3-1)^{100}.\\
\text{Let} \quad u &= \\
\text{Then} \quad y &= \\
\text{Chain Rule:} \quad \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx}\\
&= \Big(\qquad \Big)\Big(\qquad \Big)\\
&=
\end{aligned}$$

$$\frac{d}{dx}[h(x)]^n = n[h(x)]^{n-1} \cdot h'(x)$$

$$\begin{array}{rcll}
(g(h(x)))' & = & g'(h(x)) \cdot h'(x) & & \text{(notation 1)} \\
(g(u))' & = & g'(u)u' & \text{where } u = h(x) & \text{(notation 2)} \\
\frac{dy}{dx} & = & \frac{dy}{du}\frac{du}{dx} & \text{where } y = g(u) & \text{(notation 3)}
\end{array} .$$

Example (Chain Rule, Notation 1, Power Rule)

$$\begin{array}{rrcl}
\text{Differentiate} & f(x) & = & \frac{1}{\sqrt[3]{x^2+x+1}}. \\
\text{Let} & h(x) & & \\
\text{Let} & g(u) & = & \\
 & & & \\
\text{Chain Rule:} & f'(x) & = & g'(h(x))h'(x) \\
 & & = & \Bigg(\qquad \Bigg) (\qquad)
\end{array}$$

Example (Chain Rule and Quotient Rule)

Find the derivative of

$$g(t) = \left(\frac{t-2}{2t+1}\right)^9 .$$

Power Rule and Chain Rule:

$$g'(t) = 9\left(\frac{t-2}{2t+1}\right)^8 \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{t-2}{2t+1}\right)$$

Quotient Rule:

$$= 9\left(\frac{t-2}{2t+1}\right)^8 \frac{\frac{\mathrm{d}}{\mathrm{d}t}(t-2)\cdot(2t+1)-(t-2)\frac{\mathrm{d}}{\mathrm{d}t}(2t+1)}{(2t+1)^2}$$

$$= 9\left(\frac{t-2}{2t+1}\right)^8 \frac{1\cdot(2t+1)-(t-2)\cdot 2}{(2t+1)^2}$$

$$= 9\left(\frac{t-2}{2t+1}\right)^8 \frac{\cancel{2t}+1-\cancel{2t}+4}{(2t+1)^2} = \frac{45(t-2)^8}{(2t+1)^{10}}.$$

Example

Find the derivative of $y = (2x+1)^5(x^3-x+1)^4$.

Product Rule:

$$y' = \frac{\mathrm{d}}{\mathrm{d}x}\left((2x+1)^5\right)(x^3-x+1)^4 + (2x+1)^5\frac{\mathrm{d}}{\mathrm{d}x}\left((x^3-x+1)^4\right)$$

Chain Rule:

$$= \left(5(2x+1)^4\frac{\mathrm{d}}{\mathrm{d}x}(2x+1)\right)(x^3-x+1)^4 + (2x+1)^5\left(4(x^3-x+1)^3\frac{\mathrm{d}}{\mathrm{d}x}(x^3-x+1)\right)$$

$$= 5(2x+1)^4(2)(x^3-x+1)^4 + 4(2x+1)^5(x^3-x+1)^3(3x^2-1)$$

Common factor $2(2x+1)^4(x^3-x+1)^3$:

$$= 2(2x+1)^4(x^3-x+1)^3(2(2x+1)(3x^2-1)+5(x^3-x+1)))$$

$$= 2(2x+1)^4(x^3-x+1)^3(17x^3+6x^2-9x+3)$$

Example (Chain Rule, general exponential function)

$$
\begin{aligned}
\text{Differentiate} \quad y &= 2^x. \\
y &= \left(e^{\ln 2}\right)^x \\
y &= e^{x \ln 2}. \\
\text{Let} \quad u &= \\
\text{Then} \quad y &= \\
\text{Chain Rule:} \quad \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} \\
&= (\quad)(\quad) \\
&= \\
&= \\
&=
\end{aligned}
$$

Example (Chain Rule, general exponential function)

$$
\begin{aligned}
\text{Differentiate} \quad y &= a^x. \\
y &= \left(e^{\ln a}\right)^x \\
y &= e^{x \ln a}. \\
\text{Let} \quad u &= x \ln a. \\
\text{Then} \quad y &= e^u.
\end{aligned}
$$

$$
\begin{aligned}
\text{Chain Rule:} \quad \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} \\
&= (e^u)(\ln a) \\
&= \left(e^{(x \ln a)}\right)(\ln a) \\
&= \left(e^{\ln a}\right)^x (\ln a) \\
&= a^x \ln a.
\end{aligned}
$$

Theorem (The Derivative of a^x)

$$\frac{\mathrm{d}}{\mathrm{d}x}(a^x) = a^x \ln a.$$

- We can add more “links” when we use the Chain Rule.
- $y = f(u)$
- $u = g(x)$
- $x = h(t)$
- Use the Chain Rule twice:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t}$$

Example (Using the Chain Rule twice)

Differentiate: $y = \sin\sqrt{10^x+1}.$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sin\sqrt{10^x+1}\right)$$

Chain Rule: $$= \left(\qquad\qquad\right)\frac{d}{dx}\left(\qquad\qquad\right)$$

Chain Rule: $$= \left(\qquad\qquad\right)\left(\qquad\qquad\right)\frac{d}{dx}(\qquad\qquad)$$

$$= \left(\qquad\qquad\right)\left(\qquad\qquad\right)(\qquad\qquad)$$

$$=$$

Example (Using the Chain Rule twice)

Differentiate: $y = e^{\tan(\pi x)}$.

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{\tan(\pi x)}\right)$$

Chain Rule: $$= \left(\quad \right) \frac{d}{dx} (\quad)$$

Chain Rule: $$= \left(\quad \right) \left(\quad \right) \frac{d}{dx} (\quad)$$

$$= \left(\quad \right) \left(\quad \right) (\quad)$$

$$=$$

Theorem (Chain rule)

Let g-differentiable at neighborhood of a, f-diff. at neighb. of $g(a)$.

$$(f(g(x)))'_{|x=a} = f'(g(a))g'(a)$$

Proof with additional assumptions -motivation for actual proof.

Suppose that $g(x) \neq g(a)$ so long as $x \neq a$. Set $G(y) = \frac{f(y)-f(g(a))}{y-g(a)}$. $G(y)$ is continuous at $g(a) \Rightarrow G(g(x))$ is continuous at a. Furthermore $g(x)$ is continuous at a.

$$\begin{aligned}
(f\circ g)'(a) &= \lim_{x\to a} \frac{f(g(x))-f(g(a))}{x-a}\\
&= \lim_{x\to a} \left(\frac{f(g(x))-f(g(a))}{g(x)-g(a)}\right)\left(\frac{g(x)-g(a)}{x-a}\right)\\
&= \lim_{x\to a} \left(\frac{f(g(x))-f(g(a))}{g(x)-g(a)}\right)\lim_{x\to a}\left(\frac{g(x)-g(a)}{x-a}\right)\\
&= \left(\lim_{y=g(x), y\to g(a)} \frac{f(y)-f(g(a))}{y-g(a)}\right) g'(a) = f'(g(a))g'(a) \quad .
\end{aligned}$$

$\square$

Theorem (Chain rule)

g-diff. near a, f-diff. near g(a) $\Rightarrow (f(g(a)))' = f'(g(a))g'(a)$.

Proof.

Define $Q(y) = \begin{cases} \frac{f(y)-f(g(a))}{y-g(a)}, & y \neq g(a) \\ f'(g(a)), & y = g(a) \end{cases}$. $Q(g(x))$ - defined for all x near a. Therefore $f'(g(a)) = \lim_{y\to a} Q(y) = \lim_{x \to a} Q(g(x))$.

$$\begin{aligned} Q(g(x))\frac{g(x)-g(a)}{x-a} &= \begin{cases} \frac{(f(g(x))-f(g(a)))}{(g(x)-g(a))}\frac{(g(x)-g(a))}{x-a}, & g(x) \neq g(a) \\ f'(g(a))\frac{g(a)-g(a)}{x-a} = 0, & g(x) = g(a) \end{cases} \\ &= \frac{f(g(x))-f(a)}{x-a}. \end{aligned}$$

$$\begin{aligned} (f\circ g)'(a) &= \lim_{x\to a}\frac{f(g(x))-f(g(a))}{x-a} = \lim_{x\to a} Q(g(x))\frac{g(x)-g(a)}{x-a} \\ &= \lim_{x\to a} Q(g(x)) \lim_{x\to a}\frac{g(x)-g(a)}{x-a} = f'(g(a))g'(a) \quad . \end{aligned}$$

$\square$