Calculus II

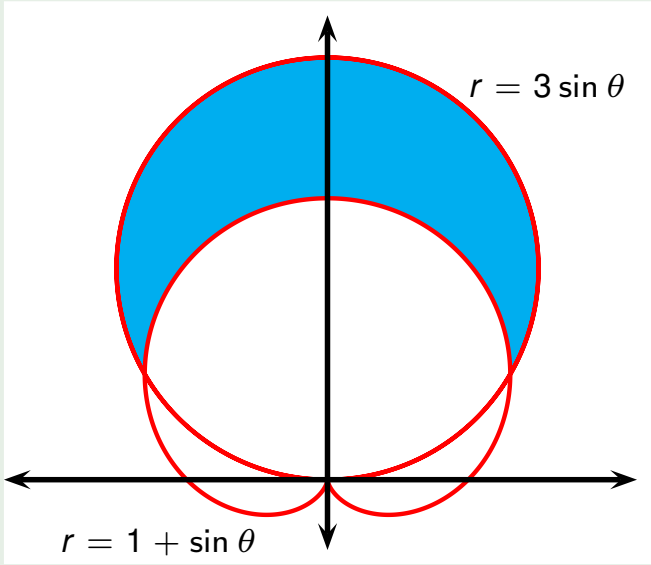
Area swept by cardioid $r = a + b\sin(\theta)$, $r = a + b\cos(n\theta)$

Todor Milev

2019

Example

Find the area that lies within the circle $r = 3\sin\theta$ and outside of the cardioid $r = 1 + \sin\theta$.



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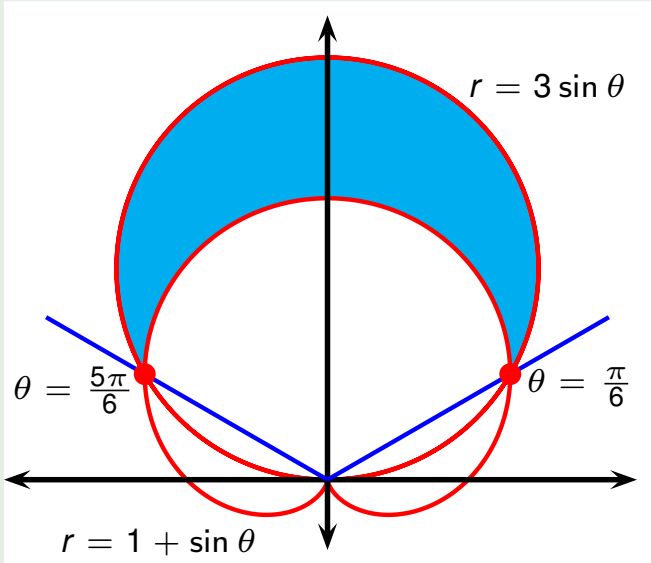

The curves meet if

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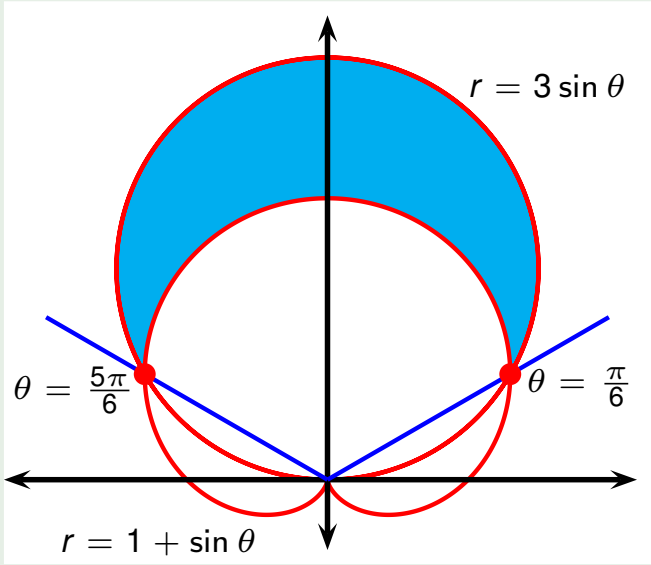

$$A = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (3\sin\theta)^2 d\theta - \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (1+\sin\theta)^2 d\theta$$

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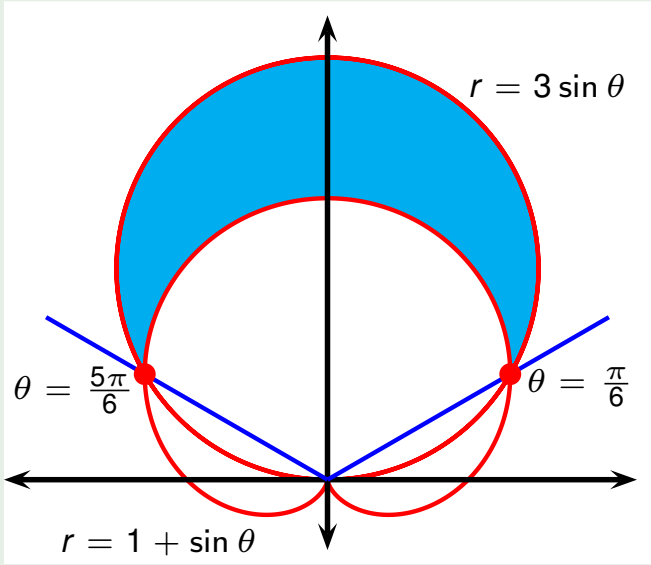

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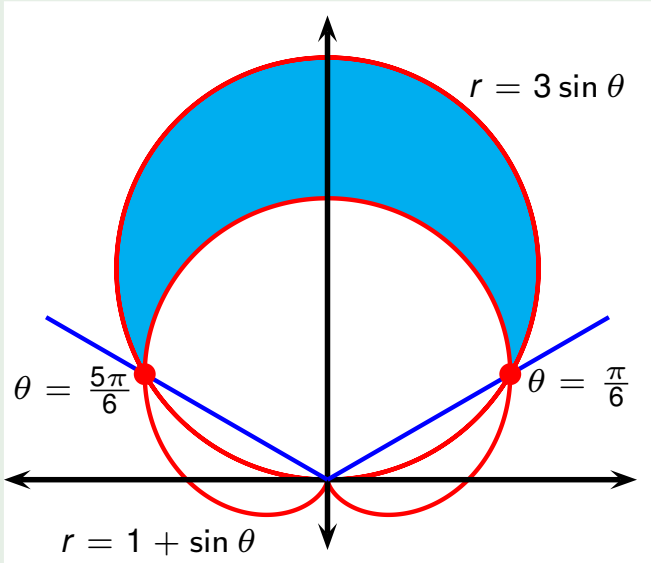

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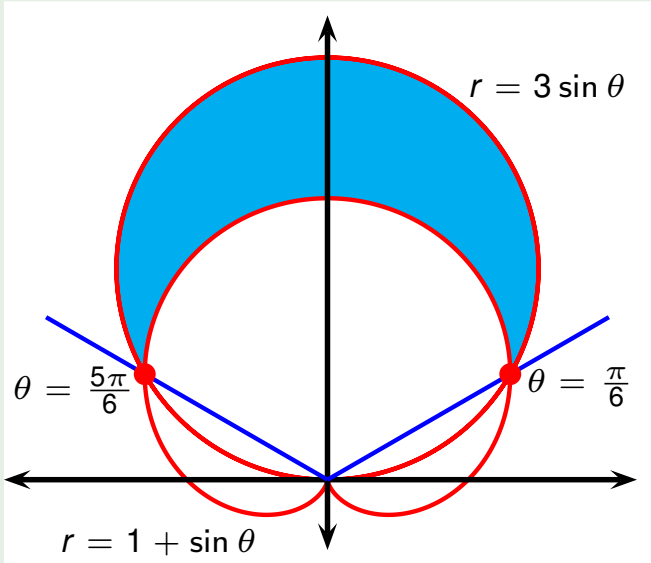

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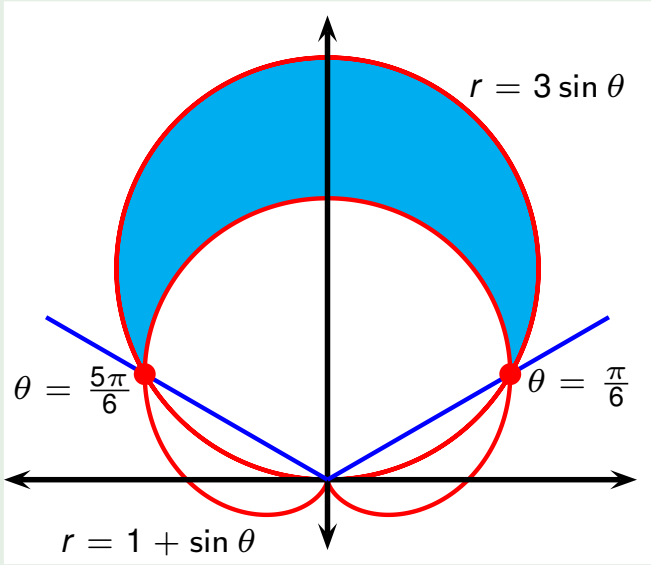

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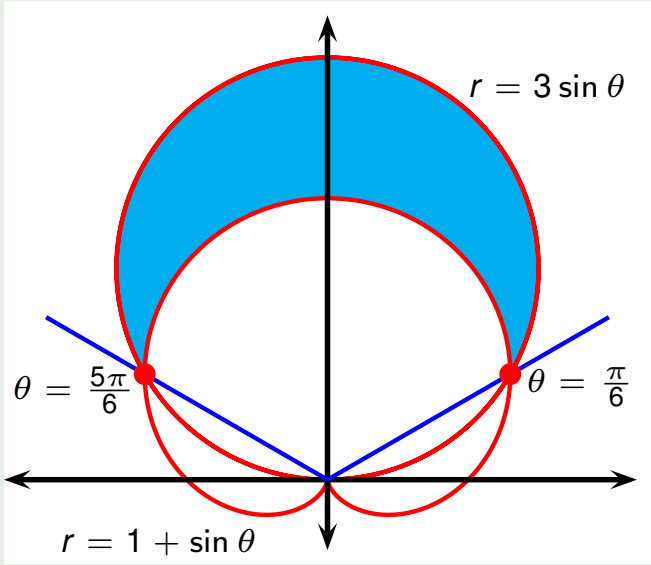

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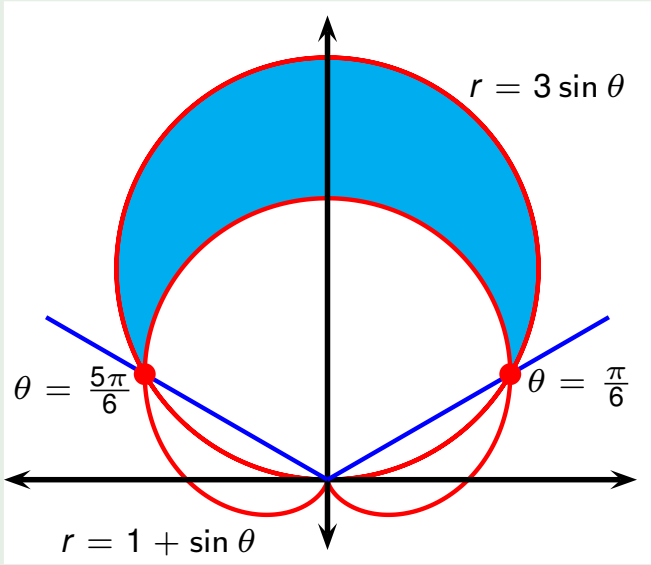

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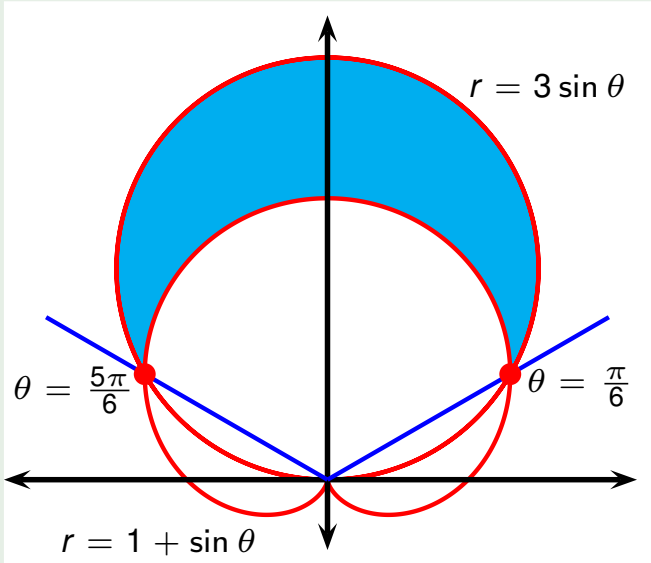

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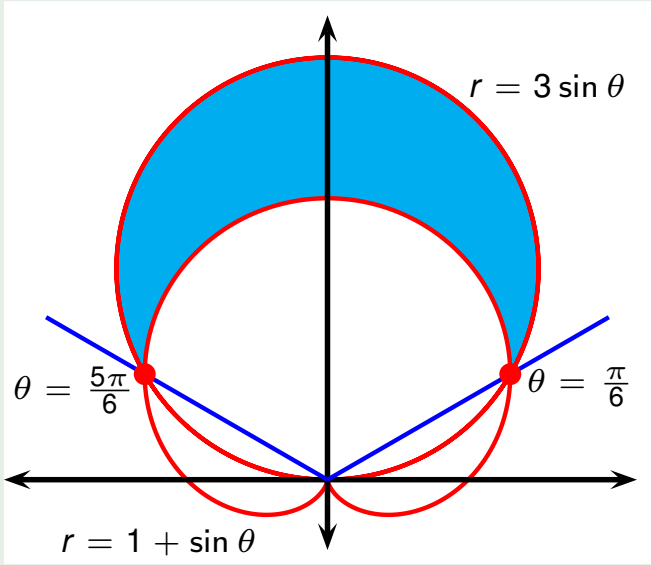

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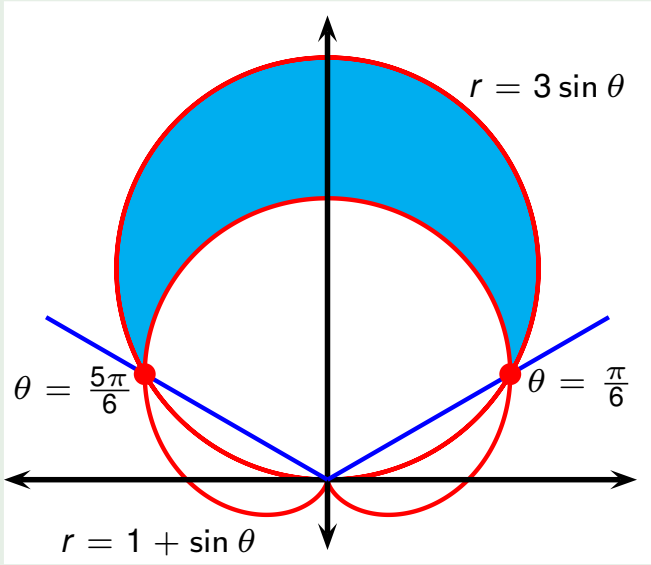

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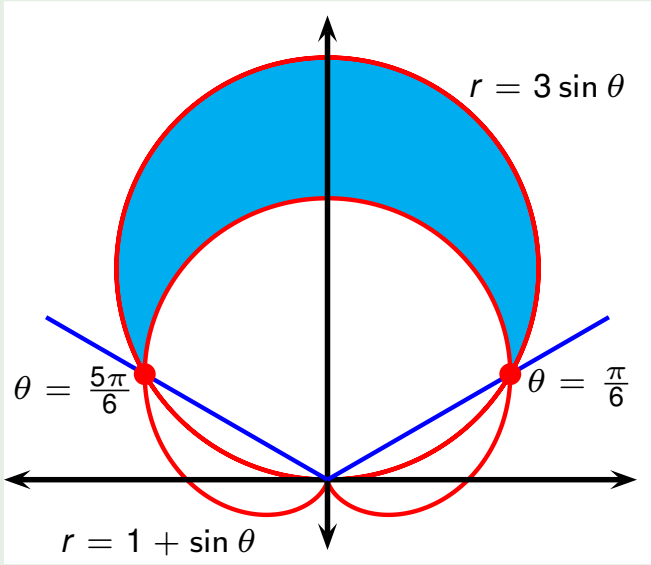

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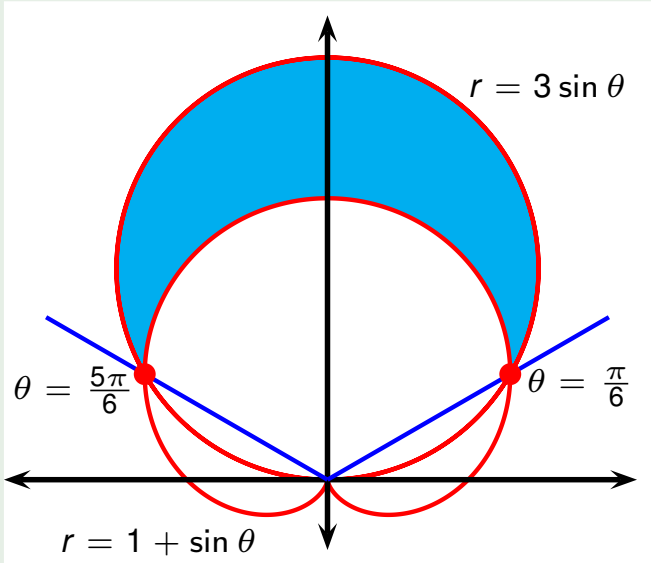

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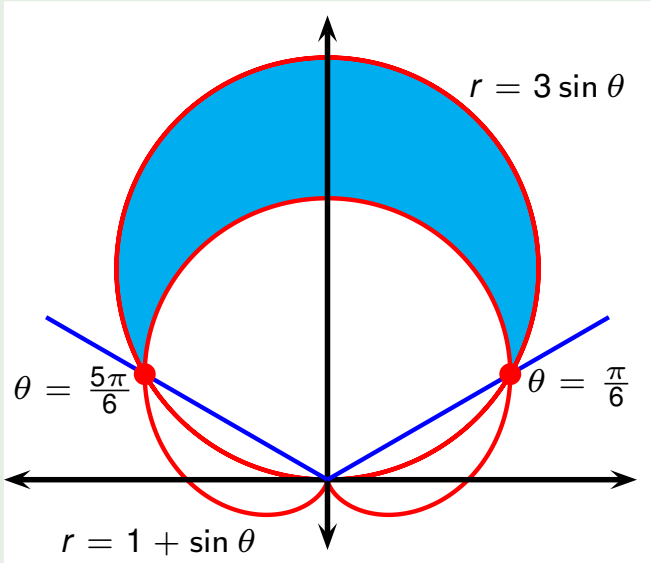

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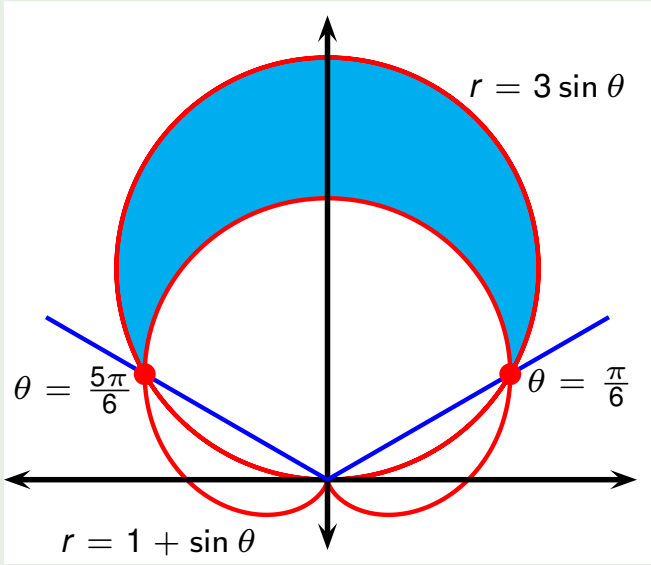

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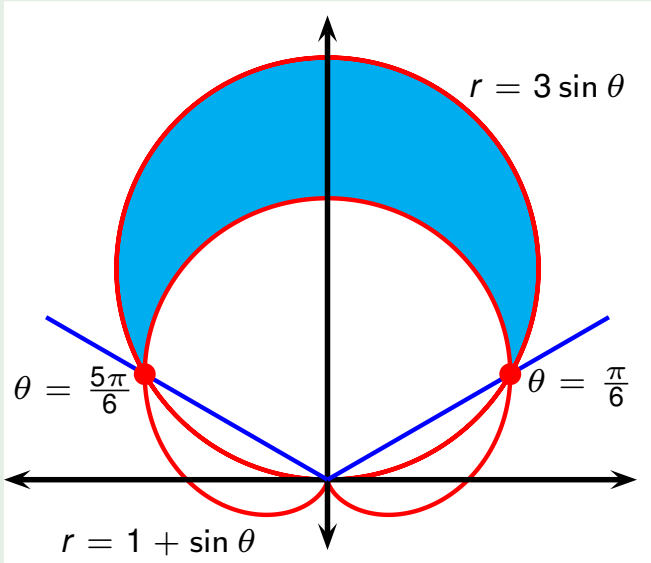

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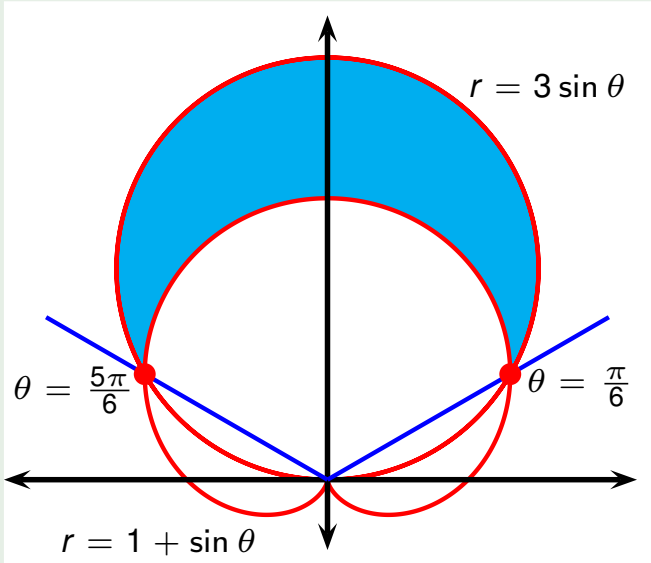
$$\begin{aligned} A &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (3\sin\theta)^2 d\theta - \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (1+\sin\theta)^2 d\theta \\ &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(9\sin^2\theta - (1 + 2\sin\theta + \sin^2\theta)\right) d\theta \\ &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(8\sin^2\theta - 1 - 2\sin\theta\right) d\theta \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(8\sin^2\theta - 1 - 2\sin\theta\right) d\theta \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (3 - 4\cos 2\theta - 2\sin\theta)\, d\theta \\ &= [3\theta - 2\sin 2\theta + 2\cos\theta]_{\frac{\pi}{6}}^{\frac{\pi}{2}} \\ &= \left(3\tfrac{\pi}{2} - 2\cdot 0 + 2\cdot 0\right) - \left(3\tfrac{\pi}{6} - 2 \quad + 2 \quad\right) \end{aligned}$$

Example

Find the area that lies within the circle $r = 3\sin\theta$ and outside of the cardioid $r = 1 + \sin\theta$.



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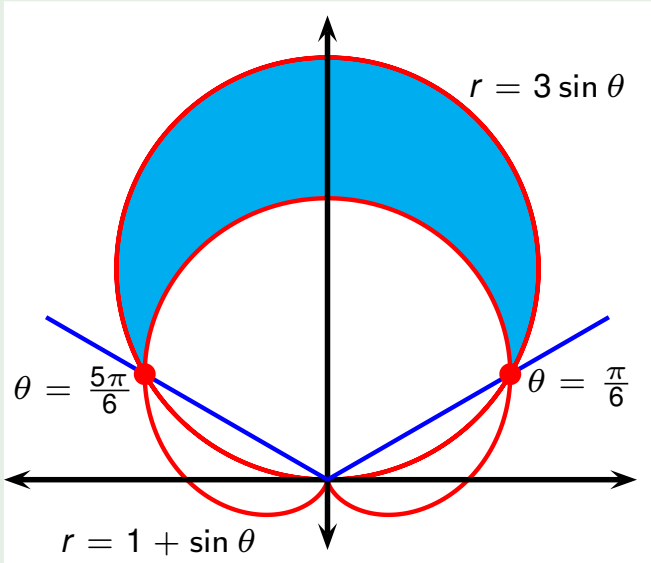
Example

Find the area that lies within the circle $r = 3\sin\theta$ and outside of the cardioid $r = 1 + \sin\theta$.



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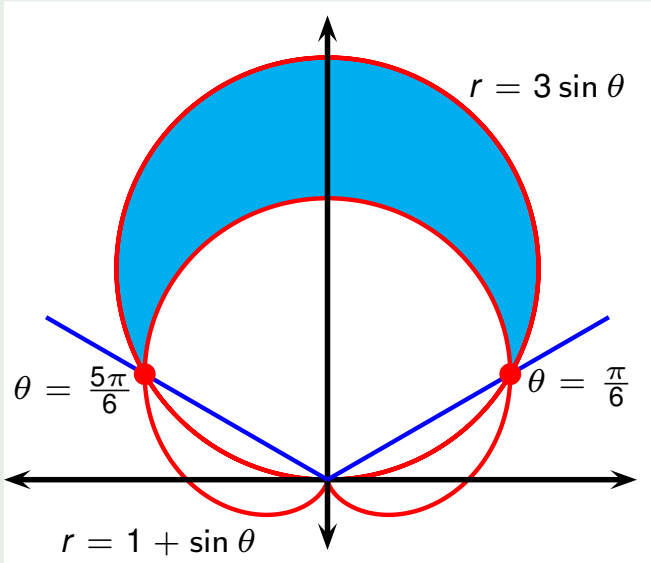
Example

Find the area that lies within the circle $r = 3\sin\theta$ and outside of the cardioid $r = 1 + \sin\theta$.



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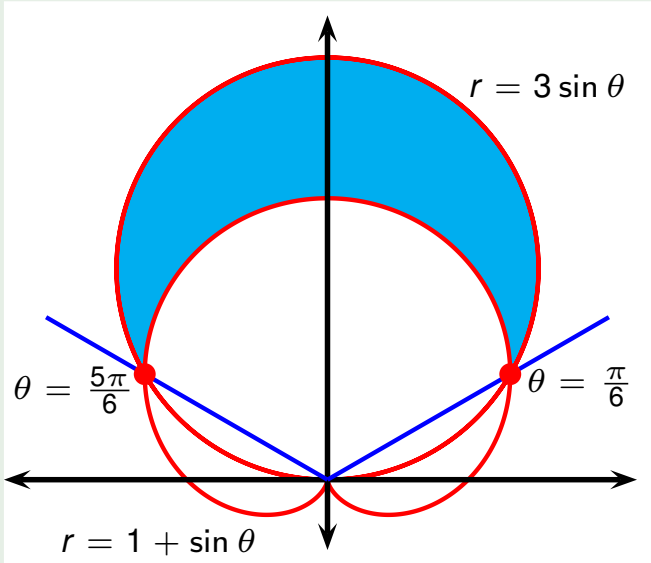
Example

Find the area that lies within the circle $r = 3\sin\theta$ and outside of the cardioid $r = 1 + \sin\theta$.



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$$\begin{aligned} A &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (3\sin\theta)^2 \mathrm{d}\theta - \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (1+\sin\theta)^2 \mathrm{d}\theta \\ &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(9\sin^2\theta - (1 + 2\sin\theta + \sin^2\theta)\right) \mathrm{d}\theta \\ &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(8\sin^2\theta - 1 - 2\sin\theta\right) \mathrm{d}\theta \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(8\sin^2\theta - 1 - 2\sin\theta\right) \mathrm{d}\theta \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (3 - 4\cos 2\theta - 2\sin\theta) \mathrm{d}\theta \\ &= [3\theta - 2\sin 2\theta + 2\cos\theta]_{\frac{\pi}{6}}^{\frac{\pi}{2}} \\ &= \left(3\tfrac{\pi}{2} - 2\cdot 0 + 2\cdot 0\right) - \left(3\tfrac{\pi}{6} - 2\tfrac{\sqrt{3}}{2} + 2\tfrac{\sqrt{3}}{2}\right) \\ &= \pi \end{aligned}$$