

# Precalculus

## Trig cofunction identities and angle-sum formulas

Todor Milev

2019

# Outline

## 1 Cofunction identities

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- 1 Cofunction identities
- 2 Trigonometric Functions of Sums of Angles

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- 1 Cofunction identities
- 2 Trigonometric Functions of Sums of Angles
- 3 Double Angle Formulas

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# Cofunction identities

## Proposition (Cofunction identities)

$$\begin{array}{llll} \sin\left(\frac{\pi}{2} - \alpha\right) & = & \cos \alpha & \sin\left(\frac{\pi}{2} + \alpha\right) & = & \cos \alpha \\ \cos\left(\frac{\pi}{2} - \alpha\right) & = & \sin \alpha & \cos\left(\frac{\pi}{2} + \alpha\right) & = & -\sin \alpha \end{array}$$

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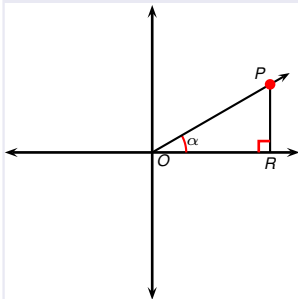
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- The proof provides intuition why the formulas are true.
- The Quadrant I part of the proof serves as a visual aid for memorization.
- There is an algebraically simpler (but theoretically advanced) way to prove the above identities through the angle sum formulas, derived in turn from Euler's formula (studied later/in another course).

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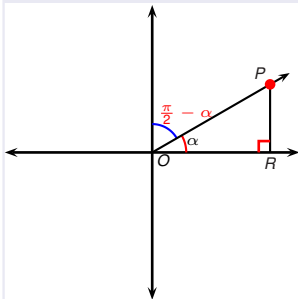


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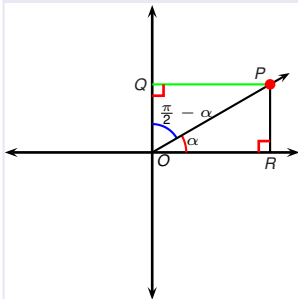


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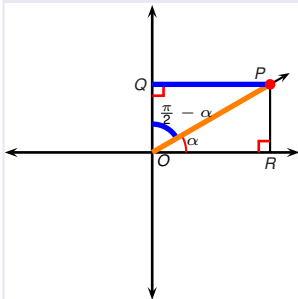


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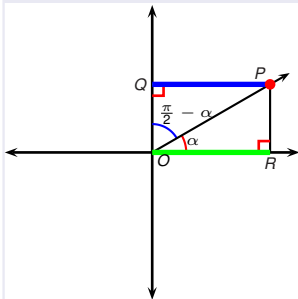


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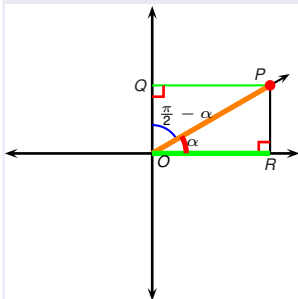


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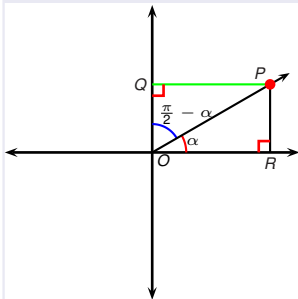


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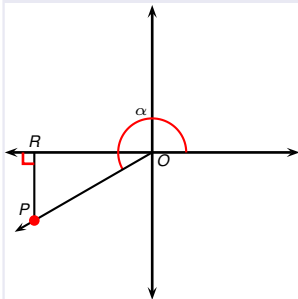


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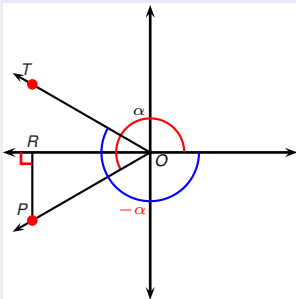


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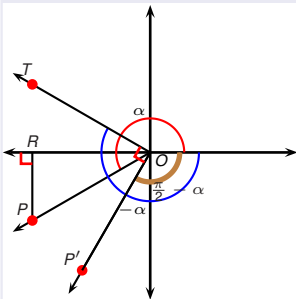


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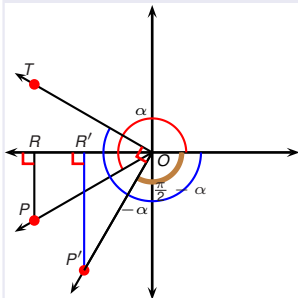


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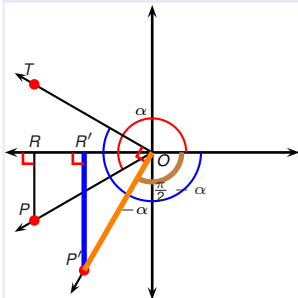


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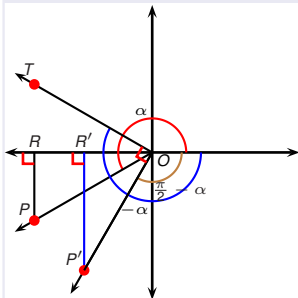


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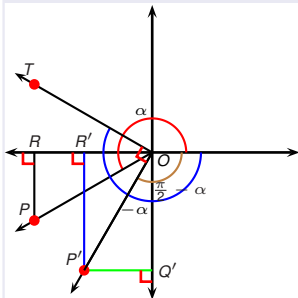


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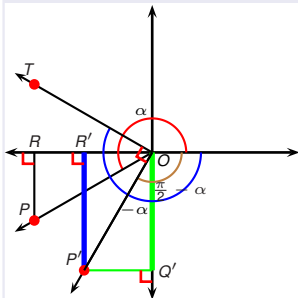


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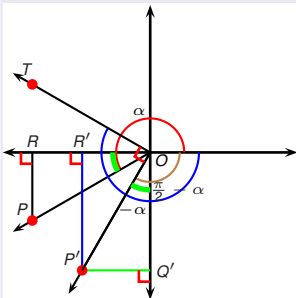


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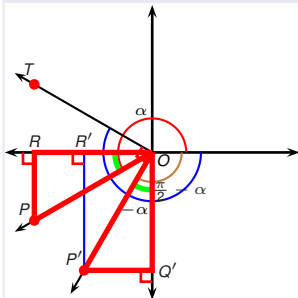
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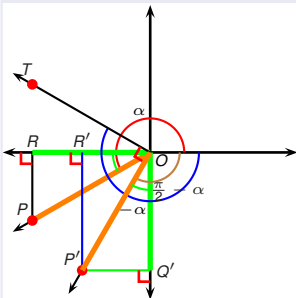
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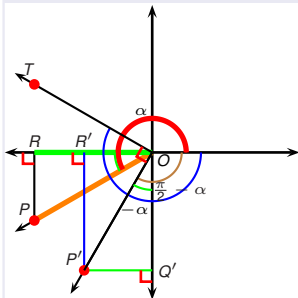


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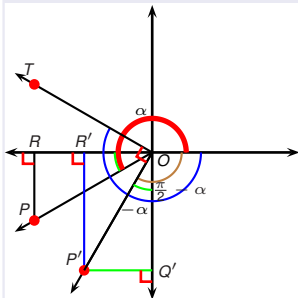


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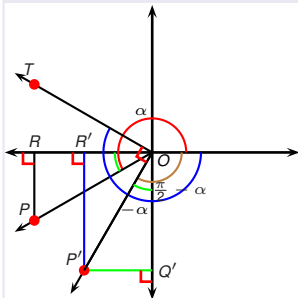


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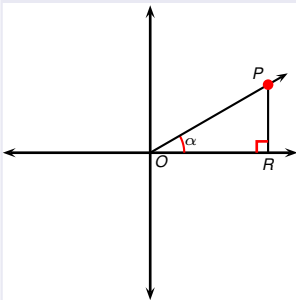


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We show  $\cos\left(\frac{\pi}{2} + \alpha\right) = -\sin \alpha$  when  $\alpha$  is in Quadrant I.

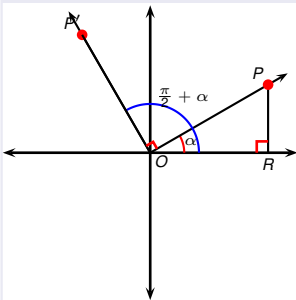


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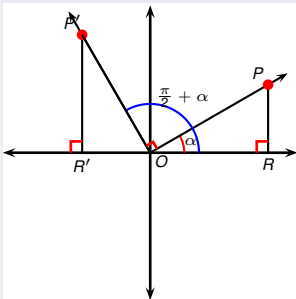


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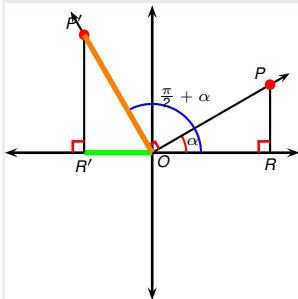


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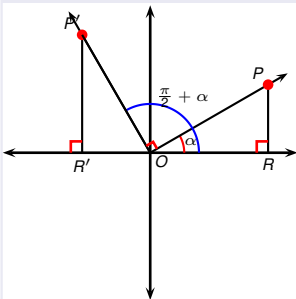


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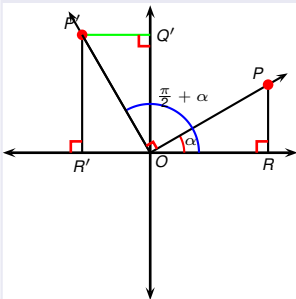


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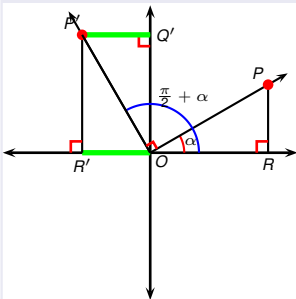


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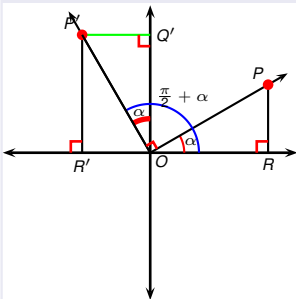


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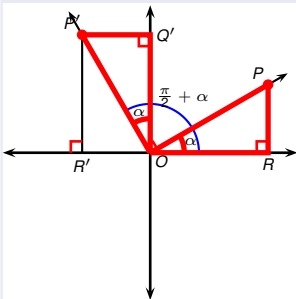


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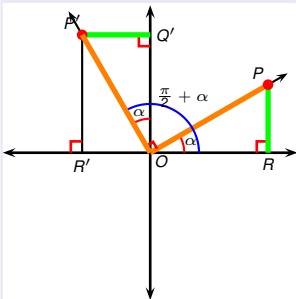


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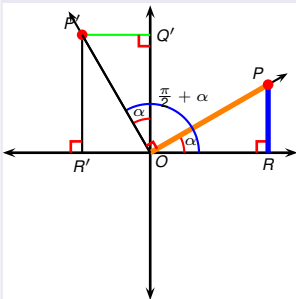
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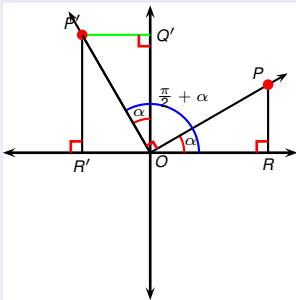
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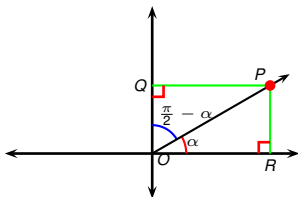
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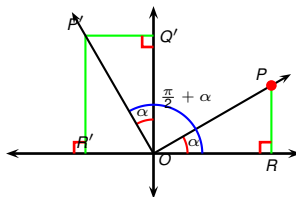
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To memorize the cofunction identities it suffices to memorize the Quadrant I case via the two diagrams below.



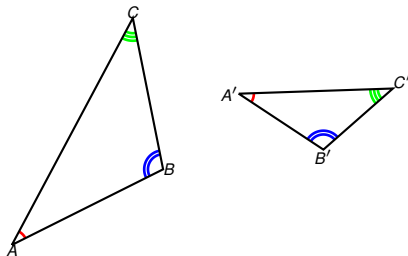
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## Definition (Similar triangles)

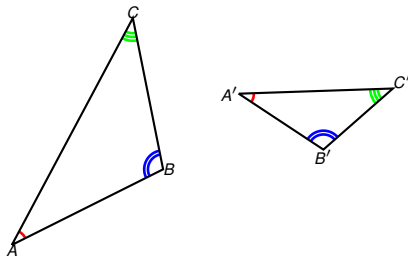
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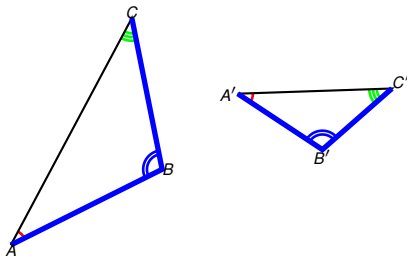




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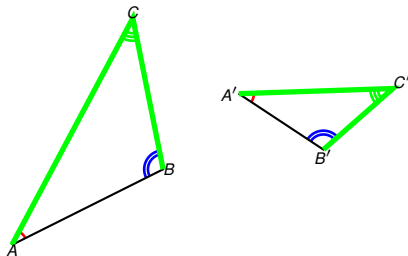
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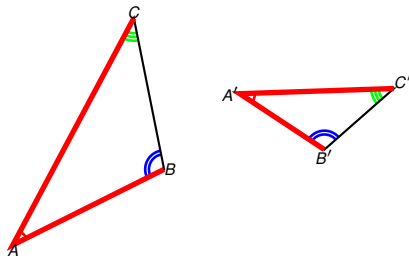
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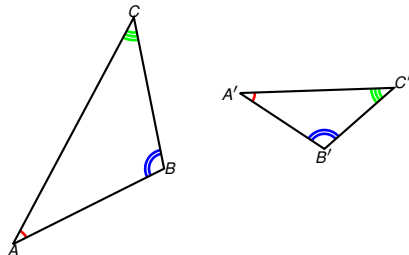


The following statement is proved in the subject of Euclidean (planar) geometry.

### Theorem (Similar triangles have equal side ratios)

*Let  $\triangle ABC$  and  $\triangle A'B'C'$  be two similar triangles. Then the ratios of the lengths of the sides of the two triangles are equal, that is*

$$\frac{|AB|}{|BC|} = \frac{|A'B'|}{|B'C'|} \quad \frac{|BC|}{|CA|} = \frac{|B'C'|}{|C'A'|} \quad \frac{|CA|}{|AB|} = \frac{|C'A'|}{|A'B'|}$$

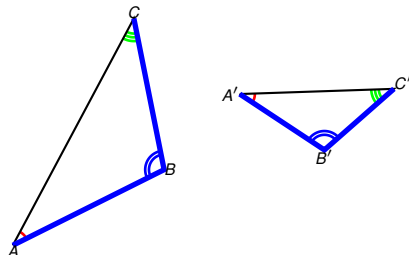


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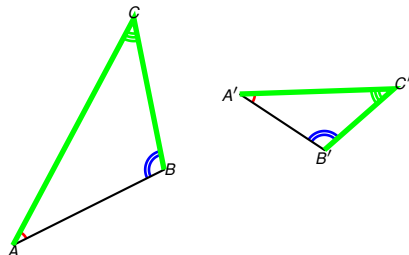


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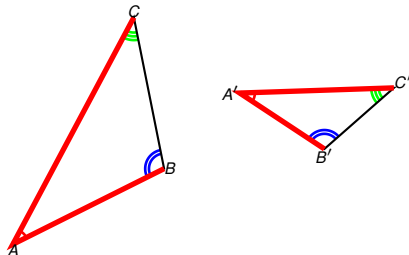


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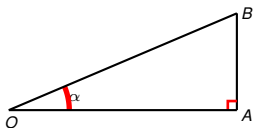
$\sin(\alpha + \beta)$ ,  $\cos(\alpha + \beta)$  via  $\sin \alpha$ ,  $\sin \beta$ ,  $\cos \alpha$ ,  $\cos \beta$

$$\sin(\alpha + \beta) = ?$$

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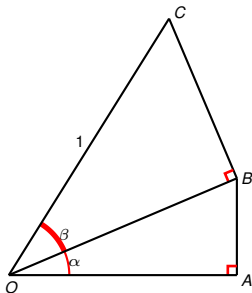
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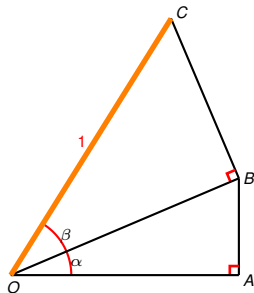
$\sin(\alpha + \beta), \cos(\alpha + \beta)$  via  $\sin \alpha, \sin \beta, \cos \alpha, \cos \beta$



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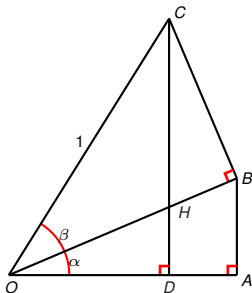
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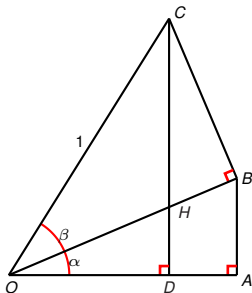
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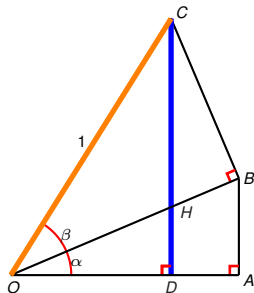
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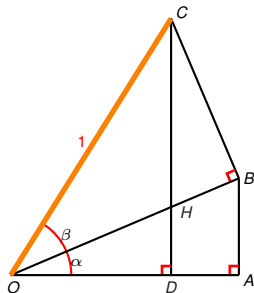
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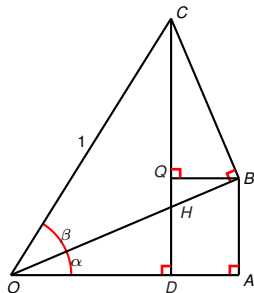
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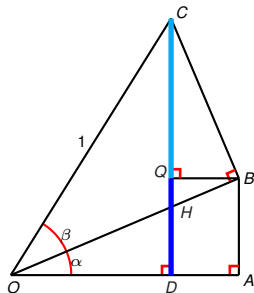


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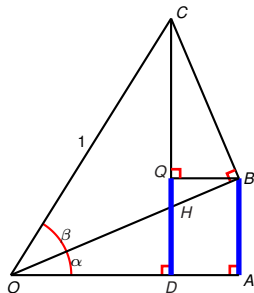
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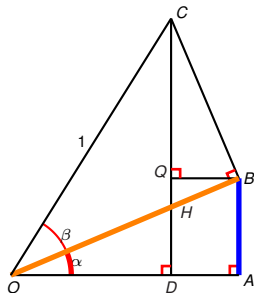
$$|QD| = |BA|$$

$$| \square DABQ$$

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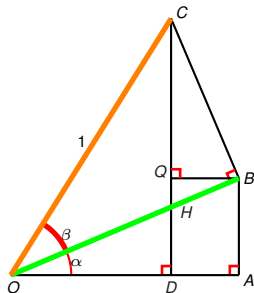
$$\begin{aligned} |QD| &= |BA| \\ &= \sin \alpha |OB| \end{aligned}$$

$$\left| \begin{array}{l} \square DABQ \\ \triangle OAB \end{array} \right|$$

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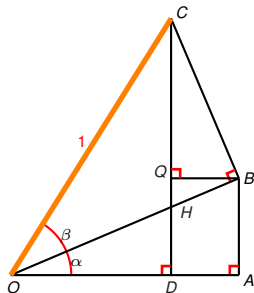


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$$\begin{aligned}|QD| &= |BA| \\ &= \sin \alpha |OB| \\ &= \sin \alpha \cos \beta |OC|\end{aligned} \left| \begin{array}{l} \square DABQ \\ \triangle OAB \\ \triangle OBC \end{array} \right.$$

$$\cos(\alpha + \beta) = ?$$

# $\sin(\alpha + \beta), \cos(\alpha + \beta)$ via $\sin \alpha, \sin \beta, \cos \alpha, \cos \beta$

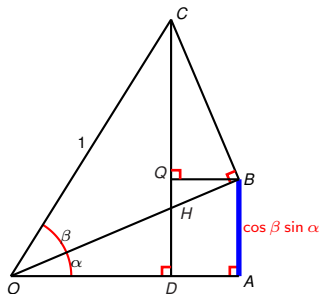


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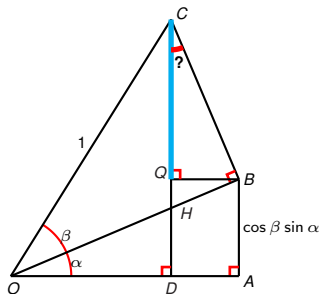


$$\begin{aligned}
 |QD| &= |BA| & \left| \begin{array}{l} \square DABQ \\ \triangle OAB \\ \triangle OBC \end{array} \right. \\
 &= \sin \alpha |OB| \\
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 &= \sin \alpha \cos \beta
 \end{aligned}$$

$$\begin{aligned}
 \sin(\alpha + \beta) &= \frac{|CD|}{|OC|} = |CD| \\
 &= |QD| + |CQ| \\
 &= \sin \alpha \cos \beta + ?
 \end{aligned}$$

$$\cos(\alpha + \beta) = ?$$

# $\sin(\alpha + \beta), \cos(\alpha + \beta)$ via $\sin \alpha, \sin \beta, \cos \alpha, \cos \beta$



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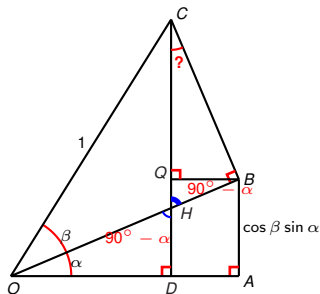
The diagram illustrates the derivation of the cosine rule using the sine rule. It shows a triangle  $ABC$  with side  $AC = 1$ . A perpendicular line segment  $CD$  is drawn from vertex  $C$  to the base  $AB$ . The angle at vertex  $A$  is  $\alpha$ , and the angle at vertex  $C$  is  $\beta$ . The diagram includes various right-angle symbols and labels for points  $Q$ ,  $D$ , and  $H$ . The angle  $90^\circ - \alpha$  is indicated at point  $H$ . The expression  $\cos \beta \sin \alpha$  is shown next to the vertical segment  $BD$ .

$$\cos(\alpha + \beta) = ?$$

$$\begin{array}{l|l} |QD| = |BA| & \square DABQ \\ = \sin \alpha |OB| & \triangle OAB \\ = \sin \alpha \cos \beta |OC| & \triangle OBC \\ = \sin \alpha \cos \beta & \\ |CQ| = & \end{array}$$



# $\sin(\alpha + \beta), \cos(\alpha + \beta)$ via $\sin \alpha, \sin \beta, \cos \alpha, \cos \beta$

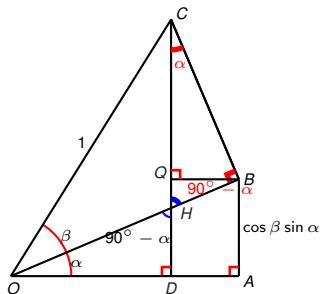


$$\begin{aligned}\sin(\alpha + \beta) &= \frac{|CD|}{|OC|} = |CD| \\ &= |QD| + |CQ| \\ &= \sin \alpha \cos \beta + ?\end{aligned}$$

$$\cos(\alpha + \beta) = ?$$

$$\begin{aligned}|QD| &= |BA| & \left| \begin{array}{l} \square DABQ \\ \triangle OAB \\ \triangle OBC \end{array} \right. \\ &= \sin \alpha |OB| \\ &= \sin \alpha \cos \beta |OC| \\ &= \sin \alpha \cos \beta \\ |CQ| &= \end{aligned}$$

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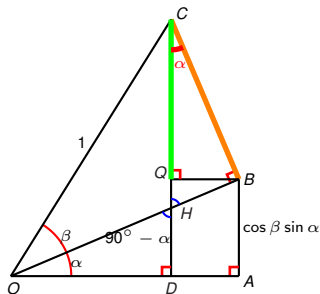


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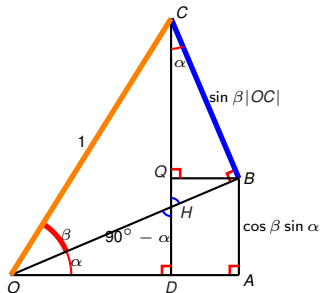


$$\begin{aligned}
 |QD| &= |BA| & \left| \begin{array}{l} \square DABQ \\ \triangle OAB \\ \triangle OBC \end{array} \right. \\
 &= \sin \alpha |OB| \\
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 &= \sin \alpha \cos \beta \\
 |CQ| &= \cos \alpha |CB| & \left| \triangle CQB \right.
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 \sin(\alpha + \beta) &= \frac{|CD|}{|OC|} = |CD| \\
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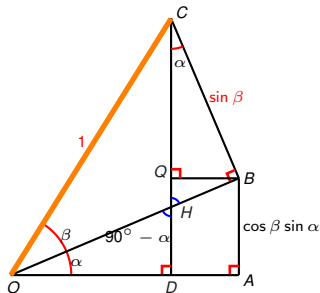


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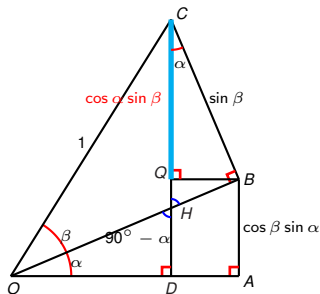


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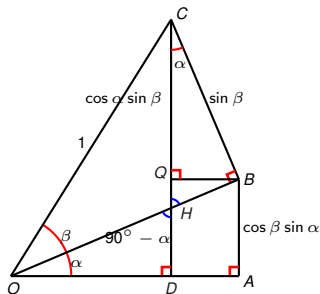


$$\begin{aligned}
 |QD| &= |BA| && \square DABQ \\
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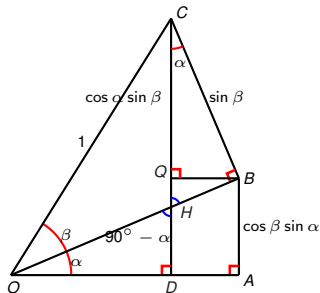
[illegible]

$$\begin{aligned}\cos(\alpha + \beta) &= \frac{|OD|}{|OC|} = |OD| \\ &= |OA| - |DA| \\ &= \cos \alpha \cos \beta - \sin \alpha \sin \beta\end{aligned}$$

$$\begin{array}{l|l} |DA| = |QB| & \square DABQ \\ = \sin \alpha |CB| & \triangle CQB \\ = \sin \alpha \sin \beta |OC| & \triangle OBC \\ = \sin \alpha \sin \beta & \end{array}$$



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# Trig Functions of Sums and Differences of Angles

## Theorem

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

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- We gave a geometric proof of the sum formulas when the two angles are acute and their sum is less than  $\pi = 90^\circ$ .

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- This can be shown by combining the preceding proof with identities such as  $\cos\left(\frac{\pi}{2} - \alpha\right) = \sin \alpha$ ,  $\cos\left(\frac{\pi}{2} + \alpha\right) = -\sin \alpha$ .

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# Trig Functions of Sums and Differences of Angles

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- We gave a geometric proof of the sum formulas when the two angles are acute and their sum is less than  $\pi = 90^\circ$ .
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- There is a theoretically more advanced (but algebraically simpler) proof using Euler's formula (to be studied later/in another course).
- The difference formulas are a consequence of the sum formulas and the fact that  $\sin$  is an odd function and  $\cos$  is even.

# Trig Functions of Differences of Angles

## Example

Prove the identities

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$$

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

from the (already demonstrated) identities

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$\sin(\alpha - \beta) = \sin(\alpha + (-\beta))$$

$$= \sin \alpha \cos(-\beta) + \cos \alpha \sin(-\beta) \quad \left| \begin{array}{l} \cos \text{ is even ,} \\ \sin \text{ is odd} \end{array} \right.$$

$$= \sin \alpha \cos \beta - \cos \alpha \sin \beta$$

$$\cos(\alpha - \beta) = \cos(\alpha + (-\beta))$$

$$= \cos \alpha \cos(-\beta) - \sin \alpha \sin(-\beta) \quad \left| \begin{array}{l} \cos \text{ is even ,} \\ \sin \text{ is odd} \end{array} \right.$$

$$= \cos \alpha \cos \beta + \sin \alpha \sin \beta$$



## Example

Find the exact value of the trigonometric function using radicals.

$$\cos(105^\circ)$$

## Example

Find the exact value of the trigonometric function using radicals.

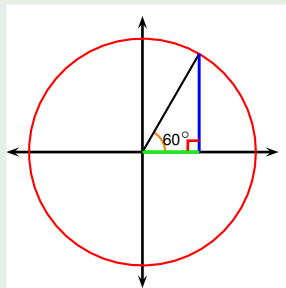
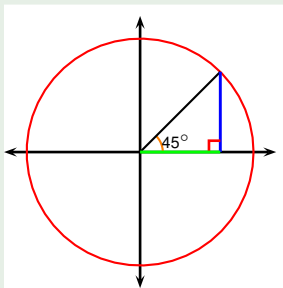
$$\cos(105^\circ) = \cos(45^\circ + 60^\circ)$$

## Example

Find the exact value of the trigonometric function using radicals.

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we know the trig  
f-ns of  $45^\circ$  and  $60^\circ$

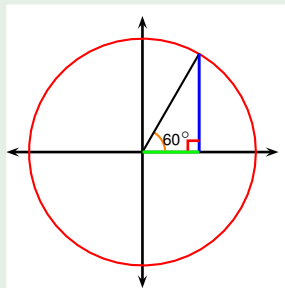
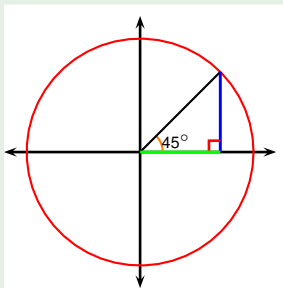


## Example

Find the exact value of the trigonometric function using radicals.

$$\cos(105^\circ) = \cos(45^\circ + 60^\circ) \\ = ?$$

we know the trig  
f-ns of  $45^\circ$  and  $60^\circ$   
**Angle sum f-la**



## Example

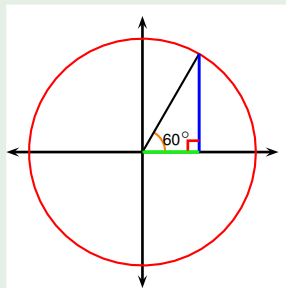
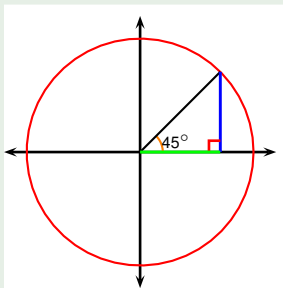
Find the exact value of the trigonometric function using radicals.

$$\cos(105^\circ) = \cos(45^\circ + 60^\circ)$$

$$= \cos(45^\circ) \cos(60^\circ) - \sin(45^\circ) \sin(60^\circ)$$

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Angle sum f-la



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Find the exact value of the trigonometric function using radicals.

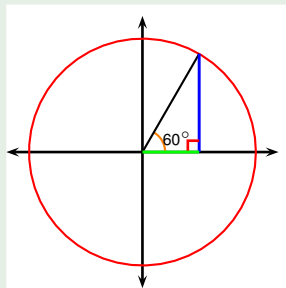
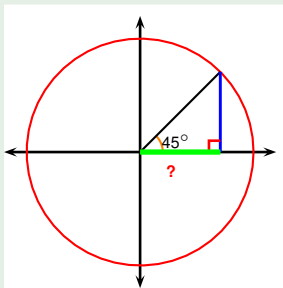
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$$= ? \cdot ? - ? \cdot ?$$

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Angle sum f-la



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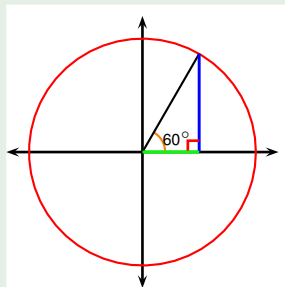
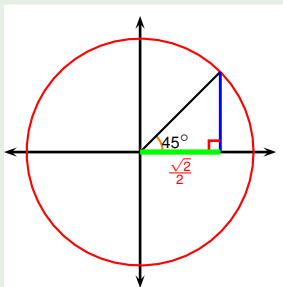
$$\cos(105^\circ) = \cos(45^\circ + 60^\circ)$$

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$$= \frac{\sqrt{2}}{2} \cdot ? - ? \cdot ?$$

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Angle sum f-la



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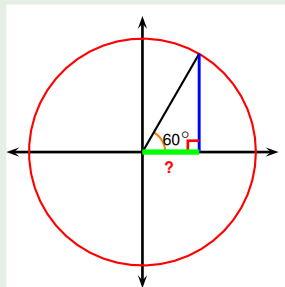
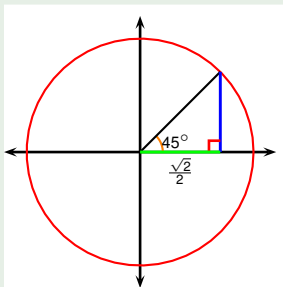
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Angle sum f-la





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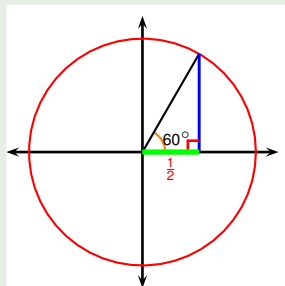
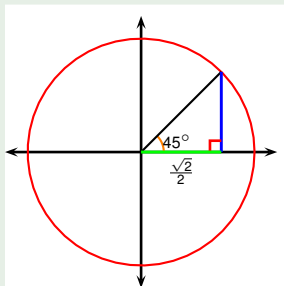
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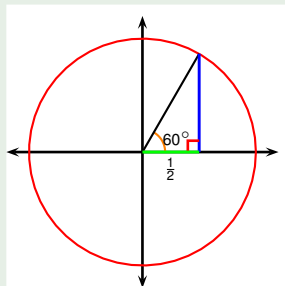
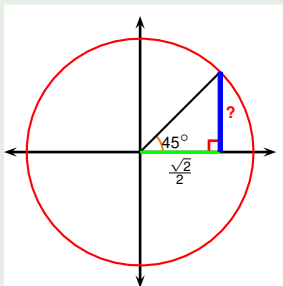
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Angle sum f-la



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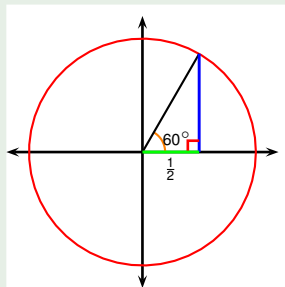
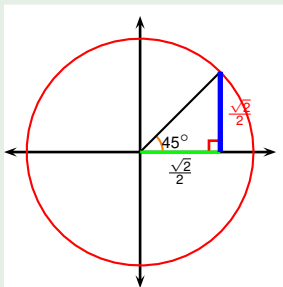
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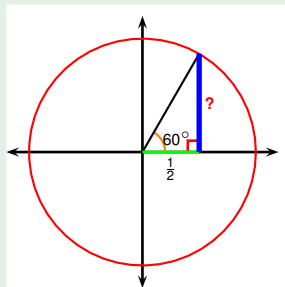
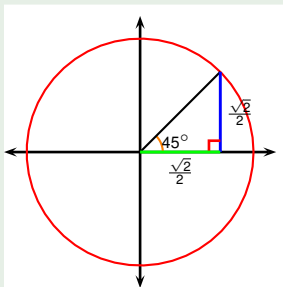
$$\cos(105^\circ) = \cos(45^\circ + 60^\circ)$$

$$= \cos(45^\circ) \cos(60^\circ) - \sin(45^\circ) \sin(60^\circ)$$

$$= \frac{\sqrt{2}}{2} \cdot \frac{1}{2} - \frac{\sqrt{2}}{2} \cdot ?$$

we know the trig  
f-ns of  $45^\circ$  and  $60^\circ$

Angle sum f-la



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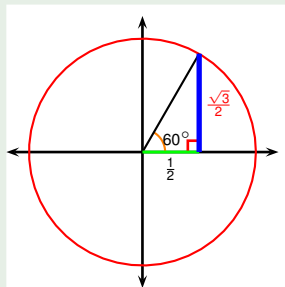
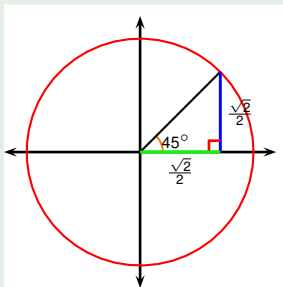
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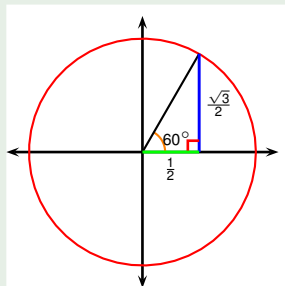
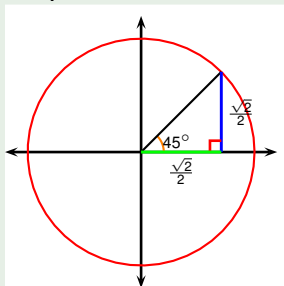
$$= \cos(45^\circ) \cos(60^\circ) - \sin(45^\circ) \sin(60^\circ)$$

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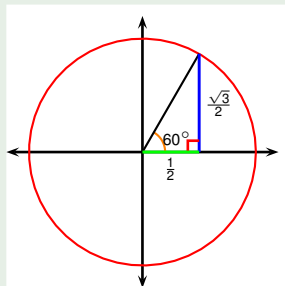
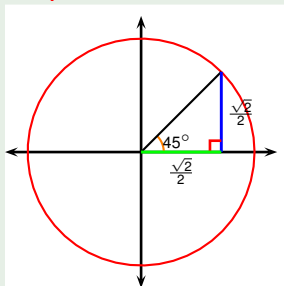
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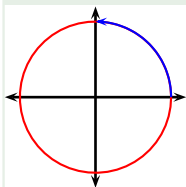
Use the angle sum/difference formulas to simplify.

$$\cos\left(\frac{\pi}{2} - x\right)$$



## Example

Use the angle sum/difference formulas to simplify.



$$\begin{aligned}\cos\left(\frac{\pi}{2} - x\right) &= \cos\left(\frac{\pi}{2}\right)\cos x + \sin\left(\frac{\pi}{2}\right)\sin x \\ &= 0 \cdot \cos(x) + 1 \cdot \sin x \\ &= \sin x\end{aligned}$$

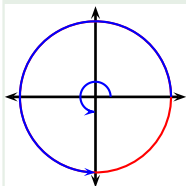
## Example

Use the angle sum/difference formulas to simplify.

$$\cot \left( \frac{3\pi}{2} + x \right)$$

## Example

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$$\begin{aligned}
 \cot \left( \frac{3\pi}{2} + x \right) &= \frac{\cos \left( \frac{3\pi}{2} + x \right)}{\sin \left( \frac{3\pi}{2} + x \right)} \\
 &= \frac{\cos \left( \frac{3\pi}{2} \right) \cos x - \sin \left( \frac{3\pi}{2} \right) \sin x}{\sin \left( \frac{3\pi}{2} \right) \cos x + \cos \left( \frac{3\pi}{2} \right) \sin x} \\
 &= \frac{0 \cdot \cos x - (-1) \sin x}{(-1) \cos x + 0 \cdot \sin x} \\
 &= \frac{-\cos x}{-\sin x} = -\frac{\sin x}{\cos x} \\
 &= -\tan x
 \end{aligned}$$

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Show that  $\tan(\pi + x) = \tan x$  using the angle sum formulas.

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as desired.

### Proposition ( $\tan, \cot$ are $\pi$ -periodic)

*The tangent and cotangent functions are  $\pi$ -periodic, in other words,*

$$\tan(\theta + \pi) = \tan \theta$$

$$\cot(\theta + \pi) = \cot \theta$$

Recall the angle sum formula  $\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$ .

### Example

Show that the Pythagorean identity  $\sin^2 \theta + \cos^2 \theta = 1$  follows from the angle difference formula.

Recall the angle sum formula  $\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$ .

### Example

Show that the Pythagorean identity  $\sin^2 \theta + \cos^2 \theta = 1$  follows from the angle difference formula.

$$\begin{aligned} 1 &= \cos 0 \\ &= \cos(\theta - \theta) \\ &= \cos \theta \cos \theta + \sin \theta \sin \theta \\ &= \cos^2 \theta + \sin^2 \theta, \end{aligned}$$

as desired.

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$$\begin{aligned}
 \tan(\alpha + \beta) &= \frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)} \\
 &= \frac{(\sin \alpha \cos \beta + \cos \alpha \sin \beta) \frac{1}{\cos \alpha \cos \beta}}{(\cos \alpha \cos \beta - \sin \alpha \sin \beta) \frac{1}{\cos \alpha \cos \beta}} \\
 &= \frac{\frac{\sin \alpha \cancel{\cos \beta}}{\cos \alpha \cancel{\cos \beta}} + \frac{\cancel{\cos \alpha} \sin \beta}{\cancel{\cos \alpha} \cos \beta}}{\frac{\cancel{\cos \alpha} \cos \beta}{\cancel{\cos \alpha} \cos \beta} - \frac{\sin \alpha \sin \beta}{\cos \alpha \cos \beta}} \\
 &= \frac{\frac{\sin \alpha}{\cos \alpha} + \frac{\sin \beta}{\cos \beta}}{1 - \frac{\sin \alpha}{\cos \alpha} \cdot \frac{\sin \beta}{\cos \beta}} \\
 &= \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}
 \end{aligned}$$



# Double angle formulas

## Proposition (Double angle formulas)

$$\begin{aligned}\sin(2\alpha) &= 2 \sin \alpha \cos \alpha \\ \cos(2\alpha) &= \cos^2 \alpha - \sin^2 \alpha \\ &= 2 \cos^2 \alpha - 1 \\ &= 1 - 2 \sin^2 \alpha\end{aligned}$$

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- The double angle formulas play a special role in integration.

## Example

Derive the double-angle formulas.

$$\sin(2\alpha) =$$

$$\cos(2\alpha) =$$

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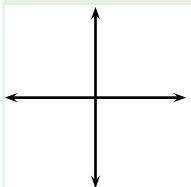
$$\begin{aligned}\sin(2\alpha) &= \sin(\alpha + \alpha) \\ &= \sin \alpha \cos \alpha + \cos \alpha \sin \alpha \\ &= 2 \sin \alpha \cos \alpha\end{aligned}$$

$$\begin{aligned}\cos(2\alpha) &= \cos(\alpha + \alpha) \\ &= \cos \alpha \cos \alpha - \sin \alpha \sin \alpha \\ &= \cos^2 \alpha - \sin^2 \alpha \\ &= \cos^2 \alpha - (1 - \cos^2 \alpha) \\ &= 2 \cos^2 \alpha - 1 \\ &= 1 - \sin^2 \alpha - \sin^2 \alpha \\ &= 1 - 2 \sin^2 \alpha\end{aligned}$$

## Example

Using radicals, find the exact value of the trigonometric expression.

$$\cos 105^\circ$$

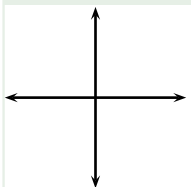


Recall the half angle formula  $\cos \alpha = \pm \sqrt{\frac{1 + \cos(2\alpha)}{2}}$ .

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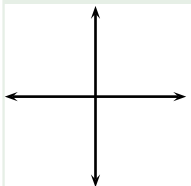


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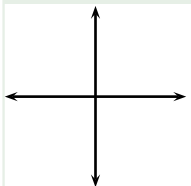


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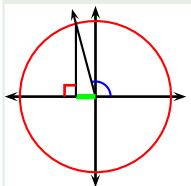


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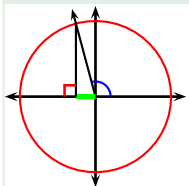


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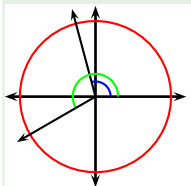


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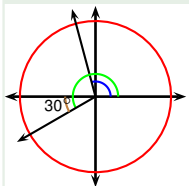


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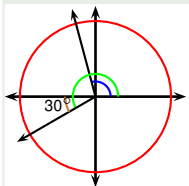


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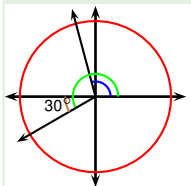


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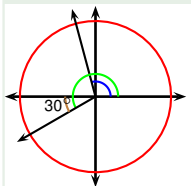


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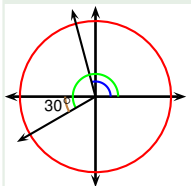


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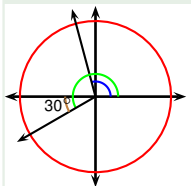


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## Proposition (Power-Reducing Formulas)

$$\sin^2 \alpha = \frac{1 - \cos(2\alpha)}{2} \quad \cos^2 \alpha = \frac{1 + \cos(2\alpha)}{2}$$

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$$\begin{aligned}\cos(2\alpha) &= 1 - 2\sin^2 \alpha \\ \color{red}{2}\sin^2 \alpha &= 1 - \cos(2\alpha) \\ \sin^2 \alpha &= \frac{1 - \cos(2\alpha)}{\color{red}{2}}\end{aligned}$$



## Proposition (Power-Reducing Formulas)

$$\sin^2 \alpha = \frac{1 - \cos(2\alpha)}{2} \quad \cos^2 \alpha = \frac{1 + \cos(2\alpha)}{2}$$

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$$\begin{aligned} \cos(2\alpha) &= 1 - 2\sin^2 \alpha & \cos(2\alpha) &= 2\cos^2 \alpha - 1 \\ 2\sin^2 \alpha &= 1 - \cos(2\alpha) \\ \sin^2 \alpha &= \frac{1 - \cos(2\alpha)}{2} \end{aligned}$$





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### Corollary

$$\sin \alpha = \pm \sqrt{\frac{1 - \cos(2\alpha)}{2}} \quad \cos \alpha = \pm \sqrt{\frac{1 + \cos(2\alpha)}{2}}$$

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$$\sin \alpha = \pm \sqrt{\frac{1 - \cos(2\alpha)}{2}} \quad \cos \alpha = \pm \sqrt{\frac{1 + \cos(2\alpha)}{2}}$$

### Corollary (Half-Angle Formulas)

$$\sin \left( \frac{\beta}{2} \right) = \pm \sqrt{\frac{1 - \cos \beta}{2}} \quad \cos \left( \frac{\beta}{2} \right) = \pm \sqrt{\frac{1 + \cos \beta}{2}}$$



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- The power reducing formulas are used to express  $\sin^k \alpha$  and  $\cos^k \alpha$  via lower powers of the sin and cos functions (applied to angles other than  $\alpha$ ).

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- The power reducing formulas are used to express  $\sin^k \alpha$  and  $\cos^k \alpha$  via lower powers of the sin and cos functions (applied to angles other than  $\alpha$ ).
- This technique will play a key role in integration (studied later/in another course).

## Example

Rewrite  $\sin^4 \alpha$  in terms of first powers of the cosines and sines of multiples of the angle  $\alpha$ .

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Recall the formulas:  $\sin^2 \beta = ?$  ,  $\cos^2 \beta = ?$  .

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$$\begin{aligned}\sin^4 \alpha &= \left( \sin^2 \alpha \right)^2 \\ &= \left( ? \right)^2\end{aligned}$$

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Rewrite  $\sin^4 \alpha$  in terms of first powers of the cosines and sines of multiples of the angle  $\alpha$ .

$$\begin{aligned} \sin^4 \alpha &= (\sin^2 \alpha)^2 \\ &= \left( \frac{1 - \cos(2\alpha)}{2} \right)^2 \\ &= \frac{1}{4} \left( ? \right) \end{aligned}$$



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 &= \frac{1}{4} (1 - 2\cos(2\alpha) + ?)
 \end{aligned}$$

Recall the formulas:  $\sin^2 \beta = \frac{1 - \cos(2\beta)}{2}$ ,  $\cos^2 \beta = \frac{\cos(2\beta) + 1}{2}$ .

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 &= \frac{1}{4} \left( 1 - 2\cos(2\alpha) + \frac{\cos(2 \cdot 2\alpha)}{2} + \frac{1}{2} \right) \\
 &= \frac{1}{4} \left( \frac{3}{2} - 2\cos(2\alpha) + \frac{\cos(4\alpha)}{2} \right)
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 &= \frac{1}{4} \left( \frac{3}{2} - 2\cos(2\alpha) + \frac{\cos(4\alpha)}{2} \right) \\
 &= \frac{1}{8} (3 - 4\cos(2\alpha) + \cos(4\alpha))
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