

Precalculus

Degree lowering formulas

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Proposition (Power-Reducing Formulas)

$$\sin^2 \alpha = \frac{1 - \cos(2\alpha)}{2} \quad \cos^2 \alpha = \frac{1 + \cos(2\alpha)}{2}$$

Proof.



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$$\cos(2\alpha) = 1 - 2\sin^2 \alpha$$



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$$\sin \alpha = \pm \sqrt{\frac{1 - \cos(2\alpha)}{2}} \quad \cos \alpha = \pm \sqrt{\frac{1 + \cos(2\alpha)}{2}}$$

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$$\sin \alpha = \pm \sqrt{\frac{1 - \cos(2\alpha)}{2}} \quad \cos \alpha = \pm \sqrt{\frac{1 + \cos(2\alpha)}{2}}$$

Corollary (Half-Angle Formulas)

$$\sin\left(\frac{\beta}{2}\right) = \pm \sqrt{\frac{1 - \cos \beta}{2}} \quad \cos\left(\frac{\beta}{2}\right) = \pm \sqrt{\frac{1 + \cos \beta}{2}}$$

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- The power reducing formulas are used to express $\sin^k \alpha$ and $\cos^k \alpha$ via lower powers of the sin and cos functions (applied to angles other than α).

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- The power reducing formulas are used to express $\sin^k \alpha$ and $\cos^k \alpha$ via lower powers of the sin and cos functions (applied to angles other than α).
- This technique will play a key role in integration (studied later/in another course).

Example

Rewrite $\sin^4 \alpha$ in terms of first powers of the cosines and sines of multiples of the angle α .

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Recall the formulas: $\sin^2 \beta = ?$, $\cos^2 \beta = ?$.

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Rewrite $\sin^4 \alpha$ in terms of first powers of the cosines and sines of multiples of the angle α .

$$\begin{aligned}\sin^4 \alpha &= \left(\sin^2 \alpha \right)^2 \\ &= \left(? \right)^2\end{aligned}$$

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$$\begin{aligned} \sin^4 \alpha &= (\sin^2 \alpha)^2 \\ &= \left(\frac{1 - \cos(2\alpha)}{2} \right)^2 \end{aligned}$$

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$$\begin{aligned} \sin^4 \alpha &= (\sin^2 \alpha)^2 \\ &= \left(\frac{1 - \cos(2\alpha)}{2} \right)^2 \\ &= \frac{1}{4} \left(? \right) \end{aligned}$$

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 &= \frac{1}{4} \left(\frac{3}{2} - 2\cos(2\alpha) + \frac{\cos(4\alpha)}{2} \right)
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 &= \frac{1}{4} \left(\frac{3}{2} - 2\cos(2\alpha) + \frac{\cos(4\alpha)}{2} \right) \\
 &= \frac{1}{8} (3 - 4\cos(2\alpha) + \cos(4\alpha))
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