

Calculus II

Trigonometric integrals

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Outline

- 1 Trigonometric Integrals
 - Integrating rational trigonometric integrals
 - Ad hoc methods for trigonometric integrals

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Integrals of the form $\int R(\cos \theta, \sin \theta) d\theta$, R

Let R be an arbitrary rational function in two variables (quotient of polynomials in two variables).

Question

Can we integrate $\int R(\cos \theta, \sin \theta) d\theta$?

- Yes. We will learn how in what follows.
- The algorithm for integration is roughly:
 - Apply the substitution $\theta = 2 \arctan t$ to transform to integral of rational function.
 - Solve as previously studied.

The rationalizing substitution $\theta = 2 \arctan t$

Let R - rational function in two variables. $\int R(\cos \theta, \sin \theta) d\theta$ can be integrated via the substitution $\theta = 2 \arctan t$. How does this transform $\sin \theta$, $\cos \theta$? How does this transform $d\theta$? How is t expressed via θ ?

$$\begin{aligned}\sin \theta &= \sin(2 \arctan t) = \frac{2 \tan(\arctan t)}{1 + \tan^2(\arctan t)} = \frac{2t}{1 + t^2} \\ \cos \theta &= \cos(2 \arctan t) = \frac{1 - \tan^2(\arctan t)}{1 + \tan^2(\arctan t)} = \frac{1 - t^2}{1 + t^2}\end{aligned}$$

Recall the expression of $\sin(2z)$, $\cos(2z)$ via $\tan z$:

$$\begin{aligned}\sin(2z) &= 2 \sin z \cos z = \frac{2 \sin z \cos z \frac{1}{\cos^2 z}}{(\cos^2 z + \sin^2 z) \frac{1}{\cos^2 z}} = \frac{2 \tan z}{1 + \tan^2 z} \\ \cos(2z) &= \cos^2 z - \sin^2 z = \frac{(\cos^2 z - \sin^2 z) \frac{1}{\cos^2 z}}{(\cos^2 z + \sin^2 z) \frac{1}{\cos^2 z}} = \frac{1 - \tan^2 z}{1 + \tan^2 z}.\end{aligned}$$

The rationalizing substitution $\theta = 2 \arctan t$

Let R - rational function in two variables. $\int R(\cos \theta, \sin \theta) d\theta$ can be integrated via the substitution $\theta = 2 \arctan t$. How does this transform $\sin \theta$, $\cos \theta$? How does this transform $d\theta$? How is t expressed via θ ?

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Theorem

The substitution given above transforms $\int R(\cos \theta, \sin \theta) d\theta$ to an integral of a rational function of t .

Example

Let $\theta = 2 \arctan t$, $\cos \theta = \frac{1-t^2}{1+t^2}$, $\sin \theta = \frac{2t}{1+t^2}$, $z = \frac{3}{\sqrt{5}} \left(t + \frac{1}{3}\right)$.

$$\begin{aligned}
 \int \frac{d\theta}{2 \sin \theta - \cos \theta + 5} &= \int \frac{2dt}{(1+t^2) \left(2 \frac{2t}{t^2+1} - \frac{(1-t^2)}{1+t^2} + 5\right)} \\
 &= \int \frac{2dt}{6t^2 + 4t + 4} \\
 &= \int \frac{dt}{3t^2 + 2t + 2} \\
 \text{(complete square)} &= \int \frac{dt}{3 \left(t^2 + 2t \frac{1}{3} + \frac{1}{9} - \frac{1}{9} + \frac{2}{3}\right)} \\
 &= \frac{1}{3} \int \frac{dt}{\left(t + \frac{1}{3}\right)^2 + \frac{5}{9}} \\
 &= \frac{1}{3} \int \frac{dt}{\frac{5}{9} \left(\frac{9}{5} \left(t + \frac{1}{3}\right)^2 + 1\right)}
 \end{aligned}$$

Example

Let $\theta = 2 \arctan t$, $\cos \theta = \frac{1-t^2}{1+t^2}$, $\sin \theta = \frac{2t}{1+t^2}$, $z = \frac{3}{\sqrt{5}} \left(t + \frac{1}{3}\right)$.

$$\begin{aligned}
 \int \frac{d\theta}{2 \sin \theta - \cos \theta + 5} &= \frac{1}{3} \int \frac{dt}{\frac{5}{9} \left(\frac{9}{5} \left(t + \frac{1}{3}\right)^2 + 1 \right)} \\
 &= \frac{3}{5} \int \frac{\frac{\sqrt{5}}{3} d\left(\frac{3}{\sqrt{5}} \left(t + \frac{1}{3}\right)\right)}{\left(\left(\frac{3}{\sqrt{5}} \left(t + \frac{1}{3}\right)\right)^2 + 1\right)} \\
 &= \frac{\sqrt{5}}{5} \int \frac{dz}{z^2 + 1} \\
 &= \frac{\sqrt{5}}{5} \arctan z + C \\
 &= \frac{\sqrt{5}}{5} \arctan \left(\frac{3}{\sqrt{5}} \left(t + \frac{1}{3}\right) \right) + C \\
 &= \frac{\sqrt{5}}{5} \arctan \left(\frac{3}{\sqrt{5}} \left(\tan \left(\frac{\theta}{2} \right) + \frac{1}{3} \right) \right) + C
 \end{aligned}$$

The integral $\int \sec \theta d\theta$ appears often in practice. A quicker solution will be shown later, but first we show the standard method.

Example

$$\text{Set } \theta = 2 \arctan t, \cos \theta = \frac{1 - \tan^2(\frac{\theta}{2})}{1 + \tan^2(\frac{\theta}{2})} = \frac{1 - t^2}{1 + t^2}, d\theta = 2 \frac{1}{1 + t^2} dt.$$

$$\begin{aligned} \int \sec \theta d\theta &= \int \frac{1}{\cos \theta} d\theta = \int \frac{1}{\left(\frac{1-t^2}{1+t^2}\right)} \frac{2}{(1+t^2)} dt \\ &= \int \frac{2}{1-t^2} dt = \int \left(\frac{1}{1-t} + \frac{1}{1+t} \right) dt \quad \left| \text{part. fractions} \right. \\ &= -\ln |1-t| + \ln |1+t| + C \\ &= \ln \left| \frac{1+t}{1-t} \right| + C \\ &= \ln \left| \frac{1 + \tan(\frac{\theta}{2})}{1 - \tan(\frac{\theta}{2})} \right| + C \end{aligned}$$

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$$\text{Set } \theta = 2 \arctan t, \cos \theta = \frac{1 - \tan^2(\frac{\theta}{2})}{1 + \tan^2(\frac{\theta}{2})} = \frac{1 - t^2}{1 + t^2}, d\theta = 2 \frac{1}{1 + t^2} dt.$$

$$\int \sec \theta d\theta = \ln \left| \frac{1 + \tan(\frac{\theta}{2})}{1 - \tan(\frac{\theta}{2})} \right| + C$$

This is a perfectly good answer, however there's a simplification:

$$\begin{aligned} \tan \theta + \sec \theta &= \frac{\sin \theta + 1}{\cos \theta} = \frac{2 \sin(\frac{\theta}{2}) \cos(\frac{\theta}{2}) + \sin^2(\frac{\theta}{2}) + \cos^2(\frac{\theta}{2})}{\cos^2(\frac{\theta}{2}) - \sin^2(\frac{\theta}{2})} \\ &= \frac{(\sin(\frac{\theta}{2}) + \cos(\frac{\theta}{2}))^2}{(\cos(\frac{\theta}{2}) - \sin(\frac{\theta}{2}))(\cos(\frac{\theta}{2}) + \sin(\frac{\theta}{2}))} \\ &= \frac{\sin(\frac{\theta}{2}) + \cos(\frac{\theta}{2})}{\cos(\frac{\theta}{2}) - \sin(\frac{\theta}{2})} = \frac{1 + \tan(\frac{\theta}{2})}{1 - \tan(\frac{\theta}{2})} \end{aligned}$$

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Trigonometric Integrals - quick ad hoc techniques

- As we saw, every rational trigonometric expression can be integrated with the substitution $\theta = 2 \arctan t$.
- This integration technique results in rather long computations.
- Particular integral types may be computable with quicker ad hoc techniques.
- We illustrate such techniques on examples.
- Examples to which our ad hoc techniques apply arise from integrals needed outside of the subject of Calculus II, so these techniques are important.
- The trigonometric integral we saw, $\int \frac{d\theta}{2 \sin \theta - \cos \theta + 5}$, will not work with any of following ad-hoc techniques, so the general method is important as well.

Example

$$\begin{aligned}\int \sin^3 x dx &= \int \sin^2 x \sin x dx \\&= \int \sin^2 x d(-\cos x) \\&= \int (-1) (1 - \cos^2 x) d(\cos x) \\&= \int (\cos^2 x - 1) d(\cos x) \\&= \int (u^2 - 1) du \\&= \frac{u^3}{3} - u + C \\&= \frac{1}{3} \cos^3 x - \cos x + C .\end{aligned}$$

Can we rewrite
 $\sin^2 x$ via $\cos x$?

Set $u = \cos x$

Example

$$\begin{aligned}
 \int \cos^5 x \sin^2 x dx &= \int \cos^4 x \sin^2 x \cos x dx \\
 &= \int \cos^4 x \sin^2 x d(\sin x) && \left| \begin{array}{l} \text{Can we rewrite} \\ \cos^4 x \text{ via } \sin x? \end{array} \right. \\
 &= \int (\cos^2 x)^2 \sin^2 x d(\sin x) \\
 &= \int (1 - \sin^2 x)^2 \sin^2 x d(\sin x) && \left| \begin{array}{l} \text{Set } u = \sin x \end{array} \right. \\
 &= \int (1 - u^2)^2 u^2 du \\
 &= \int (1 - 2u^2 + u^4) u^2 du \\
 &= \int (u^2 - 2u^4 + u^6) du \\
 &= \frac{u^3}{3} - 2\frac{u^5}{5} + \frac{u^7}{7} + C \\
 &= \frac{\sin^3 x}{3} - 2\frac{\sin^5 x}{5} + \frac{\sin^7 x}{7} + C .
 \end{aligned}$$

$$\begin{aligned}
 \int \sin^m x \cos^n x dx &= \int \sin^m x \cos^{n-1} x d(\sin x) \\
 &= \int \sin^m x (1 - \sin^2 x)^{\frac{n-1}{2}} d(\sin x) \\
 &= \int u^m (1 - u^2)^{\frac{n-1}{2}} du
 \end{aligned}$$

When n – odd:

$$\cos x dx$$

$$= d(\sin x)$$

Express $\cos x$
via $\sin x$

$$\text{Set } \sin x = u$$

$$\begin{aligned}
 \int \sin^m x \cos^n x dx &= \int \sin^{m-1} x \cos^n x d(-\cos x) \\
 &= - \int (1 - \cos^2 x)^{\frac{m-1}{2}} \cos^n x d(\cos x) \\
 &= - \int (1 - u^2)^{\frac{m-1}{2}} u^n du
 \end{aligned}$$

When m – odd:

$$\sin x dx$$

$$= d(-\cos x)$$

Express $\sin x$
via $\cos x$

$$\text{Set } \cos x = u$$

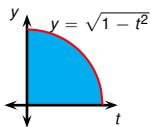
If both m, n – even, use $\left| \begin{array}{l} \sin^2 x = \frac{1 - \cos(2x)}{2} \\ \cos^2 x = \frac{\cos(2x) + 1}{2} \end{array} \right.$ and substitute $s = 2x$ to lower trig powers. Repeat above considerations.

Example

$$\begin{aligned}
 \int_0^{\frac{\pi}{2}} \sin^2 x \, dx &= \int_0^{\frac{\pi}{2}} \left(\frac{1 - \cos(2x)}{2} \right) dx && \left| \begin{array}{l} \text{express } \sin^2 x \\ \text{via } \cos(2x) \end{array} \right. \\
 &= \left[\frac{x}{2} - \frac{\sin(2x)}{4} \right]_0^{\frac{\pi}{2}} \\
 &= \left(\frac{\pi}{4} - \frac{\sin \pi}{4} \right) - \left(0 - \frac{\sin 0}{4} \right) = \frac{\pi}{4}.
 \end{aligned}$$

Example

Set $t = \cos x$, $x \in [0, \frac{\pi}{2}] \Rightarrow \sin x \geq 0$. Then
 $dt = d(\cos x) = -\sin x \, dx$.



$$\begin{aligned}
 \int_{t=0}^{t=1} \sqrt{1-t^2} \, dt &= - \int_{x=\frac{\pi}{2}}^{x=0} \sqrt{1-\cos^2 x} \sin x \, dx \\
 &= \int_{x=\frac{\pi}{2}}^{x=0} \sqrt{\sin^2 x} \sin x \, dx \\
 &= \int_0^{\frac{\pi}{2}} \sin^2 x \, dx = \frac{\pi}{4}.
 \end{aligned}$$

Example

$$\begin{aligned}
 \int \tan^8 x \sec^4 x dx &= \int \tan^8 x \sec^2 x \sec^2 x dx \\
 &= \int \tan^8 x \sec^2 x d(\tan x) && \left. \begin{array}{l} \text{Can we rewrite} \\ \sec^2 x \text{ via } \tan x? \end{array} \right\} \\
 &= \int \tan^8 x (1 + \tan^2 x) d(\tan x) && \left. \begin{array}{l} \text{Set } u = \tan x \end{array} \right\} \\
 &= \int u^8 (1 + u^2) du \\
 &= \int (u^8 + u^{10}) du \\
 &= \frac{u^9}{9} + \frac{u^{11}}{11} + C \\
 &= \frac{\tan^9 x}{9} + \frac{\tan^{11} x}{11} + C .
 \end{aligned}$$

Example

$$\begin{aligned}
 \int \tan^5 x \sec^9 x dx &= \int \tan^4 x \sec^8 x \tan x \sec x dx \\
 &= \int \tan^4 x \sec^8 x d(\sec x) && \left| \begin{array}{l} \text{Can we rewrite} \\ \tan^4 x \text{ via } \sec x? \end{array} \right. \\
 &= \int (\tan^2 x)^2 \sec^8 x d(\sec x) \\
 &= \int (\sec^2 x - 1)^2 \sec^8 x d(\sec x) && \left| \begin{array}{l} \text{Set } u = \sec x \end{array} \right. \\
 &= \int (1 - u^2)^2 u^8 du \\
 &= \int (1 - 2u^2 + u^4) u^8 du \\
 &= \int (u^8 - 2u^{10} + u^{12}) du \\
 &= \frac{u^9}{9} - 2\frac{u^{11}}{11} + \frac{u^{13}}{13} + C \\
 &= \frac{\sec^9 x}{9} - 2\frac{\sec^{11} x}{11} + \frac{\sec^{13} x}{13} + C .
 \end{aligned}$$

Partial strategy for fast evaluation of $\int \tan^m x \sec^n x dx$

$\begin{aligned}\int \tan^m x \sec^n x dx &= \int \tan^m x \sec^{n-2} x d(\tan x) \\ &= \int \tan^m x (1 + \tan^2 x)^{\frac{n-2}{2}} d(\tan x) \\ &= \int u^m (1 + u^2)^{\frac{n-2}{2}} du\end{aligned}$	$n - \text{even}, n \geq 2$ $\sec^2 x dx$ $= d(\tan x)$ Express $\sec x$ via $\tan x$ Set $u = \tan x$
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$\begin{aligned}\int \tan^m x \sec^n x dx &= \int \tan^{m-1} x \sec^{n-1} x d(\sec x) \\ &= \int (\sec^2 x - 1)^{\frac{m-1}{2}} \sec^{n-1} x d(\sec x) \\ &= \int (u^2 - 1)^{\frac{m-1}{2}} u^n du\end{aligned}$	$m - \text{odd}, n \geq 1$ $\tan x \sec x dx$ $= d(\sec x)$ Express $\tan x$ via $\sec x$ Set $u = \sec x$
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Outside of the above cases we either use more tricks or resort to the general method $x = 2 \arctan t$.

Example

$$\begin{aligned}
 \int \tan x dx &= \int \frac{\sin x}{\cos x} dx = \int \frac{1}{\cos x} d(-\cos x) && \left| \text{Set } u = \cos x \right. \\
 &= - \int \frac{du}{u} = -\ln |u| + C \\
 &= -\ln |\cos x| + C = \ln |\sec x| + C
 \end{aligned}$$

The following can be/was computed via $x = 2 \arctan t$. Alternatively:

Example

$$\begin{aligned}
 \int \sec x dx &= \int \sec x \frac{(\sec x + \tan x)}{(\sec x + \tan x)} dx \\
 &= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx \\
 &= \int \frac{d(\tan x + \sec x)}{\sec x + \tan x} && \left| \text{Set } u = \sec x + \tan x \right. \\
 &= \int \frac{du}{u} = \ln |u| + C \\
 &= \ln |\sec x + \tan x| + C.
 \end{aligned}$$

Example

$$\begin{aligned}\int \tan^3 x dx &= \int \tan x \tan^2 x dx \\&= \int \tan x (\sec^2 x - 1) dx \\&= \int \tan x \sec^2 x dx - \int \tan x dx \\&= \int \tan x d(\tan x) - \ln |\sec x| \quad \left| \text{Set } u = \tan x \right. \\&= \int u du + \ln \left| \frac{1}{\sec x} \right| \\&= \frac{u^2}{2} + \ln |\cos x| + C \\&= \frac{\tan^2 x}{2} + \ln |\cos x| + C\end{aligned}$$

Example

$$\int \sec^3 x dx = \int \sec x \sec^2 x dx$$

$$= \int \sec x d(\tan x)$$

Integrate
by parts

$$= \sec x \tan x - \int \tan x d(\sec x)$$

$$= \sec x \tan x - \int \tan^2 x \sec x dx$$

$$= \sec x \tan x - \int (\sec^2 x - 1) \sec x dx$$

$$= \sec x \tan x - \int \sec^3 x dx + \int \sec x dx$$

$$2 \int \sec^3 x dx = \sec x \tan x + \ln |\sec x + \tan x| + C$$

$$\int \sec^3 x dx = \frac{1}{2} (\sec x \tan x + \ln |\sec x + \tan x|) + K.$$

To evaluate integrals of the form

$$\textcircled{1} \int \sin(mx) \cos(nx) dx$$

$$\textcircled{2} \int \sin(mx) \sin(nx) dx$$

$$\textcircled{3} \int \cos(mx) \cos(nx) dx$$

use the corresponding identity:

$$\textcircled{1} \sin A \cos B = \frac{1}{2} [\sin(A - B) + \sin(A + B)]$$

$$\textcircled{2} \sin A \sin B = \frac{1}{2} [\cos(A - B) - \cos(A + B)]$$

$$\textcircled{3} \cos A \cos B = \frac{1}{2} [\cos(A - B) + \cos(A + B)]$$

Example

$$\begin{aligned}\int \sin(4x) \cos(5x) dx &= \int \frac{1}{2} [\sin(4x - 5x) + \sin(4x + 5x)] dx \\ &= \frac{1}{2} \int (\sin(-x) + \sin(9x)) dx \\ &= \frac{1}{2} \int (-\sin x + \sin(9x)) dx \\ &= \frac{1}{2} \left(\cos x - \frac{1}{9} \cos(9x) \right) + C\end{aligned}$$