Precalculus
Definition of the trigonometric functions and basic computations

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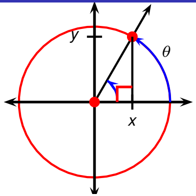
Outline

Definition of the trigonometric functions



- For an angle-measure θ we selected geometric angle with initial arm on x axis and terminal arm selected by traveling θ units on the unit circle.
- Let (x, y) be the intersection of the terminal arm of the geometric angle with the unit circle.

Definition ($\sin$ and $\cos$)

The sine and cosine functions of the angle θ, denoted by $\sin\theta$ and $\cos\theta$, are defined by

$$\cos\theta = x \qquad \sin\theta = y.$$

Definition (additional trigonometric functions)

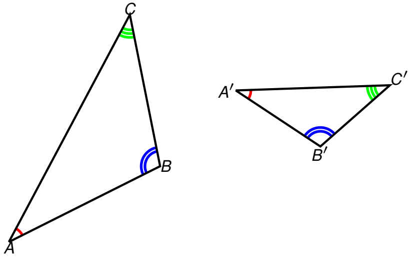
The functions tangent, cotangent, secant and cosecant of the angle θ, denoted by $\tan\theta$, $\cot\theta$, $\sec\theta$, $\csc\theta$, are defined by

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta}.$$

Definition (Similar triangles)

We say that a triangle $\triangle ABC$ is similar to a triangle $\triangle A'B'C'$ if the two triangles have equal angles.

- The equal angles are assumed given in the same order for both triangles, that is, $\angle ABC = \angle A'B'C'$, $\angle BCA = \angle B'C'A'$, $\angle CAB = \angle C'A'B'$.

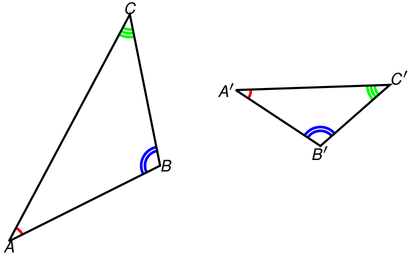


The following statement is proved in the subject of Euclidean (planar) geometry.

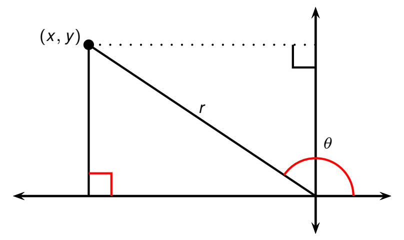
Theorem (Similar triangles have equal side ratios)

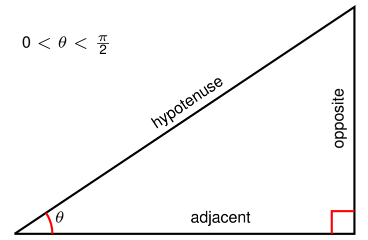
Let $\triangle ABC$ and $\triangle A'B'C'$ be two similar triangles. Then the ratios of the lengths of the sides of the two triangles are equal, that is

$$\frac{|AB|}{|BC|} = \frac{|A'B'|}{|B'C'|} \qquad \frac{|BC|}{|CA|} = \frac{|B'C'|}{|C'A'|} \qquad \frac{|CA|}{|AB|} = \frac{|C'A'|}{|A'B'|}$$



Trigonometric Functions and Right Angle Triangles

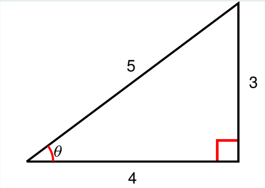




$\cos\theta = \frac{x}{r}$ $\quad$ $\sec\theta = \frac{r}{x}$ $\sin\theta = \frac{y}{r}$ $\quad$ $\csc\theta = \frac{r}{y}$ $\tan\theta = \frac{y}{x}$ $\quad$ $\cot\theta = \frac{x}{y}$	$\cos\theta = \frac{\text{adj}}{\text{hyp}}$ $\quad$ $\sec\theta = \frac{\text{hyp}}{\text{adj}}$ $\sin\theta = \frac{\text{opp}}{\text{hyp}}$ $\quad$ $\csc\theta = \frac{\text{hyp}}{\text{opp}}$ $\tan\theta = \frac{\text{opp}}{\text{adj}}$ $\quad$ $\cot\theta = \frac{\text{adj}}{\text{opp}}$
All angles	Acute angles

- The trigonometric functions can be defined without requesting that the pt. (x, y) on the terminal arm of the angle lie on the unit circle.
- To do so we rescale by the distance r from the origin.
- The trig functions of acute θ (between 0 and $\frac{\pi}{2}$) can be interpreted as ratios of sides of right angle triangle with angle θ.

Example



Let the angle θ be as indicated in the figure. Find the values of the six trigonometric functions of θ.

To find the trigonometric functions, we need to know the length of the hypotenuse.

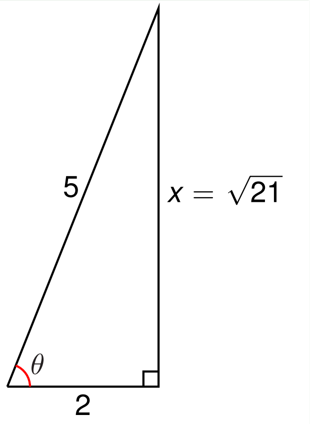
$$\text{hypotenuse} = \sqrt{4^2 + 3^2} = \sqrt{25} = 5.$$

Using the right angle triangle ratio interpretations of the trig functions, we can compute:

$$\sin\theta = \frac{3}{5} \quad \cos\theta = \frac{4}{5} \quad \tan\theta = \frac{3}{4}$$
$$\csc\theta = \frac{5}{3} \quad \sec\theta = \frac{5}{4} \quad \cot\theta = \frac{4}{3}$$

Example

If $\cos\theta = \frac{2}{5}$ and $0 < \theta < \frac{\pi}{2}$, find the other five trigonometric functions of θ.

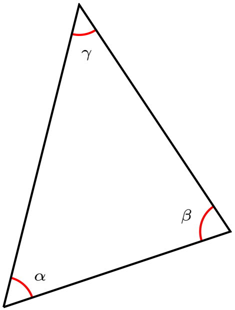


- Label the hypotenuse with length 5 and the adjacent side with length 2.
- Pythagorean theorem: $x^2 + 2^2 = 5^2$.
- Therefore $x^2 = 21$, so $x = \sqrt{21}$.

$$\sin\theta = \frac{\sqrt{21}}{5} \qquad \tan\theta = \frac{\sqrt{21}}{2}$$

$$\csc\theta = \frac{5}{\sqrt{21}} \qquad \sec\theta = \frac{5}{2}$$

$$\cot\theta = \frac{2}{\sqrt{21}}$$



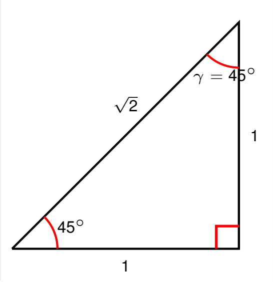
Proposition

The angles of every triangle sum up to $\pi = 180^\circ$*.*

In other words, if α, β, γ are the angles indicated in the figure, then we have:

$$\alpha + \beta + \gamma = 180^\circ.$$

Example



Find the values of $\sin 45^\circ, \cos 45^\circ, \tan 45^\circ$.

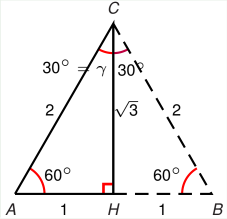
- Draw the 45° angle in right angle triangle, adjacent side of length 1.
- Let γ be the angle indicated on the plot.
- Angles in triangle sum to 180°:

$$\begin{aligned} 45^\circ + 90^\circ + \gamma &= 180^\circ \\ \gamma &= 180^\circ - 90^\circ - 45^\circ = 45^\circ. \end{aligned}$$

- Triangle has two equal angles $\Rightarrow$ is isosceles (has two equal sides).
- $\Rightarrow$ Opposite leg: length 1 $\Rightarrow$ length(hyp) $= \sqrt{1^2+1^2} = \sqrt{2}$.
- $\sin 45^\circ = \frac{\text{opp}}{\text{hyp}} = \frac{\sqrt{2}}{2}$ $\quad \cos 45^\circ = \frac{\text{adj}}{\text{hyp}} = \frac{\sqrt{2}}{2}$

  $\tan 45^\circ = \frac{\text{opp}}{\text{adj}} = \frac{1}{1} = 1.$

Example



Find the values of $\sin 60^\circ, \cos 60^\circ, \tan 60^\circ$, $\sin 30^\circ, \cos 30^\circ, \tan 30^\circ$.

Construct a right angled $\triangle AHC$ as indicated: angles $60^\circ, 90^\circ, \gamma$. Angles in $\triangle$ sum to 180°:

$$\begin{aligned} 60^\circ + 90^\circ + \gamma &= 180^\circ \\ \gamma &= 180^\circ - 90^\circ - 60^\circ = 30^\circ. \end{aligned}$$

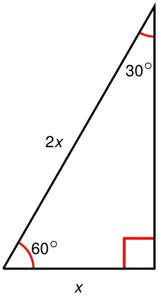
Construct $\triangle HBC$ as indicated so that $\triangle HBC \cong \triangle HAC$. $\triangle ABC$ has three equal angles ($= 60^\circ$) $\Rightarrow$ its sides are of equal length. Therefore

$$\begin{aligned} |AC| &= |AB| = 1 + 1 = 2 \\ |CH| &= \sqrt{|AC|^2 - |AH|^2} && \Big| \text{ Pythagorean theorem} \\ &= \sqrt{2^2 - 1^2} = \sqrt{3} \end{aligned}$$

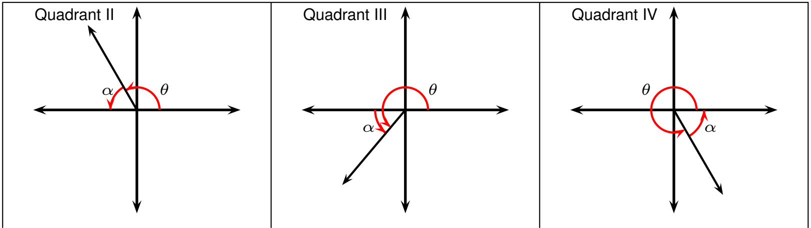
$$\begin{aligned} \sin 60^\circ &= \frac{\sqrt{3}}{2} & \cos 60^\circ &= \frac{1}{2} & \tan 60^\circ &= \frac{\sqrt{3}}{1} = \sqrt{3} \\ \sin 30^\circ &= \frac{1}{2} & \cos 30^\circ &= \frac{\sqrt{3}}{2} & \tan 30^\circ &= \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}. \end{aligned}$$

Observation

- *If the hypotenuse of a right angle triangle is twice larger than one of the sides, then the angle opposite to that side is* 30°*.*
- *Conversely, in a right angle triangle with angle* 30°*, the hypotenuse is twice longer than the shorter of the two legs.*



The computation of the reference angle α depends on the quadrant.



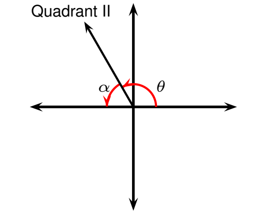
Quadrant II	Quadrant III	Quadrant IV
$\alpha = \pi - \theta$	$\alpha = \theta - \pi$	$\alpha = 2\pi - \theta$
$\alpha = 180^\circ - \theta$	$\alpha = \theta - 180^\circ$	$\alpha = 360^\circ - \theta$

To compute trigonometric functions from obtuse ($> 90^\circ$) or negative angles, we can use the following visual aid.

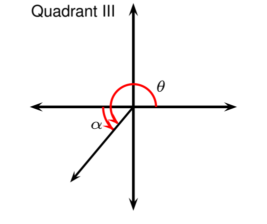
Definition (Reference Angle)

Let θ be an angle in standard position. Its reference angle is the acute positive angle formed by the terminal arm and the *x* axis.

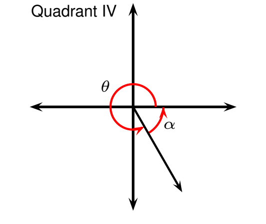
The computation of the reference angle α depends on the quadrant.



$$\alpha = \pi - \theta$$
$$\alpha = 180^\circ - \theta$$



$$\alpha = \theta - \pi$$
$$\alpha = \theta - 180^\circ$$



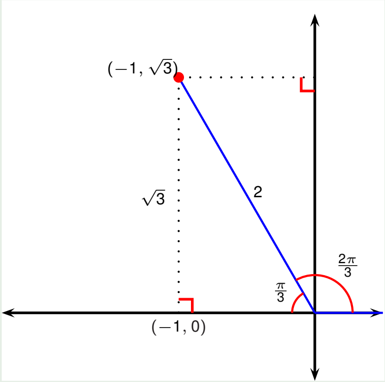
$$\alpha = 2\pi - \theta$$
$$\alpha = 360^\circ - \theta$$

Observation

One can find the value of a trigonometric function of θ as follows.

- *Find the reference angle α associated to θ.*
- *Find the trig function of α.*
- *Use the quadrant in which θ lies to affix an appropriate sign to the function value.*

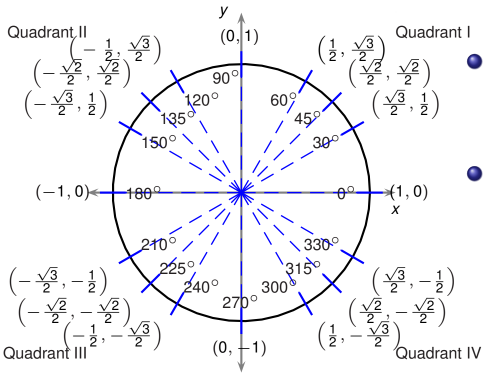
Example



Find the exact values of the trigonometric functions of $\theta = \dfrac{2\pi}{3} = 120^\circ$.

$$\sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2} \quad \cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2} \quad \tan\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{-1} = -\sqrt{3}$$

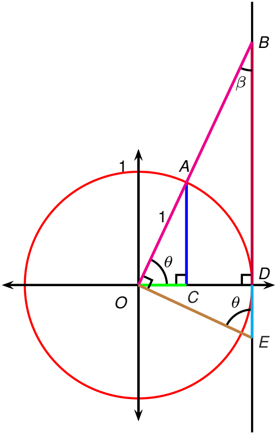
$$\csc\left(\frac{2\pi}{3}\right) = \frac{2}{\sqrt{3}} \quad \sec\left(\frac{2\pi}{3}\right) = -\frac{2}{1} = -2 \quad \cot\left(\frac{2\pi}{3}\right) = -\frac{1}{\sqrt{3}}$$



- One only needs to memorize sines and cosines in Quadrant I and on the axes.
- The remaining sines and cosines are extracted by
  - taking the sine/cosine of the reference angle
  - and adjusting the sign according to the quadrant.

Deg.	0°	30°	45°	60°	90°	120°	135°	150°	180°	270°	360°
Rad.	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π
sin	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	0	1

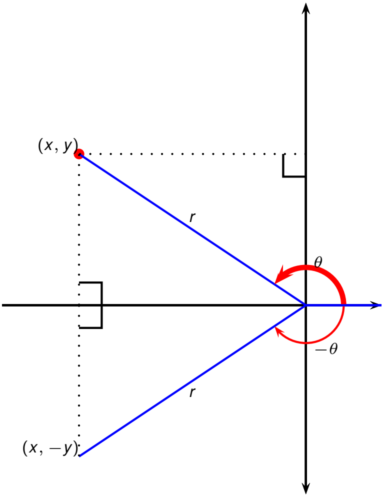
Geometric interpretation of all trigonometric functions



$$\begin{aligned}\beta &= 180^\circ - 90^\circ - \theta\\ &= 90^\circ - \theta\\ \angle OED &= 180^\circ - 90^\circ - \beta\\ &= 90^\circ - (90^\circ - \theta)\\ &= \theta\end{aligned}$$

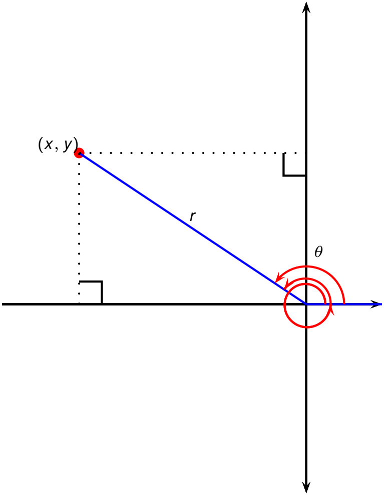
Fix unit circle, center O, coordinates $(0, 0)$. Let $\angle DOB = \theta$. Let OB intersect the circle at point A. Coordinates of A are $(\cos\theta, \sin\theta)$.

$$\begin{aligned}\sin\theta &= \frac{\text{opp}}{\text{hyp}} = \frac{|AC|}{|OA|} = \frac{|AC|}{1} = |AC|\\ \cos\theta &= \frac{\text{adj}}{\text{hyp}} = \frac{|OC|}{|OA|} = \frac{|OC|}{1} = |OC|\\ \tan\theta &= \frac{\text{opp}}{\text{adj}} = \frac{|BD|}{|OD|} = \frac{|BD|}{1} = |BD|\\ \cot\theta &= \frac{\text{adj}}{\text{opp}} = \frac{|DE|}{|OD|} = \frac{|DE|}{1} = |DE|\\ \sec\theta &= \frac{\text{hyp}}{\text{adj}} = \frac{|OB|}{|OD|} = \frac{|OB|}{1} = |OB|\\ \csc\theta &= \frac{\text{hyp}}{\text{opp}} = \frac{|OE|}{|DO|} = \frac{|OE|}{1} = |OE|\end{aligned}$$



$$\sin\theta = \frac{y}{r} \quad \csc\theta = \frac{r}{y}$$
$$\cos\theta = \frac{x}{r} \quad \sec\theta = \frac{r}{x}$$
$$\tan\theta = \frac{y}{x} \quad \cot\theta = \frac{x}{y}$$

- Positive angles are obtained by rotating counterclockwise.
- Negative angles are obtained by rotating clockwise.
- If (x, y) is on the terminal arm of the angle θ, then $(x, -y)$ is on the terminal arm of $-\theta$.
- $\sin(-\theta) = \frac{-y}{r} = -\frac{y}{r} = -\sin\theta$.
- $\cos(-\theta) = \frac{x}{r} = \cos\theta$.
- sin is an odd function.
- cos is an even function.



$\sin\theta = \frac{y}{r}$ $\csc\theta = \frac{r}{y}$
$\cos\theta = \frac{x}{r}$ $\sec\theta = \frac{r}{x}$
$\tan\theta = \frac{y}{x}$ $\cot\theta = \frac{x}{y}$

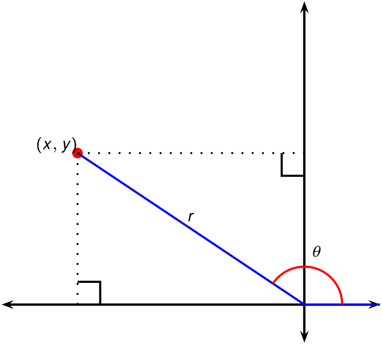
- 2π represents a full rotation.
- $\theta + 2\pi$ has the same terminal arm as θ.
- $\theta + 2\pi$ uses the same point (x, y) and the same length r.
- $\sin(\theta + 2\pi) = \sin\theta$.
- $\cos(\theta + 2\pi) = \cos\theta$.
- We say $\sin$ and $\cos$ are 2π-periodic.

Trigonometric Identities

Definition (Trigonometric Identity)

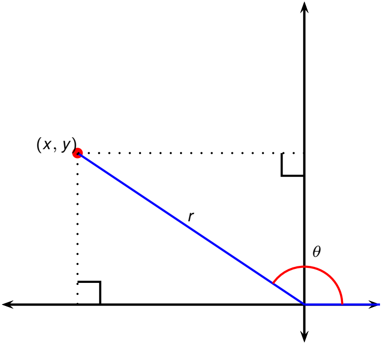
A trigonometric identity is an equality between the trigonometric functions in one or more variables that holds for all values of the involved variables in the domains of all of the expressions.

- By convention, when dealing with trigonometric identities we do not account for the domains of the involved expressions.
- For example, $\frac{\sin\theta}{\sin\theta} = 1$ is considered a valid trigonometric identity, although, when considered as a function, the left hand side is not defined for $\theta \neq 0$.



$$\sin\theta = \frac{y}{r} \quad \csc\theta = \frac{r}{y}$$
$$\cos\theta = \frac{x}{r} \quad \sec\theta = \frac{r}{x}$$
$$\tan\theta = \frac{y}{x} \quad \cot\theta = \frac{x}{y}$$

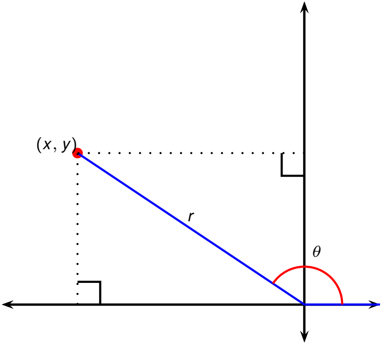
- $\csc\theta = \frac{1}{\sin\theta}$
- $\sec\theta = \frac{1}{\cos\theta}$
- $\cot\theta = \frac{1}{\tan\theta}$
- $\tan\theta = \frac{\sin\theta}{\cos\theta}$
- $\cot\theta = \frac{\cos\theta}{\sin\theta}$



$$\sin\theta = \frac{y}{r} \quad \csc\theta = \frac{r}{y}$$
$$\cos\theta = \frac{x}{r} \quad \sec\theta = \frac{r}{x}$$
$$\tan\theta = \frac{y}{x} \quad \cot\theta = \frac{x}{y}$$

$$\begin{aligned} & \sin^2\theta + \cos^2\theta \\ = & \frac{y^2}{r^2} + \frac{x^2}{r^2} \\ = & \frac{y^2 + x^2}{r^2} \\ = & \frac{r^2}{r^2} \\ = & 1 \end{aligned}$$

Therefore $\sin^2\theta + \cos^2\theta = 1$.



$$\sin\theta = \frac{y}{r} \quad \csc\theta = \frac{r}{y}$$
$$\cos\theta = \frac{x}{r} \quad \sec\theta = \frac{r}{x}$$
$$\tan\theta = \frac{y}{x} \quad \cot\theta = \frac{x}{y}$$

Example ($\tan^2\theta + 1 = \sec^2\theta$)

Prove the identity
$\tan^2\theta + 1 = \sec^2\theta$.

$$\begin{aligned} \sin^2\theta + \cos^2\theta &= 1 \\ \frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} &= \frac{1}{\cos^2\theta} \\ \tan^2\theta + 1 &= \sec^2\theta \end{aligned}$$