Calculus II
Integral of rational function with cubic denominator, part 3

Todor Milev

2019

Example

Find $\int \frac{2x^2-x+4}{x^3+4x}\mathrm{d}x$.

- $\deg(2x^2 - x + 4) < \deg(x^3 + 4x)$: don't divide.
- Factor denominator: $x^3 + 4x = x(x^2 + 4)$.

$$\begin{aligned}
\frac{2x^2 - x + 4}{x(x^2+4)} &= \frac{A}{x} + \frac{Bx + C}{(x^2+4)} \\
2x^2 - x + 4 &= A(x^2+4) + (Bx + C)x \\
2x^2 - x + 4 &= (A + B)x^2 + Cx + 4A
\end{aligned}$$

$$A = 1 \qquad C = -1 \qquad A + B = 2, \text{ therefore } B = 1$$

$$\begin{aligned}
\int \frac{2x^2 - x + 4}{x(x^2+4)}\mathrm{d}x &= \int \left(\frac{1}{x} + \frac{x-1}{x^2+4} \right) \mathrm{d}x \\
&= \int \frac{1}{x}\mathrm{d}x + \int \frac{x}{x^2+4}\mathrm{d}x - \int \frac{1}{x^2+4}\mathrm{d}x \\
&= \ln|x| + \frac{1}{2}\ln(x^2+4) - \frac{1}{2}\arctan\left(\frac{x}{2}\right) + K
\end{aligned}$$

$Q(x)$ contains quadratic factors, multiplicity 1

- Suppose $Q(x)$ contains quadratic factors $ax^2 + bx + c$ with where $b^2 - 4ac < 0$ (i.e., the factor is irreducible).
- Suppose none of the quadratic factors is repeated.
- The for each quadratic factor we need to add a partial fraction of the form
$$\frac{Ax + B}{ax^2 + bx + c}.$$