Calculus II
Tangents and curve length

Todor Milev

2019

Outline

1. Tangents to Curves
   - Tangents to Polar Curves

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   - Tangents to Polar Curves
2. Arc Length
   - Arc Length in Polar Coordinates

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Tangents

Let C be the curve $C : \left| \begin{array}{rcl} x &=& f(t) \\ y &=& g(t) \end{array} \right., t \in [a, b]$.

Definition

Suppose $f'(t)$ and $g'(t)$ are not simultaneously equal to 0.

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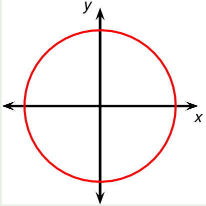
Note. When $f'(t) = g'(t) = 0$, for curves C with additional properties, natural definition(s) of tangent(s) do exist but are beyond Calc II.

Example



Find the tangent to the curve
$$\gamma : \left| \begin{array}{lcl} x &=& \cos t \\ y &=& \sin t \end{array} \right. , t \in [0, 2\pi) \text{ at } t = \frac{\pi}{4}, t = \frac{2\pi}{3}, t = \pi.$$

Recall $C : \left| \begin{array}{lcl} x & = & f(t) \\ y & = & g(t) \end{array} \right., t \in [a, b]$, tangent vector at t is $(f'(t), g'(t))$.

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If $\frac{dx}{dt} \neq 0$, *we have* $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$.

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We write informally $x = x(t), y = y(t)$ to simplify notation.

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If $\frac{\mathrm{d}x}{\mathrm{d}t} \neq 0$, *we have* $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}$.

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$$\begin{array}{rcll} y & = & F(x) & \Big| \text{ apply } \frac{\mathrm{d}}{\mathrm{d}t} \\ \frac{\mathrm{d}y}{\mathrm{d}t} & = & \frac{\mathrm{d}}{\mathrm{d}t}(F(x)) & \Big| \text{ use chain rule} \\ & = & \frac{\mathrm{d}F}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} & \Big| \text{ divide by } x'(t) \\ \frac{\mathrm{d}y}{\mathrm{d}x} & = & \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} & \end{array}$$

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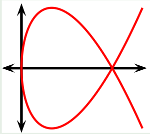
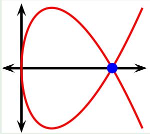


A curve C is defined by $x = t^2, y = t^3 - 3t$.

1. Show C traverses $(x, y) = (3, 0)$ for two values of t; find the tangent slopes for both of these values.
2. Find the points on C where the tangents are horizontal or vertical.
3. Find two intervals where we can write y as a function of x.
4. Determine concavity intervals of the functions found in item 3.

Example



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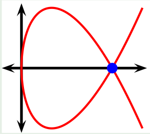

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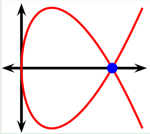

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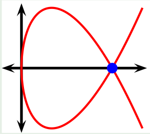

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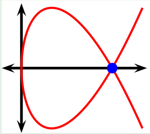

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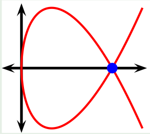

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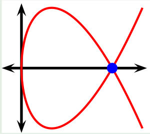

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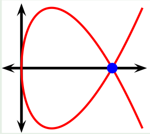

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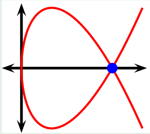

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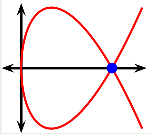

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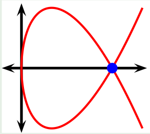

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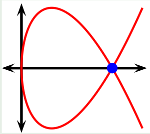

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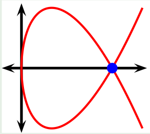

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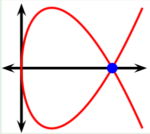

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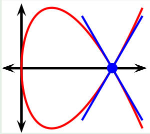

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  Therefore the tangents at $(3, 0)$ have slopes $\pm\sqrt{3}$.

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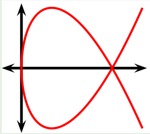

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2. Find the points on C where the tangents are horizontal or vertical.

Horizontal tangent:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 0$$

Vertical tangent:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 0$$

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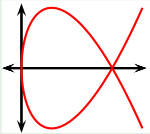

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$$\begin{aligned} \frac{dy}{dt} &= 0 \\ 3t^2 - 3 &= 0 \end{aligned}$$

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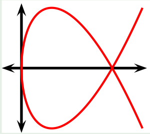

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$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \\ 3t^2 - 3 &= 0 \\ 3(t^2 - 1) &= 0 \end{aligned}$$

Vertical tangent:

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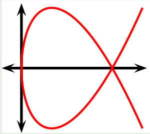

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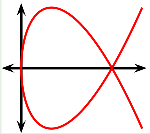

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$\frac{dx}{dt} \neq 0$ when $t = \pm 1$, so there are horizontal tangents when $t = \pm 1$.

Vertical tangent:

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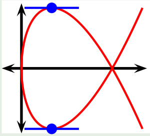

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The points are $(1, 2)$ and $(1, -2)$.

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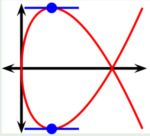
$$\frac{\mathrm{d}x}{\mathrm{d}t} = 0$$

Example



A curve C is defined by $x = t^2, y = t^3 - 3t$.

2. Find the points on C where the tangents are horizontal or vertical.

Horizontal tangent:

$$\begin{aligned} \frac{dy}{dt} &= 0 \\ 3t^2 - 3 &= 0 \\ 3(t^2 - 1) &= 0 \\ t &= \pm 1 \end{aligned}$$

$\frac{dx}{dt} \neq 0$ when $t = \pm 1$, so there are horizontal tangents when $t = \pm 1$.
The points are $(1, 2)$ and $(1, -2)$.

Vertical tangent:

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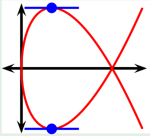

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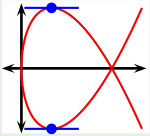

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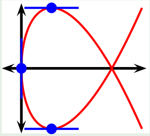
$\frac{dy}{dt} \neq 0$ when $t = 0$, so there is a vertical tangent when $t = 0$.

Example



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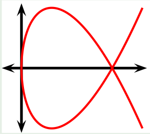
Example



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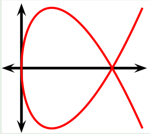
3. Find two intervals where we can write y as a function of x.

Example



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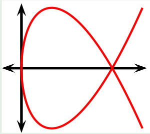

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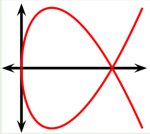
Example



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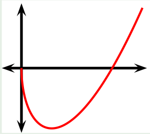

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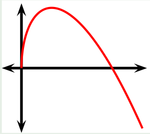
Example



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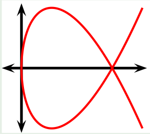
Example



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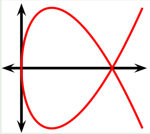
Example



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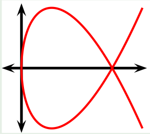

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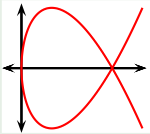

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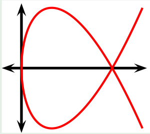

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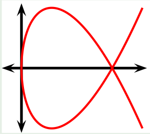

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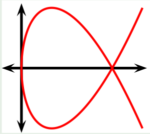

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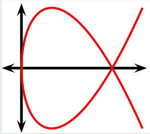

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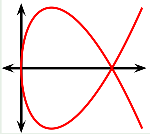

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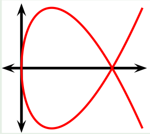

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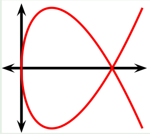

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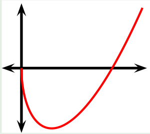

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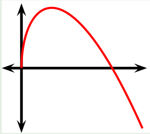
Therefore y as a function of x (which is a function of t) is concave up when $t > 0$

Example



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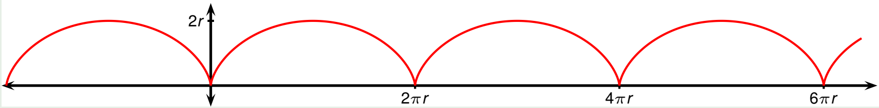
Therefore y as a function of x (which is a function of t) is concave up when $t > 0$ and concave down when $t < 0$.

Example

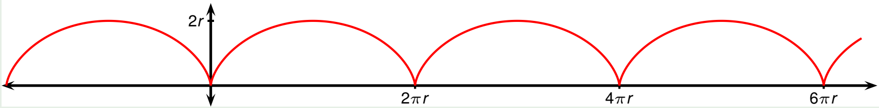
Consider the cycloid $x = r(\theta - \sin\theta)$, $y = r(1 - \cos\theta)$.



1. At what points is the tangent horizontal?
2. At what points is the tangent vertical?

Example

Consider the cycloid $x = r(\theta - \sin\theta)$, $y = r(1 - \cos\theta)$.



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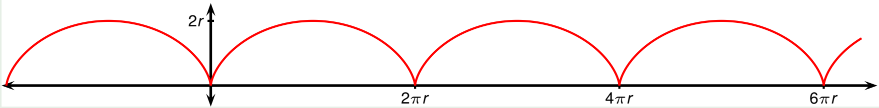

1. At what points is the tangent horizontal?

- The slope of the tangent is $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$

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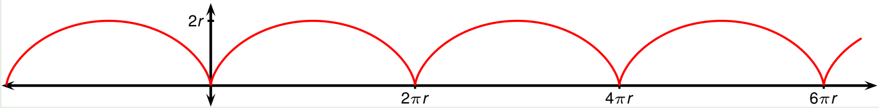

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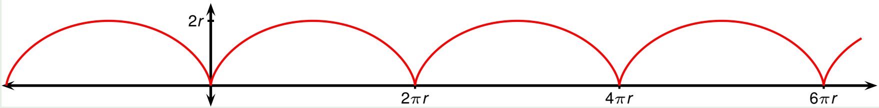

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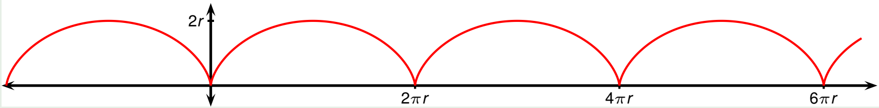

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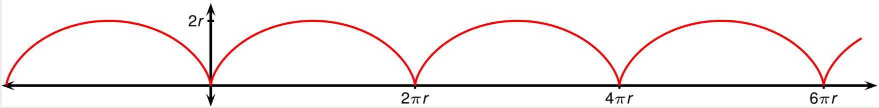

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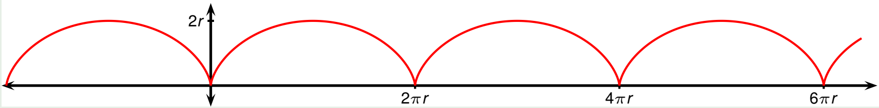

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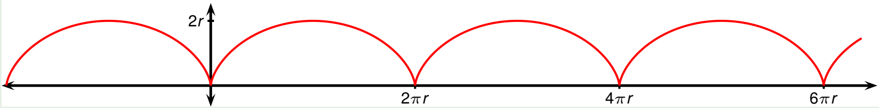

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- The tangent is horizontal when $\mathrm{d}y/\mathrm{d}x = 0$, that is, when $\mathrm{d}y/\mathrm{d}\theta = 0$ and $\mathrm{d}x/\mathrm{d}\theta \neq 0$.

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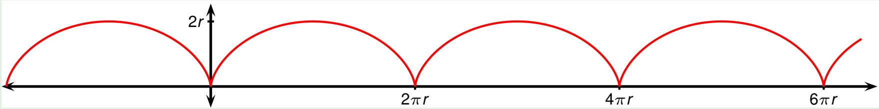

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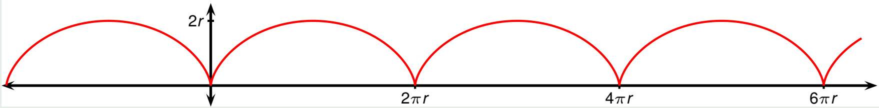

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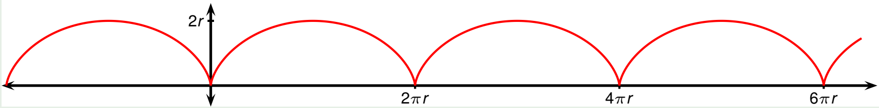

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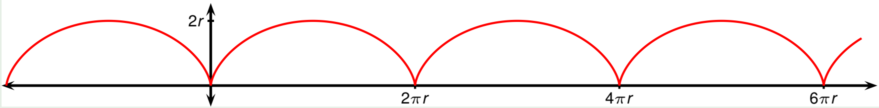

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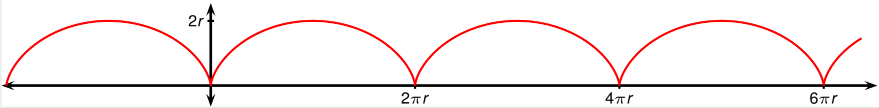

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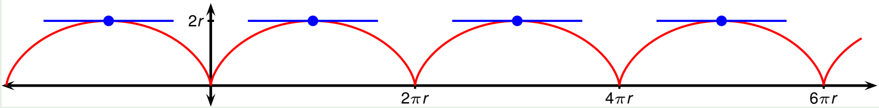

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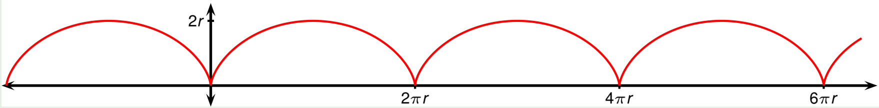

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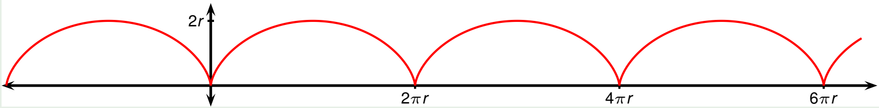

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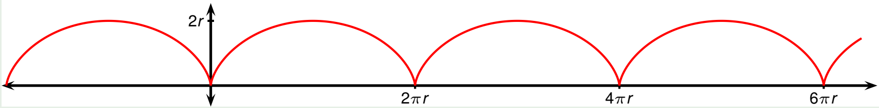

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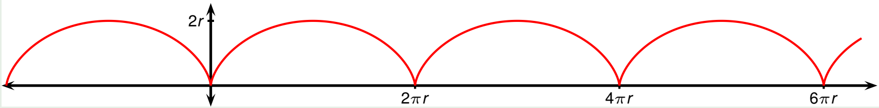

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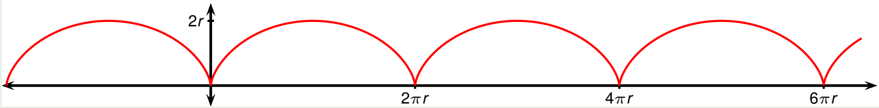

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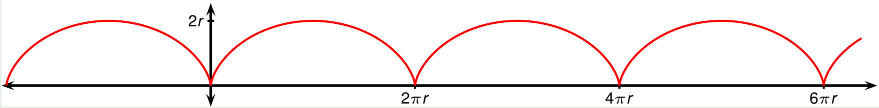

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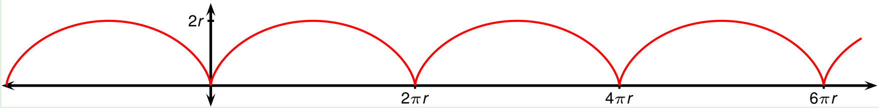

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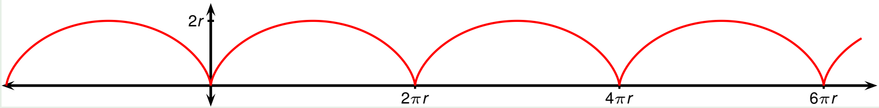

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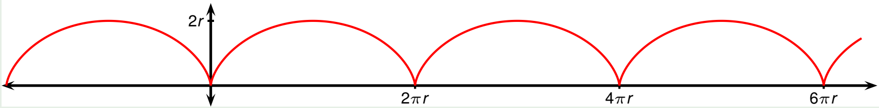

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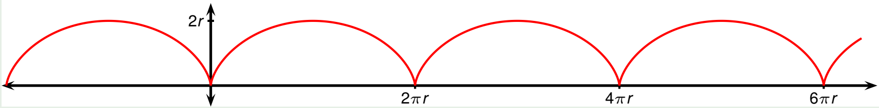

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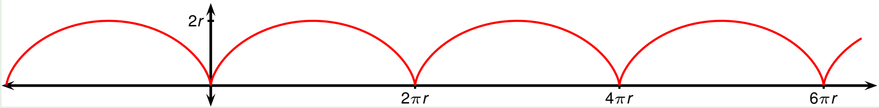

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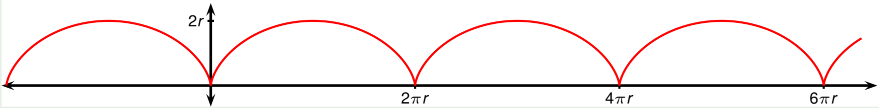

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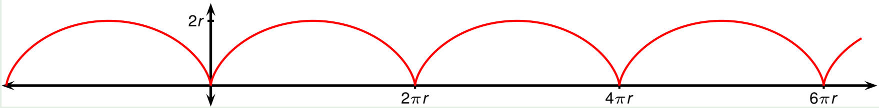

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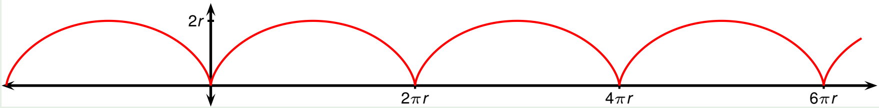

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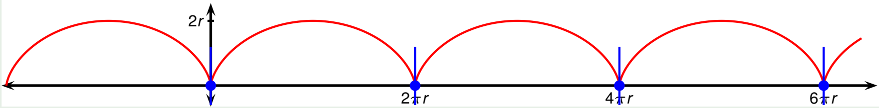

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Tangents to Polar Curves

To find the tangent line to a polar curve $r = f(\theta)$, regard θ as a parameter and write the parametric equations as

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&= \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta + f(\theta)(-\sin\theta)}\\
&= \frac{\frac{\mathrm{d}r}{\mathrm{d}\theta}\sin\theta + r\cos\theta}{\frac{\mathrm{d}r}{\mathrm{d}\theta}\cos\theta - r\sin\theta}
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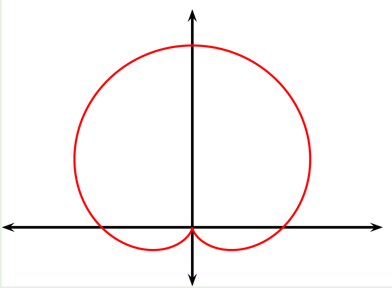
Example

Find the points on $r = 1 + \sin\theta$ where the tangent is horizontal or vertical.



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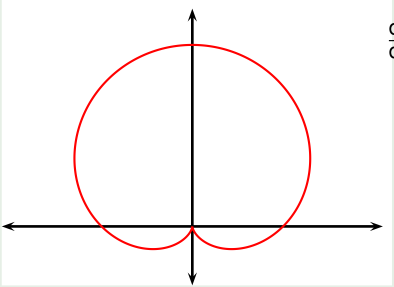

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}r}{\mathrm{d}\theta}\sin\theta + r\cos\theta}{\frac{\mathrm{d}r}{\mathrm{d}\theta}\cos\theta - r\sin\theta}$$

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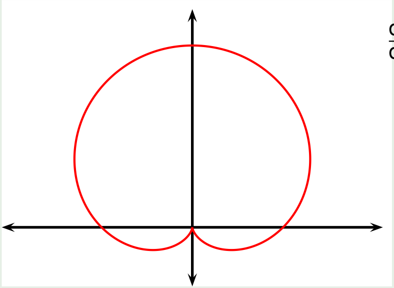

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} = \frac{\sin\theta + \qquad\qquad \cos\theta}{\cos\theta - \qquad\qquad \sin\theta}$$

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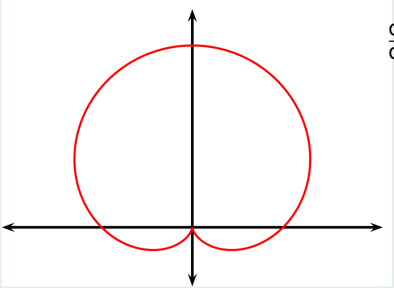

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} = \frac{\sin\theta + (1+\sin\theta)\cos\theta}{\cos\theta - (1+\sin\theta)\sin\theta}$$

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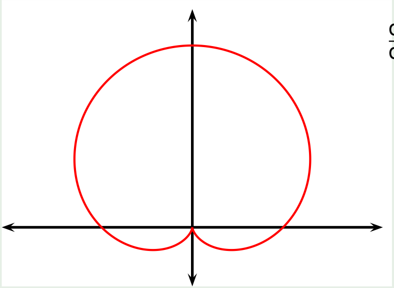

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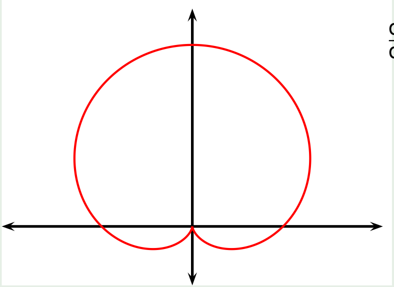

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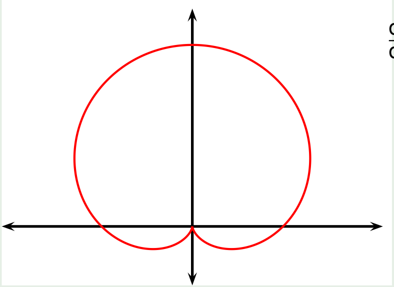

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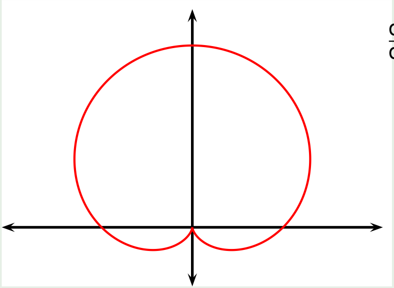

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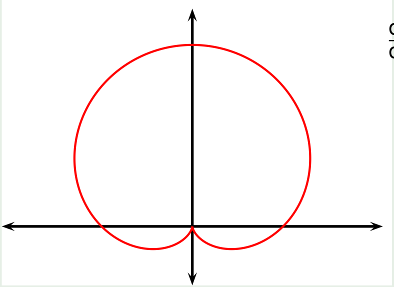

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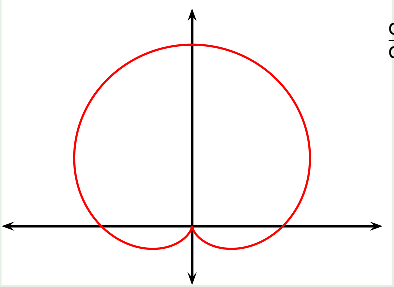

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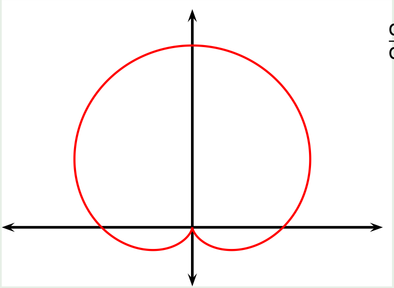

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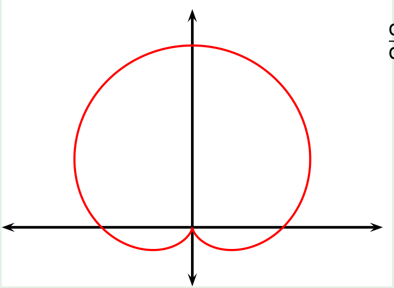

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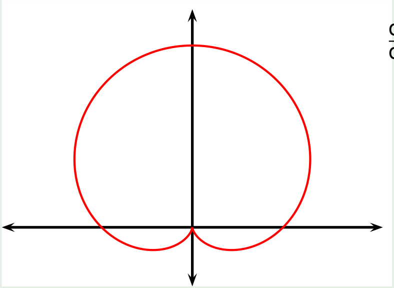

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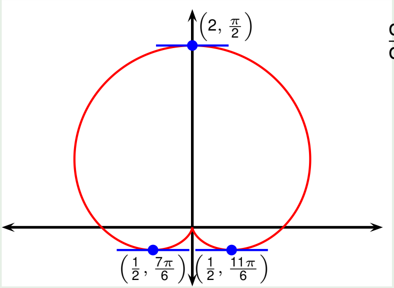

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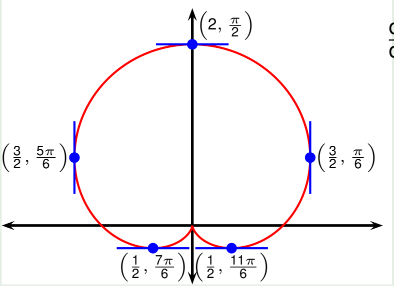

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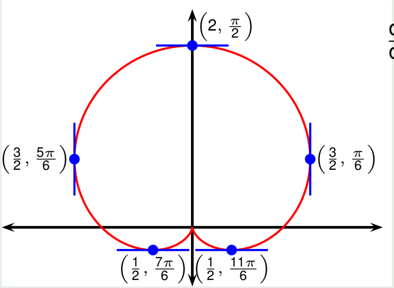

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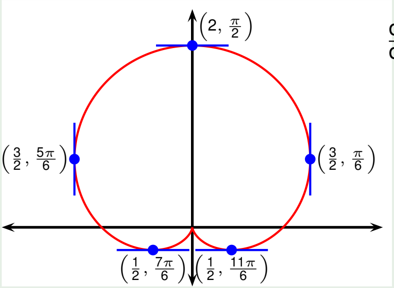

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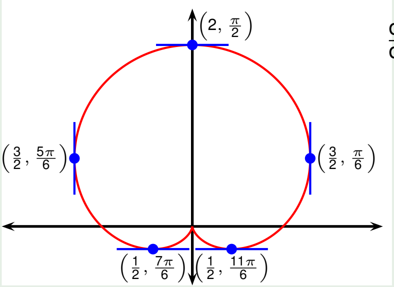

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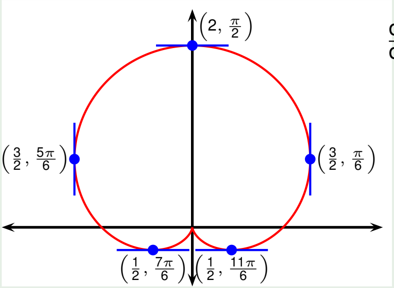

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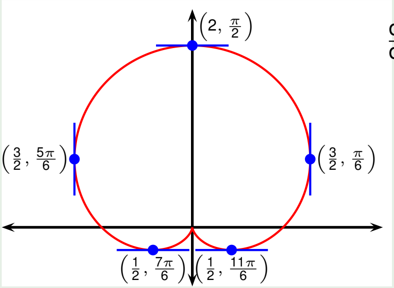

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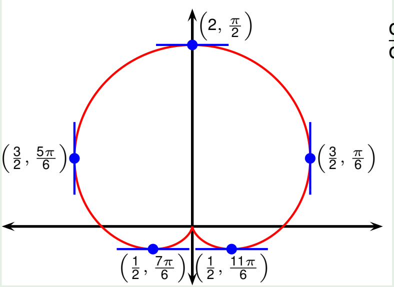

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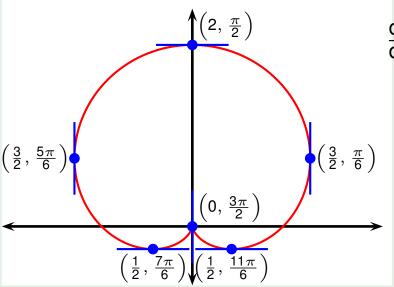

$$\begin{aligned}\frac{dy}{dx} &= \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} = \frac{\cos\theta\sin\theta + (1+\sin\theta)\cos\theta}{\cos\theta\cos\theta - (1+\sin\theta)\sin\theta} \\ &= \frac{\cos\theta(1+2\sin\theta)}{1-2\sin^2\theta-\sin\theta} = \frac{\cos\theta(1+2\sin\theta)}{(1+\sin\theta)(1-2\sin\theta)}\end{aligned}$$

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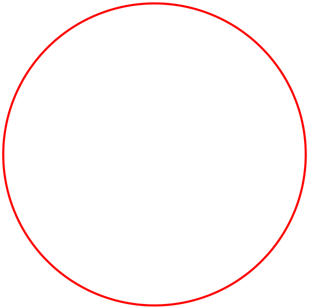
- $(1 + \sin\theta)(1 - 2\sin\theta) = 0$ when $\theta = \frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$.

- Horizontal tangents at $(2, \pi/2)$, $(1/2, 7\pi/6)$, and $(1/2, 11\pi/6)$.
- Vertical tangents at $(3/2, \pi/6)$, and $(3/2, 5\pi/6)$.
- If $\theta = 3\pi/2$, top and bottom are both 0, so use L'Hospital's Rule.

$$\lim_{\theta\to 3\pi/2^-}\frac{dy}{dx} = \lim_{\theta\to 3\pi/2^-}\frac{1+2\sin\theta}{1-2\sin\theta}\cdot\lim_{\theta\to 3\pi/2^-}\frac{\cos\theta}{1+\sin\theta} = -\frac{1}{3}\lim_{\theta\to 3\pi/2^-}\frac{-\sin\theta}{\cos\theta} = \infty$$

Arc Length



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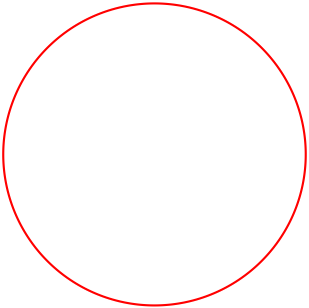

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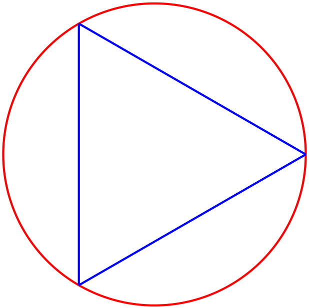

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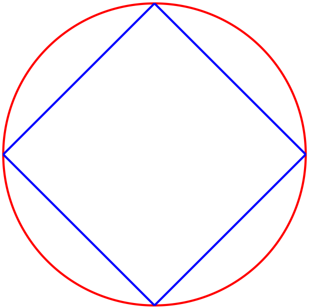

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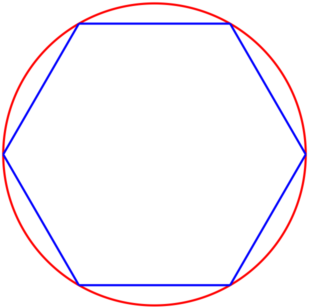

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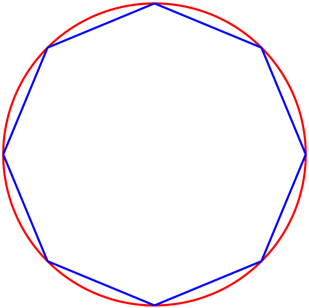

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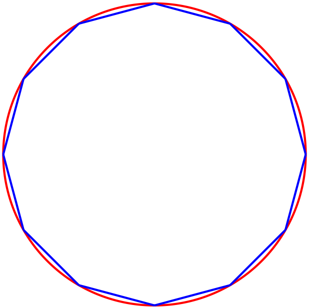

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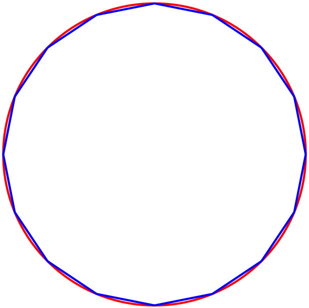

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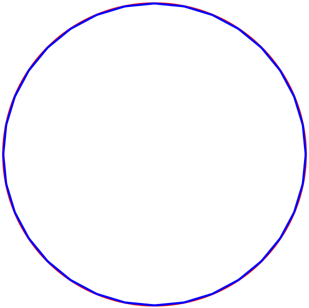

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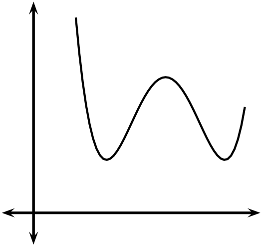

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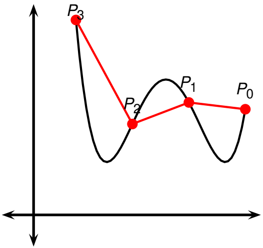

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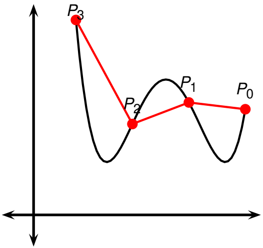

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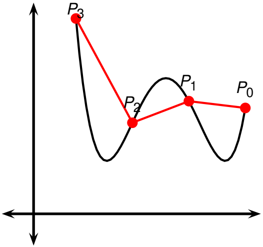

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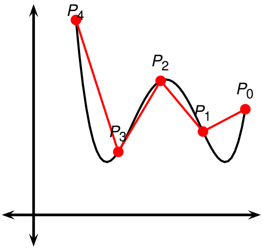

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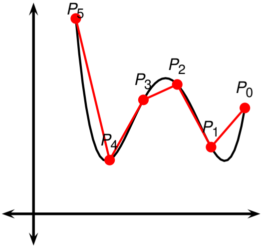

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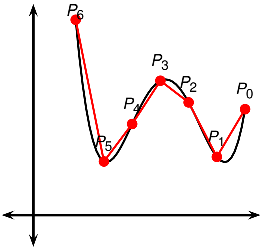

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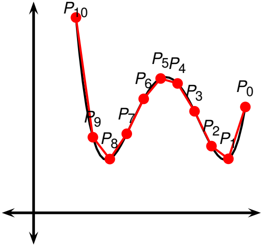

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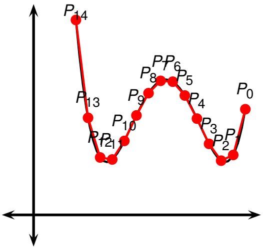

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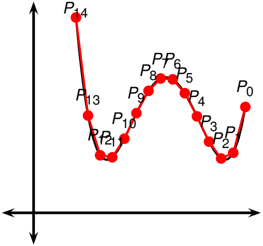

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Let $\gamma : \left| \begin{array}{rcl} x & = & x(t) \\ y & = & y(t) \end{array} \right., t \in [a, b]$.

Definition

Suppose $x'(t)$ and $y'(t)$ (exist and) are continuous on $[a, b]$. Then the length of the curve γ is defined as

$$L(\gamma) = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2} \, \mathrm{d}t$$

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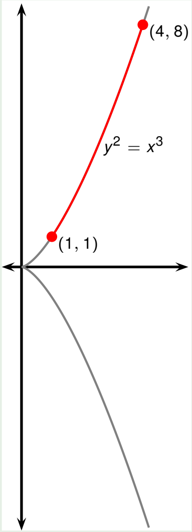
Suppose f' exists and is continuous on $[a, b]$. Then the length of the curve $y = f(x)$, $a \leq x \leq b$, is

$$\begin{aligned} L &= \int_a^b \sqrt{1 + (f'(x))^2}\, \mathrm{d}x \\ &= \int_a^b \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\, \mathrm{d}x \qquad \text{(in Leibniz notation)} \quad . \end{aligned}$$

Example

Find the length of the arc of $y^2 = x^3$ between $(1, 1)$ and $(4, 8)$.



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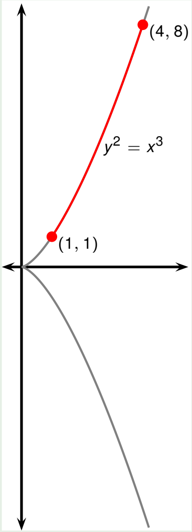

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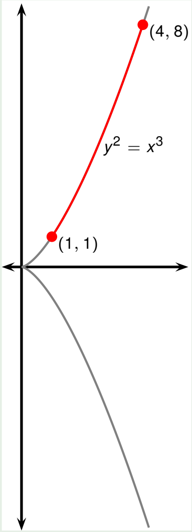

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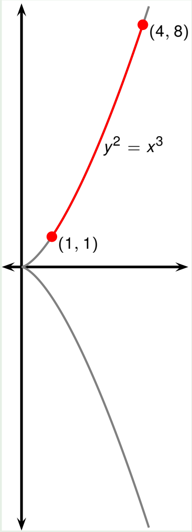

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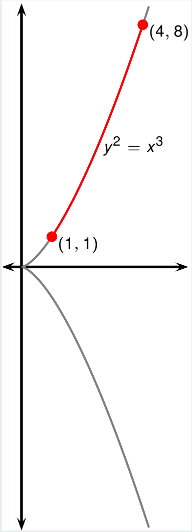

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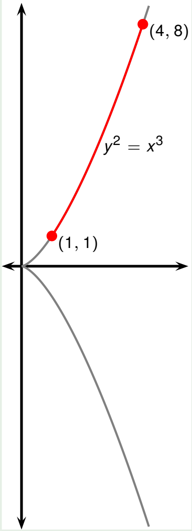

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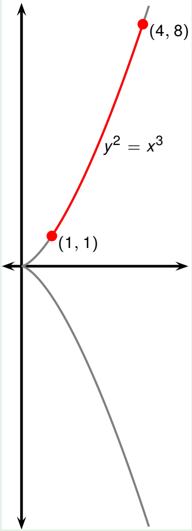

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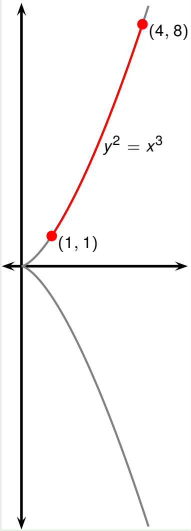

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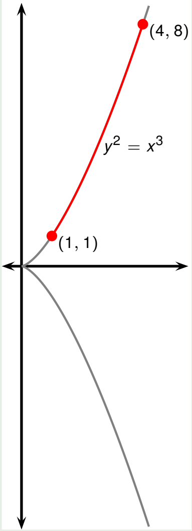

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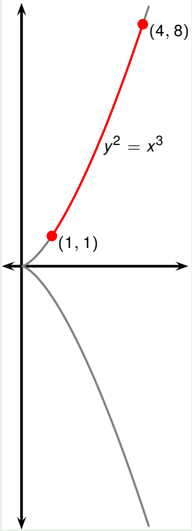

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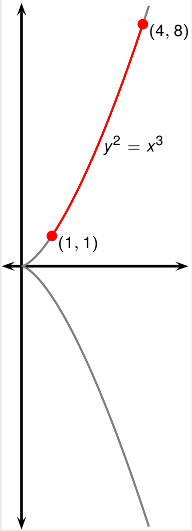

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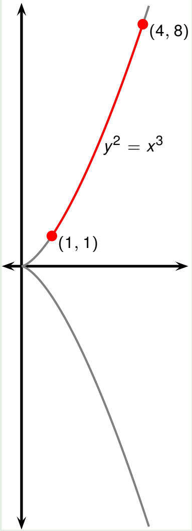

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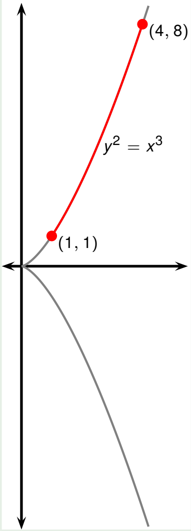

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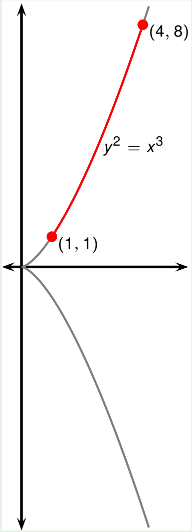

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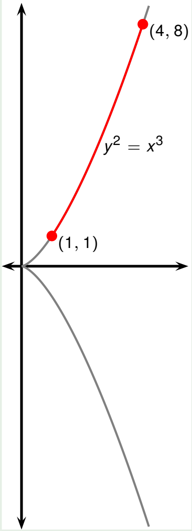

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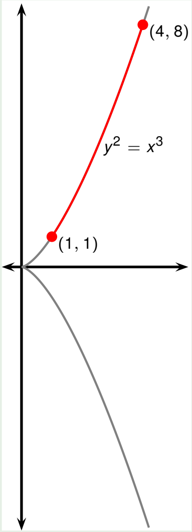

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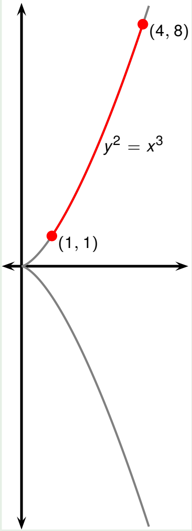
- $u = 1 + \frac{9}{4}x$ and $\mathrm{d}u = \frac{9}{4}\mathrm{d}x$.
- When $x = 1$, $u = \frac{13}{4}$.
- When $x = 4$, $u =$.

$$\begin{aligned} L &= \int_1^4 \sqrt{1 + (y')^2}\mathrm{d}x \\ &= \int_1^4 \sqrt{1 + \frac{9}{4}x}\,\mathrm{d}x = \int_{13/4} \frac{4}{9}\sqrt{u}\,\mathrm{d}u \end{aligned}$$

Example

Find the length of the arc of $y^2 = x^3$ between $(1, 1)$ and $(4, 8)$.



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- When $x = 4$, $u = 10$.

$$
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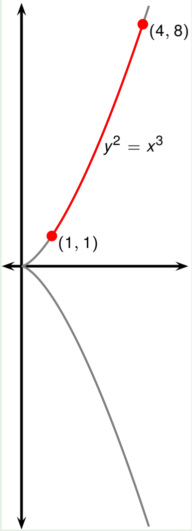

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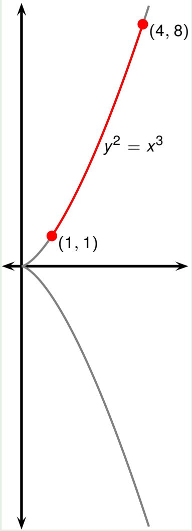

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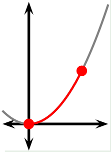
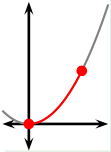
Example



Find the length of the arc of the parabola $y = x^2$ from $(0,0)$ to $(1,1)$.

$$L = \int_{x=0}^{x=1} \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} \mathrm{d}x$$

Example



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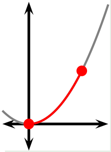

Find the length of the arc of the parabola $y = x^2$ from $(0, 0)$ to $(1, 1)$.

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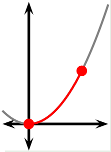

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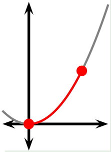

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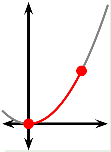

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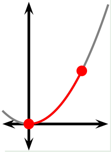

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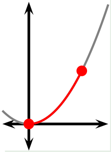

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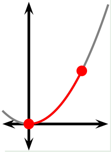

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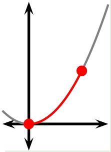

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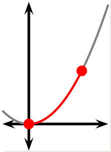

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Set $x = \frac{1}{2}\tan\theta$

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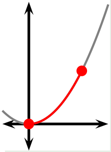
Example



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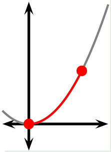
Example



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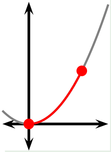
Example



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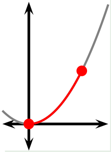
Example



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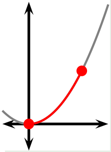

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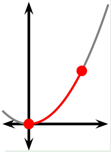
Example



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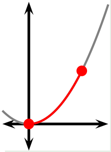
Example



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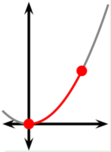
Example



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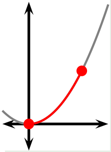
$$= \frac{1}{4}\left(\sec\alpha\tan\alpha + \ln|\sec\alpha + \tan\alpha|\right)$$

Example



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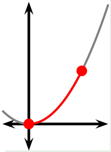
$$= \frac{1}{4}\left(\sec\alpha\,{\color{red}\tan\alpha} + \ln|\sec\alpha + {\color{red}\tan\alpha}|\right)$$

$$= \frac{1}{4}\left({\color{red}\mathbf{?}}\cdot\mathbf{?} \quad + \ln|\mathbf{?} \quad + {\color{red}\mathbf{?}}|\right)$$

Example



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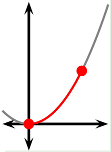
$$= \frac{1}{4}\left(2\cdot\textbf{?} \quad + \ln|\textbf{?} \quad + 2|\right)$$

Example



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$$\frac{dy}{dx} = 2x$$

$$L = \int_{x=0}^{x=1} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_{x=0}^{x=1} \sqrt{1+4x^2}\, dx \quad \Bigg| \quad \text{Set } x = \tfrac{1}{2}\tan\theta$$

$$= \int_{\theta=0}^{\theta=\arctan 2} \sqrt{1+\tan^2\theta}\ d\left(\frac{1}{2}\tan\theta\right)$$

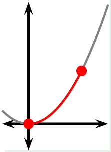
$$= \int_{\theta=0}^{\theta=\arctan 2} \sec\theta \cdot \frac{1}{2}\sec^2\theta d\theta = \frac{1}{2}\int_0^\alpha \sec^3\theta\, d\theta \quad \Bigg| \quad \text{Set } \alpha = \arctan 2$$

$$= \frac{1}{2}\cdot\left[\frac{1}{2}\left(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right)\right]_{\theta=0}^{\theta=\alpha}$$

$$= \frac{1}{4}\left(\sec\alpha\tan\alpha + \ln|\sec\alpha + \tan\alpha|\right)$$

$$= \frac{1}{4}\left(2\cdot \textbf{?} \quad + \ln|\textbf{?} \quad + 2|\right)$$

Example



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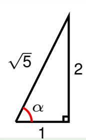

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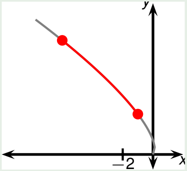

Find the length of the curve γ.

$$\gamma : \left| \begin{array}{rcl} x(t) & = & \sqrt{t} - 2t \\ y(t) & = & \frac{8}{3} t^{\frac{3}{4}} \end{array} \right. , t \in [1, 4] \quad .$$

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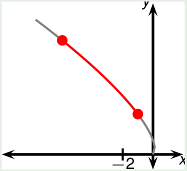

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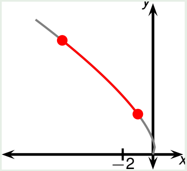

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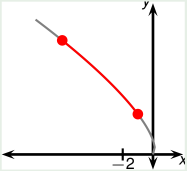

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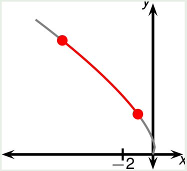

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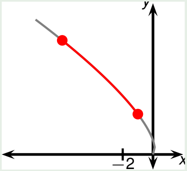

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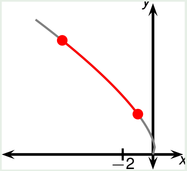

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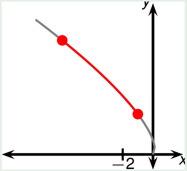

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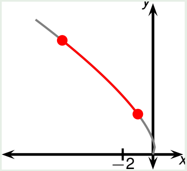

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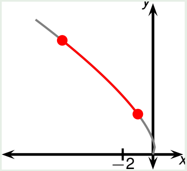

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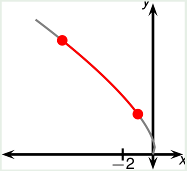

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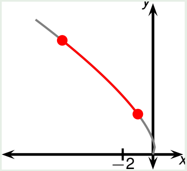

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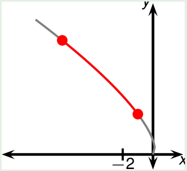

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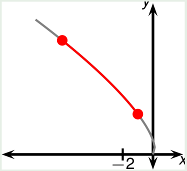

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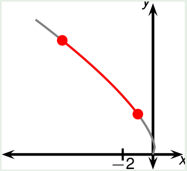

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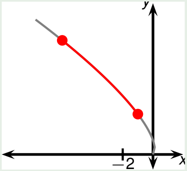

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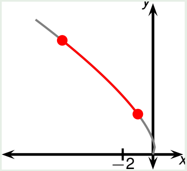

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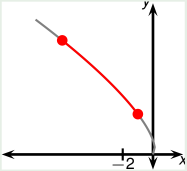

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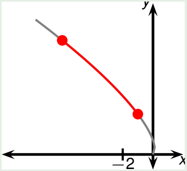

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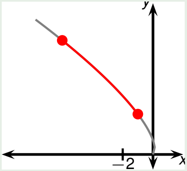
We have that $x'(t) = \frac{1}{2\sqrt{t}} - 2$ and $y'(t) = \frac{8}{3} \cdot \frac{3}{4} t^{-\frac{1}{4}} = 2t^{-\frac{1}{4}}$.

$$\begin{aligned} L(\gamma) &= \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2} \mathrm{d}t = \int_1^4 \sqrt{\left(\frac{1}{2\sqrt{t}} - 2\right)^2 + \left(2t^{-\frac{1}{4}}\right)^2} \mathrm{d}t \\ &= \int_1^4 \sqrt{\frac{1}{4t} - \frac{2}{\sqrt{t}} + 4 + \frac{4}{\sqrt{t}}} \, \mathrm{d}t \\ &= \int_1^4 \sqrt{\frac{1}{4t} + \frac{2}{\sqrt{t}} + 4} \, \mathrm{d}t = \int_1^4 \sqrt{\left(\frac{1}{2\sqrt{t}} + 2\right)^2} \mathrm{d}t \\ &= \int_1^4 \left(\frac{1}{2\sqrt{t}} + 2\right) \mathrm{d}t = \left[\sqrt{t} + \mathbf{?} \right]_1^4 \end{aligned}$$

Example



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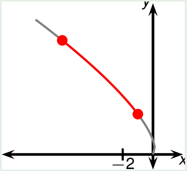
Example



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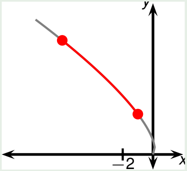
Example



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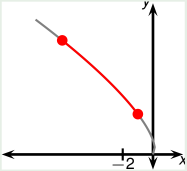
Example



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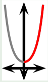
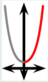
Example ($(a+b)^2$, $(a-b)^2$, $2ab = 1/2$)



Find the length of the arc of $y = \frac{1}{6}e^{3x} + \frac{1}{6}e^{-3x}$ from $x = 0$ to $x = 1$.

Example ($(a+b)^2$, $(a-b)^2$, $2ab = 1/2$)



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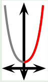

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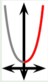

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$$y' = \frac{1}{2}e^{3x} - \frac{1}{2}e^{-3x}.$$

$$(y')^2 = \frac{1}{4}e^{6x} - \frac{1}{4}e^{3x}e^{-3x} - \frac{1}{4}e^{3x}e^{-3x} + \frac{1}{4}e^{-6x}$$

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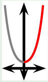

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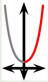

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$$\begin{aligned}(y')^2 &= \frac{1}{4}e^{6x} - \frac{1}{4}e^{3x}e^{-3x} - \frac{1}{4}e^{3x}e^{-3x} + \frac{1}{4}e^{-6x}\\ &= \frac{1}{4}e^{6x} - \frac{1}{2} + \frac{1}{4}e^{-6x}.\end{aligned}$$

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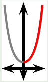

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$$
\begin{aligned}
y' &= \frac{1}{2}e^{3x} - \frac{1}{2}e^{-3x}.\\
(y')^2 &= \frac{1}{4}e^{6x} - \frac{1}{4}e^{3x}e^{-3x} - \frac{1}{4}e^{3x}e^{-3x} + \frac{1}{4}e^{-6x}\\
&= \frac{1}{4}e^{6x} - \frac{1}{2} + \frac{1}{4}e^{-6x}.\\
L &= \int_0^1 \sqrt{1+(y')^2}\mathrm{d}x
\end{aligned}
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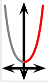

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&= \frac{1}{4}e^{6x} - \frac{1}{2} + \frac{1}{4}e^{-6x}.\\
L &= \int_0^1 \sqrt{1 + (y')^2}\mathrm{d}x = \int_0^1 \sqrt{1 + \frac{1}{4}e^{6x} - \frac{1}{2} + \frac{1}{4}e^{-6x}}\mathrm{d}x
\end{aligned}
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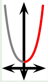
Example ($(a+b)^2$, $(a-b)^2$, $2ab = 1/2$)



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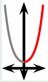
Example ($(a+b)^2$, $(a-b)^2$, $2ab = 1/2$)



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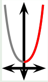
&= \int_0^1 \sqrt{\frac{1}{4}e^{6x} + \frac{1}{2} + \frac{1}{4}e^{-6x}}\mathrm{d}x = \int_0^1 \sqrt{\left(\frac{1}{2}e^{3x} + \frac{1}{2}e^{-3x}\right)^2}\mathrm{d}x
\end{aligned}$$

Example ($(a+b)^2$, $(a-b)^2$, $2ab = 1/2$)



Find the length of the arc of $y = \frac{1}{6}e^{3x} + \frac{1}{6}e^{-3x}$ from $x = 0$ to $x = 1$.

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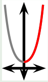
$$\begin{aligned}(y')^2 &= \frac{1}{4}e^{6x} - \frac{1}{4}e^{3x}e^{-3x} - \frac{1}{4}e^{3x}e^{-3x} + \frac{1}{4}e^{-6x}\\ &= \frac{1}{4}e^{6x} - \frac{1}{2} + \frac{1}{4}e^{-6x}.\end{aligned}$$

$$\begin{aligned}L &= \int_0^1 \sqrt{1 + (y')^2}\mathrm{d}x = \int_0^1 \sqrt{1 + \frac{1}{4}e^{6x} - \frac{1}{2} + \frac{1}{4}e^{-6x}}\mathrm{d}x\\ &= \int_0^1 \sqrt{\frac{1}{4}e^{6x} + \frac{1}{2} + \frac{1}{4}e^{-6x}}\mathrm{d}x = \int_0^1 \sqrt{\left(\frac{1}{2}e^{3x} + \frac{1}{2}e^{-3x}\right)^2}\mathrm{d}x\\ &= \int_0^1 \left(\frac{1}{2}e^{3x} + \frac{1}{2}e^{-3x}\right)\mathrm{d}x\end{aligned}$$

Example ($(a+b)^2$, $(a-b)^2$, $2ab = 1/2$)



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$$\begin{aligned}
y' &= \frac{1}{2}e^{3x} - \frac{1}{2}e^{-3x}.\\
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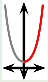
&= \int_0^1 \sqrt{\frac{1}{4}e^{6x} + \frac{1}{2} + \frac{1}{4}e^{-6x}}\mathrm{d}x = \int_0^1 \sqrt{\left(\frac{1}{2}e^{3x} + \frac{1}{2}e^{-3x}\right)^2}\mathrm{d}x\\
&= \int_0^1 \left(\frac{1}{2}e^{3x} + \frac{1}{2}e^{-3x}\right)\mathrm{d}x = \Big[\qquad \Big]_0^1
\end{aligned}$$

Example ($(a+b)^2$, $(a-b)^2$, $2ab = 1/2$)



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$$\begin{aligned}(y')^2 &= \frac{1}{4}e^{6x} - \frac{1}{4}e^{3x}e^{-3x} - \frac{1}{4}e^{3x}e^{-3x} + \frac{1}{4}e^{-6x} \\ &= \frac{1}{4}e^{6x} - \frac{1}{2} + \frac{1}{4}e^{-6x}.\end{aligned}$$

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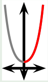
Example ($(a+b)^2$, $(a-b)^2$, $2ab = 1/2$)



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$$
\begin{aligned}
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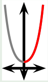
&= \int_0^1 \sqrt{\frac{1}{4}e^{6x} + \frac{1}{2} + \frac{1}{4}e^{-6x}}\mathrm{d}x = \int_0^1 \sqrt{\left(\frac{1}{2}e^{3x} + \frac{1}{2}e^{-3x}\right)^2}\mathrm{d}x\\
&= \int_0^1 \left(\frac{1}{2}e^{3x} + \frac{1}{2}e^{-3x}\right)\mathrm{d}x = \left[\frac{1}{6}e^{3x} \qquad\qquad \right]_0^1
\end{aligned}
$$

Example ($(a+b)^2$, $(a-b)^2$, $2ab = 1/2$)



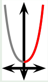
Find the length of the arc of $y = \frac{1}{6}e^{3x} + \frac{1}{6}e^{-3x}$ from $x = 0$ to $x = 1$.

$$y' = \frac{1}{2}e^{3x} - \frac{1}{2}e^{-3x}.$$

$$\begin{aligned}(y')^2 &= \frac{1}{4}e^{6x} - \frac{1}{4}e^{3x}e^{-3x} - \frac{1}{4}e^{3x}e^{-3x} + \frac{1}{4}e^{-6x}\\ &= \frac{1}{4}e^{6x} - \frac{1}{2} + \frac{1}{4}e^{-6x}.\end{aligned}$$

$$\begin{aligned}L &= \int_0^1 \sqrt{1+(y')^2}\mathrm{d}x = \int_0^1 \sqrt{1 + \frac{1}{4}e^{6x} - \frac{1}{2} + \frac{1}{4}e^{-6x}}\mathrm{d}x\\ &= \int_0^1 \sqrt{\frac{1}{4}e^{6x} + \frac{1}{2} + \frac{1}{4}e^{-6x}}\mathrm{d}x = \int_0^1 \sqrt{\left(\frac{1}{2}e^{3x} + \frac{1}{2}e^{-3x}\right)^2}\mathrm{d}x\\ &= \int_0^1 \left(\frac{1}{2}e^{3x} + \frac{1}{2}e^{-3x}\right)\mathrm{d}x = \left[\frac{1}{6}e^{3x} - \frac{1}{6}e^{-3x}\right]_0^1\end{aligned}$$

Example ($(a+b)^2$, $(a-b)^2$, $2ab = 1/2$)



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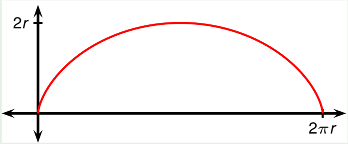
&= \int_0^1 \sqrt{\frac{1}{4}e^{6x} + \frac{1}{2} + \frac{1}{4}e^{-6x}}\mathrm{d}x = \int_0^1 \sqrt{\left(\frac{1}{2}e^{3x} + \frac{1}{2}e^{-3x}\right)^2}\mathrm{d}x\\
&= \int_0^1 \left(\frac{1}{2}e^{3x} + \frac{1}{2}e^{-3x}\right)\mathrm{d}x = \left[\frac{1}{6}e^{3x} - \frac{1}{6}e^{-3x}\right]_0^1 = \frac{e^3 - e^{-3}}{6}.
\end{aligned}
$$

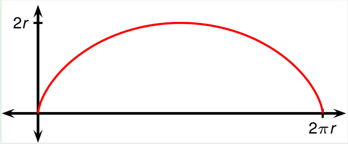
Example



Find the length of one arch of the cycloid

$$x = r(\theta - \sin\theta), \quad y = r(1 - \cos\theta).$$

Example



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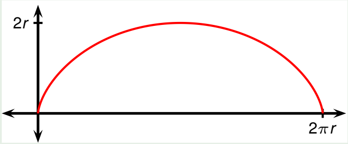
Example



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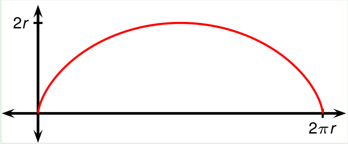

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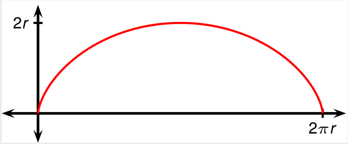

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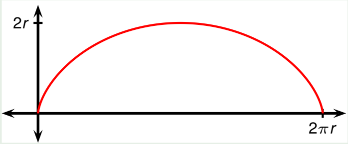

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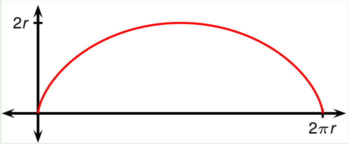

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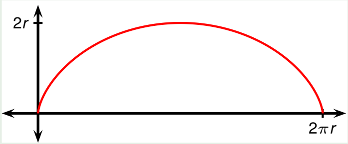

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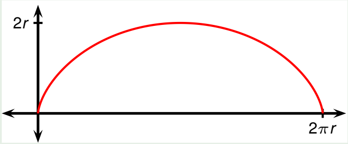

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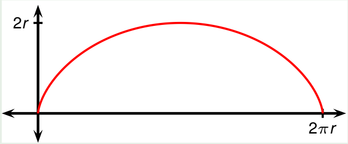
$$\sqrt{2(1-\cos\theta)}$$

Example



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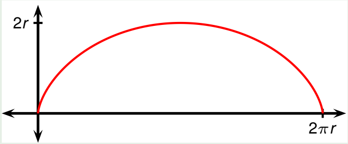
Use the identity $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$. Then

$$\sqrt{2(1 - \cos\theta)} = \sqrt{4\sin^2(\theta/2)}$$

Example



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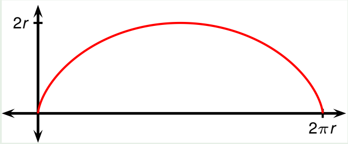

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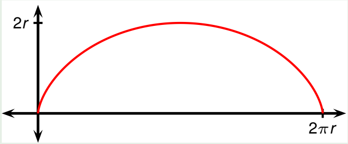

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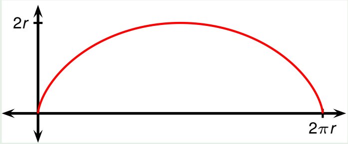

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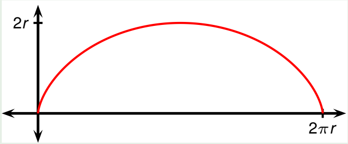
$$L = r\int_0^{2\pi} 2\sin(\theta/2) d\theta = r\left[-4\cos(\theta/2)\right]_0^{2\pi}$$

Example



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Arc Length

To find the arc length of a polar curve $r = f(\theta)$, $a \leq \theta \leq b$, regard θ as a parameter.

The arc length is

$$L = \int_a^b \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2} \mathrm{d}\theta$$

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$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = \frac{\mathrm{d}r}{\mathrm{d}\theta}\cos\theta - r\sin\theta \qquad \frac{\mathrm{d}y}{\mathrm{d}\theta} = \frac{\mathrm{d}r}{\mathrm{d}\theta}\sin\theta + r\cos\theta$$

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$$+$$

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$$+$$

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$$\begin{aligned}\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 &= \left(\frac{dr}{d\theta}\right)^2\cos^2\theta - 2r\frac{dr}{d\theta}\cos\theta\sin\theta + r^2\sin^2\theta \\ &+ \left(\frac{dr}{d\theta}\right)^2\sin^2\theta + 2r\frac{dr}{d\theta}\sin\theta\cos\theta + r^2\cos^2\theta\end{aligned}$$

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$$\begin{aligned}\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 &= \left(\frac{dr}{d\theta}\right)^2\cos^2\theta - 2r\frac{dr}{d\theta}\cos\theta\sin\theta + r^2\sin^2\theta \\ &\quad + \left(\frac{dr}{d\theta}\right)^2\sin^2\theta + 2r\frac{dr}{d\theta}\sin\theta\cos\theta + r^2\cos^2\theta \\ &= \left(\frac{dr}{d\theta}\right)^2\end{aligned}$$

The arc length is

$$L = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta$$

Arc Length

To find the arc length of a polar curve $r = f(\theta)$, $a \leq \theta \leq b$, regard θ as a parameter. Then the derivatives of the parametric equations are

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = \frac{\mathrm{d}r}{\mathrm{d}\theta}\cos\theta - r\sin\theta \qquad \frac{\mathrm{d}y}{\mathrm{d}\theta} = \frac{\mathrm{d}r}{\mathrm{d}\theta}\sin\theta + r\cos\theta$$

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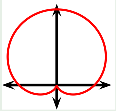
Example



Find the length of the cardioid $r = 1 + \sin\theta$.

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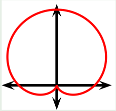

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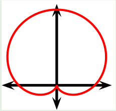

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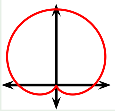

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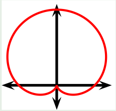

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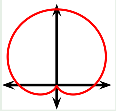

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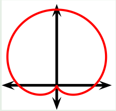

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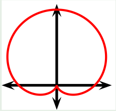

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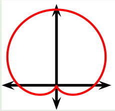

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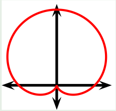

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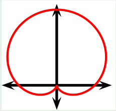

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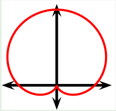

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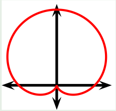
Example



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&= \int_0^{2\pi} \sqrt{2 + 2\sin\theta} \frac{\sqrt{2 - 2\sin\theta}}{\sqrt{2 - 2\sin\theta}} \mathrm{d}\theta = \int_0^{2\pi} \frac{\sqrt{4 - 4\sin^2\theta}}{\sqrt{2 - 2\sin\theta}} \mathrm{d}\theta \\
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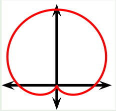

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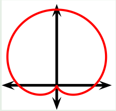
Example



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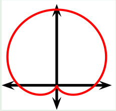

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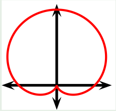

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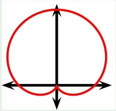

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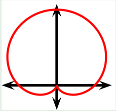
&= \int_0^{2\pi} \sqrt{2+2\sin\theta}\frac{\sqrt{2-2\sin\theta}}{\sqrt{2-2\sin\theta}}\mathrm{d}\theta = \int_0^{2\pi} \frac{\sqrt{4-4\sin^2\theta}}{\sqrt{2-2\sin\theta}}\mathrm{d}\theta \\
&= \int_0^{2\pi} \frac{\sqrt{4\cos^2\theta}}{\sqrt{2-2\sin\theta}}\mathrm{d}\theta = \int_0^{2\pi} \frac{2|\cos\theta|}{\sqrt{2-2\sin\theta}}\mathrm{d}\theta \\
&= \int_0^{\pi/2} \frac{2\cos\theta}{\sqrt{2-2\sin\theta}}\mathrm{d}\theta + \int_{\pi/2}^{3\pi/2} \frac{-2\cos\theta}{\sqrt{2-2\sin\theta}}\mathrm{d}\theta + \int_{3\pi/2}^{2\pi} \frac{2\cos\theta}{\sqrt{2-2\sin\theta}}\mathrm{d}\theta \\
&= \left[-2\sqrt{2-2\sin\theta}\right]_0^{\pi/2} + \left[2\sqrt{2-2\sin\theta}\right]_{\pi/2}^{3\pi/2} + \left[-2\sqrt{2-2\sin\theta}\right]_{3\pi/2}^{2\pi} \\
&= -2\left(0-\sqrt{2}\right) + 2\left(\quad - \quad\right) - 2\left(\quad - \quad\right)
\end{aligned}$$

Example



Find the length of the cardioid $r = 1 + \sin\theta$. The full length is given by $0 \leq \theta \leq 2\pi$.

$$L = \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta = \int_0^{2\pi} \sqrt{(1+\sin\theta)^2 + \cos^2\theta} d\theta$$

$$= \int_0^{2\pi} \sqrt{2 + 2\sin\theta} \frac{\sqrt{2-2\sin\theta}}{\sqrt{2-2\sin\theta}} d\theta = \int_0^{2\pi} \frac{\sqrt{4 - 4\sin^2\theta}}{\sqrt{2-2\sin\theta}} d\theta$$

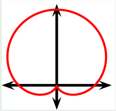
$$= \int_0^{2\pi} \frac{\sqrt{4\cos^2\theta}}{\sqrt{2-2\sin\theta}} d\theta = \int_0^{2\pi} \frac{2|\cos\theta|}{\sqrt{2-2\sin\theta}} d\theta$$

$$= \int_0^{\pi/2} \frac{2\cos\theta}{\sqrt{2-2\sin\theta}} d\theta + \int_{\pi/2}^{3\pi/2} \frac{-2\cos\theta}{\sqrt{2-2\sin\theta}} d\theta + \int_{3\pi/2}^{2\pi} \frac{2\cos\theta}{\sqrt{2-2\sin\theta}} d\theta$$

$$= \left[-2\sqrt{2-2\sin\theta}\right]_0^{\pi/2} + \left[2\sqrt{2-2\sin\theta}\right]_{\pi/2}^{3\pi/2} + \left[-2\sqrt{2-2\sin\theta}\right]_{3\pi/2}^{2\pi}$$

$$= -2\left(0 - \sqrt{2}\right) + 2\left(2 - \quad\right) - 2\left(\quad - \quad \right)$$

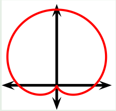
Example



Find the length of the cardioid $r = 1 + \sin\theta$. The full length is given by $0 \leq \theta \leq 2\pi$.

$$
\begin{aligned}
L &= \int_0^{2\pi} \sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\mathrm{d}\theta = \int_0^{2\pi} \sqrt{(1+\sin\theta)^2 + \cos^2\theta}\mathrm{d}\theta \\
&= \int_0^{2\pi} \sqrt{2+2\sin\theta}\frac{\sqrt{2-2\sin\theta}}{\sqrt{2-2\sin\theta}}\mathrm{d}\theta = \int_0^{2\pi} \frac{\sqrt{4-4\sin^2\theta}}{\sqrt{2-2\sin\theta}}\mathrm{d}\theta \\
&= \int_0^{2\pi} \frac{\sqrt{4\cos^2\theta}}{\sqrt{2-2\sin\theta}}\mathrm{d}\theta = \int_0^{2\pi} \frac{2|\cos\theta|}{\sqrt{2-2\sin\theta}}\mathrm{d}\theta \\
&= \int_0^{\pi/2} \frac{2\cos\theta}{\sqrt{2-2\sin\theta}}\mathrm{d}\theta + \int_{\pi/2}^{3\pi/2} \frac{-2\cos\theta}{\sqrt{2-2\sin\theta}}\mathrm{d}\theta + \int_{3\pi/2}^{2\pi} \frac{2\cos\theta}{\sqrt{2-2\sin\theta}}\mathrm{d}\theta \\
&= \left[-2\sqrt{2-2\sin\theta}\right]_0^{\pi/2} + \left[2\sqrt{2-2\sin\theta}\right]_{\pi/2}^{3\pi/2} + \left[-2\sqrt{2-2\sin\theta}\right]_{3\pi/2}^{2\pi} \\
&= -2\left(0-\sqrt{2}\right) + 2\left(2 - \quad\right) - 2\left(\quad - \quad \right)
\end{aligned}
$$

Example



Find the length of the cardioid $r = 1 + \sin\theta$. The full length is given by $0 \leq \theta \leq 2\pi$.

$$L = \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta = \int_0^{2\pi} \sqrt{(1+\sin\theta)^2 + \cos^2\theta} d\theta$$

$$= \int_0^{2\pi} \sqrt{2+2\sin\theta} \frac{\sqrt{2-2\sin\theta}}{\sqrt{2-2\sin\theta}} d\theta = \int_0^{2\pi} \frac{\sqrt{4-4\sin^2\theta}}{\sqrt{2-2\sin\theta}} d\theta$$

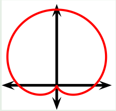
$$= \int_0^{2\pi} \frac{\sqrt{4\cos^2\theta}}{\sqrt{2-2\sin\theta}} d\theta = \int_0^{2\pi} \frac{2|\cos\theta|}{\sqrt{2-2\sin\theta}} d\theta$$

$$= \int_0^{\pi/2} \frac{2\cos\theta}{\sqrt{2-2\sin\theta}} d\theta + \int_{\pi/2}^{3\pi/2} \frac{-2\cos\theta}{\sqrt{2-2\sin\theta}} d\theta + \int_{3\pi/2}^{2\pi} \frac{2\cos\theta}{\sqrt{2-2\sin\theta}} d\theta$$

$$= \left[-2\sqrt{2-2\sin\theta}\right]_0^{\pi/2} + \left[2\sqrt{2-2\sin\theta}\right]_{\pi/2}^{3\pi/2} + \left[-2\sqrt{2-2\sin\theta}\right]_{3\pi/2}^{2\pi}$$

$$= -2\left(0 - \sqrt{2}\right) + 2(2 - 0) - 2\left(\quad - \quad \right)$$

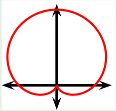
Example



Find the length of the cardioid $r = 1 + \sin\theta$. The full length is given by $0 \leq \theta \leq 2\pi$.

$$
\begin{aligned}
L &= \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta = \int_0^{2\pi} \sqrt{(1+\sin\theta)^2 + \cos^2\theta} d\theta \\
&= \int_0^{2\pi} \sqrt{2+2\sin\theta}\frac{\sqrt{2-2\sin\theta}}{\sqrt{2-2\sin\theta}} d\theta = \int_0^{2\pi} \frac{\sqrt{4-4\sin^2\theta}}{\sqrt{2-2\sin\theta}} d\theta \\
&= \int_0^{2\pi} \frac{\sqrt{4\cos^2\theta}}{\sqrt{2-2\sin\theta}} d\theta = \int_0^{2\pi} \frac{2|\cos\theta|}{\sqrt{2-2\sin\theta}} d\theta \\
&= \int_0^{\pi/2} \frac{2\cos\theta}{\sqrt{2-2\sin\theta}} d\theta + \int_{\pi/2}^{3\pi/2} \frac{-2\cos\theta}{\sqrt{2-2\sin\theta}} d\theta + \int_{3\pi/2}^{2\pi} \frac{2\cos\theta}{\sqrt{2-2\sin\theta}} d\theta \\
&= \left[-2\sqrt{2-2\sin\theta}\right]_0^{\pi/2} + \left[2\sqrt{2-2\sin\theta}\right]_{\pi/2}^{3\pi/2} + \left[-2{\color{red}\sqrt{2-2\sin\theta}}\right]_{3\pi/2}^{\color{red}2\pi} \\
&= -2\left(0-\sqrt{2}\right) + 2(2-0) - 2\left(\quad - \quad\right)
\end{aligned}
$$

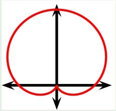
Example



Find the length of the cardioid $r = 1 + \sin\theta$. The full length is given by $0 \leq \theta \leq 2\pi$.

$$
\begin{aligned}
L &= \int_0^{2\pi} \sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\mathrm{d}\theta = \int_0^{2\pi} \sqrt{(1+\sin\theta)^2 + \cos^2\theta}\mathrm{d}\theta \\
&= \int_0^{2\pi} \sqrt{2+2\sin\theta}\frac{\sqrt{2-2\sin\theta}}{\sqrt{2-2\sin\theta}}\mathrm{d}\theta = \int_0^{2\pi} \frac{\sqrt{4-4\sin^2\theta}}{\sqrt{2-2\sin\theta}}\mathrm{d}\theta \\
&= \int_0^{2\pi} \frac{\sqrt{4\cos^2\theta}}{\sqrt{2-2\sin\theta}}\mathrm{d}\theta = \int_0^{2\pi} \frac{2|\cos\theta|}{\sqrt{2-2\sin\theta}}\mathrm{d}\theta \\
&= \int_0^{\pi/2} \frac{2\cos\theta}{\sqrt{2-2\sin\theta}}\mathrm{d}\theta + \int_{\pi/2}^{3\pi/2} \frac{-2\cos\theta}{\sqrt{2-2\sin\theta}}\mathrm{d}\theta + \int_{3\pi/2}^{2\pi} \frac{2\cos\theta}{\sqrt{2-2\sin\theta}}\mathrm{d}\theta \\
&= \left[-2\sqrt{2-2\sin\theta}\right]_0^{\pi/2} + \left[2\sqrt{2-2\sin\theta}\right]_{\pi/2}^{3\pi/2} + \left[-2\sqrt{2-2\sin\theta}\right]_{3\pi/2}^{2\pi} \\
&= -2\left(0-\sqrt{2}\right) + 2\left(2-0\right) - 2\left(\sqrt{2} - \quad\right)
\end{aligned}
$$

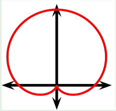
Example



Find the length of the cardioid $r = 1 + \sin\theta$. The full length is given by $0 \leq \theta \leq 2\pi$.

$$
\begin{aligned}
L &= \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta = \int_0^{2\pi} \sqrt{(1+\sin\theta)^2 + \cos^2\theta} d\theta \\
&= \int_0^{2\pi} \sqrt{2+2\sin\theta} \frac{\sqrt{2-2\sin\theta}}{\sqrt{2-2\sin\theta}} d\theta = \int_0^{2\pi} \frac{\sqrt{4-4\sin^2\theta}}{\sqrt{2-2\sin\theta}} d\theta \\
&= \int_0^{2\pi} \frac{\sqrt{4\cos^2\theta}}{\sqrt{2-2\sin\theta}} d\theta = \int_0^{2\pi} \frac{2|\cos\theta|}{\sqrt{2-2\sin\theta}} d\theta \\
&= \int_0^{\pi/2} \frac{2\cos\theta}{\sqrt{2-2\sin\theta}} d\theta + \int_{\pi/2}^{3\pi/2} \frac{-2\cos\theta}{\sqrt{2-2\sin\theta}} d\theta + \int_{3\pi/2}^{2\pi} \frac{2\cos\theta}{\sqrt{2-2\sin\theta}} d\theta \\
&= \left[-2\sqrt{2-2\sin\theta}\right]_0^{\pi/2} + \left[2\sqrt{2-2\sin\theta}\right]_{\pi/2}^{3\pi/2} + \left[-2\sqrt{2-2\sin\theta}\right]_{3\pi/2}^{2\pi} \\
&= -2\left(0-\sqrt{2}\right) + 2(2-0) - 2\left(\sqrt{2} - \quad\right)
\end{aligned}
$$

Example



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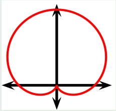
&= \int_0^{2\pi} \sqrt{2 + 2\sin\theta} \frac{\sqrt{2-2\sin\theta}}{\sqrt{2-2\sin\theta}} \mathrm{d}\theta = \int_0^{2\pi} \frac{\sqrt{4 - 4\sin^2\theta}}{\sqrt{2-2\sin\theta}} \mathrm{d}\theta \\
&= \int_0^{2\pi} \frac{\sqrt{4\cos^2\theta}}{\sqrt{2-2\sin\theta}} \mathrm{d}\theta = \int_0^{2\pi} \frac{2|\cos\theta|}{\sqrt{2-2\sin\theta}} \mathrm{d}\theta \\
&= \int_0^{\pi/2} \frac{2\cos\theta}{\sqrt{2-2\sin\theta}} \mathrm{d}\theta + \int_{\pi/2}^{3\pi/2} \frac{-2\cos\theta}{\sqrt{2-2\sin\theta}} \mathrm{d}\theta + \int_{3\pi/2}^{2\pi} \frac{2\cos\theta}{\sqrt{2-2\sin\theta}} \mathrm{d}\theta \\
&= \left[-2\sqrt{2-2\sin\theta}\right]_0^{\pi/2} + \left[2\sqrt{2-2\sin\theta}\right]_{\pi/2}^{3\pi/2} + \left[-2\sqrt{2-2\sin\theta}\right]_{3\pi/2}^{2\pi} \\
&= -2\left(0 - \sqrt{2}\right) + 2(2-0) - 2\left(\sqrt{2} - 2\right)
\end{aligned}
$$

Example



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$$
\begin{aligned}
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&= \int_0^{2\pi} \sqrt{2+2\sin\theta}\frac{\sqrt{2-2\sin\theta}}{\sqrt{2-2\sin\theta}}\mathrm{d}\theta = \int_0^{2\pi} \frac{\sqrt{4-4\sin^2\theta}}{\sqrt{2-2\sin\theta}}\mathrm{d}\theta \\
&= \int_0^{2\pi} \frac{\sqrt{4\cos^2\theta}}{\sqrt{2-2\sin\theta}}\mathrm{d}\theta = \int_0^{2\pi} \frac{2|\cos\theta|}{\sqrt{2-2\sin\theta}}\mathrm{d}\theta \\
&= \int_0^{\pi/2} \frac{2\cos\theta}{\sqrt{2-2\sin\theta}}\mathrm{d}\theta + \int_{\pi/2}^{3\pi/2} \frac{-2\cos\theta}{\sqrt{2-2\sin\theta}}\mathrm{d}\theta + \int_{3\pi/2}^{2\pi} \frac{2\cos\theta}{\sqrt{2-2\sin\theta}}\mathrm{d}\theta \\
&= \left[-2\sqrt{2-2\sin\theta}\right]_0^{\pi/2} + \left[2\sqrt{2-2\sin\theta}\right]_{\pi/2}^{3\pi/2} + \left[-2\sqrt{2-2\sin\theta}\right]_{3\pi/2}^{2\pi} \\
&= -2\left(0-\sqrt{2}\right) + 2\left(2-0\right) - 2\left(\sqrt{2}-2\right) = 8
\end{aligned}
$$