Arithmetics

Lecture 7: Floating point

reference materials, internal use

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Example

An important example is the geometric series

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}, \qquad a \neq 0$$

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- If $-1 < r < 1$, then $r^n \to 0$, so the geometric series is convergent and its sum is $a/(1-r)$.

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- If $-1 < r < 1$, then $r^n \to 0$, so the geometric series is convergent and its sum is $a/(1-r)$.
- If $r > 1$ or $r \leq -1$, then r^n is divergent, so $\sum_{n=1}^{\infty} ar^{n-1}$ diverges.

This theorem summarizes the results of the previous example.

Theorem (Convergence of Geometric Series)

The geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

is convergent if $|r| < 1$ *and its sum is*

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}.$$

If $|r| \geq 1$*, the series is divergent.*
a *is called the first term and* r *is called the common ratio.*

Example

Find the sum of the geometric series $$-2+\frac{6}{5}-\frac{18}{25}+\frac{54}{125}-\cdots$$

For $|r| < 1$, recall that the sum of a geometric series is

$$a + ar + ar^2 + ar^3 + \dots$$

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Find the sum of the geometric series $\quad -2 + \frac{6}{5} - \frac{18}{25} + \frac{54}{125} - \cdots$

- The first term is $a = -2$.

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- The first term is $a = -2$.
- The common ratio is $r =$ **?** .

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- $a = \frac{17}{10^3}$ and $r = \frac{1}{10^2}$.

$$\begin{aligned} 2.3171717\ldots &= 2.3 + \frac{\frac{17}{10^3}}{1 - \frac{1}{10^2}} = 2.3 + \frac{\frac{17}{1000}}{\frac{99}{100}} \\ &= \frac{23}{10} + \frac{17}{990} = \frac{1147}{495} \end{aligned}$$

Algorithm (Write a fraction to periodic base 10)

Input: a fraction $\frac{p}{q}$. Output: integer A and two groups of digits such that $\frac{p}{q} = A + 0.b_1 \dots b_m\overline{c_1 \dots c_n}$.

1 *Initialize* ***digits*** $= ()$ *as the empty sequence.*

2 *Initialize* ***remainders*** $= ()$ *as the empty sequence.*

2 *Divide p by q with remainder r and set A to be the quotient. Append r to remainders.*

3 *While r is not equal to zero:*

- 3.1 *Multiply r by 10.*
- 3.2 *Divide the result by q with remainder r' and quotient d.*
- 3.3 *If r' belongs to* ***remainders*** *with first occurrence at position $m+1$, slice digits into two sequences $b_1, \dots b_m$ and $c_1, \dots c_n$. Return A and $b_1, ..., b_m$, $c_1, \dots, c_n$ as the desired digits.*
- 3.4 *Append d to* ***digits*** *and r to* ***remainders***.
- 3.5 *Set $r = r'$ and go back to Step 3.*

4 *If r attained the value 0 in the execution of the loop, the fraction $\frac{p}{q}$ has a finite decimal representation given by A and* ***digits***.

Algorithm (Write a fraction to periodic base X)

Input: a fraction $\frac{p}{q}$. Output: integer A and two groups of digits such that $\frac{p}{q} = A + 0.b_1 \dots b_m\overline{c_1 \dots c_n}$.

1 *Initialize* ***digits*** $= ()$ *as the empty sequence.*

2 *Initialize* ***remainders*** $= ()$ *as the empty sequence.*

2 *Divide p by q with remainder r and set A to be the quotient. Append r to remainders.*

3 *While r is not equal to zero:*

  3.1 *Multiply r by X.*

  3.2 *Divide the result by q with remainder r' and quotient d.*

  3.3 *If r' belongs to* ***remainders*** *with first occurrence at position $m+1$, slice digits into two sequences $b_1, \dots b_m$ and $c_1, \dots c_n$. Return A and $b_1, ..., b_m$, $c_1, \dots, c_n$ as the desired digits.*

  3.4 *Append d to* ***digits*** *and r to* ***remainders****.*

  3.5 *Set $r = r'$ and go back to Step 3.*

4 *If r attained the value 0 in the execution of the loop, the fraction $\frac{p}{q}$ has a finite decimal representation given by A and* ***digits****.*

Example

Convert $\frac{86}{7}$ to repeating decimal notation.

Example

Convert $\frac{86}{7}$ to repeating decimal notation.

Divide with 86 by 7 with remainder.

$$86 = \textbf{?} \cdot 7 + \textbf{?}$$

Example

Convert $\frac{86}{7}$ to repeating decimal notation.

Divide with 86 by 7 with remainder.

$$86 = 12 \cdot 7 + 2$$

Example

Convert $\frac{86}{7}$ to repeating decimal notation.
Divide with 86 by 7 with remainder.

$$86 = 12 \cdot 7 + 2 \quad \Bigg| \quad \frac{86}{7} = 12 + \dots$$

Example

Convert $\frac{86}{7}$ to repeating decimal notation.
Divide with 86 by 7 with remainder.

$$86 = 12 \cdot 7 + 2 \quad \Bigg| \quad \frac{86}{7} = 12 + \dots$$

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Example

Convert $\frac{86}{7}$ to repeating decimal notation.
Divide with 86 by 7 with remainder.

$$86 = 12 \cdot 7 + 2 \quad \Big| \quad \frac{86}{7} = 12 + \dots$$

Example

Convert $\frac{86}{7}$ to repeating decimal notation.

Divide with 86 by 7 with remainder. Repeat using remainder ...

$$86 = 12 \cdot 7 + 2 \quad \Big| \quad \frac{86}{7} = 12 + \dots$$

2

Example

Convert $\frac{86}{7}$ to repeating decimal notation.
Divide with 86 by 7 with remainder. Repeat using remainder ...

$$\begin{array}{rcl|rcl} 86 & = & 12 \cdot 7 + 2 & \frac{86}{7} & = & 12 + \dots \\ 2 \cdot 10 & = & 20 & & & \end{array}$$

Example

Convert $\frac{86}{7}$ to repeating decimal notation.
Divide with 86 by 7 with remainder. Repeat using remainder ...

$$\begin{array}{rcl|l} 86 &=& 12 \cdot 7 + 2 & \frac{86}{7} = 12 + \dots \\ 2 \cdot 10 &=& 20 & \end{array}$$

Example

Convert $\frac{86}{7}$ to repeating decimal notation.
Divide with 86 by 7 with remainder. Repeat using remainder ...

$$\begin{array}{rcccl|ccl} & & 86 & = & 12 \cdot 7 + 2 & \frac{86}{7} & = & 12 + \dots \\ 2 \cdot 10 & = & \color{red}{20} & \color{red}{=} & \color{red}{\mathbf{?} \cdot 7 + \mathbf{?}} & & & \end{array}$$

Example

Convert $\frac{86}{7}$ to repeating decimal notation.
Divide with 86 by 7 with remainder. Repeat using remainder ...

$$
\begin{array}{rcrcl|ccc}
 & & 86 & = & 12\cdot 7+2 & \frac{86}{7} & = & 12+\dots \\
2\cdot 10 & = & \color{red}{20} & \color{red}{=} & \color{red}{2\cdot 7+6} & & &
\end{array}
$$

Example

Convert $\frac{86}{7}$ to repeating decimal notation.
Divide with 86 by 7 with remainder. Repeat using remainder ...

$$\begin{array}{rcrcl|lcl} & & 86 & = & 12\cdot 7+2 & \frac{86}{7} & = & 12+\dots \\ 2\cdot 10 & = & 20 & = & {\color{red}2}\cdot 7+6 & & = & 12.{\color{red}2}+\dots \end{array}$$

Example

Convert $\frac{86}{7}$ to repeating decimal notation.
Divide with 86 by 7 with remainder. Repeat using remainder ...

$$
\begin{array}{rcrcl|rcl}
 & & 86 & = & 12\cdot 7 + 2 & \frac{86}{7} & = & 12 + \dots \\
2\cdot 10 & = & 20 & = & 2\cdot 7 + {\color{red}6} & & = & 12.2 + \dots \\
{\color{red}6}\cdot 10 & = & 60 & & & & &
\end{array}
$$

Example

Convert $\frac{86}{7}$ to repeating decimal notation.
Divide with 86 by 7 with remainder. Repeat using remainder ...

$$
\begin{array}{rcrcl|rcl}
 & & 86 & = & 12\cdot 7+2 & \frac{86}{7} & = & 12+\dots \\
2\cdot 10 & = & 20 & = & 2\cdot 7+6 & & = & 12.2+\dots \\
6\cdot 10 & = & 60 & = & \mathbf{?}\cdot 7+\mathbf{?} & & &
\end{array}
$$

Example

Convert $\frac{86}{7}$ to repeating decimal notation.
Divide with 86 by 7 with remainder. Repeat using remainder ...

$$\begin{array}{rcrcl|lcl} & & 86 & = & 12\cdot 7+2 & \frac{86}{7} & = & 12+\dots \\ 2\cdot 10 & = & 20 & = & 2\cdot 7+6 & & = & 12.2+\dots \\ 6\cdot 10 & = & \color{red}{60} & \color{red}{=} & \color{red}{8\cdot 7+4} & & = & 12.28+\dots \end{array}$$

Example

Convert $\frac{86}{7}$ to repeating decimal notation.
Divide with 86 by 7 with remainder. Repeat using remainder ...

$$
\begin{array}{rcrcl|rcl}
 & & 86 & = & 12\cdot 7+2 & \frac{86}{7} & = & 12+\dots \\
2\cdot 10 & = & 20 & = & 2\cdot 7+6 & & = & 12.2+\dots \\
6\cdot 10 & = & 60 & = & {\color{red}8}\cdot 7+4 & & = & 12.2{\color{red}8}+\dots
\end{array}
$$

Example

Convert $\frac{86}{7}$ to repeating decimal notation.
Divide with 86 by 7 with remainder. Repeat using remainder ...

$$\begin{array}{rcrcl|rcl}
 & & 86 & = & 12\cdot 7+2 & \frac{86}{7} & = & 12+\dots \\
2\cdot 10 & = & 20 & = & 2\cdot 7+6 & & = & 12.2+\dots \\
6\cdot 10 & = & 60 & = & 8\cdot 7+\color{red}{4} & & = & 12.28+\dots \\
\color{red}{4}\cdot 10 & = & 40 & & & & &
\end{array}$$

Example

Convert $\frac{86}{7}$ to repeating decimal notation.
Divide with 86 by 7 with remainder. Repeat using remainder ...

$$
\begin{array}{rcccl|ccl}
 & & 86 & = & 12 \cdot 7 + 2 & \frac{86}{7} & = & 12 + \dots \\
2 \cdot 10 & = & 20 & = & 2 \cdot 7 + 6 & & = & 12.2 + \dots \\
6 \cdot 10 & = & 60 & = & 8 \cdot 7 + 4 & & = & 12.28 + \dots \\
4 \cdot 10 & = & 40 & = & \mathbf{?} \cdot 7 + \mathbf{?} & & &
\end{array}
$$

Example

Convert $\frac{86}{7}$ to repeating decimal notation.
Divide with 86 by 7 with remainder. Repeat using remainder ...

$$
\begin{array}{rcrcl|rcl}
 & & 86 & = & 12\cdot 7+2 & \frac{86}{7} & = & 12+\dots \\
2\cdot 10 & = & 20 & = & 2\cdot 7+6 & & = & 12.2+\dots \\
6\cdot 10 & = & 60 & = & 8\cdot 7+4 & & = & 12.28+\dots \\
4\cdot 10 & = & \color{red}{40} & \color{red}{=} & \color{red}{5\cdot 7+5} & & = & 12.285+\dots
\end{array}
$$

Example

Convert $\frac{86}{7}$ to repeating decimal notation.
Divide with 86 by 7 with remainder. Repeat using remainder ...

$$\begin{array}{rcrcl|rcl}
 & & 86 & = & 12\cdot 7+2 & \frac{86}{7} & = & 12+\dots \\
2\cdot 10 & = & 20 & = & 2\cdot 7+6 & & = & 12.2+\dots \\
6\cdot 10 & = & 60 & = & 8\cdot 7+4 & & = & 12.28+\dots \\
4\cdot 10 & = & 40 & = & 5\cdot 7+{\color{red}5} & & = & 12.285+\dots \\
{\color{red}5}\cdot 10 & = & 50 & & & & &
\end{array}$$

Example

Convert $\frac{86}{7}$ to repeating decimal notation.
Divide with 86 by 7 with remainder. Repeat using remainder ...

$$
\begin{array}{rcrcl|rcl}
 & & 86 & = & 12 \cdot 7 + 2 & \frac{86}{7} & = & 12 + \dots \\
2 \cdot 10 & = & 20 & = & 2 \cdot 7 + 6 & & = & 12.2 + \dots \\
6 \cdot 10 & = & 60 & = & 8 \cdot 7 + 4 & & = & 12.28 + \dots \\
4 \cdot 10 & = & 40 & = & 5 \cdot 7 + 5 & & = & 12.285 + \dots \\
5 \cdot 10 & = & 50 & = & \textbf{?} \cdot 7 + \textbf{?} & & &
\end{array}
$$

Example

Convert $\frac{86}{7}$ to repeating decimal notation.
Divide with 86 by 7 with remainder. Repeat using remainder ...

$$\begin{array}{rclcl|ccl}
 & & 86 & = & 12\cdot 7+2 & \frac{86}{7} & = & 12+\dots \\
2\cdot 10 & = & 20 & = & 2\cdot 7+6 & & = & 12.2+\dots \\
6\cdot 10 & = & 60 & = & 8\cdot 7+4 & & = & 12.28+\dots \\
4\cdot 10 & = & 40 & = & 5\cdot 7+5 & & = & 12.285+\dots \\
5\cdot 10 & = & \color{red}{50} & \color{red}{=} & \color{red}{7\cdot 7+1} & & = & 12.2851+\dots
\end{array}$$

Example

Convert $\frac{86}{7}$ to repeating decimal notation.
Divide with 86 by 7 with remainder. Repeat using remainder ...

$$
\begin{array}{rcrcl|ccl}
 & & 86 & = & 12\cdot 7+2 & \frac{86}{7} & = & 12+\dots \\
2\cdot 10 & = & 20 & = & 2\cdot 7+6 & & = & 12.2+\dots \\
6\cdot 10 & = & 60 & = & 8\cdot 7+4 & & = & 12.28+\dots \\
4\cdot 10 & = & 40 & = & 5\cdot 7+5 & & = & 12.285+\dots \\
5\cdot 10 & = & 50 & = & 7\cdot 7+{\color{red}1} & & = & 12.2851+\dots \\
{\color{red}1}\cdot 10 & = & 10 & & & & &
\end{array}
$$

Example

Convert $\frac{86}{7}$ to repeating decimal notation.
Divide with 86 by 7 with remainder. Repeat using remainder ...

$$\begin{array}{rcrcl|ccl}
 & & 86 & = & 12\cdot 7+2 & \frac{86}{7} & = & 12+\dots \\
2\cdot 10 & = & 20 & = & 2\cdot 7+6 & & = & 12.2+\dots \\
6\cdot 10 & = & 60 & = & 8\cdot 7+4 & & = & 12.28+\dots \\
4\cdot 10 & = & 40 & = & 5\cdot 7+5 & & = & 12.285+\dots \\
5\cdot 10 & = & 50 & = & 7\cdot 7+1 & & = & 12.2851+\dots \\
1\cdot 10 & = & \color{red}{10} & \color{red}{=} & \color{red}{\mathbf{?}\cdot 7+\mathbf{?}} & & &
\end{array}$$

Example

Convert $\frac{86}{7}$ to repeating decimal notation.

Divide with 86 by 7 with remainder. Repeat using remainder ...

$$
\begin{array}{rcrcl|rcl}
 & & 86 & = & 12\cdot 7+2 & \frac{86}{7} & = & 12+\dots \\
2\cdot 10 & = & 20 & = & 2\cdot 7+6 & & = & 12.2+\dots \\
6\cdot 10 & = & 60 & = & 8\cdot 7+4 & & = & 12.28+\dots \\
4\cdot 10 & = & 40 & = & 5\cdot 7+5 & & = & 12.285+\dots \\
5\cdot 10 & = & 50 & = & 7\cdot 7+1 & & = & 12.2851+\dots \\
1\cdot 10 & = & \color{red}{10} & \color{red}{=} & \color{red}{1\cdot 7+3} & & = & 12.28571+\dots
\end{array}
$$

Example

Convert $\frac{86}{7}$ to repeating decimal notation.
Divide with 86 by 7 with remainder. Repeat using remainder ...

$$
\begin{array}{rcrcl|rcl}
 & & 86 & = & 12\cdot 7+2 & \frac{86}{7} & = & 12+\dots \\
2\cdot 10 & = & 20 & = & 2\cdot 7+6 & & = & 12.2+\dots \\
6\cdot 10 & = & 60 & = & 8\cdot 7+4 & & = & 12.28+\dots \\
4\cdot 10 & = & 40 & = & 5\cdot 7+5 & & = & 12.285+\dots \\
5\cdot 10 & = & 50 & = & 7\cdot 7+1 & & = & 12.2851+\dots \\
1\cdot 10 & = & 10 & = & 1\cdot 7+\color{red}{3} & & = & 12.28571+\dots \\
\color{red}{3}\cdot 10 & = & 30 & & & & &
\end{array}
$$

Example

Convert $\frac{86}{7}$ to repeating decimal notation.
Divide with 86 by 7 with remainder. Repeat using remainder ...

$$\begin{array}{rcrcl|rcl}
 & & 86 & = & 12\cdot 7+2 & \frac{86}{7} & = & 12+\dots \\
2\cdot 10 & = & 20 & = & 2\cdot 7+6 & & = & 12.2+\dots \\
6\cdot 10 & = & 60 & = & 8\cdot 7+4 & & = & 12.28+\dots \\
4\cdot 10 & = & 40 & = & 5\cdot 7+5 & & = & 12.285+\dots \\
5\cdot 10 & = & 50 & = & 7\cdot 7+1 & & = & 12.2851+\dots \\
1\cdot 10 & = & 10 & = & 1\cdot 7+3 & & = & 12.28571+\dots \\
3\cdot 10 & = & 30 & = & \mathbf{?}\cdot 7+\mathbf{?} & & &
\end{array}$$

Example

Convert $\frac{86}{7}$ to repeating decimal notation.
Divide with 86 by 7 with remainder. Repeat using remainder ...

$$
\begin{array}{rclcl|lcl}
 & & 86 & = & 12\cdot 7+2 & \frac{86}{7} & = & 12+\dots \\
2\cdot 10 & = & 20 & = & 2\cdot 7+6 & & = & 12.2+\dots \\
6\cdot 10 & = & 60 & = & 8\cdot 7+4 & & = & 12.28+\dots \\
4\cdot 10 & = & 40 & = & 5\cdot 7+5 & & = & 12.285+\dots \\
5\cdot 10 & = & 50 & = & 7\cdot 7+1 & & = & 12.2851+\dots \\
1\cdot 10 & = & 10 & = & 1\cdot 7+3 & & = & 12.28571+\dots \\
3\cdot 10 & = & \color{red}{30} & \color{red}{=} & \color{red}{4\cdot 7+2} & & = & 12.285714+\dots
\end{array}
$$

Example

Convert $\frac{86}{7}$ to repeating decimal notation.
Divide with 86 by 7 with remainder. Repeat using remainder ...

$$
\begin{array}{rcrcl|rcl}
 & & 86 & = & 12\cdot 7+2 & \frac{86}{7} & = & 12+\dots \\
2\cdot 10 & = & 20 & = & 2\cdot 7+6 & & = & 12.2+\dots \\
6\cdot 10 & = & 60 & = & 8\cdot 7+4 & & = & 12.28+\dots \\
4\cdot 10 & = & 40 & = & 5\cdot 7+5 & & = & 12.285+\dots \\
5\cdot 10 & = & 50 & = & 7\cdot 7+1 & & = & 12.2851+\dots \\
1\cdot 10 & = & 10 & = & 1\cdot 7+3 & & = & 12.28571+\dots \\
3\cdot 10 & = & 30 & = & 4\cdot 7+\color{red}{2} & & = & 12.285714+\dots \\
\color{red}{2}\cdot 10 & = & 20 & & & & &
\end{array}
$$

Example

Convert $\frac{86}{7}$ to repeating decimal notation.
Divide with 86 by 7 with remainder. Repeat using remainder ...

$$
\begin{array}{rcrcl|ccl}
 & & 86 & = & 12\cdot 7+2 & \frac{86}{7} & = & 12+\dots \\
2\cdot 10 & = & 20 & = & 2\cdot 7+6 & & = & 12.2+\dots \\
6\cdot 10 & = & 60 & = & 8\cdot 7+4 & & = & 12.28+\dots \\
4\cdot 10 & = & 40 & = & 5\cdot 7+5 & & = & 12.285+\dots \\
5\cdot 10 & = & 50 & = & 7\cdot 7+1 & & = & 12.2851+\dots \\
1\cdot 10 & = & 10 & = & 1\cdot 7+3 & & = & 12.28571+\dots \\
3\cdot 10 & = & 30 & = & 4\cdot 7+2 & & = & 12.285714+\dots \\
2\cdot 10 & = & 20 & = & \dots & & = & 12.\overline{285714}+\dots
\end{array}
$$

... until the remainder repeats.

Example

Convert $\frac{86}{7}$ to repeating decimal notation.
Divide with 86 by 7 with remainder. Repeat using remainder ...

$$
\begin{array}{rcrcl|rcl}
 & & 86 & = & 12\cdot 7+2 & \frac{86}{7} & = & 12+\dots \\
2\cdot 10 & = & 20 & = & 2\cdot 7+6 & & = & 12.2+\dots \\
6\cdot 10 & = & 60 & = & 8\cdot 7+4 & & = & 12.28+\dots \\
4\cdot 10 & = & 40 & = & 5\cdot 7+5 & & = & 12.285+\dots \\
5\cdot 10 & = & 50 & = & 7\cdot 7+1 & & = & 12.2851+\dots \\
1\cdot 10 & = & 10 & = & 1\cdot 7+3 & & = & 12.28571+\dots \\
3\cdot 10 & = & 30 & = & 4\cdot 7+2 & & = & 12.285714+\dots \\
2\cdot 10 & = & 20 & = & \dots & & = & 12.\overline{285714}+\dots
\end{array}
$$

... until the remainder repeats.

Example

Convert $\frac{86}{7}$ to repeating decimal notation.
Divide with 86 by 7 with remainder. Repeat using remainder ...

$$
\begin{array}{rcrcl|lcl}
 & & 86 & = & 12\cdot 7+2 & \frac{86}{7} & = & 12+\dots \\
2\cdot 10 & = & 20 & = & 2\cdot 7+6 & & = & 12.2+\dots \\
6\cdot 10 & = & 60 & = & 8\cdot 7+4 & & = & 12.28+\dots \\
4\cdot 10 & = & 40 & = & 5\cdot 7+5 & & = & 12.285+\dots \\
5\cdot 10 & = & 50 & = & 7\cdot 7+1 & & = & 12.2851+\dots \\
1\cdot 10 & = & 10 & = & 1\cdot 7+3 & & = & 12.28571+\dots \\
3\cdot 10 & = & 30 & = & 4\cdot 7+2 & & = & 12.285714+\dots \\
2\cdot 10 & = & 20 & = & \dots & & = & 12.\overline{285714}+\dots
\end{array}
$$

... until the remainder repeats. Answer: $\frac{86}{7} = 12.\overline{285714}$.

Example

Convert $\frac{86}{7}$ to repeating decimal notation.
Divide with 86 by 7 with remainder. Repeat using remainder ...

$$
\begin{array}{rcccl|lcl}
 & & 86 & = & \color{red}{12} \cdot 7 + 2 & \frac{86}{7} & = & 12 + \dots \\
2 \cdot 10 & = & 20 & = & \color{red}{2} \cdot 7 + 6 & & = & 12.2 + \dots \\
6 \cdot 10 & = & 60 & = & \color{red}{8} \cdot 7 + 4 & & = & 12.28 + \dots \\
4 \cdot 10 & = & 40 & = & \color{red}{5} \cdot 7 + 5 & & = & 12.285 + \dots \\
5 \cdot 10 & = & 50 & = & \color{red}{7} \cdot 7 + 1 & & = & 12.2851 + \dots \\
1 \cdot 10 & = & 10 & = & \color{red}{1} \cdot 7 + 3 & & = & 12.28571 + \dots \\
3 \cdot 10 & = & 30 & = & \color{red}{4} \cdot 7 + 2 & & = & 12.285714 + \dots \\
2 \cdot 10 & = & 20 & = & \dots & & = & \color{red}{12.\overline{285714}} + \dots
\end{array}
$$

... until the remainder repeats. Answer: $\frac{86}{7} = \color{red}{12.\overline{285714}}$.

Example

Convert $\frac{2}{13}$ to repeating decimal notation.

$$
\begin{array}{rclcl|rcl}
 & & 2 & = & 0\cdot 13+2 & \frac{2}{13} & = & 0+\dots \\
2\cdot 10 & = & 20 & = & 1\cdot 13+7 & & = & 0.1+\dots \\
7\cdot 10 & = & 70 & = & 5\cdot 13+5 & & = & 0.15+\dots \\
5\cdot 10 & = & 50 & = & 3\cdot 13+11 & & = & 0.153+\dots \\
11\cdot 10 & = & 110 & = & 8\cdot 13+6 & & = & 0.1538+\dots \\
6\cdot 10 & = & 60 & = & 4\cdot 13+8 & & = & 0.15384+\dots \\
8\cdot 10 & = & 80 & = & 6\cdot 13+2 & & = & 0.153846+\dots \\
2\cdot 10 & = & 20 & = & 1\cdot 13+7 & & = & 0.1538461+\dots \\
\dots & & & & & & = & 0.\overline{153846}+\dots
\end{array}
$$

Answer: $\frac{2}{13} = 0.\overline{153846}$