Calculus I
Areas and integrals

Todor Milev

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Outline

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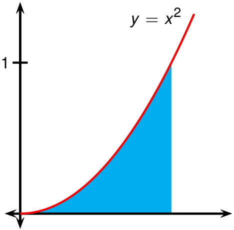
The Area Problem

- How can we find the area under $y = x^2$ between $x = 0$ and $x = 1$?



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- We can approximate it using rectangles.
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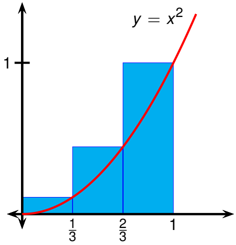

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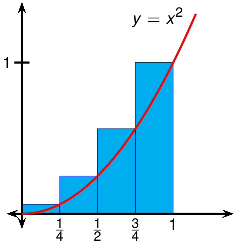

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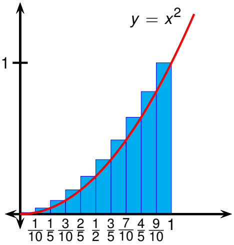

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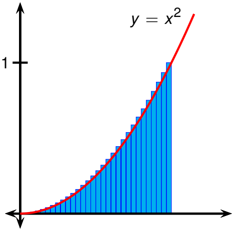

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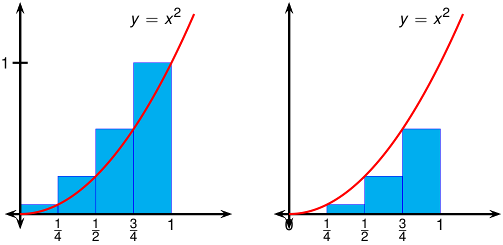
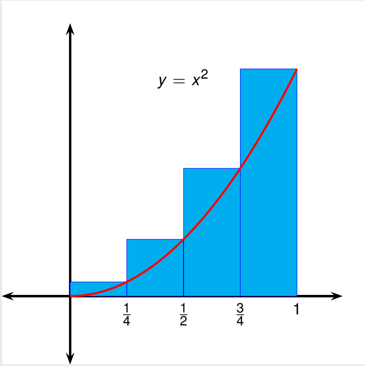
- Four strips gives a better approximation.
- We could use the left endpoints to find the heights instead.



Example

Find the sum of the areas of the four approximating rectangles obtained using right endpoints.

- Let R_4 denote the sum of the areas of the rectangles.
- Each rectangle has width $\frac{1}{4}$.
- The heights are $\left(\frac{1}{4}\right)^2$, $\left(\frac{1}{2}\right)^2$, $\left(\frac{3}{4}\right)^2$, and 1^2.



$$R_4 = \frac{1}{4} \cdot \left(\frac{1}{4}\right)^2 + \frac{1}{4} \cdot \left(\frac{1}{2}\right)^2 + \frac{1}{4} \cdot \left(\frac{3}{4}\right)^2 + \frac{1}{4} \cdot (1)^2 = \frac{15}{32} = 0.46875$$

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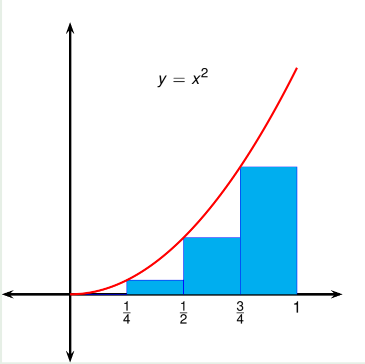
- Each rectangle has width $\frac{1}{4}$.
- The heights are $\left(\frac{1}{4}\right)^2$, $\left(\frac{1}{2}\right)^2$, $\left(\frac{3}{4}\right)^2$, and 1^2.
- A similar calculation works for L_4, the sum of the areas of the left endpoint rectangles.



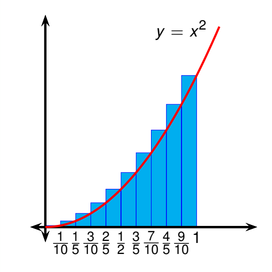
$$R_4 = \frac{1}{4}\cdot\left(\frac{1}{4}\right)^2 + \frac{1}{4}\cdot\left(\frac{1}{2}\right)^2 + \frac{1}{4}\cdot\left(\frac{3}{4}\right)^2 + \frac{1}{4}\cdot(1)^2 = \frac{15}{32} = 0.46875$$

$$L_4 = \frac{1}{4}\cdot(0)^2 + \frac{1}{4}\cdot\left(\frac{1}{4}\right)^2 + \frac{1}{4}\cdot\left(\frac{1}{2}\right)^2 + \frac{1}{4}\cdot\left(\frac{3}{4}\right)^2 = \frac{7}{32} = 0.21875$$

Example

For the region S underneath the parabola $y = x^2$ from 0 to 1, show that the area under the approximating rectangles approaches $\frac{1}{3}$, that is,

$$\lim_{n\to\infty} R_n = \frac{1}{3}.$$



- Each rectangle has width $\frac{1}{n}$.
- The heights are $\left(\frac{1}{n}\right)^2, \left(\frac{2}{n}\right)^2, \dots, \left(\frac{n}{n}\right)^2$.
- New formula:
- $1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$.

$$R_n = \frac{1}{n}\left(\frac{1}{n}\right)^2 + \frac{1}{n}\left(\frac{2}{n}\right)^2 + \dots + \frac{1}{n}\left(\frac{n}{n}\right)^2 = \frac{1}{n^3}\left(1^2 + 2^2 + \dots + n^2\right)$$

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{1}{n^3}\frac{n(n+1)(2n+1)}{6} = \lim_{n\to\infty} \frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) = \frac{1}{3}$$

Example (The $\dots$ and $\sum$ notations for series)

Let A be the sum of the positive even integers between 2 and 124. Write A using the $\dots$ notation and using the $\sum$ notation.

$$\begin{aligned} A &= 2+4+6+\dots+124 \\ &= 2+4+6+\dots+2n+\dots+124 \\ &= 2\cdot 1+2\cdot 2+2\cdot 3+\dots+2\cdot n+\dots+2\cdot 62 \\ &= \sum_{n=1}^{62} 2n \quad . \end{aligned}$$

- We aim to introduce the $\sum$ notation for series via this example.
- The $\dots$ notation is informal but easier to read.
- If the $\dots$ are too ambiguous, we should include the general term.
- To make it clearer we should rewrite all elements in the pattern of the general term.
- If that is still ambiguous we should switch to the completely unambiguous $\sum$ notation.

Example (The $\ldots$ and $\sum$ notations for series)

Let A be the sum of the positive even integers between 2 and 124. Write A using the $\ldots$ notation and using the $\sum$ notation.

$$\begin{aligned} A &= 2+4+6+\cdots+124 \\ &= 2+4+6+\cdots+2n+\cdots+124 \\ &= 2\cdot 1+2\cdot 2+2\cdot 3+\cdots+2\cdot n+\cdots+2\cdot 62 \\ &= \sum_{n=1}^{62} 2n \quad . \end{aligned}$$

- The number n is the index (counter) of the sum.
- $\sum$ tells us to add several copies of the summed term, where in each term the index is replaced by a concrete value.
- The values taken by the index are determined by the boundaries of summation.
- The index varies over all integers starting with the lower boundary and ending with upper boundary.
- In programming, what objects are similar to $\sum$?

Example (The $\ldots$ and $\sum$ notations for series)

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- To go from $\sum$ to $\ldots$ notation: substitute few values for the index. Make sure to include the last value.
- To go from $\ldots$ to $\sum$ notation:
  - figure out a pattern for the general term just as with sequences;
  - select first and last index so that your general term formula reproduces the first and last terms of the sequence.

Example (The $\ldots$ and $\sum$ notations for series)

Let A be the sum of the positive even integers between 2 and 124. Write A using the $\ldots$ notation and using the $\sum$ notation.

$$\begin{array}{rcl} A & = & 2+4+6+\cdots+124 \\ & = & 2+4+6+\cdots+2n+\cdots+124 \\ & = & 2\cdot 1+2\cdot 2+2\cdot 3+\cdots+2\cdot n+\cdots+2\cdot 62 \\ & = & \displaystyle\sum_{n=1}^{62} 2n \quad . \end{array}$$

- Bear in mind the $\ldots$ notation is informal.
  - There are infinitely many formulas that fit any single pattern.
  - Thus it is acceptable to use the $\ldots$ notation only when we believe there is a single completely obvious pattern that will be recognized by every one.
  - The pattern should be obvious not only to us, but also to our potential readers.
  - If in doubt or seeking complete rigor we should use the $\sum$ notation.

Definition

Sigma Notation: The sum of n terms $a_1, a_2, \ldots, a_n$ is written as

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$$

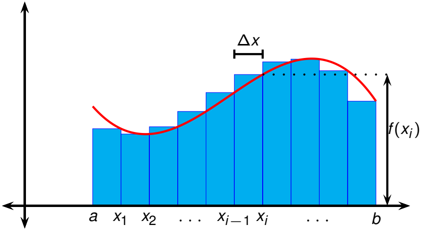
where i is the *index of summation*, a_i is the *i'th term*, and the *upper and lower bounds of summation* are n and 1 respectively.

NOTE: The lower bound doesn't have to be 1. Any integer less than or equal to the upper bound is legitimate.
The index i may be replaced with another symbol, often j or k.

Example

$$\sum_{j=3}^{7} j^2 = 9 + 16 + 25 + 36 + 49$$

Estimate the area under $y = f(x)$ between a and b using n strips.



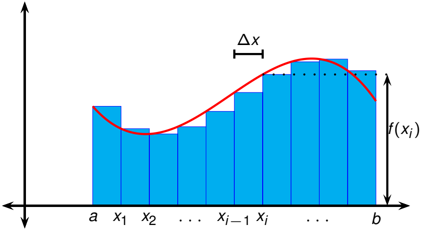
- The right endpoints of the subintervals are

$$\begin{aligned} x_1 &= a + \Delta x \\ x_2 &= a + 2\Delta x \\ x_3 &= a + 3\Delta x \\ &\vdots \end{aligned}$$

- The width of the interval is $b - a$.
- The width one strip is $\Delta x = \frac{b-a}{n}$.
- $[a, b]$ is divided into n subintervals: $[x_0, x_1]$, $[x_1, x_2]$, $\dots$, $[x_{n-1}, x_n]$, where $x_0 = a$ and $x_n = b$.
- The height of the ith rectangle is $f(x_i)$.
- The area of the ith rectangle is $f(x_i)\Delta x$.

$$R_n = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + \cdots + f(x_n)\Delta x$$

Estimate the area under $y = f(x)$ between a and b using n strips.



- The left endpoints of the subintervals are

$$\begin{aligned} x_0 &= a \\ x_1 &= a + \Delta x \\ x_2 &= a + 2\Delta x \\ &\vdots \end{aligned}$$

- The width of the interval is $b - a$.
- The width one strip is $\Delta x = \frac{b-a}{n}$.
- $[a, b]$ is divided into n subintervals: $[x_0, x_1]$, $[x_1, x_2]$, $\dots$, $[x_{n-1}, x_n]$, where $x_0 = a$ and $x_n = b$.
- The height of the ith rectangle is $f(x_{i-1})$.
- The area of the ith rectangle is $f(x_{i-1})\Delta x$.

$$L_n = f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_{n-1})\Delta x$$

Definition (Area Under a Curve)

Let $f(x) > 0$. The area of the region S that lies under $y = f(x)$ is the limit of the sum of the areas of the approximating rectangles:

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} [f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x]$$

- This limit always exists if f is continuous.
- If f is continuous, we get the same limit if we use left endpoints:
$$A = \lim_{n\to\infty} L_n = \lim_{n\to\infty} [f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x]$$
- If f is continuous, we get the same limit if we use any number x_i^* in the interval $[x_{i-1}, x_i]$. x_i^* is called a sample point.
$$A = \lim_{n\to\infty} [f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x]$$

Definition (Riemann Sum)

A Riemann sum is any sum of the form

$$f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x.$$

The Definite Integral

Definition (Definite Integral)

- Let f be a function defined for $a \leq x \leq b$.
- Divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = (b-a)/n$ nd set $x_0 = a$, $x_n = b$.
- Let x_0, $x_1, \dots, x_n$ be the endpoints of the subintervals.
- Let $x_1^*, x_2^*, \dots, x_n^*$ be any sample points in these subintervals; that is, x_i^* is in $[x_{i-1}, x_i]$.

Suppose the limit $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} f(x_i^*)\Delta x$ exists and is independent of the choice of sample points x_i^*. Then we say that f is an integrable function. If f is integrable we call the limit the integral of f over $[a, b]$ and write

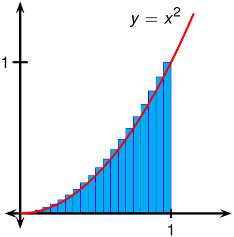
$$\int_a^b f(x)\mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x \quad .$$

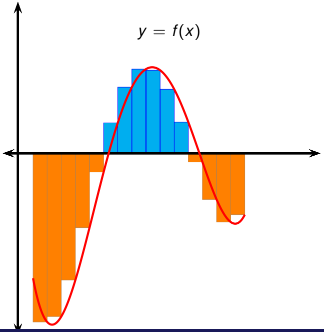
$$\int_a^b f(x)\mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x,$$

- $\int$ is called the integration sign.
- $f(x)$ is called the integrand.
- a and b are called the limits of integration.
- The definite integral is a number. It does not depend on x. We could use any variable instead of x.

$$\int_a^b f(x)\mathrm{d}x = \int_a^b f(t)\mathrm{d}t = \int_a^b f(r)\mathrm{d}r = \int_a^b f(\theta)\mathrm{d}\theta$$

- We know already that if $f(x)$ is always positive, then $\int_a^b f(x)\mathrm{d}x$ is the area under the curve.





- What if $f(x)$ is sometimes negative?

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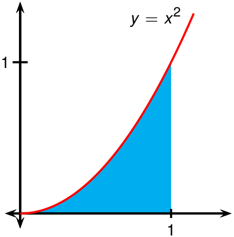
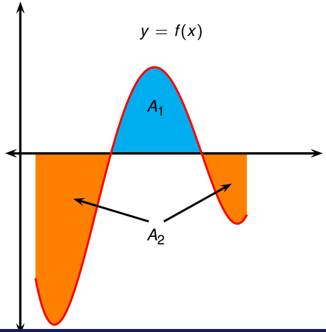




- What if $f(x)$ is sometimes negative?
- Then $\int_a^b f(x)\mathrm{d}x = A_1 - A_2$.
- A_1 is the area of the region above the x-axis and below the graph of f.
- A_2 is the area of the region below the x-axis and above the graph of f.

Theorem

Let f be a continuous function on $[a, b]$. Then f is integrable over $[a, b]$.

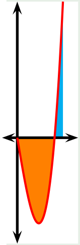
- In particular the integral does not depend the choice of sampling points so long as the intervals containing them shrink.
- The proof of this theorem is not difficult, but is outside of the scope of Calculus I and II.
- The only "difficulty" in the proof stems from the fact that we have not rigorously constructed the real numbers.
- We already (silently) assumed a construction of the real numbers when we defined limits.
- Such a construction is also (silently) assumed in most regular high school mathematics courses.
- A student interested in a proof of the theorem should google "Darboux integral".

The following **power sums** will be useful in what follows:

1. $$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$
2. $$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$
3. $$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

Example

Evaluate $\int_0^3 (x^3 - 6x)\mathrm{d}x$. $\Delta x = \frac{b-a}{n} = \frac{3}{n}$.



$$\int_0^3 (x^3 - 6x)\mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{3i}{n}\right)\frac{3}{n}$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left[\left(\frac{3i}{n}\right)^3 - 6\left(\frac{3i}{n}\right)\right] = \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left[\frac{27}{n^3} i^3 - \frac{18}{n} i\right]$$

$$= \lim_{n\to\infty} \left[\frac{81}{n^4} \sum_{i=1}^{n} i^3 - \frac{54}{n^2} \sum_{i=1}^{n} i\right]$$

$$= \lim_{n\to\infty} \left(\frac{81}{n^4}\left[\frac{n(n+1)}{2}\right]^2 - \frac{54}{n^2}\frac{n(n+1)}{2}\right)$$

$$= \lim_{n\to\infty} \left[\frac{81}{4}\left(1+\frac{1}{n}\right)^2 - 27\left(1+\frac{1}{n}\right)\right] = \frac{81}{4} - 27 = -\frac{27}{4}$$

Properties of the Definite Integral

- So far when we have calculated $\int_a^b f(x)\mathrm{d}x$, we have assumed that $a < b$.
- The definition as a limit of Riemann sums will still work even if we don't assume this.
- If we reverse a and b, then Δx changes from $\frac{b-a}{n}$ to $\frac{a-b}{n}$.

$$\int_b^a f(x)\mathrm{d}x = -\int_a^b f(x)\mathrm{d}x$$

- If $a = b$, then $\Delta x = 0$.

$$\int_a^a f(x)\mathrm{d}x = 0$$

Properties of the Integral

1. $\int_a^b c\mathrm{d}x = c(b-a)$, where c is any constant.
2. $\int_a^b [f(x)+g(x)]\mathrm{d}x = \int_a^b f(x)\mathrm{d}x + \int_a^b g(x)\mathrm{d}x$.
3. $\int_a^b cf(x)\mathrm{d}x = c\int_a^b f(x)\mathrm{d}x$, where c is any constant.
4. $\int_a^b [f(x)-g(x)]\mathrm{d}x = \int_a^b f(x)\mathrm{d}x - \int_a^b g(x)\mathrm{d}x$.

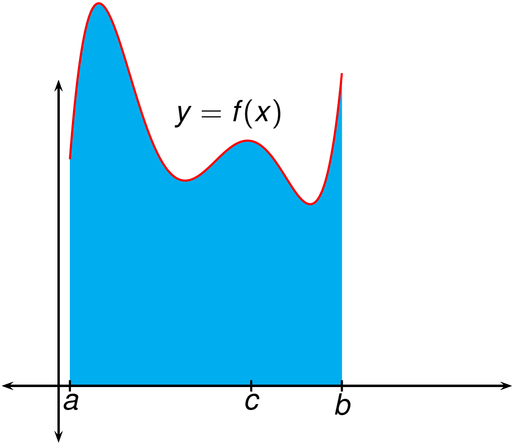
Example

Use the properties of integrals to evaluate

$$
\begin{aligned}
\int_0^1 (4+3x^2)\mathrm{d}x &= \int_0^1 4\mathrm{d}x + \int_0^1 3x^2\mathrm{d}x && \text{Property 2}\\
&= \int_0^1 4\mathrm{d}x + 3\int_0^1 x^2\mathrm{d}x && \text{Property 3}\\
&= 4(1-0) + 3\int_0^1 x^2\mathrm{d}x && \text{Property 1}\\
&= 4 + 3\cdot\frac{1}{3} && \text{From preceding lectures/slides}\\
&= 5
\end{aligned}
$$

Properties of the Integral

5. $\displaystyle\int_a^b f(x)\mathrm{d}x = \int_a^c f(x)\mathrm{d}x + \int_c^b f(x)\mathrm{d}x$



Example

If it is known that $\int_0^{10} f(x)\mathrm{d}x = 17$ and $\int_0^{8} f(x)\mathrm{d}x = 12$, then find $\int_8^{10} f(x)\mathrm{d}x$.

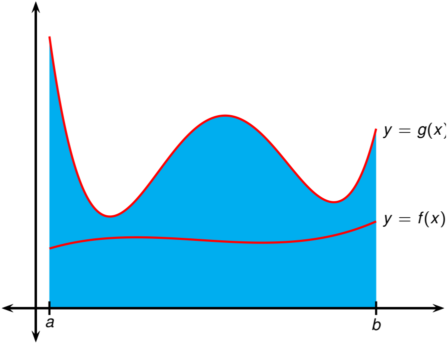
$$\begin{aligned}\int_0^{8} f(x)\mathrm{d}x + \int_8^{10} f(x)\mathrm{d}x &= \int_0^{10} f(x)\mathrm{d}x\\ \int_8^{10} f(x)\mathrm{d}x &= \int_0^{10} f(x)\mathrm{d}x - \int_0^{8} f(x)\mathrm{d}x\\ &= 17 - 12\\ &= 5\end{aligned}$$

Comparison Properties of the Integral

6. If $f(x) \geq 0$ for all $a \leq x \leq b$, then $\displaystyle\int_a^b f(x)\mathrm{d}x \geq 0$.

Comparison Properties of the Integral

7. If $f(x) \leq g(x)$ for all $a \leq x \leq b$, then $\int_a^b f(x)\mathrm{d}x \leq \int_a^b g(x)\mathrm{d}x$.



$$\int_a^b f(x)\mathrm{d}x \leq \int_a^b g(x)\mathrm{d}x$$

Comparison Properties of the Integral

8. If $m \leq f(x) \leq M$ for all $a \leq x \leq b$, then

$$m(b-a) \leq \int_a^b f(x)\mathrm{d}x \leq M(b-a)$$