Calculus II
Improper integrals

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Outline

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Improper Integrals

- The definition of $\int_a^b f(x)\mathrm{d}x$, where f is defined on $[a, b]$, has two requirements:
  1. $[a, b]$ is a finite interval.
  2. f has no infinite discontinuities in $[a, b]$.
- We are now going to relax these requirements.
  1. We allow infinite intervals, such as $(a, \infty), (-\infty, b)$, and $(-\infty, \infty)$.
  2. f might have infinite discontinuities in $[a, b]$.
- Such integrals are called improper integrals.

Definition (Improper Integral)

The integral

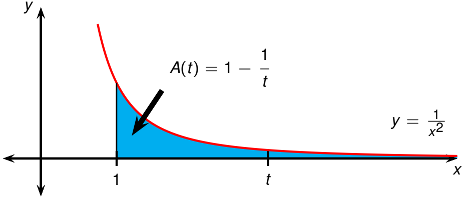
$$\int_a^b f(x)\mathrm{d}x$$

is called improper if one or more of the endpoints a and b is infinite, or if f has an infinite discontinuity on $[a, b]$.

Type I: Infinite Intervals

- Consider the region A that lies under $y = 1/x^2$, above the x-axis, and to the right of $x = 1$.
- To find its area, approximate with $A(t)$, the area of the region under $1/x^2$, above the x-axis, right of $x = 1$, and left of $x = t$.

$$A(t) = \int_1^t \frac{dx}{x^2} = \left[-\frac{1}{x}\right]_1^t = 1 - \frac{1}{t}$$



- Notice $A(t) < 1$ no matter how large t is.
- Also notice $\lim_{t\to\infty} A(t) = \lim_{t\to\infty} \left(1 - \frac{1}{t}\right) = 1$.
- We say that the area A is equal to 1 and write $\int_1^\infty \frac{1}{x^2} dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2} dx = 1$.

Definition (Improper Integral of Type I)

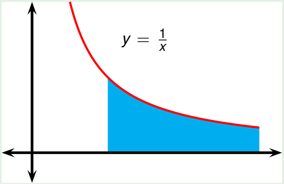
1. If $\int_a^t f(x)\mathrm{d}x$ exists for every $t \geq a$, then
$$\int_a^\infty f(x)\mathrm{d}x = \lim_{t\to\infty} \int_a^t f(x)\mathrm{d}x$$
if the limit exists.
2. If $\int_t^b f(x)\mathrm{d}x$ exists for every $t \leq b$, then
$$\int_{-\infty}^b f(x)\mathrm{d}x = \lim_{t\to-\infty} \int_t^b f(x)\mathrm{d}x$$
if the limit exists.

$\int_a^\infty f(x)\mathrm{d}x$ and $\int_{-\infty}^b f(x)\mathrm{d}x$ are called convergent if the corresponding limit exists and divergent if it doesn't exist.

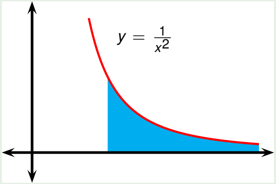
3. If both $\int_a^\infty f(x)\mathrm{d}x$ and $\int_{-\infty}^a f(x)\mathrm{d}x$ are convergent, then we define
$$\int_{-\infty}^\infty f(x)\mathrm{d}x = \int_{-\infty}^a f(x)\mathrm{d}x + \int_a^\infty f(x)\mathrm{d}x.$$

Example

Determine whether $\int_1^\infty \frac{1}{x} dx$ is convergent or divergent.



Infinite area

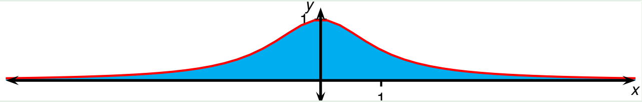


Finite area

$$\begin{aligned}\int_1^\infty \frac{1}{x} dx &= \lim_{t\to\infty} \int_1^t \frac{1}{x} dx\\ &= \lim_{t\to\infty} [\ln x]_1^t\\ &= \lim_{t\to\infty} (\ln t - \ln 1)\\ &= \lim_{t\to\infty} \ln t = \infty\end{aligned}$$

Therefore the improper integral is divergent.

Example



Evaluate $\int_{-\infty}^{\infty} \frac{1}{1+x^2} \mathrm{d}x$.

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{1}{1+x^2} \mathrm{d}x &= \int_{-\infty}^{0} \frac{1}{1+x^2} \mathrm{d}x + \int_{0}^{\infty} \frac{1}{1+x^2} \mathrm{d}x \\ &= \frac{\pi}{2} + \frac{\pi}{2} = \pi\end{aligned}$$

Evaluate the two integrals separately:

$$\begin{aligned}\int_{-\infty}^{0} \frac{1}{1+x^2} \mathrm{d}x &= \lim_{t\to-\infty} \int_{t}^{0} \frac{1}{1+x^2} \mathrm{d}x = \lim_{t\to-\infty} \left[\arctan x\right]_t^0 \\ &= \lim_{t\to-\infty} (\arctan 0 - \arctan t) = \lim_{t\to-\infty} (0 - \arctan t) \\ &= 0 - \left(-\frac{\pi}{2}\right) = \frac{\pi}{2} \\ \int_{0}^{\infty} \frac{1}{1+x^2} \mathrm{d}x &= \lim_{t\to\infty} \int_{0}^{t} \frac{1}{1+x^2} \mathrm{d}x = \lim_{t\to\infty} \left[\arctan x\right]_0^t \\ &= \lim_{t\to\infty} (\arctan t - \arctan 0) = \lim_{t\to\infty} \arctan t = \frac{\pi}{2}\end{aligned}$$

Example

For what values of p is the integral $\int_1^\infty \frac{1}{x^p} \mathrm{d}x$ convergent?

- We know from Example 1 that if $p = 1$, the integral is divergent.
- Assume $p \neq 1$.

$$\int_1^\infty \frac{1}{x^p} \mathrm{d}x = \lim_{t\to\infty} \int_1^t \frac{1}{x^p} \mathrm{d}x = \lim_{t\to\infty} \left[\frac{x^{-p+1}}{-p+1} \right]_1^t = \lim_{t\to\infty} \frac{\frac{1}{t^{p-1}} - 1}{1-p}$$

- If $p > 1$, then $p - 1 > 0$, so as $t \to \infty$, $t^{p-1} \to \infty$ and $1/t^{p-1} \to 0$.
- Therefore $\int_1^\infty \frac{1}{x^p} \mathrm{d}x = \frac{1}{p-1}$ if $p > 1$, and so the integral is convergent.
- If $p < 1$, then $p - 1 < 0$, so $\frac{1}{t^{p-1}} = t^{1-p} \to \infty$ as $t \to \infty$.
- Therefore $\int_1^\infty \frac{1}{x^p} \mathrm{d}x$ is divergent if $p < 1$.

Theorem

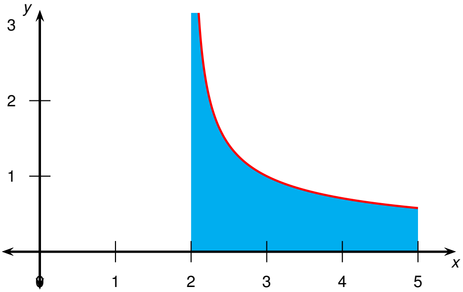
$\int_1^\infty \frac{1}{x^p} \mathrm{d}x$ *converges if* $p > 1$ *and diverges if* $p \leq 1$.

Type II: Discontinuous Integrands

We can use the same approach if the function f is discontinuous at one of the endpoints a and b in the integral $\int_a^b f(x)\mathrm{d}x$.
For example, $\frac{1}{\sqrt{x-2}}$ is discontinuous at 2, so we might wonder if the integral

$$\int_2^5 \frac{1}{\sqrt{x-2}}\mathrm{d}x$$

exists.



Definition (Improper Integral of Type II)

1. If f is continuous on $[a, b)$ and discontinuous at b, then
$$\int_a^b f(x)\mathrm{d}x = \lim_{t\to b^-} \int_a^t f(x)\mathrm{d}x$$
if the limit exists.
2. If f is continuous on $(a, b]$ and discontinuous at a, then
$$\int_a^b f(x)\mathrm{d}x = \lim_{t\to a^+} \int_t^b f(x)\mathrm{d}x$$
if the limit exists.

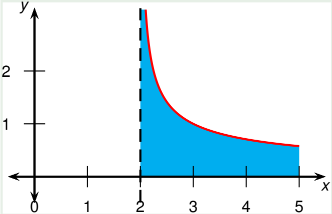
$\int_a^b f(x)\mathrm{d}x$ is called convergent if the corresponding limit exists and divergent if it doesn't exist.

3. If f has a discontinuity at c, where $a < c < b$, and both $\int_a^c f(x)\mathrm{d}x$ and $\int_c^b f(x)\mathrm{d}x$ are convergent, then we define
$$\int_a^b f(x)\mathrm{d}x = \int_a^c f(x)\mathrm{d}x + \int_c^b f(x)\mathrm{d}x$$

Example

Find $\displaystyle\int_2^5 \frac{1}{\sqrt{x-2}}\mathrm{d}x$.

Observe that $x = 2$ is a vertical asymptote for the integrand.



Area = $2\sqrt{3}$

$$
\begin{aligned}
\int_2^5 \frac{1}{\sqrt{x-2}}\mathrm{d}x &= \lim_{t\to 2^+}\int_t^5 \frac{1}{\sqrt{x-2}}\mathrm{d}x\\
&= \lim_{t\to 2^+}\left[2\sqrt{x-2}\right]_t^5\\
&= \lim_{t\to 2^+} 2\left(\sqrt{5-2}-\sqrt{t-2}\right)\\
&= 2\sqrt{3}
\end{aligned}
$$

Example

Evaluate $\displaystyle\int_0^3 \frac{1}{x-1}\mathrm{d}x$.
Observe that $x = 1$ is a vertical asymptote for the integrand.

$$\int_0^3 \frac{1}{x-1}\mathrm{d}x = \int_0^1 \frac{1}{x-1}\mathrm{d}x + \int_1^3 \frac{1}{x-1}\mathrm{d}x$$

$$\begin{aligned}\int_0^1 \frac{\mathrm{d}x}{x-1} &= \lim_{t\to 1^-}\int_0^t \frac{\mathrm{d}x}{x-1} = \lim_{t\to 1^-}\left[\ln|x-1|\right]_0^t \\ &= \lim_{t\to 1^-}\ln|t-1| - \ln 1 = -\infty\end{aligned}$$

- Therefore the integral diverges.
- If we had not noticed the vertical asymptote, we might have made the following mistake:

$$\int_0^3 \frac{\mathrm{d}x}{x-1} = \left[\ln|x-1|\right]_0^3 = \ln 2 - \ln 1 = \ln 2.$$

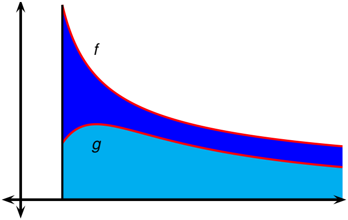
A Comparison Test for Improper Integrals

Sometimes it's impossible to find the exact value of an integral, but we still want to know if it's convergent or divergent. For such cases, we can sometimes use the following theorem.

Theorem (Comparison Theorem)

Suppose f and g are continuous and $f(x) \geq g(x) \geq 0$ for $x \geq a$.

1. *If $\int_a^\infty f(x)\mathrm{d}x$ is convergent, then $\int_a^\infty g(x)\mathrm{d}x$ is convergent.*
2. *If $\int_a^\infty g(x)\mathrm{d}x$ is divergent, then $\int_a^\infty f(x)\mathrm{d}x$ is divergent.*

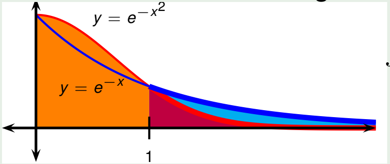


A similar theorem holds for Type II improper integrals.

Example

Show that $\int_0^\infty e^{-x^2}\mathrm{d}x$ is convergent.

- The antiderivative of e^{-x^2} isn't an elementary function.
- If integral were $\int_0^\infty e^{-x}\mathrm{d}x$, we'd have no problem integrating.
- Notice that $0 \leq e^{-x^2} \leq e^{-x}$ for $x \geq 1$ (because $-x^2 < -x$ for $x > 1$ and the exponent is an increasing function).
- Split $\int_0^\infty e^{-x^2}\mathrm{d}x = \int_0^1 e^{-x^2}\mathrm{d}x + \int_1^\infty e^{-x^2}\mathrm{d}x$.
- On the RHS, first integral is proper - no effect on convergence.



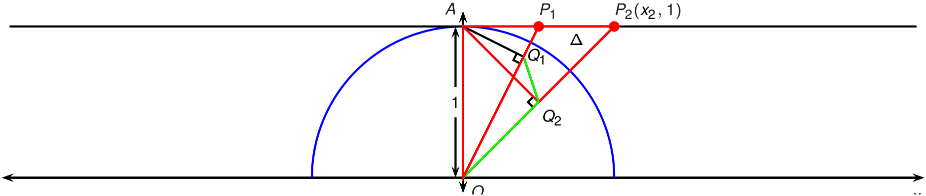
$$\begin{aligned}\int_1^\infty e^{-x^2}\mathrm{d}x &\leq \int_1^\infty e^{-x}\mathrm{d}x \\ &= \lim_{t\to\infty}\int_1^t e^{-x}\mathrm{d}x \\ &= \lim_{t\to\infty}\left[-e^{-x}\right]_1^t \\ &= \lim_{t\to\infty}\left(-e^{-t}-\left(-e^{-1}\right)\right) \\ &= e^{-1}\end{aligned}$$

By the Comparison Theorem, $\int_1^\infty e^{-x^2}\mathrm{d}x$ converges $\Rightarrow$ $\int_0^\infty e^{-x^2}\mathrm{d}x$ converges.

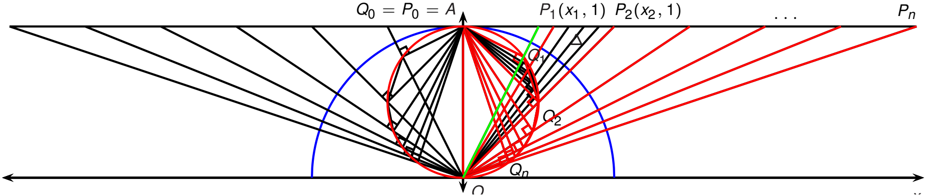
Example

Is $\displaystyle\int_1^\infty \frac{1+e^{-x}}{x} \mathrm{d}x$ convergent or divergent?

- Notice that for $x \geq 1$ we have $\dfrac{1+e^{-x}}{x} > \dfrac{1}{x}$.
- By a previously studied example, $\int_1^\infty \frac{\mathrm{d}x}{x}$ is divergent.
- Therefore $\displaystyle\int_1^\infty \frac{1+e^{-x}}{x} \mathrm{d}x$ is divergent by the Comparison Theorem.

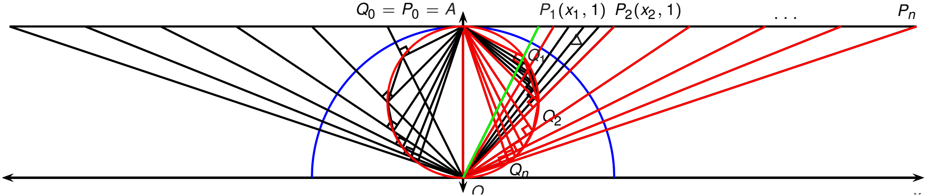


Draw a unit circle as above, let O, A be as indicated. Let P_2 be the point $(x_2, 1)$, P_1 be the point $(x_2 - \Delta, 1)$. By the Pythagorean theorem, $|OP_2|^2 = 1 + x_2^2$ and similarly $|OP_1|^2 = 1 + (x_2 - \Delta)^2$. Let Q_1, Q_2 be as indicated. Then $\triangle OP_2A$ is similar to $\triangle OAQ_2$. By Euclidean geometry, $\frac{|OA|}{|OP_2|} = \frac{|OQ_2|}{|OA|}$ and so $|OQ_2||OP_2| = |OA|^2 = 1$ and therefore $\frac{|OQ_2|}{|OP_2|} = \frac{|OQ_2||OP_2|}{|OP_2|^2} = \frac{1}{|OP_2|^2} = \frac{1}{1+x_2^2}$. Similarly conclude $|OQ_1||OP_1| = |OA|^2 = 1 = |OQ_2||OP_2|$. Therefore $\frac{|OQ_1|}{|OP_2|} = \frac{|OQ_2|}{|OP_1|}$ and so $\triangle OQ_2Q_1$ is similar to $\triangle OP_1P_2$. Therefore $\frac{|Q_1Q_2|}{|P_1P_2|} = \frac{|OQ_2|}{|OP_1|}$ and so $|Q_1Q_2| = \frac{|P_1P_2||OQ_2|}{|OP_1|} = \left(\frac{|OP_2|}{|OP_1|}\right)\frac{|OQ_2|}{|OP_2|}|P_1P_2| = \frac{|OP_2|}{|OP_1|}\frac{\Delta}{1+x_2^2}$.



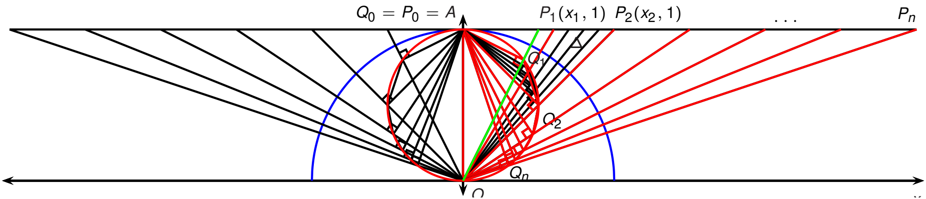
$|Q_1Q_2| = \frac{|OP_2|}{|OP_1|}\frac{\Delta}{1+x_2^2}$. For any $\varepsilon > 0$, can choose Δ: $1 < \frac{|OP_2|}{|OP_1|} < 1 + \varepsilon$.

If we let $P_2 \to P_1$, i.e., $\Delta \to 0$, we get $\frac{|OP_2|}{|OP_1|} \to 1$. In strict mathematical language: for every $\varepsilon > 0$ there exists $\delta > 0$ such that when $\Delta < \delta$ we have that $1 > \frac{|OP_2|}{|OP_1|} > 1 - \varepsilon$. Furthermore, the choice of δ can be made independent of the value of x_2: to prove that one analyzes the expression $\frac{|OP_2|}{|OP_1|} = \sqrt{\frac{1+x_2^2}{1+(x_2-\Delta)^2}}$. We leave the tedious but otherwise easy details to the interested student.



$|Q_1Q_2| = \frac{|OP_2|}{|OP_1|}\frac{\Delta}{1+x_2^2}$. For any $\varepsilon > 0$, can choose Δ: $1 < \frac{|OP_2|}{|OP_1|} < 1 + \varepsilon$.

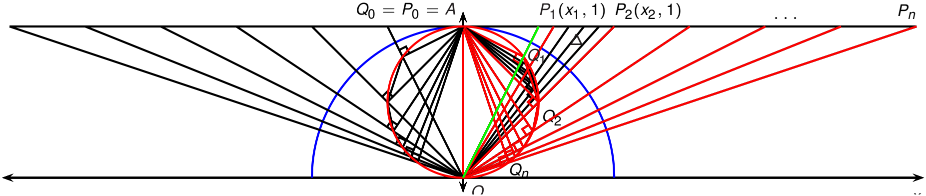
Fix a large number N and let Δ be such that $n = \frac{N}{\Delta}$ is integer. Let $P_0 = (0, 1)$, $P_1 = (\Delta, 1)$, $P_2 = (2\Delta, 1), \dots, P_n = (n\Delta, 1)$, and let $Q_0, Q_1, Q_2, \dots, Q_n$ be as indicated.



$|Q_1 Q_2| = \frac{|OP_2|}{|OP_1|}\frac{\Delta}{1+x_2^2}$. For any $\varepsilon > 0$, can choose Δ: $1 < \frac{|OP_2|}{|OP_1|} < 1 + \varepsilon$.

$$
\begin{array}{ccccc}
\frac{\Delta}{1+x_1^2} & < & |Q_0 Q_1| & < & (1+\varepsilon)\frac{\Delta}{1+x_1^2} \\
\frac{\Delta}{1+x_2^2} & < & |Q_1 Q_2| & < & (1+\varepsilon)\frac{\Delta}{1+x_2^2} \\
 & \vdots & & & \\
\frac{\Delta}{1+x_n^2} & < & |Q_{n-1} Q_n| & < & (1+\varepsilon)\frac{\Delta}{1+x_n^2} \\
\hline
\sum_{i=1}^{n} \frac{\Delta}{1+x_i^2} & < & \sum_{i=1}^{n} |Q_{i-1} Q_i| & < & (1+\varepsilon)\sum_{i=1}^{n} \frac{\Delta}{1+x_i^2} \\
\downarrow & & \downarrow & & \downarrow \\
\int_0^{\infty N} \frac{dx}{1+x^2} & = & \lim\limits_{\Delta, N, \varepsilon} \sum |Q_{i-1} Q_i| & = & (1+\varepsilon)\int_0^{\infty N} \frac{dx}{1+x^2}
\end{array}
$$

Let $\Delta \to 0$. Next take $N \to \infty$. Finally take $\varepsilon \to 0$, use squeeze thm.



$|Q_1Q_2| = \frac{|OP_2|}{|OP_1|}\frac{\Delta}{1+x_2^2}$. For any $\varepsilon > 0$, can choose Δ: $1 < \frac{|OP_2|}{|OP_1|} < 1+\varepsilon$. The points $Q_1, Q_2, \dots$ see the segment OA from an angle of $\frac{\pi}{2}$. Therefore, by Euclidean geometry, the points $Q_1, Q_2, \dots$ lie on the circle C with radius $\frac{1}{2}$ and center $(0, \frac{1}{2})$. Therefore $\sum |Q_{i-1}Q_i|$ approximates half of the circumference of the circle C. By symmetry,

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{1+x^2} = \text{ circumference of } C = 2\pi\left(\frac{1}{2}\right) = \pi,$$

as desired.