Calculus I
Definite integrals and areas between curves

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Outline

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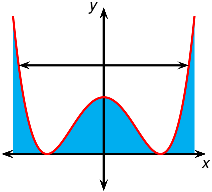
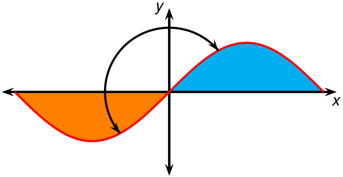
Symmetry

Theorem (Integrals of Symmetric Functions)

Suppose f is continuous on $[-a, a]$.

1. *If f is even (that is, $f(-x) = f(x)$), then* $\displaystyle\int_{-a}^{a} f(x)\,\mathrm{d}x = 2\int_{0}^{a} f(x)\,\mathrm{d}x.$
2. *If f is odd (that is, $f(-x) = -f(x)$), then*
$$\int_{-a}^{a} f(x)\,\mathrm{d}x = \int_{0}^{a} f(x)\,\mathrm{d}x + \int_{-a}^{0} f(x)\,\mathrm{d}x = 0.$$



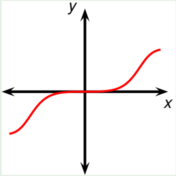


Example

Since $f(x) = x^6 + 1$ satisfies $f(-x) = f(x)$, it is even, and so

$$\begin{aligned}\int_{-2}^{2} (x^6+1)\,\mathrm{d}x &= 2\int_{0}^{2} (x^6+1)\,\mathrm{d}x \\ &= 2\left[\frac{1}{7}x^7 + x\right]_0^2 \\ &= 2\left(\frac{128}{7} + 2\right) \\ &= \frac{284}{7}.\end{aligned}$$

Example

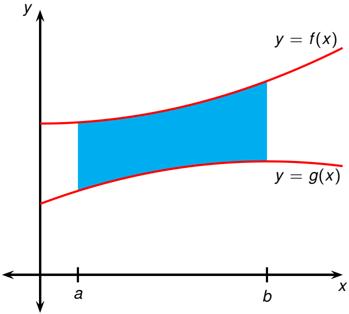


Since $f(x) = \dfrac{\tan x - x}{1 - 2x^2 + 2x^4}$ satisfies $f(-x) = -f(x)$, it is odd, and so

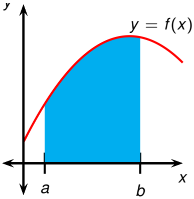
$$\int_{-1}^{1} \frac{\tan x - x}{1 - 2x^2 + 2x^4} \mathrm{d}x = 0.$$

More About Areas

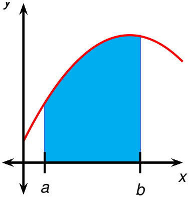
Suppose two curves, $y = f(x)$ and $y = g(x)$, are given.
How do we find the area bounded by those curves between the endpoints $x = a$ and $x = b$?



The Area Under a Curve



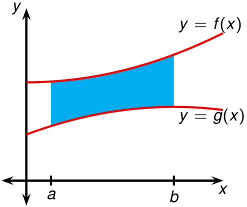
rectangle area $= f(x) \cdot \Delta x$



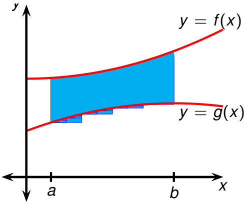
rectangles $= n \to \infty$

$A = \int_a^b f(x) dx$

The Area Between Two Curves



rectangle area $= (f(x) - g(x)) \cdot \Delta x$



rectangles $= n \to \infty$

$A = \int_a^b [f(x) - g(x)] dx$

Definition (The Area Between Two Curves)

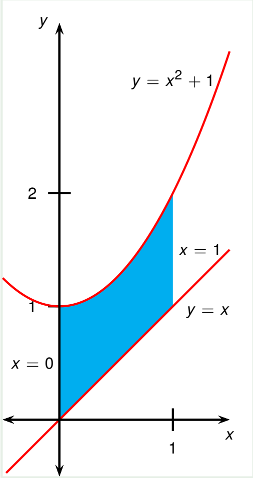
The area between two curves $y = f(x)$ and $y = g(x)$ bounded by the endpoints $x = a$ and $x = b$ is

$$\int_a^b |f(x) - g(x)| \mathrm{d}x.$$

Note that we use the absolute value, because in general we don't know which curve is above the other.

Example

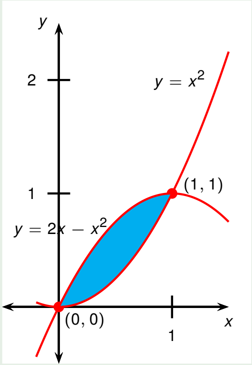
Find the area of the region bounded above by $y = x^2 + 1$, bounded below by $y = x$, and bounded on its sides by $x = 0$ and $x = 1$.



1. Graph the functions.
2. Identify the region.
3. Integrate.

$$\begin{aligned} A &= \int_0^1 |(x^2+1) - x| \mathrm{d}x \\ &= \int_0^1 (x^2 - x + 1) \mathrm{d}x \\ &= \left[\frac{x^3}{3} - \frac{x^2}{2} + x \right]_0^1 \\ &= \frac{1}{3} - \frac{1}{2} + 1 = \frac{5}{6}. \end{aligned}$$

Example



1. Find the point of intersection.
2. Graph the functions.
3. Identify the region.
4. Integrate.

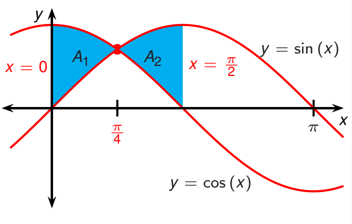
Find the area of the region enclosed by the parabolas $y = x^2$ and $y = 2x - x^2$.

$$
\begin{aligned}
x^2 &= 2x - x^2 \\
0 &= 2x - 2x^2 = 2x(1 - x) \\
x &= 0 \text{ or } 1.
\end{aligned}
$$

$$
A = \int_0^1 (2x - 2x^2) \mathrm{d}x = 2 \int_0^1 (x - x^2) \mathrm{d}x
$$

$$
= 2 \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = 2 \left(\frac{1}{2} - \frac{1}{3} \right) = \frac{1}{3}.
$$

Example



1. Find the point of intersection.
2. Graph the functions.
3. Identify the region.
4. Integrate.

Find the area of the region enclosed by the curves $y = \sin x$, $y = \cos x$, $x = 0$ and $x = \pi/2$.

The only point of intersection in the interval $[0, \pi/2]$ is $(\pi/4, 1/\sqrt{2})$.

$$\begin{aligned} A &= A_1 + A_2 \\ &= \int_0^{\pi/4} (\cos x - \sin x) dx \\ &\quad + \int_{\pi/4}^{\pi/2} (\sin x - \cos x) dx \\ &= [\sin x + \cos x]_0^{\pi/4} + [-\cos x - \sin x]_{\pi/4}^{\pi/2} \\ &= 2\sqrt{2} - 2. \end{aligned}$$