Calculus II

Integrals of the form $\int \sqrt{ax^2 + c}\,dx$, $a, c > 0$

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Euler substitution: $x = \frac{1}{2}\left(\frac{1}{t} - t\right)$, $\sqrt{x^2+1} = \frac{1}{2}\left(\frac{1}{t} + t\right)$,
$t = \sqrt{x^2+1} - x$, $\mathrm{d}x = -\frac{1}{2}\left(\frac{1}{t^2}+1\right)\mathrm{d}t$. Recall $t > 0$.

Example

$$
\begin{aligned}
\int \sqrt{x^2+1}\,\mathrm{d}x &= -\int \frac{1}{2}\left(\frac{1}{t}+t\right)\frac{1}{2}\left(\frac{1}{t^2}+1\right)\mathrm{d}t\\
&= -\frac{1}{4}\int\left(\frac{1}{t^3}+2\frac{1}{t}+t\right)\mathrm{d}t\\
&= -\frac{1}{4}\left(-\frac{t^{-2}}{2}+2\ln|t|+\frac{t^2}{2}\right)+C\\
&= \frac{1}{2}\left(\frac{1}{2}\left(t^{-1}-t\right)\frac{1}{2}\left(t^{-1}+t\right)\right)-\frac{1}{2}\ln t+C\\
&= \frac{1}{2}x\sqrt{x^2+1}-\frac{1}{2}\ln\left(\sqrt{x^2+1}-x\right)+C\\
&= \frac{1}{2}x\sqrt{x^2+1}+\frac{1}{2}\ln\frac{\sqrt{x^2+1}+x}{\left(\sqrt{x^2+1}-x\right)\left(\sqrt{x^2+1}+x\right)}+C\\
&= \frac{1}{2}x\sqrt{x^2+1}+\frac{1}{2}\ln\left(\sqrt{x^2+1}+x\right)+C
\end{aligned}
$$

Example

$$\int \frac{1}{x^2\sqrt{x^2+9}}\mathrm{d}x = \int \frac{1}{x^2 3\sqrt{\left(\frac{x}{3}\right)^2+1}}\mathrm{d}x$$

$$= \int \frac{1}{(3\cot\theta)^2 3\sqrt{\cot^2\theta+1}}\mathrm{d}(3\cot\theta)$$

$$= \int \frac{1}{27\cot^2\theta\sqrt{\csc^2\theta}}\left(-3\csc^2\theta\right)\mathrm{d}\theta$$

$$= \frac{1}{9}\int \frac{-\csc^2\theta}{\cot^2\theta\csc\theta}\mathrm{d}\theta$$

$$= \frac{1}{9}\int \frac{-\sin\theta}{\cos^2\theta}\mathrm{d}\theta = \frac{1}{9}\int \frac{1}{\cos^2\theta}\mathrm{d}(\cos\theta)$$

$$= \frac{1}{9}\int \frac{\mathrm{d}u}{u^2} = -\frac{1}{9u} + C = -\frac{\sec\theta}{9} + C$$

$$= -\frac{\sqrt{x^2+9}}{9x} + C$$

Set

$\frac{x}{3} = \cot\theta$

$x = 3\cot\theta$

$\theta \in (0,\pi)$

$\theta \in (0,\pi) \Rightarrow$

$\csc\theta > 0$

Set $u = \cos\theta$