Precalculus
, Factorization of polynomials: overview

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2019

Outline

Recall that $i^2 = -1$, $\sqrt{-1} = i$.

Example (Polynomial factorizations)

$$2x^2 + 3x - 5 \quad =$$

$$x^2 + 1 \quad =$$

$$x^4 - 1 \quad =$$

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2x^2 + 3x - 5 &= (\textbf{?} \qquad)(\textbf{?} \qquad) \\
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2x^2 + 3x - 5 &= (2x + 5)(x - 1) \\
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2x^2 + 3x - 5 &= (2x+5)(x-1) = 2\left(x - \left(-\frac{5}{2}\right)\right)(x-1) \\
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Theorem (The Fundamental Theorem of Algebra)

Every polynomial can be factored into product of linear terms

$$p(x) = a_0x^n + a_1x^{n-1} + \dots + a_n = a_0(x - x_1)\dots(x - x_n),$$

where $x_1, \dots, x_n$ are the (not necessarily different) roots of $p(x)$.

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Corollary

Every real polynomial can be factored into a product of real linear terms and real quadratic terms with no real roots, i.e., factors of form

- $(x - r)$*, where* r *is real and*
- $ax^2 + bx + c$ *with* $b^2 - 4ac < 0$ *where* a, b, c *are real.*

$$a_0x^n + a_1x^{n-1} + \dots + a_n = a_0(x - x_1)\dots(x - x_n)$$
$=$prod. real quadratics no roots & lin. terms.

Example

$$2x^2 + 3x - 5 = (2x + 5)(x - 1) = 2\left(x - \left(-\frac{5}{2}\right)\right)(x - 1)$$

$$x^2 + 1 = x^2 - (-1) = x^2 - i^2 = (x - (-i))(x - i)$$

$$x^4 - 1 = (x^2 - 1)(x^2 + 1) = (x - 1)(x + 1)(x^2 + 1)$$

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Example

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$=$prod. real quadratics no roots & lin. terms.

Example

$$2x^2 + 3x - 5 = (2x+5)(x-1) = 2\left(x - \left(-\tfrac{5}{2}\right)\right)(x-1)$$
real roots

$$x^2 + 1 = x^2 - (-1) = x^2 - i^2 = (x - (-i))(x - i)$$
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$$x^4 - 1 = (x^2-1)(x^2+1) = (x-1)(x+1)(x^2+1)$$
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- We study those for cubics with the aid of scientific calculator.