Calculus II
Add telescoping series, part 1

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Example

Show the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent and find its sum.

Is this a geometric series? No, because $\frac{a_{n+1}}{a_n} = \frac{\frac{1}{(n+1)(n+2)}}{\frac{1}{n(n+1)}} = \frac{n}{n+2}$ is not constant. Decompose a_n into partial fractions:

$$
\begin{aligned}
a_n &= \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1} \\
s_k &= \sum_{n=1}^{k} \frac{1}{n(n+1)} = \sum_{n=1}^{k} \left(\frac{1}{n} - \frac{1}{n+1} \right) \\
&= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \dots + \left(\frac{1}{k} - \frac{1}{k+1}\right) \\
&= 1 - \frac{1}{k+1}
\end{aligned}
$$

Therefore $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{k\to\infty} s_k = \lim_{k\to\infty} \left(1 - \frac{1}{k+1}\right) = 1$