Precalculus
Factoring quadratic polynomials

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Outline

Definition ((Partial) Factorization)

To (partially) factor a polynomial means to rewrite it as a product of two polynomials of smaller degree.

Example

$$\begin{aligned} x^2 - 1 &= (x-1)(x+1) \\ x^3 + 8 &= (x+2)\left(x^2 - 2x + 4\right) \end{aligned}$$

Theorem

The quadratic $ax^2 + bx + c$ factors as follows.

$$ax^2 + bx + c = a(x - x_1)(x - x_2),$$

where x_1 and x_2 are the roots of the quadratic, given by:

$$x_1, x_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$ax^2 + bx + c = a(x - x_1)(x - x_2)$, where $x_1, x_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Example

Factor the polynomial. If possible, guess the factorization.

$$\begin{aligned} 3x^2 + 8x - 11 &= (3x + 11)(x + \;\; -1) \\ &= 3\left(x - \left(-\frac{11}{3}\right)\right)(x - 1) \end{aligned}$$

- If there is a factorization using integers, it should be of the form

$$\begin{aligned} 3x^2 + 8x - 11 &= (3x + p)(x + q) \\ &= 3x^2 + 3xq + px + pq \\ &= 3x^2 + x(3q + p) + pq \end{aligned}$$

(Vieta's formulas) This means that :

$$\begin{aligned} 8 &= 3q + p \\ -11 &= pq \end{aligned}$$

p, q must be divisors of 11: $\pm 1, \pm 11$

$$\begin{aligned} p &= 11 \\ q &= -1 \end{aligned}$$

$ax^2 + bx + c = a(x - x_1)(x - x_2)$, where $x_1, x_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Example

Factor the polynomial. If possible, guess the factorization.

$$\begin{aligned} 3x^2 + 8x - 11 &= (3x + 11)(x + \ \ - 1) \\ &= 3\left(x - \left(-\tfrac{11}{3}\right)\right)(x - 1) \end{aligned}$$

- What if we can't guess the factorization?
- Use the formulas for x_1, x_2.

$$\begin{aligned} x_1, x_2 &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-8 \pm \sqrt{8^2 - 4 \cdot 3 \cdot (-11)}}{2 \cdot 3} \\ &= \frac{-8 \pm \sqrt{64 + 132}}{6} = \frac{-8 \pm \sqrt{196}}{6} \\ &= \frac{-8 \pm 14}{6} = \begin{cases} \frac{-8 + 14}{6} = \frac{6}{6} = 1 \\ \frac{-8 - 14}{6} = -\frac{22}{6} = -\frac{11}{3} \end{cases} \end{aligned}$$

Proposition (Vieta's formulas)

Let $ax^2 + bx + c$ be a quadratic functions with zeros x_1 and x_2. Then:

$$\begin{aligned} a(x - x_1)(x - x_2) &= ax^2 + bx + c \\ ax^2 - a(x_2 + x_1)x + ax_1x_2 &= ax^2 + bx + c \\ x_1x_2 &= \frac{c}{a} \\ x_1 + x_2 &= -\frac{b}{a} \end{aligned}$$

The last two formulas are called Vieta's formulas (after François Viète (1540-1603), Latinized name: Franciscus Vieta).

$$ax^2 + bx + c = a(x - x_1)(x - x_2)$$

$$\begin{aligned} x_1 + x_2 &= -\frac{b}{a} \\ x_1 x_2 &= \frac{c}{a} \end{aligned}$$

Vieta's formulas

Example

Factor the quadratic.

$$x^2 + 5x + 6 = (x + 2)(x + 3)$$

- The product of the two roots: $x_1 x_2 = 6$.
- The divisors of 6 are $\pm 1, \pm 2, \pm 3, \pm 6$.
- Therefore the pair x_1, x_2 is $\pm 1, \pm 6$ or $\pm 2, \pm 3$.
- The sum of the two roots: $x_1 + x_2 = -5$

$$ax^2 + bx + c = a(x - x_1)(x - x_2),$$

$$\left| \begin{array}{rcl} x_1 x_2 & = & \frac{c}{a} \\ x_1 + x_2 & = & -\frac{b}{a} \end{array} \right.$$

Example

Factor the quadratic.

$$x^2 + 3x + 1 = \left(x \quad - \left(\frac{-3+\sqrt{5}}{2}\right)\right) \left(x \quad - \left(\frac{-3-\sqrt{5}}{2}\right)\right)$$

- The product of the two roots: $x_1 x_2 = 1$.
- Integer options: $x_1 = 1, x_2 = 1$ and $x_1 = -1, x_2 = -1$.
- $\begin{array}{rcl} (x-1)(x-1) & = & (x-1)^2 = x^2 - 2x + 1 \\ (x+1)(x+1) & = & (x+1)^2 = x^2 + 2x + 1 \end{array}$ both don't work.
- $\Rightarrow$ No easy factorization; must use quadratic formula.

$$\begin{array}{rcl} x_1, x_2 & = & \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-3 \pm \sqrt{3^2 - 4 \cdot 1 \cdot 1}}{2 \cdot 1} \\ & = & \frac{-3 \pm \sqrt{5}}{2} \end{array}$$

$$ax^2 + bx + c = a(x - x_1)(x - x_2),$$

$$\begin{array}{rcl} x_1 x_2 & = & \frac{c}{a} \\ x_1 + x_2 & = & -\frac{b}{a} \end{array}$$

Example

Factor the quadratic, using complex numbers if needed.

$$x^2 + x + 1 = \left(x \quad - \left(\tfrac{-1+\sqrt{3}i}{2}\right)\right)\left(x \quad - \left(\tfrac{-1-\sqrt{3}i}{2}\right)\right)$$

- The product of the two roots: $x_1 x_2 = 1$.
- Integer options: $x_1 = 1, x_2 = 1$ and $x_1 = -1, x_2 = -1$.
- $\begin{array}{rcl} (x-1)(x-1) & = & (x-1)^2 = x^2 - 2x + 1 \\ (x+1)(x+1) & = & (x+1)^2 = x^2 + 2x + 1 \end{array}$ both don't work.
- $\Rightarrow$ No easy factorization; must use quadratic formula.

$$\begin{array}{rcl} x_1, x_2 & = & \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 1 \cdot 1}}{2 \cdot 1} \\ & = & \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm \sqrt{3}i}{2} \end{array}$$