Calculus I
Integral substitution rule

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Outline

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The Substitution Rule

- How do we integrate $\int 2x\sqrt{1+x^2}\,\mathrm{d}x$?
- Introduce a new variable $u = 1 + x^2$.
- Then $\mathrm{d}u = \mathrm{d}\left(1+x^2\right) = \left(1+x^2\right)'\mathrm{d}x = 2x\mathrm{d}x$.
- Substitute into the integral:

$$\int 2x\sqrt{1+x^2}\,\mathrm{d}x = \int \sqrt{u}\,\mathrm{d}u = \int u^{\frac{1}{2}}\mathrm{d}u = \frac{2}{3}u^{\frac{3}{2}} + C = \frac{2}{3}\left(1+x^2\right)^{\frac{3}{2}} + C$$

- Is this procedure justified?
- Take the derivative using the Chain Rule:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{2}{3}\left(1+x^2\right)^{\frac{3}{2}} + C\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{2}{3}u^{\frac{3}{2}}\right) = \frac{\cancel{2}}{\cancel{3}}\cdot\frac{\cancel{3}}{\cancel{2}}u^{\frac{1}{2}}\frac{\mathrm{d}u}{\mathrm{d}x} = \sqrt{1+x^2}(2x)$$

Theorem (The Substitution Rule)

Let $u = g(x)$ be a differentiable function whose range is an interval I and let f be a function continuous on I. Then

$$\int f(g(x))g'(x)\,\mathrm{d}x = \int f(u)\,\mathrm{d}u$$

This is the integration counterpart of the Chain Rule.

Example (Substitution Rule)

Find $\displaystyle\int x^3 \cos(x^4+3)\mathrm{d}x.$

$$\begin{aligned}
\text{Let}\quad u &= x^4+3. \\
\text{Then}\quad \mathrm{d}u &= 4x^3\mathrm{d}x \\
x^3\mathrm{d}x &= \frac{1}{4}\mathrm{d}u. \\
\text{Substitute:}\quad \int x^3\cos(x^4+3)\mathrm{d}x &= \int \frac{1}{4}\cos u\,\mathrm{d}u \\
&= \frac{1}{4}\sin u + C \\
&= \frac{1}{4}\sin(x^4+3) + C.
\end{aligned}$$

Example (Substitution Rule)

Find $\displaystyle\int \sqrt{2x+1}\,\mathrm{d}x.$

$$
\begin{aligned}
\text{Let} \quad u &= 2x+1. \\
\text{Then} \quad \mathrm{d}u &= 2\mathrm{d}x \\
\mathrm{d}x &= \frac{1}{2}\mathrm{d}u.
\end{aligned}
$$

$$
\begin{aligned}
\text{Substitute:} \quad \int \sqrt{2x+1}\,\mathrm{d}x &= \int \frac{1}{2}\sqrt{u}\,\mathrm{d}u \\
&= \frac{1}{\cancel{2}} \cdot \frac{u^{\frac{3}{2}}}{\frac{3}{\cancel{2}}} + C \\
&= \frac{1}{3}(2x+1)^{\frac{3}{2}} + C.
\end{aligned}
$$

Example (Substitution Rule)

$$\text{Find} \quad \int \frac{x}{\sqrt{3-4x^2}} \mathrm{d}x.$$

$$\begin{aligned}
\text{Let} \quad u &= 3 - 4x^2. \\
\text{Then} \quad \mathrm{d}u &= -8x\mathrm{d}x \\
x\mathrm{d}x &= -\frac{1}{8}\mathrm{d}u. \\
\text{Substitute:} \quad \int \frac{x}{\sqrt{3-4x^2}} \mathrm{d}x &= \int \left(-\frac{1}{8}\right) \frac{1}{\sqrt{u}} \mathrm{d}u \\
&= -\frac{1}{8} \cdot \frac{u^{\frac{1}{2}}}{\frac{1}{2}} + C \\
&= -\frac{1}{4}\sqrt{3-4x^2} + C.
\end{aligned}$$

Example (Substitution Rule)

$$\text{Find} \quad \int e^{3x} \mathrm{d}x.$$

$$\begin{aligned} \text{Let} \quad u &= 3x. \\ \text{Then} \quad \mathrm{d}u &= 3\mathrm{d}x \\ \mathrm{d}x &= \frac{1}{3}\mathrm{d}u. \\ \text{Substitute:} \quad \int e^{3x}\mathrm{d}x &= \int \frac{1}{3} e^{u} \mathrm{d}u \\ &= \frac{1}{3}e^{u} + C \\ &= \frac{1}{3}e^{3x} + C. \end{aligned}$$

Example (Substitution Rule, more factors)

$$\begin{aligned}
\text{Evaluate } \int 3x^5\sqrt{1+x^3}\mathrm{d}x &= \int 3x^2x^3\sqrt{1+x^3}\mathrm{d}x. \\
\text{Let } u &= 1+x^3. \\
\text{Then } \mathrm{d}u &= 3x^2\mathrm{d}x. \\
x^3 &= u-1. \\
\int 3x^2x^3\sqrt{1+x^3}\mathrm{d}x &= \int (u-1)\sqrt{u}\,\mathrm{d}u \\
&= \int \left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right)\mathrm{d}u \\
&= \left(\frac{u^{\frac{5}{2}}}{\frac{5}{2}} - \frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right) + C \\
&= \frac{2}{5}\left(1+x^3\right)^{\frac{5}{2}} - \frac{2}{3}\left(1+x^3\right)^{\frac{3}{2}} + C.
\end{aligned}$$

Definite Integrals

There are two ways to find a definite integral with the Substitution Rule:

1. First evaluate the indefinite integral, then use the FTC.

$$\int_0^4 \sqrt{2x+1}\,\mathrm{d}x = \left[\int \sqrt{2x+1}\,\mathrm{d}x\right]_0^4 = \left[\frac{1}{3}(2x+1)^{3/2}\right]_0^4$$
$$= \frac{1}{3}(2\cdot 4+1)^{3/2} - \frac{1}{3}(2\cdot 0+1)^{3/2}$$
$$= \frac{1}{3}(9)^{3/2} - \frac{1}{3}(1)^{3/2} = \frac{1}{3}(27-1) = \frac{26}{3}$$

2. Change the limits of integration when the variable is changed.

Theorem (The Substitution Rule for Definite Integrals)

If g' is continuous on $[a,b]$ and f is continuous on the range of g, then letting $u = g(x)$ we get

$$\int_a^b f(g(x))g'(x)\,\mathrm{d}x = \int_{g(a)}^{g(b)} f(u)\,\mathrm{d}u$$

Example

Find $\int_0^4 \sqrt{2x+1}\,\mathrm{d}x$.

- Let $u = 2x + 1$.
- Then $\mathrm{d}u = 2\mathrm{d}x$.
- Therefore $\mathrm{d}x = \frac{1}{2}\mathrm{d}u$.
- When $x = 0$, $u = 1$.
- When $x = 4$, $u = 9$.

$$\int\limits_{x=0}^{x=4} \sqrt{2x+1}\,\mathrm{d}x = \int\limits_{u=1}^{u=9} \frac{1}{2}\sqrt{u}\,\mathrm{d}u = \int_1^9 \frac{1}{2}u^{\frac{1}{2}}\mathrm{d}u$$

$$= \left[\frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{3}(u)^{\frac{3}{2}}\right]_1^9$$

$$= \frac{1}{3}(9)^{\frac{3}{2}} - \frac{1}{3}(1)^{\frac{3}{2}} = \frac{1}{3}(27-1) = \frac{26}{3}$$

Example

Find $\displaystyle\int_1^2 \frac{\mathrm{d}x}{(2-3x)^2}$.

- Let $u = 2 - 3x$.
- Then $\mathrm{d}u = \ -3\,\mathrm{d}x$.
- Therefore $\mathrm{d}x = -\frac{1}{3}\mathrm{d}u$.
- When $x = 1$, $u = -1$.
- When $x = 2$, $u = \ -4$.

$$\begin{aligned}\int_{x=1}^{x=2} \frac{\mathrm{d}x}{(2-3x)^2} &= \ -\frac{1}{3}\int_{u=-1}^{u=-4} \frac{\mathrm{d}u}{u^2} = -\frac{1}{3}\int_{-1}^{-4} u^{-2}\mathrm{d}u \\ &= -\frac{1}{3}\cdot\left[-\frac{1}{u}\right]_{-1}^{-4} = \frac{1}{3}\left[\frac{1}{u}\right]_{-1}^{-4} \\ &= \frac{1}{3}\left(\frac{1}{-4} - \frac{1}{-1}\right) = \frac{1}{3}\left(1 - \frac{1}{4}\right) = \frac{1}{4}.\end{aligned}$$