Calculus I
Homework
The Fundamental Theorem of Calculus, Part I

1. Differentiate $f(x)$ using the Fundamental Theorem of Calculus part 1.

(a) $f(x) = \int\limits_1^x \sin\left(t^2\right) \mathrm{d}t$

answer: $\sin\left(x^2\right)$

(b) $f(x) = \int_1^x \left(t - \sqrt{t}\right) \mathrm{d}t.$

answer: $x - \sqrt{x}$

(c) $f(x) = \int\limits_x^1 (2+t^4)^5 \ \mathrm{d}t$

answer: $-\left(2+x^4\right)^5$

(d) $f(x) = \int\limits_0^{x^2} t^2 \mathrm{d}t.$

answer: $f'(x) = 2x^5$

(e) $f(x) = \int\limits_{\ln x}^{e^x} t^3 \mathrm{d}t.$

answer: $f'(x) = e^{4x} - \frac{(\ln x)^3}{x}$

(f) $f(x) = \int_1^x \left(\sqrt{t} - \sqrt[3]{t}\right) \mathrm{d}t.$

answer: $\sqrt{x} - \sqrt[3]{x}$

(g) $f(x) = \int_1^{\frac{1}{x+1}} \sin\left(t^2\right) \mathrm{d}t.$

answer:

(h) $f(x) = \int_1^{\frac{1}{x+1}} \cos\left(t^2\right) \mathrm{d}t.$

answer: $-\frac{1}{(x+1)^2} \cos\left(\frac{1}{(x+1)^2}\right)$

(i) $f(x) = \int_0^{x^3} \cos^2 t \ \mathrm{d}t$

answer: $3x^2 \cos^2\left(x^3\right)$

Solution. 1.b

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_1^x \left(t - \sqrt{t}\right) \mathrm{d}t\right) \quad = \quad x - \sqrt{x}. \quad \Big| \text{ FTC, part 1}$$

Solution. 1.c We recall that the Fundamental Theorem of Calculus part 1 states that $\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_a^x h(t)dt\right) = h(x)$ where a is a constant. We can rewrite the integral so it has x as the upper limit:

$$f(x) = \int_x^1 (2+1^4)^5 \mathrm{d}t = -\int_1^x (2+1^4)^5 \mathrm{d}t \quad .$$

Therefore

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(-\int_1^x (2+t^4)^5 \mathrm{d}t\right) = -\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_1^x (2+t^4)^5 \mathrm{d}t\right) \overset{\text{FTC part 1}}{=} -(2+x^4)^5 \quad .$$

Solution. 1.e

$$\begin{aligned} f'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\int\limits_{\ln x}^{e^x} t^3 \mathrm{d}t\right) \\ &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\int\limits_{\ln x}^{0} t^3 \mathrm{d}t + \int\limits_0^{e^x} t^3 \mathrm{d}t\right) \\ &= \frac{\mathrm{d}}{\mathrm{d}x}\left(-\int\limits_0^{\ln x} t^3 \mathrm{d}t + \int\limits_0^{e^x} t^3 \mathrm{d}t\right). \end{aligned}$$

The Fundamental Theorem of Calculus part I states that for an arbitrary constant a, $\dfrac{\mathrm{d}}{\mathrm{d}u}\left(\int_a^u g(t)\mathrm{d}t\right) = g(u)$ (for a continuous g). We use this two compute the two derivatives:

$$
\begin{array}{rcll}
\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\displaystyle\int\limits_0^{\ln x} t^3\mathrm{d}t\right) & = & \dfrac{\mathrm{d}}{\mathrm{d}x}\left(\displaystyle\int\limits_0^{u} t^3\mathrm{d}t\right) & \Bigg| \text{ Set } u=\ln x \\
& = & u^3\cdot\dfrac{\mathrm{d}u}{\mathrm{d}x} & \\
& = & \dfrac{(\ln x)^3}{x} & \\
\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\displaystyle\int\limits_0^{e^x} t^3\mathrm{d}t\right) & = & \dfrac{\mathrm{d}}{\mathrm{d}x}\left(\displaystyle\int\limits_0^{w} t^3\mathrm{d}t\right) & \Bigg| \text{ Set } w=e^x \\
& = & w^3\cdot\dfrac{\mathrm{d}w}{\mathrm{d}x} & \\
& = & e^{3x}e^x = e^{4x}. &
\end{array}
$$

Finally, we combine the above computations to a single answer.

$$f'(x) = e^{4x} - \frac{(\ln x)^3}{x}.$$

Solution. 1.f

$$
\frac{\mathrm{d}}{\mathrm{d}x}\int_1^x\left(\sqrt{t}-\sqrt[3]{t}\right)\mathrm{d}t \quad = \quad \sqrt{x}-\sqrt[3]{x} \qquad \Big| \text{ FTC part I}
$$

Solution. 1.g

$$
\begin{array}{rcll}
\dfrac{\mathrm{d}}{\mathrm{d}x}\displaystyle\int_1^{\frac{1}{x+1}}\sin\left(t^2\right)\mathrm{d}t & = & \dfrac{\mathrm{d}}{\mathrm{d}x}\displaystyle\int_1^u \sin(t^2)\mathrm{d}t & \Big| \; u=\frac{1}{x+1}, \text{ use FTC part I, chain rule} \\
& = & \sin\left(u^2\right)\dfrac{\mathrm{d}u}{\mathrm{d}x} & \\
& = & \sin\left(\dfrac{1}{(x+1)^2}\right)\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{1}{x+1}\right) & \\
& = & \sin\left(\dfrac{1}{(x+1)^2}\right)\left(-\dfrac{1}{(x+1)^2}\right) & \\
& = & -\dfrac{1}{(x+1)^2}\sin\left(\dfrac{1}{(x+1)^2}\right) &
\end{array}
$$

Solution. 1.h

$$
\begin{array}{rcll}
\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\displaystyle\int_1^{\frac{1}{x+1}}\cos\left(t^2\right)\mathrm{d}t\right) & = & \dfrac{\mathrm{d}}{\mathrm{d}x}\left(\displaystyle\int_1^u \cos\left(t^2\right)\mathrm{d}t\right) & \Big| \text{ Set } \frac{1}{x+1}=u \\
& = & \cos\left(u^2\right)\dfrac{\mathrm{d}}{\mathrm{d}x}(u) & \Big| \text{ FTC part I and Chain Rule} \\
& = & -\dfrac{1}{(x+1)^2}\cos\left(\dfrac{1}{(x+1)^2}\right) &
\end{array}
$$