

Calculus II

Homework

Integration by parts

1. Let $x \in (0, 1)$. Express the following using x and $\sqrt{1 - x^2}$.

(a) $\sin(\arcsin(x))$.

(e) $\sin(2 \arccos(x))$.

(b) $\sin(2 \arcsin(x))$.

(f) $\sin(3 \arccos(x))$.

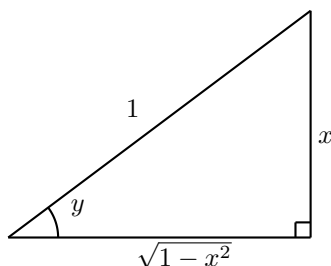
(c) $\sin(3 \arcsin(x))$.

(g) $\cos(2 \arcsin(x))$.

(d) $\sin(\arccos(x))$.

(h) $\cos(3 \arccos(x))$.

Solution. 1.b. Let $y = \arcsin x$. Then $\sin y = x$, and we can draw a right triangle with opposite side length x and hypotenuse length 1 to find the other trigonometric ratios of y .



Then $\cos y = \frac{\sqrt{1-x^2}}{1} = \sqrt{1 - x^2}$. Now we use the double angle formula to find $\sin(2 \arcsin x)$.

$$\begin{aligned} \sin(2 \arcsin x) &= \sin(2y) \\ &= 2 \sin y \cos y \\ &= 2x \sqrt{1 - x^2}. \end{aligned}$$

Solution. 1.c. Use the result of Problem 1.b. This also requires the addition formula for sine:

$$\sin(A + B) = \sin A \cos B + \sin B \cos A,$$

and the double angle formula for cosine:

$$\cos(2y) = \cos^2 y - \sin^2 y.$$

$$\begin{aligned}
\sin(3 \arcsin x) &= \sin(3y) \\
&= \sin(2y + y) \\
&= \sin(2y) \cos y + \sin y \cos(2y) && \left| \begin{array}{l} \text{Use addition formula} \\ \text{Use double angle formulas} \end{array} \right. \\
&= (2 \sin y \cos y) \cos y + \sin y (\cos^2 y - \sin^2 y) \\
&= 2 \sin y \cos^2 y + \sin y \cos^2 y - \sin^3 y \\
&= 3 \sin y \cos^2 y - \sin^3 y \\
&= 3 \sin y (1 - \sin^2 y) - \sin^3 y \\
&= 3x(1 - x^2) - x^3 \\
&= 3x - 4x^3.
\end{aligned}$$

The solution is complete. A careful look at the solution above reveals a strategy useful for problems similar to this one.

- Identify the inverse trigonometric expression- $\arcsin x, \arccos x, \arctan x, \dots$. In the present problem that was $y = \arcsin x$.
- The problem is therefore a trigonometric function of y .
- Using trig identities and algebra, rewrite the problem as a trigonometric expression involving only the trig function that transforms y to x . In the present problem we rewrote everything using $\sin y$.
- Use the fact that $\sin(\arcsin x) = x, \cos(\arccos x) = x, \dots$, etc. to simplify.

Solution. 1.f We use the same strategy outlined in the end of the solution of Problem 1.c. Set $y = \arccos x$ and so $\cos(y) = x$. Therefore:

$$\begin{aligned}
\sin(3y) &= \sin(2y + y) \\
&= \sin(2y) \cos y + \sin y \cos(2y) \\
&= 2 \sin y \cos y \cos y + \sin y (2 \cos^2 y - 1) \\
&= 2 \sin y \cos^2 y + \sin y (2 \cos^2 y - 1) \\
&= \sin y (4 \cos^2 y - 1) && \left| \begin{array}{l} \text{use } \cos y = x \\ \sin y = \sqrt{1 - x^2} \end{array} \right. \\
&= \sqrt{1 - x^2} (4x^2 - 1).
\end{aligned}$$

2. Express as the following as an algebraic expression of x . In other words, “get rid” of the trigonometric and inverse trigonometric expressions.

(a) $\cos^2(\arctan x)$.

(b) $-\sin^2(\operatorname{arccot} x)$.

(c) $\frac{1}{\cos(\arcsin x)}$.

(d) $-\frac{1}{\sin(\arccos x)}$.

Solution. 2.b. We follow the strategy outlined in the end of the solution of Problem 1.c. We set $y = \operatorname{arccot} x$. Then we need to express $-\sin^2 y$ via $\cot y$. That is a matter of algebra:

$$\begin{aligned}
-\sin^2(\operatorname{arccot} x) &= -\sin^2 y && \left| \begin{array}{l} \text{Set } y = \operatorname{arccot} x \\ \text{use } \sin^2 y + \cos^2 y = 1 \end{array} \right. \\
&= -\frac{\sin^2 y}{\sin^2 y + \cos^2 y} \\
&= -\frac{1}{\frac{\sin^2 y + \cos^2 y}{\sin^2 y}} \\
&= -\frac{1}{1 + \cot^2 y} && \left| \begin{array}{l} \text{Substitute back } \cot y = x \end{array} \right. \\
&= -\frac{1}{1 + x^2}.
\end{aligned}$$

3. Rewrite as a rational function of t . This problem will be later used to derive the Euler substitutions (an important technique for integrating).

(a) $\cos(2 \arctan t)$.

(g) $\cos(2 \operatorname{arccot} t)$.

(b) $\sin(2 \arctan t)$.

(h) $\sin(2 \operatorname{arccot} t)$.

(c) $\tan(2 \arctan t)$.

(i) $\tan(2 \operatorname{arccot} t)$.

(d) $\cot(2 \arctan t)$.

(j) $\cot(2 \operatorname{arccot} t)$.

(e) $\csc(2 \arctan t)$.

(k) $\csc(2 \operatorname{arccot} t)$.

(f) $\sec(2 \arctan t)$.

(l) $\sec(2 \operatorname{arccot} t)$.

Solution. 3.a Set $z = \arctan t$, and so $\tan z = t$. Then

$$\begin{aligned} \cos(2 \arctan t) &= \cos(2z) \\ &= \frac{\cos(2z)}{1} \\ &= \frac{\cos^2 z - \sin^2 z}{\cos^2 z + \sin^2 z} \\ &= \frac{(\cos^2 z - \sin^2 z) \frac{1}{\cos^2 z}}{(\sin^2 z + \cos^2 z) \frac{1}{\cos^2 z}} \\ &= \frac{1 - \tan^2 z}{1 + \tan^2 z} \\ &= \frac{1 - t^2}{1 + t^2} \end{aligned}$$

use double angle formulas
and $1 = \sin^2 z + \cos^2 z$
divide top and bottom by $\cos^2 z$

Solution. 3.d Set $z = \arctan t$, and so $\tan z = t$. Then

$$\begin{aligned} \cot(2 \arctan t) &= \cot(2z) \\ &= \frac{\cos(2z)}{\sin(2z)} \\ &= \frac{\cos^2 z - \sin^2 z}{2 \sin z \cos z} \\ &= \frac{1 - \tan^2 z}{2 \tan z} \\ &= \frac{1 - t^2}{2t} \end{aligned}$$

use double angle formulas

4. Compute the derivative (derive the formula).

(a) $(\arctan x)'$.

(d) $(\arccos x)'$.

(b) $(\operatorname{arccot} x)'$.

(e) Let arcsec denote the inverse of the secant function. Compute $(\operatorname{arcsec} x)'$.

(c) $(\arcsin x)'$.

5. (a) Let $a + b \neq k\pi$, $a \neq k\pi + \frac{\pi}{2}$ and $b \neq k\pi + \frac{\pi}{2}$ for any $k \in \mathbb{Z}$ (integers). Prove that

$$\frac{\tan a + \tan b}{1 - \tan a \tan b} = \tan(a + b) \quad .$$

(b) Let x and y be real. Prove that, for $xy \neq 1$, we have

$$\arctan x + \arctan y = \arctan \left(\frac{x + y}{1 - xy} \right)$$

if the left hand side lies between $(-\frac{\pi}{2}, \frac{\pi}{2})$.

Solution. 5.a We start by recalling the formulas

$$\begin{aligned}\cos(a+b) &= \cos a \cos b - \sin a \sin b \\ \sin(a+b) &= \sin a \cos b + \sin b \cos a\end{aligned}$$

These formulas have been previously studied; alternatively they follow from Euler's formula and the computation

$$\begin{aligned}\cos(a+b) + i \sin(a+b) &= e^{i(a+b)} = e^{ia} e^{ib} = (\cos a + i \sin a)(\cos b + i \sin b) \\ &= \cos a \cos b - \sin a \sin b + i(\sin a \cos b + \sin b \cos a)\end{aligned}$$

Now 5.a is done via a straightforward computation:

$$\begin{aligned}\tan(a+b) &= \frac{\sin(a+b)}{\cos(a+b)} = \frac{\sin a \cos b + \sin b \cos a}{\cos a \cos b - \sin a \sin b} = \frac{(\sin a \cos b + \sin b \cos a) \frac{1}{\cos a \cos b}}{(\cos a \cos b - \sin a \sin b) \frac{1}{\cos a \cos b}} \\ &= \frac{\tan a + \tan b}{1 - \tan a \tan b}\end{aligned}\quad (1)$$

5.b is a consequence of 5.a. Let $a = \arctan x$, $b = \arctan y$. Then (??) becomes

$$\tan(\arctan x + \arctan y) = \frac{\tan(\arctan x) + \tan(\arctan y)}{1 - \tan(\arctan x) \tan(\arctan y)} = \frac{x + y}{1 - xy},$$

where we use the fact that $\tan(\arctan w) = w$ for all w . We recall that $\arctan(\tan z) = z$ whenever $z \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Now take \arctan on both sides of the above equality to obtain

$$\arctan x + \arctan y = \arctan\left(\frac{x + y}{1 - xy}\right).$$

6. Evaluate the indefinite integral. Illustrate the steps of your solutions.

(a) $\int x \sin x dx.$

ANSWER: $-x \cos x + \sin x + C$

(f) $\int x^2 e^{-2x} dx.$

ANSWER: $-\frac{x^2}{2} e^{-2x} - \frac{x}{2} e^{-2x} - \frac{1}{4} e^{-2x} + C$

(b) $\int x e^{-x} dx.$

ANSWER: $-(x+1)e^{-x} + C$

(g) $\int x \sin(2x) dx.$

ANSWER: $-\frac{x}{2} \cos(2x) + \frac{1}{4} \sin(2x) + C$

(c) $\int x^2 e^x dx.$

ANSWER: $x^2 e^x - 2x e^x + 2e^x + C$

(h) $\int x \cos(3x) dx.$

ANSWER: $\frac{x}{3} \sin(3x) + \frac{1}{9} \cos(3x) + C$

(d) $\int x \sin(-2x) dx.$

ANSWER: $-\frac{x}{2} \cos(-2x) + \frac{1}{4} \sin(-2x) + C$

(i) $\int x^2 e^{2x} dx.$

ANSWER: $\frac{x^2}{2} e^{2x} - \frac{x}{2} e^{2x} + \frac{1}{4} e^{2x} + C$

(e) $\int x^2 \cos(3x) dx.$

ANSWER: $\frac{x}{3} \sin(3x) + \frac{1}{9} \cos(3x) - \frac{x^2}{2} \cos(3x) + C$

(j) $\int x^3 e^x dx.$

ANSWER: $x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x + C$

Solution. 6.a.

$$\int x \underbrace{\sin x dx}_{=d(-\cos x)} = -\int x d(\cos x) = -x \cos x + \int \cos x dx = -x \cos x + \sin x + C.$$

Solution. 6.c.

$$\begin{aligned}\int x^2 \underbrace{e^x dx}_{d(e^x)} &= \int x^2 de^x = x^2 e^x - \int e^x 2x dx = x^2 e^x - \int 2x de^x \\ &= x^2 e^x - 2x e^x + \int 2e^x dx = x^2 e^x - 2x e^x + 2e^x + C.\end{aligned}$$

Solution. 6.f.

$$\begin{aligned}
 \int x^2 e^{-2x} dx &= \int x^2 d\left(\frac{e^{-2x}}{-2}\right) && \left| \begin{array}{l} \text{Integrate by parts} \end{array} \right. \\
 &= -\frac{x^2 e^{-2x}}{2} - \int \left(\frac{e^{-2x}}{-2}\right) d(x^2) \\
 &= -\frac{x^2 e^{-2x}}{2} + \int x e^{-2x} dx \\
 &= -\frac{x^2 e^{-2x}}{2} + \int x d\left(\frac{e^{-2x}}{-2}\right) && \left| \begin{array}{l} \text{Integrate by parts} \end{array} \right. \\
 &= -\frac{x^2 e^{-2x}}{2} - \frac{x e^{-2x}}{2} + \frac{1}{2} \int e^{-2x} dx \\
 &= -\frac{x^2 e^{-2x}}{2} - \frac{x e^{-2x}}{2} - \frac{e^{-2x}}{4} + C.
 \end{aligned}$$

7. Evaluate the indefinite integral. Illustrate the steps of your solutions.

(a) $\int x^2 \cos(2x) dx.$

ANSWER: $\frac{1}{2} x^2 \sin(2x) - (x \cos(2x)) \frac{1}{2} + \frac{1}{4} \sin(2x) + C$

(l) $\int (\arcsin x) dx.$

ANSWER: $x \arcsin x - \sqrt{1-x^2} + C$

(b) $\int x^2 e^{ax} dx$, where a is a constant.

ANSWER: $\frac{1}{a^3} x^2 e^{ax} - \frac{2}{a^2} x e^{ax} + \frac{2}{a} e^{ax} + C$

(m) $\int (\arcsin x)^2 dx.$ (Hint: Try substituting $x = \sin y$.)

ANSWER: $\frac{1}{2} x^2 \arcsin x - \frac{1}{2} \sqrt{1-x^2} + 2x - 2 + C$

(c) $\int x^2 e^{-ax} dx$, where a is a constant.

ANSWER: $-\frac{1}{a^3} x^2 e^{-ax} - \frac{2}{a^2} x e^{-ax} - \frac{2}{a} e^{-ax} + C$

(n) $\int \arctan\left(\frac{1}{x}\right) dx.$

(o) $\int \sin x e^x dx$

ANSWER: $\frac{1}{2} (e^x \sin x - x \cos x + \cos x) + C$

(d) $\int x^2 \frac{(e^{ax} + e^{-ax})^2}{4} dx$, where a is a constant.

ANSWER: $\frac{8}{15} x^3 e^{ax} - \frac{8}{15} x^3 e^{-ax} - \frac{2}{a^2} x e^{ax} + \frac{2}{a^2} x e^{-ax} - \frac{2}{a^3} e^{ax} + \frac{2}{a^3} e^{-ax} + C$

(p) $\int \cos x e^x dx$

ANSWER: $\frac{1}{2} (e^x \sin x + x \cos x + \cos x) + C$

(e) $\int \frac{1}{\cos^2 x} dx.$ (Hint: This problem does not require integration by parts. What is the derivative of $\tan x$?)

ANSWER: $\tan x + C$

(q) $\int \sin(\ln(x)) dx.$

ANSWER: $\frac{2}{\pi} (\sin(\ln(x)) - \cos(\ln(x))) + C$

(f) $\int (\tan^2 x) dx.$ (Hint: This problem does not require integration by parts. We can use $\tan^2 x = \frac{1}{\cos^2 x} - 1$ and the previous problem.)

ANSWER: $\frac{2}{\pi} (\cos(\ln(x)) + \sin(\ln(x))) + C$

(g) $\int x \tan^2 x dx.$ (Hint: $\tan^2 x dx = d(F(x))$, where $F(x)$ is the answer from the preceding problem.)

ANSWER: $\frac{1}{2} x \tan x + x \ln |\cos x| - \frac{1}{2} \cos^2 x + C$

(r) $\int \cos(\ln(x)) dx.$

(s) $\int \ln x dx$

ANSWER: $x \ln |x| - x + C$

(t) $\int x \ln x dx.$

ANSWER: $\frac{1}{2} x^2 \ln |x| - \frac{1}{4} x^2 + C$

(h) $\int e^{-\sqrt{x}} dx.$

ANSWER: $2\sqrt{x} e^{-\sqrt{x}} - 2e^{-\sqrt{x}} + C$

(u) $\int \frac{\ln x}{\sqrt{x}} dx.$

ANSWER: $2\sqrt{x} \ln(x) - \sqrt{x} + C$

(i) $\int \cos^2 x dx.$

ANSWER: $\frac{1}{2} x + \frac{1}{4} \sin(2x) + C$

(v) $\int (\ln x)^2 dx.$

ANSWER: $x^2 \ln x - 2x \ln x + 2x - x^2 + C$

(j) $\int \frac{x}{1+x^2} dx$ (Hint: use substitution rule, don't use integration by parts)

ANSWER: $\frac{1}{2} \ln(1+x^2) + C$

(w) $\int (\ln x)^3 dx.$

ANSWER: $x^3 \ln(x) - 3x^2 \ln x + 6x - 9 + C$

(k) $\int (\arctan x) dx.$

ANSWER: $\frac{1}{2} x^2 \arctan x - \frac{1}{2} \ln(1+x^2) + C$

(x) $\int x^2 \cos^2 x dx.$ (This problem is related to Problem 7.d as $\cos x = \frac{e^{ix} + e^{-ix}}{2}$).

ANSWER: $\frac{1}{4} x^2 \sin(2x) + \frac{1}{4} x^2 \cos(2x) - \frac{1}{2} x \sin(2x) + \frac{1}{4} \cos(2x) + C$

Solution. 7.g.

$$\begin{aligned}
 \int x \tan^2 x dx &= \int x (\sec^2 x - 1) dx && \left| \text{use } \sec^2 x - 1 = \tan^2 x \right. \\
 &= \int x (\sec^2 x - 1) dx \\
 &= -\int x dx + \int x \sec^2 x dx && \left| \text{use } d(\tan x) = \sec^2 x dx \right. \\
 &= -\frac{x^2}{2} + \int x d(\tan x) && \left| \text{integrate by parts} \right. \\
 &= -\frac{x^2}{2} + x \tan x - \int \tan x dx \\
 &= -\frac{x^2}{2} + x \tan x - \int \frac{\sin x}{\cos x} dx && \left| \text{use } \sin x dx = -d(\cos x) \right. \\
 &= -\frac{x^2}{2} + x \tan x + \int \frac{d(\cos x)}{\cos x} && \left| \text{Set } y = \cos x \right. \\
 &= -\frac{x^2}{2} + x \tan x + \int \frac{1}{y} dy \\
 &= -\frac{x^2}{2} + x \tan x + \ln |y| + C && \left| \text{Substitute back } y = \cos x \right. \\
 &= -\frac{x^2}{2} + x \tan x + \ln |\cos x| + C \quad .
 \end{aligned}$$

Solution. 7.h.

$$\begin{aligned}
 \int e^{-\sqrt{x}} dx &= \int 2ye^{-y} dy && \left| \begin{array}{l} \sqrt{x} = y \\ \text{Subst.: } \frac{1}{2\sqrt{x}} dx = dy \\ dx = 2y dy \end{array} \right. \\
 &= \int 2y d(-e^{-y}) && \left| \text{int. by parts} \right. \\
 &= -2ye^{-y} + 2 \int e^{-y} dy \\
 &= -2ye^{-y} - 2e^{-y} + C \\
 &= -2\sqrt{x}e^{-\sqrt{x}} - 2e^{-\sqrt{x}} + C \quad .
 \end{aligned}$$

Solution. 7.i. Later, we shall study general methods for solving trigonometric integrals that will cover this example. Let us however show one way to solve this integral by integration by parts.

$$\begin{aligned}
 \int \cos^2 x dx &= x \cos^2 x - \int x d(\cos^2 x) \\
 &= x \cos^2 x - \int x 2 \cos x (-\sin x) dx && \left| \sin(2x) = 2 \sin x \cos x \right. \\
 &= x \cos^2 x + \int x \sin(2x) dx \\
 &= x \cos^2 x + \int x d\left(\frac{-\cos(2x)}{2}\right) \\
 &= x \cos^2 x + x \left(\frac{-\cos(2x)}{2}\right) - \int \left(\frac{-\cos(2x)}{2}\right) dx \\
 &= \frac{x}{2} (2 \cos^2 x - \cos(2x)) + \frac{\sin(2x)}{4} + C && \left| \cos(2x) = \cos^2 x - \sin^2 x \right. \\
 &= \frac{x}{2} (2 \cos^2 x - (\cos^2 x - \sin^2 x)) + \frac{\sin(2x)}{4} + C && \left| \cos^2 x + \sin^2 x = 1 \right. \\
 &= \frac{x}{2} + \frac{\sin(2x)}{4} + C \quad .
 \end{aligned}$$

Solution. 7.k

$$\begin{aligned}
 \int \arctan x dx &= x \arctan x - \int x d(\arctan x) \\
 &= x \arctan x - \int \frac{x}{x^2 + 1} dx \\
 &= x \arctan x - \int \frac{\frac{1}{2} d(x^2)}{x^2 + 1} \\
 &= x \arctan x - \int \frac{\frac{1}{2} d(x^2 + 1)}{x^2 + 1} \\
 &= x \arctan x - \frac{1}{2} \ln(x^2 + 1) + C \quad .
 \end{aligned}$$

Solution. 7.m.

$$\begin{aligned}
 \int (\arcsin x)^2 dx &= \int (\arcsin(\sin y))^2 d(\sin y) && \left| \begin{array}{l} \text{Set } x = \sin y \\ \text{Integrate by parts} \end{array} \right. \\
 &= \int y^2 \cos y dy = \int y^2 d(\sin y) \\
 &= y^2 \sin y - \int 2y \sin y dy \\
 &= y^2 \sin y + \int 2y d(\cos y) && \left| \begin{array}{l} \text{Integrate by parts} \end{array} \right. \\
 &= y^2 \sin y + 2y \cos y - 2 \int \cos y dy \\
 &= y^2 \sin y + 2y \cos y - 2 \sin y + C \\
 &= \frac{x(\arcsin x)^2}{1} && \left| \begin{array}{l} \text{Substitute } y = \arcsin x \end{array} \right. \\
 &\quad + 2\sqrt{1-x^2} \arcsin x - 2x + C \quad .
 \end{aligned}$$

Solution. 7.o

$$\begin{aligned}
 \int \sin x \underbrace{e^x dx}_{=de^x} &= \sin x e^x - \int e^x d(\sin x) = \sin x e^x - \int \cos x \underbrace{e^x dx}_{=de^x} \\
 &= \sin x e^x - e^x \cos x + \int e^x d(\cos x) \\
 &= e^x \sin x - e^x \cos x - \int e^x \sin x dx && \left| \begin{array}{l} \text{add } \int e^x \sin x dx \\ \text{to both sides} \end{array} \right. \\
 2 \int \sin x e^x dx &= \sin x e^x - e^x \cos x \\
 \int \sin x e^x dx &= \frac{1}{2} (\sin x e^x - e^x \cos x) \quad .
 \end{aligned}$$

Solution. 7.q.

$$\begin{aligned}
 \int \sin(\ln x) dx &= x \sin(\ln x) - \int x d(\sin(\ln x)) && \left| \begin{array}{l} \text{int. by parts} \end{array} \right. \\
 &= x \sin(\ln x) - \int x (\cos(\ln x)) (\ln x)' dx \\
 &= x \sin(\ln x) - \int \cos(\ln x) dx && \left| \begin{array}{l} \text{int. by parts} \end{array} \right. \\
 &= x \sin(\ln x) - \left(x \cos(\ln x) - \int x d(\cos(\ln x)) \right) \\
 &= x \sin(\ln x) - x \cos(\ln x) + \int x (-\sin(\ln x)) (\ln x)' dx \\
 &= x \sin(\ln x) - x \cos(\ln x) - \int \sin(\ln x) dx && \left| \begin{array}{l} \text{add } \int \sin(\ln x) dx \\ \text{to both sides} \end{array} \right. \\
 2 \int \sin(\ln x) dx &= x \sin(\ln x) - x \cos(\ln x) \\
 \int \sin(\ln x) dx &= \frac{x}{2} (\sin(\ln x) - \cos(\ln x)) \quad .
 \end{aligned}$$

Solution. 7.s

$$\int \ln x dx = x \ln x - \int x d(\ln x) = x \ln x - \int \frac{x}{x} dx = x \ln x - x + C \quad .$$

Solution. 7.u

$$\begin{aligned}\int \frac{\ln x}{\sqrt{x}} dx &= \int (\ln x) 2d(\sqrt{x}) && \left| \text{integrate by parts} \right. \\&= (\ln x) 2\sqrt{x} - \int 2\sqrt{x} d(\ln x) \\&= 2\sqrt{x} \ln x - 2 \int \frac{\sqrt{x}}{x} dx \\&= 2\sqrt{x} \ln x - 2 \int x^{-\frac{1}{2}} dx \\&= 2\sqrt{x} \ln x - 4\sqrt{x} + C \\&= 2\sqrt{x}(\ln x - 2) + C \quad .\end{aligned}$$

8. Compute $\int x^n e^x dx$, where n is a non-negative integer.

Solution. 8

$$\begin{aligned}\int x^n e^x dx &= \int x^n de^x \\&= x^n e^x - \int e^x dx^n \\&= x^n e^x - n \int x^{n-1} e^x dx \\&= x^n e^x - n \left(\int x^{n-1} de^x \right) \\&= x^n e^x - n \left(x^{n-1} e^x - \int (n-1) x^{n-2} e^x dx \right) \\&= x^n e^x - n x^{n-1} e^x + n(n-1) \int x^{n-2} e^x dx \\&= \dots (\text{continue above process}) \dots \\&= x^n e^x - n x^{n-1} e^x + n(n-1) x^{n-2} e^x + \dots \\&\quad + (-1)^k n(n-1)(n-2) \dots (n-k+1) x^{n-k} e^x \\&\quad + \dots + (-1)^n n! e^x + C \\&= C + \sum_{k=0}^n (-1)^k \frac{n!}{(n-k)!} x^{n-k} e^x \quad .\end{aligned}$$