Precalculus
Polynomial systems basics

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Outline

Systems of polynomial equations

Definition

A collection of one or more simultaneous polynomial equations in one or more variables is called a *polynomial system*.

- The definition includes usual one-variable polynomial equations such as:
$$x^2 + 2x - 3 = 0$$
- Typical polynomial systems have more than one variable/equation:
$$\left|\begin{array}{rcl} y^2 + xy - 4y - 2x + 4 &=& 0 \\ y^2x - 2yx - y - 2x + 4 &=& 0. \end{array}\right.$$
Here we have 2 variables (x, y), 2 equations.
- The number of variables and equations need not be equal:
$$\left|\begin{array}{rcl} x + y + z + w &=& 2 \\ y + z^2 &=& 1 \\ y + zw^2 &=& 1. \end{array}\right.$$
Here we have 4 variables (x, y, z, w), 3 equations.

Systems of polynomial equations

Definition

A collection of one or more simultaneous polynomial equations in one or more variables is called a *polynomial system*.

- Polynomial systems may have no solutions: $\left| \begin{array}{rcl} x & = & 0 \\ xy & = & 1. \end{array} \right.$

  The first equation implies $x = 0$, but then the left hand side of the second equation must equal 0.
- Polynomial systems may finitely many solutions: $\left| \begin{array}{rcl} x & = & 0 \\ y + x & = & 1. \end{array} \right.$

  The first equation implies $x = 0$, and then the second equation implies $y = 1$.
- Polynomial systems may have infinitely many solutions:

  $\left| \begin{array}{rcl} x & = & 0 \\ y + z & = & 1. \end{array} \right.$ If we set $x = 0$, $y = 1 - z$, we produce infinitely many solutions for every possible value of z.

Systems of polynomial equations

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- The branch of mathematics that studies exact solutions of polynomial systems is called Algebraic Geometry.
- The practical aspects of solving such systems are covered under the subject of Elimination Theory.
- Solving polynomial systems is an indispensable mathematical tool used in other branches of science and mathematics.
- Polynomial systems also have direct practical applications, for example kinematics - the configurations of a robotic arm can be parametrized with polynomials.

Systems of polynomial equations

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- Whether a system has finitely or infinitely many solutions and what are they can be computed with a computer algorithm.
- Algorithm: find so-called Gröbner basis (named after the Austrian W. Gröbner, 1899-1980) using the Buchberger algorithm(named after the Austrian B. Buchberger, 1942-).
- These algorithms are too advanced to cover here.
- Not well-suited for pen and paper computations: can get notoriously large and require computers/super-computers.
- A system doable by hand would typically be solved milliseconds on a modern computer.
- A system doable by hand would typically be solved easily using ad-hoc techniques.

Example

Solve the polynomial system. $\left|\begin{array}{rcl} x-4y &=& 5\\ y^2+xy &=& 10\end{array}\right.$

$$\begin{array}{rcll} x &=& 5+4y & \Big|\ \text{Solve for } x \text{ in first eq-n.}\\ y^2+xy &=& 10 & \Big|\ \text{Substitute } x \text{ away}\\ y^2+(5+4y)y &=& 10 &\\ y^2+5y+4y^2-10 &=& 0 &\\ 5y^2+5y-10 &=& 0 & \Big|\ \text{Divide by 5}\\ y^2+y-2 &=& 0 &\\ (y+2)(y-1) &=& 0 &\\ y=-2 &\text{or}& y=1 & \end{array}$$

$$\begin{array}{ll} x=5+4y & x=5+4y\\ \quad =5+4(-2)=-3 & \quad =5+4\cdot 1=9 \end{array}$$

Final answer: $x=-3, y=-2$ or $x=9, y=1$.

Example

Solve the polynomial system. $\left|\begin{array}{rcl} x-4y &=& 5\\ y^2+xy &=& 10\end{array}\right.$

Final answer: $x=-3, y=-2$ or $x=9, y=1$.

Check answer $x=-3, y=-2$:

$$\left|\begin{array}{rclcl} x-4y &=& (-3)-4(-2) &=& 5\\ y^2+xy &=& (-2)^2+(-3)(-2) &=& 10\end{array}\right.$$

Check answer $y=1, x=9$:

$$\left|\begin{array}{rclcl} x-4y &=& 9-4\cdot 1 &=& 5\\ y^2+xy &=& 1^2+9\cdot 1 &=& 10.\end{array}\right.$$

Example

The sum of two numbers x and y is 25 and the sum of their squares is 313. Given that $y \geq x$, find x and y.

$$
\begin{aligned}
x + y &= 25 && \vert \text{Solve for } y \\
y &= 25 - x \\
x^2 + y^2 &= 313 \\
x^2 + (25 - x)^2 &= 313 \\
x^2 + \left(25^2 - 2 \cdot 25 \cdot x + x^2\right) - 313 &= 0 && \left\vert (a-b)^2 = a^2 - 2ab + b^2 \right. \\
2x^2 - 50x + 625 - 313 &= 0 \\
2x^2 - 50x + 312 &= 0 && \vert \text{Divide by } 2 \\
x^2 - 25x + 156 &= 0
\end{aligned}
$$

$$
\begin{aligned}
x &= \frac{-(-25) \pm \sqrt{25^2 - 4 \cdot 1 \cdot 156}}{2 \cdot 1} \\
&= \frac{25 \pm \sqrt{625 - 624}}{2} \\
&= \frac{25 \pm 1}{2} = \begin{cases} \frac{25+1}{2} = 13 \\ \frac{25-1}{2} = 12 \end{cases}
\end{aligned}
$$

Example

The sum of two numbers x and y is 25 and the sum of their squares is 313. Given that $y \geq x$, find x and y.

$$\begin{aligned} x + y &= 25 \qquad |\text{Solve for } y \\ y &= 25 - x \\ x^2 + y^2 &= 313 \\ x^2 + (25-x)^2 &= 313 \\ x &= \frac{-(-25) \pm \sqrt{25^2 - 4 \cdot 1 \cdot 156}}{2 \cdot 1} \\ &= \frac{25 \pm \sqrt{625 - 624}}{2} \\ &= \frac{25 \pm 1}{2} = \begin{cases} \frac{25+1}{2} = 13 \\ \frac{25-1}{2} = 12 \end{cases} \\ y = 25 - x &= \begin{cases} 25 - 13 = 12 \\ 25 - 12 = 13 \end{cases} \end{aligned}$$

The two solution candidates are $x = 12, y = 13$ and $x = 13, y = 12$. Since $y \geq x$, one of the solutions needs to be discarded and our final answer is $x = 12, y = 13$.