Precalculus
, Factorization of polynomials: overview

Todor Milev

2019

Outline

Recall that $i^2 = -1$, $\sqrt{-1} = i$.

Example (Polynomial factorizations)

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\begin{aligned}
2x^2 + 3x - 5 &= \\
x^2 + 1 &= \\
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Every polynomial can be factored into product of linear terms

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Corollary

Every real polynomial can be factored into a product of real linear terms and real quadratic terms with no real roots, i.e., factors of form

- $(x - r)$, *where* r *is real and*
- $ax^2 + bx + c$ *with* $b^2 - 4ac < 0$ *where* a, b, c *are real.*

$$a_0x^n + a_1x^{n-1} + \dots + a_n = a_0(x - x_1)\dots(x - x_n)$$
$$= \text{prod. real quadratics no roots \& lin. terms.}$$

Example

$$2x^2 + 3x - 5 = (2x+5)(x-1) = 2\left(x - \left(-\tfrac{5}{2}\right)\right)(x-1)$$

$$x^2 + 1 = x^2 - (-1) = x^2 - i^2 = (x - (-i))(x - i)$$

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$$= \text{prod. real quadratics no roots \& lin. terms.}$$

Example

$$2x^2 + 3x - 5 = (2x+5)(x-1) = 2\left(x - \left(-\tfrac{5}{2}\right)\right)(x-1) \quad \text{real roots}$$

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- In theory every polynomial can be factored.

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- We study those for cubics with the aid of scientific calculator.