Precalculus
Trig cofunction identities and angle-sum formulas

Todor Milev

2019

Outline

License to use and redistribute

These lecture slides and their LaTeX source code are licensed to you under the Creative Commons license CC BY 3.0. You are free

- to Share - to copy, distribute and transmit the work,
- to Remix - to adapt, change, etc., the work,
- to make commercial use of the work,

as long as you reasonably acknowledge the original project.

- Latest version of the .tex sources of the slides:
  https://github.com/tmilev/freecalc
- Should the link be outdated/moved, search for "freecalc project".
- Creative Commons license CC BY 3.0:
  https://creativecommons.org/licenses/by/3.0/us/
  and the links therein.

Cofunction identities

Proposition (Cofunction identities)

$$
\begin{array}{rclcrcl}
\sin\left(\dfrac{\pi}{2} - \alpha\right) & = & \cos\alpha & & \sin\left(\dfrac{\pi}{2} + \alpha\right) & = & \cos\alpha \\
\cos\left(\dfrac{\pi}{2} - \alpha\right) & = & \sin\alpha & & \cos\left(\dfrac{\pi}{2} + \alpha\right) & = & -\sin\alpha
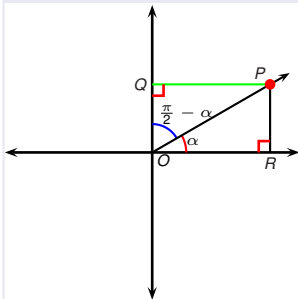\end{array}
$$

- The proof each formula is broken into 4 cases depending on which quadrant contains α.
- This makes a total of 4 formulas $\times 4$ cases per formula = 16 cases.
- We show only a few of the cases.
- The proof provides intuition why the formulas are true.
- The Quadrant I part of the proof serves as a visual aid for memorization.
- There is an algebraically simpler (but theoretically advanced) way to prove the above identities through the angle sum f-las, derived in turn from Euler's formula (studied later/in another course).

Cofunction identities

Proposition (Cofunction identities)

$$\sin\left(\frac{\pi}{2} - \alpha\right) = \cos\alpha \quad \sin\left(\frac{\pi}{2} + \alpha\right) = \cos\alpha$$
$$\cos\left(\frac{\pi}{2} - \alpha\right) = \sin\alpha \quad \cos\left(\frac{\pi}{2} + \alpha\right) = -\sin\alpha$$

Part of Proof.

We are showing $\sin\left(\frac{\pi}{2} - \alpha\right) = \cos\alpha$ when α is in quadrant I.

$$\begin{aligned}
\sin\left(\frac{\pi}{2} - \alpha\right) &= \frac{|PQ|}{|OP|} \quad \Big| \, \square ORPQ \\
&= \frac{|OR|}{|OP|} \\
&= \cos\alpha \quad \big| \text{ as desired}
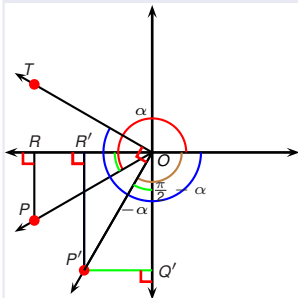\end{aligned}$$

□

Cofunction identities

Proposition (Cofunction identities)

$$
\begin{array}{rclcrcl}
\sin\left(\dfrac{\pi}{2} - \alpha\right) &=& \cos\alpha & \sin\left(\dfrac{\pi}{2} + \alpha\right) &=& \cos\alpha \\
\cos\left(\dfrac{\pi}{2} - \alpha\right) &=& \sin\alpha & \cos\left(\dfrac{\pi}{2} + \alpha\right) &=& -\sin\alpha
\end{array}
$$

Part of Proof.

We are showing $\sin\left(\frac{\pi}{2} - \alpha\right) = \cos\alpha$ when α is in Quadrant III. It follows $\frac{\pi}{2} - \alpha$ is in Quadrant III.

$$
\begin{aligned}
\sin\left(\frac{\pi}{2} - \alpha\right) &= -\frac{|P'R'|}{|OP'|} = -\frac{|OQ'|}{|OP'|} \quad \Big|\; \Box\, OR'P'Q' \\
&= -\frac{|OR|}{|OP|} \\
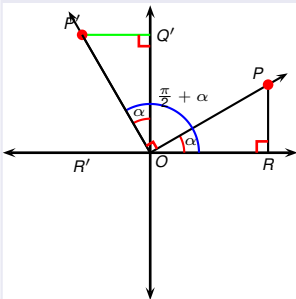&= \cos\alpha \qquad\qquad\qquad \Big|\; \text{as desired}
\end{aligned}
$$

$\square$

Cofunction identities

Proposition (Cofunction identities)

$$\sin\left(\frac{\pi}{2}-\alpha\right) = \cos\alpha \quad \sin\left(\frac{\pi}{2}+\alpha\right) = \cos\alpha$$
$$\cos\left(\frac{\pi}{2}-\alpha\right) = \sin\alpha \quad \cos\left(\frac{\pi}{2}+\alpha\right) = -\sin\alpha$$

Part of Proof.

We show $\cos\left(\frac{\pi}{2}+\alpha\right) = -\sin\alpha$ when α is in Quadrant I. It follows $\frac{\pi}{2}+\alpha$ is in Quadrant II.
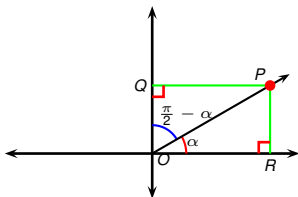
$$\begin{aligned}\cos\left(\frac{\pi}{2}+\alpha\right) &= -\frac{|OR'|}{|OP'|} \quad \Big| \ \square ORPQ \\ &= -\frac{|P'Q'|}{|OP'|} \\ &= -\frac{|PR|}{|OP|} \\ &= -\sin\alpha. \quad \Big| \text{ as desired}\end{aligned}$$

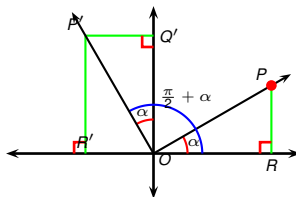$\square$

Cofunction identities

Proposition (Cofunction identities)

$$
\begin{array}{rclrcl}
\sin\left(\dfrac{\pi}{2} - \alpha\right) &=& \cos\alpha & \sin\left(\dfrac{\pi}{2} + \alpha\right) &=& \cos\alpha \\
\cos\left(\dfrac{\pi}{2} - \alpha\right) &=& \sin\alpha & \cos\left(\dfrac{\pi}{2} + \alpha\right) &=& -\sin\alpha
\end{array}
$$

To memorize the cofunction identities it suffices to memorize the Quadrant I case via the two diagrams below.

$$
\begin{array}{rcl}
\sin\left(\frac{\pi}{2} - \alpha\right) &=& \dfrac{|PQ|}{|OP|} \\
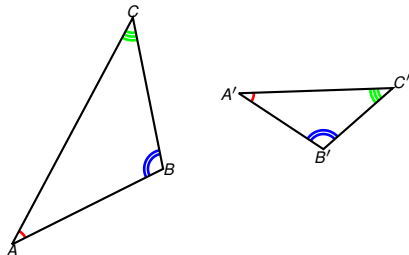\cos\left(\frac{\pi}{2} - \alpha\right) &=& \dfrac{|OQ|}{|OP|}
\end{array}
$$

$$
\begin{array}{rcl}
\sin\left(\frac{\pi}{2} + \alpha\right) &=& \dfrac{|OQ'|}{|OP|} \\
\cos\left(\frac{\pi}{2} + \alpha\right) &=& -\dfrac{|PQ'|}{|OP'|} = \dfrac{|PR|}{|OP|}
\end{array}
$$

Definition (Similar triangles)

We say that a triangle $\triangle ABC$ is similar to a triangle $\triangle A'B'C'$ if the two triangles have equal angles.

- The equal angles are assumed given in the same order for both triangles, that is, $\angle ABC = \angle A'B'C'$, $\angle BCA = \angle B'C'A'$, $\angle CAB = \angle C'A'B'$.
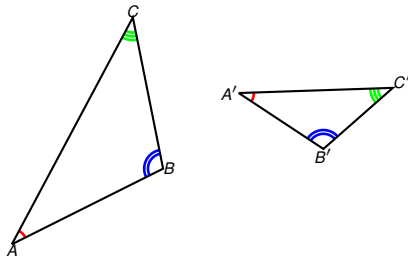
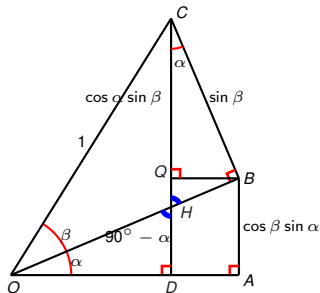The following statement is proved in the subject of Euclidean (planar) geometry.

Theorem (Similar triangles have equal side ratios)

Let $\triangle ABC$ and $\triangle A'B'C'$ be two similar triangles. Then the ratios of the lengths of the sides of the two triangles are equal, that is

$$\frac{|AB|}{|BC|} = \frac{|A'B'|}{|B'C'|} \qquad \frac{|BC|}{|CA|} = \frac{|B'C'|}{|C'A'|} \qquad \frac{|CA|}{|AB|} = \frac{|C'A'|}{|A'B'|}$$

$\sin(\alpha + \beta), \cos(\alpha + \beta)$ via $\sin \alpha, \sin \beta, \cos \alpha, \cos \beta$

$\sin(\alpha + \beta) = \dfrac{|CD|}{|OC|} = |CD|$

$= |QD| + |CQ|$

$= \sin \alpha \cos \beta + \cos \alpha \sin \beta$

$\cos(\alpha + \beta) = \dfrac{|OD|}{|OC|} = |OD|$

$= |OA| - |DA|$

$= \cos \alpha \cos \beta - \sin \alpha \sin \beta$

$\begin{aligned}
|QD| &= |BA| & &\square\,DABQ \\
&= \sin \alpha |OB| & &\triangle\,OAB \\
&= \sin \alpha \cos \beta |OC| & &\triangle\,OBC \\
&= \sin \alpha \cos \beta
\end{aligned}$

$\begin{aligned}
|CQ| &= \cos \alpha |CB| & &\triangle\,CQB \\
&= \cos \alpha \sin \beta |OC| & &\triangle\,OBC \\
&= \cos \alpha \sin \beta
\end{aligned}$

$\begin{aligned}
|OA| &= \cos \alpha |OB| & &\triangle\,OAB \\
&= \cos \alpha \cos \beta |OC| & &\triangle\,OBC \\
&= \cos \alpha \cos \beta
\end{aligned}$

$\begin{aligned}
|DA| &= |QB| & &\square\,DABQ \\
&= \sin \alpha |CB| & &\triangle\,CQB \\
&= \sin \alpha \sin \beta |OC| & &\triangle\,OBC \\
&= \sin \alpha \sin \beta
\end{aligned}$

Trig Functions of Sums and Differences of Angles

Theorem

$$
\begin{aligned}
\sin(\alpha + \beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \\
\sin(\alpha - \beta) &= \sin\alpha\cos\beta - \cos\alpha\sin\beta \\
\cos(\alpha + \beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\
\cos(\alpha - \beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta
\end{aligned}
$$

- We gave a geometric proof of the sum formulas when the two angles are acute and their sum is less than $\pi = 90°$.
- The theorem holds for all angles α, β without any restrictions.
- This can be shown by combining the preceding proof with identities such as $\cos\left(\frac{\pi}{2} - \alpha\right) = \sin\alpha$, $\cos\left(\frac{\pi}{2} + \alpha\right) = -\sin\alpha$.
- There is a theoretically more advanced (but algebraically simpler) proof using Euler's formula (to be studied later/in another course).
- The difference formulas are a consequence of the sum formulas and the fact that sin is an odd function and cos is even.

Trig Functions of Differences of Angles

Example

Prove the identities
$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$
$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$
from the (already demonstrated) identities
$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$
\begin{aligned}
\sin(\alpha - \beta) &= \sin(\alpha + (-\beta)) \\
&= \sin\alpha\cos(-\beta) + \cos\alpha\sin(-\beta) & &\left| \begin{array}{l} \cos \text{ is even ,} \\ \sin \text{ is odd} \end{array}\right. \\
&= \sin\alpha\cos\beta - \cos\alpha\sin\beta \\
\cos(\alpha - \beta) &= \cos(\alpha + (-\beta)) \\
&= \cos\alpha\cos(-\beta) - \sin\alpha\sin(-\beta) & &\left| \begin{array}{l} \cos \text{ is even ,} \\ \sin \text{ is odd} \end{array}\right. \\
&= \cos\alpha\cos\beta + \cos\alpha\sin\beta
\end{aligned}
$$

Example

Find the exact value of the trigonometric function using radicals.

$$\cos(105°) = \cos(45° + 60°)$$

we know the trig
f-ns of 45° and 60°

$$= \cos(45°)\cos(60°) - \sin(45°)\sin(60°)$$

Angle sum f-la

$$= \frac{\sqrt{2}}{2} \cdot \frac{1}{2} - \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2}$$

$$= \frac{\sqrt{2} - \sqrt{6}}{4}.$$
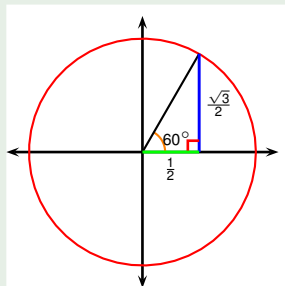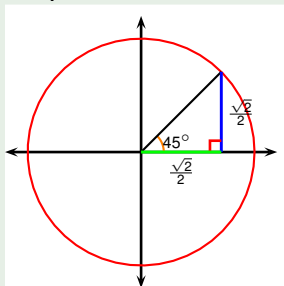
Example

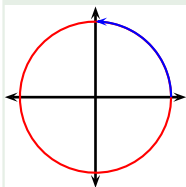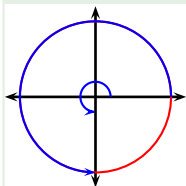Use the angle sum/difference formulas to simplify.

$$
\begin{aligned}
\cos\left(\frac{\pi}{2} - x\right) &= \cos\left(\frac{\pi}{2}\right)\cos x + \sin\left(\frac{\pi}{2}\right)\sin x \\
&= 0 \cdot \cos(x) + 1 \cdot \sin x \\
&= \sin x
\end{aligned}
$$

Example

Use the angle sum/difference formulas to simplify.

$$
\begin{aligned}
\cot\left(\frac{3\pi}{2} + x\right) &= \frac{\cos\left(\frac{3\pi}{2} + x\right)}{\sin\left(\frac{3\pi}{2} + x\right)} \\
&= \frac{\cos\left(\frac{3\pi}{2}\right)\cos x - \sin\left(\frac{3\pi}{2}\right)\sin x}{\sin\left(\frac{3\pi}{2}\right)\cos x + \cos\left(\frac{3\pi}{2}\right)\sin x} \\
&= \frac{0 \cdot \cos x - (-1)\sin x}{(-1)\cos x + 0 \cdot \sin x} \\
&= \frac{\sin x}{-\cos x} = -\frac{\sin x}{\cos x} \\
&= -\tan x
\end{aligned}
$$

Example

Show that $\tan(\pi + x) = \tan x$ using the angle sum formulas.

$$
\begin{aligned}
\tan(\pi + x) &= \frac{\sin(\pi + x)}{\cos(\pi + x)} \\
&= \frac{\sin \pi \cos x + \cos \pi \sin x}{\cos \pi \cos x - \sin \pi \sin x} \\
&= \frac{0 \cdot \cos x + (-1) \cdot \sin x}{(-1) \cdot \cos x - 0 \cdot \sin x} \\
&= \frac{-\sin x}{-\cos x} \\
&= \frac{\sin x}{\cos x} \\
&= \tan x,
\end{aligned}
$$

as desired.

Proposition ($\tan, \cot$ are π-periodic)

The tangent and cotangent functions are π-periodic, in other words,

$$\begin{aligned}
\tan(\theta + \pi) &= \tan\theta \\
\cot(\theta + \pi) &= \cot\theta
\end{aligned}$$

Recall the angle sum formula $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$.

Example

Show that the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$ follows from the angle difference formula.

$$
\begin{aligned}
1 &= \cos 0 \\
&= \cos(\theta - \theta) \\
&= \cos\theta\cos\theta + \sin\theta\sin\theta \\
&= \cos^2\theta + \sin^2\theta,
\end{aligned}
$$

as desired.

Example

Prove the angle sum formula $\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$.

$$
\begin{aligned}
\tan(\alpha + \beta) &= \frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)} \\
&= \frac{(\sin \alpha \cos \beta + \cos \alpha \sin \beta) \frac{1}{\cos \alpha \cos \beta}}{(\cos \alpha \cos \beta - \sin \alpha \sin \beta) \frac{1}{\cos \alpha \cos \beta}} \\
&= \frac{\frac{\sin \alpha \cos \beta}{\cos \alpha \cos \beta} + \frac{\cos \alpha \sin \beta}{\cos \alpha \cos \beta}}{\frac{\cos \alpha \cos \beta}{\cos \alpha \cos \beta} - \frac{\sin \alpha \sin \beta}{\cos \alpha \cos \beta}} \\
&= \frac{\frac{\sin \alpha}{\cos \alpha} + \frac{\sin \beta}{\cos \beta}}{1 - \frac{\sin \alpha}{\cos \alpha} \cdot \frac{\sin \beta}{\cos \beta}} \\
&= \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}
\end{aligned}
$$

Double angle formulas

Proposition (Double angle formulas)

$$
\begin{aligned}
\sin(2\alpha) &= 2\sin\alpha\cos\alpha \\
\cos(2\alpha) &= \cos^2\alpha - \sin^2\alpha \\
&= 2\cos^2\alpha - 1 \\
&= 1 - 2\sin^2\alpha
\end{aligned}
$$

- The double angle formulas play a special role in integration.

Example

Derive the double-angle formulas.

$$
\begin{aligned}
\sin(2\alpha) &= \sin(\alpha + \alpha) \\
&= \sin\alpha\cos\alpha + \cos\alpha\sin\alpha \\
&= 2\sin\alpha\cos\alpha \\
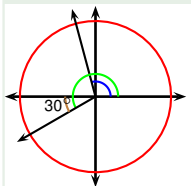\cos(2\alpha) &= \cos(\alpha + \alpha) \\
&= \cos\alpha\cos\alpha - \sin\alpha\sin\alpha \\
&= \cos^2\alpha - \sin^2\alpha \\
&= \cos^2\alpha - (1 - \cos^2\alpha) \\
&= 2\cos^2\alpha - 1 \\
&= 1 - \sin^2\alpha - \sin^2\alpha \\
&= 1 - 2\sin^2\alpha
\end{aligned}
$$

Recall the half angle formula $\cos\alpha = \pm\sqrt{\dfrac{1+\cos(2\alpha)}{2}}$.

Example

Using radicals, find the exact value of the trigonometric expression.

$$
\begin{aligned}
\cos 105^\circ &= \pm\sqrt{\frac{1+\cos\left(2\cdot 105^\circ\right)}{2}} \qquad \bigg| \cos 105^\circ < 0\\
&= -\sqrt{\frac{1+\cos\left(210^\circ\right)}{2}}\\
&= -\sqrt{\frac{1-\cos\left(30^\circ\right)}{2}}\\
&= -\sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}} = -\sqrt{\frac{2-\sqrt{3}}{2\cdot 2}}\\
&= -\frac{\sqrt{2-\sqrt{3}}}{2}
\end{aligned}
$$

Proposition (Power-Reducing Formulas)

$$\sin^2 \alpha \;=\; \frac{1 - \cos(2\alpha)}{2} \qquad \cos^2 \alpha \;=\; \frac{1 + \cos(2\alpha)}{2}$$

Proof.

$$
\begin{array}{llll}
\cos(2\alpha) & = & 1 - 2\sin^2 \alpha & \qquad \cos(2\alpha) & = & 2\cos^2 \alpha - 1 \\
2\sin^2 \alpha & = & 1 - \cos(2\alpha) & \qquad 2\cos^2 \alpha & = & 1 + \cos(2\alpha) \\
\sin^2 \alpha & = & \dfrac{1 - \cos(2\alpha)}{2} & \qquad \cos^2 \alpha & = & \dfrac{1 + \cos(2\alpha)}{2}
\end{array}
$$

□

Corollary

$$\sin \alpha \;=\; \pm\sqrt{\frac{1 - \cos(2\alpha)}{2}} \qquad \cos \alpha \;=\; \pm\sqrt{\frac{1 + \cos(2\alpha)}{2}}$$

Corollary (Half-Angle Formulas)

$$\sin \left(\frac{\beta}{2}\right) \;=\; \pm\sqrt{\frac{1 - \cos \beta}{2}} \qquad \cos \left(\frac{\beta}{2}\right) \;=\; \pm\sqrt{\frac{1 + \cos \beta}{2}}$$

Proposition (Power-Reducing Formulas)

$$\sin^2 \alpha \;=\; \frac{1 - \cos(2\alpha)}{2} \qquad \cos^2 \alpha \;=\; \frac{1 + \cos(2\alpha)}{2}$$

- The power reducing formulas are used to express $\sin^k \alpha$ and $\cos^k \alpha$ via lower powers of the sin and cos functions (applied to angles other than α).
- This technique will play a key role in integration (studied later/in another course).

Recall the formulas: $\sin^2 \beta = \frac{1-\cos(2\beta)}{2}$, $\qquad \cos^2 \beta = \frac{\cos(2\beta)+1}{2}$.

Example

Rewrite $\sin^4 \alpha$ in terms of first powers of the cosines and sines of multiples of the angle α.

$$
\begin{aligned}
\sin^4 \alpha &= \left(\sin^2 \alpha\right)^2 \\
&= \left(\frac{1-\cos(2\alpha)}{2}\right)^2 \\
&= \frac{1}{4}\left(1 - 2\cos(2\alpha) + \cos^2(2\alpha)\right) \\
&= \frac{1}{4}\left(1 - 2\cos(2\alpha) + \frac{\cos(2\cdot 2\alpha)+1}{2}\right) \\
&= \frac{1}{4}\left(1 - 2\cos(2\alpha) + \frac{\cos(2\cdot 2\alpha)}{2} + \frac{1}{2}\right) \\
&= \frac{1}{4}\left(\frac{3}{2} - 2\cos(2\alpha) + \frac{\cos(4\alpha)}{2}\right) \\
&= \frac{1}{8}\left(3 - 4\cos(2\alpha) + \cos(4\alpha)\right)
\end{aligned}
$$