

Calculus II

Simplify $\sin(k \arcsin x)$, $\cos(k \arcsin x)$,
 $\sin(k \arccos x)$, $\cos(k \arccos x)$

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Example

Rewrite $\sin(2 \arccos(x))$ as an algebraic expression of x and $\sqrt{1 - x^2}$.

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To simplify $\arccos x$ we try to use $\cos(\arccos x) = x$.

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$$\sin(2 \arccos(x)) = \sin(2y)$$

$$| \text{ Set } y = \arccos x$$

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$$\begin{aligned}\sin(2 \arccos(x)) &= \sin(2y) \\ &= ?\end{aligned}$$

Set $y = \arccos x$
Express via $\sin y, \cos y$

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$$\begin{aligned}\sin(2 \arccos(x)) &= \sin(2y) \\ &= 2 \cos y \sin y\end{aligned}$$

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Rewrite $\sin(2 \arccos(x))$ as an algebraic expression of x and $\sqrt{1 - x^2}$. To simplify $\arccos x$ we try to use $\cos(\arccos x) = x$. Therefore our aim is to **rewrite the expression only using the cos function**.

$$\begin{aligned}
 \sin(2 \arccos(x)) &= \sin(2y) \\
 &= 2 \cos y \sin y \\
 &= 2 \cos y \left(\pm \sqrt{1 - \cos^2 y} \right)
 \end{aligned}
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 \text{Set } y = \arccos x \\
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Set $y = \arccos x$
 Express via $\sin y, \cos y$
 Express $\sin y$ via $\cos y$
 $\sin y > 0$ because
 $0 \leq y \leq \pi$

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$$\cos(3 \arccos(x)) = \cos(3y) \quad \Big| \quad y = \arccos x$$

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$$\cos(3 \arccos(x)) = \cos(3y) = \cos(2y + y) \quad | \quad y = \arccos x$$

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$$\cos(3 \arccos(x)) = \cos(3y) = \cos(2y + y) \\ = ?$$

$y = \arccos x$
Angle sum f-la

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$$\begin{aligned} \cos(3 \arccos(x)) &= \cos(3y) = \cos(2y + y) \\ &= \cos(2y) \cos y - \sin(2y) \sin y \end{aligned} \quad \left| \begin{array}{l} y = \arccos x \\ \text{Angle sum f-la} \end{array} \right.$$

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 \cos(3 \arccos(x)) &= \cos(3y) = \cos(2y + y) \\
 &= \cos(2y) \cos y - \sin(2y) \sin y \\
 &= (\text{?} \quad \quad \quad) \cos y \\
 &\quad - \text{?} \quad \quad \quad \sin y
 \end{aligned}$$

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 &= \cos^3 y - 3 \sin^2 y \cos y & \text{Express } \sin y \\
 &= \cos^3 y - 3(1 - \cos^2 y) \cos y & \text{via } \cos y \\
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 &= 4\cos^3 y - 3\cos y \\
 &= 4x^3 - 3x
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