

Precalculus

Solve triangle from side and two angles

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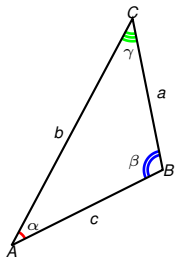
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Law of sines

Let $\triangle ABC$ have sides lengths a, b, c angles α, β, γ , as indicated: α is opposite to a , β is opposite to b , γ is opposite to c .

Proposition (Law of Sines)

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma}.$$

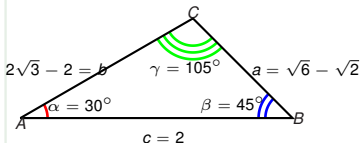


Proof.

$$\begin{aligned} \text{Area}(\triangle ABC) &= \frac{ab \sin \gamma}{2} = \frac{bc \sin \alpha}{2} & \left| \text{Div. by } \frac{b}{2} \right. \\ a \sin \gamma &= c \sin \alpha \\ \frac{a}{\sin \alpha} &= \frac{c}{\sin \gamma}. \end{aligned}$$

The remaining cases are similar. □

Example

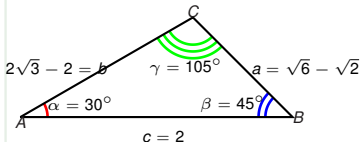


A triangle has a side of length 2cm ; the two angles adjacent to it are 30° and 45° .

- Find the other two sides of the triangle.
- Find the area of the triangle.

- Let the known side be $c = 2\text{cm}$.
- Let the known angles 30° , 45° be arranged as in the figure, and let the third angle be $\gamma = 180^\circ - 30^\circ - 45^\circ = 180^\circ - 75^\circ = 105^\circ$.
- Label the unknown sides a , b as indicated.

Example



A triangle has a side of length 2cm ; the two angles adjacent to it are 30° and 45° .

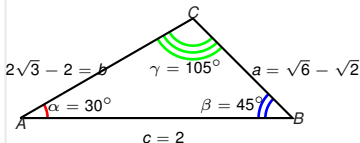
- Find the other two sides of the triangle.
- Find the area of the triangle.

$$\begin{aligned}\sin 105^\circ &= \sin(60^\circ + 45^\circ) = \sin 60^\circ \cos 45^\circ + \cos 60^\circ \sin 45^\circ \\ &= \frac{\sqrt{3}}{2} \frac{\sqrt{2}}{2} + \frac{1}{2} \frac{\sqrt{2}}{2} = \frac{\sqrt{6} + \sqrt{2}}{4}\end{aligned}$$

$$\frac{a}{\sin \alpha} = \frac{c}{\sin \gamma} \quad \text{Law of sines}$$

$$\begin{aligned}a &= \frac{c \sin \alpha}{\sin \gamma} = \frac{2 \sin 30^\circ}{\sin 105^\circ} = \frac{2 \cdot \frac{1}{2}}{\frac{\sqrt{6} + \sqrt{2}}{4}} = \frac{4(\sqrt{6} - \sqrt{2})}{(\sqrt{6} + \sqrt{2})(\sqrt{6} - \sqrt{2})} \\ &= \frac{4(\sqrt{6} - \sqrt{2})}{6 - 2} = \sqrt{6} - \sqrt{2}\end{aligned}$$

Example



A triangle has a side of length 2cm ; the two angles adjacent to it are 30° and 45° .

- Find the other two sides of the triangle.
- Find the area of the triangle.

| Law of sines

$$\frac{b}{\sin \beta} = \frac{c}{\sin \gamma}$$

$$b = \frac{c \sin \beta}{\sin \gamma} = \frac{2 \sin 45^\circ}{\sin 105^\circ} = \frac{\cancel{2} \frac{\sqrt{2}}{2}}{\frac{\sqrt{6} + \sqrt{2}}{4}} = \frac{4\sqrt{2}(\sqrt{6} - \sqrt{2})}{(\sqrt{6} + \sqrt{2})(\sqrt{6} - \sqrt{2})}$$

$$= \frac{\cancel{4}\sqrt{2}(\sqrt{6} - \sqrt{2})}{\cancel{4}} = 2\sqrt{3} - 2$$

$$\text{Area} = \frac{bc \sin \alpha}{2} = \frac{(2\sqrt{3} - 2)\cancel{2} \frac{1}{2}}{2} = \sqrt{3} - 1 \quad \text{cm}^2$$