Calculus II

Integrals of the form $\displaystyle\int \frac{a}{bx^2+c}\mathrm{d}x,\ c>0$

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Linear substitutions leading to blocks IIa and IIIa

Building block IIIa: $\int \frac{1}{u^2+1} \mathrm{d}u = \textbf{?} \qquad + C.$

Example

$$\int \frac{1}{x^2+2} \mathrm{d}x$$

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Example

$$\int \frac{1}{x^2 + 2} \mathrm{d}x = \int \frac{1}{2\left(\frac{1}{2}x^2 + 1\right)} \mathrm{d}x$$

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$$\begin{aligned}\int \frac{1}{x^2+2}\mathrm{d}x &= \int \frac{1}{2\left(\frac{1}{2}x^2+1\right)}\mathrm{d}x\\ &= \int \frac{1}{2\left(\left(\frac{x}{\sqrt{2}}\right)^2+1\right)} \quad \mathrm{d}\left(\ x \ \right)\end{aligned}$$

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Use $2 = \left(\sqrt{2}\right)^2$

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Set $\frac{x}{\sqrt{2}} = u$

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Building block IIIa: $\int \frac{1}{x^2+1} \mathrm{d}x = \arctan x + C$. **Let $a > 0$.**

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\int \frac{1}{x^2+a}\mathrm{d}x &= \int \frac{1}{a\left(\frac{1}{a}x^2+1\right)}\mathrm{d}x \\
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Use $a = \left(\sqrt{a}\right)^2$

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