Calculus II
Interval of convergence, part 3

Todor Milev

2019

Example

Find the radius of convergence and interval of convergence of the series $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$.

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(-3)^{n+1}x^{n+1}}{\sqrt{n+2}}\cdot\frac{\sqrt{n+1}}{(-3)^n x^n}\right| \\ &= \lim_{n\to\infty} 3|x|\sqrt{\frac{n+1}{n+2}}\cdot\frac{\frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} 3|x|\sqrt{\frac{1+\frac{1}{n}}{1+\frac{2}{n}}} = 3|x|\end{aligned}$$

- Ratio Test: it converges if $3|x| < 1$ and diverges if $3|x| > 1$.
- So it converges if $|x| < \frac{1}{3}$ and diverges if $|x| > \frac{1}{3}$.
- Therefore $R = \frac{1}{3}$.
- If we use $x = \frac{1}{3}$, we get $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$, which is convergent.
- If we use $x = -\frac{1}{3}$, we get $\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}}$, which is divergent.
- The interval of convergence is $(-\frac{1}{3}, \frac{1}{3}]$.