Precalculus
Additional trigonometric identity exercises

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2019

Proving the following identities is a good exercise.

1. $\sin\theta\cot\theta = \cos\theta.$
2. $(\sin\theta + \cos\theta)^2 = 1 + \sin(2\theta).$
3. $\sec\theta - \cos\theta = \tan\theta\sin\theta.$
4. $\tan^2\theta - \sin^2\theta = \tan^2\theta\sin^2\theta.$
5. $\cot^2\theta + \sec^2\theta = \tan^2\theta + \csc^2\theta.$
6. $2\csc(2\theta) = \sec\theta\csc\theta.$
7. $\tan(2\theta) = \dfrac{2\tan\theta}{1-\tan^2\theta}.$
8. $\dfrac{1}{1-\sin\theta} + \dfrac{1}{1+\sin\theta} = 2\sec^2\theta.$
9. $\tan\alpha + \tan\beta = \dfrac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta}.$
10. $\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta}.$
11. $\sin(3\theta) + \sin\theta = 2\sin(2\theta)\cos\theta.$
12. $\cos(3\theta) = 4\cos^3\theta - 3\cos\theta.$
13. $1 + \tan^2\theta = \sec^2\theta.$
14. $1 + \csc^2\theta = \cot^2\theta.$
15. $2\cos^2(2x) = 2\sin^4\theta + 2\cos^4\theta - \sin^2(2\theta).$
16. $\dfrac{1+\tan\left(\frac{\theta}{2}\right)}{1-\tan\left(\frac{\theta}{2}\right)} = \tan\theta + \sec\theta.$

Here we explicitly permit the use of the Pythagorean identities and the double angle f-las:

$$\begin{aligned} \cos^2\theta + \sin^2\theta &= 1 \\ \sin(2\theta) &= 2\sin\theta\cos\theta \\ \cos(2\theta) &= \cos^2\theta - \sin^2\theta \end{aligned}$$

Example

Prove the trigonometric identity.

$$(\sin\theta + \cos\theta)^2 = 1 + \sin(2\theta)$$

We need to transform both sides to the same expression. In this case, we choose to transform the left hand side to the right:

$$\begin{aligned} (\sin\theta + \cos\theta)^2 &= \sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta \\ &= 1 + \sin(2\theta) \end{aligned}$$

$$\left| \begin{array}{l} (A+B)^2 = \\ A^2 + 2AB + B^2 \end{array} \right.$$

Recall the formulas
$$\begin{aligned}\sin(\alpha+\beta) &= \sin\alpha\cos\beta+\cos\alpha\sin\beta\\ \cos(\alpha+\beta) &= \cos\alpha\cos\beta-\sin\alpha\sin\beta.\end{aligned}$$

Example

Express $\sin(3x)$ and $\cos(3x)$ via $\cos x$ and $\sin x$.

$$\begin{aligned}\sin(3x) &= \sin(x+2x)\\ &= \sin x\cos(2x)+\cos x\sin(2x)\\ &= \sin x\left(\cos^2 x-\sin^2 x\right)+\cos x(2\sin x\cos x)\\ &= \sin x\cos^2 x-\sin^3 x+2\sin x\cos^2 x\\ &= 3\sin x\cos^2 x-\sin^3 x\\ \cos(3x) &= \cos(x+2x)\\ &= \cos x\cos(2x)-\sin x\sin(2x)\\ &= \cos x(\cos^2 x-\sin^2 x)-\sin x(2\sin x\cos x)\\ &= \cos^3 x-\cos x\sin^2 x-2\cos x\sin^2 x\\ &= \cos^3 x-3\cos x\sin^2 x\end{aligned}$$

Example

Prove the identity $\tan\theta + \sec\theta = \dfrac{1+\tan\left(\frac{\theta}{2}\right)}{1-\tan\left(\frac{\theta}{2}\right)}$

All angles here are multiples of $\frac{\theta}{2}$, so set $\varphi = \frac{\theta}{2}$, $\theta = 2\varphi$.

$$\begin{aligned}
\tan(2\varphi)+\sec(2\varphi) &= \frac{\sin(2\varphi)}{\cos(2\varphi)} + \frac{1}{\cos(2\varphi)}\\
&= \frac{\sin(2\varphi)+1}{\cos(2\varphi)}\\
&= \frac{2\sin\varphi\cos\varphi + \sin^2\varphi + \cos^2\varphi}{\cos^2\varphi - \sin^2\varphi} && \left|\ \begin{array}{l} A^2+2AB+B^2 \\ =(A+B)^2 \\ A^2-B^2 = \\ (A-B)(A+B) \end{array}\right.\\
&= \frac{(\cos\varphi+\sin\varphi)^{\cancel{2}}}{(\cos\varphi-\sin\varphi)\cancel{(\cos\varphi+\sin\varphi)}}\\
&= \frac{(\cos\varphi+\sin\varphi)\frac{1}{\cos\varphi}}{(\cos\varphi-\sin\varphi)\frac{1}{\cos\varphi}} = \frac{1+\frac{\sin\varphi}{\cos\varphi}}{1-\frac{\sin\varphi}{\cos\varphi}}\\
&= \frac{1+\tan\varphi}{1-\tan\varphi} && \Big|\ \text{as desired.}
\end{aligned}$$