Calculus I
Derivative of $x^m e^x$

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2019

Theorem (The Product Rule)

If f and g are both differentiable, then

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x).$$

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&= \lim_{h\to 0} \left({\color{red}f(x+h)}\frac{g(x+h) - g(x)}{h} + g(x)\frac{f(x+h) - f(x)}{h} \right)
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Differentiate $f(x) = x^3 e^x$.

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