

Calculus II

Homework

Series absolute convergence, the ratio and root tests

1. Let $x \in (0, 1)$. Express the following using x and $\sqrt{1 - x^2}$.

- | | |
|----------------------------|----------------------------|
| (a) $\sin(\arcsin(x))$. | (e) $\sin(2 \arccos(x))$. |
| (b) $\sin(2 \arcsin(x))$. | (f) $\sin(3 \arccos(x))$. |
| (c) $\sin(3 \arcsin(x))$. | (g) $\cos(2 \arcsin(x))$. |
| (d) $\sin(\arccos(x))$. | (h) $\cos(3 \arccos(x))$. |

2. Express as the following as an algebraic expression of x . In other words, “get rid” of the trigonometric and inverse trigonometric expressions.

- | | |
|--|------------------------------------|
| (a) $\cos^2(\arctan x)$. | (c) $\frac{1}{\cos(\arcsin x)}$. |
| (b) $-\sin^2(\operatorname{arccot} x)$. | (d) $-\frac{1}{\sin(\arccos x)}$. |

3. Rewrite as a rational function of t . This problem will be later used to derive the Euler substitutions (an important technique for integrating).

- | | |
|---------------------------|---|
| (a) $\cos(2 \arctan t)$. | (g) $\cos(2 \operatorname{arccot} t)$. |
| (b) $\sin(2 \arctan t)$. | (h) $\sin(2 \operatorname{arccot} t)$. |
| (c) $\tan(2 \arctan t)$. | (i) $\tan(2 \operatorname{arccot} t)$. |
| (d) $\cot(2 \arctan t)$. | (j) $\cot(2 \operatorname{arccot} t)$. |
| (e) $\csc(2 \arctan t)$. | (k) $\csc(2 \operatorname{arccot} t)$. |
| (f) $\sec(2 \arctan t)$. | (l) $\sec(2 \operatorname{arccot} t)$. |

4. Compute the derivative (derive the formula).

- | | |
|------------------------------------|---|
| (a) $(\arctan x)'$. | (d) $(\arccos x)'$. |
| (b) $(\operatorname{arccot} x)'$. | (e) Let arcsec denote the inverse of the secant function. Compute $(\operatorname{arcsec} x)'$. |
| (c) $(\arcsin x)'$. | |

5. (a) Let $a + b \neq k\pi$, $a \neq k\pi + \frac{\pi}{2}$ and $b \neq k\pi + \frac{\pi}{2}$ for any $k \in \mathbb{Z}$ (integers). Prove that

$$\frac{\tan a + \tan b}{1 - \tan a \tan b} = \tan(a + b) \quad .$$

(b) Let x and y be real. Prove that, for $xy \neq 1$, we have

$$\arctan x + \arctan y = \arctan \left(\frac{x + y}{1 - xy} \right)$$

if the left hand side lies between $(-\frac{\pi}{2}, \frac{\pi}{2})$.

6. Evaluate the indefinite integral. Illustrate the steps of your solutions.

$$(a) \int x \sin x dx.$$

$$(b) \int x e^{-x} dx.$$

$$(c) \int x^2 e^x dx.$$

$$(d) \int x \sin(-2x) dx.$$

$$(e) \int x^2 \cos(3x) dx.$$

$$(f) \int x^2 e^{-2x} dx.$$

$$(g) \int x \sin(2x) dx.$$

$$(h) \int x \cos(3x) dx.$$

$$(i) \int x^2 e^{2x} dx.$$

$$(j) \int x^3 e^x dx.$$

7. Evaluate the indefinite integral. Illustrate the steps of your solutions.

$$(a) \int x^2 \cos(2x) dx.$$

$$(b) \int x^2 e^{ax} dx, \text{ where } a \text{ is a constant.}$$

$$(c) \int x^2 e^{-ax} dx, \text{ where } a \text{ is a constant.}$$

$$(d) \int x^2 \frac{(e^{ax} + e^{-ax})^2}{4} dx, \text{ where } a \text{ is a constant.}$$

$$(e) \int \frac{1}{\cos^2 x} dx. \quad (\text{Hint: This problem does not require integration by parts. What is the derivative of } \tan x?)$$

$$(f) \int (\tan^2 x) dx. \quad (\text{Hint: This problem does not require integration by parts. We can use } \tan^2 x = \frac{1}{\cos^2 x} - 1 \text{ and the previous problem.})$$

$$(g) \int x \tan^2 x dx. \quad (\text{Hint: } \tan^2 x dx = d(F(x)), \text{ where } F(x) \text{ is the answer from the preceding problem}).$$

$$(h) \int e^{-\sqrt{x}} dx.$$

$$(i) \int \cos^2 x dx.$$

$$(j) \int \frac{x}{1+x^2} dx \quad (\text{Hint: use substitution rule, don't use integration by parts})$$

$$(k) \int (\arctan x) dx.$$

$$(l) \int (\arcsin x) dx.$$

$$(m) \int (\arcsin x)^2 dx. \quad (\text{Hint: Try substituting } x = \sin y.)$$

$$(n) \int \arctan\left(\frac{1}{x}\right) dx.$$

$$(o) \int \sin x e^x dx$$

$$(p) \int \cos x e^x dx$$

$$(q) \int \sin(\ln(x)) dx.$$

$$(r) \int \cos(\ln(x)) dx.$$

$$(s) \int \ln x dx$$

$$(t) \int x \ln x dx.$$

$$(u) \int \frac{\ln x}{\sqrt{x}} dx.$$

$$(v) \int (\ln x)^2 dx.$$

$$(w) \int (\ln x)^3 dx.$$

$$(x) \int x^2 \cos^2 x dx. \quad (\text{This problem is related to Problem 7.d as } \cos x = \frac{e^{ix} + e^{-ix}}{2}).$$

8. Compute $\int x^n e^x dx$, where n is a non-negative integer.

9. Integrate. Illustrate the steps of your solution.

$$(a) \int \frac{1}{x+1} dx$$

$$(b) \int \frac{x-1}{x+1} dx$$

$$(c) \int \frac{1}{(x+1)^2} dx$$

$$(d) \int \frac{x}{(x+1)^2} dx$$

$$(e) \int \frac{1}{(2x+3)^2} dx$$

$$(f) \int \frac{x}{2x^2+3} dx$$

$$(g) \int \frac{1}{2x^2+3} dx$$

$$(h) \int \frac{x}{2x^2+x+1} dx.$$

$$(i) \int \frac{x}{2x^2+x+3} dx$$

$$(j) \int \frac{x}{x^2-x+3} dx$$

$$(k) \int \frac{1}{(x^2 + 1)^2} dx$$

$$(m) \int \frac{1}{(x^2 + 1)^3} dx$$

$$(l) \int \frac{1}{(x^2 + x + 1)^2} dx$$

10. Let a, b, c, A, B be real numbers. Suppose in addition $a \neq 0$ and $b^2 - 4ac < 0$. Integrate

$$\int \frac{Ax + B}{ax^2 + bx + c} dx \quad .$$

The purpose of this exercise is to produce a formula in form ready for implementation in a computer algebra system.

11. Let a, b, c, A, B be real numbers and let $n > 1$ be an integer. Suppose in addition $a \neq 0$ and $b^2 - 4ac < 0$. Let

$$J(n) = \int \frac{1}{(x^2 + \frac{b}{a}x + \frac{c}{a})^n} dx \quad .$$

(a) Express the integral

$$\int \frac{Ax + B}{(ax^2 + bx + c)^n} dx$$

via $J(n)$.

(b) Express $J(n)$ recursively via $J(n - 1)$

The purpose of this exercise is to produce a formula in form ready for implementation in a computer algebra system.

12. Integrate. Some of the examples require partial fraction decomposition and some do not. Illustrate the steps of your solution.

$$(a) \int \frac{1}{4x^2 + 4x + 1} dx$$

$$(h) \int \frac{x}{3x^2 + x - 2} dx$$

$$(b) \int \frac{1}{1 - x^2} dx$$

$$(i) \int \frac{x}{3x^2 + x + 2} dx$$

$$(c) \int \frac{1}{5 - x^2} dx$$

$$(j) \int \frac{x}{2x^2 + x + 1} dx$$

$$(d) \int \frac{x}{4x^2 + x + \frac{1}{16}} dx$$

$$(k) \int \frac{x}{2x^2 + x - 1} dx$$

$$(e) \int \frac{x + 1}{2x^2 + x} dx$$

$$(l) \int \frac{1}{x^2 + x + 1} dx$$

$$(f) \int \frac{x}{4x^2 + x + 5} dx$$

$$(m) \int \frac{1}{2x^2 + 5x + 1} dx$$

$$(g) \int \frac{x}{4x^2 + x - 5} dx$$

13. Evaluate the indefinite integral. Illustrate all steps of your solution.

$$(a) \int \frac{x^3 + 4}{x^2 + 4} dx$$

$$(h) \int \frac{15x^2 - 4x - 81}{(x - 3)(x + 4)(x - 1)} dx$$

$$(b) \int \frac{4x^2}{2x^2 - 1} dx$$

$$(i) \int \frac{x^4 + 10x^3 + 18x^2 + 2x - 13}{x^4 + 4x^3 + 3x^2 - 4x - 4} dx$$

$$(c) \int \frac{x^3}{x^2 + 2x - 3} dx$$

Check first that $(x - 1)(x + 2)^2(x + 1) = x^4 + 4x^3 + 3x^2 - 4x - 4$.

$$(d) \int \frac{x^3}{x^2 + 3x - 4} dx$$

$$(j) \int \frac{x^4}{(x^2 + 2)(x + 2)} dx$$

$$(e) \int \frac{x^3}{2x^2 + 3x - 5} dx$$

$$(k) \int \frac{x^5}{x^3 - 1} dx$$

$$(f) \int \frac{x^2 + 1}{(x - 3)(x - 2)^2} dx$$

$$(l) \int \frac{x^4}{(x^2 + 2)(x + 1)^2} dx$$

$$(g) \int \frac{x^4}{(x + 1)^2(x + 2)} dx$$

$$(m) \int \frac{3x^2 + 2x - 1}{(x-1)(x^2+1)} dx$$

$$(n) \int \frac{x^2 - 1}{x(x^2 + 1)^2} dx$$

14. Integrate

$$\int \frac{x^6 - x^5 + \frac{9}{2}x^4 - 4x^3 + \frac{13}{2}x^2 - \frac{7}{2}x + \frac{11}{4}}{x^5 - x^4 + 3x^3 - 3x^2 + \frac{9}{4}x - \frac{9}{4}} dx \quad .$$

15. Integrate.

$$(a) \int \frac{1}{3 + \cos x} dx.$$

$$(d) \int \frac{1}{2 + \tan x} dx. \text{ (Hint: this integral can be done simply with the substitution } x = \arctan t.)$$

$$(b) \int \frac{1}{4 + \cos x} dx.$$

$$(c) \int \frac{1}{3 + \sin x} dx.$$

$$(e) \int \frac{dx}{2 \sin x - \cos x + 5}.$$

16. Integrate. The answer key has not been proofread, use with caution.

$$(a) \int \sin(3x) \cos(2x) dx.$$

$$(b) \int \sin x \cos(5x) dx.$$

$$(c) \int \cos(3x) \sin(2x) dx.$$

$$(d) \int \sin(5x) \sin(3x) dx.$$

$$(e) \int \cos(x) \cos(3x) dx.$$

17. Integrate.

$$(a) \int \sin^2 x \cos x dx.$$

$$(c) \int \cos^3 x dx.$$

$$(b) \int \sin^2 x dx.$$

$$(d) \int \sin^3 x \cos^4 x dx.$$

18. Integrate.

$$(a) \int \sec x dx.$$

$$(b) \int \sec^3 x dx.$$

$$(c) \int \tan^3 x dx.$$

$$(d) \int \sec^2 x \tan^2 x dx.$$

19. Find a linear substitution (via completing the square) to transform the radical to a multiple of an expression of the form $\sqrt{u^2 + 1}$, $\sqrt{u^2 - 1}$ or $\sqrt{1 - u^2}$.

$$(a) \sqrt{x^2 + x + 1}.$$

$$(b) \sqrt{-2x^2 + x + 1}.$$

20. Compute the integral.

$$(a) \int \frac{\sqrt{1 + x^2}}{x^2} dx.$$

21. Compute the integral using a trigonometric substitution.

$$(a) \int \frac{\sqrt{9-x^2}}{x^2} dx \quad .$$

22. Compute the integral.

$$(a) \int \sqrt{x^2+1} dx$$

$$(b) \int \sqrt{x^2+2} dx$$

$$(c) \int \sqrt{x^2+x+1} dx$$

$$(d) \int \sqrt{(2x^2+2x+1)} dx$$

$$(e) \int \sqrt{(3x^2+2x+1)} dx$$

$$(f) \int \frac{\sqrt{x^2+1}}{x+1} dx$$

23. Let $b^2 - 4ac < 0$ and $a > 0$ be (real) numbers. Show that

$$\int \sqrt{(ax^2+bx+c)} dx = \frac{\sqrt{a}D}{2} \left(\ln \left(\sqrt{\left(\frac{2xa+b}{2\sqrt{Da}} \right)^2 + 1} + \frac{2xa+b}{2\sqrt{Da}} \right) + \frac{2xa+b}{2\sqrt{Da}} \sqrt{\left(\frac{2xa+b}{2\sqrt{Da}} \right)^2 + 1} \right) + C,$$

$$\text{where } D = \frac{4ac - b^2}{4a^2}.$$

24. Integrate

$$(a) \int \sqrt{1-x^2} dx$$

$$(b) \int \sqrt{2-x^2} dx$$

$$(c) \int \sqrt{-x^2+x+1} dx$$

$$(d) \int \sqrt{2-x-x^2} dx$$

$$(e) \int \frac{\sqrt{1-x^2}}{1+x} dx$$

$$(f) \int \frac{\sqrt{1-x^2}}{2+x} dx$$

25. Integrate

$$(a) \int \sqrt{x^2-1} dx$$

$$(b) \int \sqrt{x^2-2} dx$$

$$(c) \int \sqrt{2x^2+x-1} dx$$

$$(d) \int \sqrt{x^2+x-1} dx$$

26. (a) Express x , dx and $\sqrt{x^2+1}$ via θ and $d\theta$ for the trigonometric substitution $x = \cot \theta$, $\theta \in (0, \pi)$.

(b) Express x , dx and $\sqrt{x^2+1}$ via t and dt for the Euler substitution $x = \cot(2 \arctan t)$, $t > 0$. Express t via x .

27. Let the variables x and t be related via $\sqrt{x^2+1} = x+t$.

(a) Express x via t .

(b) Express $\sqrt{x^2+1}$ via t alone.

(c) Express dx via t and dt .

28. (a) Express x , dx and $\sqrt{x^2 + 1}$ via θ and $d\theta$ for the trigonometric substitution $x = \tan \theta$, $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$.
 (b) Express x , dx and $\sqrt{x^2 + 1}$ via t and dt for the Euler substitution $x = \tan(2 \arctan t)$, $t \in (-1, 1)$. Express t via x .
29. Let the variables x and t be related via $\sqrt{x^2 + 1} = \frac{x}{t} - 1$.
 (a) Express x via t .
 (b) Express $\sqrt{x^2 + 1}$ via t alone.
 (c) Express dx via t and dt .
30. (a) Express x , dx and $\sqrt{1 - x^2}$ via θ and $d\theta$ for the trigonometric substitution $x = \cos \theta$, $\theta \in [0, \pi]$.
 (b) Express x , dx and $\sqrt{1 - x^2}$ via t and dt for the Euler substitution $x = \cos(2 \arctan t)$, $t \geq 0$. Express t via x .
31. Let the variables x and t be related via $\sqrt{-x^2 + 1} = (1 - x)t$.
 (a) Express x via t .
 (b) Express $\sqrt{-x^2 + 1}$ via t alone.
 (c) Express dx via t and dt .
32. (a) Express x , dx and $\sqrt{1 - x^2}$ via θ and $d\theta$ for the trigonometric substitution $x = \sin \theta$, $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.
 (b) Express x , dx and $\sqrt{1 - x^2}$ via t and dt for the Euler substitution $x = \sin(2 \arctan t)$, $t \in [-1, 1]$. Express t via x .
33. Let the variables x and t be related via $\sqrt{-x^2 + 1} = 1 - xt$.
 (a) Express x via t .
 (b) Express $\sqrt{-x^2 + 1}$ via t alone.
 (c) Express dx via t and dt .
34. (a) Express x , dx and $\sqrt{x^2 - 1}$ via θ and $d\theta$ for the trigonometric substitution $x = \csc \theta$, $\theta \in [0, \frac{\pi}{2}] \cup [\pi, \frac{3\pi}{2}]$.
 (b) Express x , dx and $\sqrt{x^2 - 1}$ via t and dt for the Euler substitution $x = \sec(2 \arctan t)$, $t \in (-\infty, -1) \cup [1, 0)$. Express t via x .
35. Let the variables x and t be related via $\sqrt{x^2 - 1} = (x + 1)t$.
 (a) Express x via t .
 (b) Express $\sqrt{x^2 - 1}$ via t alone.
 (c) Express dx via t and dt .
36. (a) Express x , dx and $\sqrt{1 - x^2}$ via θ and $d\theta$ for the trigonometric substitution $x = \csc \theta$, $\theta \in [0, \frac{\pi}{2}] \cup [\pi, \frac{3\pi}{2}]$.
 (b) Express x , dx and $\sqrt{1 - x^2}$ via t and dt for the Euler substitution $x = \csc(2 \arctan t)$, $t \in (-\infty, -1) \cup [0, 1)$. Express t via x .
37. Let the variables x and t be related via $\sqrt{x^2 - 1} = \frac{1}{t} - x$.
 (a) Express x via t .
 (b) Express $\sqrt{x^2 - 1}$ via t alone.
 (c) Express dx via t and dt .
38. Compute the limits. The answer key has not been fully proofread, use with caution.

(a) $\lim_{x \rightarrow 0} \frac{\sin x}{x}$.

(b) $\lim_{x \rightarrow 0} \frac{x}{\ln(1 + x)}$.

(c) $\lim_{x \rightarrow 0} \frac{x^2}{x - \ln(1 + x)}$.

(d) $\lim_{x \rightarrow 0} \frac{x^2}{\sin x \ln(1 + x)}$.

(e) $\lim_{x \rightarrow 0} \frac{\sin^2 x}{(\ln(1 + x))^2}$.

(f) $\lim_{x \rightarrow 0} \frac{\cos x - 1}{\sin x \ln(1 + x)}$.

(g) $\lim_{x \rightarrow 0} \frac{\arctan x - x}{x^3}$.

(h) $\lim_{x \rightarrow 0} \frac{\arcsin x - x}{x^3}$.

(i) $\lim_{x \rightarrow 1} \frac{x}{x - 1} - \frac{1}{\ln x}$.

(j) $\lim_{x \rightarrow 0} \frac{\cos(nx) - \cos(mx)}{x^2}$.

$$(k) \lim_{x \rightarrow 0} \frac{\arcsin x - x - \frac{1}{6}x^3}{\sin^5 x}.$$

$$(l) \lim_{x \rightarrow 1} \frac{\sin(\pi x) \ln x}{\cos(\pi x) + 1}.$$

$$(m) \lim_{x \rightarrow 0} \frac{\sin x - x}{\arcsin x - x}.$$

$$(n) \lim_{x \rightarrow 0} \frac{\sin x - x}{\arctan x - x}.$$

$$(o) \lim_{x \rightarrow \infty} x \sin\left(\frac{2}{x}\right).$$

39. Compute the limit.

$$(a) \lim_{x \rightarrow \infty} \left(\frac{x-2}{x}\right)^x.$$

$$(b) \lim_{x \rightarrow \infty} \left(\frac{x-2}{x}\right)^{2x}$$

$$(c) \lim_{x \rightarrow \infty} \left(\frac{x}{x+3}\right)^{2x}$$

40. Find the limit.

$$(a) \lim_{x \rightarrow \infty} \left(1 - \frac{2}{x}\right)^x.$$

$$(b) \lim_{x \rightarrow 0} (1-x)^{\frac{1}{x}}.$$

$$(c) \lim_{x \rightarrow \infty} \left(\frac{x}{x-5}\right)^x.$$

$$(d) \lim_{x \rightarrow \infty} \left(\frac{x}{x-2}\right)^{3x+2}.$$

41. Determine whether the integral is convergent or divergent. Motivate your answer.

$$(a) \int_2^{\infty} \frac{1}{(x-1)^{\frac{3}{2}}} dx.$$

$$(b) \int_{-1}^1 \frac{1}{\sqrt[5]{1+x}} dx.$$

$$(c) \int_1^{\infty} \frac{1}{\sqrt[5]{1+x}} dx.$$

$$(d) \int_{-1}^{\infty} \frac{1}{\sqrt[5]{1+x}} dx.$$

$$(e) \int_{-\infty}^0 \frac{1}{2-3x} dx.$$

$$(f) \int_{-\infty}^0 \frac{1}{(2-3x)^2} dx.$$

$$(g) \int_{-\infty}^0 \frac{1}{(2-3x)^{1.00000001}} dx.$$

$$(h) \int_{-2}^{\frac{1}{2}} \frac{1}{2x-1} dx.$$

$$(i) \int_{-1}^{\infty} e^{-3x} dx.$$

$$(j) \int_{-\infty}^5 2^x dx.$$

$$(k) \int_{-\infty}^{\infty} x^3 dx.$$

$$(l) \int_{-\infty}^{\infty} x e^{-x^2} dx.$$

$$(m) \int_0^{\infty} \sqrt{x} e^{-\sqrt{x}} dx.$$

$$(n) \int_0^{\infty} \sin^2 x dx.$$

$$(o) \int_0^5 \frac{1}{x^2 + x - 2} dx.$$

$$(p) \int_0^{\infty} \frac{1}{x^2 + x + 1} dx.$$

$$(q) \int_2^{\infty} \frac{1}{x^2 - x - 1} dx.$$

$$(r) \int_0^{\infty} \frac{1}{x^2 - x - 1} dx.$$

$$(s) \int_{-\infty}^{\infty} \frac{x^2}{x^4 + 2} dx.$$

$$(t) \int_{100}^{\infty} \frac{1}{x \ln x} dx.$$

$$(u) \int_{100}^{\infty} \frac{1}{x(\ln x)^2} dx.$$

$$(v) \int_0^1 \ln x dx.$$

$$(w) \int_0^1 \frac{\ln x}{\sqrt{x}} dx.$$

$$(x) \int_0^2 x^3 \ln x dx.$$

$$(y) \int_0^1 \frac{e^{\frac{1}{x}}}{x^2} dx.$$

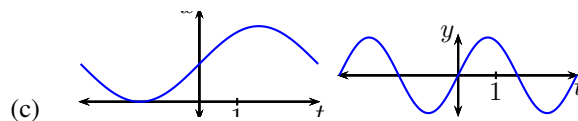
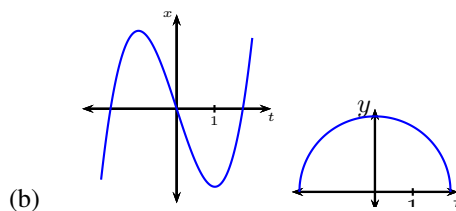
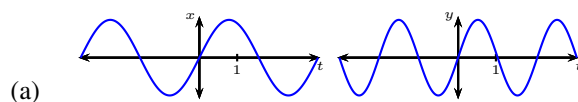
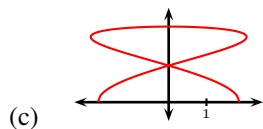
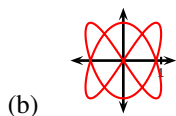
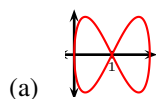
$$(z) \int_{-1}^0 \frac{e^{\frac{1}{x}}}{x^2} dx.$$

42. Determine whether the integral is convergent or divergent. Motivate your answer. The answer key has not been proofread, use with caution.

$$(a) \int_0^{\infty} \sin x^2 dx \text{ (This problem is more difficult and may re-}$$

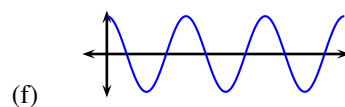
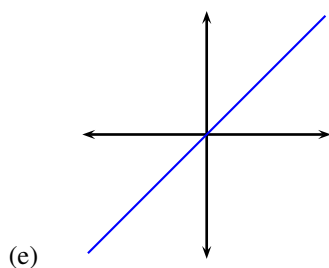
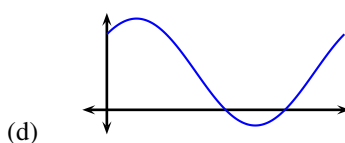
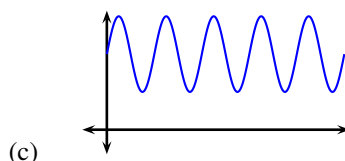
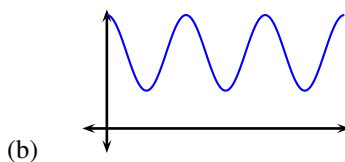
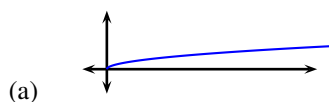
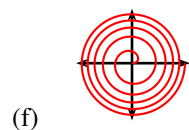
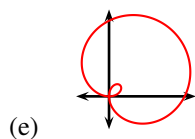
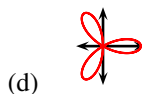
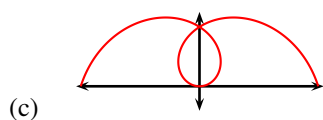
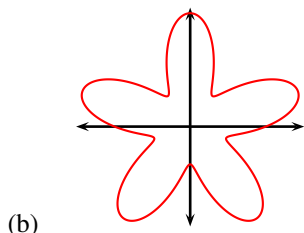
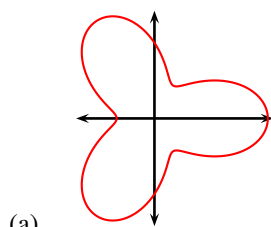
quire knowledge of sequences to solve).

43. Match the graphs of the parametric equations $x = f(t)$, $y = g(t)$ with the graph of the parametric curve $\gamma : \begin{cases} x = f(t) \\ y = g(t) \end{cases}$



44.

Match the graph of the curve to its graph in polar coordinates and to its polar parametric equations.



(i) $r = 1 + \sin(\theta) + \cos(\theta)$

(ii) $r = \theta, \theta \in [-\pi, \pi]$.

(iii) $r = \cos(3\theta), \theta \in [0, 2\pi]$.

(iv) $r = \frac{1}{4}\sqrt{\theta}, \theta \in [0, 10\pi]$.

(v) $r = 2 + \sin(5\theta)$.

(vi) $r = 2 + \cos(3\theta)$.

45.

- Sketch the curve given in polar coordinates by $r = 2 \sin \theta$. What kind of a figure is this curve? Find an equation satisfied by the curve in the (x, y) -coordinates.
- Sketch the curve given in polar coordinates by $r = 4 \cos \theta$. What kind of a figure is this curve? Find an equation satisfied by the curve in the (x, y) -coordinates.
- Sketch the curve given in polar coordinates by $r = 2 \sec \theta$. What kind of a figure is this curve? Find an equation satisfied by the curve in the (x, y) -coordinates.
- Sketch the curve given in polar coordinates by $r = 2 \csc \theta$. What kind of a figure is this curve? Find an equation satisfied by the curve in the (x, y) -coordinates.
- Sketch the curve given in polar coordinates by $r = 2 \sec(\theta + \frac{\pi}{4})$. What kind of a figure is this curve? Find an equation satisfied by the curve in the (x, y) -coordinates.
- Sketch the curve given in polar coordinates by $r = 2 \csc(\theta + \frac{\pi}{6})$. What kind of a figure is this curve? Find an equation satisfied by the curve in the (x, y) -coordinates.

46. Find the values of the parameter t for which the curve has horizontal and vertical tangents.

(a) $x = t^2 - t + 1, y = t^2 + t - 1$

(c) $x = \cos(t), y = \sin(3t)$

(b) $x = t^3 - t^2 - t + 1, y = t^2 - t - 1$.

(d) $x = \cos(t) + \sin(t), y = \sin(t)$.

47. Show that the parametric curve has multiple tangents at the point and find their slopes.

- (a) $x = \cos t, y = 2 \sin(2t)$, two tangents at $(x, y) = (0, 0)$. of tangents.
- (b) $x = \cos t \sin(3t), y = \sin(t) \sin(3t)$, six tangents at $(x, y) = (0, 0)$.
- (c) $x = \cos t, y = \sin(3t)$, find the two points at which the curve has double tangent and find the slopes of both pairs
- (d) $x = t^3 - t^2 - t + 1, y = t^2 - t - 1$, find a point where the curve has double tangent and find the slopes of the tangents.

48. Find the length of the curve.

- (a) $y = x^2, x \in [1, 2]$.
- (b) $y = \sqrt{x}, x \in [1, 2]$.
- (c) $x = \sqrt{t} - 2t$ and $y = \frac{8}{3}t^{\frac{3}{4}}$ from $t = 1$ to $t = 4$.
- (d) $\gamma : \begin{cases} x(t) = \frac{1}{t} + \frac{t^3}{3} \\ y(t) = 2t \end{cases}, t \in [1, 2]$.
- (e) $\gamma : \begin{cases} x(t) = \frac{1}{t} + t \\ y(t) = 2 \ln t \end{cases}, t \in [1, 2]$.
- (f) One arch of the cycloid

$$\gamma : \begin{cases} x(t) = t - \sin t \\ y(t) = 1 - \cos t \end{cases}, t \in [0, 2\pi]$$

- (g) The cardioid

$$\gamma : \begin{cases} x(t) = (1 + \sin t) \cos t \\ y(t) = (1 + \sin t) \sin t \end{cases}, t \in [0, 2\pi]$$

49. Set up an integral that expresses the length of the curve and find the length of the curve.

- (a) $\begin{cases} x(t) = e^t + e^{-t} \\ y(t) = 5 - 2t \end{cases}, t \in [0, 3]$
- (b) $\begin{cases} x(t) = \sin t + \cos t \\ y(t) = \sin t - \cos t \end{cases}, t \in [0, \pi]$

50. Give a geometric definition of the cycloid curve using a circle of radius 1. Using that definition, derive equations for the cycloid curve. Find area locked between one “arch” of the cycloid curve and the x axis.

51. (a) The curve given in polar coordinates by $r = 1 + \sin 2\theta$ is plotted below by computer. Find the area lying outside of this curve and inside of the circle $x^2 + y^2 = 1$.
- (b) The curve given in polar coordinates by $r = \cos(2\theta)$ is plotted below by computer. Find the area lying inside the curve and outside of the circle $x^2 + y^2 = \frac{1}{4}$.
- (c) Below is a computer generated plot of the curve $r = \sin(2\theta)$. Find the area locked inside one petal of the curve and outside of the circle $x^2 + y^2 = \frac{1}{4}$.

52. The answer key has not been proofread, use with caution.

- (a) Sketch the graph of the curve given in polar coordinates by $r = 3 \sin(2\theta)$ and find the area of one petal.
- (b) Sketch the graph of the curve given in polar coordinates by $r = 4 + 3 \sin \theta$ and find the area enclosed by the curve.

53. List the first 4 elements of the sequence.

- (a) $a_n = \frac{(-1)^n}{n}$.
- (b) $a_n = \frac{1}{n!}$.
- (c) $a_n = \cos(\pi n)$.
- (d) $a_n = \frac{(-1)^n}{2n+1}$.
- (e) $a_n = \frac{\sqrt{5}}{5} \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right)$

54. List the first 5 elements of the sequence.

$$(a) a_{n+1} = \frac{1}{2} \left(a_n + \frac{3}{a_n} \right), a_1 = 1.$$

$$(b) a_n = a_{n-1} + a_{n-2}, a_1 = 1, a_2 = 1.$$

$$(c) a_n = \frac{\left(\frac{1}{2} - n\right)}{n} a_{n-1}, a_0 = 1.$$

$$(d) a_n = a_{n-1} + 2n + 1, a_0 = 1.$$

$$(e) a_n := \frac{1}{n} a_{n-1}, a_1 = 1.$$

55. Give a simple sequence formula that matches the pattern below.

$$(a) \left(1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \dots \right).$$

$$(b) \left(-1, \frac{1}{5}, -\frac{1}{25}, \frac{1}{125}, -\frac{1}{625}, \frac{1}{3125}, \dots \right)$$

$$(c) \left(-5, 2, -\frac{4}{5}, \frac{8}{25}, -\frac{16}{125}, \frac{32}{625}, \dots \right)$$

$$(d) (4, 7, 10, 13, 16, 19, \dots)$$

$$(e) \left(-2, \frac{3}{4}, -\frac{4}{9}, \frac{5}{16}, -\frac{6}{25}, \frac{7}{36}, \dots \right)$$

$$(f) (0, -1, 0, 1, 0, -1, 0, 1, 0, -1, 0, 1, \dots)$$

56. Determine if the sequence is convergent or divergent. If convergent, find the limit of the sequence.

$$(a) a_n = n.$$

$$(b) a_n = 2^n.$$

$$(c) a_n = 1.0001^n.$$

$$(d) a_n = 0.999999^n.$$

$$(e) a_n = n - \sqrt{n+1}\sqrt{n+2}$$

$$(f) a_n = \frac{\ln n}{n}.$$

$$(g) a_n = \frac{\ln n}{\sqrt[10]{n}}.$$

$$(h) a_n = \frac{1}{n}.$$

$$(i) a_n = \frac{1}{n!}.$$

$$(j) a_n = \frac{n^n}{n!}.$$

$$(k) a_n = \cos n.$$

$$(l) a_n = \cos \left(\frac{1}{n} \right)$$

$$(m) a_n = \left(\frac{n+1}{n} \right)^n.$$

$$(n) a_n = \left(\frac{2n+1}{n} \right)^n.$$

$$(o) a_n = \left(\frac{n+1}{n} \right)^{2n}.$$

$$(p) a_n = \left(\frac{n+1}{2n} \right)^n.$$

57. Express the infinite decimal number as a rational number.

$$(a) 0.\overline{9} = 0.99999 \dots$$

$$(b) 1.\overline{6} = 1.6666 \dots$$

$$(c) 1.\overline{3} = 1.3333 \dots$$

$$(d) 1.\overline{19} = 1.191919 \dots$$

$$(e) 0.\overline{09} = 0.09090909 \dots$$

$$(f) 2.\overline{16} = 2.16161616 \dots$$

$$(g) 2014.\overline{2014} = 2014.201420142014 \dots$$

58. Express the sum of the series as a rational number.

$$(a) \sum_{n=1}^{\infty} \frac{2^n + 3^n}{5^n}$$

$$(b) \sum_{n=0}^{\infty} \frac{2^n + 5^n}{10^n}$$

$$(c) \sum_{n=1}^{\infty} \frac{5^n - 3^n}{7^n}$$

$$(d) \sum_{n=1}^{\infty} \frac{3^{n+1} + 7^{n-1}}{21^n}$$

$$(e) \sum_{n=0}^{\infty} \frac{2^{n+1} + (-3)^{n-1}}{5^n}$$

59. Sum the telescoping series (a sum is “telescoping” if it can be broken into summands so that consecutive terms cancel).

$$(a) \sum_{n=0}^{\infty} \frac{-6}{9n^2 + 3n - 2}.$$

$$(b) \sum_{n=3}^{\infty} \frac{3}{n^2 - 3n + 2}.$$

(c) $\sum_{n=2}^{\infty} \ln \left(1 - \frac{1}{n^2} \right)$. (Hint: Use the properties of the logarithm to aim for a telescoping series).

60. Use partial fractions to sum the telescoping series (a sum is “telescoping” if it can be broken into summands so that consecutive terms cancel).

(a) $\sum_{n=1}^{\infty} \frac{1}{n^2 + n}$

(c) $\sum_{n=1}^{\infty} \frac{2n}{n^4 - 3n^2 + 1}$

(b) $\sum_{n=2}^{\infty} \frac{2n + 1}{n^4 + 2n^3 - n^2 - 2n}$

(d) $\sum_{n=3}^{\infty} \frac{n^2 + n + 2}{n^4 - 5n^2 + 4}$

61. Find whether the series is convergent or divergent using an appropriate test. Some of the problems require the alternating series test. The test states the following.

Alternating series test. Suppose $b_n \searrow 0$. Then $\sum (-1)^n b_n$ is convergent.

Here, $b_n \searrow 0$ means the following.

- The sequence of numbers b_n is decreasing.
- The sequence decreases to 0, that is,

$$\lim_{n \rightarrow \infty} b_n = 0.$$

(a) $\sum_{n=1}^{\infty} (-1)^n \ln n$.

(c) $\sum_{n=2}^{\infty} \frac{n}{\ln n}$

(b) $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$.

(d) $\sum_{n=2}^{\infty} \frac{\ln n}{n}$

62. Use the integral test, the comparison test or the limit comparison test to determine whether the series is convergent or divergent. Justify your answer.

(a) $\sum_{n=1}^{\infty} \frac{1}{2n + 1}$.

(f) $\sum_{n=2}^{\infty} \frac{1}{(2n + 1) \ln(n)}$.

(b) $\sum_{n=1}^{\infty} \frac{1}{2n^2 + n^3}$.

(g) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

(c) $\sum_{n=1}^{\infty} \frac{n^2 + 3}{3n^5 + n}$

(h) $\sum_{n=2}^{\infty} \frac{1}{(2n + 1)(\ln(n))^2}$.

(d) $\sum_{n=0}^{\infty} \frac{1}{3^n + 5}$.

(i) Determine all values of p, q, r for which the series

(e) $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

$$\sum_{n=30}^{\infty} \frac{1}{n^p (\ln n)^q (\ln(\ln n))^r}$$

is convergent.

63. Establish whether the series is convergent or divergent. Use the ratio or root tests. Show all your work. The answer key has not been proofread, use with caution.

(a) $\sum_{n=0}^{\infty} (-1)^n n^2 3^{-n}$

(b) $\sum_{n=1}^{\infty} \left(\frac{n+1}{4n} \right)^n$

(c) $\sum_{n=1}^{\infty} \left(\frac{4n+1}{n} \right)^n$

$$(d) \sum_{n=1}^{\infty} \frac{n^n}{4^n n!}$$

$$(e) \sum_{n=1}^{\infty} \frac{(4n)^n}{n!}$$

64. Except for $x = \pm e$, use the ratio test to determine all real values of x for which

$$\sum_{n=0}^{\infty} x^n \frac{n!}{n^n}$$

is convergent. You are expected to use in your solution the fact that

$$\lim_{x \rightarrow 0} \left(1 + \frac{x}{n}\right)^n = e^x \quad .$$