Calculus II
Weierstrass substitution, part 2

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Example

$$\int \frac{\mathrm{d}\theta}{2\sin\theta - \cos\theta + 5}$$

Example

Let $\theta = 2\arctan t$, $\cos\theta = \frac{1-t^2}{1+t^2}$, $\sin\theta = \frac{2t}{1+t^2}$

$$\int \frac{\mathrm{d}\theta}{2\sin\theta - \cos\theta + 5} = \int \frac{2\mathrm{d}t}{(1+t^2)\left(2\frac{2t}{t^2+1} - \frac{(1-t^2)}{1+t^2} + 5\right)}$$

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$$\begin{aligned}\int \frac{\mathrm{d}\theta}{2\sin\theta - \cos\theta + 5} &= \int \frac{2\mathrm{d}t}{(1+t^2)\left(2\frac{2t}{t^2+1} - \frac{(1-t^2)}{1+t^2} + 5\right)} \\ &= \int \frac{2\mathrm{d}t}{6t^2 + 4t + 4}\end{aligned}$$

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Example

Let $\theta = 2\arctan t$, $\cos\theta = \frac{1-t^2}{1+t^2}$, $\sin\theta = \frac{2t}{1+t^2}$

$$\begin{aligned}\int \frac{\mathrm{d}\theta}{2\sin\theta - \cos\theta + 5} &= \int \frac{2\mathrm{d}t}{(1+t^2)\left(2\frac{2t}{t^2+1} - \frac{(1-t^2)}{1+t^2} + 5\right)}\\ &= \int \frac{2\mathrm{d}t}{6t^2+4t+4}\\ &= \int \frac{\mathrm{d}t}{{\color{red}3}t^2+2t+2}\\ &= \int \frac{\mathrm{d}t}{{\color{red}3}\left(t^2+2t\frac{1}{\color{red}3} \qquad\qquad + \frac{2}{\color{red}3}\right)}\end{aligned}$$

Example

Let $\theta = 2\arctan t$, $\cos\theta = \frac{1-t^2}{1+t^2}$, $\sin\theta = \frac{2t}{1+t^2}$

$$\begin{aligned}
\int \frac{\mathrm{d}\theta}{2\sin\theta - \cos\theta + 5} &= \int \frac{2\mathrm{d}t}{(1+t^2)\left(2\frac{2t}{t^2+1} - \frac{(1-t^2)}{1+t^2} + 5\right)} \\
&= \int \frac{2\mathrm{d}t}{6t^2+4t+4} \\
&= \int \frac{\mathrm{d}t}{3t^2+2t+2} \\
\text{(complete square)} \quad &= \int \frac{\mathrm{d}t}{3\left(t^2 + 2t\frac{1}{3} + \frac{1}{9} - \frac{1}{9} + \frac{2}{3}\right)}
\end{aligned}$$

Example

Let $\theta = 2\arctan t$, $\cos\theta = \frac{1-t^2}{1+t^2}$, $\sin\theta = \frac{2t}{1+t^2}$

$$\begin{aligned}
\int \frac{\mathrm{d}\theta}{2\sin\theta - \cos\theta + 5} &= \int \frac{2\mathrm{d}t}{(1+t^2)\left(2\frac{2t}{t^2+1} - \frac{(1-t^2)}{1+t^2} + 5\right)} \\
&= \int \frac{2\mathrm{d}t}{6t^2 + 4t + 4} \\
&= \int \frac{\mathrm{d}t}{3t^2 + 2t + 2} \\
&= \int \frac{\mathrm{d}t}{3\left(t^2 + 2t\frac{1}{3} + \frac{1}{9} - \frac{1}{9} + \frac{2}{3}\right)} \\
&= \frac{1}{3}\int \frac{\mathrm{d}t}{\left(t + \frac{1}{3}\right)^2 + \frac{5}{9}}
\end{aligned}$$

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Let $\theta = 2 \arctan t$, $\cos\theta = \frac{1-t^2}{1+t^2}$, $\sin\theta = \frac{2t}{1+t^2}$

$$\begin{aligned}
\int \frac{\mathrm{d}\theta}{2\sin\theta - \cos\theta + 5} &= \int \frac{2\mathrm{d}t}{(1+t^2)\left(2\frac{2t}{t^2+1} - \frac{(1-t^2)}{1+t^2} + 5\right)} \\
&= \int \frac{2\mathrm{d}t}{6t^2 + 4t + 4} \\
&= \int \frac{\mathrm{d}t}{3t^2 + 2t + 2} \\
&= \int \frac{\mathrm{d}t}{3\left(t^2 + 2t\frac{1}{3} + \frac{1}{9} - \frac{1}{9} + \frac{2}{3}\right)} \\
&= \frac{1}{3}\int \frac{\mathrm{d}t}{\left(t+\frac{1}{3}\right)^2 + \frac{5}{9}}
\end{aligned}$$

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Let $\theta = 2\arctan t$, $\cos\theta = \frac{1-t^2}{1+t^2}$, $\sin\theta = \frac{2t}{1+t^2}$

$$\begin{aligned}
\int \frac{d\theta}{2\sin\theta - \cos\theta + 5} &= \int \frac{2dt}{(1+t^2)\left(2\frac{2t}{t^2+1} - \frac{(1-t^2)}{1+t^2} + 5\right)} \\
&= \int \frac{2dt}{6t^2+4t+4} \\
&= \int \frac{dt}{3t^2+2t+2} \\
&= \int \frac{dt}{3\left(t^2+2t\frac{1}{3}+\frac{1}{9}-\frac{1}{9}+\frac{2}{3}\right)} \\
&= \frac{1}{3}\int \frac{dt}{\left(t+\frac{1}{3}\right)^2+\frac{5}{9}} \\
&= \frac{1}{3}\int \frac{dt}{\frac{5}{9}\left(\frac{9}{5}\left(t+\frac{1}{3}\right)^2+1\right)}
\end{aligned}$$

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$$
\begin{aligned}
\int \frac{\mathrm{d}\theta}{2\sin\theta - \cos\theta + 5} &= \int \frac{2\mathrm{d}t}{(1+t^2)\left(2\frac{2t}{t^2+1} - \frac{(1-t^2)}{1+t^2} + 5\right)} \\
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&= \frac{1}{3}\int \frac{\mathrm{d}t}{\left(t+\frac{1}{3}\right)^2+\frac{5}{9}} \\
&= \frac{1}{3}\int \frac{\mathrm{d}t}{\frac{5}{9}\left(\frac{9}{5}\left(t+\frac{1}{3}\right)^2+1\right)}
\end{aligned}
$$

Example

Let $\theta = 2\arctan t$, $\cos\theta = \frac{1-t^2}{1+t^2}$, $\sin\theta = \frac{2t}{1+t^2}$

$$\int \frac{\mathrm{d}\theta}{2\sin\theta - \cos\theta + 5} = \frac{1}{3}\int \frac{\mathrm{d}t}{\frac{5}{9}\left(\frac{9}{5}\left(t+\frac{1}{3}\right)^2+1\right)}$$

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\int \frac{\mathrm{d}\theta}{2\sin\theta - \cos\theta + 5} &= \frac{1}{3}\int \frac{\mathrm{d}t}{\frac{5}{9}\left(\frac{9}{5}\left(t+\frac{1}{3}\right)^2 + 1\right)} \\
&= \frac{3}{5}\int \frac{\mathrm{d}\left(\quad\left(t \quad\right)\right)}{\left(\left(\frac{3}{\sqrt{5}}\left(t+\frac{1}{3}\right)\right)^2 + 1\right)}
\end{aligned}$$

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Let $\theta = 2\arctan t$, $\cos\theta = \frac{1-t^2}{1+t^2}$, $\sin\theta = \frac{2t}{1+t^2}$

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&= \frac{3}{5}\int \frac{\mathrm{d}\left(\quad\left(t\quad\right)\right)}{\left(\left(\frac{3}{\sqrt{5}}\left(t+\frac{1}{3}\right)\right)^2 + 1\right)}
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Let $\theta = 2\arctan t$, $\cos\theta = \frac{1-t^2}{1+t^2}$, $\sin\theta = \frac{2t}{1+t^2}$, $z = \frac{3}{\sqrt{5}}\left(t+\frac{1}{3}\right)$.

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$$\begin{aligned}
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&= \frac{3}{5}\int \frac{\frac{\sqrt{5}}{3}\mathrm{d}\left(\frac{3}{\sqrt{5}}\left(t+\frac{1}{3}\right)\right)}{\left(\left(\frac{3}{\sqrt{5}}\left(t+\frac{1}{3}\right)\right)^2+1\right)} \\
&= \frac{\sqrt{5}}{5}\int \frac{\mathrm{d}z}{z^2+1} \\
&= \frac{\sqrt{5}}{5}\arctan z + C
\end{aligned}$$

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&= \frac{\sqrt{5}}{5}\int \frac{\mathrm{d}z}{z^2+1} \\
&= \frac{\sqrt{5}}{5}\arctan z + C \\
&= \frac{\sqrt{5}}{5}\arctan\left(\frac{3}{\sqrt{5}}\left(t+\frac{1}{3}\right)\right) + C
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&= \frac{\sqrt{5}}{5}\int \frac{\mathrm{d}z}{z^2+1} \\
&= \frac{\sqrt{5}}{5}\arctan z + C \\
&= \frac{\sqrt{5}}{5}\arctan\left(\frac{3}{\sqrt{5}}\left(t+\frac{1}{3}\right)\right) + C \\
&= \frac{\sqrt{5}}{5}\arctan\left(\frac{3}{\sqrt{5}}\left(\tan\left(\frac{\theta}{2}\right)+\frac{1}{3}\right)\right) + C
\end{aligned}$$