Precalculus
Graphs of trig functions; inverse trig

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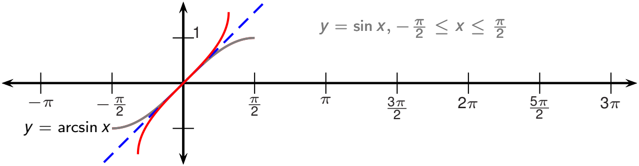
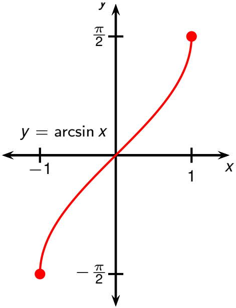
Outline

1. Inverse Trigonometric Functions
   - The arcsine function
   - The arccosine function
   - The arctangent and the remaining inverse trig functions
   - Trigonometric Functions with Inverse Trig Arguments

Inverse Trigonometric Functions



- $\sin x$ isn't one-to-one.
- It is if we restrict the domain to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.
- Then it has an inverse function.
- We call it arcsin or $\sin^{-1}$.
- $\arcsin x = y \Leftrightarrow \sin y = x$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$.



Observation

- $\arcsin y =$ *the appropriate angle whose sine equals y.*
- *Important: the output angle must lie in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.*

Example

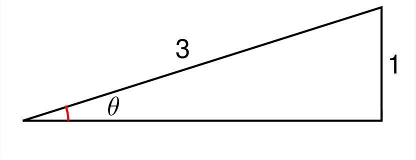
$$\text{Find } \arcsin\left(\frac{1}{2}\right).$$

- $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$.
- $-\frac{\pi}{2} \leq \frac{\pi}{6} \leq \frac{\pi}{2}$.
- Therefore $\arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$.

Example

Find $\tan\left(\arcsin\left(\frac{1}{3}\right)\right)$.

- Let $\theta = \arcsin\left(\frac{1}{3}\right)$, so $\sin\theta = \frac{1}{3}$.
- Draw a right triangle with opposite side 1 and hypotenuse 3.
- Let the angle θ be as labeled. Then $\sin\theta = \frac{1}{3}$ and so $\theta = \arcsin\left(\frac{1}{3}\right)$.
- Length of adjacent side $= \sqrt{3^2 - 1^2} = \sqrt{8} = 2\sqrt{2}$.
- Then $\tan\left(\arcsin\left(\frac{1}{3}\right)\right) = \frac{1}{2\sqrt{2}}$.



Example

Find $\arcsin(\sin(1.5))$.

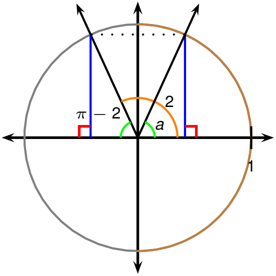
- $\frac{\pi}{2} \approx 1.57$.
- Therefore $-\frac{\pi}{2} \leq 1.5 \leq \frac{\pi}{2}$.
- Therefore $\arcsin(\sin 1.5) = 1.5$.

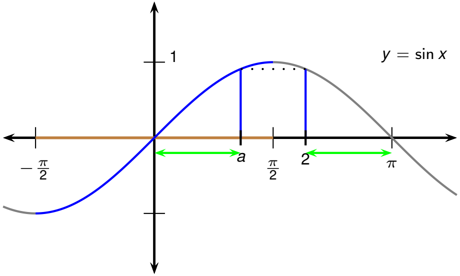
Example

Find $\arcsin(\sin 2)$.

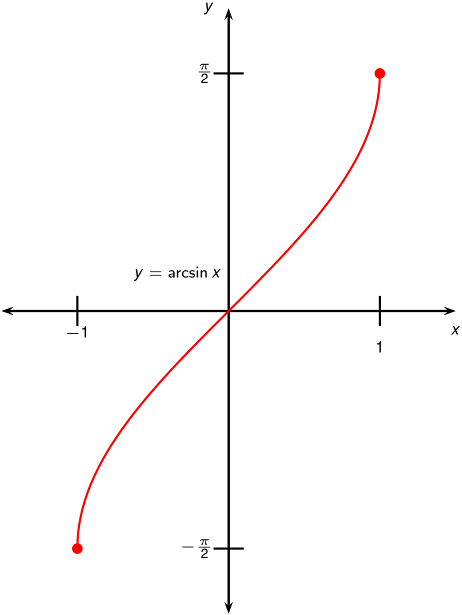
- 2 is not between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.
- We need the angle a between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ for which $\sin 2 = \sin a$.

$$\begin{array}{rrcl} & a & = & \pi - 2. \\ \text{Therefore} & \arcsin(\sin 2) & = & \arcsin(\sin a) \\ & & = & a = \pi - 2. \end{array}$$

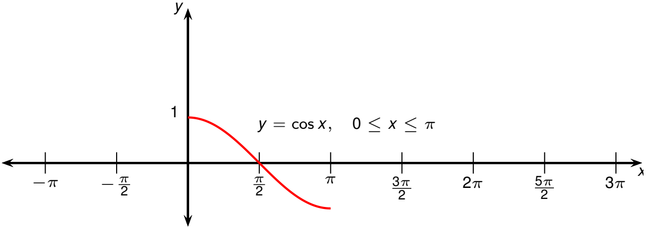




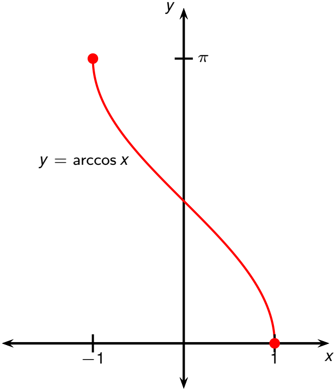
Important facts about arcsin:



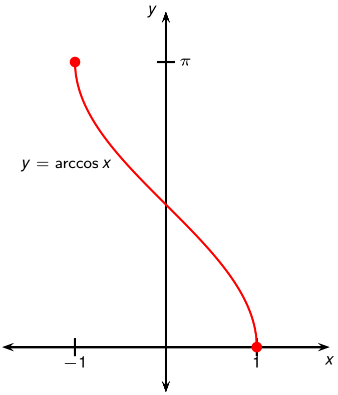
1. Domain: $[-1, 1]$.
2. Range: $[-\frac{\pi}{2}, \frac{\pi}{2}]$.
3. $\arcsin x = y \Leftrightarrow \sin y = x$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$.
4. $\arcsin(\sin x) = x$ for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.
5. $\sin(\arcsin x) = x$ for $-1 \leq x \leq 1$.
6. $\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$.



- Same for $\cos x$.
- Restrict the domain to $[0, \pi]$.
- The inverse is called arccos or $\cos^{-1}$.
- $\arccos(x) = y \Leftrightarrow \cos y = x$ and $0 \leq y \leq \pi$.



Important facts about arccos:



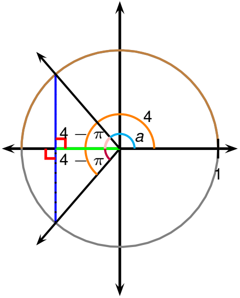
1. Domain: $[-1, 1]$.
2. Range: $[0, \pi]$.
3. $\arccos x = y \Leftrightarrow \cos y = x$ and $0 \leq y \leq \pi$.
4. $\arccos(\cos x) = x$ for $0 \leq x \leq \pi$.
5. $\cos(\arccos x) = x$ for $-1 \leq x \leq 1$.
6. $\frac{\mathrm{d}}{\mathrm{d}x}(\arccos x) = -\frac{1}{\sqrt{1-x^2}}$. (The proof is similar to the proof of the formula for the derivative of $\arcsin x$.)

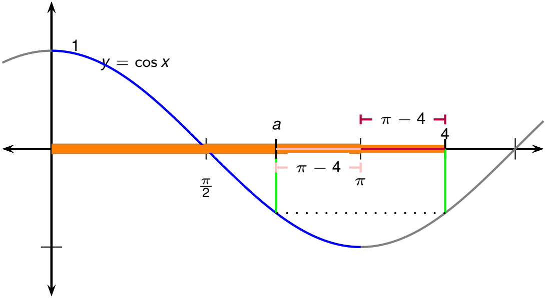
Example

Find $\arccos(\cos 4)$.

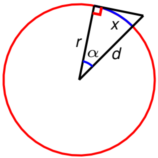
- 4 is not between 0 and π.
- We need the angle a between 0 and π for which $\cos 4 = \cos a$.

$$\begin{aligned} a &= \pi - (4-\pi) = 2\pi - 4 \\ \text{Therefore}\quad \arccos(\cos 4) &= \arccos(\cos a) \\ &= a = 2\pi - 4. \end{aligned}$$





Example

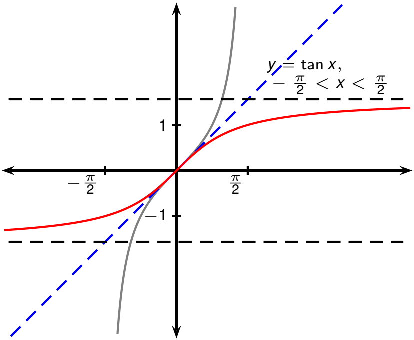


not to scale

The horizon at sea is viewed at from a ship height of 10m. How far does the ship need to travel to reach the horizon just observed? Assume earth is round with radius 6371 km and that the ship sails along the shortest curved path.

- Let d be the distance from eyes of seaman to the center of earth.
- Let r be the radius of earth. Let α be the indicated angle.
- Let the distance to the horizon be x.

$$\begin{aligned}
r &= 6371\text{km}\\
d &= 6371\text{km} + 0.01\text{km} = 6371.01\text{km}\\
\cos\alpha &= \frac{r}{d}\\
\alpha &= \arccos\left(\frac{r}{d}\right)\\
x &= r\alpha = r\arccos\left(\frac{r}{d}\right) = 6371\text{km}\arccos\left(\frac{6371\text{km}}{6371.01\text{km}}\right) \approx 11.29\text{km}
\end{aligned}$$

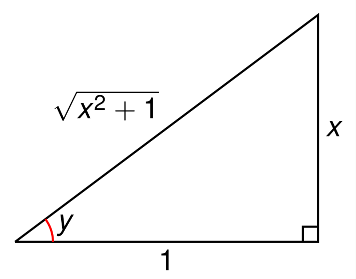


- $\tan x$ isn't one-to-one.
- Restrict the domain to $(-\frac{\pi}{2}, \frac{\pi}{2})$.
- The inverse is called $\tan^{-1}$ or $\arctan$.
- $\arctan x = y \Leftrightarrow \tan y = x$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$.
- Domain of $\arctan$: $(-\infty, \infty)$.
- Range of $\arctan$: $(-\frac{\pi}{2}, \frac{\pi}{2})$.
- $\displaystyle\lim_{x\to\infty} \arctan x = \frac{\pi}{2}$.
- $\displaystyle\lim_{x\to-\infty} \arctan x = -\frac{\pi}{2}$.

Example

Simplify the expression $\cos(\arctan x)$.

- Let $y = \arctan x$, so $\tan y = x$.
- Draw a right triangle with opposite x and adjacent 1.
- Length of hypotenuse $= \sqrt{1^2 + x^2}$.
- Then $\cos(\arctan x) = \frac{1}{\sqrt{1+x^2}}$.

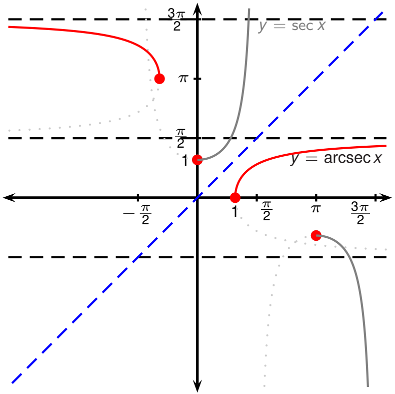


The remaining inverse trigonometric functions aren't used as often:

$$\begin{array}{llcll}
y = \operatorname{arccsc} x & (|x| \geq 1) & \Leftrightarrow & \csc y = x & \text{and} \quad y \in \left(0, \frac{\pi}{2}\right] \cup \left(\pi, \frac{3\pi}{2}\right] \\
y = \operatorname{arcsec} x & (|x| \geq 1) & \Leftrightarrow & \sec y = x & \text{and} \quad y \in \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right) \\
y = \operatorname{arccot} x & (|x| \in \mathbb{R}) & \Leftrightarrow & \cot y = x & \text{and} \quad y \in (0, \pi)
\end{array}$$

We will however make use of $\operatorname{arcsec} x$: we discuss in detail its domain.

$$y = \operatorname{arcsec} x \quad (|x| \geq 1) \quad \Leftrightarrow \quad \sec y = x \quad \text{and} \quad y \in \mathbf{?} \left[0, \tfrac{\pi}{2}\right) \cup \left[\pi, \tfrac{3\pi}{2}\right)$$



- Plot $\sec x$.
- Restrict domain to make one-to-one: Two common choices: $x \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ and $x \in \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$.
- $x \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ is good because the domain is easiest to remember: an interval without a point. **NOT our choice.**
- $x \in \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$ is good because $\tan x$ is positive on both intervals, resulting in easier differentiation and integration formulas. **Our choice.**

Example

Rewrite $\sin(2\arccos(x))$ as an algebraic expression of x and $\sqrt{1-x^2}$.
To simplify $\arccos x$ we try to use $\cos(\arccos x) = x$. Therefore our aim is to rewrite the expression only using the $\cos$ function.

$$
\begin{aligned}
\sin(2\arccos(x)) &= \sin(2y) && \text{Set } y = \arccos x \\
&= 2\cos y \sin y && \text{Express via } \sin y, \cos y \\
&= 2\cos y \left(\pm\sqrt{1-\cos^2 y}\right) && \text{Express } \sin y \text{ via } \cos y \\
&= 2\cos y \sqrt{1-\cos^2 y} && \sin y > 0 \text{ because } 0 \leq y \leq \pi \\
&= 2x\sqrt{1-x^2} && \text{use } x = \cos y
\end{aligned}
$$

Example

Rewrite $\cos(3\arccos(x))$ as an algebraic expression of x and $\sqrt{1-x^2}$.
To simplify $\arccos x$ we try to use $\cos(\arccos x) = x$. Therefore our aim is to rewrite the expression only using the $\cos$ function.

$$
\begin{array}{rll|l}
\cos(3\arccos(x)) & = \cos(3y) = \cos(2y+y) & & y = \arccos x \\
& = \cos(2y)\cos y - \sin(2y)\sin y & & \text{Angle sum f-la} \\
& = (\cos^2 y - \sin^2 y)\cos y & & \text{Express via} \\
& \quad -2\sin y \cos y \sin y & & \sin y, \cos y \\
& = \cos^3 y - \sin^2 y \cos y - 2\sin^2 y \cos y & & \\
& = \cos^3 y - 3\sin^2 y \cos y & & \text{Express } \sin y \text{ via } \cos y \\
& = \cos^3 y - 3(1-\cos^2 y)\cos y & & \\
& = 4\cos^3 y - 3\cos y & & \\
& = 4x^3 - 3x & & x = \cos y
\end{array}
$$