Calculus II
Building block integrals

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Outline

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Integrating arbitrary rational functions

Let $\frac{P(x)}{Q(x)}$ be an arbitrary rational function, i.e., a quotient of polynomials.

Question

Can we integrate $\int \frac{P(x)}{Q(x)} \mathrm{d}x$*?*

- Yes. We will learn how in what follows.
- The algorithm for integration is roughly:
  - We use algebra to split $\frac{P(x)}{Q(x)}$ into smaller pieces (“partial fractions”).
  - We use linear substitutions to transform each piece to one of 3 pairs of basic building block integrals.
  - We solve each building block integral and collect the terms.
- We study the algorithm “from the ground up”: we start with the building blocks.

The building blocks

Let n be a positive integer.

- (Building block I) The first building block integral is:
$$\int \frac{1}{x^n} \mathrm{d}x \quad .$$
- (Building block II) The second building block integral is:
$\displaystyle\int \frac{x}{(1+x^2)^n} \mathrm{d}x$. (Note: $u = 1 + x^2, x\mathrm{d}x = \frac{1}{2}\mathrm{d}u$ transforms II to I).
- (Building block III) The third building block integral is:
$$\int \frac{1}{(1+x^2)^n} \mathrm{d}x \quad .$$
- The case $n = 1$ is special for each of the building blocks:
$\displaystyle\int \frac{1}{x} \mathrm{d}x, \int \frac{x}{1+x^2} \mathrm{d}x$ and $\displaystyle\int \frac{1}{1+x^2} \mathrm{d}x$.
- The case $n = 1$ we call respectively building block Ia, IIa and IIIa. The case $n > 1$ we call respectively building block Ib, IIb and IIIb. This “building block” terminology is for our convenience, and is not a part of standard mathematical terminology.

Building block Ia

Building block Ia: $\int \frac{1}{x} \mathrm{d}x$.

Example

Integrate building block Ia

$$\int \frac{1}{x} \mathrm{d}x = \ln|x| + C$$

Linear substitutions leading to building block Ia

Building block Ia: $\displaystyle\int \frac{1}{x}\mathrm{d}x = \ln|x| + C$.

Example

Integrate

$$
\begin{aligned}
\int \frac{1}{-4x+5}\mathrm{d}x &= \int \frac{1}{(-4x+5)}\frac{\mathrm{d}(-4x)}{(-4)} \\
&= \int \frac{1}{(-4x+5)}\frac{\mathrm{d}(-4x+5)}{(-4)} && \Big|\ \text{Set } u=-4x+5 \\
&= \int \frac{1}{u}\frac{\mathrm{d}u}{(-4)} \\
&= -\frac{1}{4}\int u^{-1}\mathrm{d}u = -\frac{1}{4}\ln|u| + C \\
&= -\frac{1}{4}\ln|-4x+5| + C \quad .
\end{aligned}
$$

Lin. subst. leading to building block Ia: general case

Building block Ia: $\displaystyle\int \frac{1}{x}\mathrm{d}x = \ln|x| + C.$

Example

Integrate

$$
\begin{aligned}
\int \frac{1}{ax+b}\mathrm{d}x &= \int \frac{1}{(\ \ ax+b)}\frac{\mathrm{d}(\ \ ax)}{a} \\
&= \int \frac{1}{(\ \ ax+b)}\frac{\mathrm{d}(\ \ ax+b)}{a} && \Big|\ \text{Set } u = ax+b \\
&= \int \frac{1}{u}\frac{\mathrm{d}u}{a} \\
&= \quad \frac{1}{a}\int u^{-1}\mathrm{d}u = \quad \frac{1}{a}\ln|u| + C \\
&= \quad \frac{1}{a}\ln|ax+b| + C \quad .
\end{aligned}
$$

Building block Ib

Building block Ib: $\int \frac{1}{x^n} \mathrm{d}x = \int x^{-n} \mathrm{d}x$, $n \neq 1$.

Example (Block Ib)

$$\int \frac{1}{x^n} \mathrm{d}x = \int x^{-n} \mathrm{d}x = \frac{x^{-n+1}}{-n+1} + C$$

Linear substitutions leading to building block Ib

Building block Ib: $\displaystyle\int \frac{1}{x^n}\mathrm{d}x = \int x^{-n}\mathrm{d}x = \frac{x^{-n+1}}{-n+1} + C,\ n\neq 1.$

Example

Integrate

$$\begin{aligned}
\int \frac{1}{(3x+5)^3}\mathrm{d}x &= \int \frac{1}{(3x+5)^3}\frac{\mathrm{d}(3x)}{3}\\
&= \int \frac{1}{(3x+5)^3}\frac{\mathrm{d}(3x+5)}{3} && \Big|\ \text{Set } u = 3x+5\\
&= \int \frac{1}{u^3}\frac{\mathrm{d}u}{3}\\
&= \frac{1}{3}\int u^{-3}\mathrm{d}u = \frac{1}{3}\frac{u^{-2}}{(-2)} + C\\
&= -\frac{1}{6(3x+5)^2} + C \quad .
\end{aligned}$$

Lin. subst. leading to building block Ib: general case

Building block Ib: $\displaystyle\int \frac{1}{x^n}\mathrm{d}x = \int x^{-n}\mathrm{d}x = \frac{x^{-n+1}}{-n+1} + C,\ n \neq 1.$

Example

Let $n \neq 1$. Integrate

$$
\begin{aligned}
\int \frac{1}{(ax+b)^n}\mathrm{d}x &= \int \frac{1}{(ax+b)^n}\frac{\mathrm{d}(ax)}{a} \\
&= \int \frac{1}{(ax+b)^n}\frac{\mathrm{d}(ax+b)}{a} && \Big|\ \text{Set } u = ax+b \\
&= \int \frac{1}{u^3}\frac{\mathrm{d}u}{a} \\
&= \frac{1}{a}\int u^{-n}\mathrm{d}u = -\frac{1}{a}\frac{u^{-n+1}}{(n-1)} + C \\
&= -\frac{1}{a(n-1)(ax+b)^{n-1}} + C \quad .
\end{aligned}
$$

Building blocks IIa and IIIa

Building block IIa: $\int \frac{x}{1+x^2}\mathrm{d}x$. Building block IIIa: $\int \frac{1}{1+x^2}\mathrm{d}x$.

Example (Block IIa)

$$\begin{aligned}\int \frac{x}{1+x^2}\mathrm{d}x &= \int \frac{1}{(1+x^2)}\frac{\mathrm{d}(x^2)}{2}\\ &= \int \frac{1}{1+x^2}\frac{\mathrm{d}(1+x^2)}{2} && \Big|\ \text{Set } u = 1+x^2\\ &= \int \frac{1}{u}\frac{\mathrm{d}u}{2}\\ &= \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln\left(1+x^2\right) + C \quad .\end{aligned}$$

Example (Block IIIa)

$$\int \frac{1}{1+x^2}\mathrm{d}x = \arctan x + C$$

Linear substitutions leading to block IIa

Building block IIa: $\int \frac{x}{1+x^2} \mathrm{d}x = \frac{1}{2} \ln(1+x^2) + C$.
"Theoretical way" to solve example below: transform to IIa; this is slow.
Feel free to skip slide, we will redo in next slide with a shortcut.

Example

$$
\begin{aligned}
\int \frac{x}{2x^2+3} \mathrm{d}x &= \int \frac{x}{3\left(\frac{2}{3}x^2+1\right)} \mathrm{d}x = \int \frac{x}{3\left(\left(\sqrt{\frac{2}{3}}x\right)^2+1\right)} \mathrm{d}x \\
&= \frac{3}{2} \int \frac{\sqrt{\frac{2}{3}}x}{3\left(\left(\sqrt{\frac{2}{3}}x\right)^2+1\right)} \mathrm{d}\left(\sqrt{\frac{2}{3}}x\right) \qquad \Bigg| \text{ Set } u = \sqrt{\frac{2}{3}}x \\
&= \frac{1}{2} \int \frac{u}{u^2+1} \mathrm{d}u = \frac{1}{4} \ln(1+u^2) + C \\
&= \frac{1}{4} \ln\left(\frac{1}{3}(2x^2+3)\right) + C \\
&= \frac{1}{4} \ln(2x^2+3) + \frac{\ln\left(\frac{1}{3}\right)}{4} + C \\
&= \frac{1}{4} \ln(2x^2+3) + K \quad .
\end{aligned}
$$

Linear substitutions leading to blocks IIa

Building block IIa: $\int \frac{x}{1+x^2}\mathrm{d}x = \frac{1}{2}\ln(1+x^2)+C$.
The example below can be done directly, without transforming to block IIa.

Example

$$
\begin{array}{rcll}
\displaystyle\int \frac{x}{2x^2+3}\mathrm{d}x &=& \displaystyle\int \frac{1}{2x^2+3}\mathrm{d}\left(\frac{x^2}{2}\right) & \\
&=& \displaystyle\int \frac{1}{2x^2+3}\mathrm{d}\left(\frac{2x^2+3}{4}\right) & \left|\ \text{Set } u = 2x^2+3\right. \\
&=& \displaystyle\frac{1}{4}\int \frac{1}{u}\mathrm{d}u & \\
&=& \displaystyle\frac{1}{4}\ln|u| + C & \\
&=& \displaystyle\frac{1}{4}\ln(2x^2+3) + C &
\end{array}
$$

Linear substitutions leading to blocks IIa and IIIa

Building block IIIa: $\int \frac{1}{u^2+1} \mathrm{d}u = \arctan u + C$.

Example

$$\begin{aligned}\int \frac{1}{x^2+2}\mathrm{d}x &= \int \frac{1}{2\left(\frac{1}{2}x^2+1\right)}\mathrm{d}x \\ &= \int \frac{1}{2\left(\left(\frac{x}{\sqrt{2}}\right)^2+1\right)}\sqrt{2}\mathrm{d}\left(\frac{x}{\sqrt{2}}\right) \\ &= \frac{1}{\sqrt{2}}\int \frac{1}{u^2+1}\mathrm{d}u \\ &= \frac{1}{\sqrt{2}}\arctan(u) + C \\ &= \frac{1}{\sqrt{2}}\arctan\left(\frac{x}{\sqrt{2}}\right) + C\end{aligned}$$

Use $2 = \left(\sqrt{2}\right)^2$

Set $\frac{x}{\sqrt{2}} = u$

Linear substitutions leading to blocks IIa and IIIa

Building block IIIa: $\int \frac{1}{x^2+1}\mathrm{d}x = \arctan x + C$. **Let** $a > 0$**.**

Example

$$\begin{aligned}
\int \frac{1}{x^2+a}\mathrm{d}x &= \int \frac{1}{a\left(\frac{1}{a}x^2+1\right)}\mathrm{d}x \\
&= \int \frac{1}{a\left(\left(\frac{x}{\sqrt{a}}\right)^2+1\right)}\sqrt{a}\,\mathrm{d}\left(\frac{x}{\sqrt{a}}\right) \\
&= \frac{1}{\sqrt{a}}\int \frac{1}{u^2+1}\mathrm{d}u \\
&= \frac{1}{\sqrt{a}}\arctan(u) + C \\
&= \frac{1}{\sqrt{a}}\arctan\left(\frac{x}{\sqrt{a}}\right) + C
\end{aligned}$$

Use $a = \left(\sqrt{a}\right)^2$

Set $u = \frac{x}{\sqrt{a}}$

Linear substitutions leading to blocks IIa and IIIa

Building block IIa: $\int \frac{x}{1+x^2} \mathrm{d}x = \frac{1}{2} \ln(1+x^2) + C$.
Building block IIIa: $\int \frac{1}{1+x^2} \mathrm{d}x = \arctan x + C$.

- Let $ax^2 + bx + c$ have no real roots.
- We can find p, q so that the linear substitution $u = px + q$ transforms the quadratic to:

$$ax^2 + bx + c = r(u^2 + 1)$$

  (where r is some number to be determined).
- To find p, q, we complete the square.
- In this way, integrals of the form $\displaystyle\int \frac{Ax+B}{ax^2+bx+c} \mathrm{d}x$ are transformed to combinations of building blocks IIa and IIIa.
- We show examples; the general case is analogous and we leave it to the student.

Linear substitutions leading to blocks IIa and IIIa

Building block IIa: $\int \frac{x}{1+x^2}\mathrm{d}x = \frac{1}{2}\ln(1+x^2) + C$.
Building block IIIa: $\int \frac{1}{1+x^2}\mathrm{d}x = \arctan x + C$.

Example

No real roots $\Rightarrow$ complete the square. Let $u = x + \frac{1}{2}$, let $z = \frac{2u}{\sqrt{3}}$.

$$\begin{aligned}
\int \frac{x}{x^2+x+1}\mathrm{d}x &= \int \frac{x}{x^2 + 2\cdot\frac{1}{2}x + \frac{1}{4} - \frac{1}{4} + 1}\mathrm{d}x \\
&= \int \frac{x + \frac{1}{2} - \frac{1}{2}}{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}}\mathrm{d}\left(x + \frac{1}{2}\right) \\
&= \int \frac{u - \frac{1}{2}}{u^2 + \frac{3}{4}}\mathrm{d}u \\
&= \int \frac{u}{u^2+\frac{3}{4}}\mathrm{d}u - \frac{1}{2}\int \frac{1}{u^2+\frac{3}{4}}\mathrm{d}u
\end{aligned}$$

Linear substitutions leading to blocks IIa and IIIa

Building block IIa: $\int \frac{x}{1+x^2}\mathrm{d}x = \frac{1}{2}\ln(1+x^2) + C$.
Building block IIIa: $\int \frac{1}{1+x^2}\mathrm{d}x = \arctan x + C$.

Example

No real roots $\Rightarrow$ complete the square. Let $u = x + \frac{1}{2}$, let $z = \frac{2u}{\sqrt{3}}$.

$$\int \frac{x}{x^2+x+1}\mathrm{d}x = \int \frac{u}{u^2+\frac{3}{4}}\mathrm{d}u - \frac{1}{2}\int \frac{1}{u^2+\frac{3}{4}}\mathrm{d}u$$

$$\begin{aligned}\int \frac{1}{u^2+\frac{3}{4}}\mathrm{d}u &= \int \frac{1}{\frac{3}{4}\left(\frac{4}{3}u^2+1\right)}\mathrm{d}u\\ &= \int \frac{1}{\frac{3}{4}\left(\left(\frac{2u}{\sqrt{3}}\right)^2+1\right)}\frac{\sqrt{3}}{2}\mathrm{d}\left(\frac{2u}{\sqrt{3}}\right)\\ &= \frac{2\sqrt{3}}{3}\int \frac{1}{z^2+1}\mathrm{d}z = \frac{2\sqrt{3}}{3}\arctan z + C\end{aligned}$$

Linear substitutions leading to blocks IIa and IIIa

Building block IIa: $\int \frac{x}{1+x^2}\mathrm{d}x = \frac{1}{2}\ln(1+x^2) + C$.
Building block IIIa: $\int \frac{1}{1+x^2}\mathrm{d}x = \arctan x + C$.

Example

No real roots $\Rightarrow$ complete the square. Let $u = x + \frac{1}{2}$, let $z = \frac{2u}{\sqrt{3}}$.

$$\begin{aligned}\int \frac{x}{x^2+x+1}\mathrm{d}x &= \int \frac{u}{u^2+\frac{3}{4}}\mathrm{d}u - \frac{1}{2}\int \frac{1}{u^2+\frac{3}{4}}\mathrm{d}u\\ &= \int \frac{u}{u^2+\frac{3}{4}}\mathrm{d}u - \frac{1}{2}\frac{2\sqrt{3}}{3}\arctan z + C\end{aligned}$$

$$\begin{aligned}\int \frac{u}{u^2+\frac{3}{4}}\mathrm{d}u &= \int \frac{1}{u^2+\frac{3}{4}}\mathrm{d}\left(\frac{u^2}{2}\right)\\ &= \frac{1}{2}\int \frac{1}{u^2+\frac{3}{4}}\mathrm{d}\left(u^2+\frac{3}{4}\right) = \frac{1}{2}\ln\left(u^2+\frac{3}{4}\right) + C\end{aligned}$$

Linear substitutions leading to blocks IIa and IIIa

Building block IIa: $\int \frac{x}{1+x^2} \mathrm{d}x = \frac{1}{2}\ln(1+x^2) + C$.
Building block IIIa: $\int \frac{1}{1+x^2} \mathrm{d}x = \arctan x + C$.

Example

No real roots $\Rightarrow$ complete the square. Let $u = x + \frac{1}{2}$, let $z = \frac{2u}{\sqrt{3}}$.

$$\begin{aligned}
\int \frac{x}{x^2+x+1} \mathrm{d}x &= \int \frac{u}{u^2+\frac{3}{4}} \mathrm{d}u - \frac{1}{2} \int \frac{1}{u^2+\frac{3}{4}} \mathrm{d}u \\
&= \frac{1}{2} \ln\left(u^2+\frac{3}{4}\right) - \frac{1}{2}\frac{2\sqrt{3}}{3} \arctan z + C \\
&= \frac{1}{2} \ln\left(\left(x+\frac{1}{2}\right)^2+\frac{3}{4}\right) - \frac{\sqrt{3}}{3} \arctan\left(\frac{2u}{\sqrt{3}}\right) + C \\
&= \frac{1}{2} \ln\left(x^2+x+1\right) - \frac{\sqrt{3}}{3} \arctan\left(\frac{2x+1}{\sqrt{3}}\right) + C
\end{aligned}$$

Building blocks IIa and IIb

We solve building block IIb. For completeness, we solve block IIa again as well.

Example

$$
\begin{array}{rcl}
\displaystyle\int \frac{x}{(x^2+1)^n}\mathrm{d}x &=& \displaystyle\int \frac{1}{(x^2+1)^n}\frac{\mathrm{d}\left(x^2+1\right)}{2} \\
&=& \displaystyle\frac{1}{2}\int u^{-n}\mathrm{d}u \\
&=& \begin{cases} \displaystyle\frac{1}{2}\ln(x^2+1)+C & \text{if } n=1 \\ \displaystyle\frac{1}{2}\frac{(x^2+1)^{-n+1}}{(-n+1)}+C & \text{if } n\neq 1 \end{cases},
\end{array}
$$

where we used the substitution $u = x^2 + 1$.

Building block IIIb: example illustrating main idea

Example

Integrate $\int \frac{\mathrm{d}x}{(x^2+1)^2}$. We start with an already known integral:

$$\begin{aligned}
\arctan x + C &= \int \frac{1}{x^2+1}\mathrm{d}x \\
&= \frac{1}{x^2+1}x - \int x\mathrm{d}\left(\frac{1}{x^2+1}\right) \\
&= \frac{x}{x^2+1} - \int x\left(-\frac{2x}{(x^2+1)^2}\right)\mathrm{d}x \\
&= \frac{x}{x^2+1} + 2\int \frac{-1+x^2+1}{(x^2+1)^2}\mathrm{d}x \\
&= \frac{x}{x^2+1} + 2\int \frac{1}{x^2+1}\mathrm{d}x - 2\int \frac{1}{(x^2+1)^2}\mathrm{d}x \\
&= \frac{x}{x^2+1} + 2\arctan x - 2\int \frac{\mathrm{d}x}{(x^2+1)^2}
\end{aligned}$$

Building block IIIb: example illustrating main idea

Example

Integrate $\int \frac{\mathrm{d}x}{(x^2+1)^2}$. We start with an already known integral:

$$\arctan x + C \quad = \quad \frac{x}{x^2+1} + 2\arctan x - 2\int \frac{\mathrm{d}x}{(x^2+1)^2}$$

Rearrange terms and divide by 2 to get the desired integral:

$$\int \frac{\mathrm{d}x}{(1+x^2)^2} = \frac{1}{2}\left(\frac{x}{x^2+1} + \arctan x\right) + \quad K \quad .$$

Building block IIIb

- Building block IIIa:

$$J(1) = \int \frac{1}{(x^2+1)}\mathrm{d}x = \arctan x + C \quad .$$

- Block IIIb:

$$J(n) = \int \frac{1}{(x^2+1)^n}\mathrm{d}x$$

- Unlike other cases, IIIb is much harder than IIIa.
- Set $J(n) = \int \frac{1}{(x^2+1)^n}\mathrm{d}x$. We are looking for a formula for $J(n)$. We know $J(1) = \arctan x + C$ (this is block IIIa).
- We start by $J(n-1) = \int \frac{1}{(x^2+1)^{n-1}}\mathrm{d}x$ and integrate by parts.
- In this way we end up expressing $J(n)$ via $J(n-1)$.
- We work our way from $J(n)$ to $J(n-1)$, from $J(n-1)$ to $J(n-2)$, and so on, until we get to $J(1)$.

Example

Recall that $J(n) = \int \frac{1}{(x^2+1)^n} \mathrm{d}x$. We have that:

$$
\begin{array}{rcl}
J(n-1) & = & \displaystyle\int \frac{1}{(x^2+1)^{n-1}} \mathrm{d}x \\
& = & \displaystyle\frac{1}{(x^2+1)^{n-1}} x - \int x \mathrm{d} \left(\frac{1}{(1+x^2)^{n-1}} \right) \\
& = & \displaystyle\frac{x}{(x^2+1)^{n-1}} - \int x \left(\frac{(-n+1)2x}{(1+x^2)^n} \right) \mathrm{d}x \\
& = & \displaystyle\frac{x}{(x^2+1)^{n-1}} + 2(n-1) \int \frac{1+x^2-1}{(1+x^2)^n} \mathrm{d}x \\
& = & \displaystyle\frac{x}{(x^2+1)^{n-1}} + 2(n-1) \int \frac{1}{(1+x^2)^{n-1}} \mathrm{d}x \\
& & \displaystyle -2(n-1) \int \frac{1}{(1+x^2)^n} \mathrm{d}x \\
& = & \displaystyle\frac{x}{(x^2+1)^{n-1}} + 2(n-1)J(n-1) - 2(n-1)J(n) \quad .
\end{array}
$$

Example

Recall that $J(n) = \int \frac{1}{(x^2+1)^n} \mathrm{d}x$. We have that:

$$J(n-1) = \frac{x}{(x^2+1)^{n-1}} + 2(n-1)J(n-1) - 2(n-1)J(n) \quad .$$

Rearrange to get:

$$\begin{aligned} 2(n-1)J(n) &= \frac{x}{(x^2+1)^{n-1}} + (2n-3)J(n-1) \\ J(n) &= \frac{x}{(2n-2)(x^2+1)^{n-1}} + \frac{2n-3}{2n-2}J(n-1) \quad . \end{aligned}$$

In this way we expressed $J(n)$ using $J(n-1)$. We apply the above formula consecutively:

$J(n) = \frac{x}{(2n-2)(x^2+1)^{n-1}} + \frac{2n-3}{2n-2}\left(\frac{x}{(2n-4)(x^2+1)^{n-2}} + \frac{2n-5}{2n-4}J(n-2)\right) = \dots$

and so on. The above can be used to write a formula for the final result, but that is as complicated as the process above.

Building block integral summary

Type	a	b	Type a, lin. sub.	Type b, lin. sub
I	$\int \frac{1}{x} \mathrm{d}x$	$\int \frac{1}{x^n} \mathrm{d}x$	$\int \frac{A}{ax+b} \mathrm{d}x$	$\int \frac{A}{(ax+b)^n} \mathrm{d}x$
II	$\int \frac{x}{x^2+1} \mathrm{d}x$	$\int \frac{x}{\left(x^2+1\right)^n} \mathrm{d}x$	$\int \frac{A\left(x+\frac{b}{2a}\right)}{ax^2+bx+c} \mathrm{d}x$	$\int \frac{A\left(x+\frac{b}{2a}\right)}{\left(ax^2+bx+c\right)^n} \mathrm{d}x$
III	$\int \frac{1}{x^2+1} \mathrm{d}x$	$\int \frac{1}{\left(x^2+1\right)^n} \mathrm{d}x$	$\int \frac{B}{ax^2+bx+c} \mathrm{d}x$	$\int \frac{B}{\left(ax^2+bx+c\right)^n} \mathrm{d}x$

where A, B are arbitrary constants and a, b, c are constants with $b^2 - 4ac < 0$. The quadratics in the denominators have no real roots.

- We solved building blocks I, II and III in almost complete detail.
- The types in the remaining columns can be transformed to building block ones:
  - Block I, linear substitutions: done in full detail.
  - Block IIa, IIIa, linear substitutions: done in full detail, by means of completing the square.
  - Block IIb, IIIb, linear substitutions: done by means of completing the square; computations are analogous and we leave them for exercise.