

Calculus II

Homework

Trigonometry review

1. Let $x \in (0, 1)$. Express the following using x and $\sqrt{1 - x^2}$.

(a) $\sin(\arcsin(x))$.

(e) $\sin(2 \arccos(x))$.

(b) $\sin(2 \arcsin(x))$.

(f) $\sin(3 \arccos(x))$.

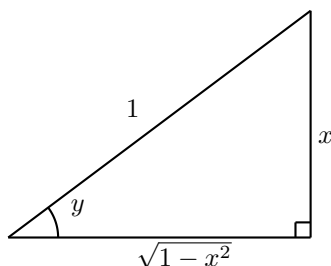
(c) $\sin(3 \arcsin(x))$.

(g) $\cos(2 \arcsin(x))$.

(d) $\sin(\arccos(x))$.

(h) $\cos(3 \arccos(x))$.

Solution. 1.b. Let $y = \arcsin x$. Then $\sin y = x$, and we can draw a right triangle with opposite side length x and hypotenuse length 1 to find the other trigonometric ratios of y .



Then $\cos y = \frac{\sqrt{1-x^2}}{1} = \sqrt{1 - x^2}$. Now we use the double angle formula to find $\sin(2 \arcsin x)$.

$$\begin{aligned} \sin(2 \arcsin x) &= \sin(2y) \\ &= 2 \sin y \cos y \\ &= 2x \sqrt{1 - x^2}. \end{aligned}$$

Solution. 1.c. Use the result of Problem 1.b. This also requires the addition formula for sine:

$$\sin(A + B) = \sin A \cos B + \sin B \cos A,$$

and the double angle formula for cosine:

$$\cos(2y) = \cos^2 y - \sin^2 y.$$

$$\begin{aligned}
\sin(3 \arcsin x) &= \sin(3y) \\
&= \sin(2y + y) \\
&= \sin(2y) \cos y + \sin y \cos(2y) && \left| \begin{array}{l} \text{Use addition formula} \\ \text{Use double angle formulas} \end{array} \right. \\
&= (2 \sin y \cos y) \cos y + \sin y (\cos^2 y - \sin^2 y) \\
&= 2 \sin y \cos^2 y + \sin y \cos^2 y - \sin^3 y \\
&= 3 \sin y \cos^2 y - \sin^3 y \\
&= 3 \sin y (1 - \sin^2 y) - \sin^3 y \\
&= 3x(1 - x^2) - x^3 \\
&= 3x - 4x^3.
\end{aligned}$$

The solution is complete. A careful look at the solution above reveals a strategy useful for problems similar to this one.

- Identify the inverse trigonometric expression- $\arcsin x, \arccos x, \arctan x, \dots$. In the present problem that was $y = \arcsin x$.
- The problem is therefore a trigonometric function of y .
- Using trig identities and algebra, rewrite the problem as a trigonometric expression involving only the trig function that transforms y to x . In the present problem we rewrote everything using $\sin y$.
- Use the fact that $\sin(\arcsin x) = x, \cos(\arccos x) = x, \dots$, etc. to simplify.

Solution. 1.f We use the same strategy outlined in the end of the solution of Problem 1.c. Set $y = \arccos x$ and so $\cos(y) = x$. Therefore:

$$\begin{aligned}
\sin(3y) &= \sin(2y + y) \\
&= \sin(2y) \cos y + \sin y \cos(2y) \\
&= 2 \sin y \cos y \cos y + \sin y (2 \cos^2 y - 1) \\
&= 2 \sin y \cos^2 y + \sin y (2 \cos^2 y - 1) \\
&= \sin y (4 \cos^2 y - 1) && \left| \begin{array}{l} \text{use } \cos y = x \\ \sin y = \sqrt{1 - x^2} \end{array} \right. \\
&= \sqrt{1 - x^2} (4x^2 - 1).
\end{aligned}$$

2. Express as the following as an algebraic expression of x . In other words, “get rid” of the trigonometric and inverse trigonometric expressions.

(a) $\cos^2(\arctan x)$.

(b) $-\sin^2(\operatorname{arccot} x)$.

(c) $\frac{1}{\cos(\arcsin x)}$.

(d) $-\frac{1}{\sin(\arccos x)}$.

Solution. 2.b. We follow the strategy outlined in the end of the solution of Problem 1.c. We set $y = \operatorname{arccot} x$. Then we need to express $-\sin^2 y$ via $\cot y$. That is a matter of algebra:

$$\begin{aligned}
-\sin^2(\operatorname{arccot} x) &= -\sin^2 y && \left| \begin{array}{l} \text{Set } y = \operatorname{arccot} x \\ \text{use } \sin^2 y + \cos^2 y = 1 \end{array} \right. \\
&= -\frac{\sin^2 y}{\sin^2 y + \cos^2 y} \\
&= -\frac{1}{\frac{\sin^2 y + \cos^2 y}{\sin^2 y}} \\
&= -\frac{1}{1 + \cot^2 y} && \left| \begin{array}{l} \text{Substitute back } \cot y = x \end{array} \right. \\
&= -\frac{1}{1 + x^2}.
\end{aligned}$$

3. Rewrite as a rational function of t . This problem will be later used to derive the Euler substitutions (an important technique for integrating).

(a) $\cos(2 \arctan t)$.

(g) $\cos(2 \operatorname{arccot} t)$.

(b) $\sin(2 \arctan t)$.

(h) $\sin(2 \operatorname{arccot} t)$.

(c) $\tan(2 \arctan t)$.

(i) $\tan(2 \operatorname{arccot} t)$.

(d) $\cot(2 \arctan t)$.

(j) $\cot(2 \operatorname{arccot} t)$.

(e) $\csc(2 \arctan t)$.

(k) $\csc(2 \operatorname{arccot} t)$.

(f) $\sec(2 \arctan t)$.

(l) $\sec(2 \operatorname{arccot} t)$.

Solution. 3.a Set $z = \arctan t$, and so $\tan z = t$. Then

$$\begin{aligned} \cos(2 \arctan t) &= \cos(2z) \\ &= \frac{\cos(2z)}{1} \\ &= \frac{\cos^2 z - \sin^2 z}{\cos^2 z + \sin^2 z} \\ &= \frac{(\cos^2 z - \sin^2 z) \frac{1}{\cos^2 z}}{(\sin^2 z + \cos^2 z) \frac{1}{\cos^2 z}} \\ &= \frac{1 - \tan^2 z}{1 + \tan^2 z} \\ &= \frac{1 - t^2}{1 + t^2} \end{aligned}$$

use double angle formulas
and $1 = \sin^2 z + \cos^2 z$
divide top and bottom by $\cos^2 z$

Solution. 3.d Set $z = \arctan t$, and so $\tan z = t$. Then

$$\begin{aligned} \cot(2 \arctan t) &= \cot(2z) \\ &= \frac{\cos(2z)}{\sin(2z)} \\ &= \frac{\cos^2 z - \sin^2 z}{2 \sin z \cos z} \\ &= \frac{1 - \tan^2 z}{2 \tan z} \\ &= \frac{1 - t^2}{2t} \end{aligned}$$

use double angle formulas

4. Compute the derivative (derive the formula).

(a) $(\arctan x)'$.

(d) $(\arccos x)'$.

(b) $(\operatorname{arccot} x)'$.

(e) Let arcsec denote the inverse of the secant function. Compute $(\operatorname{arcsec} x)'$.

(c) $(\arcsin x)'$.

5. (a) Let $a + b \neq k\pi$, $a \neq k\pi + \frac{\pi}{2}$ and $b \neq k\pi + \frac{\pi}{2}$ for any $k \in \mathbb{Z}$ (integers). Prove that

$$\frac{\tan a + \tan b}{1 - \tan a \tan b} = \tan(a + b) \quad .$$

(b) Let x and y be real. Prove that, for $xy \neq 1$, we have

$$\arctan x + \arctan y = \arctan \left(\frac{x + y}{1 - xy} \right)$$

if the left hand side lies between $(-\frac{\pi}{2}, \frac{\pi}{2})$.

Solution. 5.a We start by recalling the formulas

$$\begin{aligned}\cos(a+b) &= \cos a \cos b - \sin a \sin b \\ \sin(a+b) &= \sin a \cos b + \sin b \cos a \quad .\end{aligned}$$

These formulas have been previously studied; alternatively they follow from Euler's formula and the computation

$$\begin{aligned}\cos(a+b) + i \sin(a+b) &= e^{i(a+b)} = e^{ia} e^{ib} = (\cos a + i \sin a)(\cos b + i \sin b) \\ &= \cos a \cos b - \sin a \sin b + i(\sin a \cos b + \sin b \cos a) \quad .\end{aligned}$$

Now 5.a is done via a straightforward computation:

$$\begin{aligned}\tan(a+b) &= \frac{\sin(a+b)}{\cos(a+b)} = \frac{\sin a \cos b + \sin b \cos a}{\cos a \cos b - \sin a \sin b} = \frac{(\sin a \cos b + \sin b \cos a) \frac{1}{\cos a \cos b}}{(\cos a \cos b - \sin a \sin b) \frac{1}{\cos a \cos b}} \\ &= \frac{\tan a + \tan b}{1 - \tan a \tan b} \quad .\end{aligned} \tag{1}$$

5.b is a consequence of 5.a. Let $a = \arctan x$, $b = \arctan y$. Then (??) becomes

$$\tan(\arctan x + \arctan y) = \frac{\tan(\arctan x) + \tan(\arctan y)}{1 - \tan(\arctan x) \tan(\arctan y)} = \frac{x + y}{1 - xy} \quad ,$$

where we use the fact that $\tan(\arctan w) = w$ for all w . We recall that $\arctan(\tan z) = z$ whenever $z \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Now take \arctan on both sides of the above equality to obtain

$$\arctan x + \arctan y = \arctan \left(\frac{x + y}{1 - xy} \right) \quad .$$