Arithmetics
Long division
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- We present long integer division through examples.
- We present each algorithm step by example.

Introducing long integer division

Divide 13 by 4 with quotient and remainder. Use long integer division notation.

$$
\begin{array}{r|r}
 & 3 \\
\cline{2-2}
4 & 13 \\
 & -12 \\
\cline{2-2}
 & 1
\end{array}
$$

- Put result digit above the dividend in the position indicated.
- Divide the two-digit integers by single-digit integers by guessing.
- Multiply quotient digit by divisor, put result under current dividend.
- Subtract.
- Final answer. Dividend = 13. Divisor = 4. Quotient = 3. Remainder = 1.
- Check. $13 \stackrel{?}{=} 4 \cdot 3 + 1$

- Division by single-digit divisor is different from division by multiple-digit divisors.
- We teach division by single-digit divisors first.
- We teach then extend to divisors with multiple digits.
- We point out all steps that are different on examples.

Divide 97 by 3 with quotient and remainder.

$$\begin{array}{r|r} & 32 \\ \hline 3 & 97 \\ - & 90 \\ \hline & 7 \\ - & 6 \\ \hline & 1 \end{array}$$

- Find shortest dividend start larger than divisor. If none, we're done.
- Next quotient digit: put above last digit of that start. If this leaves gaps: fill with 0.
  - Current leading digit $>$ divisor leading digit $\Rightarrow$ divide leading digit by divisor digit. Unlike division by multi-digit divisor, don't add one to divisor leading digit. Round down if needed.
- Multiply quotient digit by divisor, put result under current dividend. Fill gaps with zeroes.
- Subtract.
- Repeat.
- Final answer. Dividend = 97. Divisor = 3. Quotient = 32. Remainder = 1.
- Check. $97 \stackrel{?}{=} 3 \cdot 32 + 1$

Divide 476 by 9 with quotient and remainder.

$$
\begin{array}{r|r}
 & 52 \\
\hline
9 & 476 \\
 & -450 \\
\cline{2-2}
 & -1 \\
 & 26 \\
 & -\ 18 \\
\cline{2-2}
 & 8
\end{array}
$$

- Find shortest dividend start larger than divisor. If none, we're done.
- Next quotient digit: put above last digit of that start. If this leaves gaps: fill with 0.
  – Larger divisor start has more digits $\Rightarrow$ divide leading two digits by divisor digit. Unlike division by multi-digit divisor, don't add one to divisor leading digit. Round down if needed.
- Multiply quotient digit by divisor, put result under current dividend. Fill gaps with zeroes.
- Subtract.
- Repeat.
- Final answer. Dividend = 476. Divisor = 9. Quotient = 52. Remainder = 8.
- Check. $476 \stackrel{?}{=} 9 \cdot 52 + 8$

Divide 8375 by 4 with quotient and remainder.

$$
\begin{array}{r|r}
 & 2093 \\
\hline
4 & 8375 \\
 & -8000 \\
\hline
 & 375 \\
 & -\ 360 \\
\hline
 & 15 \\
 & -\ 12 \\
\hline
 & 3
\end{array}
$$

- Find shortest dividend start larger than divisor. If none, we're done.
- Next quotient digit: put above last digit of that start. If this leaves gaps: fill with 0.
  - Current leading digit $>$ divisor leading digit $\Rightarrow$ divide leading digit by divisor digit. Unlike division by multi-digit divisor, don't add one to divisor leading digit. Round down if needed.
  - Larger divisor start has more digits $\Rightarrow$ divide leading two digits by divisor digit. Unlike division by multi-digit divisor, don't add one to divisor leading digit. Round down if needed.
- Multiply quotient digit by divisor, put result under current dividend. Fill gaps with zeroes.
- Subtract.
- Repeat.
- Final answer. Dividend = 8375. Divisor = 4. Quotient = 2093. Remainder = 3.
- Check. $8375 \stackrel{?}{=} 4 \cdot 2093 + 3$

- We introduce multiple-digit divisors.
- The multi-digit division algorithm is similar to the one-digit divisor, with some differences.
- The multi-digit division uses the leading digit of the divisor plus one, where as the single-digit division uses leading digit as is.
- An extra step to collect the quotient digits is needed.
- In the division step, there are three major cases:
  1. Leading digit of remaining dividend is larger than leading digit of divisor.
  2. Leading digits of remaining dividend equals the divisor leading digit and the start of the dividend is larger than the divisor.
  3. The remaining case: either:
     - leading digit of remaining dividend equals the divisor leading digit, but the dividend start is smaller than the divisor
     - leading digit of remaining dividend is smaller than the divisor leading digit.

Divisor leading digit small relative to dividend

Divide 200 by 12 with quotient and remainder.

$$
\begin{array}{rr|rr}
 & & & 1\\
 & + & & 1\\
 & & & 14\\
\cline{2-4}
 & & & 16\\
\cline{3-4}
 & & -1 & \\
12 & & 2 & 00\\
 & - & 1 & 20\\
\cline{3-4}
 & & & -1\\
 & & & 80\\
 & - & & 48\\
\cline{3-4}
 & & & 32\\
 & - & & 12\\
\cline{3-4}
 & & & -1\\
 & & & 20\\
 & - & & 12\\
\cline{3-4}
 & & & 8
\end{array}
$$

- Find shortest dividend start larger than divisor. If none, we're done.
- Next quotient digit: put above last digit of that start. If this leaves gaps: fill with 0.
  - Current leading digit $>$ divisor leading digit $\Rightarrow$ divide leading digit by divisor digit plus one. Round down if needed.
- Multiply quotient digit by divisor, put result under current dividend. Fill gaps with zeroes.
- Subtract.
- Repeat.
- If more than one quotient row, add them.
- Final answer. Dividend = 200. Divisor = 12. Quotient = 16. Remainder = 8.
- Check. $200 \stackrel{?}{=} 12 \cdot 16 + 8$

$$
\begin{aligned}
4 + 1 + 1 &= 6\\
1 &= 1
\end{aligned}
$$

Divisor leading digit large relative to dividend

Divide 200 by 77 with quotient and remainder.

$$\begin{array}{r|rrr} & & & 2 \\ \hline & {\scriptstyle -1} & {\scriptstyle -1} & \\ 77 & 2 & 0 & 0 \\ - & 1 & 5 & 4 \\ \hline & & 4 & 6 \end{array}$$

- Find shortest dividend start larger than divisor. If none, we're done.
- Next quotient digit: put above last digit of that start. If this leaves gaps: fill with 0.
  - Larger divisor start has more digits $\Rightarrow$ divide leading two digits by divisor digit plus one. Round down if needed.
- Multiply quotient digit by divisor, put result under current dividend. Fill gaps with zeroes.
- Subtract.
- Repeat.
- Final answer. Dividend = 200. Divisor = 77. Quotient = 2. Remainder = 46.
- Check. $200 \stackrel{?}{=} 77 \cdot 2 + 46$

Dividend and divisor leading digits equal

Divide 3600 by 33 with quotient and remainder.

$$
\begin{array}{rr}
 & 1 \\
+ & 1 \\
 & 1\ \ 07 \\
\hline
 & 1\ \ 09 \\
\hline
33 \big) & 3\ \ 6\ \ 00 \\
- & 3\ \ 3\ \ 00 \\
\hline
 & -1-1 \\
 & 3\ \ 00 \\
- & 2\ \ 31 \\
\hline
 & 69 \\
- & 33 \\
\hline
 & 36 \\
- & 33 \\
\hline
 & 3
\end{array}
$$

- Find shortest dividend start larger than divisor. If none, we're done.
- Next quotient digit: put above last digit of that start. If this leaves gaps: fill with 0.
  - Equal leading digits; larger divisor start has same # of digits $\Rightarrow$ set quotient digit to 1.
  - Larger divisor start has more digits $\Rightarrow$ divide leading two digits by divisor digit plus one. Round down if needed.
  - Current leading digit $>$ divisor leading digit $\Rightarrow$ divide leading digit by divisor digit plus one. Round down if needed.
- Multiply quotient digit by divisor, put result under current dividend. Fill gaps with zeroes.
- Subtract.
- Repeat.
- If more than one quotient row, add them.
- Final answer. Dividend = 3600. Divisor = 33. Quotient = 109. Remainder = 3.
- Check. $3600 \stackrel{?}{=} 33 \cdot 109 + 3$

$$
\begin{aligned}
7 + 1 + 1 &= 9 \\
0 &= 0 \\
1 &= 1
\end{aligned}
$$

Full algorithm with all cases

Divide 492619 by 46 with quotient and remainder.

$$
\begin{array}{r|rrrrr}
 & & & & 1 & 01 \\
+ & & 1 & 0 & 6 & 08 \\
\hline
 & & 1 & 0 & 7 & 09 \\
\hline
46 & 4 & 9 & 2 & 6 & 19 \\
- & 4 & 6 & 0 & 0 & 00 \\
\hline
 & & -1 & & & \\
 & & 3 & 2 & 6 & 19 \\
- & & 2 & 7 & 6 & 00 \\
\hline
 & & & -1 & & \\
 & & & 5 & 0 & 19 \\
- & & & 4 & 6 & 00 \\
\hline
 & & & & -1 & \\
 & & & & 4 & 19 \\
- & & & & 3 & 68 \\
\hline
 & & & & & -1 \\
 & & & & & 51 \\
- & & & & & 46 \\
\hline
 & & & & & 5
\end{array}
$$

- Find shortest dividend start larger than divisor. If none, we're done.
- Next quotient digit: put above last digit of that start. If this leaves gaps: fill with 0.
  - Equal leading digits; larger divisor start has same # of digits $\Rightarrow$ set quotient digit to 1.
  - Larger divisor start has more digits $\Rightarrow$ divide leading two digits by divisor digit plus one. Round down if needed.
  - Current leading digit $>$ divisor leading digit $\Rightarrow$ divide leading digit by divisor digit plus one. Round down if needed.
- Multiply quotient digit by divisor, put result under current dividend. Fill gaps with zeroes.
- Subtract.
- Repeat.
- If more than one quotient row, add them.
- Final answer. Dividend = 492619. Divisor = 46. Quotient = 10709. Remainder = 5.
- Check. $492619 \stackrel{?}{=} 46 \cdot 10709 + 5$

$$
\begin{aligned}
8 + 1 &= 9 \\
0 + 0 &= 0 \\
6 + 1 &= 7 \\
0 &= 0 \\
1 &= 1
\end{aligned}
$$

Full algorithm: a large example

Divide 619821 by 343 with quotient and remainder.

$$\begin{array}{rrrrrr}
 & & & & 1 & 00\\
+ & & & & 1 & 01\\
 & & & 1 & 6 & 06\\ \hline
 & & & 1 & 8 & 07\\ \hline
 & -1 & & & & \\
343 & 6 & 1 & 9 & 8 & 21\\
- & 3 & 4 & 3 & 0 & 00\\ \hline
 & 2 & 7 & 6 & 8 & 21\\
- & 2 & 0 & 5 & 8 & 00\\ \hline
 & & -1 & -1 & & \\
 & & 7 & 1 & 0 & 21\\
- & & 3 & 4 & 3 & 00\\ \hline
 & & 3 & 6 & 7 & 21\\
- & & 3 & 4 & 3 & 00\\ \hline
 & & & & -1 & -1\\
 & & & 2 & 4 & 21\\
- & & & 2 & 0 & 58\\ \hline
 & & & & 3 & 63\\
- & & & & 3 & 43\\ \hline
 & & & & & 20
\end{array}$$

- Find shortest dividend start larger than divisor. If none, we're done.
- Next quotient digit: put above last digit of that start. If this leaves gaps: fill with 0.
  - Current leading digit $>$ divisor leading digit $\Rightarrow$ divide leading digit by divisor digit plus one. Round down if needed.
  - Larger divisor start has more digits $\Rightarrow$ divide leading two digits by divisor digit plus one. Round down if needed.
  - Equal leading digits; larger divisor start has same # of digits $\Rightarrow$ set quotient digit to 1.
- Multiply quotient digit by divisor, put result under current dividend. Fill gaps with zeroes.
- Subtract.
- Repeat.
- If any, fill gaps in quotients with zeroes.
- If more than one quotient row, add them.
- Final answer. Dividend = 619821. Divisor = 343. Quotient = 1807. Remainder = 20.
- Check. $619821 \stackrel{?}{=} 343 \cdot 1807 + 20$

$$\begin{array}{r}
6+1+0=7\\
0+0+0=0\\
6+1+1=8\\
1=1
\end{array}$$

Full algorithm: a large example

Divide 1000000 by 2019 with quotient and remainder.

$$
\begin{array}{rr|rrrrrr}
 & & & & & & & 11\\
 & + & & & & & 1 & 21\\
 & & & & & & 3 & 63\\
\hline
 & & & & & & 4 & 95\\
\hline
 & & -1 & -1 & -1 & -1 & & \\
2019 & & 1 & 0 & 0 & 0 & 0 & 00\\
 & - & & 6 & 0 & 5 & 7 & 00\\
\hline
 & & & & & -1 & & \\
 & & & 3 & 9 & 4 & 3 & 00\\
 & - & & 2 & 0 & 1 & 9 & 00\\
\hline
 & & & & & & -1 & \\
 & & & 1 & 9 & 2 & 4 & 00\\
 & - & & 1 & 2 & 1 & 1 & 40\\
\hline
 & & & & & -1 & -1 & \\
 & & & & 7 & 1 & 2 & 60\\
 & - & & & 4 & 0 & 3 & 80\\
\hline
 & & & & & & -1 & \\
 & & & & 3 & 0 & 8 & 80\\
 & - & & & 2 & 0 & 1 & 90\\
\hline
 & & & & -1 & & & -1\\
 & & & & 1 & 0 & 6 & 90\\
 & - & & & & 6 & 0 & 57\\
\hline
 & & & & & & & -1\\
 & & & & & 4 & 6 & 33\\
 & - & & & & 2 & 0 & 19\\
\hline
 & & & & & & -1 & -1\\
 & & & & & 2 & 6 & 14\\
 & - & & & & 2 & 0 & 19\\
\hline
 & & & & & & 5 & 95
\end{array}
$$

- Find shortest dividend start larger than divisor. If none, we're done.
- Next quotient digit: put above last digit of that start. If this leaves gaps: fill with 0.
  - Larger divisor start has more digits $\Rightarrow$ divide leading two digits by divisor digit plus one. Round down if needed.
  - Current leading digit $>$ divisor leading digit $\Rightarrow$ divide leading digit by divisor digit plus one. Round down if needed.
  - Equal leading digits; larger divisor start has same # of digits $\Rightarrow$ set quotient digit to 1.
- Multiply quotient digit by divisor, put result under current dividend. Fill gaps with zeroes.
- Subtract.
- Repeat.
- If more than one quotient row, add them.
- Final answer. Dividend = 1000000. Divisor = 2019. Quotient = 495. Remainder = 595.
- Check. $1000000 \stackrel{?}{=} 2019 \cdot 495 + 595$

$$
\begin{aligned}
3+1+1&=5\\
6+2+1&=9\\
3+1&=4
\end{aligned}
$$

Example: many quotient rows

Divide 99 by 11 with quotient and remainder.

$$
\begin{array}{r|r}
 & 1 \\
+ & 1 \\
 & 1 \\
 & 2 \\
 & 4 \\
\hline
 & 9 \\
\hline
11 & 99 \\
 & -44 \\
\hline
 & 55 \\
 & -22 \\
\hline
 & 33 \\
 & -11 \\
\hline
 & 22 \\
 & -11 \\
\hline
 & 11 \\
 & -11 \\
\hline
 & 0
\end{array}
$$

- Find shortest dividend start larger than divisor. If none, we're done.
- Next quotient digit: put above last digit of that start. If this leaves gaps: fill with 0.
  - Current leading digit $>$ divisor leading digit $\Rightarrow$ divide leading digit by divisor digit plus one. Round down if needed.
  - Equal leading digits; larger divisor start has same # of digits $\Rightarrow$ set quotient digit to 1.
- Multiply quotient digit by divisor, put result under current dividend. Fill gaps with zeroes.
- Subtract.
- Repeat.
- If more than one quotient row, add them.
- Final answer. Dividend = 99. Divisor = 11. Quotient = 9. Remainder = 0.
- Check. $99 \stackrel{?}{=} 11 \cdot 9 + 0$

$4 + 2 + 1 + 1 + 1 = 9$

Division algorithm: how many quotient rows?

- When dividing with remainder, we sometimes have to collect multiple quotient rows.
- In the example of dividing 99 by 11, we saw had to collect the quotients $4, 2, 1, 1, 1$ from 5 different rows.
- Can it get any worse? No, example shows worst case scenario.

Lemma

- *With* 10 *digits, there are at most* 5 *quotient rows during division.*
- *With N digits, there are at most* $\lfloor \log_2 N \rfloor + 2$ *quotient rows during division.*

- We study counting systems that do not use 10 digits later.
- Reader not familiar with logarithms may ignore the second part.
- The lemma follows from the fact that each quotient digit that appears higher in the same column is at most half of the one below it, except possibly the highest two digits.

The Knuth optimization, part 1

- When dividing with remainder, we sometimes have to collect multiple quotient rows.
- How much of a slow down does this cause?
- In the example of dividing 99 by 11, we saw had to collect the quotients $4, 2, 1, 1, 1$ from 5 different rows.
- Can it get any worse? No, the example shows the worst case scenario.
- In fact, one can prove the following.

Lemma

- *Using* 10 *digits, there are at most* 5 *quotient rows in the division algorithm.*
- *Using* N *digits, there are at most* $\lfloor \log_2 N \rfloor + 2$ *quotient rows in the division algorithm.*

- Later on we study computations using different digit systems.

Divide 469 by 51 with quotient and remainder.

$$\begin{array}{rr} & 1\\ + & 1\\ & 7\\ \hline & 9 \end{array}$$

$$\begin{array}{r|rr} & & 469\\ 51 & - & 357\\ \cline{3-3} & & -1\\ & & 112\\ & - & 51\\ \cline{3-3} & & 61\\ & - & 51\\ \cline{3-3} & & 10 \end{array}$$

- Find shortest dividend start larger than divisor. If none, we're done.
- Next quotient digit: put above last digit of that start. If this leaves gaps: fill with 0.
  - Larger divisor start has more digits $\Rightarrow$ divide leading two digits by divisor digit plus one. Round down if needed.
  - Current leading digit $>$ divisor leading digit $\Rightarrow$ divide leading digit by divisor digit plus one. Round down if needed.
- Multiply quotient digit by divisor, put result under current dividend. Fill gaps with zeroes.
- Subtract.
- Repeat.
- If more than one quotient row, add them.
- Final answer. Dividend = 469. Divisor = 51. Quotient = 9. Remainder = 10.
- Check. $469 \stackrel{?}{=} 51 \cdot 9 + 10$
  $7 + 1 + 1 = 9$

The Knuth optimization

- Let the leading digit of the divisor be q.
- Observation: when the leading divisor digit q is large, there are fewer quotient rows.
- Observation: if we multiply the divisor and the dividend by a number s, this doesn't change the quotient and multiplies the remainder by the same factor s.
- Donald Knuth suggests the following long division optimization.
- Before division, multiply dividend & divisor by one-digit q.
- Choose q to make the divisor leading digit as large as possible.
- More precisely, for divisor leading digit d, choose q to be

$$q = \left\lfloor \frac{10}{d+1} \right\rfloor$$

- Divide integers re-scaled by q in the usual way.
- The new quotient coincides with that of the original problem; the original remainder is obtained by dividing the new one by q.

The Knuth optimization in other bases

- The Knuth optimization is intended for large examples and computations by computer.
- Thus, the Knuth optimization is not very beneficial when computing by hand.
- When using non-decimal counting systems with more than 10 digits on a computer, the Knuth optimization yields significant benefits.
- Important cases for the Knuth optimization would be the use of $2^8 = 256, 2^{16} = 65536, 2^{32} = 4294967296$ digits, as they are easily available on most modern computers.