

Calculus II

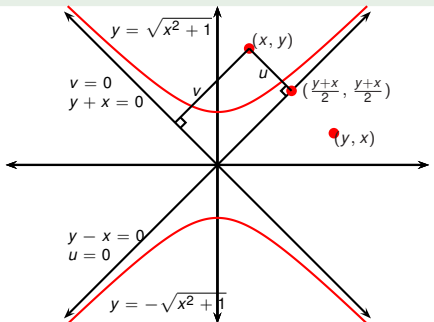
Definite integrals of the form $\int_p^q \sqrt{ax^2 + c} dx,$
 $a, c > 0$

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Example

Find the area locked b-n the hyperbolas $y = \pm\sqrt{x^2 + 1}$ and $x = \pm 2\sqrt{2}$.



Signed distance b-n (x, y) and line $u = 0$ equals

$$\begin{aligned} & \pm \sqrt{\left(x - \frac{(x+y)}{2}\right)^2 + \left(y - \frac{(x+y)}{2}\right)^2} \\ &= \pm \sqrt{\frac{1}{2}(y-x)^2} = \pm \frac{\sqrt{2}}{2}(y-x) = u. \end{aligned}$$

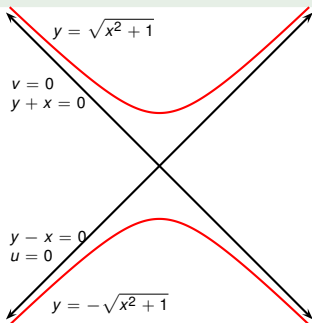
We studied $v = \frac{1}{u}$ is called a hyperbola: why do we call $y = \sqrt{x^2 + 1}$ hyperbola? Compute:

$$\begin{aligned} \sqrt{x^2 + 1} &= y \\ x^2 + 1 &= y^2 \\ y^2 - x^2 &= 1 \\ \frac{\sqrt{2}}{2}(y-x) \frac{\sqrt{2}}{2}(y+x) &= \frac{1}{2} \\ uv &= \frac{1}{2} \\ v &= \frac{1}{u}, \end{aligned}$$

where $\begin{cases} u = \frac{\sqrt{2}}{2}(y-x) \\ v = \frac{\sqrt{2}}{2}(y+x) \end{cases}$. Consider an arbitrary point (x, y) .

Example

Find the area locked b-n the hyperbolas $y = \pm\sqrt{x^2 + 1}$ and $x = \pm 2\sqrt{2}$.



Signed distance b-n (x, y) and line $u = 0$ equals u . Similarly compute that signed distance b-n (x, y) and the line $v = 0$ equals v .
 $\Rightarrow y^2 - x^2 = 1$ is the hyperbola $v = \frac{1}{2u}$ in the (u, v) -plane.

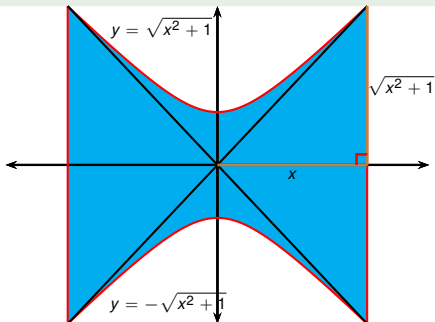
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Example

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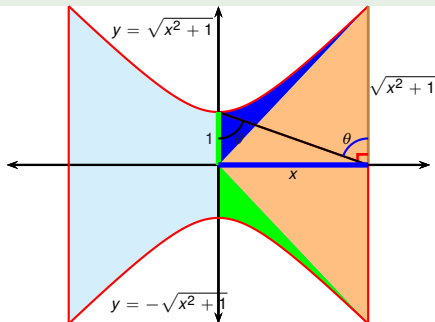


The area in question is:

$$\begin{aligned}
 & \int_{-2\sqrt{2}}^{2\sqrt{2}} 2\sqrt{x^2 + 1} dx \\
 &= 2 \left[x\sqrt{x^2 + 1} + \ln \left(\sqrt{x^2 + 1} + x \right) \right]_{-2\sqrt{2}}^{2\sqrt{2}} \\
 &= 2 \left(2\sqrt{2}\sqrt{(2\sqrt{2})^2 + 1} + \ln \left(\sqrt{(2\sqrt{2})^2 + 1} + 2\sqrt{2} \right) \right) \\
 &= 12\sqrt{2} + 2\ln(3 + 2\sqrt{2}) \\
 &\approx 20.496
 \end{aligned}$$

Example

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- Recall: integral can be solved via $x = \tan \theta$.
- Geometric interpretation of θ ?

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