

# Calculus II

## Sequences

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# Outline

## 1 Sequences

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- We are interested to study sequences such as:

$$(a_1, a_2, a_3 \dots)$$

$$(1, 2, 3, \dots)$$

$$(1, 3, 5, 7, \dots)$$

$$(1, -1, 1, -1, \dots) \quad .$$

### Definition (Most general form)

A sequence is an ordered collection of objects in which repetitions are allowed.

- The definition is too general for our purposes.
- For example it allows sequences indexed by the real numbers.
- We give a less general (but possibly easier to understand) definition shortly.
- Our less general definition will cover all uses of sequences in the present course/lectures.
- We start by a few examples.

## Example (Sequence notation)

- Consider the sequence

$$(2, 4, 6, 8, \dots).$$

- That appears to be the sequence of all positive even integers.
- We can express this sequence more compactly using the notation

$$a_n = 2n,$$

where  $a_n$  denotes the  $n$ th term.

$$a_1 = 2 \cdot 1 = 2$$

$$a_2 = 2 \cdot 2 = 4$$

$$a_3 = 2 \cdot 3 = 6$$

$$a_4 = 2 \cdot 4 = 8$$

$$\vdots$$

### Example

The sequence

$$(-1, 1, -1, 1, -1, 1, \dots)$$

can be written as  $b_n = (-1)^n$ .

### Example

The sequence

$$(1, 2, 4, 8, 16, \dots)$$

can be written as  $c_n = 2^{n-1}$ .

### Example

The sequence

$$\left(\frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, -\frac{1}{16}, \dots\right)$$

can be written as  $d_n = -\left(-\frac{1}{2}\right)^n$ .

## Definition (Sequence indexed by the integers)

A sequence is a list of numbers indexed by consecutive integers bounded below and written in a definite order

$$(a_1, a_2, a_3, a_4, \dots, a_n, \dots) \quad .$$

- In our course/lectures we assume all sequences are indexed by consecutive integers.
- Unless stated/implied otherwise
  - We assume the first index is 1.
  - Under above assumption  $a_1$  is called the first term,  $a_2$  is called the second term, and  $a_n$  is the  $n^{\text{th}}$  term.
- We often denote the sequence of elements  $(a_1, a_2, \dots)$  by

$$\{a_n\} \quad \text{and more precisely} \quad \{a_n\}_{n=1}^{\infty}$$

or by

$$(a_n) \quad \text{and more precisely} \quad (a_n)_{n=1}^{\infty}$$

- The use of  $\{\}$  versus  $()$  differs between authors and instructors.

## Definition (Sequence indexed by the integers)

A sequence is a list of numbers indexed by consecutive integers bounded below and written in a definite order

$$(a_1, a_2, a_3, a_4, \dots, a_n, \dots) \quad .$$

- To indicate a sequence labeled so the first index is not 1 write:

$$\begin{aligned} (a_n)_{n=0}^{\infty} & \text{ for } (a_0, a_1, a_2, \dots) \\ (a_n)_{n=2}^{\infty} & \text{ for } (a_2, a_3, a_4, \dots) \\ (a_n)_{n=-1}^{\infty} & \text{ for } (a_{-1}, a_0, a_1, \dots) \end{aligned}$$

## Definition

A sequence is finite if it has a finite number of elements.

- To indicate a sequence is finite either write all elements of the sequence or use indices as shown below.

$$(1, 2, 3, 4, 5, 6, 7, 8, 9, 10) \\ (a_n)_{n=1}^5 \text{ for } (a_1, a_2, a_3, a_4, a_5)$$



# Defining sequences

## Question

*How can we define a sequence of numbers  $(a_1, a_2, a_3 \dots, a_n, \dots)$ ?*

- A sequence can be interpreted as a function that takes as arguments a subset of the integers.
- Since functions can be defined in exotic and indirect ways, so can sequences.
- We will focus on the three most frequently used ways to define sequences:
  - by specifying a formula for the  $n^{\text{th}}$  term;
  - by recursion;
  - by specifying a property of integers and constructing a sequence of all integers with that property.

# Sequences via formulas

- Sequences can be defined by presenting a formula to obtain the  $n^{\text{th}}$  term  $a_n$  as a function of the index  $n$ .
- Another frequently used notation: include the formula in parenthesis and indicate the index ranges by super- and subscripts.
- There is a third informal but frequently used notation: list few terms of the sequence and let the reader guess the formula.

## Example

$a_n = \frac{n}{n+1}$	$\left(\frac{n}{n+1}\right)_{n=1}^{\infty}$	$\left(\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots\right)$
$a_n = \frac{(-1)^n(n+1)}{3^n}$	$\left(\frac{(-1)^n(n+1)}{3^n}\right)_{n=1}^{\infty}$	$\left(\frac{-2}{3}, \frac{3}{9}, \frac{-4}{27}, \frac{5}{81}, \dots\right)$
$a_n = \sqrt{n-3}, n \geq 3$	$(\sqrt{n-3})_{n=3}^{\infty}$	$(0, 1, \sqrt{2}, \sqrt{3}, \dots)$
$a_n = \cos\left(\frac{n\pi}{6}\right), n \geq 0$	$(\cos \frac{n\pi}{6})_{n=0}^{\infty}$	$\left(1, \frac{\sqrt{3}}{2}, \frac{1}{2}, 0, \dots\right)$

## Example (Sequences via formulas: find sequence terms)

Find the first five terms of each of the following sequences.

①  $a_n = 3 \cdot 2^{-n}$

②  $b_n = 1$

③  $c_n = -3(n - 1) + 5$

④  $d_n = n^2 + 1$

## Example (Sequences via f-las: guess f-la from terms)

Find a formula for the general term  $a_n$  of the sequence

$$\left(0, \frac{1}{4}, -\frac{2}{8}, \frac{3}{16}, -\frac{4}{32}, \frac{5}{64}, \dots\right)$$

$$a_1 = 0, a_2 = \frac{1}{4}, a_3 = -\frac{2}{8}, a_4 = \frac{3}{16}, a_5 = -\frac{4}{32}, a_6 = \frac{5}{64},$$

- The numerators start at 0 and go up by one with each term.
- The  $n^{\text{th}}$  term has numerator  $n - 1$ .
- The denominators start at 2 and double with each term.
- The  $n^{\text{th}}$  term has denominator  $2^n$ .
- The signs of the terms alternate between positive and negative.
- We take this into account by multiplying by  $(-1)^n$ .

$$a_n = (-1)^n \frac{n-1}{2^n}$$

## Example (Sequences via f-las: guess f-la from terms)

Find a formula for the  $n$ th term of each of the following sequences.

①  $a_n = 2 \cdot \left(\frac{1}{4}\right)^{n-1}$

$$\left(2, \frac{1}{2}, \frac{1}{8}, \frac{1}{32}, \frac{1}{128}, \dots\right)$$

②  $b_n = (-1)^n n^2$

$$-1, 4, -9, 16, -25, \dots$$

③  $c_n = -1 + 6(n-1)$

$$-1, 5, 11, 17, 23, \dots$$

# Warning about implied sequence formulas

- We found the sequence  $(0, \frac{1}{4}, -\frac{2}{8}, \frac{3}{16}, -\frac{4}{32}, \frac{5}{64}, \dots)$  can be given by:  $a_n = (-1)^n \frac{n-1}{2^n}$
- For any finite number of terms we can produce infinitely many different formulas that match them - but disagree on the terms after.
- For example the sequence above can also be obtained by:

$$a_n = \frac{27}{512}n^5 - \frac{477}{512}n^4 + \frac{3159}{512}n^3 - \frac{9651}{512}n^2 + \frac{6643}{256}n - \frac{793}{64}$$

and that produces  $a_7 = \frac{363}{32}$ .

- Bear in mind that using implied sequence formulas is **informal**.
  - It is acceptable to use the implied sequence notation only when we believe there is a single completely obvious pattern that will be recognized by every one.
  - The pattern should be obvious not only to us, but also to our potential readers.
  - If in doubt we should switch to a more rigorous notation.

# Sequences via recursion

- Sequences can be defined by recursive formulas.
- A sequence formula is recursive if it expresses the term  $a_n$  via the preceding terms  $a_1, a_2, \dots, a_{n-1}$ , rather than directly as a function of  $n$ .

## Example (Defining sequences by recursion)

Define recursively the Fibonacci sequence  $(f_n)_{n=1}^{\infty}$  by requesting that

$$f_1 = 1 \quad f_2 = 1 \quad f_n = f_{n-1} + f_{n-2}, \quad n \geq 3.$$

The first few terms are

$$1, 1,$$

- In fact the Fibonacci sequence can be described by a formula, but it is not very simple:  $a_n = \frac{\sqrt{5}}{5} \left( \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right).$

# Sequences via inclusion criterion

- A sequence can also be given by specifying a criterion to check whether a number should be included in the sequence or not.

## Example (Defining sequence by criterion)

Define  $(p_n)_{n=1}^{\infty}$  as the sequence of all primes.

$(2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, \dots)$

- We know how to check whether a number is prime.
- For example, a crude test for whether a number is prime is to check whether it is divisible by all positive numbers smaller than it.
- Our sequence is well defined; we could generate it, say, by computer.
- However, we have given no closed or even recursive formula to generate the entire sequence.



# Sequences defined indirectly

- We note that in addition to the illustrated ways to define sequences, we are also free to use for the task any well-posed statement.
- Such ways to define a sequence may be very indirect or obscure and we will not use them in our course.
- We hint the challenges that can arise by using arbitrary (but well-posed) definitions on a few examples.

## Example

- 1 Let  $a_n$  be the  $n^{\text{th}}$  digit in the decimal expansion of the number  $e$ . The first few terms of  $(a_n)$ :  
$$2, 7, 1, 8, 2, 8, 1, 8, 2, 8, 4, 5, \dots$$
- 2 Consider the sequence  $(p_n)$ , where  $p_n$  is the population of the world as of January 1 of year  $n$ .

## Definition (Arithmetic sequence)

An arithmetic sequence is one in which successive terms differ by a constant number. This constant is called the difference of the arithmetic sequence.

## Example (Which are arithmetic?)

1,	2,	3,	4,	5,	...	is arithmetic with difference 1.
23,	16,	9,	2,	-5,	...	is arithmetic with difference -7.
8,	9,	12,	17,	24,	...	is not arithmetic.
						( $9 - 8 = 1$ but $12 - 9 = 3$ .)

### Example (Which are arithmetic?)

Sequence	Arithmetic?	Difference	First term	$n$ th term
$1, -1, 1, -1, \dots$				
$\frac{1}{6}, \frac{1}{2}, \frac{5}{6}, \frac{7}{6}, \frac{3}{2}, \dots$				
$2, 2, 2, 2, \dots$				

If an arithmetic sequence has difference  $d$ , then the  $n$ th term has formula

$$a_n = a_1 + d(n - 1),$$

where  $a_1$  is the first term.

## Definition (Geometric sequence)

A geometric sequence is one in which each term is obtained by multiplying the previous one by the same constant. This constant is called the ratio of the geometric sequence.

## Example (Which are geometric?)

2,	4,	8,	16,	32,	...	is geometric with ratio 2.
1,	-3,	9,	-27,	81,	...	is geometric with ratio -3.
-42,	-14,	-21,	31,	-22,	...	is not geometric.
						$(\frac{-14}{-42} = \frac{1}{3} \text{ but } \frac{-21}{-14} = \frac{3}{2}.)$

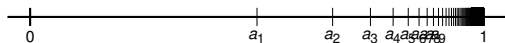
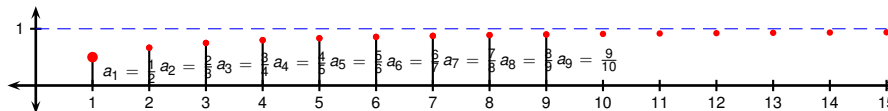
## Example (Arithmetic and geometric)

Sequence	Arithmetic/ geometric	Diff.	Ratio	$a_1$	$a_n$
$\frac{2}{3}, \frac{4}{9}, \frac{8}{27}, \frac{16}{81}, \dots$					
$7, 3, -1, -5, \dots$					
$4, 4, 4, 4, \dots$					
$\pi, -\pi^2, \pi^3, -\pi^4, \dots$					
$1, 1, 2, 2, 3, 3, \dots$					

If a geometric sequence has ratio  $r$ , then the  $n$ th term has formula

$$a_n = a_1 r^{n-1}.$$

where  $a_1$  is the first term.



- The sequence  $a_n = \frac{n}{n+1}$  can be plotted on a number line or using Cartesian coordinates.
- From the pictures, the terms in the sequence appear to approach 1 as  $n$  gets larger.
- $1 - \frac{n}{n+1} = \frac{1}{n+1}$ .
- This can be made arbitrarily small by choosing  $n$  large enough.
- We express this by writing  $\lim_{n \rightarrow \infty} \frac{n}{n+1} = 1$ .

## Definition (Limit of a Sequence)

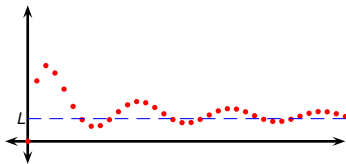
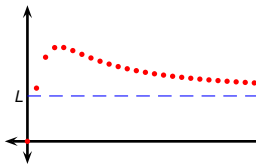
A sequence  $\{a_n\}$  has the limit  $L$ , and we write

$$\lim_{n \rightarrow \infty} a_n = L \quad \text{or} \quad a_n \rightarrow L \text{ as } n \rightarrow \infty$$

if we can make  $a_n$  as close to  $L$  as we like by taking  $n$  large enough.

## Definition (Convergent)

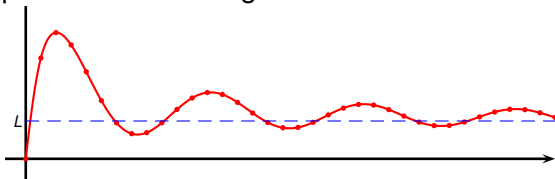
A sequence that has a limit is called convergent. A sequence that has no limit is called divergent.



If you compare the definition of the limit of a sequence with the definition of the infinite limit of a function, you'll see that the only difference between

$$\lim_{n \rightarrow \infty} a_n = L \quad \text{and} \quad \lim_{x \rightarrow \infty} f(x) = L$$

is that  $n$  is required to be an integer.



## Theorem

*If  $\lim_{x \rightarrow \infty} f(x) = L$  and  $f(n) = a_n$  for all integers  $n$ , then  $\lim_{n \rightarrow \infty} a_n = L$ .*



## Example

Find  $\lim_{n \rightarrow \infty} \frac{n}{n+1}$ .

Divide numerator and denominator by the highest power of  $n$ , and use the limit laws:

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{n}{n+1} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} &= \lim_{n \rightarrow \infty} \frac{1}{1 + \frac{1}{n}} \\ &= \frac{\lim_{n \rightarrow \infty} 1}{\lim_{n \rightarrow \infty} 1 + \lim_{n \rightarrow \infty} \frac{1}{n}} \\ &= \frac{1}{1 + 0} \\ &= 1\end{aligned}$$

Just like for functions, there is a notion of sequences tending to infinity: If  $a_n$  grows large as  $n$  becomes large, we write  $\lim_{n \rightarrow \infty} a_n = \infty$ . You can probably guess what  $\lim_{n \rightarrow \infty} a_n = -\infty$  means.

The Limit Laws for continuous functions also hold for sequences:  
If  $\{a_n\}$  and  $\{b_n\}$  are convergent sequences and  $c$  is a constant, then

$$① \quad \lim_{n \rightarrow \infty} (a_n + b_n) = \lim_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} b_n$$

$$② \quad \lim_{n \rightarrow \infty} (a_n - b_n) = \lim_{n \rightarrow \infty} a_n - \lim_{n \rightarrow \infty} b_n$$

$$③ \quad \lim_{n \rightarrow \infty} ca_n = c \lim_{n \rightarrow \infty} a_n$$

$$④ \quad \lim_{n \rightarrow \infty} (a_n b_n) = \lim_{n \rightarrow \infty} a_n \cdot \lim_{n \rightarrow \infty} b_n$$

$$⑤ \quad \lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{\lim_{n \rightarrow \infty} a_n}{\lim_{n \rightarrow \infty} b_n} \text{ if } \lim_{n \rightarrow \infty} b_n \neq 0$$

$$⑥ \quad \lim_{n \rightarrow \infty} a_n^p = \left[ \lim_{n \rightarrow \infty} a_n \right]^p \text{ if } p > 0 \text{ and } a_n > 0.$$

## Example

Calculate  $\lim_{n \rightarrow \infty} \frac{\ln n}{n}$ .

- Both  $\ln n$  and  $n$  go to  $\infty$  as  $n$  gets bigger.
- We can't use L'Hospital's Rule directly, because L'Hospital's Rule is for functions.
- Define  $f(x) = \frac{\ln x}{x}$ . Now use L'Hospital's Rule:

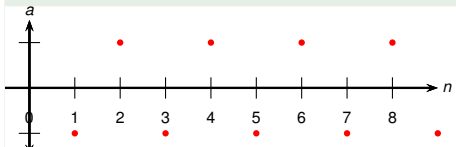
$$\lim_{x \rightarrow \infty} \frac{\ln x}{x} = \lim_{x \rightarrow \infty} \frac{1/x}{1} = 0$$

- Therefore

$$\lim_{n \rightarrow \infty} \frac{\ln n}{n} = \lim_{x \rightarrow \infty} f(x) = 0$$

## Example

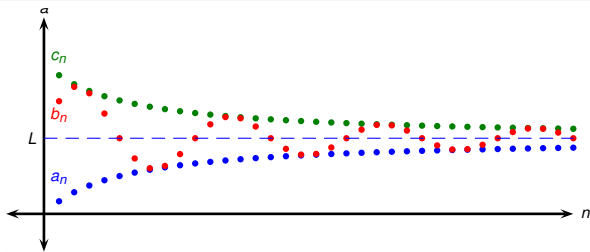
Is the sequence  $a_n = (-1)^n$  convergent or divergent?



- The terms oscillate between  $-1$  and  $1$  infinitely many times.
- Therefore the sequence doesn't approach any number.
- $\{a_n\}$  is divergent.

## Theorem (The Squeeze Theorem for Sequences)

If  $a_n \leq b_n \leq c_n$  for  $n \geq n_0$  and  $\lim_{n \rightarrow \infty} a_n = L = \lim_{n \rightarrow \infty} c_n$ , then  $\lim_{n \rightarrow \infty} b_n = L$ .



## Corollary (to the squeeze theorem)

If  $\lim_{n \rightarrow \infty} |a_n| = 0$ , then  $\lim_{n \rightarrow \infty} a_n = 0$ .

## Theorem (Squeeze theorem for functions at $\infty$ )

If  $f(x) \leq g(x) \leq h(x)$  and  $\lim_{x \rightarrow \infty} f(x) = L = \lim_{x \rightarrow \infty} h(x)$ , then  $\lim_{x \rightarrow \infty} g(x) = L$ .

## Example

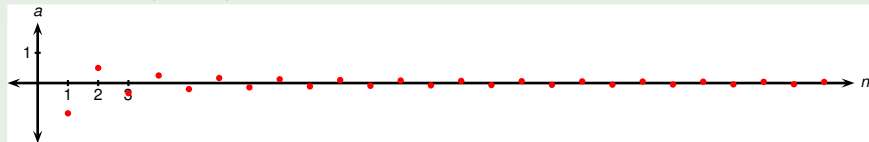
Is  $a_n = \frac{(-1)^n}{n}$  convergent or divergent?

$$\lim_{n \rightarrow \infty} \left| \frac{(-1)^n}{n} \right| = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

Therefore, by the corollary to the Squeeze Theorem,

$$\lim_{n \rightarrow \infty} \frac{(-1)^n}{n} = 0$$

Therefore  $\left\{ \frac{(-1)^n}{n} \right\}$  is convergent.



## Theorem

*If  $\lim_{n \rightarrow \infty} a_n = L$  and the function  $f$  is continuous at  $L$ , then*

$$\lim_{n \rightarrow \infty} f(a_n) = f(L)$$



## Example

Find  $\lim_{n \rightarrow \infty} \sin(\pi/n)$ .

Sine is continuous at 0.

$$\begin{aligned} & \lim_{n \rightarrow \infty} \sin(\pi/n) \\ = & \sin\left(\lim_{n \rightarrow \infty} (\pi/n)\right) \\ = & \sin 0 \\ = & 0 \end{aligned}$$

Find  $\lim_{n \rightarrow \infty} \cos(\pi/n)$ .

Cosine is continuous at 0.

$$\begin{aligned} & \lim_{n \rightarrow \infty} \cos(\pi/n) \\ = & \cos\left(\lim_{n \rightarrow \infty} (\pi/n)\right) \\ = & \cos 0 \\ = & 1 \end{aligned}$$

## Example

Discuss the convergence of the sequence  $a_n = \frac{n!}{n^n}$ , where  $n! = 1 \cdot 2 \cdot 3 \cdot \dots \cdot n$ .

- Both the top and the bottom go to infinity as  $n \rightarrow \infty$ .
- We can't use L'Hospital's Rule, because we have no function corresponding to  $n!$  ( $x!$  isn't defined if  $x$  isn't an integer).

$$a_1 = 1 \quad a_2 = \frac{1 \cdot 2}{2 \cdot 2} \quad a_3 = \frac{1 \cdot 2 \cdot 3}{3 \cdot 3 \cdot 3}$$

$$\begin{aligned} a_n &= \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot \dots \cdot n}{n \cdot n \cdot n \cdot n \cdot \dots \cdot n} \\ &= \frac{1}{n} \left( \frac{2 \cdot 3 \cdot 4 \cdot \dots \cdot n}{n \cdot n \cdot n \cdot \dots \cdot n} \right) \end{aligned}$$

- $\frac{2}{n} \leq 1, \frac{3}{n} \leq 1, \frac{4}{n} \leq 1, \dots, \frac{n}{n} \leq 1$ . Therefore  $0 \leq a_n \leq \frac{1}{n}$ .
- Since  $\frac{1}{n} \rightarrow 0$  as  $n \rightarrow \infty$ , by the Squeeze Theorem  $a_n \rightarrow 0$  as  $n \rightarrow \infty$ .

## Example

For what values of  $r$  is the sequence  $\{r^n\}$  convergent?

Consider the exponential function  $y = r^x$ .

$$\lim_{x \rightarrow \infty} r^x = \begin{cases} \infty & \text{if } r > 1 \\ 0 & \text{if } 0 < r < 1 \end{cases}$$

Therefore

$$\lim_{n \rightarrow \infty} r^n = \begin{cases} \infty & \text{if } r > 1 \\ 0 & \text{if } 0 < r < 1 \end{cases}$$

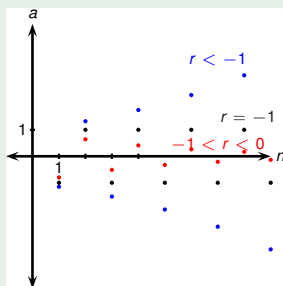
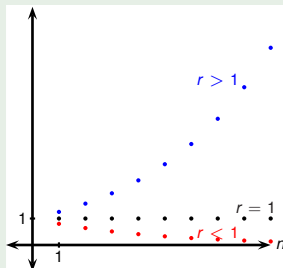
Also,  $\lim_{n \rightarrow \infty} 1^n = 1$  and  $\lim_{n \rightarrow \infty} 0^n = 0$ .

If  $-1 < r < 0$ , then  $0 < |r| < 1$ , and

$$\lim_{n \rightarrow \infty} |r^n| = \lim_{n \rightarrow \infty} |r|^n = 0$$

Therefore  $\lim_{n \rightarrow \infty} r^n = 0$ .

If  $r \leq -1$ , then  $r^n$  diverges. In particular,  $(-1)^n$  diverges.



This theorem summarizes the results of the previous example.

### Theorem (Convergence of Geometric Sequences)

*The sequence  $\{r^n\}$  is convergent if  $-1 < r \leq 1$  and divergent otherwise.*

$$\lim_{n \rightarrow \infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

## Definition (Increasing and Decreasing)

A sequence  $\{a_n\}$  is called increasing if  $a_n < a_{n+1}$  for all  $n \geq 1$ . In other words,  $\{a_n\}$  is increasing if  $a_1 < a_2 < a_3 < \dots$ .

A sequence  $\{a_n\}$  is called decreasing if  $a_n > a_{n+1}$  for all  $n \geq 1$ . In other words,  $\{a_n\}$  is decreasing if  $a_1 > a_2 > a_3 > \dots$ .

A sequence is called monotonic if it is either increasing or decreasing.

## Example

The sequence  $\left\{ \frac{1}{2n+1} \right\}$  is decreasing because

$$a_n = \frac{1}{2n+1} \quad a_{n+1} = \frac{1}{2(n+1)+1} = \frac{1}{2n+3}$$

and

$$\frac{1}{2n+1} > \frac{1}{2n+3}$$

because the denominator of the latter is bigger.

## Definition (Bounded Sequence)

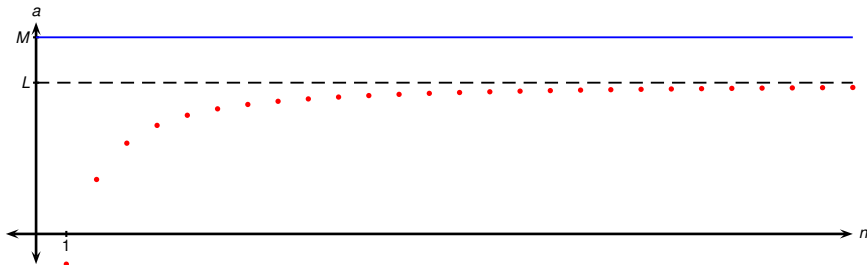
A sequence  $\{a_n\}$  is called bounded above if there exists a number  $M$  such that

$$a_n < M \quad \text{for all} \quad n \geq 1.$$

It is called bounded below if there exists a number  $M$  such that

$$a_n > M \quad \text{for all} \quad n \geq 1.$$

A bounded sequence is a sequence that is bounded below and above.



## Theorem (Monotonic Sequence Theorem)

*Every bounded, monotonic sequence is convergent.*