Calculus II

Integrals of the form $\int \frac{Ax+B}{ax^2+bx+c}\mathrm{d}x$, denominator has no real roots

Todor Milev

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Linear substitutions leading to blocks IIa and IIIa

Building block IIa: $\int \frac{x}{1+x^2}\mathrm{d}x = \frac{1}{2}\ln(1+x^2) + C$.

Building block IIIa: $\int \frac{1}{1+x^2}\mathrm{d}x = \arctan x + C$.

- Let $ax^2 + bx + c$ have no real roots.

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- Let $ax^2 + bx + c$ have no real roots.
- We can find p, q so that the linear substitution $u = px + q$ transforms the quadratic to:

$$ax^2 + bx + c = r(u^2 + 1)$$

(where r is some number to be determined).

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- In this way, integrals of the form $\int \frac{Ax + B}{ax^2 + bx + c}\mathrm{d}x$ are transformed to combinations of building blocks IIa and IIIa.
- We show examples; the general case is analogous and we leave it to the student.

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$$\int \frac{x}{x^2+x+1}\mathrm{d}x =$$

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$$\int \frac{x}{x^2+x+1}\mathrm{d}x = \int \frac{x}{x^2+2\cdot\frac{1}{2}x+\frac{1}{4}-\frac{1}{4}+1}\mathrm{d}x$$

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Linear substitutions leading to blocks IIa and IIIa

Building block IIa: $\int \frac{x}{1+x^2}\mathrm{d}x = \frac{1}{2}\ln(1+x^2) + C$.

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Example

No real roots $\Rightarrow$ complete the square. Let $u = x + \frac{1}{2}$, let $z = \frac{2u}{\sqrt{3}}$.

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