Calculus II
Homework
Series basics

1. Express the infinite decimal number as a rational number.

   (a) $0.\overline{9} = 0.99999\ldots$ answer: 1

   (b) $1.\overline{6} = 1.6666\ldots$ answer: $\frac{5}{3}$

   (c) $1.\overline{3} = 1.3333\ldots$ answer: $\frac{4}{3}$

   (d) $1.\overline{19} = 1.191919\ldots.$ answer: $\frac{118}{99}$

   (e) $0.\overline{09} = 0.0909090909\ldots.$ answer: $\frac{1}{11}$

   (f) $2.\overline{16} = 2.16161616\ldots$ answer: $\frac{214}{99}$

   (g) $2014.\overline{2014} = 2014.2014201420142014\ldots$ answer: $\frac{20140000}{9999}$

   Solution. 1.g

$$
\begin{aligned}
2014.201420142014\ldots &= 2014 + \frac{2014}{10^4} + \frac{2014}{10^8} + \ldots \\
&= 2014 + \frac{2014}{10000}\left(1 + \frac{1}{10000} + \cdots + \frac{1}{10^{4n}} + \ldots\right) \\
&= 2014 + \frac{2014}{10000}\left(\frac{1}{1-\frac{1}{10^4}}\right) \\
&= 2014 + \frac{2014}{10000}\cdot\frac{10000}{9999} \\
&= 2014 + \frac{2014}{9999} \\
&= \frac{2014\cdot 9999 + 2014}{9999} \\
&= \frac{2014\cdot 10000}{9999} \\
&= \frac{20140000}{9999}
\end{aligned}
$$

   Our answer cannot be reduced any further as the greatest common divisor of 20140000 and 9999 is 1.

2. Express the sum of the series as a rational number.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n + 3^n}{5^n}$ answer: $\frac{13}{6}$

   (b) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n + 5^n}{10^n}$ answer: $\frac{13}{4}$

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{5^n - 3^n}{7^n}$ answer: $\frac{7}{4}$

   (d) $\displaystyle\sum_{n=1}^{\infty} \frac{3^{n+1} + 7^{n-1}}{21^n}$ answer: $\frac{4}{7}$

   (e) $\displaystyle\sum_{n=0}^{\infty} \frac{2^{n+1} + (-3)^{n-1}}{5^n}$ answer: $\frac{25}{8}$

   Solution. 2.a.

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{2^n+3^n}{5^n} &= \sum_{n=1}^{\infty}\left(\frac{2}{5}\right)^n+\sum_{n=1}^{\infty}\left(\frac{3}{5}\right)^n\\ &= \frac{2}{5}\sum_{n=0}^{\infty}\left(\frac{2}{5}\right)^n+\frac{3}{5}\sum_{n=0}^{\infty}\left(\frac{3}{5}\right)^n\\ &= \frac{2}{5}\cdot\frac{1}{(1-\frac{2}{5})}+\frac{3}{5}\cdot\frac{1}{(1-\frac{3}{5})}\\ &= \frac{13}{6}\quad.\end{aligned}$$

Use geometric series sum f-la: $\sum\limits_{n=0}^{\infty} r^n=\frac{1}{1-r}$, provided $|r|<1$

Solution. 2.b.

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{2^n+5^n}{10^n} &= \sum_{n=0}^{\infty}\left(\frac{1}{5^n}+\frac{1}{2^n}\right)\\ &= \frac{1}{1-\frac{1}{2}}+\frac{1}{1-\frac{1}{5}}\\ &= \frac{13}{4}\quad.\end{aligned}$$

use $\sum\limits_{n=0}^{\infty} r^n=\frac{1}{1-r}$, for $|r|<1$

Solution. 2.d.

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{3^{n+1}+7^{n-1}}{21^n} &= \sum_{n=1}^{\infty}\left(3\cdot\frac{3^n}{21^n}+\frac{1}{7}\cdot\frac{7^n}{21^n}\right)\\ &= 3\sum_{n=1}^{\infty}\left(\frac{1}{7}\right)^n+\frac{1}{7}\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n\\ &= \frac{3}{7}\sum_{n=0}^{\infty}\left(\frac{1}{7}\right)^n+\frac{1}{21}\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n\\ &= \frac{3}{7}\cdot\frac{1}{(1-\frac{1}{7})}+\frac{1}{21}\cdot\frac{1}{(1-\frac{1}{3})}\\ &= \frac{4}{7}\quad.\end{aligned}$$

use $\sum_{n=0}^{\infty} r^n=\frac{1}{1-r}, |r|<1$

Solution. 2.e.

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{2^{n+1}+(-3)^{n-1}}{5^n} &= \sum_{n=0}^{\infty}\left(2\cdot\frac{2^n}{5^n}-\frac{1}{3}\cdot\frac{(-3)^n}{5^n}\right)\\ &= 2\sum_{n=0}^{\infty}\left(\frac{2}{5}\right)^n-\frac{1}{3}\sum_{n=0}^{\infty}\left(-\frac{3}{5}\right)^n\\ &= 2\cdot\frac{1}{(1-\frac{2}{5})}-\frac{1}{3}\cdot\frac{1}{(1-(-\frac{3}{5}))}\\ &= \frac{25}{8}\quad.\end{aligned}$$

use $\sum_{n=0}^{\infty} r^n=\frac{1}{1-r}, |r|<1$

3. Sum the telescoping series (a sum is "telescoping" if it can be broken into summands so that consecutive terms cancel).

   (a) $\displaystyle\sum_{n=0}^{\infty}\frac{-6}{9n^2+3n-2}$.

   answer: 2

   (b) $\displaystyle\sum_{n=3}^{\infty}\frac{3}{n^2-3n+2}$.

   answer: 3

   (c) $\displaystyle\sum_{n=2}^{\infty}\ln\left(1-\frac{1}{n^2}\right)$. (Hint: Use the properties of the logarithm to aim for a telescoping series).

   answer: $-\ln 2$

Solution. 3.b

$$\begin{aligned}
\sum_{n=3}^{\infty} \frac{3}{n^2-3n+2} &= \sum_{n=3}^{\infty}\left(\frac{3}{n-2}-\frac{3}{n-1}\right) && \text{use partial fractions, see below}\\
&= 3\sum_{n=3}^{\infty}\left(\frac{1}{n-2}-\frac{1}{n-1}\right)\\
&= 3\left(\underbrace{\left(1-\frac{1}{2}\right)}_{n=3}+\underbrace{\left(\frac{1}{2}-\frac{1}{3}\right)}_{n=4}+\underbrace{\left(\frac{1}{3}-\frac{1}{4}\right)}_{n=5}+\dots\right)\\
&= 3\lim_{n\to\infty}\left(1-\frac{1}{n-1}\right)=3 \quad .
\end{aligned}$$

In the above we used the partial fraction decomposition of $\frac{3}{n^2-3n+2}$. This decomposition is computed as follows.

$$\frac{3}{n^2-3n+2}=\frac{3}{(n-1)\,(n-2)}$$

We need to find A_i's so that we have the following equality of rational functions. After clearing denominators, we get the following equality.

$$3 = A_1(n-2)+A_2(n-1)$$

After rearranging we get that the following polynomial must vanish. Here, by "vanish" we mean that the coefficients of the powers of x must be equal to zero.

$$(A_2+A_1)n+(-A_2-2A_1-3)$$

In other words, we need to solve the following system.

$$\begin{array}{lll} -2A_1 & -A_2 & =3\\ A_1 & +A_2 & =0 \end{array}$$

System status	Action
$\begin{array}{lll} -2A_1 & -A_2 & =3\\ A_1 & +A_2 & =0 \end{array}$	Selected pivot column 2. Eliminated the non-zero entries in the pivot column.
$\begin{array}{lll} A_1 & +\frac{A_2}{2} & =-\frac{3}{2}\\ & \frac{A_2}{2} & =\frac{3}{2} \end{array}$	Selected pivot column 3. Eliminated the non-zero entries in the pivot column.
$\begin{array}{lll} A_1 & & =-3\\ & A_2 & =3 \end{array}$	Final result.

Therefore, the final partial fraction decomposition is the following.

$$\frac{3}{n^2-3n+2}=\frac{-3}{(n-1)}+\frac{3}{(n-2)}.$$

Solution. 3.c.

$$\begin{aligned}
\sum_{n=2}^{\infty}\ln\left(1-\frac{1}{n^2}\right) &= \sum_{n=2}^{\infty}\left(\ln\left(1-\frac{1}{n}\right)+\ln\left(1+\frac{1}{n}\right)\right)\\
&= \sum_{n=2}^{\infty}\left(\ln\left(\frac{n-1}{n}\right)+\ln\left(\frac{n+1}{n}\right)\right)\\
&= \sum_{n=2}^{\infty}\left(\ln(n-1)-2\ln(n)+\ln(n+1)\right)\\
&= (\ln 1-2\ln 2+\cancel{\ln 3})+(\ln \cancel{2-2\ln 3}+\cancel{\ln 4})\\
&\quad +(\cancel{\ln 3-2\ln 4}+\cancel{\ln 5})+\cancel{\ldots}\ldots\\
&= \lim_{n\to\infty}\left(-\ln 2-\ln n+\ln(n+1)\right)\\
&= \lim_{n\to\infty}\left(-\ln 2+\ln\left(\frac{n+1}{n}\right)\right)\\
&= -\ln 2 \quad .
\end{aligned}$$

4. Use partial fractions to sum the telescoping series (a sum is "telescoping" if it can be broken into summands so that consecutive terms cancel).

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+n}$

answer: 1

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{2n+1}{n^4+2n^3-n^2-2n}$

answer: $\frac{1}{3}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{2n}{n^4-3n^2+1}$

answer: -1

(d) $\displaystyle\sum_{n=3}^{\infty} \frac{n^2+n+2}{n^4-5n^2+4}$

answer: $\frac{3}{4}$

Solution. 4d

The partial fractions decomposition algorithm shows that

$$\frac{n^2+n+2}{n^4-5n^2+4} = \frac{1}{3}\left(\frac{2}{n-2} - \frac{2}{n-1} + \frac{1}{n+1} - \frac{1}{n+2}\right) \quad .$$

We omit the details of the partial fraction decomposition as it is quite laborious, but otherwise straightforward. Therefore

$$\begin{aligned}
\sum_{n=3}^{\infty} \frac{n^2+n+2}{n^4-5n^2+4} &= \frac{1}{3}\sum_{n=3}^{\infty}\left(\frac{2}{n-2} - \frac{2}{n-1} + \frac{1}{n+1} - \frac{1}{n+2}\right)\\
&= \frac{2}{3}\sum_{n=3}^{\infty}\left(\frac{1}{n-2} - \frac{1}{n-1}\right)\\
&\quad +\frac{1}{3}\sum_{n=3}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n+2}\right)\\
&= \frac{2}{3}\left(\left(1-\cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{2}{3}}\right) + \cancel{\dots} + \left(\cancel{\frac{1}{n-2}} - \frac{1}{n-1}\right) + \dots\right)\\
&\quad +\frac{1}{3}\left(\left(\frac{1}{4}-\cancel{\frac{1}{5}}\right) + \left(\cancel{\frac{1}{5}} - \cancel{\frac{1}{6}}\right) + \cancel{\dots} + \left(\cancel{\frac{1}{n+1}} - \frac{1}{n+2}\right) + \dots\right)\\
&= \lim_{n\to\infty}\frac{2}{3}\left(1-\frac{1}{n-1}\right) + \lim_{n\to\infty}\frac{1}{3}\left(\frac{1}{4}-\frac{1}{n+2}\right)\\
&= \frac{2}{3} + \frac{1}{3}\cdot\frac{1}{4}\\
&= \frac{3}{4} \quad .
\end{aligned}$$