Calculus I
Newton's Method

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2019

Outline

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Newton's Method

Find the roots of these equations:

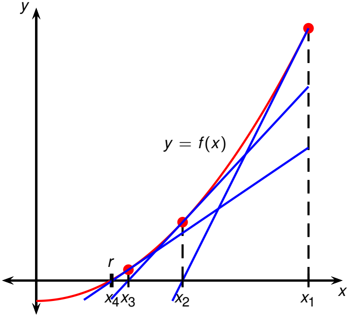
$$x^3 - 5x^2 - 6x = 0$$
$$x(x-6)(x+1) = 0$$

- Roots: $x = 0, -1$, or 6.
- No problem.

$$48x(1+x)^{60} - (1+x)^{60} + 1 = 0$$

- Problem.
- Plug it into a computer algebra system. The non-zero root is about 0.0076.
- How does the computer find the root?
- Probably using Newton's Method.

Goal: find a root r of $f(x)$.



- Pick a number x_1.
- Find the tangent to f at $(x_1, f(x_1))$.
- Call the x-intercept of this line x_2.
- Repeat the process using x_2.
- Find the tangent to f at $(x_2, f(x_2))$.
- Call the x-intercept of this line x_3, and so on.

$$\begin{aligned} x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} \\ x_3 &= x_2 - \frac{f(x_2)}{f'(x_2)} \\ &\vdots \\ x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \end{aligned}$$

$$\begin{aligned} \text{Equation:} \quad y - f(x_n) &= f'(x_n)(x - x_n) \\ x\text{-intercept:} \quad 0 - f(x_n) &= f'(x_n)(x_{n+1} - x_n) \\ f'(x_n)x_n - f(x_n) &= f'(x_n)x_{n+1} \\ x_{n+1} &= \frac{f'(x_n)x_n - f(x_n)}{f'(x_n)} \\ x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \end{aligned}$$

- Newton's Method gives us a sequence $x_1, x_2, x_3, \dots$ of approximations to a root r of a function $f(x)$.
- If the nth approximation is x_n and $f'(x_n) \neq 0$, then the $(n+1)$st approximation is
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$
- If the numbers x_n become closer and closer to r as n becomes large, we say that the sequence converges to r.
- The sequence does not always converge.

Example (Newton's Method, Example 1, p. 313)

Starting with $x_1 = 2$, find the third approximation x_3 to the root of the equation $x^3 - 2x - 5 = 0$.

$$f(x) = x^3 - 2x - 5.$$

$$f'(x) = 3x^2 - 2.$$

Newton's Method: $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 2x_n - 5}{3x_n^2 - 2}$$

$$\begin{aligned} x_2 &= x_1 - \frac{{x_1}^3 - 2x_1 - 5}{3{x_1}^2 - 2} \\ &= (2) - \frac{(2)^3 - 2(2) - 5}{3(2)^2 - 2} \\ &= 2.1. \end{aligned}$$

$$\begin{aligned} x_3 &= x_2 - \frac{{x_2}^3 - 2x_2 - 5}{3{x_2}^2 - 2} \\ &= (2.1) - \frac{(2.1)^3 - 2(2.1) - 5}{3(2.1)^2 - 2} \\ &= 2.0946. \end{aligned}$$

Example (Newton's Method)

Starting with $x_1 = 5$, use two steps of Newton's Method to approximate $\sqrt{28}$.

$$f(x) = x^2 - 28.$$
$$f'(x) = 2x.$$

Newton's Method: $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - 28}{2x_n}$$

$$\begin{aligned} x_2 &= x_1 - \frac{{x_1}^2 - 28}{2x_1} \\ &= (5) - \frac{(5)^2 - 28}{2(5)} \\ &= 5.3. \end{aligned}$$

$$\begin{aligned} x_3 &= x_2 - \frac{{x_2}^2 - 28}{2x_2} \\ &= (5.3) - \frac{(5.3)^2 - 28}{2(5.3)} \\ &= 5609/1060. \end{aligned}$$