Calculus II
Homework
Improper integrals

1. Let $x \in (0,1)$. Express the following using x and $\sqrt{1-x^2}$.

   (a) $\sin(\arcsin(x))$.

   answer: x

   (b) $\sin(2\arcsin(x))$.

   answer: $2x\sqrt{1-x^2}$

   (c) $\sin(3\arcsin(x))$.

   answer: $-4x^3+3x$

   (d) $\sin(\arccos(x))$.

   answer: $\sqrt{1-x^2}$

   (e) $\sin(2\arccos(x))$.

   answer: $2x\sqrt{1-x^2}$

   (f) $\sin(3\arccos(x))$.

   answer: $(4x^2-1)\sqrt{1-x^2}$
   $= -4\left(\sqrt{1-x^2}\right)^3 + 3\sqrt{1-x^2}$

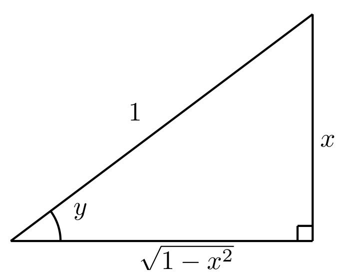
   (g) $\cos(2\arcsin(x))$.

   answer: $1-2x^2$

   (h) $\cos(3\arccos(x))$.

   answer: $4x^3-3x$

Solution. 1.b. Let $y = \arcsin x$. Then $\sin y = x$, and we can draw a right triangle with opposite side length x and hypotenuse length 1 to find the other trigonometric ratios of y.



Then $\cos y = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$. Now we use the double angle formula to find $\sin(2\arcsin x)$.

$$\begin{aligned}\sin(2\arcsin x) &= \sin(2y)\\ &= 2\sin y\cos y\\ &= 2x\sqrt{1-x^2}.\end{aligned}$$

Solution. 1.c. Use the result of Problem 1.b. This also requires the addition formula for sine:

$$\sin(A+B) = \sin A\cos B + \sin B\cos A,$$

and the double angle formula for cosine:

$$\cos(2y) = \cos^2 y - \sin^2 y.$$

$$
\begin{aligned}
\sin(3\arcsin x) &= \sin(3y) \\
&= \sin(2y+y) \\
&= \sin(2y)\cos y + \sin y\cos(2y) && \text{Use addition formula} \\
&= (2\sin y\cos y)\cos y + \sin y(\cos^2 y - \sin^2 y) && \text{Use double angle formulas} \\
&= 2\sin y\cos^2 y + \sin y\cos^2 y - \sin^3 y \\
&= 3\sin y\cos^2 y - \sin^3 y \\
&= 3\sin y(1-\sin^2 y) - \sin^3 y \\
&= 3x(1-x^2) - x^3 \\
&= 3x - 4x^3.
\end{aligned}
$$

The solution is complete. A careful look at the solution above reveals a strategy useful for problems similar to this one.

(a) Identify the inverse trigonometric expression- $\arcsin x, \arccos x, \arctan x, \dots$. In the present problem that was $y = \arcsin x$.

(b) The problem is therefore a trigonometric function of y.

(c) Using trig identities and algebra, rewrite the problem as a trigonometric expression involving only the trig function that transforms y to x. In the present problem we rewrote everything using $\sin y$.

(d) Use the fact that $\sin(\arcsin x) = x$, $\cos(\arccos x) = x, \dots$, etc. to simplify.

Solution. 1.f We use the same strategy outlined in the end of the solution of Problem 1.c. Set $y = \arccos x$ and so $\cos(y) = x$. Therefore:

$$
\begin{aligned}
\sin(3y) &= \sin(2y+y) \\
&= \sin(2y)\cos y + \sin y\cos(2y) \\
&= 2\sin y\cos y\cos y + \sin y(2\cos^2 y - 1) \\
&= 2\sin y\cos^2 y + \sin y(2\cos^2 y - 1) \\
&= \sin y(4\cos^2 y - 1) && \text{use } \begin{array}{rcl} \cos y &=& x \\ \sin y &=& \sqrt{1-x^2} \end{array} \\
&= \sqrt{1-x^2}(4x^2-1) \quad .
\end{aligned}
$$

2. Express as the following as an algebraic expression of x. In other words, "get rid" of the trigonometric and inverse trigonometric expressions.

(a) $\cos^2(\arctan x)$.

answer: $\frac{1}{1+x^2}$

(b) $-\sin^2(\operatorname{arccot} x)$.

answer: $-\frac{1}{1+x^2}$

(c) $\dfrac{1}{\cos(\arcsin x)}$.

answer: $\frac{1}{\sqrt{1-x^2}}$

(d) $-\dfrac{1}{\sin(\arccos x)}$.

answer: $-\frac{1}{\sqrt{1-x^2}}$

Solution. 2.b. We follow the strategy outlined in the end of the solution of Problem 1.c. We set $y = \operatorname{arccot} x$. Then we need to express $-\sin^2 y$ via $\cot y$. That is a matter of algebra:

$$
\begin{aligned}
-\sin^2(\operatorname{arccot} x) &= -\sin^2 y && \text{Set } y = \operatorname{arccot} x \\
&= -\frac{\sin^2 y}{\sin^2 y + \cos^2 y} && \text{use } \sin^2 y + \cos^2 y = 1 \\
&= -\frac{1}{\frac{\sin^2 y + \cos^2 y}{\sin^2 y}} \\
&= -\frac{1}{1+\cot^2 y} && \text{Substitute back } \cot y = x \\
&= -\frac{1}{1+x^2} \quad .
\end{aligned}
$$

3. Rewrite as a rational function of t. This problem will be later used to derive the Euler substitutions (an important technique for integrating).

(a) $\cos(2\arctan t)$.

answer: $\frac{1-t^2}{1+t^2}$.

(b) $\sin(2\arctan t)$.

answer: $\frac{2t}{1+t^2}$

(c) $\tan(2\arctan t)$.

answer: $\frac{2t}{1-t^2}$

(d) $\cot(2\arctan t)$.

answer: $\frac{1}{2}\left(\frac{1}{t}-t\right)$.

(e) $\csc(2\arctan t)$.

answer: $\frac{1}{2}\left(t+\frac{1}{t}\right)$

(f) $\sec(2\arctan t)$.

answer: $\frac{1+t^2}{1-t^2}$

(g) $\cos(2\operatorname{arccot} t)$.

answer: $\frac{t^2-1}{t^2+1}$

(h) $\sin(2\operatorname{arccot} t)$.

answer: $\frac{2t}{t^2+1}$

(i) $\tan(2\operatorname{arccot} t)$.

answer: $\frac{2t}{t^2-1}$

(j) $\cot(2\operatorname{arccot} t)$.

answer: $\frac{1}{2}\left(t-\frac{1}{t}\right)$

(k) $\csc(2\operatorname{arccot} t)$.

answer: $\frac{1}{2}\left(t+\frac{1}{t}\right)$

(l) $\sec(2\operatorname{arccot} t)$.

answer: $\frac{t^2+1}{t^2-1}$

Solution. 3.a Set $z = \arctan t$, and so $\tan z = t$. Then

$$
\begin{array}{rcll}
\cos(2\arctan t) &=& \cos(2z) & \\
&=& \dfrac{\cos(2z)}{1} & \left|\begin{array}{l}\text{use double angle formulas}\\ \text{and } 1=\sin^2 z+\cos^2 z\end{array}\right. \\
&=& \dfrac{\cos^2 z-\sin^2 z}{\cos^2 z+\sin^2 z} & \left|\ \text{divide top and bottom by } \cos^2 z\right. \\
&=& \dfrac{(\cos^2 z-\sin^2 z)\frac{1}{\cos^2 z}}{(\sin^2 z+\cos^2 z)\frac{1}{\cos^2 z}} & \\
&=& \dfrac{1-\tan^2 z}{1+\tan^2 z} & \\
&=& \dfrac{1-t^2}{1+t^2} & .
\end{array}
$$

Solution. 3.d Set $z = \arctan t$, and so $\tan z = t$. Then

$$
\begin{array}{rcll}
\cot(2\arctan t) &=& \cot(2z) & \\
&=& \dfrac{\cos(2z)}{\sin(2z)} & \left|\ \text{use double angle formulas}\right. \\
&=& \dfrac{\cos^2 z-\sin^2 z}{2\sin z\cos z} & \\
&=& \dfrac{1-\tan^2 z}{2\tan z} & \\
&=& \dfrac{1-t^2}{2t} & \\
&=& \dfrac{1}{2}\left(\dfrac{1}{t}-t\right) & .
\end{array}
$$

4. Compute the derivative (derive the formula).

   (a) $(\arctan x)'$.

   answer: $\frac{1}{1+x^2}$

   (b) $(\operatorname{arccot} x)'$.

   answer: $-\frac{1}{1+x^2}$

   (c) $(\arcsin x)'$.

   answer: $\frac{1}{\sqrt{1-x^2}}$

   (d) $(\arccos x)'$.

   answer: $-\frac{1}{\sqrt{1-x^2}}$

   (e) Let arcsec denote the inverse of the secant function. Compute $(\operatorname{arcsec} x)'$.

   answer: $\frac{1}{x\sqrt{x^2-1}}$

5. (a) Let $a+b \neq k\pi$, $a \neq k\pi + \frac{\pi}{2}$ and $b \neq k\pi + \frac{\pi}{2}$ for any $k \in \mathbb{Z}$ (integers). Prove that

$$\frac{\tan a + \tan b}{1 - \tan a \tan b} = \tan(a+b) \quad .$$

   (b) Let x and y be real. Prove that, for $xy \neq 1$, we have

$$\arctan x + \arctan y = \arctan\left(\frac{x+y}{1-xy}\right)$$

   if the left hand side lies between $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Solution. 5.a We start by recalling the formulas

$$\begin{array}{rcl}\cos(a+b) &=& \cos a\cos b - \sin a \sin b\\ \sin(a+b) &=& \sin a \cos b + \sin b \cos a \quad .\end{array}$$

These formulas have been previously studied; alternatively they follow from Euler's formula and the computation

$$\begin{array}{rcl}\cos(a+b)+i\sin(a+b) &=& e^{i(a+b)} = e^{ia}e^{ib} = (\cos a + i\sin a)(\cos b + i \sin b)\\ &=& \cos a\cos b - \sin a\sin b + i(\sin a\cos b + \sin b\cos a) \quad .\end{array}$$

Now 5.a is done via a straightforward computation:

$$\begin{array}{rcl}\tan(a+b) &=& \dfrac{\sin(a+b)}{\cos(a+b)} = \dfrac{\sin a\cos b + \sin b\cos a}{\cos a\cos b - \sin a\sin b} = \dfrac{(\sin a\cos b+\sin b\cos a)\frac{1}{\cos a\cos b}}{(\cos a\cos b - \sin a\sin b)\frac{1}{\cos a\cos b}}\\ &=& \dfrac{\tan a+\tan b}{1-\tan a\tan b} \quad .\end{array} \tag{1}$$

5.b is a consequence of 5.a. Let $a=\arctan x$, $b=\arctan y$. Then (**??**) becomes

$$\tan(\arctan x+\arctan y) = \frac{\tan(\arctan x)+\tan(\arctan y)}{1-\tan(\arctan x)\tan(\arctan y)} = \frac{x+y}{1-xy} \quad ,$$

where we use the fact that $\tan(\arctan w)=w$ for all w. We recall that $\arctan(\tan z)=z$ whenever $z\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$. Now take $\arctan$ on both sides of the above equality to obtain

$$\arctan x+\arctan y = \arctan\left(\frac{x+y}{1-xy}\right) \quad .$$

6. Evaluate the indefinite integral. Illustrate the steps of your solutions.

(a) $\int x\sin x \mathrm{d}x$.

answer: $-x\cos x+\sin x+C$

(b) $\int xe^{-x}\mathrm{d}x$.

answer: $-(1+x)e^{-x}+C$

(c) $\int x^2e^x\mathrm{d}x$.

answer: $x^2e^x-2xe^x+2e^x+C$

(d) $\int x\sin(-2x)\mathrm{d}x$.

answer: $\frac{x}{2}\cos(-2x)+\frac{1}{4}\sin(-2x)+C$

(e) $\int x^2\cos(3x)\mathrm{d}x$.

answer: $\frac{x^2}{3}\sin(3x)+\frac{2x}{9}\cos(3x)-\frac{2}{27}\sin(3x)+C$

(f) $\int x^2e^{-2x}\mathrm{d}x$.

answer: $-\frac{x^2e^{-2x}}{2}-\frac{xe^{-2x}}{2}-\frac{e^{-2x}}{4}+C$

(g) $\int x\sin(2x)\mathrm{d}x$.

answer: $-\frac{x}{2}\cos(2x)+\frac{1}{4}\sin(2x)+C$

(h) $\int x\cos(3x)\mathrm{d}x$.

answer: $\frac{x}{3}\sin(3x)+\frac{1}{9}\cos(3x)+C$

(i) $\int x^2e^{2x}\mathrm{d}x$.

answer: $\frac{x^2}{2}e^{2x}-\frac{x}{2}e^{2x}+\frac{e^{2x}}{4}+C$

(j) $\int x^3e^x\mathrm{d}x$.

answer: $x^3e^x-3x^2e^x+6xe^x-6e^x+C$

Solution. 6.a.

$$\int x\underbrace{\sin x\mathrm{d}x}_{=\mathrm{d}(-\cos x)} = -\int x\mathrm{d}(\cos x) = -x\cos x+\int\cos x\mathrm{d}x = -x\cos x+\sin x+C \quad .$$

Solution. 6.c.

$$\begin{array}{rcl}\int x^2\underbrace{e^x\mathrm{d}x}_{\mathrm{d}(e^x)} &=& \int x^2\mathrm{d}e^x = x^2e^x-\int e^x2x\mathrm{d}x = x^2e^x-\int 2x\mathrm{d}e^x\\ &=& x^2e^x-2xe^x+\int 2e^x\mathrm{d}x = x^2e^x-2xe^x+2e^x+C \quad .\end{array}$$

Solution. 6.f.

$$
\begin{aligned}
\int x^2 e^{-2x}\mathrm{d}x &= \int x^2 \mathrm{d}\left(\frac{e^{-2x}}{-2}\right) && \text{Integrate by parts}\\
&= -\frac{x^2e^{-2x}}{2} - \int \left(\frac{e^{-2x}}{-2}\right)\mathrm{d}\left(x^2\right)\\
&= -\frac{x^2e^{-2x}}{2} + \int xe^{-2x}\mathrm{d}x\\
&= -\frac{x^2e^{-2x}}{2} + \int x\mathrm{d}\left(\frac{e^{-2x}}{-2}\right) && \text{Integrate by parts}\\
&= -\frac{x^2e^{-2x}}{2} - \frac{xe^{-2x}}{2} + \frac{1}{2}\int e^{-2x}\mathrm{d}x\\
&= -\frac{x^2e^{-2x}}{2} - \frac{xe^{-2x}}{2} - \frac{e^{-2x}}{4} + C \quad .
\end{aligned}
$$

7. Evaluate the indefinite integral. Illustrate the steps of your solutions.

(a) $\int x^2\cos(2x)\mathrm{d}x$.

answer: $\frac{1}{2}x^2\sin(2x) + \frac{1}{2}x\cos(2x) - \frac{1}{4}\sin(2x) + C$

(b) $\int x^2e^{ax}\mathrm{d}x$, where a is a constant.

answer: $\frac{1}{a}x^2e^{ax} - \frac{2}{a^2}xe^{ax} + \frac{2}{a^3}e^{ax} + C$

(c) $\int x^2e^{-ax}\mathrm{d}x$, where a is a constant.

answer: $-\frac{1}{a}x^2e^{-ax} - \frac{2}{a^2}xe^{-ax} - \frac{2}{a^3}e^{-ax} + C$

(d) $\int x^2\frac{(e^{ax}+e^{-ax})^2}{4}\mathrm{d}x$, where a is a constant.

answer: $\frac{1}{8}\left(a^{-1}x^2e^{2ax} - a^{-1}x^2e^{-2ax} - a^{-2}xe^{2ax} - a^{-2}xe^{-2ax} + \frac{1}{2}a^{-3}e^{2ax} - \frac{1}{2}a^{-3}e^{-2ax} + \frac{2}{3}x^3\right) + C$

(e) $\int \frac{1}{\cos^2 x}\mathrm{d}x$. (Hint: This problem does not require integration by parts. What is the derivative of $\tan x$?)

answer: $\tan x + C$

(f) $\int(\tan^2 x)\mathrm{d}x$. (Hint: This problem does not require integration by parts. We can use $\tan^2 x = \frac{1}{\cos^2 x} - 1$ and the previous problem.)

(g) $\int x\tan^2 x\mathrm{d}x$. (Hint: $\tan^2 x\mathrm{d}x = \mathrm{d}(F(x))$, where $F(x)$ is the answer from the preceding problem).

answer: $-\frac{x^2}{2} + x\tan x + \ln|\cos x| + C$

(h) $\int e^{-\sqrt{x}}\mathrm{d}x$.

answer: $-2\sqrt{x}e^{-\sqrt{x}} - 2e^{-\sqrt{x}} + C$

(i) $\int\cos^2 x\ \mathrm{d}x$.

answer: $\frac{1}{4}\sin(2x) + \frac{x}{2} + C$

(j) $\int\frac{x}{1+x^2}\mathrm{d}x$ (Hint: use substitution rule, don't use integration by parts)

answer: $\frac{\ln(1+x^2)}{2} + C$

(k) $\int(\arctan x)\mathrm{d}x$.

answer: $x\arctan x - \frac{\ln(1+x^2)}{2} + C$

(l) $\int(\arcsin x)\mathrm{d}x$.

answer: $x\arcsin x + \sqrt{1-x^2} + C$

(m) $\int(\arcsin x)^2\mathrm{d}x$. (Hint: Try substituting $x = \sin y$.)

answer: $x(\arcsin x)^2 + 2\sqrt{1-x^2}\arcsin x - 2x + C$

(n) $\int\arctan\left(\frac{1}{x}\right)\mathrm{d}x$.

(o) $\int\sin xe^x\mathrm{d}x$

answer: $\frac{1}{2}(e^x\sin x - e^x\cos x) + C$

(p) $\int\cos xe^x\mathrm{d}x$

answer: $\frac{1}{2}(e^x\cos x + e^x\sin x) + C$

(q) $\int\sin(\ln(x))\mathrm{d}x$.

answer: $\frac{x}{2}(\sin(\ln x) - \cos(\ln x)) + C$

(r) $\int\cos(\ln(x))\mathrm{d}x$.

answer: $\frac{x}{2}(\cos(\ln x) + \sin(\ln x)) + C$

(s) $\int\ln x\mathrm{d}x$

answer: $x\ln|x| - x + C$

(t) $\int x\ln x\ \mathrm{d}x$.

answer: $\frac{1}{2}x^2\ln|x| - \frac{x^2}{4} + C$

(u) $\int\frac{\ln x}{\sqrt{x}}\mathrm{d}x$.

answer: $2\sqrt{x}(\ln x - 2) + C$

(v) $\int(\ln x)^2\mathrm{d}x$.

answer: $x(\ln x)^2 - 2x\ln x + 2x + C$

(w) $\int(\ln x)^3\mathrm{d}x$.

answer: $x(\ln x)^3 - 3x(\ln x)^2 + 6x\ln x - 6x + C$

(x) $\int x^2\cos^2 x\mathrm{d}x$. (This problem is related to Problem 7.d as $\cos x = \frac{e^{ix}+e^{-ix}}{2}$).

answer: $\frac{1}{4}x^2\sin(2x) + \frac{1}{4}x\cos(2x) - \frac{1}{8}\sin(2x) + \frac{1}{6}x^3$

Solution. 7.g.

$$\begin{array}{rcll}
\int x\tan^2 x\mathrm{d}x &=& \int x\left(\sec^2 x-1\right)\mathrm{d}x & \Big| \text{ use } \sec^2 x - 1 = \tan^2 x\\
&=& \int x\left(\sec^2 x-1\right)\mathrm{d}x & \\
&=& -\int x\mathrm{d}x + \int x\sec^2 x\mathrm{d}x & \Big| \text{ use } \mathrm{d}(\tan x) = \sec^2 x\mathrm{d}x\\
&=& -\frac{x^2}{2} + \int x\mathrm{d}(\tan x) & \Big| \text{ integrate by parts}\\
&=& -\frac{x^2}{2} + x\tan x - \int \tan x\mathrm{d}x & \\
&=& -\frac{x^2}{2} + x\tan x - \int \frac{\sin x}{\cos x}\mathrm{d}x & \Big| \text{ use } \sin x\mathrm{d}x = -\mathrm{d}(\cos x)\\
&=& -\frac{x^2}{2} + x\tan x + \int \frac{\mathrm{d}(\cos x)}{\cos x} & \Big| \text{ Set } y=\cos x\\
&=& -\frac{x^2}{2} + x\tan x + \int \frac{1}{y}\mathrm{d}y & \\
&=& -\frac{x^2}{2} + x\tan x + \ln|y| + C & \Big| \text{ Substitute back } y=\cos x\\
&=& -\frac{x^2}{2} + x\tan x + \ln|\cos x| + C & .
\end{array}$$

Solution. 7.h.

$$\begin{array}{rcll}
\int e^{-\sqrt{x}}\mathrm{d}x &=& \int 2ye^{-y}\mathrm{d}y & \Big| \text{ Subst.: } \begin{array}{rcl} \sqrt{x} &=& y\\ \frac{1}{2\sqrt{x}}\mathrm{d}x &=& \mathrm{d}y\\ \mathrm{d}x &=& 2y\mathrm{d}y \end{array}\\
&=& \int 2y\mathrm{d}\left(-e^{-y}\right) & \Big| \text{ int. by parts}\\
&=& -2ye^{-y} + 2\int e^{-y}\mathrm{d}y & \\
&=& -2ye^{-y} - 2e^{-y} + C & \\
&=& -2\sqrt{x}e^{-\sqrt{x}} - 2e^{-\sqrt{x}} + C & .
\end{array}$$

Solution. 7.i. Later, we shall study general methods for solving trigonometric integrals that will cover this example. Let us however show one way to solve this integral by integration by parts.

$$\begin{array}{rll}
\int \cos^2 x\mathrm{d}x &= x\cos^2 x - \int x\mathrm{d}(\cos^2 x) & \\
&= x\cos^2 x - \int x2\cos x(-\sin x)\mathrm{d}x & \Big| \sin(2x) = 2\sin x\cos x\\
&= x\cos^2 x + \int x\sin(2x)\mathrm{d}x & \\
&= x\cos^2 x + \int x\mathrm{d}\left(\frac{-\cos(2x)}{2}\right) & \\
&= x\cos^2 x + x\left(\frac{-\cos(2x)}{2}\right) - \int\left(\frac{-\cos(2x)}{2}\right)\mathrm{d}x & \\
&= \frac{x}{2}\left(2\cos^2 x - \cos(2x)\right) + \frac{\sin(2x)}{4} + C & \Big| \cos(2x) = \cos^2 x - \sin^2 x\\
&= \frac{x}{2}\left(2\cos^2 x - (\cos^2 x - \sin^2 x)\right) + \frac{\sin(2x)}{4} + C & \Big| \cos^2 x + \sin^2 x = 1\\
&= \frac{x}{2} + \frac{\sin(2x)}{4} + C & .
\end{array}$$

Solution. 7.k

$$\begin{array}{rcl}
\int \arctan x\mathrm{d}x &=& x\arctan x - \int x\mathrm{d}(\arctan x)\\
&=& x\arctan x - \int \frac{x}{x^2+1}\mathrm{d}x\\
&=& x\arctan x - \int \frac{\frac{1}{2}\mathrm{d}(x^2)}{x^2+1}\\
&=& x\arctan x - \int \frac{\frac{1}{2}\mathrm{d}(x^2+1)}{x^2+1}\\
&=& x\arctan x - \frac{1}{2}\ln(x^2+1) + C \quad .
\end{array}$$

Solution. 7.m.

$$\begin{aligned}
\int (\arcsin x)^2 \mathrm{d}x &= \int \left(\arcsin(\sin y)\right)^2 \mathrm{d}(\sin y) && \Big| \text{ Set } x=\sin y\\
&= \int y^2 \cos y \mathrm{d}y = \int y^2 \mathrm{d}(\sin y) && \Big| \text{ Integrate by parts}\\
&= y^2 \sin y - \int 2y \sin y \mathrm{d}y \\
&= y^2 \sin y + \int 2y \mathrm{d}(\cos y) && \Big| \text{ Integrate by parts}\\
&= y^2 \sin y + 2y\cos y - 2\int \cos y \mathrm{d}y \\
&= y^2 \sin y + 2y\cos y - 2\sin y + C && \Big| \text{ Substitute } y=\arcsin x\\
&= x(\arcsin x)^2 \\
&\quad +2\sqrt{1-x^2}\arcsin x - 2x + C \quad .
\end{aligned}$$

Solution. 7.o

$$\begin{aligned}
\int \sin x \underbrace{e^x \mathrm{d}x}_{=\mathrm{d}e^x} &= \sin x e^x - \int e^x \mathrm{d}(\sin x) = \sin x e^x - \int \cos x \underbrace{e^x \mathrm{d}x}_{=\mathrm{d}e^x}\\
&= \sin x e^x - e^x \cos x + \int e^x \mathrm{d}(\cos x)\\
&= e^x \sin x - e^x \cos x - \textstyle\int e^x \sin x \mathrm{d}x && \Big| \text{ add } \textstyle\int e^x \sin x \mathrm{d}x \text{ to both sides}\\
2\int \sin x e^x \mathrm{d}x &= \sin x e^x - e^x \cos x\\
\int \sin x e^x \mathrm{d}x &= \frac{1}{2}\left(\sin x e^x - e^x \cos x\right) \quad .
\end{aligned}$$

Solution. 7.q.

$$\begin{aligned}
\int \sin(\ln x)\mathrm{d}x &= x\sin(\ln x) - \int x \mathrm{d}(\sin(\ln x)) && \Big| \text{ int. by parts}\\
&= x\sin(\ln x) - \int x\left(\cos(\ln x)\right)(\ln x)' \mathrm{d}x\\
&= x\sin(\ln x) - \int \cos(\ln x)\mathrm{d}x && \Big| \text{ int. by parts}\\
&= x\sin(\ln x) - \left(x\cos(\ln x) - \int x\mathrm{d}(\cos(\ln x))\right)\\
&= x\sin(\ln x) - x\cos(\ln x) + \int x(-\sin(\ln x))(\ln x)'\mathrm{d}x\\
&= x\sin(\ln x) - x\cos(\ln x) - \int \sin(\ln x)\mathrm{d}x && \Big| \text{ add } \textstyle\int \sin(\ln x)\mathrm{d}x \text{ to both sides}\\
2\int \sin(\ln x)\mathrm{d}x &= x\sin(\ln x) - x\cos(\ln x)\\
\int \sin(\ln x)\mathrm{d}x &= \frac{x}{2}\left(\sin(\ln x) - \cos(\ln x)\right) \quad .
\end{aligned}$$

Solution. 7.s

$$\int \ln x \mathrm{d}x = x\ln x - \int x\mathrm{d}(\ln x) = x\ln x - \int \frac{x}{x}\mathrm{d}x = x\ln x - x + C \quad .$$

Solution. 7.u

$$\begin{aligned}
\int \frac{\ln x}{\sqrt{x}}\mathrm{d}x &= \int (\ln x)2\mathrm{d}\left(\sqrt{x}\right) && \Big| \text{ integrate by parts} \\
&= (\ln x)2\sqrt{x} - \int 2\sqrt{x}\mathrm{d}(\ln x) \\
&= 2\sqrt{x}\ln x - 2\int \frac{\sqrt{x}}{x}\mathrm{d}x \\
&= 2\sqrt{x}\ln x - 2\int x^{-\frac{1}{2}}\mathrm{d}x \\
&= 2\sqrt{x}\ln x - 4\sqrt{x} + C \\
&= 2\sqrt{x}(\ln x - 2) + C \quad .
\end{aligned}$$

8. Compute $\int x^n e^x \mathrm{d}x$, where n is a non-negative integer.

Solution. 8

$$\begin{aligned}
\int x^n e^x \mathrm{d}x &= \int x^n \mathrm{d}e^x \\
&= x^n e^x - \int e^x \mathrm{d}x^n \\
&= x^n e^x - n\int x^{n-1}e^x\mathrm{d}x \\
&= x^n e^x - n\left(\int x^{n-1}\mathrm{d}e^x\right) \\
&= x^n e^x - n\left(x^{n-1}e^x - \int (n-1)x^{n-2}e^x\mathrm{d}x\right) \\
&= x^n e^x - nx^{n-1}e^x + n(n-1)\int x^{n-2}e^x\mathrm{d}x \\
&= \ldots\text{(continue above process)}\ldots \\
&= x^n e^x - nx^{n-1}e^x + n(n-1)x^{n-2}e^x + \ldots \\
&\quad +(-1)^k n(n-1)(n-2)\ldots(n-k+1)x^{n-k}e^x \\
&\quad + \cdots + (-1)^n n! e^x + C \\
&= C + \sum_{k=0}^{n}(-1)^n \frac{n!}{(n-k)!}x^{n-k}e^x \quad .
\end{aligned}$$

9. Integrate. Illustrate the steps of your solution.

(a) $\int \frac{1}{x+1}\mathrm{d}x$

answer: $\ln|x+1| + C$

(b) $\int \frac{x-1}{x+1}\mathrm{d}x$

answer: $x - 2\ln|x+1| + C$

(c) $\int \frac{1}{(x+1)^2}\mathrm{d}x$

answer: $-\frac{1}{x+1} + C$

(d) $\int \frac{x}{(x+1)^2}\mathrm{d}x$

answer: $\ln|x+1| + \frac{1}{x+1} + C$

(e) $\int \frac{1}{(2x+3)^2}\mathrm{d}x$

answer: $-\frac{1}{2(2x+3)} + C$

(f) $\int \frac{x}{2x^2+3}\mathrm{d}x$

answer: $\frac{1}{4}\ln\left(2x^2+3\right) + C$

(g) $\int \frac{1}{2x^2+3}\mathrm{d}x$

answer: $\frac{\sqrt{6}}{6}\arctan\left(\sqrt{\frac{2}{3}}x\right)$

(h) $\int \frac{x}{2x^2+x+1}\mathrm{d}x$.

answer: $\frac{1}{4}\ln\left(x^2+\frac{1}{2}x+\frac{1}{2}\right) - \frac{\sqrt{7}}{14}\arctan\left(\frac{4x+1}{\sqrt{7}}\right) + C$

(i) $\int \frac{x}{2x^2+x+3}\mathrm{d}x$

answer: $\frac{1}{4}\ln\left(2x^2+x+3\right) - \frac{1}{2\sqrt{23}}\arctan\left(\frac{4x+1}{\sqrt{23}}\right) + C$

(j) $\int \frac{x}{x^2-x+3}\mathrm{d}x$

answer: $\frac{1}{2}\ln\left|x^2-x+3\right| + \frac{\sqrt{11}}{11}\arctan\left(\frac{x-\frac{1}{2}}{\frac{\sqrt{11}}{2}}\right) + C$

(k) $\int \frac{1}{\left(x^2+1\right)^2}\mathrm{d}x$

answer: $\frac{1}{2}x\left(x^2+1\right)^{-1} + \frac{1}{2}\arctan(x) + C$

(l) $\int \frac{1}{\left(x^2+x+1\right)^2}\mathrm{d}x$

answer: $\frac{2}{3}x\left(x^2+x+1\right)^{-1} + \frac{1}{3}\left(x^2+x+1\right)^{-1} + \frac{4}{9}\sqrt{3}\arctan\left(\frac{2}{3}\sqrt{3}x + \frac{\sqrt{3}}{3}\right)$

(m) $\int \frac{1}{\left(x^2+1\right)^3}\mathrm{d}x$

answer: $\frac{3}{8}x\left(x^2+1\right)^{-1} + \frac{1}{4}x\left(x^2+1\right)^{-2} + \frac{3}{8}\arctan(x) + C$

Solution. 9.h.

$$\begin{aligned}
\int \frac{x}{2x^2+x+1}\mathrm{d}x &= \int \frac{x}{2\left(x^2+2x\frac{1}{4}+\frac{1}{2}\right)}\mathrm{d}x \\
&= \int \frac{x}{2\left(x^2+2x\frac{1}{4}+\frac{1}{16}-\frac{1}{16}+\frac{1}{2}\right)}\mathrm{d}x && \text{complete square in denominator} \\
&= \frac{1}{2}\int \frac{x}{\left(x+\frac{1}{4}\right)^2+\frac{7}{16}}\mathrm{d}x \\
&= \frac{1}{2}\int \frac{x+\frac{1}{4}-\frac{1}{4}}{\left(x+\frac{1}{4}\right)^2+\frac{7}{16}}\mathrm{d}\left(x+\frac{1}{4}\right) && \text{Set } u = x+\tfrac{1}{4} \\
&= \frac{1}{2}\int \frac{u-\frac{1}{4}}{u^2+\frac{7}{16}}\mathrm{d}u \\
&= \frac{1}{2}\left(\int \frac{u}{u^2+\frac{7}{16}}\mathrm{d}u - \frac{1}{4}\int \frac{1}{u^2+\frac{7}{16}}\mathrm{d}u\right) \\
&= \frac{1}{2}\left(\frac{1}{2}\ln\left(u^2+\frac{7}{16}\right) - \frac{1}{4\sqrt{\frac{7}{16}}}\arctan\left(\frac{u}{\sqrt{\frac{7}{16}}}\right)\right)+K \\
&= \frac{1}{4}\ln\left(x^2+\frac{1}{2}x+\frac{1}{2}\right) - \frac{\sqrt{7}}{14}\arctan\left(\frac{4x+1}{\sqrt{7}}\right)+K \quad .
\end{aligned}$$

Solution. 9.l

$$\begin{aligned}
\int \frac{1}{\left(x^2+x+1\right)^2}\mathrm{d}x &= \int \frac{1}{\left(\left(x^2+2x\frac{1}{2}+\frac{1}{4}\right)-\frac{1}{4}+1\right)^2}\mathrm{d}x && \text{complete the square} \\
&= \int \frac{1}{\left(\left(x+\frac{1}{2}\right)^2+\frac{3}{4}\right)^2}\mathrm{d}\left(x+\frac{1}{2}\right) && \text{Set } w = x+\tfrac{1}{2} \\
&= \int \frac{1}{\left(w^2+\frac{3}{4}\right)^2}\mathrm{d}w \\
&= \int \frac{1}{\left(\frac{3}{4}\left(\left(\frac{2w}{\sqrt{3}}\right)^2+1\right)\right)^2}\frac{\sqrt{3}}{2}\mathrm{d}\left(\frac{2w}{\sqrt{3}}\right) && \text{Set } z = \tfrac{2w}{\sqrt{3}} \\
&= \frac{\frac{\sqrt{3}}{2}}{\left(\frac{3}{4}\right)^2}\int \frac{1}{\left(z^2+1\right)^2}\mathrm{d}z \\
&= \frac{8\sqrt{3}}{9}\int \frac{1}{\left(z^2+1\right)^2}\mathrm{d}z \quad .
\end{aligned}$$

The integral $\int \frac{1}{(z^2+1)^2}\mathrm{d}z$ was already studied; it was also given as an exercise in Problem 9.k. We leave the rest of the problem to the reader.

10. Let a, b, c, A, B be real numbers. Suppose in addition $a \neq 0$ and $b^2-4ac<0$. Integrate

$$\int \frac{Ax+B}{ax^2+bx+c}\mathrm{d}x \quad .$$

The purpose of this exercise is to produce a formula in form ready for implementation in a computer algebra system.

Solution. 10.

$$
\begin{aligned}
\int \frac{Ax+B}{ax^2+bx+c}\mathrm{d}x =& \int \frac{Ax+B}{a\left(x^2+2x\frac{b}{2a}+\frac{c}{a}\right)}\mathrm{d}x \\
=& \int \frac{Ax+B}{a\left(x^2+2x\frac{b}{2a}+\frac{b^2}{4a^2}-\frac{b^2}{4a^2}+\frac{c}{a}\right)}\mathrm{d}x && \text{complete square in denominator} \\
=& \frac{1}{a}\int \frac{Ax+B}{\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}}\mathrm{d}x && \text{Set } D=\tfrac{4ac-b^2}{4a^2} \\
=& \frac{1}{a}\int \frac{A\left(x+\frac{b}{2a}-\frac{b}{2a}\right)+B}{\left(x+\frac{b}{2a}\right)^2+D}\mathrm{d}\left(x+\frac{b}{2a}\right) && \text{Set } u=x+\tfrac{b}{2a} \\
=& \frac{1}{a}\int \frac{Au+B-\frac{Ab}{2a}}{u^2+D}\mathrm{d}u && \text{Set } C=B-\tfrac{Ab}{2a} \\
=& \frac{1}{a}\left(A\int \frac{u}{u^2+D}\mathrm{d}u + C\int \frac{1}{u^2+D}\mathrm{d}u\right) \\
=& \frac{1}{a}\left(\frac{A}{2}\ln(u^2+D)+\frac{C}{\sqrt{D}}\arctan\left(\frac{u}{\sqrt{D}}\right)\right)+K \\
=& \frac{1}{a}\left(\frac{A}{2}\ln\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)\right. \\
&\left. +\frac{C}{\sqrt{D}}\arctan\left(\frac{x+\frac{b}{2a}}{\sqrt{D}}\right)\right)+K.
\end{aligned}
$$

The solution is complete. Question to the student: where do we use $b^2-4ac<0$?

11. Let a,b,c,A,B be real numbers and let $n>1$ be an integer. Suppose in addition $a\neq 0$ and $b^2-4ac<0$. Let

$$
J(n)=\int \frac{1}{\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)^n}\mathrm{d}x \quad .
$$

(a) Express the integral

$$
\int \frac{Ax+B}{\left(ax^2+bx+c\right)^n}\mathrm{d}x
$$

via $J(n)$.

(b) Express $J(n)$ recursively via $J(n-1)$

The purpose of this exercise is to produce a formula in form ready for implementation in a computer algebra system.

Solution. 11.a.

$$
\begin{aligned}
\int \frac{Ax+B}{(ax^2+bx+c)^n}\mathrm{d}x =& \int \frac{Ax+B}{a^n\left(x^2+2x\frac{b}{2a}+\frac{c}{a}\right)^n}\mathrm{d}x \\
=& \int \frac{Ax+B}{a^n\left(x^2+2x\frac{b}{2a}+\frac{b^2}{4a^2}-\frac{b^2}{4a^2}+\frac{c}{a}\right)^n}\mathrm{d}x && \text{complete square in denominator} \\
=& \frac{1}{a^n}\int \frac{Ax+B}{\left(\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}\right)^n}\mathrm{d}x && \text{Set } D=\tfrac{4ac-b^2}{4a^2} \\
=& \frac{1}{a^n}\int \frac{A\left(x+\frac{b}{2a}-\frac{b}{2a}\right)+B}{\left(\left(x+\frac{b}{2a}\right)^2+D\right)^n}\mathrm{d}\left(x+\frac{b}{2a}\right) && \text{Set } u=x+\tfrac{b}{2a} \\
=& \frac{1}{a^n}\int \frac{Au+B-\frac{Ab}{2a}}{\left(u^2+D\right)^n}\mathrm{d}u && \text{Set } C=B-\tfrac{Ab}{2a} \\
=& \frac{1}{a^n}\left(A\int \frac{u}{\left(u^2+D\right)^n}\mathrm{d}u + C\int \frac{1}{\left(u^2+D\right)^n}\mathrm{d}u\right) \\
=& \frac{1}{a^n}\left(\frac{A}{2(1-n)}\left(u^2+D\right)^{1-n}+CJ(n)\right) \\
=& \frac{1}{a^n}\left(\frac{A}{2(1-n)}\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)^{1-n}+CJ(n)\right)
\end{aligned}
$$

Solution. 11.b. We use all notation and computations from the previous part of the problem. According to theory, in order to solve that integral, we are supposed to integrate by parts the simpler integral

$$
\begin{aligned}
J(n-1) &= \int \frac{1}{\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)^{n-1}}\mathrm{d}x = \int \frac{1}{\left(u^2+D\right)^{n-1}}\mathrm{d}u && \text{int. by parts}\\
&= \frac{u}{\left(u^2+D\right)^{n-1}} - \int u \,\mathrm{d}\left(\frac{1}{\left(u^2+D\right)^{n-1}}\right)\\
&= \frac{u}{\left(u^2+D\right)^{n-1}} + 2(n-1)\int \frac{u^2}{\left(u^2+D\right)^{n}}\mathrm{d}u\\
&= \frac{u}{\left(u^2+D\right)^{n-1}} + 2(n-1)\int \frac{u^2+D-D}{\left(u^2+D\right)^{n}}\mathrm{d}u\\
&= \frac{u}{\left(u^2+D\right)^{n-1}} + 2(n-1)J(n-1) - 2D(n-1)\int \frac{1}{\left(u^2+D\right)^{n}}\mathrm{d}u\\
&= \frac{u}{\left(u^2+D\right)^{n-1}} + 2(n-1)J(n-1) - 2D(n-1)J(n)
\end{aligned}
$$

In the above equality, we rearrange terms to get that

$$
\begin{aligned}
2D(n-1)J(n) &= \frac{u}{\left(u^2+D\right)^{n-1}} + (2n-3)J(n-1)\\
J(n) &= \frac{1}{D}\left(\frac{u}{2(n-1)\left(u^2+D\right)^{n-1}} + \frac{2n-3}{2n-2}J(n-1)\right)\\
&= \frac{1}{D}\left(\frac{x+\frac{b}{2a}}{(2n-2)\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)^{n-1}} + \frac{2n-3}{2n-2}J(n-1)\right) \quad .
\end{aligned}
$$

12. Integrate. Some of the examples require partial fraction decomposition and some do not. Illustrate the steps of your solution.

(a) $\displaystyle\int \frac{1}{4x^2+4x+1}\mathrm{d}x$

answer: $-\frac{1}{2}(2x+1)^{-1}+C$

(b) $\displaystyle\int \frac{1}{1-x^2}\mathrm{d}x$

answer: $-\frac{1}{2}\ln|x-1|+\frac{1}{2}\ln|x+1|+C$

(c) $\displaystyle\int \frac{1}{5-x^2}\mathrm{d}x$

answer: $-\frac{\sqrt{5}}{10}\ln\left|x-\sqrt{5}\right|+\frac{\sqrt{5}}{10}\ln\left|x+\sqrt{5}\right|+C$

(d) $\displaystyle\int \frac{x}{4x^2+x+\frac{1}{16}}\mathrm{d}x$

answer: $\frac{1}{4}(8x+1)^{-1}+\frac{1}{4}\ln|8x+1|+C$

(e) $\displaystyle\int \frac{x+1}{2x^2+x}\mathrm{d}x$

answer: $-\frac{1}{2}\ln|2x+1|+\ln|x|+C$

(f) $\displaystyle\int \frac{x}{4x^2+x+5}\mathrm{d}x$

answer: $\frac{1}{8}\ln\left(x^2+\frac{1}{4}x+\frac{5}{4}\right)-\frac{1}{316}\sqrt{79}\arctan\left(\frac{x+\frac{1}{8}}{\frac{1}{8}\sqrt{79}}\right)+C$

(g) $\displaystyle\int \frac{x}{4x^2+x-5}\mathrm{d}x$

answer: $\frac{5}{36}\ln\left|x+\frac{5}{4}\right|+\frac{1}{9}\ln|x-1|+C$

(h) $\displaystyle\int \frac{x}{3x^2+x-2}\mathrm{d}x$

answer: $\frac{2}{15}\ln\left|x-\frac{2}{3}\right|+\frac{1}{5}\ln|x+1|+C$

(i) $\displaystyle\int \frac{x}{3x^2+x+2}\mathrm{d}x$

answer: $\frac{1}{6}\ln\left(x^2+\frac{1}{3}x+\frac{2}{3}\right)-\frac{\sqrt{23}}{69}\arctan\left(\frac{6}{23}\sqrt{23}x+\frac{\sqrt{23}}{23}\right)+C$

(j) $\displaystyle\int \frac{x}{2x^2+x+1}\mathrm{d}x$

answer: $\frac{1}{4}\ln\left(x^2+\frac{1}{2}x+\frac{1}{2}\right)-\frac{1}{14}\sqrt{7}\arctan\left(\frac{x+\frac{1}{4}}{\frac{1}{4}\sqrt{7}}\right)+C$

(k) $\displaystyle\int \frac{x}{2x^2+x-1}\mathrm{d}x$

answer: $\frac{1}{3}\ln|x+1|+\frac{1}{6}\ln\left|x-\frac{1}{2}\right|+C$

(l) $\displaystyle\int \frac{1}{x^2+x+1}\mathrm{d}x$

answer: $\frac{2}{3}\sqrt{3}\arctan\left(\frac{x+\frac{1}{2}}{\frac{1}{2}\sqrt{3}}\right)+C$

(m) $\displaystyle\int \frac{1}{2x^2+5x+1}\mathrm{d}x$

answer: $\frac{\sqrt{17}}{17}\ln\left|x-\frac{\sqrt{17}}{4}+\frac{5}{4}\right|-\frac{\sqrt{17}}{17}\ln\left|x+\frac{\sqrt{17}}{4}+\frac{5}{4}\right|$

Solution. 12.k The quadratic in the denominator has real roots and therefore can be factored using real numbers. We therefore use partial fractions.

$$
\begin{aligned}
\int \frac{x}{2x^2+x-1}\mathrm{d}x &= \int \frac{\frac{1}{2}x}{(x+1)\left(x-\frac{1}{2}\right)}\mathrm{d}x && \text{partial fractions, see below}\\
&= \int \frac{\frac{1}{3}}{(x+1)}\mathrm{d}x + \int \frac{\frac{1}{6}}{\left(x-\frac{1}{2}\right)}\mathrm{d}x\\
&= \frac{1}{3}\ln|x+1| + \frac{1}{6}\ln\left|x-\frac{1}{2}\right| + C \quad .
\end{aligned}
$$

Except for showing how the partial fraction decomposition was obtained, our solution is complete. We proceed to compute the partial fraction decomposition used above.

We aim to decompose into partial fractions the following function (the denominator has been factored).

$$\frac{x}{2x^2+x-1} = \frac{x}{(x+1)(2x-1)} = \frac{A_1}{x+1} + \frac{A_2}{2x-1} \quad .$$

After clearing denominators, we get the following equality.

$$x = A_1(2x-1) + A_2(x+1) \quad . \tag{2}$$

Next, we need to find values for A_1 and A_2 such that the equality above becomes an identity. We show two variants to do that: the method of substitutions and the method of coefficient comparison.

Variant I. This variant relies on the fact that if substitute an arbitrary value for x in (**??**) we get a relationship that must be satisfied by the coefficients A_1 and A_2. We immediately see that setting $x = \frac{1}{2}$ (notice $x = \frac{1}{2}$ is a root of the denominator) will annihilate the term $A_1(2x-1)$ and we can immediately solve for A_2. Similarly, setting $x = -1$ ($x = -1$ is the other root of the denominator) annihilates the term $A_2(x+1)$ and we can immediately solve for A_1.

- Set $x = \frac{1}{2}$. The equation (**??**) becomes
  $$\begin{array}{rcl} \frac{1}{2} &=& A_1 \cdot 0 + A_2\left(\frac{1}{2}+1\right) \\ \frac{1}{2} &=& \frac{3}{2}A_2 \\ A_2 &=& \frac{1}{3}. \end{array}$$
- Set $x = -1$. The equation (**??**) becomes
  $$\begin{array}{rcl} -1 &=& A_1(2\cdot(-1)-1) + A_2 \cdot 0 \\ -1 &=& -3A_2 \\ A_2 &=& \frac{1}{3}. \end{array}$$

Therefore we have the partial fraction decomposition

$$\begin{array}{rcl} \frac{x}{2x^2+x-1} &=& \frac{A_1}{x+1} + \frac{A_2}{2x-1} \\ &=& \frac{\frac{1}{3}}{x+1} + \frac{\frac{1}{3}}{2x-1} \\ &=& \frac{\frac{1}{3}}{x+1} + \frac{\frac{1}{6}}{x-\frac{1}{2}} \quad . \end{array}$$

Variant II. We show the most straightforward technique for finding a partial fraction decomposition - the method of coefficient comparison. Although this technique is completely doable in practice by hand, it is often the most laborious for a human. We note that techniques such as the one given in the preceding solution Variant are faster on many (but not all) problems. The present technique is also arguably the easiest to implement on a computer. The computations below were indeed carried out by a computer program written for the purpose.

After rearranging we get that the following polynomial must vanish. Here, by "vanish" we mean that the coefficients of the powers of x must be equal to zero.

$$(A_2 + 2A_1 - 1)x + (A_2 - A_1) \quad .$$

In other words, we need to solve the following system.

$$\begin{array}{rrl} 2A_1 & +A_2 & = 1 \\ -A_1 & +A_2 & = 0 \end{array}$$

System status	Action
$\begin{array}{rrl} 2A_1 & +A_2 & = 1 \\ -A_1 & +A_2 & = 0 \end{array}$	Sel. pivot column 2. Eliminate non-pivot entries.
$\begin{array}{rrl} A_1 & +\frac{A_2}{2} & = \frac{1}{2} \\ & \frac{3}{2}A_2 & = \frac{1}{2} \end{array}$	Sel. pivot column 3. Eliminate non-pivot entries.
$\begin{array}{rrl} A_1 & & = \frac{1}{3} \\ & A_2 & = \frac{1}{3} \end{array}$	Final result.

Therefore, the final partial fraction decompocsition is:

$$\frac{\frac{x}{2}}{x^2+\frac{x}{2}-\frac{1}{2}} = \frac{\frac{1}{3}}{(x+1)} + \frac{\frac{1}{3}}{(2x-1)} \quad .$$

13. Evaluate the indefinite integral. Illustrate all steps of your solution.

(a) $\displaystyle\int \frac{x^3+4}{x^2+4}\mathrm{d}x$

answer: $\frac{x^2}{2} + 2\arctan\left(\frac{x}{2}\right) - 2\ln\left(x^2+4\right) + C$

(b) $\displaystyle\int \frac{4x^2}{2x^2-1}\mathrm{d}x$

answer: $-\frac{1}{2}\sqrt{2}\ln\left(x+\frac{1}{2}\sqrt{2}\right) + \frac{1}{2}\sqrt{2}\ln\left(x-\frac{1}{2}\sqrt{2}\right) + 2x + C$

(c) $\displaystyle\int \frac{x^3}{x^2+2x-3}\mathrm{d}x$

answer: $\frac{1}{4}\ln|x-1| + \frac{27}{4}\ln|x+3| + \frac{1}{2}x^2 - 2x$

(d) $\displaystyle\int \frac{x^3}{x^2+3x-4}\mathrm{d}x$

answer: $\frac{1}{2}x^2 - 3x + \frac{64}{5}\ln|x+4| + \frac{1}{5}\ln|x-1| + C$

(e) $\displaystyle\int \frac{x^3}{2x^2+3x-5}\mathrm{d}x$

answer: $\frac{125}{56}\ln\left(x+\frac{5}{2}\right) + \frac{1}{7}\ln(x-1) + \frac{1}{4}x^2 - \frac{3}{4}x + C$

(f) $\displaystyle\int \frac{x^2+1}{(x-3)(x-2)^2}\mathrm{d}x$

answer: $10\ln|x-3| - 9\ln|x-2| + \frac{5}{x-2} + C$

(g) $\displaystyle\int \frac{x^4}{(x+1)^2(x+2)}\mathrm{d}x$

answer: $\frac{x^2}{2} - 4x - 5\ln|x+1| + 16\ln|x+2| - \frac{1}{x+1} + C$

(h) $\displaystyle\int \frac{15x^2-4x-81}{(x-3)(x+4)(x-1)}\mathrm{d}x$

answer: $5\ln|-x-4| + 3\ln|x-3| + 7\ln|x-1|$

(i) $\displaystyle\int \frac{x^4+10x^3+18x^2+2x-13}{x^4+4x^3+3x^2-4x-4}\mathrm{d}x$

Check first that $(x-1)(x+2)^2(x+1) = x^4+4x^3+3x^2-4x-4$.

answer: $3(x+2)^{-1} + 2\ln|x+2| + \ln|x-1| + 3\ln|x+1| + x + C$

(j) $\displaystyle\int \frac{x^4}{(x^2+2)(x+2)}\mathrm{d}x$

answer: $\frac{x^2}{2} - 2x + \frac{8}{3}\ln|x+2| - \frac{1}{3}\ln\left(x^2+2\right) + \frac{2\sqrt{2}}{3}\arctan\left(\frac{\sqrt{2}}{2}x\right) + C$

(k) $\displaystyle\int \frac{x^5}{x^3-1}\mathrm{d}x$

answer: $\frac{1}{3}\ln\left|x^2+x+1\right| + \frac{1}{3}\ln|x-1| + \frac{1}{3}x^3 + C$
$= \frac{1}{3}\ln\left|x^3-1\right| + \frac{1}{3}x^3 + C$

(l) $\displaystyle\int \frac{x^4}{(x^2+2)(x+1)^2}\mathrm{d}x$

answer: $x - \frac{1}{3}(x+1)^{-1} - \frac{10}{9}\ln|x+1| - \frac{4}{9}\ln\left|x^2+2\right| - \frac{2}{9}\sqrt{2}\arctan\left(\frac{\sqrt{2}}{2}x\right) + C$

(m) $\displaystyle\int \frac{3x^2+2x-1}{(x-1)(x^2+1)}\mathrm{d}x$

answer: $2\ln|x-1| + \frac{1}{2}\ln\left(x^2+1\right) + 3\arctan x + C$

(n) $\displaystyle\int \frac{x^2-1}{x(x^2+1)^2}\mathrm{d}x$

answer: $-\left(x^2+1\right)^{-1} + \frac{1}{2}\ln\left(x^2+1\right) - \ln|x|$

Solution. 13.1 To integrate a rational function, we need to decompose it into partial fractions.

Since the numerator of the function is of degree greater than or equal to the denominator, we start the partial fraction decomposition by polynomial division.

	Remainder				
		$-2x^3$	$-3x^2$	$-4x$	-2
Divisor(s)	**Quotient(s)**				
$x^4+2x^3+3x^2+4x+2$	1				
	Dividend				
-	x^4				
	x^4	$+2x^3$	$+3x^2$	$+4x$	$+2$
		$-2x^3$	$-3x^2$	$-4x$	-2

Our next step is to factor the denominator:

$$x^4+2x^3+3x^2+4x+2 = (x+1)^2\left(x^2+2\right).$$

Next, we combine the two steps:

$$\begin{aligned}
\frac{x^4}{x^4+2x^3+3x^2+4x+2} &= 1+\frac{-2x^3-3x^2-4x-2}{x^4+2x^3+3x^2+4x+2} \\
\frac{-2x^3-3x^2-4x-2}{x^4+2x^3+3x^2+4x+2} &= \frac{-2x^3-3x^2-4x-2}{(x+1)^2(x^2+2)} \\
&= \frac{A_1}{(x+1)}+\frac{A_2}{(x+1)^2}+\frac{A_3+A_4x}{(x^2+2)}.
\end{aligned}$$

We seek to find A_i's that turn the above expression into an identity. Just as in the solution of Problem 12.k, we will use the method of coefficient comparison. We note that the solutions of Problems 13.m and 12.k provide a shortcut method.

After clearing denominators, we get the following equality.

$$\begin{aligned}
-2x^3-3x^2-4x-2 &= A_1(x+1)(x^2+2)+A_2(x^2+2) \\
&\quad +(A_3+A_4x)(x+1)^2 \\
0 &= (A_4+A_1+2)x^3 \\
&\quad +(2A_4+A_3+A_2+A_1+3)x^2 \\
&\quad +(A_4+2A_3+2A_1+4)x \\
&\quad +(A_3+2A_2+2A_1+2) \quad .
\end{aligned}$$

In order to turn the above into an identity we need to select A_i's such that the coefficients of all powers of x become zero. In other words, we need to solve the following system.

$$\begin{array}{lllll}
A_1 & & & +A_4 & =-2 \\
A_1 & +A_2 & +A_3 & +2A_4 & =-3 \\
2A_1 & & +2A_3 & +A_4 & =-4 \\
2A_1 & +2A_2 & +A_3 & & =-2 \quad .
\end{array}$$

This is a system of linear equations. There exists a standard method for solving such systems called Gaussian Elimination (this method is also known as the row-echelon form reduction method). This method is very well suited for computer implementation. We illustrate it on this particular example; for a description of the method in full generality we direct the reader to a standard course in Linear algebra.

System status	Action
$\begin{array}{lllll} A_1 & & & +A_4 & =-2 \\ A_1 & +A_2 & +A_3 & +2A_4 & =-3 \\ 2A_1 & & +2A_3 & +A_4 & =-4 \\ 2A_1 & +2A_2 & +A_3 & & =-2 \end{array}$	Sel. pivot column 2. Eliminate non-pivot entries.
$\begin{array}{lllll} A_1 & & & +A_4 & =-2 \\ & A_2 & +A_3 & +A_4 & =-1 \\ & & 2A_3 & -A_4 & =0 \\ & 2A_2 & +A_3 & -2A_4 & =2 \end{array}$	Sel. pivot column 3. Eliminate non-pivot entries.
$\begin{array}{lllll} A_1 & & & +A_4 & =-2 \\ & A_2 & +A_3 & +A_4 & =-1 \\ & & 2A_3 & -A_4 & =0 \\ & & -A_3 & -4A_4 & =4 \end{array}$	Sel. pivot column 4. Eliminate non-pivot entries.
$\begin{array}{lllll} A_1 & & & +A_4 & =-2 \\ & A_2 & & +\frac{3}{2}A_4 & =-1 \\ & & A_3 & -\frac{A_4}{2} & =0 \\ & & & -\frac{9}{2}A_4 & =4 \end{array}$	Sel. pivot column 5. Eliminate non-pivot entries.
$\begin{array}{lllll} A_1 & & & & =-\frac{10}{9} \\ & A_2 & & & =\frac{1}{3} \\ & & A_3 & & =-\frac{4}{9} \\ & & & A_4 & =-\frac{8}{9} \end{array}$	Final result.

Therefore, the final partial fraction decomposition is the following.

$$
\begin{aligned}
\frac{x^4}{x^4+2x^3+3x^2+4x+2} &= 1+\frac{-2x^3-3x^2-4x-2}{x^4+2x^3+3x^2+4x+2}\\
&= 1+\frac{-\frac{10}{9}}{(x+1)}+\frac{\frac{1}{3}}{(x+1)^2}+\frac{-\frac{8}{9}x-\frac{4}{9}}{(x^2+2)}
\end{aligned}
$$

Therefore we can integrate as follows.

$$
\begin{aligned}
\int \frac{x^4}{(x^2+2)(x+1)^2}\mathrm{d}x &= \int\left(1+\frac{-\frac{10}{9}}{(x+1)}+\frac{\frac{1}{3}}{(x+1)^2}+\frac{-\frac{8}{9}x-\frac{4}{9}}{(x^2+2)}\right)\mathrm{d}x\\
&= \int \mathrm{d}x-\frac{10}{9}\int\frac{1}{(x+1)}\mathrm{d}x+\frac{1}{3}\int\frac{1}{(x+1)^2}\mathrm{d}x\\
&\quad -\frac{8}{9}\int\frac{x}{x^2+2}\mathrm{d}x-\frac{4}{9}\int\frac{1}{x^2+2}\mathrm{d}x\\
&= x-\frac{1}{3}(x+1)^{-1}-\frac{10}{9}\log\,(x+1)\\
&\quad -\frac{4}{9}\log\,\left(x^2+2\right)-\frac{2}{9}\sqrt{2}\arctan\left(\frac{\sqrt{2}}{2}x\right)+C
\end{aligned}
$$

Solution. 13.k This problem can be solved directly with a substitution shortcut, or by the standard method.

Variant I (standard method).

$$
\begin{aligned}
\int\frac{x^5}{x^3-1}\mathrm{d}x &= \int\left(x^2+\frac{x^2}{x^3-1}\right)\mathrm{d}x && \text{Polyn. long div.}\\
&= \frac{x^3}{3}+\int\frac{x^2}{(x-1)(x^2+x+1)}\mathrm{d}x && \text{part. frac.}\\
&= \frac{x^3}{3}+\int\left(\frac{\frac{1}{3}}{x-1}+\frac{\frac{2}{3}x+\frac{1}{3}}{x^2+x+1}\right)\mathrm{d}x && \text{complete square}\\
&= \frac{x^3}{3}+\frac{1}{3}\ln|x-1|+\frac{2}{3}\int\frac{x+\frac{1}{2}}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\mathrm{d}x && \text{Set}\ \begin{array}{rcl} u &=& \left(x+\frac{1}{2}\right)^2+\frac{3}{4}\\ \frac{1}{2}\mathrm{d}u &=& \left(x+\frac{1}{2}\right)\mathrm{d}x\end{array}\\
&= \frac{x^3}{3}+\frac{1}{3}\ln|x-1|+\frac{1}{3}\int\frac{\mathrm{d}u}{u}\\
&= \frac{x^3}{3}+\frac{1}{3}\ln|x-1|+\frac{1}{3}\ln|u|+C\\
&= \frac{x^3}{3}+\frac{1}{3}\ln|x-1|+\frac{1}{3}\ln|x^2+x+1|+C
\end{aligned}
$$

Variant II (shortcut method).

$$
\begin{aligned}
\int\frac{x^5}{x^3-1}\mathrm{d}x &= \int\frac{x^5-x^2+x^2}{x^3-1}\mathrm{d}x\\
&= \int\frac{x^2(x^3-1)+x^2}{x^3-1}\mathrm{d}x\\
&= \int x^2\mathrm{d}x+\int\frac{x^2}{x^3-1}\mathrm{d}x\\
&= \frac{x^3}{3}+\int\frac{\mathrm{d}\left(\frac{x^3}{3}\right)}{x^3-1}\\
&= \frac{x^3}{3}+\frac{1}{3}\int\frac{\mathrm{d}\left(x^3-1\right)}{x^3-1} && \text{Set}\ u=x^3-1\\
&= \frac{x^3}{3}+\frac{1}{3}\int\frac{\mathrm{d}u}{u}\\
&= \frac{x^3}{3}+\frac{1}{3}\ln|u|+C\\
&= \frac{x^3}{3}+\frac{1}{3}\ln\left|x^3-1\right|+C \quad .
\end{aligned}
$$

The answers obtained in the two solution variants are of course equal since

$$
\ln|x-1|+\ln|x^2+x+1| = \ln\left|(x-1)\left(x^2+x+1\right)\right| = \ln\left|x^3-1\right| \quad .
$$

Solution. 13.m. This is a concise solution written in a form suitable for exam taking. To make this solution as short as possible we have omitted many details. On an exam, the student would be expected to carry out those omitted computations on the side. We set up the partial fraction decomposition as follows.

$$\frac{3x^2+2x-1}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1} \quad .$$

Therefore $3x^2+2x-1 = A(x^2+1)+(Bx+C)(x-1)$.

- We set $x=1$ to get $4=2A$, so $A=2$.
- We set $x=0$ to get $-1=A-C$, so $C=3$.
- Finally, set $x=2$ to get $15=5A+2B+C$, so $B=1$.

We can now compute the integral as follows.

$$\int\left(\frac{2}{x-1}+\frac{x+3}{x^2+1}\right)\mathrm{d}x = 2\ln(|x-1|)+\frac{1}{2}\ln(x^2+1)+3\arctan x+K \quad .$$

14. Integrate

$$\int \frac{x^6-x^5+\frac{9}{2}x^4-4x^3+\frac{13}{2}x^2-\frac{7}{2}x+\frac{11}{4}}{x^5-x^4+3x^3-3x^2+\frac{9}{4}x-\frac{9}{4}}\mathrm{d}x \quad .$$

Solution. 14.

Step 1. The first step of our algorithm is to reduce the fraction so that numerator has smaller degree than the denominator. This is done using polynomial long division as follows.

Variable name(s): x1 division steps total.

	Remainder						
			$\frac{3}{2}x^4$	$-x^3$	$+\frac{17}{4}x^2$	$-\frac{5}{4}x$	$+\frac{11}{4}$
Divisor(s)	**Quotient(s)**						
$x^5-x^4+3x^3-3x^2+\frac{9}{4}x-\frac{9}{4}$	x						
	Dividend						
-	x^6	$-x^5$	$+\frac{9}{2}x^4$	$-4x^3$	$+\frac{13}{2}x^2$	$-\frac{7}{2}x$	$+\frac{11}{4}$
	x^6	$-x^5$	$+3x^4$	$-3x^3$	$+\frac{9}{4}x^2$	$-\frac{9}{4}x$	
			$\frac{3}{2}x^4$	$-x^3$	$+\frac{17}{4}x^2$	$-\frac{5}{4}x$	$+\frac{11}{4}$

In other words,

$$\begin{aligned}x^6-x^5+\tfrac{9}{2}x^4-4x^3+\tfrac{13}{2}x^2-\tfrac{7}{2}x+\tfrac{11}{4}=&(x^5-x^4+3x^3-3x^2+\tfrac{9}{4}x-\tfrac{9}{4})x\\&+\tfrac{3}{2}x^4-x^3+\tfrac{17}{4}x^2-\tfrac{5}{4}x+\tfrac{11}{4} \quad ,\end{aligned}$$

and therefore

$$\begin{aligned}\frac{x^6-x^5+\frac{9}{2}x^4-4x^3+\frac{13}{2}x^2-\frac{7}{2}x+\frac{11}{4}}{x^5-x^4+3x^3-3x^2+\frac{9}{4}x-\frac{9}{4}}=&x+\frac{\frac{3}{2}x^4-x^3+\frac{17}{4}x^2-\frac{5}{4}x+\frac{11}{4}}{x^5-x^4+3x^3-3x^2+\frac{9}{4}x-\frac{9}{4}}\\=&x+\frac{6x^4-4x^3+17x^2-5x+11}{4x^5-4x^4+12x^3-12x^2+9x-9}.\end{aligned}$$

Set

$$N(x)=6x^4-4x^3+17x^2-5x+11$$

and

$$D(x)=4x^5-4x^4+12x^3-12x^2+9x-9 \quad .$$

Step 2. (Split into partial fractions). Factor the denominator $D(x)=4x^5-4x^4+12x^3-12x^2+9x-9$.

We recall from elementary algebra that there is a trick to find all rational roots of $D(x)$ on condition $D(x)$ has integer coefficients. It is well known that when $\frac{p}{q}$ is a rational number, then $\pm\frac{p}{q}$ may be a root of the integer coefficient polynomial $D(x)$ only if p is a divisor of the constant term of $D(x)$, and q is a divisor of the leading coefficient of $D(x)$. Since in our case the leading coefficient is 4 and the constant term is -9, the only possible rational roots of $D(x)$ are $\pm1,\pm3,\pm9,\pm\frac{1}{2},\pm\frac{3}{2},\pm\frac{9}{2},\pm\frac{1}{4},\pm\frac{3}{4},\pm\frac{9}{4}$. A rational number r is a root of $D(x)$ if and only if substituting $x = r$ yields 0. Direct check shows that, for example, $D(-1) = -50$. However, $D(1) = 0$ and therefore using polynomial division we get that $D(x) = (x-1)(4x^4 + 12x^2 + 9)$. We recognize that the second multiplicand is an exact square and therefore $D(x) = (x-1)(2x^2+3)^2$.

So far we got

$$\frac{N(x)}{D(x)} = \frac{6x^4 - 4x^3 + 17x^2 - 5x + 11}{(x-1)(2x^2+3)^2} \quad .$$

In order to split $\frac{N(x)}{D(x)}$ into partial fractions, we need to find numbers A, B, C, D, E such that

$$\frac{6x^4 - 4x^3 + 17x^2 - 5x + 11}{(x-1)(2x^2+3)^2} = \frac{A}{(x-1)} + \frac{Bx + C}{(2x^2+3)} + \frac{Dx + E}{(2x^2+3)^2} \quad .$$

After clearing denominators, we see that this amounts to finding A, B, C, D, E such that

$$6x^4 - 4x^3 + 17x^2 - 5x + 11 = A(2x^2+3)^2 + (Bx + C)(2x^2+3)(x-1) + (Dx + E)(x-1) \quad .$$

Plugging in $x = 1$ we see that $25 = 25A$ and so $A = 1$. We may plug back $A = 1$ and regroup to get

$$2x^4 - 4x^3 + 5x^2 - 5x + 2 = (Bx + C)(2x^2+3)(x-1) + (Dx + E)(x-1) \quad .$$

Dividing both sides by $(x-1)$ we get

$$2x^3 - 2x^2 + 3x - 2 = (Bx + C)(2x^2+3) + Dx + E \quad .$$

Regrouping we get

$$x^3(2 - 2B) + x^2(-2 - 2C) + x(3 - 3B - D) + (-2 - 3C - E) = 0 \quad .$$

As x is an indeterminate, the above expression may vanish only if all coefficients in the preceding expression vanish. Therefore we get the system

$$\left| \begin{array}{rcl} 2 - 2B &=& 0 \\ -2 - C &=& 0 \\ 3 - 3B - D &=& 0 \\ -2 - 3C - E &=& 0 \quad . \end{array} \right.$$

We may solve the above linear system using the standard algorithm for solving linear systems (the algorithm is called row reduction and is also known as Gaussian elimination). The latter algorithm is studied in any standard the Linear algebra course. Alternatively, we see from the first equations $B = 1$, $C = -1$, and substituting in the remaining equations we see $D = 0$, $E = 1$. Finally, we check that

$$\frac{x^6 - x^5 + \frac{9}{2}x^4 - 4x^3 + \frac{13}{2}x^2 - \frac{7}{2}x + \frac{11}{4}}{x^5 - x^4 + 3x^3 - 3x^2 + \frac{9}{4}x - \frac{9}{4}} = x + \frac{1}{(x-1)} + \frac{x-1}{(2x^2+3)} + \frac{1}{(2x^2+3)^2} \quad .$$

Step 3. (Find the integral of each partial fraction).

$$\begin{array}{rcl} \displaystyle\int x \mathrm{d}x &=& \displaystyle\frac{x^2}{2} + C \\ \displaystyle\int \frac{1}{x-1} \mathrm{d}x &=& \ln|x-1| + C \\ \displaystyle\int \frac{x-1}{2x^2+3} \mathrm{d}x &=& \displaystyle\int \frac{x}{2x^2+3} \mathrm{d}x - \frac{1}{3} \int \frac{1}{\frac{2}{3}x^2 + 1} \mathrm{d}x \\ &=& \displaystyle\int \frac{\mathrm{d}\left(\frac{x^2}{2}\right)}{2x^2+3} \mathrm{d}x - \frac{1}{3} \int \frac{1}{\left(\sqrt{\frac{2}{3}}x\right)^2 + 1} \mathrm{d}x \\ &=& \displaystyle\frac{1}{4} \int \frac{\mathrm{d}(2x^2+3)}{2x^2+3} \mathrm{d}x - \frac{1}{3} \int \frac{\frac{\mathrm{d}\left(\sqrt{\frac{2}{3}}x\right)}{\sqrt{\frac{2}{3}}}}{\left(\sqrt{\frac{2}{3}}x\right)^2 + 1} \\ &=& \displaystyle\frac{1}{4} \ln(2x^2+3) - \frac{\sqrt{6}}{6} \arctan\left(\sqrt{\frac{2}{3}}x\right) + C \quad . \end{array}$$

The last integral is

$$
\begin{aligned}
\int \frac{1}{(2x^2+3)^2}\mathrm{d}x &= \frac{1}{9}\int \frac{\frac{\mathrm{d}\left(\sqrt{\frac{2}{3}}x\right)}{\sqrt{\frac{2}{3}}}}{\left(\left(\sqrt{\frac{2}{3}}x\right)^2+1\right)^2} \\
&= \frac{\sqrt{6}}{18}\int \frac{\mathrm{d}\left(\sqrt{\frac{2}{3}}x\right)}{\left(\left(\sqrt{\frac{2}{3}}x\right)^2+1\right)^2} \qquad \left|\ \text{Set } y=\sqrt{\tfrac{2}{3}}x\right. \\
&= \frac{\sqrt{6}}{18}\int \frac{\mathrm{d}y}{(y^2+1)^2} \quad .
\end{aligned}
$$

The general form of the integral $\int \frac{\mathrm{d}y}{(y^2+1)^2}$ is solved in the theoretical discussion by integration by parts. As a review of the theory, we redo the computations directly.

$$
\begin{aligned}
C+\arctan y &= \int \frac{\mathrm{d}y}{y^2+1} \\
&= \frac{y}{y^2+1}+\int\frac{2y^2 dy}{(y^2+1)^2} = \frac{y}{y^2+1}+\int\frac{2(y^2+1-1)\mathrm{d}y}{(y^2+1)^2} \\
&= \frac{y}{y^2+1}+2\int\frac{\mathrm{d}y}{(y^2+1)}-2\int\frac{\mathrm{d}y}{(y^2+1)^2} \quad .
\end{aligned}
$$

Transferring summands we get

$$
\int\frac{\mathrm{d}y}{(y^2+1)^2}=\frac{1}{2}\left(\frac{y}{y^2+1}+\arctan y\right)+C \quad .
$$

We recall that $y=\sqrt{\frac{2}{3}}x$ and therefore

$$
\int\frac{\mathrm{d}x}{(2x^2+3)^2}=\frac{\sqrt{6}}{36}\left(\frac{\sqrt{\frac{2}{3}}x}{\left(\sqrt{\frac{2}{3}}x\right)^2+1}+\arctan\left(\sqrt{\frac{2}{3}}x\right)\right)+C.
$$

To get the final answer we collect all terms:

$$
\frac{1}{6}\left(\frac{x}{2x^2+3}\right)-\frac{5\sqrt{6}}{36}\arctan\left(\sqrt{\frac{2}{3}}x\right)+\frac{1}{4}\ln(2x^2+3)+\ln|x-1|+\frac{x^2}{2}+C.
$$

15. Integrate.

(a) $\displaystyle\int\frac{1}{3+\cos x}\mathrm{d}x.$

answer: $\frac{1}{\sqrt{2}}\arctan\left(\frac{1}{\sqrt{2}}\tan\left(\frac{x}{2}\right)\right)+C$

(b) $\displaystyle\int\frac{1}{4+\cos x}\mathrm{d}x.$

answer: $\frac{2}{15}\sqrt{15}\arctan\left(\frac{\sqrt{15}}{5}\tan\left(\frac{x}{2}\right)\right)+C$

(c) $\displaystyle\int\frac{1}{3+\sin x}\mathrm{d}x.$

answer: $\frac{1}{\sqrt{2}}\arctan\left(\frac{3\tan\left(\frac{x}{2}\right)+1}{2\sqrt{2}}\right)+C$

(d) $\displaystyle\int\frac{1}{2+\tan x}\mathrm{d}x.$ (Hint: this integral can be done simply with the substitution $x=\arctan t$.)

answer: $\frac{1}{5}\ln(\sin x+2\cos x)+\frac{2}{5}x+C$

(e) $\displaystyle\int\frac{\mathrm{d}x}{2\sin x-\cos x+5}.$

answer: $\frac{\sqrt{5}}{5}\arctan\left(\frac{\sqrt{5}}{3}\left(\tan\left(\frac{\theta}{2}\right)+\frac{1}{3}\right)\right)+C$

Solution. 15.a We use the standard rationalizing substitution $x=2\arctan t$, $t=\tan\left(\frac{x}{2}\right)$. We recall that from the double angle formulas it follows that

$$
\cos(2\arctan t)=\frac{\cos^2(\arctan t)-\sin^2(2\arctan t)}{\cos^2(\arctan t)+\sin^2(\arctan t)}=\frac{1-t^2}{1+t^2} \quad .
$$

Therefore we can solve the integral as follows.

$$\begin{array}{rcll}\displaystyle\int \frac{1}{3+\cos x}\mathrm{d}x &=& \displaystyle\int \frac{1}{3+\cos(2\arctan t)}\mathrm{d}\left(2\arctan t\right) & \Big| \text{ Set } x = 2\arctan t\\ &=& \displaystyle\int \frac{1}{\left(3+\frac{1-t^2}{1+t^2}\right)}\frac{2}{(1+t^2)}\mathrm{d}t & \\ &=& \displaystyle\int \frac{2}{4+2t^2}\mathrm{d}t & \\ &=& \displaystyle\int \frac{1}{2+t^2}\mathrm{d}t & \\ &=& \displaystyle\frac{\sqrt{2}}{2}\arctan\left(\frac{\sqrt{2}}{2}t\right)+C & \\ &=& \displaystyle\frac{\sqrt{2}}{2}\arctan\left(\frac{\sqrt{2}}{2}\tan\left(\frac{x}{2}\right)\right)+C \quad . & \end{array}$$

Solution. 15.d This integral is of none of the forms that can be integrated quickly. Therefore we can solve it using the standard rationalizing substitution $x = 2\arctan t$, $t = \tan\left(\frac{x}{2}\right)$. This results in somewhat long computations and we invite the reader to try it.

However, as proposed in the hint, the substitution $x = \arctan t$ works much faster:

$$\begin{array}{rcll}\displaystyle\int \frac{1}{2+\tan x}\mathrm{d}x &=& \displaystyle\int \frac{1}{2+\tan(\arctan t)}\mathrm{d}\left(\arctan t\right) & \Big| \text{ Substitute } x = \arctan t\\ &=& \displaystyle\int \frac{1}{(2+t)}\frac{1}{(1+t^2)}\mathrm{d}t & \Big| \text{ part. fractions}\\ &=& \displaystyle\int \left(\frac{\frac{1}{5}}{(t+2)}+\frac{-\frac{t}{5}+\frac{2}{5}}{(t^2+1)}\right)\mathrm{d}t & \\ &=& \displaystyle\frac{1}{5}\ln|t+2|-\frac{1}{10}\ln(t^2+1)+\frac{2}{5}\arctan t + C & \Big| \; t=\tan x\\ &=& \displaystyle\frac{1}{5}\ln|\tan x+2|-\frac{1}{10}\ln(\tan^2 x+1)+\frac{2}{5}x + C & \\ &=& \displaystyle\frac{1}{5}\ln|\tan x+2|+\frac{1}{5}\ln|\cos x|+\frac{2}{5}x + C & \\ &=& \displaystyle\frac{1}{5}\ln|(\tan x+2)\cos x|+\frac{2}{5}x + C & \\ &=& \displaystyle\frac{1}{5}\ln|\sin x+2\cos x|+\frac{2}{5}x + C. & \end{array}$$

Solution. 15.e.

Set $x = 2\arctan t$. As studied, this substitution implies $\cos x = \frac{1-t^2}{1+t^2}$, $\sin x = \frac{2t}{1+t^2}$, $\mathrm{d}x = \frac{2}{1+t^2}\mathrm{d}t$. Therefore

$$
\begin{aligned}
\int \frac{\mathrm{d}x}{2\sin x-\cos x+5} &= \int \frac{2\mathrm{d}t}{(1+t^2)\left(2\frac{2t}{t^2+1}-\frac{(-t^2+1)}{t^2+1}+5\right)} && \left|\ \text{Set } x=2\arctan t\right.\\
&= \int \frac{\mathrm{d}t}{3t^2+2t+2}\\
&= \int \frac{\mathrm{d}t}{3\left(t^2+\frac{2}{3}t+\frac{1}{9}-\frac{1}{9}+\frac{2}{3}\right)}\\
&= \int \frac{\mathrm{d}t}{3\left(\left(t+\frac{1}{3}\right)^2+\frac{5}{9}\right)}\\
&= \int \frac{\mathrm{d}t}{\frac{5}{3}\left(\left(\frac{3}{\sqrt{5}}\left(t+\frac{1}{3}\right)\right)^2+1\right)} && \left|\ \begin{array}{rl} & \text{Set}\\ w= & \frac{3}{\sqrt{5}}\left(t+\frac{1}{3}\right)\\ = & \frac{\sqrt{5}}{5}(3t+1)\\ \mathrm{d}w= & \frac{3}{\sqrt{5}}\mathrm{d}t\\ \mathrm{d}t= & \frac{\sqrt{5}}{3}\mathrm{d}w \end{array}\right.\\
&= \int \frac{\frac{\sqrt{5}}{3}dw}{\frac{5}{3}(w^2+1)}\\
&= \frac{\sqrt{5}}{5}\arctan w + C\\
&= \frac{\sqrt{5}}{5}\arctan\left(\frac{\sqrt{5}}{5}(3t+1)\right)+C\\
&= \frac{\sqrt{5}}{5}\arctan\left(\frac{\sqrt{5}}{5}\left(3\tan\left(\frac{x}{2}\right)+1\right)\right)+C \quad .
\end{aligned}
$$

16. Integrate. The answer key has not been proofread, use with caution.

(a) $\displaystyle\int \sin(3x)\cos(2x)\mathrm{d}x.$

answer: $-\frac{1}{10}\cos(5x)-\frac{1}{2}\cos x+C$

(b) $\displaystyle\int \sin x\cos(5x)\mathrm{d}x.$

answer: $-\frac{1}{12}\cos(6x)+\frac{1}{8}\cos(4x)+C$

(c) $\displaystyle\int \cos(3x)\sin(2x)\mathrm{d}x.$

answer: $-\frac{1}{10}\cos(5x)+\frac{1}{2}\cos x+C$

(d) $\displaystyle\int \sin(5x)\sin(3x)\mathrm{d}x.$

answer: $\frac{1}{4}\sin(2x)-\frac{1}{16}\sin(8x)+C$

(e) $\displaystyle\int \cos(x)\cos(3x)\mathrm{d}x.$

answer: $\frac{1}{8}\sin(4x)+\frac{1}{4}\sin(2x)+C$

17. Integrate.

(a) $\displaystyle\int \sin^2 x\cos x\mathrm{d}x.$

answer: $\frac{1}{3}\sin^3 x+C$

(b) $\displaystyle\int \sin^2 x\mathrm{d}x.$

answer: $\frac{x}{2}-\frac{1}{4}\sin(2x)+C$

(c) $\displaystyle\int \cos^3 x\mathrm{d}x.$

answer: $\sin x-\frac{1}{3}\sin^3 x+C$

(d) $\displaystyle\int \sin^3 x\cos^4 x\mathrm{d}x.$

answer: $\frac{1}{7}\cos^7 x-\frac{1}{5}\cos^5 x+C$

18. Integrate.

(a) $\displaystyle\int \sec x\mathrm{d}x.$

answer: $\ln|\sec x+\tan x| = \ln\left|\frac{1+\tan\left(\frac{x}{2}\right)}{1-\tan\left(\frac{x}{2}\right)}\right|+C$

(b) $\displaystyle\int \sec^3 x\mathrm{d}x.$

answer: $\frac{1}{2}(\sec x\tan x+\ln|\sec x+\tan x|)+C$

(c) $\displaystyle\int \tan^3 x \mathrm{d}x$.

answer: $\frac{1}{2}\tan^2 x - \ln|\sec x| + C$

(d) $\displaystyle\int \sec^2 x \tan^2 x \mathrm{d}x$.

answer: $\frac{\tan^3 x}{3} + C$

Solution. 18.a. **Variant I.**

This variant uses the standard method for solving trigonometric integrals with the substitution $x = \arctan(2t)$.

$$
\begin{array}{rcll}
\displaystyle\int \sec x \mathrm{d}x & = & \displaystyle\int \sec(2\arctan t)\mathrm{d}(2\arctan t) & \text{Set } x = 2\arctan t \\
& = & \displaystyle\int \frac{1}{\cos(2\arctan t)} \frac{2}{1+t^2}\mathrm{d}t & \text{Use } \cos(2z) = \frac{1-\tan^2 z}{1+\tan^2 z} \\
& = & \displaystyle\int \frac{1}{\frac{1-t^2}{1+t^2}} \frac{2}{1+t^2}\mathrm{d}t & \\
& = & \displaystyle\int \frac{2}{1-t^2}\mathrm{d}t & \text{part. fractions} \\
& = & \displaystyle\int \left(\frac{1}{1-t} + \frac{1}{1+t}\right)\mathrm{d}t & \\
& = & -\ln|1-t| + \ln|1+t| + C & \\
& = & \ln\left|\dfrac{1+t}{1-t}\right| & \text{Subst. } t = \tan\left(\frac{x}{2}\right) \\
& = & \ln\left|\dfrac{1+\tan\left(\frac{x}{2}\right)}{1-\tan\left(\frac{x}{2}\right)}\right| + C & \text{Last step: see below} \\
& = & \ln|\sec x + \tan x| + C \quad . &
\end{array}
$$

The expression $\ln\left|\frac{1+\tan\left(\frac{x}{2}\right)}{1-\tan\left(\frac{x}{2}\right)}\right|$ presents a perfectly good answer, which would certainly would qualify for a correct test answer. However, as shown above, it can be rewritten into the shorter form $\ln|\sec x + \tan x|$. Below we quickly prove that $\dfrac{1+\tan\left(\frac{x}{2}\right)}{1-\tan\left(\frac{x}{2}\right)}$ equals $\sec x + \tan x$.

$$
\begin{array}{rll}
\sec x + \tan x = & \dfrac{1+\sin x}{\cos x} & \begin{array}{l} \text{Use:} \\ \sin x = 2\sin\left(\frac{x}{2}\right) \\ \cos x = \cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right) \\ 1 = \cos^2\left(\frac{x}{2}\right) + \sin^2\left(\frac{x}{2}\right) \end{array} \\
= & \dfrac{\cos^2\left(\frac{x}{2}\right) + \sin^2\left(\frac{x}{2}\right) + 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}{\cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right)} & \\
= & \dfrac{\left(\sin\left(\frac{x}{2}\right) + \cos\left(\frac{x}{2}\right)\right)^2}{\left(\cos\left(\frac{x}{2}\right) - \sin\left(\frac{x}{2}\right)\right)\left(\cos\left(\frac{x}{2}\right) + \sin\left(\frac{x}{2}\right)\right)} & \\
= & \dfrac{\left(\sin\left(\frac{x}{2}\right) + \cos\left(\frac{x}{2}\right)\right)\frac{1}{\cos\left(\frac{x}{2}\right)}}{\left(\cos\left(\frac{x}{2}\right) - \sin\left(\frac{x}{2}\right)\right)\frac{1}{\cos\left(\frac{x}{2}\right)}} & \\
= & \dfrac{1+\tan\left(\frac{x}{2}\right)}{1-\tan\left(\frac{x}{2}\right)} &
\end{array}
$$

18.a. **Variant II.** This variant is based on the following observation. For an odd number $m > 0$, we studied a quick technique for integrating $\int \sin^n x \cos^m x \mathrm{d}x$: namely, use the transformation $\cos x \mathrm{d}x = \mathrm{d}(\sin x)$ and change variables $u = \sin x$. This trick relies heavily on the fact that m is odd (as we need to express the remaining even power of $\cos x$ via $\sin x$). However, the positivity of m is not essential: by multiplying top and bottom by $\cos x$ we can make this technique work also for odd negative values of m. We illustrate the technique in the solution below.

$$\begin{array}{rcll}
\displaystyle\int \sec x \mathrm{d}x &=& \displaystyle\int \frac{1}{\cos x}\mathrm{d}x & \\
&=& \displaystyle\int \frac{\cos x}{\cos^2 x}\mathrm{d}x & \\
&=& \displaystyle\int \frac{\mathrm{d}(\sin x)}{1-\sin^2 x} & \Big| \text{Set } u=\sin x \\
&=& \displaystyle\int \frac{\mathrm{d}u}{1-u^2} & \\
&=& \displaystyle\int \frac{\mathrm{d}u}{(1+u)(1-u)} & \Big| \text{part. fractions} \\
&=& \displaystyle\int \left(\frac{\frac{1}{2}}{1+u}+\frac{\frac{1}{2}}{1-u}\right)\mathrm{d}u & \\
&=& \displaystyle\frac{1}{2}\left(\ln|1+u|-\ln|1-u|\right)+C & \\
&=& \displaystyle\frac{1}{2}\ln\left|\frac{1+u}{1-u}\right|+C & \Big| \text{Subst. back } u=\sin x \\
&=& \displaystyle\frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right|+C & \Big| \text{Last step: see below} \\
&=& \ln|\sec x+\tan x|+C \quad . &
\end{array}$$

The expression $\frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right|+C$ gives a perfectly good answer (which may be the preferred answer depending on the textbook). Let us show however that $\frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right|$ equals $\ln|\sec x+\tan x|$, the answer given in the other variants.

$$\begin{array}{rcll}
\displaystyle\frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right| &=& \displaystyle\frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right| & \Big| \text{Mult. \& div by } 1+\sin x \\
&=& \displaystyle\frac{1}{2}\ln\left|\frac{(1+\sin x)^2}{(1-\sin x)(1+\sin x)}\right| & \\
&=& \displaystyle\frac{1}{2}\ln\left|\frac{(1+\sin x)^2}{\cos^2 x}\right| & \Big| \text{use } \frac{1}{2}\ln|a|=\ln|a|^{\frac{1}{2}} \\
&=& \displaystyle\ln\sqrt{\left|\frac{(1+\sin x)^2}{\cos^2 x}\right|} & \\
&=& \displaystyle\ln\left|\frac{1+\sin x}{\cos x}\right| & \\
&=& \ln|\sec x+\tan x| \quad . &
\end{array}$$

18.a. **Variant III.** This variant present a quick solution by multiplying and dividing our integrand by the multiplier $\sec x+\tan x$. Of course, the idea of using that multiplier comes from knowing the answer to the problem in advance (which can be obtained, for example, by using the preceding solution variants).

$$\begin{array}{rcll}
\displaystyle\int \sec x \mathrm{d}x &=& \displaystyle\int \sec x\frac{\sec x+\tan x}{\sec x+\tan x}\mathrm{d}x & \\
&=& \displaystyle\int \frac{\sec^2 x+\sec x\tan x}{\sec x+\tan x}\mathrm{d}x & \Big| \begin{array}{rcl}\mathrm{d}(\tan x)&=&\sec^2 x\mathrm{d}x\\ \mathrm{d}(\sec x)&=&\sec x\tan x\mathrm{d}x\end{array} \\
&=& \displaystyle\int \frac{\mathrm{d}(\sec x+\tan x)}{\sec x+\tan x} & \Big| \text{Set } u=\sec x+\tan x \\
&=& \displaystyle\int \frac{\mathrm{d}u}{u} & \\
&=& \ln|u|+C & \\
&=& \ln|\sec x+\tan x|+C \quad . &
\end{array}$$

Solution. 18.b This problem can be solved with the general method by setting $x=2\arctan t$. However, there are shorter ways to solve the integral, as we show below.

Variant I.

$$
\begin{aligned}
\int \sec^3 x \mathrm{d}x &= \int \frac{1}{\cos^3 x}\mathrm{d}x \\
&= \int \frac{\cos x}{\cos^4 x}\mathrm{d}x && \text{use } \mathrm{d}(\sin x) = \cos x \mathrm{d}x \\
&= \int \frac{1}{\cos^4 x}\mathrm{d}(\sin x) && \text{use } \cos^2 x = 1-\sin^2 x \\
&= \int \frac{1}{\left(1-\sin^2 x\right)^2}\mathrm{d}(\sin x) && \text{Set } \sin x = u \\
&= \int \frac{1}{\left(1-u^2\right)^2}\mathrm{d}u && \text{split in part. frac.} \\
&= \int \left(\frac{\frac14}{u+1} + \frac{\frac14}{(u+1)^2} + \frac{-\frac14}{u-1} + \frac{\frac14}{(u-1)^2} \right)\mathrm{d}u \\
&= \frac14 \left(\ln|u+1| - \ln|u-1| - \frac{1}{u+1} - \frac{1}{u-1} \right) + C \\
&= \frac14 \left(\ln\left|\frac{u+1}{u-1}\right| - \frac{2u}{u^2-1} \right) + C \\
&= \frac14 \left(\ln\left|\frac{\sin x+1}{\sin x-1}\right| + \frac{2\sin x}{\cos^2 x} \right) + C.
\end{aligned}
$$

Variant II. This variant uses the preceding problem to get to a solution as follows.

$$
\begin{aligned}
\int \sec^3 x \mathrm{d}x &= \int \sec x \mathrm{d}(\tan x) && \text{int. by parts} \\
&= \sec x \tan x - \int \tan x \mathrm{d}(\sec x) \\
&= \sec x \tan x - \int \sec x \tan^2 x \mathrm{d}x && \tan^2 x = \sec^2 x - 1 \\
&= \sec x \tan x - \int \sec x (\sec^2 x - 1)\mathrm{d}x \\
&= \sec x \tan x - \int \sec^3 x \mathrm{d}x + \int \sec x \mathrm{d}x && \text{Use Problem } 18.a \\
&= \sec x \tan x - \int \sec^3 x \mathrm{d}x + \ln|\sec x + \tan x| && \begin{array}{l} +\int \sec^3 x \mathrm{d}x \\ \text{to both sides} \end{array} \\
2\int \sec^3 x \mathrm{d}x &= (\sec x \tan x + \ln|\sec x + \tan x|) + C \\
\int \sec^3 x \mathrm{d}x &= \frac12 (\sec x \tan x + \ln|\sec x + \tan x|) + K \quad .
\end{aligned}
$$

19. Find a linear substitution (via completing the square) to transform the radical to a multiple of an expression of the form $\sqrt{u^2+1}$, $\sqrt{u^2-1}$ or $\sqrt{1-u^2}$.

 (a) $\sqrt{x^2+x+1}$.

 (b) $\sqrt{-2x^2+x+1}$.

Solution. 19.a

$$
\begin{aligned}
\sqrt{x^2+x+1} &= \sqrt{x^2 + 2\frac12 x + \frac14 - \frac14 + 1} \\
&= \sqrt{\left(x+\frac12\right)^2 + \frac34} \\
&= \sqrt{\frac34\left(\frac43\left(x+\frac12\right)^2 + 1\right)} \\
&= \frac{\sqrt3}{2}\sqrt{\left(\frac{2}{\sqrt3}\left(x+\frac12\right)\right)^2 + 1} \\
&= \frac{\sqrt3}{2}\sqrt{u^2+1},
\end{aligned}
$$

where $u = \frac{2}{\sqrt3}\left(x+\frac12\right) = \frac{2\sqrt3}{3}x + \frac{\sqrt3}{3}$.

Solution. 19.b

$$\begin{aligned}
\sqrt{-2x^2+x+1} &= \sqrt{-2\left(x^2-\frac{1}{2}x-\frac{1}{2}\right)} \\
&= \sqrt{-2\left(x^2-2\frac{1}{4}x+\frac{1}{16}-\frac{1}{16}-\frac{1}{2}\right)} \\
&= \sqrt{-2\left(\left(x-\frac{1}{16}\right)^2-\frac{9}{16}\right)} \\
&= \sqrt{\frac{9}{8}\left(-\frac{16}{9}\left(x-\frac{1}{16}\right)^2+1\right)} \\
&= \frac{3}{\sqrt{8}}\sqrt{-\left(\frac{4}{3}\left(x-\frac{1}{16}\right)\right)^2+1} \\
&= \frac{3}{\sqrt{8}}\sqrt{-u^2+1}
\end{aligned}$$

where $u=\frac{4}{3}\left(x-\frac{1}{16}\right)=\frac{4}{3}x-\frac{1}{12}$.

20. Compute the integral.

(a) $\displaystyle\int \frac{\sqrt{1+x^2}}{x^2}\mathrm{d}x$.

answer: $\ln\left(\sqrt{1+x^2}+x\right)-\frac{x}{\sqrt{1+x^2}}+C$

Solution. 20.a

Variant I. In this variant, we use the trigonometric substitution $x = \tan\theta$ and then solve the integral using a few algebraic tricks.

$$
\begin{array}{rcll}
\displaystyle\int \frac{\sqrt{1+x^2}}{x^2}\mathrm{d}x & = & \displaystyle\int \frac{\sqrt{1+\tan^2\theta}}{\tan^2\theta}\mathrm{d}(\tan\theta) & \left| \begin{array}{l} \text{Set} \\ x=\tan\theta \\ \theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right) \\ |\sec\theta| = \sec\theta \\ \text{for } \theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right) \end{array} \right. \\
& = & \displaystyle\int \frac{|\sec\theta|}{\tan^2\theta}\sec^2\theta\mathrm{d}\theta & \\
& = & \displaystyle\int \frac{\cos^2\theta}{\cos^3\theta\sin^2\theta}\mathrm{d}\theta & \\
& = & \displaystyle\int \frac{\cos\theta}{\cos^2\theta\sin^2\theta}\mathrm{d}\theta & \\
& = & \displaystyle\int \frac{\mathrm{d}(\sin\theta)}{(1-\sin^2\theta)\sin^2\theta} & \left| \begin{array}{l} \text{Set} \\ u=\sin\theta \\ \text{for } \theta\in\left(0,\frac{\pi}{2}\right) \\ u=\sqrt{1-\cos^2\theta} \\ u=\sqrt{1-\frac{1}{\sec^2\theta}} \\ u=\sqrt{1-\frac{1}{1+\tan^2\theta}} \\ u=\sqrt{\frac{\tan^2\theta}{1+\tan^2\theta}} \\ u=\frac{\tan\theta}{\sqrt{1+\tan^2\theta}} \\ u=\frac{x}{\sqrt{1+x^2}} \end{array} \right. \\
& = & \displaystyle\int \frac{\mathrm{d}u}{(1-u^2)u^2} & \\
& = & \displaystyle\int \frac{\mathrm{d}u}{(1-u)u^2(u+1)} & \left| \ \text{use part. frac.} \right. \\
& = & \displaystyle\int \left(\frac{\frac{1}{2}}{u+1}+\frac{-\frac{1}{2}}{u-1}+\frac{1}{u^2}\right)\mathrm{d}u & \\
& = & -\dfrac{1}{2}\ln|u-1| + \dfrac{1}{2}\ln(u+1) - u^{-1} + C & \\
& = & -\dfrac{1}{2}\ln(1-u) + \dfrac{1}{2}\ln(u+1) - u^{-1} + C & \left| \ u=\frac{x}{\sqrt{1+x^2}} < 1 \right. \\
& = & \dfrac{1}{2}\ln\left(\dfrac{1+u}{1-u}\right) - u^{-1} + C & \\
& = & \dfrac{1}{2}\ln\left(\dfrac{(1+u)}{(1-u)}\cdot\dfrac{(1+u)}{(1+u)}\right) - u^{-1} + C & \\
& = & \dfrac{1}{2}\ln\left(\dfrac{(1+u)^2}{1-u^2}\right) - u^{-1} + C & \left| \ \text{use } u=\frac{x}{\sqrt{1+x^2}} \right. \\
& = & \dfrac{1}{2}\ln\left(\dfrac{(1+u)^2}{\frac{1}{1+x^2}}\right) - \dfrac{\sqrt{1+x^2}}{x} + C & \\
& = & \dfrac{1}{2}\ln\left(\left((1+u)\sqrt{1+x^2}\right)^2\right) - \dfrac{\sqrt{1+x^2}}{x} + C & \\
& = & \ln\left(\sqrt{1+x^2}+x\right) - \dfrac{\sqrt{1+x^2}}{x} + C \quad . &
\end{array}
$$

Variant II. In this variant, we use directly the Euler substitution

$$\begin{array}{rcl} x &=& \cot(2\arctan t) \\ &=& \frac{1}{2}\left(\frac{1}{t}-t\right) \\ \mathrm{d}x &=& -\frac{1}{2}\left(\frac{1}{t^2}+1\right)\mathrm{d}t \\ \sqrt{1+x^2} &=& \frac{1}{2}\left(\frac{1}{t}+t\right) \\ t &=& \sqrt{x^2+1}-x \\ \frac{1}{t} &=& \sqrt{x^2+1}+x \quad . \end{array}$$

$$\begin{array}{rcl} \displaystyle\int \frac{\sqrt{1+x^2}}{x^2}\mathrm{d}x &=& \displaystyle\int \frac{\frac{1}{2}\left(\frac{1}{t}+t\right)}{\frac{1}{4}\left(\frac{1}{t}-t\right)^2}\left(-\frac{1}{2}\right)\left(\frac{1}{t^2}+1\right)\mathrm{d}t \\ &=& \displaystyle\int \frac{-t^4-2t^2-1}{(t-1)^2 t(t+1)^2}\mathrm{d}t \\ &=& \displaystyle\int \left(-\frac{1}{t}+\frac{1}{(t+1)^2}-\frac{1}{(t-1)^2}\right)\mathrm{d}t \\ &=& \displaystyle -\ln t - \frac{1}{t+1}+\frac{1}{t-1}+C \\ &=& \displaystyle \ln\left(\frac{1}{t}\right)+\frac{2}{t^2-1}+C \\ &=& \displaystyle \ln\left(\sqrt{1+x^2}+x\right)+\frac{1}{t\frac{1}{2}\left(t-\frac{1}{t}\right)}+C \\ &=& \displaystyle \ln\left(\sqrt{1+x^2}+x\right)-\frac{1}{t}\cdot\frac{1}{\frac{1}{2}\left(\frac{1}{t}-t\right)}+C \\ &=& \displaystyle \ln\left(\sqrt{1+x^2}+x\right)-\left(\sqrt{x^2+1}+x\right)\cdot\frac{1}{x}+C \\ &=& \displaystyle \ln\left(\sqrt{1+x^2}+x\right)-\frac{\sqrt{x^2+1}}{x}-1+C \quad . \end{array}$$

Part. frac

21. Compute the integral using a trigonometric substitution.

(a) $\displaystyle\int \frac{\sqrt{9-x^2}}{x^2}\mathrm{d}x$.

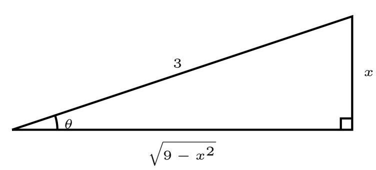
answer: $-\frac{\sqrt{9-x^2}}{x}-\arcsin\left(\frac{x}{3}\right)+C$

Solution. 21.a

$$\begin{array}{rcl} \displaystyle\int \frac{\sqrt{9-x^2}}{x^2}\mathrm{d}x &=& \displaystyle\int \frac{3\sqrt{\cos^2\theta}}{9\sin^2\theta}(3\cos\theta)\mathrm{d}\theta \\ &=& \displaystyle 9\int \frac{|\cos\theta|}{\sin^2\theta}\cos\theta\mathrm{d}\theta \\ &=& \displaystyle\int \cot^2\theta\mathrm{d}\theta \\ &=& \displaystyle\int (\csc^2\theta-1)\mathrm{d}\theta \\ &=& -\cot\theta-\theta+C \\ &=& \displaystyle -\frac{\sqrt{9-x^2}}{x}-\arcsin\left(\frac{x}{3}\right)+C, \end{array}$$

Set $x=3\sin\theta$
for $\theta\in\left[\frac{\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right]$
$\mathrm{d}x=3\cos\theta\mathrm{d}\theta$
For $\theta\in\left[\frac{\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right]$
we have $|\cos\theta|=\cos\theta$

where we expressed $\cot\theta$ via $\sin\theta$ by considering the following triangle.



22. Compute the integral.

(a) $\displaystyle\int \sqrt{x^2+1}\mathrm{d}x$

answer: $\frac{1}{2}x\sqrt{x^2+1}+\frac{1}{2}\ln\left(\sqrt{x^2+1}+x\right)+C$

(b) $\displaystyle\int \sqrt{x^2+2}\mathrm{d}x$

answer: $\ln\left(\sqrt{\frac{1}{2}x^2+1}+\frac{\sqrt{2}}{2}x\right)+\frac{\sqrt{2}}{2}x\sqrt{\frac{1}{2}x^2+1}+C$

(c) $\displaystyle\int \sqrt{x^2+x+1}\mathrm{d}x$

answer: $\frac{3}{4}\left(\frac{1}{2}\ln\left(\sqrt{\frac{4}{3}(x+\frac{1}{2})^2+1}+\frac{2}{3}\sqrt{3}\left(x+\frac{1}{2}\right)\right)+\frac{\sqrt{3}}{3}\left(x+\frac{1}{2}\right)\sqrt{\frac{4}{3}(x+\frac{1}{2})^2+1}\right)+C$

(d) $\displaystyle\int \sqrt{(2x^2+2x+1)}\mathrm{d}x$

answer: $\frac{\sqrt{2}}{4}\left(\frac{1}{2}(2x+1)\sqrt{(2x+1)^2+1}+\frac{1}{2}\ln\left(\sqrt{(2x+1)^2+1}+2x+1\right)\right)+C$

(e) $\displaystyle\int \sqrt{(3x^2+2x+1)}\mathrm{d}x$

answer: $\frac{2}{9}\sqrt{3}\left(\frac{1}{2}\ln\left(\sqrt{\frac{9}{2}(x+\frac{1}{3})^2+1}+\frac{3}{2}\sqrt{2}\left(x+\frac{1}{3}\right)\right)+\frac{3}{4}\sqrt{2}\left(x+\frac{1}{3}\right)\sqrt{\frac{9}{2}(x+\frac{1}{3})^2+1}\right)+C$

(f) $\displaystyle\int \frac{\sqrt{x^2+1}}{x+1}\mathrm{d}x$

answer: $-\sqrt{2}\ln\left(\sqrt{x^2+1}-x+\sqrt{2}-1\right)+\sqrt{2}\ln\left(\sqrt{x^2+1}-x-\sqrt{2}-1\right)+\ln\left(\sqrt{x^2+1}-x\right)+\sqrt{x^2+1}$

Solution. 22.a.

This problem can be solved both via the Euler substitution and by transforming to a trigonometric integral and solving the trigonometric integral on its own. We present both variants.

Variant I. We recall the Euler substitution for $\sqrt{x^2+1}$:

$$\begin{array}{rcl} x &=& \frac{1}{2}\left(\frac{1}{t}-t\right)\\ \sqrt{x^2+1} &=& \frac{1}{2}\left(\frac{1}{t}+t\right)\\ \mathrm{d}x &=& -\frac{1}{2}\left(\frac{1}{t^2}+1\right)\mathrm{d}t\\ t &=& \sqrt{x^2+1}-x \quad . \end{array}$$

Therefore

$$\begin{array}{rll} \displaystyle\int\sqrt{(x^2+1)}\mathrm{d}x &= \displaystyle -\int\frac{1}{4}\left(\frac{1}{t}+t\right)\left(\frac{1}{t^2}+1\right)\mathrm{d}t & \\ &= \displaystyle -\frac{1}{4}\int\left(\frac{1}{t^3}+2\frac{1}{t}+t\right)\mathrm{d}t & \\ &= \displaystyle -\frac{1}{4}\left(-\frac{t^{-2}}{2}+2\ln|t|+\frac{t^2}{2}\right)+C & \\ &= \displaystyle \frac{1}{8}\left(t^{-2}-t^2\right)-\frac{1}{2}\ln|t|+C & \left|\begin{array}{l} a^2-b^2=\\ (a-b)(a+b)\end{array}\right. \\ &= \displaystyle \frac{1}{2}\left(\underbrace{\frac{1}{2}\left(t^{-1}-t\right)}_{=x}\right)\left(\underbrace{\frac{1}{2}\left(t^{-1}+t\right)}_{=\sqrt{x^2+1}}\right)-\frac{1}{2}\ln|t|+C & \\ &= \displaystyle \frac{1}{2}x\sqrt{x^2+1}-\frac{1}{2}\ln\left|\sqrt{x^2+1}-x\right|+C & \left|\ \text{See below}\right. \\ &= \displaystyle \frac{1}{2}x\sqrt{x^2+1}+\frac{1}{2}\ln\left(\sqrt{x^2+1}+x\right)+C \quad . & \end{array}$$

Our problem is solved.

A few comments are in order. In the above expression we would have obtained a perfectly good answer if we plugged in $t=\sqrt{x^2+1}-x$ into the fourth line, however our answer would look much more complicated. Indeed, had we not used the formula $a^2-b^2=(a-b)(a+b)$ in the fourth line, the term $t^{-2}-t^2$ would be equal to $\frac{1}{(\sqrt{x^2+1}-x)^2}-(\sqrt{x^2+1}-x)^2$. In turn, the term $\frac{1}{(\sqrt{x^2+1}-x)^2}-(\sqrt{x^2+1}-x)^2$ can be simplified to $4x\sqrt{x^2+1}$ as follows. We carry out the simplifications to illustrate some of the algebraic issues arising when dealing with integrals of radicals.

$$
\begin{aligned}
t^{-2}-t^2 &= \frac{1}{(\sqrt{x^2+1}-x)^2}-(\sqrt{x^2+1}-x)^2\\
&= \frac{(\sqrt{x^2+1}+x)^2}{(\sqrt{x^2+1}-x)^2(\sqrt{x^2+1}+x)^2}\\
&\quad -(\sqrt{x^2+1}-x)^2\\
&= \frac{(\sqrt{x^2+1}+x)^2}{\underbrace{((\sqrt{x^2+1})^2-x^2)^2}_{=1}}-(\sqrt{x^2+1}-x)^2\\
&= 4x\sqrt{x^2+1} \quad .
\end{aligned}
$$

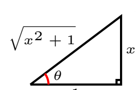
Of course, the above computations are unnecessary if we use the formula $a^2-b^2=(a-b)(a+b)$ as done in the original solution. We note that in the last transformation we transformed $\ln\left|\sqrt{x^2+1}-x\right|$ to $\ln\left(\sqrt{x^2+1}-x\right)$ because the quantity $\sqrt{x^2+1}-x$ is always positive. The proof of that fact we leave for the reader's exercise.

Finally, we note that as a last simplification to our solution, we used the transformation $\ln|t|=\ln\left(\sqrt{x^2+1}-x\right)=-\ln|\frac{1}{t}|=-\ln\left(\sqrt{x^2+1}+x\right)$. This is seen as follows.

$$
\begin{aligned}
\ln|t| &= -\ln\left|\frac{1}{t}\right| \\
&= -\ln\left(\frac{1}{\sqrt{x^2+1}-x}\right) && \Big| \text{ rationalize}\\
&= -\ln\left(\frac{\left(\sqrt{x^2+1}+x\right)}{\left(\sqrt{x^2+1}-x\right)\left(\sqrt{x^2+1}+x\right)}\right)\\
&= -\ln\left(\frac{\sqrt{x^2+1}+x}{x^2+1-x^2}\right)\\
&= -\ln\left(\sqrt{x^2+1}+x\right) \quad .
\end{aligned}
$$

Variant II. In this variant we transform to a trigonometric integral and solve it using ad-hoc methods. We recall that if we decided to solve the trigonometric integral using the standard substitution $\theta = 2\arctan t$, we would arrive at the Euler substitution given in Variant I.

$$
\begin{aligned}
\int\sqrt{x^2+1}\mathrm{d}x &= \int\sqrt{\tan^2\theta+1}\,\mathrm{d}(\tan\theta) && \begin{array}{l}\text{Set}\\ x=\tan\theta\\ \theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\end{array}\\
&= \int\sqrt{sec^2\theta}\sec^2\theta\mathrm{d}\theta && \sec\theta>0\\
&= \int\sec^3\theta\mathrm{d}\theta && \text{Problem } 18.b\\
&= \frac{1}{2}\left(\tan\theta\sec\theta+\ln|\sec\theta+\tan\theta|\right)+C
\end{aligned}
$$



$\sec\theta=\sqrt{x^2+1}$
$\tan\theta=x$

$$
= \frac{1}{2}\left(x\sqrt{x^2+1}+\ln\left(\sqrt{x^2+1}+x\right)\right)+C
$$

Solution. 22.d

$$
\begin{aligned}
\int \sqrt{(2x^2+2x+1)}\mathrm{d}x &= \int \sqrt{2}\sqrt{\left(\left(x+\frac{1}{2}\right)^2+\frac{1}{4}\right)}\mathrm{d}x && \text{complete square}\\
&= \sqrt{2}\int \sqrt{\frac{1}{4}\left(4\left(x+\frac{1}{2}\right)^2+1\right)}\mathrm{d}x\\
&= \frac{\sqrt{2}}{2}\int \sqrt{\left(4\left(x+\frac{1}{2}\right)^2+1\right)}\mathrm{d}x\\
&= \frac{\sqrt{2}}{2}\int \sqrt{\left((2x+1)^2+1\right)}\frac{1}{2}\mathrm{d}\left(2x+1\right) && \text{Set } u=2x+1\\
& && \text{Euler subst.: } u=\tfrac{1}{2}\left(\tfrac{1}{t}-t\right),\ t>0\\
&= \frac{\sqrt{2}}{4}\int \sqrt{(u^2+1)}\mathrm{d}u && \mathrm{d}u=-\tfrac{1}{2}\left(\tfrac{1}{t^2}+1\right)\mathrm{d}t\\
& && \sqrt{u^2+1}=\tfrac{1}{2}\left(\tfrac{1}{t}+t\right)\\
& && t=\sqrt{u^2+1}-u\\
&= -\frac{\sqrt{2}}{16}\int \left(\frac{1}{t}+t\right)\left(\frac{1}{t^2}+1\right)\mathrm{d}t\\
&= -\frac{\sqrt{2}}{16}\int \left(t^{-3}+2t^{-1}+t\right)\mathrm{d}t\\
&= -\frac{\sqrt{2}}{16}\left(-\frac{t^{-2}}{2}+2\ln|t|+\frac{t^2}{2}\right)+C && \text{simplify as in Problem 22.a}\\
&= \frac{\sqrt{2}}{8}\left(u\sqrt{u^2+1}+\ln\left(\sqrt{u^2+1}+u\right)\right)+C\\
&= \frac{\sqrt{2}}{8}\Big((2x+1)\sqrt{(2x+1)^2+1}\\
&\qquad +\ln\left(\sqrt{(2x+1)^2+1}+2x+1\right)\Big)+C.
\end{aligned}
$$

Solution. 22.f

$$
\begin{aligned}
\int \frac{\sqrt{x^2+1}}{x+1}\mathrm{d}x =& \int \frac{\frac{1}{2}\left(\frac{1}{t}+t\right)}{\frac{1}{2}\left(\frac{1}{t}-t\right)+1}\mathrm{d}\left(\frac{1}{2}\left(\frac{1}{t}-t\right)\right) && \left|\ \begin{array}{r} \text{Euler sub:} \\ x=\frac{1}{2}\left(\frac{1}{t}-t\right) \\ \sqrt{x^2+1}=\frac{1}{2}\left(\frac{1}{t}+t\right) \end{array}\right. \\
=& \int \left(\frac{1+t^2}{1-t^2+2t}\right)\frac{1}{2}\left(-t^{-2}-1\right)\mathrm{d}t \\
=& \int \frac{1}{2}\frac{\left(1+t^2\right)\left(-t^{-2}-1\right)}{1-t^2+2t}\mathrm{d}t \\
=& \frac{1}{2}\int \frac{t^4+2t^2+1}{t^4-2t^3-t^2}\mathrm{d}t && \left|\ \text{pol. long div.}\right. \\
=& \frac{1}{2}\int \left(1+\frac{2t^3+3t^2+1}{t^2\left(t^2-2t-1\right)}\right)\mathrm{d}t && \left|\ \text{part. fractions}\right. \\
=& \frac{1}{2}\int \left(1+\frac{2\sqrt{2}}{t-\sqrt{2}-1}+\frac{-2\sqrt{2}}{t+\sqrt{2}-1}+\frac{2}{t}+\frac{-1}{t^2}\right)\mathrm{d}t \\
=& -\sqrt{2}\ln\left|t+\sqrt{2}-1\right|+\sqrt{2}\ln\left|t-\sqrt{2}-1\right| \\
& +\frac{1}{2}t^{-1}+\ln|t|+\frac{1}{2}t+C && \left|\ t=\sqrt{x^2+1}-x\right. \\
=& -\sqrt{2}\ln\left(\sqrt{x^2+1}-x+\sqrt{2}-1\right) \\
& +\sqrt{2}\ln\left(\sqrt{x^2+1}-x-\sqrt{2}-1\right) \\
& +\ln\left(\sqrt{x^2+1}-x\right) \\
& +\frac{1}{2}\left(\sqrt{x^2+1}-x\right)^{-1}+\frac{1}{2}\sqrt{x^2+1}-\frac{1}{2}x+C && \left|\ \begin{array}{l}\text{Last 3 terms}\\ \text{simplify}\end{array}\right. \\
=& -\sqrt{2}\ln\left(\sqrt{x^2+1}-x+\sqrt{2}-1\right) \\
& +\sqrt{2}\ln\left(\sqrt{x^2+1}-x-\sqrt{2}-1\right) \\
& +\ln\left(\sqrt{x^2+1}-x\right) \\
& +\sqrt{x^2+1}+C \quad .
\end{aligned}
$$

23. Let $b^2-4ac<0$ and $a>0$ be (real) numbers. Show that

$$\int \sqrt{(ax^2+bx+c)}\mathrm{d}x = \frac{\sqrt{a}D}{2}\left(\ln\left(\sqrt{\left(\frac{2xa+b}{2\sqrt{Da}}\right)^2+1}+\frac{2xa+b}{2\sqrt{Da}}\right)+\frac{2xa+b}{2\sqrt{Da}}\sqrt{\left(\frac{2xa+b}{2\sqrt{Da}}\right)^2+1}\right)+C,$$

where $D=\dfrac{4ac-b^2}{4a^2}$.

24. Integrate

(a) $\displaystyle\int \sqrt{1-x^2}\mathrm{d}x$

answer:

(b) $\displaystyle\int \sqrt{2-x^2}\mathrm{d}x$

(c) $\displaystyle\int \sqrt{-x^2+x+1}\mathrm{d}x$

(d) $\displaystyle\int \sqrt{2-x-x^2}\mathrm{d}x$

(e) $\displaystyle\int \frac{\sqrt{1-x^2}}{1+x}\mathrm{d}x$

(f) $\displaystyle\int \frac{\sqrt{1-x^2}}{2+x}\mathrm{d}x$

Solution. 24.a

Variant I. This integral can quickly be solved using a trig substitution. The Euler substitution results in a slightly longer solution, shown in the next solution variant.

$$
\begin{aligned}
\int \sqrt{1-x^2}\mathrm{d}x &= \int \sqrt{1-\cos^2\theta}\mathrm{d}(\cos\theta) && \text{Set } x=\cos\theta, \theta\in[0,\pi] \\
&= \int \sqrt{\sin^2\theta}(-\sin\theta)\mathrm{d}\theta && \theta\in[0,\pi] \Rightarrow \sin\theta \geq 0 \\
&= -\int \sin^2\theta \mathrm{d}\theta && \sin^2\theta = \tfrac{1-\cos(2\theta)}{2} \\
&= -\int \frac{1-\cos(2\theta)}{2}\mathrm{d}\theta \\
&= -\frac{\theta}{2} + \frac{\sin(2\theta)}{4} + C \\
&= -\frac{\theta}{2} + \frac{2\sin\theta\cos\theta}{4} + C && \begin{aligned} x&=\cos\theta \\ \theta&=\arccos x \\ \sin\theta&=\sin\left(\arccos x\right) \\ &=\sqrt{1-x^2} \end{aligned} \\
&= -\frac{\arccos x}{2} + \frac{x\sqrt{1-x^2}}{2} + C \\
&= \frac{\arcsin x}{2} + \frac{x\sqrt{1-x^2}}{2} + K \quad ,
\end{aligned}
$$

where for the last equality we recall that the derivative of $\arcsin x$ is minus the derivative of $\arccos x$.

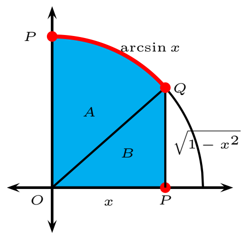
Variant II. We show how to do this integral via the Euler substitution $x = \cos(2\arctan t)$.

$$
\begin{aligned}
\int \sqrt{1-x^2}\mathrm{d}x &= \int \sqrt{1-\cos^2\theta}\mathrm{d}(\cos\theta) && \begin{aligned} &\text{Set} \\ x&=\cos(2\arctan t) \\ \tfrac{1}{2}\arccos x&=\arctan t \\ x&=\tfrac{1-t^2}{1+t^2} \\ &=\tfrac{2}{1+t^2}-1 \\ \sqrt{1-x^2}&=\tfrac{2t}{1+t^2} \end{aligned} \\
&= \int \frac{2t}{1+t^2}\mathrm{d}\left(\frac{1-t^2}{1+t^2}\right) \\
&= \int \frac{2t}{1+t^2}\left(\frac{-4t}{\left(1+t^2\right)^2}\right)\mathrm{d}t && \begin{aligned} &\text{Integral rational} \\ &\text{function} \\ &\text{we skip details} \end{aligned} \\
&= \frac{-t}{t^2+1} + \frac{2t}{\left(t^2+1\right)^2} \\
&\quad -\arctan t + C \\
&= -\frac{1}{2}\sqrt{1-x^2} + \frac{\sqrt{1-x^2}}{t^2+1} \\
&\quad -\arctan t + C \\
&= \frac{1}{2}\sqrt{1-x^2}\left(\frac{2}{t^2+1}-1\right) \\
&\quad -\arctan t + C \\
&= \frac{x\sqrt{1-x^2}}{2} - \frac{1}{2}\arccos x + C \\
&= \frac{x\sqrt{1-x^2}}{2} + \frac{1}{2}\arcsin x + K,
\end{aligned}
$$

where for the very last equality we used the fact that the derivatives of $\arcsin x$ and $\arccos x$ are negatives of one another.

Variant III. We show how to do this integral geometrically, provided already know the area of a sector of circle. Of course, here we assume we have already derived the formula for an area of a circle. We warn the reader that most methods for deriving the formula of a sector area rely on integrals, so it is possible we are making a circular reasoning argument. Since we already did the integral purely algebraically in the preceding solution variants, we can safely ignore the danger of the aforementioned circular reasoning argument. In other words, the present solution Variant is a geometric interpretation of the problem which relies on the formula for sector area of a circle (which we assumed proved elsewhere, possibly using similar integration techniques to the ones presented in Variant I and II).

By the Fundamental Theorem of Calculus, the indefinite integral measures up to a constant the area locked under the graph of $\sqrt{1-x^2}$. This graph is a part of a circle. Therefore, up to a constant, $\int \sqrt{1-t^2}\mathrm{d}t$ equals $\int_0^x \sqrt{1-t^2}\mathrm{d}t$. In turn $\int_0^x \sqrt{1-t^2}\mathrm{d}t$ is given by the area highlighted in the picture below.



$$
\begin{aligned}
\text{Area}(A) &= \frac{\text{length}\left(\widehat{PQ}\right)}{2\pi}\pi = \frac{\text{length}\left(\widehat{PQ}\right)}{2} = \frac{\arcsin x}{2}\\
\text{Area}(B) &= \text{Area}(\triangle OPQ) = \frac{x\sqrt{1-x^2}}{2}\\
\int_0^x \sqrt{1-t^2}\mathrm{d}t &= \text{Area}(A) + \text{Area}(B)\\
&= \frac{\arcsin x}{2} + \frac{x\sqrt{1-x^2}}{2}\\
&\Rightarrow\\
\int \sqrt{1-x^2}\mathrm{d}x &= \frac{\arcsin x}{2} + \frac{x\sqrt{1-x^2}}{2} + C \quad .
\end{aligned}
$$

Solution. 24.e In this problem solution we use the standard Euler substitution $x = \cos(2\arctan t)$. We recall that

$$
\begin{aligned}
x &= \cos(2\arctan t) = \frac{1-t^2}{1+t^2}\\
\arccos(x) &= 2\arctan t\\
\mathrm{d}x &= -\frac{4t}{(1+t^2)^2}\mathrm{d}t\\
\sqrt{1-x^2} &= \sin(2\arctan t) = \frac{2t}{1+t^2}\\
t &= \frac{\sqrt{1-x^2}}{x+1} \quad .
\end{aligned}
$$

$$
\begin{aligned}
\int \frac{\sqrt{1-x^2}}{1+x}\mathrm{d}x &= \int t\left(-\frac{4t}{(1+t^2)^2}\right)\mathrm{d}t && \left|\ \begin{array}{l}\text{Set } x = \frac{1-t^2}{1+t^2}\\ \text{Use f-las above}\end{array}\right.\\
&= -4\int \frac{t^2}{(1+t^2)^2}\mathrm{d}t\\
&= -4\int \frac{1+t^2-1}{(1+t^2)^2}\mathrm{d}t\\
&= -4\int \left(\frac{1}{1+t^2} - \frac{1}{(1+t^2)^2}\right)\mathrm{d}t\\
&= -4\left(\arctan t - \frac{1}{2}\left(\arctan t + \frac{t}{1+t^2}\right)\right) + C\\
&= -2\left(\arctan t - \frac{t}{1+t^2}\right) + C\\
&= -2\left(\arctan\left(\frac{\sqrt{1-x^2}}{1+x}\right) - \frac{1}{2}\sqrt{1-x^2}\right) + C\\
&= -2\arctan t + \sqrt{1-x^2} + C && \left|\ \text{Use f-las above}\right.\\
&= -\arccos x + \sqrt{1-x^2} + C\\
&= \arcsin x + \sqrt{1-x^2} + K \quad .
\end{aligned}
$$

We have included the last equality to remind the student that derivatives of $\arcsin(x)$ and $\arccos x$ are negatives of one another.

25. Integrate

(a) $\int \sqrt{x^2-1}\mathrm{d}x$

(b) $\displaystyle\int \sqrt{x^2-2}\mathrm{d}x$

(c) $\displaystyle\int \sqrt{2x^2+x-1}\mathrm{d}x$

(d) $\displaystyle\int \sqrt{x^2+x-1}\mathrm{d}x$

26. (a) Express $x, \mathrm{d}x$ and $\sqrt{x^2+1}$ via θ and $\mathrm{d}\theta$ for the trigonometric substitution $x = \cot\theta, \theta \in (0, \pi)$.
(b) Express $x, \mathrm{d}x$ and $\sqrt{x^2+1}$ via t and $\mathrm{d}t$ for the Euler substitution $x = \cot(2\arctan t), t > 0$. Express t via x.

Solution. 26.a The trigonometric substitution $x = \cot\theta$ is given by

$$\begin{array}{rcll}
\sqrt{x^2+1} &=& \sqrt{\cot^2\theta+1} & \\
&=& \sqrt{\dfrac{\cos^2\theta}{\sin^2\theta}+1} & \\
&=& \sqrt{\dfrac{\cos^2\theta+\sin^2\theta}{\sin^2\theta}} & \\
&=& \sqrt{\dfrac{1}{\sin^2\theta}} & \left|\begin{array}{l}\text{when } \theta\in(0,\pi) \text{ we have}\\ \sin\theta\geq 0 \text{ and so } \sqrt{\sin^2\theta}=\sin\theta\end{array}\right. \\
&=& \dfrac{1}{\sin\theta} = \csc\theta & .
\end{array}$$

The differential $\mathrm{d}x$ can be expressed via $\mathrm{d}\theta$ from $x = \cot\theta$. The substitution $x = \cot\theta$ can be now summarized as:

$$\begin{array}{rcl}
x &=& \cot\theta \\
\sqrt{x^2+1} &=& \dfrac{1}{\sin\theta} = \csc\theta \\
\mathrm{d}x &=& -\dfrac{\mathrm{d}\theta}{\sin^2\theta} = -\csc^2\theta\mathrm{d}\theta \\
\theta &=& \operatorname{arccot} x \quad .
\end{array}$$

Solution. 26.b We recall that the substitution $\theta = 2\arctan t$ transforms a trigonometric integral into an integral of a rational function. We now apply the substitution $\theta = 2\arctan t$ after the substitution $x = \cot\theta$:

$$\begin{array}{rcll}
x &=& \cot\theta & \left|\begin{array}{l}\text{use } \theta = 2\arctan t\\ \text{use } \cot 2z = \dfrac{\cos(2z)}{\sin(2z)} = \dfrac{1-\tan^2 z}{2\tan z}\end{array}\right. \\
&=& \cot(2\arctan t) & \\
&=& \dfrac{1-\tan^2(\arctan t)}{2\tan(\arctan t)} & \\
&=& \dfrac{1-t^2}{2t} & \\
&=& \dfrac{1}{2}\left(\dfrac{1}{t}-t\right) & .
\end{array}$$

We can furthermore compute

$$\begin{array}{rcll}
\sqrt{x^2+1} &=& \sqrt{\dfrac{1}{4}\left(\dfrac{1}{t}-t\right)^2+1} & \\
&=& \dfrac{1}{2}\sqrt{\left(\dfrac{1}{t}+t\right)^2} & \left| \sqrt{\left(\dfrac{1}{t}+t\right)^2} = \dfrac{1}{t}+t \text{ because } t>0\right. \\
&=& \dfrac{1}{2}\left(\dfrac{1}{t}+t\right) & .
\end{array} \tag{3}$$

The differential $\mathrm{d}x$ can via $\mathrm{d}x$ as follows.

$$\mathrm{d}x = \mathrm{d}\left(\frac{1}{2}\left(\frac{1}{t}-t\right)\right) = -\frac{1}{2}\left(\frac{1}{t^2}-1\right) \quad .$$

Finally, we can subtract $x = \dfrac{1}{2}\left(\dfrac{1}{t}-t\right)$ from $\sqrt{x^2+1} = \dfrac{1}{2}\left(\dfrac{1}{t}+t\right)$ to get that

$$t = \sqrt{x^2+1} - x \quad .$$

The Euler substitution $x = \cot\theta = \cot(\arctan 2t)$ can be now summarized as:

$$\begin{array}{rcl} x &=& \dfrac{1}{2}\left(\dfrac{1}{t} - t\right) \\ \sqrt{x^2+1} &=& \dfrac{1}{2}\left(\dfrac{1}{t} + t\right) \\ \mathrm{d}x &=& -\dfrac{1}{2}\left(\dfrac{1}{t^2} + 1\right)\mathrm{d}t \\ t &=& \sqrt{x^2+1} - x \quad . \end{array} \tag{4}$$

27. Let the variables x and t be related via $\sqrt{x^2+1} = x + t$.

    (a) Express x via t.

    (b) Express $\sqrt{x^2+1}$ via t alone.

    (c) Express $\mathrm{d}x$ via t and $\mathrm{d}t$.

Solution. 27.a.

$$\begin{array}{rcll} \sqrt{x^2+1} &=& x+t & \Big| \text{ square both sides} \\ x^2+1 &=& x^2+2xt+t^2 & \\ -2xt &=& t^2-1 & \\ x &=& \dfrac{1}{2}\left(\dfrac{1}{t} - t\right) \quad . & \end{array}$$

Solution. 27.b.

Use Problem 27.a to get:

$$\sqrt{x^2+1} = x + t = \frac{1}{2}\left(\frac{1}{t} - t\right) + t = \frac{1}{2}\left(\frac{1}{t} + t\right) \quad .$$

28. (a) Express $x, \mathrm{d}x$ and $\sqrt{x^2+1}$ via θ and $\mathrm{d}\theta$ for the trigonometric substitution $x = \tan\theta$, $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

    (b) Express $x, \mathrm{d}x$ and $\sqrt{x^2+1}$ via t and $\mathrm{d}t$ for the Euler substitution $x = \tan(2\arctan t)$, $t \in (-1, 1)$. Express t via x.

Solution. 28.a The trigonometric substitution $x = \tan\theta$ is given by

$$\begin{array}{rcll} \sqrt{x^2+1} &=& \sqrt{\tan^2\theta + 1} & \\ &=& \sqrt{\dfrac{\sin^2\theta}{\cos^2\theta} + 1} & \\ &=& \sqrt{\dfrac{\sin^2\theta + \cos^2\theta}{\cos^2\theta}} & \\ &=& \sqrt{\dfrac{1}{\cos^2\theta}} & \Bigg| \begin{array}{l} \text{when } \theta \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right) \text{ we have} \\ \cos\theta > 0 \text{ and so } \sqrt{\cos^2\theta} = \cos\theta \end{array} \\ &=& \dfrac{1}{\cos\theta} = \sec\theta \quad . & \end{array}$$

The differential $\mathrm{d}x$ can be expressed via $\mathrm{d}\theta$ from $x = \tan\theta$. The substitution $x = \tan\theta$ can be now summarized as:

$$\begin{array}{rcl} x &=& \tan\theta \\ \sqrt{x^2+1} &=& \dfrac{1}{\cos\theta} = \sec\theta \\ \mathrm{d}x &=& \dfrac{\mathrm{d}\theta}{\cos^2\theta} = \sec^2\theta\mathrm{d}\theta \\ \theta &=& \arctan x \quad . \end{array}$$

Solution. 28.b We recall that the substitution $\theta = 2\arctan t$ transforms a trigonometric integral into an integral of a rational function. We now apply the substitution $\theta = 2\arctan t$ after the substitution $x = \tan\theta$:

$$\begin{array}{rcll} x &=& \tan\theta & \Big| \text{ use } \theta = 2\arctan t \\ &=& \tan(2\arctan t) & \Big| \text{ use: } \tan 2z = \dfrac{\sin(2z)}{\cos(2z)} = \dfrac{2\tan z}{1-\tan^2 z} \\ &=& \dfrac{2\tan(\arctan t)}{1-\tan^2(\arctan t)} & \\ &=& \dfrac{2t}{1-t^2} & . \end{array}$$

We can furthermore compute

$$\begin{array}{rcll} \sqrt{x^2+1} &=& \sqrt{\left(\dfrac{2t}{1-t^2}\right)^2+1} & \\ &=& \sqrt{\dfrac{4t^2+(1-t^2)^2}{(1-t^2)^2}} & \\ &=& \sqrt{\dfrac{(1+t^2)^2}{(1-t^2)^2}} & \Big| \sqrt{(1-t^2)^2} = 1-t^2 \text{ because } |t|<1 \\ &=& \dfrac{1+t^2}{1-t^2} & \\ &=& \dfrac{2-(1-t^2)}{1-t^2} & \\ &=& -1+\dfrac{2}{1-t^2} & . \end{array} \tag{5}$$

From $\sqrt{x^2+1} = -1 + \dfrac{2}{1-t^2}$ and $x = \dfrac{2t}{1-t^2}$ we can express t via x:

$$\begin{array}{rcll} \sqrt{x^2+1} &=& -1+\dfrac{2}{1-t^2} & \\ &=& -1+\dfrac{1}{t}\left(\dfrac{2t}{1-t^2}\right) & \Big| \text{ use } x = \dfrac{2t}{1-t^2} \\ &=& -1+\dfrac{x}{t} & \\ 1+\sqrt{x^2+1} &=& \dfrac{x}{t} & \\ t &=& \dfrac{x}{1+\sqrt{x^2+1}} & \\ &=& \dfrac{x}{1+\sqrt{x^2+1}}\left(\dfrac{1-\sqrt{x^2+1}}{1-\sqrt{x^2+1}}\right) & \\ &=& \dfrac{x(1-\sqrt{x^2+1})}{1-x^2-1} & \\ &=& \dfrac{\sqrt{x^2+1}-1}{x} & . \end{array}$$

The differential $\mathrm{d}x$ can expressed via $\mathrm{d}t$ from $x = 1 + \dfrac{2}{t^2-1}$. The Euler substitution $x = \tan\theta = \tan(2\arctan t)$ can now be summarized as follows.

$$\begin{array}{rcll} x &=& \dfrac{2t}{1-t^2} & \\ \sqrt{x^2+1} &=& -1+\dfrac{2}{1-t^2} & \\ \mathrm{d}x &=& \dfrac{2(1+t^2)}{(1-t^2)^2}\mathrm{d}t & \\ t &=& \dfrac{\sqrt{x^2+1}-1}{x} & . \end{array} \tag{6}$$

29. Let the variables x and t be related via $\sqrt{x^2+1} = \frac{x}{t} - 1$.

(a) Express x via t.

(b) Express $\sqrt{x^2+1}$ via t alone.

(c) Express $\mathrm{d}x$ via t and $\mathrm{d}t$.

30. (a) Express $x, \mathrm{d}x$ and $\sqrt{1-x^2}$ via θ and $\mathrm{d}\theta$ for the trigonometric substitution $x = \cos\theta$, $\theta \in [0,\pi]$.
(b) Express $x, \mathrm{d}x$ and $\sqrt{1-x^2}$ via t and $\mathrm{d}t$ for the Euler substitution $x = \cos(2\arctan t)$, $t \geq 0$. Express t via x.

Solution. 30.a The trigonometric substitution $x = \cos\theta$ is given by

$$\begin{array}{rcll} \sqrt{-x^2+1} &=& \sqrt{1-\cos^2\theta} & \\ &=& \sqrt{\sin^2\theta} & \left| \begin{array}{l} \text{when } \theta \in [0,\pi] \text{ we have} \\ \sin\theta \geq 0 \text{ and so } \sqrt{\sin^2\theta} = \sin\theta \end{array} \right. \\ &=& \sin\theta \quad . & \end{array}$$

The differential $\mathrm{d}x$ can be expressed via $\mathrm{d}\theta$ from $x = \cos\theta$. The substitution $x = \cos\theta$ can be now summarized as:

$$\begin{array}{rcl} x &=& \cos\theta \\ \sqrt{-x^2+1} &=& \sin\theta \\ \mathrm{d}x &=& -\sin\theta\mathrm{d}\theta \\ \theta &=& \arccos x \quad . \end{array}$$

Solution. 30.b We recall that the substitution $\theta = 2\arctan t$ transforms a trigonometric integral into an integral of a rational function. We now apply the substitution $2\arctan t$ after the substitution $x = \cos\theta$:

$$\begin{array}{rcll} x &=& \cos\theta & \left| \begin{array}{l} \text{use } \theta = 2\arctan t \\ \text{use } \cos(2z) = \dfrac{1-\tan^2 z}{1+\tan^2 z} \end{array} \right. \\ &=& \cos(2\arctan t) & \\ &=& \dfrac{1-\tan^2(\arctan t)}{1+\tan^2(\arctan t)} & \\ &=& \dfrac{1-t^2}{1+t^2} \quad . & \end{array}$$

We can furthermore compute

$$\begin{array}{rcll} \sqrt{-x^2+1} &=& \sqrt{1-\left(\dfrac{1-t^2}{1+t^2}\right)^2} & \\ &=& \sqrt{\dfrac{(1+t^2)^2-(1-t^2)^2}{(1+t^2)^2}} & \\ &=& \sqrt{\dfrac{4t^2}{(1+t^2)^2}} & \left| \sqrt{4t^2} = 2t \text{ because } t \geq 0 \right. \\ &=& \dfrac{2t}{1+t^2} \quad . & \end{array} \tag{7}$$

The differential $\mathrm{d}x$ can be computed from $x = \frac{1-t^2}{1+t^2}$. Finally, we can express t via x with a little algebra:

$$\begin{array}{rcll} x &=& \dfrac{1-t^2}{1+t^2} & \\ (1+t^2)x &=& 1-t^2 & \\ t^2(x+1) &=& 1-x & \\ t^2 &=& \dfrac{1-x}{1+x} & \\ t &=& \sqrt{\dfrac{1-x}{1+x}} & \left| \text{here we use } t > 0 \right. \\ t &=& \dfrac{\sqrt{1-x}}{\sqrt{1+x}}\dfrac{\sqrt{1+x}}{\sqrt{1+x}} & \\ t &=& \dfrac{\sqrt{-x^2+1}}{x+1} \quad . & \end{array}$$

The Euler substitution $x=\cos(2\arctan t)$ can be now summarized as:

$$\begin{array}{rcl} x &=& \frac{1-t^2}{1+t^2}\\ \sqrt{-x^2+1} &=& \frac{2t}{1+t^2}\\ \mathrm{d}x &=& -\frac{4t}{(t^2+1)^2}\mathrm{d}t\\ t &=& \frac{\sqrt{-x^2+1}}{x+1} \quad . \end{array} \tag{8}$$

31. Let the variables x and t be related via $\sqrt{-x^2+1}=(1-x)t$.
    (a) Express x via t.
    (b) Express $\sqrt{-x^2+1}$ via t alone.
    (c) Express $\mathrm{d}x$ via t and $\mathrm{d}t$.

Solution. 31.a.

$$\begin{array}{rcll} \sqrt{-x^2+1} &=& (1-x)t & \Big| \text{ square both sides}\\ (1-x)(1+x) &=& (1-x)^2t^2 & \Big| \text{ divide by } (1-x)\\ 1+x &=& (1-x)t^2 &\\ x(1+t^2) &=& t^2-1 &\\ x &=& \frac{t^2-1}{t^2+1}=1-\frac{2}{t^2+1} \quad . & \end{array}$$

Solution. 31.b.

Use Problem 31.a to get

$$\sqrt{-x^2+1}=(1-x)t=\left(1-\left(1-\frac{2t}{t^2+1}\right)\right)t=\frac{2t}{t^2+1} \quad .$$

32. (a) Express $x, \mathrm{d}x$ and $\sqrt{1-x^2}$ via θ and $\mathrm{d}\theta$ for the trigonometric substitution $x=\sin\theta$, $\theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.
    (b) Express $x, \mathrm{d}x$ and $\sqrt{1-x^2}$ via t and $\mathrm{d}t$ for the Euler substitution $x=\sin(2\arctan t)$, $t\in[-1,1]$. Express t via x.

Solution. 32.a The trigonometric substitution $x=\sin\theta$ is given by

$$\begin{array}{rcll} \sqrt{-x^2+1} &=& \sqrt{1-\sin^2\theta} &\\ &=& \sqrt{\cos^2\theta} & \Big| \begin{array}{l}\text{when } \theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right] \text{ we have}\\ \cos\theta\geq 0 \text{ and so } \sqrt{\cos^2\theta}=\cos\theta\end{array}\\ &=& \cos\theta \quad . & \end{array}$$

The differential $\mathrm{d}x$ can be expressed via $\mathrm{d}\theta$ from $x=\sin\theta$. The substitution $x=\sin\theta$ can be now summarized as:

$$\begin{array}{rcl} x &=& \sin\theta\\ \sqrt{-x^2+1} &=& \sin\theta\\ \mathrm{d}x &=& \cos\theta\mathrm{d}\theta\\ \theta &=& \arcsin x \quad . \end{array}$$

Solution. 32.b We recall that the substitution $\theta=2\arctan t$ transforms a trigonometric integral into an integral of a rational function. We now apply the substitution $2\arctan t$ after the substitution $x=\sin\theta$:

$$\begin{array}{rcll} x &=& \sin\theta & \Big| \text{ use } \theta=2\arctan t\\ &=& \sin(2\arctan t) & \Big| \text{ use } \sin(2z)=\frac{2\tan z}{1+\tan^2 z}\\ &=& \frac{2\tan(\arctan t)}{1+\tan^2(\arctan t)} &\\ &=& \frac{2t}{1+t^2} \quad . & \end{array}$$

We can furthermore compute

$$
\begin{aligned}
\sqrt{-x^2+1} &= \sqrt{1-\left(\frac{2t}{1+t^2}\right)^2} \\
&= \sqrt{\frac{(1+t^2)^2-4t^2}{(1+t^2)^2}} \\
&= \sqrt{\frac{(1-t^2)^2}{(1+t^2)^2}} && \left| \quad \sqrt{(1-t^2)^2}=1-t^2 \text{ because } |t|\leq 1 \right. \\
&= \frac{1-t^2}{1+t^2} \\
&= \frac{2-(1+t^2)}{1+t^2} \\
&= -1+\frac{2}{1+t^2} \quad .
\end{aligned} \tag{9}
$$

The differential dx can be computed from $x=\frac{2t}{1+t^2}$. Finally, we can express t via x with a little algebra:

$$
\begin{aligned}
\sqrt{-x^2+1} &= -1+\frac{2}{1+t^2} \\
&= -1+\frac{1}{t}\left(\frac{2t}{1+t^2}\right) && \left| \quad \text{use } x=\tfrac{2t}{1+t^2} \right. \\
&= -1+\frac{x}{t} && \left| \quad +1 \text{ to both sides} \right. \\
\frac{x}{t} &= 1+\sqrt{-x^2+1} \\
t &= \frac{x}{1+\sqrt{-x^2+1}} \\
&= \frac{x}{\left(1+\sqrt{-x^2+1}\right)}\frac{\left(1-\sqrt{-x^2+1}\right)}{\left(1-\sqrt{-x^2+1}\right)} \\
&= \frac{1-\sqrt{-x^2+1}}{x} \quad .
\end{aligned}
$$

The Euler substitution $x=\sin(2\arctan t)$ can be now summarized as:

$$
\begin{aligned}
x &= \frac{2t}{1+t^2} \\
\sqrt{-x^2+1} &= -1+\frac{2}{1+t^2} \\
\mathrm{d}x &= 2\left(\frac{1-t^2}{(1+t^2)^2}\right)\mathrm{d}t \\
t &= \frac{1-\sqrt{-x^2+1}}{x} \quad .
\end{aligned}
$$

33. Let the variables x and t be related via $\sqrt{-x^2+1}=1-xt$.
    (a) Express x via t.
    (b) Express $\sqrt{-x^2+1}$ via t alone.
    (c) Express dx via t and dt.

34. (a) Express $x, \mathrm{d}x$ and $\sqrt{x^2-1}$ via θ and dθ for the trigonometric substitution $x=\csc\theta$, $\theta\in\left[0,\frac{\pi}{2}\right]\cup\left[\pi,\frac{3\pi}{2}\right)$.
    (b) Express $x, \mathrm{d}x$ and $\sqrt{1-x^2}$ via t and dt for the Euler substitution $x=\sec(2\arctan t)$, $t\in(-\infty,-1)\cup[1,0)$. Express t via x.

Solution. 34.a The trigonometric substitution $x=\sec\theta$ is given by

$$
\begin{aligned}
\sqrt{x^2-1} &= \sqrt{\sec^2\theta-1}=\sqrt{\frac{1}{\cos^2\theta}-1} \\
&= \sqrt{\frac{\sin^2\theta}{\cos^2\theta}}=\sqrt{\tan^2\theta} && \left| \quad \begin{array}{l}\text{when } \theta\in\theta\in\left[0,\frac{\pi}{2}\right)\cup\left[\pi,\frac{3\pi}{2}\right) \text{ we have} \\ \tan\theta\geq 0 \text{ and so } \sqrt{\tan^2\theta}=\tan\theta\end{array} \right. \\
&= \tan\theta \quad .
\end{aligned}
$$

The differential $\mathrm{d}x$ can be expressed via $\mathrm{d}\theta$ from $x = \sec\theta$. The substitution $x = \sec\theta$ can be now summarized as:

$$\begin{array}{rcl} x &=& \sec\theta = \dfrac{1}{\cos\theta} \\ \sqrt{x^2-1} &=& \tan\theta \\ \mathrm{d}x &=& \dfrac{\sin\theta}{\cos^2\theta}\mathrm{d}\theta = \sec\theta\tan\theta\mathrm{d}\theta \\ \theta &=& \operatorname{arcsec} x \quad . \end{array}$$

Solution. 34.b We recall that the substitution $\theta = 2\arctan t$ transforms a trigonometric integral into an integral of a rational function. We now apply the substitution $2\arctan t$ after the substitution $x = \sec\theta$:

$$\begin{array}{rcl|l} x &=& \sec\theta = \dfrac{1}{\cos\theta} & \text{use } \theta = 2\arctan t \\ &=& \dfrac{1}{\cos(2\arctan t)} & \text{use } \cos(2z) = \dfrac{1-\tan^2 z}{1+\tan^2 z} \\ &=& \dfrac{1+\tan^2(\arctan t)}{1-\tan^2(\arctan t)} & \\ &=& \dfrac{1+t^2}{1-t^2} & \\ &=& -1 + \dfrac{2}{1-t^2} \quad . & \end{array}$$

We can furthermore compute

$$\begin{array}{rcl|l} \sqrt{x^2-1} &=& \sqrt{\left(\dfrac{1+t^2}{1-t^2}\right)^2 - 1} & \\ &=& \sqrt{\dfrac{(1+t^2)^2-(1-t^2)^2}{(1-t^2)^2}} & \\ &=& \sqrt{\dfrac{4t^2}{(1-t^2)^2}} & \begin{array}{l} t \text{ and } 1-t^2 \text{ have the same} \\ \text{sign for } t\in(-\infty,-1)\cup[0,1) \end{array} \\ &=& \dfrac{2t}{1-t^2} \quad . & \end{array} \qquad (10)$$

The differential $\mathrm{d}x$ can be computed from $x = \frac{1+t^2}{1-t^2}$. Finally, we can express t via x with a little algebra:

$$\begin{array}{rcl|l} x &=& \dfrac{1+t^2}{1-t^2} & \\ (1-t^2)x &=& 1+t^2 & \\ (1+x)t^2 &=& x-1 & \\ t^2 &=& \dfrac{x-1}{x+1} & \\ t &=& \begin{cases} \sqrt{\frac{x-1}{x+1}} & x>1 \\ -\sqrt{\frac{x-1}{x+1}} & x<-1 \end{cases} & \begin{array}{l} \text{because when } x<-1, \\ \text{we have } t\in(-\infty,-1] \end{array} \\ t &=& \begin{cases} \frac{\sqrt{x^2-1}}{x+1} & x>1 \\ -\frac{\sqrt{x^2-1}}{x+1} & x<-1 \end{cases} \quad . & \end{array}$$

The Euler substitution $x = \sec(2\arctan t)$ can be now summarized as:

$$\begin{array}{rcl} x &=& \dfrac{1+t^2}{1-t^2} \\ \sqrt{x^2-1} &=& \dfrac{2t}{1-t^2} \\ \mathrm{d}x &=& \dfrac{4t}{(1-t^2)^2}\mathrm{d}t \\ t &=& \pm\dfrac{\sqrt{x^2-1}}{x+1} \quad . \end{array}$$

35. Let the variables x and t be related via $\sqrt{x^2-1} = (x+1)t$.

(a) Express x via t.

(b) Express $\sqrt{x^2-1}$ via t alone.

(c) Express $\mathrm{d}x$ via t and $\mathrm{d}t$.

Solution. 35.a.

$$\begin{array}{rcl|l} \sqrt{x^2-1} &=& (x+1)t & \text{square both sides} \\ (x-1)(x+1) &=& (x+1)^2t^2 & \text{divide by } (x+1) \\ x-1 &=& (x+1)t^2 & \\ x(1-t^2) &=& 1+t^2 & \\ x &=& \dfrac{1+t^2}{1-t^2} = -1+\dfrac{2}{1-t^2} & \end{array}$$

Solution. 35.b.

We use Problem 35.a to get

$$\sqrt{x^2-1} = (x+1)t = \left(-1+\frac{2}{1-t^2}+1\right)t = \frac{2t}{1-t^2}$$

36. (a) Express $x, \mathrm{d}x$ and $\sqrt{1-x^2}$ via θ and $\mathrm{d}\theta$ for the trigonometric substitution $x = \csc\theta$, $\theta \in \left[0, \frac{\pi}{2}\right] \cup \left[\pi, \frac{3\pi}{2}\right)$.

(b) Express $x, \mathrm{d}x$ and $\sqrt{1-x^2}$ via t and $\mathrm{d}t$ for the Euler substitution $x = \csc(2\arctan t)$, $t \in (-\infty, -1) \cup [0, 1)$. Express t via x.

Solution. 36.a The trigonometric substitution $x = \csc\theta$ is given by

$$\begin{array}{rcl|l} \sqrt{x^2-1} &=& \sqrt{\dfrac{1}{\sin^2\theta}-1} & \\ &=& \sqrt{\dfrac{\cos^2\theta}{\sin^2\theta}} = \sqrt{\cot^2\theta} & \text{when } \theta\in\theta\in\left[0,\frac{\pi}{2}\right)\cup\left[\pi,\frac{3\pi}{2}\right) \text{ we have} \\ & & & \cot\theta\geq 0 \text{ and so } \sqrt{\cot^2\theta}=\tan\theta \\ &=& \cot\theta\quad . & \end{array}$$

The differential $\mathrm{d}x$ can be expressed via $\mathrm{d}\theta$ from $x = \csc\theta$. The substitution $x = \csc\theta$ can be now summarized as:

$$\begin{array}{rcl} x &=& \csc\theta \\ \sqrt{x^2-1} &=& \cot\theta \\ \mathrm{d}x &=& -\dfrac{\cos\theta}{\sin^2\theta}\mathrm{d}\theta = -\csc\theta\cot\theta\mathrm{d}\theta \\ \theta &=& \csc^{-1}x\quad . \end{array}$$

Solution. 36.b We recall that the substitution $\theta = 2\arctan t$ transforms a trigonometric integral into an integral of a rational function. We now apply the substitution $2\arctan t$ after the substitution $x = \csc\theta$:

$$\begin{array}{rcl|l} x &=& \csc\theta = \dfrac{1}{\sin\theta} & \text{use } \theta = 2\arctan t \\ &=& \dfrac{1}{\sin(2\arctan t)} & \text{use } \sin(2z) = \dfrac{2\tan z}{1+\tan^2 z} \\ &=& \dfrac{1+\tan^2(\arctan t)}{2\tan(\arctan t)} & \\ &=& \dfrac{1+t^2}{2t} & \\ &=& \dfrac{1}{2}\left(\dfrac{1}{t}+t\right)\quad . & \end{array}$$

We can furthermore compute

$$\begin{aligned}
\sqrt{x^2-1} &= \sqrt{\left(\frac{1+t^2}{2t}\right)^2-1} \\
&= \sqrt{\frac{(1+t^2)^2-4t^2}{4t^2}} \\
&= \sqrt{\frac{(1-t^2)^2}{4t^2}} && \left| \ \frac{1-t^2}{2t} > 0 \text{ when } t\in(-\infty,-1)\cup[0,1) \right. \\
&= \frac{1-t^2}{2t} \\
&= \frac{1}{2}\left(\frac{1}{t}-t\right) \quad .
\end{aligned} \tag{11}$$

The differential $\mathrm{d}x$ can be computed from $x=\frac{1}{2}\left(\frac{1}{t}-t\right)$. Finally, we can express t via x with a little algebra:

$$\begin{aligned}
\sqrt{x^2-1} &= \frac{1-t^2}{2t} \\
\sqrt{x^2-1} &= \frac{2-(1+t^2)}{2t} && \left| \ \text{use } x=\frac{1+t^2}{2t} \right. \\
\sqrt{x^2-1} &= \frac{1}{t}-x \\
\frac{1}{t} &= \sqrt{x^2-1}+x \\
t &= \frac{1}{\sqrt{x^2-1}+x} = \frac{1}{\left(\sqrt{x^2-1}+x\right)}\frac{\left(-\sqrt{x^2-1}+x\right)}{\left(-\sqrt{x^2-1}+x\right)} \\
t &= x-\sqrt{x^2-1}
\end{aligned}$$

The Euler substitution $x=\cos(2\arctan t)$ can be now summarized as:

$$\begin{aligned}
x &= \frac{1}{2}\left(\frac{1}{t}+t\right) \\
\sqrt{-x^2+1} &= \frac{1}{2}\left(\frac{1}{t}-t\right) \\
\mathrm{d}x &= -\frac{1}{2}\left(\frac{1}{t^2}+1\right)\mathrm{d}t \\
t &= x-\sqrt{x^2-1} \quad .
\end{aligned}$$

37. Let the variables x and t be related via $\sqrt{x^2-1}=\frac{1}{t}-x$.

    (a) Express x via t.

    (b) Express $\sqrt{x^2-1}$ via t alone.

    (c) Express $\mathrm{d}x$ via t and $\mathrm{d}t$.

38. Compute the limits. The answer key has not been fully proofread, use with caution.

    (a) $\displaystyle\lim_{x\to 0}\frac{\sin x}{x}$.

    answer: 1

    (b) $\displaystyle\lim_{x\to 0}\frac{x}{\ln(1+x)}$.

    answer: 1

    (c) $\displaystyle\lim_{x\to 0}\frac{x^2}{x-\ln(1+x)}$.

    answer: 2

    (d) $\displaystyle\lim_{x\to 0}\frac{x^2}{\sin x\ln(1+x)}$.

    answer: 1

    (e) $\displaystyle\lim_{x\to 0}\frac{\sin^2 x}{(\ln(1+x))^2}$.

    answer: 1

    (f) $\displaystyle\lim_{x\to 0}\frac{\cos x-1}{\sin x\ln(1+x)}$.

    answer: $-\frac{1}{2}$

    (g) $\displaystyle\lim_{x\to 0}\frac{\arctan x-x}{x^3}$.

    answer: $-\frac{1}{3}$

    (h) $\displaystyle\lim_{x\to 0}\frac{\arcsin x-x}{x^3}$.

    answer: $\frac{1}{6}$

    (i) $\displaystyle\lim_{x\to 1}\frac{x}{x-1}-\frac{1}{\ln x}$.

    answer: $\frac{1}{2}$

    (j) $\displaystyle\lim_{x\to 0}\frac{\cos(nx)-\cos(mx)}{x^2}$.

    answer: $\frac{m^2-n^2}{2}$

(k) $\displaystyle\lim_{x\to 0} \frac{\arcsin x - x - \frac{1}{6}x^3}{\sin^5 x}$.

answer: $\frac{3}{40}$

(l) $\displaystyle\lim_{x\to 1} \frac{\sin(\pi x)\ln x}{\cos(\pi x)+1}$.

answer: $-\frac{2}{\pi}$

(m) $\displaystyle\lim_{x\to 0} \frac{\sin x - x}{\arcsin x - x}$.

answer: -1

(n) $\displaystyle\lim_{x\to 0} \frac{\sin x - x}{\arctan x - x}$.

answer: $\frac{1}{2}$

(o) $\displaystyle\lim_{x\to \infty} x\sin\left(\frac{2}{x}\right)$.

answer: 2

Solution. 38l The limit is of the form "$\frac{0}{0}$" so we are allowed to use L'Hospital's rule.

$$\begin{array}{rcll}
\displaystyle\lim_{x\to 1} \frac{\sin(\pi x)\ln x}{\cos(\pi x)+1} &=& \displaystyle\lim_{x\to 1} \frac{(\sin(\pi x)\ln x)'}{(\cos(\pi x)+1)'} & \\
&=& \displaystyle\lim_{x\to 1} \frac{\left(\pi\cos(\pi x)\ln x + \sin(\pi x)\frac{1}{x}\right)}{(-\pi\sin(\pi x))} & \left| \text{type "}\frac{0}{0}\text{", L'Hospital's rule}\right. \\
&=& \displaystyle\lim_{x\to 1} \frac{\left(\pi\cos(\pi x)\ln x + \sin(\pi x)\frac{1}{x}\right)'}{(-\pi\sin(\pi x))'} & \\
&=& \displaystyle\lim_{x\to 1} \frac{-\pi^2\sin(\pi x)\ln(x) + 2\pi\cos(\pi x)x^{-1} - \sin(\pi x)x^{-2}}{(-\pi^2\cos(\pi x))} & \\
&=& \displaystyle\frac{-\pi^2\sin(\pi)\ln(1) + 2\pi\cos(\pi) - \sin(\pi)}{(-\pi^2\cos(\pi))} & \\
&=& \displaystyle -\frac{2}{\pi} \quad . &
\end{array}$$

Solution. 38n **Solution I.**

$$\begin{array}{rcll}
\displaystyle\lim_{x\to 0} \frac{\sin x - x}{\arctan x - x} &=& \displaystyle\lim_{x\to 0} \frac{\cos x - 1}{\frac{1}{1+x^2} - 1} & \left| \text{L'Hospital rule}\right. \\
&=& \displaystyle\lim_{x\to 0} \frac{-\sin x}{\frac{-2x}{(1+x^2)^2}} & \left| \text{L'Hospital rule again}\right. \\
&=& \displaystyle\lim_{x\to 0} \frac{(1+x^2)^2}{2}\frac{\sin x}{x} & \\
&=& \displaystyle\lim_{x\to 0} \frac{(1+x^2)^2}{2} \lim_{x\to 0}\frac{\sin x}{x} & \\
&=& \displaystyle\frac{1}{2} \quad . &
\end{array}$$

Solution II.

$$\begin{array}{rcll}
\displaystyle\lim_{x\to 0} \frac{\sin x - x}{\arctan x - x} &=& \displaystyle\lim_{x\to 0} \frac{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right) - x}{\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \dots\right) - x} & \left| \text{use the Maclaurin series of } \sin, \arctan\right. \\
&=& \displaystyle\lim_{x\to 0} \frac{-\frac{x^3}{6} + x^5\left(\frac{1}{5!} - \dots\right)}{-\frac{x^3}{3} + x^5\left(\frac{1}{5} - \dots\right)} & \left| \begin{array}{l}\text{The expressions in parenthesis} \\ \quad\text{are continous functions in x}\end{array}\right. \\
&=& \displaystyle\lim_{x\to 0} \frac{-\frac{1}{6} + x^2\left(\frac{1}{5!} - \dots\right)}{-\frac{1}{3} + x^2\left(\frac{1}{5} - \dots\right)} & \\
&=& \displaystyle\frac{-\frac{1}{6} + 0}{\frac{1}{3} + 0} & \\
&=& \displaystyle\frac{1}{2} \quad . &
\end{array}$$

Solution. 38o.

$$\begin{array}{rcll}
\displaystyle\lim_{x\to \infty} x\sin\left(\frac{2}{x}\right) &=& \displaystyle\lim_{x\to \infty} \frac{\sin\left(\frac{2}{x}\right)}{\frac{1}{x}} & \left| \begin{array}{l}\text{indeterminate form} \\ \text{Use L'Hospital's rule}\end{array}\right. \\
&=& \displaystyle\lim_{x\to \infty} \frac{\cos\left(\frac{2}{x}\right)\left(-\frac{2}{x^2}\right)}{-\frac{1}{x^2}} & \\
&=& \displaystyle\lim_{x\to \infty} 2\cos\left(\frac{x}{2}\right) & \\
&=& 2 \quad . &
\end{array}$$

39. Compute the limit.

(a) $\lim\limits_{x\to\infty}\left(\frac{x-2}{x}\right)^{x}$.

answer: e^{-2}

(b) $\lim\limits_{x\to\infty}\left(\frac{x-2}{x}\right)^{2x}$

answer: e^{-4}.

(c) $\lim\limits_{x\to\infty}\left(\frac{x}{x+3}\right)^{2x}$

answer: e^{-6}

Solution. 39.a.

Variant I.

$$\begin{array}{rcll}\lim\limits_{x\to\infty}\left(\frac{x-2}{x}\right)^{x} &=& \lim\limits_{x\to\infty}\left(1-\frac{2}{x}\right)^{x} & \left|\ \text{use } \lim\limits_{x\to\infty}\left(1+\frac{k}{x}\right)^{x}=e^{k}\right.\\ &=& e^{-2}\quad . & \end{array}$$

Variant II.

$$\begin{array}{rcll}\lim\limits_{x\to\infty}\left(\frac{x-2}{x}\right)^{x} &=& \lim\limits_{x\to\infty} e^{\ln\left(\left(\frac{x-2}{x}\right)^{x}\right)} & \\ \lim\limits_{x\to\infty}\ln\left(\left(\frac{x-2}{x}\right)^{x}\right) &=& \lim\limits_{x\to\infty} x\left(\ln(x-2)-\ln(x)\right) & \\ &=& \lim\limits_{x\to\infty}\frac{\ln(x-2)-\ln(x)}{\frac{1}{x}} & \left|\ \text{L'Hospital rule}\right.\\ &=& \lim\limits_{x\to\infty}\frac{\frac{1}{x-2}-\frac{1}{x}}{-\frac{1}{x^2}} & \\ &=& \lim\limits_{x\to\infty}\frac{-2x^2}{x^2-2x}=-2 & \left|\ \text{Therefore}\right.\\ \lim\limits_{x\to\infty}\left(\frac{x-2}{x}\right)^{x} &=& \lim\limits_{x\to\infty} e^{\ln\left(\left(\frac{x-2}{x}\right)^{x}\right)} & \\ &=& e^{\lim\limits_{x\to\infty}\ln\left(\left(\frac{x-2}{x}\right)^{x}\right)} & \\ &=& e^{-2}\quad . & \end{array}$$

40. Find the limit.

(a) $\lim\limits_{x\to\infty}\left(1-\frac{2}{x}\right)^{x}$.

answer: e^{-2}

(b) $\lim\limits_{x\to 0}(1-x)^{\frac{1}{x}}$.

answer: e^{-1}

(c) $\lim\limits_{x\to\infty}\left(\frac{x}{x-5}\right)^{x}$.

answer: e^{5}

(d) $\lim\limits_{x\to\infty}\left(\frac{x}{x-2}\right)^{3x+2}$.

answer: e^{6}

41. Determine whether the integral is convergent or divergent. Motivate your answer.

(a) $\displaystyle\int_{2}^{\infty}\frac{1}{(x-1)^{\frac{3}{2}}}\mathrm{d}x$.

answer: convergent

(b) $\displaystyle\int_{-1}^{1}\frac{1}{\sqrt[5]{1+x}}\mathrm{d}x$.

answer: convergent

(c) $\displaystyle\int_{1}^{\infty}\frac{1}{\sqrt[5]{1+x}}\mathrm{d}x$.

answer: divergent

(d) $\displaystyle\int_{-1}^{\infty}\frac{1}{\sqrt[5]{1+x}}\mathrm{d}x$.

answer: divergent

(e) $\displaystyle\int_{-\infty}^{0}\frac{1}{2-3x}\mathrm{d}x$.

answer: divergent

(f) $\displaystyle\int_{-\infty}^{0}\frac{1}{(2-3x)^2}\mathrm{d}x$.

answer: convergent

(g) $\displaystyle\int_{-\infty}^{0}\frac{1}{(2-3x)^{1.00000001}}\mathrm{d}x$.

answer: convergent

(h) $\displaystyle\int_{-2}^{\frac{1}{2}} \frac{1}{2x-1}\mathrm{d}x.$

answer: divergent

(i) $\displaystyle\int_{-1}^{\infty} e^{-3x}\mathrm{d}x.$

answer: convergent, equals $\frac{e^3}{3}$

(j) $\displaystyle\int_{-\infty}^{5} 2^x\mathrm{d}x.$

answer: convergent

(k) $\displaystyle\int_{-\infty}^{\infty} x^3\mathrm{d}x.$

answer: divergent

(l) $\displaystyle\int_{-\infty}^{\infty} xe^{-x^2}\mathrm{d}x.$

answer: convergent, equals 0

(m) $\displaystyle\int_{0}^{\infty} \sqrt{x}e^{-\sqrt{x}}\mathrm{d}x.$

answer: convergent, equals 4

(n) $\displaystyle\int_{0}^{\infty} \sin^2 x\mathrm{d}x.$

answer: divergent

(o) $\displaystyle\int_{0}^{5} \frac{1}{x^2+x-2}\mathrm{d}x.$

answer: divergent

(p) $\displaystyle\int_{0}^{\infty} \frac{1}{x^2+x+1}\mathrm{d}x.$

answer: convergent

(q) $\displaystyle\int_{2}^{\infty} \frac{1}{x^2-x-1}\mathrm{d}x.$

answer: convergent

(r) $\displaystyle\int_{0}^{\infty} \frac{1}{x^2-x-1}\mathrm{d}x.$

answer: divergent

(s) $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{x^4+2}\mathrm{d}x.$

answer: convergent

(t) $\displaystyle\int_{100}^{\infty} \frac{1}{x\ln x}\mathrm{d}x.$

answer: divergent

(u) $\displaystyle\int_{100}^{\infty} \frac{1}{x(\ln x)^2}\mathrm{d}x.$

answer: convergent

(v) $\displaystyle\int_{0}^{1} \ln x\mathrm{d}x.$

answer: convergent

(w) $\displaystyle\int_{0}^{1} \frac{\ln x}{\sqrt{x}}\mathrm{d}x.$

answer: convergent

(x) $\displaystyle\int_{0}^{2} x^3\ln x\mathrm{d}x.$

answer: convergent, equals $-1+4\ln 2$

(y) $\displaystyle\int_{0}^{1} \frac{e^{\frac{1}{x}}}{x^2}\mathrm{d}x.$

answer: divergent

(z) $\displaystyle\int_{-1}^{0} \frac{e^{\frac{1}{x}}}{x^2}\mathrm{d}x.$

answer: convergent

Solution. 41.m It is possible to show that this integral is convergent by using the comparison theorem. However, we shall use direct integration instead. First, we solve the indefinite integral:

$$
\begin{array}{rcll}
\displaystyle\int \sqrt{x}e^{-\sqrt{x}}\mathrm{d}x &=& \displaystyle\int \sqrt{x}e^{-\sqrt{x}}\frac{2\sqrt{x}\mathrm{d}x}{2\sqrt{x}} & \Big| \text{ use } \mathrm{d}\sqrt{x} = \frac{\mathrm{d}x}{2\sqrt{x}} \\
&=& \displaystyle\int \sqrt{x}e^{-\sqrt{x}}\left(2\sqrt{x}\mathrm{d}\sqrt{x}\right) & \Big| \text{ Set } \sqrt{x} = u \\
&=& \displaystyle 2\int u^2 e^{-u}\mathrm{d}u & \\
&=& \displaystyle 2\left(-\int u^2 \mathrm{d}\left(e^{-u}\right)\right) & \Big| \text{ integrate by parts} \\
&=& \displaystyle 2\left(-u^2e^{-u} + \int e^{-u}\mathrm{d}\left(u^2\right)\right) & \\
&=& \displaystyle 2\left(-u^2e^{-u} + \int 2ue^{-u}\mathrm{d}u\right) & \\
&=& \displaystyle 2\left(-u^2e^{-u} - \int 2u\mathrm{d}e^{-u}\right) & \Big| \text{ integrate by parts again} \\
&=& \displaystyle 2\left(-u^2e^{-u} - 2ue^{-u} + \int 2e^{-u}\mathrm{d}u\right) & \\
&=& 2\left(-u^2e^{-u} - 2ue^{-u} - 2e^{-u}\right) + C & \\
&=& 2\left(-xe^{-\sqrt{x}} - 2\sqrt{x}e^{-\sqrt{x}} - 2e^{-\sqrt{x}}\right) + C &
\end{array}
$$

Therefore

$$
\begin{array}{rcll}
\displaystyle\int\limits_0^\infty \sqrt{x}e^{-\sqrt{x}}\mathrm{d}x &=& \displaystyle\lim_{t\to\infty} 2\left[-xe^{-\sqrt{x}} - 2\sqrt{x}e^{-\sqrt{x}} - 2e^{-\sqrt{x}}\right]_0^\infty & \\
&=& \displaystyle 4 + \lim_{t\to\infty} 4\left(-te^{-\sqrt{t}} - \sqrt{t}e^{-\sqrt{t}} - e^{-\sqrt{t}}\right) & \Big| \text{ Set } u = \sqrt{t} \\
&=& \displaystyle 4 - 4\lim_{u\to\infty}\left(u^2e^{-u} + ue^{-u} + e^{-u}\right) & \\
&=& \displaystyle 4 - 4\lim_{u\to\infty}\frac{u^2+u+1}{e^u} & \Big| \text{ use L'Hospital's rule for limit, see below} \\
&=& 4 \quad , &
\end{array}
$$

and the integral converges to 4. In the above computation we used the following limit computation

$$
\begin{array}{rcll}
\displaystyle\lim_{u\to\infty}\frac{u^2+u+1}{e^u} &=& \displaystyle\lim_{u\to\infty}\frac{2u+1}{e^u} & \Big| \text{ Apply L'Hospital's rule} \\
&=& \displaystyle\lim_{u\to\infty}\frac{2}{e^u} & \\
&=& 0 \quad . &
\end{array}
$$

Solution. 41.s The integrand is a rational function and therefore we can solve this problem by finding the indefinite integral and then computing the limit. We would need to start by factoring x^4+2 into irreducible quadratic factors - that is already quite laborious:

$$
x^4 + 2 = \left(x^2 + \sqrt[4]{8}x + \sqrt{2}\right)\left(x^2 - \sqrt[4]{8}x + \sqrt{2}\right) \quad .
$$

The problem asks us only to establish the convergence of the integral; it does not ask us to compute its actual numerical value. Therefore we can give a much simpler solution. The function is even and therefore it suffices to establish whether $\displaystyle\int\limits_0^\infty \frac{x^2}{x^4+2}\mathrm{d}x$ is convergent.

We have that

$$
\int\limits_0^\infty \frac{x^2}{x^4+2}\mathrm{d}x = \int\limits_0^1 \frac{x^2}{x^4+2}\mathrm{d}x + \int\limits_1^\infty \frac{x^2}{x^4+2}\mathrm{d}x \quad .
$$

The function $\frac{x^2}{x^4+2}$ is continuous so $\int\limits_0^1 \frac{x^2}{x^4+2}\mathrm{d}x$ integrates to a number, which does not affect the convergence of the above expression. Therefore the convergence of our integral is governed by the convergence of $\int\limits_1^\infty \frac{x^2}{x^4+2}\mathrm{d}x$. To establish that that integral is convergent, we use the comparison theorem as follows.

$$\begin{aligned}
\int\limits_1^\infty \frac{x^2}{x^4+2}\mathrm{d}x &\leq \int\limits_1^\infty \frac{x^2}{x^4}\mathrm{d}x \\
&= \int_1^\infty x^{-2}\mathrm{d}x \\
&= \lim_{t\to\infty}\left[-\frac{1}{x}\right]_1^t \\
&= \lim_{t\to\infty} 1-\frac{1}{t} \\
&= 1 \quad .
\end{aligned}$$

we have that $x^4+2 > x^4$ and therefore $\dfrac{x^2}{x^4+2} \leq \dfrac{x^2}{x^4}$

In this way we showed $\displaystyle\int_1^\infty \frac{x^2}{x^4+2}\mathrm{d}x \leq 1$. Therefore, as $\dfrac{x^2}{x^4+2} \geq 0$ is positive, we can apply the comparison theorem to get that $\displaystyle\int_1^\infty \frac{x^2}{x^4+2}\mathrm{d}x$ is convergent.

42. Determine whether the integral is convergent or divergent. Motivate your answer. The answer key has not been proofread, use with caution.

    (a) $\displaystyle\int\limits_0^\infty \sin x^2 \mathrm{d}x$ (This problem is more difficult and may require knowledge of sequences to solve).

    answer: convergent