Calculus I
Lecture 9
Product and Quotient Rule

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https://github.com/tmilev/freecalc

2020

Outline

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Theorem (The Power Rule (General Version))

If n is any real number, then

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^n) = nx^{n-1}.$$

Example (Power Rule, negative exponent)

$$\text{Differentiate} \quad y = \frac{1}{x}.$$

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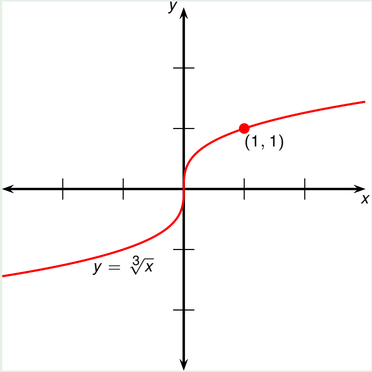
$$= -\frac{1}{x^2}.$$

Example (Calculating the tangent line using the Power Rule)

Find an equation for the tangent line to the cubic $y = \sqrt[3]{x}$ at the point $P = (1, 1)$.



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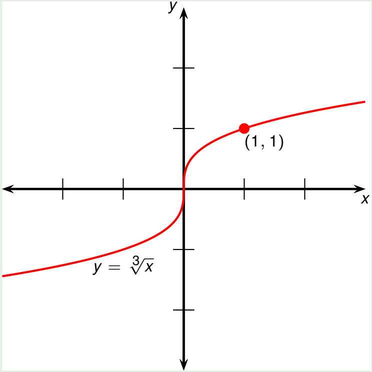

Here $a = 1$ and $f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$.

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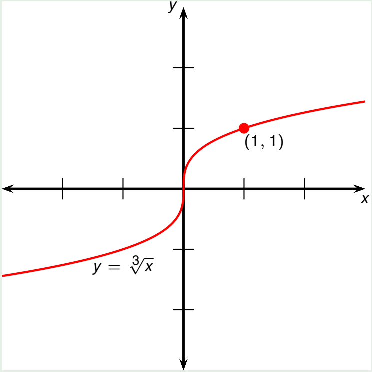

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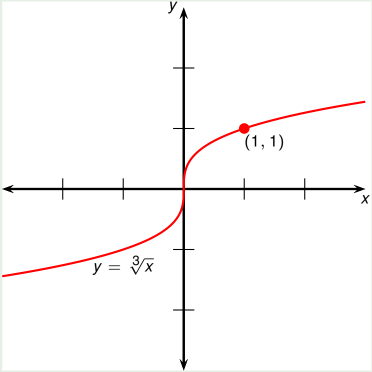

Here $a = 1$ and $f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$.

$$\begin{aligned} f'(x) &= \frac{1}{3}x^{\frac{1}{3}-1} \\ &= \frac{1}{3}x^{\frac{-2}{3}} \end{aligned}$$

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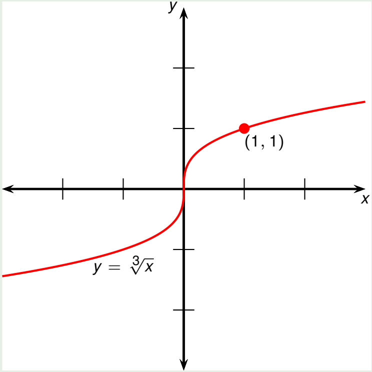

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$$
\begin{aligned}
f'(x) &= \frac{1}{3}x^{\frac{1}{3}-1} \\
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\end{aligned}
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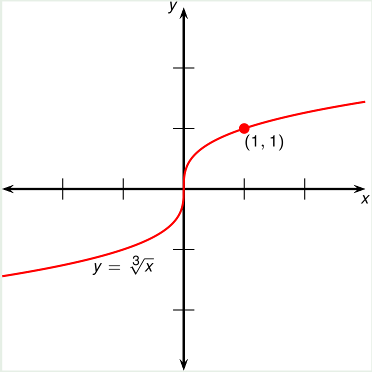

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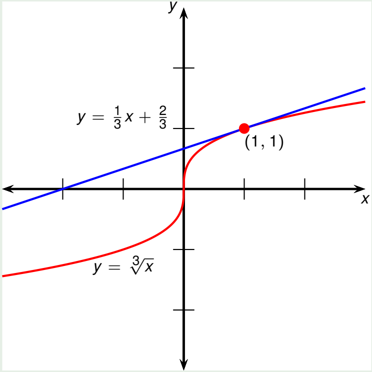

Here $a = 1$ and $f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$.

$$\begin{aligned} f'(x) &= \frac{1}{3}x^{\frac{1}{3}-1} \\ &= \frac{1}{3}x^{\frac{-2}{3}} \\ &= \frac{1}{3\sqrt[3]{x^2}}. \\ f'(1) &= \frac{1}{3}. \end{aligned}$$

Point-slope form: $y - 1 = \frac{1}{3}(x - 1)$, or $y = \frac{1}{3}x + \frac{2}{3}$.

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$$= \mathbf{?}$$

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Power Rule: $$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{5x^{-\frac{1}{6}}}{6} = \frac{5}{6\sqrt[6]{x}}.$$

Theorem (The Constant Multiple Rule)

If c is a constant and f is a differentiable function, then

$$\frac{\mathrm{d}}{\mathrm{d}x}[cf(x)] = c\frac{\mathrm{d}}{\mathrm{d}x}f(x).$$

Proof.

$\square$

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$$\text{Let} \quad g(x) = cf(x).$$

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$$\begin{aligned} \text{Let} \quad & g(x) = cf(x). \\ \text{Then} \quad & g'(x) = \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} \end{aligned}$$

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\text{Then} \quad g'(x) &= \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} \\
&= \lim_{h\to 0}\frac{{\color{red}c}f(x+h)-{\color{red}c}f(x)}{h} \\
&= \lim_{h\to 0}\frac{{\color{red}c}(f(x+h)-f(x))}{h}
\end{aligned}$$

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&= \lim_{h\to 0} \frac{cf(x+h)-cf(x)}{h} \\
&= \lim_{h\to 0} \frac{c(f(x+h)-f(x))}{h} \\
\text{Limit Law 3:} \quad &= c\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}
\end{aligned}$$

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\text{Limit Law 3:} \quad &= c\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
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\text{Limit Law 3:} \quad &= c\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\
&= c f'(x).
\end{aligned}$$

$\square$

Example (Constant Multiple Rule, Power Rule)

Find the derivative of $y = \dfrac{2x^5}{7}$.

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Constant Multiple Rule: $= \left(\frac{2}{7}\right)\frac{\mathrm{d}}{\mathrm{d}x}\left(x^5\right)$

$$= \left(\frac{2}{7}\right)\left(5x^4\right)$$

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Constant Multiple Rule: $= \left(\dfrac{2}{7}\right)\dfrac{d}{dx}\left(x^5\right)$

$$= \left(\frac{2}{7}\right)\left(5x^4\right)$$

$$= \frac{10x^4}{7}.$$

Example (Constant Multiple Rule, Power Rule)

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$$= (-1)(\mathbf{?})$$

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$$= (-1)(1)$$

Example (Constant Multiple Rule, Power Rule)

$$
\begin{aligned}
\text{Find the derivative of} \quad u &= -x. \\
u &= (-1)(x). \\
\frac{\mathrm{d}u}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}\left[(-1)(x)\right] \\
\text{Constant Multiple Rule:} \quad &= (-1)\frac{\mathrm{d}}{\mathrm{d}x}(x) \\
&= (-1)(1) \\
&= -1.
\end{aligned}
$$

Example (Constant Multiple Rule, Power Rule, Negative Exponent)

Find the derivative of $t = \dfrac{2\pi}{x^4}$.

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$$= (2\pi)\left(\textbf{?} \qquad \right)$$

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$$= (2\pi)\left(-4x^{-5}\right)$$

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\text{Constant Multiple Rule:} \quad &= (2\pi)\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{-4}\right) \\
&= (2\pi)\left(-4x^{-5}\right) \\
&= -\frac{8\pi}{x^5}.
\end{aligned}
$$

Theorem (The Sum Rule)

If f and g are both differentiable, then

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(x)+g(x)] = \frac{\mathrm{d}}{\mathrm{d}x}f(x) + \frac{\mathrm{d}}{\mathrm{d}x}g(x).$$

Proof.

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& = \lim_{h\to 0}\frac{[f(x+h)+g(x+h)]-[f(x)+g(x)]}{h}
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\text{Let}\quad F(x) &= f(x)+g(x).\\
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&= \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}+\frac{g(x+h)-g(x)}{h}\right]
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&= \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}+\frac{g(x+h)-g(x)}{h}\right]\\
\text{Limit Law 1:}\quad &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}+\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}
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\text{Limit Law 1:}\quad &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} + \lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\\
&= f'(x)+g'(x).
\end{aligned}$$

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\text{Limit Law 1:}\quad &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}+\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\\
&= f'(x)+g'(x).
\end{aligned}$$

$\square$

The Sum Rule can be extended to any number of summands. For instance, using the theorem twice, we get

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$$(f+g+h)' = [(f+g)+h]' = (f+g)' + h' = f' + g' + h'.$$

By writing $f-g$ as $f+(-1)g$ and applying the Sum Rule and the Constant Multiple Rule, we get

Theorem (The Difference Rule)

If f and g are both differentiable, then

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(x)-g(x)] = \frac{\mathrm{d}}{\mathrm{d}x}f(x) - \frac{\mathrm{d}}{\mathrm{d}x}g(x).$$

The Constant Multiple Rule, the Sum Rule, the Difference Rule, and the Power Rule can be combined to differentiate any polynomial.

Example (Derivative of a Polynomial)

$$\text{If} \quad y = x^{16} + 2\sqrt{3}x^7 - 4x^3 + \frac{x}{8} - 5,$$

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&= 16x^{15} + 14\sqrt{3}x^6 - 12x^2 + \frac{1}{8}.
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$$\begin{aligned} f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h\to 0} \frac{a^{x+h} - a^x}{h} \\ &= \lim_{h\to 0} \frac{a^x a^h - a^x}{h} \\ &= \lim_{h\to 0} \frac{{\color{red}a^x}\left(a^h - 1\right)}{h} \\ &= {\color{red}a^x} \lim_{h\to 0} \frac{a^h - 1}{h} \\ &= {\color{red}a^x} \lim_{h\to 0} \frac{a^h - a^0}{h} \end{aligned}$$

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&= a^x f'(0).
\end{aligned}
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We have shown that, if $f(x) = a^x$ is differentiable at 0, then it is differentiable everywhere, and

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exists.

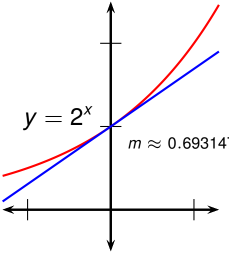
We will later show that

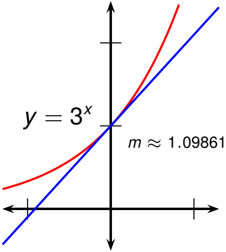
$$f'(0) = \lim_{h \to 0} \frac{a^h - 1}{h} = \ln(a).$$

Here, $\ln$ is the natural logarithm function.

$$\text{If} \quad f(x) = a^x, \quad \text{then} \quad f'(x) = f'(0)a^x.$$

The formula above is simplest when $f'(0) = 1$. Since $\lim\limits_{h\to 0} \frac{2^h - 1}{h} \approx 0.69$ and $\lim\limits_{h\to 0} \frac{3^h-1}{h} \approx 1.10$, we expect there is a number a between 2 and 3 such that $\lim\limits_{h\to 0} \frac{a^h-1}{h} = 1$.





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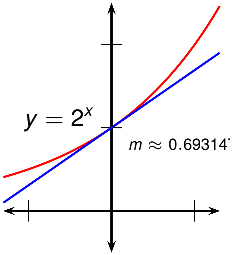
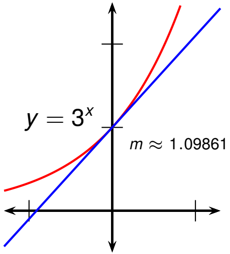
Definition (e)

e is the number such that $\lim_{h\to 0} \frac{e^h-1}{h} = 1$.





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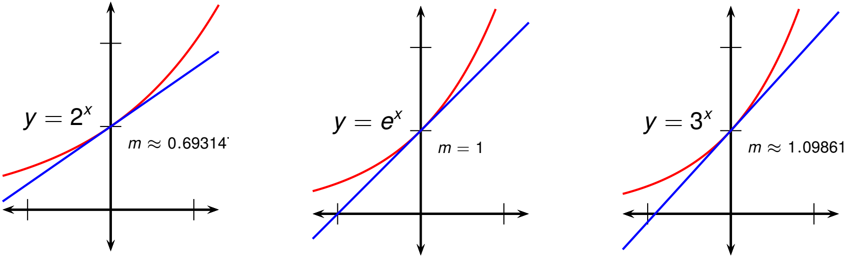

Definition (Natural Exponential Function)

e^x is called the natural exponential function. Its derivative is

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(e^x\right) = e^x.$$

Example (Derivative of a Polynomial and the Natural Exponential Function)

Differentiate $y = e^x + x^7.$

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We need a formula for the derivative of the product of two functions. One might guess that the derivative of a product is the product of the derivatives; however, this is wrong.

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The correct formula is called the Product Rule.

Theorem (The Product Rule)

If f and g are both differentiable, then

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x).$$

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&= e^x\left(x^3 + 3x^2\right).
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If f and g are differentiable and $g(x) \neq 0$, then

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- The proof of the Quotient Rule is similar to the proof of the Product Rule.
- There is an alternative algebraic proof via the Product Rule, the Power Rule and the (not yet studied) Chain Rule.

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&= \frac{\left(-5x^{10} - 2x^6 + 10x^4 + 4\right) - \left(-6x^{10} - 12x^6\right)}{(-x^6 + 2)^2} \\
&= \frac{x^{10} + 10x^6 + 10x^4 + 4}{(-x^6 + 2)^2}.
\end{aligned}$$