Calculus III
Lecture 10

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Outline

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Multivariable Chain Rule Motivation

Recall:

- f, differentiable function,
- $\mathbf{u} = (u_1, u_2, u_3)$, unit vector,
- $P(x_0, y_0, z_0)$, point.

What is the rate of change of f at P in the direction $\mathbf{u}$?
Directional derivative

$$(D_{\mathbf{u}} f)(P) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(x_0 + tu_1, y_0 + tu_2, z_0 + tu_3)$$

More general, if

- $w = w(x, y, z)$;
- $x = x(t)$, $y = y(t)$, $z = z(t)$,

and all the functions are differentiable, how do we compute $\frac{\mathrm{d}w}{\mathrm{d}t}$?

Chain Rule

Differentials

$$\mathrm{d}w = w_x(x,y,z)\mathrm{d}x + w_y(x,y,z)\mathrm{d}y + w_z(x,y,z)\mathrm{d}z$$

and

$$\mathrm{d}x = x'(t)\mathrm{d}t \qquad \mathrm{d}y = y'(t)\mathrm{d}t \qquad \mathrm{d}z = z'(t)\mathrm{d}t\ .$$

Then

$$\mathrm{d}\left(w\right) = \left(w_x x'(t) + w_y y'(t) + w_z z'(t)\right)\mathrm{d}t$$

Therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}(w(x(t),y(t),z(t))) = \frac{\partial w}{\partial x}(x,y,z)\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial w}{\partial y}(x,y,z)\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial w}{\partial z}(x,y,z)\frac{\mathrm{d}z}{\mathrm{d}t}$$

Derivative of composition of functions $\Longrightarrow$ Chain Rule

Algebra of Chain rule - Tree Diagrams

- $w = w(x, y, z)$;
- $x = x(t)$, $y = y(t)$, $z = z(t)$,

$$\frac{dw}{dt}(t) = \frac{\partial w}{\partial x}(x, y, z)\frac{dx}{dt}(t) + \frac{\partial w}{\partial y}(x, y, z)\frac{dy}{dt}(t) + \frac{\partial w}{\partial z}(x, y, z)\frac{dz}{dt}(t)$$

Alternative way of arranging terms - tree diagram:

$$\begin{tikzcd}
 & w & \\
x \arrow[ur] & y \arrow[u] & z \arrow[ul] \\
t \arrow[u] & t \arrow[u] & t \arrow[u]
\end{tikzcd}$$

More General Chain Rule

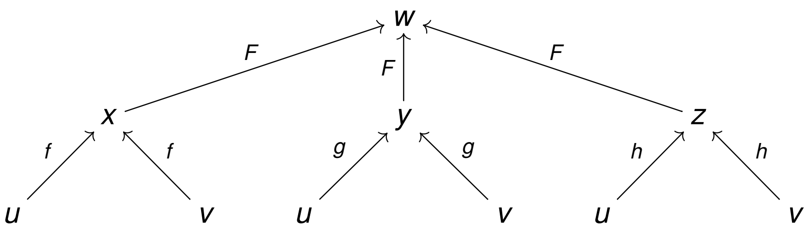
More general formula:

- $w = F(x, y, z)$;
- $x = f(u, v)$, $y = g(u, v)$, $z = h(u, v)$.

$$w = F(f(u, v), g(u, v), h(u, v)) = G(u, v)$$

To compute $\frac{\partial w}{\partial u} = \frac{\partial G}{\partial u}$:

- arrange variables in a tree diagram:



Example: powerexponential

Let $f(x) = x^x$. Compute $f'(x)$.

- Calculus I method: logarithmic differentiation or $x^x = e^{x \ln x}$.
- Calculus III method: chain rule.

Let $w = w(u, v) = u^v$ and $u = u(x) = x$, $v = v(x) = x$.

Then $f(x) = w(u(x), v(x))$ and

Directional Derivatives via the Chain Rule

- Let f differentiable function,
- Let $\mathbf{u} = (u_1, u_2, u_3)$, unit vector,
- Let $P(x_0, y_0, z_0)$, point.

What is the rate of change of f at P in the direction $\mathbf{u}$? Answer was studied: directional derivative.

$$(D_{\mathbf{u}} f)_{(x,y,z)=(x_0,y_0,z_0)} = \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} f(x_0 + tu_1, y_0 + tu_2, z_0 + tu_3)$$

$$(D_{\mathbf{u}}f)(x_0, y_0, z_0) = \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} f(x_0 + tu_1, y_0 + tu_2, z_0 + tu_3)$$

Let $w = f(x, y, z)$ and $\left|\begin{array}{rcl} x &=& x_0 + tu_1 \\ y &=& y_0 + tu_2 \\ z &=& z_0 + tu_3 \end{array}\right.$.

$$\begin{aligned}
(D_{\mathbf{u}}f)(x_0, y_0, z_0) &= \frac{\mathrm{d}w}{\mathrm{d}t}_{|t=0} \\
&= \left(\frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial f}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t}\right)_{|t=0} \\
&= \frac{\partial f}{\partial x}u_1 + \frac{\partial f}{\partial y}u_2 + \frac{\partial f}{\partial z}u_3 \\
&= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \cdot \mathbf{u} = \nabla f \cdot \mathbf{u}
\end{aligned}$$

Definition (∇f (“nabla of f”))

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right).$$

Example

Find the directional derivative of $f(x, y, z) = \ln\left(x^2 + 2y^2 - z^2\right)$ at $P(2, 1, -1)$ in the direction $\mathbf{v} = (-1, 2, 1)$.
A unit vector in the direction of $\mathbf{v}$ is

$$\mathbf{u} = \frac{1}{|\mathbf{v}|}\mathbf{v} = \frac{1}{\sqrt{6}}(-1, 2, 1)\ .$$

The partial derivatives are

$$\begin{array}{rclrcl}
\frac{\partial f}{\partial x} & = & \frac{2x}{x^2+2y^2-z^2} & \frac{\partial f}{\partial x}(2,1,-1) & = & \frac{4}{5} \\
\frac{\partial f}{\partial y} & = & \frac{4y}{x^2+2y^2-z^2} & \frac{\partial f}{\partial y}(2,1,-1) & = & \frac{4}{5} \\
\frac{\partial f}{\partial z} & = & \frac{-2z}{x^2+2y^2-z^2} & \frac{\partial f}{\partial z}(2,1,-1) & = & \frac{2}{5}
\end{array}$$

$$\nabla f_{|(x,y,z)=(2,1,-1)} = \left(\frac{4}{5}, \frac{4}{5}, \frac{2}{5}\right)$$

Example

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$$\mathbf{u} = \frac{1}{|\mathbf{v}|}\mathbf{v} = \frac{1}{\sqrt{6}}(-1, 2, 1) .$$

$$\nabla f_{|(x,y,z)=(2,1,-1)} = (f_x, f_y, f_z)_{|(x,y,z)=(2,1,-1)} = \left(\frac{4}{5}, \frac{4}{5}, \frac{2}{5}\right)$$

$$(D_{\mathbf{u}} f)_{|(x,y,z)=(2,1,-1)} = \nabla f_{|(x,y,z)=(2,1,-1)} \cdot \mathbf{u} = \frac{\sqrt{6}}{5}$$

$(D_{\mathbf{u}} f)(2, 1, -1) > 0$ implies that if we start at $(2, 1, -1)$ and move in the direction $\mathbf{u}$, then f is increasing.

Gradient

- Let f be a differentiable function.
- At a given point P, in which direction does f increase the fastest?
- What is that maximal rate of increase?
- It can be shown that if the maximal rate of increase is strictly positive, then it is achieved in exactly one direction.

Definition

The *gradient vector* of f at P is the unique vector that has

- magnitude equal to the maximal rate of increase of f at P.
- if the magnitude is not zero, then the direction is the one in which f increases the fastest.

Definition

The *gradient vector* of f at P is the unique vector such that:

- its magnitude equals the maximal rate of increase of f at P;
- if magn. $\neq 0$, its direction is the one in which f increases fastest.

- Recall that $\nabla f = (f_x, f_y, f_z)$.
- The increase of f in unit direction $\mathbf{u}$ is $D_{\mathbf{u}} f$. We have:
$$(D_{\mathbf{u}} f) = (\nabla f) \cdot \mathbf{u} = |\nabla f| \cdot |\mathbf{u}| \cos\alpha = |\nabla f| \cos\alpha \ ,$$
where α is the angle between ∇f and $\mathbf{u}$.
- If $|\nabla f| \neq 0$, then $(D_{\mathbf{u}} f)$ is maximal when $\cos\alpha = 1$, i.e., $\alpha = 0$.
- Therefore the maximum of $D_{\mathbf{u}} f$ is achieved for $\mathbf{u} = \frac{\nabla f}{|\nabla f|}$.
- The maximum of $D_{\mathbf{u}} f$ is then $|\nabla f| = |(f_x, f_y, f_z)|$.

Theorem (Coordinate Computation of gradient vector)

The gradient vector of f equals $\nabla f = (f_x, f_y, f_z)$.

- In view of preceding thm., the gradient of f is denoted by ∇f.

Covariant Derivative

- f: a differentiable function.
- Directional derivative $D_{\mathbf{u}}f$= rate of change along straight line.
- Let γ: a smooth parametric curve.
- Question: How does f change as we move along γ?

Definition

The rate of change of $f(\gamma(t))$ with respect to t is called the *covariant derivative* of f along γ and is denoted by $\nabla_{\gamma'} f$.

We can compute the covariant derivative using the chain rule:

$$(\nabla_{\gamma'(t_0)} f)(\gamma(t_0)) = \frac{\mathrm{d}}{\mathrm{d}t}_{|t=t_0} f(\gamma(t)) = (\nabla f)_{\gamma(t_0)} \cdot \gamma'(t_0) \ .$$

If $\mathbf{u}$ is a unit vector, $\gamma(t_0) = P$ and $\gamma'(t_0) = \mathbf{u}$, then:

$$(D_{\mathbf{u}} f)(P) = (\nabla f)_P \cdot \mathbf{u} = (\nabla f)_{\gamma(t_0)} \cdot \gamma'(t_0) = \frac{\mathrm{d}}{\mathrm{d}t}_{t=t_0} f(\gamma(t)) \ .$$

Gradient in Polar Coordinates

$\mathbf{e}_r = \mathbf{e}_r(P)$ and $\mathbf{e}_\theta = \mathbf{e}_\theta(P)$ are the polar fundamental directions at P

$$(\nabla f)_P = a\mathbf{e}_r + b\mathbf{e}_\theta$$

$\mathbf{e}_r$ and $\mathbf{e}_\theta$ perpendicular unit vectors $\Longrightarrow$

$$\begin{aligned} a =& (\nabla f)_P \cdot \mathbf{e}_r = (D_{\mathbf{e}_r} f)(P) \\ b =& (\nabla f)_P \cdot \mathbf{e}_\theta = (D_{\mathbf{e}_\theta} f)(P) \end{aligned}$$

To compute $(D_{\mathbf{e}_r} f)(P)$ we use the line through $P(r_0, \theta_0)$ with direction $\mathbf{e}_r$, which in polar coordinates is given by $(r, \theta) = (t, \theta_0)$. Therefore

$$a = (D_{\mathbf{e}_r} f)(P) = \left.\frac{d}{dt}\right|_{t=r_0} f(t, \theta_0) = \frac{\partial f}{\partial r}(P) .$$

To compute $(D_{\mathbf{e}_\theta} f)(P)$ we use the circle centered at the origin and passing through $P(r_0, \theta_0)$. The polar parametrization of this circle that has *unit*

Application

Let f be a function on the plane such that f depends only on the distance to a fixed point, O.
In a polar coordinate system with origin at O we get $f(P) = g(r)$

$$\nabla = \mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} .$$

$$\nabla f = g'(r)\mathbf{e}_r = g'(r)\,\widehat{\mathbf{r}} = \frac{g'(r)}{r}\,\mathbf{r} .$$

Example: $f(P) = |OP|^{-1} = r^{-1} = g(r)$. Then

$$\nabla f = g'(r)\mathbf{e}_r = -r^{-2}\mathbf{e}_r = -\frac{1}{r^3}\,\mathbf{r}$$

Problem: Let $\mathbf{X}$ be a vector field of the form

$$\mathbf{X} = h(r)\mathbf{r}$$

for some continuous function h. Show that $\mathbf{X}$ is a *gradient field*: there exists a smooth function f such that $\mathbf{X} = \nabla f$.

Gravity and Gradient

- Let an object move along surface $z = f(x, y)$.
- Let gravity $\mathbf{G}$ be constant, $\mathbf{G} = -mg\,\mathbf{k}$.
- Normal to surface:

$$\mathbf{n} = (-f_x(x_0, y_0), -f_y(x_0, y_0), 1) = -\nabla f + \mathbf{k}$$

- Let $\mathbf{F}$ be the component of $\mathbf{G}$ effectively acting on the object. Object is restricted to the surface $\Rightarrow$ $\mathbf{F}$ is the component of $\mathbf{G}$ tangent to the surface.

$$\begin{array}{rcl} \mathbf{F} & = & \mathbf{orth}_{\boldsymbol{n}}\mathbf{G} = -mg\,\mathbf{orth}_{\boldsymbol{n}}\mathbf{k} \\ \mathbf{orth}_{\boldsymbol{n}}\mathbf{k} & = & \mathbf{k} - \mathbf{proj}_{\boldsymbol{n}}\mathbf{k} = \mathbf{k} - \frac{\mathbf{k}\cdot\mathbf{n}}{|\mathbf{n}|^2}\,\mathbf{n} = \mathbf{k} - \frac{1}{|\mathbf{n}|^2}(-\nabla f + \mathbf{k}) \end{array}$$

Horizontal component of $\mathbf{F}$:

$$\frac{mg}{1 + |\nabla f|^2}(-\nabla f)$$

Gravity pulls object in the direction of fastest descent.

Differential operators definition

- Let D be an open set in the plane.
- Let $\mathcal{C}^\infty(D)$ denote the set of infinitely differentiable f-ns over D.

Definition

The two-variable differential operators $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are the maps from $\mathcal{C}^\infty(D)$ to $\mathcal{C}^\infty(D)$ given by: $\frac{\partial}{\partial x}(f) = \frac{\partial f}{\partial x}$ and $\frac{\partial}{\partial y}(f) = \frac{\partial f}{\partial y}$ for every function $f \in \mathcal{C}^\infty(D)$.

- The operator $\frac{\partial^n}{\partial x^n}\frac{\partial^m}{\partial y^m}$ is defined via

$$\frac{\partial^n}{\partial x^n}\frac{\partial^m}{\partial y^m}(f) = \underbrace{\frac{\partial}{\partial x}\dots\frac{\partial}{\partial x}}_{n \text{ times}}\underbrace{\frac{\partial}{\partial y}\dots\frac{\partial}{\partial y}}_{m \text{ times}}(f)$$

Definition (Smooth finite order differential operators)

A differential operator over D is a map from $\mathcal{C}^\infty(D)$ to $\mathcal{C}^\infty(D)$ obtained by sums of operators $\frac{\partial^n}{\partial x^n}\frac{\partial^m}{\partial y^m}$ with coefficients in $\mathcal{C}^\infty(D)$.

Differential operator notation

Definition (Smooth finite order differential operators)

A differential operator over D is a map from $\mathcal{C}^\infty(D)$ to $\mathcal{C}^\infty(D)$ obtained by sums of operators $\frac{\partial^n}{\partial x^n}\frac{\partial^m}{\partial y^m}$ with coefficients in $\mathcal{C}^\infty(D)$.

- For $n = 0, m = 0$, the operator $\frac{\partial^n}{\partial x^n}\frac{\partial^m}{\partial y^m}$ is by definition equal to 1.
- A function g in C^∞ gives rise to a differential operator via multiplication: $(g \cdot f)(x) = (gf)(x) = g(x)f(x)$.
- Functions are by definition zero-order differential operators.
- The number $m + n$ is defined to be the order of the differential operator $\frac{\partial^n}{\partial x^n}\frac{\partial^m}{\partial y^m}$.
- The order of a differential operator ξ is the largest order of the differential operators appearing in the expression of ξ via $\frac{\partial^n}{\partial x^n}\frac{\partial^m}{\partial y^m}$.
- Analogous definitions exist for functions in n variables.

Recall that $\frac{\partial z}{\partial x} = \left(\frac{\partial f}{\partial x}\right)(x, y)$ and $\frac{\partial z}{\partial y} = \left(\frac{\partial f}{\partial y}\right)(x, y)$.

Example (Derivatives in polar coordinates)

$$
\begin{tikzcd}
 & & & z & & & \\
 & x \arrow[urr] & & & & y \arrow[ull] & \\
r \arrow[ur] & & \theta \arrow[ul] & & r \arrow[ur] & & \theta \arrow[ul]
\end{tikzcd}
$$

Let $x = r\cos\theta$, $y = r\sin\theta$ and $z = f(x, y)$.

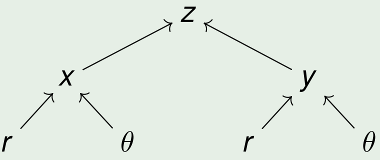
$$
\begin{aligned}
\frac{\partial z}{\partial r} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta \\
&= \cos\theta\frac{\partial z}{\partial x} + \sin\theta\frac{\partial z}{\partial y} \quad .
\end{aligned}
$$

The above is true for all differentiable $z = f(x, y)$, therefore

$$
\frac{\partial}{\partial r} = \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y} \quad .
$$

Recall that $\frac{\partial z}{\partial x} = \left(\frac{\partial f}{\partial x}\right)(x, y)$ and $\frac{\partial z}{\partial y} = \left(\frac{\partial f}{\partial y}\right)(x, y)$.

Example (Derivatives in polar coordinates)



Let $x = r\cos\theta$, $y = r\sin\theta$ and $z = f(x, y)$.

- Compute $\frac{\partial z}{\partial \theta}$ via $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.
- Express the differential operator $\frac{\partial}{\partial \theta}$ via $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$.

$$\begin{aligned}\frac{\partial z}{\partial \theta} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} = \frac{\partial f}{\partial x}(-r\sin\theta) + \frac{\partial f}{\partial y} r\cos\theta \\ &= -r\sin\theta\frac{\partial z}{\partial x} + r\cos\theta\frac{\partial z}{\partial y} \quad .\end{aligned}$$

The above is true for all differentiable $z = f(x, y)$, therefore

$$\frac{\partial}{\partial \theta} = -r\sin\theta\frac{\partial}{\partial x} + r\cos\theta\frac{\partial}{\partial y} \quad .$$

Example (Partial Derivatives in Polar Coordinates)

Express $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ via $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$, where $x = r\cos\theta, y = r\sin\theta$.
We computed previously that

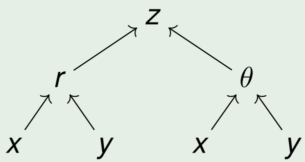
$$\begin{aligned} \frac{\partial}{\partial r} &= \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y} \\ \frac{\partial}{\partial \theta} &= -r\sin\theta\frac{\partial}{\partial x} + r\cos\theta\frac{\partial}{\partial y} \ . \end{aligned}$$

This is a linear system in $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$. To solve the system, eliminate $\frac{\partial}{\partial x}$ by multiplying the first equality by $r\sin\theta$, the second by $\cos\theta$ and adding the two. Similarly eliminate $\frac{\partial}{\partial y}$ by multiplying the first equality by $-r\cos\theta$ and the second by $\sin\theta$ and adding the two. Finally :

$$\begin{aligned} \frac{\partial}{\partial x} &= \cos\theta\frac{\partial}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial y} &= \sin\theta\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\frac{\partial}{\partial \theta} \end{aligned}$$

Example (Partial Derivatives in Polar Coordinates)

Express $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ via $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$, where $x = r\cos\theta, y = r\sin\theta$.
Suppose $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Recall that



$$\begin{aligned} \tan\theta &= \frac{r\sin\theta}{r\cos\theta} = \frac{y}{x} \\ \theta &= \arctan\left(\frac{y}{x}\right) \\ r &= \sqrt{x^2+y^2} \end{aligned}$$

$$\begin{aligned} \frac{\partial z}{\partial x} &= \frac{\partial z}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial x} = \frac{\partial z}{\partial r} \quad \frac{x}{\sqrt{x^2+y^2}} + \frac{\partial z}{\partial \theta} \quad \left(\frac{-y}{x^2+y^2}\right) \\ &= \cos\theta\frac{\partial z}{\partial r} - \frac{\sin\theta}{r}\frac{\partial z}{\partial \theta} \\ \frac{\partial z}{\partial y} &= \frac{\partial z}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial x} = \frac{\partial z}{\partial r} \quad \frac{y}{\sqrt{x^2+y^2}} + \frac{\partial z}{\partial \theta} \quad \frac{x}{x^2+y^2} \\ &= \sin\theta\frac{\partial z}{\partial r} + \frac{\cos\theta}{r}\frac{\partial z}{\partial \theta} \end{aligned}$$

The above hold for all z, therefore $\begin{aligned} \frac{\partial}{\partial x} &= \cos\theta\frac{\partial}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial y} &= \sin\theta\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\frac{\partial}{\partial \theta} \end{aligned}$.

The Laplace Operator

Definition

The *n*-variable Laplace operator is the differential operator:

$$\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} = \frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_n^2} \quad .$$

In paticular the two-variable Laplace operator is:

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

The Laplace operator is named after Pierre Laplace (1749-1827).

Example

Express the Laplace operator $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ in polar coordinates.

$$\Delta = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \,.$$

Harmonic Functions

Recall that $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

Definition (Harmonic function definition)

Functions f such that $\Delta f = 0$ are called *harmonic* functions.

Example

The function $f(x, y) = \ln\left(x^2 + y^2\right)$ is a harmonic function. Rewrite in polar coordinates: $f(x, y) = g(r, \theta) = \ln\left(r^2\right) = 2\ln r$. Then

$$
\begin{aligned}
\Delta g &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial g}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 g}{\partial \theta^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial(2\ln r)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2(2\ln r)}{\partial \theta^2} \\
&= \frac{1}{r}\frac{\partial}{\partial r}\left(r \cdot \frac{2}{r}\right) = \frac{1}{r}\frac{\partial}{\partial r}(2) = 0 .
\end{aligned}
$$

Fact: The only harmonic functions independent of θ are of the form $g(r, \theta) = c_1 \ln r + c_2$.