Calculus II
Lecture 16

Todor Milev
https://github.com/tmilev/freecalc

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Outline

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Formal Series

Definition (Formal Series)

A formal series is a list of numbers delimited by the plus sign.

$$a_1 + a_2 + a_3 + a_4 + \cdots + a_n + \cdots$$

- Recall a sequence is a list of numbers.

$$a_1, \quad a_2, \quad a_3, \quad a_4, \quad \dots, \quad a_n, \quad \dots$$

- The $+$ sign indicates our intention to attempt to sum the elements of the formal series.
- Except for the indication of that intention, formal series and sequences are essentially synonymous.
- The sum of a finite sequence/finite formal series is studied in the subject of elementary arithmetics.
- The sum, if convergent, of an infinite sequence/infinite formal series will be defined in the following slides.

Example (The … and $\sum$ notations for series)

Let A be the sum of the positive even integers between 2 and 124. Write A using the … notation and using the $\sum$ notation.

$$\begin{aligned} A &= 2+4+6+\cdots+124 \\ &= 2+4+6+\cdots+2n+\cdots+124 \\ &= 2\cdot 1+2\cdot 2+2\cdot 3+\cdots+2\cdot n+\cdots+2\cdot 62 \\ &= \sum_{n=1}^{62} 2n \quad . \end{aligned}$$

- We aim to introduce the $\sum$ notation for series via this example.
- The … notation is informal but easier to read.
- If the … are too ambiguous, we should include the general term.
- To make it clearer we should rewrite all elements in the pattern of the general term.
- If that is still ambiguous we should switch to the completely unambiguous $\sum$ notation.

Example (The ... and $\sum$ notations for series)

Let A be the sum of the positive even integers between 2 and 124. Write A using the ... notation and using the $\sum$ notation.

$$
\begin{aligned}
A &= 2+4+6+\dots+124\\
&= 2+4+6+\dots+2n+\dots+124\\
&= 2\cdot 1+2\cdot 2+2\cdot 3+\dots+2\cdot n+\dots+2\cdot 62\\
&= \sum_{n=1}^{62} 2n \quad .
\end{aligned}
$$

- The number n is the index (counter) of the sum.
- $\sum$ tells us to add several copies of the summed term, where in each term the index is replaced by a concrete value.
- The values taken by the index are determined by the boundaries of summation.
- The index varies over all integers starting with the lower boundary and ending with upper boundary.
- In programming, what objects are similar to $\sum$?

Example (The $\ldots$ and $\sum$ notations for series)

Let A be the sum of the positive even integers between 2 and 124. Write A using the $\ldots$ notation and using the $\sum$ notation.

$$\begin{aligned} A &= 2+4+6+\cdots+124 \\ &= 2+4+6+\cdots+2n+\cdots+124 \\ &= 2\cdot 1+2\cdot 2+2\cdot 3+\cdots+2\cdot n+\cdots+2\cdot 62 \\ &= \sum_{n=1}^{62} 2n \quad . \end{aligned}$$

- To go from $\sum$ to $\ldots$ notation: substitute few values for the index. Make sure to include the last value.
- To go from $\ldots$ to $\sum$ notation:
  - figure out a pattern for the general term just as with sequences;
  - select first and last index so that your general term formula reproduces the first and last terms of the sequence.

Example (The $\dots$ and $\sum$ notations for series)

Let A be the sum of the positive even integers between 2 and 124. Write A using the $\dots$ notation and using the $\sum$ notation.

$$
\begin{aligned}
A &= 2+4+6+\dots+124\\
&= 2+4+6+\dots+2n+\dots+124\\
&= 2\cdot 1+2\cdot 2+2\cdot 3+\dots+2\cdot n+\dots+2\cdot 62\\
&= \sum_{n=1}^{62} 2n \quad .
\end{aligned}
$$

- Bear in mind the $\dots$ notation is informal.
  - There are infinitely many formulas that fit any single pattern.
  - Thus it is acceptable to use the $\dots$ notation only when we believe there is a single completely obvious pattern that will be recognized by every one.
  - The pattern should be obvious not only to us, but also to our potential readers.
  - If in doubt or seeking complete rigor we should use the $\sum$ notation.

Definition (Arithmetic series)

An arithmetic series is a series whose terms are an arithmetic sequence.

Example (Sum of a small arithmetic series)

The sum of the arithmetic series $7+4+1-2-5$ is

Example (Sum of a large arithmetic series)

Find the sum of the arithmetic series
$$7+4+1-2-5-\cdots-53-56.$$
Let s denote the sum.
$$\begin{array}{rcrrrrr}
s & = & 7 & +4 & +1 & -\cdots & -56 \\
+\,s & = & -\,56 & -\,53 & -\,50 & -\cdots & +\,7 \\
\hline
2s & = & -\,49 & -\,49 & -\,49 & -\cdots & -\,49
\end{array}$$

$$\begin{aligned} \text{Therefore}\quad 2s &= (-49)(22) \\ s &= -49\cdot 22/2 = -539. \end{aligned}$$

Theorem (Sum of an arithmetic series)

The sum of a finite arithmetic series is the average of the first and last terms, multiplied by the number of terms. That is,

$$a+(a+d)+(a+2d)+\cdots+(a+(n-1)d)=\frac{a+(a+(n-1)d)}{2}n.$$

The only infinite arithmetic series with a sum is the series of all 0.

Example (Sum of an arithmetic series)

Find the sum of the arithmetic series

$$5+10+15+20+\cdots+100.$$

The series contains terms.The average of the first and last terms is

.

Therefore the sum is $\frac{}{2}$.

Definition (Geometric series)

A geometric series is a series whose terms are a geometric sequence.

Example (The sum of a finite geometric series)

Let $r \neq 1$. Find the sum of the geometric series

$$a + ar + ar^2 + ar^3 + \dots + ar^{M-1} = \sum_{n=1}^{M} ar^{n-1}.$$

Let s denote the sum.

$$\begin{array}{rrcllllll}
 & s & = & a & +ar & +ar^2 & +\cdots & +ar^{M-1} & \\
- & rs & = & & ar & +ar^2 & +\cdots & +ar^{n-1} & +ar^M \\
\hline
 & s-rs & = & a - ar^M & & & & & \\
 & s & = & \frac{a(1-r^M)}{1-r} & & & & &
\end{array}$$

Theorem (The sum of a finite geometric series)

Let $r \neq 1$. The sum of the finite geometric series $\sum_{n=1}^{M} ar^{n-1}$ is $a\frac{1-r^M}{1-r}$.

- Does it make sense to add infinitely many numbers?
- Sometimes yes, sometimes no.
- Consider the series $\sum_{n=1}^{\infty} n$.

$$1 + 2 + 3 + 4 + 5 + \cdots + n + \cdots$$

- If we add the terms, we get the partial sums 1,3,6,10,15.
- After the nth term, we get $\frac{n(n+1)}{2}$.
- This goes to ∞ as n gets bigger.
- Now consider the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$.

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots + \frac{1}{2^n} + \cdots$$

- If we add the terms, we get the partial sums $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \frac{31}{32}$.
- After the nth term, we get $1 - \frac{1}{2^n}$.
- This gets closer and closer to 1. We write $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$.

Definition (Partial Sum, Convergent, Divergent, Sum)

Given a series $\sum_{i=1}^{\infty} a_i = a_1 + a_2 + a_3 + \cdots$, let s_n denote the nth partial sum:

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$$

If the sequence $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = s$, then we say that the series $\sum_{i=1}^{\infty} a_i$ is convergent, and we write

$$\sum_{i=1}^{\infty} a_i = s.$$

In this case, we call s the sum of the series.
If the sequence $\{s_n\}$ is divergent, then we say that the series $\sum_{i=1}^{\infty} a_i$ is divergent.

Example

An important example is the geometric series

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}, \qquad a \neq 0$$

- If $r = 1$, then $s_n = a + a + \cdots + a = na \to \pm\infty$.
- Since $\lim_{n\to\infty} s_n$ doesn't exist, the series is divergent when $r = 1$.
- If $r \neq 1$, then

$$\begin{array}{rrcllllll} & s_n & = & a & +ar & +ar^2 & +\cdots & +ar^{n-1} & \\ - & rs_n & = & & ar & +ar^2 & +\cdots & +ar^{n-1} & +ar^n \\ \hline & s_n - rs_n & = & a - ar^n & & & & & \\ & s_n & = & \frac{a(1-r^n)}{1-r} & & & & & \end{array}$$

- If $-1 < r < 1$, then $r^n \to 0$, so the geometric series is convergent and its sum is $a/(1-r)$.
- If $r > 1$ or $r \leq -1$, then r^n is divergent, so $\sum_{n=1}^{\infty} ar^{n-1}$ diverges.

This theorem summarizes the results of the previous example.

Theorem (Convergence of Geometric Series)

The geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

is convergent if $|r| < 1$ *and its sum is*

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}.$$

If $|r| \geq 1$*, the series is divergent.*
a *is called the first term and* r *is called the common ratio.*

For $|r| < 1$, recall that the sum of a geometric series is

$$a + ar + ar^2 + ar^3 + \dots = a\left(1 + r + r^2 + r^3 + \dots\right) = \frac{a}{1-r}$$

alternatively

$$\sum_{n=1}^{\infty} ar^{n-1} = \sum_{m=0}^{\infty} ar^m = a\sum_{m=0}^{\infty} r^m = \frac{a}{1-r}$$

Example

Find the sum of the geometric series $-2 + \frac{6}{5} - \frac{18}{25} + \frac{54}{125} - \cdots$

- The first term is $a = -2$.
- The common ratio is $r = \frac{\frac{6}{5}}{-2} = -\frac{3}{5}$.
- Therefore the sum is

$$\sum_{n=1}^{\infty}(-2)\left(-\frac{3}{5}\right)^{n-1} = \frac{(-2)}{1-\left(-\frac{3}{5}\right)} = -\frac{2}{\frac{8}{5}} = -\frac{5}{4}$$

Example

Write the number $2.3\overline{17} = 2.3171717\ldots$ as a quotient of integers.

$$2.3171717\ldots = 2.3 + \frac{17}{10^3} + \frac{17}{10^5} + \frac{17}{10^7} + \cdots$$

- After the first term, we have a geometric series.
- $a = \frac{17}{10^3}$ and $r = \frac{1}{10^2}$.

$$\begin{aligned} 2.3171717\ldots &= 2.3 + \frac{\frac{17}{10^3}}{1 - \frac{1}{10^2}} = 2.3 + \frac{\frac{17}{1000}}{\frac{99}{100}} \\ &= \frac{23}{10} + \frac{17}{990} = \frac{1147}{495} \end{aligned}$$

Example

Show the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent and find its sum.

Is this a geometric series? No, because $\frac{a_{n+1}}{a_n} = \frac{\frac{1}{(n+1)(n+2)}}{\frac{1}{n(n+1)}} = \frac{n}{n+2}$ is not constant. Decompose a_n into partial fractions:

$$\begin{aligned}
a_n &= \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1} \\
s_k &= \sum_{n=1}^{k} \frac{1}{n(n+1)} = \sum_{n=1}^{k} \left(\frac{1}{n} - \frac{1}{n+1} \right) \\
&= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \dots + \left(\frac{1}{k} - \frac{1}{k+1}\right) \\
&= 1 - \frac{1}{k+1}
\end{aligned}$$

Therefore $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{k\to\infty} s_k = \lim_{k\to\infty} \left(1 - \frac{1}{k+1}\right) = 1$

Example

Show that the harmonic series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ diverges.

$$
\begin{aligned}
s_1 &= 1\\
s_2 &= 1 + \tfrac{1}{2}\\
s_4 &= 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} > 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{4} = 1 + \tfrac{2}{2}\\
s_8 &= 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8}\\
&> 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8}\\
&= 1 + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} = 1 + \tfrac{3}{2}\\
s_{16} &= 1 + \tfrac{1}{2} + \left(\tfrac{1}{3} + \tfrac{1}{4}\right) + \left(\tfrac{1}{5} + \cdots + \tfrac{1}{8}\right) + \left(\tfrac{1}{9} + \cdots + \tfrac{1}{16}\right)\\
&> 1 + \tfrac{1}{2} + \left(\tfrac{1}{4} + \tfrac{1}{4}\right) + \left(\tfrac{1}{8} + \cdots + \tfrac{1}{8}\right) + \left(\tfrac{1}{16} + \cdots + \tfrac{1}{16}\right)\\
&= 1 + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} = 1 + \tfrac{4}{2}\\
&\vdots\\
s_{2^n} &> 1 + \tfrac{n}{2}
\end{aligned}
$$

Therefore $s_{2^n} \to \infty$ as $n \to \infty$, so $\{s_n\}$ is divergent, so the harmonic series is divergent.