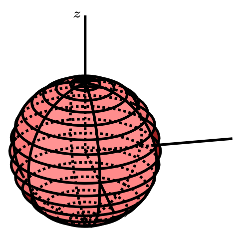
Calculus III
Homework on Lecture 15

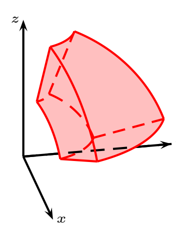
1. **Problem 1.e is of higher difficulty.**

   - Write the Jacobian matrix of the indicated variable change.
   - Set up an integral expressing the volume of the region using the indicated variable change and the multivariable integral substitution rule.
   - Integrate to find the volume of the region.

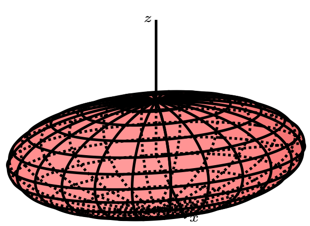
   (a) Spherical coordinates; use to find the volume of a ball of radius R.



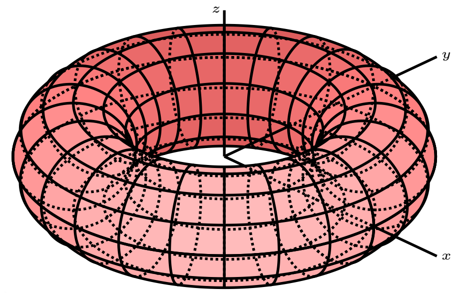
   (b) Spherical coordinates; use to find the volume of a curvilinear spherical box, given in spherical coordinates by $\rho_{min} \leq \rho \leq \rho_{max}$, $\phi_{min} \leq \phi \leq \phi_{max}$, $\theta_{min} \leq \theta \leq \theta_{max}$.



   (c) Ellipsoidal coordinates: $\mathbf{f} : \left| \begin{array}{rcl} x &=& a\rho \sin\phi \cos\theta \\ y &=& b\rho \sin\phi \sin\theta \\ z &=& c\rho \cos\phi \end{array} \right.$; use to find the volume of an ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, $a, b, c > 0$.

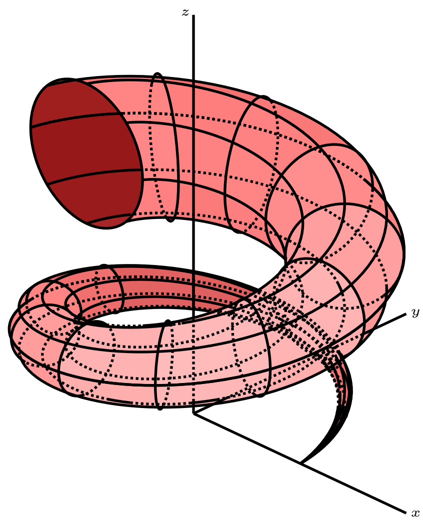


   (d) Variable change: $T : \left| \begin{array}{l} x = (R + \rho\cos\theta)\cos\phi \\ y = (R + \rho\cos\theta)\sin\phi \\ z = \rho\sin\theta \end{array} \right.$; use to find the volume of a torus with major radius R and minor radius r,

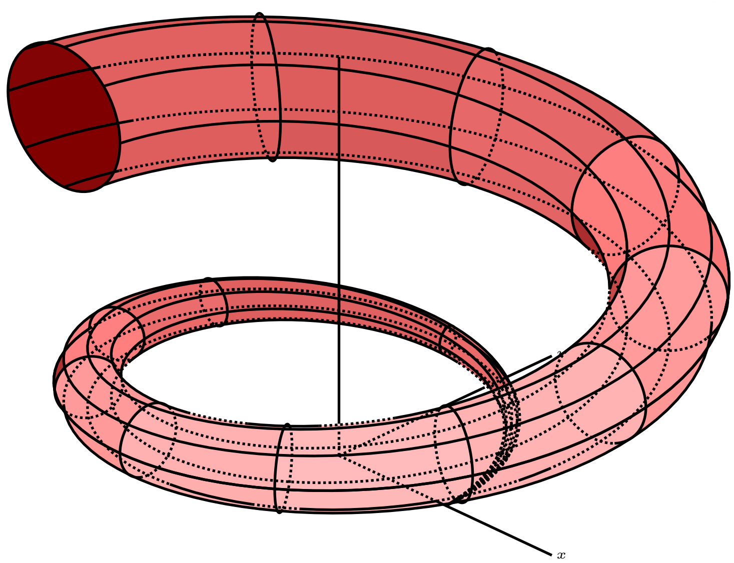


i.e., the figure given by $\rho \in [0, r], \phi \in [0, 2\pi], \theta \in [0, 2\pi]$.

(e) Variable change: $\left| \begin{array}{lcl} x & = & (2 + \rho\cos\theta)\cos\phi \\ y & = & (2 + \rho\cos\theta)\sin\phi \\ z & = & \rho\sin\theta + \frac{\phi}{3} \end{array} \right.$, use to find the volume of the horn given by $\theta \in [0, 2\pi], \phi \in [0, 3\pi], \rho \in \left[0, \frac{\phi}{9}\right]$.



(f) Variable change: $\left| \begin{array}{lcl} x & = & (2 + \phi/3 + \rho\cos\theta)\cos\phi \\ y & = & (2 + \phi/3 + \rho\cos\theta)\sin\phi \\ z & = & \rho\sin\theta + \frac{\phi}{3} \end{array} \right.$, use to find the volume of the horn given by $\theta \in [0, 2\pi], \phi \in [0, 3\pi], \rho \in \left[0, \frac{\phi}{9}\right]$.



Solution. 1.e **This solution is only partial.**

Let $\mathbf{f}$ be the map given by the variable change:

$$\left|\begin{array}{lcl} x &=& (2+\rho\cos\theta)\cos\phi \\ y &=& (2+\rho\cos\theta)\sin\phi \\ z &=& \rho\sin\theta+\frac{\phi}{3} \end{array}\right.$$

$$J_{\mathbf{f}} = \begin{pmatrix} \frac{\mathrm{d}x}{\mathrm{d}\rho} & \frac{\mathrm{d}x}{\mathrm{d}\phi} & \frac{\mathrm{d}x}{\mathrm{d}\theta} \\ \frac{\mathrm{d}y}{\mathrm{d}\rho} & \frac{\mathrm{d}y}{\mathrm{d}\phi} & \frac{\mathrm{d}y}{\mathrm{d}\theta} \\ \frac{\mathrm{d}z}{\mathrm{d}\rho} & \frac{\mathrm{d}z}{\mathrm{d}\phi} & \frac{\mathrm{d}z}{\mathrm{d}\theta} \end{pmatrix} = \begin{pmatrix} \cos\phi\cos\theta & -\rho\cos\theta\sin\phi-2\sin\phi & -\rho\cos\phi\sin\theta \\ \cos\theta\sin\phi & \rho\cos\phi\cos\theta+2\cos\phi & -\rho\sin\phi\sin\theta \\ \sin\theta & 1/3 & \rho\cos\theta \end{pmatrix}$$

Then
$$\begin{aligned} \det J_{\mathbf{f}} &= \cos^2\phi\cos\theta\sin^2\theta+\rho^2\cos\theta\sin^2\phi\sin^2\theta \\ &\quad +\rho^2\cos^3\theta\sin^2\phi+\rho^2\cos^2\phi\cos^3\theta+2\rho\sin^2\phi\sin^2\theta \\ &\quad +2\rho\cos^2\theta\sin^2\phi+2\rho\cos^2\phi\sin^2\theta+2\rho\cos^2\phi\cos^2\theta \\ &= \rho^2\cos\theta+2\rho \end{aligned}$$

The rest of the problem we leave to the student.

Solution. 1.e **This solution is only partial.**

Let $\mathbf{f}$ be the map given by the variable change:

$$\left|\begin{array}{lcl} x &=& (2+\phi/3\rho\cos\theta)\cos\phi \\ y &=& (2+\phi/3\rho\cos\theta)\sin\phi \\ z &=& \rho\sin\theta+\frac{\phi}{3} \end{array}\right.$$

$$J_{\mathbf{f}} = \begin{pmatrix} \frac{\mathrm{d}x}{\mathrm{d}\rho} & \frac{\mathrm{d}x}{\mathrm{d}\phi} & \frac{\mathrm{d}x}{\mathrm{d}\theta} \\ \frac{\mathrm{d}y}{\mathrm{d}\rho} & \frac{\mathrm{d}y}{\mathrm{d}\phi} & \frac{\mathrm{d}y}{\mathrm{d}\theta} \\ \frac{\mathrm{d}z}{\mathrm{d}\rho} & \frac{\mathrm{d}z}{\mathrm{d}\phi} & \frac{\mathrm{d}z}{\mathrm{d}\theta} \end{pmatrix} = \begin{pmatrix} \cos\phi\cos\theta & -\rho\cos\theta\sin\phi-1/3\phi\sin\phi-2\sin\phi+1/3\cos\phi & -\rho\cos\phi\sin\theta \\ \cos\theta\sin\phi & \rho\cos\phi\cos\theta+1/3\phi\cos\phi+1/3\sin\phi+2\cos\phi & -\rho\sin\phi\sin\theta \\ \sin\theta & 1/3 & \rho\cos\theta \end{pmatrix}$$

Then
$$\begin{aligned} \det J_{\mathbf{f}} &= \rho^2\cos^2\phi\cos\theta\sin^2\theta+\rho^2\cos\theta\sin^2\phi\sin^2\theta+\rho^2\cos^3\theta\sin^2\phi+\rho^2\cos^2\phi\cos^3\theta \\ &\quad +\tfrac{1}{3}\phi\rho\sin^2\phi\sin^2\theta+\tfrac{1}{3}\phi\rho\cos^2\theta\sin^2\phi+\tfrac{1}{3}\phi\rho\cos^2\phi\sin^2\theta+\tfrac{1}{3}\phi\rho\cos^2\phi\cos^2\theta \\ &\quad +2\rho\sin^2\phi\sin^2\theta+2\rho\cos^2\theta\sin^2\phi+2\rho\cos^2\phi\sin^2\theta+2\rho\cos^2\phi\cos^2\theta \\ &= \rho^2\cos\theta+\frac{\phi\rho}{3}+2\rho \end{aligned}$$

The rest of the problem we leave to the student.