Calculus I
Homework Antiderivatives and Integrals
Lecture 21

1. Find all antiderivatives of the functions.

(a) $f(x) = \sqrt{3} + \pi^2$.

answer: $x\left(\pi^2 + \sqrt{3}\right) + C$

(b) $f(x) = x - 5$.

answer: $\frac{x^2}{2} - 5x + C$

(c) $f(x) = x^2 - 2x + 6$.

answer: $\frac{x^3}{3} - x^2 + 6x + C$

(d) $f(x) = \dfrac{x(x+1)}{2}$.

answer: $\frac{1}{6}x^3 + \frac{1}{4}x^2 + C$

(e) $f(x) = x(x+1)(2x+1)$.

answer: $\frac{1}{2}x^4 + x^3 + \frac{1}{2}x^2 + C$

(f) $f(x) = 7x^{\frac{2}{7}} + x^{-\frac{4}{7}}$.

answer: $\frac{49}{9}x^{\frac{9}{7}} + \frac{7}{3}x^{\frac{3}{7}} + C$

(g) $f(x) = x^{2.4} - 2x^{\sqrt{3}-1}$.

answer: $\frac{5}{17}x^{\frac{17}{5}} - \frac{2\sqrt{3}x^{\sqrt{3}}}{3} + C$

(h) $f(x) = \dfrac{8}{x^7}$.

answer: $-\frac{4}{3}x^{-6} + C$

(i) $f(x) = \dfrac{x+1}{x^3}$.

answer: $-x^{-1} - \frac{1}{2}x^{-2} + C$

(j) $f(x) = \dfrac{1}{x}$.

answer: $\ln|x| + C$

(k) $f(x) = \dfrac{x^2+1}{x}$.

answer: $\frac{1}{2}x^2 + \ln|x| + C$

(l) $f(x) = \dfrac{5 - 4x^3 + 2x^6}{x^4}$.

answer: $\frac{2}{3}x^3 - \frac{5}{3}x^{-3} - 4\ln|x| + C$

(m) $g(x) = \dfrac{1 + \sqrt{x} + x}{\sqrt{x^3}}$.

answer: $2x^{\frac{1}{2}} - 2x^{-\frac{1}{2}} + \ln|x| + C$

(n) $f(t) = 3\sin t - 4\cos t$.

answer: $-3\cos t - 4\sin t + C$

(o) $f(\theta) = \sec^2\theta$.

answer: $\tan\theta + C$

(p) $f(\theta) = \csc^2\theta$.

answer: $-\cot\theta + C$

(q) $f(t) = \sec t\tan t + \csc t\cot t$.

answer: $\sec t - \csc t + C$

(r) $f(x) = \dfrac{2 + x\cos x}{x}$.

answer: $2\ln|x| + \sin x$

2. Verify by differentiation that the formula is correct.

(a) $\displaystyle\int \sqrt{1+x^2}\,\mathrm{d}x = \frac{1}{2}\left(x\sqrt{1+x^2} + \ln\left(x + \sqrt{1+x^2}\right) + C\right)$

(b) $\displaystyle\int \sin^2 x\,\mathrm{d}x = -\frac{1}{4}\sin(2x) + \frac{1}{2}x + C$.

(c) $\displaystyle\int \sin^3 x\,\mathrm{d}x = \frac{1}{3}\cos^3 x - \cos x + C$.

(d) $\displaystyle\int \frac{x}{\sqrt{1+x}}\mathrm{d}x = \frac{2}{3}(x-2)\sqrt{1+x} + C$

3. Evaluate the integral (definite or indefinite).

(a) $\displaystyle\int_{-2}^{3} \left(x^2 - 1\right)\mathrm{d}x$.

answer: $\left[\frac{1}{3}x^3 - x\right]_{-2}^{3} = \frac{20}{3}$

(b) $\displaystyle\int_{1}^{2} \left(4x^3 + 3x^2 + 2x + 1\right)\mathrm{d}x$.

answer: $\left[x^4 + x^3 + x^2 + x\right]_{1}^{2} = 26$

(c) $\displaystyle\int_{0}^{2} (x-1)(x^2+1)\mathrm{d}x$.

answer: $\left[\frac{1}{4}x^4 - \frac{1}{3}x^3 + \frac{1}{2}x^2 - x\right]_{0}^{2} = \frac{4}{3}$

(d) $\displaystyle\int_{-1}^{1} \left(\frac{x(x+1)}{2}\right)^2\mathrm{d}x$.

answer: $\left[\frac{1}{20}x^5 + \frac{1}{8}x^4 + \frac{1}{12}x^3\right]_{-1}^{1} = \frac{4}{15}$

(e) $\displaystyle\int_{0}^{1} (1+x^2)^3 dx$.

answer: $\left[\frac{1}{7}x^7 + \frac{3}{5}x^5 + x^3 + x\right]_{0}^{1} = \frac{96}{35}$

(f) $\displaystyle\int_{1}^{2} \left(\frac{1}{x} - \frac{4}{x^2}\right)\mathrm{d}x$.

answer: $\left[4x^{-1} + \ln x\right]_{1}^{2} = \ln 2 - 2$

(g) $\displaystyle\int_{1}^{4} \sqrt{x}(1+x)\mathrm{d}x$.

answer: $\left[\frac{2}{5}x^{\frac{5}{2}} + \frac{2}{3}x^{\frac{3}{2}}\right]_{1}^{4} = \frac{256}{15}$

(h) $\displaystyle\int_{1}^{4} \sqrt{\frac{6}{x}}\mathrm{d}x$.

answer: $\left[2\sqrt{6}\sqrt{x}\right]_{1}^{4} = 2\sqrt{6}$

(i) $\displaystyle\int_{1}^{4} \frac{\frac{1}{\sqrt{x}} + 1 + x}{\sqrt{x}}\mathrm{d}x$.

answer: $\left[\frac{2}{3}x^{\frac{3}{2}} + 2\sqrt{x} + \ln|x|\right]_{1}^{4} = \ln(4) + \frac{20}{3}$

(j) $\displaystyle\int_{1}^{8} \frac{1+x}{\sqrt[3]{x}}\mathrm{d}x$.

answer: $\left[\frac{3}{5}x^{\frac{5}{3}} + \frac{3}{2}x^{\frac{2}{3}}\right]_{1}^{8} = \frac{231}{10}$

(k) $\displaystyle\int_{1}^{64} \frac{\frac{1}{\sqrt[3]{x}} + \sqrt[3]{x}}{\sqrt{x}}\mathrm{d}x$.

answer: $\left[\frac{6}{5}x^{\frac{5}{6}} + 6x^{\frac{1}{6}}\right]_{1}^{64} = \frac{216}{5}$

(l) $\displaystyle\int_{0}^{1} \left(\sqrt[5]{x^6} + \sqrt[6]{x^5}\right)\mathrm{d}x$.

answer: $\left[\frac{5}{11}x^{\frac{11}{5}} + \frac{6}{11}x^{\frac{11}{6}}\right]_0^1 = 1$

(m) $\displaystyle\int_1^2 \left(x+\frac{1}{x}\right)^2 \mathrm{d}x.$

answer: $\left[frac13x^3 - x^{-1} + 2x\right]_1^2 = \frac{29}{6}$

(n) $\displaystyle\int_1^2 \left(x+\frac{1}{x}\right)^3 \mathrm{d}x.$

answer: $\left[\frac{1}{4}x^4 + \frac{3}{2}x^2 - \frac{1}{2}x^{-2} + 3\ln x\right]_1^2 = \frac{69}{8} + 3\ln 2$

(o) $\displaystyle\int_1^2 \left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2 \mathrm{d}x.$

answer: $\left[\frac{1}{2}x^2 + \ln x + 2x\right]_1^2 = \frac{7}{2} + \ln(2)$

(p) $\displaystyle\int_1^2 \left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^3 \mathrm{d}x.$

answer: $\left[\frac{2}{5}x^{\frac{5}{2}} + 2x^{\frac{3}{2}} + 6\sqrt{x} - 2x^{-\frac{1}{2}}\right]_1^2 = \frac{53}{5}\sqrt{2} - \frac{32}{5}$

(q) $\displaystyle\int_0^2 |x-1|\mathrm{d}x.$

answer: 1

(r) $\displaystyle\int_0^1 \left|x-\frac{1}{2}\right| \mathrm{d}x.$

answer: $\frac{1}{4}$

(s) $\displaystyle\int_{-1}^1 (x-3|x|)\mathrm{d}x.$

answer: -3

(t) $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \csc^2\theta \mathrm{d}\theta.$

answer: [] =

(u) $\displaystyle\int_0^{\frac{\pi}{4}} \frac{1-\cos^2\theta}{\cos^2\theta}\mathrm{d}\theta.$

answer: [] =

(v) $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\sin^2\theta}{\cos^2\theta}\mathrm{d}\theta.$

answer: [] =

(w) $\displaystyle\int_0^{\frac{\pi}{4}} \tan^2\theta \mathrm{d}\theta.$

answer: [] =

(x) $\displaystyle\int_0^{\frac{\pi}{3}} \frac{\sin\theta + \sin\theta\tan^2\theta}{\sec^2\theta}\mathrm{d}\theta.$

answer: [] =

(y) $\displaystyle\int_0^{\pi} (\sin\theta - \cos\theta)\mathrm{d}\theta.$

answer: [] =

(z) $\displaystyle\int_0^{\pi} |\sin x|\mathrm{d}x.$

Solution. 3.r

$$
\begin{aligned}
\int_0^1 \left|x-\frac{1}{2}\right|\mathrm{d}x &= \int_0^{\frac{1}{2}} \left|x-\frac{1}{2}\right|\mathrm{d}x + \int_{\frac{1}{2}}^1 \left|x-\frac{1}{2}\right|\mathrm{d}x \\
&= \int_0^{\frac{1}{2}} \left(\frac{1}{2}-x\right)\mathrm{d}x + \int_{\frac{1}{2}}^1 \left(x-\frac{1}{2}\right)\mathrm{d}x \\
&= \left[-\frac{x^2}{2}+\frac{x}{2}\right]_0^{\frac{1}{2}} + \left[\frac{x^2}{2}-\frac{x}{2}\right]_{\frac{1}{2}}^1 \\
&= \left(-\frac{1}{8}+\frac{1}{4}\right) + \left(\frac{1}{2}-\frac{1}{2}-\left(\frac{1}{8}-\frac{1}{4}\right)\right) \\
&= \frac{1}{4}
\end{aligned}
$$

$$
\begin{aligned}
\left|x-\tfrac{1}{2}\right| &= \tfrac{1}{2}-x \text{ when } x \leq \tfrac{1}{2} \\
\left|x-\tfrac{1}{2}\right| &= x-\tfrac{1}{2} \text{ when } x \geq \tfrac{1}{2}
\end{aligned}
$$

4. Integrate (definite or indefinite).

(a) $\displaystyle\int_1^8 \frac{t-t^{\frac{1}{3}}+2}{t^{\frac{4}{3}}}\mathrm{d}t$.

answer: $-\ln 8 + \frac{15}{2}$

(b) $\displaystyle\int_1^4 \left(x+\sqrt{x}\right)^2 dx$.

answer: $\frac{533}{10}$

(c) $\displaystyle\int \frac{\sqrt[3]{x}-x^{\frac{1}{2}}+1}{x}\mathrm{d}x.$

answer: $-2\sqrt{x} + 3x^{\frac{1}{3}} + \ln|x| + C$

(d) $\displaystyle\int \frac{\sqrt[3]{x}-1}{x}\mathrm{d}x.$

answer: $3x^{\frac{1}{3}} - \ln|x| + C$

Solution. 4c

$$
\begin{aligned}
\int \frac{\sqrt[3]{x}-x^{\frac{1}{2}}+1}{x}\mathrm{d}x &= \int \left(x^{-\frac{2}{3}} - x^{-\frac{1}{2}} + \frac{1}{x}\right)\mathrm{d}x \\
&= +3x^{\frac{1}{3}} - 2\sqrt{x} + \ln|x| + C.
\end{aligned}
$$

Solution. 4d

$$\begin{aligned}\int \frac{\sqrt[3]{x}-1}{x}\mathrm{d}x &= \int \left(x^{-\frac{2}{3}} - x^{-1}\right)\mathrm{d}x \\ &= 3x^{\frac{1}{3}} - \ln|x| + C.\end{aligned}$$