# Calculus III
## Lecture 13

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https://github.com/tmilev/freecalc

2020

# Outline

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# A Cheaper "Census"

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- Our population estimate becomes

$$\text{population}(\mathcal{R}) = \sum \text{pop.}(D_k) \simeq \sum \text{density\_near}(P_k)\text{area}(D_k).$$

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The *Riemann sum* defined by such data is $\displaystyle\sum_k f(P_k)\,\mathrm{area}(D_k)$ .

# Double Integrals

$\mathcal{R}$-region covered by $D_k$, $D_k$ don't overlap except at boundaries.

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## Definition

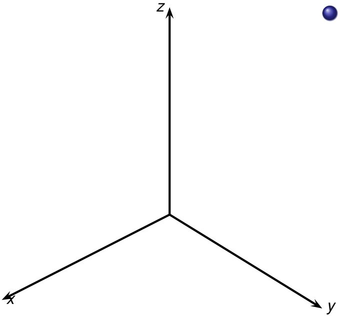
If the limit

$$\lim_{\max_k(\mathrm{diam} D_k)\to 0} \sum_k f(P_k)\,\mathrm{area}(D_k)$$

exists and is finite, then its value is called the *double integral of $f$ over $\mathcal{R}$ (with respect to area)*, and is denoted by

$$\iint_{\mathcal{R}} f(P)\mathrm{d}A \quad .$$

# Midpoint Rule

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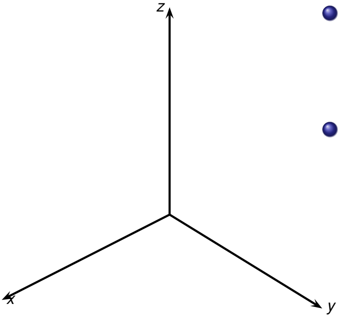


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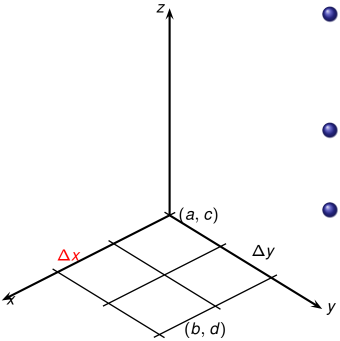


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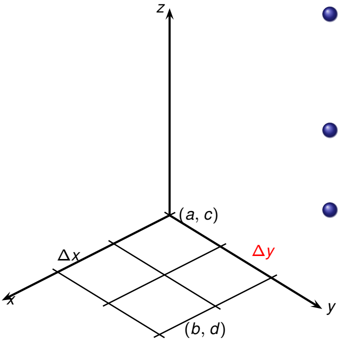


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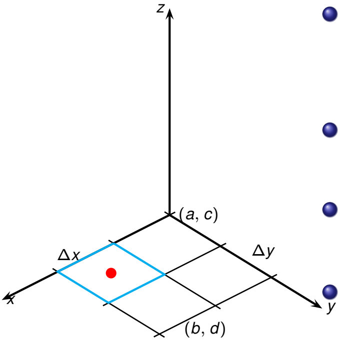


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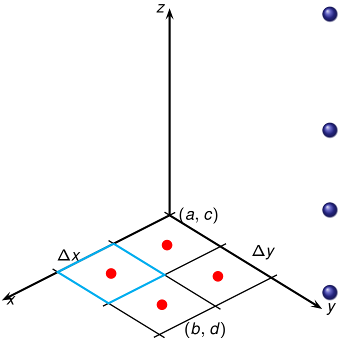


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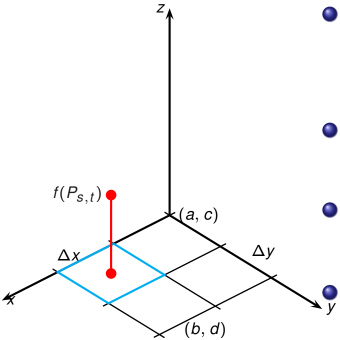


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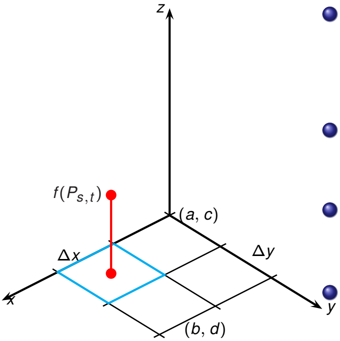


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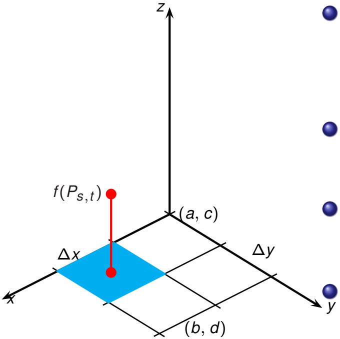


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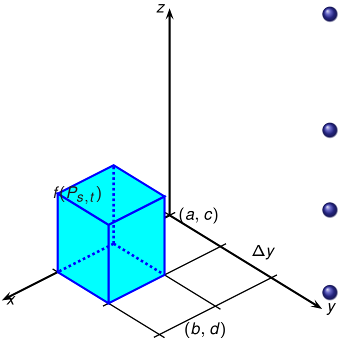


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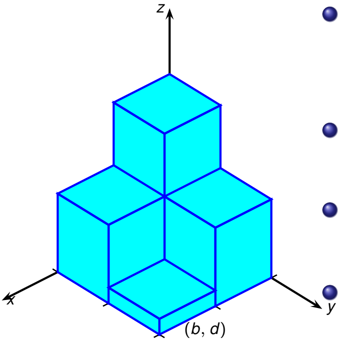


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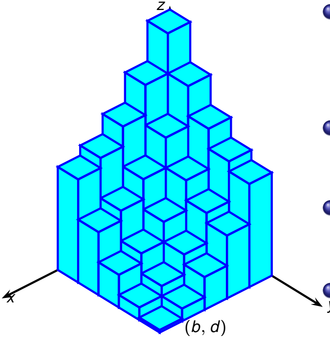


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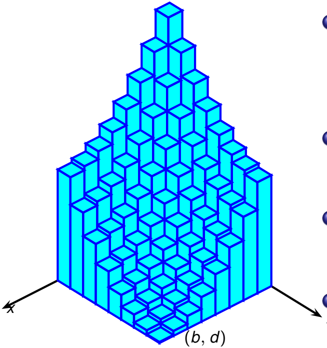


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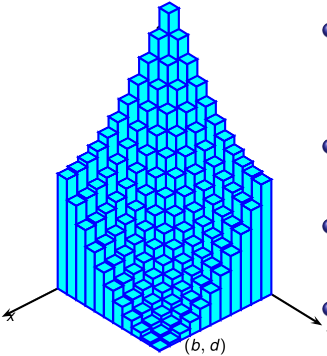


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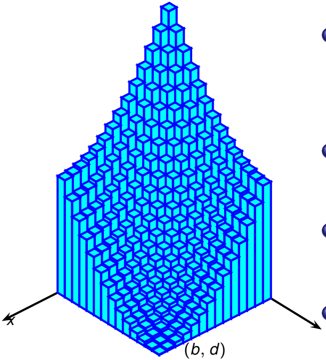


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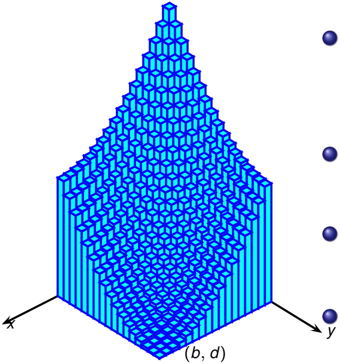


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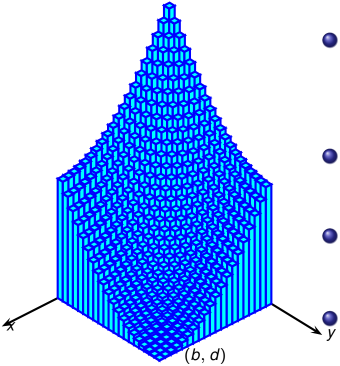


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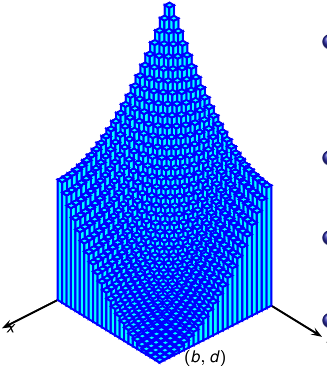


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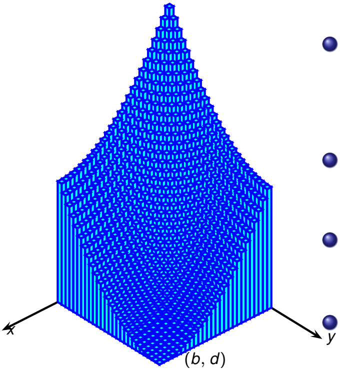


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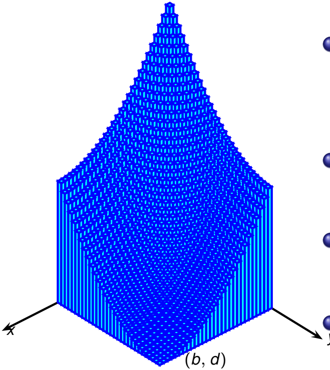


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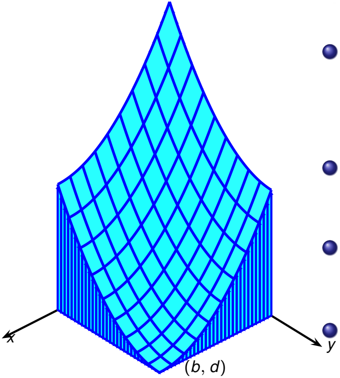


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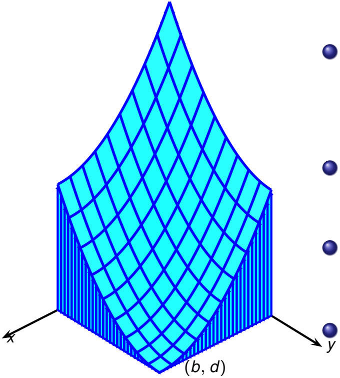


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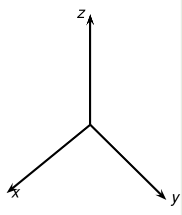
## Example

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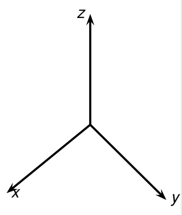


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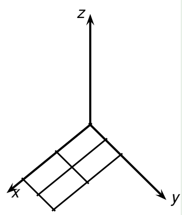


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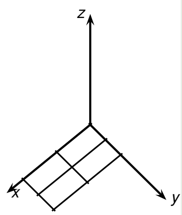


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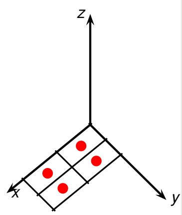


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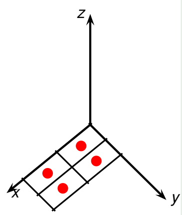


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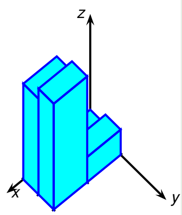


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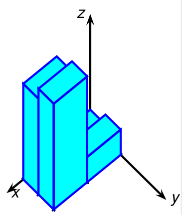


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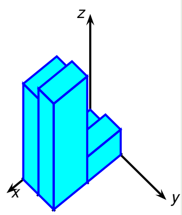


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# Theoretical Examples

- The total population over a region $\mathcal{R}$ is:

$$\text{population}(\mathcal{R}) = \iint_{\mathcal{R}} \text{density}(P)\,\mathrm{d}A \simeq \sum_k \text{density}(P_k)\,\text{area}(D_k)\ .$$

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- Area of a region:

$$\text{Area}(\mathcal{R}) = \iint_{\mathcal{R}} 1\, \mathrm{d}A \quad .$$

# Double Integral Properties

$$\iint_{\mathcal{R}} f(P)\, \mathrm{d}A = \lim_{\max_k(\mathrm{diam} D_k)\to 0} \sum_k f(P_k)\, \mathrm{area}(D_k)$$

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- Monotonicity property: If $m \leq f(P) \leq M$ for all $P$ in $\mathcal{R}$, then

$$m\,\mathrm{area}(\mathcal{R}) \leq \iint_{\mathcal{R}} f(P)\,\mathrm{d}A \leq M\,\mathrm{area}(\mathcal{R})\,.$$

# Applications

- Average value of $f$ on $\mathcal{R}$.

$$\begin{aligned}\iint_{\mathcal{R}} f(P)\,\mathrm{d}A &= \iint_{\mathcal{R}} (\text{average value of } f \text{ on } \mathcal{R})\,\mathrm{d}A\\ &= (\text{average value of } f \text{ on } \mathcal{R}) \iint_{\mathcal{R}} \mathrm{d}A\\ &= (\text{average value of } f \text{ on } \mathcal{R}) \cdot \text{area}(\mathcal{R})\\ \text{average value of } f \text{ on } \mathcal{R} &= \frac{1}{\text{area}(\mathcal{R})} \iint_{\mathcal{R}} f(P)\,\mathrm{d}A\,.\end{aligned}$$

## Theorem (Mean Value Theorem)

*If $f$ is continuous on $\mathcal{R}$, then there exists $P_0$ in $\mathcal{R}$ such that*

$$f(P_0) = \frac{1}{area(\mathcal{R})} \iint_{\mathcal{R}} f(Q)\, \mathrm{d}A$$

## Theorem (Analog of Fundamental Theorem of Calculus)

*If $f$ is continuous around $P$, then*

$$\lim_{D \to \{P\}} \frac{1}{area(D)} \iint_D f(Q) \mathrm{d}A = f(P)$$

# Vectorial Integrals

The double integral definition extends directly to f-ns with vector output.

## Definition

$$\iint_{\mathcal{R}} \mathbf{F}(P)\,\mathrm{d}A = \lim_{\mathrm{maxdiam}(\mathcal{D})\to 0} \sum_k \mathbf{F}(P_k)\,\mathrm{area}(D_k)$$

# Theoretical example: Electric force on a lamina

- Given:
  - a charge $Q$, located at the origin;
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- Recall that the attraction force exerted on a charge $Q$ located at the origin by a charge $c$ located at a point with position vector **r** is $\varepsilon Qc\frac{\mathbf{r}}{|\mathbf{r}|^3}$.

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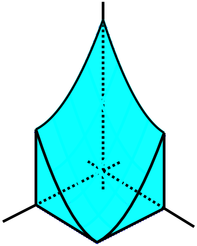

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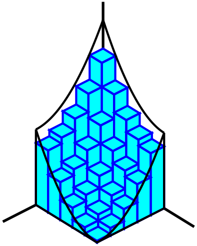

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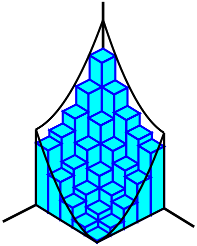

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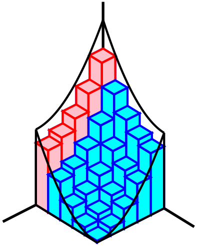

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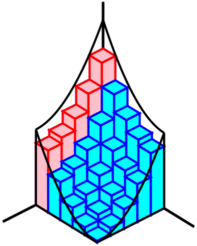

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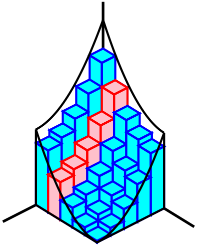

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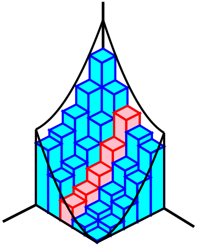

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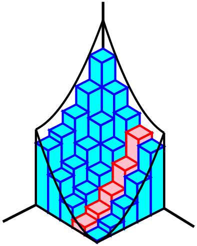

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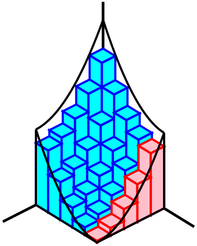

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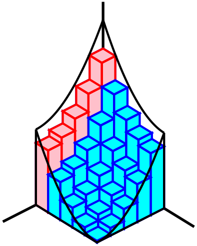

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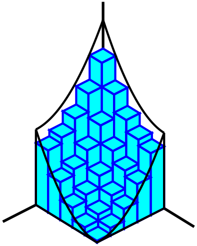

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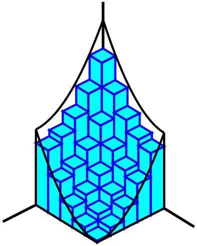

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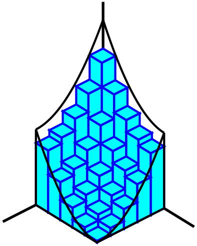

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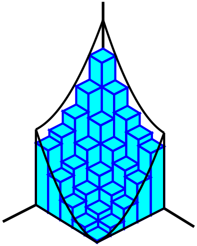

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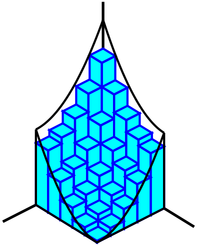

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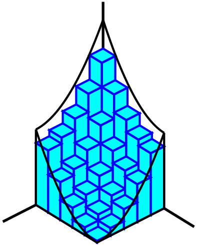

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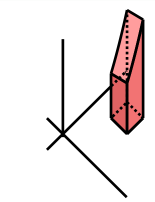
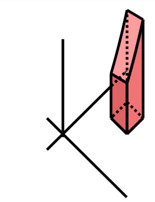
This theorem allows to integrate non-continuous functions. The term "set of measure 0" is too technical to define here; usually studied in the subject(s) "Real Analysis/Measure Theory".

## Example



Compute $\displaystyle\iint\limits_{[1,2]\times[2,3]} (2x+3y^2)\,\mathrm{d}x\mathrm{d}y.$

## Example



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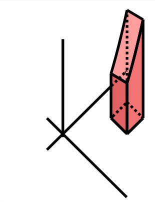

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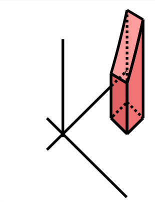

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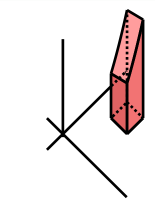

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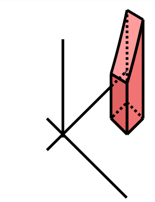

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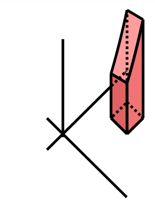

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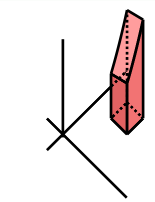

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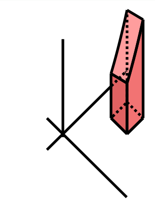

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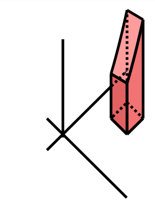

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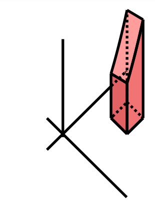

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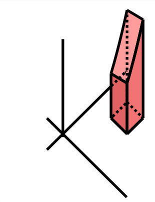

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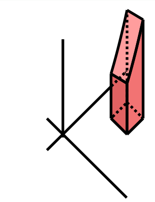

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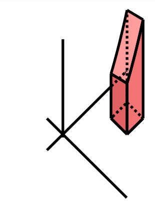

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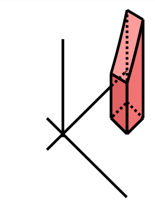

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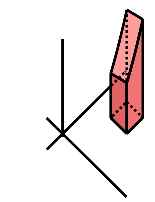

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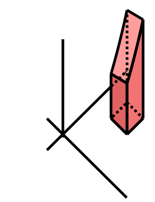

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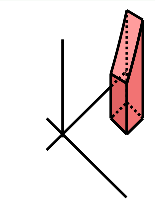

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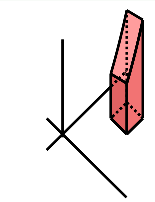

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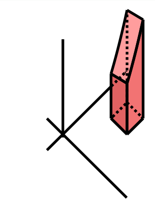

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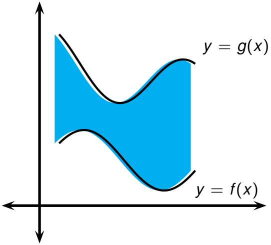
# More General Regions

What makes iterated integrals work over rectangular regions?



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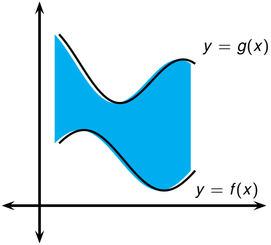

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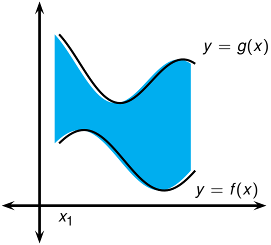


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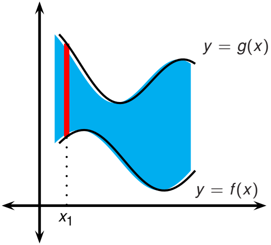


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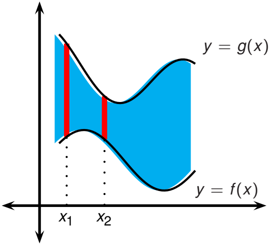


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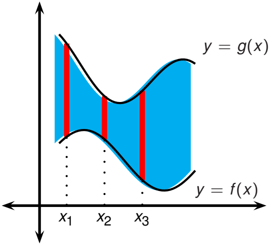


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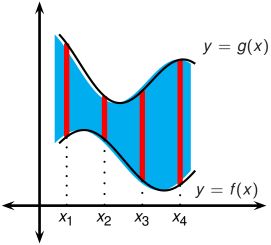


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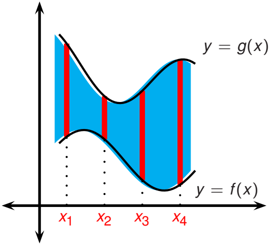


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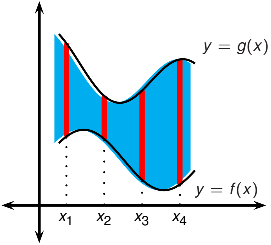


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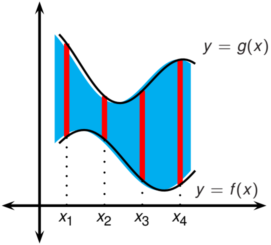


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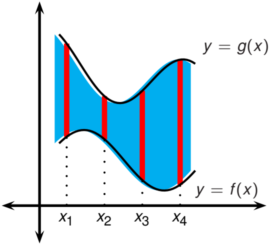


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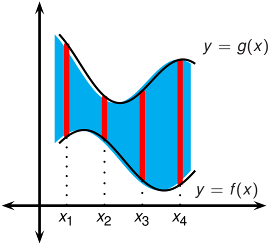


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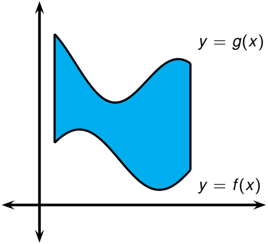
- Regions of type I: vertical slices are segments.
- Regions of type II: horizontal slices are segments.

We call such regions curvilinear trapezoids.

# Strategy: Curvilinear Trapezoids (Type I)



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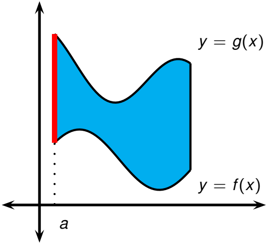


- Identify the leftmost point(s), with $x$-coordinate $x = a$ and the rightmost point(s), $x = b$.

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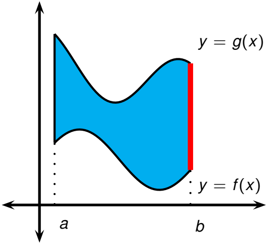


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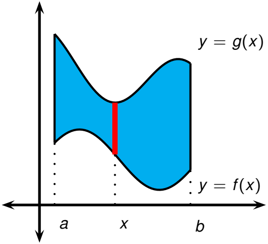


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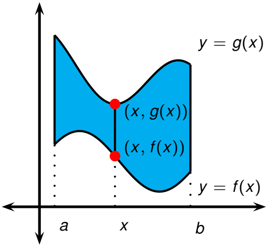


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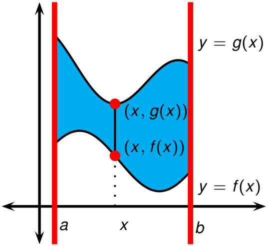


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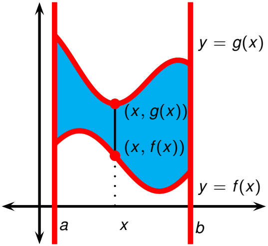


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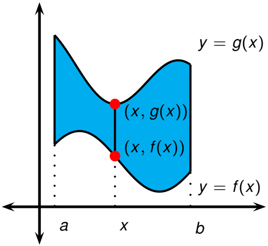


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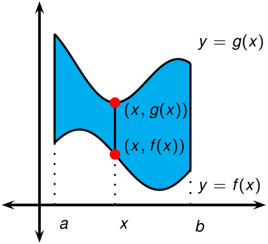


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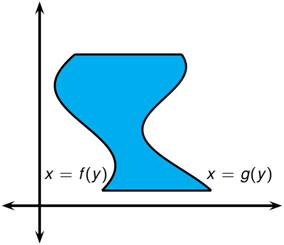

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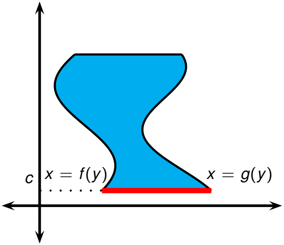


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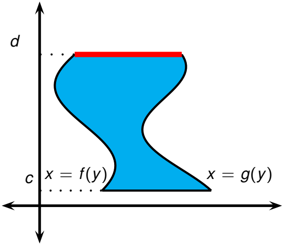


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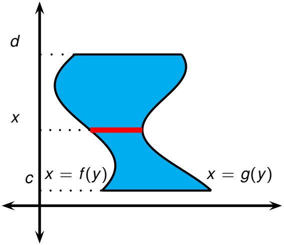


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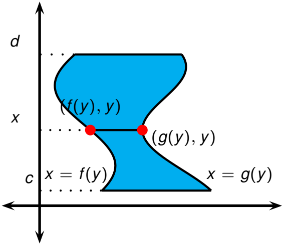


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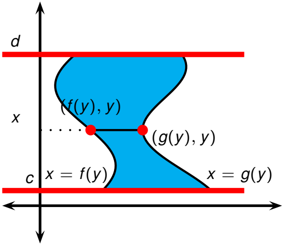


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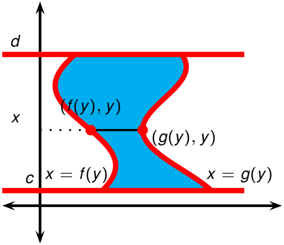


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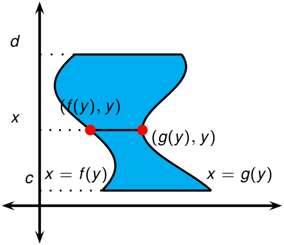


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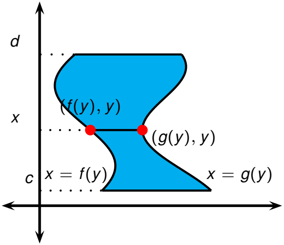


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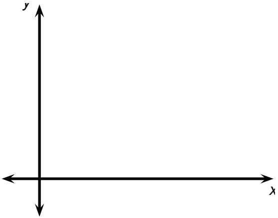
# Strategy for Computing a Double Integral




## Problem

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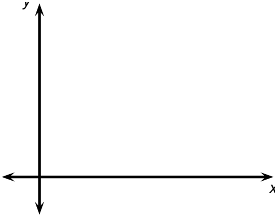


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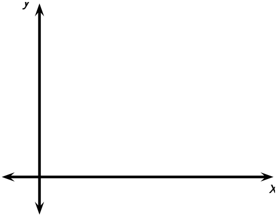


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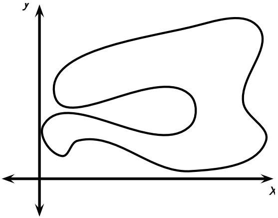


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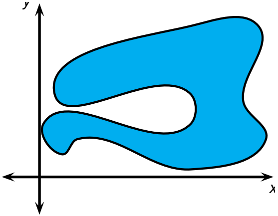


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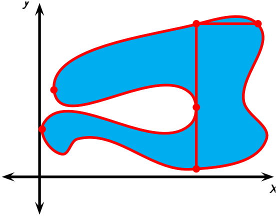


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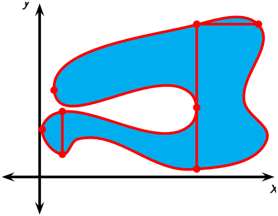


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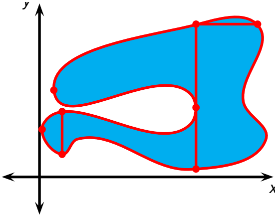


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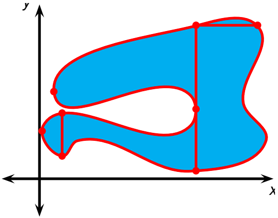


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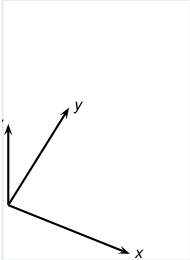
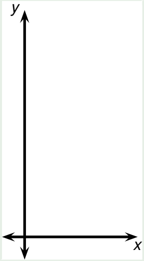
- Our strategy will be augmented/combined later with variable changes (via the multivariable substitution rule).

## Example




Let $\mathcal{R}$ be the region bounded by $y = 2x$ and $y = x^2$. Compute

$$\iint_{\mathcal{R}} \frac{1}{8}\left(x^2 + y^2\right) \mathrm{d}x\mathrm{d}y$$

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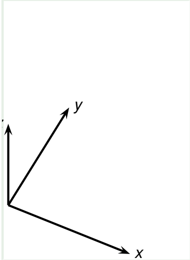


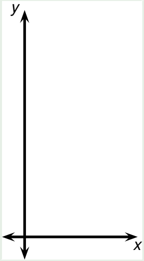


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Plot $y = 2x$.

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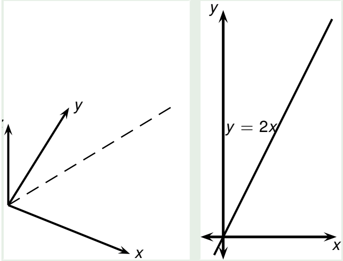


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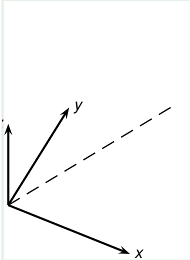


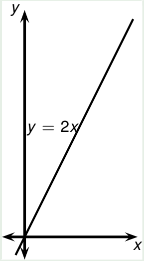


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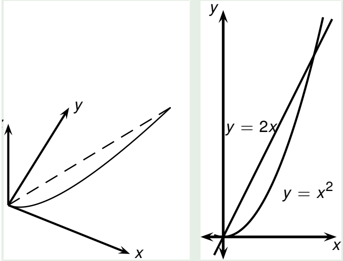


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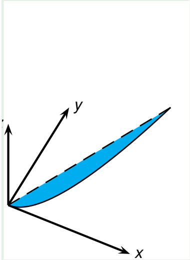


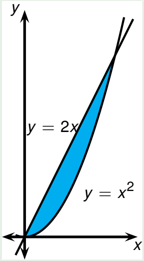


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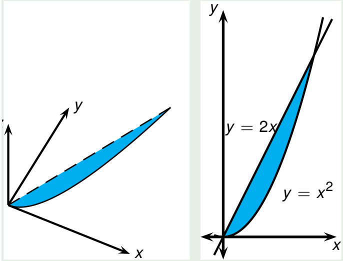


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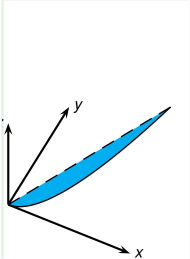
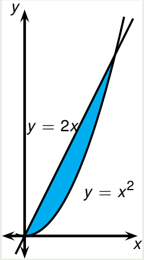
The two curves intersect when **?**

# Example

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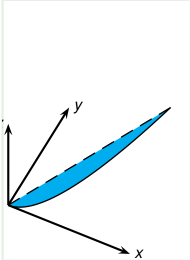
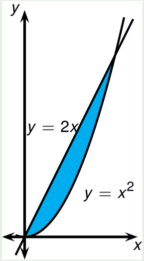
Plot $y = 2x$. Plot $y = x^2$. Identify the region.

The two curves intersect when
$$\begin{array}{rcl} x^2 &=& 2x \\ x(x-2) &=& 0 \\ x &=& 0 \text{ or } 2. \end{array}$$

## Example

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$$\iint_{\mathcal{R}} \frac{1}{8}\left(x^2 + y^2\right) dxdy$$

Plot $y = 2x$. Plot $y = x^2$. Identify the region.

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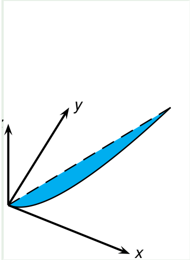


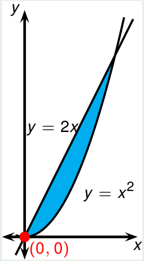


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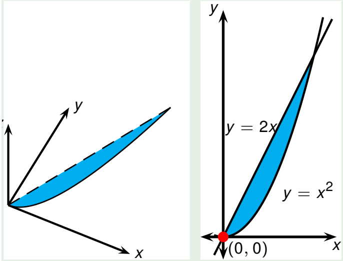


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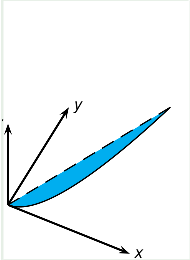

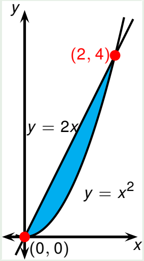


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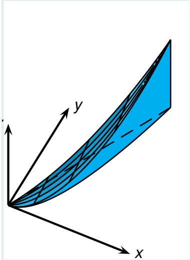


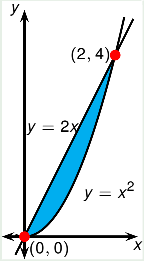


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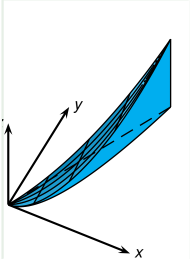


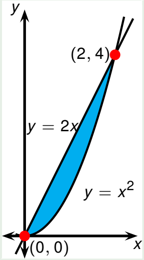


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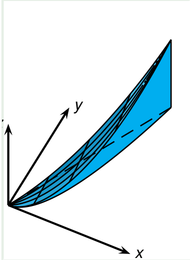
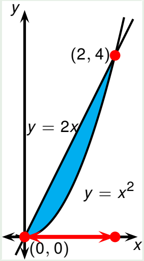
$$\int_{x=?}^{x=?} \left( \int \frac{1}{8}\left(x^2 + y^2\right) \mathrm{d}y \right) \mathrm{d}x$$

## Example

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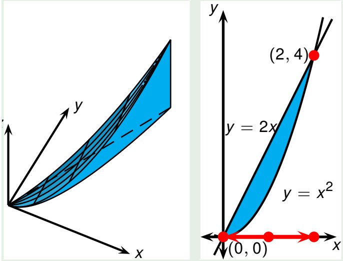


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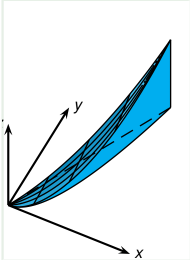


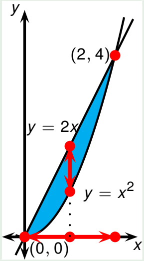


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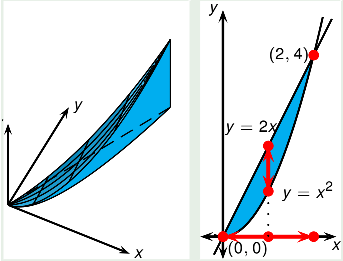


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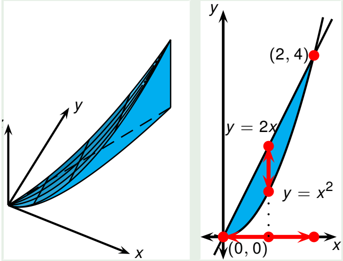


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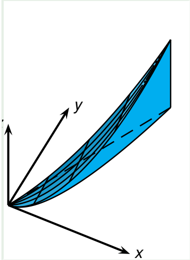


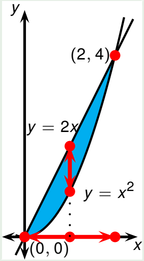


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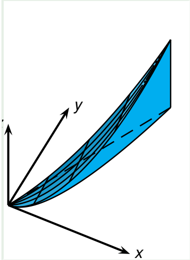


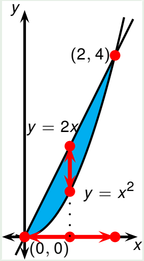


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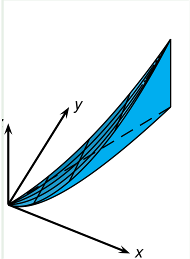
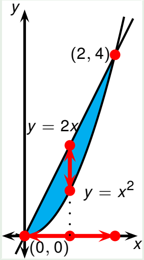
## Example

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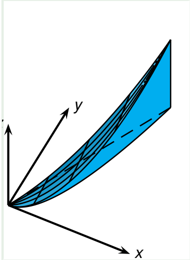

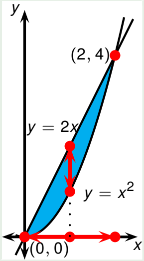


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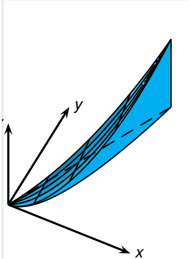


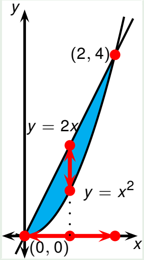


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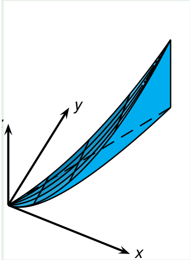


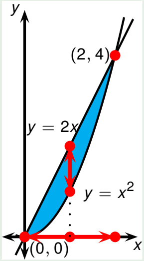


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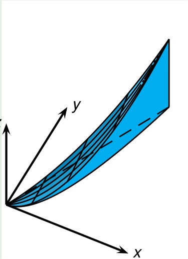
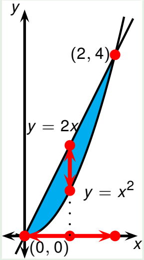
## Example

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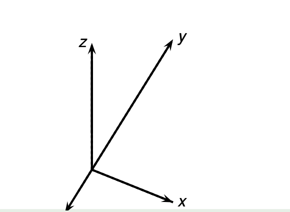
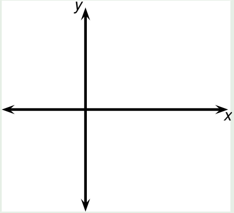
$$= \frac{27}{35}$$

## Example

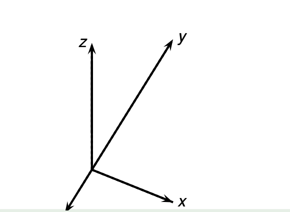


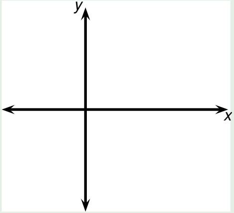




Let $\mathcal{R}$ be the region bounded by $y = x - 1$ and $y^2 = 2x + 6$. Compute

$$\iint_{\mathcal{R}} \left(2 + \frac{1}{4}xy\right) \mathrm{d}x\mathrm{d}y.$$

## Example

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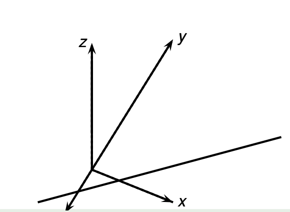
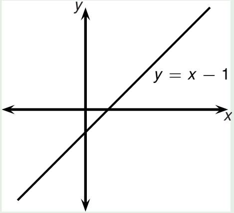
Plot $x - 1$.

## Example

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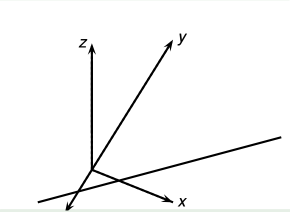


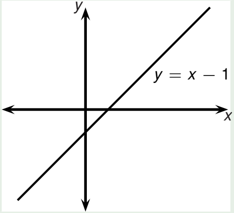


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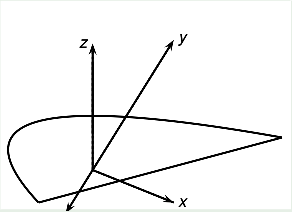


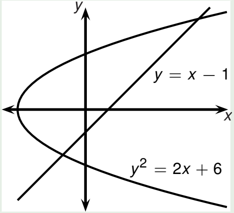


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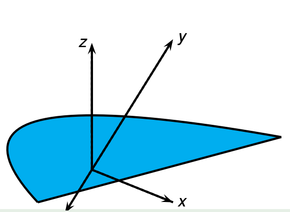


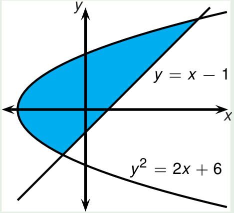


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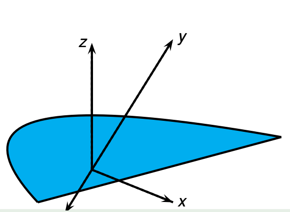


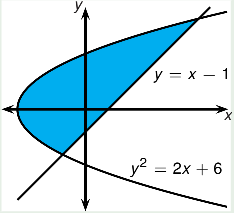


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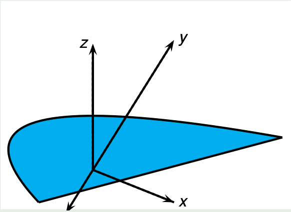


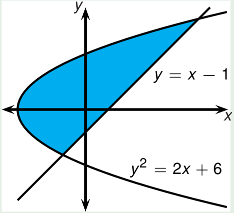


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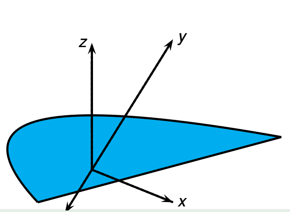


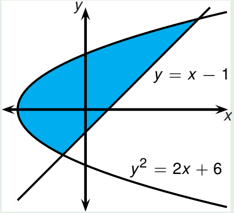


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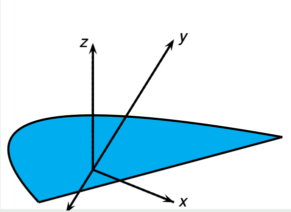


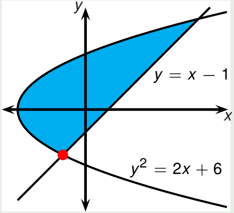


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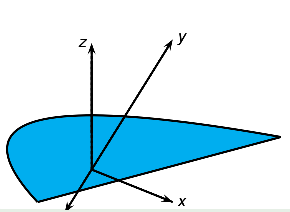


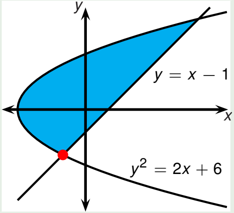


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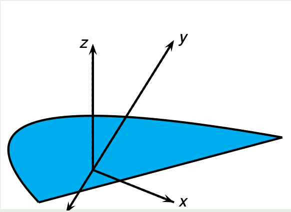


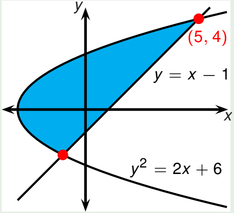


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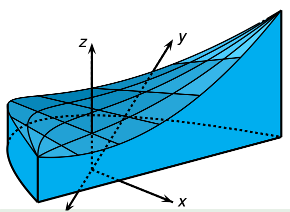


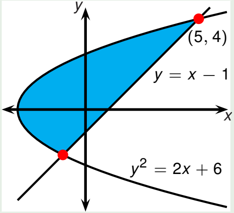


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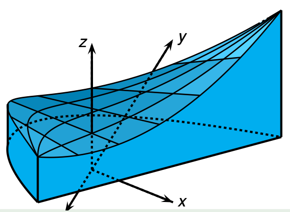


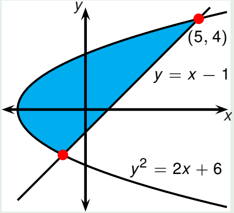


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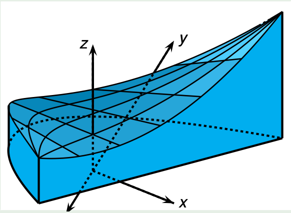


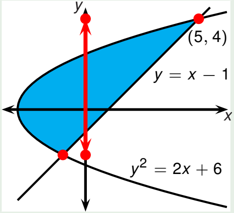


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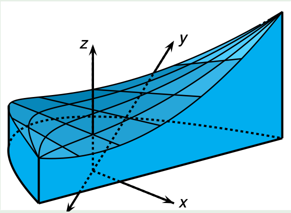


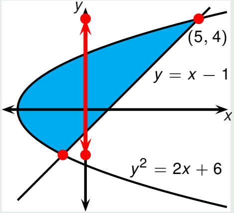


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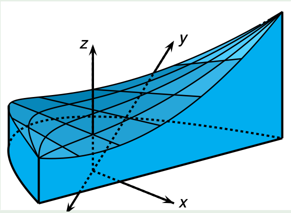


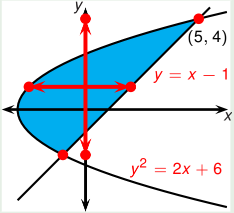


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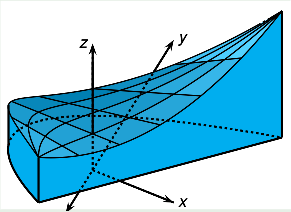


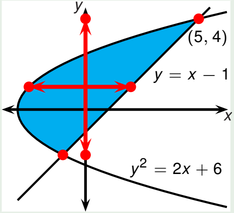


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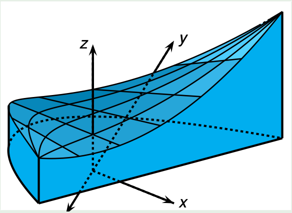


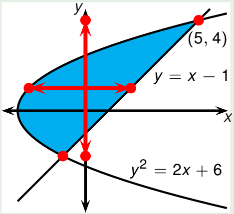


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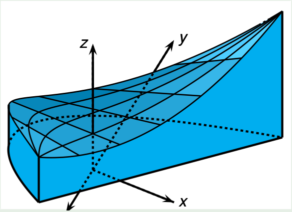


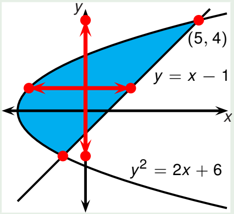


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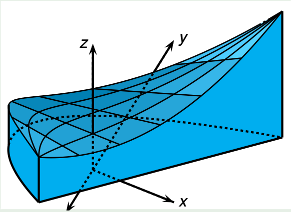


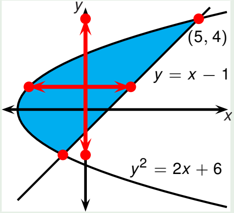


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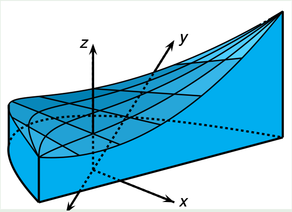


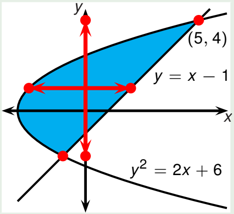


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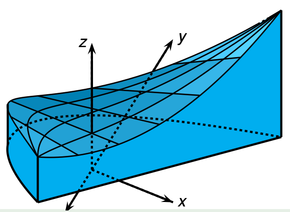


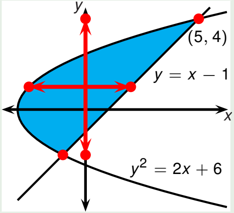


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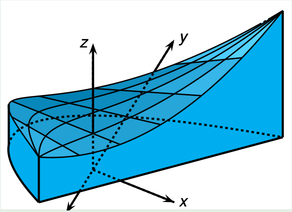
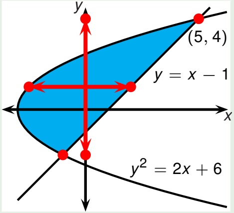
$$= \int_{y=-2}^{y=4} \left(-\frac{1}{32}y^5 + \frac{1}{2}y^3 - \frac{3}{4}y^2 + y + 8\right) \mathrm{d}y$$

## Example




Let $\mathcal{R}$ be the region bounded by $y = x - 1$ and $y^2 = 2x + 6$. Compute

$$\iint_{\mathcal{R}} \left(2 + \frac{1}{4}xy\right) dxdy.$$

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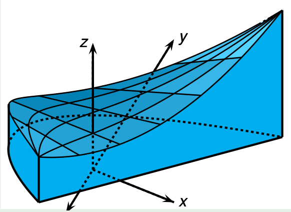
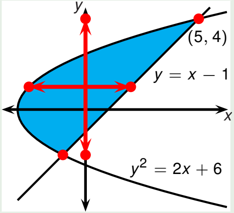
$$= \left[ ? \right]_{-2}^{4}$$

## Example

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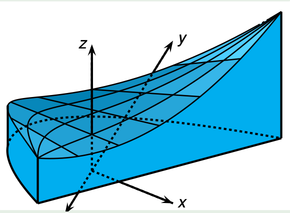
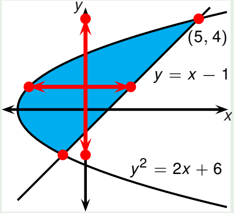
$$\begin{aligned}\int_{y=-2}^{y=4}\int_{x=\frac{y^2-6}{2}}^{x=y+1}\left(2+\frac{1}{4}xy\right)dxdy &= \int_{y=-2}^{y=4}\left[2x+\frac{x^2y}{8}\right]_{x=\frac{y^2-6}{2}}^{x=y+1}dy\\ &= \int_{y=-2}^{y=4}\left(-\frac{1}{32}y^5+\frac{1}{2}y^3 -\frac{3}{4}y^2+y+8\right)dy\\ &= \left[-\frac{1}{192}y^6+\frac{1}{8}y^4 -\frac{1}{4}y^3+\frac{1}{2}y^2+8y\right]_{-2}^{4}\end{aligned}$$

## Example

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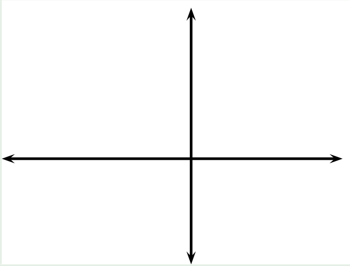
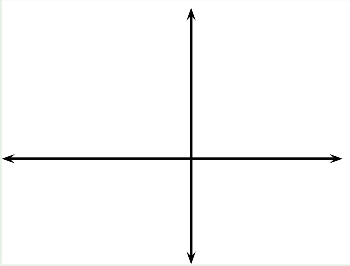
## Example



Let $\mathcal{R}$ be region bounded by $y = (x+1)^2$, $x = y - y^3$, the line $x = -1$ and the line $y = -1$. Set-up iterated integrals for

$$\iint_{\mathcal{R}} f \mathrm{d}A.$$

## Example



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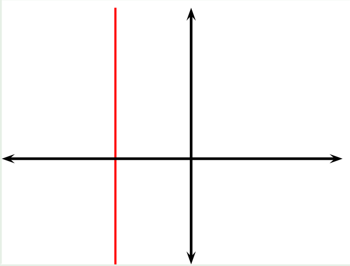
Plot $x = -1$.

## Example



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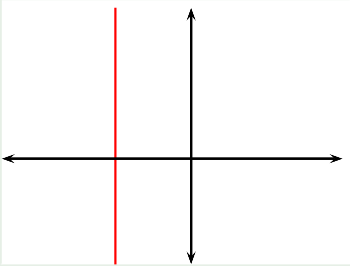

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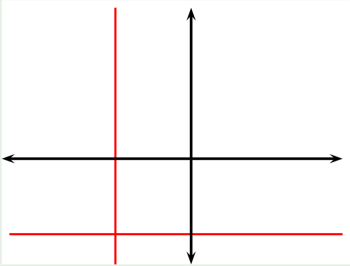

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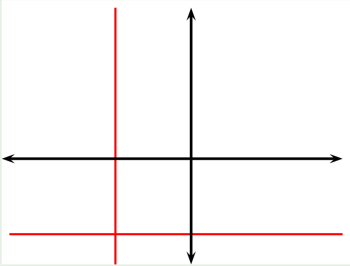

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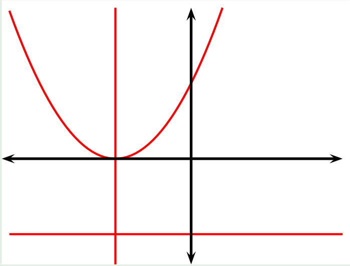

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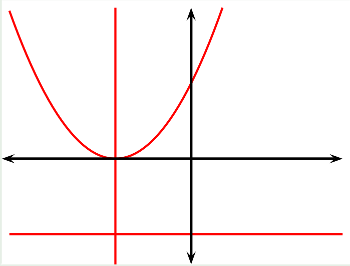

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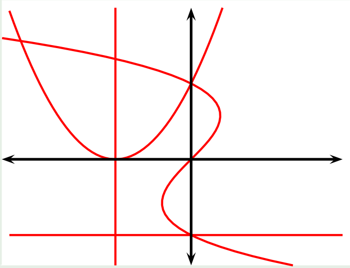

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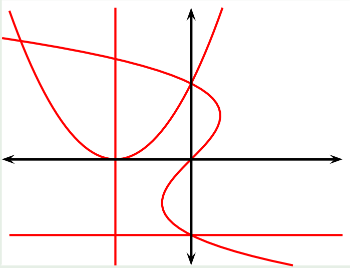

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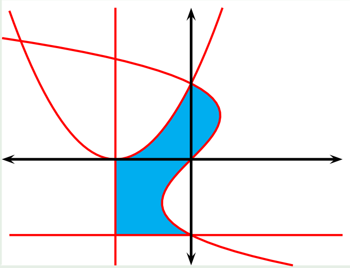
Plot $x = -1$. Plot $y = -1$. Plot $y = (x+1)^2$. Plot $x = y - y^3$. Identify the region.

## Example



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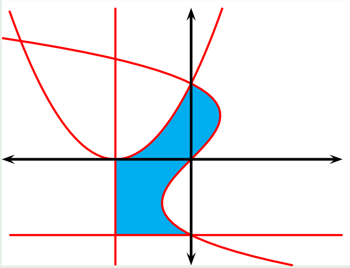

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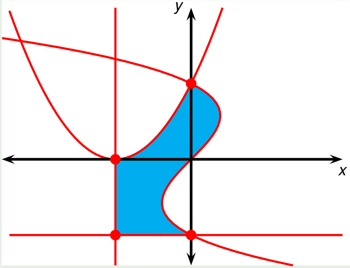
**?**

## Example

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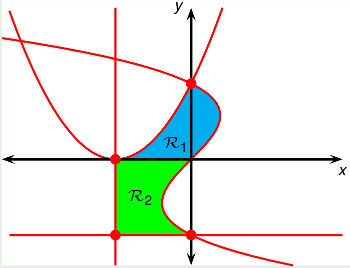
$(-1, -1), (0, -1), (-1, 0), (0, 1)$.

## Example

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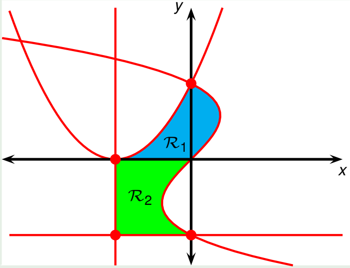
## Example

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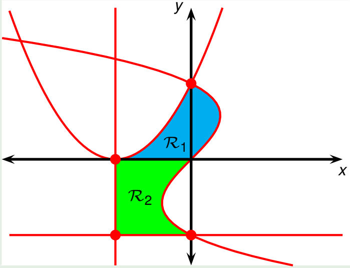
$$\iint_{\mathcal{R}_1} f dA + \iint_{\mathcal{R}_2} f dA = \int \quad \int \quad f\,\mathbf{?} \quad + \int \quad \int \quad f\,\mathbf{?}$$

## Example

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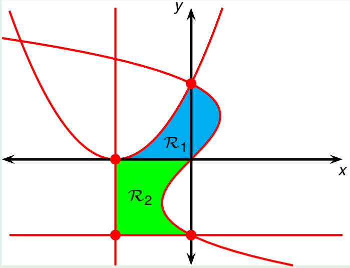


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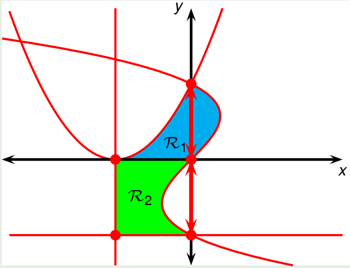
## Example

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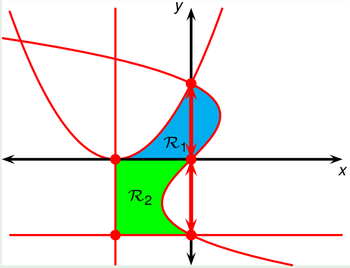
## Example

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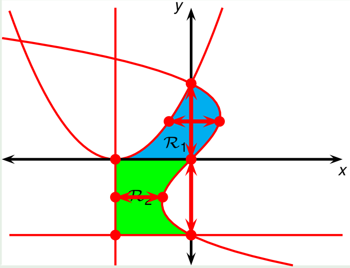
## Example

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## Example

$\iint_{[0,\infty)\times[0,\infty)} e^{-x-y} \mathrm{d}x\mathrm{d}y$

## Example

$\int_0^1 \int_{3y}^3 e^{x^2} \mathrm{d}x\mathrm{d}y$