Calculus III
Lecture 2

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https://github.com/tmilev/freecalc

2020

Outline

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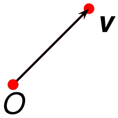
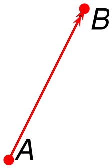
Definition of vector



- A *position vector* **v** (simply - *vector*) is a point in a space where there's a fixed preferred point O.
- Preferred point O is called the origin.
- If not given by O, vector is depicted by arrow from O to defining point.
- Vector given by origin = zero vector **0**.
- Points & vectors can be identified but:
  - use term "vector" $\Rightarrow$ space has preferred origin point;
  - if we specifically allow point/vector addition we use the term "vector" instead of "point";
  - when we do not intend to carry out addition operations we use the term "point" instead of "vector".
- We will soon equip vectors with two operations, vector addition and multiplication by scalars.

Displacement Vectors



Definition

A displacement vector is an ordered pair of points (A, B).

- When $A \neq B$, represent as arrow, A - tail B- head.
- Define displacement vector magnitude (A, B) to be the length of the segment $|AB|$.
- If $A \neq B$ the direction of the displacement vector is defined as the ray starting at A and passing through B.
- If $A = B$:
  - displacement vector has zero magnitude and non-specified direction
  - (A, A): zero displacement vector at point A.

Equality and Equivalence of Displacement Vectors

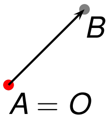
- We define two displacement vectors (A, B) and (D, C) to be equal if $A = D$ and $B = C$.
- Equal displacement vectors $\to$ same magnitude and direction.
- Same magnitude and direction $\not\to$ equal displacement vectors.
- We define two displacement vectors to be equivalent if they have the same magnitude and direction. We write $(A, B) \equiv (D, C)$.
- $$(A, B) \equiv (D, C) \iff ABCD \text{ is a parallelogram } .$$

Position vectors via displacement vectors

- Suppose we have space without chosen origin.
- To each displacement vector (A, B), assign position vector by choosing origin to be the tail A and giving the position vector by the head B.
- We are ready to give "origin-free" alternative definition/interpretation of vector.

Definition (Alternative definition/interpretation of position vector)

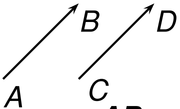
Define a position vector as the set that consists all displacement vectors equivalent to one fixed displacement vector.



- Definitions are technically different but equivalent.
- We choose which def. to use according to application.
- The set of zero displacement vectors with arbitrary tail points = zero position vector, $\mathbf{0}$.

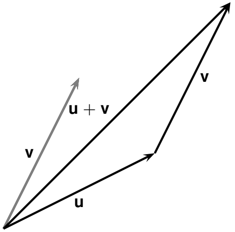
Additional notation for position vectors

- In preceding slide: each position vector $\boldsymbol{u}$ can be thought of as a set of equivalent displacement vectors.
- So we can represent position vectors via displacement vectors.
- For two points A, B define the position vector $\overrightarrow{AB}$ or $\boldsymbol{AB}$ as the vector represented by the displacement vector (A, B).
- $\Rightarrow$ it's allowed to represent position vectors as arrows with tails not necessarily at origin.



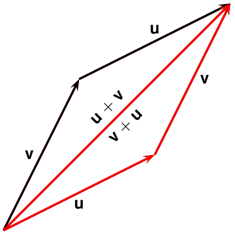
$$\boldsymbol{u} = \boldsymbol{AB} = \boldsymbol{CD}$$

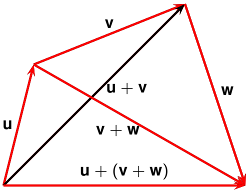
Addition of Vectors



- **Triangle Rule.** Define sum of position vectors **u** and **v** as follows.
- Attach representative displacement vectors head to tail.
- Declare the sum to be the position vector with the tail of the first displacement vector and the head of the second displacement vector.

Properties of addition





- Addition is commutative (parallelogram rule):

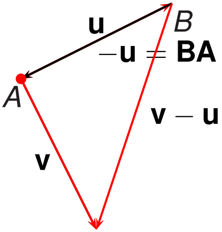
$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}.$$

- Addition is associative:

$$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$$

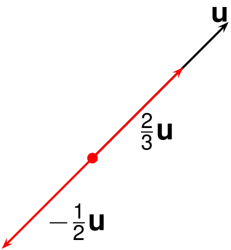
- As usual we write $\mathbf{u}+\mathbf{v}+\mathbf{w} = (\mathbf{u}+\mathbf{v})+\mathbf{w} = \mathbf{u}+(\mathbf{v}+\mathbf{w})$.

Difference of vectors



- Let $\mathbf{u} = \mathbf{AB}$.
- We define $-\mathbf{u}$ to be the vector for which $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
- Since $\mathbf{AB} + \mathbf{BA} = \mathbf{0}$, it follows $-\mathbf{u} = \mathbf{BA}$.
- In other words $-\mathbf{u}$ is depicted using the arrow opposite to **u**.
- From picture, it's evident $-\mathbf{u}$ can be chosen one way only.
- We define the difference of vectors $\mathbf{v}, \mathbf{u}$ via $\mathbf{v} - \mathbf{u} = (-\mathbf{u}) + \mathbf{v}$ (triangle rule).

Linear Combinations

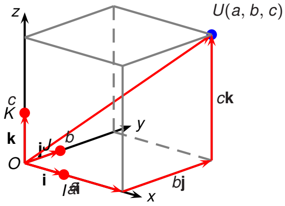


- Let $\mathbf{u}$ be vector, c be a real number (scalar).
- Define the product of the vector $\mathbf{u}$ and the scalar c as follows.
  - If $c > 0$ define $c\mathbf{u}$ as the vector:
    - with the same direction
    - with magnitude proportional with coefficient c to the magnitude of $\mathbf{u}$, i.e., $|c\mathbf{u}| = c|\mathbf{u}|$.
  - If $c < 0$ define $c\mathbf{u}$ as the vector $(-c)(-\mathbf{u})$, i.e, as the vector:
    - with opposite direction
    - with magnitude $|c\mathbf{u}| = |(-c)(-\mathbf{u})| = (-c)|-\mathbf{u}| = |c||\mathbf{u}|$
  - If $c = 0$ then define $c\mathbf{u} = 0\mathbf{u} = \mathbf{0}$.
- If $c_1, \dots, c_n$ are scalars and $\mathbf{u}_1, \dots, \mathbf{u}_n$ are vectors, we say

$$\mathbf{v} = c_1\mathbf{u}_1 + \dots + c_n\mathbf{u}_n$$

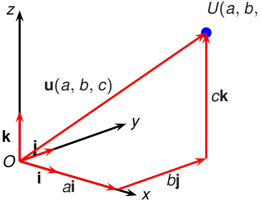
is a *linear combination* of the vectors $\mathbf{u}_1, \dots, \mathbf{u}_n$.

Vectors in Coordinates



- Fix coordinate system *Oxyz*.
- Let I, J, K be the points giving the units on the x, y, z axes as indicated.
- Define $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ to be the unit vectors $\mathbf{OI}$, $\mathbf{OJ}$, $\mathbf{OK}$.

- Let $\mathbf{u} = \mathbf{OU}$ be a vector.
- Let U have coordinates (a, b, c).
- Then $\mathbf{u} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$.
- This follows from the point-vector identification.



- From preceding: arbitrary vector $\mathbf{u} = \mathbf{OU}$ can be decomposed as $\mathbf{u} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, where (a, b, c): Cartesian coordinates of U.
- Thus $\mathbf{u}$ is identified with the triple of numbers (a, b, c).

- Under the first definition of vector, a vector is simply a point in a vector space (=space with a distinguished point).
- From now on, we assume the first definition of vector: we use the notation (a, b, c) both for points in vector spaces (vectors) and points in spaces not equipped with vector space structure.
- Under the second alternative definition of vector, there is a formal distinction between points and vectors.
- Some authors who use the second definition use the notation $\langle a, b, c\rangle$ to denote vectors and (a, b, c) to denote points.

Operations in Coordinates

- Vector magnitude is given by

$$|(u_1, u_2, u_3)| = |OP| = \sqrt{u_1^2 + u_2^2 + u_3^2}$$

- Vector addition is given by:

$$(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2) \ .$$

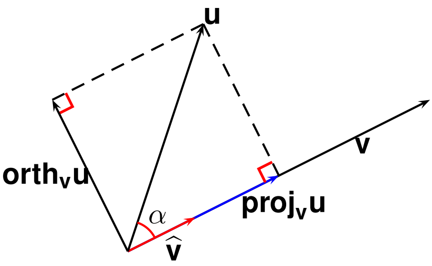
- Scalar multiple is given by:

$$c(x, y, z) = (cx, cy, cz) \ .$$

- Let $A(x_A, y_A, z_A)$ and $B(x_B, y_B, z_B)$ be points. Then

$$\mathbf{AB} = \mathbf{AO} + \mathbf{OB} = \mathbf{OB} - \mathbf{OA} = (x_B - x_A, y_B - y_A, z_B - z_A).$$

Dot Product



- Let $\mathbf{u}$, $\mathbf{v}$ vectors, $\mathbf{v} \neq \mathbf{0}$.
- Denote by $\mathbf{proj_v u}$ the projection of $\mathbf{u}$ along $\mathbf{v}$.
- Denote by $\mathrm{comp}_\mathbf{v}\mathbf{u}$ the magnitude of $\mathbf{proj_v u}$, i.e., $|\mathbf{proj_v u}| = \mathrm{comp}_\mathbf{v}\mathbf{u}$
- Denote by $\mathbf{orth_v u}$ the projection of $\mathbf{u}$ in direction orthogonal to $\mathbf{v}$.
- $\mathbf{u} = \mathbf{orth_v u} + \mathbf{proj_v u}$.
- We have $\widehat{\mathbf{v}} = \frac{1}{|\mathbf{v}|}\mathbf{v}$ is the unit vector along $\mathbf{v}$.
- Let α: angle between $\mathbf{v}$ and $\mathbf{u}$.
- Then $\mathrm{comp}_\mathbf{v}\mathbf{u} = \cos\alpha|\mathbf{u}|$. Therefore $\mathbf{proj_v u} = \cos\alpha|\mathbf{u}|\widehat{\mathbf{v}}$.
- Define dot product of $\mathbf{u}$ and $\mathbf{v}$:

$$\mathbf{u}\cdot\mathbf{v} = \cos\alpha|\mathbf{u}||\mathbf{v}|.$$

The Dot Product

- If $\mathbf{v} = \mathbf{0}$ or $\mathbf{u} = \mathbf{0}$, then $\mathbf{u} \cdot \mathbf{v} = 0$.
- If $\mathbf{u} \neq \mathbf{0} \neq \mathbf{v}$
$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos \alpha,$$
where α is any angle between $\mathbf{u}$ and $\mathbf{v}$.
- If $\mathbf{u} \neq \mathbf{0} \neq \mathbf{v}$, then
$$\mathbf{u} \cdot \mathbf{v} = 0 \iff \mathbf{u} \perp \mathbf{v} \; .$$
- The dot product is commutative: $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
- $\mathbf{u} \cdot \mathbf{v} = (\mathbf{proj_v u}) \cdot \mathbf{v}$.
- The dot product is linear in each argument:
$$(a\mathbf{u} + b\mathbf{w}) \cdot \mathbf{v} = a\mathbf{u} \cdot \mathbf{v} + b\mathbf{w} \cdot \mathbf{v}$$
$$\mathbf{u} \cdot (a\mathbf{v} + b\mathbf{w}) = a\mathbf{u} \cdot \mathbf{v} + b\mathbf{u} \cdot \mathbf{w}$$
- Dot product is positive definite:
$$\mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2 \geq 0$$
$$\mathbf{v} \cdot \mathbf{v} = 0 \Leftrightarrow \mathbf{v} = \mathbf{0}$$

Computations in Coordinates

- Let $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ unit vectors along axes.
- Distinct unit vectors are $\perp \Rightarrow \cos$ of angle b-n them is $0 = \cos \frac{\pi}{2}$.
- Therefore $\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{j} = \mathbf{i} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0$.
- $\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1$

Theorem (Can be taken as definition)

$$\mathbf{u} \cdot \mathbf{v} = (u_1, u_2, u_3) \cdot (v_1, v_2, v_3) = u_1 v_1 + u_2 v_2 + u_3 v_3 \; .$$

Proof.

$$\begin{aligned}
\mathbf{u} \cdot \mathbf{v} &= (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \cdot (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) \\
&= \quad u_1 v_1 \mathbf{i} \cdot \mathbf{i} \; + u_1 v_2 \mathbf{i} \cdot \mathbf{j} \; + u_1 v_3 \mathbf{i} \cdot \mathbf{k} \\
&\quad + u_2 v_1 \mathbf{j} \cdot \mathbf{i} \; + u_2 v_2 \mathbf{j} \cdot \mathbf{j} \; + u_2 v_3 \mathbf{j} \cdot \mathbf{k} \\
&\quad + u_3 v_1 \mathbf{k} \cdot \mathbf{i} + u_3 v_2 \mathbf{k} \cdot \mathbf{j} + u_3 v_3 \mathbf{k} \cdot \mathbf{k} \\
&= u_1 v_1 + u_2 v_2 + u_3 v_3 \quad .
\end{aligned}$$

$\square$

Length via dot product

Let $\mathbf{u} = (u_1, u_2, u_3)$. Recall $|\mathbf{u}| = \sqrt{u_1^2 + u_2^2 + u_3^2}$.

Observation

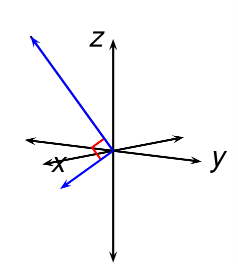
$$|\mathbf{u}| = \sqrt{\mathbf{u} \cdot \mathbf{u}} \quad .$$

$$|\mathbf{u}|^2 = \mathbf{u} \cdot \mathbf{u} \quad .$$

Example

$$(1, 2, 3) \cdot (6, 5, 4) = 1 \cdot 6 + 2 \cdot 5 + 3 \cdot 4 = 28$$

Example

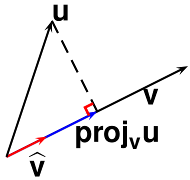


Are the vectors $(1,-2,3)\cdot(1,-1,-1)$ perpendicular?

$$(1,-2,3)\cdot(1,-1,-1) = 1\cdot1+(-1)\cdot(-2)+3\cdot(-1) = 0,$$

therefore the vectors are perpendicular. Is this apparent from the picture? Not unless the two vectors lie in a plane parallel to the surface of the page/computer screen.

Projections in coordinates



$$\begin{aligned}\mathbf{u} &= u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k} = (u_1, u_2, u_3)\\ \mathbf{v} &= v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k} = (v_1, v_2, v_3)\end{aligned}$$

Theorem

$$\begin{aligned}comp_{\mathbf{v}}\mathbf{u} &= \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|} = \frac{u_1v_1 + u_2v_2 + u_3v_3}{\sqrt{v_1^2 + v_2^2 + v_3^2}}\\ \mathbf{proj_v u} &= (comp_{\mathbf{v}}\mathbf{u})\,\widehat{\mathbf{v}} = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|}\frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\mathbf{v} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v}\quad.\end{aligned}$$

Example

Let $\mathbf{u} = (1,2,3)$, $\mathbf{v} = (6,5,4)$.

- Compute the scalar projection $\mathrm{comp}_{\mathbf{v}}\mathbf{u}$ of $\mathbf{u}$ onto $\mathbf{v}$.
- Compute the vector projection $\mathbf{proj}_{\mathbf{v}}\mathbf{u}$ of $\mathbf{u}$ onto $\mathbf{v}$.
- Compute the orthogonal component $\mathbf{orth}_{\mathbf{v}}\mathbf{u}$.

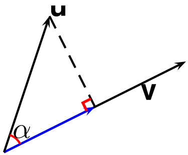
$$\mathrm{comp}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|}$$

$$\mathrm{comp}_{(6,5,4)}(1,2,3) = \frac{(6,5,4)\cdot(1,2,3)}{\sqrt{(6,5,4)\cdot(6,5,4)}} = \frac{6\cdot 1 + 5\cdot 2 + 4\cdot 3}{\sqrt{6^2+5^2+4^2}} = \frac{28}{\sqrt{77}}$$

$$\mathbf{proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\mathbf{v}$$

$$\mathbf{proj}_{(6,5,4)}(1,2,3) = \tfrac{28}{77}(6,5,4) = \left(\tfrac{24}{11}, \tfrac{20}{11}, \tfrac{16}{11}\right)$$
$$\mathbf{orth}_{(6,5,4)}(1,2,3) = (1,2,3) - \mathbf{proj}_{(6,5,4)}(1,2,3) = \left(-\tfrac{13}{11}, \tfrac{2}{11}, \tfrac{17}{11}\right)$$

Angles



$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= |\mathbf{u}||\mathbf{v}|\cos\alpha\\ \cos\alpha &= \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}\\ \alpha &= \arccos\left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}\right)\end{aligned}$$

Let $\alpha = \angle(\mathbf{u}, \mathbf{v})$.

Example

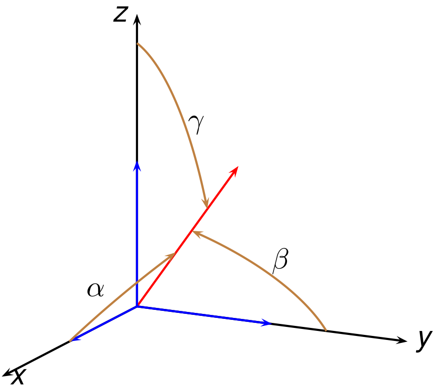
Compute the angle $\angle((1, 2, 3), (6, 5, 4))$.

$$\begin{aligned}\alpha &= \arccos\left(\frac{(1,2,3)\cdot(6,5,4)}{|(1,2,3)||(6,5,4)|}\right)\\ &= \arccos\left(\frac{28}{\sqrt{14}\sqrt{77}}\right) = \arccos\left(\frac{4}{\sqrt{22}}\right)\\ &\approx 0.549467 \approx 31.482^\circ\end{aligned}$$

Direction Angles

Definition

The direction angles α, β, γ of the vector **u** are defined as the angles between **u** and the unit vectors **i**, **j**, **k** (in the same order).



$$\begin{aligned}\mathbf{u} &= (u_1, u_2, u_3)\\ \alpha &= \angle(\mathbf{u}, \mathbf{i})\\ \beta &= \angle(\mathbf{u}, \mathbf{j})\\ \gamma &= \angle(\mathbf{u}, \mathbf{k})\ .\end{aligned}$$

$$\cos\alpha = \frac{\mathbf{u}\cdot\mathbf{i}}{|\mathbf{u}|\,|\mathbf{i}|} = \frac{u_1}{\sqrt{u_1^2+u_2^2+u_3^2}}$$

Similarly for $\cos\beta$ and $\cos\gamma$. Then:

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1\ .$$