Precalculus
Lecture 7
Trigonometric Graphs

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https://github.com/tmilev/freecalc

2020

Outline

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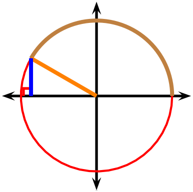
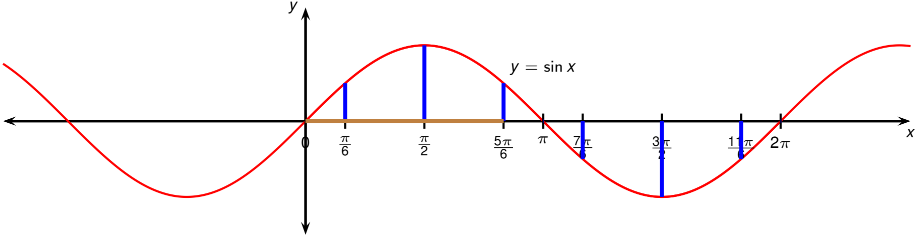
Graph of $\sin x$

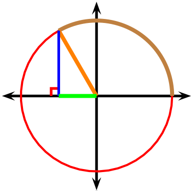


x	0	$\frac{\pi}{6}$	$\frac{\pi}{2}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{3\pi}{2}$	$\frac{11\pi}{6}$	2π
$\sin x$	0	$\frac{1}{2}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	-1	$-\frac{1}{2}$	0

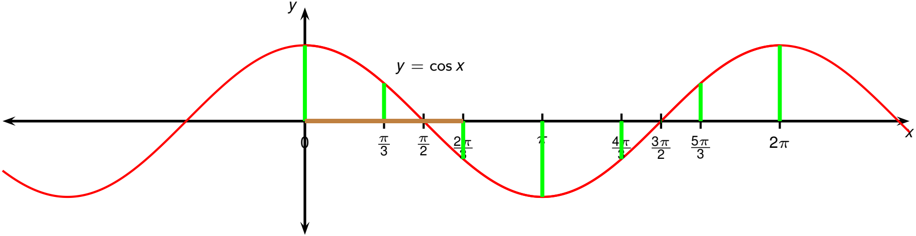


The graph of $\sin x$ is 2π-periodic so the rest of the graph can be inferred from the interval $[0, 2\pi]$.

Graph of $\cos x$



x	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	2π
$\cos x$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	0



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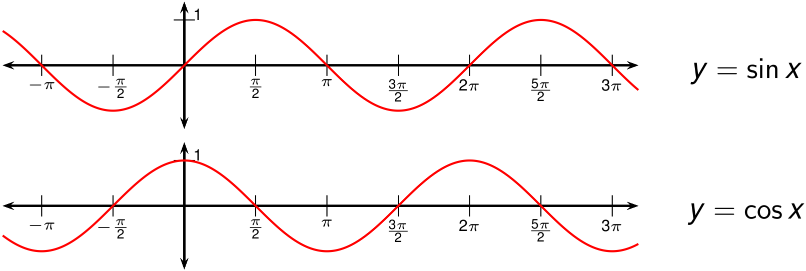
Graphs of the Trigonometric Functions



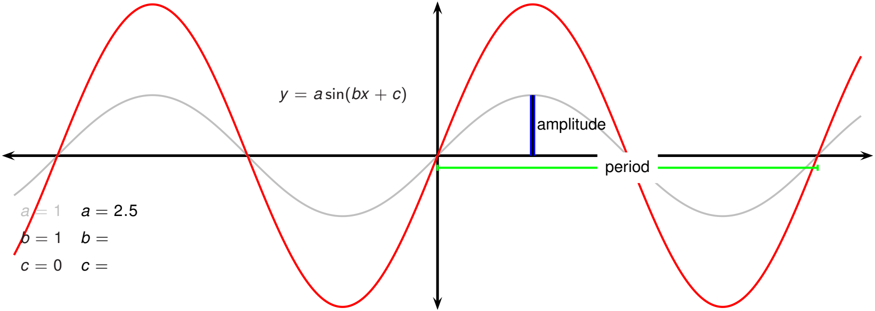
- $\sin x$ has zeroes at $n\pi$ for all integers n.
- $\cos x$ has zeroes at $\frac{\pi}{2} + n\pi$ for all integers n.
- $-1 \leq \sin x \leq 1$.
- $-1 \leq \cos x \leq 1$.
- If we translate the graph of $\cos x$ by $\frac{\pi}{2}$ units to the right we get the graph of $\sin x$. This is a consequence of $\cos\left(x - \frac{\pi}{2}\right) = \sin x$.

- The graph of $a\sin(bx+c)$ is referred to as a "wave".

Definition (Phase, period, frequency, amplitude of a wave)

In the function $a\sin(bx+c)$, the number $|a|$ is called the *amplitude* of the wave, the number $\frac{b}{2\pi}$ is called the *frequency* of the wave, the number $\frac{2\pi}{b}$ is called the *period* of the wave, the number c is called the *phase* of the wave.

- What happens when we change the amplitude? The frequency/period? The phase?



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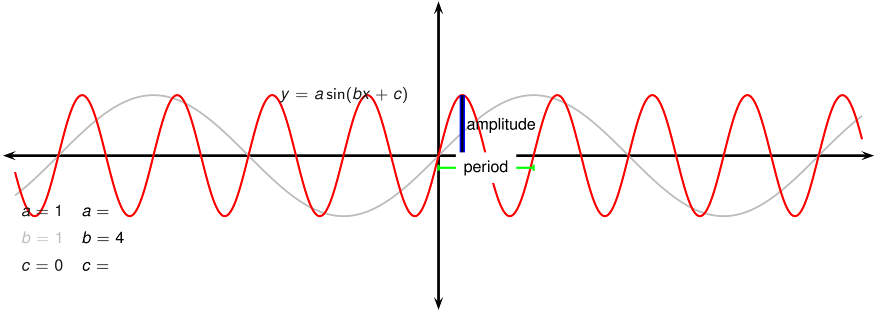

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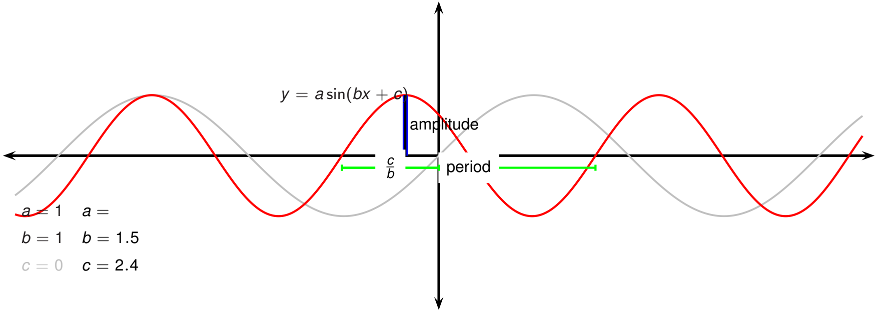
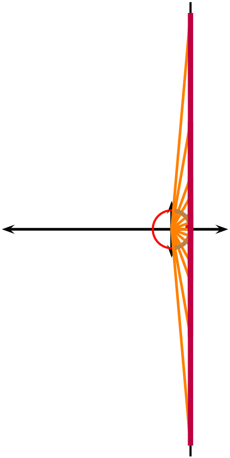
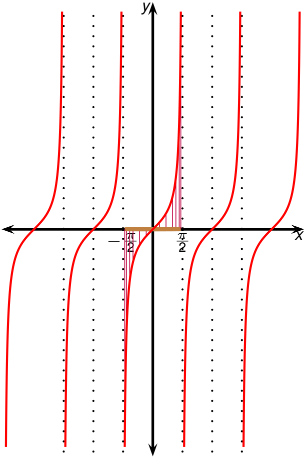
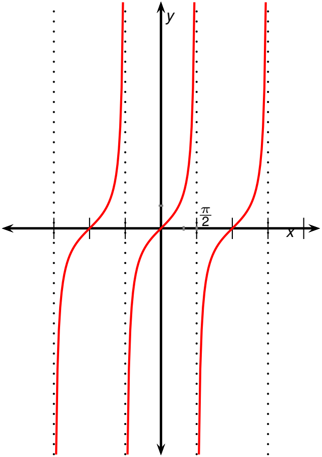


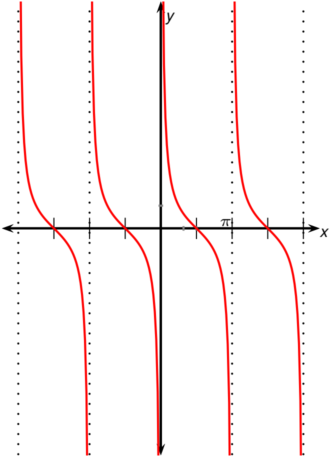
Graph of $\tan x$





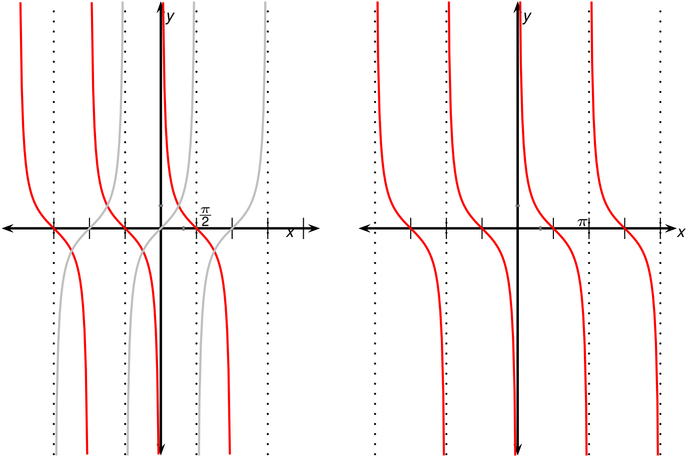
Near $\pm\frac{\pi}{2}$ the graph of $\tan x$ approaches $\pm\infty$. The graph of $\tan x$ is π-periodic so the rest of the graph can be inferred from the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.





$y = \tan x$ $\qquad$ $y = \cot x$

If we move the graph of $\tan x$ by $\frac{\pi}{2}$ units to the left (or right) and reflect across the x axis, we get the graph of $\cot x$. This follows from $\tan\left(x \pm \frac{\pi}{2}\right) = -\cot x$.



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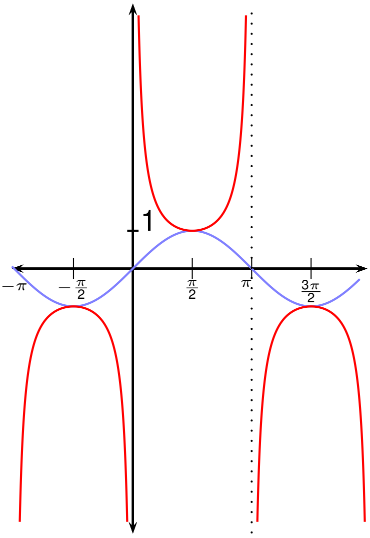
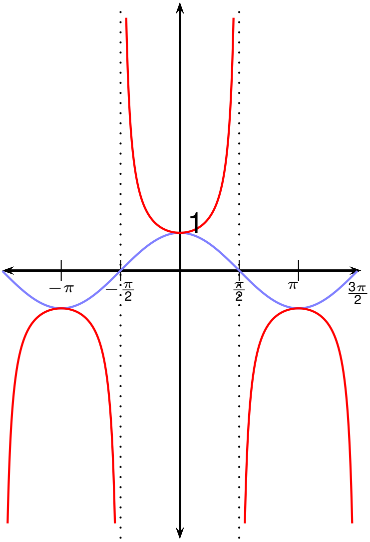
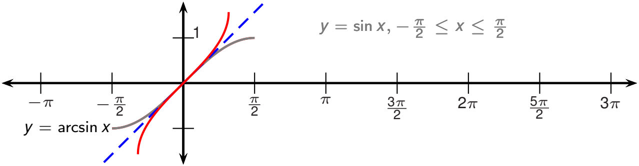

1
$-\pi$
$-\frac{\pi}{2}$
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$

$y = \csc x$

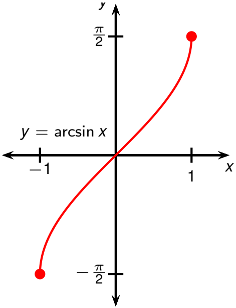

1
$-\pi$
$-\frac{\pi}{2}$
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$

$y = \sec x$

Inverse Trigonometric Functions



- $\sin x$ isn't one-to-one.
- It is if we restrict the domain to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.
- Then it has an inverse function.
- We call it arcsin or $\sin^{-1}$.
- $\arcsin x = y \Leftrightarrow \sin y = x$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.



Observation

- $\arcsin y =$ *the appropriate angle whose sine equals y.*
- *Important: the output angle must lie in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.*

Example

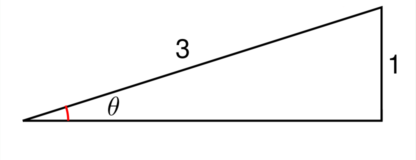
Find $\arcsin\left(\dfrac{1}{2}\right)$.

- $\sin\left(\dfrac{\pi}{6}\right) = \dfrac{1}{2}$.
- $-\dfrac{\pi}{2} \leq \dfrac{\pi}{6} \leq \dfrac{\pi}{2}$.
- Therefore $\arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$.

Example

Find $\tan\left(\arcsin\left(\frac{1}{3}\right)\right)$.

- Let $\theta = \arcsin\left(\frac{1}{3}\right)$, so $\sin\theta = \frac{1}{3}$.
- Draw a right triangle with opposite side 1 and hypotenuse 3.
- Let the angle θ be as labeled. Then $\sin\theta = \frac{1}{3}$ and so $\theta = \arcsin\left(\frac{1}{3}\right)$.
- Length of adjacent side $= \sqrt{3^2 - 1^2} = \sqrt{8} = 2\sqrt{2}$.
- Then $\tan\left(\arcsin\left(\frac{1}{3}\right)\right) = \frac{1}{2\sqrt{2}}$.



Example

Find $\arcsin(\sin(1.5))$.

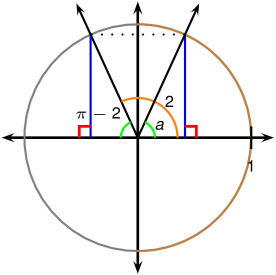
- $\frac{\pi}{2} \approx 1.57$.
- Therefore $-\frac{\pi}{2} \leq 1.5 \leq \frac{\pi}{2}$.
- Therefore $\arcsin(\sin 1.5) = 1.5$.

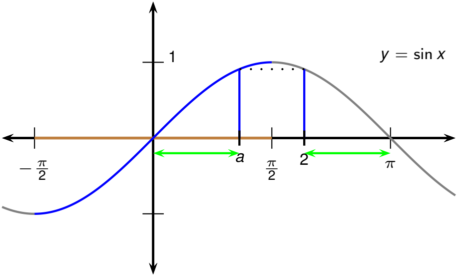
Example

Find $\arcsin(\sin 2)$.

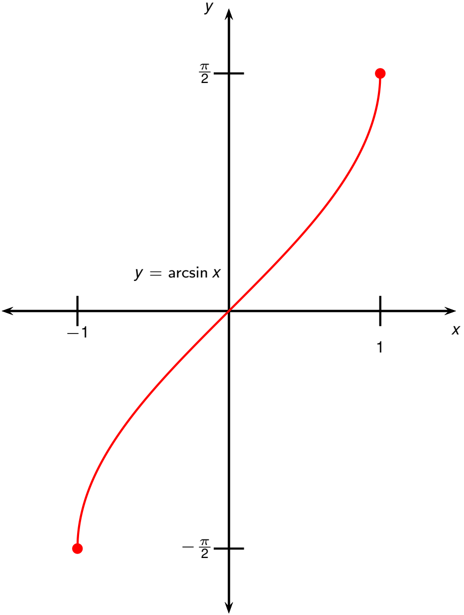
- 2 is not between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.
- We need the angle a between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ for which $\sin 2 = \sin a$.

$$\begin{array}{lrcl} & a & = & \pi - 2. \\ \text{Therefore} & \arcsin(\sin 2) & = & \arcsin(\sin a) \\ & & = & a = \pi - 2. \end{array}$$

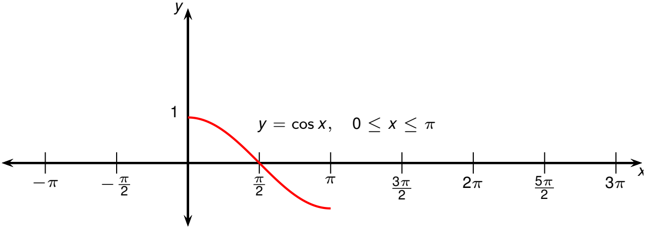


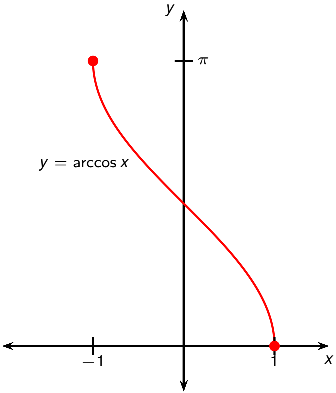


Important facts about arcsin:



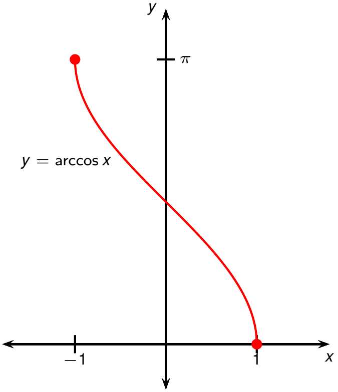
1. Domain: $[-1, 1]$.
2. Range: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.
3. $\arcsin x = y \Leftrightarrow \sin y = x$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$.
4. $\arcsin(\sin x) = x$ for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.
5. $\sin(\arcsin x) = x$ for $-1 \leq x \leq 1$.
6. $\frac{\mathrm{d}}{\mathrm{d}x}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$.





- Same for $\cos x$.
- Restrict the domain to $[0, \pi]$.
- The inverse is called $\arccos$ or $\cos^{-1}$.
- $\arccos(x) = y \Leftrightarrow \cos y = x$ and $0 \leq y \leq \pi$.

Important facts about arccos:



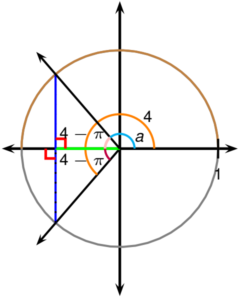
1. Domain: $[-1, 1]$.
2. Range: $[0, \pi]$.
3. $\arccos x = y \Leftrightarrow \cos y = x$ and $0 \leq y \leq \pi$.
4. $\arccos(\cos x) = x$ for $0 \leq x \leq \pi$.
5. $\cos(\arccos x) = x$ for $-1 \leq x \leq 1$.
6. $\frac{\mathrm{d}}{\mathrm{d}x}(\arccos x) = -\frac{1}{\sqrt{1 - x^2}}$.
(The proof is similar to the proof of the formula for the derivative of $\arcsin x$.)

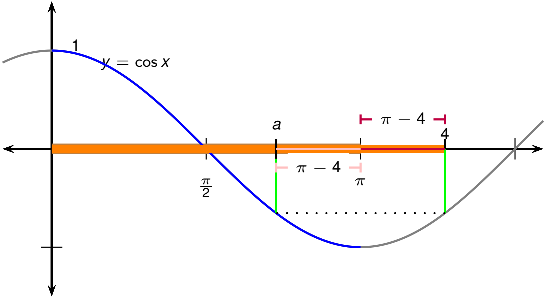
Example

Find $\arccos(\cos 4)$.

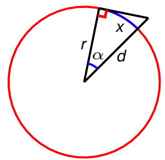
- 4 is not between 0 and π.
- We need the angle a between 0 and π for which $\cos 4 = \cos a$.

$$\begin{array}{lrcl} & a & = & \pi - (4 - \pi) = 2\pi - 4 \\ \text{Therefore} & \arccos(\cos 4) & = & \arccos(\cos a) \\ & & = & a = 2\pi - 4. \end{array}$$





Example



not to scale

The horizon at sea is viewed at from a ship height of 10m. How far does the ship need to travel to reach the horizon just observed? Assume earth is round with radius 6371 km and that the ship sails along the shortest curved path.

- Let d be the distance from eyes of seaman to the center of earth.
- Let r be the radius of earth. Let α be the indicated angle.
- Let the distance to the horizon be x.

$$\begin{aligned} r &= 6371\text{km} \\ d &= 6371\text{km} + 0.01\text{km} = 6371.01\text{km} \\ \cos\alpha &= \frac{r}{d} \\ \alpha &= \arccos\left(\frac{r}{d}\right) \\ x &= r\alpha = r\arccos\left(\frac{r}{d}\right) = 6371\text{km}\arccos\left(\frac{6371\text{km}}{6371.01\text{km}}\right) \approx 11.29\text{km} \end{aligned}$$

Example

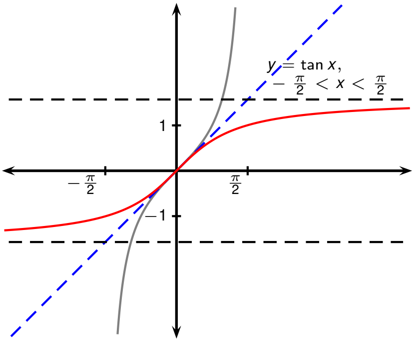
Rewrite $\sin(2\arccos(x))$ as an algebraic expression of x and $\sqrt{1-x^2}$. To simplify $\arccos x$ we try to use $\cos(\arccos x) = x$. Therefore our aim is to rewrite the expression only using the $\cos$ function.

$\sin(2\arccos(x)) = \sin(2y)$	Set $y = \arccos x$
$= 2\cos y \sin y$	Express via $\sin y, \cos y$
$= 2\cos y\left(\pm\sqrt{1-\cos^2 y}\right)$	Express $\sin y$ via $\cos y$
$= 2\cos y\sqrt{1-\cos^2 y}$	$\sin y > 0$ because $0 \leq y \leq \pi$
$= 2x\sqrt{1-x^2}$	use $x = \cos y$

Example

Rewrite $\cos(3\arccos(x))$ as an algebraic expression of x and $\sqrt{1-x^2}$. To simplify $\arccos x$ we try to use $\cos(\arccos x) = x$. Therefore our aim is to rewrite the expression only using the $\cos$ function.

$$
\begin{aligned}
\cos(3\arccos(x)) &= \cos(3y) = \cos(2y+y) && \Big|\ y = \arccos x \\
&= \cos(2y)\cos y - \sin(2y)\sin y && \Big|\ \text{Angle sum f-la} \\
&= (\cos^2 y - \sin^2 y)\cos y && \Big|\ \text{Express via} \\
&\quad - 2\sin y \cos y \sin y && \Big|\ \sin y, \cos y \\
&= \cos^3 y - \sin^2 y \cos y - 2\sin^2 y \cos y \\
&= \cos^3 y - 3\sin^2 y \cos y && \Big|\ \text{Express } \sin y \text{ via } \cos y \\
&= \cos^3 y - 3(1-\cos^2 y)\cos y \\
&= 4\cos^3 y - 3\cos y \\
&= 4x^3 - 3x && \Big|\ x = \cos y
\end{aligned}
$$

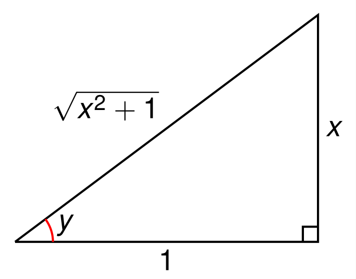


- $\tan x$ isn't one-to-one.
- Restrict the domain to $(-\frac{\pi}{2}, \frac{\pi}{2})$.
- The inverse is called $\tan^{-1}$ or arctan.
- $\arctan x = y \Leftrightarrow \tan y = x$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$.
- Domain of arctan: $(-\infty, \infty)$.
- Range of arctan: $(-\frac{\pi}{2}, \frac{\pi}{2})$.
- $\lim_{x\to\infty} \arctan x = \frac{\pi}{2}$.
- $\lim_{x\to-\infty} \arctan x = -\frac{\pi}{2}$.

Example

Simplify the expression $\cos(\arctan x)$.

- Let $y = \arctan x$, so $\tan y = x$.
- Draw a right triangle with opposite x and adjacent 1.
- Length of hypotenuse $= \sqrt{1^2 + x^2}$.
- Then $\cos(\arctan x) = \frac{1}{\sqrt{1+x^2}}$.

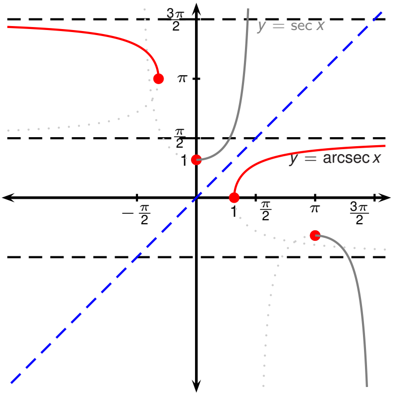


The remaining inverse trigonometric functions aren't used as often:

$$
\begin{array}{llcll}
y = \operatorname{arccsc} x & (|x| \geq 1) & \Leftrightarrow & \csc y = x & \text{and} \quad y \in \left(0, \frac{\pi}{2}\right] \cup \left(\pi, \frac{3\pi}{2}\right] \\
y = \operatorname{arcsec} x & (|x| \geq 1) & \Leftrightarrow & \sec y = x & \text{and} \quad y \in \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right) \\
y = \operatorname{arccot} x & (|x| \in \mathbb{R}) & \Leftrightarrow & \cot y = x & \text{and} \quad y \in (0, \pi)
\end{array}
$$

We will however make use of $\operatorname{arcsec} x$: we discuss in detail its domain.

$$y = \operatorname{arcsec} x \quad (|x| \geq 1) \quad \Leftrightarrow \quad \sec y = x \quad \text{and} \quad y \in \textbf{?} \left[0, \tfrac{\pi}{2}\right) \cup \left[\pi, \tfrac{3\pi}{2}\right)$$



- Plot $\sec x$.
- Restrict domain to make one-to-one: Two common choices: $x \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ and $x \in \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$.
- $x \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ is good because the domain is easiest to remember: an interval without a point. **NOT our choice.**
- $x \in \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$ is good because $\tan x$ is positive on both intervals, resulting in easier differentiation and integration formulas. **Our choice.**