Calculus III
Lecture 15

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https://github.com/tmilev/freecalc

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Outline

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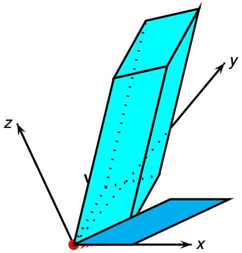
- Let $\mathbf{o}$ be a marked point. If omitted, we assume $\mathbf{o}$ is the origin.
- Let $\mathbf{v}_1 = (v_{11}, \dots, v_{1n}), \dots, \mathbf{v}_k = (v_{k1}, \dots, v_{kn})$ be k vectors in n-dimensional space, $k \leq n$.
- Let $\mathcal{R}$ be region spanned by the vectors at $\mathbf{o}$, coefficients in $[0, 1]$.
- $\mathcal{R} = \{\mathbf{o} + t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \dots + t_k\mathbf{v}_k | t_1 \in [0, 1], \dots, t_k \in [0, 1]\}$.

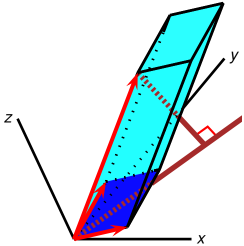
Definition (parallelotope at $\mathbf{o}$)

We call a region $\mathcal{R}$ of the above form a k-dimensional parallelotope at the point $\mathbf{o}$ in n-dimensional space.



- When $k, n, \mathbf{o}$ are clear from context we can omit them.

k	n	parallelotope name
1	any	segment (in n-dim space)
2	2	parallelogram
2	3	parallelogram in space
3	3	parallelepiped



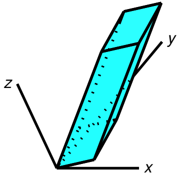
- Let $\mathbf{v}_1 = (v_{11}, \dots, v_{1n}), \dots, \mathbf{v}_n = (v_{n1}, \dots, v_{nn})$ be n-vectors in n-dimensional space.
- Let $\mathcal{R}_k$ be parallelotope spanned by $\mathbf{v}_1, \dots, \mathbf{v}_k$.
- $\mathcal{R}_k$ can be regarded as "prism" with base $\mathcal{R}_{k-1}$.
- Let h_k be the height from $\mathbf{v}_k$ to the base $\mathcal{R}_{k-1}$.

Definition (k-volume of a parallelotope)

Define $\mathrm{Vol}_1(\mathcal{R}_1) = |\mathbf{v}_1|$. For $k > 1$, define $\mathrm{Vol}_k(\mathcal{R}_k) = h_k \mathrm{Vol}_{k-1}(\mathcal{R}_{k-1})$.

- Let the height vector $\mathbf{h}_k$ be the vector of the form $\mathbf{h}_k = \mathbf{v}_k + a_1\mathbf{v}_1 + \dots a_{k-1}\mathbf{v}_{k-1}$ for which $\mathbf{h}_k \cdot \mathbf{v}_1 = 0, \dots, \mathbf{h}_k \cdot \mathbf{v}_{k-1} = 0$.
- Then h_k is computed as the length of $\mathbf{h}_k$.
- For the largest parallelotope $\mathcal{R}_n$, we already have definition of volume: the integral of 1 over $\mathcal{R}_n$.
- We will see that $\mathrm{Vol}_n(\mathcal{R}_n)$ equals that integral.

Length, Surface Area, Volume as k-volumes



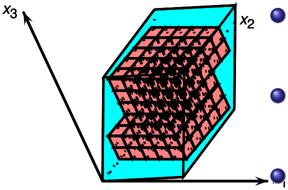
- Let $\mathbf{v}_1 = (v_{11}, \dots, v_{1n}), \dots, \mathbf{v}_n = (v_{n1}, \dots, v_{nn})$ be n-vectors in n-dimensional space.
- Let $\mathcal{R}_k$ be the parallelotope spanned by $\mathbf{v}_1, \dots, \mathbf{v}_k$.
- Let h_k be the height of $\mathcal{R}_k$ with base $\mathcal{R}_{k-1}$.

Definition (k-volume of a parallelotope)

Define $Vol_1(\mathcal{R}_1) = |\mathbf{v}_1|$. For $k > 1$, define $\mathrm{Vol}_k(\mathcal{R}_k) = h_k \mathrm{Vol}_{k-1}(\mathcal{R}_{k-1})$.

	spanned by	$\mathrm{Vol}_k(\mathcal{R}_k)$	volume name
$\mathcal{R}_1$	$\mathbf{v}_1$	$h_1 = \|\mathbf{v}_1\|$	length
$\mathcal{R}_2$	$\mathbf{v}_1, \mathbf{v}_2$	$h_1 h_2$	(surface) area
$\mathcal{R}_2$	$\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$	$h_1 h_2 h_3$	volume
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathcal{R}_k$	$\mathbf{v}_1, \dots, \mathbf{v}_k$	$h_1 \dots h_k$	k-volume

Integral and Algebraic Volume Definitions Agree



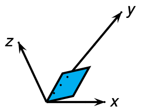
- Let $\mathbf{v}_1 = (v_{11}, \dots, v_{1n}), \dots, \mathbf{v}_n = (v_{n1}, \dots, v_{nn})$ be n-vectors in n-dimensional space.
- Let $\mathcal{R}_k$ be the parallelotope spanned by $\mathbf{v}_1, \dots, \mathbf{v}_k$.
- Let h_k be the height of $\mathcal{R}_k$ with base $\mathcal{R}_{k-1}$.

Theorem

$$\mathrm{Vol}_n(\mathcal{R}_n) = h_n \mathrm{Vol}_{n-1}(\mathcal{R}_{n-1}) = \int \cdots \int_{\mathcal{R}_n} 1 \cdot \mathrm{d}x_1 \dots \mathrm{d}x_n.$$

- Right hand side: approx. vol. with boxes, sides along coord. axes.
- Left hand side: approximate volume with slabs parallel to base.
- Theorem is fully intuitive but its proof is surprisingly laborious.

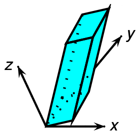
- Let $\mathbf{v}_1 = (v_{11}, \dots, v_{1n}), \dots, \mathbf{v}_n = (v_{n1}, \dots, v_{nn})$ be n-vectors.
- Let $\mathcal{R}_k$ be the parallelotope spanned by $\mathbf{v}_1, \dots, \mathbf{v}_k$.
- Recall that $\mathbf{v}_i \cdot \mathbf{v}_j = v_{i1}v_{j1} + \dots + v_{in}v_{jn}$.



Theorem (k-volume = Gram determinant)

$$\mathrm{Vol}_k(\mathcal{R}_k) = \sqrt{\begin{vmatrix} \mathbf{v}_1 \cdot \mathbf{v}_1 & \dots & \mathbf{v}_1 \cdot \mathbf{v}_k \\ \vdots & \dots & \vdots \\ \mathbf{v}_k \cdot \mathbf{v}_1 & \dots & \mathbf{v}_k \cdot \mathbf{v}_k \end{vmatrix}}.$$

Proof: studied in Linear algebra (Vol_k - defined by algebra only). $\mathrm{Vol}_n(\mathcal{R}_n)$ is a perfect square for all n.



Theorem

$$\mathrm{Vol}_n(\mathcal{R}_n) = \pm \begin{vmatrix} v_{11} & \dots & v_{n1} \\ \vdots & \dots & \vdots \\ v_{1n} & \dots & v_{nn} \end{vmatrix}.$$

Properties of determinants

- Multiplying a column of a matrix by a number changes multiplies the determinant by the same number. In precise notation:

Lemma

$$\begin{vmatrix} a_{11} & \dots & xa_{1k} & \dots & a_{1n} \\ a_{21} & \dots & xa_{2k} & \dots & a_{2n} \\ \vdots & & & & \\ a_{n1} & \dots & xa_{nk} & \dots & a_{nn} \end{vmatrix} = x \begin{vmatrix} a_{11} & \dots & a_{1k} & \dots & a_{1n} \\ a_{21} & \dots & a_{2k} & \dots & a_{2n} \\ \vdots & & & & \\ a_{n1} & \dots & a_{nk} & \dots & a_{nn} \end{vmatrix}$$

Example

Find the 1-dimensional volume (length) of the segment through the origin spanned by $\mathbf{v} = (1, 2, 3)$.

Example

Find the 1-dimensional volume (length) of the segment $\mathcal{R}_1$ through the origin spanned by $\mathbf{v} = (v_1, v_2, v_3)$.

$$\mathrm{Vol}_1 = \sqrt{\underbrace{\mathbf{v}\cdot\mathbf{v}}_{1\times 1 \text{ Gram determinant}}} = \sqrt{v_1^2 + v_2^2 + v_3^2}.$$

Example

Let $\mathcal{R}_2$ be the parallelogram in 2-dimensional space spanned by $\mathbf{v}_1 = (2, 3)$, $\mathbf{v}_2 = (5, 7)$. Find the area of $\mathcal{R}_2$.

Example

Let $\mathcal{R}_2$ be the parallelogram in 2-dimensional space spanned by $\mathbf{v}_1 = (v_{11}, v_{12})$, $\mathbf{v}_2 = (v_{21}, v_{22})$. Find the area of $\mathcal{R}_2$.

$$\begin{aligned} \mathrm{Vol}_2 &= \pm \begin{vmatrix} v_{11} & v_{21} \\ v_{12} & v_{22} \end{vmatrix} \\ \mathrm{Vol}_2 &= \sqrt{\begin{vmatrix} \mathbf{v}_1 \cdot \mathbf{v}_1 & \mathbf{v}_1 \cdot \mathbf{v}_2 \\ \mathbf{v}_2 \cdot \mathbf{v}_1 & \mathbf{v}_2 \cdot \mathbf{v}_2 \end{vmatrix}} \end{aligned}$$

Example

Find the surface area of the parallelogram spanned by $\mathbf{v}_1 = (1, 2, 3)$ and $\mathbf{v}_2 = (5, 7, 11)$.

$$\mathrm{Vol}_2 = \sqrt{\begin{vmatrix} \mathbf{v}_1 \cdot \mathbf{v}_1 & \mathbf{v}_1 \cdot \mathbf{v}_2 \\ \mathbf{v}_2 \cdot \mathbf{v}_1 & \mathbf{v}_2 \cdot \mathbf{v}_2 \end{vmatrix}} = \sqrt{\begin{vmatrix} 14 & 52 \\ 52 & 195 \end{vmatrix}} = \sqrt{26}.$$

Example

Find the surface area of the parallelogram spanned by $\mathbf{v}_1 = (v_{11}, v_{12}, v_{13})$ and $\mathbf{v}_2 = (v_{21}, v_{22}, v_{23})$.

Example

Find the volume of the parallelepiped with vertex at the origin and spanned by $\mathbf{v}_1 = (1, 2, 3)$, $\mathbf{v}_2 = (3, 5, 7)$, $\mathbf{v}_3 = (5, 7, 11)$.

$$\mathrm{Vol}_3 = \left| \det \begin{pmatrix} 1 & 2 & 3 \\ 3 & 5 & 7 \\ 5 & 7 & 11 \end{pmatrix} \right| = |-2| = 2.$$

Example

Find the volume of the parallelepiped spanned by $\mathbf{v}_1 = (v_{11}, v_{12}, v_{13})$, $\mathbf{v}_2 = (v_{21}, v_{22}, v_{23})$, $\mathbf{v}_3 = (v_{31}, v_{32}, v_{33})$.

- Recall the polar coordinate variable change

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta. \end{aligned}$$

- This variable change can be thought of as two functions: $x = h(r,\theta) = r\cos\theta$ and $y = g(r,\theta) = r\sin\theta$.
- The functions h, g map the two-dimensional plane with coordinates r, θ into the two-dimensional plane with coordinates x, y.
- Let $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ be an infinitely differentiable map.
- In other words, $\mathbf{f}$ takes n scalar inputs and produces n scalar outputs.

Definition (Infinitely Smooth Variable Change)

An infinitely differentiable map $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ is called an (infinitely) smooth variable change.

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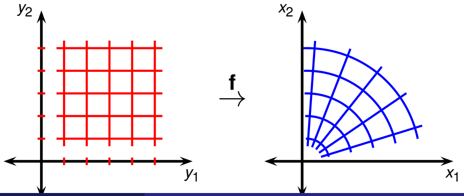
An infinitely differentiable map $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ is called an (infinitely) smooth variable change.

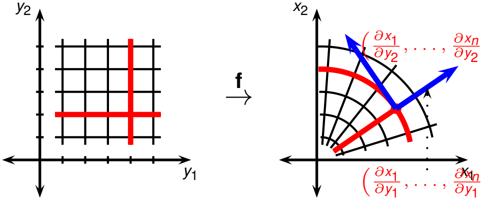
- Variable change $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ is given by f-ns $f_1, \dots, f_n$. We write:

$$\mathbf{f} : \left| \begin{array}{rcl} x_1 & = & f_1(y_1, \dots, y_n) \\ & \vdots & \\ x_n & = & f_n(y_1, \dots, y_n) \end{array} \right. .$$

- The variables $y_1, \dots, y_n$ denote coordinates in the domain of $\mathbf{f}$.
- We may include vars. $x_1, \dots, x_n$ denoting coords. in codomain of $\mathbf{f}$.
- Fix $y_2, \dots, y_n$ and view $\mathbf{f}$ as curve with respect to y_1; plot.
- Do similarly with respect to the remaining variables.



$$\mathbf{f}: \left| \begin{array}{l} x_1 = f_1(y_1, \dots, y_n) \\ \quad \vdots \\ x_n = f_n(y_1, \dots, y_n) \end{array} \right. .$$



Definition (Jacobian matrix)

The Jacobian matrix of a variable change **f** is defined as the matrix

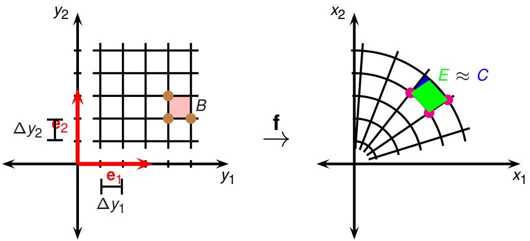
$$J_{\mathbf{f}} = \begin{pmatrix} \frac{\partial f_1}{\partial y_1} & \cdots & \frac{\partial f_1}{\partial y_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial y_1} & \cdots & \frac{\partial f_n}{\partial y_n} \end{pmatrix} = \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \cdots & \frac{\partial x_1}{\partial y_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial y_1} & \cdots & \frac{\partial x_n}{\partial y_n} \end{pmatrix}$$

- Consider curve given by **f** with parameter y_1 (other y_j's-fixed).
- Then the tangent vector of that curve is $\left(\frac{\partial x_1}{\partial y_1}, \dots, \frac{\partial x_n}{\partial y_1}\right)$.
- Similar considerations hold for $y_2, \dots, y_n$.

Variable change in multivariable integrals

$$\mathbf{f}: \left| \begin{array}{rcl} x_1 &=& f_1(y_1, \dots, y_n) \\ & \vdots & \\ x_n &=& f_n(y_1, \dots, y_n) \end{array} \right.$$

$$J_{\mathbf{f}} = \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \dots & \frac{\partial x_1}{\partial y_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial y_1} & \dots & \frac{\partial x_n}{\partial y_n} \end{pmatrix}$$

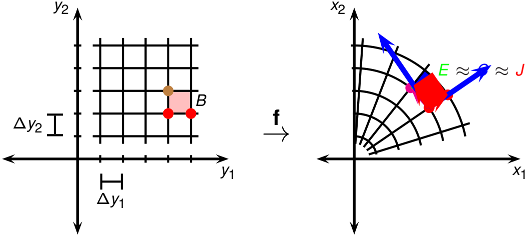


- Let $\mathbf{e}_1, \dots, \mathbf{e}_n$ be basis vectors. Fix point $\mathbf{y} = (y_1, \dots, y_n)$.
- Let $\Delta y_1, \dots, \Delta y_n$ be small numbers. Construct small box B with corner $\mathbf{y}$ spanned by the vectors $\Delta y_1 \mathbf{e}_1, \dots, \Delta y_n \mathbf{e}_n$.
- The point $\mathbf{y}$ and the corners $\mathbf{y} + \Delta y_1 \mathbf{e}_1, \dots, \mathbf{y} + \Delta y_n \mathbf{e}_n$ suffice to identify B.
- $\mathrm{Vol}(B) = \Delta y_1 \dots \Delta y_n$.
- Let the image of B be $\mathbf{f}(B) = C$. C is a "curvilinear box".
- Let E be the parrallelotope at $\mathbf{f}(\mathbf{y})$ spanned by images of the corners of B. Then $\mathrm{Vol}(C) \approx \mathrm{Vol}_n(E)$.

Variable change in multivariable integrals

$$\mathbf{f}: \left| \begin{array}{l} x_1 = f_1(y_1, \dots, y_n) \\ \quad\vdots \\ x_n = f_n(y_1, \dots, y_n) \end{array} \right. .$$

$$J_{\mathbf{f}} = \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \dots & \frac{\partial x_1}{\partial y_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial y_1} & \dots & \frac{\partial x_n}{\partial y_n} \end{pmatrix}$$

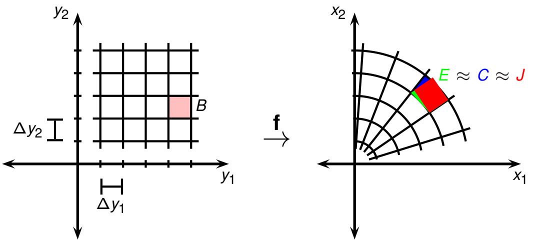


- $\mathrm{Vol}(C) \approx \mathrm{Vol}_n(E)$.
- The first edge of E corresponds to the vector
$$\mathbf{f}(\mathbf{y} + \Delta y_1 \mathbf{e}_1) - \mathbf{f}(\mathbf{y}) \approx \Delta y_1 \left(D_{\mathbf{e}_1}(\mathbf{f}(\mathbf{y}))\right) = \Delta y_1 \frac{\partial \mathbf{f}}{\partial y_1}$$
$$= \Delta y_1 \left(\frac{\partial x_1}{\partial y_1}, \dots, \frac{\partial x_n}{\partial y_1} \right) = \left(\Delta y_1 \frac{\partial x_1}{\partial y_1}, \dots, \Delta y_1 \frac{\partial x_n}{\partial y_1} \right).$$
- Similar considerations holds for the other edges of E.
- Let J be the parallelotope at $\mathbf{f}(\mathbf{y})$ spanned by the vectors
$\Delta y_1 \left(\frac{\partial x_1}{\partial y_1}, \dots, \frac{\partial x_n}{\partial y_1} \right), \dots \Delta y_n \left(\frac{\partial x_1}{\partial y_n}, \dots, \frac{\partial x_n}{\partial y_n} \right)$.
- Then $\mathrm{Vol}(C) \approx \mathrm{Vol}_n(E) \approx \mathrm{Vol}_n(J)$.

Variable change in multivariable integrals

$$\mathbf{f}:\left|\begin{array}{rcl} x_1 &=& f_1(y_1,\dots,y_n)\\ &\vdots& \\ x_n &=& f_n(y_1,\dots,y_n)\end{array}\right. .$$

$$J_{\mathbf{f}} = \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \dots & \frac{\partial x_1}{\partial y_n}\\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial y_1} & \dots & \frac{\partial x_n}{\partial y_n}\end{pmatrix}$$



- Let J be the parallelotope at $\mathbf{f}(\mathbf{y})$ spanned by the vectors $\Delta y_1 \left(\frac{\partial x_1}{\partial y_1},\dots,\frac{\partial x_n}{\partial y_1}\right), \dots, \Delta y_n \left(\frac{\partial x_1}{\partial y_n},\dots,\frac{\partial x_n}{\partial y_n}\right)$. Suppose $\det J_{\mathbf{f}} \geq 0$.
- $\mathrm{Vol}(C) \approx \mathrm{Vol}_n(E) \approx \mathrm{Vol}_n(J) = \det J_{\mathbf{f}} \Delta y_1 \dots \Delta y_n$

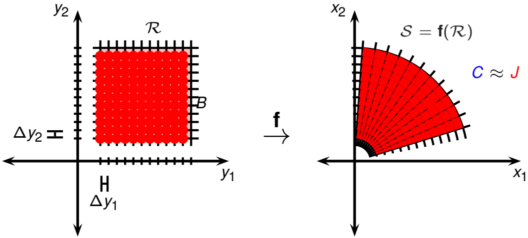
$$\mathrm{Vol}_n(J) = \pm \begin{vmatrix} \Delta y_1 \frac{\partial x_1}{\partial y_1} & \dots & \Delta y_n \frac{\partial x_1}{\partial y_n}\\ \vdots & \ddots & \vdots \\ \Delta y_1 \frac{\partial x_n}{\partial y_1} & \dots & \Delta y_n \frac{\partial x_n}{\partial y_n}\end{vmatrix} = \pm \Delta y_1 \dots \Delta y_n \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \dots & \frac{\partial x_1}{\partial y_n}\\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial y_1} & \dots & \frac{\partial x_n}{\partial y_n}\end{vmatrix}$$

$$= \pm \det(J_{\mathbf{f}}) \Delta y_1 \dots \Delta y_n \quad .$$

Variable change in multivariable integrals

$$\mathbf{f}: \left| \begin{array}{l} x_1 = f_1(y_1, \dots, y_n) \\ \quad\vdots \\ x_n = f_n(y_1, \dots, y_n) \end{array} \right. .$$

$$J_{\mathbf{f}} = \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \cdots & \frac{\partial x_1}{\partial y_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial y_1} & \cdots & \frac{\partial x_n}{\partial y_n} \end{pmatrix}$$

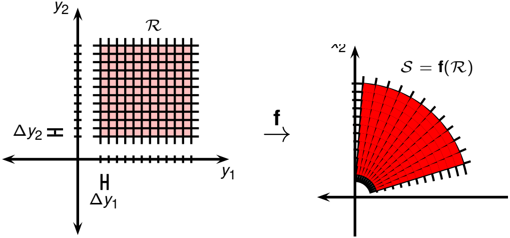


- $$\mathrm{Vol}(\mathcal{S}) = \sum_{\mathbf{y}} \mathrm{Vol}(C(\mathbf{y})) \approx \sum_{\mathbf{y}} \det\left(J_{\mathbf{f}}(\mathbf{y})\right) \Delta y_1 \dots \Delta y_n$$
  $$\int_{\mathcal{S}} 1 \cdot \mathrm{d}x_1 \dots \mathrm{d}x_n = \int \cdots \int_{\mathcal{R}} \det\left(J_{\mathbf{f}}(y)\right) \mathrm{d}y_1 \dots \mathrm{d}y_n$$
- Regard **y** as a variable and let it traverse a rectangular mesh.
- Sum over the rectangular mesh.
- Let $\Delta y_1 \to 0, \dots, \Delta y_n \to 0$.

Variable change in multivariable integrals

$$\mathbf{f}: \left| \begin{array}{lcl} x_1 &=& f_1(y_1, \dots, y_n) \\ & \vdots & \\ x_n &=& f_n(y_1, \dots, y_n) \end{array} \right. .$$

$$J_{\mathbf{f}} = \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \dots & \frac{\partial x_1}{\partial y_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial y_1} & \dots & \frac{\partial x_n}{\partial y_n} \end{pmatrix}$$



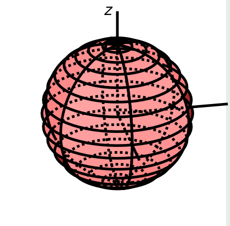
Theorem (Variable change in multivariable integrals)

Let $\mathbf{f}$ *be a smooth one to one variable change. Let* $\mathbf{f}(\mathcal{R}) = \mathcal{S}$. *Let h be an integrable function. Then*

$$\int_{\mathcal{S}} \cdots \int h(x_1, \dots, x_n) \mathrm{d}x_1 \dots \mathrm{d}x_n = \int_{\mathcal{R}} \cdots \int h(f_1, \dots, f_n) \det\left(J_{\mathbf{f}}(\mathbf{y})\right) \mathrm{d}y_1 \dots \mathrm{d}y_n,$$

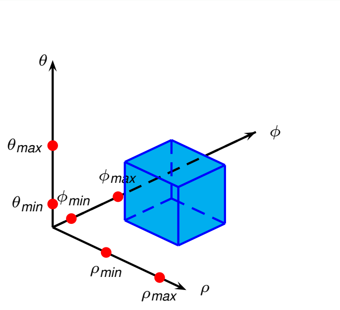
provided that $\det\left(J_{\mathbf{f}}(\mathbf{y})\right) \geq 0$ *for all* $\mathbf{y} \in \mathcal{R}$.

Example

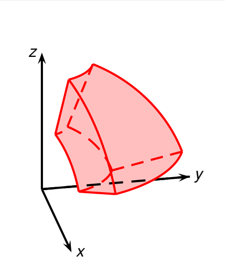


Find the volume of a ball of radius r.

Example



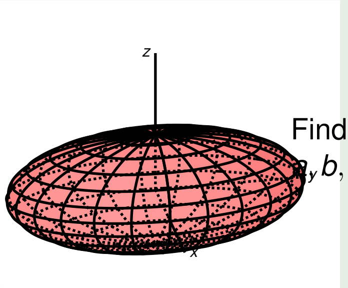
$\xrightarrow{\mathbf{f}}$



Find the volume of a spherical curvilinear box, given by the spherical coordinate inequalities

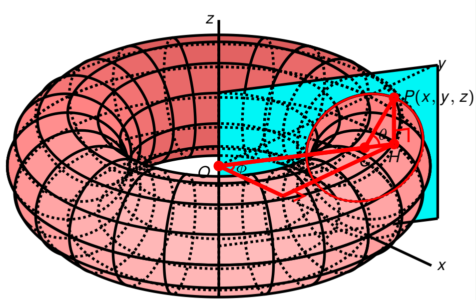
$$\rho_{min} \leq \rho \leq \rho_{max},$$
$$\phi_{min} \leq \phi \leq \phi_{max},$$
$$\theta_{min} \leq \theta \leq \theta_{max}.$$

Example



Find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{y^2}{c^2} = 1$, $a, b, c > 0$.

Example (Volume of toroid)



Find the volume of a toroid T (the inside of a torus S) with major radius R and minor radius r.

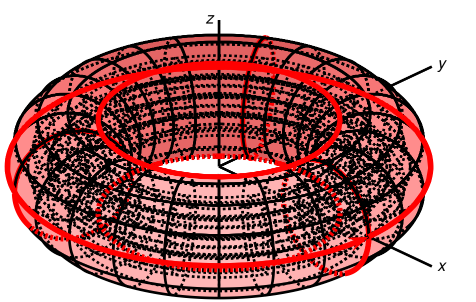
$$S: \left| \begin{array}{rcl} x &=& (R + r\cos\theta)\cos\phi \\ y &=& (R + r\cos\theta)\sin\phi \\ z &=& r\sin\theta \end{array} \right.$$

Suppose the toroid sits in space as drawn. Let $P(x, y, z) \in S$. Let $\mathcal{P}$ be the plane through the z-axis and P.

Let H be the heel of the perpendicular from P to the x, y-plane. Let C be the center of the circle cross-section of $\mathcal{P}$ with T. Let ϕ and θ be the indicated angles. We have

$$\begin{array}{rcl} |OC| &=& R \\ |PC| &=& r \\ |PH| &=& r\sin\theta \\ |OH| &=& R + r\cos\theta \end{array}.$$

Example (Volume of toroid)



Find the volume of a toroid T (the inside of a torus S) with major radius R and minor radius r.

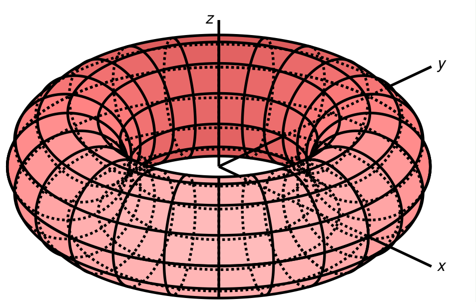
$$S: \left| \begin{array}{l} x = (R + r\cos\theta)\cos\phi \\ y = (R + r\cos\theta)\sin\phi \\ z = r\sin\theta \end{array} \right.$$

$$T: \left| \begin{array}{l} x = (R + \rho\cos\theta)\cos\phi \\ y = (R + \rho\cos\theta)\sin\phi \quad \rho \in [0, \mathbf{?}r], \phi \in [0, \mathbf{?}2\pi), \theta \in [0, \mathbf{?}2\pi). \\ z = \rho\sin\theta \end{array} \right.$$

Let $\mathbf{f}$ be the map participating in the parametrization of T.

$$\mathrm{Vol}(T) = \int_{\theta=0}^{\theta=2\pi} \int_{\phi=0}^{\phi=2\pi} \int_{\rho=0}^{\rho=r} \det\left(J_{\mathbf{f}}\right) \mathrm{d}\rho \mathrm{d}\phi \mathrm{d}\theta$$

Example (Volume of toroid)



Find volume of toroid T, major radius R minor radius r.

$$\mathbf{f}: \left| \begin{array}{l} x = (R+\rho\cos\theta)\cos\phi \\ y = (R+\rho\cos\theta)\sin\phi \\ z = \rho\sin\theta \end{array}\right.$$

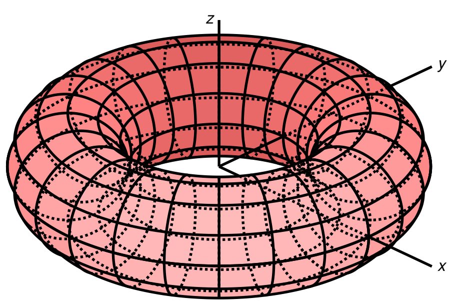
$$J_{\mathbf{f}} = \begin{pmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{pmatrix}$$

$$\mathrm{Vol}(T) = \int\limits_{\theta=0}^{\theta=2\pi} \int\limits_{\phi=0}^{\phi=2\pi} \int\limits_{\rho=0}^{\rho=r} \det(J_{\mathbf{f}})\, \mathrm{d}\rho \mathrm{d}\phi \mathrm{d}\theta = \int\limits_0^{2\pi}\int\limits_0^{2\pi}\int\limits_0^{r} \rho A \,\mathrm{d}\rho\mathrm{d}\phi\mathrm{d}\theta$$

$$J_{\mathbf{f}} = \begin{pmatrix} \cos\theta\cos\phi & -A\sin\phi & -\rho\sin\theta\cos\phi \\ \cos\theta\sin\phi & A\cos\phi & -\rho\sin\theta\sin\phi \\ \sin\theta & 0 & \rho\cos\theta \end{pmatrix}$$

where we have set $A = R + \rho\cos\theta$.

Example (Volume of toroid)



Find volume of toroid T, major radius R, minor radius r.

$$\mathbf{f}: \left| \begin{array}{l} x = (R + \rho\cos\theta)\cos\phi \\ y = (R + \rho\cos\theta)\sin\phi \\ z = \rho\sin\theta \end{array} \right.$$

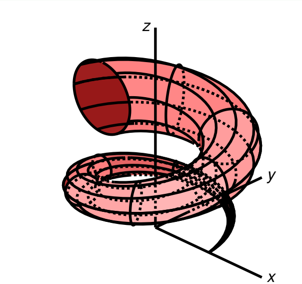
$$J_{\mathbf{f}} = \begin{pmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{pmatrix}$$

$$\mathrm{Vol}(T) = \int\limits_{\theta=0}^{\theta=2\pi} \int\limits_{\phi=0}^{\phi=2\pi} \int\limits_{\rho=0}^{\rho=r} \det(J_{\mathbf{f}})\, \mathrm{d}\rho \mathrm{d}\phi \mathrm{d}\theta = \int\limits_0^{2\pi} \int\limits_0^{2\pi} \int\limits_0^{r} \rho(R + \rho\cos\theta) \mathrm{d}\rho \mathrm{d}\phi \mathrm{d}\theta$$

$$= \int\limits_0^{2\pi} \int\limits_0^{2\pi} \left[\frac{R\rho^2}{2} + \frac{\rho^3}{3}\cos\theta \right]_{\rho=0}^{\rho=r} \mathrm{d}\phi \mathrm{d}\theta = \int\limits_0^{2\pi} \int\limits_0^{2\pi} \left(\frac{Rr^2}{2} + \frac{r^3}{3}\cos\theta \right) \mathrm{d}\phi \mathrm{d}\theta$$

$$= 2\pi \int_0^{2\pi} \left(\frac{Rr^2}{2} + \frac{r^3}{3}\cos\theta \right) \mathrm{d}\theta = 2\pi \int_0^{2\pi} \frac{Rr^2}{2} \mathrm{d}\theta = 2Rr^2\pi^2$$

Example



Find the volume of the horn given by

$$\left|\begin{array}{rcl} x &=& (2+\rho\cos\theta)\cos\phi \\ y &=& (2+\rho\cos\theta)\sin\phi \\ z &=& \rho\sin\theta + \frac{\phi}{3} \end{array}\right. ,$$

$\theta \in [0, 2\pi], \phi \in [0, 3\pi], \rho \in \left[0, \frac{\phi}{9}\right]$.

Theorem (Variable change in multivariable integrals)

f - smooth, one-to-one, $\mathbf{f}(\mathcal{R}) = \mathcal{S}$, $\det(J_{\mathbf{f}}(\mathbf{y})) \geq 0$.

$$\int_{\mathcal{S}} \cdots \int h(x_1, \ldots, x_n) \mathrm{d}x_1 \ldots \mathrm{d}x_n = \int_{\mathcal{R}} \cdots \int h(f_1, \ldots, f_n) \det(J_{\mathbf{f}}(\mathbf{y})) \mathrm{d}y_1 \ldots \mathrm{d}y_n,$$

- One-variable subst. rule: $\int_{f(a)}^{f(b)} h(x)\mathrm{d}x = \int_a^b h(f(y))f'(y)\mathrm{d}y$.
- The one-variable substitution rule is valid
  - without positivity requirements (arranged by compensating with minus sign when changing boundaries of integration)
  - and without requiring that f be one to one (compensated by neutralizing contributions arising from sign changes of $f'(y)$).
- Similarly integration can be generalized so multivar. subst. holds
  - without positivity of $\det(J_{\mathbf{f}})$ (arranged by compensating with minus sign when changing orientation of spaces),
  - without requiring that $\mathbf{f}$ be one to one (compensated by neutralizing contributions arising from sign changes of $\det J_{\mathbf{f}}$).
- When using the above generalization of $\int$, one writes