Calculus III
Homework on Lecture 12

1. Using the second derivative test, find the local minima and maxima as well as the saddle points of the function.

   (a) $f(x,y) = 1 + x^3 + y^3 - 3xy$.
   (b) $f(x,y) = x^3y + x^2 - 27y$.
   (c) $f(x,y) = e^{2y-x^2-y^2}$.
   (d) $f(x,y) = e^x \sin y$.
   (e) $f(x,y) = x^2 + y^2 + \frac{1}{x^2y^2}$.
   (f) $f(x,y) = x^2 + x^2y + y^3 - 4y$.

answer:

(x,y)	critical point type
$\left(0,-2\frac{\sqrt{3}}{3}\right)$	local maximum
$\left(0,2\frac{\sqrt{3}}{3}\right)$	local minimum
$(-1,-1)$	saddle
$(1,-1)$	saddle

Solution. 1f The critical points of f are given by:

$$\begin{array}{rcl} \frac{\partial f}{\partial x} = 0 &=& 2xy + 2x \\ \frac{\partial f}{\partial y} = 0 &=& 3y^2 + x^2 - 4. \end{array}$$

The first equality implies $x(y+1) = 0$, and we have two cases: $x = 0$ and $y = -1$.
Case 1. $x = 0$. We substitute in second equality and solve:

$$\begin{array}{rcl} 3y^2 - 4 &=& 0 \\ y^2 &=& \frac{4}{3} \\ y &=& \pm 2\frac{\sqrt{3}}{3}. \end{array}$$

Case 1 provides us with two critical points, $(x,y) = \left(0, 2\frac{\sqrt{3}}{3}\right)$ and $(x,y) = \left(0, -2\frac{\sqrt{3}}{3}\right)$.
Case 2. $x \neq 0$. It follows that $y = -1$. We substitute in the second equality and solve:

$$\begin{array}{rcl} 3 + x^2 - 4 &=& 0 \\ x^2 &=& 1 \\ x &=& \pm 1 \quad . \end{array}$$

Case 2 provides us with two additional critical points, $(x,y) = (1,-1)$ and $(x,y) = (-1,-1)$.
The Hessian matrix of f and its determinant are:

$$H = \begin{pmatrix} 2y+2 & 2x \\ 2x & 6y \end{pmatrix} \quad \det H = 12y(y+1) - 4x^2 \quad .$$

At $(x,y) = \left(0, 2\frac{\sqrt{3}}{3}\right)$, $\det H = 8\sqrt{3} + 16 > 0$, and $\frac{\partial f}{\partial x^2} > 0$ so f has a local minimum at that point. At $(x,y) = \left(0, -2\frac{\sqrt{3}}{3}\right)$, we have $\det H = -8\sqrt{3} + 16 > 0$. We further have $\frac{\partial f}{\partial x^2} = 2(\frac{2}{\sqrt{3}} - 1) < 0$ so f has a local maximum at that point. Finally at $(x,y) = (\pm 1, -1)$, we have $\det H = -4 < 0$ and so both points are saddle points of f.
Our final answer is as follows.

(x,y)	critical point type
$\left(0,-2\frac{\sqrt{3}}{3}\right)$	local maximum
$\left(0,2\frac{\sqrt{3}}{3}\right)$	local minimum
$(-1,-1)$	saddle
$(1,-1)$	saddle

2. Find the maximum of the function subject to the given restriction, or show the maximum does not exist.

   The problems don't have an answer key yet. If you think that a problem is incorrectly posed, make a clean argument why that is the case.

   (a) $f(x, y) = x^2 + 2y^2$, $xy = 1$.
   (b) $f(x, y) = 4x + 5y$, $x^2 + y^2 = 13$.
   (c) $f(x, y) = x^2 y$, $x^2 + 2y^2 = 1$.
   (d) $f(x, y) = e^{xy}$, $x^3 + y^3 = 2$.

   answer: maximum on $(x, y) = (1, 1)$, $f_{max} = f(1, 1) = e^2$

   (e) $f(x, y) = x + 3y + 5z$, $x^2 + y^2 + z^2 = 35$.
   (f) $f(x, y) = x - z$, $x^2 + 3y^2 + z^2 = 1$.
   (g) $f(x, y) = xyz$, $x^2 + 3y^2 + 5z^2 = 8$.
   (h) $f(x, y) = x^2 y^2 z^2$, $x^2 + y^2 + z^2 = 1$.
   (i) $f(x, y) = x^2 + y^2 + z^2$, $x^4 + y^4 + z^4 = 1$.
   (j) $f(x, y) = x^4 + y^4 + z^4$, $x^2 + y^2 + z^2 = 1$.
   (k) $f(x_1, \dots, x_n) = x_1 + \dots + x_n$, $x_1^2 + \dots + x_n^2 = 1$.
   (l) Find the local extrema of $f(x, y) = y + x$ when x, y satisfy the restriction $y^2 + y + x^2 + x = 1$.

   answer:

(x, y)	critical point type
$\left(\frac{-1-\sqrt{3}}{2}, \frac{-1-\sqrt{3}}{2}\right)$	minimum
$\left(\frac{-1+\sqrt{3}}{2}, \frac{-1+\sqrt{3}}{2}\right)$	maximum

Solution. 2.d The restriction is $g(x, y) = x^3 + y^3 - 2 = 0$. We use the method of Lagrange multipliers. We have that $\nabla f = (e^{x+y}, e^{x+y})$ and $\nabla g = \left(3x^2, 3y^2\right)$. We have a local extremum when $\lambda \nabla f = \nabla g$, i.e., when

$$\begin{array}{rcl} \lambda e^{x+y} &=& 3x^2 \\ \lambda e^{x+y} &=& 3y^2 \\ x^3 + y^3 &=& 2 \end{array}$$

The first two equations imply $y^2 = x^2$ which implies $y = \pm x$.

Case 1. Suppose $y = -x$. Then the last equation $x^3 + y^3 = 2$ reduces to $0 = 2$, which has no solutions; this case yields no candidates for maxima and minima.

Case 2. Suppose $y = x$. We substitute into the third equation and solve:

$$\begin{array}{rcll} 2x^3 &=& 2 & \\ x^3 - 1 &=& 0 & \\ (x-1)(x^2 + x + 1) &=& 0 & \big| \ x^2 + x + 1 \neq 0 \text{ for all real } x \\ x &=& 1 & \end{array}$$

Therefore $x = 1, y = 1$ is the only critical point obtained by the method of Lagrange multipliers. To find out whether the critical point is a maximum or minimum, we can rewrite our restriction as $y(x) = \sqrt[3]{2 - x^3}$ and so $f(x, y(x)) = e^{x + \sqrt[3]{2 - x^3}}$. Since the exponent is an increasing function, $e^{x + \sqrt[3]{2 - x^3}}$ has extrema if and only if the function $x + \sqrt[3]{2 - x^3}$ has the same type of extrema. $x + \sqrt[3]{2 - x^3}$ has second derivative $-2x^4(-x^3 + 2)^{-\frac{5}{3}} - 2x(-x^3 + 2)^{-\frac{2}{3}}$, which evaluates to -4 when $x = 1$. Therefore by the single-variable second derivative criterion $f(x, y(x)) = e^{x + \sqrt[3]{2 - x^3}}$ has a local maximum and so the critical point is a local maximum.

We point out that via the equality $f(x, y(x)) = e^{x + \sqrt[3]{2 - x^3}}$ this problem can be solved without using Lagrange multipliers, however the computations would be longer.

Solution. 2.l The restriction is $g(x, y) = y^2 + y + x^2 + x - 1 = 0$. We use the method of Lagrange multipliers. We have that $\nabla f = (1, 1)$ and $\nabla g = (2y + 1, 2x + 1)$. We have a local extremum when $\lambda \nabla f = \nabla g$, i.e., when

$$\begin{array}{rcl} \lambda &=& (2y + 1) \\ \lambda &=& (2x + 1) \\ y^2 + y + x^2 + x - 1 &=& 0 \end{array}$$

The first two equations imply $y = x$. We substitute that into the last equation to get that $2x^2 + 2x - 1 = 0$. The solutions to the latter are $x = \dfrac{-2 \pm \sqrt{2^2 - 4 \cdot 2 \cdot (-1)}}{4} = \dfrac{-1 \pm \sqrt{3}}{2}$. The only restriction on the points (x, y) is that they lie on the curve $y^2 + y + x^2 + x = 1$ (a circle). A circle is a bounded and closed set. We recall that a set in space is bounded if is contained in a ball (with finite radius) and a set in space is closed if it contains all of its boundary points. Therefore f must attain both its minimum and its maximum on it. Therefore the two critical points are maximum and minimum of f. Substitution of our answer in f shows that f attains its minimum at $(x, y) = \left(\dfrac{-1-\sqrt{3}}{2}, \dfrac{-1-\sqrt{3}}{2}\right)$ and its maximum at $(x, y) = \left(\dfrac{-1+\sqrt{3}}{2}, \dfrac{-1+\sqrt{3}}{2}\right)$. Our final answer is below.

(x, y)	max or min
$\left(\frac{-1-\sqrt{3}}{2}, \frac{-1-\sqrt{3}}{2}\right)$	minimum
$\left(\frac{-1+\sqrt{3}}{2}, \frac{-1+\sqrt{3}}{2}\right)$	maximum