Calculus III
Lecture 8

Todor Milev
https://github.com/tmilev/freecalc

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Outline

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Multivariable Limits

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- Question: What happens to $f(Q)$ as Q gets closer to P_0?

Numerical Exploration of Limits - Example

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Numerical data suggests $f(Q)$ approaches 0 as $Q \to P_0(0,0)$.

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- If the limit L exists, it is unique.

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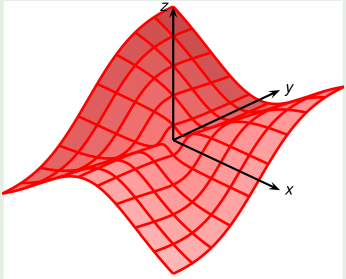

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For the last equality, we use the squeeze theorem:
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We say that the one-variable limit

$$\lim_{t \to 0} f(\mathbf{r} + t\mathbf{u})$$

is the limit of f along the direction $\mathbf{u}$.

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Theorem

If the limit $\lim_{Q\to P} f(Q)$ exists, then every directional limit $\lim_{t\to 0} f(\mathbf{r} + t\mathbf{u})$ exists and all directional limits are equal.

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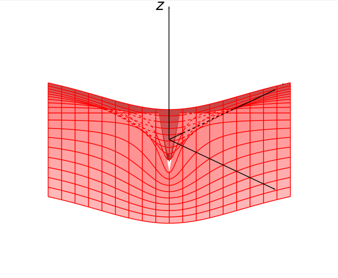

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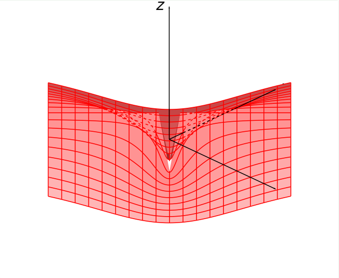

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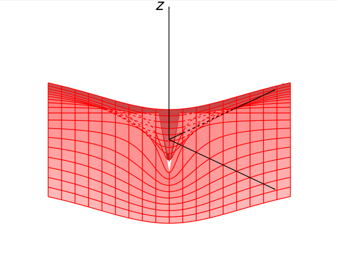

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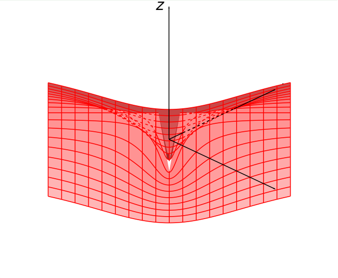

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This expression depends only on θ; as $r \to 0$ permits arbitrary behavior of θ, we'd had guessed correctly that the limit doesn't exist.

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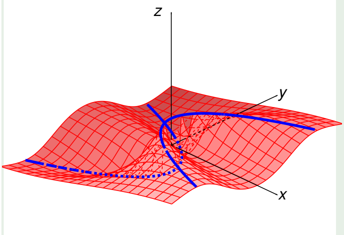
Example (All directional limits exist, limit doesn't)



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Therefore $\lim\limits_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$ does not exist.

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We say that the one-variable limit

$$\lim_{t \to 0} f(\mathbf{r}(t))$$

is the limit of f along the path $\mathbf{r}(t)$.

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If the limit $\lim_{Q \to P} f(Q)$ *exists, then every path limit exists and all path limits are equal.*

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- If we pick our path to be of the form $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{u}$, we see that the directional limit is a special case of the path limit.

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Continuity of vector fields

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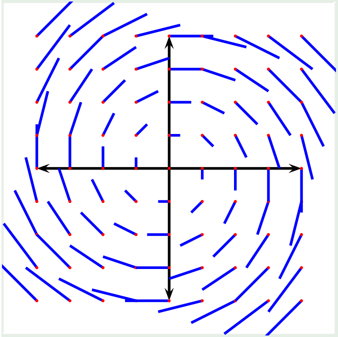

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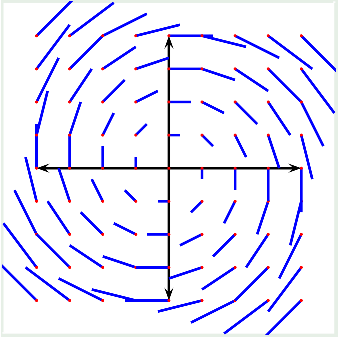

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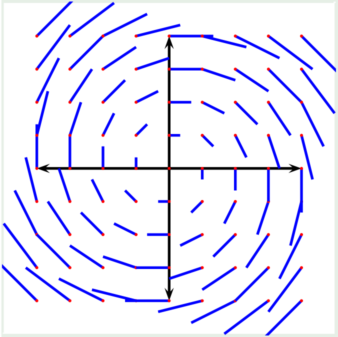

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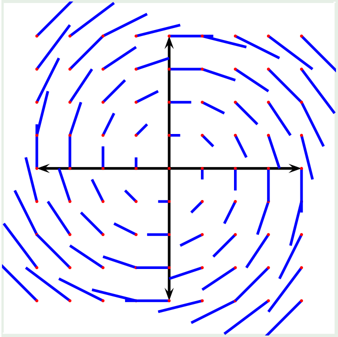

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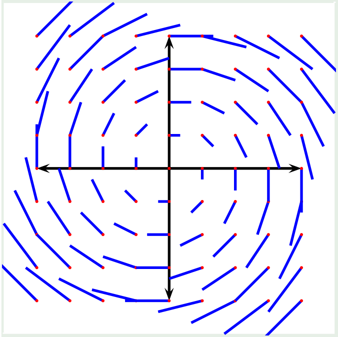

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