

# Calculus II

## Homework on Lecture 3

1. Evaluate the indefinite integral. Illustrate the steps of your solutions.

(a)  $\int x \sin x dx.$

(f)  $\int x^2 e^{-2x} dx.$

(b)  $\int x e^{-x} dx.$

(g)  $\int x \sin(2x) dx.$

(c)  $\int x^2 e^x dx.$

(h)  $\int x \cos(3x) dx.$

(d)  $\int x \sin(-2x) dx.$

(i)  $\int x^2 e^{2x} dx.$

(e)  $\int x^2 \cos(3x) dx.$

(j)  $\int x^3 e^x dx.$

**Solution.** 1.a.

$$\int x \underbrace{\sin x dx}_{=d(-\cos x)} = - \int x d(\cos x) = -x \cos x + \int \cos x dx = -x \cos x + \sin x + C.$$

**Solution.** 1.c.

$$\begin{aligned} \int x^2 \underbrace{e^x dx}_{d(e^x)} &= \int x^2 de^x = x^2 e^x - \int e^x 2x dx = x^2 e^x - \int 2x de^x \\ &= x^2 e^x - 2x e^x + \int 2e^x dx = x^2 e^x - 2x e^x + 2e^x + C. \end{aligned}$$

**Solution.** 1.f.

$$\begin{aligned} \int x^2 e^{-2x} dx &= \int x^2 d\left(\frac{e^{-2x}}{-2}\right) && \left| \begin{array}{l} \text{Integrate by parts} \end{array} \right. \\ &= -\frac{x^2 e^{-2x}}{2} - \int \left(\frac{e^{-2x}}{-2}\right) d(x^2) \\ &= -\frac{x^2 e^{-2x}}{2} + \int x e^{-2x} dx \\ &= -\frac{x^2 e^{-2x}}{2} + \int x d\left(\frac{e^{-2x}}{-2}\right) && \left| \begin{array}{l} \text{Integrate by parts} \end{array} \right. \\ &= -\frac{x^2 e^{-2x}}{2} - \frac{x e^{-2x}}{2} + \frac{1}{2} \int e^{-2x} dx \\ &= -\frac{x^2 e^{-2x}}{2} - \frac{x e^{-2x}}{2} - \frac{e^{-2x}}{4} + C. \end{aligned}$$

2. Evaluate the indefinite integral. Illustrate the steps of your solutions.

$$(a) \int x^2 \cos(2x) dx.$$

$$\text{ANSWER: } \frac{1}{2} (x \cos(2x) - (x^2) \sin(2x) + (x^2) \cos(2x) - \frac{1}{2} x^3 \sin(2x))$$

$$(b) \int x^2 e^{ax} dx, \text{ where } a \text{ is a constant.}$$

$$\text{ANSWER: } \frac{1}{a^3} (x^2 e^{ax} - 2ax e^{ax} + 2e^{ax})$$

$$(c) \int x^2 e^{-ax} dx, \text{ where } a \text{ is a constant.}$$

$$\text{ANSWER: } -\frac{1}{a^3} (x^2 e^{-ax} + 2ax e^{-ax} - 2e^{-ax})$$

$$(d) \int x^2 \frac{(e^{ax} + e^{-ax})^2}{4} dx, \text{ where } a \text{ is a constant.}$$

$$\text{ANSWER: } \frac{1}{4a^3} (x^2 (e^{2ax} + e^{-2ax}) - 2x (e^{2ax} - e^{-2ax}) + (e^{2ax} + e^{-2ax}))$$

$$(e) \int \frac{1}{\cos^2 x} dx. \quad (\text{Hint: This problem does not require integration by parts. What is the derivative of } \tan x?)$$

$$\text{ANSWER: } \tan x$$

$$(f) \int (\tan^2 x) dx. \quad (\text{Hint: This problem does not require integration by parts. We can use } \tan^2 x = \frac{1}{\cos^2 x} - 1 \text{ and the previous problem.})$$

$$(g) \int x \tan^2 x dx. \quad (\text{Hint: } \tan^2 x dx = d(F(x)), \text{ where } F(x) \text{ is the answer from the preceding problem.})$$

$$\text{ANSWER: } \frac{1}{2} x^2 \tan x - \frac{1}{4} x^2 \ln | \cos x | + \frac{1}{4} x^2$$

$$(h) \int e^{-\sqrt{x}} dx.$$

$$\text{ANSWER: } -2\sqrt{x} e^{-\sqrt{x}} - 4e^{-\sqrt{x}} + C$$

$$(i) \int \cos^2 x dx.$$

$$\text{ANSWER: } \frac{1}{2} x + \frac{1}{4} \sin(2x)$$

$$(j) \int \frac{x}{1+x^2} dx \quad (\text{Hint: use substitution rule, don't use integration by parts})$$

$$\text{ANSWER: } \frac{1}{2} \ln(1+x^2)$$

$$(k) \int (\arctan x) dx.$$

$$\text{ANSWER: } x \arctan x - \frac{1}{2} \ln(1+x^2)$$

$$(l) \int (\arcsin x) dx.$$

$$\text{ANSWER: } x \arcsin x - \sqrt{1-x^2}$$

$$(m) \int (\arcsin x)^2 dx. \quad (\text{Hint: Try substituting } x = \sin y.)$$

$$\text{ANSWER: } x^2 \arcsin x - \sqrt{1-x^2} + \frac{1}{2} x^2 \ln(x + \sqrt{1-x^2})$$

$$(n) \int \arctan\left(\frac{1}{x}\right) dx.$$

$$(o) \int \sin x e^x dx$$

$$\text{ANSWER: } e^x (\sin x - \cos x)$$

$$(p) \int \cos x e^x dx$$

$$\text{ANSWER: } e^x (\cos x + \sin x)$$

$$(q) \int \sin(\ln(x)) dx.$$

$$\text{ANSWER: } \frac{1}{2} (\sin(\ln(x)) - \cos(\ln(x)))$$

$$(r) \int \cos(\ln(x)) dx.$$

$$\text{ANSWER: } \frac{1}{2} (\cos(\ln(x)) + \sin(\ln(x)))$$

$$(s) \int \ln x dx$$

$$\text{ANSWER: } x \ln x - x$$

$$(t) \int x \ln x dx.$$

$$\text{ANSWER: } \frac{1}{2} x^2 \ln |x| - \frac{1}{4} x^2 + C$$

$$(u) \int \frac{\ln x}{\sqrt{x}} dx.$$

$$\text{ANSWER: } 2\sqrt{x} \ln(x) - \sqrt{x} + C$$

$$(v) \int (\ln x)^2 dx.$$

$$\text{ANSWER: } x(\ln x)^2 - 2x \ln x + x$$

$$(w) \int (\ln x)^3 dx.$$

$$\text{ANSWER: } x(\ln x)^3 - 3x(\ln x)^2 + 6x \ln x - 6x$$

$$(x) \int x^2 \cos^2 x dx. \quad (\text{This problem is related to Problem 2.d as } \cos x = \frac{e^{ix} + e^{-ix}}{2}).$$

$$\text{ANSWER: } \frac{1}{8} x^3 \cos(2x) + \frac{1}{4} x^2 \sin(2x) - \frac{1}{8} x^3 \sin(2x) + \frac{1}{4} x^2 \cos(2x)$$

**Solution.** 2.g.

$$\begin{aligned}
 \int x \tan^2 x \, dx &= \int x (\sec^2 x - 1) \, dx && \left| \text{use } \sec^2 x - 1 = \tan^2 x \right. \\
 &= \int x (\sec^2 x - 1) \, dx \\
 &= -\int x \, dx + \int x \sec^2 x \, dx && \left| \text{use } d(\tan x) = \sec^2 x \, dx \right. \\
 &= -\frac{x^2}{2} + \int x d(\tan x) && \left| \text{integrate by parts} \right. \\
 &= -\frac{x^2}{2} + x \tan x - \int \tan x \, dx \\
 &= -\frac{x^2}{2} + x \tan x - \int \frac{\sin x}{\cos x} \, dx && \left| \text{use } \sin x \, dx = -d(\cos x) \right. \\
 &= -\frac{x^2}{2} + x \tan x + \int \frac{d(\cos x)}{\cos x} && \left| \text{Set } y = \cos x \right. \\
 &= -\frac{x^2}{2} + x \tan x + \int \frac{1}{y} \, dy \\
 &= -\frac{x^2}{2} + x \tan x + \ln |y| + C && \left| \text{Substitute back } y = \cos x \right. \\
 &= -\frac{x^2}{2} + x \tan x + \ln |\cos x| + C \quad .
 \end{aligned}$$

**Solution.** 2.h.

$$\begin{aligned}
 \int e^{-\sqrt{x}} \, dx &= \int 2ye^{-y} \, dy && \left| \begin{array}{l} \text{Subst.: } \sqrt{x} = y \\ \frac{1}{2\sqrt{x}} \, dx = dy \\ dx = 2y \, dy \end{array} \right. \\
 &= \int 2y \, d(-e^{-y}) && \left| \text{int. by parts} \right. \\
 &= -2ye^{-y} + 2 \int e^{-y} \, dy \\
 &= -2ye^{-y} - 2e^{-y} + C \\
 &= -2\sqrt{x}e^{-\sqrt{x}} - 2e^{-\sqrt{x}} + C \quad .
 \end{aligned}$$

**Solution.** 2.i. Later, we shall study general methods for solving trigonometric integrals that will cover this example. Let us however show one way to solve this integral by integration by parts.

$$\begin{aligned}
 \int \cos^2 x \, dx &= x \cos^2 x - \int x d(\cos^2 x) \\
 &= x \cos^2 x - \int x 2 \cos x (-\sin x) \, dx && \left| \sin(2x) = 2 \sin x \cos x \right. \\
 &= x \cos^2 x + \int x \sin(2x) \, dx \\
 &= x \cos^2 x + \int x d\left(\frac{-\cos(2x)}{2}\right) \\
 &= x \cos^2 x + x \left(\frac{-\cos(2x)}{2}\right) - \int \left(\frac{-\cos(2x)}{2}\right) \, dx \\
 &= \frac{x}{2} (2 \cos^2 x - \cos(2x)) + \frac{\sin(2x)}{4} + C && \left| \cos(2x) = \cos^2 x - \sin^2 x \right. \\
 &= \frac{x}{2} (2 \cos^2 x - (\cos^2 x - \sin^2 x)) + \frac{\sin(2x)}{4} + C && \left| \cos^2 x + \sin^2 x = 1 \right. \\
 &= \frac{x}{2} + \frac{\sin(2x)}{4} + C \quad .
 \end{aligned}$$

**Solution.** 2.k

$$\begin{aligned}
 \int \arctan x \, dx &= x \arctan x - \int x d(\arctan x) \\
 &= x \arctan x - \int \frac{x}{x^2 + 1} \, dx \\
 &= x \arctan x - \int \frac{\frac{1}{2} d(x^2)}{x^2 + 1} \\
 &= x \arctan x - \int \frac{\frac{1}{2} d(x^2 + 1)}{x^2 + 1} \\
 &= x \arctan x - \frac{1}{2} \ln(x^2 + 1) + C \quad .
 \end{aligned}$$

**Solution. 2.m.**

$$\begin{aligned}
 \int (\arcsin x)^2 dx &= \int (\arcsin(\sin y))^2 d(\sin y) && \left| \begin{array}{l} \text{Set } x = \sin y \\ \text{Integrate by parts} \end{array} \right. \\
 &= \int y^2 \cos y dy = \int y^2 d(\sin y) \\
 &= y^2 \sin y - \int 2y \sin y dy \\
 &= y^2 \sin y + \int 2y d(\cos y) && \left| \begin{array}{l} \text{Integrate by parts} \end{array} \right. \\
 &= y^2 \sin y + 2y \cos y - 2 \int \cos y dy \\
 &= y^2 \sin y + 2y \cos y - 2 \sin y + C && \left| \begin{array}{l} \text{Substitute } y = \arcsin x \end{array} \right. \\
 &= \frac{x(\arcsin x)^2}{1} \\
 &\quad + 2\sqrt{1-x^2} \arcsin x - 2x + C \quad .
 \end{aligned}$$

**Solution. 2.o**

$$\begin{aligned}
 \int \sin x \underbrace{e^x dx}_{=de^x} &= \sin x e^x - \int e^x d(\sin x) = \sin x e^x - \int \cos x \underbrace{e^x dx}_{=de^x} \\
 &= \sin x e^x - e^x \cos x + \int e^x d(\cos x) \\
 &= e^x \sin x - e^x \cos x - \int e^x \sin x dx && \left| \begin{array}{l} \text{add } \int e^x \sin x dx \\ \text{to both sides} \end{array} \right. \\
 2 \int \sin x e^x dx &= \sin x e^x - e^x \cos x \\
 \int \sin x e^x dx &= \frac{1}{2} (\sin x e^x - e^x \cos x) \quad .
 \end{aligned}$$

**Solution. 2.q.**

$$\begin{aligned}
 \int \sin(\ln x) dx &= x \sin(\ln x) - \int x d(\sin(\ln x)) && \left| \begin{array}{l} \text{int. by parts} \end{array} \right. \\
 &= x \sin(\ln x) - \int x (\cos(\ln x)) (\ln x)' dx \\
 &= x \sin(\ln x) - \int \cos(\ln x) dx && \left| \begin{array}{l} \text{int. by parts} \end{array} \right. \\
 &= x \sin(\ln x) - \left( x \cos(\ln x) - \int x d(\cos(\ln x)) \right) \\
 &= x \sin(\ln x) - x \cos(\ln x) + \int x (-\sin(\ln x)) (\ln x)' dx \\
 &= x \sin(\ln x) - x \cos(\ln x) - \int \sin(\ln x) dx && \left| \begin{array}{l} \text{add } \int \sin(\ln x) dx \\ \text{to both sides} \end{array} \right. \\
 2 \int \sin(\ln x) dx &= x \sin(\ln x) - x \cos(\ln x) \\
 \int \sin(\ln x) dx &= \frac{x}{2} (\sin(\ln x) - \cos(\ln x)) \quad .
 \end{aligned}$$

**Solution. 2.s**

$$\int \ln x dx = x \ln x - \int x d(\ln x) = x \ln x - \int \frac{x}{x} dx = x \ln x - x + C \quad .$$

**Solution.** 2.u

$$\begin{aligned}\int \frac{\ln x}{\sqrt{x}} dx &= \int (\ln x) 2d(\sqrt{x}) && \left| \text{integrate by parts} \right. \\&= (\ln x) 2\sqrt{x} - \int 2\sqrt{x} d(\ln x) \\&= 2\sqrt{x} \ln x - 2 \int \frac{\sqrt{x}}{x} dx \\&= 2\sqrt{x} \ln x - 2 \int x^{-\frac{1}{2}} dx \\&= 2\sqrt{x} \ln x - 4\sqrt{x} + C \\&= 2\sqrt{x}(\ln x - 2) + C \quad .\end{aligned}$$

3. Compute  $\int x^n e^x dx$ , where  $n$  is a non-negative integer.

**Solution.** 3

$$\begin{aligned}\int x^n e^x dx &= \int x^n de^x \\&= x^n e^x - \int e^x dx^n \\&= x^n e^x - n \int x^{n-1} e^x dx \\&= x^n e^x - n \left( \int x^{n-1} de^x \right) \\&= x^n e^x - n \left( x^{n-1} e^x - \int (n-1) x^{n-2} e^x dx \right) \\&= x^n e^x - n x^{n-1} e^x + n(n-1) \int x^{n-2} e^x dx \\&= \dots (\text{continue above process}) \dots \\&= x^n e^x - n x^{n-1} e^x + n(n-1) x^{n-2} e^x + \dots \\&\quad + (-1)^k n(n-1)(n-2) \dots (n-k+1) x^{n-k} e^x \\&\quad + \dots + (-1)^n n! e^x + C \\&= C + \sum_{k=0}^n (-1)^k \frac{n!}{(n-k)!} x^{n-k} e^x \quad .\end{aligned}$$