

Calculus I

Lecture 10

Trigonometric Derivatives

Todor Milev

<https://github.com/tmilev/freecalc>

2020

Outline

1 Derivatives of Trigonometric Functions

License to use and redistribute

These lecture slides and their \LaTeX source code are licensed to you under the Creative Commons license CC BY 3.0. You are free

- to Share - to copy, distribute and transmit the work,
- to Remix - to adapt, change, etc., the work,
- to make commercial use of the work,

as long as you reasonably acknowledge the original project.

- Latest version of the .tex sources of the slides:

<https://github.com/tmilev/freecalc>

- Should the link be outdated/moved, search for “freecalc project”.
- Creative Commons license CC BY 3.0:

<https://creativecommons.org/licenses/by/3.0/us/>
and the links therein.

License to use and redistribute

These lecture slides and their \LaTeX source code are licensed to you under the Creative Commons license CC BY 3.0. You are free

- to Share - to copy, distribute and transmit the work,
- to Remix - to adapt, change, etc., the work,
- to make commercial use of the work,

as long as you reasonably acknowledge the original project.

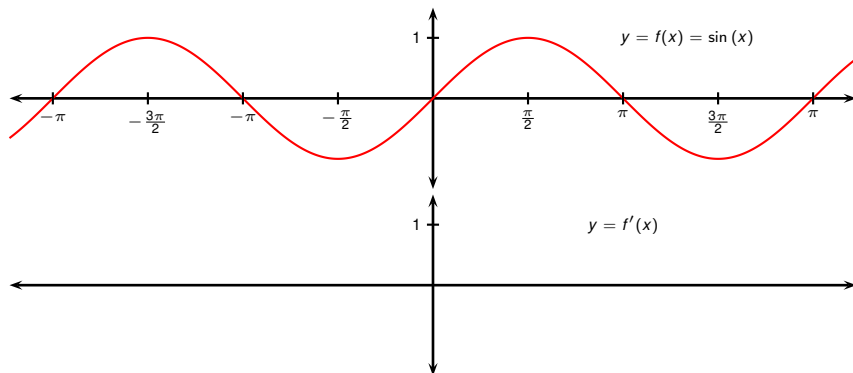
- Latest version of the .tex sources of the slides:

<https://github.com/tmilev/freecalc>

- Should the link be outdated/moved, search for “freecalc project”.
- Creative Commons license CC BY 3.0:

<https://creativecommons.org/licenses/by/3.0/us/>
and the links therein.

Derivatives of Trigonometric Functions



What is the derivative of $f(x) = \sin x$? It looks like $\cos x$.

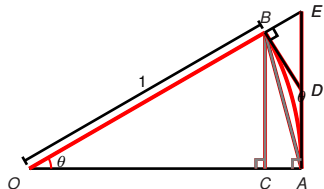
Let $f(x) = \sin x$.

$$\begin{aligned}\text{Then } f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin x}{h} \\&= \lim_{h \rightarrow 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\&= \lim_{h \rightarrow 0} \left(\frac{\sin x \cos h - \sin x}{h} + \frac{\cos x \sin h}{h} \right) \\&= \lim_{h \rightarrow 0} \left(\sin x \left(\frac{\cos h - 1}{h} \right) + \cos x \left(\frac{\sin h}{h} \right) \right) \\&= \lim_{h \rightarrow 0} \sin x \cdot \lim_{h \rightarrow 0} \left(\frac{\cos h - 1}{h} \right) + \lim_{h \rightarrow 0} \cos x \cdot \lim_{h \rightarrow 0} \left(\frac{\sin h}{h} \right) \\&= \sin x \cdot \lim_{h \rightarrow 0} \left(\frac{\cos h - 1}{h} \right) + \cos x \cdot \lim_{h \rightarrow 0} \left(\frac{\sin h}{h} \right)\end{aligned}$$

We need to do more work to find the other two limits.

Claim: $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$

Suppose $0 < \theta < \frac{\pi}{2}$. Write $\sin \theta$ using ratios of side lengths of a triangle.



$$\sin \theta = \frac{|BC|}{|OB|} = |BC| < |AB| < \text{arc}AB = \theta$$

Therefore $\sin \theta < \theta$ and therefore $\frac{\sin \theta}{\theta} < 1$.

$$\begin{aligned} \theta = \text{arc}AB &< |AD| + |DB| < |AD| + |DE| \\ &= |AE| = |OA| \tan \theta = \tan \theta \end{aligned}$$

Therefore $\theta < \tan \theta = \frac{\sin \theta}{\cos \theta}$, so $\cos \theta < \frac{\sin \theta}{\theta}$.

$$\cos \theta < \frac{\sin \theta}{\theta} < 1$$

$\lim_{\theta \rightarrow 0} \cos \theta = 1$ and $\lim_{\theta \rightarrow 0} 1 = 1$, so by the Squeeze Theorem

$\lim_{\theta \rightarrow 0^+} \frac{\sin \theta}{\theta} = 1$. $\frac{\sin \theta}{\theta}$ is even, so the left limit is also 1.

Let $f(x) = \sin x$.

$$\begin{aligned}\text{Then } f'(x) &= \lim_{h \rightarrow 0} \sin x \cdot \lim_{h \rightarrow 0} \left(\frac{\cos h - 1}{h} \right) + \lim_{h \rightarrow 0} \cos x \cdot \lim_{h \rightarrow 0} \left(\frac{\sin h}{h} \right) \\ &= \sin x \cdot \lim_{h \rightarrow 0} \left(\frac{\cos h - 1}{h} \right) + \cos x \cdot 1\end{aligned}$$

We need to find

$$\begin{aligned}\lim_{h \rightarrow 0} \frac{\cos h - 1}{h} &= \lim_{h \rightarrow 0} \frac{(\cos h - 1)}{h} \cdot \frac{(\cos h + 1)}{(\cos h + 1)} = \lim_{h \rightarrow 0} \frac{\cos^2 h - 1}{h(\cos h + 1)} \\ &= \lim_{h \rightarrow 0} \frac{-\sin^2 h}{h(\cos h + 1)} = - \lim_{h \rightarrow 0} \left(\frac{\sin h}{h} \cdot \frac{\sin h}{\cos h + 1} \right) \\ &= - \lim_{h \rightarrow 0} \frac{\sin h}{h} \cdot \lim_{h \rightarrow 0} \frac{\sin h}{\cos h + 1} = -1 \cdot \left(\frac{0}{1 + 1} \right) = 0\end{aligned}$$

Theorem (The Derivative of $\sin x$)

$$\frac{d}{dx}(\sin x) = \cos x$$

Example (Product Rule, Product Rule with Sine)

Differentiate $f(x) = x \sin x$.

$$\begin{aligned}\text{Product Rule: } f'(x) &= \frac{d}{dx}(x)(\sin x) + (x) \frac{d}{dx}(\sin x) \\ &= (1)(\sin x) + (x)(\cos x) \\ &= x \cos x + \sin x.\end{aligned}$$

Example (Quotient Rule, Natural Exponential Function and Sine)

Differentiate $y = \frac{e^x}{2 + \sin x}$.

Quotient Rule:

$$\begin{aligned}\frac{dy}{dx} &= \frac{\frac{d}{dx}(e^x)(2 + \sin x) - (e^x)\frac{d}{dx}(2 + \sin x)}{(2 + \sin x)^2} \\ &= \frac{(e^x)(2 + \sin x) - (e^x)(\cos x)}{(2 + \sin x)^2} \\ &= \frac{2e^x + e^x \sin x - e^x \cos x}{(2 + \sin x)^2} \\ &= \frac{e^x(2 + \sin x - \cos x)}{(2 + \sin x)^2}.\end{aligned}$$

Example (Trigonometric limit)

$$\begin{aligned}\text{Find } \lim_{x \rightarrow 0} \frac{2x}{\sin(9x)} &= \lim_{x \rightarrow 0} \frac{2x}{\sin(9x)} \cdot \frac{9}{9} \\ &= \lim_{x \rightarrow 0} \frac{2}{9} \cdot \frac{9x}{\sin(9x)} \\ &= \lim_{x \rightarrow 0} \frac{2}{9} \cdot \frac{1}{\frac{\sin(9x)}{9x}} = \lim_{\theta \rightarrow 0} \frac{2}{9} \cdot \frac{1}{\frac{\sin \theta}{\theta}}.\end{aligned}$$

$$\text{Let } \theta = 9x.$$

$$\text{As } x \rightarrow 0, \quad \theta \rightarrow 0.$$

$$\begin{aligned}\text{Then } \lim_{x \rightarrow 0} \frac{2x}{\sin(9x)} &= \frac{2}{9} \cdot \frac{1}{\lim_{\theta \rightarrow 0} \left(\frac{\sin \theta}{\theta} \right)} \\ &= \frac{2}{9} \cdot \frac{1}{1} = \frac{2}{9}.\end{aligned}$$

Theorem (The Derivative of $\cos x$)

$$\frac{d}{dx}(\cos x) = -\sin x$$

- This can be proved in a similar fashion as the formula for $\sin x$.
- Alternatively, this can be proved using the derivative of $\sin x$ and (the not yet studied) Implicit Differentiation and Chain Rule.

Example (Product Rule, with Cosine)

Differentiate $f(x) = x \cos x$.

$$\begin{aligned}\text{Product Rule: } f'(x) &= \frac{d}{dx}(x)(\cos x) + (x) \frac{d}{dx}(\cos x) \\ &= (1)(\cos x) + (x)(-\sin x) \\ &= -x \sin x + \cos x.\end{aligned}$$

Theorem (The Derivative of Tangent)

$$\frac{d}{dx}(\tan x) = \sec^2 x.$$

Proof.

$$\text{Let } y = \tan x = \frac{\sin x}{\cos x}.$$

Quotient Rule:

$$\begin{aligned}\frac{dy}{dx} &= \frac{\frac{d}{dx}(\sin x)(\cos x) - (\sin x)\frac{d}{dx}(\cos x)}{(\cos x)^2} \\ &= \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} \\ &= \sec^2 x.\end{aligned}$$



Derivatives of Trigonometric Functions

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

Example (Quotient Rule, Trig)

Differentiate $y = \frac{\sec x}{1 + \tan x}$.

Quotient Rule:

$$\begin{aligned}\frac{dy}{dx} &= \frac{\frac{d}{dx}(\sec x)(1 + \tan x) - (\sec x)\frac{d}{dx}(1 + \tan x)}{(1 + \tan x)^2} \\&= \frac{(\sec x \tan x)(1 + \tan x) - (\sec x)(\sec^2 x)}{(1 + \tan x)^2} \\&= \frac{\sec x(\tan x + \tan^2 x - \sec^2 x)}{(1 + \tan x)^2} \\&= \frac{\sec x(\tan x + (-1))}{(1 + \tan x)^2} = \frac{\sec x(\tan x - 1)}{(1 + \tan x)^2}.\end{aligned}$$

Example (Using the Product Rule twice)

Differentiate:

$$y = \theta e^{\theta} (\tan \theta + \sec \theta).$$

Product Rule:

$$y' = \frac{d}{d\theta} (\theta e^{\theta}) (\tan \theta + \sec \theta) + \theta e^{\theta} \frac{d}{d\theta} (\tan \theta + \sec \theta)$$

Product Rule:

$$\begin{aligned} &= \left(\theta \frac{d}{d\theta} (e^{\theta}) + \frac{d}{d\theta} (\theta) e^{\theta} \right) (\tan \theta + \sec \theta) + \theta e^{\theta} (\sec^2 \theta + \tan \theta \sec \theta) \\ &= (\theta (e^{\theta}) + (1) e^{\theta}) (\tan \theta + \sec \theta) + \theta e^{\theta} (\sec^2 \theta + \tan \theta \sec \theta) \\ &= \theta e^{\theta} \sec \theta (\sec \theta + \tan \theta) + e^{\theta} (\theta + 1) (\tan \theta + \sec \theta) \\ &= (\theta \sec \theta + \theta + 1) e^{\theta} (\tan \theta + \sec \theta). \end{aligned}$$

Example

Find the 27th derivative of $f(x) = \cos x$.

$$f'(x) = -\sin x$$

$$f''(x) = -\cos x$$

$$f'''(x) = \sin x$$

$$f^{(4)}(x) = \cos x$$

$$f^{(5)}(x) = -\sin x$$

- The derivatives repeat in a cycle of length 4.
- $f^{(24)}(x) = \cos x$.
- Differentiate three more times: $f^{(27)}(x) = \sin x$.