Precalculus
Lecture 10
Exponents

Todor Milev
https://github.com/tmilev/freecalc

2020

Outline

1. Exponents
   - Two ways to define exponents
   - Basic properties
   - The Natural Exponential Function

License to use and redistribute

These lecture slides and their LaTeX source code are licensed to you under the Creative Commons license CC BY 3.0. You are free

- to Share - to copy, distribute and transmit the work,
- to Remix - to adapt, change, etc., the work,
- to make commercial use of the work,

as long as you reasonably acknowledge the original project.

- Latest version of the .tex sources of the slides:
  https://github.com/tmilev/freecalc
- Should the link be outdated/moved, search for "freecalc project".
- Creative Commons license CC BY 3.0:
  https://creativecommons.org/licenses/by/3.0/us/
  and the links therein.

Properties of exponential expressions.

For integer x, y and bases a, b, we demonstrate the exponent rules by example.

1. $a^x a^y = a^{x+y}$
2. $\dfrac{a^x}{a^y} = a^{x-y}$
3. $(a^x)^y = a^{xy}$
4. $(ab)^x = a^x b^x$

These rules do continue to hold for all $a > 0$, $b > 0$ and arbitrary x and y. The rules do fail when $a < 0$, $b < 0$ and x, y are not integers.

$$
\begin{aligned}
7^3 \cdot 7^2 &= (7 \cdot 7 \cdot 7)(7 \cdot 7) \\
&= 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \\
&= 7^5 \\
&= 7^{3+2}.
\end{aligned}
$$

Properties of exponential expressions.

For integer x, y and bases a, b, we demonstrate the exponent rules by example.

1. $a^x a^y = a^{x+y}$
2. $\dfrac{a^x}{a^y} = a^{x-y}$
3. $(a^x)^y = a^{xy}$
4. $(ab)^x = a^x b^x$

These rules do continue to hold for all $a > 0$, $b > 0$ and arbitrary x and y. The rules do fail when $a < 0$, $b < 0$ and x, y are not integers.

$$
\begin{aligned}
\frac{7^3}{7^2} &= \frac{\cancel{7} \cdot \cancel{7} \cdot 7}{\cancel{7} \cdot \cancel{7}} \\
&= 7 \\
&= 7^1 \\
&= 7^{3-2}.
\end{aligned}
$$

Properties of exponential expressions.

For integer x, y and bases a, b, we demonstrate the exponent rules by example.

1. $a^x a^y = a^{x+y}$
2. $\dfrac{a^x}{a^y} = a^{x-y}$
3. $(a^x)^y = a^{xy}$
4. $(ab)^x = a^x b^x$

These rules do continue to hold for all $a > 0$, $b > 0$ and arbitrary x and y. The rules do fail when $a < 0$, $b < 0$ and x, y are not integers.

$$
\begin{aligned}
\left(7^2\right)^4 &= 7^2 \cdot 7^2 \cdot 7^2 \cdot 7^2 \\
&= (7 \cdot 7)(7 \cdot 7)(7 \cdot 7)(7 \cdot 7) \\
&= 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \\
&= 7^8 \\
&= 7^{2 \cdot 4}
\end{aligned}
$$

Properties of exponential expressions.

For integer x, y and bases a, b, we demonstrate the exponent rules by example.

1. $a^x a^y = a^{x+y}$
2. $\dfrac{a^x}{a^y} = a^{x-y}$
3. $(a^x)^y = a^{xy}$
4. $(ab)^x = a^x b^x$

These rules do continue to hold for all $a > 0$, $b > 0$ and arbitrary x and y. The rules do fail when $a < 0$, $b < 0$ and x, y are not integers.

$$
\begin{aligned}
(5 \cdot 7)^3 &= (5 \cdot 7)(5 \cdot 7)(5 \cdot 7) \\
&= 5 \cdot 7 \cdot 5 \cdot 7 \cdot 5 \cdot 7 \\
&= 5 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdot 7 \\
&= 5^3 \cdot 7^3
\end{aligned}
$$

Exponents overview

- For integer x, we know how to compute a^x as a function of a.
- How do we compute $f(x) = a^x$ when x is not an integer?
- We need to go back to the definition of a^x (for x non-integer).
- In what follows we give/recall an elementary way to define exponent.
- Then we give an alternative second definition.
- The second definition will be studied in sufficient depth only much later.
- The two definitions are equivalent: if we choose one definition the other becomes a theorem and the other way round.
- Choosing one definition makes some statements easier to prove and others more difficult.
- We shall discuss pros and cons of the two. In a nutshell:
  - the first elementary definition is easier to motivate;
  - the second alternative definition is easier to compute with.

Exponent definition using limits (approach I)

- For integer p we know to compute a^p.
- Therefore for integer q we know to compute
  $a^{\frac{1}{q}} = \sqrt[q]{a} = \max\{x|$ for which $x^q \le a\}$.
- Therefore we know to compute $a^{\frac{p}{q}}$ for all rational $\frac{p}{q}$.
- We can then define

$$a^x = \lim_{\substack{y \to x \\ y\text{-rational}}} a^y$$

  For example, a^π would be defined as the limit of the sequence
  $a^{3.14}, a^{3.141}, a^{3.1415}, \ldots$.

- Cons: not computationally effective; not how computers compute.
- Pros: for non-integer x and y, it is very easy to prove that
  $a^{x+y} = a^x a^y$ - this follows from the definition of limit above.
- This is the definition assumed in many elementary courses.

Exponent definition using series (approach II)

- The following formula (studied much later) can be used as alternative definition.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \dots$$

Here $n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot (n-1) \cdot n$ and is read "n factorial".

- For $|x| < 1$ define

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{n+1} x^n}{n} + \dots$$

Infinite sum studied much later.

- For arbitrary $a > 0$ define a^x as $a^x = e^{x \ln a}$.
- Cons: more difficult to prove $e^{x+y} = e^x e^y$ and $e^{\ln(1+x)} = 1 + x$, proof done later.
- Pros: this is how e^x and a^x are actually computed (by modern computers and by humans in the past).

Exponential Functions

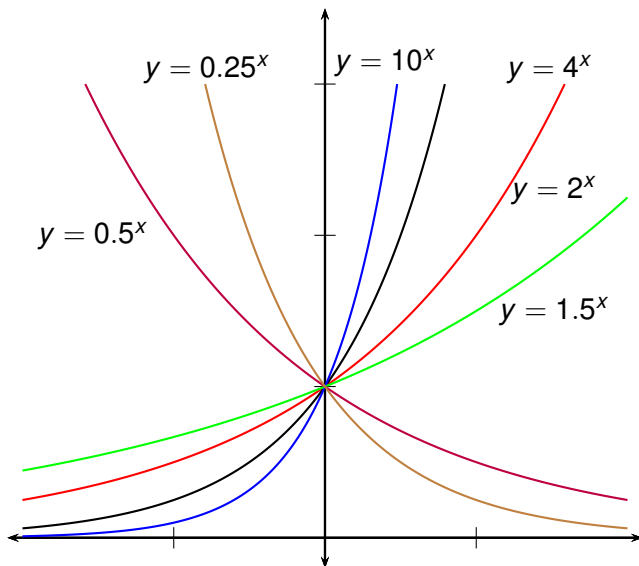The function $f(x) = 2^x$ is called an exponential function because the variable x is the exponent.

$y = 2^x$
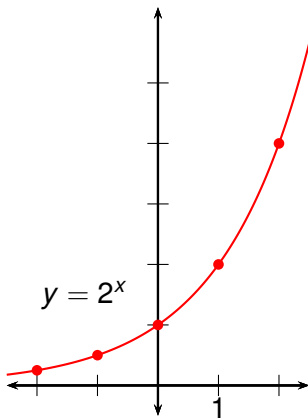
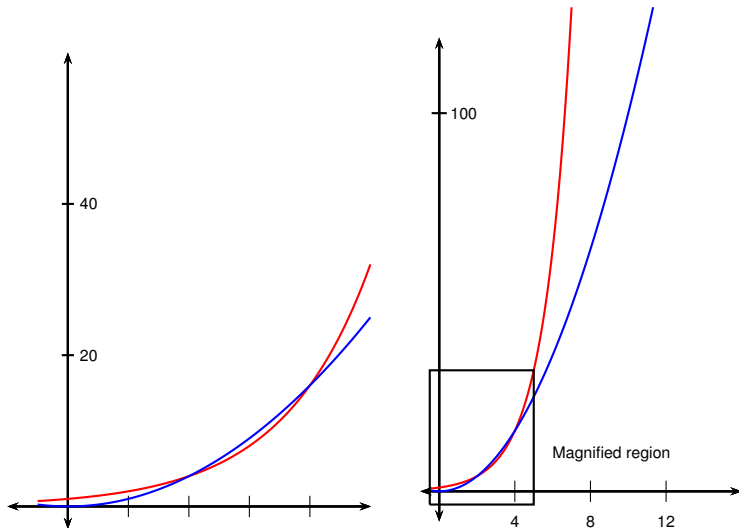| x	y
2	4
1	2
0	1
-1	$\frac{1}{2}$
-2	$\frac{1}{4}$

(Exponential Function Terminology)

An exponential function is a function of the form $f(x) = a^x$, where a is a positive constant.

Graphs of various exponential functions.

$y = 0.25^x$ $y = 10^x$ $y = 4^x$
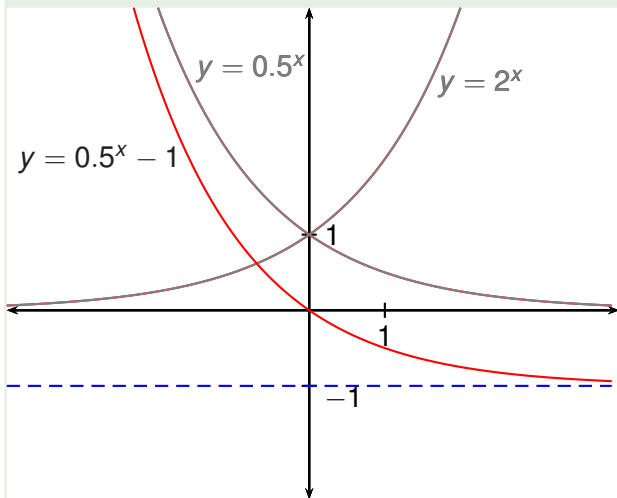
$y = 0.5^x$

$y = 2^x$

$y = 1.5^x$

Graphical comparison of $y = 2^x$ with $y = x^2$. Axes have different scales.

Example

Draw the graph of the function $y = 2^{-x} - 1 = 0.5^x - 1 = \left(\frac{1}{2}\right)^x - 1$.
Assume the graph of $y = 2^x$ given.

- Plot of 2^x assumed given.
- Plot $f(-x)$ = reflect $f(x)$ across y axis.
- Plot $g(x) - 1$ = shift graph $g(x)$ 1 unit down.

Proposition

Let $a > 0$, $a \neq 1$. Let x and y be real numbers. Then $a^x = a^y$ if and only if $x = y$.

- In other words, the exponent function a^x is one-to-one.

The Natural Exponential Function

- One base for an exponential function is especially useful.
- It has a special property: its tangent line at $x = 0$ has slope $m = 1$.
- We call this number e, known as Euler's number or Napier's constant.
- e is a number between 2 and 3.
- In fact, $e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots \approx 2.71828$.

$y = 2^x$ $m \approx 0.693147$ $y = e^x$ $m = 1$ $y = 3^x$ $m \approx 1.09861$
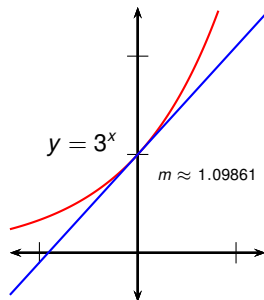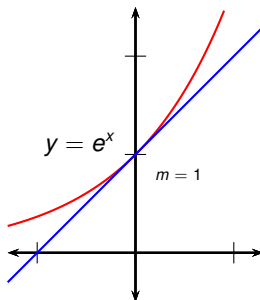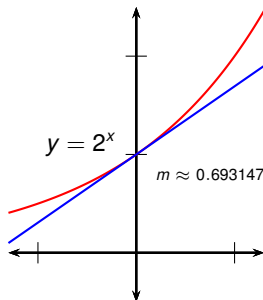
Recall that $e = 1 + \frac{1}{1} + \frac{1}{2!} + \frac{1}{3!} + \cdots \approx 2.718281828$.

Theorem (The Number e as a Limit)

For large n we have that:

$$
\begin{aligned}
e &\approx \left(1 + \frac{1}{n}\right)^n \\
&\approx (1 + n)^{\frac{1}{n}} \\
e^x &\approx \left(1 + \frac{x}{n}\right)^n
\end{aligned}
$$

All approximations become better as n increases.

- The approximation was discovered by Jacob Bernoulli (1655-1705) in order to apply to compound interest rate computations.

- In finance, compound interest is interest on a deposit which gets added automatically to the deposit so it earns additional interest from then on.
- The period in which this compounding process occurs is called compounding period.
- Annual compound interest rate of $k\%$ compounded once a year multiplies the current deposit by a factor of $\left(1 + \dfrac{k}{100}\right)$.
- Therefore n years of annual compound interest rate of $k\%$ compounded once a year multiplies the original deposit by factor:

$$\underbrace{\underbrace{\underbrace{\left(1 + \frac{k}{100}\right)}_{\text{after 1 year}} \cdot \left(1 + \frac{k}{100}\right)}_{\text{after 2 years}} \cdot \ldots \cdot \left(1 + \frac{k}{100}\right)}_{\text{after } n \text{ years}} = \left(1 + \frac{k}{100}\right)^n$$

Definition

The amount of money obtained from principal (original deposit) P after n years of annual compound interest rate of $k\%$, compounded once a year, is given by the formula

$$P \left(1 + \frac{k}{100} \right)^n.$$

Example

You have 1000 USD kept at annual rate of 5%. The interest is compounded yearly. Approximate without using a calculator the amount of money you will have after 40 years. Check your approximation with a calculator.

Example

Decide, without using a calculator, which is more profitable: earning a yearly compound interest of 2% for 150 years or earning yearly simple interest of 11% for 150 years? Check your approximation with a calculator.

Example

When quickly computing interest rate "in the head", financial advisors often use the following trick called the "rule of 72". To find the time in years t needed for a sum to double under compound interest rate of $k\%$, financial advisors simply approximate $t \approx \frac{72}{k}$.

To illustrate the rule, under an interest rate of 2%, one needs approximately $\frac{72}{2} = 36$ years for the sum to double. Under interest rate of 6%, the sum doubles after only about $\frac{72}{6} = 12$ years. In 36 years an interest of 6% would double 3 times, in other words would increase by a factor of $2^3 = 8$.

Using the approximation $e \approx \left(1 + \frac{1}{n}\right)^n$ for large n, justify the rule of 72.