Calculus III
Homework on Lecture 13

1. Evaluate the double integral.

   (a) $\displaystyle\iint_D x^3y^2\mathrm{d}x\mathrm{d}y$, $D=\{(x,y)|0\le x\le 2, -x\le y\le x\}$.

   (b) $\displaystyle\iint_D \frac{4y}{x^3+2}\mathrm{d}x\mathrm{d}y$, $D=\{(x,y)|1\le x\le 2, 0\le y\le 2x\}$.

   (c) $\displaystyle\iint_D \frac{2y}{x^2+1}\mathrm{d}x\mathrm{d}y$, $D=\{(x,y)|0\le x\le 1, 0\le y\le \sqrt{x}\}$.

   (d) $\displaystyle\iint_D e^{y^2}\mathrm{d}x\mathrm{d}y$, $D=\{(x,y)|0\le y\le 1, 0\le x\le y\}$.

   (e) $\displaystyle\iint_D x\cos y\mathrm{d}x\mathrm{d}y$, D bounded by $y=0$, $y=x^2$, $x=1$.

   (f) $\displaystyle\iint_D (x+y)\mathrm{d}x\mathrm{d}y$, D bounded by $y=\sqrt{x}$ and $y=x^2$.

   (g) $\displaystyle\iint_D y^3\mathrm{d}x\mathrm{d}y$, D - triangle with vertices $(0,2),(1,1),(3,2)$.

   (h) $\displaystyle\iint_D xy^2\mathrm{d}x\mathrm{d}y$, D enclosed by $x=0$ and $x^2+y^2=1$.

   (i) $\displaystyle\iint_D (2x-y)\mathrm{d}x\mathrm{d}y$, D bounded by circle with radius 2 centered at the origin.

   (j) $\displaystyle\iint_D 2xy\mathrm{d}x\mathrm{d}y$, D- triangular region with vertices $(0,0),(1,2),(0,3)$.

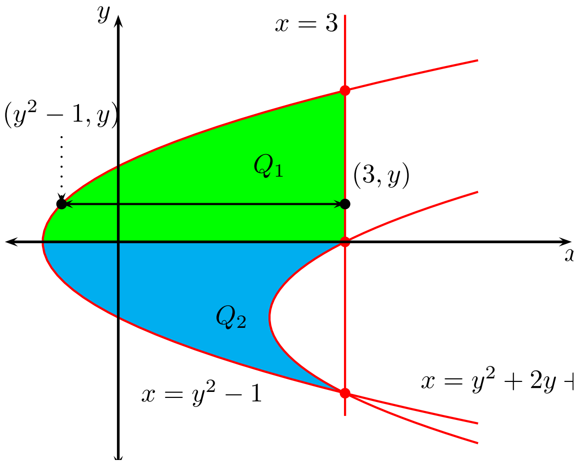
2. Evaluate the double integral. The answer key has not been proofread, use with caution.

   (a) $\iint_{\mathcal{R}} xy\mathrm{d}x\mathrm{d}y$ where $\mathcal{R}$ is bounded by the curves $x=3$, $x+1=y^2$, $x=y^2+2y+3$.

   answer: $\frac{52}{15}$

   (b) $\iint_{\mathcal{R}} xy\mathrm{d}x\mathrm{d}y$.
   where $\mathcal{R}$ is the region enclosed by $y=x^2+1$ and $y=2x^2-x-1$.

Solution. 2.a



We start by plotting the region. $x = y^2 - 1$ is a parabola symmetric across the x axis; $x = y^2 + 2y + 3$ is a parabola with vertex at $x = 3$, $y = -2$. The two parabolas intersect when

$$\begin{array}{rcl} x = y^2 - 1 &=& y^2 + 2y + 3 \\ 2y + 4 &=& 0 \\ y &=& -2 \\ x &=& y^2 - 1 = (-2)^2 - 1 = 3, \end{array}$$

i.e., when $(x, y) = (3, -2)$. The line $x = 3$ intersects $x + 1 = y^2$ when $y^2 = 3 + 1 = 4$, i.e., when $(x, y) = (3, \pm 2)$. The line $x = 3$ intersects $x = y^2 + 2y + 3$ when $3 = y^2 + 2y + 3$. This implies $y(y - 2) = 0$ and finally we conclude the intersections of $x = 3$ with $x = y^2 + 2y + 3$ are $(x, y) = (3, 0)$ and $(x, y) = (3, 2)$. We can conclude that there three curves are plotted as indicated in the figure. Of the 8 regions bounded by the curves only two are bounded, and only one of them is bounded by all three curves. Since no further instruction is given in the problem, we assume that the intended region is the one bounded by all three curves, i.e., the region indicated in the figure above (the other bounded region can be enclosed using two of the curves only). Let Q_1 and Q_2 be the regions indicated in the figure above. Those two regions are curvilinear trapezoids with vertical bases. Consider the region Q_1. Fix the y coordinate of a point in Q_1. The figure shows that, for that fixed value of y, x varies between $y^2 - 1$ and 3. For Q_2, it similarly follows that, for a fixed y, x varies between $y^2 - 1$ and $y^2 + 2y + 3$. The points in Q_1 have y coordinates in the range $y \in [0, 2]$, and similarly, in Q_2 we have that the y coordinate varies in the range $y \in [-2, 0]$. Thus our regions are parametrized as

$$\begin{array}{rcl} Q_1 &=& \{(x, y) | 0 \leq y \leq 2, y^2 - 1 \leq x \leq 3\} \\ Q_1 &=& \{(x, y) | -2 \leq y \leq 0, y^2 - 1 \leq x \leq y^2 + 2y + 3\}. \end{array}$$

Finally our integral becomes

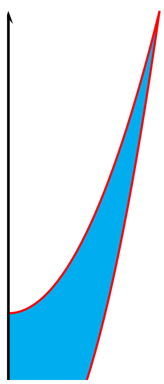
$$\begin{array}{rcl} \displaystyle\iint_{\mathcal{R}} xy \mathrm{d}x \mathrm{d}y &=& \displaystyle\iint_{Q_1} xy \mathrm{d}x \mathrm{d}y + \iint_{Q_2} \mathrm{d}x \mathrm{d}y \\ &=& \displaystyle\int_{y=0}^{y=2} \left(\int_{x=y^2-1}^{x=3} xy \mathrm{d}x \right) \mathrm{d}y + \int_{y=-2}^{y=0} \left(\int_{x=y^2-1}^{x=y^2+2y+3} xy \mathrm{d}x \right) \mathrm{d}y \\ &=& \displaystyle\int_{y=0}^{y=2} \left[\frac{x^2 y}{2} \right]_{x=y^2-1}^{x=3} \mathrm{d}y + \int_{y=0}^{y=3} \left[\frac{x^2 y}{2} \right]_{x=y^2-1}^{x=y^2+2y+3} \mathrm{d}y \\ &=& \displaystyle\int_{y=-2}^{y=0} \left(-\frac{1}{2} y (y^2 - 1)^2 + \frac{9}{2} y \right) \mathrm{d}y + \int_{y=-2}^{y=0} \left(\frac{1}{2} y (y^2 + 2y + 3)^2 - \frac{1}{2} y (y^2 - 1)^2 \right) \mathrm{d}y \\ &=& \displaystyle\int_{y=0}^{y=2} \left(-\frac{1}{2} y^5 + y^3 + 4y \right) \mathrm{d}y + \int_{y=-2}^{y=0} \left(2y^4 + 6y^3 + 6y^2 + 4y \right) \mathrm{d}y \\ &=& \displaystyle\left[-\frac{1}{12} y^6 + \frac{1}{4} y^4 + 2y^2 \right]_{y=0}^{y=2} + \left[\frac{2}{5} y^5 + \frac{3}{2} y^4 + 2y^3 + 2y^2 \right]_{y=-2}^{y=0} \\ &=& \displaystyle\frac{20}{3} - \frac{16}{5} = \frac{52}{15} \end{array}$$

Solution. 2.b

We start by plotting the region. $y = x^2 + 1$ is a parabola symmetric across the y axis; $y = 2x^2 - x - 1$ is a parabola with vertex at $x = \frac{1}{4}$, $y = -\frac{9}{8}$. The two parabolas intersect when

$$\begin{array}{rcl} x^2 + 1 &=& 2x^2 - x - 1 \\ x^2 - x - 2 &=& 0 \\ x &=& -1 \text{ or } 2 \quad . \end{array}$$

Thus the region looks as plotted below.



Therefore our region $\mathcal{R}$ is parametrized as

$$\mathcal{R} = \{(x, y) | 2x^2 - x - 1 \leq x^2 + 1\}.$$

This is a single curvilinear trapezoid. We can integrate directly as follows.

$$\begin{array}{rcl} \displaystyle\iint_{\mathcal{R}} xy \mathrm{d}x\mathrm{d}y &=& \displaystyle\int_{x=-1}^{x=2} \int_{y=2x^2-x-1}^{y=x^2+1} xy \mathrm{d}y\mathrm{d}x \\ &=& \displaystyle\int_{x=-1}^{x=2} \left[\frac{1}{2}xy^2\right]_{y=2x^2-x-1}^{y=x^2+1} \mathrm{d}y \\ &=& \displaystyle\int_{x=-1}^{x=2} \left(-\frac{3}{2}x^5 + 2x^4 + \frac{5}{2}x^3 - x^2\right) \mathrm{d}x \\ &=& \displaystyle\left[-\frac{1}{4}x^6 + \frac{2}{5}x^5 + \frac{5}{8}x^4 - \frac{1}{3}x^3\right]_{x=-1}^{x=2} \\ &=& \displaystyle\frac{153}{40} \quad . \end{array}$$

3. Integrate.

   (a) $\displaystyle\int_{y=0}^{y=\sqrt{\pi}} \int_{x=y}^{x=\sqrt{\pi}} \cos\left(x^2\right) \mathrm{d}x\mathrm{d}y.$

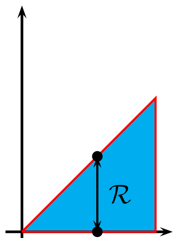
   (b) $\displaystyle\int_{y=0}^{y=1} \int_{x=\sqrt{y}}^{x=\sqrt[5]{y}} e^{-x^3} \mathrm{d}x\mathrm{d}y.$

Solution. 3a The issue with this integral is that we cannot integrate $\cos\left(x^2\right)$ with respect to x using (finitely many) elementary functions and their compositions. However, this expression is easy to integrate with respect to y. Therefore changing the order of integration (using Fubini's Theorem) could possibly help. Let the region of integration be $\mathcal{R}$. Then

$$\mathcal{R} = \left\{(x,y) | 0 \leq y \leq \sqrt{\pi}, y \leq x \leq \sqrt{\pi}\right\} .$$

We plot the region to find it is the triangle indicated in the figure below. When we fix the value of x, y varies between 0 and x. Therefore we can re-parametrize $\mathcal{R}$ via vertical slices:

$$\mathcal{R} = \left\{(x,y) | 0 \leq x \leq \sqrt{\pi}, 0 \leq y \leq x\right\} .$$



By Fubini's theorem, the iterated integral equals the double integral, which in turn can be evaluated using the iterated integral using the second parametrization of $\mathcal{R}$.

$$\begin{aligned}
\int_{y=0}^{y=\sqrt{\pi}} \int_{x=y}^{x=\sqrt{\pi}} \cos\left(x^2\right) \mathrm{d}x \mathrm{d}y &= \iint_{\mathcal{R}} \cos\left(x^2\right) \mathrm{d}x \mathrm{d}y && \text{By Fubini's Theorem} \\
&= \int_{x=0}^{\sqrt{\pi}} \int_{y=0}^{y=x} \cos\left(x^2\right) \mathrm{d}y \mathrm{d}x && \text{again by Fubini's Theorem} \\
&= \int_{x=0}^{\sqrt{\pi}} \left[y \cos\left(x^2\right)\right]_{y=0}^{y=x} \mathrm{d}x \\
&= \int_{x=0}^{\sqrt{\pi}} x \cos\left(x^2\right) \mathrm{d}x \\
&= \int_{x=0}^{\sqrt{\pi}} \cos\left(x^2\right) \frac{1}{2} \mathrm{d}\left(x^2\right) \\
&= \frac{1}{2} \left[\sin\left(x^2\right)\right]_{x=0}^{x=\sqrt{\pi}} \\
&= \frac{1}{2} \left(\sin \pi - \sin 0\right) = 0 \quad .
\end{aligned}$$

Solution. 3b This problem exploits the same idea as Problem 3a - that sometimes changing the order of integration is helpful for the algebraic manipulations.

Let the region of integration be $\mathcal{R}$. We have

$$\mathcal{R} = \{(x,y) | \sqrt{y} \leq x \leq \sqrt[5]{y}, 0 \leq y \leq 1\} .$$

$\mathcal{R}$ can be plotted as follows.

Therefore $\mathcal{R}$ can be reparametrized as follows.

$$\mathcal{R} = \left\{(x,y) | x^5 \leq y \leq x^2, 0 \leq x \leq 1\right\} .$$

By Fubini's theorem, the iterated integral equals the double integral, which in turn can be evaluated using the iterated integral with respect to the second parametrization of $\mathcal{R}$.

$$\begin{aligned}
\int_{y=0}^{y=1}\int_{x=\sqrt{y}}^{x=\sqrt[5]{y}} e^{-x^3}\mathrm{d}x\mathrm{d}y &= \iint_{\mathcal{R}} e^{-x^3}\mathrm{d}x\mathrm{d}y && \text{By Fubini's Theorem}\\
&= \int_{x=0}^{x=1}\int_{y=x^5}^{y=x^2} e^{-x^3}\mathrm{d}y\mathrm{d}x && \text{again by Fubini's Theorem}\\
&= \int_{x=0}^{x=1}\left[ye^{-x^3}\right]_{y=x^5}^{y=x^2}\mathrm{d}x\\
&= \int_{x=0}^{x=1} x^2(1-x^3)e^{-x^3}\mathrm{d}x\\
&= \int_{x=0}^{x=1}(1-x^3)e^{-x^3}\frac{1}{3}\mathrm{d}\left(x^3\right) && \text{Set } z=x^3\\
&= \frac{1}{3}\int_{z=0}^{z=1}(1-z)e^{-z}\mathrm{d}z\\
&= \frac{1}{3}\left[ze^{-z}\right]_{z=0}^{z=1}\\
&= \frac{1}{3}e^{-1} = \frac{1}{3e} \quad .
\end{aligned}$$