Precalculus
Lecture 20

Todor Milev
https://github.com/tmilev/freecalc

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Outline

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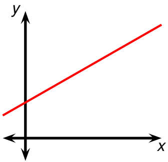
Linear Functions

Definition (Linear Function)

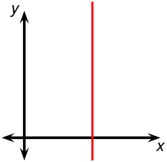
A linear function is a function the graph of which is a line. We can write any linear function in slope-intercept form:

$$f(x) = mx + b.$$

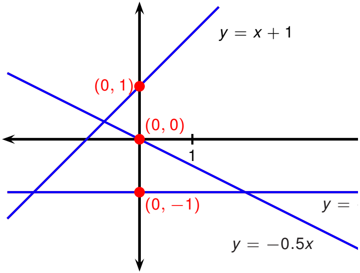
m is called the slope, and b is called the y-intercept.



- Any non-vertical line arises as the graph of a linear function.



- Vertical lines fail the vertical line test and therefore are not graphs of a function of x.



$f(x)$	Direction	y-intercept
$x + 1$	$\nearrow$	1
$-0.5x + 0$	$\searrow$	0
-1	$\rightarrow$	-1

- $m > 0$ means the graph of f points up ($\nearrow$).
- $m < 0$ means the graph of f points down ($\searrow$).
- $m = 0$ means the graph of f is horizontal ($\rightarrow$).
- b tells us the height of the point where the graph hits the y-axis.

Polynomials

Definition (Polynomial Function)

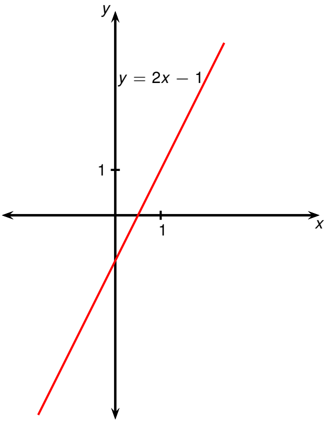
A polynomial function is a function f of the form

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1} + a_n x^n,$$

where n is a non-negative integer and $a_0, \ldots, a_n$ are real numbers, called the coefficients. If $a_n \neq 0$ the integer n is called the degree of f.

$f(x)$	Polynomial?	Degree	a_0	a_1	a_2
$x^4 - x + 1$	Yes	4	1	-1	0
6	Yes	0	6	0	0
$3x^2 - \frac{1}{2}x + \sqrt{x}$	No				
$3x^2 - \frac{1}{2}x + \sqrt{2}$	Yes	2	$\sqrt{2}$	$-\frac{1}{2}$	3
$3x^2 - \frac{1}{2x} + \sqrt{2}$	No				

- Linear functions are polynomial (functions).
- So are quadratic functions. Their graphs are parabolas.
- And there are many more.



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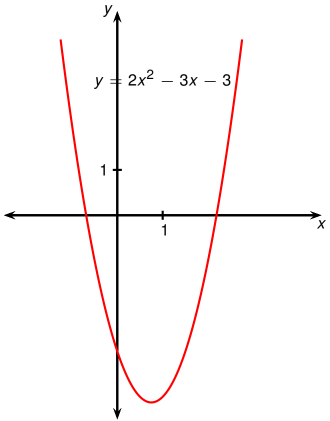

Quadratic

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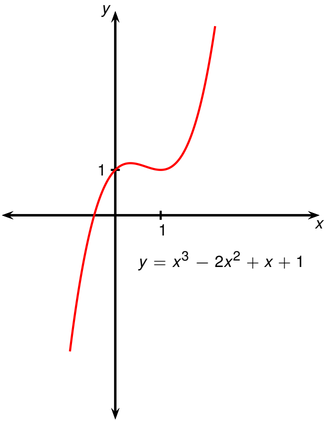

Cubic

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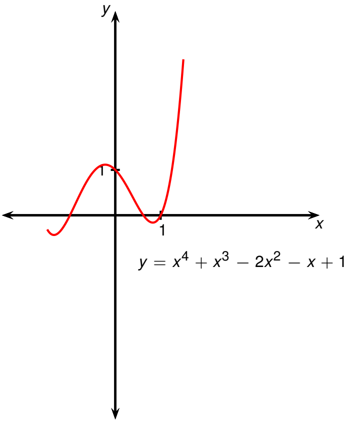

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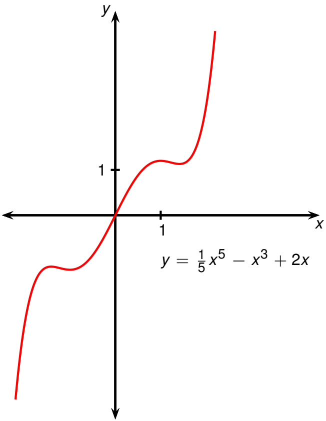


Quintic

Power Functions

Definition (Power Function)

Let $x > 0$, a - arbitrary real number. The power function is defined as

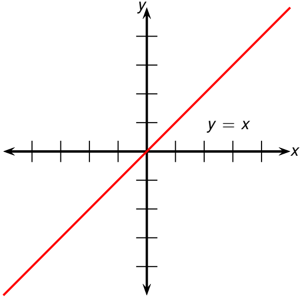
$$f(x) = e^{a \ln x} = x^a \quad .$$

x = base. a = exponent or power. First equality = one of ways to define for non-integer a (we study $\ln x$, e^x later).

If a - positive integer $(1, 2, 3, \dots)$ then x^a = polynomial function.

$x^n = \underbrace{x \dots x}_{n \text{ times}}$ when n-integer.

$$\begin{aligned} (x^a)^b &= x^{ab} \\ (xy)^b &= x^b y^b \\ x^{a+b} &= x^a x^b \\ x^{-a} &= \frac{1}{x^a} \end{aligned}$$



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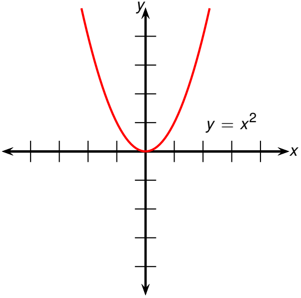

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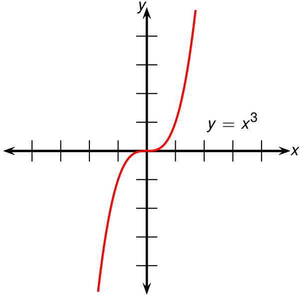

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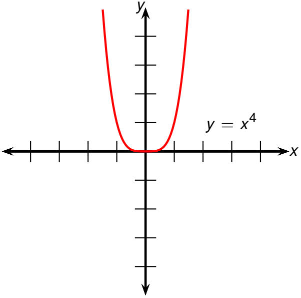

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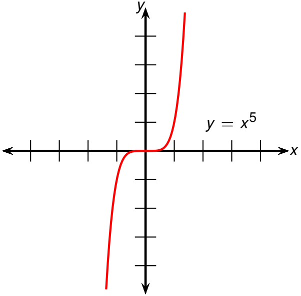
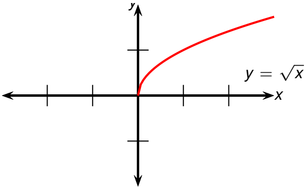
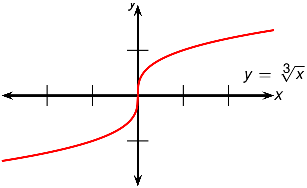


- n - positive integer, $f(x) = x^{\frac{1}{n}} = \sqrt[n]{x}$ = the n^{th} root function. $\sqrt[n]{x} \geq 0$ for $x \geq 0$.
- For $n = 2$, we get the square root $\sqrt{x}$; for $n = 3$ we get the cube root $\sqrt[3]{x}$, and so on.
- Let $x > 0$. For $n = 2m + 1$-odd, we can extend the definition of n^{th} root to negative numbers by $\sqrt[2m+1]{-x} := -\sqrt[2m+1]{x}$.
- In this course, even roots of negative numbers are not defined.
- The graph of $\sqrt{x}$ is the top half of the parabola $x = y^2$. Similarly for $y = \sqrt[2m]{x}$, we graph top of $x = y^{2m}$.
- The graph of the cube root $f(x) = \sqrt[3]{x}$ is the graph of the polynomial $x = y^3$. Similarly for $y = \sqrt[2m+1]{x}$, we graph $x = y^{2m+1}$.





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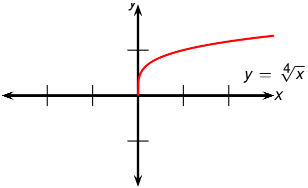
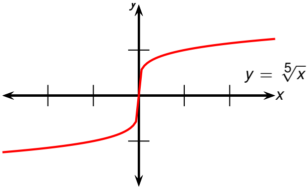
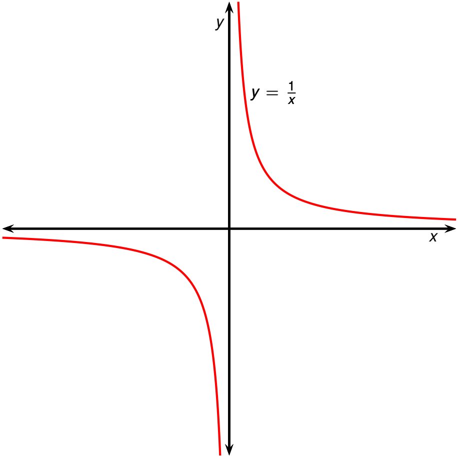




$f(x) = x^{-1} = \frac{1}{x}$ is called the reciprocal function. Its graph has equation $y = \frac{1}{x}$, or $xy = 1$, and is an hyperbola with the coordinate axes as its



asymptotes.

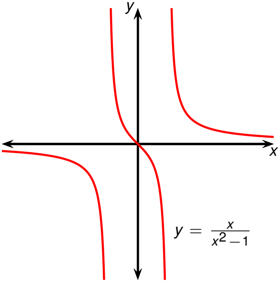
Rational Functions

Definition (Rational Function)

A rational function is a quotient of two polynomials; that is, a function of the form

$$f(x) = \frac{g(x)}{h(x)},$$

where g and h are polynomials.



Example ($x/(x^2-1)$)

The function

$$f(x) = \frac{x}{x^2-1}$$

is a rational function.

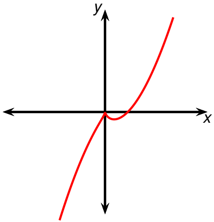
Algebraic Functions

(Algebraic Function)

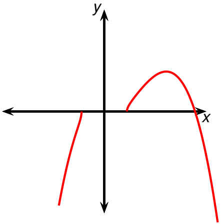
A function in x that can be constructed using x, constants, and finitely many of the operations $+, -, *, /$, and $\sqrt[n]{\ }$ is an algebraic function.

Outside of present course: function $f(x)$ = algebraic if it satisfies a polynomial equation with polynomial coefficients, i.e., $a_0(x) + a_1(x)f(x) + \cdots + a_n(x)\left(f(x)\right)^n = 0$ for some polynomials $a_i(x)$.

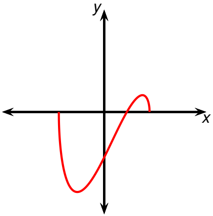
Examples.



$y = (x-1)\sqrt[3]{x^2}$



$y = \frac{1}{5}(4x - x^2)\sqrt{x^2-1}$



$y = (x-1)\sqrt{4-x^2}$

Transcendental Functions

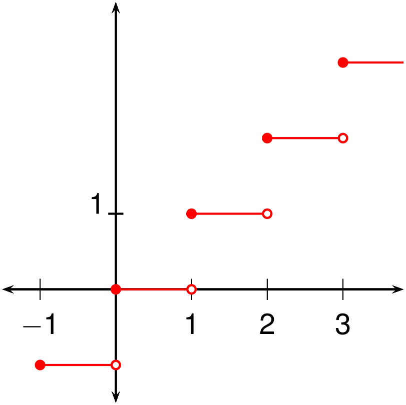
Transcendental functions include many classes of functions.

- Trigonometric functions such as $\cos x, \sin x, \tan x$, etc.
- Exponential functions such as 2^x, $\left(\frac{1}{2}\right)^x$, 5^x, e^x, etc.
- The logarithm function $\ln x$.
- And many more.
- Outside of the present course: by definition, a function is transcendental if it is not algebraic, i.e., if it satisfies no polynomial equation with polynomial coefficients.

Definition (Greatest Integer Function)

The *greatest integer function* $\lfloor x \rfloor$ is defined as the largest integer that is less than or equal to x.

In computer science this function is called the *floor* function.



$$\begin{aligned} \lfloor 4 \rfloor &= 4 \\ \lfloor 4.8 \rfloor &= 4 \\ \lfloor \pi \rfloor &= 3 \\ \lfloor \sqrt{2} \rfloor &= 1 \\ \left\lfloor -\frac{1}{2} \right\rfloor &= -1 \\ \lfloor -\pi \rfloor &= -4 \end{aligned}$$