

# Calculus II

## Lecture 4

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# Outline

- 1 Integration of Rational Functions
  - Building block integrals

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# Integrating arbitrary rational functions

Let  $\frac{P(x)}{Q(x)}$  be an arbitrary rational function, i.e., a quotient of polynomials.

## Question

*Can we integrate  $\int \frac{P(x)}{Q(x)} dx$ ?*

- Yes. We will learn how in what follows.
- The algorithm for integration is roughly:
  - We use algebra to split  $\frac{P(x)}{Q(x)}$  into smaller pieces (“partial fractions”).
  - We use linear substitutions to transform each piece to one of 3 pairs of basic building block integrals.
  - We solve each building block integral and collect the terms.
- We study the algorithm “from the ground up”: we start with the building blocks.

# The building blocks

Let  $n$  be a positive integer.

- (Building block I) The first building block integral is:

$$\int \frac{1}{x^n} dx \quad .$$

- (Building block II) The second building block integral is:

$$\int \frac{x}{(1+x^2)^n} dx. \quad (\text{Note: } u = 1+x^2, xdx = \frac{1}{2}du \text{ transforms II to I}).$$

- (Building block III) The third building block integral is:

$$\int \frac{1}{(1+x^2)^n} dx \quad .$$

- The case  $n = 1$  is special for each of the building blocks:

$$\int \frac{1}{x} dx, \int \frac{x}{1+x^2} dx \text{ and } \int \frac{1}{1+x^2} dx.$$

- The case  $n = 1$  we call respectively building block Ia, IIa and IIIa.  
The case  $n > 1$  we call respectively building block Ib, IIb and IIIb.  
This “building block” terminology is for our convenience, and is not a part of standard mathematical terminology.

# Building block Ia

Building block Ia:  $\int \frac{1}{x} dx$ .

## Example

Integrate building block Ia

$$\int \frac{1}{x} dx = \ln |x| + C$$

# Linear substitutions leading to building block Ia

Building block Ia:  $\int \frac{1}{x} dx = \ln |x| + C.$

## Example

Integrate

$$\begin{aligned}\int \frac{1}{-4x+5} dx &= \int \frac{1}{(-4x+5)} \frac{d(-4x)}{(-4)} \\ &= \int \frac{1}{(-4x+5)} \frac{d(-4x+5)}{(-4)} && \left| \text{Set } u = -4x+5 \right. \\ &= \int \frac{1}{u} \frac{du}{(-4)} \\ &= -\frac{1}{4} \int u^{-1} du = -\frac{1}{4} \ln |u| + C \\ &= -\frac{1}{4} \ln |-4x+5| + C.\end{aligned}$$

# Lin. subst. leading to building block 1a: general case

Building block 1a:  $\int \frac{1}{x} dx = \ln |x| + C.$

## Example

Integrate

$$\begin{aligned}\int \frac{1}{ax+b} dx &= \int \frac{1}{(ax+b)} \frac{d(ax)}{a} \\ &= \int \frac{1}{(ax+b)} \frac{d(ax+b)}{a} && \left| \text{Set } u = ax+b \right. \\ &= \int \frac{1}{u} \frac{du}{a} \\ &= \frac{1}{a} \int u^{-1} du = \frac{1}{a} \ln |u| + C \\ &= \frac{1}{a} \ln |ax+b| + C.\end{aligned}$$



# Building block Ib

Building block Ib:  $\int \frac{1}{x^n} dx = \int x^{-n} dx, n \neq 1.$

## Example (Block Ib)

$$\int \frac{1}{x^n} dx = \int x^{-n} dx = \frac{x^{-n+1}}{-n+1} + C$$

# Linear substitutions leading to building block Ib

Building block Ib:  $\int \frac{1}{x^n} dx = \int x^{-n} dx = \frac{x^{-n+1}}{-n+1} + C, n \neq 1.$

## Example

Integrate

$$\begin{aligned} \int \frac{1}{(3x+5)^3} dx &= \int \frac{1}{(3x+5)^3} \frac{d(3x)}{3} \\ &= \int \frac{1}{(3x+5)^3} \frac{d(3x+5)}{3} && \left| \text{Set } u = 3x+5 \right. \\ &= \int \frac{1}{u^3} \frac{du}{3} \\ &= \frac{1}{3} \int u^{-3} du = \frac{1}{3} \frac{u^{-2}}{(-2)} + C \\ &= -\frac{1}{6(3x+5)^2} + C. \end{aligned}$$

# Lin. subst. leading to building block Ib: general case

Building block Ib:  $\int \frac{1}{x^n} dx = \int x^{-n} dx = \frac{x^{-n+1}}{-n+1} + C, n \neq 1.$

## Example

Let  $n \neq 1$ . Integrate

$$\begin{aligned} \int \frac{1}{(ax+b)^n} dx &= \int \frac{1}{(ax+b)^n} \frac{d(ax)}{a} \\ &= \int \frac{1}{(ax+b)^n} \frac{d(ax+b)}{a} && \left| \text{Set } u = ax+b \right. \\ &= \int \frac{1}{u^n} \frac{du}{a} \\ &= \frac{1}{a} \int u^{-n} du = -\frac{1}{a} \frac{u^{-n+1}}{n-1} + C \\ &= -\frac{1}{a(n-1)(ax+b)^{n-1}} + C. \end{aligned}$$

# Building blocks IIa and IIIa

Building block IIa:  $\int \frac{x}{1+x^2} dx$ . Building block IIIa:  $\int \frac{1}{1+x^2} dx$ .

## Example (Block IIa)

$$\begin{aligned}\int \frac{x}{1+x^2} dx &= \int \frac{1}{(1+x^2)} \frac{d(x^2)}{2} \\ &= \int \frac{1}{1+x^2} \frac{d(1+x^2)}{2} \\ &= \int \frac{1}{u} \frac{du}{2} \\ &= \frac{1}{2} \ln |u| + C = \frac{1}{2} \ln (1+x^2) + C \quad .\end{aligned} \quad \left| \begin{array}{l} \text{Set } u = 1 + x^2 \end{array} \right.$$

## Example (Block IIIa)

$$\int \frac{1}{1+x^2} dx = \arctan x + C$$

# Linear substitutions leading to block IIa

Building block IIa:  $\int \frac{x}{1+x^2} dx = \frac{1}{2} \ln(1+x^2) + C$ .

“Theoretical way” to solve example below: transform to IIa; this is slow.

**Feel free to skip slide, we will redo in next slide with a shortcut.**

## Example

$$\int \frac{x}{2x^2+3} dx = \int \frac{x}{3\left(\frac{2}{3}x^2+1\right)} dx = \int \frac{x}{3\left(\left(\sqrt{\frac{2}{3}}x\right)^2+1\right)} dx$$

$$= \frac{3}{2} \int \frac{\sqrt{\frac{2}{3}}x}{3\left(\left(\sqrt{\frac{2}{3}}x\right)^2+1\right)} d\left(\sqrt{\frac{2}{3}}x\right)$$

$$\left| \text{Set } u = \sqrt{\frac{2}{3}}x \right.$$

$$= \frac{1}{2} \int \frac{u}{u^2+1} du = \frac{1}{4} \ln(1+u^2) + C$$

$$= \frac{1}{4} \ln\left(\frac{1}{3}(2x^2+3)\right) + C$$

$$= \frac{1}{4} \ln(2x^2+3) + \frac{\ln\left(\frac{1}{3}\right)}{4} + C$$

$$= \frac{1}{4} \ln(2x^2+3) + K$$

# Linear substitutions leading to blocks IIa

Building block IIa:  $\int \frac{x}{1+x^2} dx = \frac{1}{2} \ln(1+x^2) + C$ .

The example below can be done directly, without transforming to block IIa.

## Example

$$\begin{aligned} \int \frac{x}{2x^2+3} dx &= \int \frac{1}{2x^2+3} d\left(\frac{x^2}{2}\right) \\ &= \int \frac{1}{2x^2+3} d\left(\frac{2x^2+3}{4}\right) && \left| \text{Set } u = 2x^2+3 \right. \\ &= \frac{1}{4} \int \frac{1}{u} du \\ &= \frac{1}{4} \ln|u| + C \\ &= \frac{1}{4} \ln(2x^2+3) + C \end{aligned}$$

# Linear substitutions leading to blocks IIa and IIIa

Building block IIIa:  $\int \frac{1}{u^2+1} du = \arctan u + C$ .

## Example

$$\begin{aligned}\int \frac{1}{x^2+2} dx &= \int \frac{1}{2\left(\frac{1}{2}x^2+1\right)} dx && \left| \begin{array}{l} \text{Use } 2 = (\sqrt{2})^2 \\ \text{Set } \frac{x}{\sqrt{2}} = u \end{array} \right. \\ &= \int \frac{1}{2\left(\left(\frac{x}{\sqrt{2}}\right)^2+1\right)} \sqrt{2} d\left(\frac{x}{\sqrt{2}}\right) \\ &= \frac{1}{\sqrt{2}} \int \frac{1}{u^2+1} du \\ &= \frac{1}{\sqrt{2}} \arctan(u) + C \\ &= \frac{1}{\sqrt{2}} \arctan\left(\frac{x}{\sqrt{2}}\right) + C\end{aligned}$$

# Linear substitutions leading to blocks IIa and IIIa

Building block IIIa:  $\int \frac{1}{x^2+1} dx = \arctan x + C$ . **Let**  $a > 0$ .

## Example

$$\begin{aligned}\int \frac{1}{x^2+a} dx &= \int \frac{1}{a\left(\frac{1}{a}x^2+1\right)} dx \\ &= \int \frac{1}{a\left(\left(\frac{x}{\sqrt{a}}\right)^2+1\right)} \sqrt{a} d\left(\frac{x}{\sqrt{a}}\right) \\ &= \frac{1}{\sqrt{a}} \int \frac{1}{u^2+1} du \\ &= \frac{1}{\sqrt{a}} \arctan(u) + C \\ &= \frac{1}{\sqrt{a}} \arctan\left(\frac{x}{\sqrt{a}}\right) + C\end{aligned}$$

Use  $a = (\sqrt{a})^2$

Set  $u = \frac{x}{\sqrt{a}}$



# Linear substitutions leading to blocks IIa and IIIa

Building block IIa:  $\int \frac{x}{1+x^2} dx = \frac{1}{2} \ln(1+x^2) + C.$

Building block IIIa:  $\int \frac{1}{1+x^2} dx = \arctan x + C.$

- Let  $ax^2 + bx + c$  have no real roots.
- We can find  $p, q$  so that the linear substitution  $u = px + q$  transforms the quadratic to:

$$ax^2 + bx + c = r(u^2 + 1)$$

(where  $r$  is some number to be determined).

- To find  $p, q$ , we complete the square.
- In this way, integrals of the form  $\int \frac{Ax + B}{ax^2 + bx + c} dx$  are transformed to combinations of building blocks IIa and IIIa.
- We show examples; the general case is analogous and we leave it to the student.

# Linear substitutions leading to blocks IIa and IIIa

Building block IIa:  $\int \frac{x}{1+x^2} dx = \frac{1}{2} \ln(1+x^2) + C.$

Building block IIIa:  $\int \frac{1}{1+x^2} dx = \arctan x + C.$

## Example

No real roots  $\Rightarrow$  complete the square. Let  $u = x + \frac{1}{2}$ , let  $z = \frac{2u}{\sqrt{3}}$ .

$$\begin{aligned} \int \frac{x}{x^2 + x + 1} dx &= \int \frac{x}{x^2 + 2 \cdot \frac{1}{2}x + \frac{1}{4} - \frac{1}{4} + 1} dx \\ &= \int \frac{x + \frac{1}{2} - \frac{1}{2}}{\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}} d\left(x + \frac{1}{2}\right) \\ &= \int \frac{u - \frac{1}{2}}{u^2 + \frac{3}{4}} du \\ &= \int \frac{u}{u^2 + \frac{3}{4}} du - \frac{1}{2} \int \frac{1}{u^2 + \frac{3}{4}} du \end{aligned}$$

# Linear substitutions leading to blocks IIa and IIIa

Building block IIa:  $\int \frac{x}{1+x^2} dx = \frac{1}{2} \ln(1+x^2) + C.$

Building block IIIa:  $\int \frac{1}{1+x^2} dx = \arctan x + C.$

## Example

No real roots  $\Rightarrow$  complete the square. Let  $u = x + \frac{1}{2}$ , let  $z = \frac{2u}{\sqrt{3}}$ .

$$\begin{aligned} \int \frac{x}{x^2 + x + 1} dx &= \int \frac{u}{u^2 + \frac{3}{4}} du - \frac{1}{2} \int \frac{1}{u^2 + \frac{3}{4}} du \\ \int \frac{1}{u^2 + \frac{3}{4}} du &= \int \frac{1}{\frac{3}{4} \left( \frac{4}{3} u^2 + 1 \right)} du \\ &= \int \frac{1}{\frac{3}{4} \left( \left( \frac{2u}{\sqrt{3}} \right)^2 + 1 \right)} \frac{\sqrt{3}}{2} d\left( \frac{2u}{\sqrt{3}} \right) \\ &= \frac{2\sqrt{3}}{3} \int \frac{1}{z^2 + 1} dz = \frac{2\sqrt{3}}{3} \arctan z + C \end{aligned}$$

# Linear substitutions leading to blocks IIa and IIIa

Building block IIa:  $\int \frac{x}{1+x^2} dx = \frac{1}{2} \ln(1+x^2) + C.$

Building block IIIa:  $\int \frac{1}{1+x^2} dx = \arctan x + C.$

## Example

No real roots  $\Rightarrow$  complete the square. Let  $u = x + \frac{1}{2}$ , let  $z = \frac{2u}{\sqrt{3}}$ .

$$\begin{aligned} \int \frac{x}{x^2 + x + 1} dx &= \int \frac{u}{u^2 + \frac{3}{4}} du - \frac{1}{2} \int \frac{1}{u^2 + \frac{3}{4}} du \\ &= \int \frac{u}{u^2 + \frac{3}{4}} du - \frac{1}{2} \frac{2\sqrt{3}}{3} \arctan z + C \end{aligned}$$

$$\begin{aligned} \int \frac{u}{u^2 + \frac{3}{4}} du &= \int \frac{1}{u^2 + \frac{3}{4}} d\left(\frac{u^2}{2}\right) \\ &= \frac{1}{2} \int \frac{1}{u^2 + \frac{3}{4}} d\left(u^2 + \frac{3}{4}\right) = \frac{1}{2} \ln\left(u^2 + \frac{3}{4}\right) + C \end{aligned}$$

# Linear substitutions leading to blocks IIa and IIIa

Building block IIa:  $\int \frac{x}{1+x^2} dx = \frac{1}{2} \ln(1+x^2) + C.$

Building block IIIa:  $\int \frac{1}{1+x^2} dx = \arctan x + C.$

## Example

No real roots  $\Rightarrow$  complete the square. Let  $u = x + \frac{1}{2}$ , let  $z = \frac{2u}{\sqrt{3}}$ .

$$\begin{aligned} \int \frac{x}{x^2 + x + 1} dx &= \int \frac{u}{u^2 + \frac{3}{4}} du - \frac{1}{2} \int \frac{1}{u^2 + \frac{3}{4}} du \\ &= \frac{1}{2} \ln \left( u^2 + \frac{3}{4} \right) - \frac{1}{2} \frac{2\sqrt{3}}{3} \arctan z + C \\ &= \frac{1}{2} \ln \left( \left( x + \frac{1}{2} \right)^2 + \frac{3}{4} \right) - \frac{\sqrt{3}}{3} \arctan \left( \frac{2u}{\sqrt{3}} \right) + C \\ &= \frac{1}{2} \ln \left( x^2 + x + 1 \right) - \frac{\sqrt{3}}{3} \arctan \left( \frac{2x+1}{\sqrt{3}} \right) + C \end{aligned}$$

# Building blocks IIa and IIb

We solve building block IIb. For completeness, we solve block IIa again as well.

## Example

$$\begin{aligned}\int \frac{x}{(x^2 + 1)^n} dx &= \int \frac{1}{(x^2 + 1)^n} \frac{d(x^2 + 1)}{2} \\ &= \frac{1}{2} \int u^{-n} du \\ &= \begin{cases} \frac{1}{2} \ln(x^2 + 1) + C & \text{if } n = 1 \\ \frac{1}{2} \frac{(x^2 + 1)^{-n+1}}{(-n+1)} + C & \text{if } n \neq 1 \end{cases},\end{aligned}$$

where we used the substitution  $u = x^2 + 1$ .

# Building block IIIb: example illustrating main idea

## Example

Integrate  $\int \frac{dx}{(x^2+1)^2}$ . We start with an already known integral:

$$\begin{aligned}\arctan x + C &= \int \frac{1}{x^2+1} dx \\&= \frac{1}{x^2+1} x - \int x d\left(\frac{1}{x^2+1}\right) \\&= \frac{x}{x^2+1} - \int x \left(-\frac{2x}{(x^2+1)^2}\right) dx \\&= \frac{x}{x^2+1} + 2 \int \frac{-1 + x^2 + 1}{(x^2+1)^2} dx \\&= \frac{x}{x^2+1} + 2 \int \frac{1}{x^2+1} dx - 2 \int \frac{1}{(x^2+1)^2} dx \\&= \frac{x}{x^2+1} + 2 \arctan x - 2 \int \frac{dx}{(x^2+1)^2}\end{aligned}$$

# Building block IIIb: example illustrating main idea

## Example

Integrate  $\int \frac{dx}{(x^2+1)^2}$ . We start with an already known integral:

$$\arctan x + C = \frac{x}{x^2+1} + 2 \arctan x - 2 \int \frac{dx}{(x^2+1)^2}$$

Rearrange terms and divide by 2 to get the desired integral:

$$\int \frac{dx}{(1+x^2)^2} = \frac{1}{2} \left( \frac{x}{x^2+1} + \arctan x \right) + K.$$



# Building block IIIb

- Building block IIIa:

$$J(1) = \int \frac{1}{(x^2 + 1)} dx = \arctan x + C \quad .$$

- Block IIIb:

$$J(n) = \int \frac{1}{(x^2 + 1)^n} dx$$

- Unlike other cases, IIIb is much harder than IIIa.
- Set  $J(n) = \int \frac{1}{(x^2+1)^n} dx$ . We are looking for a formula for  $J(n)$ . We know  $J(1) = \arctan x + C$  (this is block IIIa).
- We start by  $J(n-1) = \int \frac{1}{(x^2+1)^{n-1}} dx$  and integrate by parts.
- In this way we end up expressing  $J(n)$  via  $J(n-1)$ .
- We work our way from  $J(n)$  to  $J(n-1)$ , from  $J(n-1)$  to  $J(n-2)$ , and so on, until we get to  $J(1)$ .

## Example

Recall that  $J(n) = \int \frac{1}{(x^2+1)^n} dx$ . We have that:

$$\begin{aligned}
 J(n-1) &= \int \frac{1}{(x^2+1)^{n-1}} dx \\
 &= \frac{1}{(x^2+1)^{n-1}} x - \int x d\left(\frac{1}{(1+x^2)^{n-1}}\right) \\
 &= \frac{x}{(x^2+1)^{n-1}} - \int x \left(\frac{(-n+1)2x}{(1+x^2)^n}\right) dx \\
 &= \frac{x}{(x^2+1)^{n-1}} + 2(n-1) \int \frac{1+x^2-1}{(1+x^2)^n} dx \\
 &= \frac{x}{(x^2+1)^{n-1}} + 2(n-1) \int \frac{1}{(1+x^2)^{n-1}} dx \\
 &\quad - 2(n-1) \int \frac{1}{(1+x^2)^n} dx \\
 &= \frac{x}{(x^2+1)^{n-1}} + 2(n-1)J(n-1) - 2(n-1)J(n) .
 \end{aligned}$$

## Example

Recall that  $J(n) = \int \frac{1}{(x^2+1)^n} dx$ . We have that:

$$J(n-1) = \frac{x}{(x^2+1)^{n-1}} + 2(n-1)J(n-1) - 2(n-1)J(n) \quad .$$

Rearrange to get:

$$\begin{aligned} 2(n-1)J(n) &= \frac{x}{(x^2+1)^{n-1}} + (2n-3)J(n-1) \\ J(n) &= \frac{x}{(2n-2)(x^2+1)^{n-1}} + \frac{2n-3}{2n-2}J(n-1) \quad . \end{aligned}$$

In this way we expressed  $J(n)$  using  $J(n-1)$ . We apply the above formula consecutively:

$$J(n) = \frac{x}{(2n-2)(x^2+1)^{n-1}} + \frac{2n-3}{2n-2} \left( \frac{x}{(2n-4)(x^2+1)^{n-2}} + \frac{2n-5}{2n-4}J(n-2) \right) = \dots$$

and so on. The above can be used to write a formula for the final result, but that is as complicated as the process above.

# Building block integral summary

Type	a	b	Type a, lin. sub.	Type b, lin. sub
I	$\int \frac{1}{x} dx$	$\int \frac{1}{x^n} dx$	$\int \frac{A}{ax+b} dx$	$\int \frac{A}{(ax+b)^n} dx$
II	$\int \frac{x}{x^2+1} dx$	$\int \frac{x}{(x^2+1)^n} dx$	$\int \frac{A(x+\frac{b}{2a})}{ax^2+bx+c} dx$	$\int \frac{A(x+\frac{b}{2a})}{(ax^2+bx+c)^n} dx$
III	$\int \frac{1}{x^2+1} dx$	$\int \frac{1}{(x^2+1)^n} dx$	$\int \frac{B}{ax^2+bx+c} dx$	$\int \frac{B}{(ax^2+bx+c)^n} dx$

where  $A, B$  are arbitrary constants and  $a, b, c$  are constants with  $b^2 - 4ac < 0$ . The quadratics in the denominators have no real roots.

- We solved building blocks I, II and III in almost complete detail.
- The types in the remaining columns can be transformed to building block ones:
  - Block I, linear substitutions: done in full detail.
  - Block IIa, IIIa, linear substitutions: done in full detail, by means of completing the square.
  - Block IIb, IIIb, linear substitutions: done by means of completing the square; computations are analogous and we leave them for exercise.