

Calculus I

Lecture 14

Logarithmic Differentiation

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Outline

- 1 Derivatives of Logarithmic Functions
- 2 Derivative of $a(x)^{b(x)}$
- 3 Logarithmic Differentiation
 - The Number e as a Limit

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Derivatives of Logarithmic Functions

Theorem (The Derivative of $\log_a x$)

$$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}.$$

Proof.

Let $y = \log_a x$.

Then $a^y = x$.

Differentiate implicitly: $a^y (\ln a) y' = 1$

$$\begin{aligned} y' &= \frac{1}{a^y \ln a} \\ &= \frac{1}{x \ln a}. \end{aligned}$$



Example (Chain Rule)

Differentiate $f(x) = \log_3(5^x + 1)$.

Let $h(x) =$

Let $g(x) = \log_3 x$.

Theorem (The Derivative of $\log_a x$)

$$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}.$$

$\ln x = \log_e x$. Therefore when we set $a = e$ we get the derivative of $\ln x$:

$$\begin{aligned}\frac{d}{dx}(\ln x) &= \frac{1}{x \ln e} \\ &= \frac{1}{x(1)} \\ &= \frac{1}{x}.\end{aligned}$$

Theorem (The Derivative of $\ln x$)

$$\frac{d}{dx}(\ln x) = \frac{1}{x}.$$

Example (Chain Rule, Natural Logarithm)

Differentiate $y = \ln(e^x \sec x)$.

$$y = \ln e^x + \ln(\sec x)$$

$$= x + \ln(\sec x).$$

$$\frac{dy}{dx} = 1 + \frac{d}{dx}(\ln(\sec x))$$

Let $u =$

Then $\ln(\sec x) =$

Chain Rule:
$$\begin{aligned} \frac{dy}{dx} &= 1 + \frac{d}{du}(\quad) \frac{du}{dx} \\ &= 1 + \left(\quad \right) \left(\quad \right) \\ &= \\ &= \end{aligned}$$

Example

Find $f'(x)$ if $f(x) = \ln |x|$.

$$f(x) = \begin{cases} \ln x & \text{if } x > 0 \\ \ln(-x) & \text{if } x < 0 \end{cases}.$$

$$f'(x) = \begin{cases} \frac{1}{x} & \text{if } x > 0 \\ \frac{1}{-x}(-1) & \text{if } x < 0 \end{cases}$$

$$= \begin{cases} \frac{1}{x} & \text{if } x > 0 \\ \frac{1}{x} & \text{if } x < 0 \end{cases}$$

$$= \frac{1}{x} \text{ if } x \neq 0.$$

Example

Differentiate $x^{\tan x}$, where $x > 0$.

$$\frac{d}{dx} (x^{\tan x}) = \frac{d}{dx} \left((e^{\ln x})^{\tan x} \right)$$

Convert base to e ?

$$= \frac{d}{dx} (e^{(\ln x) \tan x})$$

$$= \frac{d}{dx} (e^u)$$

Set $(\ln x) \tan x = u$

$$= \frac{d}{du} (e^u) \frac{du}{dx}$$

Chain rule

$$= e^u \frac{d}{dx} ((\ln x) \tan x)$$

$$= e^{(\ln x) \tan x} ((\ln x)' \tan x + (\ln x) (\tan x)')$$

Prod. rule

$$= x^{\tan x} \left(\frac{1}{x} \tan x + (\ln x) \sec^2 x \right)$$

Example

Differentiate $(3x + 1)^{\ln x}$, where $3x + 1 > 0$.

$$\begin{aligned}
 \frac{d}{dx} \left((3x + 1)^{\ln x} \right) &= \frac{d}{dx} \left(\left(e^{\ln(3x+1)} \right)^{\ln x} \right) && \left| \text{Convert base to } e? \right. \\
 &= \frac{d}{dx} \left(e^{\ln(3x+1) \ln x} \right) \\
 &= \frac{d}{dx} (e^u) = \frac{d}{du} (e^u) \frac{du}{dx} && \left| \text{Set } \ln(3x + 1) \ln x = u \right. \\
 &= e^u \frac{d}{dx} (\ln(3x + 1) \ln x) \\
 &= e^{\ln(3x+1) \ln x} \left((\ln(3x + 1))' \ln x + \ln(3x + 1) (\ln x)' \right) \\
 &= (3x + 1)^{\ln x} \left(\frac{(3x + 1)'}{3x + 1} \ln x + \ln(3x + 1) \frac{1}{x} \right) \\
 &= (3x + 1)^{\ln x} \left(\frac{3 \ln x}{3x + 1} + \ln(3x + 1) \frac{1}{x} \right)
 \end{aligned}$$

Example

Differentiate $(3x + 1)^{\ln x}$, where $3x + 1 > 0$.

$$\frac{d}{dx} \left((3x + 1)^{\ln x} \right) = (3x + 1)^{\ln x} \left(\frac{3 \ln x}{3x + 1} + \ln(3x + 1) \frac{1}{x} \right)$$

Theorem

$$\frac{d}{dx} \left((a(x))^{b(x)} \right) = (a(x))^{b(x)} \left(\frac{a'(x)}{a(x)} b(x) + \ln(a(x)) b'(x) \right), \quad a(x) > 0$$

Example (Logarithmic Differentiation)

Differentiate $y = \frac{(x-1)^{5/3} \sin^3 x}{\sqrt{e^x + 1}}$.

Take the natural logarithm of both sides:

$$\ln y = \ln \frac{(x-1)^{5/3} \sin^3 x}{\sqrt{e^x + 1}}$$

$$\ln y = (5/3) \ln(x-1) + 3 \ln(\sin x) - (1/2) \ln(e^x + 1).$$

$$\frac{d}{dx}(\ln y) = \frac{d}{dx} ((5/3) \ln(x-1) + 3 \ln(\sin x) - (1/2) \ln(e^x + 1))$$

$$\frac{1}{y} \left(\frac{dy}{dx} \right) = \left(\frac{5}{3} \left(\frac{1}{x-1} \right) \right) + \left(\frac{3 \cos x}{\sin x} \right) - \left(\frac{1}{2} \left(\frac{e^x}{e^x + 1} \right) \right)$$

$$\frac{dy}{dx} = \left(\frac{5}{3(x-1)} + 3 \cot x - \frac{e^x}{2(e^x + 1)} \right) y$$

$$= \left(\frac{5}{3(x-1)} + 3 \cot x - \frac{e^x}{2(e^x + 1)} \right) \frac{(x-1)^{5/3} \sin^3 x}{\sqrt{e^x + 1}}.$$

Steps in Logarithmic Differentiation

- 1 Take natural logarithms of both sides of an equation $y = f(x)$.
- 2 Use the properties of logarithms to simplify.
- 3 Differentiate implicitly with respect to x .
- 4 Solve the resulting equation for y' .

Note: If $f(x) < 0$, then we use $\ln |f(x)|$ instead as $\ln(f(x))$ is not defined. We computed the derivative of $\ln |f(x)|$ in the previous lecture.

Example (Variable base and exponent)

Differentiate $y = (3x + 1)^{\ln x}$.

Take logarithms of both sides:

$$\ln y = \ln(3x + 1)^{\ln x}$$

$$\ln y = \ln x \ln(3x + 1).$$

Differentiate implicitly with respect to x :

$$\frac{1}{y} y' = (\ln x) \frac{d}{dx} (\ln(3x + 1)) + (\ln(3x + 1)) \frac{d}{dx} (\ln x)$$

$$\frac{1}{y} y' = (\ln x) \left(\frac{1}{3x + 1} \cdot 3 \right) + (\ln(3x + 1)) \left(\frac{1}{x} \right)$$

$$\begin{aligned} y' &= y \left(\frac{3 \ln x}{3x + 1} + \frac{\ln(3x + 1)}{x} \right) \\ &= (3x + 1)^{\ln x} \left(\frac{3 \ln x}{3x + 1} + \frac{\ln(3x + 1)}{x} \right). \end{aligned}$$

Example (Variable base and exponent)

Differentiate $y = x^{\tan x}$.

Take logarithms of both sides:

$$\ln y = \ln x^{\tan x}$$

$$\ln y = \tan x \ln x.$$

Differentiate implicitly with respect to x :

$$\frac{1}{y}y' = (\tan x) \frac{d}{dx}(\ln x) + (\ln x) \frac{d}{dx}(\tan x)$$

$$\frac{1}{y}y' = (\tan x) \left(\frac{1}{x} \right) + (\ln x) (\sec^2 x)$$

$$\begin{aligned} y' &= y \left(\frac{\tan x}{x} + (\ln x) \sec^2 x \right) \\ &= x^{\tan x} \left(\frac{\tan x}{x} + (\ln x) \sec^2 x \right). \end{aligned}$$

Theorem (The Number e as a Limit)

$$e = \lim_{x \rightarrow 0} (1 + x)^{\frac{1}{x}} = \lim_{y \rightarrow \infty} \left(1 + \frac{1}{y}\right)^y.$$

Proof.

Let $f(x) = \ln x$. Then $f'(x) = \frac{1}{x}$, so $f'(1) = 1$.

$$\begin{aligned} 1 = f'(1) &= \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} = \lim_{x \rightarrow 0} \frac{f(1+x) - f(1)}{x} \\ &= \lim_{x \rightarrow 0} \frac{\ln(1+x) - \ln(1)}{x} = \lim_{x \rightarrow 0} \frac{1}{x} \ln(1+x) \\ &= \lim_{x \rightarrow 0} \ln(1+x)^{\frac{1}{x}}. \end{aligned}$$

Then use the fact that the exponential function is continuous:

$$e = e^1 = e^{\lim_{x \rightarrow 0} \ln(1+x)^{\frac{1}{x}}} = \lim_{x \rightarrow 0} e^{\ln(1+x)^{\frac{1}{x}}} = \lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}}. \quad \square$$