Calculus II
Homework Review problems for the final
This is a subset of the Master Problem Sheet

1. Problems that have appeared past final(s):
   (a) Problem 2.m.
   (b) Problem 4.a.
   (c) Problem 6.e (the problem was formulated slightly differently - as an improper integral).
   (d) Problem 8.b.
   (e) Problem 9.c.
   (f) Problem 10.a.
   (g) Problem 10.b.
   (h) Problem 12.c.
   (i) Problem 13.a.
   (j) Problem 14.c.
   (k) Problem 17.c.
   (l) Problem 16.c.
2. Evaluate the indefinite integral. Illustrate all steps of your solution.
   (a) $\displaystyle\int \frac{x^3+4}{x^2+4}\mathrm{d}x$
   (b) $\displaystyle\int \frac{4x^2}{2x^2-1}\mathrm{d}x$
   (c) $\displaystyle\int \frac{x^3}{x^2+2x-3}\mathrm{d}x$
   (d) $\displaystyle\int \frac{x^3}{x^2+3x-4}\mathrm{d}x$
   (e) $\displaystyle\int \frac{x^3}{2x^2+3x-5}\mathrm{d}x$
   (f) $\displaystyle\int \frac{x^2+1}{(x-3)(x-2)^2}\mathrm{d}x$
   (g) $\displaystyle\int \frac{x^4}{(x+1)^2(x+2)}\mathrm{d}x$
   (h) $\displaystyle\int \frac{15x^2-4x-81}{(x-3)(x+4)(x-1)}\mathrm{d}x$
   (i) $\displaystyle\int \frac{x^4+10x^3+18x^2+2x-13}{x^4+4x^3+3x^2-4x-4}\mathrm{d}x$

      Check first that $(x-1)(x+2)^2(x+1) = x^4+4x^3+3x^2-4x-4$.
   (j) $\displaystyle\int \frac{x^4}{(x^2+2)(x+2)}\mathrm{d}x$
   (k) $\displaystyle\int \frac{x^5}{x^3-1}\mathrm{d}x$
   (l) $\displaystyle\int \frac{x^4}{(x^2+2)(x+1)^2}\mathrm{d}x$
   (m) $\displaystyle\int \frac{3x^2+2x-1}{(x-1)(x^2+1)}\mathrm{d}x$
   (n) $\displaystyle\int \frac{x^2-1}{x(x^2+1)^2}\mathrm{d}x$
3. Compute the integral.
   (a) $\displaystyle\int \frac{\sqrt{1+x^2}}{x^2}\mathrm{d}x.$
4. Compute the integral using a trigonometric substitution.
   (a) $\displaystyle\int \frac{\sqrt{9-x^2}}{x^2}\mathrm{d}x \quad .$
5. Evaluate the indefinite integral. Illustrate the steps of your solutions.

(a) $\int x \sin x \mathrm{d}x$.

(b) $\int x e^{-x} \mathrm{d}x$.

(c) $\int x^2 e^x \mathrm{d}x$.

(d) $\int x \sin(-2x) \mathrm{d}x$.

(e) $\int x^2 \cos(3x) \mathrm{d}x$.

(f) $\int x^2 e^{-2x} \mathrm{d}x$.

(g) $\int x \sin(2x) \mathrm{d}x$.

(h) $\int x \cos(3x) \mathrm{d}x$.

(i) $\int x^2 e^{2x} \mathrm{d}x$.

(j) $\int x^3 e^x \mathrm{d}x$.

6. Use the integral test, the comparison test or the limit comparison test to determine whether the series is convergent or divergent. Justify your answer.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n+1}$.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n^2+n^3}$.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+3}{3n^5+n}$

(d) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{3^n+5}$.

(e) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

(f) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(2n+1)\ln(n)}$.

(g) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

(h) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(2n+1)(\ln(n))^2}$.

(i) Determine all values of p, q r for which the series

$$\sum_{n=30}^{\infty} \frac{1}{n^p(\ln n)^q(\ln(\ln n))^r}$$

is convergent.

7. Compute the limits. The answer key has not been fully proofread, use with caution.

(a) $\displaystyle\lim_{x\to 0} \frac{\sin x}{x}$.

(b) $\displaystyle\lim_{x\to 0} \frac{x}{\ln(1+x)}$.

(c) $\displaystyle\lim_{x\to 0} \frac{x^2}{x-\ln(1+x)}$.

(d) $\displaystyle\lim_{x\to 0} \frac{x^2}{\sin x \ln(1+x)}$.

(e) $\displaystyle\lim_{x\to 0} \frac{\sin^2 x}{(\ln(1+x))^2}$.

(f) $\displaystyle\lim_{x\to 0} \frac{\cos x - 1}{\sin x \ln(1+x)}$.

(g) $\displaystyle\lim_{x\to 0} \frac{\arctan x - x}{x^3}$.

(h) $\displaystyle\lim_{x\to 0} \frac{\arcsin x - x}{x^3}$.

(i) $\displaystyle\lim_{x\to 1} \frac{x}{x-1} - \frac{1}{\ln x}$.

(j) $\displaystyle\lim_{x\to 0} \frac{\cos(nx)-\cos(mx)}{x^2}$.

(k) $\displaystyle\lim_{x\to 0} \frac{\arcsin x - x - \frac{1}{6}x^3}{\sin^5 x}$.

(l) $\displaystyle\lim_{x\to 1} \frac{\sin(\pi x)\ln x}{\cos(\pi x)+1}$.

(m) $\displaystyle\lim_{x\to 0} \frac{\sin x - x}{\arcsin x - x}$.

(n) $\displaystyle\lim_{x\to 0} \frac{\sin x - x}{\arctan x - x}$.

(o) $\displaystyle\lim_{x\to\infty} x \sin\left(\frac{2}{x}\right)$.

8. Express the sum of the series as a rational number.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n+3^n}{5^n}$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n+5^n}{10^n}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{5^n-3^n}{7^n}$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{3^{n+1}+7^{n-1}}{21^n}$

(e) $\displaystyle\sum_{n=0}^{\infty} \frac{2^{n+1}+(-3)^{n-1}}{5^n}$

9. Sum the telescoping series (a sum is "telescoping" if it can be broken into summands so that consecutive terms cancel).

   (a) $\displaystyle\sum_{n=0}^{\infty} \frac{-6}{9n^2+3n-2}$.

   (b) $\displaystyle\sum_{n=3}^{\infty} \frac{3}{n^2-3n+2}$.

   (c) $\displaystyle\sum_{n=2}^{\infty} \ln\left(1-\frac{1}{n^2}\right)$. (Hint: Use the properties of the logarithm to aim for a telescoping series).

10. Find whether the series is convergent or divergent using an appropriate test. Some of the problems require the alternating series test. The test states the following.

    Alternating series test. Suppose $b_n \searrow 0$. Then $\sum(-1)^n b_n$ is convergent.

    Here, $b_n \searrow 0$ means the following.

    - The sequence of numbers b_n is decreasing.
    - The sequence decreases to 0, that is,

    $$\lim_{n\to\infty} b_n = 0 \quad .$$

    (a) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \ln n.$

    (b) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}.$

    (c) $\displaystyle\sum_{n=2}^{\infty} \frac{n}{\ln n}$

    (d) $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n}$

11. For each of the items below, do the following.

    - Find the Maclaurin series of the function (i.e., the power series representation of the function around $a = 0$).
    - Find the radius of convergence of the series you found in the preceding point. You are not asked to find the entire interval of convergence, but just the radius.

    (a) e^x.

    (b) xe^{-2x}.

    (c) e^{2x}.

    (d) e^{x^2}.

    (e) e^{-3x^2}.

    (f) x^2e^{2x}.

    (g) $\sin x$.

    (h) $\cos x$.

    (i) $\sin(2x)$.

    (j) $\cos(2x)$.

    (k) $\cos^2(x)$.

    (l) $x \sin x$.

12. Find the Taylor series of the function at the indicated point.

    (a) $\frac{1}{x^2}$ at $a = -1$.

    (b) $\ln\left(\sqrt{x^2-2x+2}\right)$ at $a = 1$.

    (c) Write the Taylor series of the function $\ln x$ around $a = 2$.

13. Determine the interval of convergence for the following power series.

    (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(x-2)^n}{3\sqrt{n+1}}.$

    (b) $\displaystyle\sum_{n=1}^{\infty} \frac{10^n x^n}{n^3}.$

    (c) $\displaystyle\sum_{n=1}^{\infty} \frac{10^n (x-1)^n}{n^3}.$

    (d) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{(x+1)^n}{2n+1}.$

(e) $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{(x-3)^n}{2n+1}$.

(f) $\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!}$.

(g) $\displaystyle\sum_{n=0}^{\infty}(n+1)x^n$.

(h) $\displaystyle\sum_{n=1}^{\infty}\frac{x^n}{n}$.

(i) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$.

(j) $\displaystyle\sum_{n=1}^{\infty}\binom{\frac{1}{2}}{n}x^n$, where we recall that the binomial coefficient $\displaystyle\binom{q}{n}$ stands for $\displaystyle\frac{q(q-1)\dots(q-n+1)}{n!}$.

14. Find the length of the curve.

(a) $y=x^2$, $x\in[1,2]$.

(b) $y=\sqrt{x}$, $x\in[1,2]$.

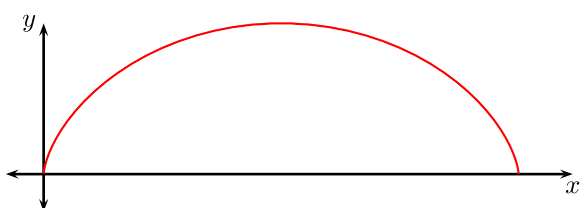
(c) $x=\sqrt{t}-2t$ and $y=\dfrac{8}{3}t^{\frac{3}{4}}$ from $t=1$ to $t=4$.

(d) $\gamma:\left|\begin{array}{rcl} x(t) & = & \frac{1}{t}+\frac{t^3}{3} \\ y(t) & = & 2t \end{array}\right.,t\in[1,2]$.

(e) $\gamma:\left|\begin{array}{rcl} x(t) & = & \frac{1}{t}+t \\ y(t) & = & 2\ln t \end{array}\right.,t\in[1,2]$.

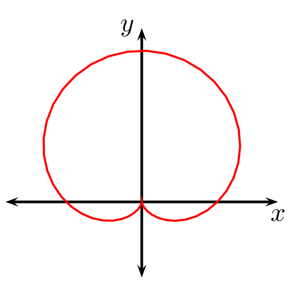
(f) One arch of the cycloid

$$\gamma:\left|\begin{array}{rcl} x(t) & = & t-\sin t \\ y(t) & = & 1-\cos t \end{array}\right.,t\in[0,2\pi]$$



(g) The cardioid

$$\gamma:\left|\begin{array}{rcl} x(t) & = & (1+\sin t)\cos t \\ y(t) & = & (1+\sin t)\sin t \end{array}\right.,t\in[0,2\pi]$$



15. Determine if the sequence is convergent or divergent. If convergent, find the limit of the sequence.

(a) $a_n=n$.

(b) $a_n=2^n$.

(c) $a_n=1.0001^n$.

(d) $a_n=0.999999^n$.

(e) $a_n=n-\sqrt{n+1}\sqrt{n+2}$

(f) $a_n=\dfrac{\ln n}{n}$.

(g) $a_n=\dfrac{\ln n}{\sqrt[10]{n}}$.

(h) $a_n = \dfrac{1}{n}$.

(i) $a_n = \dfrac{1}{n!}$.

(j) $a_n = \dfrac{n^n}{n!}$.

(k) $a_n = \cos n$.

(l) $a_n = \cos\left(\dfrac{1}{n}\right)$

(m) $a_n = \left(\dfrac{n+1}{n}\right)^n$.

(n) $a_n = \left(\dfrac{2n+1}{n}\right)^n$.

(o) $a_n = \left(\dfrac{n+1}{n}\right)^{2n}$.

(p) $a_n = \left(\dfrac{n+1}{2n}\right)^n$.

16. (a)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 - 1 \quad . \tag{1}$$

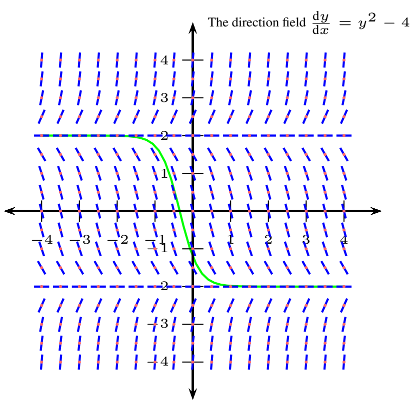
i. Find all solutions of the differential equation above.

ii. Find a solution for which $y(0) = -\frac{3}{5}$.

(b) i. Find the general solution to the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 - 4 \quad .$$

Below is a computer-generated plot of the direction field $\dfrac{\mathrm{d}y}{\mathrm{d}x} = y^2 - 4$, you may use it to get a feeling for what your answer should look like.



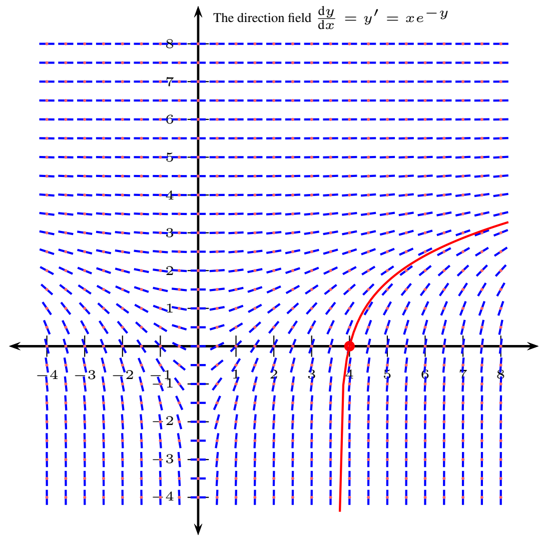
ii. Find a solution of the above equation for which $y(0) = -\frac{6}{5}$.

(c) Solve the initial-value differential equation $y' = y^2(1+x)$, $y(0) = 3$.

(d) Solve the initial-value differential equation problem

$$y' = xe^{-y} \quad , \qquad y(4) = 0.$$

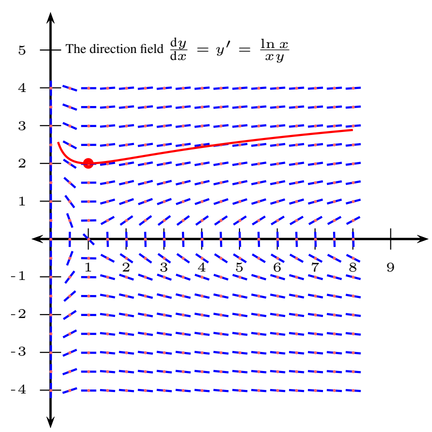
Below is a computer-generated plot of the corresponding direction field, you may use it to get a feeling for what your answer should look like.



(e) Solve the initial-value differential equation problem

$$y' = \frac{\ln x}{xy} \quad , \qquad y(1) = 2.$$

Below is a computer-generated plot of the corresponding direction field, you may use it to get a feeling for what your answer should look like.

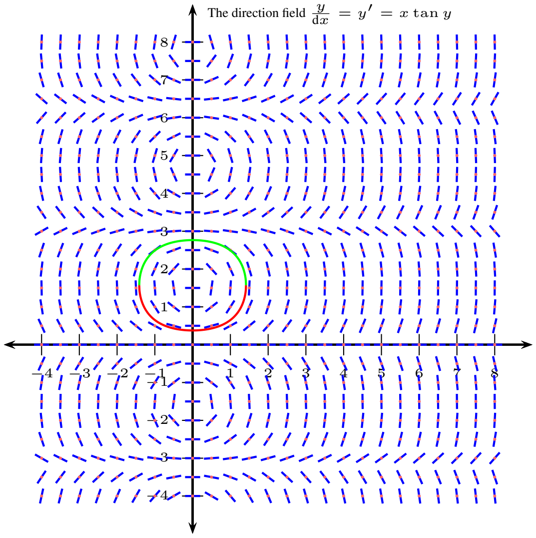


(f) i. Solve the initial-value differential equation problem

$$y' = x \tan y \quad , \qquad y(0) = \arcsin\left(\frac{1}{e}\right) \approx 0.376728.$$

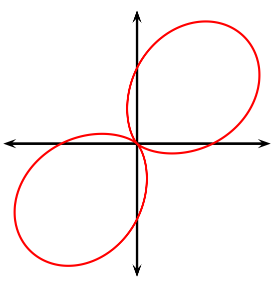
ii. Solve the same differential equation with initial condition $y(0) = \pi + \arcsin\left(-\frac{1}{e}\right) \approx 2.764865$.
Below is a computer-generated plot of corresponding direction field, you may use it to get a feeling for

what your answer should look like.

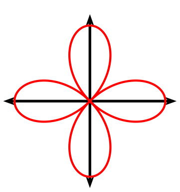


17. **This problem type will appear on the final as a bonus. We have not studied the material for this problem type.**

    (a) The curve given in polar coordinates by $r = 1 + \sin 2\theta$ is plotted below by computer. Find the area lying outside of this curve and inside of the circle $x^2 + y^2 = 1$.

    (b) The curve given in polar coordinates by $r = \cos(2\theta)$ is plotted below by computer. Find the area lying inside the curve and outside of the circle $x^2 + y^2 = \frac{1}{4}$.

    (c) Below is a computer generated plot of the curve $r = \sin(2\theta)$. Find the area locked inside one petal of the curve and outside of the circle $x^2 + y^2 = \dfrac{1}{4}$.