

Calculus I

Lecture 1

Construcing Functions

Todor Milev

<https://github.com/tmilev/freecalc>

2020

Outline

- 1 A Catalog of Essential Functions
 - Linear Functions
 - Polynomials
 - Power Functions
 - Rational Functions
 - Algebraic Functions
 - Transcendental Functions
- 2 New Functions from Old Functions

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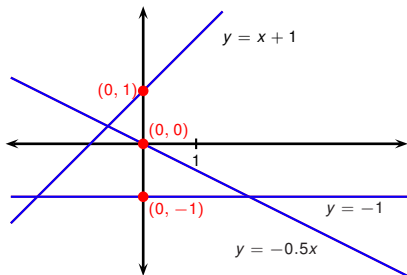
Linear Functions

Definition (Linear Function)

A linear function is a function the graph of which is a line. We can write any linear function in slope-intercept form:

$$f(x) = mx + b.$$

m is called the slope, and b is called the y -intercept.



$f(x)$	Direction	y-intercept
$x + 1$	\nearrow	1
$-0.5x + 0$	\searrow	0
-1	\rightarrow	-1

- $m > 0$ means the graph of f points up (\nearrow).
- $m < 0$ means the graph of f points down (\searrow).
- $m = 0$ means the graph of f is horizontal (\rightarrow).
- b tells us the height of the point where the graph hits the y-axis.

Polynomials

Definition (Polynomial Function)

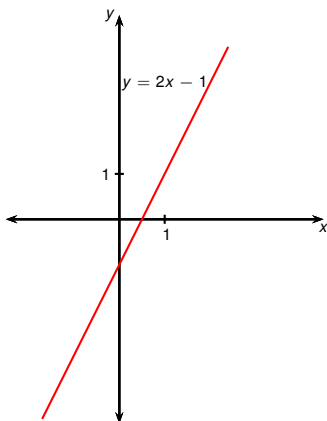
A polynomial function is a function f of the form

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1} + a_nx^n,$$

where n is a non-negative integer and a_0, \dots, a_n are real numbers, called the coefficients. If $a_n \neq 0$ the integer n is called the degree of f .

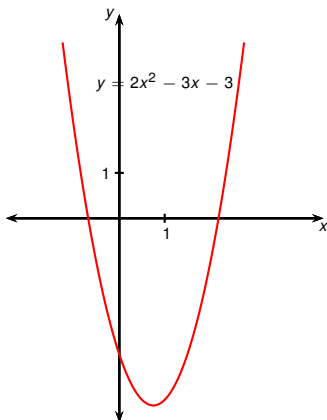
$f(x)$	Polynomial?	Degree	a_0	a_1	a_2
$x^4 - x + 1$	Yes	4	1	-1	0
6	Yes	0	6	0	0
$3x^2 - \frac{1}{2}x + \sqrt{x}$	No				
$3x^2 - \frac{1}{2}x + \sqrt{2}$	Yes	2	$\sqrt{2}$	$-\frac{1}{2}$	3
$3x^2 - \frac{1}{2x} + \sqrt{2}$	No				

- Linear functions are polynomial (functions).
- So are quadratic functions. Their graphs are parabolas.
- And there are many more.



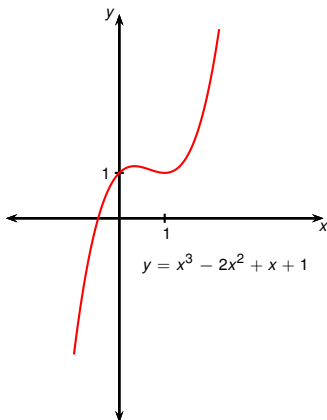
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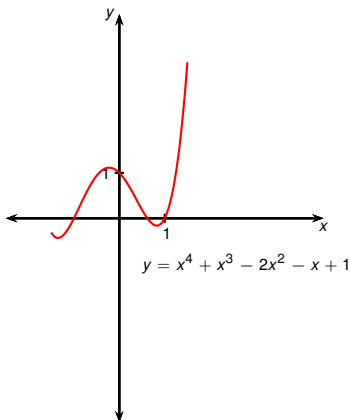
Quadratic

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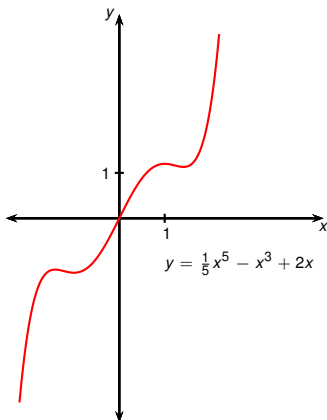
Cubic

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Quartic

- Linear functions are polynomial (functions).
- So are quadratic functions. Their graphs are parabolas.
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Quintic

Power Functions

Definition (Power Function)

Let $x > 0$, a - arbitrary real number. The power function is defined as

$$f(x) = e^{a \ln x} = x^a.$$

x = base. a = exponent or power. First equality = one of ways to define for non-integer a (we study $\ln x$, e^x later).

If a - positive integer ($1, 2, 3, \dots$)

then x^a = polynomial function.

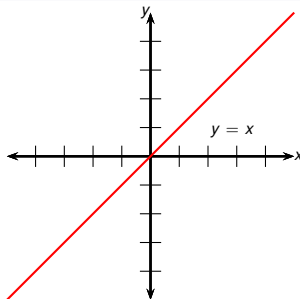
$x^n = \underbrace{x \dots x}_{n \text{ times}}$ when n -integer.

$$(x^a)^b = x^{ab}$$

$$(xy)^b = x^b y^b$$

$$x^{a+b} = x^a x^b$$

$$x^{-a} = \frac{1}{x^a}$$



Power Functions

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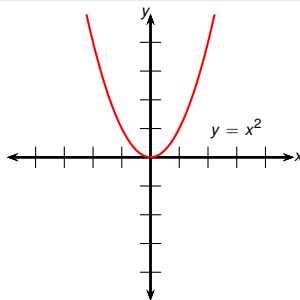
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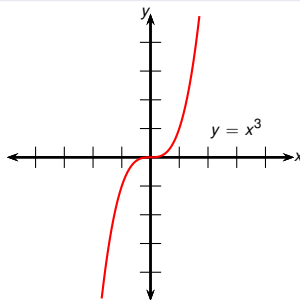
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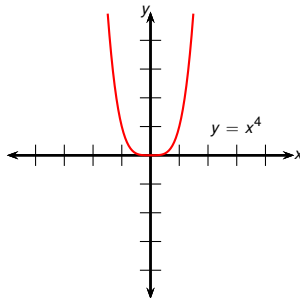
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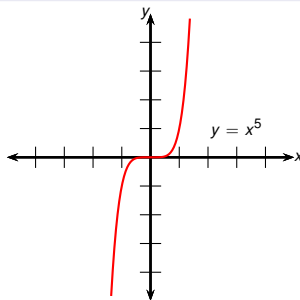
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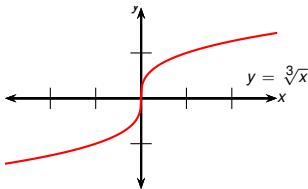
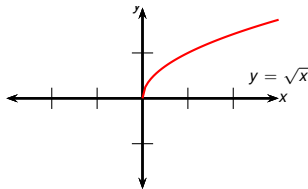
$$(xy)^b = x^b y^b$$

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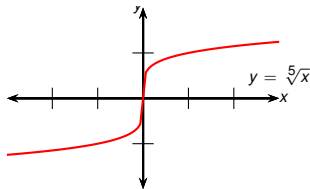
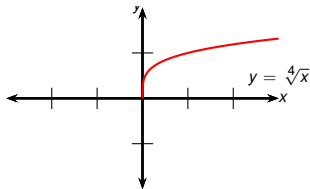
$$x^{-a} = \frac{1}{x^a}$$



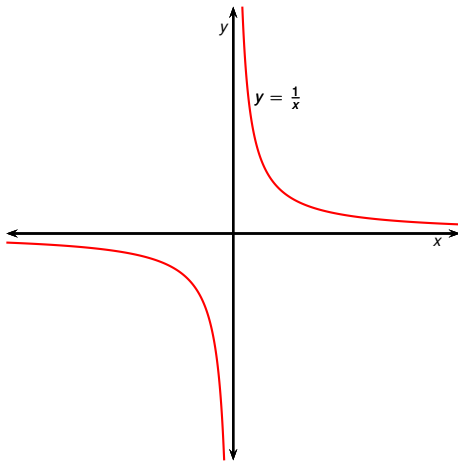
- n - positive integer, $f(x) = x^{\frac{1}{n}} = \sqrt[n]{x} =$ the n^{th} root function.
 $\sqrt[n]{x} \geq 0$ for $x \geq 0$.
- For $n = 2$, we get the square root \sqrt{x} ; for $n = 3$ we get the cube root $\sqrt[3]{x}$, and so on.
- Let $x > 0$. For $n = 2m + 1$ -odd, we can extend the definition of n^{th} root to negative numbers by $^{2m+1}\sqrt{-x} := -^{2m+1}\sqrt{x}$.
- In this course, even roots of negative numbers are not defined.
- The graph of \sqrt{x} is the top half of the parabola $x = y^2$. Similarly for $y = ^{2m}\sqrt{x}$, we graph top of $x = y^{2m}$.
- The graph of the cube root $f(x) = \sqrt[3]{x}$ is the graph of the polynomial $x = y^3$. Similarly for $y = ^{2m+1}\sqrt{x}$, we graph $x = y^{2m+1}$.



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- Let $x > 0$. For $n = 2m + 1$ -odd, we can extend the definition of n^{th} root to negative numbers by $\sqrt[n]{-x} := -\sqrt[n]{x}$.
- In this course, even roots of negative numbers are not defined.
- The graph of \sqrt{x} is the top half of the parabola $x = y^2$. Similarly for $y = \sqrt[2m]{x}$, we graph top of $x = y^{2m}$.
- The graph of the cube root $f(x) = \sqrt[3]{x}$ is the graph of the polynomial $x = y^3$. Similarly for $y = \sqrt[2m+1]{x}$, we graph $x = y^{2m+1}$.



$f(x) = x^{-1} = \frac{1}{x}$ is called the reciprocal function. Its graph has equation $y = \frac{1}{x}$, or $xy = 1$, and is an hyperbola with the coordinate axes as its



asymptotes.

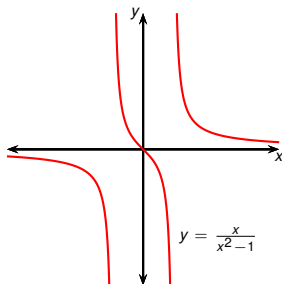
Rational Functions

Definition (Rational Function)

A rational function is a quotient of two polynomials; that is, a function of the form

$$f(x) = \frac{g(x)}{h(x)},$$

where g and h are polynomials.



Example ($x/(x^2 - 1)$)

The function

$$f(x) = \frac{x}{x^2 - 1}$$

is a rational function.

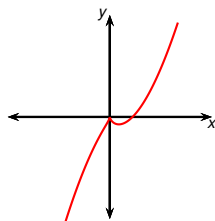
Algebraic Functions

(Algebraic Function)

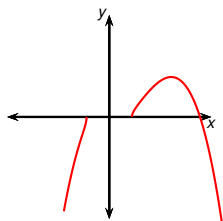
A function in x that can be constructed using x , constants, and finitely many of the operations $+$, $-$, $*$, $/$, and $\sqrt[n]{}$ is an algebraic function.

Outside of present course: function $f(x)$ = algebraic if it satisfies a polynomial equation with polynomial coefficients, i.e., $a_0(x) + a_1(x)f(x) + \cdots + a_n(x)(f(x))^n = 0$ for some polynomials $a_i(x)$.

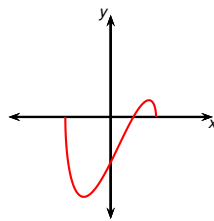
Examples.



$$y = (x-1)^{3/2}$$



$$y = \frac{1}{5}(4x - x^2)^{1/2}$$



$$y = (x-1)^{1/2}$$

Transcendental Functions

Transcendental functions include many classes of functions.

- Trigonometric functions such as $\cos x$, $\sin x$, $\tan x$, etc.
- Exponential functions such as 2^x , $\left(\frac{1}{2}\right)^x$, 5^x , e^x , etc.
- The logarithm function $\ln x$.
- And many more.
- Outside of the present course: by definition, a function is transcendental if it is not algebraic, i.e., if it satisfies no polynomial equation with polynomial coefficients.

Combinations of Functions

Two functions f and g can be combined to form new functions $f + g$, $f - g$, $f \cdot g$, and $\frac{f}{g}$:

$$\begin{aligned} (f + g)(x) &= f(x) + g(x) \\ (f - g)(x) &= f(x) - g(x) \\ (f \cdot g)(x) &= f(x) \cdot g(x) \\ \left(\frac{f}{g}\right)(x) &= \frac{f(x)}{g(x)} \quad \Bigg| \quad \text{for } g(x) \neq 0 \end{aligned}$$

Let $\text{Dom}(f)$ denote the domain of f . The function $f + g$ is defined only if both f and g are defined, and similarly for the others. Therefore

$$\begin{aligned} \text{Dom}(f + g) &= \text{Dom}(f) \cap \text{Dom}(g) \\ \text{Dom}(f - g) &= \text{Dom}(f) \cap \text{Dom}(g) \\ \text{Dom}(f \cdot g) &= \text{Dom}(f) \cap \text{Dom}(g) \\ \text{Dom}\left(\frac{f}{g}\right) &= \text{Dom}(f) \cap \text{Dom}(g) \cap \{x \mid g(x) \neq 0\} \end{aligned}$$

\cap stands for set intersection

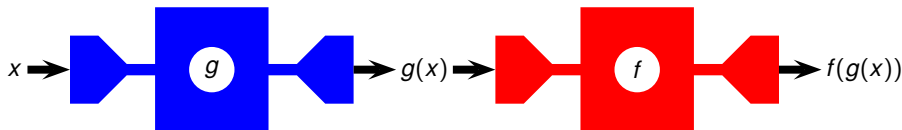
 $\{x \mid \text{right expr.}\}$ stands for set where $g(x) \neq 0$

Definition (Composition of f and g)

If f and g are two functions, then the composition of f and g is written $f \circ g$ and is defined by the formula

$$(f \circ g)(x) = f(g(x)).$$

Imagine f and g as machines taking some input and producing some output. Then $f \circ g$ corresponds to attaching both machines end-to-end so that the output of g becomes the input of f .



The domain of $f \circ g$ is the set of all numbers x in the domain of g such that $g(x)$ is in the domain of f . If the domain of f is A and the domain of g is B , we write this as

$$\{x \in B \mid g(x) \in A\}.$$

Example

Find $f \circ g$, $g \circ f$, $g \circ g$ and their domains, where $f(x) = \sqrt{x}$ and $g(x) = \sqrt{3-x}$.

$$(f \circ g)(x) = f(g(x)) = f(\sqrt{3-x}) = \sqrt{\sqrt{3-x}} = \sqrt[4]{3-x}$$

Domain:

$$3-x \geq 0$$

$$-x \geq -3$$

$$x \leq 3$$

$$x \in (-\infty, 3].$$

$$(g \circ f)(x) = g(f(x)) = g(\sqrt{x}) = \sqrt{3-\sqrt{x}}$$

Domain :

$$x \geq 0$$

$$3-\sqrt{x} \geq 0$$

$$-\sqrt{x} \geq -3$$

$$\sqrt{x} \leq 3$$

$$x \leq 9$$

$$x \in [0, 9]$$

Example

Find $f \circ g, g \circ f, g \circ g$ and their domains, where $f(x) = \sqrt{x}$ and $g(x) = \sqrt{3-x}$.

$$(g \circ g)(x) = g(g(x)) = g(\sqrt{3-x}) = \sqrt{3 - \sqrt{3-x}}$$

Domain :

$$3 - x \geq 0$$

$$-x \geq -3$$

$$x \leq 3$$

$$3 - \sqrt{3-x} \geq 0$$

$$-\sqrt{3-x} \geq -3$$

$$\sqrt{3-x} \leq 3$$

$$3 - x \leq 9$$

$$-x \leq 6$$

$$x \geq -6$$

$$x \in [-6, 3].$$

Example

Give simplified f-las for $f \circ g$, $f \circ f$, $g \circ f$, $g \circ g$. Find the implied domains.

$$f(x) = \frac{2x-1}{x+2}$$

$$x \neq -2$$

$$g(x) = \frac{2x+3}{5x-7}$$

$$x \neq \frac{7}{5}$$

$$(f \circ g)(x) = f(g(x)) = f\left(\frac{2x+3}{5x-7}\right) = \frac{2\left(\frac{2x+3}{5x-7}\right) - 1}{\frac{2x+3}{5x-7} + 2}$$

$$= \frac{\frac{2(2x+3)}{5x-7} - \frac{5x-7}{5x-7}}{\frac{2x+3}{5x-7} + \frac{2(5x-7)}{5x-7}} = \frac{\frac{4x+6-(5x-7)}{5x-7}}{\frac{2x+3+(10x-14)}{5x-7}} = \frac{-x+13}{12x-11}$$

$$x \neq \frac{11}{12}, \frac{7}{5}$$

$$(f \circ f)(x) = f(f(x)) = f\left(\frac{2x-1}{x+2}\right) = \frac{2\left(\frac{2x-1}{x+2}\right) - 1}{\frac{2x-1}{x+2} + 2}$$

$$= \frac{3x-4}{4x+3}$$

$$x \neq -2, -\frac{3}{4}$$

$$(g \circ f)(x) = \frac{7x+4}{3x-19}$$

$$x \neq -2, \frac{19}{3}$$

$$(g \circ g)(x) = \frac{19x-15}{-25x+64}$$

$$x \neq \frac{7}{5}, \frac{64}{25}$$