Calculus III
Lecture 20

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Outline

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Theorem (Divergence Theorem)

Let D be a compact set in space with boundary S a piecewise smooth parametrized surface, oriented by the outward normal, and let $\mathbf{X}$ be a smooth vector field on D given by

$$\mathbf{X}(x,y,z) = P(x,y,z)\mathbf{i} + Q(x,y,z)\,\mathbf{j} + R(x,y,z)\,\mathbf{k} \quad .$$

Then

$$\iint_S \mathbf{X}\cdot \mathrm{d}\mathbf{S} = \iiint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) \mathrm{d}V$$

Corollary (May serve as alternative definition of div)

$$\begin{aligned}(\operatorname{div}\mathbf{X})(p) &= \lim_{D\to\{p\}} \frac{1}{vol(D)} \iint_S \mathbf{X}\cdot \mathrm{d}\mathbf{S}\\ &= \lim_{D\to\{p\}} \frac{1}{vol(D)} \iiint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) \mathrm{d}V\\ &= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.\end{aligned}$$

Divergence Theorem

- Let $\mathbf{X} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$.
- Recall our notation

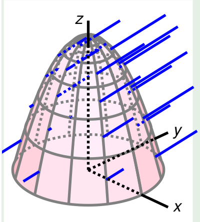
$$\begin{aligned}\operatorname{div}\mathbf{X} &= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = (\partial_x, \partial_y, \partial_z)\cdot(P, Q, R)\\ \operatorname{div}\mathbf{X} &= \nabla\cdot\mathbf{X}.\end{aligned}$$

Theorem (Divergence Theorem)

$$\iint_S \mathbf{X}\cdot \mathrm{d}\mathbf{S} = \iiint_D \operatorname{div}\mathbf{X}\,\mathrm{d}V$$

- If $(\operatorname{div}\mathbf{X})(p) > 0$, then p acts as a source;
- If $(\operatorname{div}\mathbf{X})(p) < 0$, then p acts as a sink;
- If $\operatorname{div}\mathbf{X} \equiv 0$ on some domain D, then $\mathbf{X}$ is incompressible on D.

Example



Let S be the part of the paraboloid $z = 4 - x^2 - y^2$ above the $xy-$plane, oriented upward, and $\mathbf{X} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$. Use the Divergence Theorem to compute $\iint_S \mathbf{X} \cdot d\mathbf{S}$.

The surface S does not enclose a region in space. However, we add the disk D of radius 2 centered at the origin in the plane $z = 0$ to make it closed. R orients D with the downward normal, hence

$$\begin{aligned} \iint_{S\uparrow \cup D\downarrow} \mathbf{X} \cdot d\mathbf{S} &= \iiint_R \operatorname{div} \mathbf{X} \, dV = 0 \, , \\ \iint_{S\uparrow} \mathbf{X} \cdot d\mathbf{S} &= \iint_{D\uparrow} \mathbf{X} \cdot d\mathbf{S} \quad . \end{aligned}$$

The upward normal to D is $\mathbf{k}$, hence $\mathbf{X} \cdot d\mathbf{S} = \mathbf{X} \cdot \mathbf{k} \, dS = c \, dS$. Therefore

$$\iint_S \mathbf{X} \cdot d\mathbf{S} = \iint_D \mathbf{X} \cdot d\mathbf{S} = \iint_D c \, dS = c \cdot \operatorname{area}(D) = 4\pi c \, .$$

Balloon Pressure Equilibrium

- Let $\mathbf{F}$ be the total displacement force due pressure difference between interior and exterior of inflated balloon:

$$\mathbf{F} = \iint_S \mathrm{d}\mathbf{F} = \iint_S p\mathbf{N}\,\mathrm{d}S = \iint_S p\,\mathrm{d}\mathbf{S}\,.$$

- For every unit vector $\mathbf{u}$ we have

$$\mathbf{F}\cdot\mathbf{u} = \left(\iint_S p\mathbf{N}\,\mathrm{d}S\right)\cdot\mathbf{u} = \iint_S p\,\mathbf{u}\cdot\mathbf{N}\,\mathrm{d}S = \iiint_D \mathrm{div}(p\,\mathbf{u})\mathrm{d}V = 0$$

  because $\mathrm{div}(p\mathbf{u}) = 0$ since the vector field $\mathbf{X} = p\,\mathbf{u}$ is constant on D.
- Therefore $\mathbf{F}\cdot\mathbf{u} = 0$ for every unit vector $\mathbf{u}$;
- Which implies $\mathbf{F} = \mathbf{0}$.

Archemedes' Law from the Divergence Theorem

A solid body is submerged into a tank containing a liquid of constant density ρ. What is the buoyant force?

- Body occupies a region D, exterior boundary S;
- Unit outward normal field $\mathbf{N}$;
- Magnitude of pressure at depth a below the surface is $p_0 + \rho a g$, where
  - g is the magnitude of the gravitational acceleration.
  - p_0 is the pressure at surface of liquid
- Infinitesimal force acting on S is $\mathrm{d}\mathbf{F} = -(p_0 + \rho a g)\,\mathbf{N}\,\mathrm{d}S$,
- The total force is

$$\boldsymbol{F} = \iint_S \mathrm{d}\mathbf{F} = \iint_S -(p_0+\rho a g)\,\mathbf{N}\,\mathrm{d}S = \iint_S -\rho a g\,\boldsymbol{N}\,\mathrm{d}S = \iint_S \rho g z\,\boldsymbol{N}$$

- EC: Use the Divergence Theorem to show that $\mathbf{F} = \rho V g\,\mathbf{k}$ (V: volume of the region enclosed by S.)

Curl

Let $\mathbf{X} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ be a smooth vector field.

Definition (Curl, coordinate definition)

The *curl* of a vector field $\mathbf{X}$, denoted by $\mathbf{curl}\,\mathbf{X}$, is defined by

$$\mathbf{curl}\,\mathbf{X} = (\partial_y R - \partial_z Q)\,\mathbf{i} + (\partial_z P - \partial_x R)\,\mathbf{j} + (\partial_x Q - \partial_y P)\,\mathbf{k}.$$

$$\mathbf{curl}\,\mathbf{X} = \left|\begin{array}{ccc} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ P & Q & R \end{array}\right| = \nabla \times \mathbf{X}\,.$$

- Just like div, **curl** can be equipped with a coordinate-free definition (in this case the above definition becomes a theorem).

Induced Orientation on a Boundary Curve

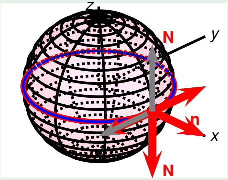
- Let S be smooth surface, oriented by unit normal vector $\mathbf{n}$.
- Let D be region in S, bounded by a curve $C = \partial D$.
- Let $\mathbf{N}$ denote the unit vector field on C which is
  - tangent to S;
  - normal to C;
  - pointing outward of D.
- Let $\mathbf{T}$ be unit tangent vector to C (and hence tangent to S).
- Then $\mathbf{N}$ orients the tangents of C and thus C itself.

Definition

We say that $\mathbf{T}$ is *positively oriented* if the triple $\{\mathbf{n}, \mathbf{N}, \mathbf{T}\}$ is positively oriented in space.

- Since $\mathbf{T}, \mathbf{n}, \mathbf{N}$ are pairwise orthogonal unit vectors, positive orientation is equivalent to $\mathbf{T} = \mathbf{n} \times \mathbf{N}$.
- If we view the plane tangent to S from the tip of $\mathbf{n}$, then $\{\mathbf{N}, \mathbf{T}\}$ is positively oriented in that plane.

Example (Orientation of the equator of a sphere)



Let S be the unit sphere $x^2 + y^2 + z^2 = 1$ oriented by the outward normal $\mathbf{n}$, $D = S \cap \{z \geq 0\}$ be the upper hemisphere. Introduce an orientation on the boundary $C = \partial D$.

- At the point $(1, 0, 0)$ the normal to the surface $\mathbf{n}$ equals $\mathbf{i}$.
- Let $\mathbf{T}$ be a unit tangent to C at $(1, 0, 0)$; then $\mathbf{T} = \mathbf{j}$ or $-\mathbf{j}$.
- Let $\mathbf{N}$ be unit vector perpendicular to $\mathbf{n}$ and $\mathbf{T}$, pointing outwards from $D \Rightarrow \mathbf{N}$ equals $-\mathbf{k}$.
- $\Rightarrow$ positively oriented tangent to C is $\mathbf{T} = \mathbf{n} \times \mathbf{N} = \mathbf{i} \times (-\mathbf{k}) = \mathbf{j}$.
- A viewer, standing along $\mathbf{n}$ with feet on surface, and facing in the direction of the tangent, has the surface to the left.
- Change D to be lower hemisphere: we get $\mathbf{N} = \mathbf{k}$, $\mathbf{T} = \mathbf{n} \times \mathbf{N} = -\mathbf{j}$.
- A viewer, standing along $\mathbf{n}$ with feet on surface, facing in the direction of the tangent, has the surface again to the left.

- Let S be a smooth surface, oriented by the unit normal field $\mathbf{n}$.
- Let D be a region on S, bounded by the piecewise smooth curve $C = \partial D$.
- Let C have unit tangent $\mathbf{T}$ positively oriented by $\mathbf{n}$.
- Let $\mathbf{X} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ be a smooth vector field defined in a open set around S.
- Recall that $\mathbf{curl}\, X = (R_y - Q_z)\,\mathbf{i} + (P_z - R_x)\,\mathbf{j} + (Q_x - P_y)\,\mathbf{k}$.
- Recall that $\mathbf{X} \cdot \mathrm{d}\mathbf{r} = \mathbf{T}\mathrm{d}s$ and $\mathrm{d}\mathbf{S} = \mathbf{n}\mathrm{d}S$.

Theorem (Stokes)

$$\oint_C \mathbf{X} \cdot \mathrm{d}\mathbf{r} = \iint_D \mathbf{curl}\, \mathbf{X} \cdot \mathrm{d}\mathbf{S} \quad .$$

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Idea of proof:

- Use a parametrization of S to get integrals in the parameter plane.
- Apply Green's Theorem in the parameter plane.

We can use Stokes' theorem to:

- Evaluate line integrals by computing a surface integral, or
- Evaluate a surface integral by computing a line integral.

Example

Vector Potential

Given a smooth vector field $\mathbf{X}$, one can ask:

- Is $\mathbf{X}$ the **curl** of a vector field?
- Any field $\mathbf{G}$ such that $\mathbf{X} = \mathbf{curl}\,\mathbf{G}$ is called a *vector potential* for $\mathbf{X}$.
- If $\mathbf{X} = \nabla \times \mathbf{G}$ is a curl field, then $\operatorname{div} \mathbf{X} = 0$.
- Two vector potentials differ by a gradient field.

Surface D the part of the paraboloid $z = 4 - x^2 - y^2$ above the $xy-$plane, oriented upward, $\mathbf{X} = a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}$

$$\iint_D \mathbf{X} \cdot \mathrm{d}\mathbf{S}$$

$\operatorname{div} \mathbf{X} = 0$, hence $\mathbf{X}$ may be the curl of a vector field $\mathbf{G} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$.

$$Q_x - P_y = c, \qquad P_z - R_x = b, \qquad R_y - Q_z = a\,.$$

One solution is $Q = cx$, $P = bz$, $R = ay$, hence $\mathbf{G} = bz\,\mathbf{i} + cx\,\mathbf{j} + ay\,\mathbf{k}$ is a vector potential for $\mathbf{X}$. Then $\mathbf{X} = \mathbf{curl}\,\mathbf{G}$ and therefore

$$\iint_D \mathbf{X} \cdot \mathrm{d}\mathbf{S} = \iint_D \mathbf{curl}\,\mathbf{G} \cdot \mathrm{d}\mathbf{S} = \oint_C \mathbf{G} \cdot \mathbf{dr} = \oint_C bz\,dx + cx\,dy + ay\,dz\,,$$

where $C = \partial D$, the circle of radius 2 centered at the origin, oriented counterclockwise;$x = 2\cos t$, $y = 2\sin t$, $z = 0$, with $0 \leqslant t \leqslant 2\pi$ is an orientation-compatible parametrization of C

$$\oint_C \mathbf{G} \cdot \mathbf{dr} = \int_0^{2\pi} 2c\cos t\,(2\cos t)\,dt = 4c\int_0^{2\pi} \cos^2 t\,dt = 4\pi\,c\,.$$

Div, Curl, Grad

$$\operatorname{div}(\mathbf{curl}\,\mathbf{X}) = \nabla\cdot(\nabla\times\mathbf{X}) = 0\ .$$

B: ball centered at p, with boundary a sphere S centered at p.

$$\iiint_B \operatorname{div}(\mathbf{curl}\,\mathbf{X})\,dV = \iint_{S=\partial B} \mathbf{curl}\,\mathbf{X}\cdot \mathrm{d}\mathbf{S} = \oint_{\partial S} \mathbf{X}\cdot\mathbf{dr} = 0\ ,$$

$$\operatorname{div}(\mathbf{curl}\,\mathbf{X})(p) = \lim_{B\to\{p\}} \frac{1}{\operatorname{vol}(B)} \iiint_B \operatorname{div}(\mathbf{curl}\,\mathbf{X})\,dV = 0\ .$$

$$\mathbf{curl}(\mathbf{grad} f) = \nabla\times(\nabla f) = \mathbf{0}\ .$$

D: disk centered at p, in the plane normal to $\mathbf{n}$ at p, and $C=\partial D$

$$\iint_D \mathbf{curl}\,(\mathbf{grad} f)\cdot\mathbf{n}\,\mathrm{d}S = \iint_D \mathbf{curl}\,(\mathbf{grad} f)\cdot \mathrm{d}\mathbf{S} = \oint_C \mathbf{grad} f\cdot\mathbf{dr} = 0\ ,$$

$$\mathbf{curl}\,(\mathbf{grad} f)(p)\cdot\mathbf{n} = \lim_{D\to\{p\}} \frac{1}{\operatorname{area}(D)} \iint_D \mathbf{curl}(\mathbf{grad} f)\cdot\mathbf{n}\,\mathrm{d}S = 0\ ;$$

since this is valid for all unit vectors $\mathbf{n}$, we conclude that