Calculus III
Homework on Lecture 4

1. Write vectorial and scalar equations of the line L passing through the given point and with the given direction.

   (a) $P_0 = (1, 2, 3)$, $\mathbf{u} = (-3, -2, -1)$.

   answer: $L : (1,2,3) + t(-3,-2,-1)$, $L : \left|\begin{array}{rcl} x &=& 1-3t \\ y &=& 2-2t \\ z &=& 3-t \end{array}\right., t \in \mathbb{R}$

   (b) $P_0 = (3, 5, 7)$, $\mathbf{u} = (2, 3, 4)$.

   answer: $L : (3,5,7) + t(2,3,4)$, $L : \left|\begin{array}{rcl} x &=& 3+2t \\ y &=& 5+3t \\ z &=& 7+4t \end{array}\right., t \in \mathbb{R}$

2. Write vectorial and scalar equations of the line passing L through the given points.

   (a) $(2, 3, 5)$ and $(3, 5, 7)$.

   answer: $L : (2,3,5) + t(1,2,2)$ or $L : \left|\begin{array}{rcl} x &=& 2+t \\ y &=& 3+2t \\ z &=& 5+2t \end{array}\right., t \in \mathbb{R}$

   (b) $(-1, -1, 1)$ and $(-1, 1, -1)$.

   answer: $L : (-1,0,0) + t(0,1,-1)$ or $L : \left|\begin{array}{rcl} x &=& -1 \\ y &=& t \\ z &=& -t \end{array}\right., t \in \mathbb{R}$

3. We recall that the 8 points $(1, 1, 1), (-1, 1, 1), (1, -1, 1), (-1, -1, 1), (1, 1, -1), (-1, 1, -1), (1, -1, -1), (-1, -1, -1)$ (all possible sign combinations) give the vertices of a cube with edge 2 units.

   Find equations for all lines connecting two vertices in the cube above that pass through the origin (how many connecting two vertices of a cube are there? How many of them are edges?).

   answer: There are 4 such edges. See the solution below for their equations.

   Solution. 3. A cube has a total of 8 vertices. A line is given by two (distinct) points, therefore there are $\binom{8}{2} = \frac{8\cdot 7}{2} = 28$ total lines connecting two distinct vertices of a cube. Of those 12 lines are cube edges, $12 = 6 \cdot 2$ are diagonals of cube faces, and 4 are inner diagonals. All four inner diagonals contain the origin. A justification for this can undoubtedly be given by writing all 28 line equations. However, the origin is in the center of the cube, and we know from our every-day geometric intuition that only the inner diagonals contain the center of a cube; we give no further justification.

   The 4 inner diagonals of the cube, call them L_1, L_2, L_3, L_4 pass through the points

$$\begin{array}{rcl} (1,1,1), (-1,-1,-1) &\in& L_1 \\ (1,1,-1), (-1,-1,1) &\in& L_2 \\ (1,-1,1), (-1,1,-1) &\in& L_3 \\ (-1,1,1), (1,-1,-1) &\in& L_4 \end{array} .$$

   Therefore equations for these lines are given by:

$$\begin{array}{ll} L_1 : & t(1,1,1) \\ L_2 : & t(1,1,-1) \\ L_3 : & t(1,-1,1) \\ L_4 : & t(-1,1,1) \end{array} .$$

4. Find an equation of the plane $\mathcal{P}$ passing through the given point and with the given normal. Find parametric vectorial equations of the plane.

   (a) $P_0(1, 2, 3)$, $\mathbf{n} = (4, 5, 6)$.

   answer: $\mathcal{P} : 4x + 5y + 6z = 32$, $\mathcal{P} : (1,2,3) + t(5,-4,0) + s(3,0,-2)$

(b) $P_0(2,3,5)$, $\mathbf{n} = (-3,-5,-7)$.

answer: $\mathcal{P}: -3x - 5y - 7z = -56$. $\mathcal{P}: (2,3,5) + t(5,-3,0) + s(7,0,-3)$

(c) $P_0(1,1,1)$, $\mathbf{n} = (1,1,1)$.

answer: $\mathcal{P}: x + y + z = 3$. $\mathcal{P}: (1,1,1) + t(1,-1,0) + s(1,0,-1)$

Solution. 4.a As studied, an equation passing through $(1,2,3)$ and with normal $(4,5,6)$ has equation:

$$\begin{array}{rcl} (x,y,z)\cdot(4,5,6) &=& (4,5,6)\cdot(1,2,3) \\ 4x+5y+6z &=& 23 \end{array}$$

To find parametric equations of the plane, we need to find two directions, $\mathbf{u}, \mathbf{v}$, that can be added to the base point to obtain all points in the plane. This means that a direction vector $\mathbf{u}$ has to be perpendicular to $\mathbf{n}$. Equivalently, a direction vector $\mathbf{u}$ lies in the plane passing through the origin and orthogonal to $\mathbf{n}$. This means $\mathbf{u}(u_1, u_2, u_3)$ satisfies the equation:

$$\begin{array}{rcl} \mathbf{u}\cdot\mathbf{n} &=& 0 \\ 4u_1+5u_2+6u_3 &=& 0. \end{array} \qquad (1)$$

There are infinitely many solutions to that equation - in fact, for each point in the plane passing through the origin and orthogonal to $\mathbf{n}$ there is one solution. However, we only need to find two such non-colinear solutions, and declare them to be our vectors $\mathbf{u}$ and $\mathbf{v}$. It is very easy to do that: if we set u_1 and u_2 to be arbitrary, then u_3 can always be chosen so as to make the equation above hold. There are a number of accepted ways to choose u_1 and u_2 in a not-so-arbitrary fashion. For reasons outside of the scope of this homework, such ways to choose u_1 and u_2 may be preferable to the choosing at random. Our scheme for choosing a vector $\mathbf{u}$ will be to choose $u_1 = 1$ and $u_2 = 0$ (or the other way round for $\mathbf{v}$), and then to rescale the resulting vector so all coordinates are integers and the first non-zero coordinate is positive. In other words, we select $\mathbf{u}$ to be proportional to $(1,0,-\frac{4}{6})$, and $\mathbf{v}$ to be proportional to $(0,1,-\frac{5}{6})$, i.e., we select

$$\begin{array}{rcl} \mathbf{u} &=& (3,0,-2) \\ \mathbf{v} &=& (0,6,-5) \end{array}.$$

Finally we get that a parametric equation of the plane is given by:

$$(1,2,3) + s(3,0,-2) + t(0,6,-5) \quad . \qquad (2)$$

The above equations are not unique; our problem has many correct answers.

How do we check if two plane parametrizations are equivalent? Equivalently, how do we check that equation (2) gives a plane that coincides with the plane in given in (1)? Here's what we need to do to make sure our answer is correct (we leave the justification for that to the reader):

- Check that our $\mathbf{u}, \mathbf{v}$ are orthogonal to $\mathbf{n}$.
- Check that our $\mathbf{u}, \mathbf{v}$ are not proportional to one another.
- Check that the base point of our equation is in the original plane.

5. Find an equation of plane $\mathcal{P}$ passing through the point and parallel to the given directions.

   (a) $P_0(1,2,3)$, $\mathbf{u} = (2,3,5)$, $\mathbf{v} = (3,5,7)$.

   answer: $\mathcal{P}: z + y - 4x - 1 = 0$

   (b) $P_0(1,1,1)$, $\mathbf{u} = (1,-1,0)$, $\mathbf{v} = (0,1,-1)$.

   answer: $\mathcal{P}: z + y + x - 3 = 0$

6. Find an equation of the plane $\mathcal{P}$ passing through the given points.

   (a) $P_0(2,3,5)$, $P_1(3,5,7)$, $P_2(5,7,11)$.

   answer: $\mathcal{P}: -z + 2x + 1 = 0$

   (b) $P_0(1,1,1)$, $P_1(1,-1,-1)$, $P_2(-1,-1,1)$.

   answer: $\mathcal{P}: -z + y - x + 1 = 0$

7. Find the distance between the line and the point.

   (a) The line passing through $P_0(1,1,1)$ and $P_1(-1,-1,-1)$ and the point $Q(1,0,0)$.

   answer: $\frac{\sqrt{6}}{3}$

   (b) The line passing through $P_0(-2,3,-5)$ and $P_1(3,4,5)$ and the point $Q(2,-2,2)$.

   answer: $\frac{\sqrt{57610}}{42}$

8. Find the distance between the plane and the point.

   (a) The plane passing through $P_0(1,2,3)$, $P_1(2,3,5)$ and $P_2(3,5,7)$ and the point $Q(2,-2,2)$. answer: $\frac{3}{5}\sqrt{5}$

   (b) The plane passing through $P_0(1,2,3)$, $P_1(2,3,5)$ and $P_2(3,5,7)$ and the point $Q(5,7,11)$. answer: 0

   (c) The plane passing through the points $P_0(1,1,1)$, $P_1(1,-1,-1)$, $P_2(-1,-1,1)$ and the point $Q(-1,1,-1)$. answer: $\frac{4}{3}\sqrt{3}$

9. Recall that a regular tetrahedron can be realized using 4 vertices of a cube.

   (a) In a regular tetrahedron, find the angle between two edges that share a common vertex. answer: $\frac{\pi}{3} = 60^\circ$.

   (b) In a regular tetrahedron, find the angle between two edges that share a common vertex. answer: $\frac{\pi}{2} = 90^\circ$.

10. Recall that a regular tetrahedron can be realized using 4 vertices of a cube.
    Find the distance between two opposite edges of a regular tetrahedron inscribed in a 2x2x2 cm cube.

    answer: $2cm$

11. Find the distance between the lines.

    (a) The line passing through $Q_0(1,2,3)$ and $Q_1(6,5,4)$ and the line passing through $P_0(1,3,5)$ and $P_1(2,4,6)$. answer: 0

    (b) The line passing through $Q_0(1,2,3)$ and $Q_1(2,3,5)$ and the line passing through $P_0(3,5,7)$ and $P_1(5,7,11)$.

    answer: $\frac{\sqrt{30}}{6}$

    (c) The line passing through $Q_0(1,1,1)$ and $Q_1(-1,-1,-1)$ and the line passing through $P_0(1,-1,-1)$ and $P_1(-1,1,-1)$.

    answer: $\frac{\sqrt{6}}{3}$

    (d) The line passing through $(1,3,4)$ and $(2,3,1)$ and the line passing through $(1,2,2)$ and $(0,2,5)$.

    answer: $\frac{\sqrt{35}}{5}$

    (e) The line passing through $(1,3,4)$ and $(2,3,1)$ and the line passing through $(1,2,2)$ and $(0,2,4)$.

    answer: 1

Solution. 11.a We need to first establish whether the two lines are parallel. Let $\mathbf{u}$ be the direction vector of the first line given by

$$\mathbf{u} = \mathbf{Q}_0\mathbf{Q}_1 = (6,5,4) - (1,2,3) = (5,3,1)$$

and let $\mathbf{v}$ be the direction vector of the second line given by

$$\mathbf{v} = \mathbf{P}_0\mathbf{P}_1 = (2,4,6) - (1,3,5) = (1,1,1).$$

Now it is straightforward to see that the two lines are not parallel - indeed, one immediately sees that $\mathbf{u} = (5,3,1)$ is not a scalar multiple of $\mathbf{v} = (1,1,1)$. Since the two lines are not parallel, the two direction vectors determine a plane through the origin whose normal vector is given by

$$\mathbf{n} = \mathbf{u} \times \mathbf{v} = (5,3,1) \times (1,1,1) = \left| \begin{array}{ccc} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & 3 & 1 \\ 1 & 1 & 1 \end{array} \right| = 2\mathbf{i} - 4\mathbf{j} + 2\mathbf{k} = (2,-4,2) \quad .$$

We note that if the vectors $\mathbf{u}, \mathbf{v}$ were parallel, then the cross product above would had been zero. Now the distance between the two lines is obtained by taking an arbitrary vector with tail on one line and head on the other, and computing the length of its projection onto $\mathbf{n}$. We use the vector $\mathbf{r} = \mathbf{Q}_0\mathbf{P}_0$. Then the distance d between the two lines is given by:

$$d = \frac{|\mathbf{r} \cdot \mathbf{n}|}{|\mathbf{n}|} = \frac{|((1,3,5) - (1,2,3)) \cdot \mathbf{n}|}{|\mathbf{n}|} = \frac{|(0,1,2) \cdot (2,-4,2)|}{|\mathbf{n}|} = 0.$$

Therefore the distance between the two lines is zero. This completes our solution.

We note that since the distance between the lines is zero, they must intersect. As a consistency check for our work, let us verify that the two lines do indeed intersect. The first line is parametrized by $(1,2,3) + t(5,3,1)$ i.e., has parametric equations

$$\left| \begin{array}{rcl} x & = & 1 + 5t \\ y & = & 2 + 3t \\ z & = & 3 + t \end{array} \right. \quad .$$

Similarly, the second line is given by the equations

$$\left|\begin{array}{rcl} x &=& 1+s \\ y &=& 3+s \\ z &=& 5+s \end{array}\right. .$$

Therefore to find an intersection of the two lines, we need to solve the system

$$\left|\begin{array}{rcl} 1+5t &=& 1+s \\ 2+3t &=& 3+s \\ 3+t &=& 5+s \end{array}\right. .$$

From the first equality we get that $s = 5t$. We substitute that into the second equality to get that $t = -\frac{1}{2}$. Therefore the intersection of the two lines is the point

$$(1,2,3) - \frac{1}{2}(5,3,1) = \left(-\frac{3}{2}, \frac{1}{2}, \frac{5}{2}\right) = (1,3,5) - \frac{5}{2}(1,1,1) \quad ;$$

all our error checks have been successful.

Solution. 11.d We present a solution in a concise form suitable for exam taking.

Let L_1, L_2 be the two lines.

$$\begin{array}{rcll}
\mathbf{u} &=& (2,3,1)-(1,3,4) = (1,0,-3) & \big| \text{ direction vector } L_1 \\
\mathbf{v} &=& (0,2,5)-(1,2,2) = (-1,0,3) = -\mathbf{u} & \big| \text{ direction vector } L_2 \\
&& \text{Therefore } L_1 \parallel L_2 & \\
\mathbf{r} &=& (2,3,1)-(0,2,5) = (2,1,-4) & \big| \text{ arbitrary vector connecting } L_1, L_2 \\
L_1 \parallel L_2 \Rightarrow && & \\
\mathrm{dist}(L_1, L_2) &=& |\mathbf{orth}_{\mathbf{u}}\mathbf{r}| & \\
&=& |\mathbf{r} - \mathbf{proj}_{\mathbf{u}}\mathbf{r}| & \\
&=& \left|\mathbf{r} - \frac{\mathbf{r}\cdot\mathbf{u}}{|\mathbf{u}|^2}\mathbf{u}\right| & \\
&=& \left|(2,1,-4) - \frac{(2,1,-4)\cdot(1,0,-3)}{1^2+0^2+(-3)^2}(1,0,-3)\right| & \\
&=& \left|\left(\frac{3}{5}, 1, \frac{1}{5}\right)\right| & \\
&=& \sqrt{\left(\frac{3}{5}\right)^2 + 1^2 + \left(\frac{1}{5}\right)^2} & \\
&=& \frac{\sqrt{35}}{5} \quad . &
\end{array}$$

Solution. 11.e We present a solution in a concise form suitable for exam taking.

$$\begin{array}{rcll}
\mathbf{u} &=& (2,3,1)-(1,3,4) = (1,0,-3) & \big| \text{ direction vector } L_1 \\
\mathbf{v} &=& (0,2,4)-(1,2,2) = (-1,0,2) = -\mathbf{u} & \big| \text{ direction vector } L_2 \\
\mathbf{r} &=& (1,3,4)-(1,2,2) = (0,1,2) & \big| \text{ arbitrary vector connecting } L_1, L_2 \\
\mathbf{n} &=& u \times v = (0,1,0) & \big| \neq 0 \Rightarrow L_1 \nparallel L_2 \\
L_1 \nparallel L_2 \Rightarrow && & \\
\mathrm{dist}(L_1, L_2) &=& |\mathbf{proj}_{\mathbf{n}}\mathbf{r}| & \\
&=& \left|\mathbf{r} \cdot \frac{\mathbf{n}}{|\mathbf{n}|}\right| & \\
&=& |(0,1,2)\cdot(0,1,0)| & \\
&=& 1 \quad . &
\end{array}$$

12. Find the angle between the line and the plane.

    (a) The line passing through $(-1,-1,-1)$ and $(1,1,1)$ and the plane with equation $z = -1$.

    answer: $\arcsin\left(\frac{\sqrt{3}}{3}\right) \approx 0.61548 \approx 35.2644^\circ$

    (b) The line passing through $(2,3,5)$ and $(3,5,7)$ and the plane passing through $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$. r: $\arcsin\left(\frac{5}{9}\sqrt{3}\right) \approx 1.295 \approx 74.21^\circ$

13. Recall that a regular tetrahedron can be realized using 4 vertices of a cube. Find the angle between an edge of a regular tetrahedron and one of the two sides of the tetrahedron not containing the edge.

    answer: $\arcsin\left(\frac{\sqrt{6}}{3}\right) \approx 0.955 \approx 54.736^\circ$

14. Recall that a regular tetrahedron can be realized using 4 vertices of a cube.
    Find the angle between two faces of a regular tetrahedron.

answer: $\arccos\left(\frac{1}{3}\right) \approx 1.230959 \approx 70.5287555^\circ$