Calculus III
Lecture 10

Todor Milev
https://github.com/tmilev/freecalc

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Outline

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Multivariable Chain Rule Motivation

Recall:

- f, differentiable function,
- $\mathbf{u} = (u_1, u_2, u_3)$, unit vector,
- $P(x_0, y_0, z_0)$, point.

What is the rate of change of f at P in the direction $\mathbf{u}$?

Multivariable Chain Rule Motivation

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Directional derivative

$$(D_{\mathbf{u}} f)(P) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(x_0 + tu_1, y_0 + tu_2, z_0 + tu_3)$$

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More general, if

- $w = w(x, y, z)$;
- $x = x(t)$, $y = y(t)$, $z = z(t)$,

and all the functions are differentiable, how do we compute $\frac{\mathrm{d}w}{\mathrm{d}t}$?

Chain Rule

Differentials

$$\mathrm{d}w = w_x(x,y,z)\mathrm{d}x + w_y(x,y,z)\mathrm{d}y + w_z(x,y,z)\mathrm{d}z$$

and

$$\mathrm{d}x = x'(t)\mathrm{d}t \qquad \mathrm{d}y = y'(t)\mathrm{d}t \qquad \mathrm{d}z = z'(t)\mathrm{d}t\ .$$

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Then

$$\mathrm{d}\left(w\right) = \left(w_x x'(t) + w_y y'(t) + w_z z'(t)\right)\mathrm{d}t$$

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Then

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Therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}(w(x(t),y(t),z(t))) = \frac{\partial w}{\partial x}(x,y,z)\frac{\mathrm{d}x}{\mathrm{d}t}+\frac{\partial w}{\partial y}(x,y,z)\frac{\mathrm{d}y}{\mathrm{d}t}+\frac{\partial w}{\partial z}(x,y,z)\frac{\mathrm{d}z}{\mathrm{d}t}$$

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Derivative of composition of functions $\Longrightarrow$ Chain Rule

Algebra of Chain rule - Tree Diagrams

- $w = w(x, y, z)$;
- $x = x(t)$, $y = y(t)$, $z = z(t)$,

$$\frac{dw}{dt}(t) = \frac{\partial w}{\partial x}(x, y, z)\frac{dx}{dt}(t) + \frac{\partial w}{\partial y}(x, y, z)\frac{dy}{dt}(t) + \frac{\partial w}{\partial z}(x, y, z)\frac{dz}{dt}(t)$$

Alternative way of arranging terms - tree diagram:

$$\begin{tikzcd}
 & w & \\
x \arrow[ur] & y \arrow[u] & z \arrow[ul] \\
t \arrow[u] & t \arrow[u] & t \arrow[u]
\end{tikzcd}$$

More General Chain Rule

More general formula:

- $w = F(x, y, z)$;
- $x = f(u, v)$, $y = g(u, v)$, $z = h(u, v)$.

$$w = F(f(u, v), g(u, v), h(u, v)) = G(u, v)$$

To compute $\frac{\partial w}{\partial u} = \frac{\partial G}{\partial u}$:

- arrange variables in a tree diagram:

$$\begin{tikzcd}
 & & & & w & & & & \\
 & x \arrow[urrr, "F"] & & & y \arrow[u, "F"] & & & z \arrow[ulll, "F"'] & \\
u \arrow[ur, "f"] & & v \arrow[ul, "f"'] & u \arrow[ur, "g"] & & v \arrow[ul, "g"'] & u \arrow[ur, "h"] & & v \arrow[ul, "h"']
\end{tikzcd}$$

Example: powerexponential

Let $f(x) = x^x$. Compute $f'(x)$.

- Calculus I method: logarithmic differentiation or $x^x = e^{x \ln x}$.
- Calculus III method: chain rule.

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- Calculus III method: chain rule.

Let $w = w(u, v) = u^v$ and $u = u(x) = x$, $v = v(x) = x$.

Then $f(x) = w(u(x), v(x))$ and

Directional Derivatives via the Chain Rule

- Let f differentiable function,
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Directional Derivatives via the Chain Rule

- Let f differentiable function,
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What is the rate of change of f at P in the direction $\mathbf{u}$? Answer was studied: directional derivative.

$$(D_{\mathbf{u}} f)_{(x,y,z)=(x_0,y_0,z_0)} = \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} f(x_0 + tu_1, y_0 + tu_2, z_0 + tu_3)$$

$$(D_{\mathbf{u}}f)(x_0, y_0, z_0) = \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} f(x_0 + tu_1, y_0 + tu_2, z_0 + tu_3)$$

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Let $w = f(x, y, z)$ and $\left| \begin{array}{lll} x & = & x_0 + tu_1 \\ y & = & y_0 + tu_2 \\ z & = & z_0 + tu_3 \end{array} \right.$.

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Definition (∇f (“nabla of f”))

$$\nabla f=\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}\right).$$

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Example

Find the directional derivative of $f(x, y, z) = \ln\left(x^2 + 2y^2 - z^2\right)$ at $P(2, 1, -1)$ in the direction $\mathbf{v} = (-1, 2, 1)$.

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A unit vector in the direction of $\mathbf{v}$ is

$$\mathbf{u} = \frac{1}{|\mathbf{v}|}\mathbf{v} = \frac{1}{\sqrt{6}}(-1, 2, 1)\ .$$

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The partial derivatives are

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$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{2x}{x^2+2y^2-z^2} \\ \frac{\partial f}{\partial y} &= \frac{4y}{x^2+2y^2-z^2} \\ \frac{\partial f}{\partial z} &= \frac{-2z}{x^2+2y^2-z^2} \end{aligned}$$

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$$(D_{\mathbf{u}} f)_{|(x,y,z)=(2,1,-1)} = \nabla f_{|(x,y,z)=(2,1,-1)} \cdot \mathbf{u} = \frac{\sqrt{6}}{5}$$

$(D_{\mathbf{u}} f)(2, 1, -1) > 0$ implies that if we start at $(2, 1, -1)$ and move in the direction $\mathbf{u}$, then f is increasing.

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- At a given point P, in which direction does f increase the fastest?

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- magnitude equal to the maximal rate of increase of f at P.
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- Recall that $\nabla f = (f_x, f_y, f_z)$.
- The increase of f in unit direction $\mathbf{u}$ is $D_{\mathbf{u}} f$. We have:

$$(D_{\mathbf{u}} f) = (\nabla f) \cdot \mathbf{u}$$

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- In view of preceding thm., the gradient of f is denoted by ∇f.

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If $\mathbf{u}$ is a unit vector, $\gamma(t_0) = P$ and $\gamma'(t_0) = \mathbf{u}$, then:

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Gradient in Polar Coordinates

$\mathbf{e}_r = \mathbf{e}_r(P)$ and $\mathbf{e}_\theta = \mathbf{e}_\theta(P)$ are the polar fundamental directions at P

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To compute $(D_{\mathbf{e}_r} f)(P)$ we use the line through $P(r_0, \theta_0)$ with direction $\mathbf{e}_r$, which in polar coordinates is given by $(r, \theta) = (t, \theta_0)$. Therefore

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To compute $(D_{\mathbf{e}_\theta} f)(P)$ we use the circle centered at the origin and passing through $P(r_0, \theta_0)$. The polar parametrization of this circle that has *unit* tangent at P is given by $(r, \theta) = (r_0, \frac{1}{r_0} t)$. Therefore

$$b = (D_{\mathbf{e}_\theta} f)(P) = \left.\frac{d}{dt}\right|_{t=\theta_0} f\left(r_0, \frac{1}{r_0} t\right) = \frac{1}{r_0}\frac{\partial f}{\partial \theta}(P) .$$

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From the previous computations:

$$(\nabla f)_P = \frac{\partial f}{\partial r}(P)\mathbf{e}_r + \frac{1}{r_0}\frac{\partial f}{\partial \theta}(P)\mathbf{e}_\theta \ ,$$

or

$$\nabla = \mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} \ .$$

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Let f be a function on the plane such that f depends only on the distance to a fixed point, O.

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Problem: Let $\mathbf{X}$ be a vector field of the form

$$\mathbf{X} = h(r)\mathbf{r}$$

for some continuous function h. Show that $\mathbf{X}$ is a *gradient field*: there exists a smooth function f such that $\mathbf{X} = \nabla f$.

Gravity and Gradient

- Let an object move along surface $z = f(x, y)$.
- Let gravity $\mathbf{G}$ be constant, $\mathbf{G} = -mg\,\mathbf{k}$.
- Normal to surface:

$$\mathbf{n} = (-f_x(x_0, y_0), -f_y(x_0, y_0), 1) = -\nabla f + \mathbf{k}$$

- Let $\mathbf{F}$ be the component of $\mathbf{G}$ effectively acting on the object. Object is restricted to the surface $\Rightarrow$ $\mathbf{F}$ is the component of $\mathbf{G}$ tangent to the surface.

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Gravity pulls object in the direction of fastest descent.

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The two-variable differential operators $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are the maps from $\mathcal{C}^\infty(D)$ to $\mathcal{C}^\infty(D)$ given by: $\frac{\partial}{\partial x}(f) = \frac{\partial f}{\partial x}$ and $\frac{\partial}{\partial y}(f) = \frac{\partial f}{\partial y}$ for every function $f \in \mathcal{C}^\infty(D)$.

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A differential operator over D is a map from $\mathcal{C}^\infty(D)$ to $\mathcal{C}^\infty(D)$ obtained by sums of operators $\frac{\partial^n}{\partial x^n}\frac{\partial^m}{\partial y^m}$ with coefficients in $\mathcal{C}^\infty(D)$.

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- Analogous definitions exist for functions in n variables.

Example (Derivatives in polar coordinates)

$$\begin{tikzcd}
 & & & z & & & \\
 & x \arrow[urr] & & & & y \arrow[ull] & \\
r \arrow[ur] & & \theta \arrow[ul] & & r \arrow[ur] & & \theta \arrow[ul]
\end{tikzcd}$$

Let $x = r\cos\theta$, $y = r\sin\theta$ and $z = f(x, y)$.

- Compute $\frac{\partial z}{\partial r}$ via $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.
- Express the differential operator $\frac{\partial}{\partial r}$ via $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$.

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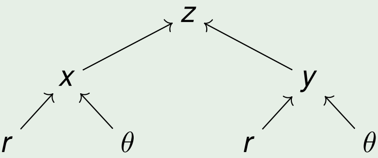

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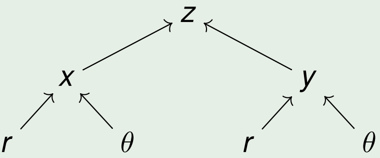
$$\frac{\partial z}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta$$

Example (Derivatives in polar coordinates)



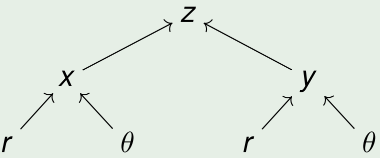
Let $x = r\cos\theta$, $y = r\sin\theta$ and $z = f(x, y)$.

- Compute $\frac{\partial z}{\partial r}$ via $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.
- Express the differential operator $\frac{\partial}{\partial r}$ via $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$.

$$\frac{\partial z}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta$$

Recall that $\frac{\partial z}{\partial x} = \left(\frac{\partial f}{\partial x}\right)(x, y)$ and $\frac{\partial z}{\partial y} = \left(\frac{\partial f}{\partial y}\right)(x, y)$.

Example (Derivatives in polar coordinates)



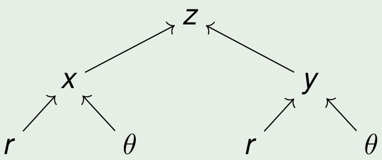
Let $x = r\cos\theta$, $y = r\sin\theta$ and $z = f(x, y)$.

- Compute $\frac{\partial z}{\partial r}$ via $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.
- Express the differential operator $\frac{\partial}{\partial r}$ via $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$.

$$
\begin{aligned}
\frac{\partial z}{\partial r} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta \\
&= \cos\theta\frac{\partial z}{\partial x} + \sin\theta\frac{\partial z}{\partial y} \quad .
\end{aligned}
$$

Recall that $\frac{\partial z}{\partial x} = \left(\frac{\partial f}{\partial x}\right)(x, y)$ and $\frac{\partial z}{\partial y} = \left(\frac{\partial f}{\partial y}\right)(x, y)$.

Example (Derivatives in polar coordinates)



Let $x = r\cos\theta$, $y = r\sin\theta$ and $z = f(x, y)$.

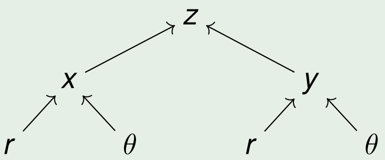
- Compute $\frac{\partial z}{\partial r}$ via $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.
- Express the differential operator $\frac{\partial}{\partial r}$ via $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$.

$$
\begin{aligned}
\frac{\partial z}{\partial r} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta \\
&= \cos\theta\frac{\partial z}{\partial x} + \sin\theta\frac{\partial z}{\partial y} \quad .
\end{aligned}
$$

The above is true for all differentiable $z = f(x, y)$,

Recall that $\frac{\partial z}{\partial x} = \left(\frac{\partial f}{\partial x}\right)(x, y)$ and $\frac{\partial z}{\partial y} = \left(\frac{\partial f}{\partial y}\right)(x, y)$.

Example (Derivatives in polar coordinates)



Let $x = r\cos\theta$, $y = r\sin\theta$ and $z = f(x, y)$.

- Compute $\frac{\partial z}{\partial r}$ via $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.
- Express the differential operator $\frac{\partial}{\partial r}$ via $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$.

$$\begin{aligned}\frac{\partial z}{\partial r} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta \\ &= \cos\theta\frac{\partial z}{\partial x} + \sin\theta\frac{\partial z}{\partial y} \quad .\end{aligned}$$

The above is true for all differentiable $z = f(x, y)$, therefore

$$\frac{\partial}{\partial r} = \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y} \quad .$$

Example (Derivatives in polar coordinates)

$$
\begin{tikzcd}
 & & & z & & & \\
 & x \arrow[urr] & & & & y \arrow[ull] & \\
r \arrow[ur] & & \theta \arrow[ul] & & r \arrow[ur] & & \theta \arrow[ul]
\end{tikzcd}
$$

Let $x = r\cos\theta$, $y = r\sin\theta$ and $z = f(x, y)$.

- Compute $\frac{\partial z}{\partial \theta}$ via $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.
- Express the differential operator $\frac{\partial}{\partial \theta}$ via $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$.

Example (Derivatives in polar coordinates)

Let $x = r\cos\theta$, $y = r\sin\theta$ and $z = f(x, y)$.

- Compute $\frac{\partial z}{\partial \theta}$ via $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.
- Express the differential operator $\frac{\partial}{\partial \theta}$ via $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$.

$$
\begin{tikzcd}[column sep=small]
 & & & z & & & \\
 & x \arrow[urr] & & & & y \arrow[ull] & \\
r \arrow[ur] & & \theta \arrow[ul] & & r \arrow[ur] & & \theta \arrow[ul]
\end{tikzcd}
$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta}$$

Example (Derivatives in polar coordinates)

$$
\begin{tikzcd}
& & & z & & & \\
& x \arrow[urr] & & & & y \arrow[ull] & \\
r \arrow[ur] & & \theta \arrow[ul] & & r \arrow[ur] & & \theta \arrow[ul]
\end{tikzcd}
$$

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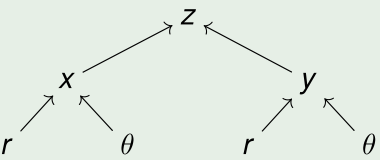
- Express the differential operator $\frac{\partial}{\partial \theta}$ via $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$.

$$\frac{\partial z}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} = \frac{\partial f}{\partial x}(-r\sin\theta) + \frac{\partial f}{\partial y} r\cos\theta$$

Example (Derivatives in polar coordinates)



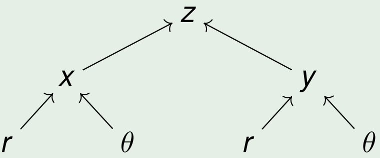
Let $x = r\cos\theta$, $y = r\sin\theta$ and $z = f(x, y)$.

- Compute $\frac{\partial z}{\partial \theta}$ via $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.
- Express the differential operator $\frac{\partial}{\partial \theta}$ via $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$.

$$\frac{\partial z}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} = \frac{\partial f}{\partial x}(-r\sin\theta) + \frac{\partial f}{\partial y} r\cos\theta$$

Recall that $\frac{\partial z}{\partial x} = \left(\frac{\partial f}{\partial x}\right)(x, y)$ and $\frac{\partial z}{\partial y} = \left(\frac{\partial f}{\partial y}\right)(x, y)$.

Example (Derivatives in polar coordinates)



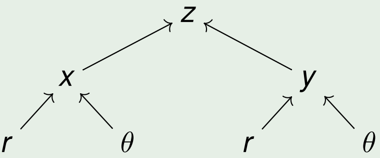
Let $x = r\cos\theta$, $y = r\sin\theta$ and $z = f(x, y)$.

- Compute $\frac{\partial z}{\partial \theta}$ via $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.
- Express the differential operator $\frac{\partial}{\partial \theta}$ via $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$.

$$
\begin{aligned}
\frac{\partial z}{\partial \theta} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} = \frac{\partial f}{\partial x}(-r\sin\theta) + \frac{\partial f}{\partial y} r\cos\theta \\
&= -r\sin\theta\frac{\partial z}{\partial x} + r\cos\theta\frac{\partial z}{\partial y} \quad .
\end{aligned}
$$

Recall that $\frac{\partial z}{\partial x} = \left(\frac{\partial f}{\partial x}\right)(x, y)$ and $\frac{\partial z}{\partial y} = \left(\frac{\partial f}{\partial y}\right)(x, y)$.

Example (Derivatives in polar coordinates)



Let $x = r\cos\theta$, $y = r\sin\theta$ and $z = f(x, y)$.

- Compute $\frac{\partial z}{\partial \theta}$ via $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.
- Express the differential operator $\frac{\partial}{\partial \theta}$ via $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$.

$$
\begin{aligned}
\frac{\partial z}{\partial \theta} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} = \frac{\partial f}{\partial x}(-r\sin\theta) + \frac{\partial f}{\partial y} r\cos\theta \\
&= -r\sin\theta\frac{\partial z}{\partial x} + r\cos\theta\frac{\partial z}{\partial y} \quad .
\end{aligned}
$$

The above is true for all differentiable $z = f(x, y)$,

Recall that $\frac{\partial z}{\partial x} = \left(\frac{\partial f}{\partial x}\right)(x, y)$ and $\frac{\partial z}{\partial y} = \left(\frac{\partial f}{\partial y}\right)(x, y)$.

Example (Derivatives in polar coordinates)

$$
\begin{tikzcd}
 & & & z & & & \\
 & x \arrow[urr] & & & & y \arrow[ull] & \\
r \arrow[ur] & & \theta \arrow[ul] & & r \arrow[ur] & & \theta \arrow[ul]
\end{tikzcd}
$$

Let $x = r\cos\theta$, $y = r\sin\theta$ and $z = f(x, y)$.

- Compute $\frac{\partial z}{\partial \theta}$ via $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.
- Express the differential operator $\frac{\partial}{\partial \theta}$ via $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$.

$$
\begin{aligned}
\frac{\partial z}{\partial \theta} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} = \frac{\partial f}{\partial x}(-r\sin\theta) + \frac{\partial f}{\partial y} r\cos\theta \\
&= -r\sin\theta\frac{\partial z}{\partial x} + r\cos\theta\frac{\partial z}{\partial y} \quad .
\end{aligned}
$$

The above is true for all differentiable $z = f(x, y)$, therefore

$$
\frac{\partial}{\partial \theta} = -r\sin\theta\frac{\partial}{\partial x} + r\cos\theta\frac{\partial}{\partial y} \quad .
$$

Example (Partial Derivatives in Polar Coordinates)

Express $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ via $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$, where $x = r\cos\theta, y = r\sin\theta$.

Example (Partial Derivatives in Polar Coordinates)

Express $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ via $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$, where $x = r\cos\theta, y = r\sin\theta$.
We computed previously that

$$\begin{array}{rcl}\frac{\partial}{\partial r} &=& \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}\\ \frac{\partial}{\partial \theta} &=& -r\sin\theta\frac{\partial}{\partial x} + r\cos\theta\frac{\partial}{\partial y}\end{array} .$$

Example (Partial Derivatives in Polar Coordinates)

Express $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ via $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$, where $x = r\cos\theta, y = r\sin\theta$.
We computed previously that

$$\begin{array}{rcl}\frac{\partial}{\partial r} & = & \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}\\ \frac{\partial}{\partial \theta} & = & -r\sin\theta\frac{\partial}{\partial x} + r\cos\theta\frac{\partial}{\partial y}\end{array} .$$

This is a linear system in $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$.

Example (Partial Derivatives in Polar Coordinates)

Express $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ via $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$, where $x = r\cos\theta, y = r\sin\theta$.
We computed previously that

$$\begin{array}{rcl} \frac{\partial}{\partial r} & = & \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y} \\ \frac{\partial}{\partial \theta} & = & -r\sin\theta\frac{\partial}{\partial x} + r\cos\theta\frac{\partial}{\partial y} \ . \end{array}$$

This is a linear system in $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$. To solve the system, eliminate $\frac{\partial}{\partial x}$ by multiplying the first equality by $r\sin\theta$, the second by $\cos\theta$ and adding the two.

Example (Partial Derivatives in Polar Coordinates)

Express $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ via $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$, where $x = r\cos\theta, y = r\sin\theta$.
We computed previously that

$$\begin{array}{rcl}\frac{\partial}{\partial r} & = & \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}\\ \frac{\partial}{\partial \theta} & = & -r\sin\theta\frac{\partial}{\partial x} + r\cos\theta\frac{\partial}{\partial y}\ .\end{array}$$

This is a linear system in $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$. To solve the system, eliminate $\frac{\partial}{\partial x}$ by multiplying the first equality by $r\sin\theta$, the second by $\cos\theta$ and adding the two. Similarly eliminate $\frac{\partial}{\partial y}$ by multiplying the first equality by $-r\cos\theta$ and the second by $\sin\theta$ and adding the two.

Example (Partial Derivatives in Polar Coordinates)

Express $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ via $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$, where $x = r\cos\theta, y = r\sin\theta$. We computed previously that

$$\begin{array}{rcl} \frac{\partial}{\partial r} & = & \cos\theta \frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y} \\ \frac{\partial}{\partial \theta} & = & -r\sin\theta\frac{\partial}{\partial x} + r\cos\theta\frac{\partial}{\partial y} \end{array} .$$

This is a linear system in $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$. To solve the system, eliminate $\frac{\partial}{\partial x}$ by multiplying the first equality by $r\sin\theta$, the second by $\cos\theta$ and adding the two. Similarly eliminate $\frac{\partial}{\partial y}$ by multiplying the first equality by $-r\cos\theta$ and the second by $\sin\theta$ and adding the two. Finally :

$$\begin{array}{rcl} \frac{\partial}{\partial x} & = & \cos\theta\frac{\partial}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial y} & = & \sin\theta\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\frac{\partial}{\partial \theta} \end{array}$$

Example (Partial Derivatives in Polar Coordinates)

Express $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ via $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$, where $x = r\cos\theta, y = r\sin\theta$.

Example (Partial Derivatives in Polar Coordinates)

Express $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ via $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$, where $x = r\cos\theta, y = r\sin\theta$.

Recall that

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

Example (Partial Derivatives in Polar Coordinates)

Express $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ via $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$, where $x = r\cos\theta, y = r\sin\theta$.

Recall that

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Example (Partial Derivatives in Polar Coordinates)

Express $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ via $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$, where $x = r\cos\theta$, $y = r\sin\theta$.

Recall that

$$\tan\theta = \frac{r\sin\theta}{r\cos\theta} = \frac{y}{x}$$

Example (Partial Derivatives in Polar Coordinates)

Express $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ via $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$, where $x = r\cos\theta$, $y = r\sin\theta$.

Recall that

$$\tan\theta = \frac{r\sin\theta}{r\cos\theta} = \frac{y}{x}$$

Example (Partial Derivatives in Polar Coordinates)

Express $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ via $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$, where $x = r\cos\theta, y = r\sin\theta$.
Suppose $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Recall that

$$\begin{aligned}\tan\theta &= \frac{r\sin\theta}{r\cos\theta} = \frac{y}{x}\\ \theta &= \arctan\left(\frac{y}{x}\right)\end{aligned}$$

Example (Partial Derivatives in Polar Coordinates)

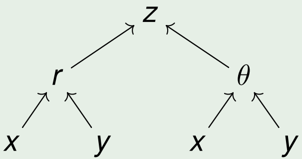
Express $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ via $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$, where $x = r\cos\theta, y = r\sin\theta$. Suppose $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Recall that

$$\begin{aligned}
\tan\theta &= \frac{r\sin\theta}{r\cos\theta} = \frac{y}{x} \\
\theta &= \arctan\left(\frac{y}{x}\right) \\
r &= \sqrt{x^2 + y^2}
\end{aligned}$$

Example (Partial Derivatives in Polar Coordinates)

Express $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ via $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$, where $x = r\cos\theta, y = r\sin\theta$.

Suppose $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Recall that

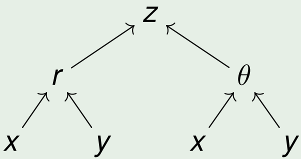


$$\begin{aligned} \tan\theta &= \frac{r\sin\theta}{r\cos\theta} = \frac{y}{x} \\ \theta &= \arctan\left(\frac{y}{x}\right) \\ r &= \sqrt{x^2+y^2} \end{aligned}$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial x}$$

Example (Partial Derivatives in Polar Coordinates)

Express $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ via $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$, where $x = r\cos\theta, y = r\sin\theta$.
Suppose $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Recall that



$$\begin{aligned}
\tan\theta &= \frac{r\sin\theta}{r\cos\theta} = \frac{y}{x}\\
\theta &= \arctan\left(\frac{y}{x}\right)\\
r &= \sqrt{x^2+y^2}
\end{aligned}$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial x} = \frac{\partial z}{\partial r}\textbf{?} \qquad + \frac{\partial z}{\partial \theta}$$

Example (Partial Derivatives in Polar Coordinates)

Express $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ via $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$, where $x = r\cos\theta, y = r\sin\theta$. Suppose $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Recall that

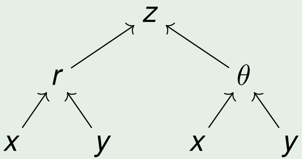
$$
\begin{tikzcd}
 & & & z & & & \\
 & r \arrow[urr] & & & & \theta \arrow[ull] & \\
x \arrow[ur] & & y \arrow[ul] & & x \arrow[ur] & & y \arrow[ul]
\end{tikzcd}
$$

$$
\begin{aligned}
\tan\theta &= \frac{r\sin\theta}{r\cos\theta} = \frac{y}{x}\\
\theta &= \arctan\left(\frac{y}{x}\right)\\
r &= \sqrt{x^2+y^2}
\end{aligned}
$$

$$
\frac{\partial z}{\partial x} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial x} = \frac{\partial z}{\partial r}\,\frac{x}{\sqrt{x^2+y^2}} + \frac{\partial z}{\partial \theta}
$$

Example (Partial Derivatives in Polar Coordinates)

Express $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ via $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$, where $x = r\cos\theta, y = r\sin\theta$.
Suppose $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Recall that



$$\begin{aligned}
\tan\theta &= \frac{r\sin\theta}{r\cos\theta} = \frac{y}{x}\\
\theta &= \arctan\left(\frac{y}{x}\right)\\
r &= \sqrt{x^2+y^2}
\end{aligned}$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial x} = \frac{\partial z}{\partial r}\frac{x}{\sqrt{x^2+y^2}} + \frac{\partial z}{\partial \theta}\,?$$

Example (Partial Derivatives in Polar Coordinates)

Express $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ via $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$, where $x = r\cos\theta, y = r\sin\theta$.
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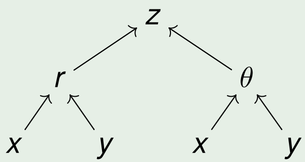
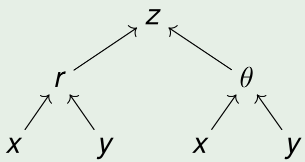


$$
\begin{aligned}
\tan\theta &= \frac{r\sin\theta}{r\cos\theta} = \frac{y}{x}\\
\theta &= \arctan\left(\frac{y}{x}\right)\\
r &= \sqrt{x^2+y^2}
\end{aligned}
$$

$$
\frac{\partial z}{\partial x} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial x} = \frac{\partial z}{\partial r}\ \frac{x}{\sqrt{x^2+y^2}} + \frac{\partial z}{\partial \theta}\ \left(\frac{-y}{x^2+y^2}\right)
$$

Example (Partial Derivatives in Polar Coordinates)

Express $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ via $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$, where $x = r\cos\theta, y = r\sin\theta$. Suppose $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Recall that

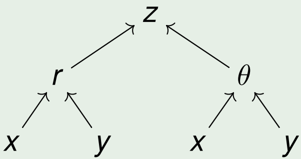


$$\begin{aligned} \tan\theta &= \frac{r\sin\theta}{r\cos\theta} = \frac{y}{x} \\ \theta &= \arctan\left(\frac{y}{x}\right) \\ r &= \sqrt{x^2+y^2} \end{aligned}$$

$$\begin{aligned} \frac{\partial z}{\partial x} &= \frac{\partial z}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial x} = \frac{\partial z}{\partial r}\frac{x}{\sqrt{x^2+y^2}} + \frac{\partial z}{\partial \theta}\left(\frac{-y}{x^2+y^2}\right) \\ &= \cos\theta\frac{\partial z}{\partial r} - \frac{\sin\theta}{r}\frac{\partial z}{\partial \theta} \end{aligned}$$

Example (Partial Derivatives in Polar Coordinates)

Express $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ via $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$, where $x = r\cos\theta, y = r\sin\theta$.
Suppose $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Recall that



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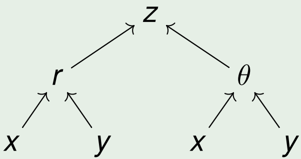

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$$

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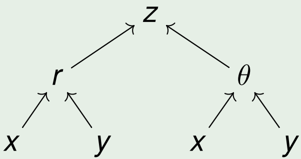

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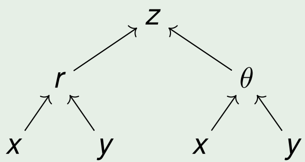

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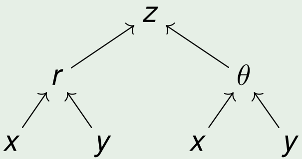

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The above hold for all z,

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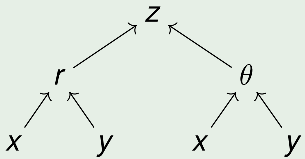

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\end{aligned}
$$

The above hold for all z, therefore
$$
\begin{aligned}
\frac{\partial}{\partial x} &= \cos\theta\frac{\partial}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial}{\partial \theta}\\
\frac{\partial}{\partial y} &= \sin\theta\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\frac{\partial}{\partial \theta}
\end{aligned}.
$$

The Laplace Operator

Definition

The *n*-variable Laplace operator is the differential operator:

$$\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \quad .$$

In paticular the two-variable Laplace operator is:

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

The Laplace operator is named after Pierre Laplace (1749-1827).

Example

Express the Laplace operator $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ in polar coordinates.

$$\Delta = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} .$$

Harmonic Functions

Recall that $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

Definition (Harmonic function definition)

Functions f such that $\Delta f = 0$ are called *harmonic* functions.

Example

The function $f(x, y) = \ln\left(x^2 + y^2\right)$ is a harmonic function.

Harmonic Functions

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The function $f(x, y) = \ln\left(x^2 + y^2\right)$ is a harmonic function. Rewrite in polar coordinates: $f(x, y) = g(r, \theta) = \ln\left(r^2\right) = 2\ln r$. Then

$$\begin{aligned}
\Delta g &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial g}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 g}{\partial \theta^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial(2\ln r)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2(2\ln r)}{\partial \theta^2} \\
&= \frac{1}{r}\frac{\partial}{\partial r}\left(r\cdot\frac{2}{r}\right) = \frac{1}{r}\frac{\partial}{\partial r}(2) = 0 .
\end{aligned}$$

Harmonic Functions

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Fact: The only harmonic functions independent of θ are of the form $g(r, \theta) = c_1 \ln r + c_2$.