Calculus I
Lecture 4
Continuity

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Outline

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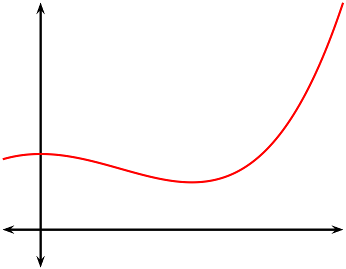
Continuity

- Let f be a function and a be a point in its domain.
- Suppose $\lim_{x\to a} f(x)$ exists.

Definition (Continuous at a Number)

We say that f is continuous at a if

$$\lim_{x\to a} f(x) = f(a).$$



Definition (Discontinuous at a Number)

Suppose f is defined at a. We say f is discontinuous at a if it is not continuous at a.

Physical phenomena are often continuous. The majority of the physical phenomena that are understood are continuous. Examples:

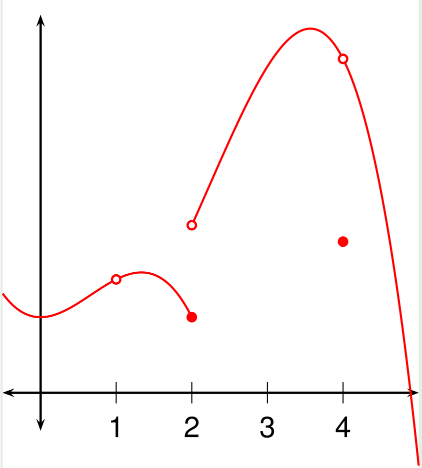
- Motion of a vehicle with respect to time without sudden brakes.
- Orbits of planets and celestial bodies with respect to time.
- A person's height with respect to time.
- And many more.

Discontinuous phenomena examples:

- Particle velocities during collisions and explosions.
- Electric current phenomena, gating events in porins (the event of a molecule passing in and out of a cell).
- Particle physics phenomena.
- And many more.

Example

The picture below shows a graph of a function f. At which numbers is f either discontinuous or not defined? Why?

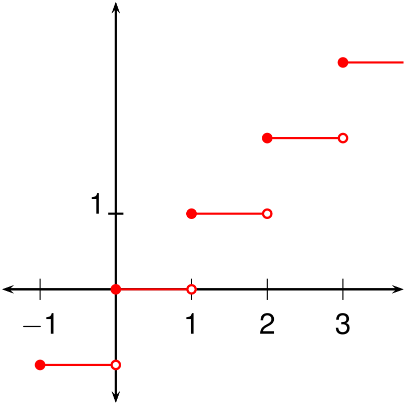


- Not defined at 1:
- $\lim_{x\to 1} f(x)$ exists.
- $f(1)$ is not defined.
- Discontinuous at 2:
- $f(2)$ is defined.
- $\lim_{x\to 2} f(x)$ doesn't exist.
- Discontinuous at 4:
- $f(4)$ is defined.
- $\lim_{x\to 4} f(x)$ exists.
- $\lim_{x\to 4} f(x) \neq f(4)$.

Definition (Greatest Integer Function)

The *greatest integer function* $\lfloor x \rfloor$ is defined as the largest integer that is less than or equal to x.

In computer science this function is called the *floor* function.

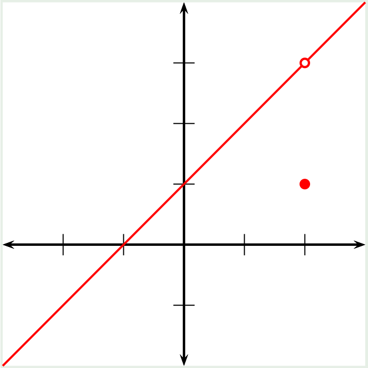


$$\begin{aligned} \lfloor 4 \rfloor &= 4 \\ \lfloor 4.8 \rfloor &= 4 \\ \lfloor \pi \rfloor &= 3 \\ \lfloor \sqrt{2} \rfloor &= 1 \\ \left\lfloor -\frac{1}{2} \right\rfloor &= -1 \\ \lfloor -\pi \rfloor &= -4 \end{aligned}$$

Example

Where is this function discontinuous?

$$f(x) = \begin{cases} \frac{x^2-x-2}{x-2} & \text{if} \quad x \neq 2 \\ 1 & \text{if} \quad x = 2 \end{cases}$$

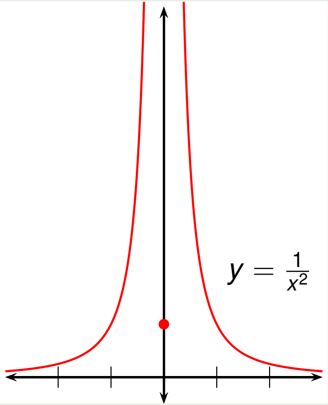


- $f(2)$ is defined ($f(2) = 1$).
- $\lim_{x\to 2} f(x)$ exists ($= 3$).
- $\lim_{x\to 2} f(x) \neq f(2)$.
- Discontinuous at 2.
- This is called a removable discontinuity because we can redefine f at one point to make f continuous.

Example

Where is this function discontinuous?

$$f(x) = \begin{cases} \frac{1}{x^2} & \text{if} \quad x \neq 0 \\ 1 & \text{if} \quad x = 0 \end{cases}$$

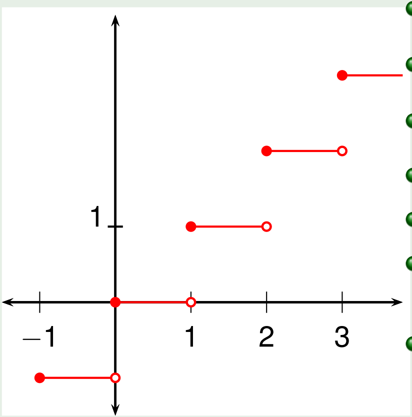


- $f(0)$ is defined ($f(0) = 1$).
- $\lim\limits_{x \to 0} f(x)$ doesn't exist (∞).
- Discontinuous at 0.
- This is called an infinite discontinuity.

Example

Where is this function discontinuous?

$$f(x) = \lfloor x \rfloor$$



- $f(1)$ exists ($f(1) = 1$).
- $\lim\limits_{x\to 1^+} f(x) = 1.$
- $\lim\limits_{x\to 1^-} f(x) = 0.$
- $\lim\limits_{x\to 1} f(x)$ doesn't exist.
- Discontinuous at 1.
- Discontinuous at every integer n.
- The left and right limits both exist but are not equal.
- Such discontinuities are called jump discontinuities (the function appears to "jump").

Definition (Continuous from the Right or Left)

A function f is continuous from the right at a number a if

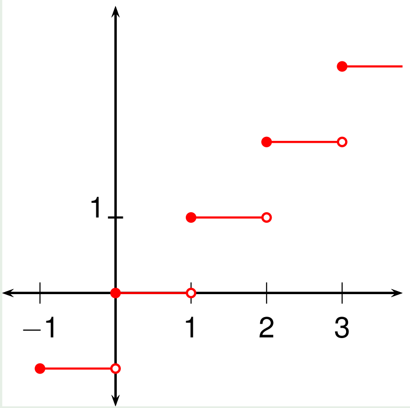
$$\lim_{x \to a^+} f(x) = f(a)$$

and f is continuous from the left at a if

$$\lim_{x \to a^-} f(x) = f(a).$$

Example

Consider $f(x) = \lfloor x \rfloor$, and pick any integer n.



- $f(n) = n$.
- $\lim\limits_{x\to n^+} f(x) = n$.
- Continuous from the right at n.
- $\lim\limits_{x\to n^-} f(x) = n - 1$.
- Discontinuous from the left at n.

Definition (Continuous on an Interval)

A function f is continuous on an interval if it is continuous at every number in the interval.

- If f is defined at the right left endpoint of an interval, continuous means continuous from the left right.
- Think of a function that is continuous on an interval as a function that has no breaks in its graph, and so can be drawn “without lifting your pen”.

Theorem (Algebra of Continuous Functions)

If f and g are continuous at a and c is a constant, then the following functions are also continuous at a:

1. $f+g$
2. $f-g$
3. cf
4. fg
5. $\frac{f}{g}$ *if* $g(a)\neq 0$.

Proof.

$$\lim_{x\to a} f(x) = f(a) \text{ and } \lim_{x\to a} g(x) = g(a).$$

$$\begin{aligned}\lim_{x\to a}(f+g)(x) &= \lim_{x\to a}[f(x)+g(x)] \\ &= \lim_{x\to a} f(x) + \lim_{x\to a} g(x) \qquad \text{(by Law 1)} \\ &= f(a)+g(a) = (f+g)(a)\end{aligned}$$

This shows $f+g$ is continuous at a. The other parts are similar. □

Theorem (Classes of Continuous Functions)

The following types of functions are continuous at every number in their domains:

polynomials	*rational functions*
root functions	*trigonometric functions*

Theorem (Compositions of Continuous Functions)

If g is continuous at a and f is continuous at $g(a)$, then the composition function $f \circ g$ given by $(f \circ g)(x) = f(g(x))$ is continuous at a.

Example

Find $\lim\limits_{x\to -2} \frac{x^3+2x^2-1}{5-3x}$.

The function $f(x) = \frac{x^3+2x^2-1}{5-3x}$ is rational, so is continuous on its domain. Its domain is given by $x \neq \frac{5}{3}$.

$$\begin{aligned}
\lim_{x\to -2} \frac{x^3+2x^2-1}{5-3x} &= \lim_{x\to -2} f(x) \\
&= f(-2) \\
&= \frac{(-2)^3+2(-2)^2-1}{5-3(-2)} \\
&= -\frac{1}{11}
\end{aligned}$$

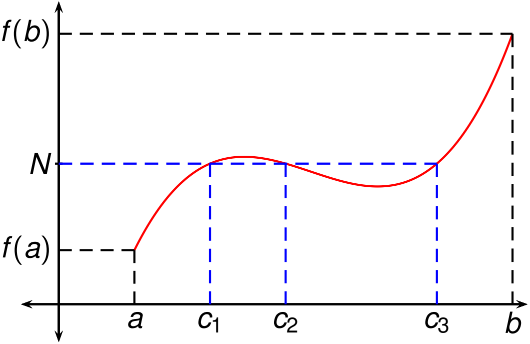
Example

Where is the function $F(x) = \frac{1}{\sqrt{x^2+7}-4}$ continuous?

- We can write F as the composition of 4 functions:
- $F = f \circ g \circ h \circ k$, or $F(x) = f(g(h(k(x))))$.
- $k(x) = x^2 + 7$.
- $h(u) = \sqrt{u}$.
- $g(v) = v - 4$.
- $f(w) = \frac{1}{w}$.
- These functions are continuous on their domains, so F is continuous on its domain.
- Its domain is given by $x \neq 3$ and $x \neq -3$.
- Therefore F is continuous on $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$.

Theorem (The Intermediate Value Theorem)

Suppose f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.



Example

Show that there is a root of the equation
$$4x^3 - 6x^2 + 3x - 2 = 0$$
between 1 and 2.

- Let $f(x) = 4x^3 - 6x^2 + 3x - 2$.
- f is continuous.
- Use the IVT with $a = 1$, $b = 2$, and $N = 0$.
- $f(1) = -1$.
- $f(2) = 12$.
- $f(1) < 0 < f(2)$.
- Therefore there is a c between 1 and 2 such that $f(c) = 0$.