Calculus I
Lecture 0
Representing Functions

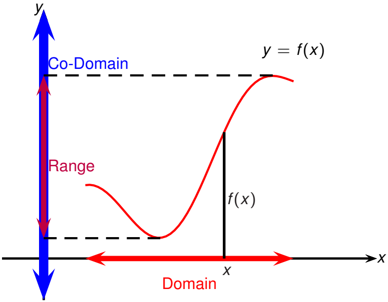
Todor Milev
https://github.com/tmilev/freecalc

2020

Outline

License to use and redistribute

License to use and redistribute



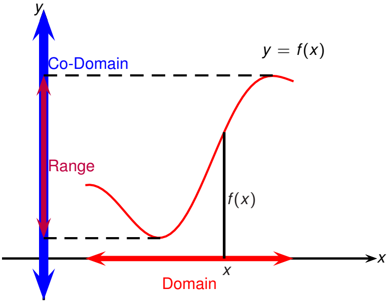
Definition (Function)

A function f is a rule that assigns to each element x in a set D exactly one element, called $f(x)$, in a set E.

- Functions are also synonymously called “maps”.

Definition (Domain)

The set D in the definition of f is called the domain of f.

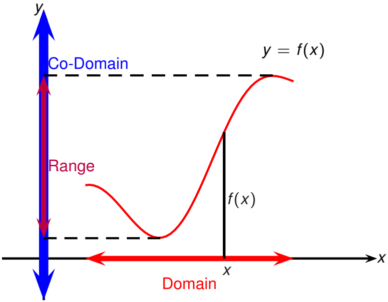


Definition (Function)

A function f is a rule that assigns to each element x in a set D exactly one element, called $f(x)$, in a set E.

Definition (Co-domain)

The set E in the definition of f is called the co-domain of f.



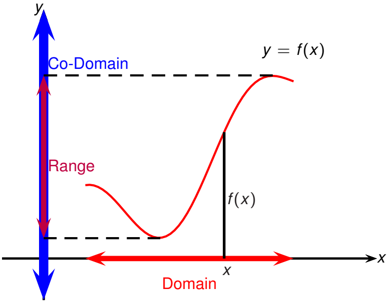
Definition (Function)

A function f is a rule that assigns to each element x in a set D exactly one element, called $f(x)$, in a set E.

Definition (Value of f at x)

The number $f(x)$ is called *the value of f at x* and is read "f of x".

- The value of f at x is also called the image of x under the map f.
- In the expression $f(x)$, x is referred to as the *argument* of f.



Definition (Function)

A function f is a rule that assigns to each element x in a set D exactly one element, called $f(x)$, in a set E.

Definition (Range)

The set of all possible values taken by $f(x)$ as the element x runs over elements of D is called the range of f.

The Vertical Line Test

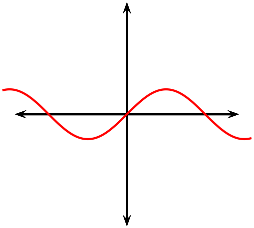
Question

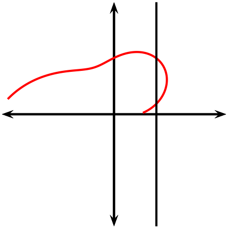
Given a curve in the plane, is it the graph of a function or not?

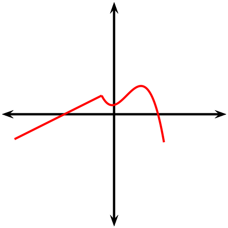
The answer is as follows.

Proposition (The Vertical Line Test)

A curve in the plane is the graph of a function if and only if no vertical line intersects it more than once.





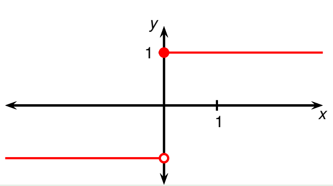


Piecewise Defined Functions

Definition (Piecewise Defined Function)

A piecewise defined function is a function that is defined by different algebraic formulas on different subsets of its domain.

Example



$$f(x) = \begin{cases} 1 & \text{if} \quad x \geq 0 \\ -1 & \text{if} \quad x < 0 \end{cases}$$

The filled red circle means $(0, 1)$ is on the curve.

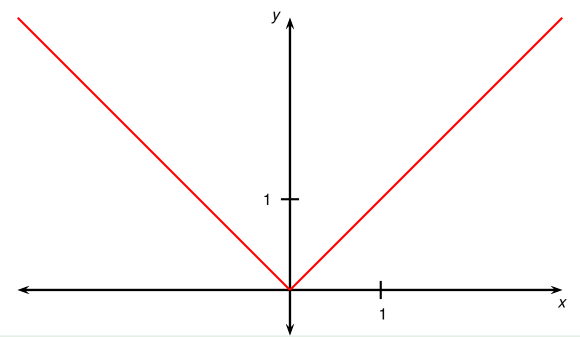
The open circle means $(0, -1)$ is not on the curve.

Example

The absolute value $|x|$ of a number a is defined to be

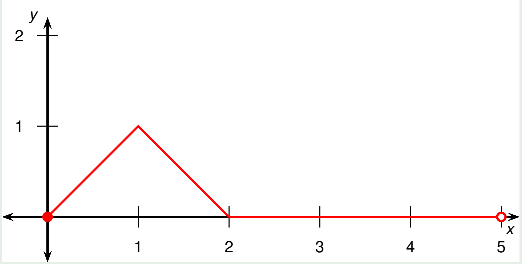
$$|x| = \begin{cases} x & \text{if} \quad x \geq 0 \\ -x & \text{if} \quad x < 0. \end{cases}$$

Sketch a graph of the function $f(x) = |x|$.



Example

Find a formula for the function f whose graph is given below.

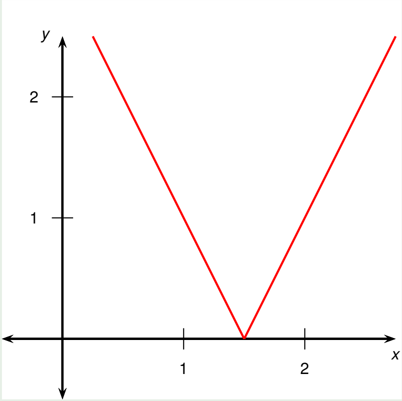


Different formulas on $[0, 1)$, $[1, 2)$, and $[2, 5)$.

$$f(x) = \begin{cases} x & \text{if } 0 \leq x < 1 \\ 2 - x & \text{if } 1 \leq x < 2 \\ 0 & \text{if } 2 \leq x < 5 \end{cases}$$

Example

Sketch the function $f(x) = |2x - 3|$.

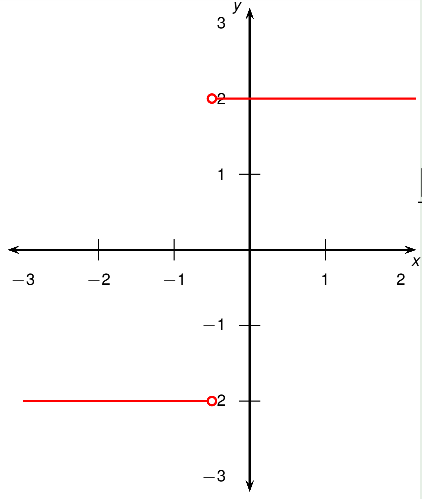


$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

$$\begin{aligned} |2x-3| &= \begin{cases} 2x-3 & \text{if } 2x-3 \geq 0 \\ -(2x-3) & \text{if } 2x-3 < 0 \end{cases} \\ &= \begin{cases} 2x-3 & \text{if } 2x \geq 3 \\ -2x+3 & \text{if } 2x < 3 \end{cases} \\ &= \begin{cases} 2x-3 & \text{if } x \geq 3/2 \\ -2x+3 & \text{if } x < 3/2. \end{cases} \end{aligned}$$

Example

Sketch the function $f(x) = \dfrac{|4x+2|}{2x+1}$.



$$|u| = \begin{cases} u & \text{if } u \geq 0 \\ -u & \text{if } u < 0. \end{cases}$$

$$\frac{|4x+2|}{2x+1} = \begin{cases} \frac{4x+2}{2x+1} & \text{if } 4x+2 > 0 \\ \frac{-(4x+2)}{2x+1} & \text{if } 4x+2 < 0 \end{cases}$$

$$= \begin{cases} \frac{2\cancel{(2x+1)}}{\cancel{2x+1}} & \text{if } 4x > -2 \\ \frac{-2\cancel{(2x+1)}}{\cancel{2x+1}} & \text{if } 4x < -2 \end{cases}$$

$$= \begin{cases} 2 & \text{if } x > -\frac{1}{2} \\ -2 & \text{if } x < -\frac{1}{2}. \end{cases}$$

Symmetry

Definition (Even and Odd Functions)

A function f is called even if $f(-x) = f(x)$ for all x in its domain. A function f is called odd if $f(-x) = -f(x)$ for all x in its domain.

Example (x^2 is Even, x^3 is Odd)

The function $f(x) = x^2$ is even:

$$f(-x) = (-x)^2 = x^2 = f(x).$$

The function $g(x) = x^3$ is odd:

$$g(-x) = (-x)^3 = -x^3 = -g(x).$$

Definition (Even and Odd Functions)

A function f is called even if $f(-x) = f(x)$ for all x in its domain. A function f is called odd if $f(-x) = -f(x)$ for all x in its domain.

Example

Determine whether each of the following functions is even, odd, or neither even nor odd.

$$f(x) = x^5 + x$$

$$\begin{aligned} f(-x) &= (-x)^5 + (-x) \\ &= -x^5 - x \\ &= -(x^5 + x) \\ &= -f(x) \end{aligned}$$

Therefore f is odd.

$$g(x) = 1 - x^4$$

$$\begin{aligned} g(-x) &= 1 - (-x)^4 \\ &= 1 - x^4 \\ &= g(x) \end{aligned}$$

Therefore g is even.

$$h(x) = 2x - 1$$

$$\begin{aligned} h(-x) &= 2(-x) - 1 \\ &= -2x - 1 \\ &\neq h(x), -h(x) \end{aligned}$$

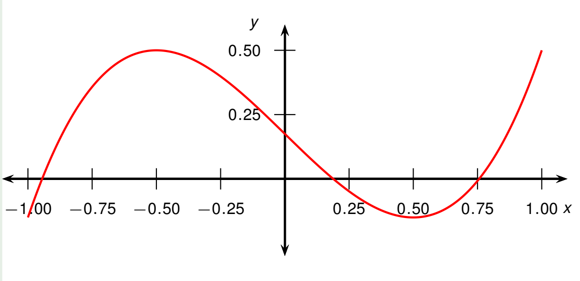
Therefore h is neither even nor odd.

Increasing and Decreasing Functions

Definition (Increasing and Decreasing Functions)

A function f is called increasing on an interval I if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ in I.
It is called decreasing on the interval I if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$ in I.

Example (Increasing and Decreasing)



- f is increasing on $[-1, -\frac{1}{2}]$.
- f is decreasing on $[-\frac{1}{2}, \frac{1}{2}]$.
- f is increasing on $[\frac{1}{2}, 1]$.

A Note on Domains of Functions

If the domain of a function isn't specified, it is implied to be all numbers x for which the formula $f(x)$ is defined. There are some restrictions to consider:

- Can't divide by 0.
- Even roots of a negative number are not defined in this course ($\sqrt{-1}$, $\sqrt[4]{-2053}$, $\sqrt[6]{-15}$ ... not allowed).
- Taking $\log x$ if $x \leq 0$ is not allowed in this course; taking $\log 0$ is not allowed in any course.