Calculus III
Lecture 16

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Outline

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Double Integrals in Polar Coordiantes

- Polar coordinates: $\mathbf{f}: \left| \begin{array}{lcl} x & = & r\cos\theta \\ y & = & r\sin\theta \end{array} \right.$.
- Jacobian: $J_{\mathbf{f}} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$.

Theorem (Integral Variable Change in Polar Coordinates)

$$\iint_{\mathbf{f}(\mathcal{R})} h(x,y)\mathrm{d}x\mathrm{d}y = \iint_{\mathcal{R}} h(r\cos\theta, r\sin\theta)\ r\ \mathrm{d}r\mathrm{d}\theta.$$

Proof.

Apply the variable change theorem:

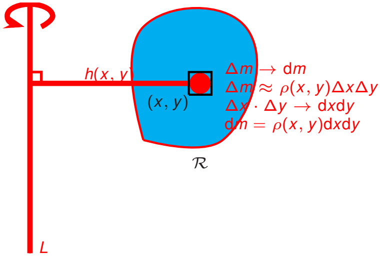
$$\begin{aligned} \iint_{\mathbf{f}(\mathcal{R})} h(x,y)\mathrm{d}x\mathrm{d}y &= \iint_{\mathcal{R}} h(r\cos\theta, r\sin\theta)\det(J_{\mathbf{f}})\mathrm{d}r\mathrm{d}\theta \\ &= \iint_{\mathcal{R}} h(r\cos\theta, r\sin\theta)\, r\mathrm{d}r\mathrm{d}\theta \quad . \end{aligned}$$

$\square$

Moment of Inertia

Definition (Moment of inertia of a mass point)

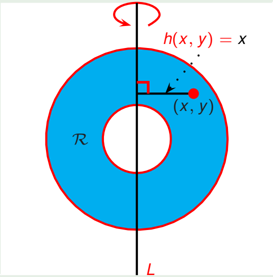
The moment of inertia I of a point mass with mass m rotating about an axis L is defined the quantity $I = h^2 m$, where h is the distance between the point and the axis.



- Fix an (x, y)-coordinate system.
- Let $\mathcal{R}$ be infinitely thin lamina in the plane with variable density $\rho(x, y)$.
- Let $h(x, y)$ be the distance between the point (x, y) and L.
- Let I_{total} denote the total moment of inertia of the lamina.

$$I_{total} = \iint_{\mathcal{R}} \mathrm{d}\left(I(x, y)\right) = \iint_{\mathcal{R}} h^2(x, y)\mathrm{d}m = \iint_{\mathcal{R}} h^2(x, y)\rho(x, y)\mathrm{d}x\mathrm{d}y \quad .$$

Example



Find the moment of inertia of a ring-like lamina with inner radius R_1, outer radius R_2 and constant density ρ, rotating about an axis that passes through its center and lying in the same plane. Let $\mathcal{S}$ = parametr. of $\mathcal{R}$ in polar coordinates.

$$\mathbf{f} \quad : \quad \left| \begin{array}{lcl} x & = & r\cos\theta \\ y & = & r\sin\theta \end{array} \right.$$

$$\begin{array}{lcl} \mathcal{R} & = & \mathbf{f}(\mathcal{S}) \\ \mathcal{S} & = & \{(r,\theta) | R_1 \leq r \leq R_2\} \end{array} \quad .$$

$$I_{total} = \iint_{\mathcal{R}} \rho\, h^2(x,y) \mathrm{d}x\mathrm{d}y = \rho \iint_{\mathcal{R}} x^2 \mathrm{d}x\mathrm{d}y = \rho \iint_{\mathcal{S}} \mathbf{?}(r^2\cos^2\theta) r\mathrm{d}r\mathrm{d}\theta$$

$$= \rho \int_{\theta=0}^{\theta=2\pi} \int_{r=R_1}^{r=R_2} r^3 \cos^2\theta \mathrm{d}r\mathrm{d}\theta = \rho \frac{(R_2^4 - R_1^4)}{4}\pi$$

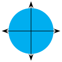
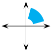
$$= \frac{(R_1^2 + R_2^2)}{4} \rho\pi(R_2^2 - R_1^2) = \left(\frac{R_1^2 + R_2^2}{4}\right) m,$$

where m is the mass of the lamina.

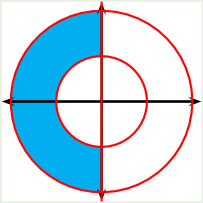
When Do We Use Polar Coordinates?

- Polar coordinates $\mathbf{f}$: $\left| \begin{array}{rcl} x & = & r\cos\theta \\ y & = & r\sin\theta \end{array} \right.$
- Let $\mathcal{S}$= region in (x, y)-coordinates, let $\mathcal{R} = \mathbf{f}(\mathcal{S})$. Recall:

$$\iint_{\mathcal{S}} f(x,y)\mathrm{d}x\mathrm{d}y = \iint_{\mathcal{R}} f(r\cos\theta, r\sin\theta) r \mathrm{d}r \mathrm{d}\theta \ .$$

- The integral may be easier in polar coordinates when:
  - $\mathcal{R} = [0, R] \times [0, 2\pi]$. Here $\mathcal{R}$ is a disk. 
  - $\mathcal{R} = [R_1, R_2] \times [0, 2\pi]$ Here $\mathcal{R}$ is an annulus (ring). 
  - $\mathcal{R} = [0, R] \times [\theta_1, \theta_2]$. Here $\mathcal{R}$ is a sector of disk. 
  - $\mathcal{R} = [R_1, R_2] \times [\theta_1, \theta_2]$. Here $\mathcal{R}$ is a sector of an annulus. 
- The integral may be easier in polar coordinates when $rf(r\cos\theta, r\sin\theta)$ is easier to integrate than $f(x, y)$.

Example



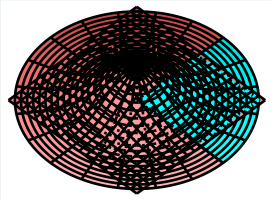
Let $\mathcal{S}$ be the region left of the $y-$axis and between the circles $x^2+y^2=1$ and $x^2+y^2=4$. Compute $\iint_{\mathcal{S}}(x+y)\mathrm{d}x\mathrm{d}y$.

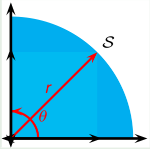
$\mathbf{f}:\left|\begin{array}{rcl} x &=& r\cos\theta \\ y &=& r\sin\theta \end{array}\right.$. Let $\mathcal{R}$ = parametr. of $\mathcal{S}$ in polar coord., $\mathbf{f}(\mathcal{R})=\mathcal{S}$.

$$\mathcal{R}=\left\{(r,\theta)\middle| 1\leq r\leq 2, \frac{\pi}{2}\leq\theta\leq\frac{3\pi}{2}\right\}=[1,2]\times\left[\frac{\pi}{2},\frac{3\pi}{2}\right].$$

$$\begin{aligned}
\iint_{\mathcal{S}}(x+y)\mathrm{d}x\mathrm{d}y &= \iint_{\mathcal{R}} \mathbf{?} r(\sin\theta+\cos\theta)r\,\mathrm{d}r\mathrm{d}\theta \\
&= \int_{\theta=\frac{\pi}{2}}^{\theta=\frac{3\pi}{2}}\int_{r=1}^{r=2} r^2(\sin\theta+\cos\theta)\mathrm{d}r\mathrm{d}\theta \\
&= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}(\sin\theta+\cos\theta)\left[\frac{r^3}{3}\right]_{r=1}^{r=2}\mathrm{d}\theta \\
&= \frac{7}{3}\left[-\cos\theta+\sin\theta\right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}} = -\frac{14}{3}.
\end{aligned}$$

Example (Improper Integrals in Polar Coordinates)





$\mathcal{S}$: first quadrant, $[0,\infty)\times[0,\infty)$.
Compute $\iint_{\mathcal{S}} e^{-x^2-y^2}\,\mathrm{d}x\mathrm{d}y$ and use it to compute $\int_{-\infty}^{\infty} e^{-x^2}\mathrm{d}x$.

$$\iint_{\mathcal{S}} e^{-x^2-y^2}\mathrm{d}x\mathrm{d}y = \int_{\theta=\mathbf{?}0}^{\theta=\mathbf{?}\frac{\pi}{2}} \int_{r=\mathbf{?}0}^{r\to\mathbf{?}\infty} e^{\mathbf{?}-r^2}\, r\mathrm{d}r\mathrm{d}\theta$$

$$= \left(\int_{\theta=0}^{\theta=\frac{\pi}{2}} \mathrm{d}\theta\right)\left(\int_{r=0}^{r\to\infty} re^{-r^2}\mathrm{d}r\right) = \frac{\pi}{2}\left[\frac{-e^{-r^2}}{2}\right]_{r=0}^{r\to\infty} = \frac{\pi}{4}$$

$$\frac{\pi}{4} = \iint_{[0,\infty)\times[0,\infty)} e^{-x^2}e^{-y^2}\mathrm{d}x\mathrm{d}y$$

$$= \left(\int_0^\infty e^{-x^2}\mathrm{d}x\right)\left(\int_0^\infty e^{-y^2}\mathrm{d}y\right) = \left(\int_0^\infty e^{-x^2}\mathrm{d}x\right)^2$$

$$\int_{-\infty}^{\infty} e^{-x^2}\mathrm{d}x = 2\int_0^\infty e^{-x^2}\mathrm{d}x = 2\left(\iint_{\mathcal{S}} e^{-x^2-y^2}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{2}} = 2\sqrt{\frac{\pi}{4}}$$

$$= \sqrt{\pi}.$$

Triple Integrals in Cylindrical Coordinates

- Cylindrical coordinates: $\mathbf{f}: \left| \begin{array}{lcl} x & = & r\cos\theta \\ y & = & r\sin\theta \\ z & = & z \end{array} \right. .$
- Jacobian: $J_{\mathbf{f}} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$

Theorem (Integral Variable Change in Polar Coordinates)

$$\iiint_{\mathbf{f}(\mathcal{R})} h(x,y,z) \mathrm{d}x\mathrm{d}y\mathrm{d}z = \iiint_{\mathcal{R}} h(r\cos\theta, r\sin\theta, z) \ \ r \ \ \mathrm{d}r\mathrm{d}\theta\mathrm{d}z.$$

Proof.

$$\begin{array}{rcl} \iiint_{\mathbf{f}(\mathcal{R})} h(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z & = & \iiint_{\mathcal{R}} h(r\cos\theta, r\sin\theta, z) \det\left(J_{\mathbf{f}}\right) \mathrm{d}r\mathrm{d}\theta\mathrm{d}z \\ & = & \iiint_{\mathcal{R}} h(r\cos\theta, r\sin\theta, z)\, r \mathrm{d}r\mathrm{d}\theta\mathrm{d}z \quad . \end{array}$$

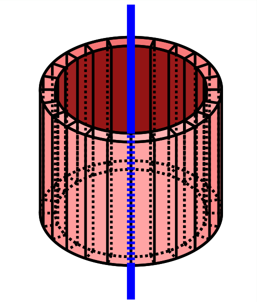
$\square$

Moment of Inertia in Cylindrical Coordinates

- Recall moment of inertia (w.r.t axis L): $I = \text{mass} \cdot \text{distance}_L^2$.
- Introduce Cartesian coordinate system so L is the z-axis.
- Convert to cylindrical coordinates $\mathbf{f}: \left| \begin{array}{lcl} x &=& r\cos\theta \\ y &=& r\sin\theta \\ z &=& z \end{array} \right.$.
- Recall that r is the distance from a point (x, y, z) to the axis L.
- Therefore

$$\begin{array}{rll} I_{total} &= \iiint_{\mathcal{R}} \mathrm{d}I = \iiint_{\mathcal{R}} r^2 \mathrm{d}m & \\ &= \iiint_{\mathcal{R}} r^2 \rho \mathrm{d}V & \\ &= \int_{z=0}^{z=H} \left(\iint_{D_z} \rho r^2 \mathrm{d}x\mathrm{d}y \right) \mathrm{d}z & \Big| \; D_z = \text{horiz. cross-section } \mathcal{R} \\ &= \int_{z=0}^{z=H} \left(\iint_{R_z} \rho r^3 \mathrm{d}r\mathrm{d}\theta \right) \mathrm{d}z & \Big| \; \mathbf{f}(R_z) = D_z \end{array}$$

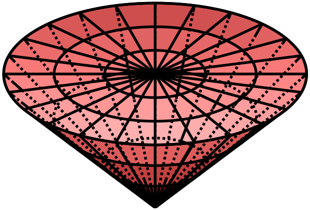
Example (Moment of inertia of cylindrical shell)



Find the moment of inertia of a cylindrical shell whose axis is the z-axis, is of height H, has inner radius R_1, outer radius R_2, and is rotating about the z-axis.

$$
\begin{aligned}
I &= \int_{z=0}^{z=H}\left(\iint_{[R_1,R_2]\times[0,2\pi]} \rho r^2 r\mathrm{d}r\,\mathrm{d}\theta\right)\mathrm{d}z \\
&= \int_{z=0}^{z=H}\left(\int_{r=R_1}^{r=R_2}\left(\int_{\theta=0}^{\theta=2\pi} \rho r^3 \mathrm{d}\theta\right)\mathrm{d}r\right) dz = 2\pi\rho H\frac{R_2^4-R_1^4}{4} \\
&= \rho\pi(R_2^2-R_1^2)H\cdot\frac{R_1^2+R_2^2}{2} = \frac{m(R_1^2+R_2^2)}{2}\ .
\end{aligned}
$$

Example (Center of Mass of Regular Circular Cone)



Find the center of mass of a solid conical body $\mathcal{S}$ of radius R, height H and density $\rho\colon \mathcal{S} \to \mathbb{R}$ proportional to the distance to the axis.

The position vector of the center of mass is

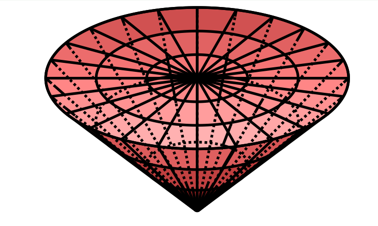
$$\mathbf{r}_C = \frac{1}{M} \iiint_{\mathcal{R}} \mathbf{r} \mathrm{d} m = \frac{1}{M} \iiint_{\mathcal{R}} \mathbf{r} \rho \mathrm{d} V,$$

where

$$M = \iiint_{\mathcal{R}} \mathrm{d} m = \iiint_{\mathcal{R}} \rho(x, y, z) \mathrm{d} V$$

is the mass of the body. It appears that the problem is well suited for a description in cylindrical coordinates.

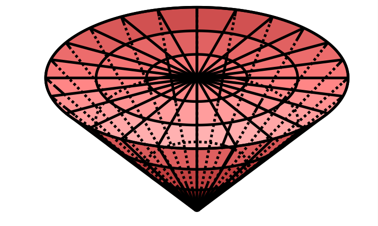
Example (Center of Mass of Regular Circular Cone)



Find the center of mass of a solid conical body $\mathcal{S}$ of radius R, height H and density $\rho\colon \mathcal{S} \to \mathbb{R}$ proportional to the distance to the axis.

- Choose Cartesian coord. system so origin = cone vertex, positive $z-$axis is along axis of the cone.
- Fix cylindrical coordinate system, $\mathbf{f} : \left| \begin{array}{rcl} x &=& r\cos\theta \\ y &=& r\sin\theta \\ z &=& z \end{array} \right.$.
- Let $\mathcal{R}$ be the re-parametrization of $\mathcal{S}$ in cylindrical coordinates: $\mathcal{R} = \{(r,\theta,z) | \mathbf{f}(r,\theta,z) \in \mathcal{S}\}$. We aim to describe $\mathcal{R}$.
- Cone base = disk D, center on z-axis, radius R, in the plane $z = H$. $\Rightarrow \mathcal{R} = \left\{(r,\theta,z) \mid 0 \leq \theta \leq 2\pi, 0 \leq r \leq R, \frac{H}{R} r \leq z \leq H\right\}$.

Example (Center of Mass of Regular Circular Cone)

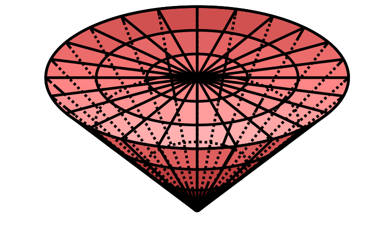


Find the center of mass of a solid conical body $\mathcal{S}$ of radius R, height H and density $\rho\colon \mathcal{S} \to \mathbb{R}$ proportional to the distance to the axis.

We have

$$\begin{aligned} M &= \iiint_{\mathcal{S}} \rho(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z = \iiint_{\mathcal{R}} cr \cdot r \,\mathrm{d}r\, \mathrm{d}\theta \mathrm{d}z \\ &= \iint_{[0,R]\times[0,2\pi]} \left(\int_{z=\frac{Hr}{R}}^{z=H} cr^2 \mathrm{d}z \right) \mathrm{d}r\mathrm{d}\theta \\ &= \int_{\theta=0}^{\theta=2\pi} \left(\int_{r=0}^{r=R} \left(\int_{z=\frac{Hr}{R}}^{z=H} cr^2\, \mathrm{d}z \right) \mathrm{d}r \right) \mathrm{d}\theta = \frac{\pi c H R^3}{6} \ . \end{aligned}$$

Example (Center of Mass of Regular Circular Cone)



Find the center of mass of a solid conical body $\mathcal{S}$ of radius R, height H and density $\rho\colon \mathcal{S} \to \mathbb{R}$ proportional to the distance to the axis.

$M = \frac{\pi c H R^3}{6}$.

The region is symmetric with respect to the axis of the cone and the distribution of mass is symmetric with respect to axis of cone. Therefore the center of mass is also on this axis.

$$z_C = \frac{1}{M}\iiint_{\mathcal{R}} z\rho(P)\mathrm{d}V = \frac{1}{M}\int_{\theta=0}^{\theta=2\pi}\left(\int_{r=0}^{r=R}\left(\int_{z=\frac{Hr}{R}}^{z=H} cr^2 z\mathrm{d}z\right)\mathrm{d}r\right)\mathrm{d}\theta$$

$$= \frac{6}{\pi c H R^3}\cdot\frac{2\pi c H^2 R^3}{15} = \frac{4}{5}H\,.$$

Triple Integrals in Spherical Coordinates

Spherical coordinates: $\mathbf{f}: \left| \begin{array}{rcl} x &=& \rho\sin\phi\cos\theta \\ y &=& \rho\sin\phi\sin\theta \\ z &=& \rho\cos\phi \end{array} \right.$.

$$J_{\mathbf{f}} = \begin{pmatrix} \frac{\partial x}{\partial\rho} & \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial\rho} & \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial\rho} & \frac{\partial z}{\partial\theta} & \frac{\partial z}{\partial z} \end{pmatrix} = \begin{pmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \cos\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{pmatrix}$$

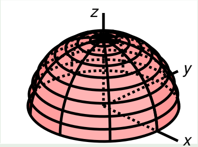
Theorem (Integral Variable Change in Polar Coordinates)

$\iiint_{\mathbf{f}(\mathcal{R})} h(x, y, z)\mathrm{d}x\mathrm{d}y\mathrm{d}z =$
$\iiint_{\mathcal{R}} h(\rho\cos\phi\cos\theta, \rho\cos\phi\sin\theta, \rho\sin\phi)\ \ \rho^2\sin\phi\ \ \mathrm{d}\rho\mathrm{d}\phi\mathrm{d}\theta.$

Proof.

$\iiint_{\mathbf{f}(\mathcal{R})} h(x, y, z)\mathrm{d}x\mathrm{d}y\mathrm{d}z = \iiint_{\mathcal{R}} h(r\cos\theta, r\sin\theta, z)\det(J_{\mathbf{f}})\,\mathrm{d}\rho\mathrm{d}\phi\mathrm{d}\theta =$
$\iiint_{\mathcal{R}} h(\rho\cos\phi\cos\theta, \rho\cos\phi\sin\theta, \rho\sin\phi)\,\rho^2\sin\phi\mathrm{d}\rho\mathrm{d}\phi\mathrm{d}\theta$. $\square$

Example (Centroid of (filled) hemisphere)

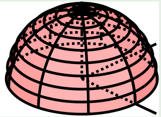


$$\mathbf{f}: \left| \begin{array}{l} x = \rho \sin\phi \cos\theta \\ y = \rho \sin\phi \sin\theta \\ z = \rho \cos\phi \end{array} \right.$$

Find the centroid (geometric center) of a (filled) hemisphere.

- Introduce Cartesian coordinates as illustrated.
- Let the coordinates of the centroid be (x_C, y_C, z_C).
- Region is symmetric with respect to the $z-$axis $\Rightarrow$ centroid is on z-axis $\Rightarrow x_C = y_C = 0$.
- z_C, is the "average" of the z coordinates of the figure:
$$z_C = \frac{1}{\mathrm{Vol}(\mathcal{S})} \iiint_{\mathcal{S}} z \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$
- Let $\mathcal{R}$ be the reparametrization of the region in spherical coordinates, $\mathbf{f}(\mathcal{R}) = \mathcal{S}$.
$\mathcal{R} = \{(\rho, \phi, \theta) | 0 \leq \rho \leq R, 0 \leq \phi \leq \frac{\pi}{2}, 0 \leq \theta \leq 2\pi\}$.

Example (Centroid of (filled) hemisphere)



$$\mathbf{f}: \left| \begin{array}{l} x = \rho \sin\phi \cos\theta \\ y = \rho \sin\phi \sin\theta \\ z = \rho\cos\phi \end{array} \right.$$

Find the centroid (geometric center) of a (filled) hemisphere.

$\mathcal{R} = \{(\rho, \phi, \theta) | 0 \leq \rho \leq R, 0 \leq \phi \leq \frac{\pi}{2}, 0 \leq \theta \leq 2\pi\}$.. Therefore

$$\begin{aligned}
z_C &= \frac{1}{\mathrm{Vol}(\mathcal{R})} \iiint_{\mathcal{R}} z \,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \frac{3}{2\pi R^3} \iiint_{\mathcal{R}} \rho\cos\phi \cdot \rho^2 \sin\phi \,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta \\
&= \frac{3}{2\pi R^3} \int_{\theta=0}^{\theta=2\pi} \left(\int_{\phi=0}^{\phi=\pi/2} \left(\int_{\rho=0}^{\rho=R} \rho^3 \sin\phi\cos\phi \,\mathrm{d}\rho \right) \mathrm{d}\phi \right) \mathrm{d}\theta \\
&= \frac{3}{2\pi R^3} \left(\int_{\theta=0}^{\theta=2\pi} \mathrm{d}\theta \right) \left(\int_{\phi=0}^{\phi=\pi/2} \sin\phi\cos\phi \,\mathrm{d}\phi \right) \left(\int_{\rho=0}^{\rho=R} \rho^3 \,\mathrm{d}\rho \right) \\
&= \frac{3}{2\pi R^3} \cdot 2\pi \cdot \left(\frac{1}{2} \sin^2\phi \Big|_{\phi=0}^{\phi=\pi/2} \right) \cdot \left(\frac{\rho^4}{4} \Big|_{\rho=0}^{\rho=R} \right) = \frac{3}{8} R\,.
\end{aligned}$$