# Calculus II
## Lecture 13

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https://github.com/tmilev/freecalc

2020

# Outline

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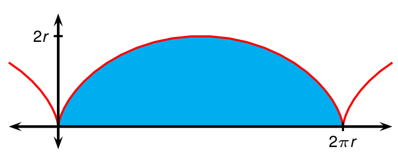
- How do we know where to put $\alpha$ and $\beta$?
- When $x = a$, $t$ will be either $\alpha$ or $\beta$. When $x = b$, $t$ will take the other value.

## Example

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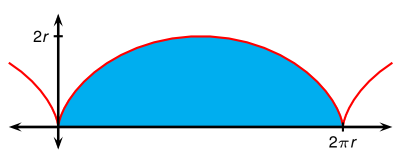


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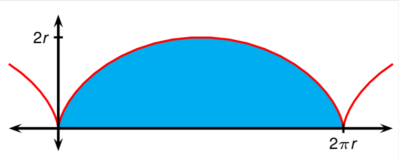


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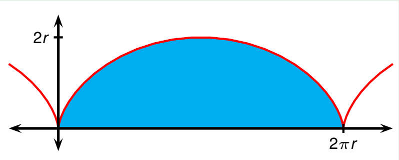


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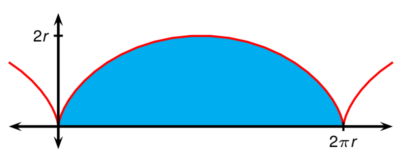


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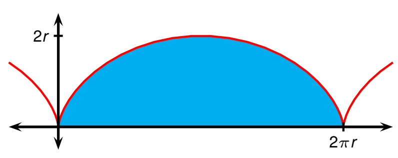


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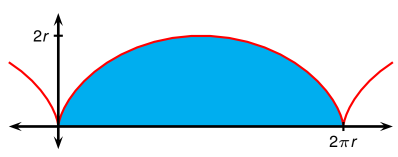


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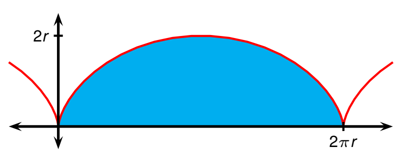


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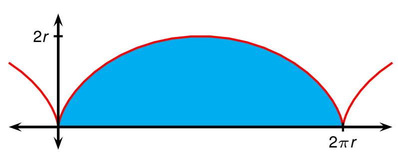


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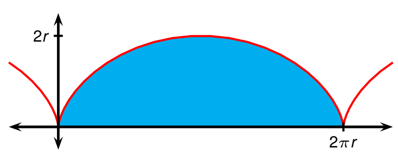


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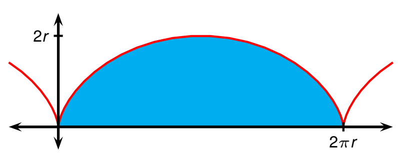


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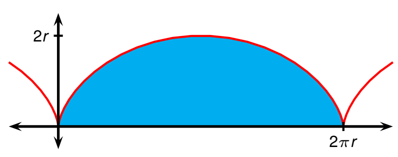


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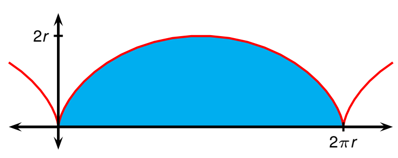


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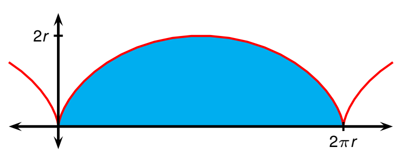


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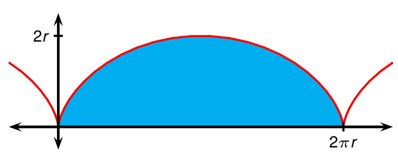


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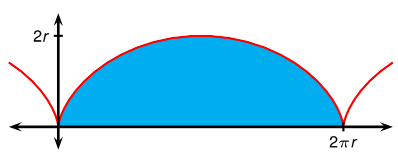


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# Areas in Polar Coordinates

Suppose we have a polar curve $r = f(\theta)$, $a \leq \theta \leq b$.

## Definition

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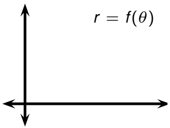

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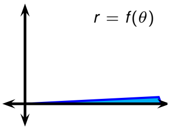

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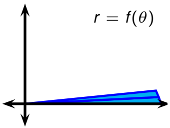

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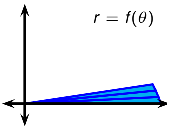

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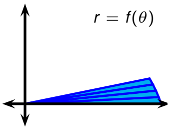

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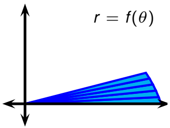

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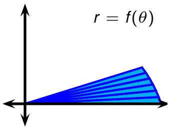

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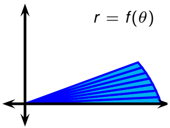

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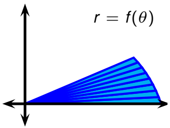

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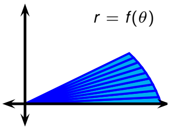

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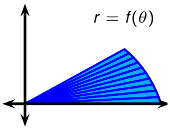

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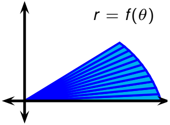

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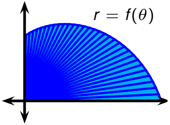

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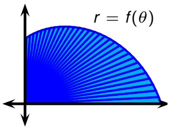
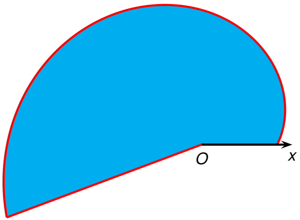



## Theorem

*Suppose no two points on the curve lie on the same ray from the origin. Then the area swept by the curve equals* $A = \displaystyle\int_a^b \frac{1}{2} (f(\theta))^2 \,\mathrm{d}\theta.$

# Area swept by a polar curve: justification



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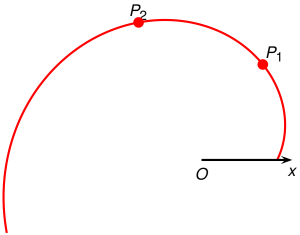
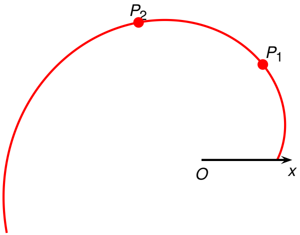



Split $[a, b]$ into $N$ equal segments via points $a = \theta_0 \leq \theta_1 \leq \cdots \leq \theta_{N-1} \leq \theta_N = b$.

# Area swept by a polar curve: justification

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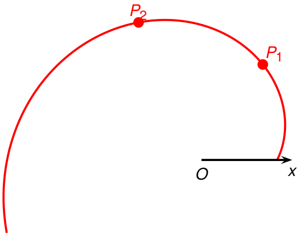


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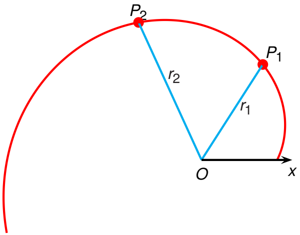


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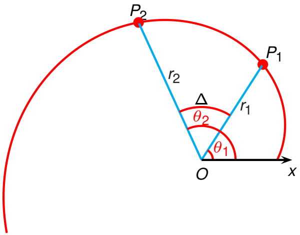


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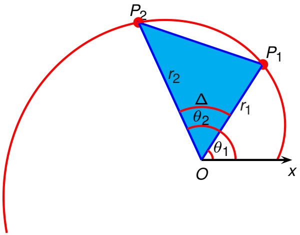


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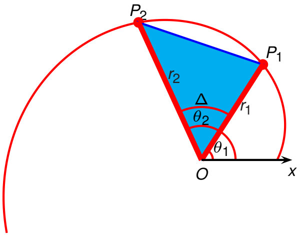


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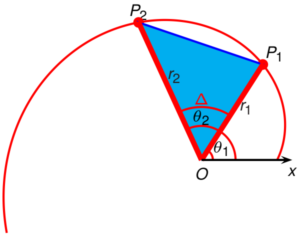


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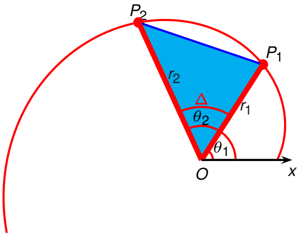


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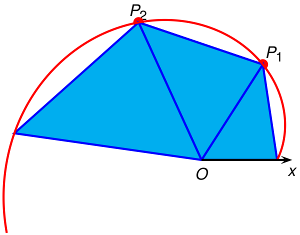


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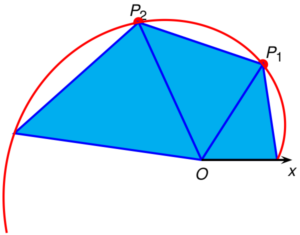


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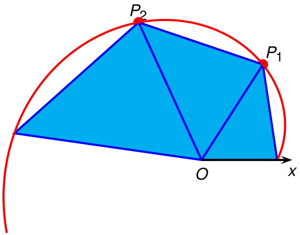


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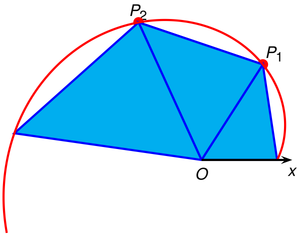


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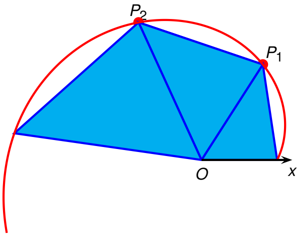


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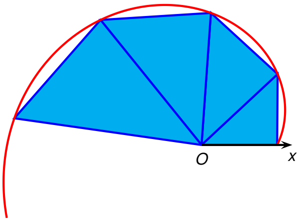


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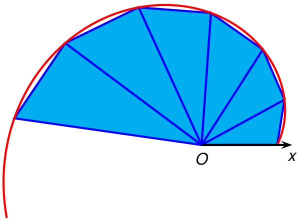


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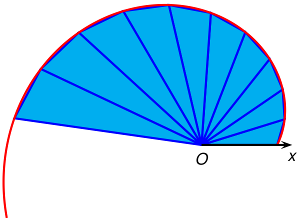


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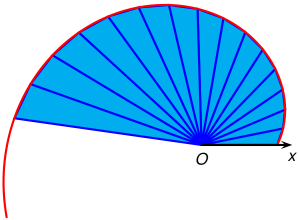


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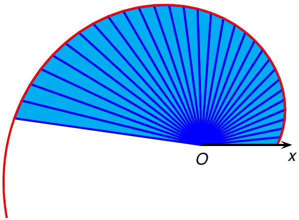


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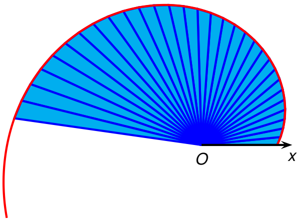


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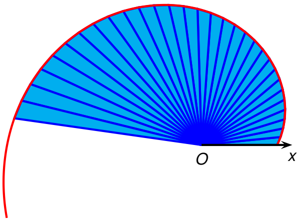


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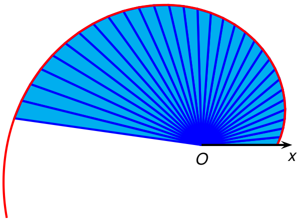


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&= \lim_{\Delta\to 0} \frac{\sin\Delta}{\Delta} \lim_{\Delta\to 0} \sum_{i=0}^{N-1} \frac{f(\theta_i)f(\theta_i+\Delta)\Delta}{2} = 1 \cdot \lim_{\Delta\to 0} \sum_{i=0}^{N-1} \frac{f(\theta_i)f(\theta_i)\Delta}{2}
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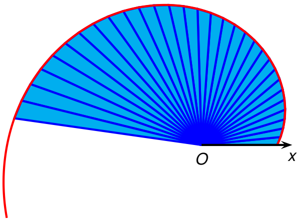


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\text{(can be proved)} &= \lim_{\Delta\to 0} \frac{\sin\Delta}{\Delta} \lim_{\Delta\to 0} \sum_{i=0}^{N-1} \frac{f(\theta_i) f(\theta_i + \Delta)\Delta}{2} = 1 \cdot \lim_{\Delta\to 0} \sum_{i=0}^{N-1} \frac{f(\theta_i) f(\theta_i)\Delta}{2}
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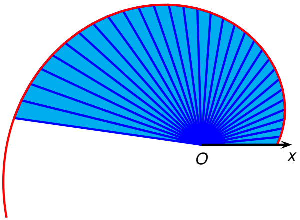


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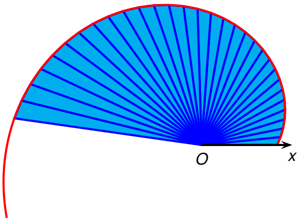


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\begin{aligned}
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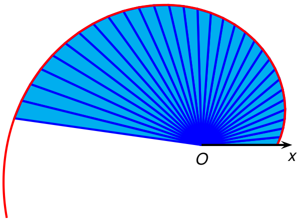
&= \lim_{\Delta \to 0} \frac{\sin \Delta}{\Delta} \lim_{\Delta \to 0} \sum_{i=0}^{N-1} \frac{f(\theta_i) f(\theta_i + \Delta) \Delta}{2} = 1 \cdot \lim_{\Delta \to 0} \sum_{i=0}^{N-1} \frac{f(\theta_i) f(\theta_i) \Delta}{2} \\
&= \lim_{\Delta \to 0} \sum_{i=0}^{N-1} \frac{f^2(\theta_i) \Delta}{2} = ?
\end{aligned}
$$

# Area swept by a polar curve: justification



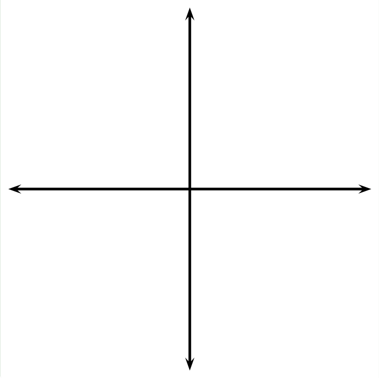

Split $[a, b]$ into $N$ equal segments via points $a = \theta_0 \leq \theta_1 \leq \cdots \leq \theta_{N-1} \leq \theta_N = b$. The length of each segment is $\Delta = \frac{b-a}{N}$. Let $r_i = f(\theta_i)$. Then each $\theta_i$ gives a point $P_i$ with polar coordinates $(r_i, \theta_i)$.

Therefore the area swept by the curve equals the limit of the sum:

$$
\begin{aligned}
A &= \lim_{\Delta\to 0} \sum_{i=0}^{N-1} \frac{f(\theta_i) f(\theta_{i+1}) \sin\Delta}{2} = \lim_{\Delta\to 0} \frac{\sin\Delta}{\Delta} \sum_{i=0}^{N-1} \frac{f(\theta_i) f(\theta_i+\Delta)\Delta}{2} \\
&= \lim_{\Delta\to 0} \frac{\sin\Delta}{\Delta} \lim_{\Delta\to 0} \sum_{i=0}^{N-1} \frac{f(\theta_i) f(\theta_i+\Delta)\Delta}{2} = 1 \cdot \lim_{\Delta\to 0} \sum_{i=0}^{N-1} \frac{f(\theta_i) f(\theta_i)\Delta}{2} \\
\text{(Riemann sum)} &= \lim_{\Delta\to 0} \sum_{i=0}^{N-1} \frac{f^2(\theta_i)\Delta}{2} = \int_a^b \frac{f^2(\theta)}{2} \mathrm{d}\theta
\end{aligned}
$$

## Example

Find the area enclosed by one loop of the four-leaved rose $r = \cos 2\theta$.



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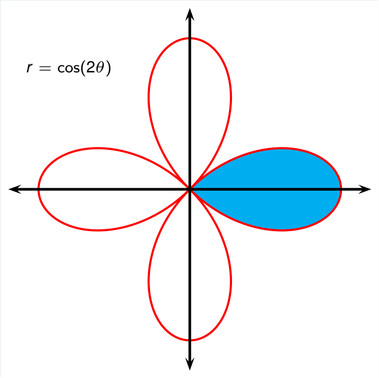


The region enclosed by the right loop corresponds to points whose $\theta$ polar coordinate lies in the interval

$\leq \theta \leq$ **?** .

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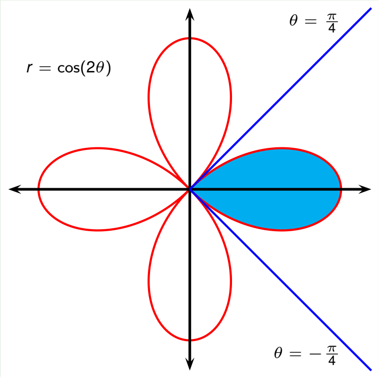


The region enclosed by the right loop corresponds to points whose $\theta$ polar coordinate lies in the interval $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$.

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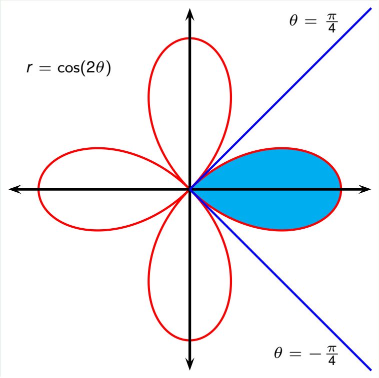


$$A = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2} r^2 \mathrm{d}\theta$$

The region enclosed by the right loop corresponds to points whose $\theta$ polar coordinate lies in the interval $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$.

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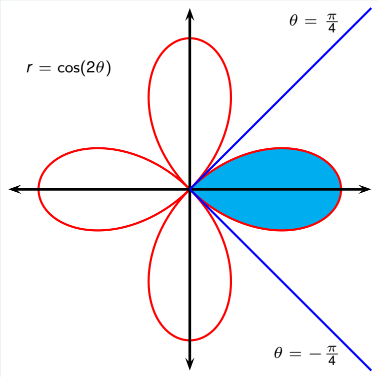


$$\begin{aligned} A &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2} r^2 d\theta \\ &= \frac{1}{2} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^2(2\theta) d\theta \end{aligned}$$

The region enclosed by the right loop corresponds to points whose $\theta$ polar coordinate lies in the interval $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$.

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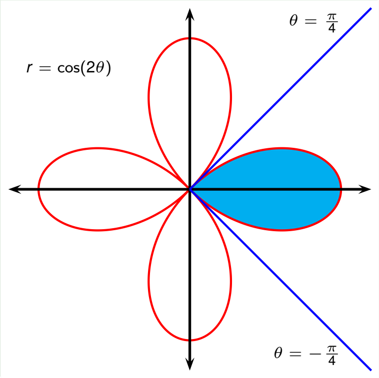


$$
\begin{aligned}
A &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2} r^2 \mathrm{d}\theta \\
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&= \int_{0}^{\frac{\pi}{4}} \cos^2(2\theta) \mathrm{d}\theta
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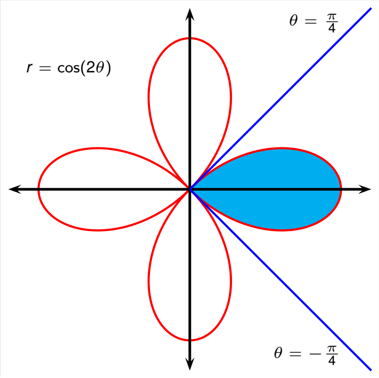


$$
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&= \int_0^{\frac{\pi}{4}} \cos^2(2\theta) \mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{4}} \frac{1}{2}(1 + \cos(4\theta)) \mathrm{d}\theta
\end{aligned}
$$

The region enclosed by the right loop corresponds to points whose $\theta$ polar coordinate lies in the interval $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$.

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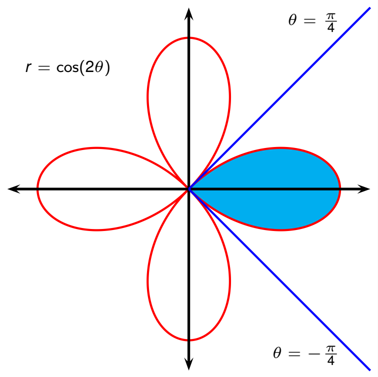


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$$\begin{aligned} A &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2} r^2 \mathrm{d}\theta \\ &= \frac{1}{2} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^2(2\theta) \mathrm{d}\theta \\ &= \int_0^{\frac{\pi}{4}} \cos^2(2\theta) \mathrm{d}\theta \\ &= \int_0^{\frac{\pi}{4}} \frac{1}{2}(1 + \cos(4\theta)) \mathrm{d}\theta \\ &= \frac{1}{2} \left[ \theta + \frac{1}{4} \sin(4\theta) \right]_0^{\frac{\pi}{4}} \end{aligned}$$

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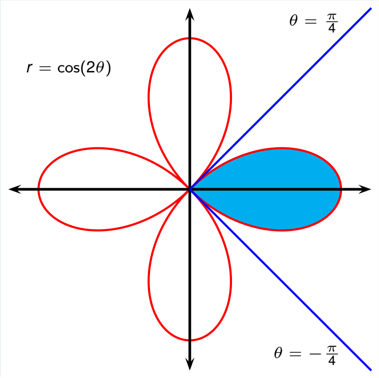


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$$
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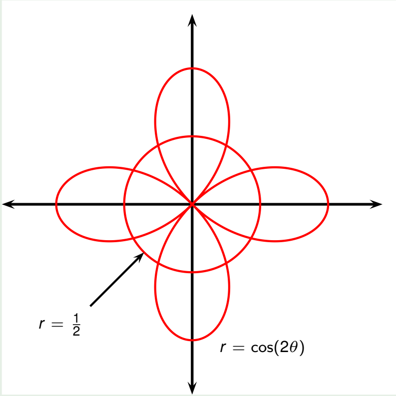
&= \int_{0}^{\frac{\pi}{4}} \frac{1}{2}(1 + \cos(4\theta)) \mathrm{d}\theta \\
&= \frac{1}{2} \left[ \theta + \frac{1}{4} \sin(4\theta) \right]_{0}^{\frac{\pi}{4}} \\
&= \frac{\pi}{8}
\end{aligned}
$$

## Example

Find all points of intersection of the polar curves $r = \frac{1}{2}$ and $r = \cos(2\theta)$.



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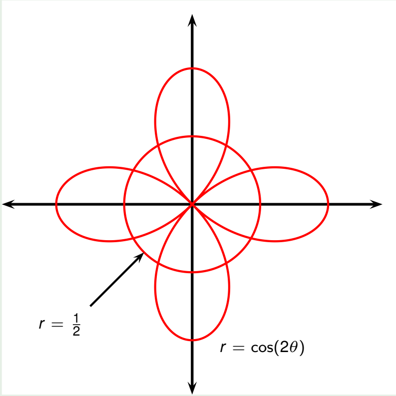


$$\cos 2\theta = \frac{1}{2}$$

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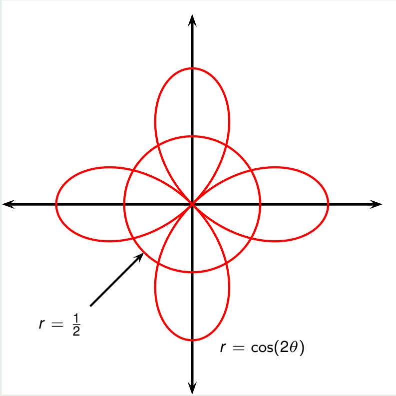


$$\begin{aligned} \cos 2\theta &= \frac{1}{2} \\ 2\theta &= \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3} \end{aligned}$$

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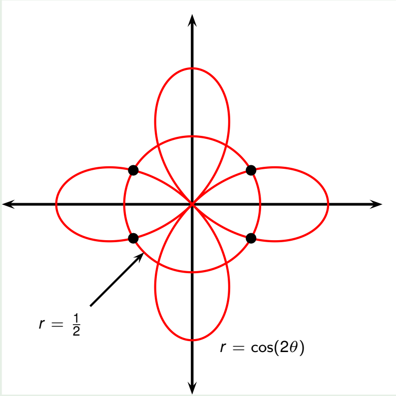


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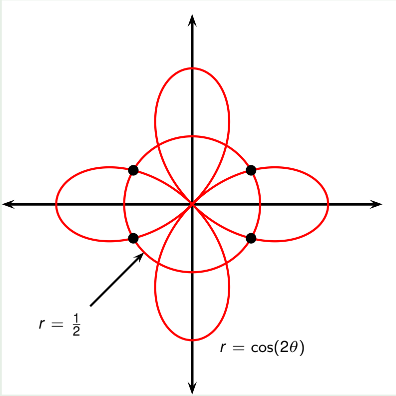


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- This only gives four points.

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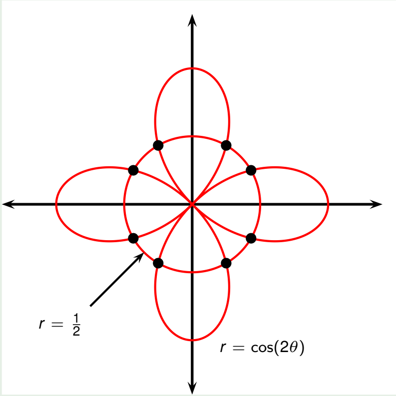


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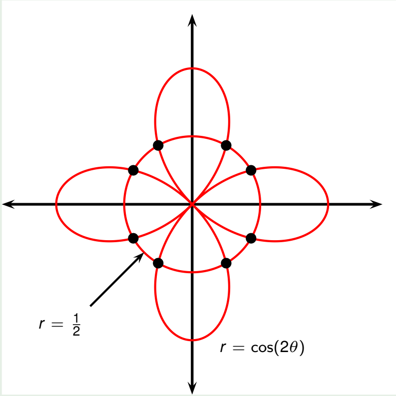


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- This only gives four points.
- There are actually eight.
- The circle $r = \frac{1}{2}$ also has polar equation $r = -\frac{1}{2}$.

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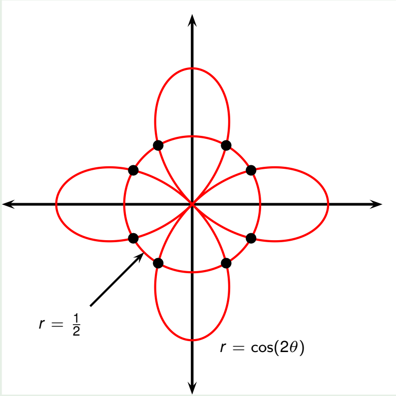


$$\begin{aligned}
\cos 2\theta &= \frac{1}{2} \\
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- This only gives four points.
- There are actually eight.
- The circle $r = \frac{1}{2}$ also has polar equation $r = -\frac{1}{2}$.
- To find all eight points, solve $\cos(2\theta) = \frac{1}{2}$ and $\cos(2\theta) = -\frac{1}{2}$.

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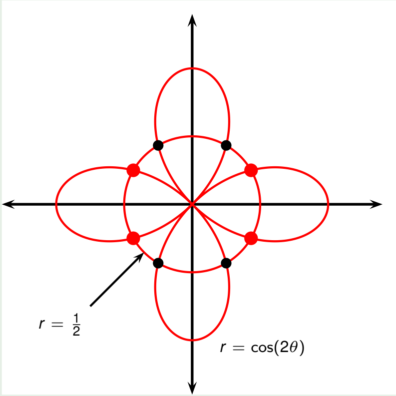


$$\begin{aligned} \cos 2\theta &= \frac{1}{2} \\ 2\theta &= \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3} \\ \theta &= \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6} \end{aligned}$$

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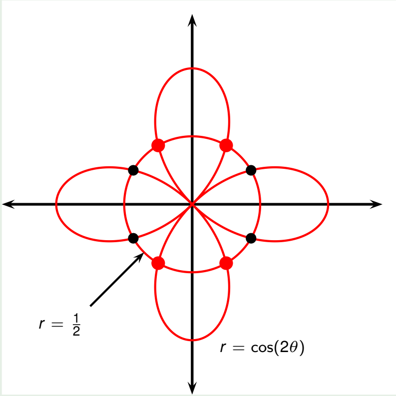


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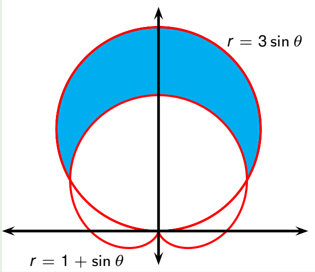
## Example

Find the area that lies within the circle $r = 3\sin\theta$ and outside of the cardioid $r = 1 + \sin\theta$.



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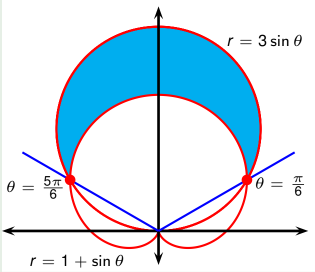


The curves meet if

$$\begin{aligned} 3\sin\theta &= 1 + \sin\theta \\ \sin\theta &= \frac{1}{2} \\ \theta &= \frac{\pi}{6}, \frac{5\pi}{6} \end{aligned}$$

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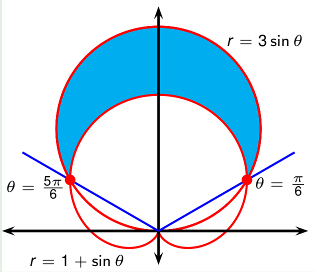


$$A = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (3\sin\theta)^2 d\theta - \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (1+\sin\theta)^2 d\theta$$

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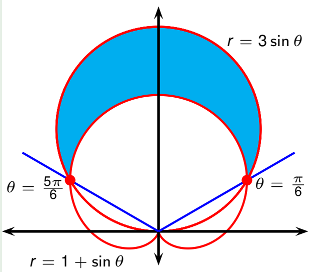


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$$= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(9\sin^2\theta - (1 + 2\sin\theta + \sin^2\theta)\right) d\theta$$

The curves meet if

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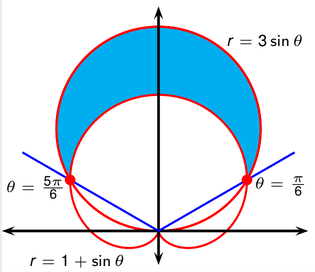


$$
\begin{aligned}
A &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (3\sin\theta)^2 \mathrm{d}\theta - \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (1+\sin\theta)^2 \mathrm{d}\theta \\
&= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(9\sin^2\theta - (1 + 2\sin\theta + \sin^2\theta)\right) \mathrm{d}\theta \\
&= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(8\sin^2\theta - 1 - 2\sin\theta\right) \mathrm{d}\theta
\end{aligned}
$$

The curves meet if

$$
\begin{aligned}
3\sin\theta &= 1 + \sin\theta \\
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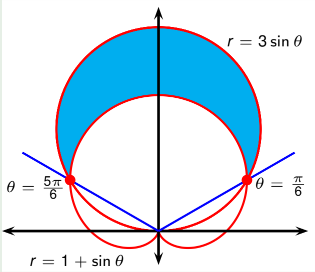


$$
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&= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(8\sin^2\theta - 1 - 2\sin\theta\right) d\theta
\end{aligned}
$$

The curves meet if

$$
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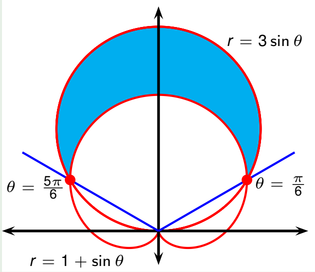


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$$\begin{aligned} A &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (3\sin\theta)^2 d\theta - \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (1+\sin\theta)^2 d\theta \\ &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(9\sin^2\theta - (1 + 2\sin\theta + \sin^2\theta)\right) d\theta \\ &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(8\sin^2\theta - 1 - 2\sin\theta\right) d\theta \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(8\sin^2\theta - 1 - 2\sin\theta\right) d\theta \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(3 - 4\cos 2\theta - 2\sin\theta\right) d\theta \end{aligned}$$

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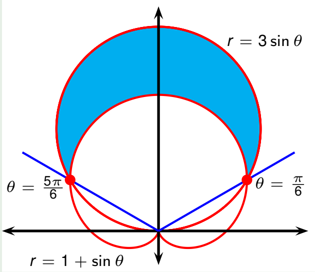


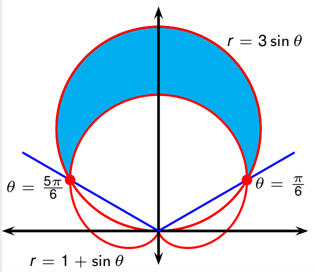

The curves meet if

$$\begin{aligned} 3\sin\theta &= 1 + \sin\theta \\ \sin\theta &= \frac{1}{2} \\ \theta &= \frac{\pi}{6}, \frac{5\pi}{6} \end{aligned}$$

$$\begin{aligned} A &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (3\sin\theta)^2 \mathrm{d}\theta - \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (1+\sin\theta)^2 \mathrm{d}\theta \\ &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(9\sin^2\theta - (1 + 2\sin\theta + \sin^2\theta)\right) \mathrm{d}\theta \\ &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(8\sin^2\theta - 1 - 2\sin\theta\right) \mathrm{d}\theta \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(8\sin^2\theta - 1 - 2\sin\theta\right) \mathrm{d}\theta \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (3 - 4\cos 2\theta - 2\sin\theta) \mathrm{d}\theta \\ &= \left[3\theta - 2\sin 2\theta + 2\cos\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} \end{aligned}$$

## Example

Find the area that lies within the circle $r = 3\sin\theta$ and outside of the cardioid $r = 1 + \sin\theta$.




The curves meet if

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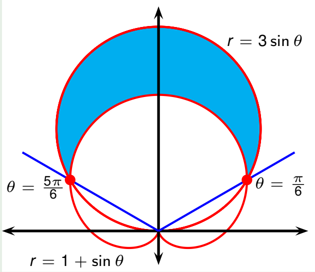
$$\begin{aligned} A &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (3\sin\theta)^2 \mathrm{d}\theta - \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (1+\sin\theta)^2 \mathrm{d}\theta \\ &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(9\sin^2\theta - (1 + 2\sin\theta + \sin^2\theta)\right) \mathrm{d}\theta \\ &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(8\sin^2\theta - 1 - 2\sin\theta\right) \mathrm{d}\theta \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(8\sin^2\theta - 1 - 2\sin\theta\right) \mathrm{d}\theta \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (3 - 4\cos 2\theta - 2\sin\theta)\, \mathrm{d}\theta \\ &= \left[3\theta - 2\sin 2\theta + 2\cos\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} \\ &= \left(3 \quad - 2\cdot \quad + 2\cdot \quad\right) - \left(3 \quad - 2 \quad + 2 \quad\right) \end{aligned}$$

## Example

Find the area that lies within the circle $r = 3\sin\theta$ and outside of the cardioid $r = 1 + \sin\theta$.



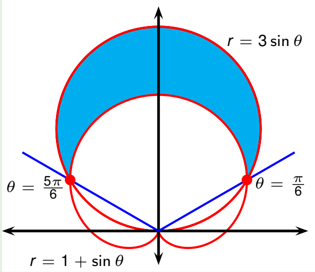

The curves meet if

$$\begin{aligned} 3\sin\theta &= 1+\sin\theta \\ \sin\theta &= \frac{1}{2} \\ \theta &= \frac{\pi}{6}, \frac{5\pi}{6} \end{aligned}$$

$$\begin{aligned} A &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (3\sin\theta)^2 d\theta - \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (1+\sin\theta)^2 d\theta \\ &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(9\sin^2\theta - (1 + 2\sin\theta + \sin^2\theta)\right) d\theta \\ &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(8\sin^2\theta - 1 - 2\sin\theta\right) d\theta \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(8\sin^2\theta - 1 - 2\sin\theta\right) d\theta \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (3 - 4\cos 2\theta - 2\sin\theta)\, d\theta \\ &= \left[3\theta - 2\sin 2\theta + 2\cos\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} \\ &= \left(3\frac{\pi}{2} - 2\cdot \quad + 2\cdot \quad\right) - \left(3 \quad - 2 \quad + 2 \quad\right) \end{aligned}$$

## Example

Find the area that lies within the circle $r = 3\sin\theta$ and outside of the cardioid $r = 1 + \sin\theta$.

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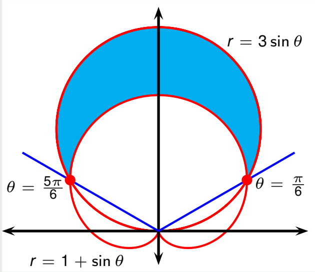
$$\begin{aligned} A &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (3\sin\theta)^2 \mathrm{d}\theta - \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (1+\sin\theta)^2 \mathrm{d}\theta \\ &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(9\sin^2\theta - (1 + 2\sin\theta + \sin^2\theta)\right) \mathrm{d}\theta \\ &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(8\sin^2\theta - 1 - 2\sin\theta\right) \mathrm{d}\theta \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(8\sin^2\theta - 1 - 2\sin\theta\right) \mathrm{d}\theta \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (3 - 4\cos 2\theta - 2\sin\theta) \mathrm{d}\theta \\ &= \left[3\theta - 2\sin 2\theta + 2\cos\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} \\ &= \left(3\frac{\pi}{2} - 2\cdot \quad + 2\cdot \quad\right) - \left(3 \quad - 2 \quad + 2 \quad\right) \end{aligned}$$

## Example

Find the area that lies within the circle $r = 3\sin\theta$ and outside of the cardioid $r = 1 + \sin\theta$.



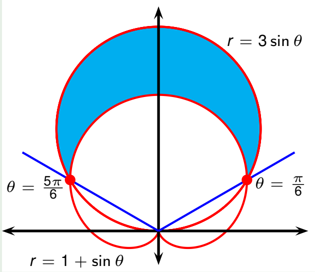

The curves meet if

$$\begin{aligned} 3\sin\theta &= 1 + \sin\theta \\ \sin\theta &= \frac{1}{2} \\ \theta &= \frac{\pi}{6}, \frac{5\pi}{6} \end{aligned}$$

$$\begin{aligned} A &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (3\sin\theta)^2 \mathrm{d}\theta - \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (1+\sin\theta)^2 \mathrm{d}\theta \\ &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(9\sin^2\theta - (1 + 2\sin\theta + \sin^2\theta)\right) \mathrm{d}\theta \\ &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(8\sin^2\theta - 1 - 2\sin\theta\right) \mathrm{d}\theta \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(8\sin^2\theta - 1 - 2\sin\theta\right) \mathrm{d}\theta \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (3 - 4\cos 2\theta - 2\sin\theta) \mathrm{d}\theta \\ &= \left[3\theta - 2\sin 2\theta + 2\cos\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} \\ &= \left(3\frac{\pi}{2} - 2\cdot 0 + 2\cdot \quad\right) - \left(3 \quad - 2 \quad + 2 \quad\right) \end{aligned}$$

## Example

Find the area that lies within the circle $r = 3\sin\theta$ and outside of the cardioid $r = 1 + \sin\theta$.



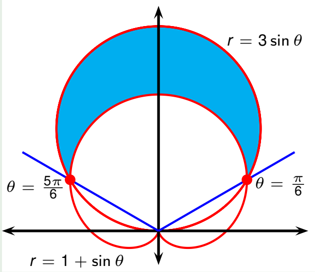

The curves meet if

$$\begin{aligned} 3\sin\theta &= 1 + \sin\theta \\ \sin\theta &= \frac{1}{2} \\ \theta &= \frac{\pi}{6}, \frac{5\pi}{6} \end{aligned}$$

$$\begin{aligned} A &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (3\sin\theta)^2 \mathrm{d}\theta - \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (1+\sin\theta)^2 \mathrm{d}\theta \\ &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(9\sin^2\theta - (1 + 2\sin\theta + \sin^2\theta)\right) \mathrm{d}\theta \\ &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(8\sin^2\theta - 1 - 2\sin\theta\right) \mathrm{d}\theta \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(8\sin^2\theta - 1 - 2\sin\theta\right) \mathrm{d}\theta \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (3 - 4\cos 2\theta - 2\sin\theta)\, \mathrm{d}\theta \\ &= \left[3\theta - 2\sin 2\theta + 2\cos\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} \\ &= \left(3\tfrac{\pi}{2} - 2\cdot 0 + 2\cdot \quad\right) - \left(3 \quad - 2 \quad + 2 \quad\right) \end{aligned}$$

## Example

Find the area that lies within the circle $r = 3\sin\theta$ and outside of the cardioid $r = 1 + \sin\theta$.




The curves meet if

$$\begin{aligned} 3\sin\theta &= 1 + \sin\theta \\ \sin\theta &= \frac{1}{2} \\ \theta &= \frac{\pi}{6}, \frac{5\pi}{6} \end{aligned}$$

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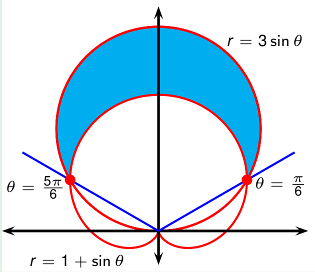
## Example

Find the area that lies within the circle $r = 3\sin\theta$ and outside of the cardioid $r = 1 + \sin\theta$.



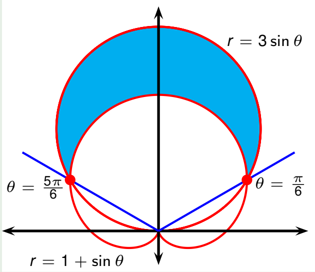

The curves meet if

$$\begin{aligned} 3\sin\theta &= 1 + \sin\theta \\ \sin\theta &= \frac{1}{2} \\ \theta &= \frac{\pi}{6}, \frac{5\pi}{6} \end{aligned}$$

$$\begin{aligned} A &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (3\sin\theta)^2 d\theta - \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (1+\sin\theta)^2 d\theta \\ &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(9\sin^2\theta - (1 + 2\sin\theta + \sin^2\theta)\right) d\theta \\ &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(8\sin^2\theta - 1 - 2\sin\theta\right) d\theta \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(8\sin^2\theta - 1 - 2\sin\theta\right) d\theta \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (3 - 4\cos 2\theta - 2\sin\theta)\, d\theta \\ &= [3\theta - 2\sin 2\theta + 2\cos\theta]_{\frac{\pi}{6}}^{\frac{\pi}{2}} \\ &= \left(3\frac{\pi}{2} - 2\cdot 0 + 2\cdot 0\right) - \left(3 \quad - 2 \quad + 2 \quad\right) \end{aligned}$$

## Example

Find the area that lies within the circle $r = 3\sin\theta$ and outside of the cardioid $r = 1+\sin\theta$.



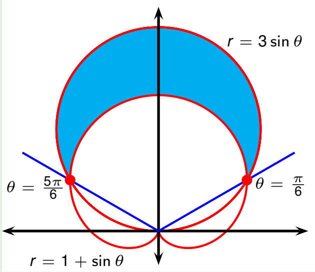

The curves meet if

$$\begin{aligned} 3\sin\theta &= 1+\sin\theta\\ \sin\theta &= \frac{1}{2}\\ \theta &= \frac{\pi}{6}, \frac{5\pi}{6}\end{aligned}$$

$$\begin{aligned} A &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (3\sin\theta)^2 d\theta - \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (1+\sin\theta)^2 d\theta\\ &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(9\sin^2\theta - (1+2\sin\theta+\sin^2\theta)\right) d\theta\\ &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(8\sin^2\theta - 1 - 2\sin\theta\right) d\theta\\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(8\sin^2\theta - 1 - 2\sin\theta\right) d\theta\\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (3 - 4\cos 2\theta - 2\sin\theta)\, d\theta\\ &= [3\theta - 2\sin 2\theta + 2\cos\theta]_{\frac{\pi}{6}}^{\frac{\pi}{2}}\\ &= \left(3\frac{\pi}{2} - 2\cdot 0 + 2\cdot 0\right) - \left(3\frac{\pi}{6} - 2 \qquad + 2 \qquad\right)\end{aligned}$$

## Example

Find the area that lies within the circle $r = 3\sin\theta$ and outside of the cardioid $r = 1 + \sin\theta$.



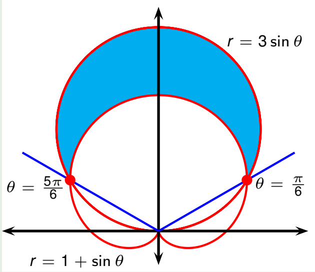

The curves meet if

$$\begin{aligned} 3\sin\theta &= 1 + \sin\theta \\ \sin\theta &= \frac{1}{2} \\ \theta &= \frac{\pi}{6}, \frac{5\pi}{6} \end{aligned}$$

$$\begin{aligned} A &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (3\sin\theta)^2 d\theta - \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (1+\sin\theta)^2 d\theta \\ &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(9\sin^2\theta - (1 + 2\sin\theta + \sin^2\theta)\right) d\theta \\ &= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(8\sin^2\theta - 1 - 2\sin\theta\right) d\theta \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(8\sin^2\theta - 1 - 2\sin\theta\right) d\theta \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (3 - 4\cos 2\theta - 2\sin\theta)\, d\theta \\ &= \left[3\theta - 2\sin 2\theta + 2\cos\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} \\ &= \left(3\frac{\pi}{2} - 2\cdot 0 + 2\cdot 0\right) - \left(3\frac{\pi}{6} - 2 \qquad + 2 \qquad \right) \end{aligned}$$

## Example

Find the area that lies within the circle $r = 3\sin\theta$ and outside of the cardioid $r = 1 + \sin\theta$.

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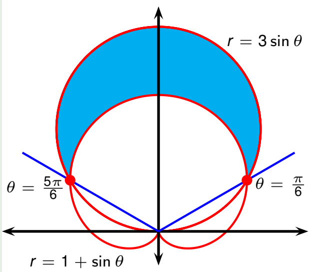
## Example

Find the area that lies within the circle $r = 3\sin\theta$ and outside of the cardioid $r = 1 + \sin\theta$.




The curves meet if

$$\begin{aligned} 3\sin\theta &= 1+\sin\theta \\ \sin\theta &= \frac{1}{2} \\ \theta &= \frac{\pi}{6}, \frac{5\pi}{6} \end{aligned}$$

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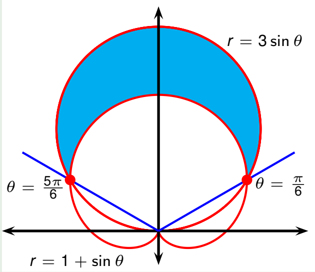
Find the area that lies within the circle $r = 3\sin\theta$ and outside of the cardioid $r = 1 + \sin\theta$.

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## Example

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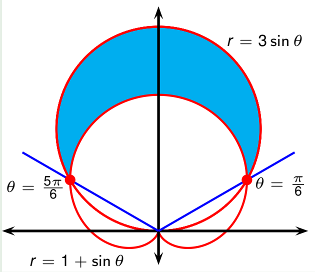


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