Precalculus
Lecture 12
Equations with Logarithms and Exponents

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Outline

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Example

Solve the equation.

$$
\begin{array}{rcll}
\log_3(2x^2+1) & = & 2 & \Big| \text{ Exponentiate base 3} \\
3^{\log_3(2x^2+1)} & = & 3^2 & \\
2x^2+1 & = & 9 & \\
2x^2 & = & 8 & \\
x^2 & = & \dfrac{8}{2} = 4 & \\
x & = & \pm\sqrt{4} = \pm 2 & \\
x = 2 & \text{or} & x = -2 & \Big| \text{ final answer}
\end{array}
$$

The logarithmic property $\log_a(xy) = \log_a x + \log_a y$ holds only for positive x, y. Failure to check the positivity of x, y can result in extraneous (fake) solutions to logarithmic equations.

Example

Solve the equation.

$$
\begin{array}{rcl|l}
\log_2(x+2) + \log_2(x-1) &=& 2 & \text{Domain: } x > 1 \\
\log_2\left((x+2)(x-1)\right) &=& 2 & \text{Exponentiate base 2} \\
(x+2)(x-1) &=& 2^2 & \\
x^2 + x - 2 &=& 4 & \\
x^2 + x - 6 &=& 0 & \\
(x-2)(x+3) &=& 0 &
\end{array}
$$

$$x = 2 \quad \text{or} \quad \cancel{x = -3}$$

$x = -3$ not a solution (outside of domain)

Example (Solve exponential equation without logarithms)

Solve for t.

Find a common base:

$$\begin{array}{rcl} 16^{4t} &=& 8^{t-2} \\ \left(2^4\right)^{4t} &=& \left(2^3\right)^{t-2} \\ 2^{16t} &=& 2^{3t-6} \\ 16t &=& 3t-6 \\ 13t &=& -6 \\ t &=& -\dfrac{6}{13}. \end{array}$$

Example

Solve the equation.

$$
\begin{array}{rcll}
2^{1-5x} & = & 12 & \Big| \text{ apply } \log_2 \\
\log_2(2^{1-5x}) & = & \log_2 12 & \\
1-5x & = & \log_2 12 = \log_2(4\cdot 3) & \\
1-5x & = & \log_2 4 + \log_2 3 & \\
1-5x & = & 2+\log_2 3 & \\
5x & = & 1-(2+\log_2 3) & \\
x & = & \dfrac{-1-\log_2 3}{5} & \\
\text{Calculator:} \quad x & \approx & -0.516993. &
\end{array}
$$

Example

Solve the equation.

$$
\begin{array}{rcll}
e^{x-3} &=& 2e^{2x-1} & \Big|\ \text{Divide by } e^{2x-1} \\
\dfrac{e^{x-3}}{e^{2x-1}} &=& 2 & \\
e^{x-3-(2x-1)} &=& 2 & \\
e^{-x-2} &=& 2 & \Big|\ \text{Apply } \ln \\
-x-2 &=& \ln 2 & \\
-x &=& \ln 2 + 2 & \\
x &=& -(\ln 2 + 2) & \\
x &=& -\ln 2 - 2 & \Big|\ \text{Final answer} \\
x &\approx& -2.693 & \Big|\ \text{Calculator}
\end{array}
$$

Example

Solve.

$$\begin{aligned}
3^{2x+5} &= 5\cdot 2^{-x+1} && \text{Common base} \\
\left(2^{\log_2 3}\right)^{2x+5} &= 5\cdot 2^{-x+1} && a = b^{\log_b a} \\
2^{(\log_2 3)(2x+5)} &= 5\cdot 2^{-x+1} \\
\frac{2^{(\log_2 3)(2x+5)}}{2^{-x+1}} &= 5 \\
2^{(\log_2 3)(2x+5)-(-x+1)} &= 5 && \text{Apply } \log_2 \\
(\log_2 3)(2x+5)+x-1 &= \log_2 5 \\
x\left(2\log_2 3+1\right)+5\log_2 3 &= \log_2 5+1 \\
x &= \frac{\log_2 5+1-5\log_2 3}{2\log_2 3+1} \\
x &\approx -1.1038 && \text{Calculator}
\end{aligned}$$

Example

Solve the equation.

$$
\begin{array}{rrcll}
 & e^{5-3x} & = & 10 & \text{apply } \ln \\
 & & = & & \\
 & & = & & \\
 & x & = & & \\
\text{Calculator:} & x & \approx & 0.8991. &
\end{array}
$$

Example (Solving an exponential word problem)

A farmer buys 48 chickens and 6 rabbits. The chicken population doubles each year, and the rabbit population doubles every six months. When does the farmer have the same number of chickens as rabbits?

Let $c(t)$ denote the number of chickens after t years, and let $r(t)$ denote the number of rabbits after t years.

Solve for t:

$$
\begin{aligned}
&= \\
&= \\
&= \\
&= \\
&= \\
&= \\
t &=
\end{aligned}
$$

Example (Solving a quadratic exponential equation)

Solve for x.

$$\begin{aligned} 9^x &= 2\cdot 3^x + 63 \\ 9^x - 2\cdot 3^x - 63 &= 0 && \Big|\ \text{Substitute } u = 3^x \\ u^2 - 2u - 63 &= 0 \\ (u-9)(u+7) &= 0 \end{aligned}$$

$$\begin{array}{ccccccc} u & = & 9 & \text{or} & u & = & -7 \\ 3^x & = & 9 & \text{or} & 3^x & = & -7 \\ x & = & 2 & & & \text{no real solution} & \end{array}$$

Example

Solve the equation

$$e^{2x} - 3e^{x} - 4 = 0$$

Set $e^{x} = u$. Then $e^{2x} = u^2$.

$$\begin{aligned} u^2 - 3u - 4 &= 0 \\ (u - 4)(u + 1) &= 0 \end{aligned}$$

$$\begin{array}{rcr} u = 4 & \text{or} & u = -1 \\ e^{x} = 4 & \text{or} & e^{x} = -1 \\ x = \ln 4 & \text{or} & \text{no real solution} \\ x \approx 1.3863 & & \end{array}$$

Example

Solve the equation

$$4^{x+1} - 2^{x+2} - 3 = 0$$

Set $u = 2^x$. Then $4^{x+1} = 4u^2$, $2^{x+2} = 4u$.

$$\begin{aligned} 4u^2 - 4u - 3 &= 0 \\ (2u-3)(2u+1) &= 0 \\ 2u - 3 = 0 \quad &\text{or} \quad 2u + 1 = 0 \\ u = \frac{3}{2} \quad &\text{or} \quad u = -\frac{1}{2} \\ 2^x = \frac{3}{2} \quad &\text{or} \quad 2^x = -\frac{1}{2} \end{aligned}$$

$$x = \log_2\left(\frac{3}{2}\right) = \frac{\ln\left(\frac{3}{2}\right)}{\ln 2} \approx 0.58496 \quad \text{or} \quad \text{no real solution}$$

Example (Exponential equation that reduces to quadratic)

Solve the equation.

$$
\begin{aligned}
3^{2x} &= 2 + 63 \cdot 3^{-2x} \\
3^{2x} - 2 - 63 \cdot 3^{-2x} &= 0 \\
u - 2 - 63u^{-1} &= 0 \\
u^2 - 2u - 63 &= 0 \\
(u - 9)(u + 7) &= 0 \\
u - 9 = 0 \quad &\text{or} \quad u + 7 = 0 \\
u = 9 \quad &\text{or} \quad u = -7 \\
3^{2x} = 9 \quad &\text{or} \quad \text{no real solution} \\
2x = \log_3 9 & \\
2x = 2 & \\
x = 1 &
\end{aligned}
$$

>2 terms $\Rightarrow$ transfer one side

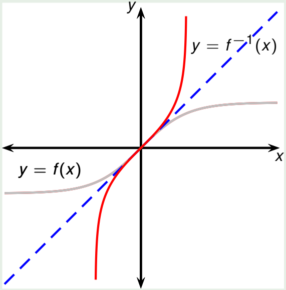
$3^{2x} = u$

$3^{-2x} = \left(3^{2x}\right)^{-1} = u^{-1}$

Multiply $\cdot u$

Example

Find $f^{-1}(x)$ for
$f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$.
$f = \tanh$ = hyperbolic tangent function.



Final answer, relabeled:
$$f^{-1}(x) = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right)$$

$$\frac{e^x - e^{-x}}{e^x + e^{-x}} = y$$

Set $u = e^x$
$e^{-x} = \frac{1}{e^x} = \frac{1}{u}$

$$\frac{\left(u - \frac{1}{u}\right) u}{\left(u + \frac{1}{u}\right) u} = y$$
$$\frac{u^2 - 1}{u^2 + 1} = y$$
$$u^2 - 1 = y(u^2 + 1)$$
$$u^2(1 - y) = 1 + y$$
$$u^2 = \frac{1+y}{1-y}$$
$$(e^x)^2 = \frac{1+y}{1-y}$$
$$e^{2x} = \frac{1+y}{1-y}$$

Take ln

$$x = \frac{1}{2} \ln\left(\frac{1+y}{1-y}\right)$$

Example

Solve the inequality.

$$\begin{aligned} 2^{-3x-5} &< 7 \\ \log_2 2^{-3x-5} &< \log_2 7 \\ -3x-5 &< \log_2 7 \\ -3x &< \log_2 7 + 5 \\ x &> -\frac{\log_2 7 + 5}{3} \\ x &\in \left(-\frac{5+\log_2 7}{3}, \infty\right) \end{aligned}$$

Logarithms preserve inequalities: apply $\log_2$

Division by negative number flips inequalities