

Calculus I

Lecture 13

Implicit Differentiation

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<https://github.com/tmilev/freecalc>

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Outline

1 Implicit Differentiation

2 Related Rates

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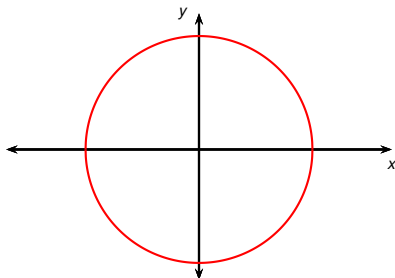
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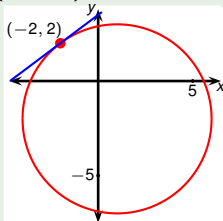
Implicit Differentiation

- So far, we have seen functions with formulas that express one variable explicitly in terms of the other.
- $y = \sqrt{x^3 + 1}$, $y = x \sin x$, etc.
- Some functions are given implicitly by a relation between x and y .
- $x^2 + y^2 = 1$ isn't the equation of any one function.
- Implicitly it gives two functions: $y = \sqrt{1 - x^2}$ and $y = -\sqrt{1 - x^2}$.
- How do we differentiate these functions?
- Differentiate both sides with respect to x , and then solve for y' .



Example

Find an equation of the tangent line to $(x - 1)^2 + (y + 2)^2 = 25$ at $(-2, 2)$.



Find $\frac{dy}{dx}$, given $(x - 1)^2 + (y + 2)^2 = 25$:

$$\frac{d}{dx} \left((x - 1)^2 \right) + \frac{d}{dx} \left((y + 2)^2 \right) = \frac{d}{dx} (25)$$

$$2(x - 1) \frac{d}{dx} (x - 1) + 2(y + 2) \frac{d}{dx} (y + 2) = 0$$

$$2(x - 1)(1) + 2(y + 2) \left(\frac{dy}{dx} \right) = 0$$

$$2(y + 2) \left(\frac{dy}{dx} \right) = 2(1 - x)$$

$$\frac{dy}{dx} = \frac{1 - x}{y + 2}$$

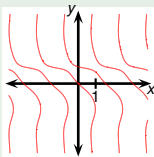
Plug in $(-2, 2)$:

$$\frac{dy}{dx} = \frac{1 - (-2)}{2 + 2} = \frac{3}{4}$$

Point-slope form:

$$y - 2 = \frac{3}{4}(x + 2)$$

Example



Find y' as an expression of x and y .

$$\sin(2(x+y)) = y^2 \cos(2x).$$

$$\frac{d}{dx}(\sin(2(x+y))) = \frac{d}{dx}(y^2 \cos(2x))$$

$$\cos(2(x+y)) \frac{d}{dx}(2(x+y)) = \frac{d}{dx}(y^2) \cos(2x) + (y^2) \frac{d}{dx}(\cos(2x))$$

$$\cos(2(x+y)) (2 + 2y') = 2yy' \cos(2x) + y^2 (-\sin(2x)) \frac{d}{dx}(2x)$$

$$2 \cos(2(x+y)) (1 + y') = 2yy' \cos(2x) - y^2 \sin(2x) 2$$

$$\cos(2(x+y)) + y' \cos(2(x+y)) = yy' \cos(2x) - y^2 \sin(2x)$$

$$y' \cos(2(x+y)) - yy' \cos(2x) = -\cos(2(x+y)) - y^2 \sin(2x)$$

$$y'(\cos(2(x+y)) - y \cos(2x)) = -\cos(2(x+y)) - y^2 \sin(2x)$$

$$y' = \frac{-\cos(2(x+y)) - y^2 \sin(2x)}{\cos(2(x+y)) - y \cos(2x)}.$$

Example

Let $x^4 + y^4 = 16$. Find y'' .

$$4x^3 + 4y^3y' = 0$$

$$y' = -\frac{x^3}{y^3}.$$

$$\begin{aligned}y'' &= \frac{d}{dx} \left(-\frac{x^3}{y^3} \right) = -\frac{\frac{d}{dx}(x^3) y^3 - x^3 \frac{d}{dx}(y^3)}{(y^3)^2} \\&= -\frac{(3x^2)y^3 - x^3(3y^2y')}{y^6} = -\frac{3x^2y^3 - 3x^3y^2\left(-\frac{x^3}{y^3}\right)}{y^6} \\&= -\frac{3x^2\left(y^3 + \frac{x^4}{y}\right)}{y^6} = -\frac{3x^2\left(\frac{y^4 + x^4}{y}\right)}{y^6} \\&= -\frac{3x^2(y^4 + x^4)}{y^7} = -\frac{3x^2(16)}{y^7} = -48\frac{x^2}{y^7}.\end{aligned}$$

Related Rates

- Suppose we are pumping a balloon with air.
- The balloon's volume is increasing.
- The balloon's radius is increasing.
- The rates of increase of these quantities are related to one another.
- It is easier to measure the rate of increase of volume.
- In a related rates problem, we compute the rate of change of one quantity in terms of the rate of change of another (which may be more easily measured).
- Procedure:
 - 1 Find an equation relating the two quantities.
 - 2 Use the Chain Rule to differentiate both sides with respect to time.

Example

Air is being pumped into a balloon such that its volume changes at a rate of $100 \text{ cm}^3/\text{s}$. How fast is the radius of the balloon increasing when the diameter is 50 cm?

- Let V denote the balloon's volume.

$$V = \frac{4}{3}\pi r^3$$

- Let r denote its radius.

$$\frac{dV}{dt} = \frac{d}{dt} \left(\frac{4}{3}\pi r^3 \right)$$

- Given: $\frac{dV}{dt} = 100 \text{ cm}^3/\text{s}$.

$$\frac{dV}{dt} = \frac{d}{dr} \left(\frac{4}{3}\pi r^3 \right) \frac{dr}{dt}$$

- Unknown: $\frac{dr}{dt}$ when $r = 25$ cm.

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

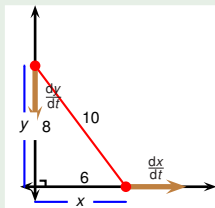
- Find an equation relating the two quantities.

$$\frac{dr}{dt} = \frac{1}{4\pi r^2} \frac{dV}{dt}$$

- Use the Chain Rule to differentiate both sides.

$$\frac{dr}{dt} = \frac{1}{4\pi(25\text{cm})^2} 100 \frac{\text{cm}^3}{\text{s}} = \frac{1}{25\pi} \text{cm/s}$$

Example



10 ft ladder rests against a vertical wall. The bottom of the ladder slides away from the wall at a rate of 1 ft/s. How fast is the ladder top sliding down when the bottom is 6 ft from the wall?

- Let y = dist. from top to ground.
- Let x = dist. from bottom to wall.
- Given: $\frac{dx}{dt} = 1$ ft/s.
- Unknown: $\frac{dy}{dt}$ when $x = 6$ ft.
- Pythagorean Theorem:
 $y = \sqrt{10^2 - 6^2} = 8$.
- Relationship b/n quantities.
- Differentiate (use Chain Rule).

$$\begin{aligned}
 x^2 + y^2 &= 10^2 = 100 \\
 2x \frac{dx}{dt} + 2y \frac{dy}{dt} &= 0 \\
 \frac{dy}{dt} &= -\frac{x}{y} \frac{dx}{dt} \\
 \frac{dy}{dt} &= -\frac{6 \text{ ft}}{8 \text{ ft}} \cdot 1 \text{ ft/s} \\
 &= -3/4 \text{ ft/s.}
 \end{aligned}$$

Therefore the top of the ladder is falling at a rate of 3/4 ft/s.