Calculus II
Lecture 19

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https://github.com/tmilev/freecalc

2020

Outline

1 Power Series

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1. Power Series
2. Power Series as Functions
   - Differentiation and Integration of Power Series

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2. Power Series as Functions
   - Differentiation and Integration of Power Series
3. Taylor and Maclaurin Series

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Power Series

Definition (Power Series)

A power series is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

where x is a variable and the c_n's are constants called the coefficients of the series.

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whose domain is the set of all x for which the series converges.
- f resembles a polynomial, except it has infinitely many terms.

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A series of the form

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- We use the convention that $(x-a)^0 = 1$, even if $x = a$.
- If $x = a$, then all terms are 0 for $n \geq 1$, so the series always converges when $x = a$.

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- Therefore by the Ratio Test the series diverges for all $x \neq 0$.
- Therefore the series only converges for $x = 0$.

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- Therefore by the Ratio Test the series converges for all x.
- Therefore the domain of the function is $(-\infty, \infty)$, or $\mathbb{R}$.

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Theorem (Convergence of Power Series)

For a power series $\sum c_n(x-a)^n$, there are three possibilities:

1. *The series converges only when $x = a$.*
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Definition (Interval of Convergence)

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3. In the third case, the inequality $|x-a| < R$ can be rewritten $a - R < x < a + R$.

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- In general, the Ratio Test (or Root Test) should be used to find the radius of convergence R.
- The Ratio and Root Tests will always fail when x is an endpoint $a - R$ or $a + R$, so the endpoints must be checked with another test.

Example

Find the radius of convergence and interval of convergence of the series $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$.

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- So it converges if $|x| < \frac{1}{3}$ and diverges if $|x| > \frac{1}{3}$.

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- Therefore $R = \frac{1}{3}$.
- If we use $x = \frac{1}{3}$, we get $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$, which is
- If we use $x = -\frac{1}{3}$, we get $\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}}$, which is

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$$= \lim_{n\to\infty} 3|x|\sqrt{\frac{n+1}{n+2}}\cdot\frac{\frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} 3|x|\sqrt{\frac{1+\frac{1}{n}}{1+\frac{2}{n}}} = 3|x|$$

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Recall the geometric series formula:

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Write $\frac{1}{1+x^2}$ as a power series and find the interval of convergence.

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&= \frac{1}{2}\cdot\frac{1}{\left(1-\left(-\frac{x}{2}\right)\right)} = \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n \quad \Bigg| \quad \text{if \& only if } \left|-\tfrac{x}{2}\right| < 1 \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^n \\
&= \frac{1}{2}-\frac{x}{4}+\frac{x^2}{8}-\frac{x^3}{16}+\dots
\end{aligned}$$

To find interval of convergence:

$$\begin{aligned}
\left|-\frac{x}{2}\right| &< 1 \\
|x| &< 2
\end{aligned}$$

Example

Find a power series representation for $\frac{1}{x+2}$.

$$\begin{aligned}\frac{1}{2+x} &= \frac{1}{2\left(1+\frac{x}{2}\right)}\\ &= \frac{1}{2}\cdot\frac{1}{\left(1-\left(-\frac{x}{2}\right)\right)} = \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n \quad \Bigg| \quad \text{if \& only if } \left|-\frac{x}{2}\right| < 1\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^n\\ &= \frac{1}{2}-\frac{x}{4}+\frac{x^2}{8}-\frac{x^3}{16}+\dots\end{aligned}$$

To find interval of convergence:

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Therefore the interval of convergence is $x \in (-2, 2)$.

Example

Find a power series representation for $\frac{x^3}{x+2}$.

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if & only if $|x| < 2$

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&= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^{n+3}
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&= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^{n+3} \\
&= \frac{x^3}{2} - \frac{x^4}{4} + \frac{x^5}{8} - \frac{x^6}{16} + \cdots
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- Another way to write this is $\frac{x^3}{x+2} = \sum_{n=3}^{\infty} \frac{(-1)^{n-1}}{2^{n-2}} x^n$.

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- Another way to write this is $\frac{x^3}{x+2} = \sum_{n=3}^{\infty} \frac{(-1)^{n-1}}{2^{n-2}} x^n$.
- The interval of convergence is again $x \in (-2, 2)$.

Differentiation and Integration of Power Series

Theorem (Differentiation and Integration of Power Series)

If a power series $\sum c_n(x-a)^n$ *has radius of convergence* $R > 0$, *then the function* f *defined by*

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ *and*

1. $$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}.$$
2. $$\int f(x)\,\mathrm{d}x = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots$$ $= C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$.

- This is called term-by-term differentiation and integration.
- Another way of saying it is

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left[\sum_{n=0}^{\infty} c_n(x-a)^n\right] &= \sum_{n=0}^{\infty} \frac{\mathrm{d}}{\mathrm{d}x}\left[c_n(x-a)^n\right] \\
\int \left[\sum_{n=0}^{\infty} c_n(x-a)^n\right] \mathrm{d}x &= \sum_{n=0}^{\infty} \int \left[c_n(x-a)^n\right] \mathrm{d}x
\end{aligned}$$

- We can treat power series like polynomials with infinitely many terms.

Example

Find the derivative of the Bessel function

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

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$$\begin{aligned} J_0(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2} \\ J_0'(x) &= \sum_{n=0}^{\infty} \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}\right) \end{aligned}$$

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- $J_0(x)$ is defined

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- $J_0(x)$ is defined everywhere.

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- $J_0(x)$ is defined everywhere.
- Therefore its derivative $J_0'(x)$ is also defined everywhere.

Example

Find a power series for $\ln(1-x)$ and state its radius of convergence.

$\ln(1-x)$

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$$\begin{aligned} \ln(1-x) &= \int d(\ln(1-x)) = \int (\ln(1-x))' dx \quad \Big| \text{ up to const.} \\ &= \int \textbf{?} \qquad\qquad dx \end{aligned}$$

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$$\begin{aligned} \ln(1-x) &= \int \mathrm{d}(\ln(1-x)) = \int (\ln(1-x))' \mathrm{d}x \quad \Big| \text{ up to const.} \\ &= \int \left(-\frac{1}{1-x}\right) \mathrm{d}x \end{aligned}$$

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- To find C, plug in $x=0$: $C=$ **?**

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- To find C, plug in $x = 0$: $C = 0$.
- Therefore the theorem on integrating power series implies that

  $$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}, \text{ for } |x| < 1.$$

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- To find C, plug in $x = 0$: $C = 0$.
- Therefore the theorem on integrating power series implies that
  $\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}$, for $|x| < 1$.
- By the same theorem, the radius of convergence remains $R = 1$.

Example

Find a power series for $\arctan x$ and state its radius of convergence.

$\arctan(x)$

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$$\begin{array}{rcll} \arctan(x) &=& \int \mathrm{d}(\arctan x) = \int (\arctan x)' \mathrm{d}x & \Big| \text{ up to const.} \\ &=& \int \left(\frac{1}{1+x^2} \right) \mathrm{d}x = \int \left(\frac{1}{1-(-x^2)} \right) \mathrm{d}x & \end{array}$$

Example

Find a power series for $\arctan x$ and state its radius of convergence.

$$\begin{array}{rcll} \arctan(x) & = & \displaystyle\int \mathrm{d}(\arctan x) = \int (\arctan x)' \mathrm{d}x & \Big| \text{ up to const.} \\ & = & \displaystyle\int \left(\frac{1}{1+x^2} \right) \mathrm{d}x = \int \left(\frac{1}{1-(-x^2)} \right) \mathrm{d}x & \\ & = & \displaystyle\int \left(\mathbf{?} \qquad\qquad\qquad \right) \mathrm{d}x & \Big| \text{ for } |x| < 1 \end{array}$$

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$$\begin{aligned}
\arctan(x) &= \int \mathrm{d}(\arctan x) = \int (\arctan x)' \mathrm{d}x && \Big| \text{ up to const.} \\
&= \int \left(\frac{1}{1+x^2}\right) \mathrm{d}x = \int \left(\frac{1}{1-(-x^2)}\right) \mathrm{d}x \\
&= \int \left(1 - x^2 + x^4 - x^6 + \dots\right) \mathrm{d}x && \Big| \text{ for } |x| < 1
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&= \int \left(1 - x^2 + x^4 - x^6 + \cdots \right) \mathrm{d}x && \Big| \text{ for } |x| < 1 \\
&= \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \right)
\end{aligned}
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\end{aligned}$$

Example

Find a power series for $\arctan x$ and state its radius of convergence.

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\arctan(x) &= \int \mathrm{d}(\arctan x) = \int (\arctan x)' \mathrm{d}x && \Big|\ \text{up to const.}\\
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- By the same theorem, the radius of convergence remains $R = 1$.

Taylor and Maclaurin Series

- Let f be a function that can be represented by a power series:
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- Therefore $c_n = \frac{f^{(n)}(a)}{n!}$.

Theorem (Coefficients of a Power Series)

If f has a power series representation at a, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n, \qquad |x-a| < R,$$

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Definition (Taylor Series)

This series is called the Taylor series of f.

The case when $a = 0$ is special enough to have its own name:

Definition (Maclaurin Series)

The Maclaurin series of f is the Taylor series of f centered at $a = 0$. In other words, it is the series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \cdots$$

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Find the Maclaurin series of $f(x) = e^x$ and its radius of convergence.

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$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

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Find the sum of the series

$$\sum_{n=0}^{\infty}(-1)^n\frac{1}{2^n n!} = 1-\frac{1}{2\cdot 1!}+\frac{1}{4\cdot 2!}-\frac{1}{8\cdot 3!}+\cdots$$

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- Just like the Maclaurin series, this series also represents e^x.

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Recall that $e^y = \sum_{n=0}^{\infty} \frac{y^n}{n!}$

Set $y = x - 3$

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The radius of convergence was already computed to be $R = \infty$.

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Therefore $R =$.

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Therefore $R = \infty$.

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Therefore $R = \infty$. It can be shown that this series sums to $\sin x$.

Example

Find the sum of the series

$$\sum_{n=0}^{\infty}(-1)^n\frac{\pi^{2n+1}}{2^{2n+1}(2n+1)!}=\frac{\pi}{2}-\frac{\pi^3}{8\cdot 3!}+\frac{\pi^5}{32\cdot 5!}-\frac{\pi^7}{128\cdot 7!}+\cdots$$

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Find the sum of the series

$$\sum_{n=0}^{\infty}(-1)^n\frac{\pi^{2n+1}}{2^{2n+1}(2n+1)!} = \frac{\pi}{2} - \frac{\pi^3}{8\cdot 3!} + \frac{\pi^5}{32\cdot 5!} - \frac{\pi^7}{128\cdot 7!} + \cdots$$

$$\sin x = \sum_{n=0}^{\infty}$$

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$$\sum_{n=0}^{\infty}(-1)^n\frac{\pi^{2n+1}}{2^{2n+1}(2n+1)!} = \sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)!}\left(\quad\right)^{2n+1}$$

Example

Find the sum of the series

$$\sum_{n=0}^{\infty}(-1)^n\frac{\pi^{2n+1}}{2^{2n+1}(2n+1)!}=\frac{\pi}{2}-\frac{\pi^3}{8\cdot 3!}+\frac{\pi^5}{32\cdot 5!}-\frac{\pi^7}{128\cdot 7!}+\cdots$$

$$\sin x = \sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)!}x^{2n+1}$$

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Example

Find the sum of the series

$$\sum_{n=0}^{\infty}(-1)^n\frac{\pi^{2n+1}}{2^{2n+1}(2n+1)!} = \frac{\pi}{2} - \frac{\pi^3}{8\cdot 3!} + \frac{\pi^5}{32\cdot 5!} - \frac{\pi^7}{128\cdot 7!} + \cdots$$

$$\sin x = \sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)!}x^{2n+1}$$

$$\sum_{n=0}^{\infty}(-1)^n\frac{\pi^{2n+1}}{2^{2n+1}(2n+1)!} = \sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)!}\left(\frac{\pi}{2}\right)^{2n+1}$$

$$= \sin\frac{\pi}{2}$$

Example

Find the sum of the series

$$\sum_{n=0}^{\infty}(-1)^n\frac{\pi^{2n+1}}{2^{2n+1}(2n+1)!}=\frac{\pi}{2}-\frac{\pi^3}{8\cdot 3!}+\frac{\pi^5}{32\cdot 5!}-\frac{\pi^7}{128\cdot 7!}+\cdots$$

$$\begin{aligned}
\sin x &= \sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)!}x^{2n+1}\\
\sum_{n=0}^{\infty}(-1)^n\frac{\pi^{2n+1}}{2^{2n+1}(2n+1)!} &= \sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)!}\left(\frac{\pi}{2}\right)^{2n+1}\\
&= \sin\frac{\pi}{2}\\
&= 1
\end{aligned}$$

Example

Find the Maclaurin series for $\cos x$.

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$$\cos x = \frac{d}{dx}\left(\quad \right)$$

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Example

Find the Maclaurin series for $\cos x$.

$$\begin{aligned}\cos x &= \frac{d}{dx}\left(\sin x\right)\\ &= \frac{d}{dx}\left(\qquad\qquad\qquad \right)\end{aligned}$$

Example

Find the Maclaurin series for $\cos x$.

$$\begin{aligned}\cos x &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\sin x\right)\\ &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}\right)\end{aligned}$$

Example

Find the Maclaurin series for $\cos x$.

$$\begin{aligned}\cos x &= \frac{\mathrm{d}}{\mathrm{d}x}(\sin x)\\ &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}\right)\\ &= \sum_{n=0}^{\infty}\frac{\mathrm{d}}{\mathrm{d}x}\left((-1)^n\frac{x^{2n+1}}{(2n+1)!}\right)\end{aligned}$$

Example

Find the Maclaurin series for $\cos x$.

$$\begin{aligned}
\cos x &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\sin x\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}\right) \\
&= \sum_{n=0}^{\infty}\frac{\mathrm{d}}{\mathrm{d}x}\left((-1)^n\frac{x^{2n+1}}{(2n+1)!}\right) \\
&= \sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)x^{2n}}{(2n+1)!}
\end{aligned}$$

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\begin{aligned}
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&= \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{}
\end{aligned}
$$

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&= \sum_{n=0}^{\infty}\frac{\mathrm{d}}{\mathrm{d}x}\left((-1)^n\frac{x^{2n+1}}{(2n+1)!}\right)\\
&= \sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)x^{2n}}{(2n+1)!}\\
&= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}
\end{aligned}$$

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&= \sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)x^{2n}}{(2n+1)!}\\
&= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}\\
&= 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots
\end{aligned}$$

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&= 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots
\end{aligned}$$

The series for $\sin x$ converges everywhere, so the series for $\cos x$ does too.

Example

Find the Maclaurin series for $x \cos x$.

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$$x \cos x \quad = \quad x$$

Example

Find the Maclaurin series for $x \cos x$.

$$x \cos x \quad = \quad x \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

Example

Find the Maclaurin series for $x \cos x$.

$$\begin{aligned} x\cos x &= x\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} \\ &= \sum_{n=0}^{\infty}(-1)^n\frac{\phantom{x^{2n+1}}}{(2n)!} \end{aligned}$$

Example

Find the Maclaurin series for $x \cos x$.

$$
\begin{aligned}
x \cos x &= x \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n)!}
\end{aligned}
$$

Example

Find the Maclaurin series for $x \cos x$.

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\begin{aligned}
x \cos x &= x \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n)!} \\
&= x - \frac{x^3}{2!} + \frac{x^5}{4!} - \frac{x^7}{6!} + \cdots
\end{aligned}
$$

Here is a table of some important Maclaurin series we have learned:

Function		Series	R
$\frac{1}{1-x}$	$=$	$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$	1
$\arctan x$	$=$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$	1
e^x	$=$	$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$	∞
$\sin x$	$=$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$	∞
$\cos x$	$=$	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$	∞

Example

Use a power series to find $\lim\limits_{x \to 0} \dfrac{e^x - 1 - x}{x^2}$.

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$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Example

Use a power series to find $\lim_{x\to 0} \frac{e^x - 1 - x}{x^2}$.

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \\ e^x - 1 - x &= \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \end{aligned}$$

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Use a power series to find $\lim\limits_{x\to 0} \dfrac{e^x - 1 - x}{x^2}$.

$$
\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \\
e^x - 1 - x &= \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \\
\frac{e^x - 1 - x}{x^2} &= \frac{1}{2!} + \frac{x}{3!} + \frac{x^2}{4!} + \cdots
\end{aligned}
$$

Example

Use a power series to find $\lim\limits_{x\to 0} \dfrac{e^x - 1 - x}{x^2}$.

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\begin{aligned}
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e^x - 1 - x &= \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \\
\frac{e^x - 1 - x}{x^2} &= \frac{1}{2!} + \frac{x}{3!} + \frac{x^2}{4!} + \cdots \\
\lim_{x\to 0} \frac{e^x - 1 - x}{x^2} &= \lim_{x\to 0} \left(\frac{1}{2} + \frac{x}{3!} + \frac{x^2}{4!} + \cdots \right) \qquad .
\end{aligned}
$$

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e^x - 1 - x &= \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \\
\frac{e^x - 1 - x}{x^2} &= \frac{1}{2!} + \frac{x}{3!} + \frac{x^2}{4!} + \cdots \\
\lim_{x\to 0} \frac{e^x - 1 - x}{x^2} &= \lim_{x\to 0} \left(\frac{1}{2} + \frac{x}{3!} + \frac{x^2}{4!} + \cdots \right) = \qquad .
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$$

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Use a power series to find $\lim\limits_{x\to 0} \dfrac{e^x - 1 - x}{x^2}$.

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \\ e^x - 1 - x &= \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \\ \frac{e^x - 1 - x}{x^2} &= \frac{1}{2!} + \frac{x}{3!} + \frac{x^2}{4!} + \cdots \\ \lim_{x\to 0} \frac{e^x - 1 - x}{x^2} &= \lim_{x\to 0}\left(\frac{1}{2} + \frac{x}{3!} + \frac{x^2}{4!} + \cdots\right) = \frac{1}{2} \quad . \end{aligned}$$

Example

Use a power series to find $\lim\limits_{x\to 0} \dfrac{x - \sin x}{x^3}$.

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Use a power series to find $\lim_{x\to 0} \frac{x - \sin x}{x^3}$.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

Example

Use a power series to find $\lim\limits_{x\to 0} \dfrac{x - \sin x}{x^3}$.

$$\begin{aligned} \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\ -\sin x &= -x + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \cdots \end{aligned}$$

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\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\
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x - \sin x &= \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \cdots
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\frac{x - \sin x}{x^3} &= \frac{1}{3!} - \frac{x^2}{5!} + \frac{x^4}{7!} - \cdots
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\frac{x - \sin x}{x^3} &= \frac{1}{3!} - \frac{x^2}{5!} + \frac{x^4}{7!} - \cdots \\
\lim_{x \to 0} \frac{x - \sin x}{x^3} &= \lim_{x \to 0} \left(\frac{1}{6} - \frac{x^2}{5!} + \frac{x^4}{7!} - \cdots \right)
\end{aligned}
$$

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Use a power series to find $\lim\limits_{x\to 0} \dfrac{x - \sin x}{x^3}$.

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\lim_{x\to 0} \frac{x - \sin x}{x^3} &= \lim_{x\to 0}\left(\frac{1}{6} - \frac{x^2}{5!} + \frac{x^4}{7!} - \cdots\right) =
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\lim_{x\to 0} \frac{x - \sin x}{x^3} &= \lim_{x\to 0}\left(\frac{1}{6} - \frac{x^2}{5!} + \frac{x^4}{7!} - \cdots\right) = \frac{1}{6}
\end{aligned}$$