Precalculus
Lecture 19

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https://github.com/tmilev/freecalc

2020

Outline

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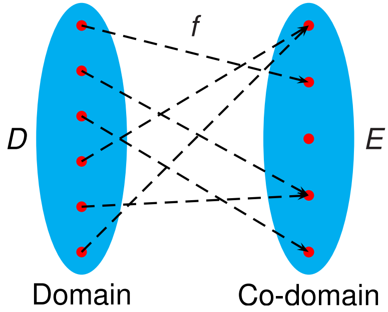


- A function has domain $D \Rightarrow$ there is exactly one arrow starting at each element of D.
- An element of the co-domain can be at the tip of more than one arrow.
- It is allowed to have an element in the co-domain without arrows pointing to it.

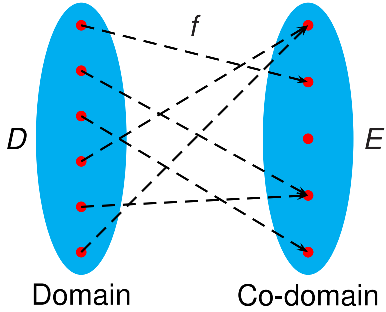
Definition (Function)

A function f is a rule that assigns to each element x in a set D exactly one element, called $f(x)$, in a set E.

- Functions are also synonymously called “maps”.
- In the picture above, f is represented via the arrows.

Definition (Domain)

The set D in the definition of f is called the domain of f.



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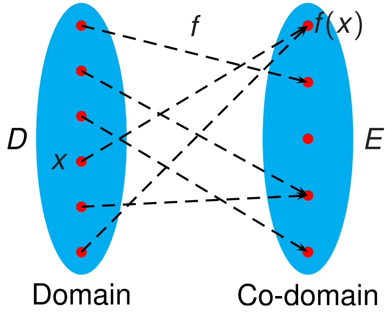

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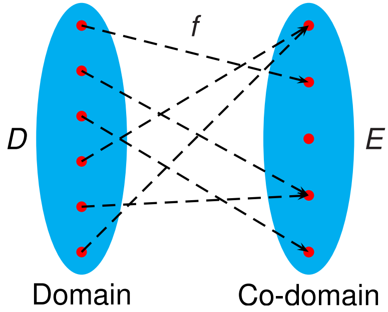
Definition (Value of f at x)

The number $f(x)$ is called *the value of f at x* and is read "f of x".

- The value of f at x is also called the image of x under the map f.
- In the expression $f(x)$, x is referred to as the *argument* of f.



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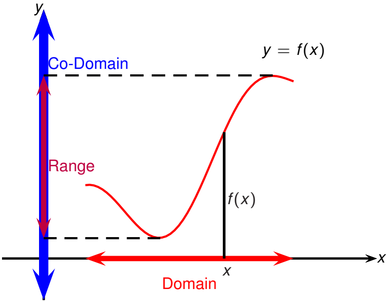
A function f is a rule that assigns to each element x in a set D exactly one element, called $f(x)$, in a set E.

Definition (Range)

The set of all possible values taken by $f(x)$ as the element x runs over elements of D is called the range of f.



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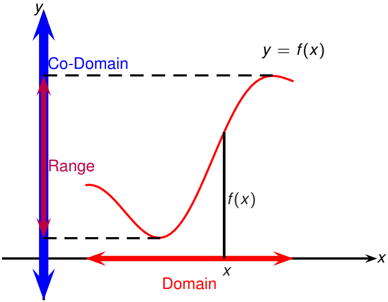

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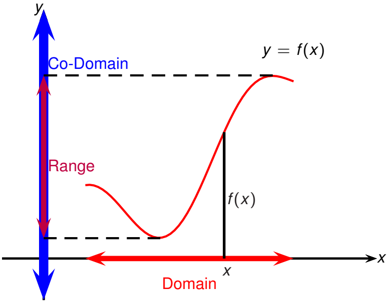

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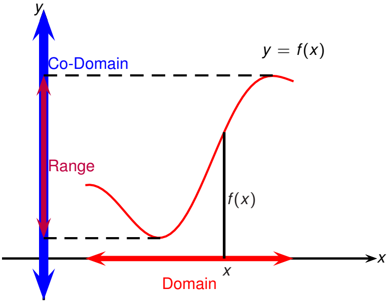

Definition (Function)

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Definition (Range)

The set of all possible values taken by $f(x)$ as the element x runs over elements of D is called the range of f.

- The notation $f(x)$ for the image of x was introduced by Leonhard Euler.
- Expressions such as $a(x+y)$ may either refer to
  - the function a applied to the argument $x+y$ or
  - the number a multiplied by $x+y$.
- Which of the two cases is at hand should be clarified with English language.
- However if no such clarification is present (as often is the case in mathematical exercises/tests), the matter is up to the reader's intelligent interpretation.

Functions via formulas

- When we want to define a function f whose domain (input) is a number, we often use algebraic formulas, for example:
$$f(x) = 2x^2 + x + 1.$$
- In the notation above, x is an independent, bounded (dummy, placeholder) variable - it denotes a substitution pattern.
- We could think of x as a placeholder - instead of $f(x) = 2x^2 + x + 1$ we could write $f(\square) = 2\square^2 + \square + 1$.
- Here, $\square$ denotes our ability to substitute $f(\square)$ by $2\square^2 + \square + 1$.
- For example $f(1) = 2 \cdot 1^2 + 1 + 1$.
- The word independent refers to the fact that x is no relation with any of the other variables in the text.

Functions via formulas

- When we want to define a function f whose domain (input) is a number, we often use algebraic formulas, for example:
$$f(x) = 2x^2 + x + 1.$$
- In the notation above, x is an independent, bounded (dummy, placeholder) variable - it denotes a substitution pattern.
- Another example is $f(x^2) = 2\left(x^2\right)^2 + x^2 + 1$.
- This example illustrates the meaning of the word bounded (dummy, placeholder): the dummy variable x is only a convenient placeholder label, and is a distinct mathematical object from the variable x which has meaning outside of the expression $f(x^2)$.
- If we omit the clarification colors, it is no longer clear whether $f(x)$ refers to the defining expression for $f(x)$, or to an expression $f(x)$ where x has meaning outside of the definition of f.

Functions via formulas

- When we want to define a function f whose domain (input) is a number, we often use algebraic formulas, for example:
$$f(t) = 2t^2 + t + 1.$$
- In the notation above, x is an independent, bounded (dummy, placeholder) variable - it denotes a substitution pattern.
- Computer algebra systems will “keep track of the colors” and will not confuse the dummy x with the non-dummy variable x.
- For humans however the danger of confusion is real.
- In case of human confusion, clarification should be sought through renaming variables, as illustrated above.
- The relabeling of the dummy variable to t removes any confusion about the meaning of $f(x^2)$.
- In computer programming, the issues described here are addressed via “variable scope rules”.

Example

Let $f(x) = x^2 - x - 1$. Evaluate the difference quotient and simplify your answer.

$$
\begin{aligned}
\frac{f(2+h) - f(2)}{h} &= \frac{\left((2+h)^2 - (2+h) - 1\right) - (2^2 - 2 - 1)}{h} \\
&= \frac{\cancel{2^2} + 2 \cdot 2h + h^2 - \cancel{2} - h - \cancel{1} - \cancel{2^2} + \cancel{2} + \cancel{1}}{h} \\
&= \frac{h^2 + 3h}{h} \\
&= \frac{\cancel{h}(h+3)}{\cancel{h}} \\
&= h + 3
\end{aligned}
$$

A Note on Domains of Functions

If the domain of a function isn't specified, it is implied to be all numbers x for which the formula $f(x)$ is defined. There are some restrictions to consider:

- Can't divide by 0.
- Even roots of a negative number are not defined in this course ($\sqrt{-1}$, $\sqrt[4]{-2053}$, $\sqrt[6]{-15}\dots$ not allowed).
- Taking $\log x$ if $x \leq 0$ is not allowed in this course; taking $\log 0$ is not allowed in any course.

Example

Find the implied domains of the given functions.

$$f(x) = \sqrt[4]{x-2} + \sqrt[3]{6-x}$$

- Any risk of dividing by 0? No.
- Any risk of taking the even root of a negative number? Yes.
- $x-2$ must not be negative.

$$\begin{aligned} x-2 &\geq 0 \\ x &\geq 2 \end{aligned}$$

Domain is all real numbers greater than or equal to 2; that is, $[2, \infty)$.

$$g(x) = \frac{x^2-9}{x^2-x-6}$$

- Any risk of dividing by 0? Yes.
- Any risk of taking the even root of a negative number? No.
- x^2-x-6 must not equal 0.

$$\begin{aligned} x^2-x-6 &\neq 0 \\ (x-3)(x+2) &\neq 0 \\ x &\neq 3 \text{ or } -2 \end{aligned}$$

Domain is all real numbers except 3 and -2; that is, $(-\infty, -2) \cup (-2, 3) \cup (3, \infty)$.

The Vertical Line Test

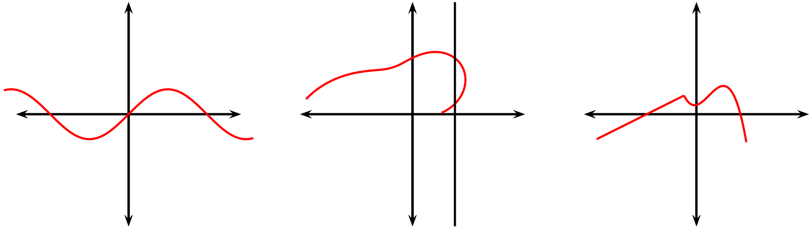
Question

Given a curve in the plane, is it the graph of a function or not?

The answer is as follows.

Proposition (The Vertical Line Test)

A curve in the plane is the graph of a function if and only if no vertical line intersects it more than once.

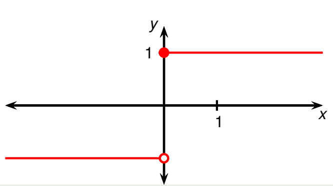


Piecewise Defined Functions

Definition (Piecewise Defined Function)

A piecewise defined function is a function that is defined by different algebraic formulas on different subsets of its domain.

Example



$$f(x) = \begin{cases} 1 & \text{if} \quad x \geq 0 \\ -1 & \text{if} \quad x < 0 \end{cases}$$

The filled red circle means $(0, 1)$ is on the curve.

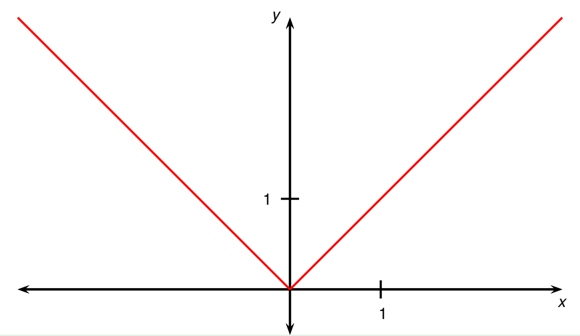
The open circle means $(0, -1)$ is not on the curve.

Example

The absolute value $|x|$ of a number a is defined to be

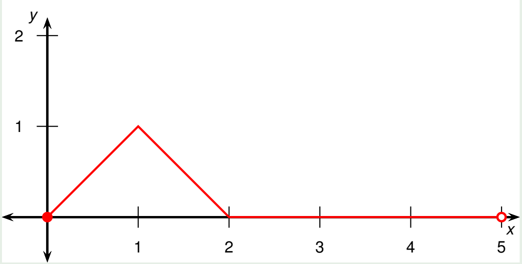
$$|x| = \begin{cases} x & \text{if} \quad x \geq 0 \\ -x & \text{if} \quad x < 0. \end{cases}$$

Sketch a graph of the function $f(x) = |x|$.



Example

Find a formula for the function f whose graph is given below.

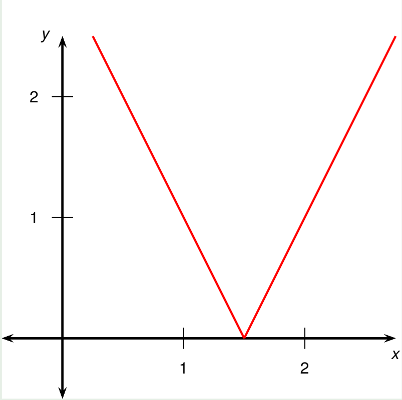


Different formulas on $[0, 1)$, $[1, 2)$, and $[2, 5)$.

$$f(x) = \begin{cases} x & \text{if } 0 \leq x < 1 \\ 2 - x & \text{if } 1 \leq x < 2 \\ 0 & \text{if } 2 \leq x < 5 \end{cases}$$

Example

Sketch the function $f(x) = |2x - 3|$.

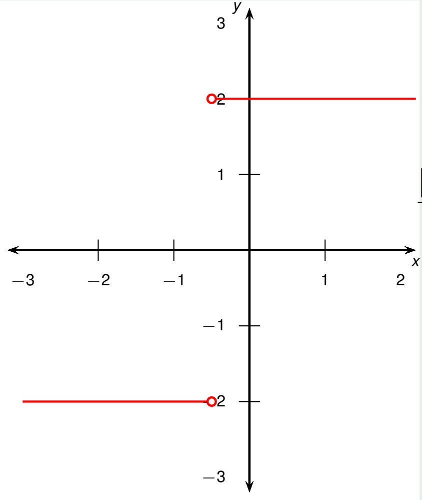


$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

$$\begin{aligned} |2x - 3| &= \begin{cases} 2x - 3 & \text{if } 2x - 3 \geq 0 \\ -(2x - 3) & \text{if } 2x - 3 < 0 \end{cases} \\ &= \begin{cases} 2x - 3 & \text{if } 2x \geq 3 \\ -2x + 3 & \text{if } 2x < 3 \end{cases} \\ &= \begin{cases} 2x - 3 & \text{if } x \geq 3/2 \\ -2x + 3 & \text{if } x < 3/2. \end{cases} \end{aligned}$$

Example

Sketch the function $f(x) = \dfrac{|4x+2|}{2x+1}$.



$$|u| = \begin{cases} u & \text{if } u \geq 0 \\ -u & \text{if } u < 0. \end{cases}$$

$$\frac{|4x+2|}{2x+1} = \begin{cases} \frac{4x+2}{2x+1} & \text{if } 4x+2 > 0 \\ \frac{-(4x+2)}{2x+1} & \text{if } 4x+2 < 0 \end{cases}$$

$$= \begin{cases} \frac{2\cancel{(2x+1)}}{\cancel{2x+1}} & \text{if } 4x > -2 \\ \frac{-2\cancel{(2x+1)}}{\cancel{2x+1}} & \text{if } 4x < -2 \end{cases}$$

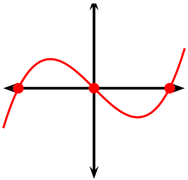
$$= \begin{cases} 2 & \text{if } x > -\frac{1}{2} \\ -2 & \text{if } x < -\frac{1}{2}. \end{cases}$$

Definition

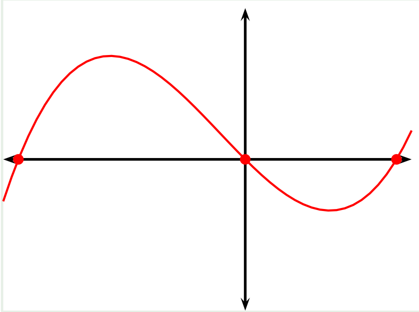
The zeros of a function f are the values of the argument x for which $f(x) = 0$.

Observation

The zeros of a function are the x-coordinates of the x intercepts of the graph of the function.



Example



Find the zeroes of
$$f(x) = \frac{1}{6}x^3 + \frac{1}{6}x^2 - x.$$

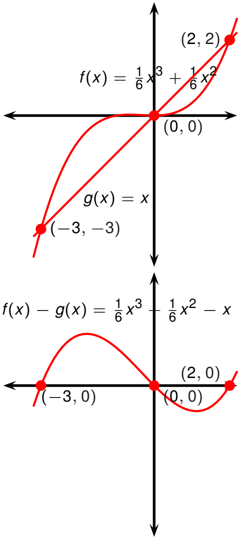
Example

- Find when $f(x) = g(x)$, where

$$f(x) = \frac{1}{6}x^3 + \frac{1}{6}x^2 \qquad\qquad g(x) = x$$

- Find the intersections of the graphs of f and g.

Let g of x and f of x be functions.



Observation

- *To solve $f(x) = g(x)$ means to find the x coordinates of the intersections of the graphs of f and g.*
- *To solve $f(x) = g(x)$ is equivalent to solving the equation $f(x) - g(x) = 0$.*
- *To solve $f(x) = g(x)$ means to find the zeroes of $f(x) - g(x)$.*
- *The x coordinates of the intersections of $f(x)$ and $g(x)$ coincide with the x coordinates of the x intercepts of $f(x) - g(x)$.*

Symmetry

Definition (Even and Odd Functions)

A function f is called even if $f(-x) = f(x)$ for all x in its domain. A function f is called odd if $f(-x) = -f(x)$ for all x in its domain.

Example (x^2 is Even, x^3 is Odd)

The function $f(x) = x^2$ is even:

$$f(-x) = (-x)^2 = x^2 = f(x).$$

The function $g(x) = x^3$ is odd:

$$g(-x) = (-x)^3 = -x^3 = -g(x).$$

Definition (Even and Odd Functions)

A function f is called even if $f(-x) = f(x)$ for all x in its domain. A function f is called odd if $f(-x) = -f(x)$ for all x in its domain.

Example

Determine whether each of the following functions is even, odd, or neither even nor odd.

$$f(x) = x^5 + x$$

$$\begin{aligned} f(-x) &= (-x)^5 + (-x) \\ &= -x^5 - x \\ &= -(x^5 + x) \\ &= -f(x) \end{aligned}$$

Therefore f is odd.

$$g(x) = 1 - x^4$$

$$\begin{aligned} g(-x) &= 1 - (-x)^4 \\ &= 1 - x^4 \\ &= g(x) \end{aligned}$$

Therefore g is even.

$$h(x) = 2x - 1$$

$$\begin{aligned} h(-x) &= 2(-x) - 1 \\ &= -2x - 1 \\ &\neq h(x), -h(x) \end{aligned}$$

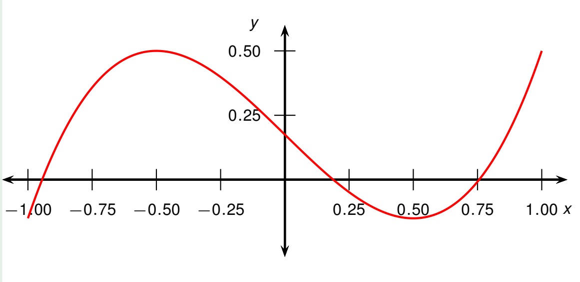
Therefore h is neither even nor odd.

Increasing and Decreasing Functions

Definition (Increasing and Decreasing Functions)

A function f is called increasing on an interval I if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ in I.
It is called decreasing on the interval I if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$ in I.

Example (Increasing and Decreasing)



- f is increasing on $[-1, -\frac{1}{2}]$.
- f is decreasing on $[-\frac{1}{2}, \frac{1}{2}]$.
- f is increasing on $[\frac{1}{2}, 1]$.