Calculus III
Lecture 1

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Outline

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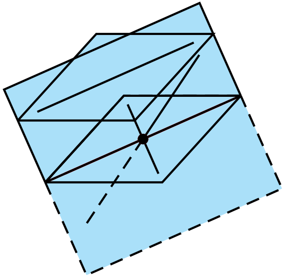
Configurations of Lines and Planes

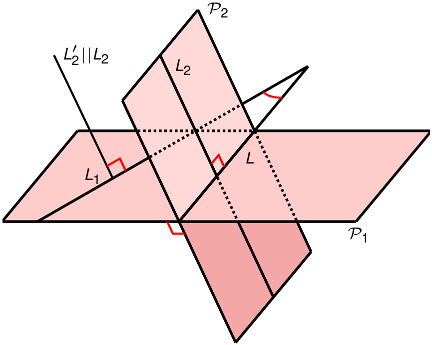
	pair of objects	intersection	parallelism	co-planar?
2 lines	intersecting lines	one point	not parallel	yes
	parallel lines	empty	parallel	yes
	skew lines	none	not parallel	no
line & plane	line intersecting plane	one point	not parallel	no
	line parallel to a plane	none	parallel	no
	line lying in plane	line	-	yes
2 planes	intersecting planes	line	not parallel	-
	parallel planes	none	parallel	-



Distances and Angles

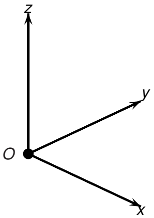
- In the preceding slides/lectures we saw distinguishing configurations of lines and planes does not require the notion of angle or distance. Nonetheless the latter are fundamental.
- Distance is a function that assigns to two points A, B the non-negative number $|AB|$ that quantifies/measures how close/far apart are the points. We denote distance also by $d(A, B)$.
- From elementary Euclidean geometry: if we know the lengths of the sides of a triangle, we know the magnitude of its angles.
- So the notion of magnitude of angle follows from that of distance.
- We note that knowing distances determines magnitudes of angles but not their signs.
- Signs of angles are a manifestation of the fundamental concept of orientation, which we will study later.
- We recall two intersecting lines are perpendicular when the angle between them is $\pm\frac{\pi}{2}$.

Line/Plane Configurations and Distances and Angles



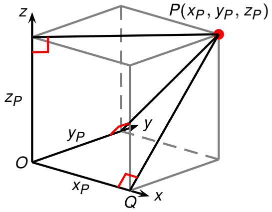
- The planes $\mathcal{P}_1$ and $\mathcal{P}_2$ are perpendicular on each other.
- The lines L_2 and L are coplanar and perpendicular to each other.
- The lines L_1 and L_2 are skew and perpendicular to each other.
- The lines L_1 and L are coplanar and not perpendicular.
- The line L_2 is perpendicular to the plane $\mathcal{P}_1$.
- The line L_1 is not perpendicular to the plane $\mathcal{P}_2$.

Rectangular/Cartesian Coordinates



- A Cartesian coordinate system is given by fixing:
  - a point O (called the origin),
  - 3 pairwise perpendicular lines intersecting at the origin,
  - a direction in each of the coordinate axis.
- The three lines are labeled as x-axis, y-axis and z-axis.

Rectangular/Cartesian Coordinates



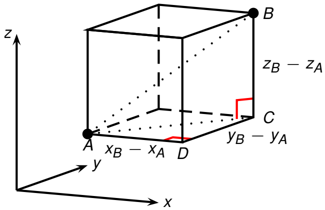
- P -point. We assign to it triple (x_P, y_P, z_P).
- Assignment will be such that distinct points are assigned distinct triples.
- $Q =$ base of perpendicular from P to x-axis.
- Define x_P as signed distance b-n O and Q.
- Take distance with $+$ sign if OQ points in direction of x-axis, $-$ sign else.
- Definitions of y_P, z_P are similar.
- (x_P, y_P, z_P) = Cartesian coordinates of P.
- x_P is called the x-coordinate of P, and so on for other axes.
- (x_P, y_P, z_P) = singed lengths of edges of the rectangular box indicated in the picture.

Euclidean Distance in Coordinates

Theorem (Can be taken as definition)

The distance b-n the points $A(x_A, y_A, z_A)$ and $B(x_B, y_B, z_B)$ is given by:

$$d(A,B) = |AB| = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}$$



Why is this so? Geometric explanation:

$$\begin{aligned} |AC|^2 &= |AD|^2 + |DC|^2 && \triangle ADC \\ |AB|^2 &= |BC|^2 + |AC|^2 && \triangle ACB \\ &= |BC|^2 + |AD|^2 + |DC|^2 \\ &= (x_B - x_A)^2 + (y_B - y_A)^2 \\ &\quad + (z_B - z_A)^2 \quad , \end{aligned}$$

Example:

$$d(P(3,1,2), Q(1,2,3)) = \sqrt{(1-3)^2 + (2-1)^2 + (3-2)^2} = \sqrt{6} \quad .$$

Example

Find the distance between the points with coordinates $(1, -2, 3)$ and $(-4, 5, 6)$.
$d((1, -2, 3), (-4, 5, 6)) = \sqrt{(-4 - 1)^2 + (5 - (-2))^2 + (6 - 3)^2} = \sqrt{83}$.

Example

A cube has edge 3 cm. Find the distance between a vertex of the cube and the midpoint of one of the three opposing sides.

Sets in Space

X subset of a set Y:

$$X = \{A \text{ in } Y | A \text{ has property } \mathcal{P}\} \subset Y$$

Examples (Fixed point Q, fixed $r > 0$):

$$X = \{A \text{ in Space } | d(A, Q) = r\} = S_r(Q) \ ,$$

Sphere of radius r centered at Q.

$$B_r(Q) = \{A \text{ in Space } | d(A, Q) < r\} \ ,$$

Open ball of radius r centered at Q.

$$\overline{B}_r(Q) = \{A \text{ in Space } | d(A, Q) \leqslant r\} \ ,$$

Closed ball of radius r centered at Q.

Equation(s) of Subsets

$$X = \{(x, y, z) | x, y, z \text{ satisfy certain relation(s) } \} .$$

Examples:
$\{(x, y, z)|x^2 + y^2 + z^2 = 1\}$:
sphere of radius $r = 1$ centered at the origin $(0, 0, 0)$
Also refered to as: sphere $x^2 + y^2 + z^2 = 1$

$\{(x, y, z)|x = 0\}$: coordinate Left-Up plane

$\{(x, y, z)|x = 0 \text{ and } y = 0\}$:
intersection of coordinate planes $\to$ coordinate axis

Can be given by only one equation:
$x^2 + y^2 = 0 \to x = 0, y = 0$, and z arbitrary $\to$
vertical axis above $(0, 0)$ in $(x, y)-$plane

Important: Equations in Plane vs. Space.

Recognizing Spheres from Equations

$Q(x_0, y_0, z_0)$, $r > 0$, $A(x, y, z)$. Remark: $d(A, Q) = r \longleftrightarrow d^2(A, Q) = r^2$

$$S_r(Q): \quad (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$$

Example:

$$(x - 2)^2 + (y - 0)^2 + (z + 1)^2 = 3^2$$
$$x^2 + y^2 + z^2 - 4x + 2z - 4 = 0$$

- no mixed terms xy, xz, or yz;
- quadratic terms x^2, y^2, and z^2 with the same coefficient.

Examples:

$$x^2 + y^2 + z^2 - 4x + 2y = 0$$

Complete the square:

$$(x - 2)^2 + (y + 1)^2 + z^2 = 5$$

Sphere of radius $\sqrt{5}$ centered at $(2, -1, 0)$.
How about $x^2 + y^2 + z^2 - 4x + 2y = -6$? Passes both tests, but ...