Calculus II
Lecture 21

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https://github.com/tmilev/freecalc

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Outline

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Definition (Complex numbers)

The set of complex numbers $\mathbb{C}$ is defined as the set

$$\{a + bi | a, b - \text{real numbers}\},$$

where the number i is a number for which

$$i^2 = -1 \quad .$$

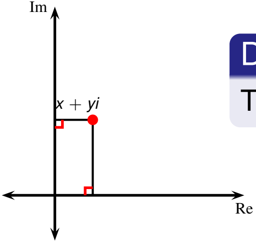
The number i is called the imaginary unit. By definition, $\sqrt{-1} = i$.

- Complex addition/subtraction

$$(a + bi) \pm (c + di) = (a \pm c) + (b \pm d)i \quad .$$

- Complex multiplication

$$\begin{aligned}(a + bi)(c + di) &= ac + adi + bci + bdi^2 = ac + adi + bci - bd \\ &= (ac - bd) + i(ad + bc)\end{aligned}$$



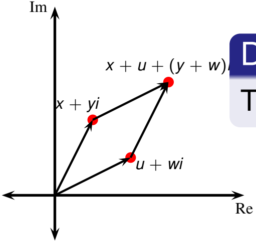
Definition (Complex numbers)

The complex numbers are the set $\{x + yi | x, y \in \mathbb{R}\}$.

- Real numbers are usually denoted by $\mathbb{R}$.
- Complex numbers are usually denoted by $\mathbb{C}$.

Consider $z = x + yi$.

- x is called the real part of z, y is called the imaginary part of z. We write $x = \mathrm{Re} z = \mathrm{Re}(x + yi)$, $y = \mathrm{Im} z = \mathrm{Im}(x + yi)$.
- Real & imaginary part of z can be used as x, y-coords. to depict z.
- In this way we view complex number $x + iy$ as the point (position vector) (x, y) in a two-dimensional space.
- The addition of complex numbers corresponds to vector addition.
- Multiplication by a real number c corresponds to vector scalar multiplication by c (scaling).
- The space the complex numbers is referred to as the complex plane (sometimes alternatively called the complex line).



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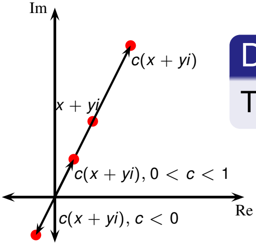

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Let $u = 2+3i$, $v = 5-7i$.

Example (Addition)

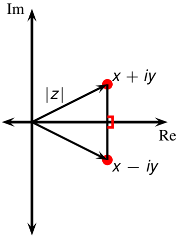
$u+v = (2+3i)+(5-7i) = (2+5)+(3-7)i = 7-4i.$

Example (Subtraction)

$u-v = (2+3i)-(5-7i) = (2-5)+(3-(-7))i = -3+10i.$

Example (Multiplication)

$$\begin{aligned} u\cdot v &= (2+3i)\cdot(5-7i) \\ &= 2\cdot 5 + 2\cdot(-7)i + 3i\cdot 5 + 3i(-7i) \\ &= 10 - 14i + 15i - 21i^2 \\ &= 10 + i - (-21) \\ &= 31 + i \end{aligned}$$



Let $z = x + iy$ be a complex number.

Definition (Complex conjugation)

We say that $\bar{z} = \overline{(x+iy)} = x - iy$ is the *complex conjugate of* z. The transformation that maps z to $\bar{z}$ is called *complex conjugation*.

In the complex plane, complex conj. = reflection across real axis.

Theorem

$z\bar{z}$ is a non-negative real number. $z\bar{z}$ equals 0 if and only if $z = 0$.

Proof.

$z\bar{z} = (x+iy)(x-iy) = x^2 - (iy)^2 = x^2 + y^2$ is real and non-negative.
$z\bar{z} = 0$ implies $x^2 + y^2 = 0$, which implies $x = y = 0$. $\square$

Definition

The quantity $|z| = \sqrt{z\bar{z}} = \sqrt{x^2+y^2}$ is called the *absolute* value of z.

Theorem (Conjugation preserves $+, \cdot$)

1. $\overline{z\cdot w} = \overline{z}\cdot\overline{w}$
2. $\overline{z+w} = \overline{z}+\overline{w}$

Proof.

Let $z = x+yi$, $w = u+vi$.

- $$\begin{aligned} \overline{z}\cdot\overline{w} &= \overline{(x+yi)}\cdot\overline{(u+vi)} = (x-yi)(u-vi) \\ &= (xu-yv)-(xv+yu)i \\ \overline{z\cdot w} &= \overline{(x+yi)(u+iv)} = \overline{xu-yv+(xv+yu)i} \\ &= (xu-yv)-(xv+yu)i \quad . \end{aligned}$$
- $$\begin{aligned} \overline{z}+\overline{w} &= \overline{(x+yi)}+\overline{(u+vi)} = (x-yi)+(u-vi) \\ &= (x+u)-(y+v)i \\ \overline{z+w} &= \overline{(x+yi)+(u+iv)} = \overline{(x+u)+(y+v)i} \\ &= (x+u)-(y+v)i \quad . \end{aligned}$$

$\square$

In the preceding slide we proved the following.

Theorem (Conjugation preserves $\cdot$)

$\overline{z \cdot w} = \overline{z} \cdot \overline{w}$.

Corollary

$|zw| = |z||w|$.

Proof.

$|zw| = \sqrt{zw\overline{zw}} = \sqrt{zw\bar{z}\bar{w}} = \sqrt{z\bar{z}}\sqrt{w\bar{w}} = |z||w|$. □

Corollary

$\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$, $w \neq 0$.

Let $u = 2 + 3i$, $v = 5 - 7i$.

Example (Division)

$$
\begin{aligned}
\frac{u}{v} &= \frac{2+3i}{5-7i} \\
&= \frac{(2+3i)}{(5-7i)}\frac{(5+7i)}{(5+7i)} \\
&= \frac{(2+3i)(5+7i)}{5^2-(7i)^2} \\
&= \frac{10+15i+14i+21i^2}{5^2+7^2} \\
&= \frac{10-21+29i}{25+49} \\
&= \frac{-11+29i}{74} \\
&= -\frac{11}{74}+\frac{29}{74}i
\end{aligned}
$$

Multiply and divide by complex conjugate of denominator

Let $u = a + bi$, $v = c + di$, $v \neq 0$.

Example (Complex number division)

$$\begin{aligned}
\frac{u}{v} &= \frac{a+bi}{c+di} \\
&= \frac{(a+bi)}{(c+di)}\frac{(c-di)}{(c-di)} \\
&= \frac{(a+bi)(c-di)}{c^2-(di)^2} \\
&= \frac{ac-adi+cbi-bdi^2}{c^2+d^2} \\
&= \frac{ac+bd+(bc-ad)i}{c^2+d^2} \\
&= \frac{ac+bd}{c^2+d^2}+\frac{(bc-ad)}{c^2+d^2}i
\end{aligned}$$

Multiply and divide by complex conjugate of denominator

Definition (Complex number division)

The quotient $\frac{u}{v}$, $v \neq 0$ is defined via the formula above.

Theorem

Let $e(z) = \sum\limits_{n=0}^{\infty} \frac{z^n}{n!}$, $z \in \mathbb{C}$. *Then* $e(z)e(w) = e(z+w)$.

Power series over $\mathbb{C}$ are defined similarly to power series over $\mathbb{R}$. The following proof lies outside of scope Calc II. Details are omitted and get filled in a course of Complex Analysis. You will not be tested on it.

Proof.

$$
\begin{aligned}
e(z)e(w) &= \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{m=0}^{\infty} \frac{w^m}{m!} = \sum_{s=0}^{\infty} \sum_{k=0}^{s} \frac{z^k w^{s-k}}{k!(s-k)!} \\
&= \sum_{s=0}^{\infty} \sum_{k=0}^{s} \frac{z^k w^{s-k}}{s!} \frac{s!}{k!(s-k)!} = \sum_{s=0}^{\infty} \frac{(z+w)^s}{s!} = e(z+w).
\end{aligned}
$$

□

Lemma (Newton Binomial formula)

$(z+w)^s = \sum_{k=0}^{s} z^k w^{s-k} \frac{s!}{k!(s-k)!}$.

Definition (Real exponent, Definition I)

Let $z \in \mathbb{R}$. The real exponent e^z is defined as $\lim_{\substack{p \to x \\ p \text{ is rational}}} e^p$.

Definition (Exponent, Definition II)

Let $z \in \mathbb{C}$. The complex exponent e^z is defined by $e^z = e(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$.

- For real z, e^z may be defined via Definition I.
- For complex z, e^z is defined via Definition II.
- Real numbers are complex numbers (with zero imaginary part). Thus Definition II is also valid when z is a real number, and therefore Definition II is more general.
- A calculus course may be built by presenting Definition II first and proving Definition I as a theorem.
- Alternatively, a calculus course may be built by first presenting Definition I, and then expanding it to Definition II.

Definition (Real exponent, Definition I)

Let $z \in \mathbb{R}$. The real exponent e^z is defined as $\lim\limits_{\substack{p \to x \\ p \text{ is rational}}} e^p$.

Definition (Exponent, Definition II)

Let $z \in \mathbb{C}$. The complex exponent e^z is defined by $e^z = e(z) = \sum\limits_{n=0}^{\infty} \frac{z^n}{n!}$.

Theorem

When $z \in \mathbb{R}$, Definition I and Definition II are equivalent.

Sketch of Proof. Definition I implies Definition II over $\mathbb{R}$.

Under Definition I the Maclaurin series of e^z was computed to be $\sum\limits_{n=0}^{\infty} \frac{z^n}{n!}$. Under Definition I, it can be shown that e^z equals its Maclaurin series, which is the defining expression for Definition II. $\square$

Definition (Real exponent, Definition I)

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Definition (Exponent, Definition II)

Let $z \in \mathbb{C}$. The complex exponent e^z is defined by $e^z = e(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$.

Theorem

When $z \in \mathbb{R}$, Definition I and Definition II are equivalent.

Sketch of Proof. Definition II implies Definition I over $\mathbb{R}$.

We showed that $e(z+w) = e(z)e(w)$. Using that statement alone, one can show that the two definitions agree over the rational numbers. Two continuous functions are equal if they are equal over the rationals, and the theorem follows. $\square$

Euler's Formula

Theorem (Euler's Formula)

$$e^{ix} = \cos x + i \sin x,$$

where $e \approx 2.71828$ is Euler's/Napier's constant .

Proof.

Recall $n! = 1 \cdot 2 \cdot 3 \cdot \dots \cdot (n-1) \cdot n$. Borrow from Calc II the f-las:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + \frac{(-1)^n x^{2n}}{(2n)!} + \dots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots$$

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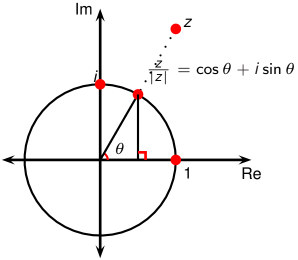
where $e \approx 2.71828$ *is Euler's/Napier's constant .*

Proof.

Recall $n! = 1 \cdot 2 \cdot 3 \cdot \dots \cdot (n-1) \cdot n$. Borrow from Calc II the f-las:

$$\begin{array}{rclcccccc}
i\sin x & = & & ix & & -i\frac{x^3}{3!} & & +i\frac{x^5}{5!} & -\dots \\
\cos x & = & 1 & & -\frac{x^2}{2!} & & +\frac{x^4}{4!} & & +\dots \\
\hline
e^{ix} & = & 1 & +ix & -\frac{x^2}{2!} & -i\frac{x^3}{3!} & +\frac{x^4}{4!} & +i\frac{x^5}{5!} & -\dots
\end{array}$$

Rearrange. Plug-in $z = ix$. Use $i^2 = -1$. Multiply $\sin x$ by i. Add to get $e^{ix} = \cos x + i \sin x$. $\square$



Theorem (Euler's formula)

$e^{i\theta} = \cos\theta + i\sin\theta.$

Lemma

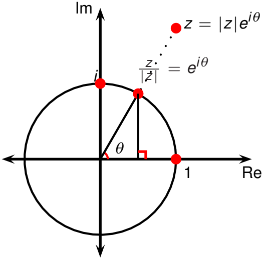
$$\left|\frac{z}{|z|}\right| = \frac{|z|}{|z|} = 1.$$

- Let $z = x + iy$ be a non-zero complex number.
- Then 0, z, $\frac{z}{|z|}$ lie on a ray and $\frac{z}{|z|}$ lies on the unit circle.
- Let θ - angle between the real axis and the ray between 0 and z.
- Then $\frac{z}{|z|} = \cos\theta + i\sin\theta = e^{i\theta}$.

Definition (Polar form of complex numbers)

Let $z \neq 0$. $z = |z|(\cos\theta + i\sin\theta) = |z|e^{i\theta}$ is called polar form of z.

- Let $\rho = \ln|z| = \ln\sqrt{x^2 + y^2} = \frac{1}{2}\ln(x^2 + y^2)$.
- Then $z = |z|(\cos\theta + i\sin\theta) = e^{\rho}(\cos\theta + i\sin\theta) = e^{\rho}e^{i\theta} = e^{\rho + i\theta}$.



Definition (Polar form of complex numbers)

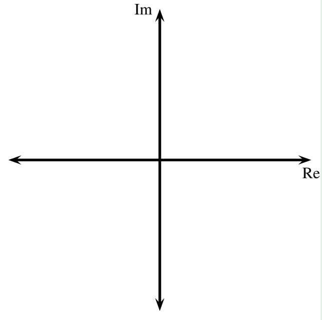
Let $z \neq 0$. Then $z = |z|(\cos\theta + i\sin\theta)$ is called polar form of z.

- θ is called an argument of z. We write

$$\theta = \arg z.$$

- If θ is an argument of z, so is $\theta + 2k\pi$ for all integers k.
- If $\theta \in (-\pi, \pi]$, we say that θ is the principal argument of z.
- If we write $\theta = \arg z$ without clarifying the choice of the argument, it is implied that θ is the principal argument of z, $\theta \in (-\pi, \pi]$.
- One should never write $\theta = \arg z$ without clarifying the choice of argument.

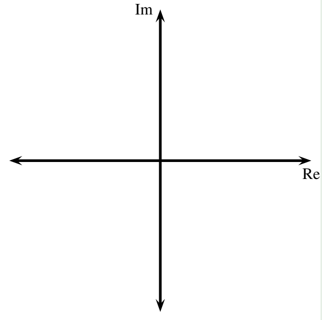
Example



Plot the number z. Write z in polar form, using the principal value of the argument of z (polar angle).
Recall that θ is the principal argument $\Rightarrow$ $\theta \in (-\pi, \pi]$.

z	$\|z\|$	θ	$\|z\|(\cos\theta + i\sin\theta)$
1	1	0	$\cos 0 + i \sin 0$
i	1	$\frac{\pi}{2}$	$\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)$
-1	1	π	$\cos\pi + i\sin\pi$
$-i$	1	$-\frac{\pi}{2}$	$\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)$

Example



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Recall that θ is the principal argument $\Rightarrow$ $\theta \in (-\pi, \pi]$.

z	$\lvert z \rvert$	θ	$\lvert z \rvert(\cos\theta + i\sin\theta)$
$\frac{1}{2} + \frac{\sqrt{3}}{2} i$	1	$\frac{\pi}{3}$	$\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)$
$1 + i$	2	$\frac{\pi}{4}$	$2\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)$
$1 - i$	2	$-\frac{\pi}{4}$	$2\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$
$-\sqrt{3} - i$	2	$-\frac{2\pi}{3}$	$2\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right)$
$\frac{3}{5} + \frac{4}{5} i$	5	$\arctan\left(\frac{4}{3}\right)$	$5\left(\cos\left(\arctan\left(\frac{4}{3}\right)\right) + i\sin\left(\arctan\left(\frac{4}{3}\right)\right)\right)$

Definition (Real exponent)

Let $\rho \in \mathbb{R}$. The real exponent e^ρ is defined as $\lim\limits_{\substack{p \to \rho \\ p \text{ is rational}}} e^p$.

Definition (Extension to $\mathbb{C}$)

Let $\rho, \theta \in \mathbb{R}$. Define the complex exponent $e^{\rho + i\theta}$ via
$e^{\rho+i\theta} = e^\rho(\cos\theta + i\sin\theta)$

- For the duration of this slide, assume Definition I of real exponent.
- Extend this def. to complex numbers (motivation: Euler's f-la).

Theorem

(a) Let $\alpha, \beta \in \mathbb{R}$. Then $e^{i\alpha}e^{i\beta} = e^{i\alpha+i\beta} = e^{i(\alpha+\beta)}$.
(b) Let $z, w \in \mathbb{C}$. Then $e^z e^w = e^{z+w}$.

Proof of (a).

$e^{i\alpha}e^{i\beta} = (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) = (\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\cos\alpha\sin\beta + \sin\alpha\cos\beta) = \cos(\alpha+\beta) + i\sin(\alpha+\beta) = e^{i(\alpha+\beta)}$. □

- The trig. f-las used above need separate (relatively long) proof.

Definition (Exponent, Def. II)

$$e^z = e(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Definition (Polar form)

$$|z| = e^\rho, \; z = e^\rho(\cos\theta + i\sin\theta)$$

- For the duration of this slide, assume Definition II of exponent.
- In preceding slides/lectures, by algebraic manipulations of series, we showed that $e(z)e(w) = e^z e^w = e^{z+w} = e(z+w)$.

Theorem

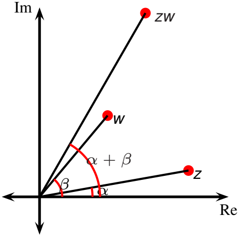
$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + i\sin\beta\cos\alpha$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta, \qquad \text{where } \alpha,\beta\in\mathbb{R}$$

Proof.

$$\begin{aligned}\cos(\alpha+\beta) + i\sin(\alpha+\beta) &= e^{i(\alpha+\beta)} = e^{i\alpha+i\beta} = e^{i\alpha}e^{i\beta}\\ &= (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)\\ &= (\cos\alpha\cos\beta - \sin\alpha\sin\beta)\\ &\quad + i(\cos\alpha\sin\beta + \sin\alpha\cos\beta).\end{aligned}$$

Compare real and imaginary part to get the desired trig identities. □

Geometric interpretation of complex multiplication



- Let $z, w \neq 0$ and let
$$|z| = e^{\rho}, \quad |w| = e^{\sigma}.$$
- Let α, β be arguments of z, w.
$$\begin{aligned} z &= e^{\rho}(\cos\alpha + i\sin\alpha) = e^{\rho+i\alpha} \\ w &= e^{\sigma}(\cos\beta + i\sin\beta) = e^{\sigma+i\beta}. \end{aligned}$$

Theorem (Summary)

$$\begin{aligned} zw &= |z|(\cos\alpha + i\sin\alpha)|w|(\cos\beta + i\sin\beta) \\ &= e^{\rho}(\cos\alpha + i\sin\alpha)e^{\sigma}(\cos\beta + i\sin\beta) = e^{\rho+i\alpha}e^{\sigma+i\beta} \\ &= e^{\rho+\sigma+i(\alpha+\beta)} = |z||w|(\cos(\alpha+\beta) + i\sin(\alpha+\beta)). \end{aligned}$$

- An argument (polar angle) of zw is $\alpha+\beta$.
- $\Rightarrow$ Multiplying complex numbers adds arguments (polar angles).
- Multiplying complex numbers multiplies absolute values.

Theorem (de Moivre's formula)

$(\cos\alpha + i\sin\alpha)^n = \cos(n\alpha) + i\sin(n\alpha).$

Proof.

$(\cos\alpha + i\sin\alpha)^n = \left(e^{i\alpha}\right)^n = e^{in\alpha} = \cos(n\alpha) + i\sin(n\alpha).$ $\square$

The formula is named after the French mathematician A. de Moivre (1667-1754).

Polar form $z = |z|(\cos\theta + i\sin\theta)$.

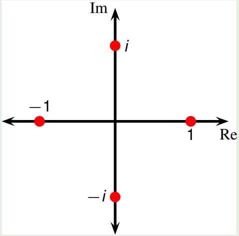
Example

Compute $\left(\sqrt{3}+i\right)^{2014}$ and its polar form.

Write $\sqrt{3}+i$ in polar form: $\sqrt{3}+i = 2\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right) = 2e^{i\frac{\pi}{6}}$.

$$\begin{aligned}
\left(\sqrt{3}+i\right)^{2014} &= \left(2e^{i\frac{\pi}{6}}\right)^{2014} \\
&= 2^{2014}e^{i2014\cdot\frac{\pi}{6}} \\
&= 2^{2014}\left(e^{i\left(335+\frac{2}{3}\right)\pi}\right) \\
&= 2^{2014}e^{i335\pi}e^{i\frac{2}{3}\pi} \\
&= 2^{2014}(-1)\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right) \\
&= -2^{2014}\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) \\
&= 2^{2013}\left(1-\sqrt{3}i\right).
\end{aligned}$$

Example



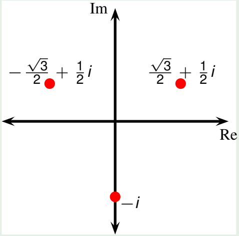
Find all complex solutions of the equation $z^4 = 1$.

Let $z = |z|(\cos\theta + i\sin\theta)$ be the polar form of z with $\theta \in (-\pi, \pi]$. Since $|z|^4 = |z^4| = 1$ it follows that $|z| = 1$ and so $z = \cos\theta + i\sin\theta$.

By de Moivre's equality $z^4 = \cos(4\theta) + i\sin(4\theta) = 1$. This implies $\sin(4\theta) = 0$, $\cos(4\theta) = 1$ and so $4\theta = 2k\pi$, k-integer. Therefore $\theta = k\frac{\pi}{2}$. Among those values, $\theta = -\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi$ belong to $(-\pi, \pi]$. We may discard the other values of θ as do not give rise to new points. Therefore the equation $z^4 = 1$ has 4 roots given by

$$\begin{aligned} z &= \cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right) = -i \\ z &= \cos 0 + i\sin 0 = 1 \\ z &= \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) = i \\ z &= \cos\pi + i\sin\pi = -1 \quad . \end{aligned}$$

Example



Find all complex numbers z such that $z^3 = i$. Let $z = |z|(\cos\theta + i\sin\theta)$ be the polar form of z for which $\theta \in (-\pi, \pi]$. We have that $1 = |i| = |z^3| = |z|^3$. Since $|z|$ is a positive real number, $|z|^3 = 1$ implies $|z| = 1$.

$$\begin{array}{rcl|l} i &=& (\cos(\theta) + i\sin(\theta))^3 & \text{de Moivre} \\ \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) &=& \cos(3\theta) + i\sin(3\theta) & \text{Polar form } i \\ 3\theta &=& \frac{\pi}{2} + 2k\pi & k \text{ any integer} \\ \theta &=& \frac{\pi}{6} + \frac{2k}{3}\pi & k \text{ any integer} \end{array}$$

Values of θ that differ by even multiple of π produce the same value for $z \Rightarrow$ restrict our attention to $\theta \in (-\pi, \pi]$, i.e. $k = 0, 1, -1 \Rightarrow$ $\theta = \frac{\pi}{6}, \frac{5\pi}{6}, -\frac{\pi}{2}$. Our final answer is **to be continued.**