Calculus III
Lecture 6

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Outline

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$\mathbb{R}^2, \mathbb{R}^3, \mathbb{R}^n$

Definition ($\mathbb{R}^2$)

The set of ordered pairs of real numbers is denoted by $\mathbb{R}^2$.

Definition ($\mathbb{R}^3$)

The set of ordered triples of real numbers is denoted by $\mathbb{R}^3$.

Definition ($\mathbb{R}^n$)

The set of ordered n-tuples of real numbers is denoted by $\mathbb{R}^n$.

Example

$(1, -2, 3) \in \mathbb{R}^3$
$(0, 5) \in \mathbb{R}^2$
$(0, 5, -2, 4, 0) \in \mathbb{R}^5$
$(0, 1, 2, 3, \dots, n) \in \mathbb{R}^{n+1}$

Parametric Equations of a Line Segment

- Recall parametric vector equation of line L:

$$\begin{aligned}
\mathbf{r} &= \mathbf{r}_0 + t\mathbf{u}, && t \text{ real number.}\\
\mathbf{r} &= \mathbf{r}_0 + t(\mathbf{r}_1 - \mathbf{r}_0), && t \text{ real number.}\\
\mathbf{r} &= (1-t)\mathbf{r}_0 + t\mathbf{r}_1, && t \text{ real number.}
\end{aligned}$$

- Parametric scalar equations:

$$\left|\begin{array}{l} x = x_0 + tu_1 \\ y = y_0 + tu_2 \\ z = z_0 + tu_3 \end{array}\right. \Leftrightarrow \left|\begin{array}{l} x = x_0 + t(x_1 - x_0) \\ y = y_0 + t(y_1 - y_0) \\ z = z_0 + t(z_1 - z_0) \end{array}\right. \Leftrightarrow \left|\begin{array}{l} x = (1-t)x_0 + tx_1 \\ y = (1-t)y_0 + ty_1 \\ z = (1-t)z_0 + tz_1 \end{array}\right.$$

- Segment with endpoints $P_0(\mathbf{r}_0)$ and $P_1(\mathbf{r}_1)$:

$$\mathbf{r} = (1-t)\mathbf{r}_0 + t\mathbf{r}_1, \quad 0 \leqslant t \leqslant 1$$

Example

Parametrize

- the line L passing through $P_0(1,2,3)$ and $P_1(5,2,1)$;
- the line segment S connecting $P_0(1,2,3)$ and $P_1(5,2,1)$.

Direction of L: $\mathbf{u} = \mathbf{r}_1 - \mathbf{r}_0 = (4,0,-2)$.Parametric vectorial equations of L:

$$\mathbf{r} = (1,2,3) + t(4,0-2) \Leftrightarrow \mathbf{r} = (1+4t, 2, 3-2t) \ .$$

Parametric scalar equations of line L:

$$\left\{\begin{array}{ll} x & = 1+4t \\ y & = 2 \\ z & = 3-2t \end{array}\right. , \quad t \text{ real number.}$$

Parametric vectorial equation of segment S:

$$\mathbf{r} = t(1,2,3) + (1-t)(5,2,1) \quad t \in [0,1] \quad .$$

Parametrized Curves

- A curve parametrization is a function $\mathbf{r} : [a, b] \to \mathbb{R}^2$ $\mathbf{r} : [a, b] \to \mathbb{R}^3$, or $\mathbf{r} : [a, b] \to \mathbb{R}^n$ in general.
- Input is scalar (parameter).
- Output is (position) vector.
- The image of $\mathbf{r}$ is a set of points in $\mathbb{R}^n$; we call that set curve image.
- The term “curve” is ambiguous and either means a curve parametrization or a curve image.
- $\mathbf{r}\colon [a, b] \to \mathbb{R}^2$ plane curve.
- $\mathbf{r}\colon [a, b] \to \mathbb{R}^3$ space curve.
- Function $\mathbf{r}$: parametrization of the curve image.
- t: parameter of the curve parametrization.
- $\mathbf{r}(t) = (x(t), y(t), z(t))$, $a \leq t \leq b$.
- $x, y, z\colon [a, b] \to \mathbb{R}$, coordinate functions.

Example

- $\mathbf{r}\colon \mathbb{R} \to \mathbb{R}^3$,

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{u}$$

- Line through $P(1, 4, 3)$ direction $\mathbf{u} = (-1, 2, 0)$:

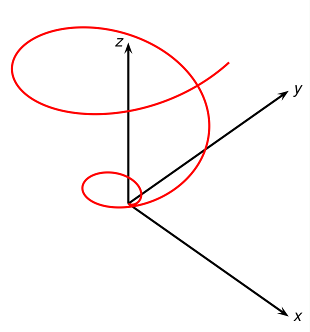
$$\begin{array}{rcl} \mathbf{r}(t) & = & (1 - t, 4 + 2t, 3) \\ \mathbf{p}(s) & = & (s, 6 - s, 3) \end{array}$$

- Curve parametrization is not unique!
- For example $t = 1 - s = \varphi(s)$ is a reparametrization.

Example

Describe the curve:

$$\begin{aligned} x(t) &= 3t\cos(2t) \\ y(t) &= 3t\sin(2t) \\ z(t) &= t^2 \end{aligned}$$



- Cylindrical coordinates:

$$\begin{aligned} r(t) &= 3t \\ \theta(t) &= 2t \\ z(t) &= t^2 \end{aligned}$$

- "Tornado".

Limits

Definition

We say that

$$\lim_{t \to a} \mathbf{r}(t) = \mathbf{u}$$

if by selecting that $t \neq a$ be close enough to a we can guarantee that $\mathbf{r}(t)$ is as close to $\mathbf{u}$ as we want.

In strict mathematical language: $\lim_{t\to a} \mathbf{r}(t) = \mathbf{u}$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all t with $0 < |t - a| < \delta$ we have that $|\mathbf{r}(t) - u| < \varepsilon$.

- We define the "postman distance" between (x_1, y_1, z_1) and (x_2, y_2, z_2) to be the number $\max(|x_1 - x_2|, |y_1 - y_2|, |z_1 - z_2|)$.
- Two points in Euclidean distance are close if and only if they are close in "postman distance".
- Unlike higher dimensions, in dimension 1 postman distance coincides with Euclidean distance.
- Let $\mathbf{r}(t) = (x(t), y(t), z(t))$ and $\mathbf{u} = (u_1, u_2, u_3)$.
- Then

$$\lim_{t\to a} \mathbf{r}(t) = \mathbf{u} \iff \left| \begin{array}{l} \lim\limits_{t\to a} x(t) = u_1 \\ \lim\limits_{t\to a} y(t) = u_2 \\ \lim\limits_{t\to a} z(t) = u_3 \end{array} \right. .$$

Continuity

Definition

Suppose

- $\mathbf{r}$ is defined at t_0
- $\lim_{t \to t_0} \mathbf{r}(t)$ exists.

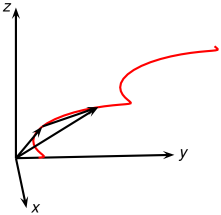
Then we say that $\mathbf{r} \colon [a, b] \to \mathbb{R}^3$ is continuous at t_0 if

$$\lim_{t \to t_0} \mathbf{r}(t) = \mathbf{r}(t_0) \quad .$$

Observation

$\mathbf{r}(t) = (x(t), y(t), z(t))$ is continuous at $t_0 \iff x(t), y(t), z(t)$ are all continuous at t_0.

Derivatives



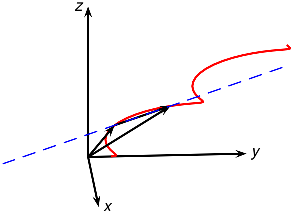
$$
\begin{array}{rcl}
\mathbf{f}\colon [a,b] & \to & \mathbb{R}^3 \\
\mathbf{f}(t) & = & (x(t), y(t), z(t)) \\
\\
\mathbf{f}'(t) & = & (x'(t), y'(t), z'(t)) \\
\mathbf{f}''(t) & = & (x''(t), y''(t), z''(t)) \\
& \vdots &
\end{array}
$$

- **Average Velocity** $= \dfrac{\textbf{change in position}}{\text{change in time}} = \dfrac{\mathbf{f}(t) - \mathbf{f}(t_0)}{t - t_0}$.
- Instantaneous rate of change:

$$\mathbf{f}'(t_0) = \lim_{t \to t_0} \frac{\mathbf{f}(t) - \mathbf{f}(t_0)}{t - t_0} = \lim_{h \to 0} \frac{\mathbf{f}(t_0 + h) - \mathbf{f}(t_0)}{h}.$$

- $\mathbf{f}(t)$ vector $\Longrightarrow$ $\mathbf{f}'(t)$ vector.
- Higher order derivatives: $\mathbf{f}'(t)$, $\mathbf{f}''(t) = (\mathbf{f}'(t))'$ (acceleration), **...**

Tangent Lines



$$\begin{aligned} \mathbf{f}\colon [a,b] &\to \mathbb{R}^3 \\ \mathbf{f}(t) &= (x(t), y(t), z(t)) \\ \mathbf{f}'(t) &= \lim_{h\to 0} \frac{\mathbf{f}(t+h)-\mathbf{f}(t)}{h} \\ &= (x'(t), y'(t), z'(t)) \end{aligned}$$

- $\mathbf{f}'(t_0)$: direction of tangent line through $\mathbf{f}(t_0)$.
- Tangent equation at $\mathbf{f}(t_0)$:
$$\mathbf{r}(t) = \mathbf{f}(t_0) + t\mathbf{f}'(t_0).$$
- Linear approximation:
$$\mathbf{f}(t) \approx L_{\mathbf{f},t_0}(t) = \mathbf{f}(t_0) + t\mathbf{f}'(t_0).$$
- A linear approximation is good if:
$$\lim_{t\to 0} \left| \frac{\mathbf{f}(t) - L_{\mathbf{f},t_0}(t)}{t} \right| = 0.$$
- Differentials:
$$\mathrm{d}\mathbf{f} = \mathbf{f}'\mathrm{d}t = (x', y', z')\mathrm{d}t.$$

Example

Let $\mathbf{r}(t)$ be the coordinate curves for the spherical coordinates, i.e., let

$$\begin{array}{rcl}\mathbf{e}_\rho(t) & = & (t\sin\phi\cos\theta, t\sin\phi\sin\theta, t\cos\phi)\\ \mathbf{e}_\phi(t) & = & (\rho\sin t\cos\theta, \rho\sin t\sin\theta, \rho\cos t)\\ \mathbf{e}_\theta(t) & = & (\rho\sin\phi\cos t, \rho\sin\phi\sin t, \rho\cos\phi)\end{array}$$

where ρ, ϕ, θ are regarded as constants and t as the curve parameter. Find $\mathbf{e}'_\rho(t)$, $\mathbf{e}'_\phi(t)$, $\mathbf{e}'_\theta(t)$. Compute $(\mathbf{e}'_\rho(\rho)\times\mathbf{e}'_\theta(\theta))\cdot\mathbf{e}'_\phi(\phi)$.

$$\begin{array}{rcl}\mathbf{e}'_\rho(t) & = & (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)\\ \mathbf{e}'_\phi(t) & = & (\rho\cos t\cos\theta, \rho\cos t\sin\theta, -\rho\sin t)\\ \mathbf{e}'_\theta(t) & = & (-\rho\sin\phi\sin t, \rho\sin\phi\cos t, 0)\\ (\mathbf{e}'_\rho(\rho)\times\mathbf{e}'_\phi(\phi))\cdot\mathbf{e}'_\theta(\theta) & = & \left|\begin{array}{ccc}\sin\phi\cos\theta & \sin\phi\sin\theta & \cos\phi\\ \rho\cos\phi\cos\theta & \rho\cos\phi\sin\theta & -\rho\sin\phi\\ -\rho\sin\phi\sin\theta & \rho\sin\phi\cos\theta & 0\end{array}\right|\\ & = & \rho^2\sin\phi\end{array}$$

Differentiation Rules

Component-wise operation $\implies$ same rules as for scalar output

Product Rules:

$$
\begin{aligned}
[f(t)\mathbf{r}(t)]' &= f'(t)\mathbf{r}(t) + f(t)\mathbf{r}'(t)\\
[\mathbf{u}(t)\cdot\mathbf{v}(t)]' &= \mathbf{u}'(t)\cdot\mathbf{v}(t) + \mathbf{u}(t)\cdot\mathbf{v}'(t)\\
[\mathbf{u}(t)\times\mathbf{v}(t)]' &= \mathbf{u}'(t)\times\mathbf{v}(t) + \mathbf{u}(t)\times\mathbf{v}'(t)
\end{aligned}
$$

Chain Rule:

$$[\mathbf{r}(f(t))]' = f'(t)\mathbf{r}'(f(t))$$

Example:

$$\frac{\mathrm{d}|\mathbf{r}(t)|}{\mathrm{d}t} = [\sqrt{\mathbf{r}(t)\cdot\mathbf{r}(t)}]' = [\sqrt{\square}]' = \frac{1}{2\sqrt{\square}}\square' = \frac{1}{2\sqrt{\mathbf{r}(t)\cdot\mathbf{r}(t)}}[\mathbf{r}(t)\cdot\mathbf{r}(t)]' =$$

$$= \frac{1}{2|\mathbf{r}(t)|}[\mathbf{r}'(t)\cdot\mathbf{r}(t) + \mathbf{r}(t)\cdot\mathbf{r}'(t)] = \frac{\mathbf{r}(t)\cdot\mathbf{r}'(t)}{|\mathbf{r}(t)|}$$

Application

$$|\mathbf{r}|' = \frac{\mathbf{r} \cdot \mathbf{r}'}{|\mathbf{r}|}$$

- Suppose a point has a trajectory on a sphere with center at origin. Therefore:

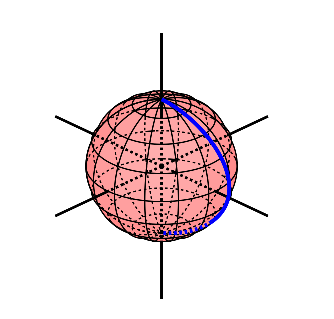
$$\begin{array}{rcl} |\mathbf{r}(t)| & = & \text{constant} \\ |\mathbf{r}(t)|' & = & 0 \\ \mathbf{r}(t) \cdot \mathbf{r}'(t) & = & 0 \end{array}$$

- In other words, $\mathbf{r}(t) =$ constant implies $\mathbf{r}(t) \perp \mathbf{r}'(t)$.
- Velocity $\perp$ Position $\iff$ Velocity vector $\mathbf{r}'(t)$ tangent to sphere.
- What can we say about constant acceleration?

$$\mathbf{r} \cdot \mathbf{r}' \equiv 0 \implies [\mathbf{r} \cdot \mathbf{r}']' = 0 \iff \mathbf{r}' \cdot \mathbf{r}' + \mathbf{r} \cdot \mathbf{r}'' = 0 \implies \mathbf{r} \cdot \mathbf{r}'' = -|\mathbf{r}'|^2 \leq 0$$

Acceleration vector $\mathbf{r}''$ points inside the sphere.

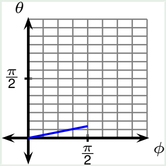
Example



Compute the acceleration vector when traversing the loxodrome curve below.

$$\begin{aligned} x &= \rho \sin(at)\cos(bt) \\ y &= \rho \sin(at)\sin(bt) \\ z &= \rho \cos(at) \end{aligned}$$

Spherical coordinates:



$$\begin{aligned} x &= \rho \sin\phi \cos\theta \\ y &= \rho \sin\phi \sin\theta \\ z &= \rho \cos\phi \end{aligned}$$

Line Integrals

$\mathbf{r}\colon [a,b] \to \mathbb{R}^3$

- Division $a = t_0 < t_1 < t_2 < \cdots < t_k < t_{k+1} < \cdots < t_n = b$
- Sample points $t_k \leq s_k \leq t_{k+1}$
- Riemann sum:

$$\sum_{k=0}^{n-1} (t_{k+1} - t_k)\mathbf{r}(s_k)$$

- Definite integral:

$$\int_{t=a}^{t=b} \mathbf{r}(t)\, dt = \lim \sum_{k=0}^{n-1} (t_{k+1} - t_k)\mathbf{r}(s_k)$$

- Result: a vector.

Line Integral Properties

- Component-wise: if $\mathbf{r}(t) = (x(t), y(t), z(t))$, then
  $$\int_a^b \mathbf{r}(t)\,\mathrm{d}t = \left(\int_a^b x(t)\,\mathrm{d}t, \int_a^b y(t)\,\mathrm{d}t, \int_a^b z(t)\,\mathrm{d}t\right)$$
- Substitution Rule:
  $$\int_a^b f'(t)\mathbf{r}(f(t))\,\mathrm{d}t = \int_{f(a)}^{f(b)} \mathbf{r}(\tau)\,d\tau$$
- Fundamental Theorem of Calculus: If $\mathbf{u}'(t) \equiv \mathbf{r}(t)$, then $\displaystyle\int_{t=a}^{t=b} \mathbf{r}(t)\,\mathrm{d}t = \mathbf{u}(b) - \mathbf{u}(a)$
- Derivative $\implies$ Total change
  $$\mathbf{r}(t) = \mathbf{r}(t_0) + \int_{t_0}^{t} \mathbf{r}'(\tau)\,\mathrm{d}\tau$$

Example

An object thrown from initial position $\mathbf{r}_0$ with initial velocity $\mathbf{v}_0$. Describe the trajectory of the object (ignore air resistance).

Total force: gravity: $m\mathbf{a} = \mathbf{F} = -mg\mathbf{k} \implies \mathbf{a} = -g\mathbf{k} \implies \mathbf{r}''(t) = -g\mathbf{k}$

$$\mathbf{r}'(t) = \mathbf{r}'(0) + \int_0^t -g\mathbf{k}\, d\tau = \mathbf{v}_0 + \left(\int_0^t -g\, d\tau\right)\mathbf{k} = \mathbf{v}_0 - gt\mathbf{k}$$

$$\mathbf{r}(t) = \mathbf{r}(0) + \int_0^t (\mathbf{v}_0 - g\tau\mathbf{k})\, d\tau = \mathbf{r}_0 + \left(\int_0^t d\tau\right)\mathbf{v}_0 - \left(\int_0^t g\tau\, d\tau\right)\mathbf{k} \implies$$

$$\boxed{\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}_0 - \frac{1}{2}gt^2\mathbf{k}}$$

Parabola in the plane determined by $\mathbf{v}_0$ and $\mathbf{k}$.

Arclength

- $\mathbf{r}\colon [a,b] \to \mathbb{R}^3$: piecewise smooth function
- Distance traveled $=$ Speed $\cdot$ Time

$$\mathrm{d}L = |\mathbf{r}'(t)|\,\mathrm{d}t \Longrightarrow L = \int_{t=a}^{t=b} |\mathbf{r}'(t)|\,\mathrm{d}t$$

- Let Δs = distance traveled along curve in short time Δt:

$$\Delta s \approx \sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2} = \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2 + \left(\frac{\Delta z}{\Delta t}\right)^2}\,\Delta t\ .$$

- Infinitesimal element of arclength ($\Delta t \to 0$):

$$\mathrm{d}s = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\,\mathrm{d}t = |\mathbf{r}'(t)|\,\mathrm{d}t$$

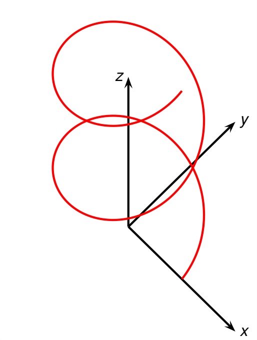
- Length of parametrized curve:

$$L = \int_{t=a}^{t=b} \mathrm{d}s = \int_a^b |\mathbf{r}'(t)|\,\mathrm{d}t\ .$$

Arclength Function

- Fix $t = a$ as starting point. For $t \geqslant a$, let
- $L(t) =$ distance traveled between a and t = length of the piece of the curve corresponding to values of the parameter between a and t.
$$L(t) = \int_a^t |\mathbf{r}'(\tau)| \, \mathrm{d}\tau$$
- The function $L\colon [a, b] \to \mathbb{R}$ is called the *arclength function*.

Example



Let $\mathbf{r}(t) = (\cos t, \sin t, t)$ (the curve is called a helix - (not a spiral)).

- Do you know the name of this curve?
- Find the arclength function.
- Find the length of the segment of the curve given by $t \in [0, 2\pi]$.

$$\begin{aligned}
\mathbf{r}'(t) &= (-\sin t, \cos t, 1) \\
\mathbf{r}'(t)| &= \sqrt{2} \quad . \\
L(t) &= \int_0^t |\mathbf{r}'(\tau)| d\tau \\
&= t\sqrt{2} \quad .
\end{aligned}$$

Parametrization by Arclength

- C: piecewise smooth parametrized curve joining points A and B;
- $\mathbf{r}\colon I \to C$: parametrization of C,

$$\mathbf{r}(t) = \text{position at time } t$$

  Not canonically defined: "depends who is driving".
- $\mathbf{p}\colon [0, L] \to C$:

$$\mathbf{p}(s) = \text{position at distance } s \text{ from } A \text{ along } C$$

  Canonically defined: "distance markers along the road".
- $\mathbf{p}$: parametrization by arclength

$$s = L(s) = \int_{\sigma=a}^{\sigma=s} |\mathbf{p}'(\sigma)|\, \mathrm{d}\sigma \implies 1 = L'(s) = |\mathbf{p}'(s)|\ .$$

Curve Reparametrizations

- Let C be a piecewise smooth curve joining pts A and B.
- Let $\mathbf{r}\colon [a,b] \to C$ be parametrization of C.
- Let $\mathbf{p}\colon [0,L] \to C$ be arclength parametrization.
- Question: How do we get $\mathbf{p}$ from $\mathbf{r}$?
- $Q = \mathbf{r}(t) = \mathbf{p}(s)$: point on curve C.

$$s = \text{distance from } A \text{ to } Q \text{ along } C = \int_{\tau=a}^{\tau=t} |\mathbf{r}'(\tau)|\, d\tau = \varphi(t)$$

- φ invertible and φ^{-1} smooth $\Longleftarrow \varphi'(t) \neq 0 \Longleftrightarrow \mathbf{r}'(t) \neq \mathbf{0}$.
- Regular parametrization: $\mathbf{r}'(t) \neq 0$ for all t
- $t = \varphi^{-1}(s) \Longrightarrow \mathbf{p}(s) = \mathbf{r}(\varphi^{-1}(s))$.

Example

Reparametrize $\mathbf{r}(t) = (\cos t, \sin t, t)$ via arclength.

- $|\mathbf{r}'(t)| = \sqrt{2} \Longrightarrow$

$$\varphi(t) = \int_{\tau=0}^{\tau=t} |\mathbf{r}'(\tau)| \, \mathrm{d}\tau = t\sqrt{2}$$

- $\varphi^{-1}(s) = \frac{s}{\sqrt{2}}$

$$\mathbf{p}(s) = \left(\cos\left(\frac{s}{\sqrt{2}}\right), \sin\left(\frac{s}{\sqrt{2}}\right), \left(\frac{s}{\sqrt{2}}\right)\right)$$

is a parametrization by arclength.

Curvature

- Question: How can we *measure* how bent a curve is?
- Answer: Measure change in tangent direction with respect to arclength.
- $\mathbf{p} = \mathbf{p}(s)$: parametrization by arclength of smooth curve C;
- $\mathbf{T}(s)$: unit tangent vector.
- $\mathbf{v}$: fixed direction (unit vector); $\alpha(s)$: angle between $\mathbf{T}(s)$ and $\mathbf{v}$.
- Define the *curvature* of C at $\mathbf{p}(s)$ to be

$$\kappa(s) = \left|\alpha'(s)\right| \quad .$$

- Alternative formulas:

$$\kappa(s) = |\mathbf{T}'(s)| = \left|\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s}\right| \quad .$$

- $\mathbf{r} = \mathbf{r}(t)$: smooth parametrization, not necessary by arclength.

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{|\mathbf{r}''(t) \times \mathbf{r}'(t)|}{|\mathbf{r}'(t)|^3} \quad .$$

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| \Rightarrow \kappa = \frac{|\mathbf{r}''(t)\times\mathbf{r}'(t)|}{|\mathbf{r}'(t)|^3}$$

$s(t) = \int_{t_0}^{t} |\mathbf{r}'(x)| dx$-curve (arc)length function.

$$|\mathbf{v}|' = \left(\sqrt{\mathbf{v}\cdot\mathbf{v}}\right)' = \frac{(\mathbf{v}\cdot\mathbf{v})'}{2\sqrt{\mathbf{v}\cdot\mathbf{v}}} = \frac{2\mathbf{v}'\cdot\mathbf{v}}{2|\mathbf{v}|} = \frac{\mathbf{v}'\cdot\mathbf{v}}{|\mathbf{v}|}. \quad \frac{ds}{dt} = |\mathbf{r}'(t)| \quad \frac{d\mathbf{T}}{ds}\frac{ds}{dt} = \frac{d\mathbf{T}}{dt}$$

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \left|\frac{\frac{d\mathbf{T}}{dt}}{\frac{ds}{dt}}\right| = \left|\frac{\mathbf{T}'(t)}{\mathbf{r}'(t)}\right| = \frac{\left|\left(\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}\right)'\right|}{|\mathbf{r}'(t)|} = \frac{\frac{\left|\mathbf{r}''|\mathbf{r}'|-\mathbf{r}'(|\mathbf{r}'|)'\right|}{|\mathbf{r}'|^2}}{|\mathbf{r}'|}$$

$$= \frac{\left|\mathbf{r}''|\mathbf{r}'| - \mathbf{r}'\frac{\mathbf{r}''\cdot\mathbf{r}'}{|\mathbf{r}'|}\right|}{|\mathbf{r}'|^3} = \frac{\sqrt{\left(\mathbf{r}''|\mathbf{r}'| - \mathbf{r}'\frac{\mathbf{r}''\cdot\mathbf{r}'}{|\mathbf{r}'|}\right)\cdot\left(\mathbf{r}''|\mathbf{r}'| - \mathbf{r}'\frac{\mathbf{r}''\cdot\mathbf{r}'}{|\mathbf{r}'|}\right)}}{|\mathbf{r}'|^3}$$

$$= \frac{\sqrt{|\mathbf{r}''|^2|\mathbf{r}'|^2 - 2\mathbf{r}''\cdot\mathbf{r}'\cancel{|\mathbf{r}'|}\frac{\mathbf{r}''\cdot\mathbf{r}'}{\cancel{|\mathbf{r}'|}} + \frac{\cancel{|\mathbf{r}'|^2}(\mathbf{r}''\cdot\mathbf{r}')^2}{\cancel{|\mathbf{r}|^2}}}}{|\mathbf{r}'|^3} = \frac{\sqrt{|\mathbf{r}''|^2|\mathbf{r}'|^2 - (\mathbf{r}''\cdot\mathbf{r}')^2}}{|\mathbf{r}'|^3}$$

$$= \frac{\sqrt{|\mathbf{r}''|^2|\mathbf{r}'|^2 - \mathbf{r}''|^2|\mathbf{r}'|^2\cos^2\alpha}}{|\mathbf{r}'|^3} = \frac{\sqrt{|\mathbf{r}''|^2|\mathbf{r}'|^2\sin^2\alpha}}{|\mathbf{r}'|^3} = \frac{|\mathbf{r}''(t)\times\mathbf{r}'(t)|}{|\mathbf{r}'(t)|^3}$$

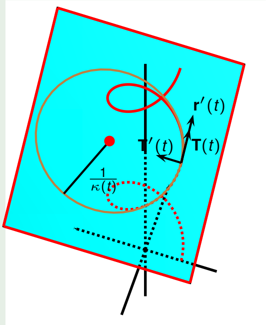
Notation: **T**- unit tangent vector, **r**- position vector, κ-curvature.

Example

Compute the curvature of $\mathbf{r}(t) = (\cos t, \sin t, t)$.

$$\begin{aligned}
\mathbf{r}'(t) &= (-\sin t, \cos t, 1) \\
|\mathbf{r}'(t)| &= \sqrt{2} \\
\mathbf{T}(t) &= \frac{1}{\sqrt{2}}(-\sin t, \cos t, 1)
\end{aligned}$$

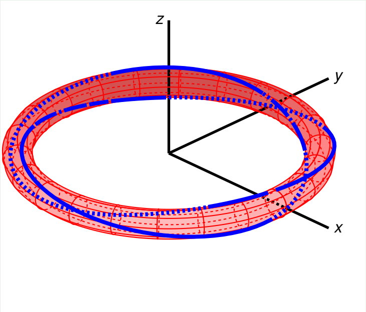
$$\begin{aligned}
\mathbf{T}'(t) &= \tfrac{1}{\sqrt{2}}(-\cos t, -\sin t, 0) \\
|\mathbf{T}'| &= \tfrac{1}{\sqrt{2}} \\
\kappa(t) &= \tfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \tfrac{1}{2} \quad .
\end{aligned}$$



Alternatively:

$$\begin{aligned}
\mathbf{r}'' &= (-\cos t, -\sin t, 0) \\
\mathbf{r}'' \times \mathbf{r}' &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\cos t & -\sin t & 0 \\ -\sin t & \cos t & 1 \end{vmatrix} \\
&= -\sin t\,\mathbf{i} + \cos t\,\mathbf{j} - \mathbf{k} \\
\kappa(t) &= \frac{|\mathbf{r}''(t) \times \mathbf{r}'(t)|}{|\mathbf{r}'(t)|^3} = \frac{\sqrt{2}}{(\sqrt{2})^3} = \frac{1}{2} \ .
\end{aligned}$$

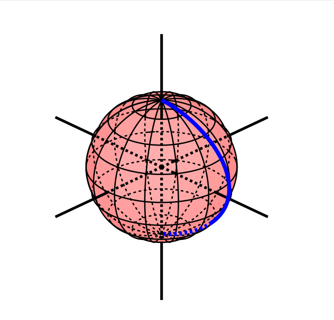
Example



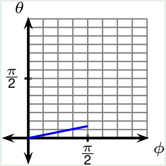
Compute the curvature of the (torus) trefoil curve

$$
\begin{aligned}
x &= (R + r\sin(3t))\cos(2t)\\
y &= (R + r\sin(3t))\sin(2t)\\
z &= r\cos(3t)
\end{aligned}
$$

Example



Spherical coordinates:



$x = \rho \sin\phi \cos\theta$
$y = \rho \sin\phi \sin\theta$
$z = \rho \cos\phi$

Compute the curvature of the loxodrome

$$\begin{aligned} x &= \rho \sin(at)\cos(bt) \\ y &= \rho \sin(at)\sin(bt) \\ z &= \rho\cos(at). \end{aligned}$$

Components of Acceleration

- Object moves through space, $\mathbf{r} = \mathbf{r}(t)$ position vector at time t;
- Velocity vector $\mathbf{v}(t) = \mathbf{r}'(t)$;
- Tangent direction: $\mathbf{T}(t) = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|}$;
- Speed is $v(t) = |\mathbf{v}(t)|$;
- Acceleration $\mathbf{a}(t) = \mathbf{r}''(t)$;
- Tangential component $\mathbf{a}_T(t)$:

$$\mathbf{a}_T(t) = \mathrm{proj}_{\mathbf{T}(t)}\mathbf{a}(t) = \frac{\mathbf{a} \cdot \mathbf{T}}{|\mathbf{T}|}\mathbf{T} = \frac{\mathbf{v}' \cdot \mathbf{v}}{|\mathbf{v}|}\mathbf{T} = |\mathbf{v}|'\mathbf{T} = v'\mathbf{T} ,$$

$$a_T(t) = |\mathbf{a}_T(t)| = |v'(t)| .$$

- Normal component $\mathbf{a}_N(t) = \mathrm{orth}_{\mathbf{T}(t)}\mathbf{a}(t)$,

$$a_N(t) = |\mathbf{a}_N(t)| = |\mathrm{orth}_{\mathbf{T}}\mathbf{a}| = |\mathbf{a}\times\mathbf{T}| = \frac{|\mathbf{r}'' \times \mathbf{r}'|}{|\mathbf{r}'|} = \kappa|\mathbf{r}'|^2 = \kappa(t)v^2(t) .$$