

# Precalculus

## Lecture 3

### Angle Sum Formulas

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<https://github.com/tmilev/freecalc>

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# Outline

- 1 Cofunction identities
- 2 Trigonometric Functions of Sums of Angles
- 3 Double Angle Formulas

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# Cofunction identities

## Proposition (Cofunction identities)

$$\begin{aligned}\sin\left(\frac{\pi}{2} - \alpha\right) &= \cos \alpha & \sin\left(\frac{\pi}{2} + \alpha\right) &= \cos \alpha \\ \cos\left(\frac{\pi}{2} - \alpha\right) &= \sin \alpha & \cos\left(\frac{\pi}{2} + \alpha\right) &= -\sin \alpha\end{aligned}$$

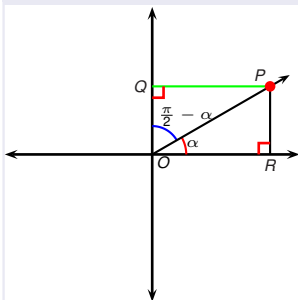
- The proof each formula is broken into 4 cases depending on which quadrant contains  $\alpha$ .
- This makes a total of 4 formulas  $\times$  4 cases per formula = 16 cases.
- We show only a few of the cases.
- The proof provides intuition why the formulas are true.
- The Quadrant I part of the proof serves as a visual aid for memorization.
- There is an algebraically simpler (but theoretically advanced) way to prove the above identities through the angle sum formulas, derived in turn from Euler's formula (studied later/in another course).

# Cofunction identities

## Proposition (Cofunction identities)

$$\begin{aligned}\sin\left(\frac{\pi}{2} - \alpha\right) &= \cos \alpha & \sin\left(\frac{\pi}{2} + \alpha\right) &= \cos \alpha \\ \cos\left(\frac{\pi}{2} - \alpha\right) &= \sin \alpha & \cos\left(\frac{\pi}{2} + \alpha\right) &= -\sin \alpha\end{aligned}$$

## Part of Proof.



We are showing  $\sin\left(\frac{\pi}{2} - \alpha\right) = \cos \alpha$  when  $\alpha$  is in quadrant I.

$$\begin{aligned}\sin\left(\frac{\pi}{2} - \alpha\right) &= \frac{|PQ|}{|OP|} & \left| \square ORPQ \right. \\ &= \frac{|OR|}{|OP|} \\ &= \cos \alpha & \left. \text{as desired} \right.\end{aligned}$$

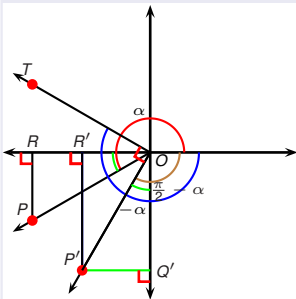


# Cofunction identities

## Proposition (Cofunction identities)

$$\begin{aligned}\sin\left(\frac{\pi}{2} - \alpha\right) &= \cos \alpha & \sin\left(\frac{\pi}{2} + \alpha\right) &= \cos \alpha \\ \cos\left(\frac{\pi}{2} - \alpha\right) &= \sin \alpha & \cos\left(\frac{\pi}{2} + \alpha\right) &= -\sin \alpha\end{aligned}$$

## Part of Proof.



We are showing  $\sin\left(\frac{\pi}{2} - \alpha\right) = \cos \alpha$  when  $\alpha$  is in Quadrant III. It follows  $\frac{\pi}{2} - \alpha$  is in Quadrant III.

$$\begin{aligned}\sin\left(\frac{\pi}{2} - \alpha\right) &= -\frac{|P'R'|}{|OP'|} = -\frac{|OQ'|}{|OP'|} \quad \square OR'P'Q' \\ &= -\frac{|OR|}{|OP|} \\ &= \cos \alpha\end{aligned}$$

as desired

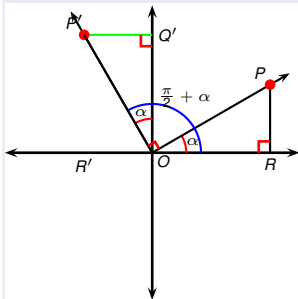


# Cofunction identities

## Proposition (Cofunction identities)

$$\begin{aligned}\sin\left(\frac{\pi}{2} - \alpha\right) &= \cos \alpha & \sin\left(\frac{\pi}{2} + \alpha\right) &= \cos \alpha \\ \cos\left(\frac{\pi}{2} - \alpha\right) &= \sin \alpha & \cos\left(\frac{\pi}{2} + \alpha\right) &= -\sin \alpha\end{aligned}$$

## Part of Proof.



We show  $\cos\left(\frac{\pi}{2} + \alpha\right) = -\sin \alpha$  when  $\alpha$  is in Quadrant I. It follows  $\frac{\pi}{2} + \alpha$  is in Quadrant II.

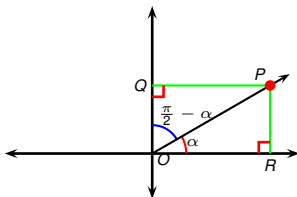
$$\begin{aligned}\cos\left(\frac{\pi}{2} + \alpha\right) &= -\frac{|OR'|}{|OP'|} && \square ORPQ \\ &= -\frac{|P'Q'|}{|OP'|} \\ &= -\frac{|PR|}{|OP|} \\ &= -\sin \alpha. && \text{as desired } \square\end{aligned}$$

# Cofunction identities

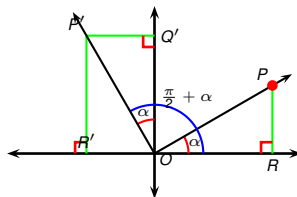
## Proposition (Cofunction identities)

$$\begin{aligned}\sin\left(\frac{\pi}{2} - \alpha\right) &= \cos \alpha & \sin\left(\frac{\pi}{2} + \alpha\right) &= \cos \alpha \\ \cos\left(\frac{\pi}{2} - \alpha\right) &= \sin \alpha & \cos\left(\frac{\pi}{2} + \alpha\right) &= -\sin \alpha\end{aligned}$$

To memorize the cofunction identities it suffices to memorize the Quadrant I case via the two diagrams below.



$$\begin{aligned}\sin\left(\frac{\pi}{2} - \alpha\right) &= \frac{|PQ|}{|OP|} \\ \cos\left(\frac{\pi}{2} - \alpha\right) &= \frac{|OQ|}{|OP|}\end{aligned}$$



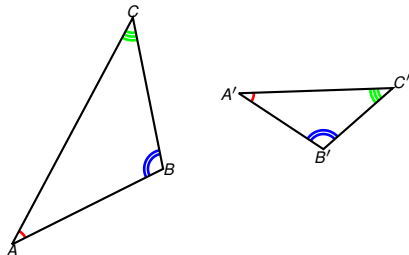
$$\begin{aligned}\sin\left(\frac{\pi}{2} + \alpha\right) &= \frac{|OQ'|}{|OP|} \\ \cos\left(\frac{\pi}{2} + \alpha\right) &= -\frac{|PQ'|}{|OP'|} = \frac{|PR|}{|OP|}\end{aligned}$$



## Definition (Similar triangles)

We say that a triangle  $\triangle ABC$  is similar to a triangle  $\triangle A'B'C'$  if the two triangles have equal angles.

- The equal angles are assumed given in the same order for both triangles, that is,  $\angle ABC = \angle A'B'C'$ ,  $\angle BCA = \angle B'C'A'$ ,  $\angle CAB = \angle C'A'B'$ .

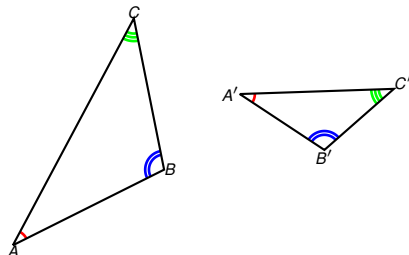


The following statement is proved in the subject of Euclidean (planar) geometry.

### Theorem (Similar triangles have equal side ratios)

*Let  $\triangle ABC$  and  $\triangle A'B'C'$  be two similar triangles. Then the ratios of the lengths of the sides of the two triangles are equal, that is*

$$\frac{|AB|}{|BC|} = \frac{|A'B'|}{|B'C'|} \quad \frac{|BC|}{|CA|} = \frac{|B'C'|}{|C'A'|} \quad \frac{|CA|}{|AB|} = \frac{|C'A'|}{|A'B'|}$$



$$\begin{array}{l|l} |DA| = |QB| & \square DABQ \\ = \sin \alpha |CB| & \triangle CQB \\ = \sin \alpha \sin \beta |OC| & \triangle OBC \\ = \sin \alpha \sin \beta & \end{array}$$

# Trig Functions of Sums and Differences of Angles

## Theorem

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$$

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

- We gave a geometric proof of the sum formulas when the two angles are acute and their sum is less than  $\pi = 90^\circ$ .
- The theorem holds for all angles  $\alpha, \beta$  without any restrictions.
- This can be shown by combining the preceding proof with identities such as  $\cos\left(\frac{\pi}{2} - \alpha\right) = \sin \alpha$ ,  $\cos\left(\frac{\pi}{2} + \alpha\right) = -\sin \alpha$ .
- There is a theoretically more advanced (but algebraically simpler) proof using Euler's formula (to be studied later/in another course).
- The difference formulas are a consequence of the sum formulas and the fact that  $\sin$  is an odd function and  $\cos$  is even.

# Trig Functions of Differences of Angles

## Example

Prove the identities

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$$

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

from the (already demonstrated) identities

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$\sin(\alpha - \beta) = \sin(\alpha + (-\beta))$$

$$= \sin \alpha \cos(-\beta) + \cos \alpha \sin(-\beta) \quad \left| \begin{array}{l} \cos \text{ is even ,} \\ \sin \text{ is odd} \end{array} \right.$$

$$= \sin \alpha \cos \beta - \cos \alpha \sin \beta$$

$$\cos(\alpha - \beta) = \cos(\alpha + (-\beta))$$

$$= \cos \alpha \cos(-\beta) - \sin \alpha \sin(-\beta) \quad \left| \begin{array}{l} \cos \text{ is even ,} \\ \sin \text{ is odd} \end{array} \right.$$

$$= \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

## Example

Find the exact value of the trigonometric function using radicals.

$$\cos(105^\circ) = \cos(45^\circ + 60^\circ)$$

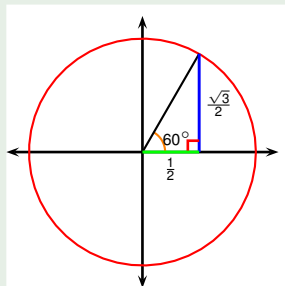
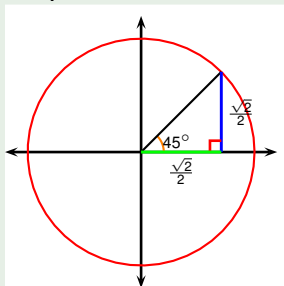
$$= \cos(45^\circ) \cos(60^\circ) - \sin(45^\circ) \sin(60^\circ)$$

$$= \frac{\sqrt{2}}{2} \cdot \frac{1}{2} - \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2}$$

$$= \frac{\sqrt{2} - \sqrt{6}}{4}$$

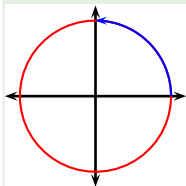
we know the trig  
f-ns of  $45^\circ$  and  $60^\circ$

Angle sum f-la



## Example

Use the angle sum/difference formulas to simplify.

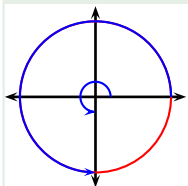


$$\begin{aligned}\cos\left(\frac{\pi}{2} - x\right) &= \cos\left(\frac{\pi}{2}\right)\cos x + \sin\left(\frac{\pi}{2}\right)\sin x \\ &= 0 \cdot \cos(x) + 1 \cdot \sin x \\ &= \sin x\end{aligned}$$

## Example

Use the angle sum/difference formulas to simplify.

$$\begin{aligned}
 \cot \left( \frac{3\pi}{2} + x \right) &= \frac{\cos \left( \frac{3\pi}{2} + x \right)}{\sin \left( \frac{3\pi}{2} + x \right)} \\
 &= \frac{\cos \left( \frac{3\pi}{2} \right) \cos x - \sin \left( \frac{3\pi}{2} \right) \sin x}{\sin \left( \frac{3\pi}{2} \right) \cos x + \cos \left( \frac{3\pi}{2} \right) \sin x} \\
 &= \frac{0 \cdot \cos x - (-1) \sin x}{(-1) \cos x + 0 \cdot \sin x} \\
 &= \frac{-\cos x}{-\sin x} = -\frac{\sin x}{\cos x} \\
 &= -\tan x
 \end{aligned}$$





## Example

Show that  $\tan(\pi + x) = \tan x$  using the angle sum formulas.

$$\begin{aligned}\tan(\pi + x) &= \frac{\sin(\pi + x)}{\cos(\pi + x)} \\&= \frac{\sin \pi \cos x + \cos \pi \sin x}{\cos \pi \cos x - \sin \pi \sin x} \\&= \frac{0 \cdot \cos x + (-1) \cdot \sin x}{(-1) \cdot \cos x - 0 \cdot \sin x} \\&= \frac{-\sin x}{-\cos x} \\&= \frac{\sin x}{\cos x} \\&= \tan x,\end{aligned}$$

as desired.

### Proposition ( $\tan, \cot$ are $\pi$ -periodic)

*The tangent and cotangent functions are  $\pi$ -periodic, in other words,*

$$\tan(\theta + \pi) = \tan \theta$$

$$\cot(\theta + \pi) = \cot \theta$$

Recall the angle sum formula  $\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$ .

### Example

Show that the Pythagorean identity  $\sin^2 \theta + \cos^2 \theta = 1$  follows from the angle difference formula.

$$\begin{aligned} 1 &= \cos 0 \\ &= \cos(\theta - \theta) \\ &= \cos \theta \cos \theta + \sin \theta \sin \theta \\ &= \cos^2 \theta + \sin^2 \theta, \end{aligned}$$

as desired.

## Example

Prove the angle sum formula  $\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$ .

$$\begin{aligned}
 \tan(\alpha + \beta) &= \frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)} \\
 &= \frac{(\sin \alpha \cos \beta + \cos \alpha \sin \beta) \frac{1}{\cos \alpha \cos \beta}}{(\cos \alpha \cos \beta - \sin \alpha \sin \beta) \frac{1}{\cos \alpha \cos \beta}} \\
 &= \frac{\frac{\sin \alpha \cancel{\cos \beta}}{\cos \alpha \cancel{\cos \beta}} + \frac{\cancel{\cos \alpha} \sin \beta}{\cancel{\cos \alpha} \cos \beta}}{\frac{\cancel{\cos \alpha} \cos \beta}{\cancel{\cos \alpha} \cos \beta} - \frac{\sin \alpha \sin \beta}{\cos \alpha \cos \beta}} \\
 &= \frac{\frac{\sin \alpha}{\cos \alpha} + \frac{\sin \beta}{\cos \beta}}{1 - \frac{\sin \alpha}{\cos \alpha} \cdot \frac{\sin \beta}{\cos \beta}} \\
 &= \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}
 \end{aligned}$$

# Double angle formulas

## Proposition (Double angle formulas)

$$\begin{aligned}\sin(2\alpha) &= 2 \sin \alpha \cos \alpha \\ \cos(2\alpha) &= \cos^2 \alpha - \sin^2 \alpha \\ &= 2 \cos^2 \alpha - 1 \\ &= 1 - 2 \sin^2 \alpha\end{aligned}$$

- The double angle formulas play a special role in integration.

## Example

Derive the double-angle formulas.

$$\begin{aligned}\sin(2\alpha) &= \sin(\alpha + \alpha) \\ &= \sin \alpha \cos \alpha + \cos \alpha \sin \alpha \\ &= 2 \sin \alpha \cos \alpha\end{aligned}$$

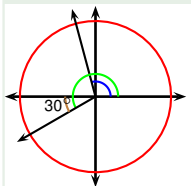
$$\begin{aligned}\cos(2\alpha) &= \cos(\alpha + \alpha) \\ &= \cos \alpha \cos \alpha - \sin \alpha \sin \alpha \\ &= \cos^2 \alpha - \sin^2 \alpha \\ &= \cos^2 \alpha - (1 - \cos^2 \alpha) \\ &= 2 \cos^2 \alpha - 1 \\ &= 1 - \sin^2 \alpha - \sin^2 \alpha \\ &= 1 - 2 \sin^2 \alpha\end{aligned}$$

Recall the half angle formula  $\cos \alpha = \pm \sqrt{\frac{1 + \cos(2\alpha)}{2}}$ .

## Example

Using radicals, find the exact value of the trigonometric expression.

$$\begin{aligned}
 \cos 105^\circ &= \pm \sqrt{\frac{1 + \cos(2 \cdot 105^\circ)}{2}} && \left| \cos 105^\circ < 0 \right. \\
 &= -\sqrt{\frac{1 + \cos(210^\circ)}{2}} \\
 &= -\sqrt{\frac{1 - \cos(30^\circ)}{2}} \\
 &= -\sqrt{\frac{1 - \frac{\sqrt{3}}{2}}{2}} = -\sqrt{\frac{2 - \sqrt{3}}{2 \cdot 2}} \\
 &= -\frac{\sqrt{2 - \sqrt{3}}}{2}
 \end{aligned}$$



## Proposition (Power-Reducing Formulas)

$$\sin^2 \alpha = \frac{1 - \cos(2\alpha)}{2} \quad \cos^2 \alpha = \frac{1 + \cos(2\alpha)}{2}$$

### Proof.

$$\begin{aligned} \cos(2\alpha) &= 1 - 2\sin^2 \alpha & \cos(2\alpha) &= 2\cos^2 \alpha - 1 \\ 2\sin^2 \alpha &= 1 - \cos(2\alpha) & 2\cos^2 \alpha &= 1 + \cos(2\alpha) \\ \sin^2 \alpha &= \frac{1 - \cos(2\alpha)}{2} & \cos^2 \alpha &= \frac{1 + \cos(2\alpha)}{2} \end{aligned}$$



### Corollary

$$\sin \alpha = \pm \sqrt{\frac{1 - \cos(2\alpha)}{2}} \quad \cos \alpha = \pm \sqrt{\frac{1 + \cos(2\alpha)}{2}}$$

### Corollary (Half-Angle Formulas)

$$\sin\left(\frac{\beta}{2}\right) = \pm \sqrt{\frac{1 - \cos \beta}{2}} \quad \cos\left(\frac{\beta}{2}\right) = \pm \sqrt{\frac{1 + \cos \beta}{2}}$$



## Proposition (Power-Reducing Formulas)

$$\sin^2 \alpha = \frac{1 - \cos(2\alpha)}{2} \quad \cos^2 \alpha = \frac{1 + \cos(2\alpha)}{2}$$

- The power reducing formulas are used to express  $\sin^k \alpha$  and  $\cos^k \alpha$  via lower powers of the sin and cos functions (applied to angles other than  $\alpha$ ).
- This technique will play a key role in integration (studied later/in another course).

Recall the formulas:  $\sin^2 \beta = \frac{1 - \cos(2\beta)}{2}$ ,  $\cos^2 \beta = \frac{\cos(2\beta) + 1}{2}$ .

## Example

Rewrite  $\sin^4 \alpha$  in terms of first powers of the cosines and sines of multiples of the angle  $\alpha$ .

$$\begin{aligned}\sin^4 \alpha &= (\sin^2 \alpha)^2 \\&= \left( \frac{1 - \cos(2\alpha)}{2} \right)^2 \\&= \frac{1}{4} (1 - 2\cos(2\alpha) + \cos^2(2\alpha)) \\&= \frac{1}{4} \left( 1 - 2\cos(2\alpha) + \frac{\cos(2 \cdot 2\alpha) + 1}{2} \right) \\&= \frac{1}{4} \left( 1 - 2\cos(2\alpha) + \frac{\cos(2 \cdot 2\alpha)}{2} + \frac{1}{2} \right) \\&= \frac{1}{4} \left( \frac{3}{2} - 2\cos(2\alpha) + \frac{\cos(4\alpha)}{2} \right) \\&= \frac{1}{8} (3 - 4\cos(2\alpha) + \cos(4\alpha))\end{aligned}$$