Precalculus
Lecture 11
Logarithms

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Outline

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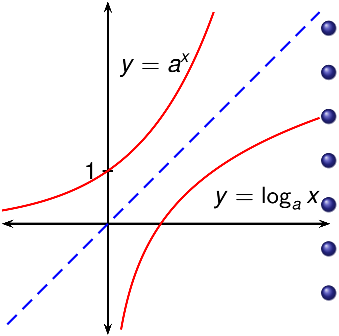
Logarithmic Functions



- Suppose $a > 0$, $a \neq 1$.
- Let $f(x) = a^x$.
- Then f is either increasing or decreasing.
- Therefore f is one-to-one.
- Therefore f has an inverse function, f^{-1}.
- The graph shows $y = a^x$ for $a > 1$.
- The graph of $y = \log_a x$ is the reflection of this in the line $y = x$.

Definition ($\log_a x$)

The inverse function of $f(x) = a^x$ is called the logarithmic function with base a, and is written $\log_a x$. It is defined by the formula

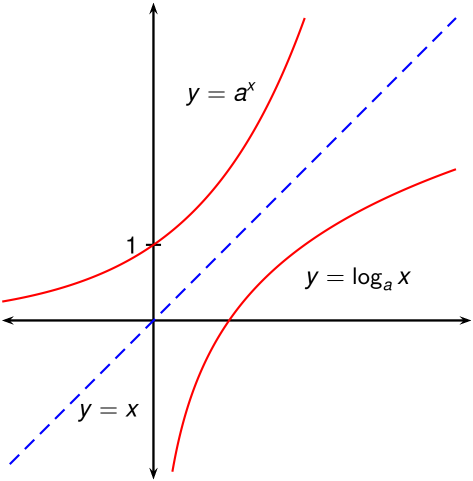
$$\log_a x = y \qquad \Leftrightarrow \qquad a^y = x.$$

If $x > 0$, then $\log_a x$ is the exponent to which the base a must be raised to give x.

Example

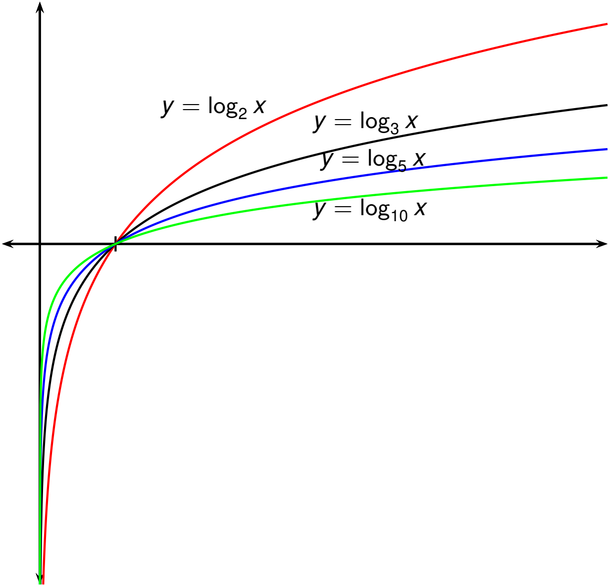
Evaluate:

1. $\log_3 81 = 4$ because $3^4 = 81$.
2. $\log_{25} 5 = \frac{1}{2}$ because $25^{\frac{1}{2}} = \sqrt{25} = 5$.
3. $\log_{10} 0.001 = -3$ because $10^{-3} = 0.001$.



- Suppose $a > 1$.
- Domain of a^x: $\mathbb{R}$.
- Range of a^x: $(0, \infty)$.
- Domain of $\log_a x$: $(0, \infty)$.
- Range of $\log_a x$: $\mathbb{R}$.
- $\log_a(a^x) = x$ for $x \in \mathbb{R}$.
- $a^{\log_a x} = x$ for $x > 0$.

Graphs of various logarithmic functions with $a > 1$

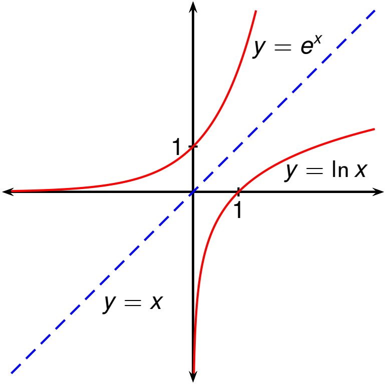


Natural Logarithms

Definition ($\ln x$)

The logarithm with base e is called the natural logarithm, and has a special notation:

$$\log_e x = \ln x.$$



- $\ln x = y \quad \Leftrightarrow \quad e^y = x$.
- $\ln(e^x) = x$ for $x \in \mathbb{R}$.
- $e^{\ln x} = x$ for $x > 0$.

What does $\log x$ stand for? **WARNING:** there are **two different** accepted uses for $\log x$.

- In some texts/applications $\log x$ stands for

$$\log x = \log_{10} x \quad .$$

- Used in many engineering texts.
- Used in many natural sciences texts.
- Used in many high school textbooks.
- Used in old math textbooks.

- In other texts/applications $\log x$ stands for (the principal branch of the) **complex logarithm**

$$\log x = \begin{cases} \ln x = \log_e x & \text{if } x > 0 \\ \ln(-x) + \pi i & \text{if } x < 0 \\ \textbf{?} & \text{for } x \notin \mathbb{R} \end{cases} \quad .$$

  - Used in mathematical, many computer science texts.
  - Used in many natural science texts.
  - Used in most computer algebra systems.
  - This is the notation accepted by most mathematicians.
- $\log$ and $\ln$ have different domains but else coincide: $\ln$ is defined for positive reals, and $\log$ - for non-zero complex.

- *In the present course we shall abstain from using the notation $\log x$.*
- *When we need logarithms base 10 we will always write $\log_{10}$.*

- Within this course, we request that the student abstain from using $\log x$ and use instead the unambiguous $\log_{10} x$.
- Outside of this course, we recommend that the student continue avoiding the notation $\log$.
- Should our recommendation contradict the commonly accepted conventions in the field of study of the student, we expect the student to honor the conventions of their fields of study.

Summary of logarithm notation conventions

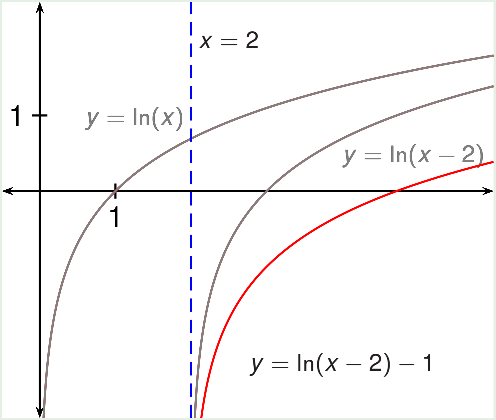
	Name	ISO notation	Other notation	Used in
$\log_2(x)$	binary logarithm	$\mathrm{lb}(x)$	$\mathrm{ld}(x)$, $\log(x)$, $\lg(x)$	computer science, information theory, music theory, photography
$\log_e(x)$	natural logarithm	$\ln(x)$	$\log(x)$	mathematics, physics, chemistry, statistics, economics, information theory, and engineering
$\log_{10}(x)$	common logarithm	$\lg(x)$	$\log(x)$	various engineering, logarithm tables, handheld calculators, spectroscopy

Table source: Wikipedia

- Standardized in ISO_31-11 (International Standards Organization).

Example

Draw the graph of $y = \ln(x - 2) - 1$.



- Graph $y = \ln(x)$ assumed known.
- $f(x - 2)$ shifts graph 2 units to the right.
- $g(x) - 1$ shifts graph 1 unit down.

Theorem (Properties of Logarithmic Functions)

If $a > 1$, the function $f(x) = \log_a x$ is a one-to-one, continuous, increasing function with domain $(0, \infty)$ and range $\mathbb{R}$. If $x, y, a, b > 0$ and r is any real number, then

1. $\log_a(xy) = \log_a x + \log_a y$.
2. $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$.
3. $\log_a(x^r) = r \log_a x$.
4. $\log_a(x) = \log_b x \log_a b = \frac{\log_b x}{\log_b a} = \frac{\ln x}{\ln a}$.

Recall that $\log_a(x) = \log_b x \log_a b = \frac{\log_b x}{\log_b a} = \frac{\ln x}{\ln a}$.

Example

Using only the ln and arithmetic operations of your calculator, compute $\log_5(13)$. Confirm your answer by exponentiation.

$$\log_5(13) = \frac{\ln 13}{\ln 5} \approx \frac{2.564949357}{1.609437912} \approx 1.593693.$$

As a check of our computations, we compute by calculator:
$13 = 5^{\log_5 13} \approx 5^{1.593693} \approx 13.000007508$, and our computations check out.

Use the properties of logarithms to evaluate the following.

Example

$$\begin{aligned} \log_4 2 + \log_4 32 &= \log_4(2 \cdot 32) \\ &= \log_4(64) \\ &= 3 \\ & \text{(because } 4^3 = 64.) \end{aligned}$$

Example

$$\begin{aligned} \log_2 80 - \log_2 5 &= \log_2\left(\frac{80}{5}\right) \\ &= \log_2(16) \\ &= 4 \\ & \text{(because } 2^4 = 16.) \end{aligned}$$

Example

Compute the exact value of the expression as a rational number.

$$\begin{aligned}\log_7 \sqrt[3]{49} &= \log_7 \left(49^{\frac{1}{3}}\right)\\ &= \frac{1}{3}\log_7 49\\ &= \frac{1}{3}\log_7 7^2\\ &= \frac{2}{3}\log_7 7\\ &= \frac{2}{3}\end{aligned}$$

Example

Fully expand the expression to a sum of logarithms. Your answer should not contain logarithms of products or logarithms of exponents.

$$\begin{aligned}\ln\left(\frac{y\sqrt{1+x}}{z^2}\right) &= \ln\left(y\sqrt{1+x}\right)-\ln\left(z^2\right)\\ &= \ln y+\ln\sqrt{1+x}-2\ln z\\ &= \ln y+\frac{1}{2}\ln(1+x)-2\ln z\end{aligned}$$

The inverse hyperbolic function $\operatorname{arcsinh} = \ln\left(x+\sqrt{1+x^2}\right)$ is used when studying hyperbolas (types of curves in the plane).

Example

Demonstrate that $-\ln\left(\sqrt{1+x^2}-x\right) = \ln\left(x+\sqrt{1+x^2}\right)$.

$$\begin{aligned}
-\ln\left(\sqrt{1+x^2}-x\right) &= \ln\left(\frac{1}{\sqrt{x^2+1}-x}\right) \quad \Bigg| \text{ rationalize} \\
&= \ln\left(\frac{\left(\sqrt{x^2+1}+x\right)}{\left(\sqrt{x^2+1}-x\right)\left(\sqrt{x^2+1}+x\right)}\right) \\
&= \ln\left(\frac{\sqrt{x^2+1}+x}{x^2+1-x^2}\right) \\
&= \ln\left(\sqrt{x^2+1}+x\right) \quad .
\end{aligned}$$

Proposition (Additional Properties of Logarithmic Functions)

If $a, b > 0$*, then*

1. $\log_{\frac{1}{a}} x = -\log_a x$
2. $\log_a b = \frac{1}{\log_b a}$.
3. $\log_{a^k} b = \frac{1}{k} \log_a b$.

Example

Compute as a rational number, without using calculator.

$$\log_{\frac{1}{\sqrt[3]{49}}} \sqrt{343} \quad =$$

Example

Compute as a rational number, without using calculator.

$$
\begin{aligned}
\log_7(24)+\log_{\frac{1}{7}}(3)-\log_{49}(64) &= \log_7(24)+\frac{\log_7(3)}{\log_7\left(\frac{1}{7}\right)}-\frac{\log_7(64)}{\log_7(49)}\\
&= \log_7(24)+\frac{\log_7(3)}{-1}-\frac{\log_7(64)}{2}\\
&= \log_7(24)-\log_7(3)-\frac{1}{2}\log_7(64)\\
\left[\begin{aligned}&\log_a x-\log_a y=\log_a\left(\tfrac{x}{y}\right)\\ &\log_a x^r = r\log_a x\end{aligned}\right] \qquad &= \log_7\left(\frac{24}{3}\right)-\log_7\left(64^{\frac{1}{2}}\right)\\
&= \log_7(8)-\log_7\left(\sqrt{64}\right)\\
&= \log_7 8-\log_7 8 = 0\\
\text{[alternatively:]} \qquad &= \log_7\left(\frac{8}{8}\right)=\log_7(1)=0.
\end{aligned}
$$

Example

Prove the logarithmic properties.

1. $\log_a(xy) = \log_a x + \log_a y$.
2. $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$.
3. $\log_a(x^r) = r\log_a x$.
4. $\log_a(x) = \log_b x \log_a b = \frac{\log_b x}{\log_b a} = \frac{\ln x}{\ln a}$.