

# Precalculus

## Lecture 6

### Inverse Functions

Todor Milev

<https://github.com/tmilev/freecalc>

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# Outline

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## Inverse Functions

- One-to-one Functions
- The Definition of the Inverse of  $f$

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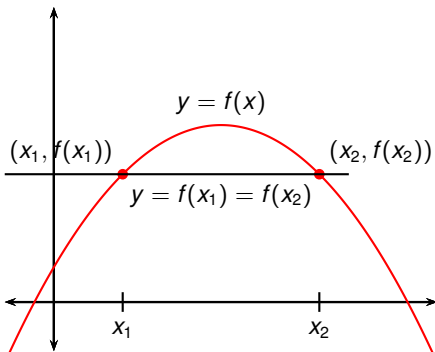
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# One-to-one Functions

## Definition (One-to-one Function)

A function  $f$  is a one-to-one function if it never takes on the same value twice; that is,

$$f(x_1) \neq f(x_2) \text{ whenever } x_1 \neq x_2.$$



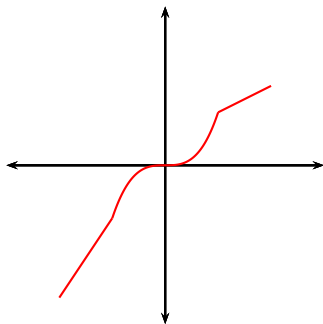
← This function is not one-to-one.

Question: How can we tell from the graph of a function whether it is one-to-one or not?

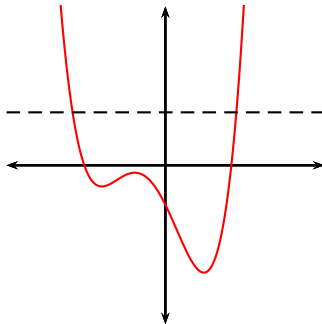
Answer: Use the horizontal line test.

### The Horizontal Line Test.

A function is one-to-one if and only if no horizontal line intersects it more than once.



One-to-one



Not one-to-one

# The Definition of the Inverse of $f$

## Definition ( $f^{-1}$ )

Let  $f$  be a one-to-one function with domain  $A$  and range  $B$ . Then the inverse of  $f$  is the function  $f^{-1}$  that has domain  $B$  and range  $A$  and is defined by

$$f^{-1}(y) = x \quad \Leftrightarrow \quad f(x) = y$$

for all  $y$  in  $B$ .

Note:

- Only one-to-one functions have inverses.
- $f^{-1}$  reverses the effect of  $f$ .
- domain of  $f^{-1}$  = range of  $f$ .
- range of  $f^{-1}$  = domain of  $f$ .

## Example ( $f(x) = x^3$ )

The inverse of  $f(x) = x^3$  is  $f^{-1}(x) = \sqrt[3]{x}$ . This is because if  $y = x^3$ , then

$$f^{-1}(y) = \sqrt[3]{y} = \sqrt[3]{x^3} = x.$$

The inverse of  $f$  is denoted as  $f^{-1}$ . This notation is one of the most frequent causes of student confusion. **WARNING:**

$$f^{-1}(x) \text{ does not mean } (f(x))^{-1} = \frac{1}{f(x)}.$$

The notations are different: the superscript  $-1$  has different positions.

- $f^{-1}$  is the compositional inverse of  $f$ .
- $\frac{1}{f(x)}$  is the multiplicative inverse of  $f(x)$ .
- $f^2(x)$  is an abbreviation for  $(f(x))^2$ ,  $f^3(x)$  is an abbreviation of  $(f(x))^3$ , and so on.
- However,  $f^{-1}(x)$  is not the abbreviation of  $(f(x))^{-1}$  and does not follow this pattern.

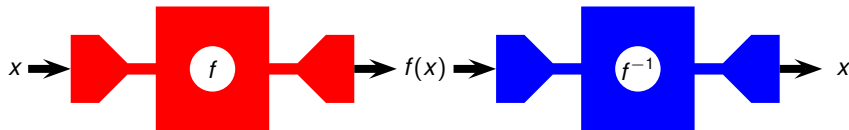
$$f^n(x) = \begin{cases} \text{stands for } (f(x))^n & \text{when } n = 1, 2, 3, \dots \\ \text{stands for inverse of } f \text{ applied to } x & \text{when } n = -1 \\ \text{should be avoided} & \text{when } n \neq -1, 1, 2, 3, \dots \end{cases}$$

To reduce confusion, if possible, use  $\frac{1}{f(x)}$  instead of  $(f(x))^{-1}$ .

$$f^{-1}(y) = x \quad \Leftrightarrow \quad f(x) = y.$$

Therefore

$$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = f^{-1}(y) = x.$$



Switch the roles of  $x$  and  $y$ :

$$f^{-1}(x) = y \quad \Leftrightarrow \quad f(y) = x.$$

Therefore

$$(f \circ f^{-1})(x) = f(f^{-1}(x)) = f(y) = x.$$



# How to Find the Inverse of a One-to-one Function

- 1 Write  $y = f(x)$ .
- 2 Solve this equation for  $x$  in terms of  $y$  (if possible).

## Example

If  $f(x) = x^3 + 2$ , find a formula for  $f^{-1}(y)$ .

$$y = x^3 + 2$$

$$x^3 = y - 2$$

$$x = \sqrt[3]{y - 2}$$

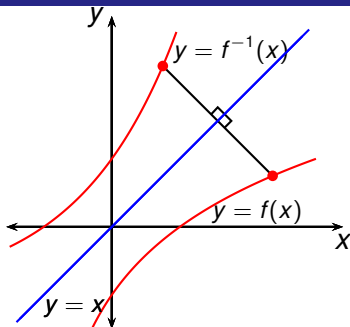
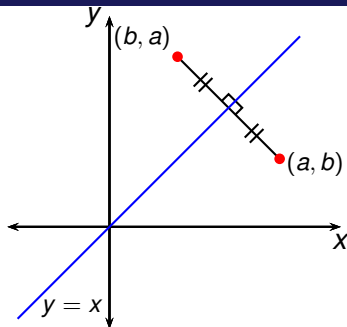
Therefore  $x = f^{-1}(y) = \sqrt[3]{y - 2}$ . Sometimes we relabel  $x$  and  $y$  and write  $f^{-1}(x) = \sqrt[3]{x - 2}$ . Whenever in doubt, do not relabel anything.

## Example (Guess and Check)

If  $f(x) = 2x + \sin 2x + e^{\frac{x}{2}}$ , find  $f^{-1}(1)$ . You do not need to show that  $f$  has an inverse.

$$\begin{aligned} f(\quad) &= 2(\quad) + \sin 2(\quad) + e^{\frac{(\quad)}{2}} \\ &= \\ &= 1. \end{aligned}$$

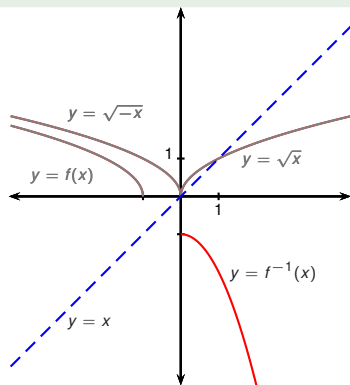
Therefore  $f^{-1}(1) =$



Interchanging  $x$  and  $y$  suggests relation between the graphs of  $f^{-1}$  and  $f$ :

- Suppose  $(a, b)$  is on the graph of  $f$ .
- Then  $f(a) = b$ .
- Then  $f^{-1}(b) = a$ .
- Then  $(b, a)$  is on the graph of  $f^{-1}$ .
- $(b, a)$  is the reflection of  $(a, b)$  in the line  $y = x$ .
- Thus the graph of  $f^{-1}$  is obtained by reflecting the graph of  $f$  in the line  $y = x$ .

## Example

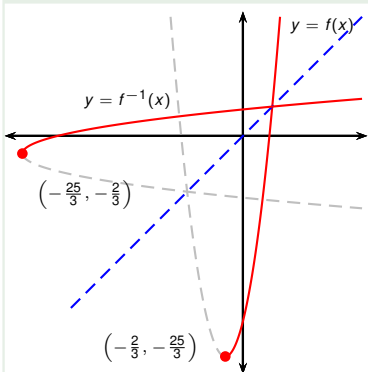


Sketch the graph of  $f(x) = \sqrt{-x-1}$  and its inverse function.

- Draw the graph of  $y = \sqrt{x}$ .
- $y = \sqrt{-x}$  is the reflection of  $y = \sqrt{x}$  in the  $y$ -axis.
- $y = f(x) = \sqrt{-(x+1)} = \sqrt{-x-1}$  is the shift of  $y = \sqrt{-x}$  one unit to the left.
- $y = f^{-1}(x)$  is the reflection of  $y = f(x)$  across the line  $y = x$ .

# Example ( )

Given:  $f(x) = 3x^2 + 4x - 7$  with domain  $x \geq -\frac{2}{3}$ . Find  $f^{-1}(x)$ .



$$\begin{aligned} 3x^2 + 4x - 7 &= y \\ 3x^2 + 4x + (-7 - y) &= 0 \end{aligned}$$

That's a quadratic equation in  $x$ . Solve:

$$\begin{aligned} &\frac{-4 \pm \sqrt{4^2 - 4 \cdot 3 \cdot (-y - 7)}}{2 \cdot 3} \\ &= -\frac{2 \pm \sqrt{25 + 3y}}{3} = -\frac{2}{3} \pm \frac{\sqrt{25 + 3y}}{3} \end{aligned}$$

Final answer, relabelled:

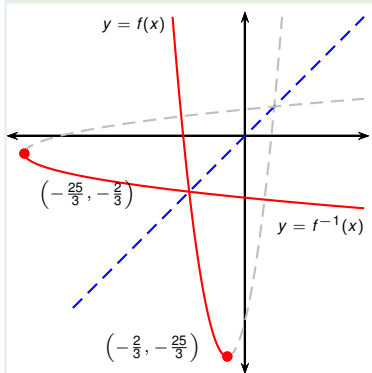
$$f^{-1}(x) = -\frac{2}{3} + \frac{\sqrt{25 + 3x}}{3}$$

We are given  $x \geq -\frac{2}{3}$ , therefore

$$x = -\frac{2}{3} + \frac{\sqrt{25+3y}}{3} = f^{-1}(y).$$

## Example (What if we change the problem to $x \leq -\frac{2}{3}$ ?)

Given:  $f(x) = 3x^2 + 4x - 7$  with domain  $x \leq -\frac{2}{3}$ . Find  $f^{-1}(x)$ .



$$\begin{aligned} 3x^2 + 4x - 7 &= y \\ 3x^2 + 4x + (-7 - y) &= 0 \end{aligned}$$

That's a quadratic equation in  $x$ . Solve:

$$\begin{aligned} &\frac{-4 \pm \sqrt{4^2 - 4 \cdot 3 \cdot (-y - 7)}}{2 \cdot 3} \\ &= -\frac{2 \pm \sqrt{25 + 3y}}{3} = -\frac{2}{3} \pm \frac{\sqrt{25 + 3y}}{3} \end{aligned}$$

Final answer, relabelled:

$$f^{-1}(x) = -\frac{2}{3} - \frac{\sqrt{25 + 3x}}{3}$$

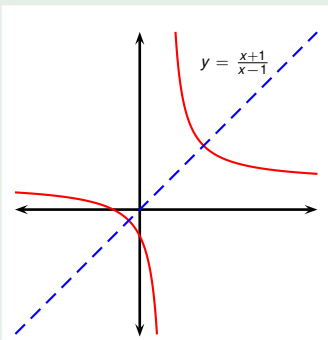
We are given  $x \leq -\frac{2}{3}$ , therefore

$$x = -\frac{2}{3} - \frac{\sqrt{25+3y}}{3} = f^{-1}(y).$$

## Example

Find  $f^{-1}(x)$  where  $f(x) = \frac{x+1}{x-1}$ .

We deal with domains and ranges later:



Answer:  $f^{-1}(x) = \frac{x+1}{x-1}$ ,  
 $x \neq 1$ .

$$\begin{array}{rcll}
 y & = & \frac{x+1}{x-1} & \left| \begin{array}{l} \text{mult. by } (x-1) \\ \text{div. by } (y-1) \\ \text{relabel } x, y \end{array} \right. \\
 y(x-1) & = & x+1 & \\
 x(y-1) & = & y+1 & \\
 f^{-1}(y) = x & = & \frac{y+1}{y-1} & \\
 f^{-1}(x) & = & \frac{x+1}{x-1} & 
 \end{array}$$

We divided by  $y - 1$  so  $y \neq 1$ . Therefore the domain of  $f^{-1}$  is all real numbers except 1.

Can a non-identity function be its own inverse? Yes,  $f$  is.

What does it mean for  $f$  to be its own inverse?  
 Graph of  $f$  is symmetric across  $y = x$ .