Calculus II
Homework on Lecture 19

1. Determine the interval of convergence for the following power series.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(x-2)^n}{3\sqrt{n+1}}$.

   answer: $x \in [1,3)$.

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{10^n x^n}{n^3}$.

   answer: $x \in \left[-\frac{1}{10}, \frac{1}{10}\right]$.

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{10^n (x-1)^n}{n^3}$.

   answer: $x \in [0.9, 1.1]$.

   (d) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{(x+1)^n}{2n+1}$.

   answer: $x \in (-2, 0]$.

   (e) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{(x-3)^n}{2n+1}$.

   answer: $x \in (2, 4]$.

   (f) $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

   answer: converges for all x.

   (g) $\displaystyle\sum_{n=0}^{\infty} (n+1)x^n$.

   answer: converges for $|x| < 1$.

   (h) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}$.

   answer: converges for $x \in [-1, 1)$.

   (i) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$.

   answer: converges for $x \in [-1, 1]$.

   (j) $\displaystyle\sum_{n=1}^{\infty} \binom{\frac{1}{2}}{n} x^n$, where we recall that the binomial coefficient $\displaystyle\binom{q}{n}$ stands for $\displaystyle\frac{q(q-1)\dots(q-n+1)}{n!}$.

   answer: converges for $x \in (-1, 1]$.

   Solution. 1.a. We apply the Ratio Test to get that $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x-2|$. Therefore the power series converges at least in the interval $x \in (1,3)$. When $x = 3$, the series becomes $\sum_{n=1}^{\infty} \frac{1}{3\sqrt{n+1}}$, which diverges - this can be seen, for example, by comparing to the p-series $\frac{1}{\sqrt{n}}$. When $x = 1$, the series becomes $\sum_{n=1}^{\infty} \frac{(-1)^n}{3\sqrt{n+1}}$, which converges by the Alternating Series Test. Our final answer $x \in [1,3)$.

2. (a) Find the Maclaurin series for xe^{x^3}.

   answer: $\displaystyle\sum_{n=0}^{\infty} \frac{x^{3n+1}}{n!}$

(b) Use your series to find the Maclaurin series of $\int x e^{x^3} \mathrm{d}x$.

answer: $C + \sum_{n=0}^{\infty} \frac{x^{3n+2}}{(3n+2)n!}$ note the integral can't be integrated with elementary functions.

3. For each of the items below, do the following.

- Find the Maclaurin series of the function (i.e., the power series representation of the function around $a = 0$).
- Find the radius of convergence of the series you found in the preceding point. You are not asked to find the entire interval of convergence, but just the radius.

(a) e^x.

answer: $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

(b) xe^{-2x}.

answer: $\sum_{n=0}^{\infty} (-1)^n 2^n x^{n+1} = \sum_{n=1}^{\infty} (-1)^{n-1} 2^{n-1} x^n$ converges for $x \in ()$

(c) e^{2x}.

answer: $e^{2x} = \sum_{n=0}^{\infty} \frac{2^n x^n}{n!}$

(d) e^{x^2}.

answer: $e^{x^2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$

(e) e^{-3x^2}.

answer: $e^{-3x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n 3^n x^{2n}}{n!}$

(f) $x^2 e^{2x}$.

answer: $x^2 e^{2x} = \sum_{n=0}^{\infty} \frac{2^n x^{n+2}}{n!}$

(g) $\sin x$.

answer: $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

(h) $\cos x$.

answer: $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

(i) $\sin(2x)$.

answer: $\sin(2x) = \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n+1} x^{2n+1}}{(2n+1)!}$

(j) $\cos(2x)$.

answer: $\cos(2x) = \sum_{n=0}^{\infty} (-1)^n 2^{2n} \frac{x^{2n}}{(2n)!}$

(k) $\cos^2(x)$.

answer: $\cos^2 x = \frac{1}{2} + \sum_{n=0}^{\infty} (-1)^n 2^{2n-1} \frac{x^{2n}}{(2n)!}$

(l) $x \sin x$.

answer: $x \sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{(2n+1)!}$

4. For each of the items below, do the following.

- Find the Maclaurin series of the function (i.e., the power series representation of the function around $a = 0$).
- Find the radius of convergence of the series you found in the preceding point.

(a) $\frac{1}{3-x}$.

answer: $\sum_{n=0}^{\infty} \frac{x^n}{3^{n+1}}$ $R = 3$ converges for $x \in (-3, 3)$

(b) $\frac{1}{3-2x}$.

answer: $\sum_{n=0}^{\infty} \frac{2^n}{3^{n+1}} x^n$ $R = \frac{3}{2}$ converges for $x \in \left(-\frac{3}{2}, \frac{3}{2}\right)$

(c) $\frac{1}{2x+3}$.

answer: $\frac{1}{3}\left(1 - \frac{2x}{3} + \left(\frac{2x}{3}\right)^2 - \left(\frac{2x}{3}\right)^3 + \dots\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{3} \left(\frac{2}{3}\right)^n x^n$ $R = \frac{3}{2}$ converges for $x \in \left(-\frac{3}{2}, \frac{3}{2}\right)$

(d) $\frac{1}{1+x^2}$.

answer: $\sum_{n=0}^{\infty} (-1)^n x^{2n}$ $R = 1$ converges for $x \in (-1, 1)$

(e) $\frac{1}{1-2x^2}$.

answer: $\sum_{n=0}^{\infty} 2^n x^{2n}$ $R = \frac{1}{2}$ converges for $x \in \left(-\frac{1}{2}, \frac{1}{2}\right)$

(f) $\frac{1}{x^2-1}$.

answer: $-\sum_{n=0}^{\infty} x^{2n}$ $R = 1$ converges for $x \in (-1, 1)$

(g) $\frac{\frac{1}{2}}{x-1} - \frac{\frac{1}{2}}{x+1}$.

answer: same as 4.f

(h) $\frac{1}{(1-x)^2}$.

answer: $1 + 2x + 3x^2 + 4x^3 + \dots = \sum_{n=0}^{\infty} (n+1) x^n$ $R = 1$ converges for $x \in (-1, 1)$

(i) $\frac{1}{(1-x)^3}$.

answer: $\frac{1}{2}\left(2 + 6x + 12x^2 + \dots + n(n-1)x^{n-2} + \dots\right) = \sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2} x^n$ $R = 1$ converges for $x \in (-1, 1)$

(j) $\ln(1+x)$.

answer: $\sum\limits_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}$ $R=1$ converges for $x\in(-1,1]$

(k) $\ln(1-x)$.

answer: $-\sum\limits_{n=1}^{\infty}\frac{x^n}{n}$ $R=1$ converges for $x\in[-1,1)$

(l) $\ln(1-3x)$.

answer: $-\sum\limits_{n=0}^{\infty}\frac{3^n x^n}{n}$ $R=\frac{1}{3}$ converges for $x\in\left(-\frac{1}{3},\frac{1}{3}\right]$

(m) $\ln(1-3x^2)$.

answer: $-\sum\limits_{n=1}^{\infty}\frac{3^n x^{2n}}{n}$ $R=\frac{1}{\sqrt{3}}$ converges for $x\in\left(-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$

(n) $\ln(3-2x^2)$.

answer: $\ln 3-\sum\limits_{n=1}^{\infty}\left(\frac{2}{3}\right)^n\frac{x^{2n}}{n}$ $R=\sqrt{\frac{3}{2}}$ converges for $x\in\left(-\sqrt{\frac{3}{2}},\sqrt{\frac{3}{2}}\right)$

(o) $x\ln(3-2x^2)$.

answer: $x\ln 3-\sum\limits_{n=1}^{\infty}\left(\frac{2}{3}\right)^n\frac{x^{2n+1}}{n}$ $R=\sqrt{\frac{3}{2}}$ converges for $x\in\left(-\sqrt{\frac{3}{2}},\sqrt{\frac{3}{2}}\right)$

(p) $\arctan x$.

answer: $\sum\limits_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$ $R=1$ converges for $x\in(-1,1]$

(q) $\arctan(2x)$.

answer: $\sum\limits_{n=0}^{\infty}\frac{(-1)^n 2^{2n+1}x^{2n+1}}{2n+1}$ $R=\frac{1}{2}$ converges for $x\in\left(-\frac{1}{2},\frac{1}{2}\right]$

(r) $\arctan\left(2x^2\right)$.

answer: $\sum\limits_{n=0}^{\infty}(-1)^n\frac{2^{2n+1}x^{4n+2}}{2n+1}$ $R=\frac{1}{\sqrt{2}}$ converges for $x\in\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right]$

Solution. 4.h

$$
\begin{array}{rcll}
\dfrac{1}{1-x} & = & \dfrac{\mathrm{d}}{\mathrm{d}x}\left(1+x+x^2+x^3+\dots\right) & \left|\begin{array}{l}\text{geometric series,}\\ \text{converges if and only if}\\ |x|<1\end{array}\right.\\
\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{1}{1-x}\right) & = & \dfrac{\mathrm{d}}{\mathrm{d}x}\left(1+x+x^2+x^3+\dots\right) & \left|\ \text{apply } \frac{\mathrm{d}}{\mathrm{d}x}\right.\\
-\dfrac{(1-x)'}{(1-x)^2}=\dfrac{1}{(1-x)^2} & = & 1+2x+3x^2+\dots & \\
\dfrac{1}{(1-x)^2} & = & \displaystyle\sum_{n=0}^{\infty}(n+1)x^n & \left|\ \text{rewrite in } \sum \text{ notation.}\right.
\end{array}
$$

The radius of convergence of the geometric series is 1. Differentiating does not change the radius of convergence. We have that the radius of convergence of $1+x+x^2+\dots$ is 1 and therefore we have that $\frac{1}{(1-x)^2}=\sum\limits_{n=0}^{\infty}(n+1)x^n$ converges for $|x|<1$ and the radius of convergence is $R=1$.

The problem does not ask us to determine the interval of convergence, however let us do it for exercise. The endpoints of the interval of convergence are -1 and 1. The series is divergent for both of them: indeed at $x=-1$ the series becomes $\sum\limits_{n=0}^{\infty}(-1)^n(n+1)x^n$ and at $x=1$ the series becomes $\sum\limits_{n=0}^{\infty}(n+1)x^n$. Both of these series are divergent as their terms do not tend to zero as n tends to infinity. Thus the interval of convergence is $(-1,1)$.

We generalize this problem in Problem 5.

Solution. 4.k

$$
\begin{array}{rcll}
\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\ln(1-x)\right) &=& \dfrac{-1}{1-x} & \text{expand as geometric series}\\
&=& -\left(1+x+x^2+x^3+\dots\right) & \text{for } |x|<1 \\
&&& \text{Integrate indefinitely, } |x|<1\\
&&& \text{For power series,}\\
&&& \text{integral of infinite}\\
\displaystyle\int \frac{\mathrm{d}}{\mathrm{d}x}(\ln(1-x))\mathrm{d}x &=& \displaystyle -\int\left(1+x+x^2+x^3+\dots\right)\mathrm{d}x & \text{sum equals}\\
&&& \text{infinite sum of integrals}\\
&&& \text{inside the convergence radius}\\
\ln(1-x) &=& -\left(x+\dfrac{x^2}{2}+\dfrac{x^3}{3}+\dots\right)+C & \text{To find } C \text{ set } x=0\\
0=\ln 1 &=& -0+C=C &\\
\ln(1-x) &=& -\left(x+\frac{x^2}{2}+\frac{x^3}{3}+\dots\right) &\\
&=& \displaystyle -\sum_{n=1}^{\infty}\frac{x^n}{n} \quad . &
\end{array}
$$

The radius of convergence of the geometric series $1+x+x^2+\dots$ is 1. Since the series for $\ln(1-x)$ is obtained from the geometric series via integration, its radius of convergence is again 1.

We note that the interval of convergence for the series $-\sum\limits_{n=1}^{\infty}\frac{x^n}{n}$ is $[-1,1)$ - the series is convergent at $x=-1$ by the alternating series test and divergent at $x=1$ (at $x=1$ the series is minus the harmonic series). This shows that integration of power series can change convergence at the endpoints of the interval of convergence.

Solution. 4.n. We solve this problem by reducing it to Problem 4.k, which asserts the power series expansion $\ln(1-y)=-\sum\limits_{n=1}^{\infty}\dfrac{y^n}{n}$ for $|y|<1$.

$$
\begin{array}{rcll}
\ln\left(3-2x^2\right) &=& \ln\left(3\left(1-\dfrac{2}{3}x^2\right)\right) &\\
&=& \ln 3+\ln\left(1-\dfrac{2}{3}x^2\right) & \text{Set } y=\frac{2}{3}x^2\\
&=& \ln 3+\ln(1-y) &\\
&&& \ln(1-y)=-\sum\limits_{n=1}^{\infty}\frac{y^n}{n} \text{ for } |y|<1\\
&=& \ln 3-\displaystyle\sum_{n=1}^{\infty}\frac{y^n}{n} & \text{above does not hold for } |y|>1\\
&&& \text{above may (not) hold for } y=\pm 1\\
&=& \ln 3-\displaystyle\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n\frac{x^{2n}}{n} \quad . & \text{Substituted back } y=\frac{2}{3}x^2 \quad .
\end{array}
$$

As indicated above, the equality $\ln(1-y)=-\sum\limits_{n=1}^{\infty}\frac{y^n}{n}$ holds for $|y|<1$ and fails for $|y|>1$ (for $|y|>1$ the series $\sum\limits_{n=1}^{\infty}\frac{y^n}{n}$ diverges). Therefore interval of convergence is given by

$$
\begin{array}{rcll}
|y| &<& 1 & \text{use } y=\frac{2}{3}x^2\\
\left|\frac{2}{3}x^2\right| &<& 1 &\\
\left|x^2\right| &<& \frac{3}{2} &\\
|x| &<& \sqrt{\frac{3}{2}}, &
\end{array}
$$

i.e., the radius of convergence is $R=\sqrt{\frac{3}{2}}$.

5. Compute the Maclaurin series of

$$
\left(\frac{1}{(1-x)^k}\right) \quad ,
$$

where $n\geq 1$ is an integer.

Solution. 5 We have that

$$\begin{array}{lll}
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{1-x}\right) = \frac{(1-x)'}{(1-x)^2} & = & \frac{1}{(1-x)^2} \\
\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{1}{1-x}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{(1-x)^2}\right) = -2\frac{(1-x)'}{(1-x)^3} & = & \frac{2}{(1-x)^3} \\
\frac{\mathrm{d}^3}{\mathrm{d}x^3}\left(\frac{1}{1-x}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{2}{(1-x)^3}\right) = 2(-3)\frac{(1-x)'}{(1-x)^4} & = & \frac{2\cdot 3}{(1-x)^4} \\
\vdots & & \\
\frac{\mathrm{d}^{k-2}}{\mathrm{d}x^{k-2}}\left(\frac{1}{1-x}\right) & = & \frac{(k-2)!}{(1-x)^{k-1}} \\
\frac{\mathrm{d}^{k-2}}{\mathrm{d}x^{k-2}}\left(\frac{1}{1-x}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{(k-2)!}{(1-x)^{k-1}}\right) & = & \frac{(k-1)!}{(1-x)^k} \\
\vdots & &
\end{array}$$

We can now compute Maclaurin series as follows:

$$\begin{aligned}
\mathrm{Mc}\left(\frac{1}{(1-x)^k}\right) &= \mathrm{Mc}\left(\frac{1}{(k-1)!}\frac{\mathrm{d}^{k-1}}{\mathrm{d}x^{k-1}}\left(\frac{1}{(1-x)}\right)\right) \\
&= \frac{1}{(k-1)!}\frac{\mathrm{d}^{k-1}}{\mathrm{d}x^{k-1}}\left(\mathrm{Mc}\left(\frac{1}{1-x}\right)\right) \\
&= \frac{1}{(k-1)!}\frac{\mathrm{d}^{k-1}}{\mathrm{d}x^{k-1}}\left(\sum_{n=0}^{\infty} x^n\right) \\
&= \frac{1}{(k-1)!}\left(\sum_{n=0}^{\infty} n(n-1)\dots(n-k+2)x^{n-k+1}\right) && \text{Recall } \binom{n}{k} = \tfrac{n(n-1)\dots(n-k+1)}{k!} \\
&= \sum_{n=0}^{\infty}\binom{n}{k-1}x^{n-k+1} && \text{Set } n-k+1=m \\
&= \sum_{m=-k+1}^{\infty}\binom{m+k-1}{k-1}x^m && \text{first } k-2 \text{ summands are zero} \\
&= \sum_{m=0}^{\infty}\binom{m+k-1}{k-1}x^m
\end{aligned}$$

6. Compute the Maclaurin series of

$$(1+x)^q \quad ,$$

where $q \in \mathbb{R}$ is an arbitrary real number.

Solution. 6 Since q does not have to be an integer, we cannot directly relate its power series to the power series of $\frac{1}{1+x}$ or its derivatives. We therefore compute the Maclaurin series directly using their definition.

$$\begin{array}{lll}
\frac{\mathrm{d}}{\mathrm{d}x}\left((1+x)^q\right) & = & q(1+x)^{q-1} \\
\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left((1+x)^q\right) & = & q(q-1)(1+x)^{q-2} \\
\vdots & & \\
\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left((1+x)^q\right) & = & q(q-1)(q-2)\dots(q-n+1)(1+x)^{q-n} \quad .
\end{array}$$

Therefore $\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left((1+x)^q\right)_{|x=0} = q(q-1)(q-2)\dots(q-n+1)(1+0)^{q-n} = q(q-1)(q-2)\dots(q-n+1)$. Therefore

$$\begin{aligned}
\mathrm{Mc}\left((1+x)^q\right) &= \sum_{n=0}^{\infty}\frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left((1+x)^q\right)_{|x=0} x^n \\
&= \sum_{n=0}^{\infty}\frac{q(q-1)(q-2)\dots(q-n+1)}{n!}x^n = \sum_{n=0}^{\infty}\binom{q}{n}x^n \quad .
\end{aligned} \tag{1}$$

For the last equality we recall the definition of binomial coefficient $\binom{q}{n} = \frac{q(q-1)\dots(q-n+1)}{n!}$ and that it allows for q to be an arbitrary complex number . The above formula is a generalization of the Newton binomial formula.

7. Compute the Maclaurin series of the function.

(a) $\sqrt{1+x}$.

answer: $\sum\limits_{n=0}^{\infty} \binom{\frac{1}{2}}{n} x^n$

(b) $\dfrac{1}{\sqrt{1+x}}$.

answer: $\sum\limits_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} x^n$

(c) $\dfrac{1}{\sqrt{1-x^2}}$.

answer: $\sum\limits_{n=0}^{\infty} (-1)^n \binom{-\frac{1}{2}}{n} x^{2n}$

(d) $\arcsin x$.

answer: $\sum\limits_{n=0}^{\infty} (-1)^n \binom{-\frac{1}{2}}{n} \frac{x^{2n+1}}{2n+1}$

Solution. 7.a This problem follows directly from the formula $(1+x)^q = \sum\limits_{n=0}^{\infty} \binom{q}{n} x^n$.

$$\mathrm{Mc}\left(\sqrt{1+x}\right) = \mathrm{Mc}\left((1+x)^{\frac{1}{2}}\right) = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} x^n \quad .$$

Solution. 7.b This problem can be solved by computing the derivative of the preceding problem. However, it is easier to simply apply the generalized Newton Binomial formula.

$$\mathrm{Mc}\left((1+x)^{-\frac{1}{2}}\right) = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} x^n \quad .$$

Solution. 7.c This problem is solved by replacing x with $-x^2$ in Problem 7.b. To avoid the possible confusion, we carry out the substitution by introducing an intermediate variable y.

$$\begin{array}{rcll}
\mathrm{Mc}\left(\left(1-x^2\right)^{-\frac{1}{2}}\right) & = & \mathrm{Mc}\left((1+y)^{-\frac{1}{2}}\right) & \left| \text{Set } y=-x^2 \right. \\
& = & \sum\limits_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} y^n & \left| \text{Substitute back } y=-x^2 \right. \\
& = & \sum\limits_{n=0}^{\infty} (-1)^n \binom{-\frac{1}{2}}{n} x^{2n} \quad . &
\end{array}$$

Solution. 7.d We have that $\frac{\mathrm{d}}{\mathrm{d}x}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$, and the Maclaurin series of $\frac{1}{\sqrt{1-x^2}}$ were computed in Problem 7.c. The power series of $\arcsin x$ are therefore obtained via integration.

$$\begin{array}{rcll}
\dfrac{\mathrm{d}}{\mathrm{d}x}\mathrm{Mc}(\arcsin x) & = & \mathrm{Mc}\left(\dfrac{\mathrm{d}}{\mathrm{d}x}(\arcsin x)\right) & \\
& = & \mathrm{Mc}\left(\dfrac{1}{\sqrt{1-x^2}}\right) & \left| \text{use Problem 7.}c \right. \\
& = & \sum\limits_{n=0}^{\infty} (-1)^n \binom{-\frac{1}{2}}{n} x^{2n} & \\
\mathrm{Mc}(\arcsin x) & = & \displaystyle\int \left(\sum\limits_{n=0}^{\infty} (-1)^n \binom{-\frac{1}{2}}{n} x^{2n}\right) \mathrm{d}x & \\
& = & C + \sum\limits_{n=0}^{\infty} (-1)^n \binom{-\frac{1}{2}}{n} \displaystyle\int x^{2n} \mathrm{d}x & \\
& = & C + \sum\limits_{n=0}^{\infty} (-1)^n \binom{-\frac{1}{2}}{n} \dfrac{x^{2n+1}}{2n+1} & \left| C = 0 \text{ since } \arcsin 0 = 0 \right. \\
& = & \sum\limits_{n=0}^{\infty} (-1)^n \binom{-\frac{1}{2}}{n} \dfrac{x^{2n+1}}{2n+1} \quad . &
\end{array}$$

8. Find the Taylor series of the function at the indicated point.

(a) $\frac{1}{x^2}$ at $a=-1$.

answer: $1 + 2(x+1) + 3(x+1)^2 + \dots = \sum\limits_{n=0}^{\infty} (n+1)(x+1)^n$

(b) $\ln\left(\sqrt{x^2-2x+2}\right)$ at $a=1$.

answer: $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-1)^{2n}}{2n}$

(c) Write the Taylor series of the function $\ln x$ around $a=2$.

answer: $\ln 2+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n}(x-2)^n$

Solution. 8.b

$$\begin{aligned}\ln\left(\sqrt{x^2-2x+2}\right) &= \frac{1}{2}\ln\left((x-1)^2+1\right) && \Big|\ \text{use } \ln(1+y)=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{y^n}{n},\ |y|<1\\ &= \frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\left((x-1)^2\right)^n}{n}\\ &= \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-1)^{2n}}{2n} \quad .\end{aligned}$$

Although the problem does not ask us to do this, we will determine the interval of convergence of the series for exercise. If we use the fact that $\ln(1+y)=\sum\limits_{n=1}^{\infty}(-1)^{n+1}\frac{y^n}{n}$ holds for $-1<y\leq 1$, it follows immediately that the above equality holds for $0<(x-1)^2\leq 1$, which holds for $x\in[0,2]$. Let us however compute the interval of convergence without using the aforementioned fact.

Let a_n be the n^{th} term of our series, i.e., let

$$a_n=(-1)^{n+1}\frac{(x-1)^{2n}}{2n} \quad .$$

We use the ratio test:

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(-1)^{n+2}(x-1)^{2n+2}}{(2n+2)}\frac{2n}{(-1)^{n+1}(x-1)^{2n}}\right|\\ &= \lim_{n\to\infty}(x-1)^2\frac{n}{n+1}\\ &= (x-1)^2 \quad .\end{aligned}$$

By the ratio test, the series is divergent for $(x-1)^2>1$, i.e., for $|x-1|>1$, and convergent for $(x-1)^2<1$, i.e., for $|x-1|<1$. The ratio test is inconclusive at only two points: $x-1=1$, i.e., $x=2$ and $x-1=-1$, i.e., $x=0$. At both points the series becomes $\sum\limits_{n=1}^{\infty}(-1)^{n+1}\frac{2^{2n}}{2n}$ and the series is convergent at both points by the alternating series test.

Solution. 8.c This solution is similar to the solution of 8.b, but we have written it in a concise fashion suitable for test taking. Denote Taylor series at a by T_a and recall that the Maclaurin series of are just T_0, the Taylor series at 0.

$$\begin{aligned}T_2(\ln x) &= T_2(\ln((x-2)+2))\\ &= T_2\left(\ln\left(2\left(\frac{x-2}{2}+1\right)\right)\right)\\ &= T_2\left(\ln 2+\ln\left(1+\frac{x-2}{2}\right)\right) && \Big|\ T_0(\ln(1+y))=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}y^n}{n}\\ &= \ln 2+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(\frac{x-2}{2}\right)}{n}\\ &= \ln 2+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2^n}(x-2)^n \quad .\end{aligned}$$

9. Find the Taylor series around the indicated point. The answer key has not been proofread, use with caution.

(a) $\frac{1}{x}$ at $a=1$.

answer: $1-(x-1)+(x-1)^2-(x-1)^3+\cdots=\sum_{n=0}^{\infty}(-1)^n(x-1)^n$

(b) $\frac{1}{x^2}$ at $a=1$.

answer: $1-2(x-1)+3(x-1)^2-4(x-1)^3+\cdots=\sum_{n=0}^{\infty}(n+1)(-1)^n(x-1)^n$

10. **(This problem is of higher difficulty, it will not appear on the quiz.)** Let $f(x)$ be defined as

$$f(x):=\begin{cases} e^{-\frac{1}{x^2}} & \text{if } x>0\\ 0 & \text{otherwise.}\end{cases}$$

(a) Prove that if $R(x)$ is an arbitrary rational function,

$$\lim_{\substack{x \to 0 \\ x > 0}} R(x) e^{-\frac{1}{x^2}} = 0$$

(b) Prove that $f(x)$ is differentiable at 0 and $f'(0) = 0$.

(c) Prove that the Maclaurin series of $f(x)$ are 0 (but $f(x)$ is clearly a non-zero function).