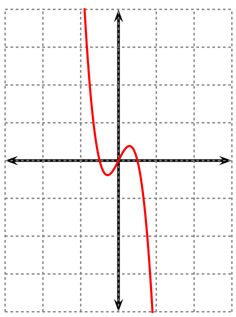
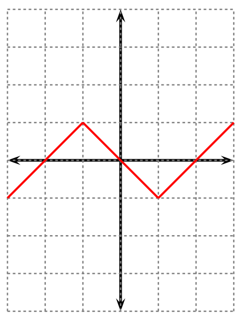
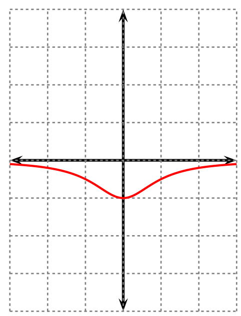
Calculus I
Homework Curve Sketching
Lecture 17

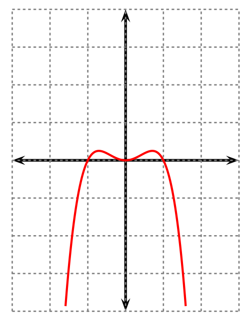
1.

Match each of the following function plots:

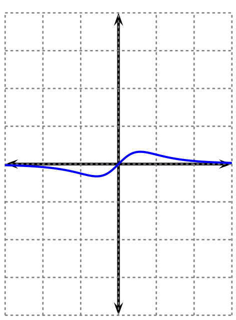
1.


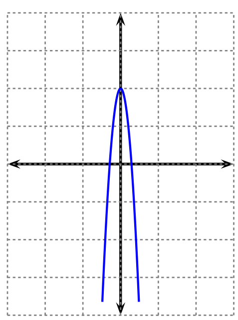
2.


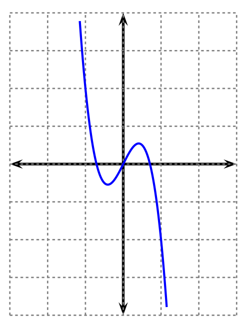
3.


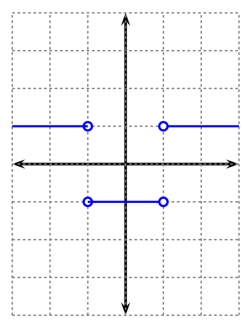
4.


to their derivative plots:

(a)


(b)


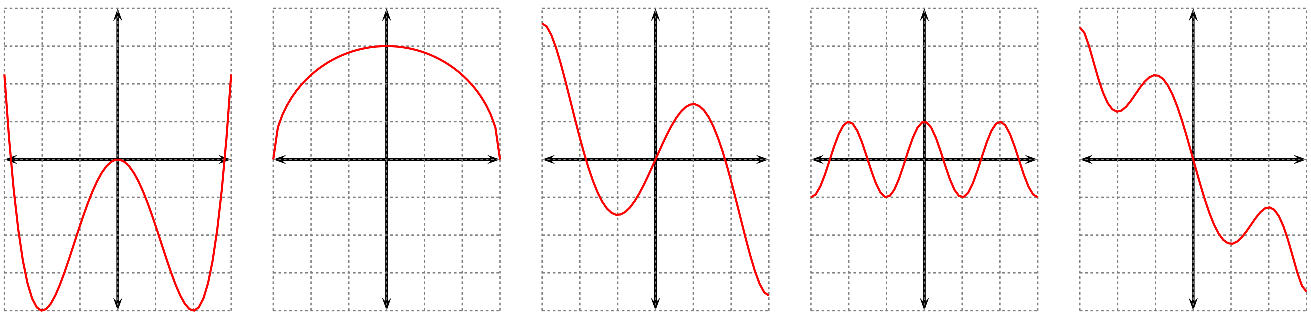
(c)


(d)


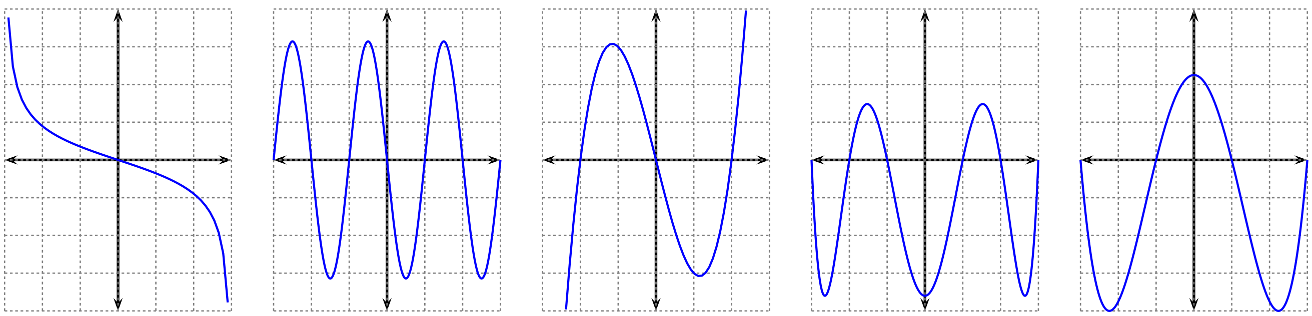
Give reasons for your choices. Can you guess formulas that would give a similar (or precisely the same) graph, and confirm visually your guess using a graphing device?

2.

Match each of the following function plots:



to their derivative plots:



Solution. 2

(1) matches (c) because (1) has three local extrema and (c) is the only derivative graph with three zeros.

(2) matches (a) because (2) has one local extrema and (a) is the only derivative graph with one zero.

(3) matches (e) because (3) has four local extrema (± 1 and ± 3) and (e) is the only derivative graph with four zeros.

(4) matches (b) because (4) has seven local extrema and (b) is the only derivative graph with seven zeros.

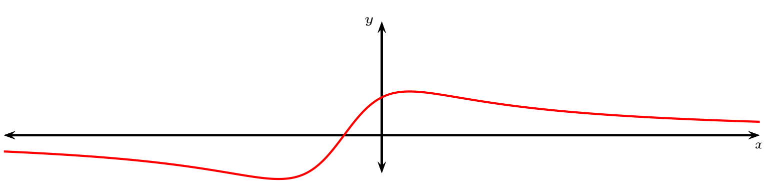
(5) matches (d) because (5) has six local extrema and (d) is the only derivative graph with six zeros.

3. Find the

- the implied domain of f,
- x and y intercepts of f,
- horizontal and vertical asymptotes,
- intervals of increase and decrease,
- local and global minima, maxima,
- intervals of concavity,
- points of inflection.

Label all relevant points on the graph. Show all of your computations.

(a) $f(x) = \dfrac{x + \frac{1}{2}}{x^2 + x + 1}$



answer:

y-intercept: $\frac{1}{2}$, x-intercept: $-\frac{1}{2}$

Horizontal asymptote: $y = 0$, vertical: none

local and global min at $x = \frac{-1-\sqrt{3}}{2}$, local and global max at $x = \frac{-1+\sqrt{3}}{2}$

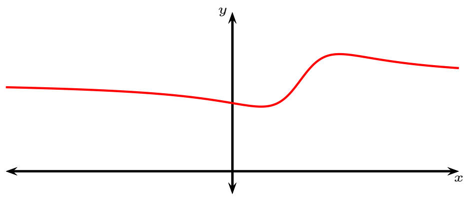
Intervals of decrease: $\left(-\infty, \frac{-1-\sqrt{3}}{2}\right) \cup \left(\frac{-1+\sqrt{3}}{2}, \infty\right)$, intervals of decrease $\left(\frac{-1-\sqrt{3}}{2}, \frac{-1+\sqrt{3}}{2}\right)$

Concave down on $(-\infty, -2) \cup \left(-\frac{1}{2}, 1\right)$, concave up on $\left(-2, -\frac{1}{2}\right) \cup (1, \infty)$

Inflection points at: $x = -2, x = -\frac{1}{2}, x = 1$

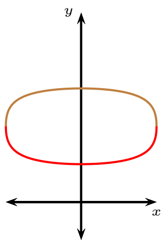
(b) $f(x) = \dfrac{2x^2 - 5x + \frac{9}{2}}{x^2 - 3x + 3}$. **For this problem, indicate only the x-coordinates of the local maxima/minima and inflection points; you do not need to compute the y-coordinates of those points.**

Computation shows that $f'(x) = \dfrac{-x^2 + 3x - \frac{3}{2}}{(x^2 - 3x + 3)^2}$ and that $f''(x) = \dfrac{(2x - 3)x(x - 3)}{(x^2 - 3x + 3)^3}$; you may use those computations without further justification.



answer:
y-intercept: $\frac{3}{2}$
horizontal asymptote: $y = 2$, vertical: none
increasing on $\left(\frac{3-\sqrt{3}}{2}, \frac{3+\sqrt{3}}{2}\right)$, decreasing on $\left(-\infty, \frac{3-\sqrt{3}}{2}\right) \cup \left(\frac{3+\sqrt{3}}{2}, \infty\right)$
local and global min at $x = \frac{3-\sqrt{3}}{2}$, local and global max at $x = \frac{3+\sqrt{3}}{2}$
concave up on $\left(0, \frac{3}{2}\right) \cup (3, \infty)$, concave down $(-\infty, 0) \cup \left(\frac{3}{2}, 3\right)$
inflection points at $x = 0$, $x = \frac{3}{2}$, $x = 3$

(c) $f(x) = \dfrac{2\sqrt{-x^2+1}+1}{\sqrt{-x^2+1}+1}$, $f(x) = \dfrac{1}{\sqrt{-x^2+1}+1}$

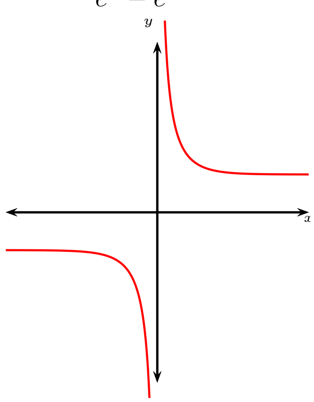


The two functions are plotted simultaneously in the x, y-plane. Indicate which part of the graph is the graph of which function.

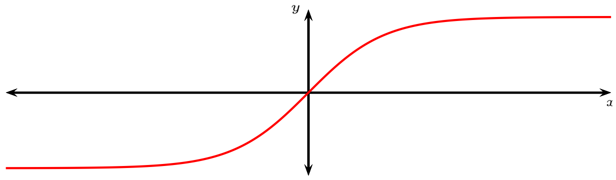
answer:
For $f(x) = \frac{2\sqrt{-x^2+1}+1}{\sqrt{-x^2+1}+1}$:
y-intercept: $x = \frac{3}{2}$, no x intercept
no asymptotes
increasing on $[-1, 0]$, decreasing on $[0, 1]$
global and local max at $x = 0$, global and local min at $x = \pm 1$.
concave down on $[-1, 1]$
no inflection points

answer:
For $f(x) = \frac{1}{\sqrt{-x^2+1}+1}$:
y-intercept: $x = \frac{1}{2}$, no x intercept
no asymptotes
decreasing on $[-1, 0]$, increasing on $[0, 1]$
global and local min at $x = 0$, global and local max at $x = \pm 1$.
concave up on $[-1, 1]$
no inflection points

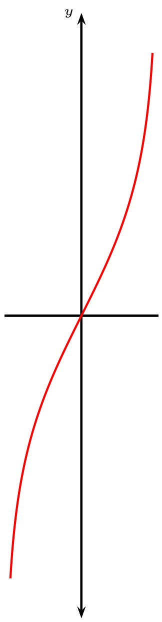
(d) $f(x) = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$



(e) $f(x) = \dfrac{-e^{-x}+e^{x}}{e^{-x}+e^{x}}$

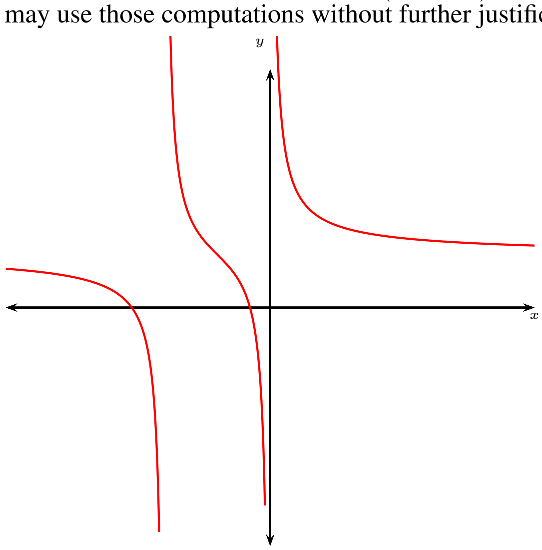


(f) $f(x) = \ln\left(\dfrac{x+1}{-x+1}\right)$

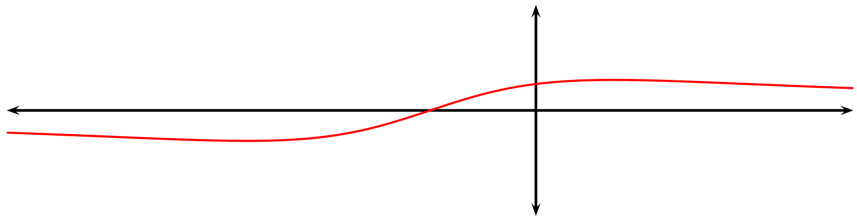


(g) $f(x) = \dfrac{x^2+3x+1}{x^2+2x}$. **For this problem, indicate only the x-coordinates of the local maxima/minima and inflection points; you do not need to compute the y-coordinates of those points.**

Computation shows that $f'(x) = \dfrac{-x^2-2x-2}{\left(x^2+2x\right)^2}$ and that $f''(x) = \dfrac{2x^3+6x^2+12x+8}{\left(x^2+2x\right)^3} = \dfrac{(x+1)\left(2x^2+4x+8\right)}{\left(x^2+2x\right)^3}$; you may use those computations without further justification.



answer:
y-intercept: none, x-intercepts: $\frac{-3\pm\sqrt{5}}{2}$
horizontal asymptote: $y = 1$, vertical: $x = -2$ and $x = 0$
always decreasing
no local/global minima/maxima
concave down on $(-\infty, -2) \cup (-1, 0)$, concave up on $(-2, -1) \cup (0, \infty)$
inflection point at $x = -1$

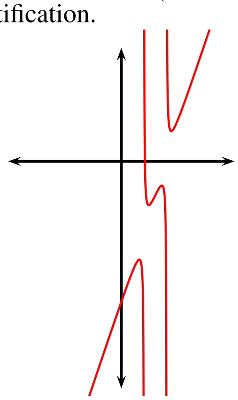


(h) $f(x) = \dfrac{x+1}{x^2+2x+4}$

answer:
y-intercept: $\frac{1}{4}$, x-intercept: -1
horizontal asymptote: $y = 0$, vertical: none
increasing on $\left(-1-\sqrt{3}, -1+\sqrt{3}\right)$, decreasing on $\left(-\infty, -1-\sqrt{3}\right) \cup \left(-1+\sqrt{3}, \infty\right)$
local and global min at $x = -1-\sqrt{3}$, local and global max at $x = -1+\sqrt{3}$
concave up on $(-4, -1) \cup (2, \infty)$, concave down $(-\infty, -4) \cup (-1, 2)$
inflection points at $x = -4$, $x = -1$, $x = 2$

(i) $f(x) = \dfrac{3x^3 - 30x^2 + 97x - 99}{x^2 - 6x + 8}$. **For this problem, do not find the x-intercepts of the function. Indicate only the x-coordinates of the local maxima/minima and inflection points; you do not need to compute the y-coordinates of those points.**

Computation shows that $f'(x) = \dfrac{3x^4 - 36x^3 + 155x^2 - 282x + 182}{\left(x^2 - 6x + 8\right)^2} = \dfrac{\left(x^2 - 6x + 7\right)\left(3x^2 - 18x + 26\right)}{\left(x^2 - 6x + 8\right)^2}$ and that $f''(x) = \dfrac{2x^3 - 18x^2 + 60x - 72}{\left(x^2 - 6x + 8\right)^3} = \dfrac{(x-3)\left(2x^2 - 12x + 24\right)}{\left(x^2 - 6x + 8\right)^3}$; you may use those computations without further justification.



answer:
y-intercept $\frac{-99}{8}$, x-intercept: not requested
horizontal asymptote: none, vertical: $x = 2$ and $x = 4$
increasing on $\left(-\infty, -\sqrt{2}+3\right) \cup \left(-\frac{\sqrt{3}}{3}+3, \frac{\sqrt{3}}{3}+3\right) \cup \left(\sqrt{2}+3, \infty\right)$, decreasing on the complement intervals
local maxima at $x = -\sqrt{2}+3$, $x = \frac{\sqrt{3}}{3}+3$
local minima at $x = -\frac{\sqrt{3}}{3}+3$, $x = \sqrt{2}+3$
concave up on $((2, 3) \cup (4, \infty))$
concave down on $(-\infty, 2) \cup (3, 4)$
inflection point at $x = 3$

Solution. 3.b

Domain. We have that f is not defined only when we have division by zero, i.e., if $x^2 - 3x + 3$ equals zero. However, the roots of $x^2 - 3x + 3$ are not real numbers: they are $\frac{3 \pm \sqrt{3^2 - 4 \cdot 3}}{2} = \frac{3 \pm \sqrt{-3}}{2}$, and therefore $x^2 - 3x + 3$ cannot equal zero (for real x). Alternatively, completing the square shows that the denominator is always positive:

$$x^2 - 3x + 3 = x^2 - 2 \cdot \frac{3}{2}x + \frac{9}{4} - \frac{9}{4} + 3 = \left(x - \frac{3}{2}\right)^2 + \frac{3}{4} > 0$$

Therefore the domain of f is all real numbers.

x, y-intercepts. The y-intercept of f equals by definition $f(0) = \dfrac{2 \cdot 0^2 - 5 \cdot 0 + \frac{9}{2}}{0^2 - 3 \cdot 0 + 3} = \dfrac{\frac{9}{2}}{3} = \dfrac{3}{2}$. The x intercept of f is those values of x for which $f(x) = 0$. The graph of f shows no such x, and that is confirmed by solving the equation $f(x) = 0$:

$$\begin{array}{rcll} f(x) & = & 0 & \\ \dfrac{2x^2 - 5x + \frac{9}{2}}{x^2 - 3x + 3} & = & 0 & \Big| \text{ Mult. by } x^2 - 3x + 3 \\ 2x^2 - 5x + \dfrac{9}{2} & = & 0 & \\ x_1, x_2 & = & \dfrac{5 \pm \sqrt{25 - 4 \cdot 2 \cdot \frac{9}{2}}}{4} = \dfrac{5 \pm \sqrt{-9}}{4} & , \end{array}$$

so there are no real solutions (the number $\sqrt{-9}$ is not real).

Asymptotes. Since f is defined for all real numbers, its graph has no vertical asymptotes. To find the horizontal asymptote(s), we need to compute the limits $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$. The two limits are equal, as the direct computation below shows:

$$\begin{array}{rcll}\lim\limits_{x\to\pm\infty}\dfrac{2x^2-5x+\frac{9}{2}}{x^2-3x+3} &=& \lim\limits_{x\to\pm\infty}\dfrac{\left(2x^2-5x+\frac{9}{2}\right)\frac{1}{x^2}}{\left(x^2-3x+3\right)\frac{1}{x^2}} & \left| \text{Divide by leading monomial in denominator}\right.\\ &=& \lim\limits_{x\to\pm\infty}\dfrac{2-\frac{5}{x}+\frac{9}{2x^2}}{1-\frac{3}{x}+\frac{3}{x^2}} & \\ &=& \dfrac{2-0+0}{1-0+0} & \\ &=& 2 & \end{array}$$

Therefore the graph of $f(x)$ has a single horizontal asymptote at $y=2$.

Intervals of increase and decrease. The intervals of increase and decrease of f are governed by the sign of f'. We compute:

$$\begin{array}{rcl} f'(x) &=& \left(\dfrac{2x^2-5x+\frac{9}{2}}{x^2-3x+3}\right)' \\ &=& \dfrac{\left(2x^2-5x+\frac{9}{2}\right)'\left(x^2-3x+3\right)-\left(2x^2-5x+\frac{9}{2}\right)\left(x^2-3x+3\right)'}{\left(x^2-3x+3\right)^2} \\ &=& \dfrac{-x^2+3x-\frac{3}{2}}{\left(x^2-3x+3\right)^2} \end{array}$$

As the denominator is a square, the sign of f' is governed by the sign of $-x^2+3x-\frac{3}{2}$. To find where $-x^2+3x-\frac{3}{2}$ changes sign, we compute the zeroes of this expression:

$$\begin{array}{rcll} -x^2+3x-\dfrac{3}{2} &=& 0 & \left| \text{Mult. by } -2\right. \\ 2x^2-6x+3 &=& 0 & \\ x_1,x_2 &=& \dfrac{6\pm\sqrt{36-24}}{4}=\dfrac{6\pm\sqrt{12}}{4} & \\ x_1,x_2 &=& \dfrac{3\pm\sqrt{3}}{2} & \end{array}$$

Therefore the quadratic $-x^2+3x-\frac{3}{2}$ factors as

$$-\left(x-x_1\right)\left(x-x_2\right)=-\left(x-\left(\frac{3-\sqrt{3}}{2}\right)\right)\left(x-\left(\frac{3+\sqrt{3}}{2}\right)\right) \qquad (1)$$

The points x_1,x_2 split the real line into three intervals: $\left(-\infty,\frac{3-\sqrt{3}}{2}\right)$, $\left(\frac{3-\sqrt{3}}{2},\frac{3+\sqrt{3}}{2}\right)$ and $\left(\frac{3+\sqrt{3}}{2},\infty\right)$, and each of the factors of (1) has constant sign inside each of the intervals. If we choose x to be a very negative number, it follows that $-(x-x_1)(x-x_2)$ is a negative, and therefore $f'(x)$ is negative for $x\in(-\infty,\frac{3-\sqrt{3}}{2})$. For $x\in(\frac{3-\sqrt{3}}{2},\frac{3+\sqrt{3}}{2})$, exactly one factor of f' changes sign and therefore $f'(x)$ is positive in that interval; finally only one factor of $f'(x)$ changes sign in the last interval so $f'(x)$ is negative on $(\frac{3+\sqrt{3}}{2},\infty)$.

Our computations can be summarized in the following table.

Interval	$f'(x)$	$f(x)$
$\left(-\infty,\frac{3-\sqrt{3}}{2}\right)$	$-$	$\searrow$
$\left(\frac{3-\sqrt{3}}{2},\frac{3+\sqrt{3}}{2}\right)$	$+$	$\nearrow$
$\left(\frac{3+\sqrt{3}}{2},\infty\right)$	$-$	$\searrow$

Local and global minima and maxima. The table above shows that $f(x)$ changes from decreasing to increasing at $x=x_1=\frac{3-\sqrt{3}}{2}$ and therefore f has a local minimum at that point. The table also shows that $f(x)$ changes from increasing to decreasing at $x=x_2=\frac{3+\sqrt{3}}{2}$ and therefore f has a local maximum at that point. The so found local maximum and local minimum turn out to be global: there are two things to consider here. First, no other finite point is critical and thus cannot be maximum or minimum - however this leaves out the possibility of a maximum/minimum "at infinity". This possibility can be quickly ruled out by looking at the graph of f. To do so via algebra, compute first $f(x_1)$ and $f(x_2)$:

$$
\begin{aligned}
f(x_1)=f\left(\frac{3-\sqrt{3}}{2}\right) &= \frac{2\left(\frac{3-\sqrt{3}}{2}\right)^2-5\left(\frac{3-\sqrt{3}}{2}\right)+\frac{9}{2}}{\left(\frac{3-\sqrt{3}}{2}\right)^2-3\left(\frac{3-\sqrt{3}}{2}\right)+3}=2-\frac{\sqrt{3}}{3}\\
f(x_2)=f\left(\frac{3+\sqrt{3}}{2}\right) &= \frac{2\left(\frac{3+\sqrt{3}}{2}\right)^2-5\left(\frac{3+\sqrt{3}}{2}\right)+\frac{9}{2}}{\left(\frac{3+\sqrt{3}}{2}\right)^2-3\left(\frac{3+\sqrt{3}}{2}\right)+3}=2+\frac{\sqrt{3}}{3} \quad .
\end{aligned}
$$

On the other hand, while computing the horizontal asymptotes, we established that $\lim_{x\to\pm\infty} f(x) = 2$. This implies that all x sufficiently far away from $x = 0$, we have that $f(x)$ is close to 2. Therefore $f(x)$ is larger than $f(x_1)$ and smaller than $f(x_2)$ for all sufficiently far away from $x = 0$. This rules out the possibility for a maximum or a minimum "at infinity", as claimed above.

Intervals of concavity. The intervals of concavity of f are governed by the sign of f''. The second derivative of f is:

$$
\begin{array}{rcll}
f''(x) &=& (f'(x))' = \left(\dfrac{-x^2+3x-\frac{3}{2}}{(x^2-3x+3)^2}\right)' & \\
&=& \left(-x^2+3x-\dfrac{3}{2}\right)'\left(\dfrac{1}{(x^2-3x+3)^2}\right)+\left(-x^2+3x-\dfrac{3}{2}\right)\left(\dfrac{1}{(x^2-3x+3)^2}\right)' & \text{second differentiation: chain rule}\\
&=& (-2x+3)\left(\dfrac{1}{(x^2-3x+3)^2}\right)+\left(-x^2+3x-\dfrac{3}{2}\right)(-2)\dfrac{(x^2-3x+3)'}{(x^2-3x+3)^3} & \\
&=& (-2x+3)\left(\dfrac{1}{(x^2-3x+3)^2}\right)+\left(2x^2-6x+3\right)\dfrac{(2x-3)}{(x^2-3x+3)^3} & \text{factor out } \frac{(2x-3)}{(x^2-3x+3)^2}\\
&=& \dfrac{(2x-3)}{(x^2-3x+3)^2}\left(-1+\dfrac{(2x^2-6x+3)}{(x^2-3x+3)}\right) & \\
&=& \dfrac{(2x-3)}{(x^2-3x+3)^2}\left(\dfrac{-\left(x^2-3x+3\right)+(2x^2-6x+3)}{(x^2-3x+3)}\right) & \\
&=& \dfrac{(2x-3)(x^2-3x)}{(x^2-3x+3)^3} & \\
&=& \dfrac{(2x-3)x(x-3)}{(x^2-3x+3)^3} &
\end{array}
$$

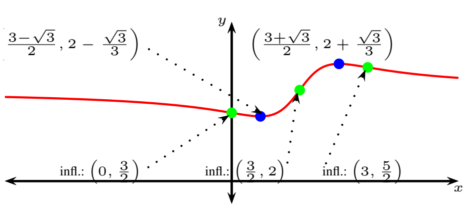
When computing the domain of f, we established that the denominator of the above expression is always positive. Therefore $f''(x)$ changes sign when the terms in the numerator change sign, namely, at $x = 0$, $x = \frac{3}{2}$ and $x = 3$.

Our computations can be summarized in the following table. In the table, we use the $\cup$ symbol to denote that the function is concave up in the indicated interval, and $\cap$ to denote that the function is concave down.

Interval	$f''(x)$	$f(x)$
$(-\infty, 0)$	$-$	$\cap$
$(0, \frac{3}{2})$	$+$	$\cup$
$(\frac{3}{2}, 3)$	$-$	$\cap$
$(3, \infty)$	$+$	$\cup$

Points of inflection. The preceding table shows that $f''(x)$ changes sign at $0, \frac{3}{2}, 3$ and therefore the points of inflection are located at $x = 0, x = \frac{3}{2}$ and $x = 3$, i.e., the points of inflection are $(0, f(0)) = \left(0, \frac{3}{2}\right)$, $\left(\frac{3}{2}, f\left(\frac{3}{2}\right)\right) = \left(\frac{3}{2}, 2\right)$, $(3, f(3)) = \left(3, \frac{5}{2}\right)$.

We can command our graphing device to use the so computed information to label the graph of the function. Finally, we can confirm visually that our function does indeed behave in accordance with our computations.



Solution. 3.h

This problem is very similar to Problem 3.b. We recommend to the student to solve the problem first "with closed textbook" and only then to compare with the present solution.

Domain. As f is a quotient of two polynomials (rational function), its implied domain is all x except those for which we get division by zero for f. Consequently the domain of f is all x for which $x^2+2x+4=0$. However, the polynomial x^2+2x+4 has no real roots - its roots are $\dfrac{-2\pm\sqrt{4-16}}{2}=-1\pm\sqrt{-3}$, and therefore the domain of f is all real numbers. Alternatively, we can complete the square: $x^2+2x+4=(x+1)^2+3$ and so x^2+2x+4 is positive for all values of x.

x**,** y**-intercepts.** The y-intercept of f equals by definition $f(0)=\dfrac{0+1}{0^2+2\cdot 0+4}=\dfrac{1}{4}$. The x intercept of f is those values of x for which $f(x)=0$. We compute

$$\begin{array}{rcl} f(x) &=& 0\\ \dfrac{x+1}{x^2+2x+4} &=& 0\\ x+1 &=& 0\\ x &=& -1 \quad , \end{array}$$

and the x-intercept of f is $x=-1$.

Asymptotes. The line $x=a$ is a vertical asymptote when $\lim\limits_{x\to a^{\pm}} f(x)=\pm\infty$; as f is defined for all real numbers, this implies that there are no vertical asymptotes.

The line $y=L$ is a horizontal asymptote if $\lim\limits_{x\to\pm\infty} f(x)$ exists and equals L. We compute:

$$\lim_{x\to\infty} f(x)=\lim_{x\to\infty}\frac{(x+1)\frac{1}{x^2}}{(x^2+2x+4)\frac{1}{x^2}}=\lim_{x\to\infty}\frac{\frac{1}{x}+\frac{1}{x^2}}{1+\frac{2}{x}+\frac{4}{x^2}}=\frac{0+0}{1+0+0}=0$$

Therefore $y=0$ is a horizontal asymptote for f. An analogous computation shows that $\lim\limits_{x\to\pm\infty} f(x)=0$ and so $y=0$ is the only horizontal asymptote of f.

Intervals of increase and decrease. The intervals of increase and decrease of f are governed by the sign of f'. We compute:

$$\begin{array}{rcll} f'(x) &=& \left(\dfrac{x+1}{x^2+2x+4}\right)' & \Big|\ \text{qutotient rule}\\ &=& \dfrac{(x+1)'\left(x^2+2x+4\right)-(x+1)\left(x^2+2x+4\right)'}{\left(x^2+2x+4\right)^2} &\\ &=& \dfrac{x^2+2x+4-(x+1)(2x+2)}{\left(x^2+2x+4\right)^2} &\\ &=& \dfrac{x^2+2x+4-\left(2x^2+4x+2\right)}{\left(x^2+2x+4\right)^2} &\\ &=& \dfrac{-x^2-2x+2}{\left(x^2+2x+4\right)^2} & \end{array}$$

As x^2+2x+4 is positive, the sign of f' is governed by the sign of $-x^2+2x+2$. To find out where $-x^2+2x+2$ changes sign, we compute the zeroes of this expression:

$$\begin{array}{rcll} -x^2-2x+2 &=& 0 &\\ x^2+2x-2 &=& 0 & \Big|\ \text{use the quadratic formula}\\ x_1,x_2 &=& -1\pm\sqrt{3} \quad . & \end{array}$$

Therefore the quadratic $-x^2+2x+2$ factors as

$$-(x-x_1)(x-x_2)=-\left(x-\left(-1-\sqrt{3}\right)\right)\left(x-\left(-1+\sqrt{3}\right)\right) \qquad (2)$$

The points x_1,x_2 split the real line into three intervals: $(-\infty,-1-\sqrt{3})$, $(-1-\sqrt{3},-1+\sqrt{3})$ and $(-1+\sqrt{3},\infty)$, and each of the factors of (2) has constant sign inside each of the intervals. If we choose x to be a very negative number, it follows that $-(x-x_1)(x-x_2)$ is a negative, and therefore $f'(x)$ is negative for $x\in(-\infty,-1-\sqrt{3})$. For $x\in(-1-\sqrt{3},-1+\sqrt{3})$, exactly one factor of f' changes sign and therefore $f'(x)$ is positive in that interval; finally only one factor of $f'(x)$ changes sign in the last interval so $f'(x)$ is negative on $(-1+\sqrt{3},\infty)$.

Our computations can be summarized in the following table.

Interval	$f'(x)$	$f(x)$
$(-\infty, -1-\sqrt{3})$	$-$	$\searrow$
$(-1-\sqrt{3}, -1+\sqrt{3})$	$+$	$\nearrow$
$(-1+\sqrt{3}, \infty)$	$-$	$\searrow$

Local and global minima and maxima. The table above shows that $f(x)$ changes from decreasing to increasing at $x = x_1 = -1-\sqrt{3}$ and therefore f has a local minimum at that point. The table also shows that $f(x)$ changes from increasing to decreasing at $x = x_2 = -1+\sqrt{3}$ and therefore f has a local maximum at that point. The so found local maximum and local minimum turn out to be global: indeed, no other finite point is critical and thus cannot be maximum or minimum; on the other hand $\lim\limits_{x\to\pm\infty} f(x) = 1$ and this implies that all x sufficiently far away from $x = 0$ have that $f(x)$ is close to 0, and therefore $f(x)$ is larger than $f(x_1)$ and smaller than $f(x_2)$ for all x.

Intervals of concavity. The intervals of concavity of f are governed by the sign of f''. The second derivative of f is:

$$
\begin{aligned}
f''(x) &= (f'(x))' = \left(\frac{-x^2-2x+2}{(x^2+2x+4)^2}\right)' \\
&= (-x^2-2x+2)'\left(\frac{1}{(x^2+2x+4)^2}\right) + (-x^2-2x+2)\left(\frac{1}{(x^2+2x+4)^2}\right)' && \text{use chain rule for second differentiation} \\
&= (-2x-2)\left(\frac{1}{(x^2+2x+4)^2}\right) + (-x^2-2x+2)(-2)\frac{(x^2+2x+4)'}{(x^2+2x+4)^3} \\
&= -(2x+2)\left(\frac{1}{(x^2+2x+4)^2}\right) + (2x^2+4x-4)\frac{(2x+2)}{(x^2+2x+4)^3} && \text{factor out } \tfrac{(2x+2)}{(x^2+2x+4)^2} \\
&= \frac{(2x+2)}{(x^2+2x+4)^2}\left(-1+\frac{(2x^2+4x-4)}{(x^2+2x+4)}\right) \\
&= \frac{(2x+2)}{(x^2+2x+4)^2}\left(\frac{-(x^2+2x+4)+(2x^2+4x-4)}{(x^2+2x+4)}\right) \\
&= \frac{(2x+2)(x^2+2x-8)}{(x^2+2x+4)^3} && \text{factor } (x^2+2x-8) \\
&= \frac{(2x+2)(x+4)(x-2)}{(x^2+2x+4)^3}
\end{aligned}
$$

As we previously established, the denominator of the above expression is always positive. Therefore the expression above changes sign when the terms in the numerator change sign, namely, at $x=-1$, $x=-4$ and $x=2$.

Our computations can be summarized in the following table.

Interval	$f''(x)$	$f(x)$
$(-\infty,-4)$	$-$	$\cap$
$(-4,-1)$	$+$	$\cup$
$(-1,2)$	$-$	$\cap$
$(2,\infty)$	$+$	$\cup$

Points of inflection. The preceding table shows that $f''(x)$ changes sign at $-4,-1,2$ and therefore the points of inflection are located at $x=-4, x=-1$ and $x=2$, i.e., the points of inflection are $\left(-4,-\frac{1}{4}\right)$, $(-1,0)$, $\left(2,\frac{1}{4}\right)$.

4. (a) Sketch the graph of $y = x^4 - 8x^2 + 8$ by determining the intervals of increase and decrease, finding the local mins and maxes, determining where the graph is concave up and concave down, and plotting a few key points.

answer: Check your graph with a calculator or online graphing program.
Local max at 0, local mins at 2 and -2. Concave down between $-\sqrt{4/3}$ and $\sqrt{4/3}$, and concave up otherwise.

(b) Sketch the graph of $y = \frac{x-1}{x^2-9}$ by graphing any vertical and horizontal asymptotes, finding the x- and y-intercepts, and then sketching a graph that fits this information.

answer: Check your graph with a calculator or online graphing program.
Vertical asymptotes at $x=3$ and $x=-3$.
Horizontal asymptote at $y=0$.
y-intercept of $\frac{1}{9}$; x-intercept of 1.

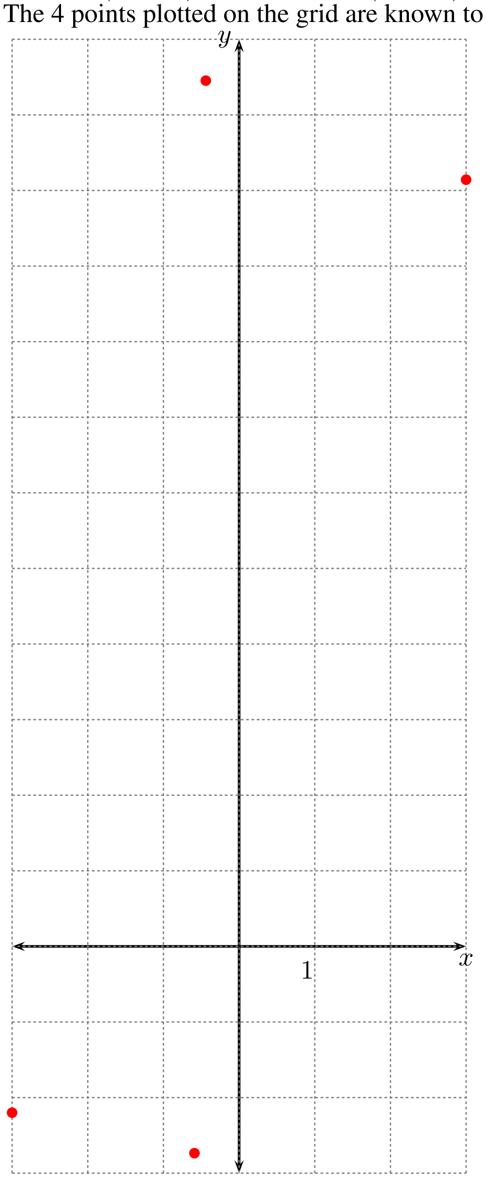
(c) Consider the function $f(x) = \dfrac{4x^2+10x+5}{2x+1}$. Computation shows that $f'(x) = \dfrac{8x^2+8x}{(2x+1)^2}$ and $f''(x) = \dfrac{8}{(2x+1)^3}$.

- Find the intervals of increase and intervals of decrease of f.
- Find the local maxima and minima of f.
- Find where the function is concave up and where it is concave down.

- Sketch the function $f(x)$ roughly by hand. Make sure that your plot matches your computations from the preceding parts of the problem.

  You may use the provided grid and coordinate system. From the previous page, we recall that $f(x) = \dfrac{4x^2+10x+5}{2x+1}$, $f'(x) = \dfrac{8x^2+8x}{(2x+1)^2}$ and $f''(x) = \dfrac{8}{(2x+1)^3}$.

  The 4 points plotted on the grid are known to lie on the curve.



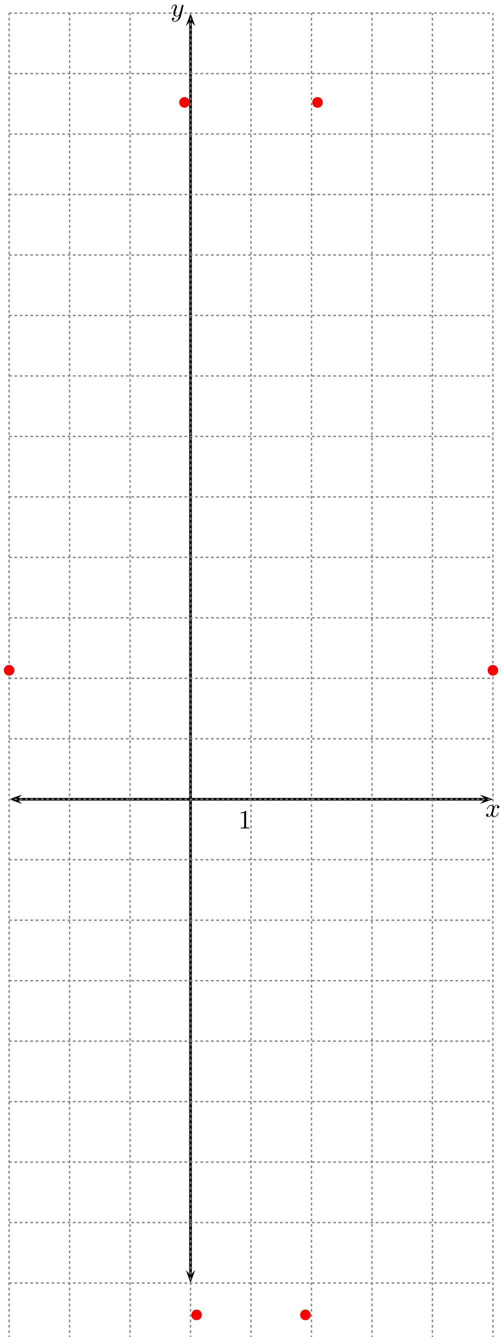
(d) Consider the function $f(x) = \dfrac{2x^2-4x+2}{x^2-2x}$.

- Find the vertical asymptotes of f. **For this particular sub-question, and for this sub-question alone, no justification is required (just write the answer).**
- Computation shows that $f'(x) = \dfrac{-4x+4}{(x^2-2x)^2}$. Find the intervals of increase and decrease of f.
- Find the local maxima and minima of f.
- Computation shows that $f''(x) = \dfrac{12x^2-24x+16}{(x^2-2x)^3}$. Find where the function is concave up and where it is concave down.
- Sketch the function $f(x)$ roughly by hand. Make sure that your plot matches your computations from the preceding parts of the problem.

  You may use the provided grid and coordinate system. We recall that

  $f(x) = \dfrac{2x^2-4x+2}{x^2-2x}$,

$f'(x) = \dfrac{-4x+4}{(x^2-2x)^2}$,
$f''(x) = \dfrac{12x^2-24x+16}{(x^2-2x)^3}$.
The points plotted below are known to lie on the curve.



Solution. 4c

Intervals of increase and decrease. The intervals of increase and decrease of $f(x)$ are determined by the intervals where $f'(x)$ does not change sign. The candidates for the endpoints of these intervals are the critical points of $f(x)$, i.e., the points for which $f'(x) = 0$ and the points for which $f'(x)$ is not defined. Since $f'(x) = \frac{8x(x+1)}{(2x+1)^2}$, it follows that $f'(x)$ may change sign the critical points $x = 0$, $x = -1$ and $x = -\frac{1}{2}$. However $f'(x)$ does not change sign near $x = -\frac{1}{2}$ as the term $(2x+1)$ is raised to an even power. Therefore the intervals of increase and decrease are given in the following table.

	$(-\infty, -1)$	$(-1, 0)$	$(0, \infty)$
$f'(x)$	$+$	$-$	$+$
$f(x)$	$\nearrow$	$\searrow$	$\nearrow$

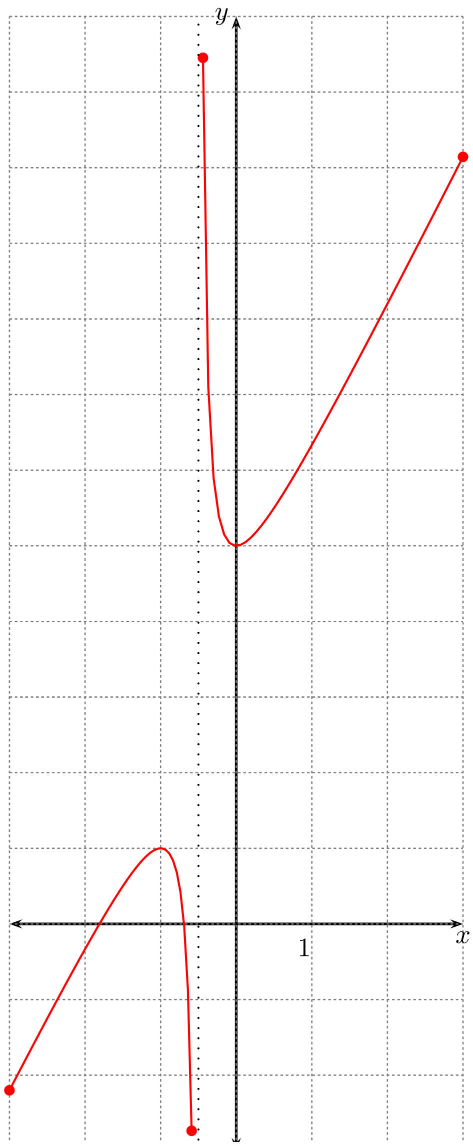
Local maxima and minima. A local maximum occurs where f changes from increasing to decreasing or the other way around.

Therefore the preceding point implies that f has local maximum at $x = -1$ equal to $f(-1) = 1$ and local minimum at $x = 0$ equal to 5.

Intervals of concavity. The intervals of concavity are determined by the points where $f''(x) = \frac{8}{(2x+1)^3}$ changes sign. The denominator of $f''(x)$ changes sign near $x = -\frac{1}{2}$, so the intervals of concavity are $\left(-\infty, -\frac{1}{2}\right)$ and $\left(-\frac{1}{2}, \infty\right)$. The concavity of $f(x)$ is then determined by the following table.

	$\left(-\infty, -\frac{1}{2}\right)$	$\left(-\frac{1}{2}, \infty\right)$
$f'(x)$	$-$	$+$
$f(x)$	$\cap$	$\cup$

Curve sketching. Please note that $x = -\frac{1}{2}$ is a vertical asymptote of $f(x)$. Together with the data computed above this makes it relatively easy to quickly produce a relatively accurate plot of $f(x)$ by hand. A computer generated plot is included below.



Solution. 4d

Vertical asymptotes. The only candidates for vertical asymptotes are the points where f is not defined, i.e., the points where $x^2 - 2x = 0$. In other words, the candidates for vertical asymptotes are 0 and 2. A short computation (not presented here as the problem requests that we omit it) shows that both $x = 0$ and $x = 2$ are vertical asymptotes.

Intervals of increase and decrease. The denominator of f' is a square so its sign is dictated by its numerator. The numerator of f' is positive for $x < 1$ and negative for $x > 1$. Therefore f increases for $x \in (-\infty, 0) \cup (0, 1)$ decreases for $x \in (1, 2) \cup (2, \infty)$.

Local maxima and minima. From the preceding point, the only local extremum is located where the function changes from increasing to decreasing, i.e., the only local extremum is the local maximum at $x = 1, y = \frac{2-4+2}{1-2} = 0$.

Intervals of concavity. The equation $12x^2 - 24x + 16 = 0$ simplifies to $3x^2 - 6x + 4 = 0$ and that has no real solutions (the solutions are $\frac{6 \pm \sqrt{36-48}}{6} = \frac{3 \pm \sqrt{3}i}{3}$). Since $12x^2 - 24x + 16 = 0$ is a parabola without real roots that opens up, it is strictly positive. Therefore the sign of f'' is determined by the sign of its denominator. The denominator is negative for $x \in (0, 2)$ and positive otherwise. Therefore f is concave up for $x \in (-\infty 0) \cup (2, \infty)$ and concave down for $x \in (0, 2)$.

Curve sketch. A computer generated plot is included below.