Precalculus
Lecture 18

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https://github.com/tmilev/freecalc

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Outline

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$\mathbb{R}^2, \mathbb{R}^3, \mathbb{R}^n$

Definition ($\mathbb{R}^2$)

The set of ordered pairs of real numbers is denoted by $\mathbb{R}^2$.

Definition ($\mathbb{R}^3$)

The set of ordered triples of real numbers is denoted by $\mathbb{R}^3$.

Definition ($\mathbb{R}^n$)

The set of ordered n-tuples of real numbers is denoted by $\mathbb{R}^n$.

Example

$(1, -2, 3) \in \mathbb{R}^3$
$(0, 5) \in \mathbb{R}^2$
$(0, 5, -2, 4, 0) \in \mathbb{R}^5$
$(0, 1, 2, 3, \dots, n) \in \mathbb{R}^{n+1}$

Definition

- An equation of the form $ax + by + c = 0$, where a, b, c are constants such that a and b are not simultaneously zero, is called a linear equation.
- A set of pairs of numbers (x, y) is called a line in $\mathbb{R}^2$ if it is the set of solutions to some linear equation.
- A set of points in the plane will be called a line if it is the graph of some linear equation.

- To introduce the Cartesian coordinate system we used informal, intuitive notions of point, lines and the plane.
- We could (and often do in more advanced subjects) remove this informality by *defining* $\mathbb{R}^2$ to be the Euclidean plane.

Example

Find an equation of the line passing through $(1, 3)$ and $(2, 6)$.

$$(2-1)(y-3)=(6-3)(x-1)$$

- It suffices to manufacture a linear equation such that when we plug in $(1, 3)$ and $(2, 6)$ we get an identity.
- A (very simple) equation satisfied by $x=1$, $y=3$ is:
$$y-3=x-1.$$
This is so because both sides become zero when $x=1$, $y=3$.
- If we plug in $x=2$ and $y=6$ in the above we don't get an identity, but that can be easily fixed:
$$(2-1)(6-3)=(6-3)(2-1)$$
- Perhaps the last modification caused $x=1$, $y=3$ to no longer be solutions? No - both sides are still zero when $x=1$, $y=3$.

Example

Find an equation of the line passing through (x_1, y_1) and (x_2, y_2).

$$(x_2 - x_1)(y - y_1) = (y_2 - y_1)(x - x_1)$$

- It suffices to manufacture a linear equation such that when we plug in (x_1, y_1) and (x_2, y_2) we get an identity.
- A (very simple) equation satisfied by $x = x_1$, $y = y_2$ is:
  $$y - y_2 = x - x_1.$$
  This is so because both sides become zero when $x = x_1$, $y = y_1$.
- If we plug in $x = x_2$ and $y = y_2$ in the above we don't get an identity (necessarily), but that can be easily fixed:
  $$(x_2 - x_1)(y_2 - y_1) = (y_2 - y_1)(x_2 - x_1)$$
- Perhaps the last modification caused $x = x_1$, $y = y_1$ to no longer be solutions? No - both sides are still zero when $x = x_1$, $y = y_1$.

Proposition

Let (x_1, y_1) and (x_2, y_2) be points and L be the line between them. Then L has equation

$$(x_2 - x_1)(y - y_1) = (y_2 - y_1)(x - x_1).$$

If $x_1 \neq x_2$, set $m = \frac{y_2 - y_1}{x_2 - x_1}$; then L has also equations

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$y - y_1 = m(x - x_1) \qquad \text{point-slope form}$$

$$y = m(x - x_1) + y_1$$

$$y = mx + y_1 - mx_1$$

Set $b = y_1 - mx_1$

$$y = mx + b \qquad \text{slope-intercept form}$$

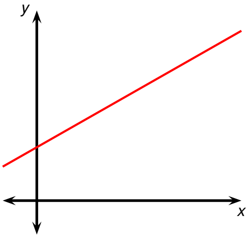
Definition (non-vertical line, slope-intercept form)

A line that is the graph of an equation of the form

$$y = mx + b$$

is called a *non-vertical* line.

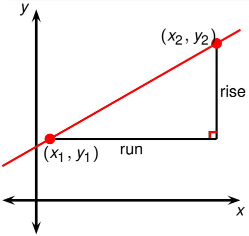
- The equation above is called the slope-intercept form of the (non-vertical) line.
- The number m is called the slope of the line.
- The number b is the y intercept of the line.

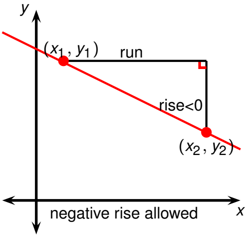


Geometric Interpretation of Slope

Definition (non-vertical line, slope-intercept form)

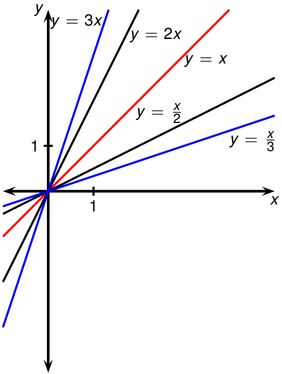
$y = mx + b$, m - is called slope, b is called y-intercept.



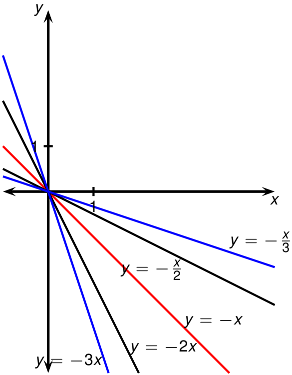


- Fix pts. (x_1, y_1), (x_2, y_2) on the line with $x_2 > x_1$.
- Call $x_2 - x_1$ the run of the line between the points. The run is assumed positive.
- Call $y_2 - y_1$ the rise of the line between the two points. Negative rise is allowed.

$$
\begin{array}{rrcl}
 & y_2 & = & mx_2 + b \\
- & y_1 & = & mx_1 + b \\
\hline
 & y_2 - y_1 & = & mx_2 + \cancel{b} - mx_1 - \cancel{b} \\
 & y_2 - y_1 & = & m(x_2 - x_1) \\
 & m & = & \dfrac{y_2 - y_1}{x_2 - x_1} \\
 & m & = & \dfrac{\text{rise}}{\text{run}}
\end{array}
$$



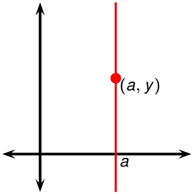
- If two linear functions have positive slopes, the one with the bigger slope increases faster.
- $y = 2x$ increases twice as fast as $y = x$.
- $y = 3x$ increases three times as fast as $y = x$.
- $y = \frac{x}{2}$ increases half as fast as $y = x$.
- $y = \frac{x}{3}$ increases one third as fast as $y = x$.



- If two linear functions have negative slopes, the one with the lower slope decreases faster.
- $y = -2x$ decreases twice as fast as $y = -x$.
- $y = -3x$ decreases three times as fast as $y = -x$.
- $y = -\frac{1}{2}x$ decreases half as fast as $y = -x$.
- $y = -\frac{1}{3}x$ decreases one third as fast as $y = -x$.

Definition (Vertical line)

A line of the form $x = a$ is called a vertical line.



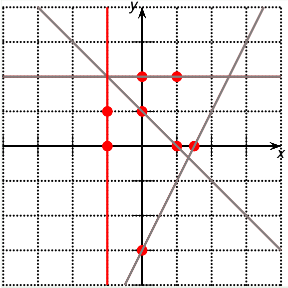
- y does not participate directly in the equation $x = a$.
- Therefore the equation cannot be rewritten in slope-intercept form ($y = \mathbf{?}x + \mathbf{?}$).
- Consequently the notion of a slope is not undefined for vertical lines.

Plotting Lines from line equation

To plot a line from its equation $ax + by = c$ do the following.

- If $b = 0$, the line is vertical through $x = \frac{c}{a}$. Suppose $b \neq 0$.
  - Plug in arbitrary number for x and find y from $y = \frac{c-ax}{b}$.
  - Use same procedure to find a second point on the line.
  - If $a \neq 0$: can also plug in values for y to find x.
  - Draw a line between the two dots.

Example



Plot the line with the given equation.

equation	pt.	another pt.
$x + y = 1$	$(1, 0)$	$(0, 1)$
$2x - y = 3$	$(0, -3)$	$\left(\frac{3}{2}, 0\right)$
$y = 2$	$(0, 2)$	$(1, 2)$
$x = -1$	$(-1, 0)$	$(-1, 1)$

Other points can be used as well.

Example

Find an equation of a line passing though the indicated pairs of points.

- $(1, 2)$ and $(2, -1)$.
- $(1, 1)$ and $(2, -2)$.
- $(0, 1)$ and $(1, 0)$.
- $(3, 5)$ and $(7, -11)$.

Example

Find an equation of the line passing through $(1, 2)$ with slope $-\frac{1}{2}$.

To find the intersection of two lines (if they do intersect) with equations $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ we need to solve the system of equations

$$\left| \begin{array}{rcl} a_1x + b_1y + c_1 & = & 0 \\ a_2x + b_2y + c_2 & = & 0 \end{array} \right.$$

Example

Find the intersection of the following lines.

1. $x - y = 3$ and $x + 2y = 10$.
2. $3x - y = 3$ and $x = 1 - 3y$.
3. Line $x = 3$ and $x = 1 - 2y$
4. Line through $(2, 0)$ and $(1, 2)$ and line through $(3, 7)$ and $(2, 5)$.
5. Line through $(3, -1)$ and $(-1, 3)$ and line through $(1, 1)$ and $(2, 3)$.

Definition

Two lines are parallel if they have no common point.

Proposition

Two non-vertical lines are parallel if and only if they have equal slopes and different y intercepts.

Proof $\Leftarrow$.

- Suppose the two lines have different y intercepts and have the same slope m.
- Then the lines have equations as shown below.

$$\left|\begin{array}{rcl} y & = & mx + b_1 \\ y & = & mx + b_2 \end{array}\right.$$

- System has no solutions as $b_1 \neq b_2 \Rightarrow$ the lines don't intersect. $\square$

Definition

Two lines are parallel if they have no common point.

Proposition

Two non-vertical lines are parallel if and only if they have equal slopes and different y intercepts.

Proof $\Rightarrow$.

- Suppose the two lines have different slopes.
- Suppose the lines have equations as shown below.

$$
\begin{array}{rll}
- \quad y=& m_1 x + b_1 & \\
y=& m_2 x + b_2 & \\
\hline
0=& (m_1 - m_2)x + b_1 - b_2 & \\
(m_1 - m_2)x=& b_2 - b_1 & \Big| \text{ Div. by } m_1 - m_2 \neq 0 \\
x=& \dfrac{b_2 - b_1}{m_1 - m_2} &
\end{array}
$$

- The system has solution $x = \frac{b_2 - b_1}{m_1 - m_2}$, $y = m_1 \frac{b_2 - b_1}{m_1 - m_2} + b_1 \Rightarrow$ the lines intersect. $\square$