

Calculus I

Lecture 7

Exponents and Logarithms

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<https://github.com/tmilev/freecalc>

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Outline

1 Exponential Functions

- Two ways to define exponents
- Basic properties
- The Natural Exponential Function

2 Logarithmic Functions

- Logarithm basics
- Natural Logarithms

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Properties of exponential expressions.

For integer x, y and bases a, b , we demonstrate the exponent rules by example.

$$\textcircled{1} \quad a^x a^y = a^{x+y}$$

$$\textcircled{2} \quad \frac{a^x}{a^y} = a^{x-y}$$

$$\textcircled{3} \quad (a^x)^y = a^{xy}$$

$$\textcircled{4} \quad (ab)^x = a^x b^x$$

These rules do continue to hold for all $a > 0$, $b > 0$ and arbitrary x and y . The rules do fail when $a < 0$, $b < 0$ and x, y are not integers.

$$\begin{aligned} 7^3 \cdot 7^2 &= (7 \cdot 7 \cdot 7)(7 \cdot 7) \\ &= 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \\ &= 7^5 \\ &= 7^{3+2}. \end{aligned}$$

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$$\begin{aligned} \frac{7^3}{7^2} &= \frac{\cancel{7} \cdot \cancel{7} \cdot 7}{\cancel{7} \cdot \cancel{7}} \\ &= 7 \\ &= 7^1 \\ &= 7^{3-2}. \end{aligned}$$

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These rules do continue to hold for all $a > 0$, $b > 0$ and arbitrary x and y . The rules do fail when $a < 0$, $b < 0$ and x, y are not integers.

$$\begin{aligned} (7^2)^4 &= 7^2 \cdot 7^2 \cdot 7^2 \cdot 7^2 \\ &= (7 \cdot 7)(7 \cdot 7)(7 \cdot 7)(7 \cdot 7) \\ &= 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \\ &= 7^8 \\ &= 7^{2 \cdot 4} \end{aligned}$$

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$$\begin{aligned}(5 \cdot 7)^3 &= (5 \cdot 7)(5 \cdot 7)(5 \cdot 7) \\ &= 5 \cdot 7 \cdot 5 \cdot 7 \cdot 5 \cdot 7 \\ &= 5 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdot 7 \\ &= 5^3 \cdot 7^3\end{aligned}$$

Exponents overview

- For integer x , we know how to compute a^x as a function of a .
- How do we compute $f(x) = a^x$ when x is not an integer?
- We need to go back to the definition of a^x (for x non-integer).
- In what follows we give/recall an elementary way to define exponent.
- Then we give an alternative second definition.
- The second definition will be studied in sufficient depth only much later.
- The two definitions are equivalent: if we choose one definition the other becomes a theorem and the other way round.
- Choosing one definition makes some statements easier to prove and others more difficult.
- We shall discuss pros and cons of the two. In a nutshell:
 - the first elementary definition is easier to motivate;
 - the second alternative definition is easier to compute with.

Exponent definition using limits (approach I)

- For integer p we know to compute a^p .
- Therefore for integer q we know to compute $a^{\frac{1}{q}} = \sqrt[q]{a} = \max\{x \mid x^q \leq a\}$.
- Therefore we know to compute $a^{\frac{p}{q}}$ for all rational $\frac{p}{q}$.
- We can then define

$$a^x = \lim_{\substack{y \rightarrow x \\ y\text{-rational}}} a^y$$

For example, a^π would be defined as the limit of the sequence $a^{3.14}, a^{3.141}, a^{3.1415}, \dots$

- Cons: not computationally effective; not how computers compute.
- Pros: for non-integer x and y , it is very easy to prove that $a^{x+y} = a^x a^y$ - this follows from the definition of limit above.
- This is the definition assumed in many elementary courses.

Exponent definition using series (approach II)

- The following formula (studied much later) can be used as alternative definition.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

Here $n! = 1 \cdot 2 \cdot 3 \cdot \dots \cdot (n-1) \cdot n$ and is read “ n factorial”.

- For $|x| < 1$ define

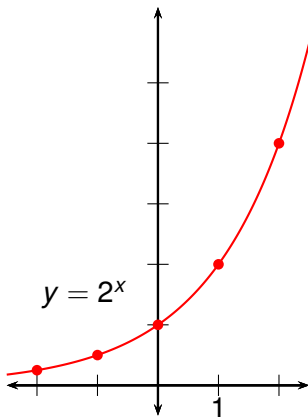
$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{n+1} x^n}{n} + \cdots$$

Infinite sum studied much later.

- For arbitrary $a > 0$ define a^x as $a^x = e^{x \ln a}$.
- Cons: more difficult to prove $e^{x+y} = e^x e^y$ and $e^{\ln(1+x)} = 1+x$, proof done later.
- Pros: this is how e^x and a^x are actually computed (by modern computers and by humans in the past).

Exponential Functions

The function $f(x) = 2^x$ is called an exponential function because the variable x is the exponent.

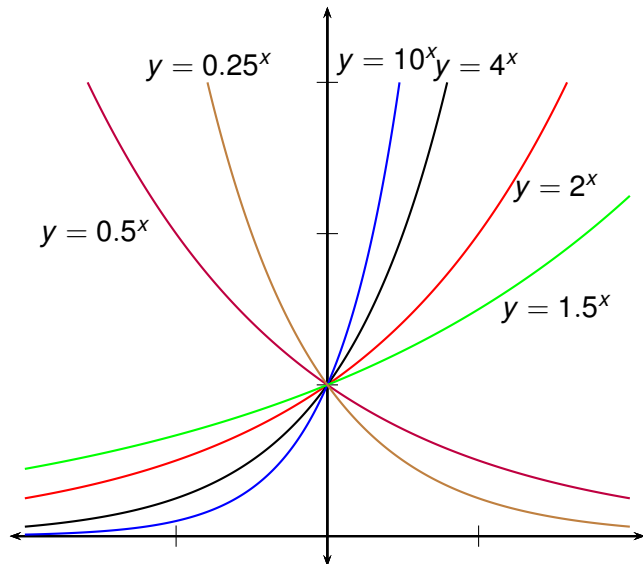


x	y
2	4
1	2
0	1
-1	$\frac{1}{2}$
-2	$\frac{1}{4}$

(Exponential Function Terminology)

An exponential function is a function of the form $f(x) = a^x$, where a is a positive constant.

Graphs of various exponential functions.



Observations

- a^x is always positive.
- $a^0 = 1$ for all a .

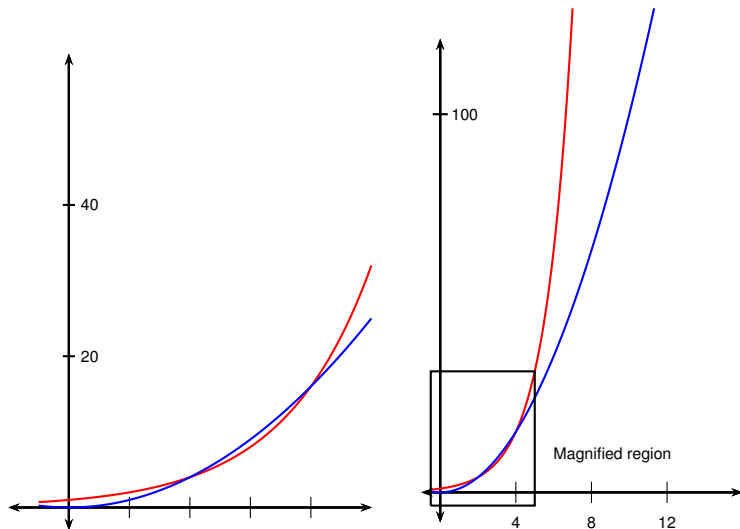
For $a > 1$:

- $\lim_{x \rightarrow \infty} a^x = \infty$.
- $\lim_{x \rightarrow -\infty} a^x = 0$.

$a < 1$:

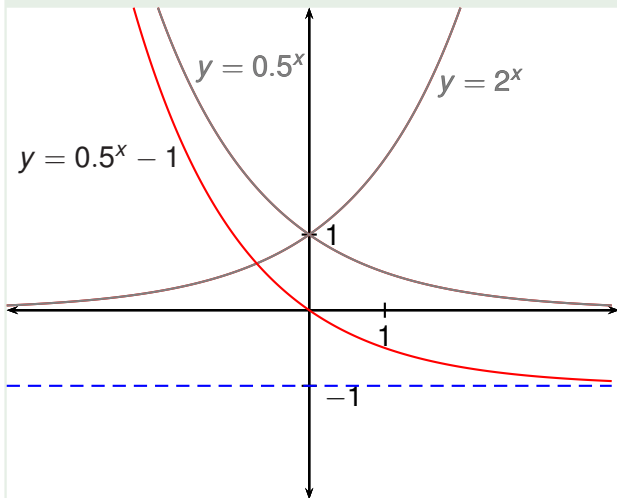
- $\lim_{x \rightarrow \infty} a^x = 0$.
- $\lim_{x \rightarrow -\infty} a^x = \infty$.

Graphical comparison of $y = 2^x$ with $y = x^2$. Axes have different scales.



Example

Draw the graph of the function $y = 2^{-x} - 1 = 0.5^x - 1 = \left(\frac{1}{2}\right)^x - 1$. Assume the graph of $y = 2^x$ given.



- Plot of 2^x assumed given.
- Plot $f(-x) =$ reflect $f(x)$ across y axis.
- Plot $g(x) - 1 =$ shift graph $g(x)$ 1 unit down.

Example (Solve exponential equation without logarithms)

Solve for t .

Find a common base:

$$\begin{aligned}16^{4t} &= 8^{t-2} \\ (2^4)^{4t} &= (2^3)^{t-2} \\ 2^{16t} &= 2^{3t-6} \\ 16t &= 3t - 6 \\ 13t &= -6 \\ t &= -\frac{6}{13}.\end{aligned}$$

Example (Solving a quadratic exponential equation)

Solve for x .

$$\begin{aligned}9^x &= 2 \cdot 3^x + 63 \\9^x - 2 \cdot 3^x - 63 &= 0 & \left| \text{Substitute } u = 3^x \right. \\u^2 - 2u - 63 &= 0 \\(u - 9)(u + 7) &= 0\end{aligned}$$

$$\begin{aligned}u &= 9 \quad \text{or} \quad u = -7 \\3^x &= 9 \quad \text{or} \quad 3^x = -7 \\x &= 2 & \text{no real solution}\end{aligned}$$

Example (Solving an exponential word problem)

A farmer buys 48 chickens and 6 rabbits. The chicken population doubles each year, and the rabbit population doubles every six months. When does the farmer have the same number of chickens as rabbits?

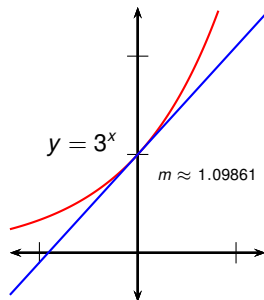
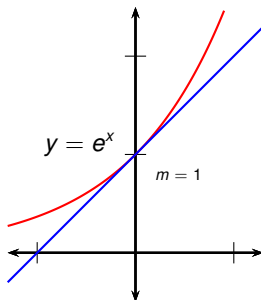
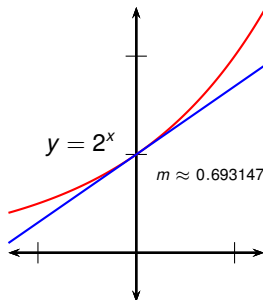
Let $c(t)$ denote the number of chickens after t years, and let $r(t)$ denote the number of rabbits after t years.

Solve for t : $=$

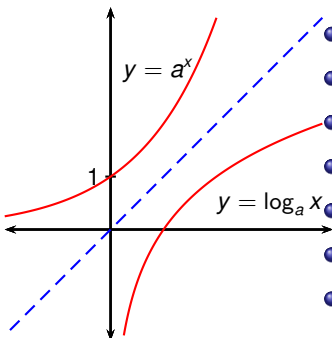
 $=$ $=$ $=$ $=$ $=$ $t =$

The Natural Exponential Function

- One base for an exponential function is especially useful.
- It has a special property: its tangent line at $x = 0$ has slope $m = 1$.
- We call this number e , known as Euler's number or Napier's constant.
- e is a number between 2 and 3.
- In fact, $e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots \approx 2.71828$.



Logarithmic Functions



- Suppose $a > 0$, $a \neq 1$.
- Let $f(x) = a^x$.
- Then f is either increasing or decreasing.
- Therefore f is one-to-one.
- Therefore f has an inverse function, f^{-1} .
- The graph shows $y = a^x$ for $a > 1$.
- The graph of $y = \log_a x$ is the reflection of this in the line $y = x$.

Definition ($\log_a x$)

The inverse function of $f(x) = a^x$ is called the logarithmic function with base a , and is written $\log_a x$. It is defined by the formula

$$\log_a x = y \quad \Leftrightarrow \quad a^y = x.$$

If $x > 0$, then $\log_a x$ is the exponent to which the base a must be raised to give x .

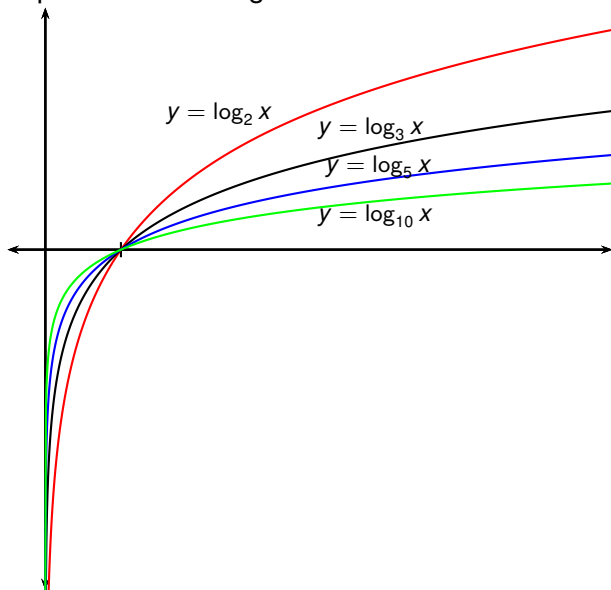
Example

Evaluate:

- 1 $\log_3 81 = 4$ because $3^4 = 81$.
- 2 $\log_{25} 5 = \frac{1}{2}$ because $25^{\frac{1}{2}} = \sqrt{25} = 5$.
- 3 $\log_{10} 0.001 = -3$ because $10^{-3} = 0.001$.



- Suppose $a > 1$.
- Domain of a^x : \mathbb{R} .
- Range of a^x : $(0, \infty)$.
- Domain of $\log_a x$: $(0, \infty)$.
- Range of $\log_a x$: \mathbb{R} .
- $\log_a(a^x) = x$ for $x \in \mathbb{R}$.
- $a^{\log_a x} = x$ for $x > 0$.

Graphs of various logarithmic functions with $a > 1$ 

Theorem (Properties of Logarithmic Functions)

If $a > 1$, the function $f(x) = \log_a x$ is a one-to-one, continuous, increasing function with domain $(0, \infty)$ and range \mathbb{R} . If $x, y, a, b > 0$ and r is any real number, then

$$\textcircled{1} \log_a(xy) = \log_a x + \log_a y.$$

$$\textcircled{2} \log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y.$$

$$\textcircled{3} \log_a(x^r) = r \log_a x.$$

$$\textcircled{4} \log_a(x) = \log_b x \log_a b = \frac{\log_b x}{\log_b a} = \frac{\ln x}{\ln a}.$$

Use the properties of logarithms to evaluate the following.

Example

$$\begin{aligned}\log_4 2 + \log_4 32 &= \log_4(2 \cdot 32) \\ &= \log_4(64) \\ &= 3 \\ &\quad (\text{because } 4^3 = 64.)\end{aligned}$$

Example

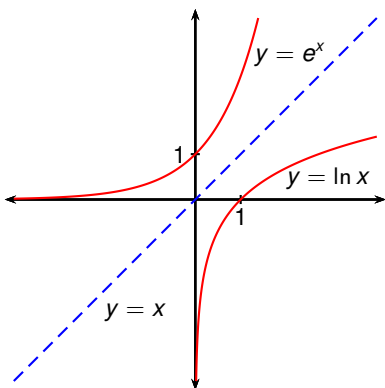
$$\begin{aligned}\log_2 80 - \log_2 5 &= \log_2 \left(\frac{80}{5} \right) \\ &= \log_2(16) \\ &= 4 \\ &\quad (\text{because } 2^4 = 16.)\end{aligned}$$

Natural Logarithms

Definition ($\ln x$)

The logarithm with base e is called the natural logarithm, and has a special notation:

$$\log_e x = \ln x.$$



- $\ln x = y \quad \Leftrightarrow \quad e^y = x.$
- $\ln(e^x) = x$ for $x \in \mathbb{R}.$
- $e^{\ln x} = x$ for $x > 0.$

Summary of logarithm notation conventions

	Name	ISO notation	Other notation	Used in
$\log_2(x)$	binary logarithm	$\text{lb}(x)$	$\text{ld}(x)$, $\log(x)$, $\text{lg}(x)$	computer science, information theory, music theory, photography
$\log_e(x)$	natural logarithm	$\ln(x)$	$\log(x)$	mathematics, physics, chemistry, statistics, economics, information theory, and engineering
$\log_{10}(x)$	common logarithm	$\text{lg}(x)$	$\log(x)$	various engineering, logarithm tables, handheld calculators, spectroscopy

Table source: Wikipedia

- Standardized in ISO 31-11 (International Standards Organization).

Example

Solve the equation.

$$\begin{aligned}e^{5-3x} &= 10 \\ \ln(e^{5-3x}) &= \ln 10 \\ 5 - 3x &= \ln 10 \\ 3x &= 5 - \ln 10 \\ x &= \frac{5 - \ln 10}{3} \\ \text{Calculator: } x &\approx 0.8991.\end{aligned}$$

Example

Solve the equation

$$e^{2x} - 3e^x - 4 = 0$$

Set $e^x = u$. Then $e^{2x} = u^2$.

$$u^2 - 3u - 4 = 0$$

$$(u - 4)(u + 1) = 0$$

$$u = 4$$

or

$$u = -1$$

$$e^x = 4$$

or

$$e^x = -1$$

$$x = \ln 4$$

or

no real solution

$$x \approx 1.3863$$

Example

Solve the equation

$$4^{x+1} - 2^{x+2} - 3 = 0$$

Set $u = 2^x$. Then $4^{x+1} = 4u^2$, $2^{x+2} = 4u$.

$$4u^2 - 4u - 3 = 0$$

$$(2u - 3)(2u + 1) = 0$$

$$2u - 3 = 0 \quad \text{or} \quad 2u + 1 = 0$$

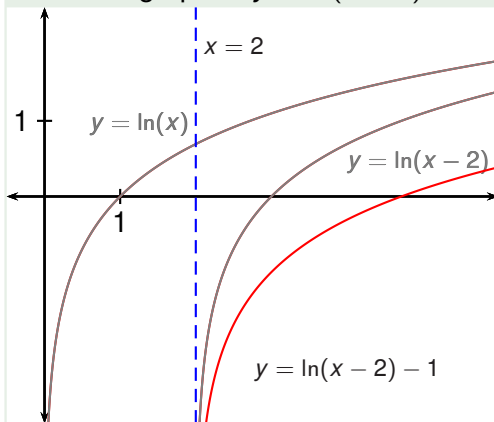
$$u = \frac{3}{2} \quad \text{or} \quad u = -\frac{1}{2}$$

$$2^x = \frac{3}{2} \quad \text{or} \quad 2^x = -\frac{1}{2}$$

$$x = \log_2 \left(\frac{3}{2} \right) = \frac{\ln \left(\frac{3}{2} \right)}{\ln 2} \approx 0.58496 \quad \text{or} \quad \text{no real solution}$$

Example

Draw the graph of $y = \ln(x - 2) - 1$.



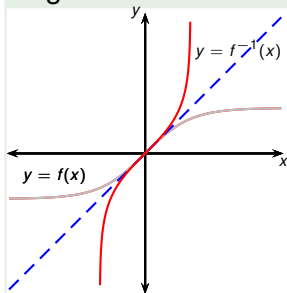
- Graph $y = \ln(x)$ assumed known.
- $f(x - 2)$ shifts graph 2 units to the right.
- $g(x) - 1$ shifts graph 1 unit down.

Example

Find $f^{-1}(x)$ for

$$f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

$f = \tanh =$ hyperbolic tangent function.



Final answer, relabeled:

$$f^{-1}(x) = \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right)$$

$$\frac{e^x - e^{-x}}{e^x + e^{-x}} = y$$

$$\frac{(u - \frac{1}{u})u}{(u + \frac{1}{u})u} = y$$

$$\frac{u^2 - 1}{u^2 + 1} = y$$

$$u^2 - 1 = y(u^2 + 1)$$

$$u^2(1 - y) = 1 + y$$

$$u^2 = \frac{1 + y}{1 - y}$$

$$(e^x)^2 = \frac{1 + y}{1 - y}$$

$$e^{2x} = \frac{1 + y}{1 - y}$$

$$x = \frac{1}{2} \ln \left(\frac{1 + y}{1 - y} \right)$$

$$\begin{aligned} \text{Set } u &= e^x \\ e^{-x} &= \frac{1}{e^x} = \frac{1}{u} \end{aligned}$$

Take \ln