Calculus III
Lecture 15

Todor Milev
https://github.com/tmilev/freecalc

2020

Outline

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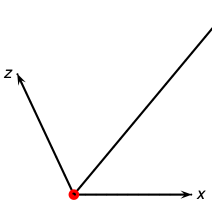

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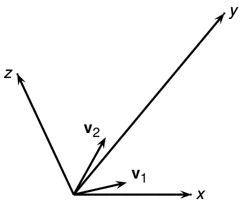

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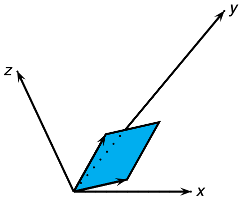

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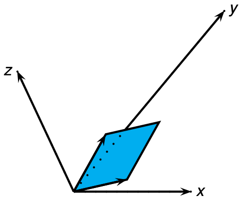

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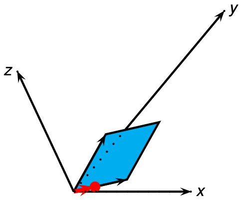

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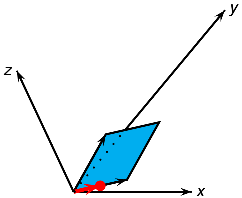

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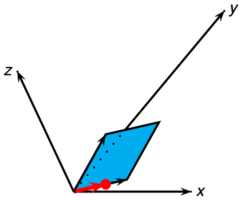

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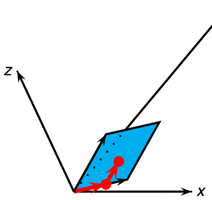

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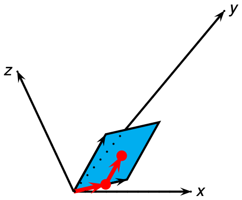

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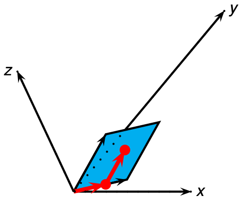

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Definition (parallelotope at $\mathbf{o}$)

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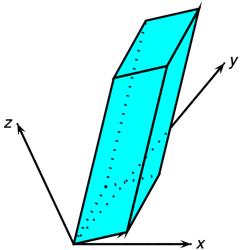

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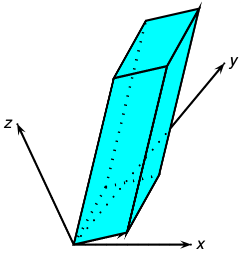

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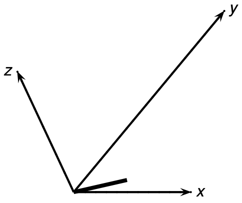

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k	n	parallelotope name
1	any	?

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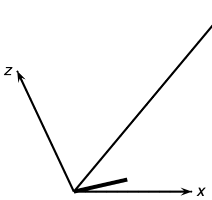

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2		

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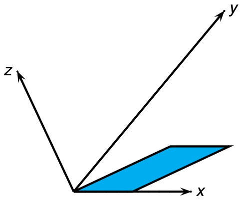

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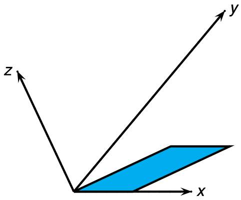

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k	n	parallelotope name
1	any	segment (in n-dim space)
2	2	parallelogram

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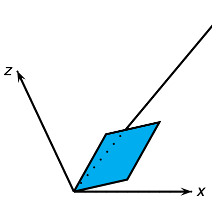

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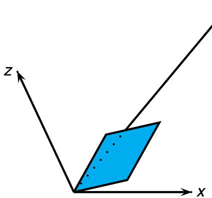

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2	2	parallelogram
2	3	parallelogram in space

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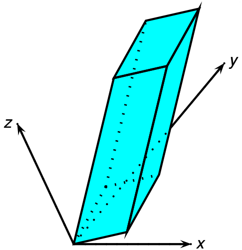

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2	2	parallelogram
2	3	parallelogram in space
3	3	?

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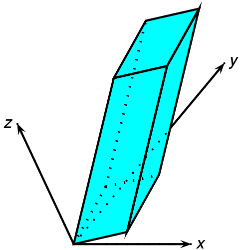

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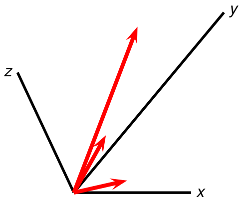
2	2	parallelogram
2	3	parallelogram in space
3	3	parallelepiped



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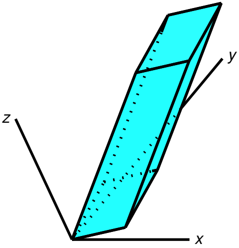

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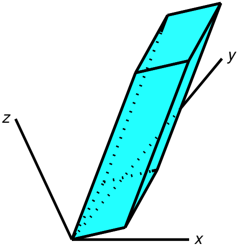

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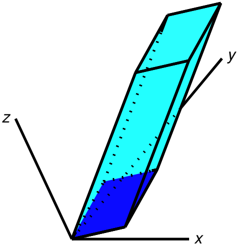

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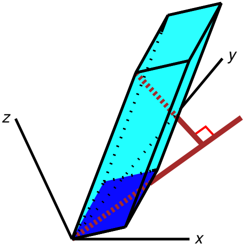

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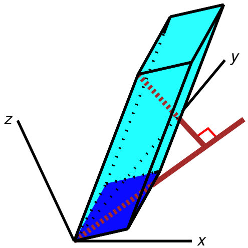

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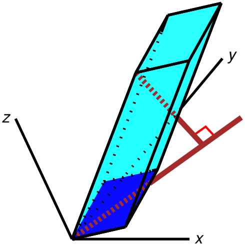
Definition (k-volume of a parallelotope)

Define $\mathrm{Vol}_1(\mathcal{R}_1) = |\mathbf{v}_1|$.



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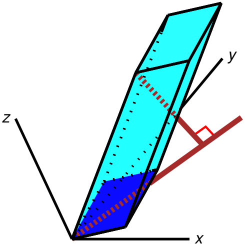

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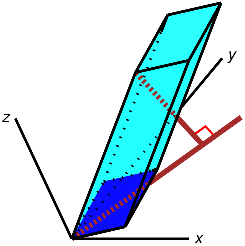

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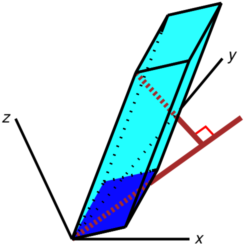

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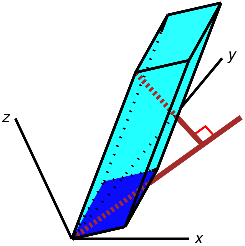

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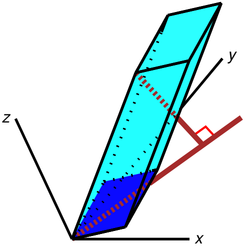

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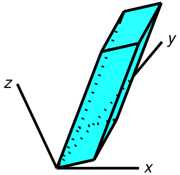
- We will see that $\mathrm{Vol}_n(\mathcal{R}_n)$ equals that integral.

Length, Surface Area, Volume as k-volumes



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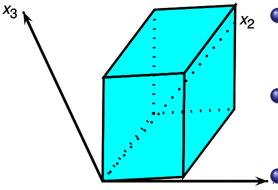
	spanned by	$\mathrm{Vol}_k(\mathcal{R}_k)$	volume name
$\mathcal{R}_1$	$\mathbf{v}_1$	$h_1 = \lvert\mathbf{v}_1\rvert$	length
$\mathcal{R}_2$	$\mathbf{v}_1, \mathbf{v}_2$	$h_1 h_2$	(surface) area
$\mathcal{R}_2$	$\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$	$h_1 h_2 h_3$	volume
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathcal{R}_k$	$\mathbf{v}_1, \dots, \mathbf{v}_k$	$h_1 \dots h_k$	k-volume

Integral and Algebraic Volume Definitions Agree



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$$\mathrm{Vol}_n(\mathcal{R}_n) = h_n \mathrm{Vol}_{n-1}(\mathcal{R}_{n-1}) = \int \cdots \int_{\mathcal{R}_n} 1 \cdot \mathrm{d}x_1 \dots \mathrm{d}x_n .$$

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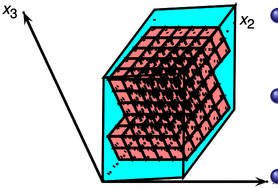

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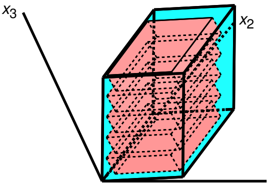

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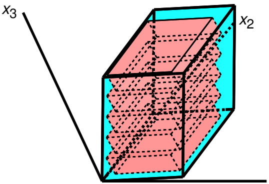

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- Right hand side: approx. vol. with boxes, sides along coord. axes.
- Left hand side: approximate volume with slabs parallel to base.
- Theorem is fully intuitive but its proof is surprisingly laborious.

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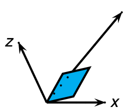

Theorem (k-volume = Gram determinant)

$$\mathrm{Vol}_k(\mathcal{R}_k) = \sqrt{\begin{vmatrix} \mathbf{v}_1 \cdot \mathbf{v}_1 & \dots & \mathbf{v}_1 \cdot \mathbf{v}_k \\ \vdots & \dots & \vdots \\ \mathbf{v}_k \cdot \mathbf{v}_1 & \dots & \mathbf{v}_k \cdot \mathbf{v}_k \end{vmatrix}}.$$

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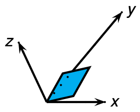

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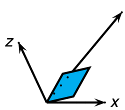

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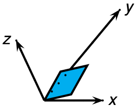

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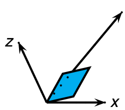

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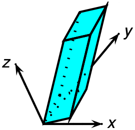

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Properties of determinants

- Multiplying a column of a matrix by a number changes multiplies the determinant by the same number. In precise notation:

Lemma

$$\begin{vmatrix} a_{11} & \dots & xa_{1k} & \dots & a_{1n} \\ a_{21} & \dots & xa_{2k} & \dots & a_{2n} \\ \vdots & & & & \\ a_{n1} & \dots & xa_{nk} & \dots & a_{nn} \end{vmatrix} = x \begin{vmatrix} a_{11} & \dots & a_{1k} & \dots & a_{1n} \\ a_{21} & \dots & a_{2k} & \dots & a_{2n} \\ \vdots & & & & \\ a_{n1} & \dots & a_{nk} & \dots & a_{nn} \end{vmatrix}$$

Example

Find the 1-dimensional volume (length) of the segment through the origin spanned by $\mathbf{v} = (1, 2, 3)$.

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$$\mathrm{Vol}_1 = \sqrt{\underbrace{\mathbf{v}\cdot\mathbf{v}}_{1\times 1 \text{ Gram determinant}}} = \sqrt{v_1^2 + v_2^2 + v_3^2}.$$

Example

Let $\mathcal{R}_2$ be the parallelogram in 2-dimensional space spanned by $\mathbf{v}_1 = (2, 3)$, $\mathbf{v}_2 = (5, 7)$. Find the area of $\mathcal{R}_2$.

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Let $\mathcal{R}_2$ be the parallelogram in 2-dimensional space spanned by $\mathbf{v}_1 = (v_{11}, v_{12})$, $\mathbf{v}_2 = (v_{21}, v_{22})$. Find the area of $\mathcal{R}_2$.

$$\begin{aligned} \mathrm{Vol}_2 &= \pm \begin{vmatrix} v_{11} & v_{21} \\ v_{12} & v_{22} \end{vmatrix} \\ \mathrm{Vol}_2 &= \sqrt{\begin{vmatrix} \mathbf{v}_1 \cdot \mathbf{v}_1 & \mathbf{v}_1 \cdot \mathbf{v}_2 \\ \mathbf{v}_2 \cdot \mathbf{v}_1 & \mathbf{v}_2 \cdot \mathbf{v}_2 \end{vmatrix}} \end{aligned}$$

Example

Find the surface area of the parallelogram spanned by $\mathbf{v}_1 = (1, 2, 3)$ and $\mathbf{v}_2 = (5, 7, 11)$.

$$\mathrm{Vol}_2 = \sqrt{\begin{vmatrix} \mathbf{v}_1 \cdot \mathbf{v}_1 & \mathbf{v}_1 \cdot \mathbf{v}_2 \\ \mathbf{v}_2 \cdot \mathbf{v}_1 & \mathbf{v}_2 \cdot \mathbf{v}_2 \end{vmatrix}} = \sqrt{\begin{vmatrix} 14 & 52 \\ 52 & 195 \end{vmatrix}} = \sqrt{26}.$$

Example

Find the surface area of the parallelogram spanned by $\mathbf{v}_1 = (v_{11}, v_{12}, v_{13})$ and $\mathbf{v}_2 = (v_{21}, v_{22}, v_{23})$.

Example

Find the volume of the parallelepiped with vertex at the origin and spanned by $\mathbf{v}_1 = (1, 2, 3)$, $\mathbf{v}_2 = (3, 5, 7)$, $\mathbf{v}_3 = (5, 7, 11)$.

$$\mathrm{Vol}_3 = \left| \det \begin{pmatrix} 1 & 2 & 3 \\ 3 & 5 & 7 \\ 5 & 7 & 11 \end{pmatrix} \right| = |-2| = 2.$$

Example

Find the volume of the parallelepiped spanned by $\mathbf{v}_1 = (v_{11}, v_{12}, v_{13})$, $\mathbf{v}_2 = (v_{21}, v_{22}, v_{23})$, $\mathbf{v}_3 = (v_{31}, v_{32}, v_{33})$.

- Recall the polar coordinate variable change

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta. \end{aligned}$$

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- The functions h, g map the two-dimensional plane with coordinates r, θ into the two-dimensional plane with coordinates x, y.
- Let $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ be an infinitely differentiable map.

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- This variable change can be thought of as two functions: $x = h(r,\theta) = r\cos\theta$ and $y = g(r,\theta) = r\sin\theta$.
- The functions h, g map the two-dimensional plane with coordinates r, θ into the two-dimensional plane with coordinates x, y.
- Let $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ be an infinitely differentiable map.
- In other words, $\mathbf{f}$ takes n scalar inputs and produces n scalar outputs.

- Recall the polar coordinate variable change

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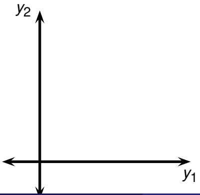

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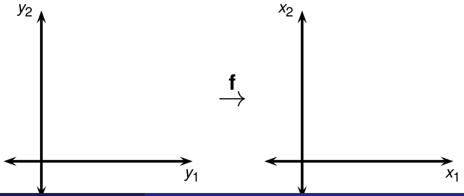

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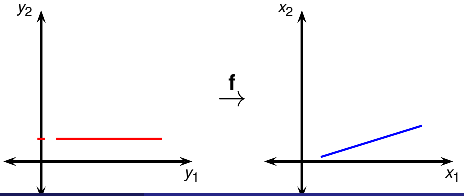

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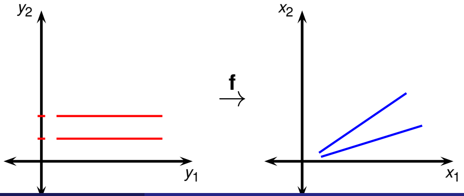

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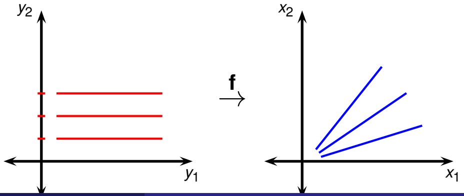

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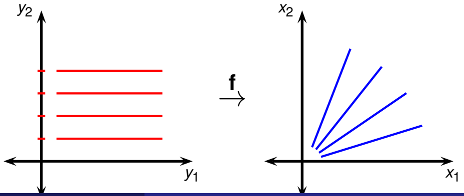

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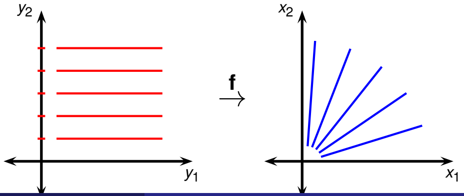

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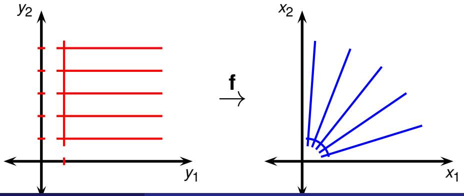

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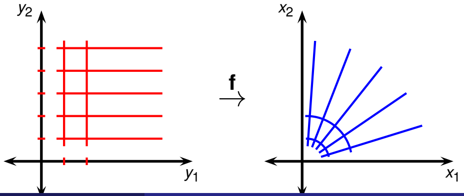

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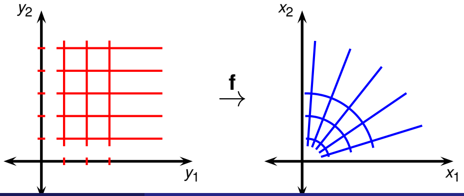

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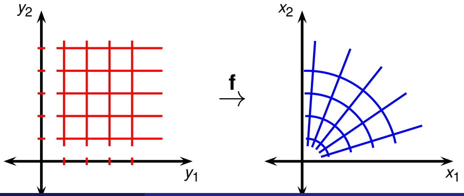

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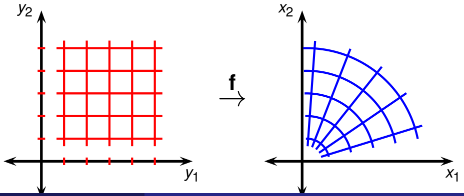

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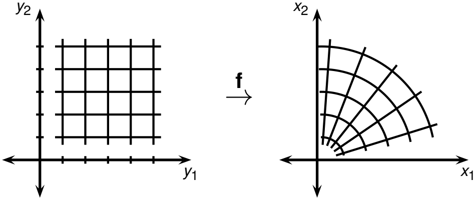

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The Jacobian matrix of a variable change **f** is defined as the matrix

$$J_{\mathbf{f}} = \begin{pmatrix} \frac{\partial f_1}{\partial y_1} & \dots & \frac{\partial f_1}{\partial y_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial y_1} & \dots & \frac{\partial f_n}{\partial y_n} \end{pmatrix}$$

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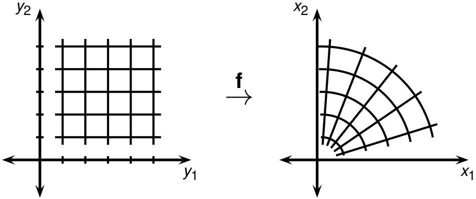

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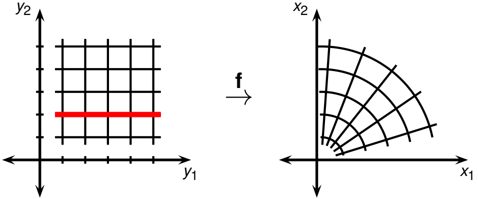

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- Consider curve given by **f** with parameter y_1 (other y_j's-fixed).

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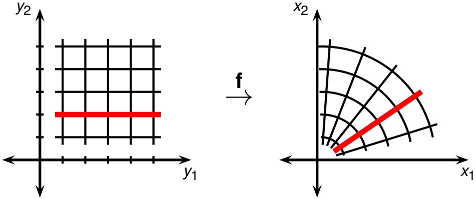

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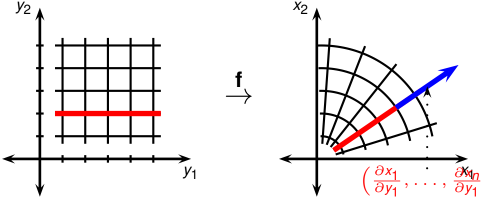

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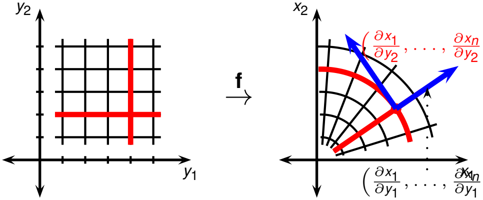

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- Similar considerations hold for y_2

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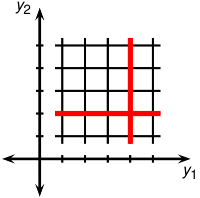
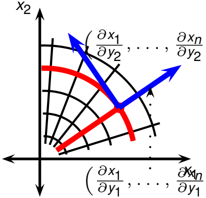


$\xrightarrow{\mathbf{f}}$



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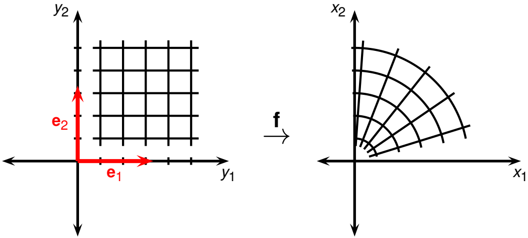
- Similar considerations hold for $y_2, \dots, y_n$.

Variable change in multivariable integrals

$$\mathbf{f}: \left| \begin{array}{rcl} x_1 & = & f_1(y_1, \dots, y_n) \\ & \vdots & \\ x_n & = & f_n(y_1, \dots, y_n) \end{array} \right.$$

$$J_{\mathbf{f}} = \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \dots & \frac{\partial x_1}{\partial y_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial y_1} & \dots & \frac{\partial x_n}{\partial y_n} \end{pmatrix}$$

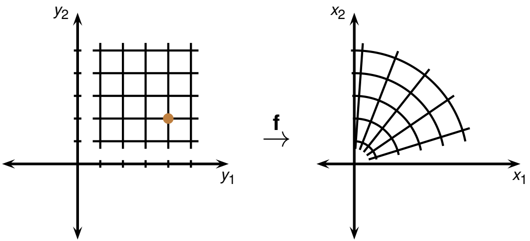


- Let $\mathbf{e}_1, \dots, \mathbf{e}_n$ be basis vectors.

Variable change in multivariable integrals

$$\mathbf{f}: \left| \begin{array}{rcl} x_1 &=& f_1(y_1, \dots, y_n) \\ &\vdots& \\ x_n &=& f_n(y_1, \dots, y_n) \end{array} \right.$$

$$J_{\mathbf{f}} = \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \dots & \frac{\partial x_1}{\partial y_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial y_1} & \dots & \frac{\partial x_n}{\partial y_n} \end{pmatrix}$$

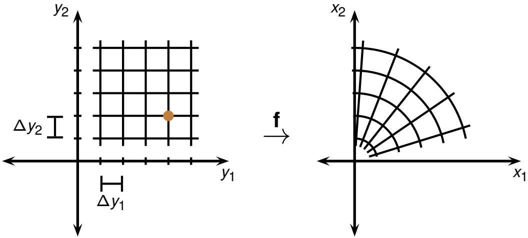


- Let $\mathbf{e}_1, \dots, \mathbf{e}_n$ be basis vectors. Fix point $\mathbf{y} = (y_1, \dots, y_n)$.

Variable change in multivariable integrals

$$\mathbf{f}: \left| \begin{array}{l} x_1 = f_1(y_1, \dots, y_n) \\ \quad\vdots \\ x_n = f_n(y_1, \dots, y_n) \end{array} \right.$$

$$J_{\mathbf{f}} = \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \dots & \frac{\partial x_1}{\partial y_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial y_1} & \dots & \frac{\partial x_n}{\partial y_n} \end{pmatrix}$$

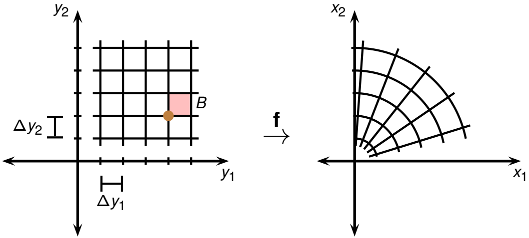


- Let $\mathbf{e}_1, \dots, \mathbf{e}_n$ be basis vectors. Fix point $\mathbf{y} = (y_1, \dots, y_n)$.
- Let $\Delta y_1, \dots, \Delta y_n$ be small numbers.

Variable change in multivariable integrals

$$\mathbf{f}: \left| \begin{array}{rcl} x_1 &=& f_1(y_1, \dots, y_n) \\ & \vdots & \\ x_n &=& f_n(y_1, \dots, y_n) \end{array} \right.$$

$$J_{\mathbf{f}} = \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \dots & \frac{\partial x_1}{\partial y_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial y_1} & \dots & \frac{\partial x_n}{\partial y_n} \end{pmatrix}$$

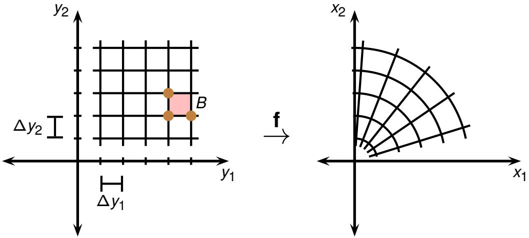


- Let $\mathbf{e}_1, \dots, \mathbf{e}_n$ be basis vectors. Fix point $\mathbf{y} = (y_1, \dots, y_n)$.
- Let $\Delta y_1, \dots, \Delta y_n$ be small numbers. Construct small box B with corner $\mathbf{y}$ spanned by the vectors $\Delta y_1 \mathbf{e}_1, \dots, \Delta y_n \mathbf{e}_n$.

Variable change in multivariable integrals

$$\mathbf{f}: \left| \begin{array}{l} x_1 = f_1(y_1, \dots, y_n) \\ \quad\vdots \\ x_n = f_n(y_1, \dots, y_n) \end{array} \right.$$

$$J_{\mathbf{f}} = \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \cdots & \frac{\partial x_1}{\partial y_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial y_1} & \cdots & \frac{\partial x_n}{\partial y_n} \end{pmatrix}$$



- Let $\mathbf{e}_1, \dots, \mathbf{e}_n$ be basis vectors. Fix point $\mathbf{y} = (y_1, \dots, y_n)$.
- Let $\Delta y_1, \dots, \Delta y_n$ be small numbers. Construct small box B with corner $\mathbf{y}$ spanned by the vectors $\Delta y_1 \mathbf{e}_1, \dots, \Delta y_n \mathbf{e}_n$.
- The point $\mathbf{y}$ and the corners $\mathbf{y} + \Delta y_1 \mathbf{e}_1, \dots, \mathbf{y} + \Delta y_n \mathbf{e}_n$ suffice to identify B.

Variable change in multivariable integrals

$$\mathbf{f}: \left| \begin{array}{rcl} x_1 &=& f_1(y_1, \dots, y_n) \\ & \vdots & \\ x_n &=& f_n(y_1, \dots, y_n) \end{array} \right.$$

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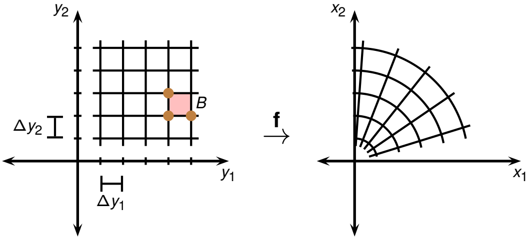
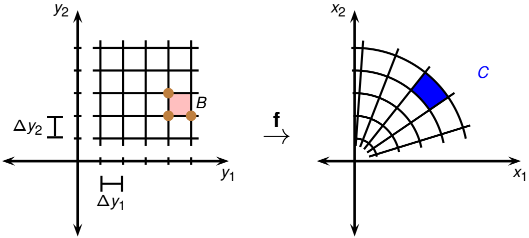


- Let $\mathbf{e}_1, \dots, \mathbf{e}_n$ be basis vectors. Fix point $\mathbf{y} = (y_1, \dots, y_n)$.
- Let $\Delta y_1, \dots, \Delta y_n$ be small numbers. Construct small box B with corner $\mathbf{y}$ spanned by the vectors $\Delta y_1 \mathbf{e}_1, \dots, \Delta y_n \mathbf{e}_n$.
- The point $\mathbf{y}$ and the corners $\mathbf{y} + \Delta y_1 \mathbf{e}_1, \dots, \mathbf{y} + \Delta y_n \mathbf{e}_n$ suffice to identify B.
- $\mathrm{Vol}(B) = \Delta y_1 \dots \Delta y_n$.

Variable change in multivariable integrals

$$\mathbf{f} : \left| \begin{array}{l} x_1 = f_1(y_1, \dots, y_n) \\ \quad \vdots \\ x_n = f_n(y_1, \dots, y_n) \end{array} \right.$$

$$J_{\mathbf{f}} = \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \dots & \frac{\partial x_1}{\partial y_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial y_1} & \dots & \frac{\partial x_n}{\partial y_n} \end{pmatrix}$$

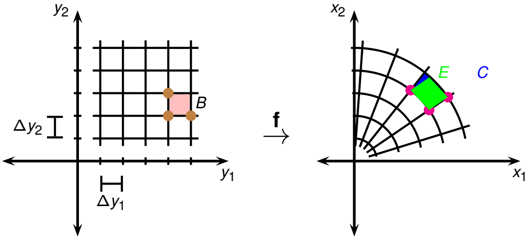


- Let $\mathbf{e}_1, \dots, \mathbf{e}_n$ be basis vectors. Fix point $\mathbf{y} = (y_1, \dots, y_n)$.
- Let $\Delta y_1, \dots, \Delta y_n$ be small numbers. Construct small box B with corner $\mathbf{y}$ spanned by the vectors $\Delta y_1 \mathbf{e}_1, \dots, \Delta y_n \mathbf{e}_n$.
- The point $\mathbf{y}$ and the corners $\mathbf{y} + \Delta y_1 \mathbf{e}_1, \dots, \mathbf{y} + \Delta y_n \mathbf{e}_n$ suffice to identify B.
- $\mathrm{Vol}(B) = \Delta y_1 \dots \Delta y_n$.
- Let the image of B be $\mathbf{f}(B) = C$. C is a "curvilinear box".

Variable change in multivariable integrals

$$\mathbf{f}: \left| \begin{array}{l} x_1 = f_1(y_1, \dots, y_n) \\ \quad\vdots \\ x_n = f_n(y_1, \dots, y_n) \end{array} \right.$$

$$J_{\mathbf{f}} = \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \dots & \frac{\partial x_1}{\partial y_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial y_1} & \dots & \frac{\partial x_n}{\partial y_n} \end{pmatrix}$$

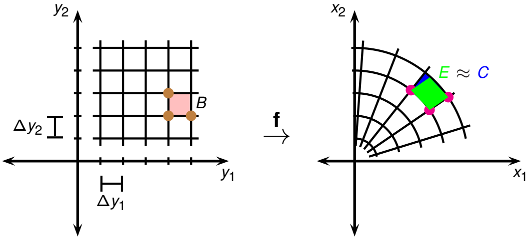


- Let $\mathbf{e}_1, \dots, \mathbf{e}_n$ be basis vectors. Fix point $\mathbf{y} = (y_1, \dots, y_n)$.
- Let $\Delta y_1, \dots, \Delta y_n$ be small numbers. Construct small box B with corner $\mathbf{y}$ spanned by the vectors $\Delta y_1 \mathbf{e}_1, \dots, \Delta y_n \mathbf{e}_n$.
- The point $\mathbf{y}$ and the corners $\mathbf{y} + \Delta y_1 \mathbf{e}_1, \dots, \mathbf{y} + \Delta y_n \mathbf{e}_n$ suffice to identify B.
- $\mathrm{Vol}(B) = \Delta y_1 \dots \Delta y_n$.
- Let the image of B be $\mathbf{f}(B) = C$. C is a “curvilinear box”.
- Let E be the parrallelotope at $\mathbf{f}(\mathbf{y})$ spanned by images of the corners of B.

Variable change in multivariable integrals

$$\mathbf{f} : \left| \begin{array}{l} x_1 = f_1(y_1, \dots, y_n) \\ \quad\vdots \\ x_n = f_n(y_1, \dots, y_n) \end{array} \right.$$

$$J_{\mathbf{f}} = \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \dots & \frac{\partial x_1}{\partial y_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial y_1} & \dots & \frac{\partial x_n}{\partial y_n} \end{pmatrix}$$

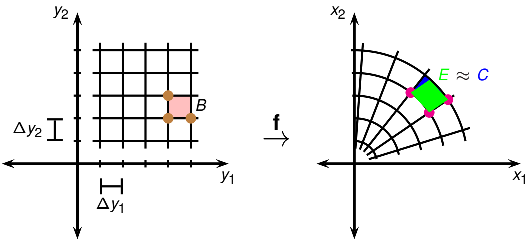


- Let $\mathbf{e}_1, \dots, \mathbf{e}_n$ be basis vectors. Fix point $\mathbf{y} = (y_1, \dots, y_n)$.
- Let $\Delta y_1, \dots, \Delta y_n$ be small numbers. Construct small box B with corner $\mathbf{y}$ spanned by the vectors $\Delta y_1 \mathbf{e}_1, \dots, \Delta y_n \mathbf{e}_n$.
- The point $\mathbf{y}$ and the corners $\mathbf{y} + \Delta y_1 \mathbf{e}_1, \dots, \mathbf{y} + \Delta y_n \mathbf{e}_n$ suffice to identify B.
- $\mathrm{Vol}(B) = \Delta y_1 \dots \Delta y_n$.
- Let the image of B be $\mathbf{f}(B) = C$. C is a "curvilinear box".
- Let E be the parrallelotope at $\mathbf{f}(\mathbf{y})$ spanned by images of the corners of B. Then $\mathrm{Vol}(C) \approx \mathrm{Vol}_n(E)$.

Variable change in multivariable integrals

$$\mathbf{f}: \left| \begin{array}{rcl} x_1 &=& f_1(y_1, \dots, y_n) \\ & \vdots & \\ x_n &=& f_n(y_1, \dots, y_n) \end{array} \right.$$

$$J_{\mathbf{f}} = \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \cdots & \frac{\partial x_1}{\partial y_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial y_1} & \cdots & \frac{\partial x_n}{\partial y_n} \end{pmatrix}$$



- Let $\mathbf{e}_1, \dots, \mathbf{e}_n$ be basis vectors. Fix point $\mathbf{y} = (y_1, \dots, y_n)$.
- Let $\Delta y_1, \dots, \Delta y_n$ be small numbers. Construct small box B with corner $\mathbf{y}$ spanned by the vectors $\Delta y_1 \mathbf{e}_1, \dots, \Delta y_n \mathbf{e}_n$.
- The point $\mathbf{y}$ and the corners $\mathbf{y} + \Delta y_1 \mathbf{e}_1, \dots, \mathbf{y} + \Delta y_n \mathbf{e}_n$ suffice to identify B.
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- Let the image of B be $\mathbf{f}(B) = C$. C is a "curvilinear box".
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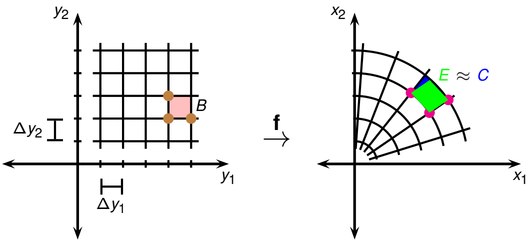
Variable change in multivariable integrals

$$\mathbf{f}: \left| \begin{array}{l} x_1 = f_1(y_1, \dots, y_n) \\ \vdots \\ x_n = f_n(y_1, \dots, y_n) \end{array} \right. .$$

$$J_{\mathbf{f}} = \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \cdots & \frac{\partial x_1}{\partial y_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial y_1} & \cdots & \frac{\partial x_n}{\partial y_n} \end{pmatrix}$$

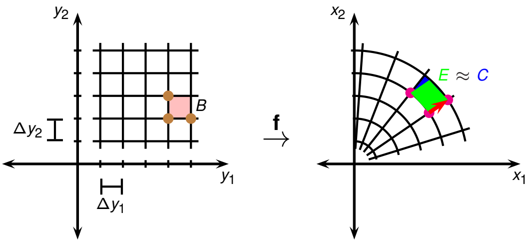


- $\mathrm{Vol}(C) \approx \mathrm{Vol}_n(E)$.

Variable change in multivariable integrals

$$\mathbf{f}: \left| \begin{array}{rcl} x_1 &=& f_1(y_1, \dots, y_n) \\ & \vdots & \\ x_n &=& f_n(y_1, \dots, y_n) \end{array} \right. .$$

$$J_{\mathbf{f}} = \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \dots & \frac{\partial x_1}{\partial y_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial y_1} & \dots & \frac{\partial x_n}{\partial y_n} \end{pmatrix}$$

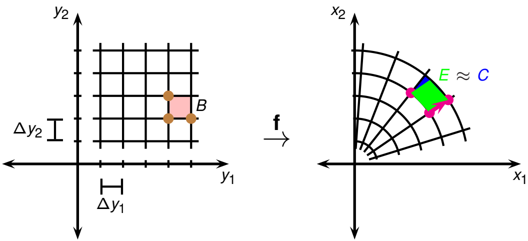


- $\mathrm{Vol}(C) \approx \mathrm{Vol}_n(E)$.
- The first edge of E

Variable change in multivariable integrals

$$\mathbf{f}: \left| \begin{array}{rcl} x_1 &=& f_1(y_1, \dots, y_n) \\ & \vdots & \\ x_n &=& f_n(y_1, \dots, y_n) \end{array} \right. .$$

$$J_{\mathbf{f}} = \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \dots & \frac{\partial x_1}{\partial y_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial y_1} & \dots & \frac{\partial x_n}{\partial y_n} \end{pmatrix}$$

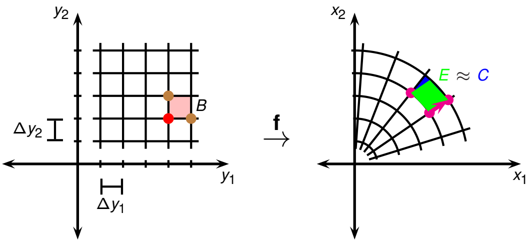


- $\mathrm{Vol}(C) \approx \mathrm{Vol}_n(E)$.
- The first edge of E corresponds to the vector $\mathbf{f}(\mathbf{y} + \Delta y_1 \mathbf{e}_1) - \mathbf{f}(\mathbf{y})$

Variable change in multivariable integrals

$$\mathbf{f}: \left| \begin{array}{rcl} x_1 &=& f_1(y_1,\dots,y_n) \\ &\vdots& \\ x_n &=& f_n(y_1,\dots,y_n) \end{array} \right. .$$

$$J_{\mathbf{f}} = \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \dots & \frac{\partial x_1}{\partial y_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial y_1} & \dots & \frac{\partial x_n}{\partial y_n} \end{pmatrix}$$



- $\mathrm{Vol}(C) \approx \mathrm{Vol}_n(E)$.
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Variable change in multivariable integrals

$$\mathbf{f}: \left| \begin{array}{rcl} x_1 &=& f_1(y_1, \dots, y_n) \\ & \vdots & \\ x_n &=& f_n(y_1, \dots, y_n) \end{array} \right. .$$

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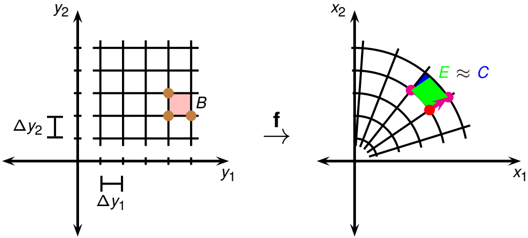

- $\mathrm{Vol}(C) \approx \mathrm{Vol}_n(E)$.
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Variable change in multivariable integrals

$$\mathbf{f}: \left| \begin{array}{lcl} x_1 &=& f_1(y_1, \dots, y_n) \\ & \vdots & \\ x_n &=& f_n(y_1, \dots, y_n) \end{array} \right. .$$

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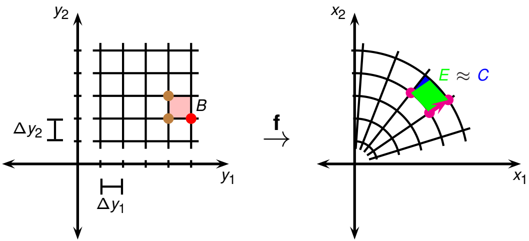

- $\mathrm{Vol}(C) \approx \mathrm{Vol}_n(E)$.
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Variable change in multivariable integrals

$$\mathbf{f}: \left| \begin{array}{rcl} x_1 &=& f_1(y_1,\dots,y_n) \\ &\vdots& \\ x_n &=& f_n(y_1,\dots,y_n) \end{array} \right. .$$

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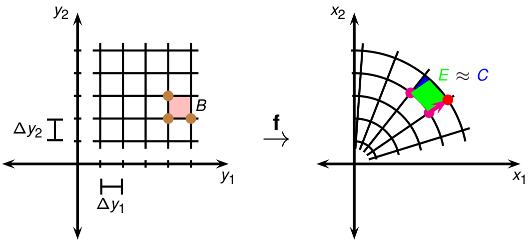

- $\mathrm{Vol}(C) \approx \mathrm{Vol}_n(E)$.
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Variable change in multivariable integrals

$$\mathbf{f}: \left| \begin{array}{rcl} x_1 &=& f_1(y_1, \dots, y_n) \\ & \vdots & \\ x_n &=& f_n(y_1, \dots, y_n) \end{array} \right. .$$

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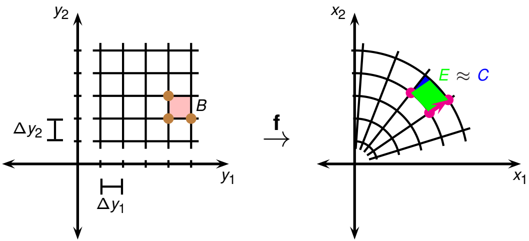
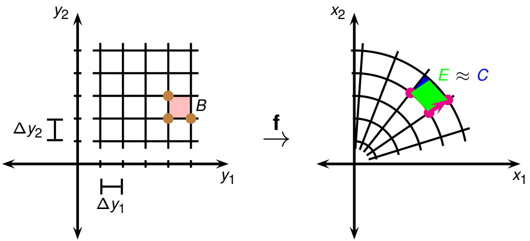


- $\mathrm{Vol}(C) \approx \mathrm{Vol}_n(E)$.
- The first edge of E corresponds to the vector $\mathbf{f}(\mathbf{y} + \Delta y_1 \mathbf{e}_1) - \mathbf{f}(\mathbf{y}) \approx \Delta y_1 \left(D_{\mathbf{e}_1}\left(\mathbf{f}(\mathbf{y})\right)\right)$

Variable change in multivariable integrals

$$\mathbf{f}: \left| \begin{array}{l} x_1 = f_1(y_1, \dots, y_n) \\ \quad \vdots \\ x_n = f_n(y_1, \dots, y_n) \end{array} \right. .$$

$$J_{\mathbf{f}} = \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \dots & \frac{\partial x_1}{\partial y_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial y_1} & \dots & \frac{\partial x_n}{\partial y_n} \end{pmatrix}$$



- $\mathrm{Vol}(C) \approx \mathrm{Vol}_n(E)$.
- The first edge of E corresponds to the vector
  $\mathbf{f}(\mathbf{y} + \Delta y_1 \mathbf{e}_1) - \mathbf{f}(\mathbf{y}) \approx \Delta y_1 \left(D_{\mathbf{e}_1}(\mathbf{f}(\mathbf{y}))\right) = \Delta y_1 \frac{\partial \mathbf{f}}{\partial y_1}$

Variable change in multivariable integrals

$$\mathbf{f}: \left| \begin{array}{rcl} x_1 &=& f_1(y_1,\dots,y_n) \\ &\vdots& \\ x_n &=& f_n(y_1,\dots,y_n) \end{array} \right. .$$

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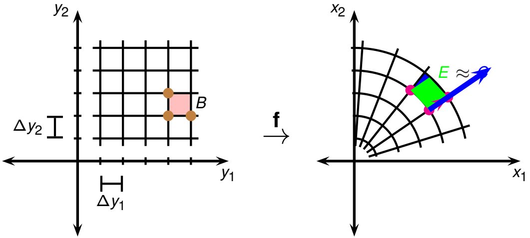

- $\mathrm{Vol}(C) \approx \mathrm{Vol}_n(E)$.
- The first edge of E corresponds to the vector

$$\mathbf{f}(\mathbf{y} + \Delta y_1 \mathbf{e}_1) - \mathbf{f}(\mathbf{y}) \approx \Delta y_1 \left(D_{\mathbf{e}_1} \left(\mathbf{f}(\mathbf{y}) \right) \right) = \Delta y_1 \frac{\partial \mathbf{f}}{\partial y_1}$$

$$= \Delta y_1 \left(\frac{\partial x_1}{\partial y_1}, \dots, \frac{\partial x_n}{\partial y_1} \right)$$

Variable change in multivariable integrals

$$\mathbf{f}: \left| \begin{array}{rcl} x_1 &=& f_1(y_1, \dots, y_n) \\ &\vdots& \\ x_n &=& f_n(y_1, \dots, y_n) \end{array} \right. .$$

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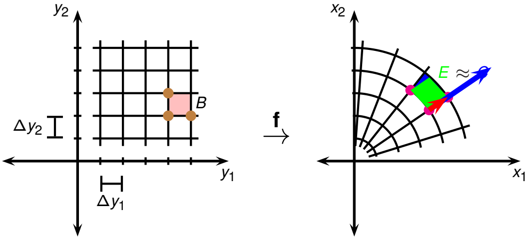


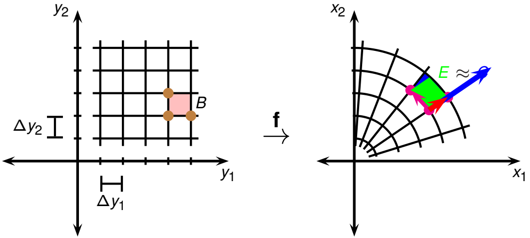
- $\mathrm{Vol}(C) \approx \mathrm{Vol}_n(E)$.
- The first edge of E corresponds to the vector

$$\mathbf{f}(\mathbf{y} + \Delta y_1 \mathbf{e}_1) - \mathbf{f}(\mathbf{y}) \approx \Delta y_1 \left(D_{\mathbf{e}_1} \left(\mathbf{f}(\mathbf{y}) \right) \right) = \Delta y_1 \frac{\partial \mathbf{f}}{\partial y_1}$$
$$= \Delta y_1 \left(\frac{\partial x_1}{\partial y_1}, \dots, \frac{\partial x_n}{\partial y_1} \right) = \left(\Delta y_1 \frac{\partial x_1}{\partial y_1}, \dots, \Delta y_1 \frac{\partial x_n}{\partial y_1} \right) .$$

Variable change in multivariable integrals

$$\mathbf{f}: \left| \begin{array}{rcl} x_1 &=& f_1(y_1, \dots, y_n) \\ &\vdots& \\ x_n &=& f_n(y_1, \dots, y_n) \end{array} \right. .$$

$$J_{\mathbf{f}} = \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \dots & \frac{\partial x_1}{\partial y_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial y_1} & \dots & \frac{\partial x_n}{\partial y_n} \end{pmatrix}$$



- $\mathrm{Vol}(C) \approx \mathrm{Vol}_n(E)$.
- The first edge of E corresponds to the vector
$$\begin{aligned} \mathbf{f}(\mathbf{y} + \Delta y_1 \mathbf{e}_1) - \mathbf{f}(\mathbf{y}) &\approx \Delta y_1 \left(D_{\mathbf{e}_1} (\mathbf{f}(\mathbf{y})) \right) = \Delta y_1 \frac{\partial \mathbf{f}}{\partial y_1} \\ &= \Delta y_1 \left(\frac{\partial x_1}{\partial y_1}, \dots, \frac{\partial x_n}{\partial y_1} \right) = \left(\Delta y_1 \frac{\partial x_1}{\partial y_1}, \dots, \Delta y_1 \frac{\partial x_n}{\partial y_1} \right) . \end{aligned}$$
- Similar considerations holds for the other edges of E.

Variable change in multivariable integrals

$$\mathbf{f}: \left| \begin{array}{l} x_1 = f_1(y_1, \dots, y_n) \\ \quad\vdots \\ x_n = f_n(y_1, \dots, y_n) \end{array} \right. .$$

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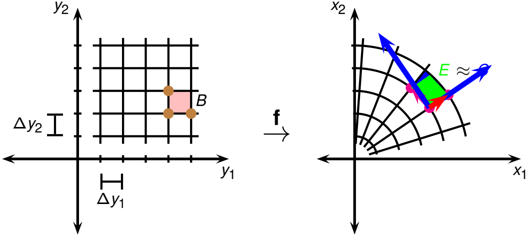

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$$= \Delta y_1 \left(\frac{\partial x_1}{\partial y_1}, \dots, \frac{\partial x_n}{\partial y_1} \right) = \left(\Delta y_1 \frac{\partial x_1}{\partial y_1}, \dots, \Delta y_1 \frac{\partial x_n}{\partial y_1} \right) .$$
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Variable change in multivariable integrals

$$\mathbf{f}:\left|\begin{array}{l} x_1 = f_1(y_1,\dots,y_n) \\ \quad\vdots \\ x_n = f_n(y_1,\dots,y_n) \end{array}\right. .$$

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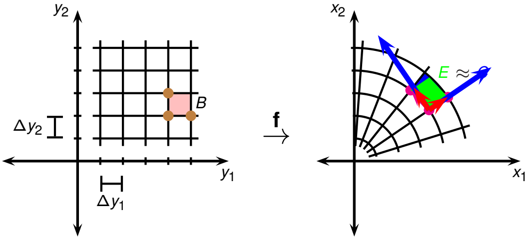

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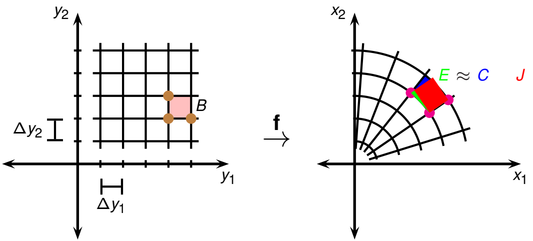
Variable change in multivariable integrals

$$\mathbf{f}: \left| \begin{array}{rcl} x_1 &=& f_1(y_1, \dots, y_n) \\ &\vdots& \\ x_n &=& f_n(y_1,\dots,y_n) \end{array} \right. .$$

$$J_{\mathbf{f}} = \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \dots & \frac{\partial x_1}{\partial y_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial y_1} & \dots & \frac{\partial x_n}{\partial y_n} \end{pmatrix}$$



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$$\begin{aligned}\mathbf{f}(\mathbf{y} + \Delta y_1 \mathbf{e}_1) - \mathbf{f}(\mathbf{y}) \approx \Delta y_1 \left(D_{\mathbf{e}_1}(\mathbf{f}(\mathbf{y}))\right) &= \Delta y_1 \frac{\partial \mathbf{f}}{\partial y_1} \\ = \Delta y_1 \left(\frac{\partial x_1}{\partial y_1}, \dots, \frac{\partial x_n}{\partial y_1}\right) &= \left(\Delta y_1 \frac{\partial x_1}{\partial y_1}, \dots, \Delta y_1 \frac{\partial x_n}{\partial y_1}\right).\end{aligned}$$
- Similar considerations holds for the other edges of E.
- Let J be the parallelotope at $\mathbf{f}(\mathbf{y})$ spanned by the vectors
$$\Delta y_1 \left(\frac{\partial x_1}{\partial y_1}, \dots, \frac{\partial x_n}{\partial y_1}\right), \dots \Delta y_n \left(\frac{\partial x_1}{\partial y_n}, \dots, \frac{\partial x_n}{\partial y_n}\right).$$

Variable change in multivariable integrals

$$\mathbf{f}: \left| \begin{array}{rcl} x_1 &=& f_1(y_1,\dots,y_n) \\ & \vdots & \\ x_n &=& f_n(y_1,\dots,y_n) \end{array} \right. .$$

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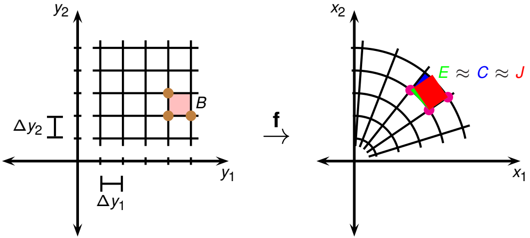

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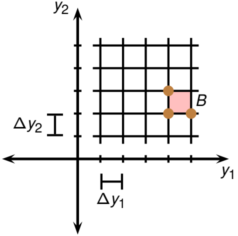
- Then $\mathrm{Vol}(C) \approx \mathrm{Vol}_n(E) \approx \mathrm{Vol}_n(J)$.

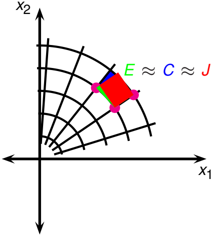
Variable change in multivariable integrals

$$\mathbf{f}:\left|\begin{array}{rcl} x_1 &=& f_1(y_1,\dots,y_n)\\ &\vdots& \\ x_n &=& f_n(y_1,\dots,y_n)\end{array}\right. .$$

$$J_{\mathbf{f}} = \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \dots & \frac{\partial x_1}{\partial y_n}\\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial y_1} & \dots & \frac{\partial x_n}{\partial y_n}\end{pmatrix}$$



$\xrightarrow{\mathbf{f}}$



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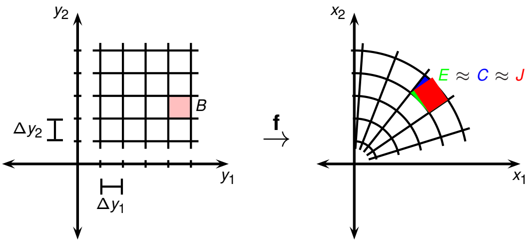
$$\begin{aligned}\mathbf{f}(\mathbf{y}+\Delta y_1\mathbf{e}_1)-\mathbf{f}(\mathbf{y}) \approx \Delta y_1\left(D_{\mathbf{e}_1}\left(\mathbf{f}(\mathbf{y})\right)\right) &= \Delta y_1 \frac{\partial \mathbf{f}}{\partial y_1}\\ = \Delta y_1\left(\frac{\partial x_1}{\partial y_1},\dots,\frac{\partial x_n}{\partial y_1}\right) &= \left(\Delta y_1 \frac{\partial x_1}{\partial y_1},\dots,\Delta y_1\frac{\partial x_n}{\partial y_1}\right).\end{aligned}$$
- Similar considerations holds for the other edges of E.
- Let J be the parallelotope at $\mathbf{f}(\mathbf{y})$ spanned by the vectors $\Delta y_1\left(\frac{\partial x_1}{\partial y_1},\dots,\frac{\partial x_n}{\partial y_1}\right), \dots \Delta y_n\left(\frac{\partial x_1}{\partial y_n},\dots,\frac{\partial x_n}{\partial y_n}\right)$.
- Then $\mathrm{Vol}(C) \approx \mathrm{Vol}_n(E) \approx \mathrm{Vol}_n(J)$.

Variable change in multivariable integrals

$$\mathbf{f}: \left| \begin{array}{rcl} x_1 &=& f_1(y_1,\dots,y_n) \\ &\vdots& \\ x_n &=& f_n(y_1,\dots,y_n) \end{array} \right. .$$

$$J_{\mathbf{f}} = \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \cdots & \frac{\partial x_1}{\partial y_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial y_1} & \cdots & \frac{\partial x_n}{\partial y_n} \end{pmatrix}$$



- Let J be the parallelotope at $\mathbf{f}(\mathbf{y})$ spanned by the vectors $\Delta y_1 \left(\frac{\partial x_1}{\partial y_1},\dots,\frac{\partial x_n}{\partial y_1}\right),\dots,\Delta y_n\left(\frac{\partial x_1}{\partial y_n},\dots,\frac{\partial x_n}{\partial y_n}\right)$.
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Variable change in multivariable integrals

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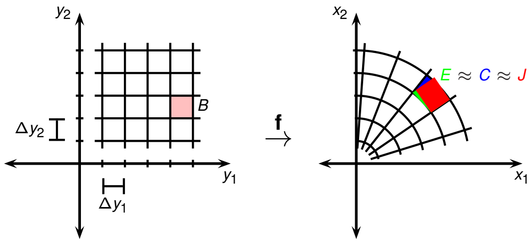

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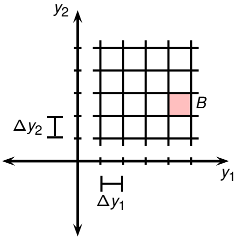
$$\mathrm{Vol}_n(J) = \pm \begin{vmatrix} \Delta y_1 \frac{\partial x_1}{\partial y_1} & \cdots & \Delta y_n \frac{\partial x_1}{\partial y_n} \\ \vdots & \ddots & \vdots \\ \Delta y_1 \frac{\partial x_n}{\partial y_1} & \cdots & \Delta y_n \frac{\partial x_n}{\partial y_n} \end{vmatrix}$$

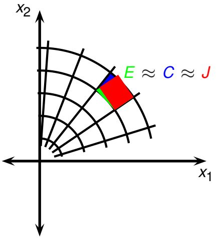
Variable change in multivariable integrals

$$\mathbf{f}: \left| \begin{array}{l} x_1 = f_1(y_1,\dots,y_n) \\ \quad\vdots \\ x_n = f_n(y_1,\dots,y_n) \end{array} \right. .$$

$$J_{\mathbf{f}} = \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \cdots & \frac{\partial x_1}{\partial y_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial y_1} & \cdots & \frac{\partial x_n}{\partial y_n} \end{pmatrix}$$



$\xrightarrow{\mathbf{f}}$



- Let J be the parallelotope at $\mathbf{f}(\mathbf{y})$ spanned by the vectors $\Delta y_1 \left(\frac{\partial x_1}{\partial y_1},\dots,\frac{\partial x_n}{\partial y_1}\right),\dots,\Delta y_n\left(\frac{\partial x_1}{\partial y_n},\dots,\frac{\partial x_n}{\partial y_n}\right)$.
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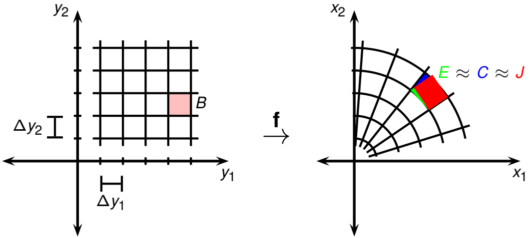
$$\mathrm{Vol}_n(J) = \pm \begin{vmatrix} \Delta y_1 \frac{\partial x_1}{\partial y_1} & \cdots & \Delta y_n \frac{\partial x_1}{\partial y_n} \\ \vdots & \ddots & \vdots \\ \Delta y_1 \frac{\partial x_n}{\partial y_1} & \cdots & \Delta y_n \frac{\partial x_n}{\partial y_n} \end{vmatrix}$$

Variable change in multivariable integrals

$$\mathbf{f}: \left| \begin{array}{l} x_1 = f_1(y_1, \dots, y_n) \\ \quad\vdots \\ x_n = f_n(y_1, \dots, y_n) \end{array} \right. .$$

$$J_{\mathbf{f}} = \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \cdots & \frac{\partial x_1}{\partial y_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial y_1} & \cdots & \frac{\partial x_n}{\partial y_n} \end{pmatrix}$$



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Variable change in multivariable integrals

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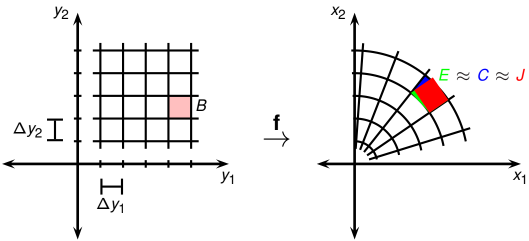

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Variable change in multivariable integrals

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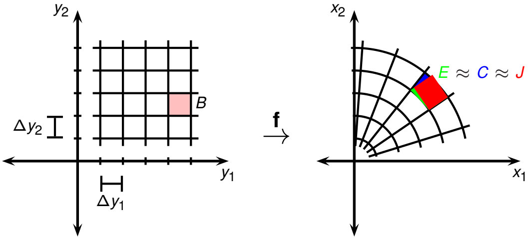

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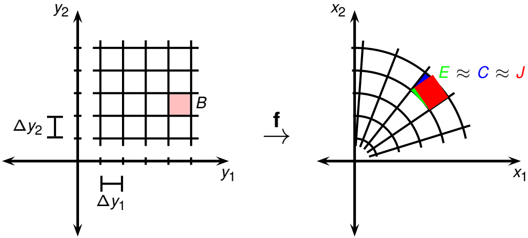
$$= \pm \det(J_{\mathbf{f}})\, \Delta y_1 \dots \Delta y_n \quad .$$

Variable change in multivariable integrals

$$\mathbf{f}: \left| \begin{array}{rcl} x_1 &=& f_1(y_1,\dots,y_n) \\ &\vdots& \\ x_n &=& f_n(y_1,\dots,y_n) \end{array} \right. .$$

$$J_{\mathbf{f}} = \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \dots & \frac{\partial x_1}{\partial y_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial y_1} & \dots & \frac{\partial x_n}{\partial y_n} \end{pmatrix}$$



- Let J be the parallelotope at $\mathbf{f}(\mathbf{y})$ spanned by the vectors $\Delta y_1 \left(\frac{\partial x_1}{\partial y_1},\dots,\frac{\partial x_n}{\partial y_1}\right),\dots,\Delta y_n \left(\frac{\partial x_1}{\partial y_n},\dots,\frac{\partial x_n}{\partial y_n}\right)$. Suppose $\det J_{\mathbf{f}} \geq 0$.
- $\mathrm{Vol}(C) \approx \mathrm{Vol}_n(E) \approx \mathrm{Vol}_n(J)$

$$\mathrm{Vol}_n(J) = \pm \begin{vmatrix} \Delta y_1 \frac{\partial x_1}{\partial y_1} & \dots & \Delta y_n \frac{\partial x_1}{\partial y_n} \\ \vdots & \ddots & \vdots \\ \Delta y_1 \frac{\partial x_n}{\partial y_1} & \dots & \Delta y_n \frac{\partial x_n}{\partial y_n} \end{vmatrix} = \pm \Delta y_1 \dots \Delta y_n \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \dots & \frac{\partial x_1}{\partial y_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial y_1} & \dots & \frac{\partial x_n}{\partial y_n} \end{vmatrix}$$
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Variable change in multivariable integrals

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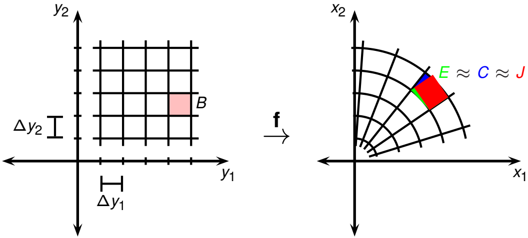


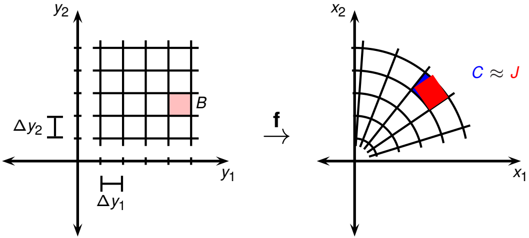
- Let J be the parallelotope at $\mathbf{f}(\mathbf{y})$ spanned by the vectors $\Delta y_1 \left(\frac{\partial x_1}{\partial y_1}, \dots, \frac{\partial x_n}{\partial y_1} \right), \dots, \Delta y_n \left(\frac{\partial x_1}{\partial y_n}, \dots, \frac{\partial x_n}{\partial y_n} \right)$. Suppose $\det J_{\mathbf{f}} \geq 0$.
- $\mathrm{Vol}(C) \approx \mathrm{Vol}_n(E) \approx \mathrm{Vol}_n(J) = \det J_{\mathbf{f}} \Delta y_1 \dots \Delta y_n$

$$\mathrm{Vol}_n(J) = \pm \begin{vmatrix} \Delta y_1 \frac{\partial x_1}{\partial y_1} & \cdots & \Delta y_n \frac{\partial x_1}{\partial y_n} \\ \vdots & \ddots & \vdots \\ \Delta y_1 \frac{\partial x_n}{\partial y_1} & \cdots & \Delta y_n \frac{\partial x_n}{\partial y_n} \end{vmatrix} = \pm \Delta y_1 \dots \Delta y_n \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \cdots & \frac{\partial x_1}{\partial y_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial y_1} & \cdots & \frac{\partial x_n}{\partial y_n} \end{vmatrix}$$
$$= \pm \det(J_{\mathbf{f}}) \Delta y_1 \dots \Delta y_n \quad .$$

Variable change in multivariable integrals

$$\mathbf{f}: \left| \begin{array}{rcl} x_1 & = & f_1(y_1, \dots, y_n) \\ & \vdots & \\ x_n & = & f_n(y_1, \dots, y_n) \end{array} \right. .$$

$$J_{\mathbf{f}} = \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \dots & \frac{\partial x_1}{\partial y_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial y_1} & \dots & \frac{\partial x_n}{\partial y_n} \end{pmatrix}$$



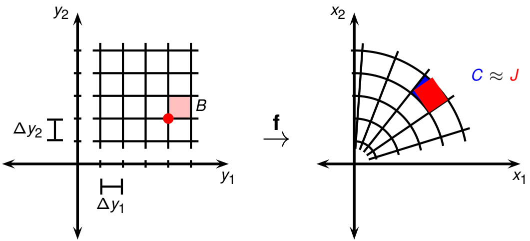
$$\mathrm{Vol}(C\quad) \approx \det(J_{\mathbf{f}}\quad)\Delta y_1 \dots \Delta y_n$$

-

Variable change in multivariable integrals

$$\mathbf{f}: \left| \begin{array}{l} x_1 = f_1(y_1, \dots, y_n) \\ \quad\vdots \\ x_n = f_n(y_1, \dots, y_n) \end{array} \right. .$$

$$J_{\mathbf{f}} = \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \dots & \frac{\partial x_1}{\partial y_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial y_1} & \dots & \frac{\partial x_n}{\partial y_n} \end{pmatrix}$$



$$\mathrm{Vol}(C(\mathbf{y})) \approx \det(J_{\mathbf{f}}(\mathbf{y}))\,\Delta y_1 \dots \Delta y_n$$

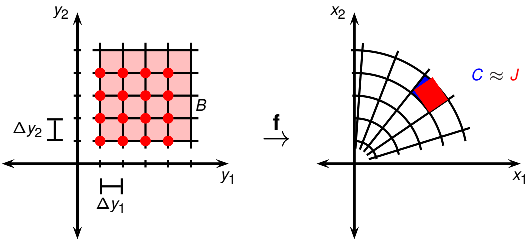
-

- Regard **y** as a variable

Variable change in multivariable integrals

$$\mathbf{f}:\left|\begin{array}{rcl} x_1 &=& f_1(y_1,\dots,y_n)\\ &\vdots& \\ x_n &=& f_n(y_1,\dots,y_n)\end{array}\right. .$$

$$J_{\mathbf{f}} = \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \dots & \frac{\partial x_1}{\partial y_n}\\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial y_1} & \dots & \frac{\partial x_n}{\partial y_n}\end{pmatrix}$$



$$\mathrm{Vol}(C(\mathbf{y})) \approx \det\left(J_{\mathbf{f}}(\mathbf{y})\right)\Delta y_1\dots\Delta y_n$$

-

- Regard **y** as a variable and let it traverse a rectangular mesh.

Variable change in multivariable integrals

$$\mathbf{f}:\left|\begin{array}{rcl} x_1 &=& f_1(y_1,\dots,y_n)\\ &\vdots& \\ x_n &=& f_n(y_1,\dots,y_n)\end{array}\right. .$$

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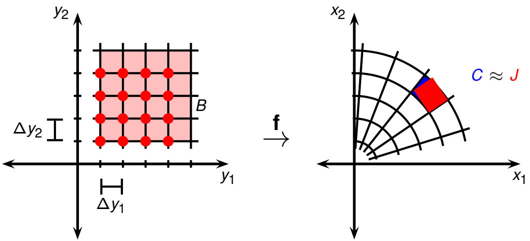

- $$\sum_{\mathbf{y}} \mathrm{Vol}(C(\mathbf{y})) \quad \approx \quad \sum_{\mathbf{y}} \det\left(J_{\mathbf{f}}(\mathbf{y})\right) \Delta y_1 \dots \Delta y_n$$
- Regard **y** as a variable and let it traverse a rectangular mesh.
- Sum over the rectangular mesh.

Variable change in multivariable integrals

$$\mathbf{f}:\left|\begin{array}{l} x_1 = f_1(y_1,\dots,y_n)\\ \quad\vdots \\ x_n = f_n(y_1,\dots,y_n)\end{array}\right. .$$

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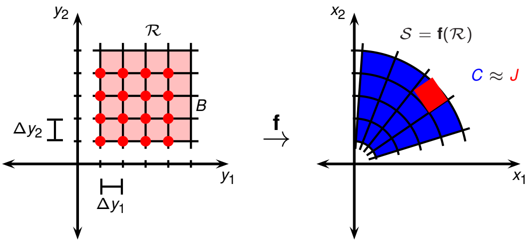

$$\mathrm{Vol}(\mathcal{S}) = \sum_{\mathbf{y}} \mathrm{Vol}(C(\mathbf{y})) \quad \approx \quad \sum_{\mathbf{y}} \det\left(J_{\mathbf{f}}(\mathbf{y})\right)\Delta y_1\dots\Delta y_n$$

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Variable change in multivariable integrals

$$\mathbf{f}: \left| \begin{array}{l} x_1 = f_1(y_1, \dots, y_n) \\ \quad \vdots \\ x_n = f_n(y_1, \dots, y_n) \end{array} \right. .$$

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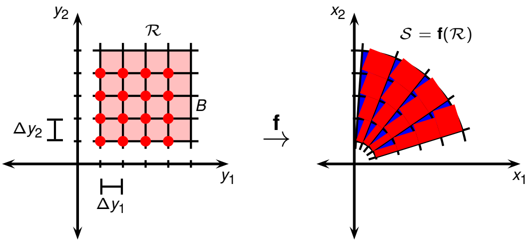

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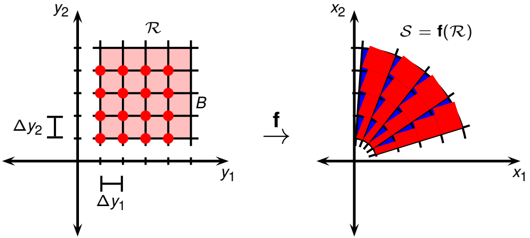

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- $$\int_{\mathcal{S}} 1 \cdot \mathrm{d}x_1 \dots \mathrm{d}x_n$$
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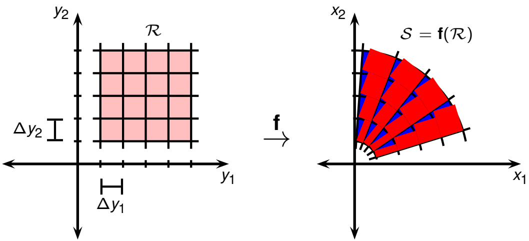

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Variable change in multivariable integrals

$$\mathbf{f}: \left| \begin{array}{l} x_1 = f_1(y_1, \dots, y_n) \\ \quad \vdots \\ x_n = f_n(y_1, \dots, y_n) \end{array} \right. .$$

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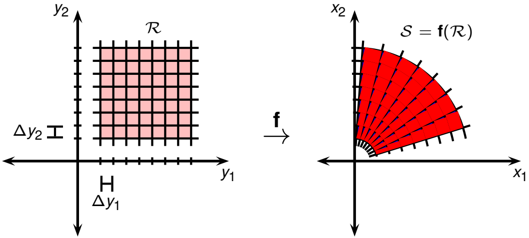

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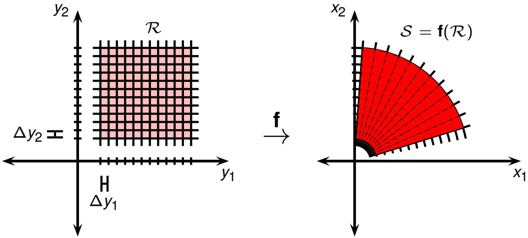

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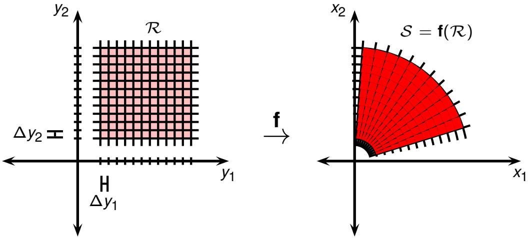

- $$\begin{array}{ccc} \mathrm{Vol}(\mathcal{S}) = \sum\limits_{\mathbf{y}} \mathrm{Vol}(C(\mathbf{y})) & \approx & \sum\limits_{\mathbf{y}} \det\left(J_{\mathbf{f}}(\mathbf{y})\right)\Delta y_1 \dots \Delta y_n \\ \int_{\mathcal{S}} 1 \cdot \mathrm{d}x_1 \dots \mathrm{d}x_n & & \int \cdots \int_{\mathcal{R}} \det\left(J_{\mathbf{f}}(y)\right) \, \mathrm{d}y_1 \dots \mathrm{d}y_n \end{array}$$
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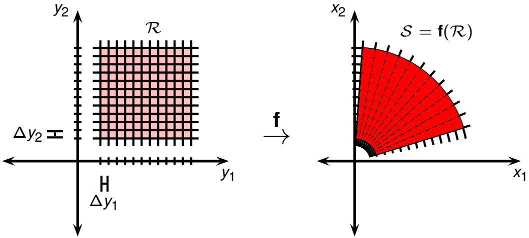

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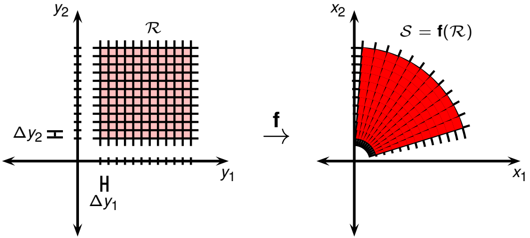

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Variable change in multivariable integrals

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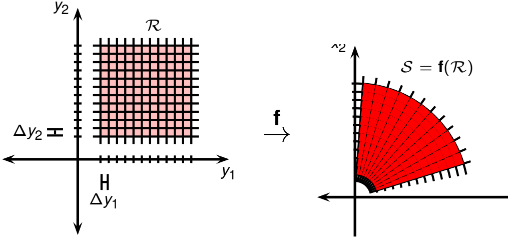

Theorem (Variable change in multivariable integrals)

Let $\mathbf{f}$ *be a smooth one to one variable change. Let* $\mathbf{f}(\mathcal{R}) = \mathcal{S}$*.*

Then

$$\int \cdots \int_{\mathcal{S}} \qquad \mathrm{d}x_1 \dots \mathrm{d}x_n = \int \cdots \int_{\mathcal{R}} \qquad \det\left(J_{\mathbf{f}}(\mathbf{y})\right) \mathrm{d}y_1 \dots \mathrm{d}y_n,$$

provided that $\det\left(J_{\mathbf{f}}(\mathbf{y})\right) \geq 0$ *for all* $\mathbf{y} \in \mathcal{R}$*.*

Variable change in multivariable integrals

$$\mathbf{f}: \left| \begin{array}{lcl} x_1 & = & f_1(y_1, \dots, y_n) \\ & \vdots & \\ x_n & = & f_n(y_1, \dots, y_n) \end{array} \right. .$$

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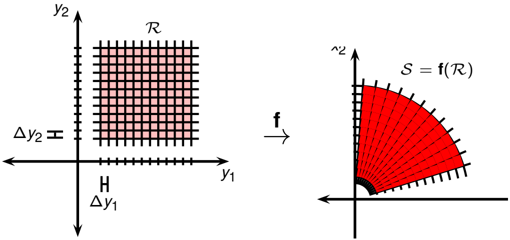


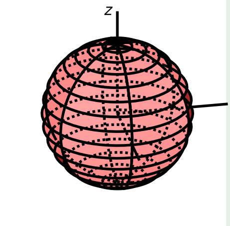
Theorem (Variable change in multivariable integrals)

Let $\mathbf{f}$ *be a smooth one to one variable change. Let* $\mathbf{f}(\mathcal{R}) = \mathcal{S}$. *Let h be an integrable function. Then*

$$\int_{\mathcal{S}} \cdots \int h(x_1, \dots, x_n) \mathrm{d}x_1 \dots \mathrm{d}x_n = \int_{\mathcal{R}} \cdots \int h(f_1, \dots, f_n) \det \left(J_{\mathbf{f}}(\mathbf{y}) \right) \mathrm{d}y_1 \dots \mathrm{d}y_n,$$

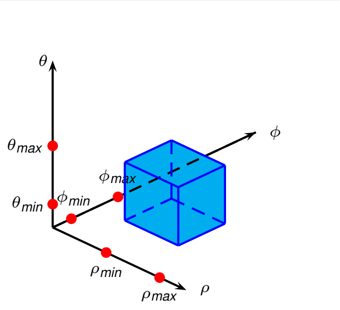
provided that $\det \left(J_{\mathbf{f}}(\mathbf{y}) \right) \geq 0$ *for all* $\mathbf{y} \in \mathcal{R}$.

Example

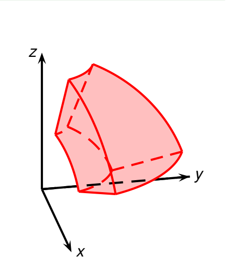


Find the volume of a ball of radius r.

Example



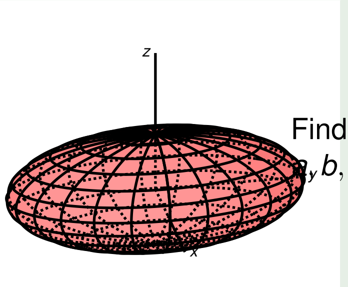
$\xrightarrow{\mathbf{f}}$



Find the volume of a spherical curvilinear box, given by the spherical coordinate inequalities

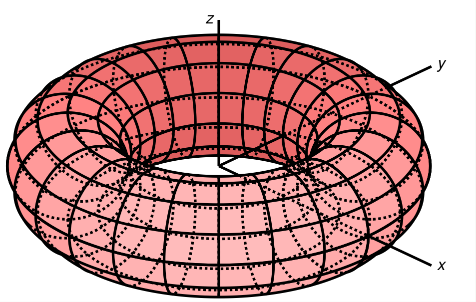
$\rho_{min} \leq \rho \leq \rho_{max}$,
$\phi_{min} \leq \phi \leq \phi_{max}$,
$\theta_{min} \leq \theta \leq \theta_{max}$.

Example



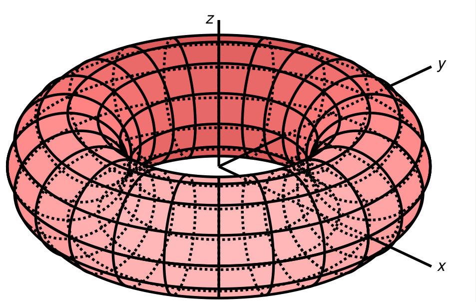
Find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{y^2}{c^2} = 1$, $a, b, c > 0$.

Example (Volume of toroid)



Find the volume of a toroid T (the inside of a torus S) with major radius R and minor radius r.

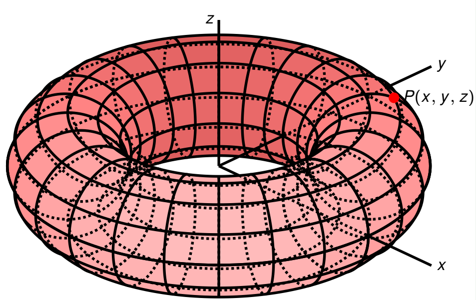
Example (Volume of toroid)



Find the volume of a toroid T (the inside of a torus S) with major radius R and minor radius r.

Suppose the toroid sits in space as drawn.

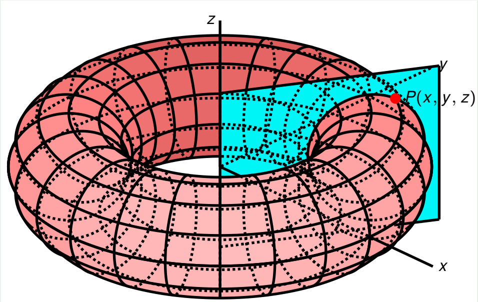
Example (Volume of toroid)



Find the volume of a toroid T (the inside of a torus S) with major radius R and minor radius r.

Suppose the toroid sits in space as drawn. Let $P(x, y, z) \in S$.

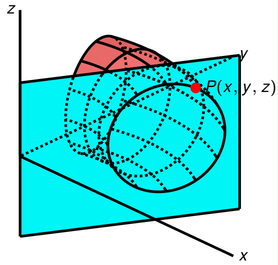
Example (Volume of toroid)



Find the volume of a toroid T (the inside of a torus S) with major radius R and minor radius r.

Suppose the toroid sits in space as drawn. Let $P(x, y, z) \in S$. Let $\mathcal{P}$ be the plane through the z-axis and P.

Example (Volume of toroid)



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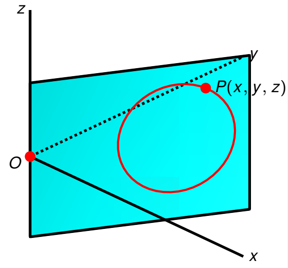

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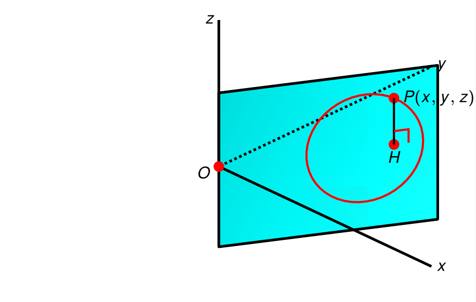
Example (Volume of toroid)



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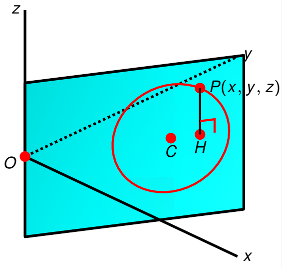

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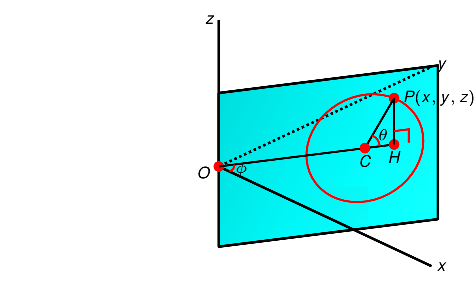

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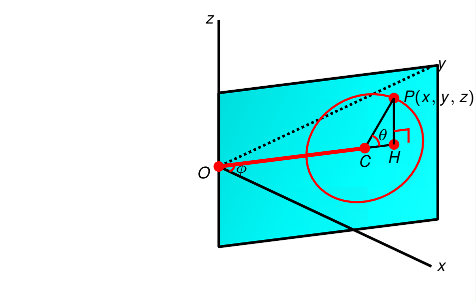

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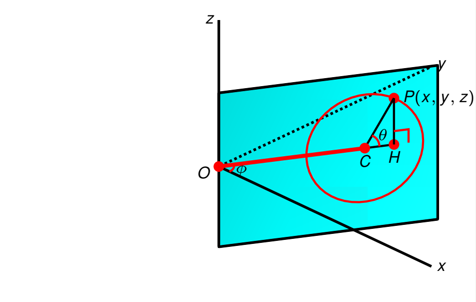

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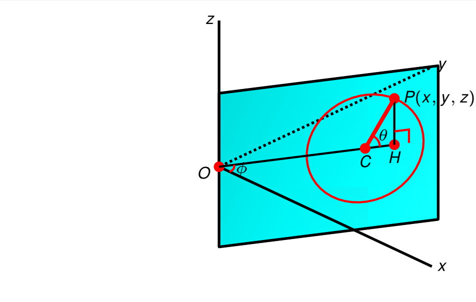

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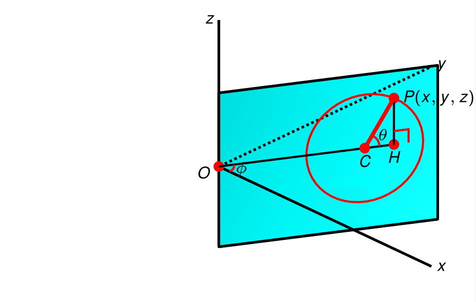

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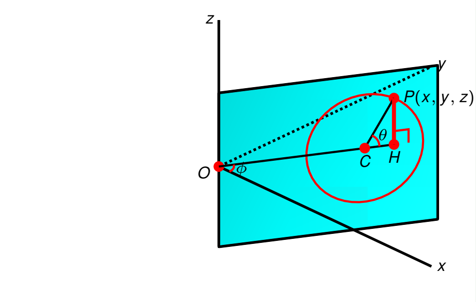

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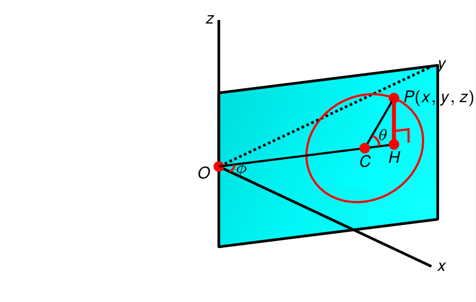

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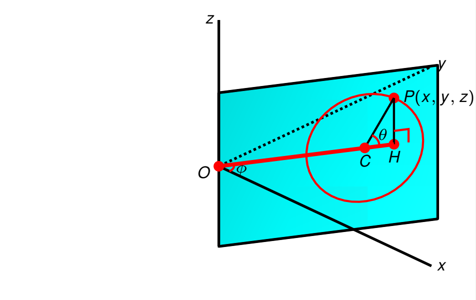

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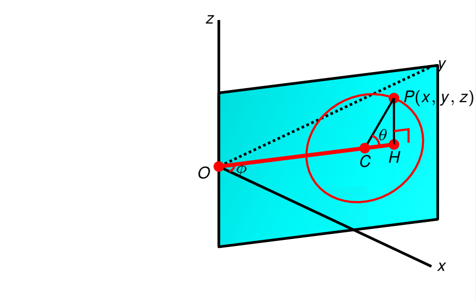

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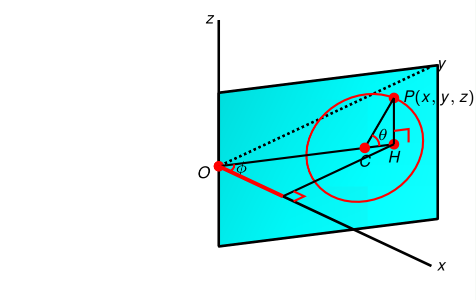

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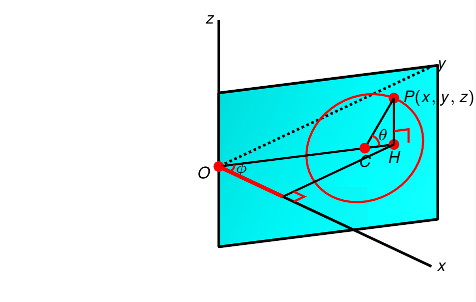

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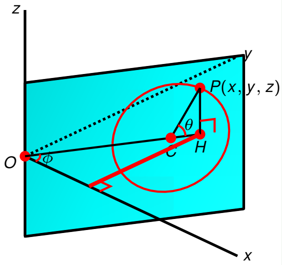

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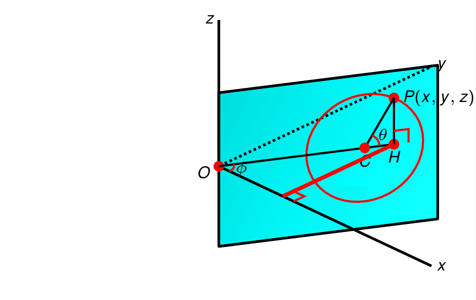

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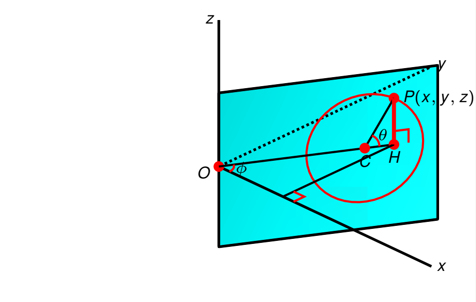

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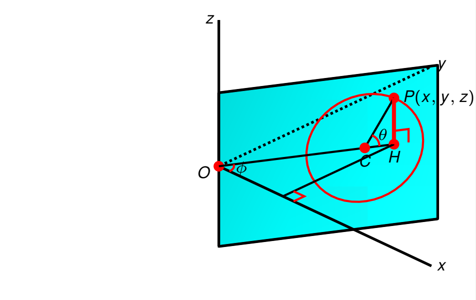

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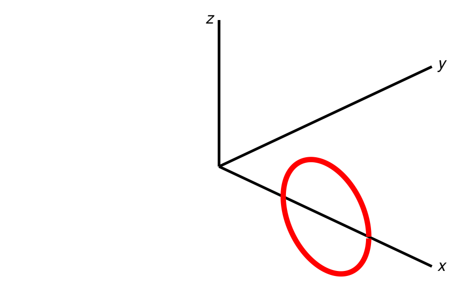

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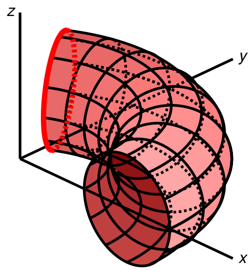

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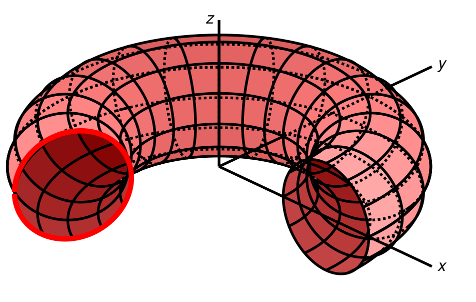

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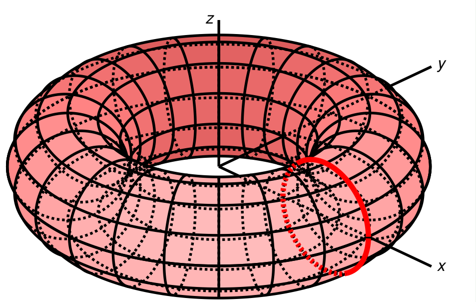

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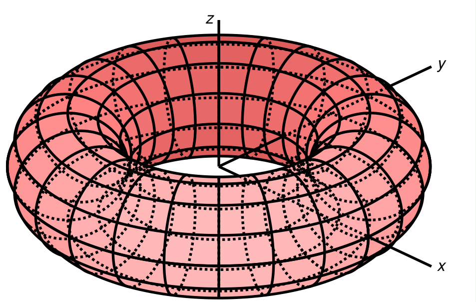

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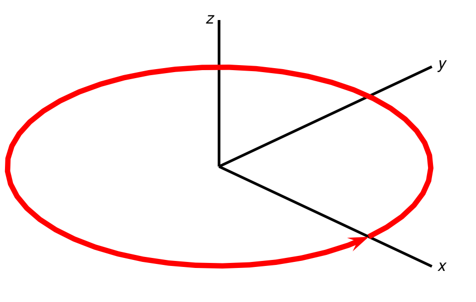

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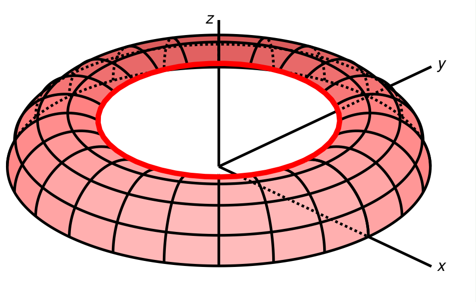

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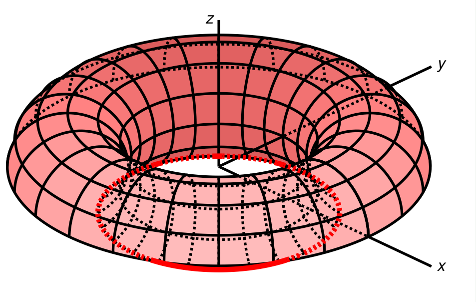

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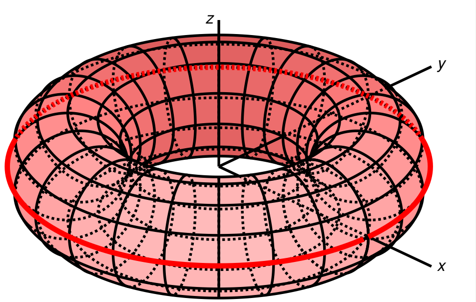

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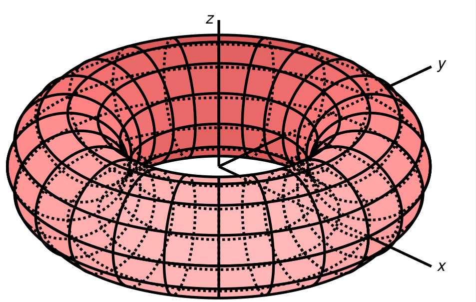

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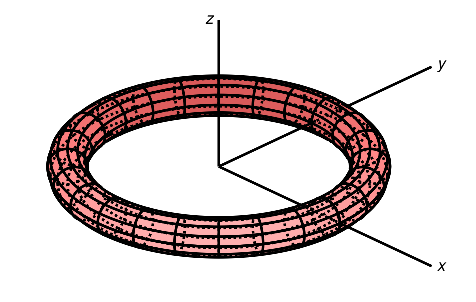

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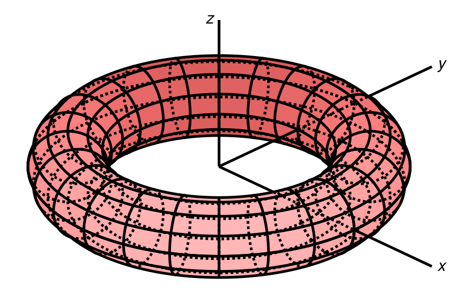

Find the volume of a toroid T (the inside of a torus S) with major radius R and minor radius r.

$$S: \left| \begin{array}{l} x=(R+r\cos\theta)\cos\phi \\ y=(R+r\cos\theta)\sin\phi \\ z=r\sin\theta \end{array} \right.$$

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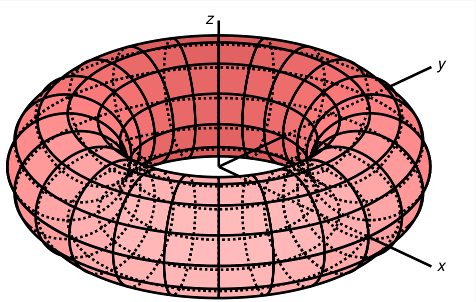

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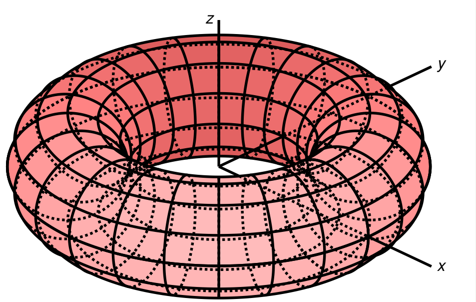

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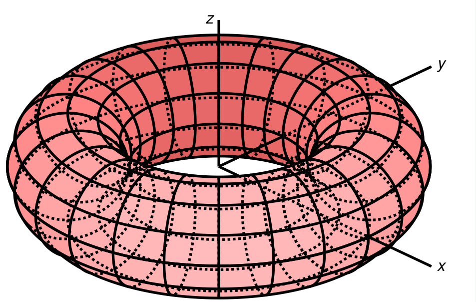

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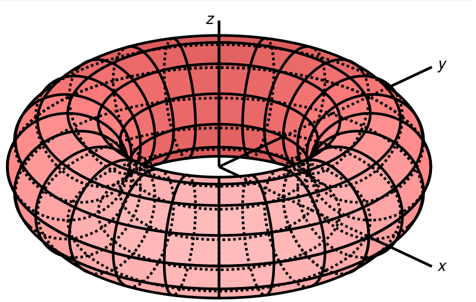
Let $\mathbf{f}$ be the map participating in the parametrization of T.

Example (Volume of toroid)



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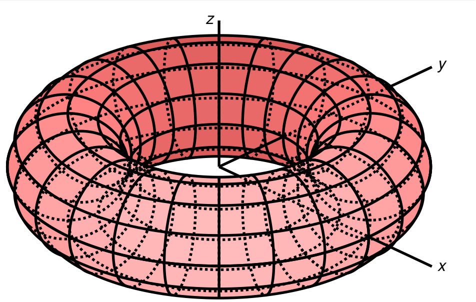
Let $\mathbf{f}$ be the map participating in the parametrization of T.

$$\mathrm{Vol}(T) = \int\limits_{\theta=\mathbf{?}}^{\theta=\mathbf{?}} \int\limits_{\phi=\mathbf{?}}^{\phi=\mathbf{?}} \int\limits_{\rho=\mathbf{?}}^{\rho=\mathbf{?}} \det\left(J_{\mathbf{f}}\right) \mathrm{d}\rho \mathrm{d}\phi \mathrm{d}\theta$$

Example (Volume of toroid)



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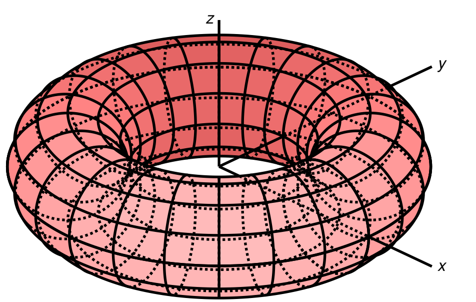
Example (Volume of toroid)



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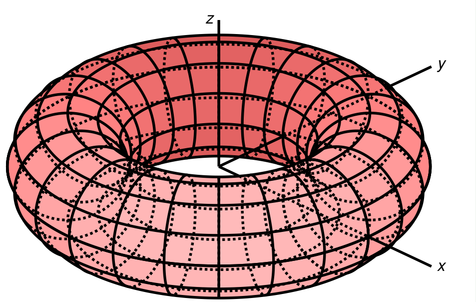
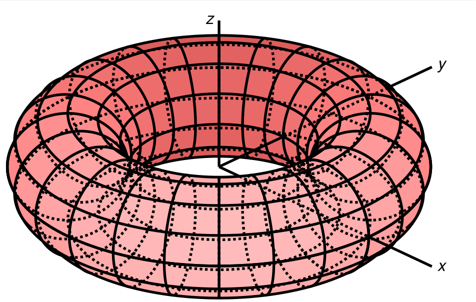


Find volume of toroid T, major radius R minor radius r.

$$\mathbf{f}: \left| \begin{array}{l} x=(R+\rho\cos\theta)\cos\phi \\ y=(R+\rho\cos\theta)\sin\phi \\ z=\rho\sin\theta \end{array} \right.$$

$$\mathrm{Vol}(T)=\int\limits_{\theta=0}^{\theta=2\pi}\int\limits_{\phi=0}^{\phi=2\pi}\int\limits_{\rho=0}^{\rho=r}\det\left(J_{\mathbf{f}}\right)\mathrm{d}\rho\mathrm{d}\phi\mathrm{d}\theta$$

Example (Volume of toroid)



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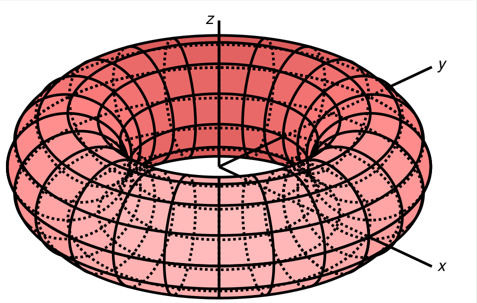
Example (Volume of toroid)



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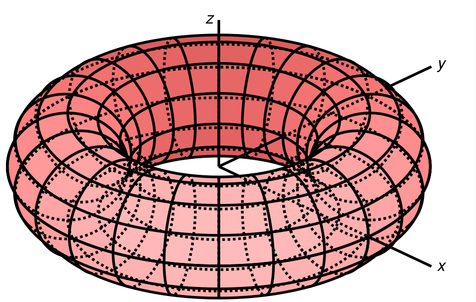

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$$\mathrm{Vol}(T) = \int_{\theta=0}^{\theta=2\pi} \int_{\phi=0}^{\phi=2\pi} \int_{\rho=0}^{\rho=r} \det(J_{\mathbf{f}})\, \mathrm{d}\rho \mathrm{d}\phi \mathrm{d}\theta$$

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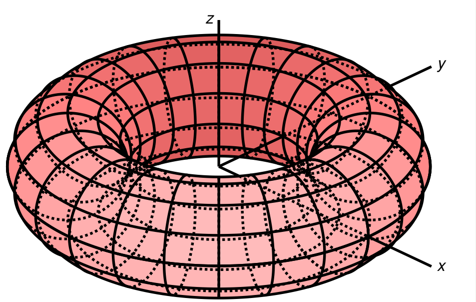

Find volume of toroid T, major radius R minor radius r.

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$$J_{\mathbf{f}} = \begin{pmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{pmatrix}$$

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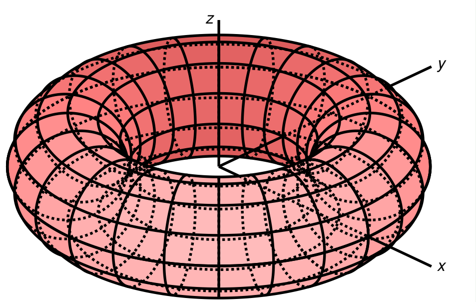

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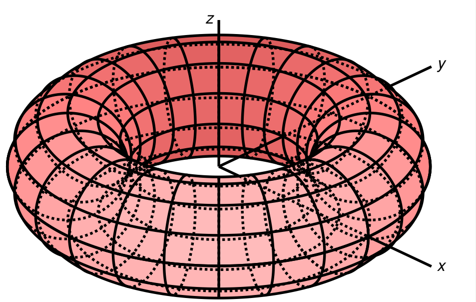

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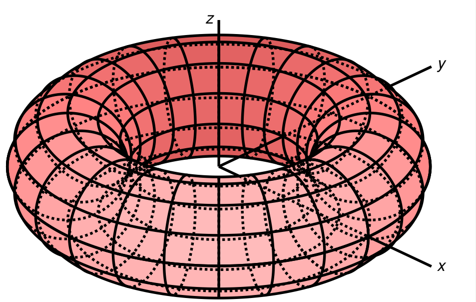
$$\mathrm{Vol}(T)=\int_{\theta=0}^{\theta=2\pi}\int_{\phi=0}^{\phi=2\pi}\int_{\rho=0}^{\rho=r} \det\left(J_{\mathbf{f}}\right) \mathrm{d}\rho \mathrm{d}\phi \mathrm{d}\theta$$

Example (Volume of toroid)



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$$\mathrm{Vol}(T) = \int\limits_{\theta=0}^{\theta=2\pi} \int\limits_{\phi=0}^{\phi=2\pi} \int\limits_{\rho=0}^{\rho=r} \det(J_{\mathbf{f}})\, d\rho d\phi d\theta$$

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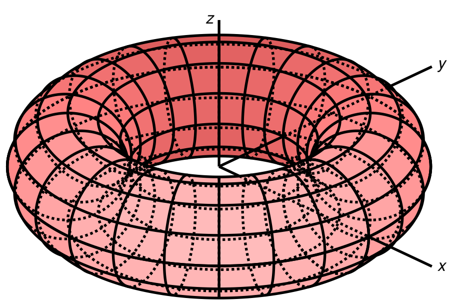

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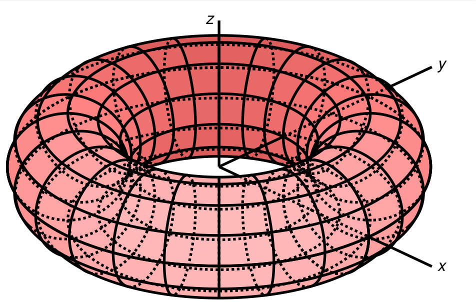
$$J_{\mathbf{f}} = \begin{pmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{pmatrix}$$

Example (Volume of toroid)



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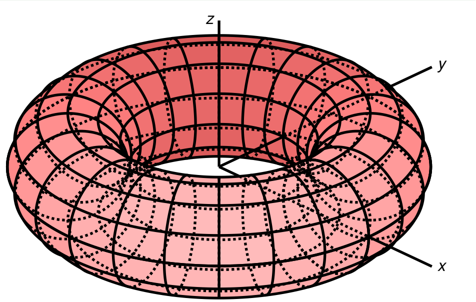
$$J_{\mathbf{f}} = \begin{pmatrix} \cos\theta\cos\phi & -A\sin\phi & -\rho\sin\theta\cos\phi \\ \cos\theta\sin\phi & A\cos\phi & -\rho\sin\theta\sin\phi \\ \sin\theta & 0 & \rho\cos\theta \end{pmatrix}$$

where we have set $A = R + \rho\cos\theta$.

Example (Volume of toroid)



Find volume of toroid T, major radius R minor radius r.

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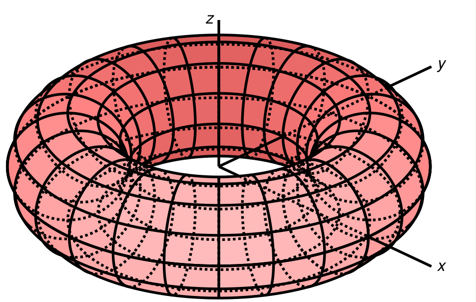
$$\mathrm{Vol}(T)=\int\limits_{\theta=0}^{\theta=2\pi}\int\limits_{\phi=0}^{\phi=2\pi}\int\limits_{\rho=0}^{\rho=r}\det(J_{\mathbf{f}})\,\mathrm{d}\rho\mathrm{d}\phi\mathrm{d}\theta=\int\limits_{0}^{2\pi}\int\limits_{0}^{2\pi}\int\limits_{0}^{r} \mathbf{?}\quad \mathrm{d}\rho\mathrm{d}\phi\mathrm{d}\theta$$

$$J_{\mathbf{f}}=\begin{pmatrix} \cos\theta\cos\phi & -A\sin\phi & -\rho\sin\theta\cos\phi \\ \cos\theta\sin\phi & A\cos\phi & -\rho\sin\theta\sin\phi \\ \sin\theta & 0 & \rho\cos\theta \end{pmatrix}$$

where we have set $A=R+\rho\cos\theta$.

Example (Volume of toroid)



Find volume of toroid T, major radius R minor radius r.

$$\mathbf{f}: \left| \begin{array}{l} x = (R+\rho\cos\theta)\cos\phi \\ y = (R+\rho\cos\theta)\sin\phi \\ z = \rho\sin\theta \end{array} \right.$$

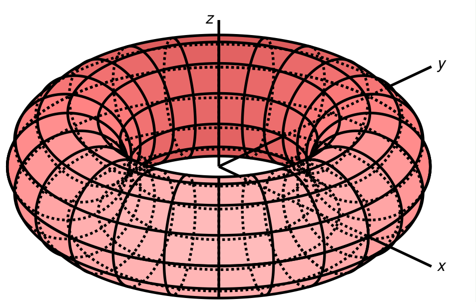
$$J_{\mathbf{f}} = \begin{pmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{pmatrix}$$

$$\mathrm{Vol}(T) = \int\limits_{\theta=0}^{\theta=2\pi} \int\limits_{\phi=0}^{\phi=2\pi} \int\limits_{\rho=0}^{\rho=r} \det(J_{\mathbf{f}})\, d\rho d\phi d\theta = \int\limits_{0}^{2\pi} \int\limits_{0}^{2\pi} \int\limits_{0}^{r} {\color{red}\rho A}\, d\rho d\phi d\theta$$

$$J_{\mathbf{f}} = \begin{pmatrix} \cos\theta\cos\phi & -A\sin\phi & -\rho\sin\theta\cos\phi \\ \cos\theta\sin\phi & A\cos\phi & -\rho\sin\theta\sin\phi \\ \sin\theta & 0 & \rho\cos\theta \end{pmatrix}$$

where we have set ${\color{red}A = R + \rho\cos\theta}$.

Example (Volume of toroid)



Find volume of toroid T, major radius R, minor radius r.

$$\mathbf{f}: \left| \begin{array}{l} x = (R + \rho\cos\theta)\cos\phi \\ y = (R + \rho\cos\theta)\sin\phi \\ z = \rho\sin\theta \end{array} \right.$$

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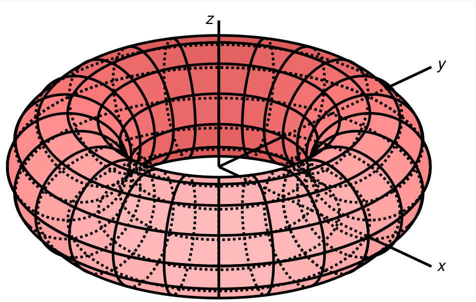
$$\mathrm{Vol}(T) = \int\limits_{\theta=0}^{\theta=2\pi} \int\limits_{\phi=0}^{\phi=2\pi} \int\limits_{\rho=0}^{\rho=r} \det(J_{\mathbf{f}})\, \mathrm{d}\rho \mathrm{d}\phi \mathrm{d}\theta = \int\limits_{0}^{2\pi} \int\limits_{0}^{2\pi} \int\limits_{0}^{r} \rho(R + \rho\cos\theta) \mathrm{d}\rho \mathrm{d}\phi \mathrm{d}\theta$$

Example (Volume of toroid)



Find volume of toroid T, major radius R, minor radius r.

$$\mathbf{f}: \left| \begin{array}{l} x=(R+\rho\cos\theta)\cos\phi \\ y=(R+\rho\cos\theta)\sin\phi \\ z=\rho\sin\theta \end{array} \right.$$

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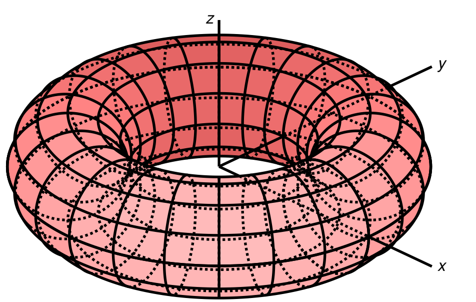
Example (Volume of toroid)



Find volume of toroid T, major radius R, minor radius r.

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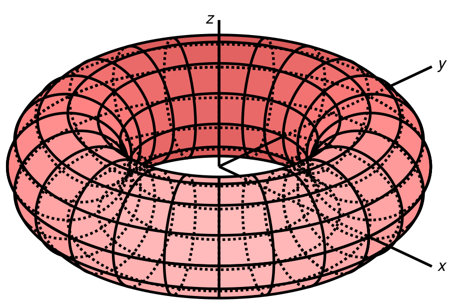

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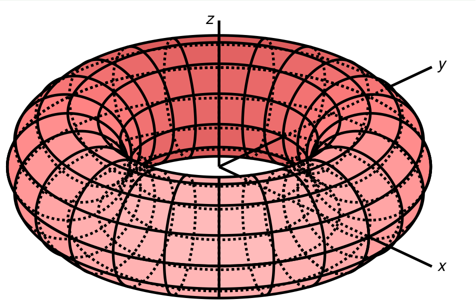

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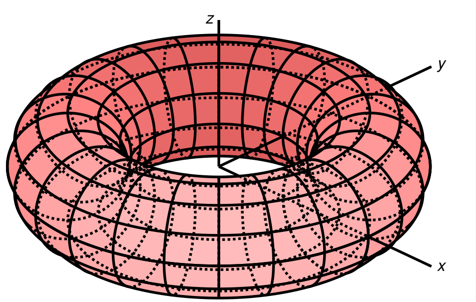

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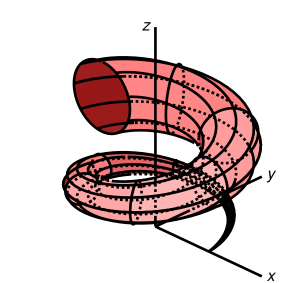
Example



Find the volume of the horn given by

$$\left|\begin{array}{rcl} x &=& (2+\rho\cos\theta)\cos\phi \\ y &=& (2+\rho\cos\theta)\sin\phi \\ z &=& \rho\sin\theta+\frac{\phi}{3} \end{array}\right. ,$$

$\theta\in[0,2\pi], \phi\in[0,3\pi], \rho\in\left[0,\frac{\phi}{9}\right]$.

Theorem (Variable change in multivariable integrals)

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- However we will learn to compute with $\wedge$ in practice.