Calculus II
Lecture 16

Todor Milev
https://github.com/tmilev/freecalc

2020

Outline

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Formal Series

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- The sum, if convergent, of an infinite sequence/infinite formal series will be defined in the following slides.

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- If that is still ambiguous we should switch to the completely unambiguous $\sum$ notation.

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- In programming, what objects are similar to $\sum$?

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  - select first and last index so that your general term formula reproduces the first and last terms of the sequence.

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  - If in doubt or seeking complete rigor we should use the $\sum$ notation.

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An arithmetic series is a series whose terms are an arithmetic sequence.

Example (Sum of a small arithmetic series)

The sum of the arithmetic series $7+4+1-2-5$ is 5.

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$$\text{Therefore} \quad \begin{aligned} 2s &= (-49)(22) \\ s &= -49\cdot 22/2 = -539. \end{aligned}$$

Theorem (Sum of an arithmetic series)

The sum of a finite arithmetic series is the average of the first and last terms, multiplied by the number of terms. That is,

$$a + (a + d) + (a + 2d) + \cdots + (a + (n - 1)d) = \frac{a + (a + (n - 1)d)}{2} n.$$

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The series contains 20 terms.The average of the first and last terms is $\frac{5+100}{2}$.
Therefore the sum is $\frac{5+100}{2} \cdot 20 = 105 \cdot 10 = 1050$.

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Theorem (The sum of a finite geometric series)

Let $r \neq 1$. The sum of the finite geometric series $\sum_{n=1}^{M} ar^{n-1}$ is $a\frac{1-r^M}{1-r}$.

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- If we add the terms, we get the partial sums $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \frac{31}{32}$.
- After the nth term, we get $1 - \frac{1}{2^n}$.

- Does it make sense to add infinitely many numbers?
- Sometimes yes, sometimes no.
- Consider the series $\sum_{n=1}^{\infty} n$.

$$1 + 2 + 3 + 4 + 5 + \cdots + n + \cdots$$

- If we add the terms, we get the partial sums 1,3,6,10,15.
- After the nth term, we get $\frac{n(n+1)}{2}$.
- This goes to ∞ as n gets bigger.
- Now consider the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$.

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- This gets closer and closer to 1. We write $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$.

Definition (Partial Sum, Convergent, Divergent, Sum)

Given a series $\sum_{i=1}^{\infty} a_i = a_1 + a_2 + a_3 + \cdots$, let s_n denote the nth partial sum:

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$$

If the sequence $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = s$, then we say that the series $\sum_{i=1}^{\infty} a_i$ is convergent, and we write

$$\sum_{i=1}^{\infty} a_i = s.$$

In this case, we call s the sum of the series.
If the sequence $\{s_n\}$ is divergent, then we say that the series $\sum_{i=1}^{\infty} a_i$ is divergent.

Example

An important example is the geometric series

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}, \qquad a \neq 0$$

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- If $-1 < r < 1$, then $r^n \to 0$, so the geometric series is convergent and its sum is $a/(1-r)$.
- If $r > 1$ or $r \leq -1$, then r^n is divergent, so $\sum_{n=1}^{\infty} ar^{n-1}$ diverges.

This theorem summarizes the results of the previous example.

Theorem (Convergence of Geometric Series)

The geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

is convergent if $|r| < 1$ *and its sum is*

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}.$$

If $|r| \geq 1$*, the series is divergent.*
a *is called the first term and* r *is called the common ratio.*

Example

Find the sum of the geometric series $$-2+\frac{6}{5}-\frac{18}{25}+\frac{54}{125}-\cdots$$

For $|r| < 1$, recall that the sum of a geometric series is

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$$2.3171717\ldots = 2.3 + \frac{17}{10^3} + \frac{17}{10^5} + \frac{17}{10^7} + \cdots$$

- After the first term, we have a geometric series.
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$$2.3171717\ldots \quad = \quad 2.3 + \frac{\frac{17}{10^3}}{1 - \frac{1}{10^2}} = 2.3 + \frac{\frac{17}{1000}}{\frac{99}{100}}$$

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Show the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent and find its sum.

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Therefore $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{k\to\infty} s_k$

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Therefore $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{k\to\infty} s_k = \lim_{k\to\infty} \left(1 - \frac{1}{k+1}\right)$

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Therefore $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{k\to\infty} s_k = \lim_{k\to\infty} \left(1 - \frac{1}{k+1}\right) = 1$

Example

Show that the harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ diverges.

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s_1 &= 1\\
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Show that the harmonic series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n} = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$ diverges.

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s_{16} & = & 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \cdots + \frac{1}{8}\right) + \left(\frac{1}{9} + \cdots + \frac{1}{16}\right)
\end{array}
$$

Example

Show that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ diverges.

$$
\begin{aligned}
s_1 &= 1 \\
s_2 &= 1 + \tfrac{1}{2} \\
s_4 &= 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} > 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{4} = 1 + \tfrac{2}{2} \\
s_8 &= 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8} \\
&> 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} \\
&= 1 + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} = 1 + \tfrac{3}{2} \\
s_{16} &= 1 + \tfrac{1}{2} + \left(\tfrac{1}{3} + \tfrac{1}{4}\right) + \left(\tfrac{1}{5} + \cdots + \tfrac{1}{8}\right) + \left(\tfrac{1}{9} + \cdots + \tfrac{1}{16}\right) \\
&> 1 + \tfrac{1}{2} + \left(\tfrac{1}{4} + \tfrac{1}{4}\right) + \left(\tfrac{1}{8} + \cdots + \tfrac{1}{8}\right) + \left(\tfrac{1}{16} + \cdots + \tfrac{1}{16}\right)
\end{aligned}
$$

Example

Show that the harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ diverges.

$$
\begin{aligned}
s_1 &= 1 \\
s_2 &= 1 + \tfrac{1}{2} \\
s_4 &= 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} > 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{4} = 1 + \tfrac{2}{2} \\
s_8 &= 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8} \\
&> 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} \\
&= 1 + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} = 1 + \tfrac{3}{2} \\
s_{16} &= 1 + \tfrac{1}{2} + \left(\tfrac{1}{3} + \tfrac{1}{4}\right) + \left(\tfrac{1}{5} + \cdots + \tfrac{1}{8}\right) + \left(\tfrac{1}{9} + \cdots + \tfrac{1}{16}\right) \\
&> 1 + \tfrac{1}{2} + \left(\tfrac{1}{4} + \tfrac{1}{4}\right) + \left(\tfrac{1}{8} + \cdots + \tfrac{1}{8}\right) + \left(\tfrac{1}{16} + \cdots + \tfrac{1}{16}\right)
\end{aligned}
$$

Example

Show that the harmonic series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n} = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$ diverges.

$$\begin{aligned}
s_1 &= 1\\
s_2 &= 1+\tfrac{1}{2}\\
s_4 &= 1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4} > 1+\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{4} = 1+\tfrac{2}{2}\\
s_8 &= 1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\tfrac{1}{5}+\tfrac{1}{6}+\tfrac{1}{7}+\tfrac{1}{8}\\
&> 1+\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{4}+\tfrac{1}{8}+\tfrac{1}{8}+\tfrac{1}{8}+\tfrac{1}{8}\\
&= 1+\tfrac{1}{2}+\tfrac{1}{2}+\tfrac{1}{2} = 1+\tfrac{3}{2}\\
s_{16} &= 1+\tfrac{1}{2}+\left(\tfrac{1}{3}+\tfrac{1}{4}\right)+\left(\tfrac{1}{5}+\cdots+\tfrac{1}{8}\right)+\left(\tfrac{1}{9}+\cdots+\tfrac{1}{16}\right)\\
&> 1+\tfrac{1}{2}+\left(\tfrac{1}{4}+\tfrac{1}{4}\right)+\left(\tfrac{1}{8}+\cdots+\tfrac{1}{8}\right)+\left(\tfrac{1}{16}+\cdots+\tfrac{1}{16}\right)
\end{aligned}$$

Example

Show that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ diverges.

$$
\begin{aligned}
s_1 &= 1 \\
s_2 &= 1 + \frac{1}{2} \\
s_4 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1 + \frac{2}{2} \\
s_8 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \\
&> 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} \\
&= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1 + \frac{3}{2} \\
s_{16} &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \cdots + \frac{1}{8}\right) + \left(\frac{1}{9} + \cdots + \frac{1}{16}\right) \\
&> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \cdots + \frac{1}{8}\right) + \left(\frac{1}{16} + \cdots + \frac{1}{16}\right) \\
&= 1 + \frac{1}{2}
\end{aligned}
$$

Example

Show that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ diverges.

$$
\begin{aligned}
s_1 &= 1 \\
s_2 &= 1 + \tfrac{1}{2} \\
s_4 &= 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} > 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{4} = 1 + \tfrac{2}{2} \\
s_8 &= 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8} \\
&> 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} \\
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s_{16} &= 1 + \tfrac{1}{2} + \left(\tfrac{1}{3} + \tfrac{1}{4}\right) + \left(\tfrac{1}{5} + \cdots + \tfrac{1}{8}\right) + \left(\tfrac{1}{9} + \cdots + \tfrac{1}{16}\right) \\
&> 1 + \tfrac{1}{2} + \left(\tfrac{1}{4} + \tfrac{1}{4}\right) + \left(\tfrac{1}{8} + \cdots + \tfrac{1}{8}\right) + \left(\tfrac{1}{16} + \cdots + \tfrac{1}{16}\right) \\
&= 1 + \tfrac{1}{2} + \tfrac{1}{2}
\end{aligned}
$$

Example

Show that the harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ diverges.

$$\begin{aligned}
s_1 &= 1\\
s_2 &= 1 + \tfrac{1}{2}\\
s_4 &= 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} > 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{4} = 1 + \tfrac{2}{2}\\
s_8 &= 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8}\\
&> 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8}\\
&= 1 + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} = 1 + \tfrac{3}{2}\\
s_{16} &= 1 + \tfrac{1}{2} + \left(\tfrac{1}{3} + \tfrac{1}{4}\right) + \left(\tfrac{1}{5} + \cdots + \tfrac{1}{8}\right) + \left(\tfrac{1}{9} + \cdots + \tfrac{1}{16}\right)\\
&> 1 + \tfrac{1}{2} + \left(\tfrac{1}{4} + \tfrac{1}{4}\right) + \left(\tfrac{1}{8} + \cdots + \tfrac{1}{8}\right) + \left(\tfrac{1}{16} + \cdots + \tfrac{1}{16}\right)\\
&= 1 + \tfrac{1}{2} + \tfrac{1}{2}
\end{aligned}$$

Example

Show that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ diverges.

$$
\begin{array}{rcl}
s_1 & = & 1 \\
s_2 & = & 1 + \frac{1}{2} \\
s_4 & = & 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1 + \frac{2}{2} \\
s_8 & = & 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \\
 & > & 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} \\
 & = & 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1 + \frac{3}{2} \\
s_{16} & = & 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \cdots + \frac{1}{8}\right) + \left(\frac{1}{9} + \cdots + \frac{1}{16}\right) \\
 & > & 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \cdots + \frac{1}{8}\right) + \left(\frac{1}{16} + \cdots + \frac{1}{16}\right) \\
 & = & 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}
\end{array}
$$

Example

Show that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ diverges.

$$
\begin{aligned}
s_1 &= 1 \\
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s_8 &= 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8} \\
&> 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} \\
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s_{16} &= 1 + \tfrac{1}{2} + \left(\tfrac{1}{3} + \tfrac{1}{4}\right) + \left(\tfrac{1}{5} + \cdots + \tfrac{1}{8}\right) + \left(\tfrac{1}{9} + \cdots + \tfrac{1}{16}\right) \\
&> 1 + \tfrac{1}{2} + \left(\tfrac{1}{4} + \tfrac{1}{4}\right) + \left(\tfrac{1}{8} + \cdots + \tfrac{1}{8}\right) + \left(\tfrac{1}{16} + \cdots + \tfrac{1}{16}\right) \\
&= 1 + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2}
\end{aligned}
$$

Example

Show that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ diverges.

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\begin{aligned}
s_1 &= 1 \\
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s_8 &= 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8} \\
&> 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} \\
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s_{16} &= 1 + \tfrac{1}{2} + \left(\tfrac{1}{3} + \tfrac{1}{4}\right) + \left(\tfrac{1}{5} + \cdots + \tfrac{1}{8}\right) + \left(\tfrac{1}{9} + \cdots + \tfrac{1}{16}\right) \\
&> 1 + \tfrac{1}{2} + \left(\tfrac{1}{4} + \tfrac{1}{4}\right) + \left(\tfrac{1}{8} + \cdots + \tfrac{1}{8}\right) + \left(\tfrac{1}{16} + \cdots + \tfrac{1}{16}\right) \\
&= 1 + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2}
\end{aligned}
$$

Example

Show that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ diverges.

$$
\begin{aligned}
s_1 &= 1\\
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s_8 &= 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8}\\
&> 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8}\\
&= 1 + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} = 1 + \tfrac{3}{2}\\
s_{16} &= 1 + \tfrac{1}{2} + \left(\tfrac{1}{3} + \tfrac{1}{4}\right) + \left(\tfrac{1}{5} + \cdots + \tfrac{1}{8}\right) + \left(\tfrac{1}{9} + \cdots + \tfrac{1}{16}\right)\\
&> 1 + \tfrac{1}{2} + \left(\tfrac{1}{4} + \tfrac{1}{4}\right) + \left(\tfrac{1}{8} + \cdots + \tfrac{1}{8}\right) + \left(\tfrac{1}{16} + \cdots + \tfrac{1}{16}\right)\\
&= 1 + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} = 1 + \tfrac{4}{2}
\end{aligned}
$$

Example

Show that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ diverges.

$$
\begin{aligned}
s_1 &= 1\\
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s_{16} &= 1 + \tfrac{1}{2} + \left(\tfrac{1}{3} + \tfrac{1}{4}\right) + \left(\tfrac{1}{5} + \cdots + \tfrac{1}{8}\right) + \left(\tfrac{1}{9} + \cdots + \tfrac{1}{16}\right)\\
&> 1 + \tfrac{1}{2} + \left(\tfrac{1}{4} + \tfrac{1}{4}\right) + \left(\tfrac{1}{8} + \cdots + \tfrac{1}{8}\right) + \left(\tfrac{1}{16} + \cdots + \tfrac{1}{16}\right)\\
&= 1 + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} = 1 + \tfrac{4}{2}\\
&\vdots\\
s_{2^n} &>
\end{aligned}
$$

Example

Show that the harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ diverges.

$$
\begin{aligned}
s_1 &= 1\\
s_2 &= 1 + \tfrac{1}{2}\\
s_4 &= 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} > 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{4} = 1 + \tfrac{2}{2}\\
s_8 &= 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8}\\
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&= 1 + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} = 1 + \tfrac{3}{2}\\
s_{16} &= 1 + \tfrac{1}{2} + \left(\tfrac{1}{3} + \tfrac{1}{4}\right) + \left(\tfrac{1}{5} + \cdots + \tfrac{1}{8}\right) + \left(\tfrac{1}{9} + \cdots + \tfrac{1}{16}\right)\\
&> 1 + \tfrac{1}{2} + \left(\tfrac{1}{4} + \tfrac{1}{4}\right) + \left(\tfrac{1}{8} + \cdots + \tfrac{1}{8}\right) + \left(\tfrac{1}{16} + \cdots + \tfrac{1}{16}\right)\\
&= 1 + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} = 1 + \tfrac{4}{2}\\
&\vdots\\
s_{2^n} &> 1 + \tfrac{n}{2}
\end{aligned}
$$

Example

Show that the harmonic series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n} = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$ diverges.

$$
\begin{aligned}
s_1 &= 1\\
s_2 &= 1+\tfrac{1}{2}\\
s_4 &= 1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4} > 1+\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{4} = 1+\tfrac{2}{2}\\
s_8 &= 1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\tfrac{1}{5}+\tfrac{1}{6}+\tfrac{1}{7}+\tfrac{1}{8}\\
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s_{16} &= 1+\tfrac{1}{2}+\left(\tfrac{1}{3}+\tfrac{1}{4}\right)+\left(\tfrac{1}{5}+\cdots+\tfrac{1}{8}\right)+\left(\tfrac{1}{9}+\cdots+\tfrac{1}{16}\right)\\
&> 1+\tfrac{1}{2}+\left(\tfrac{1}{4}+\tfrac{1}{4}\right)+\left(\tfrac{1}{8}+\cdots+\tfrac{1}{8}\right)+\left(\tfrac{1}{16}+\cdots+\tfrac{1}{16}\right)\\
&= 1+\tfrac{1}{2}+\tfrac{1}{2}+\tfrac{1}{2}+\tfrac{1}{2} = 1+\tfrac{4}{2}\\
&\vdots\\
s_{2^n} &> 1+\tfrac{n}{2}
\end{aligned}
$$

Therefore $s_{2^n}\to\infty$ as $n\to\infty$, so $\{s_n\}$ is divergent, so the harmonic series is divergent.