

Calculus I

Lecture 23

The Fundamental Theorem Of Calculus Part II

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<https://github.com/tmilev/freecalc>

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Outline

- 1 The Fundamental Theorem of Calculus
 - Proof of FTC, part 1
- 2 The Net Change Theorem

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The Fundamental Theorem of Calculus

- The Fundamental Theorem of Calculus has two parts.
- Part 2 of the FTC roughly says “integration undoes differentiation.”
- Part 2 of the FTC was already studied as the Evaluation Theorem. It allows us to compute integrals by finding antiderivatives, without writing limits of Riemann sums.
- Part 1 of the FTC roughly says “differentiation undoes integration.”
- Part 1 of the FTC deals with functions of the form

$$g(x) = \int_a^x f(t)dt$$

where f is a continuous function on $[a, b]$ and x varies between a and b .

$$g(x) = \int_a^x f(t)dt$$

- g depends only on x .
- If x is a fixed number, then $\int_a^x f(t)dt$ is a fixed number.
- If we let x vary, then $\int_a^x f(t)dt$ varies.
- If f is positive, then g can be interpreted as the area under f from a to x .

Example (FTC Part 1)

If $g(x) = \int_1^x (e^t + 2t)dt$, find $g'(x)$.

$$\begin{aligned} g(x) &= \left[e^t + t^2 \right]_1^x \\ &= (e^x + x^2) - (e^1 + 1^2) \\ &= e^x + x^2 - e - 1. \end{aligned}$$

$$\begin{aligned} g'(x) &= \frac{d}{dx}(e^x + x^2 - e - 1) \\ &= e^x + 2x - 0 - 0 \\ &= e^x + 2x. \end{aligned}$$

Theorem (The Fundamental Theorem of Calculus, Part 1)

If f is continuous on $[a, b]$, then the function g defined by

$$g(x) = \int_a^x f(t)dt$$

is continuous on $[a, b]$ and differentiable on (a, b) , and $g'(x) = f(x)$.

Example

Find the derivative of $g(x) = \int_0^x \sqrt{1+t^2} dt$.

- $f(t) = \sqrt{1+t^2}$ is continuous.
- By the FTC, Part 1,

$$g'(x) = \sqrt{1+x^2}$$

Example (FTC, Part 1)

For each formula $g(x)$, find the derivative $g'(x)$.

$g(x)$	$g'(x)$
$\int_0^x \sin(t^2 + 1) \cos(t^3 + 2) dt$	$\sin(x^2 + 1) \cos(x^3 + 2)$
$\int_{35}^x \frac{1 + r^2 + 4r^3}{1 - r^4} dr$	$\frac{1 + x^2 + 4x^3}{1 - x^4}$
$\int_{-1}^x \frac{\cos 2\theta + 1}{1 + \sin^2 \theta} d\theta$	$\frac{\cos 2x + 1}{1 + \sin^2 x}$

Example (Chain Rule, FTC Part 1)

Differentiate $y = \int_0^{x^4} \sec t dt.$

Let $u = x^4.$

Then $y = \int_0^u \sec t dt.$

Chain Rule:
$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dx} \\ &= (\sec u) (4x^3) \\ &= 4x^3 \sec(x^4).\end{aligned}$$

Theorem (The Fundamental Theorem of Calculus)

Suppose f is continuous on $[a, b]$. Then

- ① *If $G(x) = \int_a^x f(t)dt$, then $G'(x) = f(x)$.*
- ② *$\int_a^b f(x)dx = F(b) - F(a)$, where F is any antiderivative of f .*

We already studied part 2 of the FTC as the Evaluation Theorem.

Theorem

Let A, B -numbers, $a(x), b(x)$ -differentiable functions with $A < a(x) < B, A < b(x) < B$. Let f - continuous on $[A, B]$ and $G(x) = \int_{a(x)}^{b(x)} f(t)dt$. Then $G'(x) = f(b(x))b'(x) - f(a(x))a'(x)$.

Proof.

Let $c \in (A, B)$. Set $h(u) = \int_c^u f(t)dt$. FTC part 1 states that $h'(u) = f(u)$.

$$\begin{aligned} G(x) &= \int_{a(x)}^{b(x)} f(t)dt = \int_c^{b(x)} f(t)dt + \int_{a(x)}^c f(t)dt \\ &= \int_c^{b(x)} f(t)dt - \int_c^{a(x)} f(t)dt = h(b(x)) - h(a(x)) \quad . \end{aligned}$$

Then using the chain rule we get

$$G'(x) = (h(b(x)) - h(a(x)))' = h'(b(x))b'(x) - h'(a(x))a'(x) = f(b(x))b'(x) - f(a(x))a'(x), \text{ as desired.}$$



Problems similar to the following often appear on Calculus I exams.

Example

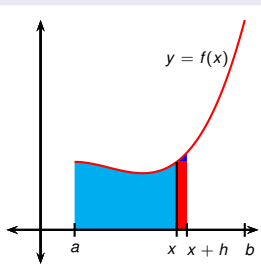
Let $G(x) = \int_{\sqrt{x}}^{x^2} \ln t dt$, $x > 0$. Find $G'(x)$.

$$G'(x) = (\ln x^2)(x^2)' - (\ln \sqrt{x})(\sqrt{x})' = \left(4x - \frac{1}{4}x^{-\frac{1}{2}}\right) \ln x.$$

Theorem (The Fundamental Theorem of Calculus part 1)

Let f be a function continuous on $[a, b]$ and let $G(x) = \int_a^x f(t)dt$ for all $x \in [a, b]$. Then G is differentiable and $G'(x) = f(x)$.

Proof.



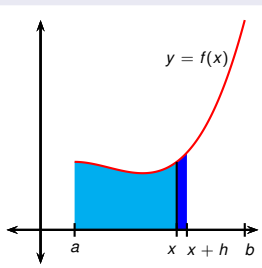
Let $\varepsilon > 0$. There exists δ such that $|f(t) - f(x)| < \varepsilon$ for all t for which $|x - t| < \delta$. Then for all $0 < h < \delta$:

$$\begin{aligned}
 \varepsilon &> f(t) - f(x) > -\varepsilon & \left| \begin{array}{l} \text{integrate} \\ \text{divide by } h \end{array} \right. \\
 h\varepsilon &> \int_x^{x+h} (f(t) - f(x))dt > -h\varepsilon \\
 \varepsilon &> \frac{\int_x^{x+h} (f(t) - f(x))dt}{h} > -\varepsilon \\
 \varepsilon &> \frac{\int_x^{x+h} f(t)dt}{h} - \frac{hf(x)}{h} > -\varepsilon \\
 \varepsilon &> \left| \frac{\int_x^{x+h} f(t)dt}{h} - f(x) \right|
 \end{aligned}$$

Theorem (The Fundamental Theorem of Calculus part 1)

Let f be a function continuous on $[a, b]$ and let $G(x) = \int_a^x f(t)dt$ for all $x \in [a, b]$. Then G is differentiable and $G'(x) = f(x)$.

Proof.



In analogous fashion we can handle the case $h < 0$, to prove: for any $\varepsilon > 0$ there exists $\delta > 0$ so that for all $0 < h$ $|h| < \delta$ we have $\left| \frac{\int_x^{x+h} f(t)dt}{h} - f(x) \right| < \varepsilon$.

$$\begin{aligned} G'(x) &= \lim_{h \rightarrow 0} \frac{G(x+h) - G(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\int_a^{x+h} f(t)dt - \int_a^x f(t)dt}{h} \\ &= \lim_{h \rightarrow 0} \frac{\int_x^{x+h} f(t)dt}{h} = f(x) \end{aligned}$$

- The Evaluation Theorem says that, if f is continuous on $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a),$$

where $F(x)$ is an antiderivative of $f(x)$.

- This means $F' = f$, so

$$\int_a^b F'(x)dx = F(b) - F(a),$$

- $F'(x)$ is the rate of change of $y = F(x)$ with respect to x .
- $F(b) - F(a)$ is the net change in y as x changes from a to b .

Theorem (The Net Change Theorem)

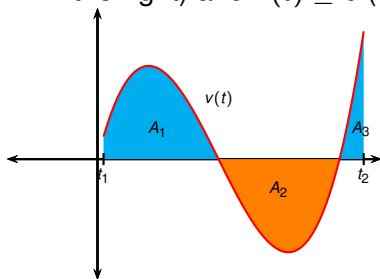
The integral of the rate of change is the net change:

$$\int_a^b F'(x)dx = F(b) - F(a).$$

- If an object moves along a straight line with position function $s(t)$, then its velocity is $v(t) = s'(t)$.
- In this case, the Net Change Theorem says

$$\int_{t_1}^{t_2} v(t) dt = s(t_2) - s(t_1).$$

- This is the displacement, or net change of position.
- If we want to calculate the distance the object travels, we have to consider separately the intervals where $v(t) \geq 0$ (object moves to the right) and $v(t) \leq 0$ (object moves to the left).



$$\begin{aligned} \text{displacement} &= \int_{t_1}^{t_2} v(t) dt \\ &= A_1 - A_2 + A_3 \end{aligned}$$

$$\begin{aligned} \text{distance} &= \int_{t_1}^{t_2} |v(t)| dt \\ &= A_1 + A_2 + A_3 \end{aligned}$$

Example

A particle moves along a line so that its velocity at time t is $v(t) = t^2 - t - 6$ (measured in meters per second).

- 1 Find the displacement of the particle during the time period $1 \leq t \leq 4$.
- 2 Find the distance traveled during this time period.

Example

A particle moves along a line so that its velocity at time t is $v(t) = t^2 - t - 6$ (measured in meters per second).

- Find the displacement of the particle during the time period $1 \leq t \leq 4$.

The displacement is

$$\begin{aligned} s(4) - s(1) &= \int_1^4 v(t) dt = \int_1^4 (t^2 - t - 6) dt \\ &= \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t \right]_1^4 \\ &= \left(\frac{4^3}{3} - \frac{4^2}{2} - 6 \cdot 4 \right) - \left(\frac{1^3}{3} - \frac{1^2}{2} - 6 \cdot 1 \right) \\ &= -\frac{9}{2}. \end{aligned}$$

Therefore the particle moves 4.5m to the left.

Example

A particle moves along a line so that its velocity at time t is $v(t) = t^2 - t - 6$ (measured in meters per second).

- ② Find the distance traveled during this time period.

$v(t) = t^2 - t - 6 = (t - 3)(t + 2)$ and so $v(t) \leq 0$ on the interval $[1, 3]$ and $v(t) \geq 0$ on the interval $[3, 4]$.

The distance is

$$\begin{aligned}\int_1^4 |v(t)| dt &= \int_1^3 [-v(t)] dt + \int_3^4 v(t) dt \\&= \int_1^3 (-t^2 + t + 6) dt + \int_3^4 (t^2 - t - 6) dt \\&= \left[-\frac{t^3}{3} + \frac{t^2}{2} + 6t \right]_1^3 + \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t \right]_3^4 \\&= \frac{61}{6} \approx 10.17\text{m}\end{aligned}$$

Rectilinear Motion

- Suppose a particle is moving in a straight line, with position function $s(t)$.
- Its velocity is $v(t) = s'(t)$.
- Its acceleration is $a(t) = v'(t)$.
- Position is the antiderivative of velocity.
- Velocity is the antiderivative of acceleration.
- If we know the acceleration and the initial values $s(0)$ and $v(0)$ for position and velocity, then we can find $s(t)$ by antidifferentiating twice.

An object near the Earth is subject to a gravitational force that produces a downward acceleration of 32 ft/s^2 (or 9.8 m/s^2).

Example

A ball is thrown upward with a speed of 48 ft/s from the edge of a cliff 432 ft above the ground. Find its height above the ground t seconds later.

To find C , use the fact that $v(0) = 48$.

$$v'(t) = a(t) = -32$$

$$v(t) = -32t + C$$

$$= -32t + 48$$

$$v(0) = 48$$

$$-32 \cdot 0 + C = 48$$

$$C = 48$$

$$s'(t) = -32t + 48$$

$$s(t) = -16t^2 + 48t + D$$

$$= -16t^2 + 48t + 432$$

To find D , use the fact that $s(0) = 432$.

$$s(0) = 432$$

$$-16 \cdot 0^2 + 48 \cdot 0 + D = 432$$

$$D = 432$$