Calculus III
Lecture 14

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https://github.com/tmilev/freecalc

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Outline

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Density and Mass

Question

Let $\mathcal{R}$ be a region in space. Suppose we know the density of $\mathcal{R}$ at every point. Can we find the mass of $\mathcal{R}$?

- Partition the region $\mathcal{R}$ into regions with $D_1, \dots, D_k$ with small diameter.
- Choose a sample point P_k inside each D_k. Then $\mathrm{mass}(D_k) \approx \rho(P_k)\mathrm{vol}(D_k)$.
- Sum the above approximations to get an approximation for mass$\mathcal{R}$: $\mathrm{mass}(\mathcal{R}) \approx \sum_{k=1}^{N} \rho(P_k)\mathrm{vol}(D_k)$.
- Take the limit as the diameter of the partitions tends to zero:

$$\mathrm{mass}(\mathcal{R}) = \lim_{\max_k \mathrm{diam}(D_k) \to 0} \sum_{k=1}^{N} \rho(P_k)\mathrm{vol}(D_k) \ .$$

Triple Integrals

Let f be a scalar or vector-valued function on region $\mathcal{R}$.

Definition

If the limit

$$\lim_{\max_k \operatorname{diam}(D_k)\to 0} \sum_{k=1}^{N} f(P_k)\operatorname{vol}(D_k)$$

exists and is finite, its value is called *the integral of f on $\mathcal{R}$ with respect to volume* and is denoted by

$$\iiint_{\mathcal{R}} f(P)\mathrm{d}V \quad .$$

- If f is a scalar function, then the value of the integral is a scalar.
- If f is a vector-valued function, then the integral is a vector.
- If f is continuous, the limit is guaranteed to exist. If f is not continuous, the limit may fail to exist.

Theoretical Examples

- The volume of a region is defined via a triple integral.

$$\mathrm{vol}(\mathcal{R}) = \iiint_{\mathcal{R}} 1 \cdot \mathrm{d}V$$

- The mass of a body can be computed via a triple integral.

$$\mathrm{mass}(\mathcal{R}) = \iiint_{\mathcal{R}} \mathrm{density}(P) \cdot \mathrm{d}V \ .$$

- Average value of function f (with respect to volume) is given by:

$$\text{average value of } f = \frac{1}{\mathrm{vol}(\mathcal{R})} \iiint_{\mathcal{R}} f(P) \cdot \mathrm{d}V.$$

- The average value of a function f with respect to mass distribution:

$$\text{av. value of } f = \frac{1}{\mathrm{m}(\mathcal{R})} \iiint_{\mathcal{R}} f(P)\,\mathrm{d}m = \frac{1}{\mathrm{m}(\mathcal{R})} \iiint_{\mathcal{R}} f(P)\rho(P)\,\mathrm{d}V \ .$$

Iterated Integrals

- To compute a triple integral over $\mathcal{R}$ one reduces to iterated integrals.
- One reduces to
  - a single integral of a double integral
  - or double integral of a single integral.
- Single integral of a double integral: decomposition into slices.
  - Project the body on an axis.
  - Look at 2D slices perpendicular to that axis (CT-scan).

$$\iiint_{\mathcal{R}} f(P)\mathrm{d}V = \int_{\text{location of slice}} \left(\iint_{\text{slice}} f(P)\mathrm{d}A \right) \mathrm{d}h$$

- Double integral of a single integral: decomposition into rods.
  - Project the body on a plane.
  - Look at 1D slices perpendicular to that plane (rods).

$$\iiint_{\mathcal{R}} f(P)\mathrm{d}V = \iint_{\text{location of rod}} \left(\int_{\text{rod}} f(P)\mathrm{d}h \right) \mathrm{d}A$$

Example: Moment of Inertia

- Problem: compute the moment of inertia I
  - of a rectangular box with sides $2a$, $2b$, and $2c$
  - rotating about axis L through center that is perpendicular to a face.
  - The box has constant density ρ. Therefore it's mass is $m = 8\rho abc$.
- Coord. system: rotation axis = z-axis, x, y axes along box sides.

$$I = \iiint_{\mathcal{R}} \rho \operatorname{dist}^2(P, L) \mathrm{d}V = \iiint_{\mathcal{R}} \rho(x^2 + y^2) \,\mathrm{d}x\mathrm{d}y\mathrm{d}z \;.$$

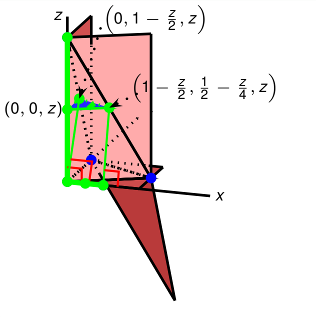
- Decompose into slices as follows.
  - Project $\mathcal{R}$ onto the $z-$axis to get segment from $z = -c$ to $z = c$.

$$\iiint_{\mathcal{R}} \rho(x^2 + y^2)\mathrm{d}x\mathrm{d}y\mathrm{d}z = \int_{z=-c}^{z=c} \left(\iint_{S_z} \rho(x^2 + y^2) \,\mathrm{d}x\mathrm{d}y \right) \mathrm{d}z$$

  - For a fixed z, the slice S_z is: $-a \leq x \leq a$, $-b \leq y \leq b$.

$$I_L = \int_{z=-c}^{z=c} \left(\int_{x=-a}^{x=a} \left(\int_{y=-b}^{y=b} \rho(x^2 + y^2)\mathrm{d}y \right) \mathrm{d}x \right) \mathrm{d}z = \frac{m(a^2 + b^2)}{3}.$$

Example (Decomposition into slices)



Compute the volume of the region $\mathcal{R}$ bounded by $x+2y+z=2$, $x=2y$, $x=0$, $z=0$.

$$\mathrm{vol}(\mathcal{R}) = \iiint_{\mathcal{R}} 1 \cdot \mathrm{d}V.$$

$\mathcal{R}$ is **?** a tetrahedron with vertices at $(0,0,0)$, $(0,1,0)$, $(0,0,2)$, and $\left(1, \frac{1}{2}, 0\right)$.

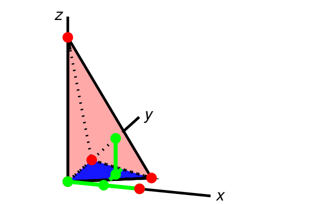
Project $\mathcal{R}$ onto the $z-$axis to get segment from $z=0$ to $z=2$. Fix a value for z to get the slice S_z shown in the picture.

$$\mathrm{vol}(\mathcal{R}) = \iiint_{\mathcal{R}} 1 \cdot \mathrm{d}V = \int_{z=0}^{z=2} \left(\iint_{S_z} 1 \cdot \mathrm{d}x\mathrm{d}y \right) \mathrm{d}z$$

Project S_z onto x-axis to get segment from $x=0$ to $x=1-\frac{z}{2}$. Fix $x \in [0, 1-\frac{z}{2}]$. Vertical slice: segment from $y=\frac{x}{2}$ to $y=1-\frac{z}{2}-\frac{x}{2}$.

$$\mathrm{vol}(\mathcal{R}) = \iiint 1 \cdot \mathrm{d}V = \int^{z=2} \left(\int^{x=1-\frac{z}{2}} \left(\int^{y=1-\frac{z}{2}-\frac{x}{2}} 1 \cdot \mathrm{d}y \right) \mathrm{d}x \right) \mathrm{d}z$$

Example (Decomposition into rods)



Compute the volume of the region $\mathcal{R}$ bounded by $x+2y+z=2$, $x=2y$, $x=0$, $z=0$.

$$\mathrm{vol}(\mathcal{R}) = \iiint_{\mathcal{R}} 1\cdot \mathrm{d}V$$

$\mathcal{R}$ is a tetrahedron with vertices at $(0,0,0)$, $(0,1,0)$, $(0,0,2)$, and $\left(1,\frac{1}{2},0\right)$.

$$\begin{aligned}\mathrm{vol}(\mathcal{R}) &= \iiint_{\mathcal{R}} 1\cdot \mathrm{d}V = \iint_D \left(\int_{z=0}^{z=2-x-2y} 1\cdot \mathrm{d}z\right)\mathrm{d}x\mathrm{d}y\\ &= \iint_D (2-x-2y)\mathrm{d}x\mathrm{d}y = \int_{x=0}^{x=1}\left(\int_{y=\frac{x}{2}}^{y=1-\frac{x}{2}}(2-x-2y)\mathrm{d}y\right)\mathrm{d}x\\ &= \int_0^1\left(\left[(2-x)y-y^2\right]_{y=\frac{x}{2}}^{y=1-\frac{x}{2}}\right)\mathrm{d}x = \textbf{?}\int_0^1 (x^2-2x+1)\mathrm{d}x = \frac{1}{3}.\end{aligned}$$

Project the region onto the $xy-$plane to get triangle D with vertices $(0,0,0)$, $(0,1,0)$ and $\left(1,\frac{1}{2},0\right)$. Fix $(x,y)\in D$; the vertical rod is segment with endpoints $z=0$ and $z=2-x-2y$. Project D on the $x-$axis to get segment from $x=0$ to $x=1$. Fix x in that range; the