Calculus III
Homework on Lecture 10

1. Recall that the directional derivative $D_{\mathbf{u}}$ in the direction $\mathbf{u}$ is defined as the covariant derivative $D_{\mathbf{u}} f = \nabla_{\frac{\mathbf{u}}{|\mathbf{u}|}} f$. Find the covariant derivative $\nabla_{\mathbf{u}} f$ and the directional derivative $D_{\mathbf{u}} f$ at the indicated point.

   (a) $f(x, y) = x^2 + y^2$, $\mathbf{u} = (1, 2)$, $(x, y) = P = (2, 1)$.

   answer: $\nabla_{\mathbf{u}} f(P) = 8$, $D_{\mathbf{u}} f(P) = \frac{8}{5}\sqrt{5}$

   (b) $f(x, y) = e^{x+y}$, $\mathbf{u} = (1, 1)$, $(x, y) = P = (0, 0)$.

   answer: $\nabla_{\mathbf{u}} f(P) = 2$, $D_{\mathbf{u}} f(P) = \sqrt{2}$

   (c) $f(x, y, z) = \ln\sqrt{x^2 + y^2 + z^2}$, $\mathbf{u} = (1, -1, 1)$, $(x, y, z) = P = (1, 1, 1)$.

   answer: $\nabla_{\mathbf{u}} f(P) = \frac{1}{3}$, $D_{\mathbf{u}} f(P) = \frac{\sqrt{3}}{9}$

   (d) $f(x, y, z) = \ln\sqrt{x^2 - 2y^2 + z^2}$, $\mathbf{u} = (1, -1, 2)$, $(x, y, z) = (1, 1, 2)$

   answer: $\nabla_{\mathbf{u}} f(P) = \frac{7}{3}$, $D_{\mathbf{u}} f(P) = \frac{7}{18}\sqrt{6}$

   (e) $f(x, y, z) = xyz$, $\mathbf{u} = (-1, -2, 3)$, $(x, y, z) = (1, 1, 1)$.

   answer: $\nabla_{\mathbf{u}} f(P) = 0$, $D_{\mathbf{u}} f(P) = 0$

2. (a) Let the variables b, c, x_1, x_2 be related via $b = -x_1 - x_2$ and $c = x_1 x_2$.

   i. Express the differential operators $\frac{\partial}{\partial c}$ and $\frac{\partial}{\partial b}$ via $\frac{\partial}{\partial x_1}$ and $\frac{\partial}{\partial x_2}$.

   ii. Express the differential operators $\frac{\partial}{\partial x_1}$ and $\frac{\partial}{\partial x_2}$ via $\frac{\partial}{\partial c}$ and $\frac{\partial}{\partial b}$.

   (b) Let x, y, z and ρ, ϕ, θ be related via the usual spherical coordinates equations i.e., $x =$

   i. Express the differential operators $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ via $\frac{\partial}{\partial \rho}, \frac{\partial}{\partial \phi}, \frac{\partial}{\partial \theta}$.

   answer:
$$\begin{aligned}
\frac{\partial}{\partial x} &= \sin\phi\cos\theta\frac{\partial}{\partial \rho} + \frac{\cos\phi\cos\theta}{\rho}\frac{\partial}{\partial \phi} - \frac{\sin\theta}{\rho\sin\phi}\frac{\partial}{\partial \theta}\\
\frac{\partial}{\partial y} &= \sin\phi\sin\theta\frac{\partial}{\partial \rho} + \frac{\cos\phi\sin\theta}{\rho}\frac{\partial}{\partial \phi} + \frac{\cos\theta}{\rho\sin\phi}\frac{\partial}{\partial \theta}\\
\frac{\partial}{\partial z} &= \cos\phi\frac{\partial}{\partial \rho} - \frac{\sin\phi}{\rho}\frac{\partial}{\partial \phi}
\end{aligned}$$

   ii. Express the differential operators $\frac{\partial}{\partial \rho}, \frac{\partial}{\partial \phi}, \frac{\partial}{\partial \theta}$ via $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$.

   answer:
$$\begin{aligned}
\frac{\partial}{\partial \rho} &= \frac{x}{\rho}\frac{\partial}{\partial x} + \frac{y}{\rho}\frac{\partial}{\partial y} + \frac{z}{\rho}\frac{\partial}{\partial z} = \frac{x}{\sqrt{x^2+y^2+z^2}}\frac{\partial}{\partial x} + \frac{y}{\sqrt{x^2+y^2+z^2}}\frac{\partial}{\partial y} + \frac{z}{\sqrt{x^2+y^2+z^2}}\frac{\partial}{\partial z}\\
\frac{\partial}{\partial \phi} &= \frac{xz}{\sqrt{x^2+y^2}}\frac{\partial}{\partial x} + \frac{yz}{\sqrt{x^2+y^2}}\frac{\partial}{\partial y} - \sqrt{x^2+y^2}\frac{\partial}{\partial z}\\
\frac{\partial}{\partial \theta} &= -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}
\end{aligned}$$

   iii. Express the Laplace differential operator $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ via $\frac{\partial}{\partial \rho}, \frac{\partial}{\partial \phi}, \frac{\partial}{\partial \theta}$ (in other words, write the 3 dimensional Laplace operator in spherical coordinates).

   answer:
$$\begin{aligned}
\frac{\partial^2}{\partial x^2} &=\\
\frac{\partial^2}{\partial y^2} &=\\
\frac{\partial^2}{\partial z^2} &=\\
\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} &=
\end{aligned}$$

Solution. 2(b)i **To be written.**