Calculus III
Lecture 4

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Outline

1. Equations of Lines
   - Line from point and direction
   - Line from two points
2. Equations of planes
   - Plane from point and normal
   - Plane from two directions
   - Plane from three points
3. Distances, Angles, Parallelism, Incidence
   - Distance Between Point and Line
   - Parallel Lines
   - Angle Between Lines
   - Distance Between Skew Lines
   - Distance Between Plane and Parallel Line
   - Angle Between Plane and Line
   - Parallel Planes
   - Angle Between Planes

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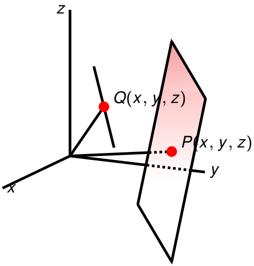
Main Questions



What condition(s) should

- the position vector
- the coordinates

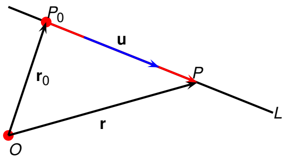
of a point satisfy for it to be on a specific

- line L
- plane $\mathcal{P}$?

Condition(s) in terms of:

- position vector $\Rightarrow$ vector (system of) equations;
- coordinates $\Rightarrow$ scalar equations.

Line from Point and Direction



- Suppose we have line L that passes through point P_0 and has non-zero direction $\mathbf{u}$.
- Denote by $\mathbf{r}_0 = \mathbf{OP}_0$ the position vector of P_0.
- P with position vector $\mathbf{r}$ is on $L \Leftrightarrow$
- $\mathbf{P}_0\mathbf{P}$ has the same direction as $\mathbf{u} \Leftrightarrow$
- $\mathbf{P}_0\mathbf{P}$ is a scalar multiple of $\mathbf{u} \Leftrightarrow$
- $\mathbf{r} - \mathbf{r}_0 = t\mathbf{u}$ for some real number t.

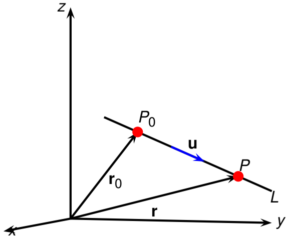
Definition

The equation

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{u}$$

is called a parametric equation of the the line L.

Line from Point and Direction



L- line with direction $\mathbf{u}$ passing through P_0

- Point $P_0(x_0, y_0, z_0)$, $\mathbf{r}_0 = (x_0, y_0, z_0)$;
- Direction $\mathbf{u} = (u_1, u_2, u_3)$. $P(x, y, z)$ with position vector $\mathbf{r}$ is on $L \Leftrightarrow$

  $\mathbf{r} = \mathbf{r}_0 + t\mathbf{u} \Leftrightarrow$

  $(x, y, z) = (x_0, y_0, z_0) + t(u_1, u_2, u_3) \Leftrightarrow$

Definition

The equations

$$\left| \begin{array}{ll} x & = x_0 + tu_1 \\ y & = y_0 + tu_2 \\ z & = z_0 + tu_3 \end{array} \right. , t \in \mathbb{R}$$

are called parametric scalar equations of the line L.

$$\left|\begin{array}{ll} x & = x_0 + tu_1 \\ y & = y_0 + tu_2 \\ z & = z_0 + tu_3 \end{array}\right. \implies \boxed{\frac{x - x_0}{u_1} = \frac{y - y_0}{u_2} = \frac{z - z_0}{u_3}}$$ Symmetric equations

- Caution! Symmetric equations are valid for $u_1, u_2, u_3 \neq 0$. For example if $u_2 = 0$ the equations should be:

$$\frac{x - x_0}{u_1} = \frac{z - z_0}{u_3} \quad \text{and} \quad y = y_0$$

Example

L - line with direction $\mathbf{u} = (4, 5, 6)$ passing through $P_0(1, 2, 3)$. Find

- a parametric vectorial equation of L;
- a parametric scalar equation of L;
- symmetric equations of L.

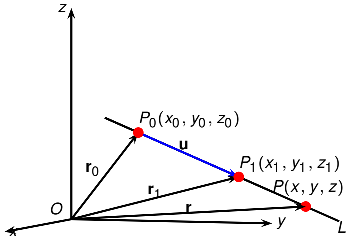
Parametric vectorial equation:
$\mathbf{r} = (1, 2, 3) + t(4, 5, 6) \leftrightarrow \mathbf{r} = (1 + 4t, 2 + 5t, 3 + 6t)$
Parametric scalar equations:

$$\left| \begin{array}{rl} x = & 1 + 4t \\ y = & 2 + 5t \\ z = & 3 + 6t \end{array} \right. , \quad t \text{ real number.}$$

Symmetric equations:

$$\frac{x-1}{4} = \frac{y-2}{5} = \frac{z-3}{6} .$$

Line from Two Points



- Given: distinct points P_0 and P_1, position vectors $\mathbf{r}_0$ and $\mathbf{r}_1$.
- Goal: write equations of line L through P_0 and P_1.
- Direction of L: $\mathbf{u} = \mathbf{r}_1 - \mathbf{r}_0$.
- $\mathbf{u} = (x_1 - x_0, y_1 - y_0, z_1 - z_0)$.

Definition

Parametric equation of a line L:

$$\mathbf{r} = \mathbf{r}_0 + t(\mathbf{r}_1 - \mathbf{r}_0) \quad \Leftrightarrow \quad \mathbf{r} = (1-t)\mathbf{r}_0 + t\mathbf{r}_1$$

Parametric scalar equations of a line L:

$$\left|\begin{array}{ll} x &= x_0 + t(x_1 - x_0) \\ y &= y_0 + t(y_1 - y_0) \\ z &= z_0 + t(z_1 - z_0) \end{array}\right. \Leftrightarrow \left|\begin{array}{ll} x &= (1-t)x_0 + tx_1 \\ y &= (1-t)y_0 + ty_1 \\ z &= (1-t)z_0 + tz_1 \end{array}\right. , \quad t \text{ real number.}$$

Example

Write the equations of line L through $P_0(1,2,3)$ and $P_1(5,2,1)$.

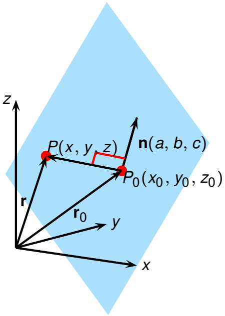
- Direction $\mathbf{u}$ of L: $\mathbf{u} = \mathbf{r}_1 - \mathbf{r}_0 = (4, 0, -2)$.
- Parametric vector equation:
  $\mathbf{r} = (1,2,3) + t(4,0,-2) \Leftrightarrow \mathbf{r} = (1+4t, 2, 3-2t)$.
- Parametric scalar equations:

$$\left| \begin{array}{ll} x & = 1+4t \\ y & = 2 \\ z & = 3-2t \end{array} \right. , \quad t \text{ real number.}$$

- Symmetric equations:

$$\frac{x-1}{4} = \frac{z-3}{-2} \quad \text{and} \quad y = 2 .$$

Plane from Point and Normal



- Point P_0, with position vector $\mathbf{r}_0$; $\mathbf{r}_0 = (x_0, y_0, z_0)$
- Direction $\mathbf{n}$, non-zero vector. $\mathbf{n} = (a, b, c)$
- Goal: describe plane passing through P_0 and orthogonal to $\mathbf{n}$.
- Point P with position $\mathbf{r}$ is on $\mathcal{P} \Leftrightarrow$
- $\mathbf{P}_0\mathbf{P} = \mathbf{r} - \mathbf{r}_0$ is orthogonal (normal) to $\mathbf{n} \Leftrightarrow$
- Implicit vectorial equation: $(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n} = 0$.
- A point $P(x, y, z)$ is on $\mathcal{P} \Leftrightarrow$

Definition (Implicit scalar equation)

$$(x - x_0, y - y_0, z - z_0) \cdot (a, b, c) = 0 \Leftrightarrow$$
$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Example

Find an equation of the plane

- passing through $P_0(1, 2, 3)$
- and perpendicular (normal) to the direction $\mathbf{n} = (6, 5, 4)$.

$$6(x-1)+5(y-2)+4(z-3)=0$$

$$6x+5y+4z=28$$

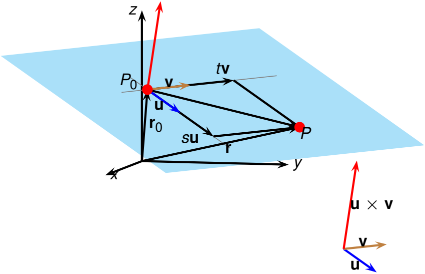
The general equation of a plane is given by:

$$ax+by+cz=d.$$

The coefficients a, b, c are the components of normal to the plane,

$$\mathbf{n}=(a, b, c)\ .$$

Plane from Point and two Directions



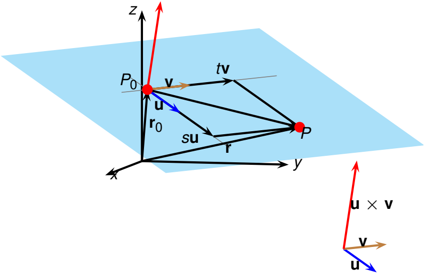
- Given: point P_0 with position vector $\mathbf{r}_0$.
- Non-parallel directions $\mathbf{u}$ and $\mathbf{v}$.
- Goal: give equations of plane $\mathcal{P}$ through P_0 and parallel to both $\mathbf{u}$ and $\mathbf{v}$.

Normal direction $\mathbf{n} = \mathbf{u} \times \mathbf{v} \neq \mathbf{0}$.
Implicit equation: $P(\mathbf{r})$ is on $\mathcal{P} \iff (\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n} = 0$ Interpretation:

$$\mathrm{Vol}(R(\mathbf{r} - \mathbf{r}_0, \mathbf{u}, \mathbf{v})) = 0$$

$P(\mathbf{r})$ is on the plane $\mathcal{P} \Leftrightarrow$
$\mathbf{P}_0\mathbf{P}$ is a combination of $\mathbf{u}$, $\mathbf{v} \Leftrightarrow$
There are scalars s, t such that $\mathbf{r} - \mathbf{r}_0 = s\mathbf{u} + t\mathbf{v} \Leftrightarrow$

Plane from Point and two Directions



- Given: point P_0 with position vector $\mathbf{r}_0$.
- Non-parallel directions $\mathbf{u}$ and $\mathbf{v}$.
- Goal: give equations of plane $\mathcal{P}$ through P_0 and parallel to both $\mathbf{u}$ and $\mathbf{v}$.

Parametric equation:

$$\mathbf{r} = \mathbf{r}_0 + s\mathbf{u} + t\mathbf{v}$$

Let $P_0(x_0, y_0, z_0)$, $P(x, y, z)$ $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$. $\Rightarrow$
Parametric scalar equations:

$$\left|\begin{array}{rl} x = & x_0 + su_1 + tv_1 \\ y = & y_0 + su_2 + tv_2 \\ z = & z_0 + su_3 + tv_3 \end{array}\right. \text{, for } s, t \text{ real parameters.}$$

Example

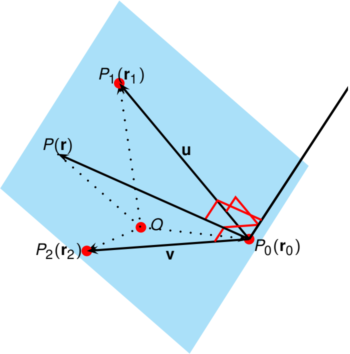
Find equations of a plane passing through $P_0(1,2,3)$ and parallel to the vectors $\mathbf{u} = (-1,0,2)$, $\mathbf{v} = (0,-2,1)$.

$$\mathbf{n} = \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 0 & 2 \\ 0 & -2 & 1 \end{vmatrix} = 4\mathbf{i} + \mathbf{j} + 2\mathbf{k}$$

$\Rightarrow$ implicit scalar equation given by:
$4(x-1) + 1(y-2) + 2(z-3) = 0 \iff 4x + y + 2z = 12$
Parametric vectorial equation:
$(x,y,z) = (1,2,3) + s(-1,0,2) + t(0,-2,1)$
Parametric scalar equations:

$$\left| \begin{array}{ll} x & = 1 - s \\ y & = 2 - 2t \\ z & = 3 + 2s + t \end{array} \right. \qquad s, t \text{ real parameters.}$$

Plane from Three Points



- Given: three non-collinear points $P_0(\mathbf{r}_0)$, $P_1(\mathbf{r}_1)$, $P_2(\mathbf{r}_2)$.
- Goal: find equations fo plane $\mathcal{P}$ passing through P_0, P_1, and P_2.
- The plane is parallel to $\mathbf{u} = \mathbf{P}_0\mathbf{P}_1 = \mathbf{r}_1 - \mathbf{r}_0$ and passing through $P_0 \Rightarrow$ this problem was solved previously.

Normal $\mathbf{n} = \mathbf{u} \times \mathbf{v} = (\mathbf{r}_1 - \mathbf{r}_0) \times (\mathbf{r}_2 - \mathbf{r}_0)$
Implicit equation:

$$(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n} = 0$$

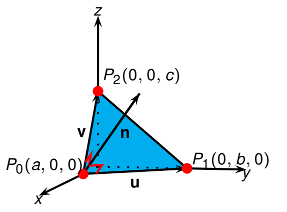
$$\boxed{(\mathbf{r} - \mathbf{r}_0) \cdot [(\mathbf{r}_1 - \mathbf{r}_0) \times (\mathbf{r}_2 - \mathbf{r}_0)] = 0}$$

$$\mathrm{Vol}(R(\mathbf{P}_0\mathbf{P}, \mathbf{P}_0\mathbf{P}_1, \mathbf{P}_0\mathbf{P}_2)) = 0$$

Example

Let $P_0(a,0,0)$, $P_1(0,b,0)$ $P_2(0,0,c)$ be three points, $a,b,c \neq 0$. Find plane $\mathcal{P}$ passing through P_0, P_1, P_2 (i.e., plane with prescribed x, y, z-intercepts).

$\mathcal{P}$: parallel to
$\mathbf{P}_0\mathbf{P}_1 = (-a,b,0), \mathbf{P}_0\mathbf{P}_2 = (-a,0,c)$.
Normal:



$$\mathbf{n} = \mathbf{P}_0\mathbf{P}_1 \times \mathbf{P}_0\mathbf{P}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a & b & 0 \\ -a & 0 & c \end{vmatrix} = bc\mathbf{i} + ac\mathbf{j} + ab\mathbf{k}.$$

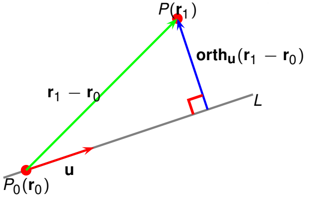
Implicit scalar equation of plane:

$$\begin{aligned} (x-a,y,z)\cdot(bc,ac,ab) &= 0 \\ bcx + acy + abz &= abc \\ \frac{x}{a} + \frac{y}{b} + \frac{z}{c} &= 1 \end{aligned}$$

Relationships betwee points lines and planes

- So far we studied the following geometric objects/
  - Points: $P(\mathbf{r})$.
  - Lines: L: $\mathbf{r} = \mathbf{r}_0 + t\mathbf{u}$
  - Planes: $\mathcal{P}$: $(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n} = 0$
- We investigate the following relationships/geometric quantities:
  - Parallelism
  - Perpendicularity
  - Angles
  - Distances
  - Intersections

Point and line



- Given: Point $P(\mathbf{r}_1)$,
- line $L: \quad \mathbf{r} = \mathbf{r}_0 + t\mathbf{u}$.
- Goal: find the distance between P and L.

Distance from P to L:

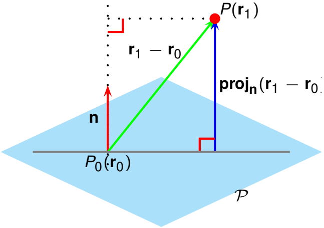
$$d(P, L) = |\mathbf{orth}_{\mathbf{u}}(\mathbf{r}_1 - \mathbf{r}_0)|$$

$$d(P, L) = \left| \mathbf{r}_1 - \mathbf{r}_0 - \frac{(\mathbf{r}_1 - \mathbf{r}_0) \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}} \mathbf{u} \right|$$

$$d(P, L) = \frac{|(\mathbf{r}_1 - \mathbf{r}_0) \times \mathbf{u}|}{|\mathbf{u}|}$$

Valid only in 3 dimensions!

Distance between point and plane



- Given: point $P(\mathbf{r}_1) = (x_1, y_1, z_1)$,
- plane $\mathcal{P}: \quad (\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n} = 0$.
- Goal: find the distance between P and $\mathcal{P}$.

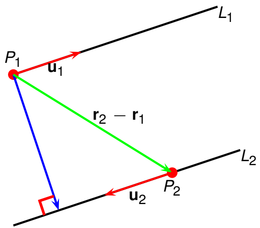
Distance from P to $\mathcal{P}$:

$$\begin{aligned} d(P, \mathcal{P}) &= |\mathbf{proj_n}(\mathbf{r}_1 - \mathbf{r}_0)| \\ d(P, \mathcal{P}) &= \frac{|(\mathbf{r}_1 - \mathbf{r}_0) \cdot \mathbf{n}|}{|\mathbf{n}|} \end{aligned}$$

Scalar equation:

$$\begin{aligned} \mathcal{P}: \quad ax + by + cz + d = (\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n} &= 0 \\ \mathbf{n} &= (a, b, c) \\ d(P, \mathcal{P}) = \frac{|(\mathbf{r}_1 - \mathbf{r}_0) \cdot \mathbf{n}|}{|\mathbf{n}|} &= \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}} \end{aligned}$$

Parallel lines



- Given: lines $\begin{array}{rcl} L_1: \mathbf{r} & = & \mathbf{r}_1 + t\mathbf{u}_1 \\ L_2: \mathbf{r} & = & \mathbf{r}_2 + s\mathbf{u}_2 \end{array}$.
- Goal: distance between lines.

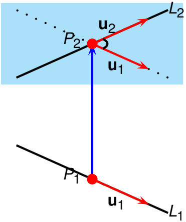
Parallel lines $L_1 || L_2 \iff \mathbf{u}_1, \mathbf{u}_2$ collinear $\iff \boxed{\mathbf{u}_1 \times \mathbf{u}_2 = \mathbf{0}}$ Distance:

$$d = d(L_1, L_2) = d(P_1, L_2) = d(P_2, L_1)$$

$$d = d(L_1, L_2) = |\mathbf{orth}_{\mathbf{u}_1}(\mathbf{r}_2 - \mathbf{r}_1)|$$

$$\boxed{d = \frac{|(\mathbf{r}_2 - \mathbf{r}_1) \times \mathbf{u}_1|}{|\mathbf{u}_1|} = \frac{|(\mathbf{r}_2 - \mathbf{r}_1) \times \mathbf{u}_2|}{|\mathbf{u}_2|}}$$

Angle between lines



- Given: lines $\begin{array}{lcl} L_1: & \mathbf{r} & = & \mathbf{r}_1 + t\mathbf{u}_1 \\ L_2: & \mathbf{r} & = & \mathbf{r}_2 + s\mathbf{u}_2 \end{array}$.
- Goal: find angle between L_1 and L_2.

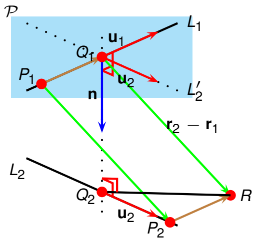
Perpendicular lines $L_1 \perp L_2 \iff \mathbf{u}_1 \perp \mathbf{u}_2$
$\iff$

$$\boxed{\mathbf{u}_1 \cdot \mathbf{u}_2 = 0}$$

Angle between lines α: angle between L_1, $L_2 \iff \alpha$: acute angle $\mathbf{u}_1$, $\mathbf{u}_2 \iff$

$$\boxed{\alpha = \arccos\left(\frac{|\mathbf{u}_1 \cdot \mathbf{u}_2|}{|\mathbf{u}_1|\,|\mathbf{u}_2|}\right)}$$

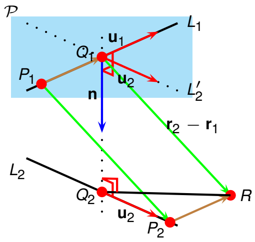
Distance between non-parallel lines



- Given: lines $\begin{array}{lcl} L_1 : & \mathbf{r} & = & \mathbf{r}_1 + t\mathbf{u}_1 \\ L_2 : & \mathbf{r} & = & \mathbf{r}_2 + s\mathbf{u}_2 \end{array}$
- The lines are skew or intersecting, i.e., $\mathbf{n} = \mathbf{u}_1 \times \mathbf{u}_2 \neq \mathbf{0}$.
- Goal: find distance between the lines = $d(L_1, L_2)$ = shortest distance b-n points on the two lines.

- Construct plane $\mathcal{P}$ with directions $\mathbf{u}_1$, $\mathbf{u}_2$ and passing through L_1.
- Distance b-n L_2 and points on $\mathcal{P}$ is constant.
- Project L_2 orthogonally on $\mathcal{P}$; let the projection be L_2'.
- Let L_2' and L_1 intersect in point Q_1.
- Let Q_2 be the heel of the perpendicular from Q_1 onto Q_2.
- $\Rightarrow Q_1 Q_2 = d(L_1, L_2)$.
- $|Q_1 Q_2| = d(L_1, L_2)$.

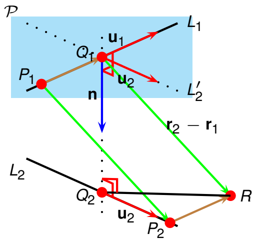
Distance between non-parallel lines



- Given: lines $\begin{array}{rcl} L_1: \mathbf{r} &=& \mathbf{r}_1 + t\mathbf{u}_1 \\ L_2: \mathbf{r} &=& \mathbf{r}_2 + s\mathbf{u}_2 \end{array}$
- The lines are skew or intersecting, i.e., $\mathbf{n} = \mathbf{u}_1 \times \mathbf{u}_2 \neq \mathbf{0}$.
- Goal: find distance between the lines = $d(L_1, L_2)$ = shortest distance b-n points on the two lines.

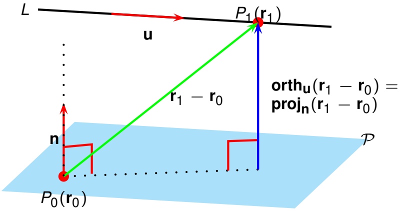
- $\mathbf{Q}_1\mathbf{Q}_2 \perp L_1, L_2 \Rightarrow \mathbf{Q}_1\mathbf{Q}_2$ is proportional to $\mathbf{n} = \mathbf{u}_1 \times \mathbf{u}_2$.
- Pick arbitrary points on L_1, L_2 - say, the base points $P_1(\mathbf{r}_1), P_2(\mathbf{r}_2)$.
- Let R be such that $\mathbf{Q}_1\mathbf{R} = \mathbf{P}_1\mathbf{P}_2 = \mathbf{r}_2 - \mathbf{r}_1$.
- Then $\mathbf{P}_2\mathbf{R}$ is proportional to $\mathbf{u}_1$.
- $\Rightarrow \mathbf{Q}_2\mathbf{R} = \mathbf{Q}_2\mathbf{P}_2 + \mathbf{P}_2\mathbf{R}$ is perpendicular to $\mathbf{n}$.

Distance between non-parallel lines



- Given: lines $\begin{array}{lcl} L_1: \mathbf{r} & = & \mathbf{r}_1 + t\mathbf{u}_1 \\ L_2: \mathbf{r} & = & \mathbf{r}_2 + s\mathbf{u}_2 \end{array}$
- The lines are skew or intersecting, i.e., $\mathbf{n} = \mathbf{u}_1 \times \mathbf{u}_2 \neq \mathbf{0}$.
- Goal: find distance between the lines = $d(L_1, L_2)$ = shortest distance b-n points on the two lines.
- $\Rightarrow \mathbf{Q}_1\mathbf{Q}_2 = \mathbf{proj}_\mathbf{n}(\mathbf{r}_2 - \mathbf{r}_1)$.
- $d(L_1, L_2) = |\mathbf{proj}_\mathbf{n}(\mathbf{r}_2 - \mathbf{r}_1)| = \boxed{\dfrac{|(\mathbf{r}_2 - \mathbf{r}_1) \cdot \mathbf{n}|}{|\mathbf{n}|}} = \frac{|(\mathbf{r}_2 - \mathbf{r}_1) \cdot (\mathbf{u}_1 \times \mathbf{u}_2)|}{|\mathbf{u}_1 \times \mathbf{u}_2|}$
- If lines are intersecting we know $d(L_1, L_2) = 0$. Since the lines intersect L_2 and L_2' coincide. $\Rightarrow (\mathbf{r}_2 - \mathbf{r}_1) \cdot (\mathbf{u}_1 \times \mathbf{u}_2) = 0 \Rightarrow$ the formula $d(L_1, L_2) = \frac{|(\mathbf{r}_2 - \mathbf{r}_1) \cdot (\mathbf{u}_1 \times \mathbf{u}_2)|}{|\mathbf{u}_1 \times \mathbf{u}_2|} = 0$ produces the expected result.

Distance between parallel line and plane



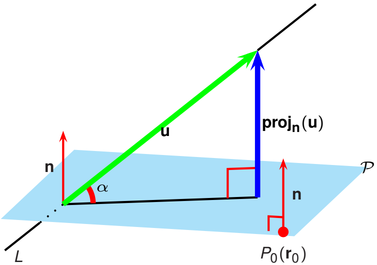
- Given: line $L: \quad \mathbf{r} = \mathbf{r}_1 + t\mathbf{u}$,
- plane $\mathcal{P}: \quad (\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n} = 0$.
- The plane and the line are parallel, i.e. $\mathbf{u} \cdot \mathbf{n} = 0$.
- Goal: find distance between the the two.

Distance from L to $\mathcal{P}$: $d(L, \mathcal{P}) = d(P_1, \mathcal{P})$

$$d(L, \mathcal{P}) = |\mathbf{orth_u}(\mathbf{r}_1 - \mathbf{r}_0)| = \mathbf{proj_n}(\mathbf{r}_1 - \mathbf{r}_0)|$$

$$d(L, \mathcal{P}) = \frac{|(\mathbf{r}_1 - \mathbf{r}_0) \times \mathbf{u}|}{|\mathbf{u}|} = \frac{|(\mathbf{r}_1 - \mathbf{r}_0) \cdot \mathbf{n}|}{|\mathbf{n}|}$$

Angle between line and plane



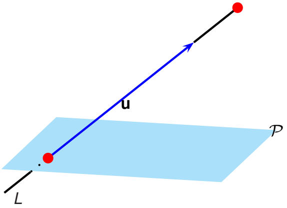
- Given: line $L: \quad \mathbf{r} = \mathbf{r}_1 + t\mathbf{u}$,
- plane $\mathcal{P}: \quad (\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n} = 0$.
- Goal: Find/define angle between line and plane.

Line perpendicular to plane $\Leftrightarrow \mathbf{u} || \mathbf{n} \Leftrightarrow \boxed{\mathbf{u} \times \mathbf{n} = \mathbf{0}}$

Angle between line and plane α: angle between L, $\mathcal{P}$.

$$\begin{aligned} \sin\alpha &= \frac{|\mathrm{\mathbf{proj}}_{\mathbf{n}}\mathbf{u}|}{|\mathbf{u}|} = \frac{|\mathbf{u}\cdot\mathbf{n}|}{|\mathbf{n}||\mathbf{u}|} \\ \alpha &= \arcsin\left(\frac{|\mathbf{u}\cdot\mathbf{n}|}{|\mathbf{u}||\mathbf{n}|}\right) \end{aligned}$$

Intersection between line and plane

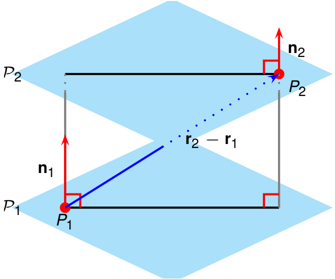


- Given: line L : $\quad \mathbf{r} = \mathbf{r}_1 + t\mathbf{u}$,
- $\mathbf{r}_1 = (x_1, y_1, z_1)$, $u = (p, q, r)$,
- plane $\mathcal{P}$: $ax + by + cz - d = 0$.
- Goal: find the intersection between line and plane.

Let $P_0(\mathbf{r}_0)$ be a point on the plane. Then a point $P(\mathbf{r})$, $\mathbf{r} = (x, y, z)$ is on the plane if $(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n} = 0$. A point $P(\mathbf{r})$ on the line is of the form $\mathbf{r} = \mathbf{r}_1 + t\mathbf{u}$, therefore P lies on both the line and the plane if:

$$
\begin{array}{rcl}
(\mathbf{r}_1 + t\mathbf{u} - \mathbf{r}_0) \cdot \mathbf{n} & = & 0 \\
(\mathbf{r}_1 - \mathbf{r}_0) \cdot \mathbf{n} + t\mathbf{u} \cdot \mathbf{n} & = & 0 \\
t & = & -\frac{(\mathbf{r}_1 - \mathbf{r}_0) \cdot \mathbf{n}}{\mathbf{u} \cdot \mathbf{n}} \\
\mathbf{r} & = & \mathbf{r}_1 - \frac{(\mathbf{r}_1 - \mathbf{r}_0) \cdot \mathbf{n}}{\mathbf{u} \cdot \mathbf{n}} \mathbf{u} \\
 & = & (x_1, y_1, z_1) - \frac{ax_1 + by_1 + cz_1 - d}{ap + bq + cr} (p, q, r)
\end{array}
$$

Parallel planes



- Given: planes
$$\begin{array}{rcl} \mathcal{P}_1: & (\mathbf{r}-\mathbf{r}_1)\cdot\mathbf{n}_1 &=& 0 \\ \mathcal{P}_2: & (\mathbf{r}-\mathbf{r}_2)\cdot\mathbf{n}_2 &=& 0 \end{array}.$$
- Goal: Establish whether planes are parallel, find distance b-n planes.

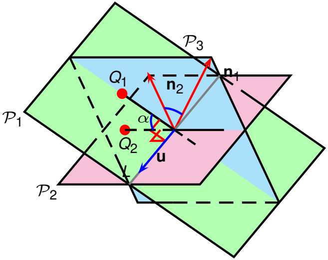
Planes are parallel $\mathcal{P}_1||\mathcal{P}_2 \Leftrightarrow \mathbf{n}_1, \mathbf{n}_2$ collinear $\Leftrightarrow \boxed{\mathbf{n}_1 \times \mathbf{n}_2 = \mathbf{0}}$.

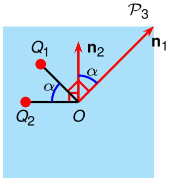
Distance: $d(\mathcal{P}_1, \mathcal{P}_2) = |\mathbf{proj}_{\mathbf{n}_1}(\mathbf{r}_2 - \mathbf{r}_1)| = \boxed{\dfrac{|(\mathbf{r}_2-\mathbf{r}_1)\cdot\mathbf{n}_1|}{|\mathbf{n}_1|}}$

Assume $\mathbf{n}_1 = \mathbf{n}_2 = (a, b, c) \Rightarrow$ plane eq-ns: $\begin{array}{l} \mathcal{P}_1: ax+by+cz = d_1 \\ \mathcal{P}_2: ax+by+cz = d_2 \end{array}.$

$$\Rightarrow \boxed{d(\mathcal{P}_1, \mathcal{P}_2) = \frac{|d_2 - d_1|}{\sqrt{a^2+b^2+c^2}}}$$

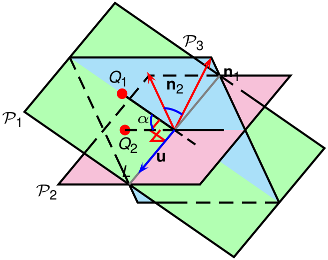
Angle between planes

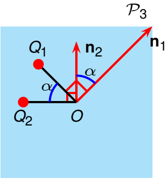




- Given: planes $\begin{aligned} \mathcal{P}_1 : \quad (\mathbf{r} - \mathbf{r}_1) \cdot \mathbf{n}_1 &= 0 \\ \mathcal{P}_2 : \quad (\mathbf{r} - \mathbf{r}_2) \cdot \mathbf{n}_2 &= 0 \end{aligned}$
- Goal: define and find the angle between the two planes.
- Let L - intersection line of two planes.
- In $\mathcal{P}_1$, drop perpendicular from arbitrary point Q_1 to L.
- In $\mathcal{P}_2$, raise a perpendicular from the perpendicular heel.
- Define angle α b-n $\mathcal{P}_1, \mathcal{P}_2$ = acute angle b-n two perpendiculars.
- Consider the plane $\mathcal{P}_3$ spanned by the two constructed perpendiculars.

Angle between planes





- Given: planes $\begin{array}{lrcl} \mathcal{P}_1: & (\mathbf{r}-\mathbf{r}_1)\cdot\mathbf{n}_1 & = & 0 \\ \mathcal{P}_2: & (\mathbf{r}-\mathbf{r}_2)\cdot\mathbf{n}_2 & = & 0 \end{array}$
- Goal: define and find the angle between the two planes.
- Consider the plane $\mathcal{P}_3$ spanned by the two constructed perpendiculars.
- $\mathcal{P}_3$ is orthogonal to L.
- $\Rightarrow \mathcal{P}_3$ contains the normal vectors $\mathbf{n}_1, \mathbf{n}_2$.
- $\mathbf{n}_1 \perp \mathbf{OQ}_1$ and $\mathbf{n}_2 \perp \mathbf{OQ}_2$.
- $\begin{array}{rcl} \alpha & = & \text{acute}\angle(\mathbf{n}_1, \mathbf{n}_2) \\ \alpha & = & \arccos\left(\frac{|\mathbf{n}_1\cdot\mathbf{n}_2|}{|\mathbf{n}_1||\mathbf{n}_2|}\right) \end{array}$
- $\perp$ planes: $\Rightarrow \alpha = \frac{\pi}{2} \iff \boxed{\mathbf{n}_1\cdot\mathbf{n}_2 = 0}$.
- Direction of L is $\mathbf{u} = \mathbf{n}_1 \times \mathbf{n}_2$.