Calculus II
Lecture 12

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Outline

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Tangents

Let C be the curve $C : \left| \begin{array}{rcl} x &=& f(t) \\ y &=& g(t) \end{array} \right., t \in [a, b]$.

Definition

Suppose $f'(t)$ and $g'(t)$ are not simultaneously equal to 0.

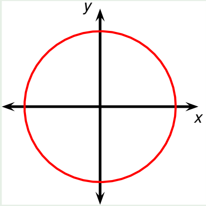
- We define $(f'(t), g'(t))$ to be the *tangent vector* to C at t.
- We define the line passing through $(f(t), g(t))$ with direction vector equal to the tangent vector to be *tangent line* to C at t. In other words, the tangent line has equation

$$(x - f(t))g'(t) = (y - g(t))f'(t) \quad .$$

- We say that the tangent to C at t is vertical if $f'(t) = 0$ (and therefore $g'(t) \neq 0$).

Note. When $f'(t) = g'(t) = 0$, for curves C with additional properties, natural definition(s) of tangent(s) do exist but are beyond Calc II.

Example



Find the tangent to the curve

$$\gamma : \left| \begin{array}{rcl} x & = & \cos t \\ y & = & \sin t \end{array} \right., t \in [0, 2\pi) \text{ at } t = \frac{\pi}{4}, t = \frac{2\pi}{3}, t = \pi.$$

Recall $C : \left| \begin{array}{rcl} x & = & x(t) \\ y & = & y(t) \end{array} \right.$, $t \in [a, b]$, tangent vector at t is $(x'(t), y'(t))$.

We write informally $x = x(t), y = y(t)$ to simplify notation.

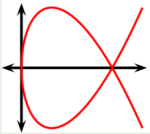
- Suppose we could eliminate the parameter t and write $y = F(x)$ for some function F near the point $(x, y) = (x(t), y(t))$.
- Suppose in $x'(t) \neq 0$ for some t.

$$\begin{array}{rcl|l} y & = & F(x) & \text{apply } \frac{\mathrm{d}}{\mathrm{d}t} \\ \frac{\mathrm{d}y}{\mathrm{d}t} & = & \frac{\mathrm{d}}{\mathrm{d}t}(F(x)) & \text{use chain rule} \\ & = & \frac{\mathrm{d}F}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} & \text{divide by } x'(t) \\ \frac{\mathrm{d}y}{\mathrm{d}x} & = & \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} & \end{array}$$

Observation

If $\frac{\mathrm{d}x}{\mathrm{d}t} \neq 0$, *we have* $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}$.

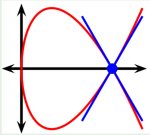
Example



A curve C is defined by $x = t^2, y = t^3 - 3t$.

1. Show C traverses $(x, y) = (3, 0)$ for two values of t; find the tangent slopes for both of these values.
2. Find the points on C where the tangents are horizontal or vertical.
3. Find two intervals where we can write y as a function of x.
4. Determine concavity intervals of the functions found in item 3.

Example



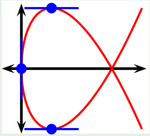
A curve C is defined by $x = t^2, y = t^3 - 3t$.

1. Show C traverses $(x, y) = (3, 0)$ for two values of t; find the tangent slopes for both of these values.

- $3 = x = t^2$ if $t = \pm\sqrt{3}$.
- $0 = y = t^3 - 3t = t(t^2 - 3)$ if $t = 0$ or $\pm\sqrt{3}$.
- Therefore the point $(3, 0)$ is traversed when t equals $\sqrt{3}$ or $-\sqrt{3}$.
- $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t} = \frac{3t^2-3}{2t}$.
- Plug in $t = \pm\sqrt{3}$: $\frac{\mathrm{d}y}{\mathrm{d}x}_{|t=\pm\sqrt{3}} = \frac{3(\pm\sqrt{3})^2 - 3}{2(\pm\sqrt{3})} = \pm\frac{6}{2\sqrt{3}} = \pm\sqrt{3}$

  Therefore the tangents at $(3, 0)$ have slopes $\pm\sqrt{3}$.

Example



A curve C is defined by $x = t^2, y = t^3 - 3t$.

2. Find the points on C where the tangents are horizontal or vertical.

Horizontal tangent:

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \\ 3t^2 - 3 &= 0 \\ 3(t^2 - 1) &= 0 \\ t &= \pm 1 \end{aligned}$$

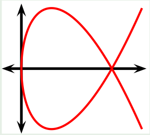
$\frac{\mathrm{d}x}{\mathrm{d}t} \neq 0$ when $t = \pm 1$, so there are horizontal tangents when $t = \pm 1$. The points are $(1, 2)$ and $(1, -2)$.

Vertical tangent:

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= 0 \\ 2t &= 0 \\ t &= 0 \end{aligned}$$

$\frac{\mathrm{d}y}{\mathrm{d}t} \neq 0$ when $t = 0$, so there is a vertical tangent when $t = 0$. The points is $(0, 0)$.

Example

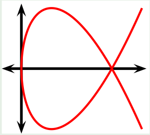


A curve C is defined by $x = t^2, y = t^3 - 3t$.

3. Find two intervals where we can write y as a function of x.

From $x = t^2$ we have that $t = \pm\sqrt{x}$. Therefore, when $t > 0$, we have that $t = \sqrt{x}$. Since that determines uniquely t via x, this means that for $t > 0$ y is a function of x. In other words, for $t > 0$, the curve satisfies the vertical line test. Similarly we conclude that when $t < 0$, y is a function of x.

Example



A curve C is defined by $x = t^2, y = t^3 - 3t$.

4. Determine the concavity intervals of the functions found in item 3.

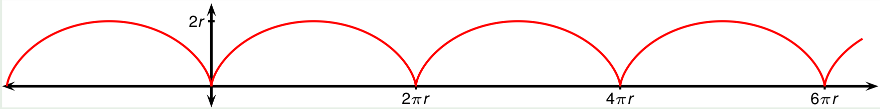
Find the second derivative:

$$\begin{aligned}
\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} &= \frac{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)}{\frac{\mathrm{d}x}{\mathrm{d}t}} = \frac{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{3t^2-3}{2t}\right)}{2t} \\
&= \frac{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{3}{2}\left(t - \frac{1}{t}\right)\right)}{2t} = \frac{\frac{3}{2} + \frac{3}{2t^2}}{2t} \\
&= \frac{\frac{3t^2+3}{2t^2}}{2t} = \frac{3(t^2+1)}{4t^3}
\end{aligned}$$

Therefore y as a function of x (which is a function of t) is concave up when $t > 0$ and concave down when $t < 0$.

Example

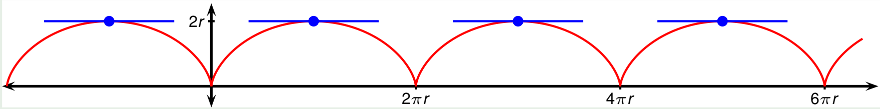
Consider the cycloid $x = r(\theta - \sin\theta)$, $y = r(1 - \cos\theta)$.



1. At what points is the tangent horizontal?
2. At what points is the tangent vertical?

Example

Consider the cycloid $x = r(\theta - \sin\theta)$, $y = r(1 - \cos\theta)$.

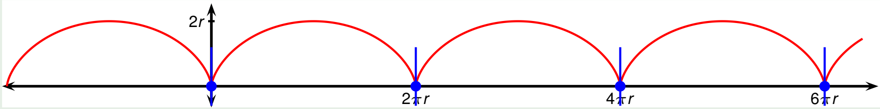


1. At what points is the tangent horizontal?

- The slope of the tangent is $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r\sin\theta}{r(1-\cos\theta)} = \frac{\sin\theta}{1-\cos\theta}$
- The tangent is horizontal when $dy/dx = 0$, that is, when $dy/d\theta = 0$ and $dx/d\theta \neq 0$.
- $r\sin\theta = dy/d\theta = 0$ if $\theta = n\pi$, where n is any integer.
- $r(1 - \cos\theta) = dx/d\theta = 0$ if $\theta = 2n\pi$, where n is any integer.
- Therefore there is a horizontal tangent when $\theta = (2n + 1)\pi$.

Example

Consider the cycloid $x = r(\theta - \sin\theta)$, $y = r(1 - \cos\theta)$.



2. At what points is the tangent vertical?

- When $\theta = 2n\pi$ both $\mathrm{d}y/\mathrm{d}\theta$ and $\mathrm{d}x/\mathrm{d}\theta$ are 0.
- To see if there is a vertical tangent, use L'Hospital's Rule.
$$\lim_{\theta\to 2n\pi^+} \frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\theta\to 2n\pi^+} \frac{\sin\theta}{1-\cos\theta} = \lim_{\theta\to 2n\pi^+} \frac{\cos\theta}{\sin\theta} \begin{matrix}\to\\ \to\end{matrix} \frac{1}{0^+}$$
- Therefore $\lim_{\theta\to 2n\pi^+}(\mathrm{d}y/\mathrm{d}x) = \infty$.
- A similar argument shows $\lim_{\theta\to 2n\pi^-}(\mathrm{d}y/\mathrm{d}x) = -\infty$.
- Therefore there is a vertical tangent when $\theta = 2n\pi$.

Tangents to Polar Curves

To find the tangent line to a polar curve $r = f(\theta)$, regard θ as a parameter and write the parametric equations as

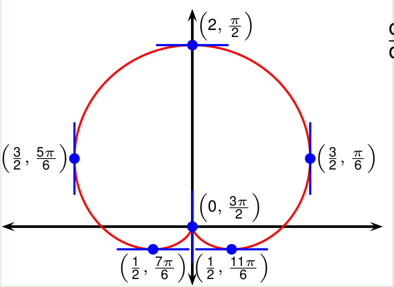
$$x = r\cos\theta = f(\theta)\cos\theta \qquad y = r\sin\theta = f(\theta)\sin\theta$$

Then use the formula for the slope of a parametric curve:

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\frac{\mathrm{d}y}{\mathrm{d}\theta}}{\frac{\mathrm{d}x}{\mathrm{d}\theta}}\\
&= \frac{\frac{\mathrm{d}}{\mathrm{d}\theta}\left(f(\theta)\sin\theta\right)}{\frac{\mathrm{d}}{\mathrm{d}\theta}\left(f(\theta)\cos\theta\right)}\\
&= \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta + f(\theta)(-\sin\theta)}\\
&= \frac{\frac{\mathrm{d}r}{\mathrm{d}\theta}\sin\theta + r\cos\theta}{\frac{\mathrm{d}r}{\mathrm{d}\theta}\cos\theta - r\sin\theta}
\end{aligned}$$

Example

Find the points on $r = 1 + \sin\theta$ where the tangent is horizontal or vertical.



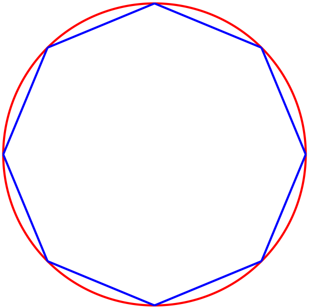
$$\begin{aligned}\frac{dy}{dx} &= \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} = \frac{\cos\theta\sin\theta + (1+\sin\theta)\cos\theta}{\cos\theta\cos\theta - (1+\sin\theta)\sin\theta}\\ &= \frac{\cos\theta(1+2\sin\theta)}{1-2\sin^2\theta - \sin\theta} = \frac{\cos\theta(1+2\sin\theta)}{(1+\sin\theta)(1-2\sin\theta)}\end{aligned}$$

- $\cos\theta(1+2\sin\theta) = 0$ when $\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$.
- $(1+\sin\theta)(1-2\sin\theta) = 0$ when $\theta = \frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$.

- Horizontal tangents at $(2, \pi/2)$, $(1/2, 7\pi/6)$, and $(1/2, 11\pi/6)$.
- Vertical tangents at $(3/2, \pi/6)$, and $(3/2, 5\pi/6)$.
- If $\theta = 3\pi/2$, top and bottom are both 0, so use L'Hospital's Rule.

$$\lim_{\theta\to 3\pi/2^-}\frac{dy}{dx} = \lim_{\theta\to 3\pi/2^-}\frac{1+2\sin\theta}{1-2\sin\theta}\cdot\lim_{\theta\to 3\pi/2^-}\frac{\cos\theta}{1+\sin\theta} = -\frac{1}{3}\lim_{\theta\to 3\pi/2^-}\frac{-\sin\theta}{\cos\theta} = \infty$$

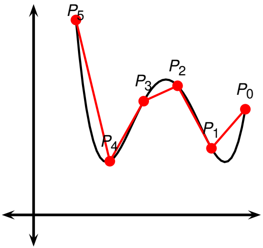
Arc Length



- What do we mean by the length of a curve?
- The length of a polygon is easy to compute: add up the length of the line segments that form the polygon.
- If the curve is a circle, approximate it by a polygon.
- Then take the limit as the number of segments of the polygon goes to ∞.

Let γ be the curve $\gamma : \left| \begin{array}{l} x = x(t) \\ y = y(t) \end{array} \right. , t \in [a, b]$

- Divide $[a, b]$ into n subintervals with endpoints $t_0, t_1, \ldots, t_n$ and equal width Δt.
- The points $P_i = (x(t_i), y(t_i))$ lie on the curve γ. The lengths of the segments with endpoints with consecutive indices from $P_0, P_1, \ldots, P_n$ approximate the length of the curve γ.
- The length L of the curve γ is the limit of the lengths of these segments as $n \to \infty$.



$$L = \lim_{n \to \infty} \sum_{i=1}^{n} |P_{i-1} P_i|$$

Let γ be the curve $\gamma : \left| \begin{array}{l} x = x(t) \\ y = y(t) \end{array} \right., t \in [a, b]$

$$\begin{aligned} L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}P_i| &= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{(x'(s_i))^2 + (y'(r_i))^2}\, \Delta t \\ &= \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, \mathrm{d}t \end{aligned}$$

- If f has continuous derivative, we can compute the above limit.
- Let $\left| \begin{array}{l} x_i = x(t_i) \\ y_i = y(t_i) \end{array} \right.$, and $\left| \begin{array}{lll} \Delta x &=& x_i - x_{i-1} = x(t_i) - x(t_{i-1}) \\ \Delta y &=& y_i - y_{i-1} = y(t_i) - y(t_{i-1}) \end{array} \right..$
- Then $|P_iP_{i-1}| = \sqrt{(\Delta x)^2 + (\Delta y)^2}$.
- Mean Value Theorem: there exist numbers s_i and r_i between t_{i-1} and t_i such that $x(t_i) - x(t_{i-1}) = x'(s_i)(t_i - t_{i-1})$ and $y(t_i) - y(t_{i-1}) = y'(r_i)(t_i - t_{i-1})$.
- $\Delta x = x'(s_i)\Delta t$, $\Delta y = y'(r_i)\Delta t$.

$$\begin{aligned} |P_{i-1}P_i| &= \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(x'(s_i)\Delta t)^2 + (y'(r_i)\Delta t)^2} \\ &= \sqrt{(x'(s_i))^2 + (y'(r_i))^2}\sqrt{(\Delta t)^2} = \sqrt{(x'(s_i))^2 + (y'(r_i))^2}\, \Delta t \end{aligned}$$

The Arc Length Formula

Let $\gamma : \left| \begin{array}{lcl} x & = & x(t) \\ y & = & y(t) \end{array} \right., t \in [a, b]$.

Definition

Suppose $x'(t)$ and $y'(t)$ (exist and) are continuous on $[a, b]$. Then the length of the curve γ is defined as

$$\begin{array}{rcl} L(\gamma) & = & \displaystyle\int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, \mathrm{d}t \\ & = & \displaystyle\int_a^b \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\, \mathrm{d}t \qquad \text{in Leibniz notation .} \end{array}$$

Arc length of graph of a function

Question

What is the length of the graph of the curve given by the graph of $y = f(x)$?

- The graph of $y = f(x)$ is written as a curve as

$$\gamma : \left| \begin{array}{ccc} x & = & t \\ y & = & f(t) \end{array} \right., t \in [a, b] \quad .$$

- In other words, the question asks what is the length $L(\gamma)$ of γ. That is a straightforward computation:

$$L(\gamma) \quad = \quad \int \sqrt{(x'(t))^2 + (y'(t))^2} \mathrm{d}t = \int \sqrt{1 + (f'(t))^2} \mathrm{d}t$$

The Arc Length Formula

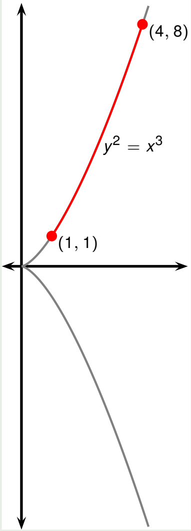
Definition

Suppose f' exists and is continuous on $[a, b]$. Then the length of the curve $y = f(x)$, $a \leq x \leq b$, is

$$\begin{aligned} L &= \int_a^b \sqrt{1 + (f'(x))^2}\, \mathrm{d}x \\ &= \int_a^b \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\, \mathrm{d}x \qquad \text{(in Leibniz notation)} \quad . \end{aligned}$$

Example

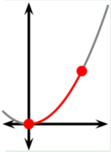
Find the length of the arc of $y^2 = x^3$ between $(1,1)$ and $(4,8)$.



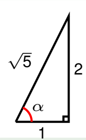
- For the top half of the curve we have:
- $y = x^{3/2}$ and $y' = \frac{3}{2}x^{1/2}$.
- $u = 1 + \frac{9}{4}x$ and $\mathrm{d}u = \frac{9}{4}\mathrm{d}x$.
- When $x = 1$, $u = \frac{13}{4}$.
- When $x = 4$, $u = 10$.

$$\begin{aligned} L &= \int_1^4 \sqrt{1+(y')^2}\mathrm{d}x \\ &= \int_1^4 \sqrt{1+\frac{9}{4}x}\,\mathrm{d}x = \int_{13/4}^{10} \frac{4}{9}\sqrt{u}\,\mathrm{d}u \\ &= \frac{4}{9}\left[\frac{2}{3}u^{3/2}\right]_{13/4}^{10} = \frac{8}{27}\left(10^{3/2} - \left(\frac{13}{4}\right)^{3/2}\right) \end{aligned}$$

Example



Find the length of the arc of the parabola $y = x^2$ from $(0,0)$ to $(1,1)$.



$$\frac{dy}{dx} = 2x$$

$$L = \int_{x=0}^{x=1} \sqrt{1+\left(\frac{dy}{dx}\right)^2} dx = \int_{x=0}^{x=1} \sqrt{1+4x^2}\, dx \quad \Big| \text{ Set } x = \tfrac{1}{2}\tan\theta$$

$$= \int_{\theta=0}^{\theta=\arctan 2} \sqrt{1+\tan^2\theta}\ d\left(\frac{1}{2}\tan\theta\right)$$

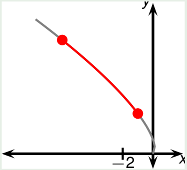
$$= \int_{\theta=0}^{\theta=\arctan 2} \sec\theta \cdot \frac{1}{2}\sec^2\theta d\theta = \frac{1}{2}\int_0^{\alpha} \sec^3\theta\, d\theta \quad \Big| \text{ Set } \alpha = \arctan 2$$

$$= \frac{1}{2}\cdot\left[\frac{1}{2}\left(\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|\right)\right]_{\theta=0}^{\theta=\alpha}$$

$$= \frac{1}{4}\left(\sec\alpha\tan\alpha + \ln|\sec\alpha+\tan\alpha|\right)$$

$$= \frac{1}{4}\left(2\cdot\sqrt{5} + \ln|\sqrt{5}+2|\right)$$

Example



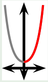
Find the length of the curve γ.

$$\gamma : \left| \begin{array}{rcl} x(t) & = & \sqrt{t} - 2t \\ y(t) & = & \frac{8}{3} t^{\frac{3}{4}} \end{array} \right. , t \in [1,4] \quad .$$

We have that $x'(t) = \frac{1}{2\sqrt{t}} - 2$ and $y'(t) = \frac{8}{3} \cdot \frac{3}{4} t^{-\frac{1}{4}} = 2t^{-\frac{1}{4}}$.

$$\begin{aligned} L(\gamma) &= \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2} \mathrm{d}t = \int_1^4 \sqrt{\left(\frac{1}{2\sqrt{t}} - 2\right)^2 + \left(2t^{-\frac{1}{4}}\right)^2} \mathrm{d}t \\ &= \int_1^4 \sqrt{\frac{1}{4t} - \frac{2}{\sqrt{t}} + 4 + \frac{4}{\sqrt{t}}} \; \mathrm{d}t \\ &= \int_1^4 \sqrt{\frac{1}{4t} + \frac{2}{\sqrt{t}} + 4} \; \mathrm{d}t = \int_1^4 \sqrt{\left(\frac{1}{2\sqrt{t}} + 2\right)^2} \mathrm{d}t \\ &= \int_1^4 \left(\frac{1}{2\sqrt{t}} + 2\right) \mathrm{d}t = \left[\sqrt{t} + 2t\right]_1^4 = \sqrt{4} + 2 \cdot 4 - \left(\sqrt{1} + 2 \cdot 1\right) = 7. \end{aligned}$$

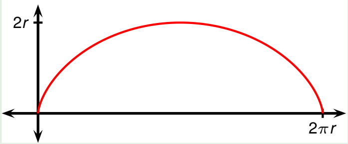
Example ($(a+b)^2$, $(a-b)^2$, $2ab = 1/2$)



Find the length of the arc of $y = \frac{1}{6}e^{3x} + \frac{1}{6}e^{-3x}$ from $x = 0$ to $x = 1$.

$$
\begin{aligned}
y' &= \frac{1}{2}e^{3x} - \frac{1}{2}e^{-3x}.\\
(y')^2 &= \frac{1}{4}e^{6x} - \frac{1}{4}e^{3x}e^{-3x} - \frac{1}{4}e^{3x}e^{-3x} + \frac{1}{4}e^{-6x}\\
&= \frac{1}{4}e^{6x} - \frac{1}{2} + \frac{1}{4}e^{-6x}.\\
L &= \int_0^1 \sqrt{1+(y')^2}\mathrm{d}x = \int_0^1 \sqrt{1 + \frac{1}{4}e^{6x} - \frac{1}{2} + \frac{1}{4}e^{-6x}}\mathrm{d}x\\
&= \int_0^1 \sqrt{\frac{1}{4}e^{6x} + \frac{1}{2} + \frac{1}{4}e^{-6x}}\mathrm{d}x = \int_0^1 \sqrt{\left(\frac{1}{2}e^{3x} + \frac{1}{2}e^{-3x}\right)^2}\mathrm{d}x\\
&= \int_0^1 \left(\frac{1}{2}e^{3x} + \frac{1}{2}e^{-3x}\right)\mathrm{d}x = \left[\frac{1}{6}e^{3x} - \frac{1}{6}e^{-3x}\right]_0^1 = \frac{e^3 - e^{-3}}{6}.
\end{aligned}
$$

Example



Find the length of one arch of the cycloid

$$x = r(\theta - \sin\theta), \quad y = r(1 - \cos\theta).$$

The first arch is $0 \leq \theta \leq 2\pi$.

$$L = \int_0^{2\pi} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2}\mathrm{d}\theta = \int_0^{2\pi} \sqrt{(r(1-\cos\theta))^2 + (r\sin\theta)^2}\mathrm{d}\theta$$

$$= \int_0^{2\pi} \sqrt{r^2(1 - 2\cos\theta + \cos^2\theta + \sin^2\theta)}\mathrm{d}\theta = r\int_0^{2\pi} \sqrt{2(1-\cos\theta)}\mathrm{d}\theta$$

Use the identity $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$. Then

$$\sqrt{2(1-\cos\theta)} = \sqrt{4\sin^2(\theta/2)} = 2|\sin(\theta/2)| = 2\sin(\theta/2)$$

$$L = r\int_0^{2\pi} 2\sin(\theta/2)\mathrm{d}\theta = r\left[-4\cos(\theta/2)\right]_0^{2\pi} = 8r$$

Arc Length

To find the arc length of a polar curve $r = f(\theta)$, $a \leq \theta \leq b$, regard θ as a parameter. Then the derivatives of the parametric equations are

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta \qquad \frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta$$

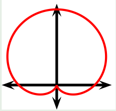
and

$$\begin{aligned}\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 &= \left(\frac{dr}{d\theta}\right)^2\cos^2\theta - 2r\frac{dr}{d\theta}\cos\theta\sin\theta + r^2\sin^2\theta \\ &\quad + \left(\frac{dr}{d\theta}\right)^2\sin^2\theta + 2r\frac{dr}{d\theta}\sin\theta\cos\theta + r^2\cos^2\theta \\ &= \left(\frac{dr}{d\theta}\right)^2 + r^2\end{aligned}$$

The arc length is

$$L = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Example



Find the length of the cardioid $r = 1 + \sin\theta$. The full length is given by $0 \leq \theta \leq 2\pi$.

$$
\begin{aligned}
L &= \int_0^{2\pi} \sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\mathrm{d}\theta = \int_0^{2\pi} \sqrt{(1+\sin\theta)^2 + \cos^2\theta}\mathrm{d}\theta \\
&= \int_0^{2\pi} \sqrt{2 + 2\sin\theta}\frac{\sqrt{2-2\sin\theta}}{\sqrt{2-2\sin\theta}}\mathrm{d}\theta = \int_0^{2\pi} \frac{\sqrt{4-4\sin^2\theta}}{\sqrt{2-2\sin\theta}}\mathrm{d}\theta \\
&= \int_0^{2\pi} \frac{\sqrt{4\cos^2\theta}}{\sqrt{2-2\sin\theta}}\mathrm{d}\theta = \int_0^{2\pi} \frac{2|\cos\theta|}{\sqrt{2-2\sin\theta}}\mathrm{d}\theta \\
&= \int_0^{\pi/2} \frac{2\cos\theta}{\sqrt{2-2\sin\theta}}\mathrm{d}\theta + \int_{\pi/2}^{3\pi/2} \frac{-2\cos\theta}{\sqrt{2-2\sin\theta}}\mathrm{d}\theta + \int_{3\pi/2}^{2\pi} \frac{2\cos\theta}{\sqrt{2-2\sin\theta}}\mathrm{d}\theta \\
&= \left[-2\sqrt{2-2\sin\theta}\right]_0^{\pi/2} + \left[2\sqrt{2-2\sin\theta}\right]_{\pi/2}^{3\pi/2} + \left[-2\sqrt{2-2\sin\theta}\right]_{3\pi/2}^{2\pi} \\
&= -2\left(0-\sqrt{2}\right) + 2\left(2-0\right) - 2\left(\sqrt{2}-2\right) = 8
\end{aligned}
$$