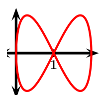
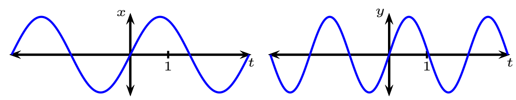
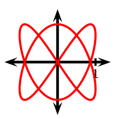
Calculus II
Homework on Lecture 11

1. Match the graphs of the parametric equations $x = f(t)$, $y = g(t)$ with the graph of the parametric curve $\gamma : \left| \begin{array}{lcl} x & = & f(t) \\ y & = & g(t) \end{array} \right.$

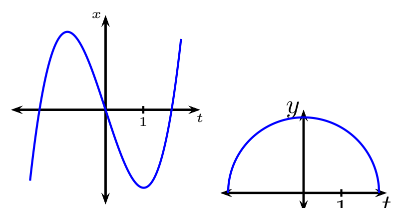
(a) 

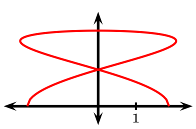
answer: matches to 1c

(a) 

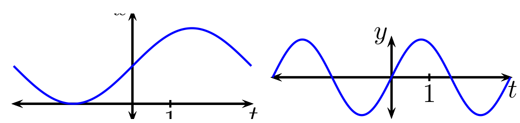
(b) 

answer: matches to 1a

(b) 

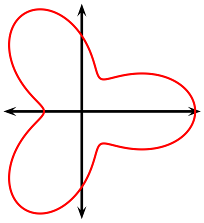
(c) 

answer: matches to 1b

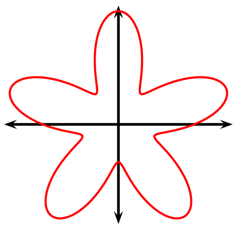
(c) 

2.

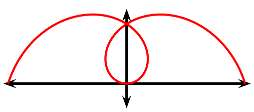
Match the graph of the curve to its graph in polar coordinates and to its polar parametric equations.

(a) 

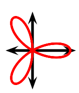
answer: matches 2b, 2vi

(b) 

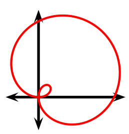
answer: matches 2c, 2v

(c) 

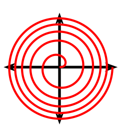
answer: matches 2e, 2ii

(d) 

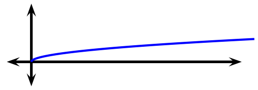
answer: matches 2f, 2iii

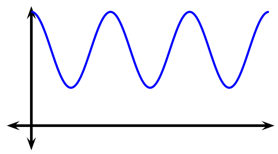
(e) 

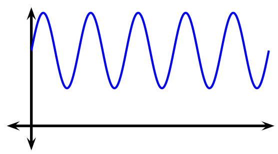
answer: matches 2d, 2i

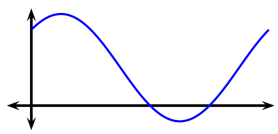
(f) 

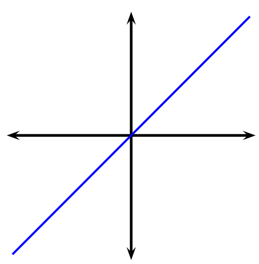
answer: matches 2a, 2iv

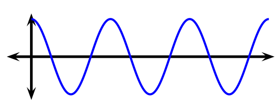
(a) 

(b) 

(c) 

(d) 

(e) 

(f) 

(i) $r = 1 + \sin(\theta) + cos(\theta)$

(ii) $r = \theta, \theta \in [-\pi, \pi]$.

(iii) $r = \cos(3\theta), \theta \in [0, 2\pi]$.

(iv) $r = \frac{1}{4}\sqrt{\theta}, \theta \in [0, 10\pi]$.

(v) $r = 2 + \sin(5\theta)$.

(vi) $r = 2 + cos(3\theta)$.

3.

(a) Sketch the curve given in polar coordinates by $r = 2\sin\theta$. What kind of a figure is this curve? Find an equation satisfied by the curve in the (x, y)-coordinates.

(b) Sketch the curve given in polar coordinates by $r = 4\cos\theta$. What kind of a figure is this curve? Find an equation satisfied by the curve in the (x, y)-coordinates.

(c) Sketch the curve given in polar coordinates by $r = 2\sec\theta$. What kind of a figure is this curve? Find an equation satisfied by the curve in the (x, y)-coordinates.

answer: the curve is the line $x = 2$

(d) Sketch the curve given in polar coordinates by $r = 2\csc\theta$. What kind of a figure is this curve? Find an equation satisfied by the curve in the (x, y)-coordinates.

(e) Sketch the curve given in polar coordinates by $r = 2\sec\left(\theta + \frac{\pi}{4}\right)$. What kind of a figure is this curve? Find an equation satisfied by the curve in the (x, y)-coordinates.

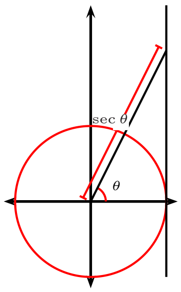
answer: the curve is the line $y = x - 2\sqrt{2}$

(f) Sketch the curve given in polar coordinates by $r = 2\csc\left(\theta + \frac{\pi}{6}\right)$. What kind of a figure is this curve? Find an equation satisfied by the curve in the (x, y)-coordinates.

Solution. 3.c. Recall from trigonometry that if we draw a unit circle as shown below, $\sec\theta$ is given by the signed distance as indicated on the figure. Therefore it is clear that the curve given in polar coordinates by $y = \sec\theta$ is the vertical line passing through $x = 1$. Analogous considerations can be made for a circle of radius 2, from where it follows that $y = 2\sec\theta$ is the vertical line passing through $x = 2$.

Alternatively, we can find an equation in the (x, y)-coordinates of the cuve by the direct computation:

$$x = r\cos\theta = 2\sec\theta\cos\theta = 2 \quad .$$



Solution. 3.e.

Approach I. Adding an angle α to the angle polar coordinate of a point corresponds to rotating that point counterclockwise at an angle α about the origin. Therefore a point P with polar coordinates $P\left(2\sec\left(\theta+\frac{\pi}{4}\right), \theta\right)$ is obtained by rotating at an angle $-\frac{\pi}{4}$ the point Q with polar coordinates $Q\left(2\sec\left(\theta+\frac{\pi}{4}\right), \theta+\frac{\pi}{4}\right)$. The point P lies on the curve with equation $r = 2\sec\left(\theta+\frac{\pi}{4}\right)$ and the point Q lies on the curve with equation $r = 2\sec\theta$ - the latter curve is the curve from problem 3.c. Thus the curve in the current problem is obtained by rotating the curve from 3.c at an angle of $-\frac{\pi}{4}$. As the curve in Problem 3.c is the vertical line $x = 2$, the curve in the present problem is also a line. Rotation at an angle of $-\frac{\pi}{4}$ of a vertical line yields a line with slope 1. When $\theta = 0$, $x = \frac{2}{\frac{\sqrt{2}}{2}} = 2\sqrt{2}$, $y = 0$ and the curve passes through $(2\sqrt{2}, 0)$. We know the slope of a line and a point through which it passes; therefore the (x, y)-coordinates of our curve satisfy

$$y = x - 2\sqrt{2} \quad .$$

Approach II. We compute

$$\begin{array}{rcl|l}
x &=& r\cos\theta = \dfrac{2\cos\theta}{\cos(\theta+\frac{\pi}{4})} & \text{multiply by } \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \\
y &=& r\sin\theta = \dfrac{2\sin\theta}{\cos(\theta+\frac{\pi}{4})} & \text{multiply by } -\sin\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2} \\
\hline
 & & & \text{add the above} \\
x\cos\left(\frac{\pi}{4}\right) - y\sin\left(\frac{\pi}{4}\right) &=& 2\dfrac{\cos\theta\cos\left(\frac{\pi}{4}\right) - \sin\theta\sin\left(\frac{\pi}{4}\right)}{\cos\left(\theta+\frac{\pi}{4}\right)} & \text{use } \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \\
\frac{\sqrt{2}}{2}(x-y) &=& 2\dfrac{\cos\left(\theta+\frac{\pi}{4}\right)}{\cos\left(\theta+\frac{\pi}{4}\right)} = 2 & \\
y &=& x - 2\sqrt{2}, &
\end{array}$$

and therefore our curve is the line given by the equation above.