Calculus III
Lecture 19

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Outline

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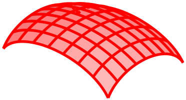
Surface Integral Motivation



- Let S be a surface in space and let $\mathrm{d}S$ denote the element of surface area.

- If ρ is the density of the surface, then $\mathrm{d}m = \rho \mathrm{d}S$ is the element of mass, and the total mass is
$$M = \iint_S \mathrm{d}m = \iint_S \rho \mathrm{d}S \, .$$
- If $\mathbf{p}$ is the pressure function - the density of force with respect to surface area - then the element of force is $\mathrm{d}\mathbf{F} = \mathrm{d}\mathbf{p}\mathrm{d}S$. The total force exerted by pressure on the surface is then
$$\mathbf{F} = \iint_S \mathrm{d}\mathbf{F} = \iint_S \mathbf{p}\mathrm{d}S.$$
- How do we compute surface integrals?

Theorem

Let $\mathbf{u}, \mathbf{v}$ *be two 3-dimensional vectors. Then* $|\mathbf{u}\times\mathbf{v}| = \sqrt{\left|\begin{array}{cc} \mathbf{u}\cdot\mathbf{u} & \mathbf{u}\cdot\mathbf{v} \\ \mathbf{v}\cdot\mathbf{u} & \mathbf{v}\cdot\mathbf{v}\end{array}\right|}$.

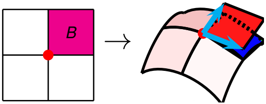
Proof.

$$
\begin{array}{rcl}
|u\times v|^2 &=& \left|\left(\left|\begin{array}{cc} u_2 & u_3\\ v_2 & v_3\end{array}\right|, -\left|\begin{array}{cc} u_1 & u_3\\ v_1 & v_3\end{array}\right|, \left|\begin{array}{cc} u_1 & u_2\\ v_1 & v_2\end{array}\right|\right)\right|^2\\
&=& (u_2v_3-u_3v_2)^2+(u_1v_3-u_3v_1)^2+(u_1v_2-u_2v_1)^2\\
&=& -2u_2u_3v_2v_3-2u_1u_3v_1v_3-2u_1u_2v_1v_2\\
&& +u_3^2v_2^2+u_3^2v_1^2+u_2^2v_3^2+u_2^2v_1^2+u_1^2v_3^2+u_1^2v_2^2
\end{array}
$$

$$
\begin{array}{rcl}
\left|\begin{array}{cc} \mathbf{u}\cdot\mathbf{u} & \mathbf{u}\cdot\mathbf{v}\\ \mathbf{u}\cdot\mathbf{u} & \mathbf{v}\cdot\mathbf{v}\end{array}\right| &=& \left|\begin{array}{cc} u_1^2+u_2^2+u_3^2 & u_1v_1+u_2v_2+u_3v_3\\ u_1v_1+u_2v_2+u_3v_3 & v_1^2+v_2^2+v_3^2\end{array}\right|\\
&=& -2u_2u_3v_2v_3-2u_1u_3v_1v_3-2u_1u_2v_1v_2\\
&& +u_3^2v_2^2+u_3^2v_1^2+u_2^2v_3^2+u_2^2v_1^2+u_1^2v_3^2+u_1^2v_2^2
\end{array}
$$

$\square$

Surface Area



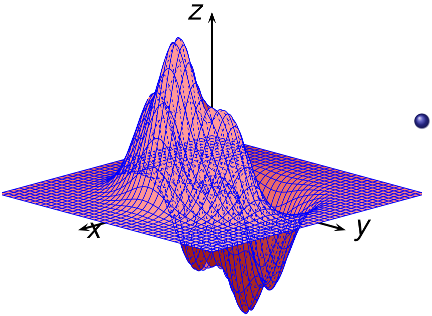
- Let $\mathbf{f}: D \to \mathbb{R}^3$ be (local) surface parametrization.
- Let (u, v) be a point in the parameter space and $P = \mathbf{f}(u, v)$.
- Let $B = [u, u + \Delta u] \times [v, v + \Delta v]$ be a small rectangle.
- Let $C = \mathbf{f}(B)$ be corresp. curvilinear patch ("2D-box") on surface.
- C is approximated by the parallelotope E at $\mathbf{f}(u, v)$ with vertices at $\mathbf{f}(u + \Delta u, v)$ and $\mathbf{f}(u, v + \Delta v)$.
- E - approx. by parallelotope J at $\mathbf{f}(u, v)$ spanned by $\frac{\partial \mathbf{f}}{\partial u}\Delta u, \frac{\partial \mathbf{f}}{\partial v}\Delta v$.

$$
\begin{aligned}
\mathrm{area}(C) \approx \mathrm{Vol}_2(J) &= |\mathbf{f}_u \times \mathbf{f}_v| \Delta u \Delta v \\
\mathrm{d}S &= |\mathbf{f}_u \times \mathbf{f}_v| \mathrm{d}u \mathrm{d}v
\end{aligned}
$$

$$
\mathrm{d}S = |\mathbf{f}_u(u, v) \times \mathbf{f}_v(u, v)| \mathrm{d}u \mathrm{d}v
$$

Definition

Surface area of graph surface



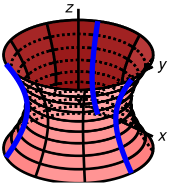
- Suppose $\mathbf{f}\colon D \to \mathbb{R}^3$ is a graph surface, i.e., is of the form $\mathbf{f}(u, v) = (u, v, g(u, v))$ for some scalar function $g(u, v)$.

$$\begin{aligned}
\mathbf{f}_u &= (1, 0, g_u)\\
\mathbf{f}_v &= (0, 1, g_v)\\
\mathrm{d}S &= |\mathbf{f}_u \times \mathbf{f}_v|\mathrm{d}u\mathrm{d}v = \sqrt{1 + (g_u)^2 + (g_v)^2}\mathrm{d}u\mathrm{d}v
\end{aligned}$$

- Surface area of the graph surface:

$$\mathrm{Area}(S) = \iint_B \sqrt{1 + (f_u)^2 + (f_v)^2}\mathrm{d}u\mathrm{d}v$$

Surface Area of Surface of Revolution



- Let C be parametrized curve in the $x > 0$ half plane of the $xz-$plane, parametrized by $x = h(u)$, $z = g(u)$, $a \leq u \leq b$.
- Let S be the surface obtained by revolving C about the $z-$axis.

Parametrize S: $\left| \begin{array}{lcl} x &=& h(v)\cos u \\ y &=& h(v)\sin u \\ z &=& g(v) \end{array} \right.$ $, u \in [0, 2\pi], v \in [a, b]$

$$\begin{aligned} \mathrm{d}S &= |\mathbf{f}_u \times \mathbf{f}_v| \mathrm{d}u\mathrm{d}v = h(v)\sqrt{(h'(v)^2 + (g'(v))^2}\mathrm{d}u\mathrm{d}v \\ \mathrm{Area}(S) &= \iint\limits_{[a,b]\times[0,2\pi]} h(v)\sqrt{(h'(v))^2 + (g'(v))^2}\mathrm{d}u\mathrm{d}v \\ &= 2\pi \int_{[a,b]} h(v)\sqrt{(h'(v)^2 + (g'(v))^2}\mathrm{d}v = 2\pi \int_C x\mathrm{d}s \end{aligned}$$

Pappus' First Centroid Theorem

$$\mathrm{Area}(S) = L(C) 2\pi \frac{1}{L(C)} \int_C x \mathrm{d}s$$

Theorem (Pappus' First Centroid Theorem)

The area of a surface of revolution is the product between the distance traveled by the centroid of the curve and the length of the revolved curve.

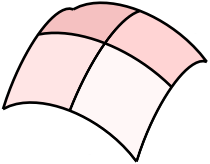
Example

Use Pappus' theorem to find the surfacea area of a torus of major radius R and minor radius r.
For a torus
the length of the revolved circle is $2\pi r$;
the centroid is at $(R, 0)$;
hence the surface area of a torus is $4\pi^2 Rr$.

Example

Use Pappus' theorem to find the surfacea area of a sphere of radius R.
C: semicircle of radius R, rotated about axis joining endpoints
resulting surface is a sphere of radius R and area $4\pi R^2$.
length of C is πR;
the centroid travels a distance of $4R$;
the centroid is at a distance of $\frac{2R}{\pi}$ from the axis.

Surface Integral Definition



- Let S be a surface in space.
- Let $\mathbf{f}\colon D \to S \subset \mathbb{R}^3$ be global parametrization of S.
- Let h be a continuous (scalar or vector-valued) function in 3 variables that is defined on S.

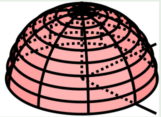
Definition

$$\iint\limits_{S=\mathbf{f}(D)} h\,\mathrm{d}S = \iint\limits_{D} h(\mathbf{f}(u,v))\,|\mathbf{f}_u \times \mathbf{f}_v| \mathrm{d}u\mathrm{d}v = \iint\limits_{D} h(\mathbf{f}(u,v)) \sqrt{\left|\begin{array}{cc} \mathbf{f}_u \cdot \mathbf{f}_u & \mathbf{f}_u \cdot \mathbf{f}_v \\ \mathbf{f}_v \cdot \mathbf{f}_u & \mathbf{f}_v \cdot \mathbf{f}_v \end{array}\right|}\, \mathrm{d}u\mathrm{d}v.$$

We extend the definition to surfaces S that don't necessarily have a global parametrization as follows.

- Suppose S can be split into finitely many pieces $S_1, \ldots S_N$ with non-overlapping interiors such that each piece S_k has a global parametrization.
- We compute the surface integral over each S_i and sum.

Example



Find the centroid of a hemisphere S of radius R.
By definition, the centroid of surface is $\frac{1}{\text{Area(S)}} \iint_S \mathbf{f} dS$, where $\mathbf{f}(u, v)$ is the position vector of S.

By symmetry, the centroid is on the z-axis; let its z-coordinate be h.

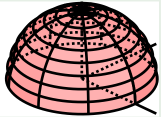
$$\text{Parametrize } S: \left| \begin{array}{l} \mathbf{f}(x, y) = \left(x, y, \sqrt{R^2 - x^2 - y^2}\right) \\ (x, y) \in D = \text{disk radius } R \end{array} \right. .$$

$$dS = |\mathbf{f}_x \times \mathbf{f}_y| dxdy = \sqrt{1 + z_x^2 + z_y^2} dxdy = \frac{R}{\sqrt{R^2 - x^2 - y^2}} dxdy$$

$$h = \frac{1}{\text{area}(S)} \iint_S z \, dS = \frac{1}{2\pi R^2} \iint_D \frac{R\sqrt{R^2 - x^2 - y^2}}{\sqrt{R^2 - x^2 - y^2}} dxdy$$

$$= \frac{1}{2\pi R} \iint_D dxdy = \frac{\pi R^2}{2\pi R} = \frac{R}{2} \quad .$$

Example



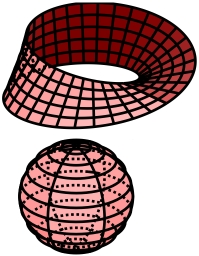
Find the centroid of a hemisphere S of radius R. By definition, the centroid of surface is $\frac{1}{\text{Area(S)}} \iint_S \mathbf{f} dS$, where $\mathbf{f}(u, v)$ is the position vector of S.

The hemisphere can be also obtained by revolving the quarter of circle $(x, z) = (f(u), g(u)) = (R \cos u, R \sin u)$, $0 \leq u \leq \frac{\pi}{2}$ about the $z-$axis.

$$S : (R \cos u \cos v, R \cos u \sin v, R \sin u), u \in \left[0, \frac{\pi}{2}\right], v \in [0, 2\pi].$$

$$\begin{aligned}
dS &= |f|\sqrt{|f'|^2 + |g'|^2} = R^2 \cos u \\
\iint_S z dS &= \iint_D R \sin u R^2 \cos u du dv \\
&= R^3 \left(\int_{u=0}^{u=\frac{\pi}{2}} \sin u \cos u \, du \right) \left(\int_{v=0}^{v=2\pi} dv \right) \\
&= 2\pi R^3 \left. \frac{\sin^2 u}{2} \right|_{u=0}^{u=\frac{\pi}{2}} = \pi R^3 . \\
\text{centroid } z - \text{coord} &= \frac{1}{\text{area}(S)} \iint_S z \, dS = \frac{1}{2\pi R^2} \pi R^3 = \frac{R}{2}
\end{aligned}$$

Orientations of Surfaces



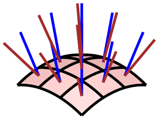
- Let S be a smooth surface, not necessarily a boundary of an open 3D set.
- To orient a surface means to make a consistent choice of normal direction on S, i.e., select a continuous unit vector field $\mathbf{N}$ normal to S.
- Such a normal field doesn't always exist! (Möebius band).

Definition

- S is *orientable* if it has a continuous normal unit vector field.
- Each choice of such a normal field endows S with an *orientation*.
- An *oriented surface* is a surface with a predetermined orientation.

- If the surface S bounds a domain in space:
  - outward normal gives the *positive* orientation;
  - inward normal gives the *negative* orientation.

Flux



Let S be an oriented surface with orientation given by the unit normal field $\mathbf{N}$. Let $\mathbf{F}$ be a smooth vector field on S.

Definition

The flux of $\mathbf{F}$ across S is

$$\iint_S \mathbf{F} \cdot \mathbf{N} \mathrm{d}S = \iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S}.$$

Divergence

Definition (May be taken as theorem)

The divergence of $\mathbf{X}$ at p is the density of flux at p

$$(\operatorname{div}\mathbf{X})(p) = \lim_{D\to\{p\}} \frac{1}{\operatorname{vol}(D)} \iint_S \mathbf{X}\cdot\mathbf{N}\mathrm{d}S\ ,$$

if the limit exists.

Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ be a smooth vector field.

Theorem (May be taken as definition of div)

The divergence of $\mathbf{F}$ *is defined as* $\operatorname{div}\mathbf{F} = \frac{\mathrm{d}P}{\mathrm{d}x} + \frac{\mathrm{d}Q}{\mathrm{d}y} + \frac{\mathrm{d}R}{\mathrm{d}y}$

- Recall that $\nabla g = \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\right)$.
- We can write $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$, $\mathbf{F} = (P, Q, R)$.
- We can write the formal equality $\nabla\cdot\mathbf{F} = \operatorname{div}\mathbf{F}$.

Flux and Divergence

- Let S be an oriented surface with orientation given by the unit normal field $\mathbf{N}$. Let $\mathbf{X}$ be a smooth vector field on S.
- Recall that the flux of $\mathbf{X}$ across S is given by

$$\iint_S \mathbf{X} \cdot \mathbf{N} \mathrm{d}S = \iint_S \mathbf{X} \cdot \mathrm{d}\mathbf{S}.$$

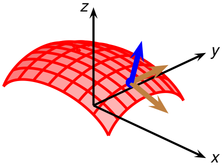
Definition (May be theorem if using alternative definition)

The divergence of $\mathbf{X}$ at p is the density of flux at p

$$(\operatorname{div} \mathbf{X})(p) = \lim_{D \to \{p\}} \frac{1}{\operatorname{vol}(D)} \iint_S \mathbf{X} \cdot \mathbf{N} \mathrm{d}S \ ,$$

if the limit exists.

Computations Using Parametrizations

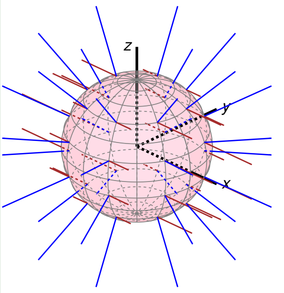


- Let S be orientable with normal field $\mathbf{N}$.
- Let $\mathbf{f}\colon D \to S$ be a smooth parametrization of S.
- $\mathbf{f}_u$ and $\mathbf{f}_v$ are tangent vectors.
- $\mathbf{f}_u \times \mathbf{f}_v$ is a normal vector.

- We say the parametrization $\mathbf{f}$ is *compatible* with the orientation given by $\mathbf{N}$ if $\mathbf{f}_u \times \mathbf{f}_v$ and $\mathbf{N}$ point in the same direction.
- Equivalently: the frame $(\mathbf{f}_u, \mathbf{f}_v, \mathbf{N})$ is positively oriented.
- If $\mathbf{f}$ is a parametrization compatible with the orientation, then

$$\begin{aligned}
\mathbf{N} &= \frac{\mathbf{f}_u\times\mathbf{f}_v}{|\mathbf{f}_u\times\mathbf{f}_v|}\\
\mathrm{d}\mathbf{S} &= \mathbf{N}\mathrm{d}S = \frac{\mathbf{f}_u\times\mathbf{f}_v}{|\mathbf{f}_u\times\mathbf{f}_v|}\cdot|\mathbf{f}_u\times\mathbf{f}_v|\mathrm{d}u\mathrm{d}v = \mathbf{f}_u\times\mathbf{f}_v\mathrm{d}u\mathrm{d}v\\
\iint_S \mathbf{X}\cdot\mathbf{N}\mathrm{d}S &= \iint_S \mathbf{X}\cdot\mathrm{d}\mathbf{S} = \iint_D \mathbf{X}\cdot(\mathbf{f}_u\times\mathbf{f}_v)\mathrm{d}u\mathrm{d}v\ .
\end{aligned}$$

Example



Compute the flux of $\mathbf{X} = ax\mathbf{i}$ across the sphere S of radius R centered at the origin, positively oriented.

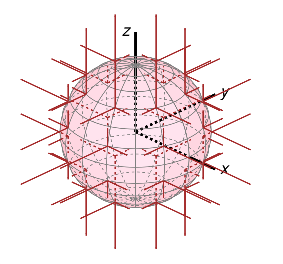
Parametrize S:

$$\begin{aligned} \mathbf{f} &= (R\sin u\cos v, R\sin u\sin v, R\cos u) \\ \mathbf{f}_u &= (R\cos u\cos v, R\cos u\sin v, -R\sin u)\ , \\ \mathbf{f}_v &= (-R\sin u\sin v, R\sin u\cos v, 0) \end{aligned}$$

$u \in [0,\pi]$, $v \in [0, 2\pi]$.

$$\begin{aligned} \mathbf{f}_u \times \mathbf{f}_v &= \left(R^2\sin^2 u\cos v, R^2\sin^2 u\sin v, R^2\sin u\cos u\right) \\ &= R\sin u\mathbf{f} = R^2\sin u\,\mathbf{N} \\ \iint_S \mathbf{X}\cdot \mathrm{d}\mathbf{S} &= \iint_D \mathbf{X}\cdot(\mathbf{f}_u\times\mathbf{f}_v)\,\mathrm{d}u\,\mathrm{d}v \\ &= \int_{u=0}^{u=\pi}\int_{v=0}^{v=2\pi} aR\sin u\cos v\, R^2\sin^2 u\cos v\,\mathrm{d}u\,\mathrm{d}v \\ &= aR^3\left(\int_{u=0}^{u=\pi}\sin^3 u\,\mathrm{d}u\right)\left(\int_{v=0}^{v=2\pi}\cos^2 v\,\mathrm{d}v\right) \\ &= aR^3\cdot\frac{4}{3}\cdot\pi = \frac{4\pi aR^3}{3}\ . \end{aligned}$$

Example



Compute the fluxes of $\mathbf{X} = ax\mathbf{i}$, $\mathbf{Y} = by\mathbf{j}$, $\mathbf{Z} = cz\mathbf{k}$ across the sphere S of radius R centered at the origin, positively oriented.

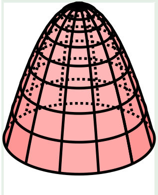
Compute the flux of $\mathbf{X} = ax\,\mathbf{i} + by\,\mathbf{j} + cz\,\mathbf{k}$.

$\iint_S ax\mathbf{i} \cdot d\mathbf{S} = \frac{4\pi a R^3}{3}$, $\iint_S by\mathbf{j} \cdot d\mathbf{S} = \frac{4\pi b R^3}{3}$, $\iint_S cz\mathbf{k} \cdot d\mathbf{S} = \frac{4\pi c R^3}{3}$

$\iint_S \mathbf{X} \cdot d\mathbf{S} = \frac{4\pi R^3}{3}(a+b+c) = (a+b+c)\,\mathrm{Vol}(Ball_R)$.

$$\operatorname{div}\mathbf{X}(0) = \lim_{R\to 0} \frac{3}{4\pi R^3} \iint_{S_R(0)} \mathbf{X} \cdot d\mathbf{S} = \lim_{R\to 0}(a+b+c) = a+b+c\,.$$

Example



Let S be the part of the paraboloid $z = 4 - x^2 - y^2$ above the $xy-$plane, oriented upward, and $\mathbf{X} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$. Compute $\iint_S \mathbf{X} \cdot \mathrm{d}\mathbf{S}$.

Parametrization: $\mathbf{f}\colon B \to \mathbb{R}^3$, $\mathbf{f}(u, v) = (u, v, 4 - u^2 - v^2)$.

$$
\begin{aligned}
\mathbf{f}_u \times \mathbf{f}_v &= (1, 0, -2u) \times (0, 1, -2v) = \left| \begin{array}{ccc} i & j & k \\ 0 & 1 & -2u \\ 1 & 0 & -2v \end{array} \right| \\
&= 2u\,\mathbf{i} + 2v\,\mathbf{j} + \mathbf{k} \\
\mathbf{N} &= \tfrac{\mathbf{f}_u \times \mathbf{f}_v}{|\mathbf{f}_u \times \mathbf{f}_v|} \ . \\
\mathbf{X} \cdot \mathrm{d}\mathbf{S} &= \mathbf{X} \cdot \mathbf{n}\, \mathrm{d}S = \mathbf{X} \cdot \left(-\tfrac{\mathbf{f}_u \times \mathbf{f}_v}{|\mathbf{f}_u \times \mathbf{f}_v|} \right) |\mathbf{f}_u \times \mathbf{f}_v| \mathrm{d}u\mathrm{d}v = \\
&= (a\mathbf{i} + b\,\mathbf{j} + c\mathbf{k}) \cdot (2u\mathbf{i} + 2v\mathbf{j} + \mathbf{k}) \mathrm{d}u\mathrm{d}v \\
&= (2au + 2bv + c) \mathrm{d}u\mathrm{d}v \\
\iint_S \mathbf{X} \cdot \mathrm{d}\mathbf{S} &= \iint_B (2au + 2bv + c)\, \mathrm{d}u\, \mathrm{d}v = c \iint_B \mathrm{d}u\mathrm{d}v = c \cdot 4\pi = 4\pi\, c \ .
\end{aligned}
$$