

Calculus II

Homework on Lecture 16

1. Express the infinite decimal number as a rational number.

(a) $0.\overline{9} = 0.99999 \dots$

ANSWER: $\frac{9}{10}$

(b) $1.\overline{6} = 1.6666 \dots$

ANSWER: $\frac{11}{6}$

(c) $1.\overline{3} = 1.3333 \dots$

ANSWER: $\frac{21}{10}$

(d) $1.\overline{19} = 1.191919 \dots$

ANSWER: $\frac{20140000}{6666}$

(e) $0.\overline{09} = 0.09090909 \dots$

ANSWER: $\frac{1}{11}$

(f) $2.\overline{16} = 2.16161616 \dots$

ANSWER: $\frac{5}{3}$

(g) $2014.\overline{2014} = 2014.201420142014 \dots$

ANSWER: $\frac{3}{4}$

Solution. 1.g

$$\begin{aligned}
 2014.201420142014 \dots &= 2014 + \frac{2014}{10^4} + \frac{2014}{10^8} + \dots \\
 &= 2014 + \frac{2014}{10000} \left(1 + \frac{1}{10000} + \dots + \frac{1}{10^{4n}} + \dots \right) \\
 &= 2014 + \frac{2014}{10000} \left(\frac{1}{1 - \frac{1}{10^4}} \right) \\
 &= 2014 + \frac{2014}{10000} \cdot \frac{10000}{9999} \\
 &= 2014 + \frac{9999}{2014 \cdot 9999 + 2014} \\
 &= \frac{9999}{2014 \cdot 10000} \\
 &= \frac{9999}{20140000} \\
 &= \frac{9999}{9999}
 \end{aligned}$$

Our answer cannot be reduced any further as the greatest common divisor of 20140000 and 9999 is 1.

2. Express the sum of the series as a rational number.

(a) $\sum_{n=1}^{\infty} \frac{2^n + 3^n}{5^n}$

ANSWER: $\frac{7}{2}$

(b) $\sum_{n=0}^{\infty} \frac{2^n + 5^n}{10^n}$

ANSWER: $\frac{4}{3}$

(c) $\sum_{n=1}^{\infty} \frac{5^n - 3^n}{7^n}$

ANSWER: $\frac{8}{25}$

(d) $\sum_{n=1}^{\infty} \frac{3^{n+1} + 7^{n-1}}{21^n}$

ANSWER: $\frac{6}{13}$

(e) $\sum_{n=0}^{\infty} \frac{2^{n+1} + (-3)^{n-1}}{5^n}$

ANSWER: $\frac{4}{13}$

Solution. 2.a.

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{2^n + 3^n}{5^n} &= \sum_{n=1}^{\infty} \left(\frac{2}{5}\right)^n + \sum_{n=1}^{\infty} \left(\frac{3}{5}\right)^n \\
&= \frac{2}{5} \sum_{n=0}^{\infty} \left(\frac{2}{5}\right)^n + \frac{3}{5} \sum_{n=0}^{\infty} \left(\frac{3}{5}\right)^n \quad \left| \begin{array}{l} \text{Use geometric series sum f-la:} \\ \sum_{n=0}^{\infty} r^n = \frac{1}{1-r}, \\ \text{provided } |r| < 1 \end{array} \right. \\
&= \frac{2}{5} \cdot \frac{1}{\left(1 - \frac{2}{5}\right)} + \frac{3}{5} \cdot \frac{1}{\left(1 - \frac{3}{5}\right)} \\
&= \frac{13}{6} .
\end{aligned}$$

Solution. 2.b.

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{2^n + 5^n}{10^n} &= \sum_{n=0}^{\infty} \left(\frac{1}{5^n} + \frac{1}{2^n} \right) \quad \left| \text{use } \sum_{n=0}^{\infty} r^n = \frac{1}{1-r}, \text{ for } |r| < 1 \right. \\
&= \frac{1}{1 - \frac{1}{2}} + \frac{1}{1 - \frac{1}{5}} \\
&= \frac{13}{4} .
\end{aligned}$$

Solution. 2.d.

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{3^{n+1} + 7^{n-1}}{21^n} &= \sum_{n=1}^{\infty} \left(3 \cdot \frac{3^n}{21^n} + \frac{1}{7} \cdot \frac{7^n}{21^n} \right) \\
&= 3 \sum_{n=1}^{\infty} \left(\frac{1}{7}\right)^n + \frac{1}{7} \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n \\
&= \frac{3}{7} \sum_{n=0}^{\infty} \left(\frac{1}{7}\right)^n + \frac{1}{21} \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n \quad \left| \text{use } \sum_{n=0}^{\infty} r^n = \frac{1}{1-r}, |r| < 1 \right. \\
&= \frac{3}{7} \cdot \frac{1}{\left(1 - \frac{1}{7}\right)} + \frac{1}{21} \cdot \frac{1}{\left(1 - \frac{1}{3}\right)} \\
&= \frac{4}{7} .
\end{aligned}$$

Solution. 2.e.

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{2^{n+1} + (-3)^{n-1}}{5^n} &= \sum_{n=0}^{\infty} \left(2 \cdot \frac{2^n}{5^n} - \frac{1}{3} \cdot \frac{(-3)^n}{5^n} \right) \\
&= 2 \sum_{n=0}^{\infty} \left(\frac{2}{5}\right)^n - \frac{1}{3} \sum_{n=0}^{\infty} \left(-\frac{3}{5}\right)^n \quad \left| \text{use } \sum_{n=0}^{\infty} r^n = \frac{1}{1-r}, |r| < 1 \right. \\
&= 2 \cdot \frac{1}{\left(1 - \frac{2}{5}\right)} - \frac{1}{3} \cdot \frac{1}{\left(1 - \left(-\frac{3}{5}\right)\right)} \\
&= \frac{25}{8} .
\end{aligned}$$

3. Sum the telescoping series (a sum is “telescoping” if it can be broken into summands so that consecutive terms cancel).

(a) $\sum_{n=0}^{\infty} \frac{-6}{9n^2 + 3n - 2} .$

ANSWER: 2

(b) $\sum_{n=3}^{\infty} \frac{3}{n^2 - 3n + 2} .$

ANSWER: 3

(c) $\sum_{n=2}^{\infty} \ln \left(1 - \frac{1}{n^2} \right) .$ (Hint: Use the properties of the logarithm to aim for a telescoping series).

ANSWER: 1 in 2

Solution. 3.b

$$\begin{aligned}
 \sum_{n=3}^{\infty} \frac{3}{n^2 - 3n + 2} &= \sum_{n=3}^{\infty} \left(\frac{3}{n-2} - \frac{3}{n-1} \right) && \left| \text{use partial fractions, see below} \right. \\
 &= 3 \sum_{n=3}^{\infty} \left(\frac{1}{n-2} - \frac{1}{n-1} \right) \\
 &= 3 \left(\left(1 - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) + \dots \right) \\
 &= 3 \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n-1} \right) = 3.
 \end{aligned}$$

In the above we used the partial fraction decomposition of $\frac{3}{n^2 - 3n + 2}$. This decomposition is computed as follows.

$$\frac{3}{n^2 - 3n + 2} = \frac{3}{(n-1)(n-2)}$$

We need to find A_i 's so that we have the following equality of rational functions. After clearing denominators, we get the following equality.

$$3 = A_1(n-2) + A_2(n-1)$$

After rearranging we get that the following polynomial must vanish. Here, by “vanish” we mean that the coefficients of the powers of x must be equal to zero.

$$(A_2 + A_1)n + (-A_2 - 2A_1 - 3)$$

In other words, we need to solve the following system.

$$\begin{array}{rcl}
 -2A_1 & -A_2 & = 3 \\
 A_1 & +A_2 & = 0
 \end{array}$$

System status	Action
$ \begin{array}{rcl} -2A_1 & -A_2 & = 3 \\ A_1 & +A_2 & = 0 \end{array} $	Selected pivot column 2. Eliminated the non-zero entries in the pivot column.
$ \begin{array}{rcl} A_1 & +\frac{A_2}{2} & = -\frac{3}{2} \\ & \frac{A_2}{2} & = \frac{3}{2} \end{array} $	Selected pivot column 3. Eliminated the non-zero entries in the pivot column.
$ \begin{array}{rcl} A_1 & & = -3 \\ & A_2 & = 3 \end{array} $	Final result.

Therefore, the final partial fraction decomposition is the following.

$$\frac{3}{n^2 - 3n + 2} = \frac{-3}{(n-1)} + \frac{3}{(n-2)}.$$

Solution. 3.c.

$$\begin{aligned}
 \sum_{n=2}^{\infty} \ln \left(1 - \frac{1}{n^2} \right) &= \sum_{n=2}^{\infty} \left(\ln \left(1 - \frac{1}{n} \right) + \ln \left(1 + \frac{1}{n} \right) \right) \\
 &= \sum_{n=2}^{\infty} \left(\ln \left(\frac{n-1}{n} \right) + \ln \left(\frac{n+1}{n} \right) \right) \\
 &= \sum_{n=2}^{\infty} (\ln(n-1) - \ln(n) + \ln(n+1)) \\
 &= (\ln 1 - 2\ln 2 + \ln 3) + (\ln 2 - 2\ln 3 + \ln 4) \\
 &\quad + (\ln 3 - 2\ln 4 + \ln 5) + \dots \\
 &= \lim_{n \rightarrow \infty} (-\ln 2 - \ln n + \ln(n+1)) \\
 &= \lim_{n \rightarrow \infty} \left(-\ln 2 + \ln \left(\frac{n+1}{n} \right) \right) \\
 &= -\ln 2.
 \end{aligned}$$

4. **Problems b, c, d will not appear on the quiz.** Use partial fractions to sum the telescoping series (a sum is “telescoping” if it can be broken into summands so that consecutive terms cancel).

$$(a) \sum_{n=1}^{\infty} \frac{1}{n^2 + n}$$

$$(c) \sum_{n=1}^{\infty} \frac{2n}{n^4 - 3n^2 + 1}$$

$$(b) \sum_{n=2}^{\infty} \frac{2n + 1}{n^4 + 2n^3 - n^2 - 2n}$$

$$(d) \sum_{n=3}^{\infty} \frac{n^2 + n + 2}{n^4 - 5n^2 + 4}$$

Solution. 4d

The partial fractions decomposition algorithm shows that

$$\frac{n^2 + n + 2}{n^4 - 5n^2 + 4} = \frac{1}{3} \left(\frac{2}{n-2} - \frac{2}{n-1} + \frac{1}{n+1} - \frac{1}{n+2} \right) .$$

We omit the details of the partial fraction decomposition as it is quite laborious, but otherwise straightforward. Therefore

$$\begin{aligned} \sum_{n=3}^{\infty} \frac{n^2 + n + 2}{n^4 - 5n^2 + 4} &= \frac{1}{3} \sum_{n=3}^{\infty} \left(\frac{2}{n-2} - \frac{2}{n-1} + \frac{1}{n+1} - \frac{1}{n+2} \right) \\ &= \frac{2}{3} \sum_{n=3}^{\infty} \left(\frac{1}{n-2} - \frac{1}{n-1} \right) \\ &\quad + \frac{1}{3} \sum_{n=3}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2} \right) \\ &= \frac{2}{3} \left(\left(1 - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \cancel{\left(\frac{1}{3} - \frac{1}{4} \right)} + \left(\frac{1}{n-2} - \frac{1}{n-1} \right) + \dots \right) \\ &\quad + \frac{1}{3} \left(\left(\frac{1}{4} - \frac{1}{5} \right) + \left(\frac{1}{5} - \frac{1}{6} \right) + \cancel{\left(\frac{1}{6} - \frac{1}{7} \right)} + \left(\frac{1}{n+1} - \frac{1}{n+2} \right) + \dots \right) \\ &= \lim_{n \rightarrow \infty} \frac{2}{3} \left(1 - \frac{1}{n-1} \right) + \lim_{n \rightarrow \infty} \frac{1}{3} \left(\frac{1}{4} - \frac{1}{n+2} \right) \\ &= \frac{2}{3} + \frac{1}{3} \cdot \frac{1}{4} \\ &= \frac{3}{4} . \end{aligned}$$