Precalculus
Lecture 10
Exponents

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https://github.com/tmilev/freecalc

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Outline

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Properties of exponential expressions.

For integer x, y and bases a, b, we demonstrate the exponent rules by example.

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$$\begin{aligned} 7^3 \cdot 7^2 &= (7\cdot 7\cdot 7)(7\cdot 7)\\ &= 7\cdot 7\cdot 7\cdot 7\cdot 7\\ &= 7^5\\ &= 7^{3+2}. \end{aligned}$$

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$$\begin{aligned}(5\cdot 7)^3 &= (5\cdot 7)(5\cdot 7)(5\cdot 7)\\ &= 5\cdot 7\cdot 5\cdot 7\cdot 5\cdot 7\\ &= \color{red}{5\cdot 5\cdot 5}\cdot 7\cdot 7\cdot 7\\ &= \color{red}{\textbf{?}}\ \cdot \textbf{?}\end{aligned}$$

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These rules do continue to hold for all $a > 0$, $b > 0$ and arbitrary x and y. The rules do fail when $a < 0$, $b < 0$ and x, y are not integers.

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  - the second alternative definition is easier to compute with.

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$$a^x = \lim_{\substack{y \to x \\ y\text{-rational}}} a^y$$
For example, a^π would be defined as the limit of the sequence $a^{3.14}, a^{3.141}, a^{3.1415}, \dots$.

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- Pros: for non-integer x and y, it is very easy to prove that $a^{x+y} = a^x a^y$ - this follows from the definition of limit above.
- This is the definition assumed in many elementary courses.

Exponent definition using series (approach II)

- The following formula (studied much later) can be used as alternative definition.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots$$

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- For $|x| < 1$ define

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + \frac{(-1)^{n+1} x^n}{n} + \dots$$

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Infinite sum studied much later.

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$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots$$

  Here $n! = 1 \cdot 2 \cdot 3 \cdot \dots \cdot (n-1) \cdot n$ and is read "n factorial".
- For $|x| < 1$ define

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + \frac{(-1)^{n+1} x^n}{n} + \dots$$

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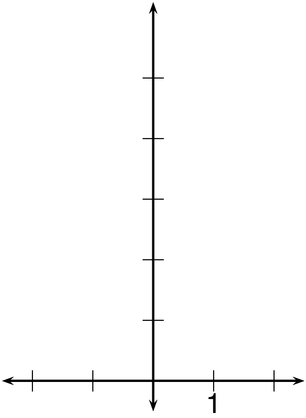
- Pros: this is how e^x and a^x are actually computed (by modern computers and by humans in the past).

Exponential Functions

The function $f(x) = 2^x$ is called an exponential function because the variable x is the exponent.



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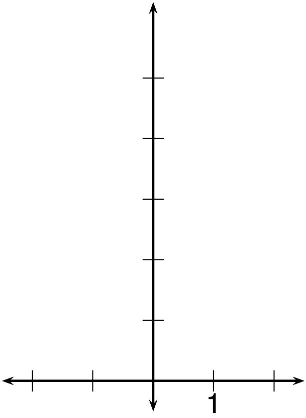

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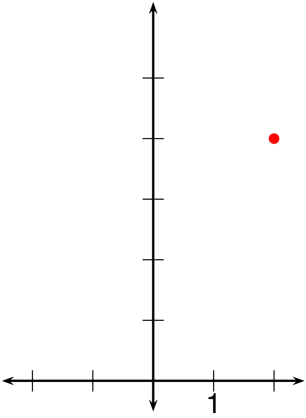

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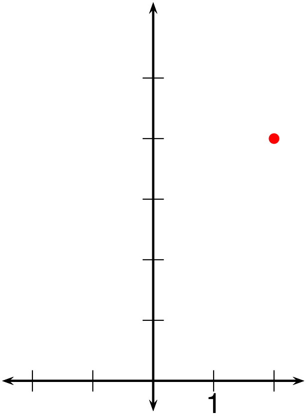

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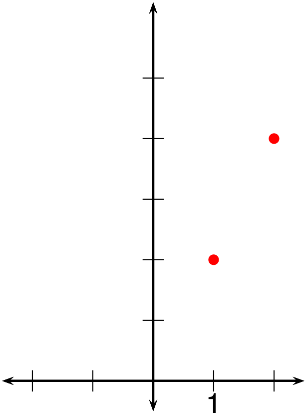

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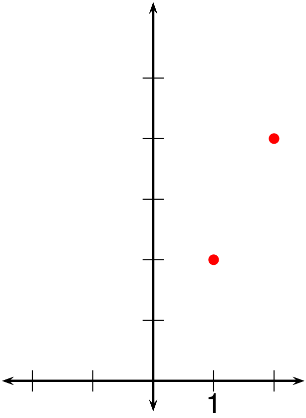

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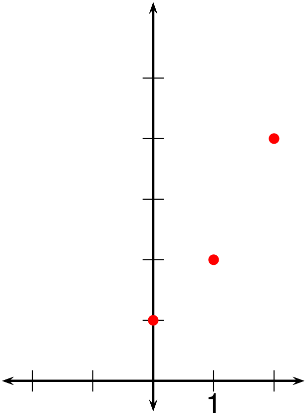

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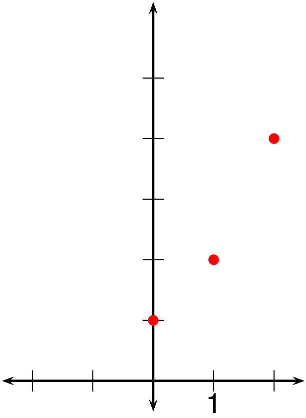

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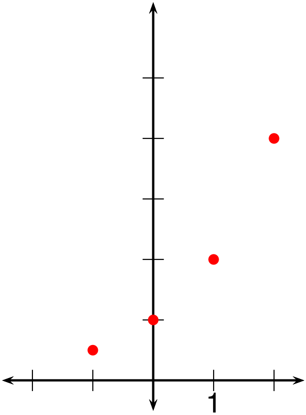

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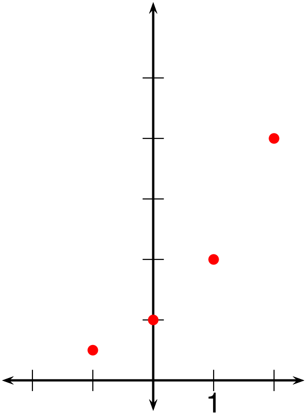

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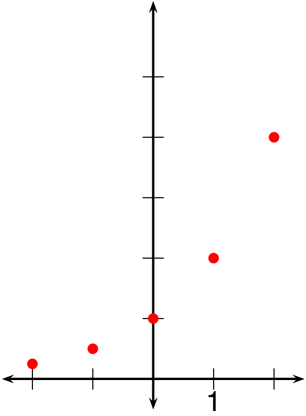

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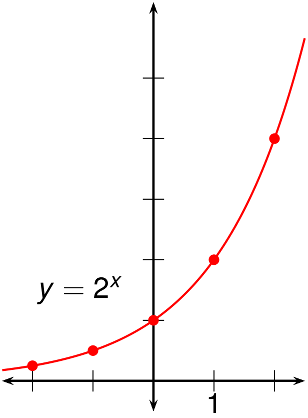

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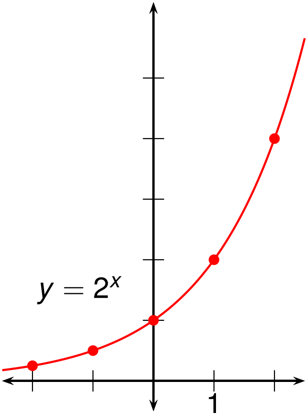

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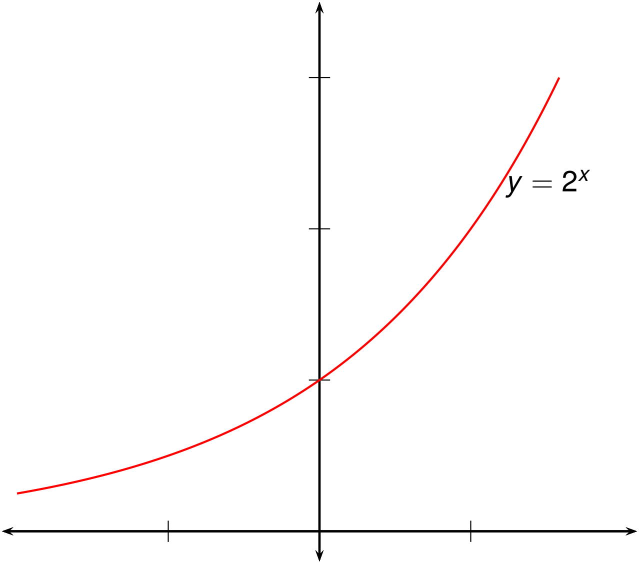
(Exponential Function Terminology)

An exponential function is a function of the form $f(x) = a^x$, where a is a positive constant.

Graphs of various exponential functions.



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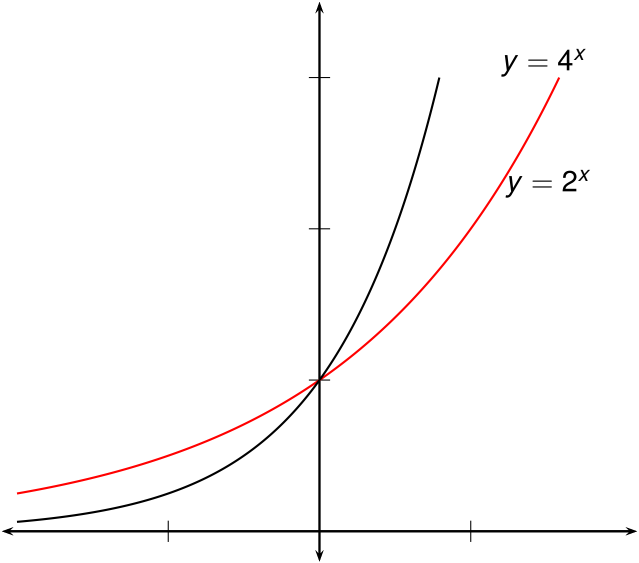

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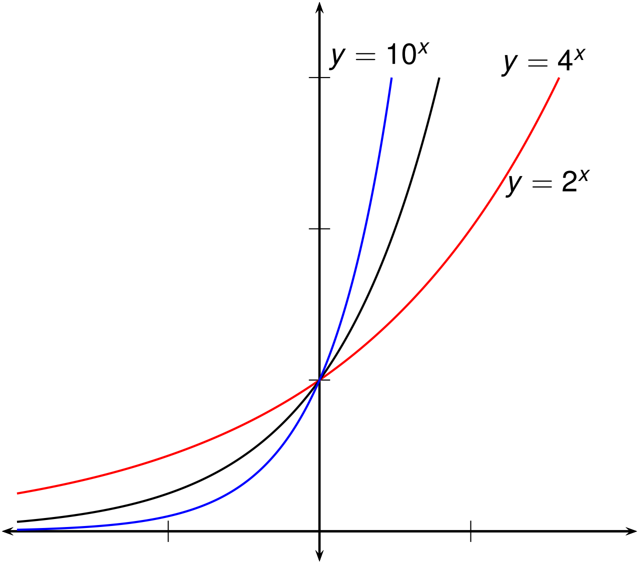

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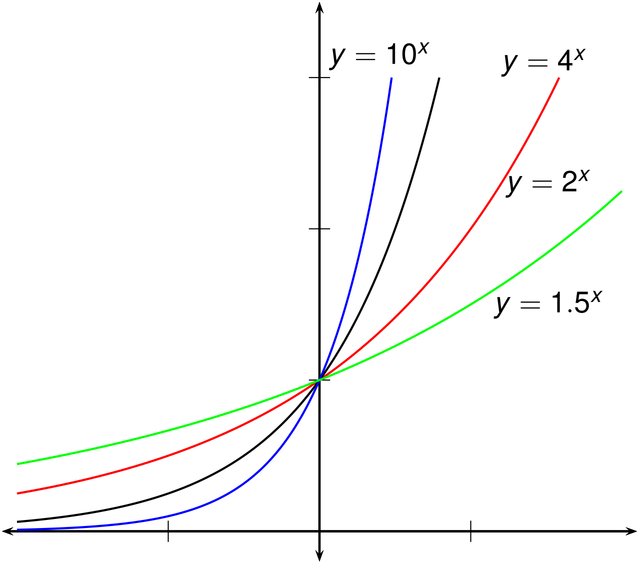

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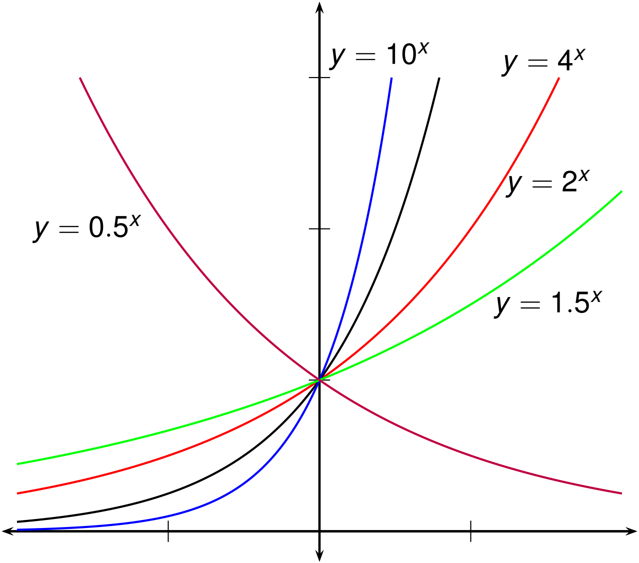

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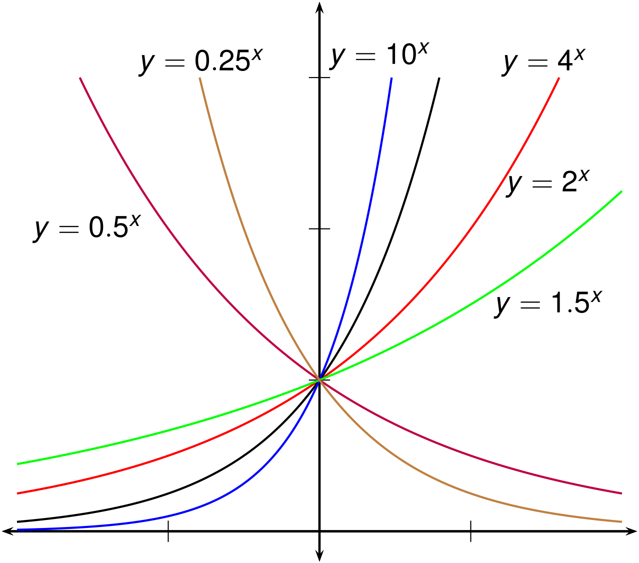
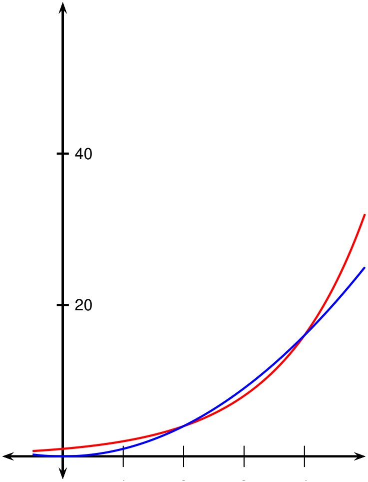


Graphical comparison of $y = 2^x$ with $y = x^2$. Axes have different scales.



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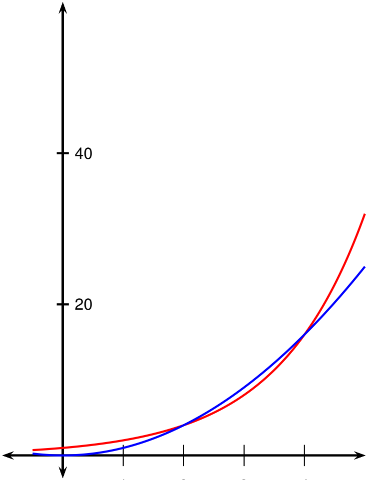
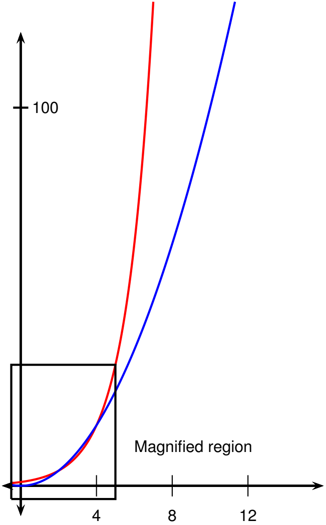
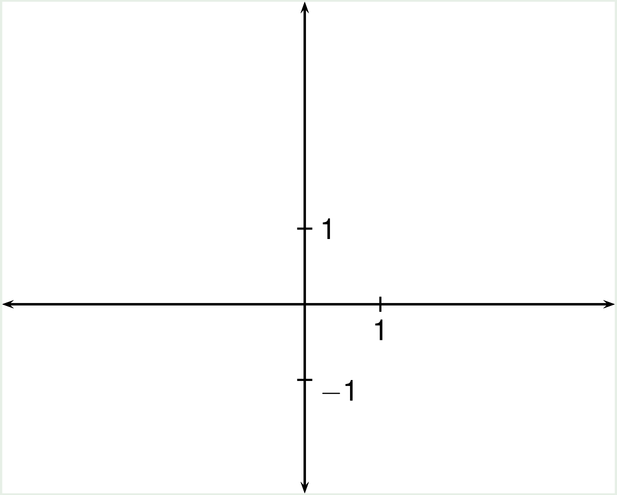




Example

Draw the graph of the function $y = 2^{-x} - 1 = 0.5^x - 1 = \left(\frac{1}{2}\right)^x - 1$.
Assume the graph of $y = 2^x$ given.



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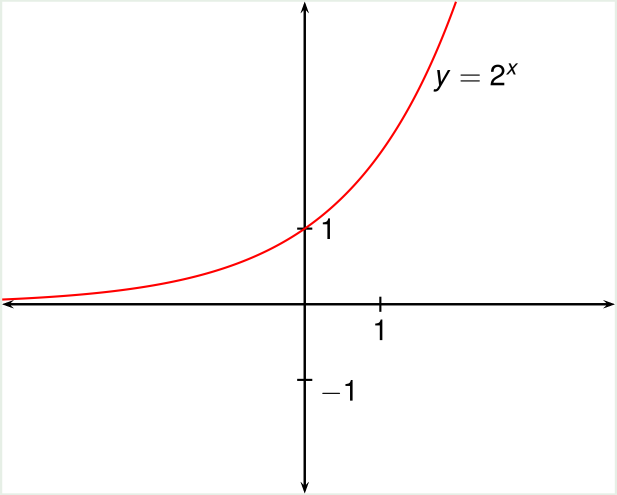

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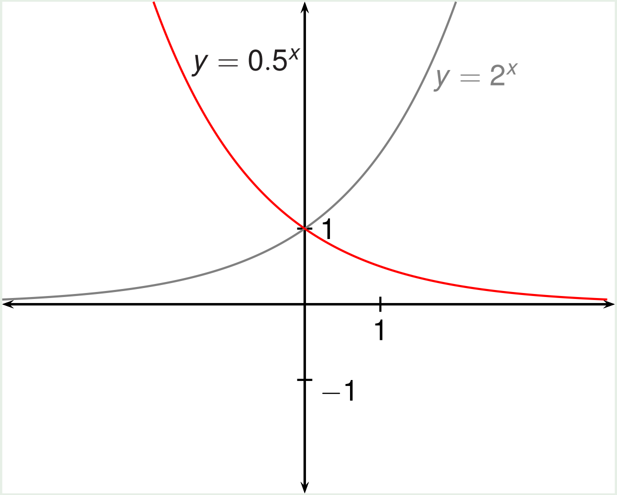

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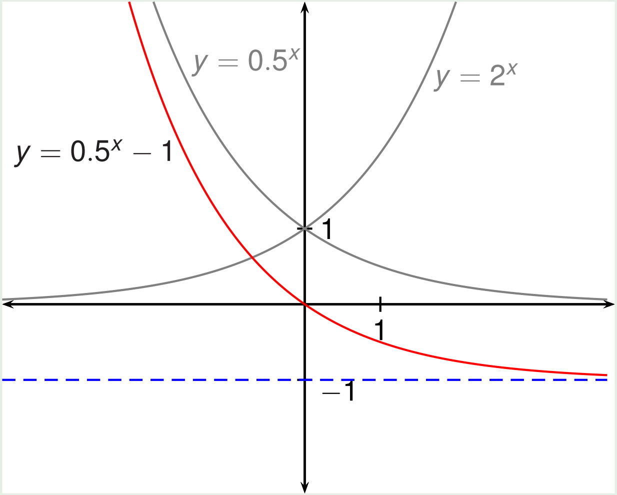


- Plot of 2^x assumed given.
- Plot $f(-x)$ = reflect $f(x)$ across y axis.
- Plot $g(x) - 1$ = shift graph $g(x)$ 1 unit down.

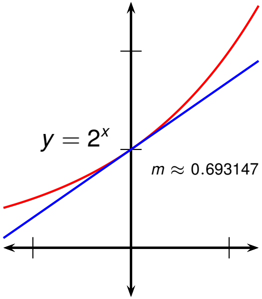
Proposition

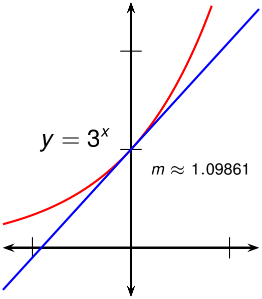
Let $a > 0$, $a \neq 1$. Let x and y be real numbers. Then $a^x = a^y$ if and only if $x = y$.

- In other words, the exponent function a^x is one-to-one.

The Natural Exponential Function

- One base for an exponential function is especially useful.





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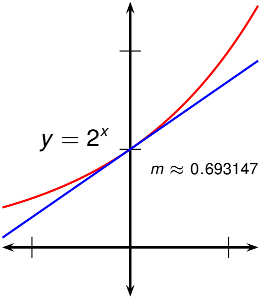

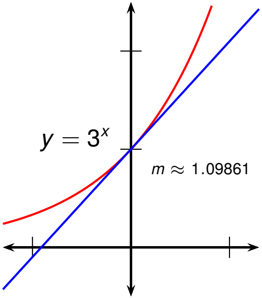

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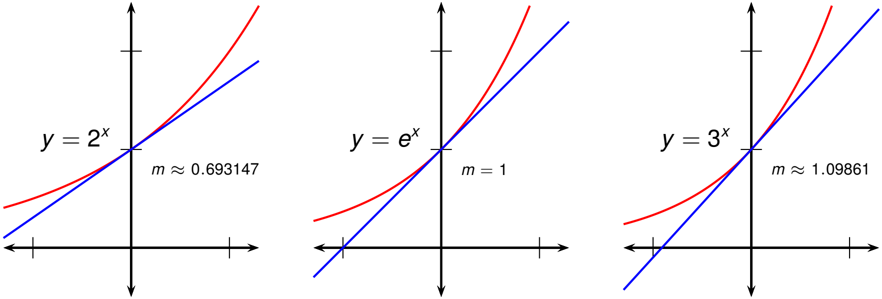

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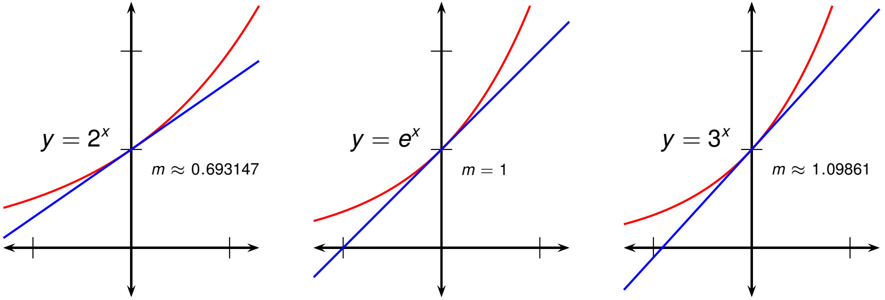

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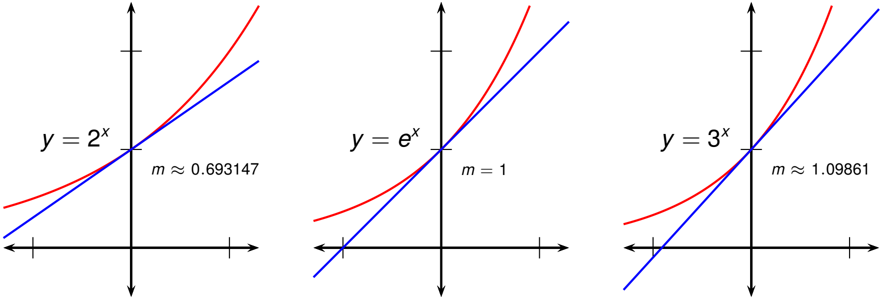
- e is a number between 2 and 3.
- In fact, $e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots \approx 2.71828$.



Recall that $e = 1 + \frac{1}{1} + \frac{1}{2!} + \frac{1}{3!} + \dots \approx 2.718281828$.

Theorem (The Number e as a Limit)

For large n we have that:

$$\begin{aligned} e &\approx \left(1 + \frac{1}{n}\right)^n \\ &\approx (1+n)^{\frac{1}{n}} \\ e^x &\approx \left(1 + \frac{x}{n}\right)^n \end{aligned}$$

All approximations become better as n increases.

- The approximation was discovered by Jacob Bernoulli (1655-1705) in order to apply to compound interest rate computations.

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Definition

The amount of money obtained from principal (original deposit) P after n years of annual compound interest rate of $k\%$, compounded once a year, is given by the formula

$$P\left(1+\frac{k}{100}\right)^n.$$

Example

You have 1000 USD kept at annual rate of 5%. The interest is compounded yearly. Approximate without using a calculator the amount of money you will have after 40 years. Check your approximation with a calculator.

Example

Decide, without using a calculator, which is more profitable: earning a yearly compound interest of 2% for 150 years or earning yearly simple interest of 11% for 150 years? Check your approximation with a calculator.

Example

When quickly computing interest rate “in the head”, financial advisors often use the following trick called the “rule of 72”. To find the time in years t needed for a sum to double under compound interest rate of $k\%$, financial advisors simply approximate $t \approx \frac{72}{k}$.
To illustrate the rule, under an interest rate of 2%, one needs approximately $\frac{72}{2} = 36$ years for the sum to double. Under interest rate of 6%, the sum doubles after only about $\frac{72}{6} = 12$ years. In 36 years an interest of 6% would double 3 times, in other words would increase by a factor of $2^3 = 8$.
Using the approximation $e \approx \left(1 + \frac{1}{n}\right)^n$ for large n, justify the rule of 72.