Calculus III
Lecture 9

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Outline

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Rate of Change

- O: fixed point in space. Define $f(P) = |OP|^2$.
- Question: How does f change around a point P_0 in space?

$$\Delta f = f(P) - f(P_0)$$

- Quantitative question. What is the rate of change of f at P_0?
- The question is ambiguous: rate of change of f with respect to what?

$$\text{Rate of change } = \frac{f(P) - f(P_0)}{\textbf{?}}$$

- Naive answer: with respect to distance from P_0: $\frac{f(P)-f(P_0)}{|P_0P|}$.
- Problem with naive answer: the instantaneous rate of change may fail to exist: $\lim_{P\to P_0} \frac{f(P)-f(P_0)}{|P_0P|}$.

Rates of Change along Lines

- Let L be a line through $P_0(\mathbf{r}_0)$.

  How does $f(\mathbf{r}) = |\mathbf{r}|^2$ change along L?

- Let $\mathbf{r} : \mathbb{R} \to L$: smooth parametrization of L, $\mathbf{r}(0) = \mathbf{r}_0$

$$g\colon \mathbb{R} \to \mathbb{R}, \quad g(t) = f(\mathbf{r}(t))$$

Rate of change of f along L = rate of change of g.

- With respect to t:

$$\lim_{t\to 0} \frac{f(\mathbf{r}(t)) - f(\mathbf{r}(0))}{t} = \lim_{t\to 0} \frac{g(t) - g(0)}{t} = g'(0)$$

- Still ambiguous: depends on the parametrization $\mathbf{r}$.
- To resolve that use arclength parametrization.
- Almost solves the problem: orientation still matters.

Directional Derivatives

- Let $f\colon D \to \mathbb{R}$, $P_0(\mathbf{r}_0)$ in D, $\mathbf{u}$ nonzero vector.
- Let L be line through P_0 with direction $\mathbf{u}$, oriented by $\mathbf{u}$.

$$\mathbf{r}\colon \mathbb{R} \to L, \quad \mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{u}$$

Definition (Covariant derivative $\nabla_{\mathbf{u}} f$)

Let $\mathbf{u}$ -nonzero vector. Define the covariant derivative $(\nabla_{\mathbf{u}} f)$ via $(\nabla_{\mathbf{u}} f)(P_0) = \lim\limits_{t\to 0} \frac{f(\mathbf{r}_0+t\mathbf{u})-f(\mathbf{r}_0)}{t}$

- If $\mathbf{r}$ is parametrized via arclength we have $|\mathbf{u}| = 1$.

Definition (Directional derivative)

Let $\mathbf{u}$ be a unit vector. Define the directional derivative $D_{\mathbf{u}} f$ via $(D_{\mathbf{u}} f)(P_0) = (\nabla_{\mathbf{u}} f)(u) = \lim\limits_{t\to 0} \frac{f(\mathbf{r}_0+t\mathbf{u})-f(\mathbf{r}_0)}{t}$.

- Define $(D_{\mathbf{u}} f)(P_0)$ to be the instantaneous rate of change of f along the line L.

Partial Derivatives

- Let $f\colon D \to \mathbb{R}$, $P_0(x_0, y_0)$ inside D.
- Consider the line $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{i} = (x_0 + t, y_0)$.
- Set $g(t) = f(\mathbf{r}(t)) = f(x_0 + t, y_0)$.
- Then $(D_{\mathbf{i}} f)(x_0, y_0) = \lim\limits_{t \to 0} \frac{g(t) - g(0)}{t}$.
- Define $\frac{\partial}{\partial x}$ to be the differential operator $D_{\mathbf{i}}$, and similarly define $\frac{\partial}{\partial y}$ to be the differential operator $D_{\mathbf{j}}$.

Definition (partial derivatives)

The partial derivatives $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ of f are defined as the directional derivatives of f in the direction of the unit vector along the x, y axes, i.e.,

$$\begin{aligned} \frac{\partial}{\partial x}(f) &= (D_{\mathbf{i}})(f) \\ \frac{\partial}{\partial y}(f) &= (D_{\mathbf{j}})(f) \quad . \end{aligned}$$

- Just as with one-variable derivatives, a number of notations are used/accepted.
- Notations for partial derivatives:

$$
\begin{aligned}
(D_{\mathbf{i}} f)(x_0, y_0) &= \frac{\partial f}{\partial x}(x_0, y_0) \\
&= f_x(x_0, y_0) \\
&= (\partial_x f)(x_0, y_0) \\
(D_{\mathbf{j}} f)(x_0, y_0) &= \frac{\partial f}{\partial y}(x_0, y_0) \\
&= f_y(x_0, y_0) \\
&= (\partial_y f)\,(x_0, y_0)
\end{aligned}
$$

- By convention, the notation $\frac{\mathrm{d}}{\mathrm{d}x}$ implies we are working with one variable only; $\frac{\partial}{\partial x}$ implies we are working with more than one.
- If in doubt about the number of variables - for example, if you intend to convert a parameter of the system into a variable - use the $\frac{\partial}{\partial x}$ notation.
- To compute a partial derivative with respect to a variable:
  - consider all other variables as constants and
  - apply the rules for differentiation for single variable functions.

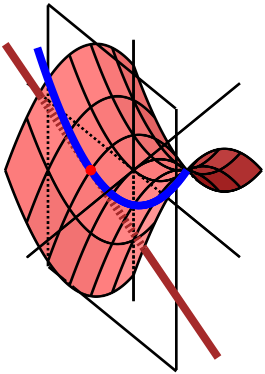
Example

Compute f_x, f_y, where $f(x,y) = y^2 \ln(2x+y) - e^y$.

$$
\begin{aligned}
f_x &= \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(y^2 \ln(2x+y) - e^y\right) = \frac{\partial}{\partial x}\left(y^2 \ln(2x+y)\right) - \frac{\partial}{\partial x}\left(e^y\right) \\
&= y^2 \frac{\partial}{\partial x}\left(\ln(2x+y)\right) - 0 = y^2 \cdot \frac{1}{2x+y} \cdot \frac{\partial}{\partial x}(2x+y) = \frac{2y^2}{2x+y} .
\end{aligned}
$$

$$
\begin{aligned}
f_y &= \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(y^2 \ln(2x+y) - e^y\right) = \frac{\partial}{\partial y}\left(y^2 \ln(2x+y)\right) - \frac{\partial}{\partial y}\left(e^y\right) \\
&= y^2 \frac{\partial}{\partial y}\left(\ln(2x+y)\right) + \frac{\partial}{\partial y}\left(y^2\right) \ln(2x+y) - e^y \\
&= 2y \ln(2x+y) + y^2 \cdot \frac{1}{2x+y} \cdot \frac{\partial}{\partial y}(2x+y) - e^y \\
&= \frac{y^2}{2x+y} + 2y \ln(2x+y) - e^y .
\end{aligned}
$$

Graphical Interpretation



- Recall the graph of f is the surface whose points are $\{(x, y, f(x, y))\}$.
- The vertical plane containing the line $\mathbf{r} = \mathbf{r}_0 + t\mathbf{i}$ is the plane $y = y_0$.
- Intersection of graph with the plane $y = y_0$ is the curve

$$\gamma(t) = (t, y_0, f(t, y_0)).$$

- The image of $\gamma(t)$ is the graph of $z = h(x) = f(x, y_0)$ in the $y = y_0$ plane.
- The direction of tangent line to γ is: $\gamma'(x_0) = (1, 0, f_x(x_0, y_0))$
- In the $xz-$plane $y = y_0$, the slope of this line is $h'(x_0) = f_x(x_0, y_0)$.

Higher Order Derivatives

- The partial derivatives of the partial derivatives f_x and f_y are called *the second order partial derivatives* of f.
- The partial derivatives of the second order derivatives are the *third order* derivatives, and so on.

$$f(x,y) \to \begin{cases} \dfrac{\partial f}{\partial x} = f_x \to \begin{cases} \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial^2 f}{\partial x^2} = f_{xx} \\ \\ \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial^2 f}{\partial y\,\partial x} = f_{xy} \end{cases} \\ \\ \dfrac{\partial f}{\partial y} = f_y \to \begin{cases} \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right) = \dfrac{\partial^2 f}{\partial x\,\partial y} = f_{yx} \\ \\ \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial y}\right) = \dfrac{\partial^2 f}{\partial y^2} = f_{yy} \end{cases} \end{cases}$$

Example

$f(x, y) = x^2y^3$. Then

$$\begin{aligned} f_x(x,y) &= 2xy^3 \\ f_{xx}(x,y) &= (2xy^3)_x = 2y^3 \\ f_{yx}(x,y) &= (3x^2y^2)_x = 6xy^2 \end{aligned} \qquad \begin{aligned} f_y(x,y) &= 3x^2y^2 \\ f_{xy}(x,y) &= (2xy^3)_y = 6xy^2 \\ f_{yy}(x,y) &= (3x^2y^2)_y = 6x^2y \end{aligned}$$

Notice that $f_{xy} = f_{yx}$. That is not a coincidence.

Theorem (Clairaut, (1713-1765))

If the second order derivatives f_{xy} and f_{yx} are continuous on an open set, then they are equal everywhere on that set.

- An open set is a connected set that contains a small open disk around all of its points, for example an open disk.
- An analogous theorem is valid in n dimensions.

Linearizations

$$\boxed{z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)} .$$

Definition

The function

$$L_{f,(x_0,y_0)}(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

is called the is called the linearization of f at (x_0, y_0).

Differentiability

If $y = h(x)$ is a function of one variable, then

$$L_{h,x_0}(x) = h(x_0) + h'(x_0)(x - x_0)$$

$$\lim_{x \to x_0} \frac{|h(x) - L_{h,x_0}(x)|}{|x - x_0|} = \lim_{x \to x_0} \left| \frac{h(x) - h(x_0)}{x - x_0} - h'(x_0) \right| = 0$$

One variable: the linear approximation is a good approximation.
Several variables:
$f_x(x_0, y_0)$, $f_y(x_0, y_0)$ exist $\implies$ f has a linear approximation $L_{f,(x_0,y_0)}$.
But is that a *good* linear approximation? Unfortunately, not always!

Multivariable Differentiability Definition

- Let (x_0, y_0) be a fixed point and a and b be arbitrary numbers.
- Define $\varepsilon_{f,a,b}(x, y) = f(x, y) - f(x_0, y_0) - a(x - x_0) - b(y - y_0)$.
- $\varepsilon_{f,a,b}$ measures how well does $f(x_0, y_0) + a(x - x_0) + b(y - y_0)$ approximate f.

For the particular case: $a = \dfrac{\partial f}{\partial x}(x_0, y_0) \qquad b = \dfrac{\partial f}{\partial y}(x_0, y_0)$ we have:

$$\varepsilon_{f,a,b}(x, y) = f(x, y) - f(x_0, y_0) - \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) - \frac{\partial f}{\partial x}(x_0, y_0)(y - y_0).$$

Definition

f is called *differentiable at* (x_0, y_0) if there exist a and b such that

$$\lim_{(x,y)\to(0,0)} \frac{\varepsilon_{f,a,b}(x, y)}{|(x - x_0, y - y_0)|} = 0$$

Remark. If a function f is differentiable, then the numbers a and b equal $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$.

Example: $f(x, y) = x^2 + xy + 2y^2$ is differentiable at $(4, 1)$.

Total Differential

If f is differentiable at (x_0, y_0), then

$$f(x, y) \simeq f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$$\Delta f \simeq f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y$$

For infinitesimally small Δx and Δy we get:
Definition: The total differential $\mathrm{d}f$ at (x_0, y_0) is

$$(\mathrm{d}f)|_{(x_0,y_0)} = f_x(x_0, y_0)\mathrm{d}x + f_y(x_0, y_0)\mathrm{d}y$$

Alternatively:

$$\mathrm{d}f = f_x\mathrm{d}x + f_y\mathrm{d}y \qquad \text{or} \qquad \mathrm{d}f = f_x\mathrm{d}x + f_y\mathrm{d}y + f_z\mathrm{d}z$$

$$\begin{aligned} \Delta f &: \text{actual change in } f \\ \mathrm{d}f \simeq \Delta f &: \text{infinitesimal change in } f \\ f_x(x_0, y_0), f_y(x_0, y_0) &: \text{error propagation factors} \end{aligned}$$

Example

A cylinder has radius $r = 3cm$ and height $h = 5cm$. The error in measuring the radius is $\pm 1mm$, and the error in measuring the height is $\pm 1mm$. Estimate the error in the volume of the cylinder.
$V(r,h) = \pi r^2 h$. The actual volume: $V(3,5) = 45\pi\, cm^3$.
The error in volume, ΔV, is estimated by dV:

$$\Delta V \simeq \mathrm{d}V = V_r(3,5)\mathrm{d}r + V_h(3,5)\mathrm{d}h \simeq V_r(3,5)\Delta r + V_h(3,5)\Delta h\,.$$

$$V_r(r,h) = 2\pi rh \Longrightarrow V_r(3,5) = 30\pi$$

$$V_h(r,h) = \pi r^2 \Longrightarrow V_h(3,5) = 9\pi$$

$$\Delta V \simeq (30\pi)(\pm 0.1) + ((9\pi)(\pm 0.1) \Longrightarrow V(r,h) \simeq V(3,5) \pm 3.9\pi\, cm^3$$

The error in volume is $\pm 3.9\pi\, cm^3$. Relative error:
$\frac{\Delta V}{V} \simeq \pm \frac{3.9\pi}{45\pi} \simeq \pm 8.6\%$
Remark: Since $V_r(3,5) > V_h(3,5)$, the result is more sensitive to errors in r than to errors in h.