Calculus III
Lecture 13

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Outline

1. Double Integrals
   - Riemann Sums, Double Integral Definition
   - Double integral properties
   - Iterated integrals

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A Cheaper "Census"

Imagine we want cheap procedure to estimate population in region $\mathcal{R}$.

- Decompose $\mathcal{R}$ into pairwise non-overlapping smaller regions D_k (states, counties, finer division...).

$$\text{population}(\mathcal{R}) = \sum_k \text{population}(D_k) = \sum_k \text{density}(D_k) \cdot \text{area}(D_k)$$

- To find the population density in D_k we need to count everyone (what an actual census does).
- Instead, we estimate the population density as follows.
  - We pick a sample point P_k in each region D_k.
  - We estimate the population density density(D_k) by counting people in a small region around P_k (density_near(P_k)).
- Our population estimate becomes

$$\text{population}(\mathcal{R}) = \sum \text{pop.}(D_k) \simeq \sum \text{density\_near}(P_k)\text{area}(D_k).$$

Riemann sum in two variables

Let $\mathcal{R}$ be a compact (closed, bounded) region in the plane, and let $f\colon \mathcal{R} \to \mathbb{R}$ be a function on $\mathcal{R}$. Let $\{D_k\}$ be finite set of regions covering $\mathcal{R}$ with the following properties.

- Each D_k is a compact set.
- The boundary of each D_k is a collection of smooth curves.
- Two regions D_i and D_j may overlap only on their boundaries.

Let P_k be a collection of sampling points with $P_k \in D_k$ for all k.

Definition (Riemann sum)

The *Riemann sum* defined by such data is $\sum_k f(P_k) \operatorname{area}(D_k)$.

Double Integrals

$\mathcal{R}$-region covered by D_k, D_k don't overlap except at boundaries.

Definition (Riemann sum)

The *Riemann sum* defined by such data is $\sum_k f(P_k)\,\mathrm{area}(D_k)$.

Definition

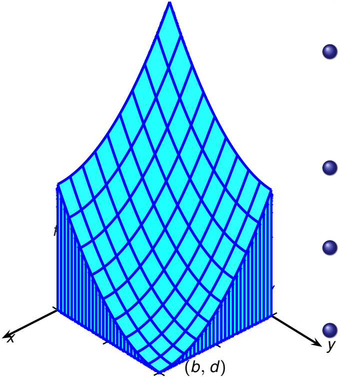
If the limit

$$\lim_{\max_k(\mathrm{diam} D_k)\to 0} \sum_k f(P_k)\,\mathrm{area}(D_k)$$

exists and is finite, then its value is called the *double integral of f over $\mathcal{R}$ (with respect to area)*, and is denoted by

$$\iint_{\mathcal{R}} f(P)\mathrm{d}A \quad .$$

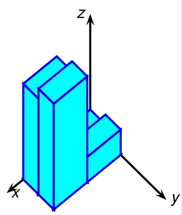
Midpoint Rule



- Suppose region of integration $\mathcal{R}$ is rectangle, i.e., $\mathcal{R} = [a, b] \times [c, d]$, integration w.r.t. $dA = dxdy$. $\iint_{\mathcal{R}} f(P)dA = \iint_{[a,b]\times[c,d]} f(x, y)\, dx\, dy$.
- If integral exists: approximate by fine enough Riemann sum.
- Simplest way: divide $\mathcal{R}$ into $n \times n$ equal pieces, sides $\Delta x = \frac{b-a}{n}$, $\Delta y = \frac{d-c}{n}$.
- For $(s, t)^{th}$ rectangle D_{st}, sample at midpoint $P_{s,t} = \left(a + \left(s - \frac{1}{2}\right)\Delta x, c + \left(t - \frac{1}{2}\right)\Delta y\right)$.

$$\iint\limits_{\mathcal{R}} f(x, y)dxdy = \lim_{n\to\infty} \sum_{1\leq s,t\leq n} f\left(P_{s,t}\right) \mathrm{area}(D_{st})$$
$$\approx \sum_{1\leq i,j\leq n} f\left(P_{s,t}\right) \Delta x \Delta y \quad .$$

Example



Use the Midpoint Rule to approximate $\iint_{[0,4]\times[0,2]} x^2 y dxdy$, with each side divided into $n = 2$ pieces.

The small rectangles have dimensions $\frac{4-0}{2} \cdot \frac{2-0}{2} = 2 \cdot 1$ and area 2. The midpoints are

$$P_{11} = \left(1, \frac{1}{2}\right), \quad P_{12} = \left(1, \frac{3}{2}\right), \quad P_{21} = \left(3, \frac{1}{2}\right), \quad P_{22} = \left(3, \frac{3}{2}\right).$$

$$\begin{aligned}\iint\limits_{[0,4]\times[0,2]} x^2 y \, dxdy &\approx 2\left(f\left(1, \frac{1}{2}\right) + f\left(3, \frac{1}{2}\right) + f\left(1, \frac{3}{2}\right) + f\left(3, \frac{3}{2}\right)\right)\\ &= 1 \cdot \frac{1}{2} \cdot 2 + 9 \cdot \frac{1}{2} \cdot 2 + 1 \cdot \frac{3}{2} \cdot 2 + 9 \cdot \frac{3}{2} \cdot 2\\ &= 1 + 9 + 3 + 27 = 40 .\end{aligned}$$

Theoretical Examples

- The total population over a region $\mathcal{R}$ is:

$$\text{population}(\mathcal{R}) = \iint_{\mathcal{R}} \text{density}(P)\, \mathrm{d}A \simeq \sum_k \text{density}(P_k)\, \text{area}(D_k)\ .$$

- Mass is the double integral of density with respect to area:

$$\text{mass}(\mathcal{R}) = \iint_{\mathcal{R}} \text{density}(P)\, \mathrm{d}A\ .$$

- Volume under the graph of $h\colon \mathcal{R} \to [0, \infty)$

$$\text{Volume} = \iint_{\mathcal{R}} h(P)\, \mathrm{d}A\ .$$

- Area of a region:

$$\text{Area}(\mathcal{R}) = \iint_{\mathcal{R}} 1\ \mathrm{d}A \quad .$$

Double Integral Properties

$$\iint_{\mathcal{R}} f(P)\,\mathrm{d}A = \lim_{\max_k(\operatorname{diam} D_k)\to 0} \sum_k f(P_k)\,\operatorname{area}(D_k)$$

- If f is bounded and continuous, except maybe on a finite number of smooth curves, then the limit exists and is finite.
- Linearity

$$\iint_{\mathcal{R}} [\lambda f(P) + \mu g(P)]\,\mathrm{d}A = \lambda \iint_{\mathcal{R}} f(P)\,\mathrm{d}A + \mu \iint_{\mathcal{R}} g(P)\,\mathrm{d}A\,.$$

- Domain additivity: if $\mathcal{R}_1$ and $\mathcal{R}_2$ intersect only along boundaries:

$$\iint_{\mathcal{R}_1\cup\mathcal{R}_2} f(P)\,\mathrm{d}A = \iint_{\mathcal{R}_1} f(P)\,\mathrm{d}A + \iint_{\mathcal{R}_2} f(P)\,\mathrm{d}A$$

- Monotonicity property: If $m \leq f(P) \leq M$ for all P in $\mathcal{R}$, then

$$m\,\operatorname{area}(\mathcal{R}) \leq \iint_{\mathcal{R}} f(P)\,\mathrm{d}A \leq M\,\operatorname{area}(\mathcal{R})\,.$$

Applications

- Average value of f on $\mathcal{R}$.

$$
\begin{aligned}
\iint_{\mathcal{R}} f(P)\,\mathrm{d}A &= \iint_{\mathcal{R}} (\text{average value of } f \text{ on } \mathcal{R})\,\mathrm{d}A \\
&= (\text{average value of } f \text{ on } \mathcal{R}) \iint_{\mathcal{R}} \mathrm{d}A \\
&= (\text{average value of } f \text{ on } \mathcal{R}) \cdot \text{area}(\mathcal{R}) \\
\text{average value of } f \text{ on } \mathcal{R} &= \frac{1}{\text{area}(\mathcal{R})} \iint_{\mathcal{R}} f(P)\,\mathrm{d}A\,.
\end{aligned}
$$

Theorem (Mean Value Theorem)

If f is continuous on $\mathcal{R}$, then there exists P_0 in $\mathcal{R}$ such that

$$f(P_0) = \frac{1}{area(\mathcal{R})} \iint_{\mathcal{R}} f(Q)\,\mathrm{d}A$$

Theorem (Analog of Fundamental Theorem of Calculus)

If f is continuous around P, then

$$\lim_{D\to\{P\}} \frac{1}{area(D)} \iint_{D} f(Q)\mathrm{d}A = f(P)$$

Vectorial Integrals

The double integral definition extends directly to f-ns with vector output.

Definition

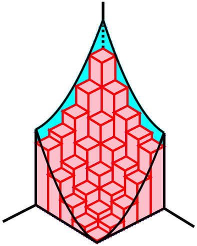
$$\iint_{\mathcal{R}} \mathbf{F}(P)\,\mathrm{d}A = \lim_{\mathrm{maxdiam}(\mathcal{D})\to 0} \sum_{k} \mathbf{F}(P_k)\,\mathrm{area}(D_k)$$

Theoretical example: Electric force on a lamina

- Given:
  - a charge Q, located at the origin;
  - charge q, uniformly distributed on a planar lamina $\mathcal{R}$.
- What is the resulting (total) force $\mathbf{F}$ on Q?
- Recall that the attraction force exerted on a charge Q located at the origin by a charge c located at a point with position vector $\mathbf{r}$ is $\varepsilon Qc\frac{\mathbf{r}}{|\mathbf{r}|^3}$.

$$
\begin{aligned}
\mathrm{d}q &= (\text{density of charge})\mathrm{d}A = \frac{q}{A(\mathcal{R})}\mathrm{d}A \\
\mathrm{d}\mathbf{F} &= \varepsilon Q\frac{\mathbf{r}}{|\mathbf{r}|^3}\mathrm{d}q = \varepsilon\frac{Qq}{A(\mathcal{R})}\frac{\mathbf{r}}{|\mathbf{r}|^3}\mathrm{d}A \\
\mathbf{F} &= \iint_{\mathcal{R}}\mathrm{d}\mathbf{F} = \iint_{\mathcal{R}}\varepsilon\frac{Qq}{A(\mathcal{R})}\frac{\mathbf{r}}{|\mathbf{r}|^3}\mathrm{d}A \\
&= \varepsilon\frac{Qq}{A(\mathcal{R})}\iint_{\mathcal{R}}\frac{\mathbf{r}}{|\mathbf{r}|^3}\mathrm{d}A
\end{aligned}
$$

Iterated Integrals



$$\iint\limits_{[a,b]\times[c,d]} f(x,y)\mathrm{d}x\mathrm{d}y \approx \sum_{1\leq i,j\leq n} f(x_i,y_j)\Delta x\Delta y = \sum_{j=1}^{n}\left(\sum_{i=1}^{n} f(x_i,y_j)\Delta x\right)\Delta y\ .$$

The j^{th} summand is a Riemann sum for $g(y_j) = \int_{x=a}^{x=b} f(x,y_j)\mathrm{d}x$.

$$\sum_{j=1}^{n}\left(\sum_{i=1}^{n} f(x_i,y_j)\Delta x\right)\Delta y \approx \sum_{j=1}^{n} g(y_j)\Delta y \approx \int_{y=c}^{y=d} g(y)\mathrm{d}y$$

$$\iint\limits_{[a,b]\times[c,d]} f(x,y)\mathrm{d}x\mathrm{d}y = \int\limits_{y=c}^{y=d} g(y)\mathrm{d}y = \int\limits_{y=c}^{y=d}\left(\int\limits_{x=a}^{x=b} f(x,y)\mathrm{d}x\right)\mathrm{d}y$$

Theorem

If f is continuous the double integral $\iint_{[a,b]\times[c,d]} f(x,y)\mathrm{d}x\mathrm{d}y$ *exists.*

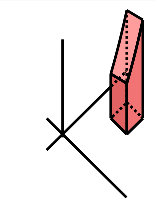
Theorem (Fubini's Theorem)

Suppose the double integral of f exists. Then, except at a set of measure 0*, the iterated integrals exist and*

$$\iint\limits_{[a,b]\times[c,d]} f(x,y)\,\mathrm{d}x\mathrm{d}y = \int\limits_{y=c}^{y=d}\left(\int\limits_{x=a}^{x=b} f(x,y)\,\mathrm{d}x\right)\mathrm{d}y = \int\limits_{x=a}^{x=b}\left(\int\limits_{y=c}^{y=d} f(x,y)\mathrm{d}y\right)\mathrm{d}x\,.$$

This theorem allows to integrate non-continuous functions. The term "set of measure 0" is too technical to define here; usually studied in the subject(s) "Real Analysis/Measure Theory".

Example



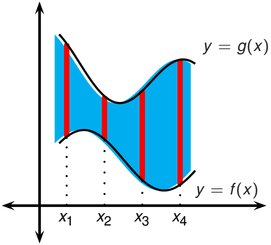
Compute $\displaystyle\iint_{[1,2]\times[2,3]} (2x+3y^2)\,\mathrm{d}x\mathrm{d}y.$

For (x, y) in $[1,2]\times[2,3]$, y takes values between $c = 2$ and $d = 3$.For a fixed value $y = y_0$, x takes values between $a = 1$ and $b = 2$.

$$
\begin{aligned}
\iint_{[1,2]\times[2,3]} (2x+3y^2)\mathrm{d}x\mathrm{d}y &= \int_{y=2}^{y=3}\left(\int_{x=1}^{x=2}(2x+3y^2)\mathrm{d}x\right)\mathrm{d}y\\
&= \int_{y=2}^{y=3}\left[x^2+3y^2x\right]_{?x=1}^{?x=2}\mathrm{d}y\\
&= \int_{y=2}^{y=3}\left((4+6y^2)-(1+3y^2)\right)\mathrm{d}y\\
&= \int_{y=2}^{y=3}(3+3y^2)\mathrm{d}y = \left[3y+y^3\right]_{y=2}^{y=3}\\
&= 36-14=22\ .
\end{aligned}
$$

More General Regions

What makes iterated integrals work over rectangular regions? Slices with respect to one variable are intervals in the other. If variable is x:



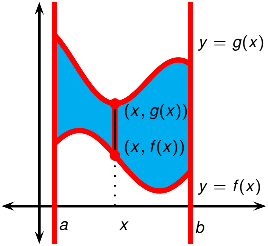
- fix x,
- integrate with respect to y,
- to obtain function that depends only on x,
- then integrate the so obtained function in x.

So far used rectangular regions; this also works if slices are intervals whose endpoints depend continuously on the location of the slice.

- Regions of type I: vertical slices are segments.
- Regions of type II: horizontal slices are segments.

We call such regions curvilinear trapezoids.

Strategy: Curvilinear Trapezoids (Type I)



- Identify the leftmost point(s), with x-coordinate $x = a$ and the rightmost point(s), $x = b$.
- Draw a vertical slice at a value x between a and b.
- Find the lowest point on that slice, $(x, f(x))$ and the highest point, $(x, g(x))$.

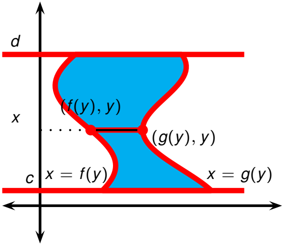
The region is the region bounded by:

- the vertical lines $x = a$ and $x = b$;
- the graphs of $y = f(x)$ and $y = g(x)$, with $f, g\colon [a, b] \to \mathbb{R}$.

$$\mathcal{R} = \{(x, y) | a \leq x \leq b, f(x) \leq y \leq g(x)\}.$$

$$\iint_{\mathcal{R}} f(x, y)\, \mathrm{d}x\mathrm{d}y = \int_{x=a}^{x=b} \left(\int_{y=f(x)}^{y=g(x)} f(x, y) \mathrm{d}y \right) \mathrm{d}x$$

Strategy: Curvilinear Trapezoids (Type II)



- Identify the lowest point(s), with y-coordinate $y = c$ and the topmost point(s), $y = d$.
- Draw a generic horizontal slice at some value y between c and d.
- Find the lowest point on that slice, $(f(y), y)$ and the topmost point, $(g(y), y)$.

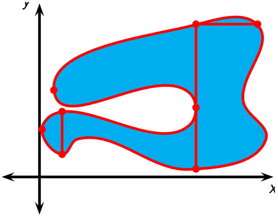
The region is bounded by:

- horizontal lines $y = c$ and $y = d$
- graphs of $x = f(y)$ and $x = g(y)$, with $f, g\colon [c, d] \to \mathbb{R}$:

$$\mathcal{R} = \{(x, y) \mid c \leq y \leq d \,,\; f(y) \leq x \leq g(y)\}\,.$$

$$\iint_{\mathcal{R}} f(x, y)\,\mathrm{d}x\mathrm{d}y = \int_{y=c}^{y=d} \left(\int_{x=f(y)}^{x=g(y)} f(x, y)\mathrm{d}x \right) \mathrm{d}y$$

Strategy for Computing a Double Integral

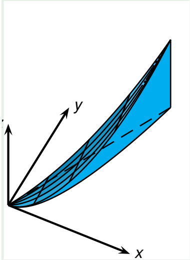


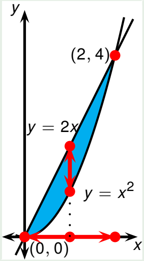
Problem

Find the integral $\int\int_{\mathcal{R}} f(x, y)\mathrm{d}x\mathrm{d}y$ *over a region* $\mathcal{R}$ *enclosed by a set of smooth curves.*

- We present a strategy for approaching the above problem.
- The tractability of this strategy depends on the concrete description of f and the enclosing curves.
  - Plot the curve(s) enclosing $\mathcal{R}$.
  - Identify the region $\mathcal{R}$.
  - Chop $\mathcal{R}$ into curvilinear trapezoids; the trapezoids are allowed to intersect only on their boundaries.
  - By possible subdivision ensure trapezoids have smooth boundaries.
  - Integrate f over the obtained curvilinear trapezoids & collect terms.
- Our strategy will be augmented/combined later with variable changes (via the multivariable substitution rule).

Example





Let $\mathcal{R}$ be the region bounded by $y = 2x$ and $y = x^2$. Compute

$$\iint_{\mathcal{R}} \frac{1}{8}\left(x^2 + y^2\right) \mathrm{d}x\mathrm{d}y$$

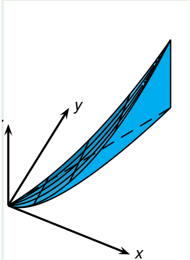
Plot $y = 2x$. Plot $y = x^2$. Identify the region.

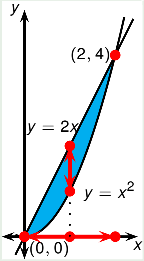
The two curves intersect when
$$\begin{array}{rcl} x^2 &=& 2x \\ x(x-2) &=& 0 \\ x &=& 0 \text{ or } 2. \end{array}$$

The intersection points are therefore $(0, 0)$ and $(2, 4)$. We can plot the function $\frac{1}{8}\left(x^2 + y^2\right)$ as above. Our integral is

$$\int_{x=0}^{x=2} \left(\int_{y=x^2}^{y=2x} \frac{1}{8}\left(x^2 + y^2\right) \mathrm{d}y \right) \mathrm{d}x$$

Example





Let $\mathcal{R}$ be the region bounded by $y = 2x$ and $y = x^2$. Compute

$$\iint_{\mathcal{R}} \frac{1}{8}\left(x^2 + y^2\right) \mathrm{d}x\mathrm{d}y$$

Plot $y = 2x$. Plot $y = x^2$. Identify the region.

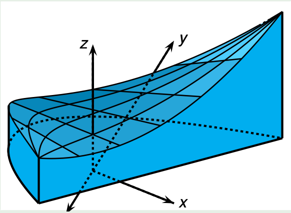
$$\int\limits_{x=0}^{x=2} \left(\int\limits_{y=x^2}^{y=2x} \frac{1}{8}\left(x^2+y^2\right)\mathrm{d}y \right)\mathrm{d}x = \frac{1}{8}\int_{x=0}^{x=2}\left[x^2y + \frac{y^3}{3}\right]_{y=x^2}^{y=2x}\mathrm{d}x$$

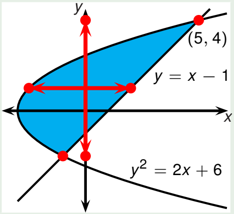
$$= \frac{1}{8}\int_0^2\left(2x^3 + \frac{8}{3}x^3 - x^4 - \frac{x^6}{3}\right)\mathrm{d}x$$

$$= \frac{1}{8}\left[-\frac{1}{21}x^7 - \frac{1}{5}x^5 + \frac{7}{6}x^4\right]_{x=0}^{x=2}$$

$$= \frac{27}{35}$$

Example





Let $\mathcal{R}$ be the region bounded by $y = x - 1$ and $y^2 = 2x + 6$. Compute

$$\iint_{\mathcal{R}} \left(2 + \frac{1}{4}xy\right) \mathrm{d}x\mathrm{d}y.$$

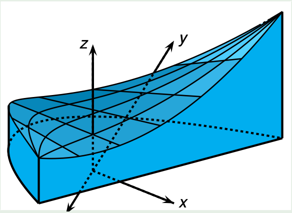
Plot $x - 1$. Plot $y^2 = 2x + 6$. Identify the region. The two curves intersect when

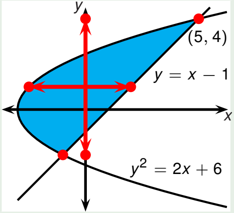
$$\begin{aligned} (x-1)^2 &= 2x+6 \\ x^2 - 2x + 1 &= 2x + 6 \\ x^2 - 4x - 5 &= 0 \\ x &= -1 \text{ or } 5. \end{aligned}$$

The two intersection points are $(-1, -2)$ and $(5, 4)$. The function can be plotted as above. The integral becomes:

$$\int_{y=-2}^{y=4} \int_{x=\frac{y^2-6}{2}}^{x=y+1} \left(2 + \frac{1}{4}xy\right) \mathrm{d}x\mathrm{d}y$$

Example





Let $\mathcal{R}$ be the region bounded by $y = x - 1$ and $y^2 = 2x + 6$. Compute

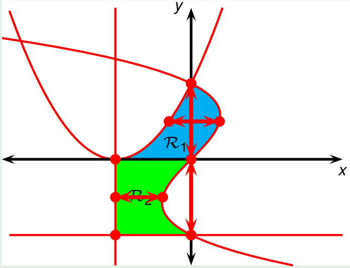
$$\iint_{\mathcal{R}} \left(2 + \frac{1}{4}xy\right) dxdy.$$

$$\int_{y=-2}^{y=4}\int_{x=\frac{y^2-6}{2}}^{x=y+1}\left(2+\frac{1}{4}xy\right)dxdy = \int_{y=-2}^{y=4}\left[2x+\frac{x^2y}{8}\right]_{x=\frac{y^2-6}{2}}^{x=y+1}dy$$

$$= \int_{y=-2}^{y=4}\left(-\frac{1}{32}y^5+\frac{1}{2}y^3 - \frac{3}{4}y^2+y+8\right)dy$$

$$= \left[-\tfrac{1}{192}y^6+\tfrac{1}{8}y^4 - \tfrac{1}{4}y^3+\tfrac{1}{2}y^2+8y\right]_{-2}^{4} = 45$$

Example



Let $\mathcal{R}$ be region bounded by $y = (x+1)^2$, $x = y - y^3$, the line $x = -1$ and the line $y = -1$. Set-up iterated integrals for

$$\iint_{\mathcal{R}} f dA.$$

Plot $x = -1$. Plot $y = -1$. Plot $y = (x+1)^2$. Plot $x = y - y^3$. Identify the region. Compute the intersection points: the four points lying on the boundary of our region have coordinates: $(-1, -1), (0, -1), (-1, 0), (0, 1)$. Split into two curvilinear trapezoids: $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$, where $\mathcal{R}_1, \mathcal{R}_2$ are as indicated. The integral becomes:

$$\iint\limits_{\mathcal{R}_1} f dA + \iint\limits_{\mathcal{R}_2} f dA = \int\limits_{y=0}^{y=1} \int\limits_{x=\sqrt{y}-1}^{x=y-y^3} f \,dx dy + \int\limits_{y=-1}^{y=0} \int\limits_{x=-1}^{x=y-y^3} f \,dx dy$$

Example

$\iint_{[0,\infty)\times[0,\infty)} e^{-x-y} \mathrm{d}x\mathrm{d}y$

Example

$\int_0^1 \int_{3y}^3 e^{x^2} \mathrm{d}x\mathrm{d}y$