# Calculus III
## Lecture 17

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# Outline

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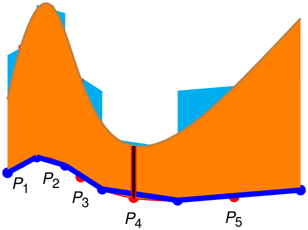
# Overview of Integrals Covered so Far

- We have so far studied integrals over:
  - intervals on a line;
  - planar regions in the plane;
  - solid regions in space.
- All studied integrals were over regions that have the same dimension as their ambient space.
- Can we make sense of an integral over region that has lower dimension then the ambient space?
- We can for arbitrary $k$-dimensional surface in $n$ dimensional space. We will only consider the examples of
  - a curve (1D region) embedded in a plane (2D)
  - a curve (1D region) embedded in space (3D)
  - a surface (2D region) embedded in space (3D).

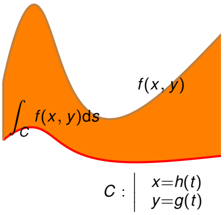
# Riemann Sums for Line Integrals




- Let $C$ be a piecewise smooth curve (endpoints included) in space.
- Let $f$ be a scalar or vector-valued function defined on $C$.
- We aim to define the integral of $f$ on $C$ with respect to arclength.
- Divide $C$ into pieces $D_1, \ldots, D_N$ with non-overlapping interiors;
- Pick a sample point $P_k$ in each $D_k$.
- The accumulation on $D_k$ is approximated by $f(P_k) \cdot \text{length}(D_k)$.
- The integral (total accum.): approximated by the Riemann sum

$$\sum_{k=1}^{N} f(P_k) \cdot \text{length}(D_k)$$

# Definition of Line Integral




## Definition

Suppose the limit

$$\lim_{\max_k(\text{segment length})\to 0} \sum_{k=1}^{N} f(P_k) \cdot \text{length}(D_k)$$

exists and is finite. Then we call this limit the *line integral of f on C with respect to arclength*, and we denote it by

$$\int_C f(x, y)ds \quad .$$

The line integral is guaranteed to exists if $f$ is a continuous functions or is bounded and continuous except at a finite number of points.

# Parametrizations and Computations

Let $\mathbf{r}\colon [a,b] \to C$ be a regular, piecewise smooth parametrization of $C$. Then $\int_C f(x,y)\mathrm{d}s$ is computed as follows.

$$\begin{aligned} \mathrm{d}s &= |\mathbf{r}'(t)|\mathrm{d}t \\ \int_{(x,y)\in C} f(x,y)\mathrm{d}s &= \int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)|\mathrm{d}t. \end{aligned}$$

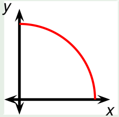
The result is independent of the parametrization of $C$ we use.

$$\mathbf{r}\colon [a,b] \to \mathbb{R}^2 \quad , \quad \mathbf{r}(t) = (x(t), y(t))$$

$$\mathrm{d}s = |\mathbf{r}'(t)|\mathrm{d}t = \sqrt{(x'(t))^2 + (y'(t))^2}\,\mathrm{d}t$$

$$\int_C f(x,y)\mathrm{d}s = \int_a^b f(x(t), y(t))\sqrt{(x'(t))^2 + (y'(t))^2}\mathrm{d}t\ .$$

## Example




Compute $\int_C x^2 y ds$, where $C$ is the first quadrant part of the circle of radius 2 centered at origin and d$s$ is the arclength form.

A parametrization of $C$ can be given as
$\mathbf{r}(t) = (x(t), y(t)) = (2\cos t, 2\sin t)$, $0 \leq t \leq \frac{\pi}{2}$.

$$
\begin{aligned}
\mathrm{d}s &= |\mathbf{r}'(t)|\mathrm{d}t = \left|\left(-2\sin t, 2\cos t\right)\right|\mathrm{d}t \\
&= 2\sqrt{(-\sin t)^2 + (\cos t)^2}\mathrm{d}t = 2\mathrm{d}t \\
\int_C x^2 y \, \mathrm{d}s &= \int_{t=0}^{t=\frac{\pi}{2}} \left(8\cos^2 t \sin t\right) 2\mathrm{d}t \\
&= 16\left[\frac{-\cos^3 t}{3}\right]_{t=0}^{t=\frac{\pi}{2}} = \frac{16}{3} .
\end{aligned}
$$

# Line Integrals from Vector Fields

- Let $C$ be piecewise smooth, *oriented* curve, parametrized via $\mathbf{r}(t)$.
- Let $\mathrm{d}s$ be the element of arclength.
- Let $\mathbf{F}$ be a continuous vector defined on $C$.
- Let $\mathbf{T}$ be unit tangent vector on $C$ compatible with orientation.
- Let $\mathbf{N}$ be unit vector perpendicular to $\mathbf{T}$ (only for planar curves).

## Definition

In any dimension, define the line integral of $\mathbf{F}$ along $C$ as

$$\int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \int_C \mathbf{F}\cdot\mathbf{T}\,\mathrm{d}s, \text{ where } \mathrm{d}\mathbf{r} = \mathbf{T}\,\mathrm{d}s\ .$$

In dimension 2, define the line integral of $\mathbf{F}$ across $C$ as

$$\int_C \mathbf{F}\cdot\mathbf{N}\,\mathrm{d}s = \int_C \mathbf{F}\cdot \mathrm{d}\mathbf{n}, \text{ where } \mathrm{d}\mathbf{n} = \mathbf{N}\,\mathrm{d}s\ .$$

- Line integral = work done by force $\mathbf{F}$ on particle moving along $C$.
- Line integral across $C$ = flux across a membrane: $\mathbf{F}\cdot\mathbf{N}$ is the normal component of $\mathbf{F}$.

# Line Integral Computations

$$\int_C \mathbf{F} \cdot \mathbf{dr} = \int_C \mathbf{F} \cdot \mathbf{T}\, \mathrm{d}s$$

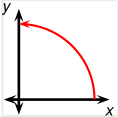
$\mathbf{r}\colon [a, b] \to C$: regular parametrization compatible with the orientation. Recall that

$$\mathbf{T} = \frac{1}{|\mathbf{r}'(t)|}\mathbf{r}'(t), \quad \mathrm{d}s = |\mathbf{r}'(t)|\, \mathrm{d}t \ \Rightarrow \mathbf{T}\, \mathrm{d}s = \mathbf{r}'(t)\, \mathrm{d}t = \mathbf{dr}\ .$$

Let $\mathbf{F}$ be given by $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$. Then we can compute the line integral as follows.

$$\begin{aligned}
\mathbf{r}'(t) &= (x'(t), y'(t)) = x'(t)\mathbf{i} + y'(t)\mathbf{j}\\
\mathbf{F} \cdot \mathbf{dr} &= \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, \mathrm{d}t = \left(P(x(t), y(t))x'(t) + Q(x(t), y(t))y'(t)\right) \mathrm{d}t\\
\int_C \mathbf{F} \cdot \mathbf{dr} &= \int_{t=a}^{t=b} \left(P(x(t), y(t))x'(t) + Q(x(t), y(t))y'(t)\right) \mathrm{d}t
\end{aligned}$$

## Example




Find the work done by the force $\mathbf{F} = (x, -y) = x\,\mathbf{i} - y\,\mathbf{j}$ on a particle moving from $(1, 0)$ to $(0, 1)$ along the quarter of the unit circle contained in the first quadrant.

A parametrization of $C$ compatible with the given orientation is $\mathbf{r}(t) = (\cos t, \sin t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}, t \in \left[0, \frac{\pi}{2}\right]$.

$$\begin{aligned}
\mathbf{F} &= (\cos t)\mathbf{i} - (\sin t)\mathbf{j} \\
\mathbf{r}'(t) &= -(\sin t)\mathbf{i} + (\cos t)\mathbf{j} \\
\mathbf{F}\cdot\mathbf{r}'(t) &= -2\sin t\cos t \\
W &= \int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \int_{t=0}^{t=\frac{\pi}{2}} \mathbf{F}\cdot\mathbf{r}'(t)\mathrm{d}t \\
&= \int_{t=0}^{t=\frac{\pi}{2}} -2\sin t\cos t\,\mathrm{d}t = \left[\cos^2 t\right]_{t=0}^{t=\frac{\pi}{2}} = -1\ .
\end{aligned}$$

What if the parametrization is not compatible with the orientation?

# Differential 1-Forms

Consider the expression $\mathbf{F}\cdot \mathrm{d}\mathbf{r}$. Since $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j}$, we have $\mathrm{d}\mathbf{r} = \mathbf{i}\mathrm{d}x + \mathbf{j}\mathrm{d}y$. If $\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$, then

$$\mathbf{F}\cdot \mathrm{d}\mathbf{r} = P(x,y)\mathrm{d}x + Q(x,y)\mathrm{d}y\ .$$

## Definition (Differential 1-form)

An expression of the type

$$\begin{array}{rcl|l} \omega &=& P(x)\,\mathrm{d}x & \text{(in 1D)} \\ \omega &=& P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}y & \text{(in 2D)} \\ \omega &=& P(x,y,z)\,\mathrm{d}x + Q(x,y,z)\,\mathrm{d}y + R(x,y,z)\mathrm{d}z & \text{(in 3D)} \end{array}$$

is called a *1-form*.

- $\int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r}$ is the integral of a 1-form over the oriented curve $C$.
- The definite integral $\int_a^b P(x)\,\mathrm{d}x$ actually means
  - the integral of the 1-form $\omega = P(x)\,\mathrm{d}x$
  - on the segment with endpoints $a$ and $b$
  - oriented from $a$ to $b$.

# Integrals of 1-Forms

Let $\omega = P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}y$ be a 1-form, let $C$ be an oriented curve. Let $\mathbf{r}\colon [a,b] \to C$, $\mathbf{r}(t) = (x(t), y(t))$ be an orientation-compatible parametrization. Consider

$$\int_C \omega = \int_C P(x,y)\mathrm{d}x + Q(x,y)\mathrm{d}y \quad .$$

We compute this integral as follows.

$$\begin{aligned} P(x,y)\mathrm{d}x + Q(x,y)\mathrm{d}y &\overset{\substack{\mathrm{d}x = x'(t)\mathrm{d}t \\ \mathrm{d}y = y'(t)\mathrm{d}t}}{=} \left(P(x(t),y(t))x'(t) + Q(x(t),y(t))y'(t)\right)\mathrm{d}t \\ \int_C P(x,y)\mathrm{d}x + Q(x,y)\mathrm{d}y &= \int_a^b \left(P(x(t),y(t))x'(t) + Q(x(t),y(t))y'(t)\right)\mathrm{d}t \end{aligned}$$

If we re-parametrize the curve, the substitution rule and the multivariable chain rule imply that the integral doesn't change.

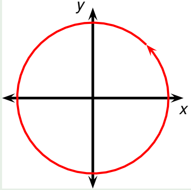
# The Notation $\oint$: Closed Path Integrals

- Suppose that the curve image $C$, parametrized by $\mathbf{r}(t) : (x(t), y(t)), t \in [a, b]$ is a closed curve.
- That is, the start point and the end point coincide.
- In other words $(x(a), y(a)) = (x(b), y(b))$.
- Let $\omega$ be a 1-form.
- Then we sometimes use the notation

$$\oint_C \omega = \int_C \omega.$$

- The circle around the first integral simply indicates the path is closed.
- The notation is mostly useful when we are integrating an closed 1-form. (Definition of closed form is/will be studied separately).

## Example




Let $C$ be a circle of radius $R$ centered at the origin, oriented counterclockwise. Compute the integral

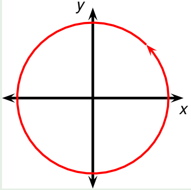
$$\oint_C -\frac{y}{x^2+y^2}\mathrm{d}x + \frac{x}{x^2+y^2}\mathrm{d}y$$

Parametrize: $C: \left| \begin{array}{rcl} x &=& R\cos t \\ y &=& R\sin t \\ \mathrm{d}x &=& (-R\sin t)\mathrm{d}t \\ \mathrm{d}y &=& (R\cos t)\mathrm{d}t \end{array} \right., 0 \leq t \leq 2\pi.$

$$\begin{aligned} -\frac{y}{x^2+y^2}\mathrm{d}x + \frac{x}{x^2+y^2}\mathrm{d}y &= -\frac{R\sin t(-R\sin t\mathrm{d}t)}{R^2} + \frac{R\cos t(R\cos t\mathrm{d}t)}{R^2} \\ &= (\cos^2 t + \sin^2 t)\mathrm{d}t = \mathrm{d}t \end{aligned}$$

$$\oint_C -\frac{y}{x^2+y^2}\mathrm{d}x + \frac{x}{x^2+y^2}\mathrm{d}y = \int_0^{2\pi} \mathrm{d}t = [t]_0^{2\pi} = 2\pi .$$

## Example




Let $C$ be a circle of radius $R$ centered at the origin, oriented counterclockwise. Compute the integral

$$\oint_C \frac{x}{x^2+y^2}\mathrm{d}x + \frac{y}{x^2+y^2}\mathrm{d}y$$

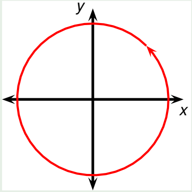
Parametrize: $C : \left| \begin{array}{rcl} x &=& R\cos t \\ y &=& R\sin t \\ \mathrm{d}x &=& (-R\sin t)\mathrm{d}t \\ \mathrm{d}y &=& (R\cos t)\mathrm{d}t \end{array} \right.$, $0 \leq t \leq 2\pi$.

$$\begin{aligned}\frac{x}{x^2+y^2}\mathrm{d}x + \frac{y}{x^2+y^2}\mathrm{d}y &= \frac{R\cos t(-R\sin t\mathrm{d}t)}{R^2} + \frac{R\sin t(R\cos t\mathrm{d}t)}{R^2} \\ &= 0\end{aligned}$$

$$\oint_C \frac{x}{x^2+y^2}\mathrm{d}x - \frac{y}{x^2+y^2}\mathrm{d}x = 0 \quad .$$

# 1-Forms in Polar Coordinates

## Example




Let $C$ be a circle of radius $R$ centered at the origin, oriented counterclockwise. Compute the integral

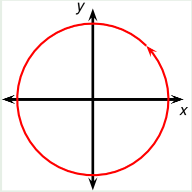
$$\oint_C -\frac{y}{x^2+y^2}\mathrm{d}x + \frac{x}{x^2+y^2}\mathrm{d}y$$

In polar coord.: $\left| \begin{array}{rcl} x &=& r\cos\theta \\ y &=& r\sin\theta \end{array} \right. \Rightarrow \left| \begin{array}{l} \mathrm{d}x = \cos\theta\mathrm{d}r - r\sin\theta\mathrm{d}\theta \\ \mathrm{d}y = \sin\theta\mathrm{d}r + r\cos\theta\mathrm{d}\theta \end{array} \right.$ .

$$\begin{array}{rcl} -\dfrac{y}{x^2+y^2}\mathrm{d}x + \dfrac{x}{x^2+y^2}\mathrm{d}y &=& -\dfrac{r\sin\theta}{r^2}(\cos\theta\mathrm{d}r - r\sin\theta\mathrm{d}\theta) \\ && +\dfrac{r\cos\theta}{r^2}(\sin\theta\mathrm{d}r + r\cos\theta\mathrm{d}\theta) \\ &=& (\cos^2\theta + \sin^2\theta)\mathrm{d}\theta \\ &=& \mathrm{d}\theta \end{array}$$

# 1-Forms in Polar Coordinates

## Example



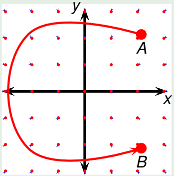

Let $C$ be a circle of radius $R$ centered at the origin, oriented counterclockwise. Compute the integral

$$\oint_C \frac{x}{x^2+y^2}\mathrm{d}x + \frac{y}{x^2+y^2}\mathrm{d}y$$

In polar coord.: $\left|\begin{array}{rcl} x &=& r\cos\theta \\ y &=& r\sin\theta \end{array}\right. \Rightarrow \left|\begin{array}{l} \mathrm{d}x = \cos\theta\mathrm{d}r - r\sin\theta\mathrm{d}\theta \\ \mathrm{d}y = \sin\theta\mathrm{d}r + r\cos\theta\mathrm{d}\theta \end{array}\right.$ .

$$\begin{aligned} \frac{x}{x^2+y^2}\mathrm{d}x + \frac{y}{x^2+y^2}\mathrm{d}y &= \frac{r\cos\theta}{r^2}(\cos\theta\mathrm{d}r - r\sin\theta\mathrm{d}\theta) \\ &\quad + \frac{r\sin\theta}{r^2}(\sin\theta\mathrm{d}r + r\cos\theta\mathrm{d}\theta) \\ &= \frac{1}{r}(\cos^2\theta + \sin^2\theta)\mathrm{d}r \\ &= \frac{1}{r}\mathrm{d}r = \mathrm{d}(\ln r) \end{aligned}$$

## Example (Work Done by Point Mass Gravity Field)




Let **F** be the vector field

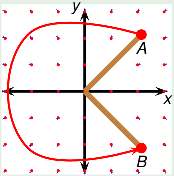
$$\mathbf{F}(\mathbf{v}) = -\frac{1}{|\mathbf{v}|^3}\mathbf{v} \quad ,$$

Let $C$ be a smooth curve with endpoints $A$ and $B$.
What is the work $W$ done by the field **F** on a particle moving from $A$ to $B$ along $C$?

Let $\mathbf{r}\colon [a,b] \to C$ be a parametrization of $C$ with $A = \mathbf{r}(a)$ and $B = \mathbf{r}(b)$.

$$W = \int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot \mathbf{r}'(t)\,\mathrm{d}t = \int_a^b \left(-\frac{1}{|\mathbf{r}(t)|^3}\mathbf{r}(t)\cdot \mathbf{r}'(t)\right)\mathrm{d}t \,.$$

## Example (Work Done by Point Mass Gravity Field)




Let **F** be the vector field

$$\mathbf{F}(\mathbf{v}) = -\frac{1}{|\mathbf{v}|^3}\mathbf{v} \quad ,$$

Let $C$ be a smooth curve with endpoints $A$ and $B$. What is the work $W$ done by the field **F** on a particle moving from $A$ to $B$ along $C$?

$$\mathbf{r}(t)\cdot\mathbf{r}'(t) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r}(t)\cdot\mathbf{r}(t)) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{r}(t)|^2.$$

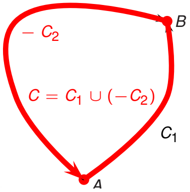
Set $u = |\mathbf{r}(t)|^2 \Rightarrow \frac{1}{2}\mathrm{d}u = \mathbf{r}(t)\cdot\mathbf{r}'(t)\mathrm{d}t$.

$$W = \int_C \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\mathrm{d}t = \int_a^b \left(-\frac{1}{|\mathbf{r}(t)|^3}\mathbf{r}(t)\cdot\mathbf{r}'(t)\right)\mathrm{d}t\,.$$

$$= \int_{u=|\mathbf{r}(a)|^2}^{u=|\mathbf{r}(b)|^2}\left(-\frac{1}{u^{\frac{3}{2}}}\right)\frac{1}{2}\mathrm{d}u = \left[u^{-\frac{1}{2}}\right]_{u=|\mathbf{r}(a)|^2}^{u=|\mathbf{r}(b)|^2} = \frac{1}{|\mathbf{r}(b)|} - \frac{1}{|\mathbf{r}(a)|}$$

In this example, we established that the line integral depends only on the endpoints $A$ and $B$ but not on the connecting path.

# Conservative Fields




## Definition

A vector field $\mathbf{F}$ is called *conservative* if for any two points $A$ and $B$ and any two paths $C_1$ and $C_2$ from $A$ to $B$ we have $\displaystyle\int_{C_1} \mathbf{F}\cdot \mathbf{dr} = \int_{C_2} \mathbf{F}\cdot \mathbf{dr}$ .

## Lemma (alternative definition)

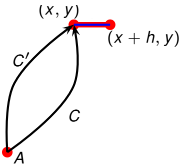
*A vector field* $\mathbf{F}$ *is conservative if and only if every point A every path C starting and ending at A we have* $\displaystyle\oint_{C} \mathbf{F}\cdot \mathbf{dr} = 0$ .

## Proof.

The path $C = C_1 \cup (-C_2)$ starts and ends at $A$ and therefore

$$0 = \oint_C \mathbf{F}\cdot \mathbf{dr} = \int_{C_1} \mathbf{F}\cdot \mathbf{dr} - \int_{C_2} \mathbf{F}\cdot \mathbf{dr}\ . \Leftrightarrow \int_{C_1} \mathbf{F}\cdot \mathbf{dr} = \int_{C_2} \mathbf{F}\cdot \mathbf{dr}. \quad \square$$

# Conservative Field $\Rightarrow$ Gradient Field




Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ be a smooth conservative field. Fix pt. $A$ inside the domain of $\mathbf{F}$. Define $f$ by $f(B) = \int_C \mathbf{F} \cdot \mathbf{dr}$, where $C$ is any piecewise smooth curve from $A$ to $B$.

## Theorem

$$\mathbf{F} = \nabla f.$$

## Proof.

Let $h > 0$; for $h$ small, the segment $S$ from $(x, y)$ to $(x+h, y)$ is in the domain of $\mathbf{F}$. $S$ is given by $\mathbf{r}(t) = (x+t)\mathbf{i} + y\mathbf{j}$, $t \in [0, h]$. On $S$, $\mathrm{d}\mathbf{r} = \mathbf{i}\mathrm{d}t$.

$$\begin{aligned}\frac{\partial}{\partial x}(f) &= \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h} = \lim_{h\to 0} \frac{1}{h}\left(\int_{C+S} \mathbf{F}\cdot\mathbf{dr} - \int_C \mathbf{F}\cdot\mathbf{dr}\right)\\ &= \lim_{h\to 0} \frac{1}{h}\int_S \mathbf{F}\cdot\mathbf{dr}\\ &= \lim_{h\to 0} \frac{1}{h}\int_{t=0}^{t=h} (P(x+t,y)\mathbf{i} + Q(x+t,y)\mathbf{j})\cdot\mathbf{i}\mathrm{d}t\\ &= \lim_{h\to 0} \frac{1}{h}\int_{t=0}^{h} P(x+h,y)\mathrm{d}t = P(x,y),\end{aligned}$$

where the last equality is the single-variable Fundamental Theorem of Calculus. Similarly it follows that $\frac{\partial}{\partial y}(f) = Q(x, y)$. $\square$

# Gradient Field $\Rightarrow$ Conservative Field

## Theorem (Fundamental Theorem of Calculus for Line Integrals)

$$\int_C (\nabla f)\cdot \mathrm{d}\mathbf{r} = f(B) - f(A)\ ,$$ *for every smooth curve $C$ from $A$ to $B$.*

## Proof.

$$\begin{aligned}\int_C (\nabla f)\cdot \mathrm{d}\mathbf{r} &= \int_C f_x \mathrm{d}x + f_y \mathrm{d}y = \int_C \left(f_x x'(t) + f_y y'(t)\right)\mathrm{d}t\\ &= \int_a^b \frac{\mathrm{d}}{\mathrm{d}t}(f(\mathbf{r}(t))\mathrm{d}t = f(B) - f(A).\end{aligned}$$ $\square$

## Definition

If $\mathbf{F} = \nabla f$ then $f$ is called *scalar potential* of $\mathbf{F}$; $\mathbf{F}$ is called *gradient field*.

Let $\mathbf{F} = \nabla f$ be gradient field. For a curve $C$ joining points $A$ and $B$

$$\int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \int_C (\nabla f)\cdot \mathrm{d}\mathbf{r} = f(B) - f(A)$$

depends only on $A$ and $B$, but not on $C$ $\Rightarrow$ $\mathbf{F}$ is conservative.

# A Criterion for Conservative (Gradient) Fields

- Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ be a smooth conservative (gradient) field.
- Then for some $f$, $\mathbf{F} = \nabla f$, hence

$$P = f_x, \qquad Q = f_y .$$

- Since mixed partial derivatives are equal, it follows that

$$P_y = (f_x)_y = f_{xy} = f_{yx} = (f_y)_x = Q_x .$$

**Proposition**

*If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ is a gradient field, then $P_y = Q_x$ .*

- A similar consideration in 3 dimensions shows the following.

**Proposition**

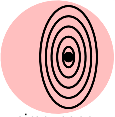
*If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ is a gradient field, then*

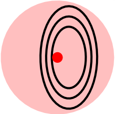
$$P_y = Q_x, \quad P_z = R_x, \quad Q_z = R_y .$$

# Simply Connected Regions

If $P_y(x,y) \neq Q_x(x,y)$, then **F** is not a gradient field.
If $P_y(x,y) = Q_x(x,y)$, is **F** necessarily a gradient field? No:

$$\begin{aligned} \mathbf{F} &= -\frac{y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j} = P\,\mathbf{i} + Q\,\mathbf{j} \\ P_y &= \frac{y^2-x^2}{(x^2+y^2)^2} = Q_x \\ \oint_{C=\text{circ. around } (0,0)} \mathbf{F}\cdot\mathrm{d}\mathbf{r} &= \oint_C \left(-\frac{y}{x^2+y^2}\mathrm{d}y + \frac{x}{x^2+y^2}\mathrm{d}y\right) = 2\pi \neq 0\,. \end{aligned}$$


simp. conn.


not simp. conn.

## Definition

A domain $D$ is called *simply connected* if every closed loop in $D$ can be deformed ("lassoed") to a point inside $D$.

## Theorem

*Let* $\mathbf{F} = P\,\mathbf{i} + Q\mathbf{j}$ *and* $P_y = Q_x$. *Suppose* **F** *is defined over a simply connected open set. Then* **F** *is a gradient field.*

## Example

Show the field $\mathbf{F} = (3 + 2xy)\mathbf{i} + (x^2 - 3y^2)\mathbf{j}$ is gradient and find a scalar potential. Compute $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$, where $C$ is any curve from $(1, 0)$ to $(0, 1)$.

$$\begin{array}{rcl}
P &=& 3 + 2xy \\
Q &=& x^2 - 3y^2 \\
P_y &=& 2x = Q_x \\
\mathbf{F} &=& \nabla f = (f_x, f_y) \\
f_x &=& P = 3 + 2xy \\
f_y &=& Q = x^2 - 3y^2 \\
f &=& \int P\mathrm{d}x = 3x + x^2y + C(y) \\
Q &=& f_y = x^2 + C'(y) \\
C'(y) &=& -3y^2 \\
C(y) &=& \int C'(y)\mathrm{d}y = \int(-3y^2)\mathrm{d}y = -y^3 + K \\
f &=& 3x + x^2y - y^3 + K \\
\int_C \mathbf{F} \cdot \mathrm{d}r &=& f(0, 1) - f(1, 0) \\
&=& (-1 + K) - (3 + K) = -4 \quad .
\end{array}$$

$\mathbb{R}^2$ – simp. conn.
$\Rightarrow$ potential exists

$C(y)$ depends on $y$

# Exact 1-Forms

Let $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ be a vector field and $\omega = \mathbf{F} \cdot \mathrm{d}\mathbf{r} = P\mathrm{d}x + Q\mathrm{d}y$ be the corresponding 1-form.

$$
\begin{array}{rcll}
\mathbf{F} & = & \nabla f & \Rightarrow \\
P & = & f_x & \\
Q & = & f_y & \Rightarrow \\
P\mathrm{d}x + Q\mathrm{d}y & = & f_x\mathrm{d}x + f_y\mathrm{d}y & \Rightarrow \\
\omega & = & \mathrm{d}f &
\end{array}
$$

## Definition

1-forms that are (total) differentials of functions are called *exact*.

$\mathbf{F}$ is a gradient field $\Rightarrow$ the 1-form $\omega = \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ is exact

## Theorem (Net Change Theorem for Line Integrals)

*If $C$ is a curve from $A$ and $B$, then* $\displaystyle\int_C \mathrm{d}f = \int_C (\nabla f) \cdot \mathrm{d}\mathbf{r} = f(B) - f(A)$.