Precalculus
Homework Lecture 14

1. Express each of the following as a single power.

   (a) $\dfrac{2^5 \cdot 2^7}{2\sqrt{2}}$

   answer: $2^{10.5} = 2^{\frac{21}{2}}$

   (b) $\dfrac{3^2 \cdot 3^{-1}}{3^3 \cdot \sqrt{3^3}}$

   answer: $3^{-\frac{7}{2}}$

   (c) $\dfrac{\pi^3}{\pi^{-1}\sqrt{\pi^5}}$

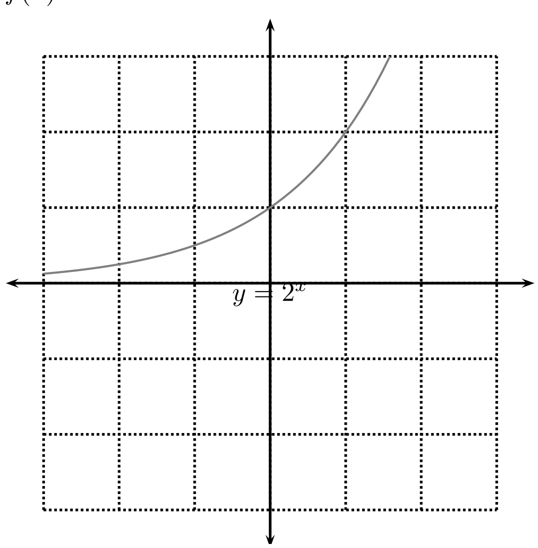
   answer: $\pi^{\frac{3}{2}}$

   Solution. 1.b.

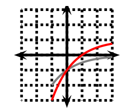
$$
\begin{aligned}
\frac{3^2 \cdot 3^{-1}}{3^3 \cdot \sqrt{3^3}} &= \frac{3^2 \cdot 3^{-1}}{3^3 \cdot (3^3)^{\frac{1}{2}}} \\
&= \frac{3^2 \cdot 3^{-1}}{3^3 \cdot 3^{\frac{3}{2}}} \\
&= \frac{3^{2-1}}{3^{3+\frac{3}{2}}} \\
&= \frac{3^1}{3^{\frac{9}{2}}} \\
&= 3^{1-\frac{9}{2}} \\
&= 3^{-\frac{7}{2}}.
\end{aligned}
$$

2. Sketch by hand approximately the given function. The function is obtained by transforming linearly the graph of a known function. The known function has been sketched for you by computer.

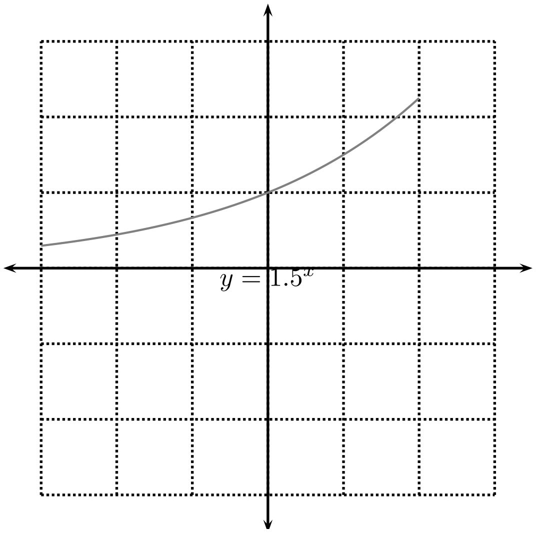
   (a) $f(x) = 2^{x+1} - 1$.



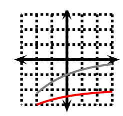
answer:



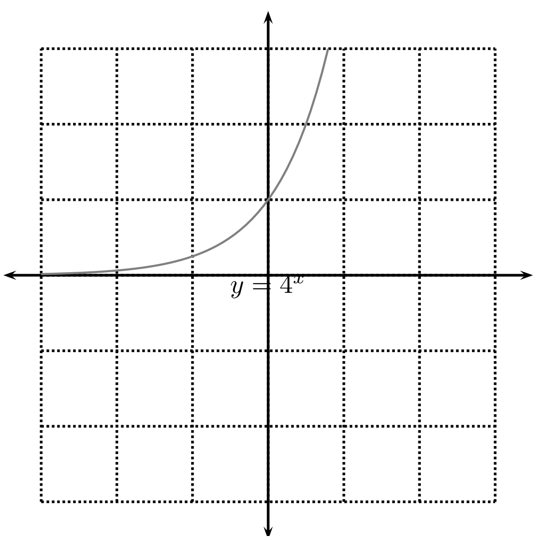
(b) $f(x) = 1.5^{x-2} + 2$.



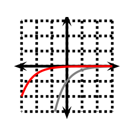
answer:



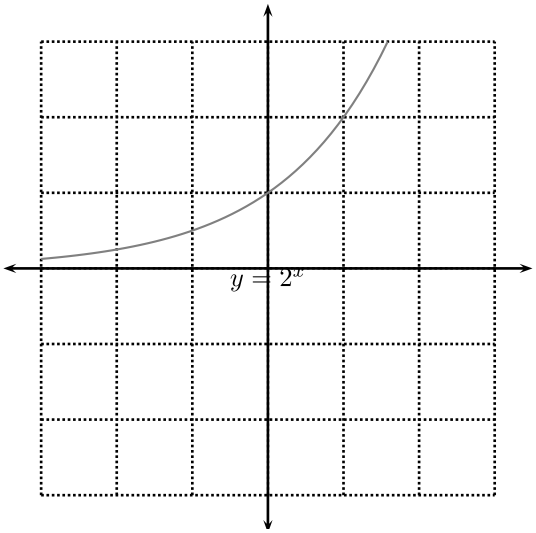
(c) $f(x) = 2^{2x-5}$.

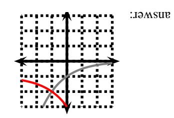


answer:

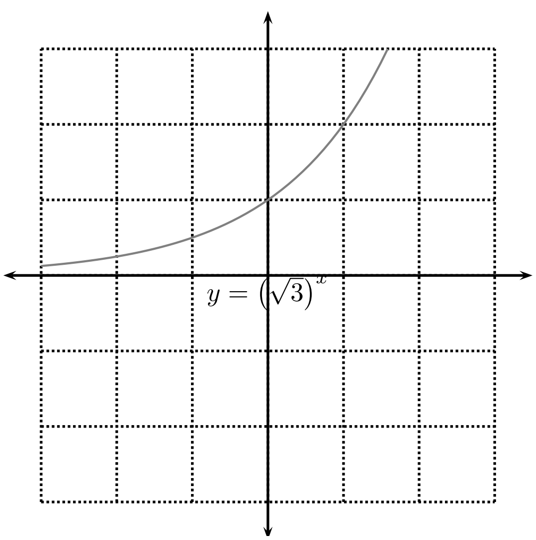


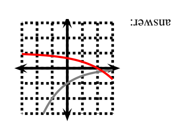
(d) $f(x) = \dfrac{1}{2^{x-1}} + 1$.





(e) $f(x) = \dfrac{1}{3^{\frac{1}{2}x+1}} - 1$.





3. (a) A sum is held under a yearly compound interest of 1%. Make an approximation by hand (no calculators allowed) by what factor will have the money increased after 200 years. Can you do the computation in your head?

(b) Decide, without using a calculator, which is more profitable: earning a yearly compound interest of 2% for 150 years or earning yearly simple interest of 11% for 150 years?

Solution. 3.a Each year, the sum increases by a factor of $\left(1+\frac{1}{100}\right)$. Therefore in 200 years the sum will have increased by

$$\begin{array}{rcll}\left(1+\frac{1}{100}\right)^{200} & = & \left(\left(1+\frac{1}{100}\right)^{100}\right)^2 & \Big| \text{ equals } \left(\left(1+\frac{1}{n}\right)^n\right)^2 \text{ for } n = 100 \\ & \approx & e^2. & \end{array}$$

As a rough estimate for e we can take $e \approx 2.7$, and so $e^2 \approx 2.7^2 = 7.29$. Our sum will have increased approximately 7.3 times. A calculator computation shows that

$$\left(1+\frac{1}{100}\right)^{200} \approx 7.316018,$$

so our "in the head" estimate is fairly accurate. Notice that the calculator computation is on its own an approximation - it was carried using double floating point precision arithmetics, which does introduce some minimal errors. Such round off errors, of course, are also present in modern banking transactions, so we do not need to adjust for those.

Solution. 3.b Simple interest of 11% per 150 years a profit of

$$0.11 * 150 = 15 + 1.5 = 16.5,$$

or altogether 17.5-fold increase of our initial sum. A 2% compound interest for 150 years yields a

$$\begin{aligned} \left(1+\tfrac{2}{100}\right)^{150} &= \left(\left(1+\tfrac{1}{50}\right)^{50}\right)^{3} \\ &\approx e^3 \end{aligned}$$

-fold increase of our sum. To establish which of the two options yields more money, we need to compare e^3 to 17.5 (without using a calculator). In the solution of 3.a we established that $e^2 \approx 7.3$, so $e^3 \approx e \cdot 7.3 \approx 2.7 \cdot 7.3 = 2 \cdot 7 + 2 \cdot 0.3 + 0.7 \cdot 7 + 0.7 \cdot 0.3 = 14 + 0.6 + 4.9 + 0.21 = 19.71 \approx 19.7$. We can say that the compound interest results in approximately 19.7-fold increase of the initial sum, so the compound interest is more profitable. A calculator computation shows that

$$\left(1+\frac{2}{100}\right)^{150} \approx 19.499603 \quad .$$

Our error of approximately 0.2 was not optimal, yet fairly accurate for an "in the head" computation.