Calculus III
Lecture 18

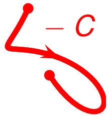
Todor Milev
https://github.com/tmilev/freecalc

2020

Outline

License to use and redistribute

Curve orientation



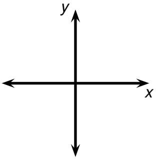
- Let C be curve image (not equipped with parametrization).
- Suppose C can be equipped with some one-to-one continuous parametrization of the form $\mathbf{r}(t)$, $t \in [a, b]$ so that $A = \mathbf{r}(a)$ (starting point), $B = \mathbf{r}(b)$ (endpoint), $A \neq B$.

Definition (Curve orientation, endpoints are distinct)

- We say the parametrization $\mathbf{r}$ *orients* the curve C.
- We say that two one-to-one parametrizations of C have *the same orientation* if they determine the same starting and endpoints.
- To orient a curve image C means to specify which of the two endpoints is a starting point and which - endpoint.

- Alternatively, to orient a curve means to specify the order in which its points are traversed (“direction of flow”).
- The definition can be extended to when the parametr. is not one-to-one (allowing $A = B$). Requires 1-dimensional manifolds.

Orientation of 2D space and Pairs of 2D Vectors



- When selecting Cartesian coord. system in the plane, the letters x and y are *a priori* equivalent.
- To select an orientation means to declare an order on the variables x and y.
- One such order is implicitly assumed when we write coordinates as by convention we write x coord. first and y-coord. - second.
- Unless stated otherwise, we assume x is first and y-second.

Definition (Orientation of pair of vectors in 2D)

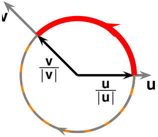
Let $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$. We say that the ordered pair of vectors $(\mathbf{u}, \mathbf{v})$ is *positively oriented* if $\begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix} > 0$.

- The definition uses the orientation of space as the coordinates of $\mathbf{u}$ and $\mathbf{v}$ are listed in the order implied by the orientation.

Orientation of 2D space and Clock Direction

Definition (Orientation of pair of vectors in 2D)

Let $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$. We say that the ordered pair of vectors $(\mathbf{u}, \mathbf{v})$ is *positively oriented* if $\begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix} > 0$.



Definition (Clock direction)

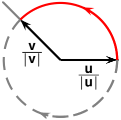
We say the vector $\mathbf{v}$ stands counterclockwise from $\mathbf{u}$ if $(\mathbf{u}, \mathbf{v})$ is a positively oriented pair of vectors.

- Multiplying det. column by positive number does not change sign.
- $\Rightarrow \frac{\mathbf{u}}{|\mathbf{u}|}, \frac{\mathbf{v}}{|\mathbf{v}|}$ have same orientation as $\mathbf{u}, \mathbf{v}$.
- Suppose $(\mathbf{u}, \mathbf{v})$ are positively oriented.
- Of the two unit circle arcs from $\frac{\mathbf{u}}{|\mathbf{u}|}$ to $\frac{\mathbf{v}}{|\mathbf{v}|}$, choose the shorter one.

Orientation of 2D space and Clock Direction

Definition (Orientation of pair of vectors in 2D)

Let $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$. We say that the ordered pair of vectors $(\mathbf{u}, \mathbf{v})$ is *positively oriented* if $\begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix} > 0$.

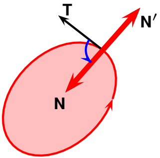


Definition (Clock direction)

We say the vector $\mathbf{v}$ stands in the positive direction from $\mathbf{u}$ if $(\mathbf{u}, \mathbf{v})$ is a positively oriented pair of vectors.

- This arc (oriented from $\frac{\mathbf{u}}{|\mathbf{u}|}$ to $\frac{\mathbf{v}}{|\mathbf{v}|}$) corresponds to positive direction.
- The positive direction is "counterclockwise", provided that
  - the orientation of space is the conventional: x first, y second;
  - the x axis is drawn horizontally to the right, the y-axis - up;
  - in case of transparent sheet of paper, we view from the "up" side.
- If any of the above changes, the notion of pos. direction may fail to correspond to the everyday use of the word "counterclockwise".

The Boundary Operator, Closed Curve Orientation



- Let D be an open set and C a closed piecewise smooth curve with parametrization $\mathbf{r}(t)$.
- Suppose the boundary of D equals C.
- Let $\mathbf{T} = \frac{\mathbf{r}}{|\mathbf{r}|}$, ($\mathbf{T}$ is the unit vector compatible with the orientation of $\mathbf{C}$).
- Let $\mathbf{N}$ be a unit vector perpendicular to $\mathbf{T}$. There are two choices for $\mathbf{N}$; we select that which points towards D as indicated.

Definition (boundary)

We say that the oriented curve C is the boundary of D if the pair of vectors $(\mathbf{T}, \mathbf{N})$ is positively oriented. We write

$$C = \partial D \quad .$$

The symbol ∂ above is called the *boundary operator*.

- When walking along the boundary ∂D, D is to the walker's left.

Green's Theorem

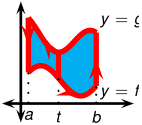
Let D be a set in the plane whose boundary $C = \partial D$ is a piecewise smooth oriented curve. Suppose P and Q functions in the plane that have continuous partial derivatives in an open region around D.

Theorem (Green)

$$\oint_C (P\mathrm{d}x + Q\mathrm{d}y) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}x\mathrm{d}y \ .$$

Companion formula:

$$\oint_C P\mathrm{d}y - Q\mathrm{d}x = \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) \mathrm{d}x\mathrm{d}y \ .$$



Theorem (Green)

$$\oint_{\partial D} (P\mathrm{d}x + Q\mathrm{d}y) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}x\mathrm{d}y \ .$$

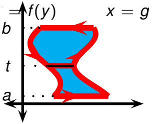
When D= representable by curv. trapezoids in both directions.

Suppose D - curv. trapezoid, vertical bases. Then ∂D is the union of:

Curve	Parametrization	parameter interval	$\mathrm{d}x$
C_1	$(t, f(t))$	$t \in [a, b]$	$\mathrm{d}t$
C_2	(b, t)	$t \in [f(b), g(b)]$	0
C_3	$(t, g(t))$	$t \in [b, a]$	$\mathrm{d}t$
C_4	(a, t)	$t \in [g(a), f(a)]$	0

.

$$\begin{aligned}
\oint_{\partial D} P\mathrm{d}x &= \int_{C_1+C_2+C_3+C_4} P\mathrm{d}x \\
&= \int_{t=a}^{t=b} P(t, f(t))\mathrm{d}t + \int_{t=b}^{t=a} P(t, g(t))\mathrm{d}t \\
&= \int_{t=a}^{t=b} \left(P(t, f(t)) - P(t, g(t))\right) \mathrm{d}t && \text{Use FTC} \\
&= \int_{t=a}^{t=b} \left(\int_{u=f(t)}^{u=g(t)} (-P_y(t, u))\, \mathrm{d}u\right) \mathrm{d}t && \text{relabel } t, u \text{ to } x, y \\
&= \iint_D \left(-\frac{\partial P}{\partial y}\right) \mathrm{d}x\mathrm{d}y.
\end{aligned}$$



Theorem (Green)

$$\oint_{\partial D} (P\mathrm{d}x + Q\mathrm{d}y) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}x\mathrm{d}y \ .$$

When D= representable by curv. trapezoids in both directions.

Suppose D - curv. trapezoid, horiz. bases. Then ∂D is the union of:

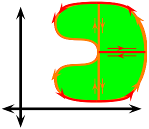
Curve	Parametrization	parameter interval	dy
C_1	$(f(t), t)$	$t \in [b, a]$	dt
C_2	(a, t)	$t \in [f(a), g(a)]$	0
C_3	$(g(t), t)$	$t \in [a, b]$	dt
C_4	(b, t)	$t \in [g(b), f(b)]$	0

.

$$\begin{aligned}
\oint_{\partial D} Q\mathrm{d}y &= \int_{C_1+C_2+C_3+C_4} Q\mathrm{d}y \\
&= \int_{t=b}^{t=a} Q(f(t), t)\mathrm{d}t + \int_{t=a}^{t=b} Q(g(t), t)\mathrm{d}t \\
&= \int_{t=a}^{t=b} \left(-Q(f(t), t) + Q(g(t), t)\right)\mathrm{d}t \\
&= \int_{t=a}^{t=b} \left(\int_{u=f(t)}^{u=g(t)} \left(Q_x(u, t)\right)\mathrm{d}u \right) \mathrm{d}t \\
&= \iint_D \left(\frac{\partial Q}{\partial x} \right) \mathrm{d}x\mathrm{d}y.
\end{aligned}$$

Use FTC

relabel t, u to x, y



Theorem (Green)

$$\oint_{\partial D} (P\mathrm{d}x + Q\mathrm{d}y) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}x\mathrm{d}y \ .$$

When D= representable by curv. trapezoids in both directions.

So far, we demonstrated that

$$\left.\begin{array}{rcl|l} \oint_{\partial D} P\mathrm{d}x &=& \iint_D \left(-\frac{\partial P}{\partial y}\right) \mathrm{d}x\mathrm{d}y & \text{curv. trapezoids vert. bases} \\ \oint_{\partial D} Q\mathrm{d}y &=& \iint_D \frac{\partial Q}{\partial x} \mathrm{d}x\mathrm{d}y & \text{curv. trapezoids horiz. bases} \end{array}\right. .$$

- Suppose D = union of curvilinear trapezoids with vertical bases, pairwise intersecting on their boundaries only. The first equality holds over each curvilinear trapezoid $\Rightarrow$ it holds over the entire D as contributions of extra line integrals cancel one another.
- Similarly if D can be represented as union of curvilinear trapezoids with horizontal bases, the second equality holds.
- Adding the two equalities proves the theorem for regions that can be decomposed by curvilinear trapezoids in both directions.

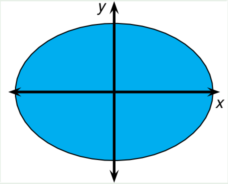
Areas using Green's Theorem

Theorem (Green)

$$\oint_{\partial D} P\mathrm{d}x + Q\mathrm{d}y = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}x\mathrm{d}y\ .$$

- One use of Green's theorem is for relating areas to certain line integrals.
- Suppose $Q_x - P_y = 1$. Then
  $\mathrm{Area}(D) = \iint_D 1\mathrm{d}x\mathrm{d}y = \iint_D (Q_x - P_y)\mathrm{d}x\mathrm{d}y = \oint_{C=\partial D} P\mathrm{d}x + Q\mathrm{d}y\ .$
- There are many ways to have $Q_x - P_y$, for example:
  - $P(x, y) = -y$ and $Q(x, y) = 0$,
  - $P(x, y) = 0$ and $Q(x, y) = y$,
  - $P(x, y) = -\frac{y}{2}$ and $Q(x, y) = \frac{x}{2}$.

Example (Areas via line integrals)

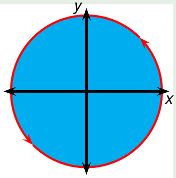


Use Green's theorem to compute the area of the region D enclosed by the ellipse C: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Let $C = \partial D$; C is parametrized by $C : \left| \begin{array}{lcl} x & = & a\cos t \\ y & = & b\sin t \end{array} \right.$, $t \in [0, 2\pi]$.

$$
\begin{aligned}
\mathrm{Area}(D) &= \iint \mathrm{d}A = \int_C x\mathrm{d}y = \int_{t=0}^{t=2\pi} a\cos t\mathrm{d}(b\sin t) \quad \Bigg| \ \text{Green's Thm.} \\
&= \int_{t=0}^{t=2\pi} a\cos(t)b\cos(t)\mathrm{d}t = ab\int_{t=0}^{t=2\pi} \cos^2 t\mathrm{d}t \\
&= \int_{t=0}^{2\pi} \left(\frac{1+\cos(2t)}{2}\right)\mathrm{d}t \\
&= ab\left[\frac{\theta}{2} + \frac{\sin(2t)}{4}\right]_{t=0}^{2\pi} = ab\pi.
\end{aligned}
$$

Example



Integrate

$$\int_C \left(y^3 + e^{\arctan x}\right) \mathrm{d}x + \left(-x^3 + \ln\left(\cos y + y + 4\right)\right) \mathrm{d}y,$$

where C is the oriented boundary of the disk D with radius 2 and centered at the origin.

Direct computation of the line integral appears intractable. Since P, Q are smooth over D we can use Green's theorem. This makes sense as P_y, Q_x are simple expressions.

$$\begin{aligned}
\int_C P\mathrm{d}x + Q\mathrm{d}y &= \int_D (Q_x - P_y)\mathrm{d}x\mathrm{d}y && \text{Green's Thm.}\\
&= \int_D \left(-3x^2 - 3y^2\right) \mathrm{d}x\mathrm{d}y && \text{use polar coords.}\\
&= \int_{\theta=0}^{2\pi} \int_{r=0}^{2} (-3r^2) r\mathrm{d}r\mathrm{d}\theta\\
&= \int_{\theta=0}^{2\pi} \left[\frac{3}{4} r^4\right]_{r=0}^{r=2} \mathrm{d}\theta = 24\pi \quad .
\end{aligned}$$

Example (Line integrals of dθ using Green's theorem)

Let C be a closed curve, enclosing an open set D, and not passing through $(0,0)$. Compute

$$\oint_C \frac{-y}{x^2+y^2}\mathrm{d}x + \frac{x}{x^2+y^2}\mathrm{d}y$$

provided that D does not contain the origin.

Since D does not contain the origin we can use Green's theorem:

$$\begin{aligned}\oint_C \tfrac{-y}{x^2+y^2}\mathrm{d}x + \tfrac{x}{x^2+y^2}\mathrm{d}y &= \oint_D \left(\tfrac{y^2-x^2}{(x^2+y^2)^2} - \tfrac{x^2-y^2}{(x^2+y^2)}\right)\mathrm{d}x\mathrm{d}y\\ &= 0.\end{aligned}$$

Example (Line integrals of dθ using Green's theorem)

Let C be a closed curve, enclosing an open set D, and not passing through $(0,0)$. Compute

$$\oint_C \frac{-y}{x^2+y^2}\mathrm{d}x + \frac{x}{x^2+y^2}\mathrm{d}y$$

provided that D contains the origin.

We cannot use Green's theorem with respect to D because the resulting double integral involve a function which is not defined at $(0,0)$. Instead we cut off a small circle at $(0,0)$.