Calculus II
Lecture 18

Todor Milev
https://github.com/tmilev/freecalc

2020

Outline

License to use and redistribute

These lecture slides and their LATEX source code are licensed to you under the Creative Commons license CC BY 3.0. You are free

- to Share - to copy, distribute and transmit the work,
- to Remix - to adapt, change, etc., the work,
- to make commercial use of the work,

as long as you reasonably acknowledge the original project.

- Latest version of the .tex sources of the slides:
  https://github.com/tmilev/freecalc
- Should the link be outdated/moved, search for "freecalc project".
- Creative Commons license CC BY 3.0:
  https://creativecommons.org/licenses/by/3.0/us/
  and the links therein.

Alternating Series

Definition (Alternating Series)

An alternating series is a series whose terms are alternately positive and negative.

Examples

Here are two examples:

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$$

$$-\tfrac{1}{2} + \tfrac{2}{3} - \tfrac{3}{4} + \tfrac{4}{5} - \tfrac{5}{6} + \tfrac{6}{7} - \cdots = \sum_{n=1}^{\infty} (-1)^{n} \frac{n}{n+1}$$

The nth term of an alternating series has the form

$$a_n = (-1)^{n-1} b_n \qquad \text{or} \qquad a_n = (-1)^n b_n$$

where b_n is positive.

Theorem (The Alternating Series Test)

If the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1 - b_2 + b_3 - b_4 + b_5 - \cdots, \qquad b_n > 0$$

satisfies

1. $b_{n+1} \leq b_n$ *for all n and*
2. $\lim_{n\to\infty} b_n = 0$

then the series is convergent.

Example

The alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

satisfies

1. $b_{n+1} < b_n$ because $\frac{1}{n+1} < \frac{1}{n}$.

2. $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} = 0$.

Therefore the series is convergent by the Alternating Series Test.

Example

The series $\sum_{n=1}^{\infty} (-1)^n \frac{3n}{4n-1}$ is alternating, but

$$\lim_{n \to \infty} b_n = \lim_{n \to \infty} \frac{3n}{4n-1} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} = \lim_{n \to \infty} \frac{3}{4 - \frac{1}{n}} = \frac{3}{4}$$

Therefore the series is divergent by the basic Divergence Test.

Estimating Sums

This theorem allows us to estimate the size of the remainder $R_n = s - s_n$ in an alternating series.

Theorem (Alternating Series Estimation Theorem)

Let $\sum (-1)^{n-1} b_n$ be the sum of an alternating series that satisfies

1. $0 \leq b_{n+1} \leq b_n$ *and*
2. $\lim_{n \to \infty} b_n = 0$.

Then the size of the error is less than the first omitted term; that is,

$$|R_n| = |s - s_n| \leq b_{n+1}.$$

Example

Find the sum of $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ correct to three decimal places. ($0! = 1$.)

1. $b_{n+1} = \dfrac{1}{(n+1)!} = \dfrac{1}{n!(n+1)} < \dfrac{1}{n!} = b_n$.

2. $0 < \dfrac{1}{n!} < \dfrac{1}{n} \to 0$, so $b_n \to 0$ as $n \to \infty$.

- Therefore the series converges by the Alternating Series Test.

$$
\begin{aligned}
s &= \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \cdots \\
&= 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} - \frac{1}{5040} + \cdots
\end{aligned}
$$

- $|s - s_6| \le b_7 = \frac{1}{5040} < 0.0002$.
- $s_6 = 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} \approx 0.368056$.
- The error of less than 0.0002 doesn't affect the third decimal place, so $s \approx s_6 \approx 0.368$.

Absolute Convergence and the Ratio and Root Tests

In this section, we start with any series $\sum a_n$ and consider the corresponding series

$$\sum |a_n| = |a_1| + |a_2| + |a_3| + \cdots$$

consisting of the absolute values of the terms of the original series.

Absolute Convergence

Definition (Absolutely Convergent)

A series $\sum a_n$ is called absolutely convergent if the series of absolute values $\sum |a_n|$ is convergent.

If $\sum a_n$ is a series with all positive terms, then $|a_n| = a_n$ and absolute convergence is the same thing as convergence in this case.

Example

The series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

is absolutely convergent because

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n^2} \right| = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

is a convergent *p*-series with $p = 2$.

Example

The alternating harmonic series
$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$
is convergent (by the alternating series test, as already demonstrated).

- Is it absolutely convergent?
$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right| = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

- This is a *p*-series with $p = 1$.
- Therefore $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right|$ is divergent.
- Therefore $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is not absolutely convergent.

Definition (Conditionally Convergent)

A series $\sum a_n$ is called conditionally convergent if it is convergent but not absolutely convergent.

- The alternating harmonic series is conditionally convergent.
- Therefore it is possible for a series to be convergent but not absolutely convergent.
- Question: Is it possible for a series to be absolutely convergent but not convergent?
- Answer: No. This is the content of the next theorem.

Theorem (Absolute Convergence Implies Convergence)

If a series is absolutely convergent, then it is convergent.

Example

Determine whether
$$\sum_{n=1}^{\infty} \frac{\cos n}{n^2} = \frac{\cos 1}{1^2} + \frac{\cos 2}{2^2} + \frac{\cos 3}{3^2} + \frac{\cos 4}{4^2} + \cdots$$
is convergent or divergent.

- The series has positive and negative terms, but is not alternating.
- Use the Comparison Test:
$$\begin{array}{ccccc} 0 & \leq & |\cos n| & \leq & 1 \\ 0 & \leq & \dfrac{|\cos n|}{n^2} & \leq & \dfrac{1}{n^2} \end{array}$$
- $\sum \frac{1}{n^2}$ is a *p*-series with $p = 2$.
- Therefore $\sum \frac{1}{n^2}$ is convergent, and so by the Comparison Test, $\sum \frac{|\cos n|}{n^2}$ is also convergent.
- Therefore $\sum \frac{\cos n}{n^2}$ is absolutely convergent.
- Therefore by the previous theorem, $\sum \frac{\cos n}{n^2}$ is convergent.

The Ratio Test

Theorem (The Ratio Test)

1. If $\lim\limits_{n \to \infty} \left| \dfrac{a_{n+1}}{a_n} \right| = L < 1$, then the series $\sum a_n$ is absolutely convergent (and therefore convergent).

2. If $\lim\limits_{n \to \infty} \left| \dfrac{a_{n+1}}{a_n} \right| = L > 1$ or $\lim\limits_{n \to \infty} \left| \dfrac{a_{n+1}}{a_n} \right| = \infty$, then the series $\sum a_n$ is divergent.

3. If $\lim\limits_{n \to \infty} \left| \dfrac{a_{n+1}}{a_n} \right| = L = 1$, then the Ratio Test is inconclusive.

The Ratio Test is inconclusive if $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = 1$.

Example

$\displaystyle\sum_{n=1}^{\infty}\dfrac{1}{n^2}$

- This is a *p*-series with $p = 2$.
- Therefore it is convergent.

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \frac{n^2}{(n+1)^2}\cdot\frac{\frac{1}{n^2}}{\frac{1}{n^2}} = \frac{1}{\left(1+\frac{1}{n}\right)^2} \to 1 \qquad \text{as } n\to\infty$$

Example

$\displaystyle\sum_{n=1}^{\infty}\dfrac{1}{n}$

- This is a *p*-series with $p = 1$.
- Therefore it is divergent.

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{1}{n+1}}{\frac{1}{n}} = \frac{n}{n+1}\cdot\frac{\frac{1}{n}}{\frac{1}{n}} = \frac{1}{1+\frac{1}{n}} \to 1 \qquad \text{as } n\to\infty$$

Example

Test the series $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$ for absolute convergence.

$$
\begin{aligned}
\left| \frac{a_{n+1}}{a_n} \right| &= \left| \frac{(-1)^{n+1} \frac{(n+1)^3}{3^{n+1}}}{(-1)^n \frac{n^3}{3^n}} \right| \\
&= \frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3} \\
&= \frac{1}{3} \left(\frac{n+1}{n} \right)^3 \\
&= \frac{1}{3} \left(1 + \frac{1}{n} \right)^3 \\
&\rightarrow \frac{1}{3} < 1
\end{aligned}
$$

Therefore the series is absolutely convergent by the Ratio Test.

Example

Test the convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}$.

$$
\begin{aligned}
\left| \frac{a_{n+1}}{a_n} \right| &= \left| \frac{\frac{(n+1)^{n+1}}{(n+1)!}}{\frac{n^n}{n!}} \right| \\
&= \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} \\
&= \frac{(n+1)(n+1)^n}{(n+1)n!} \cdot \frac{n!}{n^n} \\
&= \left(\frac{n+1}{n} \right)^n = \left(1 + \frac{1}{n} \right)^n \\
&\to e > 1
\end{aligned}
$$

Therefore the series is divergent by the Ratio Test.

The Root Test

Theorem (The Root Test)

1. If $\lim_{n \to \infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum a_n$ is absolutely convergent (and therefore convergent).

2. If $\lim_{n \to \infty} \sqrt[n]{|a_n|} = L > 1$ or $\lim_{n \to \infty} \sqrt[n]{|a_n|} = \infty$, then the series $\sum a_n$ is divergent.

3. If $\lim_{n \to \infty} \sqrt[n]{|a_n|} = L = 1$, then the Root Test is inconclusive.

If $L = 1$ in the Ratio Test, don't try the Root Test, because it will be inconclusive too.
If $L = 1$ in the Root Test, don't try the Ratio Test, because it will be inconclusive too.

Example

Test convergence of the series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$.

$$
\begin{aligned}
a_n &= \left(\frac{2n+3}{3n+2}\right)^n \\
\sqrt[n]{|a_n|} &= \frac{2n+3}{3n+2} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} \\
&= \frac{2+\frac{3}{n}}{3+\frac{2}{n}} \\
&\to \frac{2}{3} < 1
\end{aligned}
$$

Therefore the series is absolutely convergent by the Root Test.