Calculus II
Homework Review problems for the final
This is a subset of the Master Problem Sheet

1. Problems that have appeared past final(s):
   (a) Problem 2.m.
   (b) Problem 4.a.
   (c) Problem 6.e (the problem was formulated slightly differently - as an improper integral).
   (d) Problem 8.b.
   (e) Problem 9.c.
   (f) Problem 10.a.
   (g) Problem 10.b.
   (h) Problem 12.c.
   (i) Problem 13.a.
   (j) Problem 14.c.
   (k) Problem 17.c.
   (l) Problem 16.c.

2. Evaluate the indefinite integral. Illustrate all steps of your solution.

   (a) $\displaystyle\int \frac{x^3+4}{x^2+4}\mathrm{d}x$

   answer: $\frac{x^2}{2} + 2\arctan\left(\frac{x}{2}\right) - 2\ln\left(x^2+4\right) + C$

   (b) $\displaystyle\int \frac{4x^2}{2x^2-1}\mathrm{d}x$

   answer: $-\frac{1}{2}\sqrt{2}\ln\left(x+\frac{1}{2}\sqrt{2}\right) + \frac{1}{2}\sqrt{2}\ln\left(x-\frac{1}{2}\sqrt{2}\right) + 2x + C$

   (c) $\displaystyle\int \frac{x^3}{x^2+2x-3}\mathrm{d}x$

   answer: $\frac{1}{4}\ln|x-1| + \frac{27}{4}\ln|x+3| + \frac{1}{2}x^2 - 2x$

   (d) $\displaystyle\int \frac{x^3}{x^2+3x-4}\mathrm{d}x$

   answer: $\frac{1}{2}x^2 - 3x + \frac{64}{5}\ln|x+4| + \frac{1}{5}\ln|x-1| + C$

   (e) $\displaystyle\int \frac{x^3}{2x^2+3x-5}\mathrm{d}x$

   answer: $\frac{125}{56}\ln\left(x+\frac{5}{2}\right) + \frac{1}{7}\ln(x-1) + \frac{1}{4}x^2 - \frac{3}{4}x + C$

   (f) $\displaystyle\int \frac{x^2+1}{(x-3)(x-2)^2}\mathrm{d}x$

   answer: $10\ln|x-3| - 9\ln|x-2| + \frac{5}{x-2} + C$

   (g) $\displaystyle\int \frac{x^4}{(x+1)^2(x+2)}\mathrm{d}x$

   answer: $\frac{x^2}{2} - 4x - 5\ln|x+1| + 16\ln|x+2| - \frac{1}{x+1} + C$

   (h) $\displaystyle\int \frac{15x^2-4x-81}{(x-3)(x+4)(x-1)}\mathrm{d}x$

   answer: $5\ln|-x-4| + 3\ln|x-3| + 7\ln|x-1|$

   (i) $\displaystyle\int \frac{x^4+10x^3+18x^2+2x-13}{x^4+4x^3+3x^2-4x-4}\mathrm{d}x$

   Check first that $(x-1)(x+2)^2(x+1) = x^4+4x^3+3x^2-4x-4$.

   answer: $3(x+2)^{-1} + 2\ln|x+2| + \ln|x-1| + 3\ln|x+1| + x + C$

   (j) $\displaystyle\int \frac{x^4}{(x^2+2)(x+2)}\mathrm{d}x$

   answer: $\frac{x^2}{2} - 2x + \frac{8}{3}\ln|x+2| - \frac{1}{3}\ln\left(x^2+2\right) + \frac{2\sqrt{2}}{3}\arctan\left(\frac{\sqrt{2}}{2}x\right) + C$

   (k) $\displaystyle\int \frac{x^5}{x^3-1}\mathrm{d}x$

   answer: $\frac{1}{3}\ln\left|x^2+x+1\right| + \frac{1}{3}\ln|x-1| + \frac{1}{3}x^3 + C$
   $= \frac{1}{3}\ln\left|x^3-1\right| + \frac{1}{3}x^3 + C$

   (l) $\displaystyle\int \frac{x^4}{(x^2+2)(x+1)^2}\mathrm{d}x$

   answer: $x - \frac{1}{3}(x+1)^{-1} - \frac{10}{9}\ln|x+1| - \frac{4}{9}\ln\left|x^2+2\right| - \frac{2}{9}\sqrt{2}\arctan\left(\frac{\sqrt{2}}{2}x\right) + C$

   (m) $\displaystyle\int \frac{3x^2+2x-1}{(x-1)(x^2+1)}\mathrm{d}x$

   answer: $2\ln|x-1| + \frac{1}{2}\ln\left(x^2+1\right) + 3\arctan x + C$

   (n) $\displaystyle\int \frac{x^2-1}{x(x^2+1)^2}\mathrm{d}x$

   answer: $-\left(x^2+1\right)^{-1} + \frac{1}{2}\ln\left(x^2+1\right) - \ln|x|$

Solution. 2.l To integrate a rational function, we need to decompose it into partial fractions.

Since the numerator of the function is of degree greater than or equal to the denominator, we start the partial fraction decomposition by polynomial division.

	Remainder				
		$-2x^3$	$-3x^2$	$-4x$	-2
Divisor(s)	**Quotient(s)**				
$x^4+2x^3+3x^2+4x+2$	1				
	Dividend				
-	x^4				
	x^4	$+2x^3$	$+3x^2$	$+4x$	$+2$
		$-2x^3$	$-3x^2$	$-4x$	-2

Our next step is to factor the denominator:

$$x^4+2x^3+3x^2+4x+2=(x+1)^2\left(x^2+2\right).$$

Next, we combine the two steps:

$$\begin{aligned}
\frac{x^4}{x^4+2x^3+3x^2+4x+2} &= 1+\frac{-2x^3-3x^2-4x-2}{x^4+2x^3+3x^2+4x+2}\\
\frac{-2x^3-3x^2-4x-2}{x^4+2x^3+3x^2+4x+2} &= \frac{-2x^3-3x^2-4x-2}{(x+1)^2\left(x^2+2\right)}\\
&= \frac{A_1}{(x+1)}+\frac{A_2}{(x+1)^2}+\frac{A_3+A_4x}{(x^2+2)}.
\end{aligned}$$

We seek to find A_i's that turn the above expression into an identity. Just as in the solution of Problem **??**, we will use the method of coefficient comparison. We note that the solutions of Problems 2.m and **??** provide a shortcut method.

After clearing denominators, we get the following equality.

$$\begin{aligned}
-2x^3-3x^2-4x-2 &= A_1(x+1)(x^2+2)+A_2(x^2+2)\\
&\quad +(A_3+A_4x)(x+1)^2\\
0 &= (A_4+A_1+2)x^3\\
&\quad +(2A_4+A_3+A_2+A_1+3)x^2\\
&\quad +(A_4+2A_3+2A_1+4)x\\
&\quad +(A_3+2A_2+2A_1+2) \quad .
\end{aligned}$$

In order to turn the above into an identity we need to select A_i's such that the coefficients of all powers of x become zero. In other words, we need to solve the following system.

$$\begin{array}{lllll}
A_1 & & & +A_4 & =-2\\
A_1 & +A_2 & +A_3 & +2A_4 & =-3\\
2A_1 & & +2A_3 & +A_4 & =-4\\
2A_1 & +2A_2 & +A_3 & & =-2 \quad .
\end{array}$$

This is a system of linear equations. There exists a standard method for solving such systems called Gaussian Elimination (this method is also known as the row-echelon form reduction method). This method is very well suited for computer implementation. We illustrate it on this particular example; for a description of the method in full generality we direct the reader to a standard course in Linear algebra.

System status	Action
$\begin{array}{lllll} A_1 & & & +A_4 & =-2\\ A_1 & +A_2 & +A_3 & +2A_4 & =-3\\ 2A_1 & & +2A_3 & +A_4 & =-4\\ 2A_1 & +2A_2 & +A_3 & & =-2 \end{array}$	Sel. pivot column 2. Eliminate non-pivot entries.

$\begin{array}{rrrrl} A_1 & & & +A_4 & =-2 \\ & A_2 & +A_3 & +A_4 & =-1 \\ & & 2A_3 & -A_4 & =0 \\ & 2A_2 & +A_3 & -2A_4 & =2 \end{array}$	Sel. pivot column 3. Eliminate non-pivot entries.
$\begin{array}{rrrrl} A_1 & & & +A_4 & =-2 \\ & A_2 & +A_3 & +A_4 & =-1 \\ & & 2A_3 & -A_4 & =0 \\ & & -A_3 & -4A_4 & =4 \end{array}$	Sel. pivot column 4. Eliminate non-pivot entries.
$\begin{array}{rrrrl} A_1 & & & +A_4 & =-2 \\ & A_2 & & +\frac{3}{2}A_4 & =-1 \\ & & A_3 & -\frac{A_4}{2} & =0 \\ & & & -\frac{9}{2}A_4 & =4 \end{array}$	Sel. pivot column 5. Eliminate non-pivot entries.
$\begin{array}{rrrrl} A_1 & & & & =-\frac{10}{9} \\ & A_2 & & & =\frac{1}{3} \\ & & A_3 & & =-\frac{4}{9} \\ & & & A_4 & =-\frac{8}{9} \end{array}$	Final result.

Therefore, the final partial fraction decomposition is the following.

$$\begin{aligned} \frac{x^4}{x^4+2x^3+3x^2+4x+2} &= 1+\frac{-2x^3-3x^2-4x-2}{x^4+2x^3+3x^2+4x+2} \\ &= 1+\frac{-\frac{10}{9}}{(x+1)}+\frac{\frac{1}{3}}{(x+1)^2}+\frac{-\frac{8}{9}x-\frac{4}{9}}{(x^2+2)} \end{aligned}$$

Therefore we can integrate as follows.

$$\begin{aligned} \int \frac{x^4}{(x^2+2)(x+1)^2}\mathrm{d}x &= \int\left(1+\frac{-\frac{10}{9}}{(x+1)}+\frac{\frac{1}{3}}{(x+1)^2}+\frac{-\frac{8}{9}x-\frac{4}{9}}{(x^2+2)}\right)\mathrm{d}x \\ &= \int \mathrm{d}x-\frac{10}{9}\int\frac{1}{(x+1)}\mathrm{d}x+\frac{1}{3}\int\frac{1}{(x+1)^2}\mathrm{d}x \\ &\quad -\frac{8}{9}\int\frac{x}{x^2+2}\mathrm{d}x-\frac{4}{9}\int\frac{1}{x^2+2}\mathrm{d}x \\ &= x-\frac{1}{3}(x+1)^{-1}-\frac{10}{9}\log\,(x+1) \\ &\quad -\frac{4}{9}\log\,\left(x^2+2\right)-\frac{2}{9}\sqrt{2}\arctan\left(\frac{\sqrt{2}}{2}x\right)+C \end{aligned}$$

Solution. 2.k This problem can be solved directly with a substitution shortcut, or by the standard method.

Variant I (standard method).

$$\begin{aligned} \int\frac{x^5}{x^3-1}\mathrm{d}x &= \int\left(x^2+\frac{x^2}{x^3-1}\right)\mathrm{d}x && \text{Polyn. long div.} \\ &= \frac{x^3}{3}+\int\frac{x^2}{(x-1)(x^2+x+1)}\mathrm{d}x && \text{part. frac.} \\ &= \frac{x^3}{3}+\int\left(\frac{\frac{1}{3}}{x-1}+\frac{\frac{2}{3}x+\frac{1}{3}}{x^2+x+1}\right)\mathrm{d}x && \text{complete square} \\ &= \frac{x^3}{3}+\frac{1}{3}\ln|x-1|+\frac{2}{3}\int\frac{x+\frac{1}{2}}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\mathrm{d}x && \text{Set } \begin{array}{rcl} u &=& \left(x+\frac{1}{2}\right)^2+\frac{3}{4} \\ \frac{1}{2}\mathrm{d}u &=& \left(x+\frac{1}{2}\right)\mathrm{d}x \end{array} \\ &= \frac{x^3}{3}+\frac{1}{3}\ln|x-1|+\frac{1}{3}\int\frac{\mathrm{d}u}{u} \\ &= \frac{x^3}{3}+\frac{1}{3}\ln|x-1|+\frac{1}{3}\ln|u|+C \\ &= \frac{x^3}{3}+\frac{1}{3}\ln|x-1|+\frac{1}{3}\ln|x^2+x+1|+C \end{aligned}$$

Variant II (shortcut method).

$$
\begin{aligned}
\int \frac{x^5}{x^3-1}\mathrm{d}x &= \int \frac{x^5-x^2+x^2}{x^3-1}\mathrm{d}x \\
&= \int \frac{x^2(x^3-1)+x^2}{x^3-1}\mathrm{d}x \\
&= \int x^2\mathrm{d}x + \int \frac{x^2}{x^3-1}\mathrm{d}x \\
&= \frac{x^3}{3} + \int \frac{\mathrm{d}\left(\frac{x^3}{3}\right)}{x^3-1} \\
&= \frac{x^3}{3} + \frac{1}{3}\int \frac{\mathrm{d}\left(x^3-1\right)}{x^3-1} && \Big| \text{ Set } u = x^3-1 \\
&= \frac{x^3}{3} + \frac{1}{3}\int \frac{\mathrm{d}u}{u} \\
&= \frac{x^3}{3} + \frac{1}{3}\ln|u| + C \\
&= \frac{x^3}{3} + \frac{1}{3}\ln\left|x^3-1\right| + C \quad .
\end{aligned}
$$

The answers obtained in the two solution variants are of course equal since

$$\ln|x-1| + \ln|x^2+x+1| = \ln\left|(x-1)\left(x^2+x+1\right)\right| = \ln\left|x^3-1\right| \quad .$$

Solution. 2.m. This is a concise solution written in a form suitable for exam taking. To make this solution as short as possible we have omitted many details. On an exam, the student would be expected to carry out those omitted computations on the side. We set up the partial fraction decomposition as follows.

$$\frac{3x^2+2x-1}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1} \quad .$$

Therefore $3x^2+2x-1 = A(x^2+1) + (Bx+C)(x-1)$.

- We set $x=1$ to get $4 = 2A$, so $A = 2$.
- We set $x=0$ to get $-1 = A - C$, so $C = 3$.
- Finally, set $x = 2$ to get $15 = 5A + 2B + C$, so $B = 1$.

We can now compute the integral as follows.

$$\int \left(\frac{2}{x-1} + \frac{x+3}{x^2+1}\right)\mathrm{d}x = 2\ln(|x-1|) + \frac{1}{2}\ln(x^2+1) + 3\arctan x + K \quad .$$

3. Compute the integral.

   (a) $\displaystyle\int \frac{\sqrt{1+x^2}}{x^2}\mathrm{d}x.$

answer: $\ln\left(\sqrt{1+x^2}+x\right) - \frac{x}{\sqrt{1+x^2}} + C$

Solution. 3.a

Variant I. In this variant, we use the trigonometric substitution $x = \tan\theta$ and then solve the integral using a few algebraic tricks.

$$\begin{aligned}
\int \frac{\sqrt{1+x^2}}{x^2}\mathrm{d}x &= \int \frac{\sqrt{1+\tan^2\theta}}{\tan^2\theta}\mathrm{d}(\tan\theta) \\
&= \int \frac{|\sec\theta|}{\tan^2\theta}\sec^2\theta \mathrm{d}\theta \\
&= \int \frac{\cos^2\theta}{\cos^3\theta\sin^2\theta}\mathrm{d}\theta \\
&= \int \frac{\cos\theta}{\cos^2\theta\sin^2\theta}\mathrm{d}\theta
\end{aligned}$$

Set $x = \tan\theta$, $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $|\sec\theta| = \sec\theta$ for $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$= \int \frac{\mathrm{d}(\sin\theta)}{(1-\sin^2\theta)\sin^2\theta}$$

Set $u = \sin\theta$ for $\theta \in \left(0, \frac{\pi}{2}\right)$

$u = \sqrt{1-\cos^2\theta}$

$u = \sqrt{1-\frac{1}{\sec^2\theta}}$

$u = \sqrt{1-\frac{1}{1+\tan^2\theta}}$

$u = \sqrt{\frac{\tan^2\theta}{1+\tan^2\theta}}$

$u = \frac{\tan\theta}{\sqrt{1+\tan^2\theta}}$

$u = \frac{x}{\sqrt{1+x^2}}$

$$\begin{aligned}
&= \int \frac{\mathrm{d}u}{(1-u^2)u^2} \\
&= \int \frac{\mathrm{d}u}{(1-u)u^2(u+1)} && \text{use part. frac.} \\
&= \int \left(\frac{\frac{1}{2}}{u+1} + \frac{-\frac{1}{2}}{u-1} + \frac{1}{u^2}\right)\mathrm{d}u \\
&= -\frac{1}{2}\ln|u-1| + \frac{1}{2}\ln(u+1) - u^{-1} + C \\
&= -\frac{1}{2}\ln(1-u) + \frac{1}{2}\ln(u+1) - u^{-1} + C && u = \tfrac{x}{\sqrt{1+x^2}} < 1 \\
&= \frac{1}{2}\ln\left(\frac{1+u}{1-u}\right) - u^{-1} + C \\
&= \frac{1}{2}\ln\left(\frac{(1+u)}{(1-u)}\cdot\frac{(1+u)}{(1+u)}\right) - u^{-1} + C \\
&= \frac{1}{2}\ln\left(\frac{(1+u)^2}{1-u^2}\right) - u^{-1} + C && \text{use } u = \tfrac{x}{\sqrt{1+x^2}} \\
&= \frac{1}{2}\ln\left(\frac{(1+u)^2}{\frac{1}{1+x^2}}\right) - \frac{\sqrt{1+x^2}}{x} + C \\
&= \frac{1}{2}\ln\left(\left((1+u)\sqrt{1+x^2}\right)^2\right) - \frac{\sqrt{1+x^2}}{x} + C \\
&= \ln\left(\sqrt{1+x^2}+x\right) - \frac{\sqrt{1+x^2}}{x} + C \quad .
\end{aligned}$$

Variant II. In this variant, we use directly the Euler substitution

$$
\begin{aligned}
x &= \cot(2\arctan t)\\
&= \tfrac{1}{2}\left(\tfrac{1}{t}-t\right)\\
\mathrm{d}x &= -\tfrac{1}{2}\left(\tfrac{1}{t^2}+1\right)\mathrm{d}t\\
\sqrt{1+x^2} &= \tfrac{1}{2}\left(\tfrac{1}{t}+t\right)\\
t &= \sqrt{x^2+1}-x\\
\tfrac{1}{t} &= \sqrt{x^2+1}+x \quad .
\end{aligned}
$$

$$
\begin{aligned}
\int \frac{\sqrt{1+x^2}}{x^2}\mathrm{d}x &= \int \frac{\frac{1}{2}\left(\frac{1}{t}+t\right)}{\frac{1}{4}\left(\frac{1}{t}-t\right)^2}\left(-\frac{1}{2}\right)\left(\frac{1}{t^2}+1\right)\mathrm{d}t\\
&= \int \frac{-t^4-2t^2-1}{(t-1)^2t(t+1)^2}\mathrm{d}t && \Big|\ \text{Part. frac}\\
&= \int \left(-\frac{1}{t}+\frac{1}{(t+1)^2}-\frac{1}{(t-1)^2}\right)\mathrm{d}t\\
&= -\ln t-\frac{1}{t+1}+\frac{1}{t-1}+C\\
&= \ln\left(\frac{1}{t}\right)+\frac{2}{t^2-1}+C\\
&= \ln\left(\sqrt{1+x^2}+x\right)+\frac{1}{t\frac{1}{2}\left(t-\frac{1}{t}\right)}+C\\
&= \ln\left(\sqrt{1+x^2}+x\right)-\frac{1}{t}\cdot\frac{1}{\frac{1}{2}\left(\frac{1}{t}-t\right)}+C\\
&= \ln\left(\sqrt{1+x^2}+x\right)-\left(\sqrt{x^2+1}+x\right)\cdot\frac{1}{x}+C\\
&= \ln\left(\sqrt{1+x^2}+x\right)-\frac{\sqrt{x^2+1}}{x}-1+C \quad .
\end{aligned}
$$

4. Compute the integral using a trigonometric substitution.

(a) $\displaystyle\int \frac{\sqrt{9-x^2}}{x^2}\mathrm{d}x \quad .$

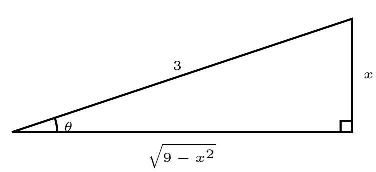
answer: $-\frac{\sqrt{9-x^2}}{x}-\arcsin\left(\frac{x}{3}\right)+C$

Solution. 4.a

$$
\begin{aligned}
\int \frac{\sqrt{9-x^2}}{x^2}\mathrm{d}x &= \int \frac{3\sqrt{\cos^2\theta}}{9\sin^2\theta}(3\cos\theta)\mathrm{d}\theta\\
&= 9\int \frac{|\cos\theta|}{\sin^2\theta}\cos\theta\mathrm{d}\theta\\
&= \int \cot^2\theta\mathrm{d}\theta\\
&= \int (\csc^2\theta-1)\mathrm{d}\theta\\
&= -\cot\theta-\theta+C\\
&= -\frac{\sqrt{9-x^2}}{x}-\arcsin\left(\frac{x}{3}\right)+C,
\end{aligned}
$$

Set $x = 3\sin\theta$
for $\theta \in \left[\frac{\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right]$
$\mathrm{d}x = 3\cos\theta\mathrm{d}\theta$
For $\theta \in \left[\frac{\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right]$
we have $|\cos\theta| = \cos\theta$

where we expressed $\cot\theta$ via $\sin\theta$ by considering the following triangle.



5. Evaluate the indefinite integral. Illustrate the steps of your solutions.

(a) $\displaystyle\int x\sin x\mathrm{d}x.$

answer: $-x\cos x+\sin x+C$

(b) $\displaystyle\int xe^{-x}\mathrm{d}x.$

answer: $-(1+x)e^{-x}+C$

(c) $\displaystyle\int x^2e^{x}\mathrm{d}x.$

answer: $x^2e^x-2xe^x+2e^x+C$

(d) $\int x \sin(-2x)\mathrm{d}x.$

answer: $\frac{x}{2}\cos(-2x) + \frac{1}{4}\sin(-2x) + C$

(e) $\int x^2 \cos(3x)\mathrm{d}x.$

answer: $\frac{x^2}{3}\sin(3x) + \frac{2x}{9}\cos(3x) - \frac{2}{27}\sin(3x) + C$

(f) $\int x^2 e^{-2x}\mathrm{d}x.$

answer: $-\frac{x^2 e^{-2x}}{2} - \frac{xe^{-2x}}{2} - \frac{e^{-2x}}{4} + C$

(g) $\int x \sin(2x)\mathrm{d}x.$

answer: $-\frac{x}{2}\cos(2x) + \frac{1}{4}\sin(2x) + C$

(h) $\int x \cos(3x)\mathrm{d}x.$

answer: $\frac{x}{3}\sin(3x) + \frac{1}{9}\cos(3x) + C$

(i) $\int x^2 e^{2x}\mathrm{d}x.$

answer: $\frac{x^2}{2}e^{2x} - \frac{x}{2}e^{2x} + \frac{e^{2x}}{4} + C$

(j) $\int x^3 e^{x}\mathrm{d}x.$

answer: $x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C$

Solution. 5.a.

$$\int x \underbrace{\sin x \mathrm{d}x}_{=\mathrm{d}(-\cos x)} = -\int x \mathrm{d}(\cos x) = -x\cos x + \int \cos x \mathrm{d}x = -x\cos x + \sin x + C \quad .$$

Solution. 5.c.

$$\begin{aligned} \int x^2 \underbrace{e^x \mathrm{d}x}_{\mathrm{d}(e^x)} &= \int x^2 \mathrm{d}e^x = x^2e^x - \int e^x 2x \mathrm{d}x = x^2e^x - \int 2x\mathrm{d}e^x \\ &= x^2e^x - 2xe^x + \int 2e^x\mathrm{d}x = x^2e^x - 2xe^x + 2e^x + C \quad . \end{aligned}$$

Solution. 5.f.

$$\begin{aligned} \int x^2 e^{-2x}\mathrm{d}x &= \int x^2 \mathrm{d}\left(\frac{e^{-2x}}{-2}\right) && \Big| \text{ Integrate by parts} \\ &= -\frac{x^2e^{-2x}}{2} - \int \left(\frac{e^{-2x}}{-2}\right)\mathrm{d}\left(x^2\right) \\ &= -\frac{x^2e^{-2x}}{2} + \int xe^{-2x}\mathrm{d}x \\ &= -\frac{x^2e^{-2x}}{2} + \int x\mathrm{d}\left(\frac{e^{-2x}}{-2}\right) && \Big| \text{ Integrate by parts} \\ &= -\frac{x^2e^{-2x}}{2} - \frac{xe^{-2x}}{2} + \frac{1}{2}\int e^{-2x}\mathrm{d}x \\ &= -\frac{x^2e^{-2x}}{2} - \frac{xe^{-2x}}{2} - \frac{e^{-2x}}{4} + C \quad . \end{aligned}$$

6. Use the integral test, the comparison test or the limit comparison test to determine whether the series is convergent or divergent. Justify your answer.

(a) $\sum_{n=1}^{\infty} \frac{1}{2n+1}.$

answer: divergent

(b) $\sum_{n=1}^{\infty} \frac{1}{2n^2+n^3}.$

answer: convergent, compare to $\sum_{n=1}^{\infty} \frac{1}{2n^2}$

(c) $\sum_{n=1}^{\infty} \frac{n^2+3}{3n^5+n}$

answer: convergent, can use limit comparison test

(d) $\sum_{n=0}^{\infty} \frac{1}{3^n+5}.$

answer: convergent, compare to $\sum_{n=0}^{\infty} \frac{1}{3^n}$

(e) $\sum_{n=2}^{\infty} \frac{1}{n\ln n}$

answer: divergent, integral test

(f) $\sum_{n=2}^{\infty} \frac{1}{(2n+1)\ln(n)}.$

answer: divergent

(g) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

answer: convergent, can use integral test

(h) $\sum_{n=2}^{\infty} \frac{1}{(2n+1)(\ln(n))^2}.$

answer: convergent

(i) Determine all values of p, q r for which the series

$$\sum_{n=30}^{\infty} \frac{1}{n^p(\ln n)^q(\ln(\ln n))^r}$$

is convergent.

Solution. 6.e. The function $\frac{1}{x\ln x}$ is decreasing, as for $x>2$, it is the quotient of 1 by increasing positive functions. $\frac{1}{x\ln x}$ tends to 0 as $x\to\infty$, and therefore the integral criterion implies that $\sum\limits_{n=2}^{\infty}\frac{1}{n\ln n}$ is convergent/divergent if and only if $\int_2^\infty \frac{1}{x\ln x}\mathrm{d}x$ is.

$$\begin{array}{rcl}\displaystyle\int_2^\infty \frac{1}{x\ln x}\mathrm{d}x &=& \displaystyle\lim_{t\to\infty}\int_2^t \frac{1}{x\ln x}\mathrm{d}x\\ &=& \displaystyle\lim_{t\to\infty}\int_2^t \frac{1}{\ln x}\mathrm{d}(\ln x)\\ &=& \displaystyle\lim_{t\to\infty}\int_2^t \mathrm{d}(\ln(\ln x))\\ &=& \displaystyle\lim_{t\to\infty}\left[\ln(\ln x)\right]_{x=2}^{x=t}\\ &=& \displaystyle\lim_{t\to\infty}\left(\ln(\ln t)-\ln(\ln 2)\right)\\ &=& \infty.\end{array}$$

The integral is divergent (and diverges to $+\infty$) and therefore, by the integral criterion, so is the sum.

Solution. 6.f The integral criterion appears to be of little help: the improper integral $\int\frac{1}{(2x+1)\ln x}\mathrm{d}x$ cannot be integrated algebraically with any of the techniques we have studied so far. Therefore it makes sense to try to solve this problem using a comparison test.

We present two solution variants. In Variant I we use the limit-comparison test. This is an easier (but slightly longer) solution. In Variant II we use the comparison test - this solution is harder as it requires algebraic intuition to select a series to compare to.

Variant I. This variant uses the limit comparison test.

The “dominant term”[1] of the denominator of $\frac{1}{(2n+1)\ln n}=\frac{1}{2n\ln n+\ln n}$ is $2n\ln n$. Therefore it makes sense to compare - or limit-compare - with $\frac{1}{n\ln n}$.

We will use the Limit Comparison Test for the series $\sum\limits_{n=2}^{\infty}a_n=\sum\limits_{n=2}^{\infty}\frac{1}{(2n+1)\ln n}$ and $\sum\limits_{n=2}^{\infty}b_n=\sum\limits_{n=2}^{\infty}\frac{1}{n\ln n}$. Both a_n and b_n are positive (for $n>2$) and therefore the Limit Comparison Test applies.

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{\frac{1}{(2n+1)\ln n}}{\frac{1}{n\ln n}}=\lim_{n\to\infty}\frac{n}{2n+1}=\lim_{n\to\infty}\frac{1}{2+\frac{1}{n}}=\frac{1}{2}.$$

Since $\lim\limits_{n\to\infty}\frac{a_n}{b_n}=\frac12\neq 0$, the Limit Comparison Test implies that the series $\sum\limits_{n=2}^{\infty}a_n$ has same convergence/divergence properties as the series $\sum\limits_{n=2}^{\infty}b_n$. In Problem 6.e we demonstrated that the series $\sum\limits_{n=2}^{\infty}b_n$ is divergent; therefore the series $\sum\limits_{n=2}^{\infty}a_n=\sum\limits_{n=2}^{\infty}\frac{1}{(2n+1)\ln n}$ is divergent as well.

Variant II. This variant uses directly the comparison test. It is slightly shorter than the preceding variant but requires more intuition.

Let $a_n=\frac{1}{(2n+1)\ln n}$. Consider the series $\sum\limits_{n=2}^{\infty}b_n$ for $b_n=\frac{1}{3n\ln n}$. We have that

$$\begin{array}{rcll}3n &\geq& 2n+1 & \Bigg|\ \text{for } n\geq 1\\ \dfrac{1}{3n} &\leq& \dfrac{1}{2n+1} \quad . & \Bigg|\ \begin{array}{l}\text{Inverting positive}\\ \text{quantities reverses}\\ \text{inequalities}\end{array}\end{array}$$

Therefore $b_n\geq a_n$. In Problem 6.e we illustrated (using the integral test) that $\sum\limits_{n=2}^{\infty}(3b_n)$ is divergent and therefore so is its constant multiple $\sum\limits_{n=2}^{\infty}b_n$. Therefore $\sum\limits_{n=2}^{\infty}\frac{1}{(2n+1)\ln n}$ is divergent by the comparison test.

7. Compute the limits. The answer key has not been fully proofread, use with caution.

(a) $\lim\limits_{x\to 0}\dfrac{\sin x}{x}$.

answer: 1

(b) $\lim\limits_{x\to 0}\dfrac{x}{\ln(1+x)}$.

answer: 1

[1] since we do not speak of rational functions, here the expression “dominant term” is used informally

(c) $\displaystyle\lim_{x\to 0} \frac{x^2}{x - \ln(1+x)}$.

answer: 2

(d) $\displaystyle\lim_{x\to 0} \frac{x^2}{\sin x \ln(1+x)}$.

answer: 1

(e) $\displaystyle\lim_{x\to 0} \frac{\sin^2 x}{(\ln(1+x))^2}$.

answer: 1

(f) $\displaystyle\lim_{x\to 0} \frac{\cos x - 1}{\sin x \ln(1+x)}$.

answer: $-\frac{1}{2}$

(g) $\displaystyle\lim_{x\to 0} \frac{\arctan x - x}{x^3}$.

answer: $-\frac{1}{3}$

(h) $\displaystyle\lim_{x\to 0} \frac{\arcsin x - x}{x^3}$.

answer: $\frac{1}{6}$

(i) $\displaystyle\lim_{x\to 1} \frac{x}{x-1} - \frac{1}{\ln x}$.

answer: $\frac{1}{2}$

(j) $\displaystyle\lim_{x\to 0} \frac{\cos(nx) - \cos(mx)}{x^2}$.

answer: $\frac{m^2-n^2}{2}$

(k) $\displaystyle\lim_{x\to 0} \frac{\arcsin x - x - \frac{1}{6}x^3}{\sin^5 x}$.

answer: $\frac{3}{40}$

(l) $\displaystyle\lim_{x\to 1} \frac{\sin(\pi x)\ln x}{\cos(\pi x) + 1}$.

answer: $-\frac{2}{\pi}$

(m) $\displaystyle\lim_{x\to 0} \frac{\sin x - x}{\arcsin x - x}$.

answer: -1

(n) $\displaystyle\lim_{x\to 0} \frac{\sin x - x}{\arctan x - x}$.

answer: $\frac{1}{2}$

(o) $\displaystyle\lim_{x\to \infty} x \sin\left(\frac{2}{x}\right)$.

answer: 2

8. Express the sum of the series as a rational number.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n + 3^n}{5^n}$

answer: $\frac{13}{6}$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n + 5^n}{10^n}$

answer: $\frac{13}{4}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{5^n - 3^n}{7^n}$

answer: $\frac{7}{4}$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{3^{n+1} + 7^{n-1}}{21^n}$

answer: $\frac{4}{7}$

(e) $\displaystyle\sum_{n=0}^{\infty} \frac{2^{n+1} + (-3)^{n-1}}{5^n}$

answer: $\frac{25}{8}$

Solution. 8.a.

$$
\begin{aligned}
\sum_{n=1}^{\infty} \frac{2^n + 3^n}{5^n} &= \sum_{n=1}^{\infty} \left(\frac{2}{5}\right)^n + \sum_{n=1}^{\infty} \left(\frac{3}{5}\right)^n \\
&= \frac{2}{5}\sum_{n=0}^{\infty} \left(\frac{2}{5}\right)^n + \frac{3}{5}\sum_{n=0}^{\infty} \left(\frac{3}{5}\right)^n \\
&= \frac{2}{5} \cdot \frac{1}{\left(1-\frac{2}{5}\right)} + \frac{3}{5} \cdot \frac{1}{\left(1-\frac{3}{5}\right)} \\
&= \frac{13}{6} \quad .
\end{aligned}
$$

Use geometric series sum f-la: $\sum\limits_{n=0}^{\infty} r^n = \frac{1}{1-r}$, provided $|r| < 1$

Solution. 8.b.

$$
\begin{aligned}
\sum_{n=0}^{\infty} \frac{2^n + 5^n}{10^n} &= \sum_{n=0}^{\infty} \left(\frac{1}{5^n} + \frac{1}{2^n}\right) \\
&= \frac{1}{1-\frac{1}{2}} + \frac{1}{1-\frac{1}{5}} \\
&= \frac{13}{4} \quad .
\end{aligned}
$$

use $\sum\limits_{n=0}^{\infty} r^n = \frac{1}{1-r}$, for $|r| < 1$

Solution. 8.d.

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{3^{n+1}+7^{n-1}}{21^n} &= \sum_{n=1}^{\infty}\left(3\cdot\frac{3^n}{21^n}+\frac{1}{7}\cdot\frac{7^n}{21^n}\right)\\
&= 3\sum_{n=1}^{\infty}\left(\frac{1}{7}\right)^n+\frac{1}{7}\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n\\
&= \frac{3}{7}\sum_{n=0}^{\infty}\left(\frac{1}{7}\right)^n+\frac{1}{21}\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n && \Big|\ \text{use } \textstyle\sum_{n=0}^{\infty} r^n=\frac{1}{1-r}, |r|<1\\
&= \frac{3}{7}\cdot\frac{1}{\left(1-\frac{1}{7}\right)}+\frac{1}{21}\cdot\frac{1}{\left(1-\frac{1}{3}\right)}\\
&= \frac{4}{7}\quad.
\end{aligned}$$

Solution. 8.e.

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{2^{n+1}+(-3)^{n-1}}{5^n} &= \sum_{n=0}^{\infty}\left(2\cdot\frac{2^n}{5^n}-\frac{1}{3}\cdot\frac{(-3)^n}{5^n}\right)\\
&= 2\sum_{n=0}^{\infty}\left(\frac{2}{5}\right)^n-\frac{1}{3}\sum_{n=0}^{\infty}\left(-\frac{3}{5}\right)^n && \Big|\ \text{use } \textstyle\sum_{n=0}^{\infty} r^n=\frac{1}{1-r}, |r|<1\\
&= 2\cdot\frac{1}{\left(1-\frac{2}{5}\right)}-\frac{1}{3}\cdot\frac{1}{\left(1-\left(-\frac{3}{5}\right)\right)}\\
&= \frac{25}{8}\quad.
\end{aligned}$$

9. Sum the telescoping series (a sum is "telescoping" if it can be broken into summands so that consecutive terms cancel).

(a) $\displaystyle\sum_{n=0}^{\infty}\frac{-6}{9n^2+3n-2}$. answer: 2

(b) $\displaystyle\sum_{n=3}^{\infty}\frac{3}{n^2-3n+2}$. answer: 3

(c) $\displaystyle\sum_{n=2}^{\infty}\ln\left(1-\frac{1}{n^2}\right)$. (Hint: Use the properties of the logarithm to aim for a telescoping series).

answer: $-\ln 2$

Solution. 9.b

$$\begin{aligned}
\sum_{n=3}^{\infty}\frac{3}{n^2-3n+2} &= \sum_{n=3}^{\infty}\left(\frac{3}{n-2}-\frac{3}{n-1}\right) && \Big|\ \text{use partial fractions, see below}\\
&= 3\sum_{n=3}^{\infty}\left(\frac{1}{n-2}-\frac{1}{n-1}\right)\\
&= 3\left(\underbrace{\left(1-\frac{1}{2}\right)}_{n=3}+\underbrace{\left(\frac{1}{2}-\frac{1}{3}\right)}_{n=4}+\underbrace{\left(\frac{1}{3}-\frac{1}{4}\right)}_{n=5}+\dots\right)\\
&= 3\lim_{n\to\infty}\left(1-\frac{1}{n-1}\right)=3\quad.
\end{aligned}$$

In the above we used the partial fraction decomposition of $\dfrac{3}{n^2-3n+2}$. This decomposition is computed as follows.

$$\frac{3}{n^2-3n+2}=\frac{3}{(n-1)(n-2)}$$

We need to find A_i's so that we have the following equality of rational functions. After clearing denominators, we get the following equality.

$$3=A_1(n-2)+A_2(n-1)$$

After rearranging we get that the following polynomial must vanish. Here, by "vanish" we mean that the coefficients of the powers of x must be equal to zero.

$$(A_2 + A_1)n + (-A_2 - 2A_1 - 3)$$

In other words, we need to solve the following system.

$$\begin{array}{lll} -2A_1 & -A_2 & = 3 \\ A_1 & +A_2 & = 0 \end{array}$$

System status	Action
$\begin{array}{lll} -2A_1 & -A_2 & = 3 \\ A_1 & +A_2 & = 0 \end{array}$	Selected pivot column 2. Eliminated the non-zero entries in the pivot column.
$\begin{array}{lll} A_1 & +\frac{A_2}{2} & = -\frac{3}{2} \\ & \frac{A_2}{2} & = \frac{3}{2} \end{array}$	Selected pivot column 3. Eliminated the non-zero entries in the pivot column.
$\begin{array}{lll} A_1 & & = -3 \\ & A_2 & = 3 \end{array}$	Final result.

Therefore, the final partial fraction decomposition is the following.

$$\frac{3}{n^2 - 3n + 2} = \frac{-3}{(n-1)} + \frac{3}{(n-2)}.$$

Solution. 9.c.

$$\begin{array}{rcl} \displaystyle\sum_{n=2}^{\infty} \ln\left(1 - \frac{1}{n^2}\right) & = & \displaystyle\sum_{n=2}^{\infty} \left(\ln\left(1 - \frac{1}{n}\right) + \ln\left(1 + \frac{1}{n}\right)\right) \\ & = & \displaystyle\sum_{n=2}^{\infty} \left(\ln\left(\frac{n-1}{n}\right) + \ln\left(\frac{n+1}{n}\right)\right) \\ & = & \displaystyle\sum_{n=2}^{\infty} \left(\ln(n-1) - 2\ln(n) + \ln(n+1)\right) \\ & = & (\ln 1 - 2\ln 2 + \cancel{\ln 3}) + (\ln 2 \cancel{- 2\ln 3} + \cancel{\ln 4}) \\ & & + (\cancel{\ln 3 - 2\ln 4} + \cancel{\ln 5}) + \cancel{\ldots}\ldots \\ & = & \displaystyle\lim_{n\to\infty} \left(-\ln 2 - \ln n + \ln(n+1)\right) \\ & = & \displaystyle\lim_{n\to\infty} \left(-\ln 2 + \ln\left(\frac{n+1}{n}\right)\right) \\ & = & -\ln 2 \quad . \end{array}$$

10. Find whether the series is convergent or divergent using an appropriate test. Some of the problems require the alternating series test. The test states the following.

Alternating series test. Suppose $b_n \searrow 0$. Then $\sum(-1)^n b_n$ is convergent.

Here, $b_n \searrow 0$ means the following.

- The sequence of numbers b_n is decreasing.
- The sequence decreases to 0, that is,

$$\lim_{n\to\infty} b_n = 0 \quad .$$

(a) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \ln n.$

answer: diverges, basic divergence test

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}.$

answer: converges, alternating series test

(c) $\displaystyle\sum_{n=2}^{\infty} \frac{n}{\ln n}$

answer: diverges, basic divergence test

(d) $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n}$

answer: converges, alternating series test

Solution. 10.a. $\lim\limits_{n\to\infty} (-1)^n \ln n$ does not exist and therefore the sum is not convergent.

Solution. 10.b. For $n > 2$, we have that $\ln n$ is a positive increasing function and therefore $\frac{1}{\ln n}$ is a decreasing positive function. Furthermore $\lim\limits_{n\to\infty} \dfrac{1}{\ln n} = 0$. Therefore the series is convergent by the alternating series test.

11. For each of the items below, do the following.
    - Find the Maclaurin series of the function (i.e., the power series representation of the function around $a = 0$).
    - Find the radius of convergence of the series you found in the preceding point. You are not asked to find the entire interval of convergence, but just the radius.

    (a) e^x.

    answer: $\sum\limits_{n=0}^{\infty} \dfrac{x^n}{n!}$

    (b) xe^{-2x}.

    answer: $\sum\limits_{n=0}^{\infty} (-1)^n 2^n x^{n+1} = \sum\limits_{n=1}^{\infty} (-1)^{n-1} 2^{n-1} x^n$ converges for $x \in ()$

    (c) e^{2x}.

    answer: $e^{2x} = \sum\limits_{n=0}^{\infty} \dfrac{2^n x^n}{n!}$

    (d) e^{x^2}.

    answer: $e^{x^2} = \sum\limits_{n=0}^{\infty} \dfrac{x^{2n}}{n!}$

    (e) e^{-3x^2}.

    answer: $e^{-3x^2} = \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n 3^n x^{2n}}{n!}$

    (f) $x^2 e^{2x}$.

    answer: $x^2 e^{2x} = \sum\limits_{n=0}^{\infty} \dfrac{2^n x^{n+2}}{n!}$

    (g) $\sin x$.

    answer: $\sin x = \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{(2n+1)!}$

    (h) $\cos x$.

    answer: $\cos x = \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{x^{2n}}{(2n)!}$

    (i) $\sin(2x)$.

    answer: $\sin(2x) = \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{2^{2n+1} x^{2n+1}}{(2n+1)!}$

    (j) $\cos(2x)$.

    answer: $\cos(2x) = \sum\limits_{n=0}^{\infty} (-1)^n 2^{2n} \dfrac{x^{2n}}{(2n)!}$

    (k) $\cos^2(x)$.

    answer: $\cos^2 x = \dfrac{1}{2} + \sum\limits_{n=0}^{\infty} (-1)^n 2^{2n-1} \dfrac{x^{2n}}{(2n)!}$

    (l) $x \sin x$.

    answer: $x \sin x = \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+2}}{(2n+1)!}$

12. Find the Taylor series of the function at the indicated point.

    (a) $\frac{1}{x^2}$ at $a = -1$.

    answer: $1 + 2(x+1) + 3(x+1)^2 + \dots = \sum\limits_{n=0}^{\infty} (n+1)(x+1)^n$

    (b) $\ln\left(\sqrt{x^2 - 2x + 2}\right)$ at $a = 1$.

    answer: $\sum\limits_{n=1}^{\infty} (-1)^{n+1} \dfrac{(x-1)^{2n}}{2n}$

    (c) Write the Taylor series of the function $\ln x$ around $a = 2$.

    answer: $\ln 2 + \sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{2n} (x-2)^n$

Solution. 12.b

$$\begin{array}{rcl|l}
\ln\left(\sqrt{x^2 - 2x + 2}\right) & = & \dfrac{1}{2} \ln\left((x-1)^2 + 1\right) & \text{use } \ln(1+y) = \sum\limits_{n=1}^{\infty} (-1)^{n+1} \frac{y^n}{n}, |y| < 1 \\
& = & \dfrac{1}{2} \sum\limits_{n=1}^{\infty} (-1)^{n+1} \dfrac{\left((x-1)^2\right)^n}{n} & \\
& = & \sum\limits_{n=1}^{\infty} (-1)^{n+1} \dfrac{(x-1)^{2n}}{2n} \quad . &
\end{array}$$

Although the problem does not ask us to do this, we will determine the interval of convergence of the series for exercise. If we use the fact that $\ln(1+y) = \sum\limits_{n=1}^{\infty} (-1)^{n+1} \frac{y^n}{n}$ holds for $-1 < y \leq 1$, it follows immediately that the above equality holds for $0 < (x-1)^2 \leq 1$, which holds for $x \in [0, 2]$. Let us however compute the interval of convergence without using the aforementioned fact.

Let a_n be the n^{th} term of our series, i.e., let

$$a_n = (-1)^{n+1} \frac{(x-1)^{2n}}{2n} \quad .$$

We use the ratio test:

$$
\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(-1)^{n+2}(x-1)^{2n+2}}{(2n+2)}\frac{2n}{(-1)^{n+1}(x-1)^{2n}}\right| \\
&= \lim_{n\to\infty}(x-1)^2\frac{n}{n+1} \\
&= (x-1)^2 \quad .
\end{aligned}
$$

By the ratio test, the series is divergent for $(x-1)^2 > 1$, i.e., for $|x-1| > 1$, and convergent for $(x-1)^2 < 1$, i.e., for $|x-1| < 1$. The ratio test is inconclusive at only two points: $x-1=1$, i.e., $x=2$ and $x-1=-1$, i.e., $x=0$. At both points the series becomes $\sum\limits_{n=1}^{\infty}(-1)^{n+1}\frac{2^{2n}}{2n}$ and the series is convergent at both points by the alternating series test.

Solution. 12.c This solution is similar to the solution of 12.b, but we have written it in a concise fashion suitable for test taking. Denote Taylor series at a by T_a and recall that the Maclaurin series of are just T_0, the Taylor series at 0.

$$
\begin{aligned}
T_2(\ln x) &= T_2(\ln((x-2)+2)) \\
&= T_2\left(\ln\left(2\left(\frac{x-2}{2}+1\right)\right)\right) \\
&= T_2\left(\ln 2 + \ln\left(1+\frac{x-2}{2}\right)\right) \qquad \left|\ T_0(\ln(1+y)) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}y^n}{n}\right. \\
&= \ln 2 + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(\frac{x-2}{2}\right)}{n} \\
&= \ln 2 + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2^n}(x-2)^n \quad .
\end{aligned}
$$

13. Determine the interval of convergence for the following power series.

(a) $\sum\limits_{n=1}^{\infty}\frac{(x-2)^n}{3\sqrt{n+1}}$.

answer: $x \in [1, 3)$.

(b) $\sum\limits_{n=1}^{\infty}\frac{10^n x^n}{n^3}$.

answer: $x \in \left[-\frac{1}{10}, \frac{1}{10}\right]$.

(c) $\sum\limits_{n=1}^{\infty}\frac{10^n (x-1)^n}{n^3}$.

answer: $x \in [0.9, 1.1]$.

(d) $\sum\limits_{n=0}^{\infty}(-1)^n\frac{(x+1)^n}{2n+1}$.

answer: $x \in (-2, 0]$.

(e) $\sum\limits_{n=0}^{\infty}(-1)^n\frac{(x-3)^n}{2n+1}$.

answer: $x \in (2, 4]$.

(f) $\sum\limits_{n=0}^{\infty}\frac{x^n}{n!}$.

answer: converges for all x.

(g) $\sum\limits_{n=0}^{\infty}(n+1)x^n$.

answer: converges for $|x| < 1$.

(h) $\sum\limits_{n=1}^{\infty}\frac{x^n}{n}$.

answer: converges for $x \in [-1, 1)$.

(i) $\sum\limits_{n=1}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$.

answer: converges for $x \in [-1, 1]$.

(j) $\sum_{n=1}^{\infty} \binom{\frac{1}{2}}{n} x^n$, where we recall that the binomial coefficient $\binom{q}{n}$ stands for $\dfrac{q(q-1)\dots(q-n+1)}{n!}$.

answer: converges for $x \in (-1, 1]$.

Solution. 13.a. We apply the Ratio Test to get that $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = |x-2|$. Therefore the power series converges at least in the interval $x \in (1,3)$. When $x = 3$, the series becomes $\sum_{n=1}^{\infty} \frac{1}{3\sqrt{n+1}}$, which diverges - this can be seen, for example, by comparing to the p-series $\frac{1}{\sqrt{n}}$. When $x = 1$, the series becomes $\sum_{n=1}^{\infty} \frac{(-1)^n}{3\sqrt{n+1}}$, which converges by the Alternating Series Test. Our final answer $x \in [1,3)$.

14. Find the length of the curve.

(a) $y = x^2$, $x \in [1,2]$.

answer: $L = \sqrt{17} + \frac{1}{4}\log\left(\sqrt{17}+4\right) - \frac{1}{4}\log\left(\sqrt{5}+2\right) - \frac{\sqrt{5}}{2} \approx 3.167841$

(b) $y = \sqrt{x}$, $x \in [1,2]$.

answer: $L = \frac{1}{8}\left(12\sqrt{2} + \ln(17+12\sqrt{2}) - 4\sqrt{5} - \ln(9+4\sqrt{5})\right) \approx 1.083$

(c) $x = \sqrt{t} - 2t$ and $y = \dfrac{8}{3}t^{\frac{3}{4}}$ from $t = 1$ to $t = 4$.

answer: $L = 7$

(d) $\gamma : \left| \begin{array}{lcl} x(t) & = & \frac{1}{t} + \frac{t^3}{3} \\ y(t) & = & 2t \end{array} \right. , t \in [1,2]$.

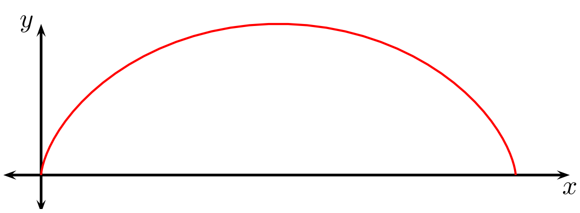
answer: $L = \frac{17}{6}$

(e) $\gamma : \left| \begin{array}{lcl} x(t) & = & \frac{1}{t} + t \\ y(t) & = & 2\ln t \end{array} \right. , t \in [1,2]$.

answer: $L = \frac{3}{2}$

(f) One arch of the cycloid

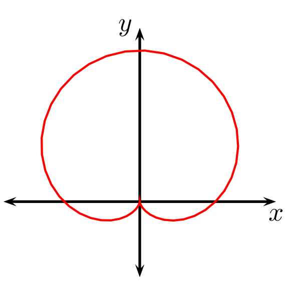
$$\gamma : \left| \begin{array}{lcl} x(t) & = & t - \sin t \\ y(t) & = & 1 - \cos t \end{array} \right. , t \in [0, 2\pi]$$



answer: $L = 8$

(g) The cardioid

$$\gamma : \left| \begin{array}{lcl} x(t) & = & (1+\sin t)\cos t \\ y(t) & = & (1+\sin t)\sin t \end{array} \right. , t \in [0, 2\pi]$$



answer: $L = 8$

Solution. 14.a The length of the parametric curve is given by

$$\begin{aligned}
L &= \int_1^2 \sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\mathrm{d}x \\
&= \int_{x=1}^{x=2} \sqrt{1+4x^2}\mathrm{d}x \\
&= \int_{u=2}^{u=4} \sqrt{u^2+1}\left(\frac{1}{2}\mathrm{d}u\right) \\
&= \frac{1}{2}\int_{u=2}^{u=4} \sqrt{u^2+1}\mathrm{d}u \\
&= \tfrac{1}{4}\left[u\sqrt{u^2+1}+\ln\left(u+\sqrt{u^2+1}\right)\right]_2^4 \\
&= \sqrt{17}+\tfrac{1}{4}\log\left(\sqrt{17}+4\right)-\tfrac{1}{4}\log\left(\sqrt{5}+2\right)-\tfrac{\sqrt{5}}{2} \\
&\approx 3.167841
\end{aligned}$$

Substitute $2x = u$, $\mathrm{d}x = \frac{1}{2}\mathrm{d}u$

$\int \sqrt{u^2+1}\mathrm{d}u$
$= \frac{1}{2}\left(u\sqrt{u^2+1}+\ln\left(u+\sqrt{u^2+1}\right)\right)+C$
previously studied

Solution. 14.b

Solution I. The curve can be rewritten in the form $x=y^2$, $y\in[1,\sqrt{2}]$.

$$\begin{aligned}
L &= \int_1^{\sqrt{2}} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2+1}\ \mathrm{d}y \\
&= \int_{y=1}^{y=\sqrt{2}} \sqrt{4y^2+1}\ \mathrm{d}y \\
&= \int_{u=2}^{u=2\sqrt{2}} \sqrt{u^2+1}\left(\frac{1}{2}\mathrm{d}u\right) \\
&= \frac{1}{2}\int \sqrt{u^2+1}\mathrm{d}u \\
&= \frac{1}{4}\left[u\sqrt{u^2+1}+\ln\left(u+\sqrt{u^2+1}\right)\right]_2^{2\sqrt{2}} \\
&= \frac{3}{2}\sqrt{2}+\frac{1}{4}\ln\left(2\sqrt{2}+3\right)-\frac{1}{4}\ln\left(\sqrt{5}+2\right)-\frac{\sqrt{5}}{2} \\
&\approx 1.083
\end{aligned}$$

Substitute $2y = u$, $\mathrm{d}y = \frac{1}{2}\mathrm{d}u$

$\int \sqrt{u^2+1}\mathrm{d}u$
$= \frac{1}{2}\left(u\sqrt{u^2+1}+\ln\left(u+\sqrt{u^2+1}\right)\right)+C$
previously studied

Solution II. The length of the parametric curve is given by

$$
\begin{aligned}
L &= \int_1^2 \sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\mathrm{d}x \\
&= \int_1^2 \sqrt{1+\left(\frac{1}{2\sqrt{x}}\right)^2}\mathrm{d}x \\
&= \int_{x=1}^{x=2} \sqrt{1+\frac{1}{4x}}\mathrm{d}x && \left|\ \begin{array}{rcl} \text{Substitute } 4x &=& u \\ \mathrm{d}x &=& \frac14 \mathrm{d}u \end{array}\right. \\
&= \int_{u=4}^{u=8} \sqrt{1+\frac{1}{u}}\left(\frac14 \mathrm{d}u\right) \\
&= \frac14\int_4^8 \sqrt{\frac{u+1}{u}}\mathrm{d}u \\
&= \frac14\int_4^8 \sqrt{\frac{u(u+1)}{u^2}}\mathrm{d}u \\
&= \frac14\int_4^8 \frac{\sqrt{u^2+u}}{u}\mathrm{d}u \\
&= \frac14\int_4^8 \frac{\sqrt{u^2+u+\frac14-\frac14}}{u}\mathrm{d}u \\
&= \frac14\int_4^8 \frac{\sqrt{\left(u+\frac12\right)^2-\frac14}}{u}\mathrm{d}u \\
&= \frac14\int_4^8 \frac{\sqrt{\frac14\left((2u+1)^2-1\right)}}{u}\mathrm{d}u \\
&= \frac18\int_{u=4}^{u=8} \frac{\sqrt{(2u+1)^2-1}}{u}\mathrm{d}u && \left|\ \begin{array}{rcl} \text{Substitute } 2u+1 &=& z \\ u &=& \frac{z-1}{2} \\ \mathrm{d}u &=& \frac12 \mathrm{d}z \end{array}\right. \\
&= \frac18\int_{z=9}^{z=17} \frac{\sqrt{z^2-1}}{\frac{z-1}{2}}\frac12\mathrm{d}z \\
&= \frac18\int_{z=9}^{z=17} \frac{\sqrt{z^2-1}}{z-1}\mathrm{d}z && \left|\ \begin{array}{rcl} \text{Trig. subst.: } z &=& \sec\theta \\ \sqrt{z^2-1} &=& \tan\theta \\ \mathrm{d}z &=& \tan\theta\sec\theta\mathrm{d}\theta \\ \text{Set } \alpha &=& \operatorname{arcsec}(9) \\ \text{Set } \beta &=& \operatorname{arcsec}(17) \\ \multicolumn{3}{l}{\text{Use } \tan^2\theta = \sec^2\theta - 1} \end{array}\right. \\
&= \frac18\int_{\theta=\operatorname{arcsec}(9)}^{\theta=\operatorname{arcsec}(17)} \frac{\tan\theta}{\sec\theta-1}\sec\theta\tan\theta\mathrm{d}\theta \\
&= \frac18\int_\alpha^\beta \frac{\tan^2\theta}{\sec\theta-1}\sec\theta\mathrm{d}\theta \\
&= \frac18\int_\alpha^\beta \frac{\sec^2\theta-1}{\sec\theta-1}\sec\theta\mathrm{d}\theta \\
&= \frac18\int_\alpha^\beta \frac{\cancel{(\sec\theta-1)}(\sec\theta+1)}{\cancel{\sec\theta-1}}\sec\theta\mathrm{d}\theta \\
&= \frac18\int_\alpha^\beta \left(\sec^2\theta+\sec\theta\right)\mathrm{d}\theta && \left|\ \begin{array}{rcl} \multicolumn{3}{l}{\int \sec\theta\mathrm{d}\theta = \ln|\sec\theta+\tan\theta| + C} \\ \multicolumn{3}{l}{\text{previously studied}} \\ \tan\theta &=& \sqrt{\sec^2\theta-1}, \theta\in\left[0,\frac{\pi}{2}\right) \\ \tan\alpha &=& \sqrt{9^2-1} = 4\sqrt5 \\ \tan\beta &=& \sqrt{17^2-1} = 12\sqrt2 \end{array}\right. \\
&= \frac18\left[\tan\theta+\ln|\sec\theta+\tan\theta|\right]_\alpha^\beta \\
&= \frac18\left(12\sqrt2+\ln(17+12\sqrt2)-4\sqrt5-\ln(9+4\sqrt5)\right) \\
&= \frac18\ln\left(12\sqrt2+17\right)-\frac18\ln\left(4\sqrt5+9\right)-\frac{\sqrt5}{2}+\frac32\sqrt2 \\
&\approx 1.083 \quad .
\end{aligned}
$$

The two answers are both approximately 1.083, so that serves to cross verify our two solutions against one another.

Comparing the two answers we notice that the logarithmic parts in the two answers look different (yet they must be equal). It follows that

$$\frac18\ln\left(12\sqrt2+17\right)-\frac18\ln\left(4\sqrt5+9\right)=\frac14\ln\left(2\sqrt2+3\right)-\frac14\ln\left(\sqrt5+2\right).$$

A short computation (which computation?), left to the reader, confirms that indeed those two expressions are equal.

Solution. 14.c. The length of the parametric curve is given by

$$L = \int_1^4 \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2} \mathrm{d}t \quad .$$

We have that

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= \frac{1}{2\sqrt{t}} - 2 \\
\frac{\mathrm{d}y}{\mathrm{d}t} &= 2t^{-\frac{1}{4}} \\
\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 &= \frac{1}{4t} - \frac{2}{\sqrt{t}} + 4 \\
\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 &= 4t^{-\frac{1}{2}} = \frac{4}{\sqrt{t}} \\
\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 &= \frac{1}{4t} + 2\frac{1}{\sqrt{t}} + 4 = \left(\frac{1}{2\sqrt{t}} + 2\right)^2 \quad .
\end{aligned}$$

$\frac{1}{2\sqrt{t}} + 2$ is positive and $\sqrt{\left(\frac{1}{2\sqrt{t}} + 2\right)^2} = \frac{1}{2\sqrt{t}} + 2$. So the integral becomes

$$L = \int_1^4 \left(\frac{1}{2\sqrt{t}} + 2\right) \mathrm{d}t = \left[\sqrt{t} + 2t\right]_{t=1}^{t=4} = (2+8) - (1+2) = 7 \quad .$$

15. Determine if the sequence is convergent or divergent. If convergent, find the limit of the sequence.

(a) $a_n = n$.

answer: divergent

(b) $a_n = 2^n$.

answer: divergent

(c) $a_n = 1.0001^n$.

answer: divergent

(d) $a_n = 0.999999^n$.

answer: convergent, $\lim_{n\to\infty} a_n = 0$

(e) $a_n = n - \sqrt{n+1}\sqrt{n+2}$

answer: convergent, $\lim_{n\to\infty} a_n = -\frac{3}{2}$

(f) $a_n = \dfrac{\ln n}{n}$.

answer: convergent, $\lim_{n\to\infty} a_n = 0$

(g) $a_n = \dfrac{\ln n}{\sqrt[10]{n}}$.

answer: convergent, $\lim_{n\to\infty} a_n = 0$

(h) $a_n = \dfrac{1}{n}$.

answer: convergent, $\lim_{n\to\infty} a_n = 0$

(i) $a_n = \dfrac{1}{n!}$.

answer: convergent, $\lim_{n\to\infty} a_n = 0$

(j) $a_n = \dfrac{n^n}{n!}$.

answer: divergent

(k) $a_n = \cos n$.

answer: divergent

(l) $a_n = \cos\left(\dfrac{1}{n}\right)$

answer: convergent, $\lim_{n\to\infty} a_n = 1$

(m) $a_n = \left(\dfrac{n+1}{n}\right)^n$.

answer: convergent, $\lim_{n\to\infty} a_n = e$

(n) $a_n = \left(\dfrac{2n+1}{n}\right)^n$.

answer: divergent

(o) $a_n = \left(\dfrac{n+1}{n}\right)^{2n}$.

answer: convergent, $\lim_{n\to\infty} a_n = e^2$

(p) $a_n = \left(\dfrac{n+1}{2n}\right)^n$.

answer: convergent, $\lim_{n\to\infty} a_n = 0$

Solution. 15m.

Consider $f(x) = \left(\frac{x+1}{x}\right)^x$, where x is a positive number. We will now show that $\lim\limits_{x\to\infty} f(x)$ exists. Since the limit is of the form 1^∞, we will start by finding the limit of the logarithm $\ln(f(x))$. We will then exponentiate that limit to find the limit of $f(x)$.

$$
\begin{array}{rcll}
\lim\limits_{x\to\infty} \ln\left(\left(\dfrac{x+1}{x}\right)^x\right) & = & \lim\limits_{x\to\infty} x\ln\left(\dfrac{x+1}{x}\right) & \\
& = & \lim\limits_{x\to\infty} \dfrac{\ln\left(\frac{x+1}{x}\right)}{\frac{1}{x}} & \\
& = & \lim\limits_{x\to\infty} \dfrac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} & \left| \begin{array}{l}\text{Form “}\frac{0}{0}\text{”} \\ \text{L'Hospital rule}\end{array}\right. \\
& = & \lim\limits_{x\to\infty} \dfrac{\frac{1}{1+\frac{1}{x}}\left(1+\frac{1}{x}\right)'}{-\frac{1}{x^2}} & \\
& = & \lim\limits_{x\to\infty} \dfrac{\frac{1}{\left(1+\frac{1}{x}\right)}\cancel{\left(-\frac{1}{x^2}\right)}}{\cancel{\left(-\frac{1}{x^2}\right)}} & \\
& = & \lim\limits_{x\to\infty} \dfrac{1}{1+\frac{1}{x}} & \\
& = & 1 & \\
\lim\limits_{x\to\infty} \left(\dfrac{x+1}{x}\right)^x & = & \lim\limits_{x\to\infty} e^{\ln\left(\left(\frac{x+1}{x}\right)^x\right)} & \left|\ \text{The exponent is continuous}\right. \\
& = & e^{\lim\limits_{x\to\infty} \ln\left(\left(\frac{x+1}{x}\right)^x\right)} & \\
& = & e^1 & \left|\ \text{use preceding}\right. \\
& = & e \quad . &
\end{array}
$$

Therefore $\lim\limits_{\substack{n\to\infty \\ n-\text{ integer}}} \left(\dfrac{n+1}{n}\right)^n = \lim\limits_{\substack{x\to\infty \\ x-\text{ real}}} \left(\dfrac{x+1}{x}\right)^x = e$ and the sequence converges (to e).

Solution. 15n.

This problem can be solved in fashion similar to Problem 15m. However there is a much simpler solution:

$$
\begin{array}{rcl|l}
\dfrac{2n+1}{n} & \geq & 2 & \text{for } n>0 \\
\lim\limits_{n\to\infty} \left(\dfrac{2n+1}{n}\right)^n & \geq & \lim\limits_{n\to\infty} 2^n & \begin{array}{l}\text{limits respect non-strict inequalities} \\ \lim\limits_{n\to\infty} 2^n \text{ computed in Problem 15b}\end{array} \\
\lim\limits_{n\to\infty} \left(\dfrac{2n+1}{n}\right)^n & = & \infty \quad . &
\end{array}
$$

16. (a)

$$
\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 - 1 \quad . \qquad (1)
$$

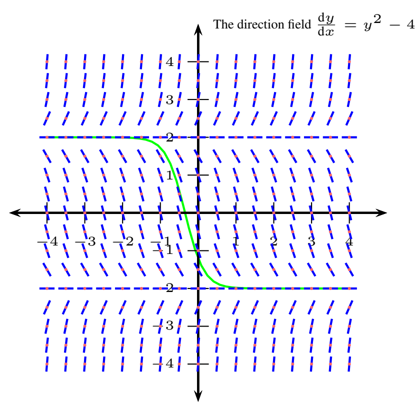
i. Find all solutions of the differential equation above.

ii. Find a solution for which $y(0) = -\frac{3}{5}$.

(b) i. Find the general solution to the differential equation

$$
\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 - 4 \quad .
$$

Below is a computer-generated plot of the direction field $\dfrac{\mathrm{d}y}{\mathrm{d}x} = y^2 - 4$, you may use it to get a feeling for what your answer should look like.



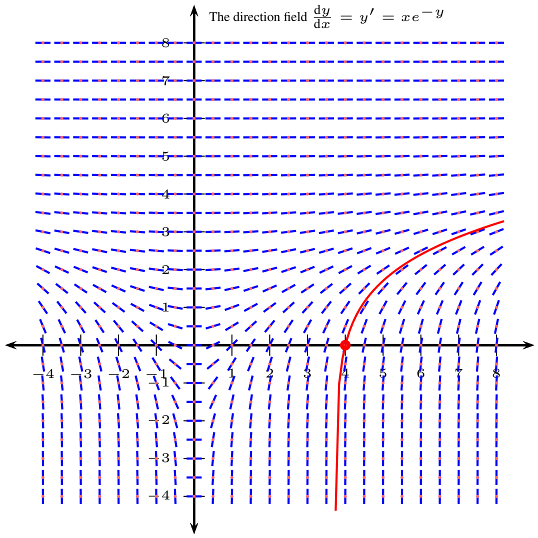
ii. Find a solution of the above equation for which $y(0) = -\frac{6}{5}$.

(c) Solve the initial-value differential equation $y' = y^2(1+x)$, $y(0) = 3$.

(d) Solve the initial-value differential equation problem

$$
y' = xe^{-y} \quad , \qquad y(4) = 0.
$$

Below is a computer-generated plot of the corresponding direction field, you may use it to get a feeling for what your answer should look like.

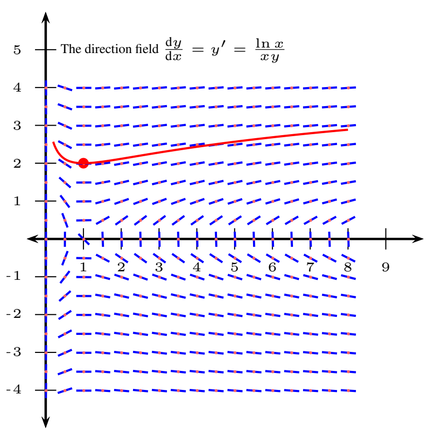


answer: $y(x) = \ln\left(\frac{x^2}{2} - 7\right)$

(e) Solve the initial-value differential equation problem

$$y' = \frac{\ln x}{xy} \quad , \qquad y(1) = 2.$$

Below is a computer-generated plot of the corresponding direction field, you may use it to get a feeling for what your answer should look like.



answer: $y(x) = \sqrt{(\ln x)^2 + 4}$

(f) i. Solve the initial-value differential equation problem

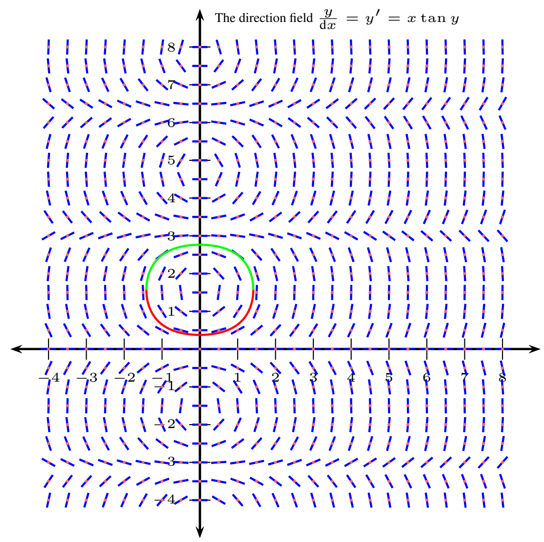
$$y' = x\tan y \quad , \qquad y(0) = \arcsin\left(\frac{1}{e}\right) \approx 0.376728.$$

answer: $y(x) = \arcsin\left(e^{\frac{x^2}{2}-1}\right)$

ii. Solve the same differential equation with initial condition $y(0) = \pi + \arcsin\left(-\frac{1}{e}\right) \approx 2.764865$.

answer: $y(x) = \pi + \arcsin\left(-e^{\frac{x^2}{2}-1}\right)$

Below is a computer-generated plot of corresponding direction field, you may use it to get a feeling for what your answer should look like.



Solution. 16.a.i. There are two variants for solving this problem. The first variant uses indefinite integration and is slightly informal, but easier to apply and remember. The second variant is more rigorous but more difficult to write up. Both solutions are acceptable for full credit in a Calculus exam. Variant I is recommended when taking exams and Variant II is recommended when writing scientific texts.

Variant I

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= y^2 - 1 && \Big| \text{ Suppose } y^2 - 1 \neq 0 \\
\frac{\frac{\mathrm{d}y}{\mathrm{d}x}}{y^2-1} &= 1 \\
\int \frac{1}{y^2-1} \underbrace{\frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x}_{=\mathrm{d}y} &= \int \mathrm{d}x \\
\int \frac{\mathrm{d}y}{y^2-1} &= x + C \\
\int \left(\frac{\frac{1}{2}}{y-1} - \frac{\frac{1}{2}}{y+1} \right) \mathrm{d}y &= x + C \\
\frac{1}{2} \ln \left| \frac{y-1}{y+1} \right| &= x + C \\
\ln \left| \frac{y-1}{y+1} \right| &= 2x + 2C \\
\left| \frac{y-1}{y+1} \right| &= e^{2x+2C} \\
\frac{y-1}{y+1} &= \pm e^{2x+2C} \\
y - 1 &= \pm e^{2x+2C}(y+1) \\
y(1 \mp e^{2x+2C}) &= 1 \pm e^{2x+2C} \\
y &= \frac{1 \pm e^{2x+2C}}{1 \mp e^{2x+2C}} \\
y &= \frac{1 \pm e^{2C} e^{2x}}{1 \mp e^{2C} e^{2x}} && \Big| \text{ Set } D = \pm e^{2C} \\
y &= \frac{1 + De^{2x}}{1 - De^{2x}} \quad .
\end{aligned}$$

The above solution works on condition that $y^2 - 1 \neq 0$. So the only case not covered is that of $y^2 - 1 = 0$, which yields the two solutions $y = \pm 1$.

Our final answer is

$$y(x) = \frac{1 + De^{2x}}{1 - De^{2x}} \quad \text{or} \quad y(x) = -1,$$

where D is an arbitrary real number. Notice that in the above answer, by allowing $D = 0$, we have covered the case $y(x) = 1$. Finally, we note that if we let $D \to \infty$, the solution $y(x) = \frac{1+De^{2x}}{1-De^{2x}}$ tends to the solution $y(x) = -1$ (here we fix a value of x before we let $D \to \infty$).

Variant II

Case 1. Suppose there exists a number x_0 such that $(y(x_0))^2 - 1 \neq 0$. Since y is a differentiable function of x, it is also continuous. Therefore for some t sufficiently close to x_0, all numbers x in the interval between t and x_0 satisfy $y(x)^2 - 1 \neq 0$.

$$\begin{array}{rcl|l}
\dfrac{\frac{\mathrm{d}y}{\mathrm{d}x}}{y^2-1} &=& 1 & \\
\displaystyle\int\limits_{x=x_0}^{x=t} \frac{1}{y^2-1} \underbrace{\frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x}_{=\mathrm{d}(y(x))} &=& \displaystyle\int\limits_{x=x_0}^{x=t} \mathrm{d}x & \text{can integrate as } y(x)^2-1\neq 0 \\
\displaystyle\int\limits_{t=x_0}^{x=t} \frac{\mathrm{d}(y(x))}{(y(x))^2-1} &=& x|_{x=x_0}^{x=t} & \text{set } z = y(x) \\
\displaystyle\int\limits_{z=y(x_0)}^{z=y(t)} \frac{\mathrm{d}z}{z^2-1} &=& t - x_0 & \\
\displaystyle\int\limits_{z=y(x_0)}^{z=y(t)} \left(\frac{\frac{1}{2}}{z-1} - \frac{\frac{1}{2}}{z+1}\right)\mathrm{d}z &=& t - x_0 & \\
\displaystyle\frac{1}{2}\ln\left|\frac{z-1}{z+1}\right|\Bigg]_{z=y(x_0)}^{z=y(t)} &=& t - x_0 & \text{Set } C = 2x_0 - \ln\left|\frac{y(x_0)-1}{y(x_0)+1}\right| \\
\ln\left|\dfrac{y(t)-1}{y(t)+1}\right| &=& 2t - C & \text{relabel dummy variable } t \text{ to } x \\
\ln\left|\dfrac{y(x)-1}{y(x)+1}\right| &=& 2x - C &
\end{array}$$

Set

$$D = e^{-C} \quad .$$

By the assumption of our case, $(y(x_0))^2 - 1 \neq 0$, so there are two remaining cases: $(y(x_0))^2 - 1 > 0$ and $(y(x_0))^2 - 1 < 0$.

Case 1.1. Suppose $\dfrac{y(x_0)-1}{y(x_0)+1} > 0$. As the function $y(x)$ is differentiable, it is also continuous. Therefore $\dfrac{y(x)-1}{y(x)+1} > 0$ for all x near x_0. Then we can remove the absolute values in the equality above to get that for all x close to x_0 we have that

$$\begin{array}{rcl|l}
\ln\left(\dfrac{y(x)-1}{y(x)+1}\right) &=& 2x - C & \text{exponentiate, recall } D = e^{-C} \\
\dfrac{y(x)-1}{y(x)+1} &=& De^{2x} & \\
y(x) - 1 &=& De^{2x}(y(x)+1) & \\
y(x)\left(1 - De^{2x}\right) &=& De^{2x} + 1 & \\
y(x) &=& \dfrac{1+De^{2x}}{1-De^{2x}} \quad . &
\end{array}$$

The solution $y(x)$ given above satisfies $\dfrac{y(x)-1}{y(x)+1} = De^{2x}$ for all x. As $D > 0$, this implies that $\dfrac{y(x)-1}{y(x)+1} > 0$. Therefore the considerations above are valid for all x, rather than only for those x near x_0. Therefore our first case yields the solution

$$y(x) = \frac{1+De^{2x}}{1-De^{2x}} \quad .$$

Case 1.2. Suppose $\dfrac{y(x_0)-1}{y(x_0)+1} < 0$. Then for all x near x_0 we get $\ln\left|\dfrac{y(x)-1}{y(x)+1}\right| = \ln\left(\dfrac{1-y(x)}{y(x)+1}\right)$ and, similarly to Case 1, we get

$$\begin{array}{rcl}
\dfrac{1-y(x)}{y(x)+1} &=& De^{2x} \\
1 - y(x) &=& De^{2x}(y(x)+1) \\
y(x)\left(1 + De^{2x}\right) &=& 1 - De^{2x} \\
y(x) &=& \dfrac{1-De^{2x}}{1+De^{2x}} \quad .
\end{array}$$

Since D is a positive constant, we conclude in a fashion analogous to Case 1 that $y(x) < 0$ for all x.

Case 2. Suppose $(y(x_0))^2 - 1 = 0$. Then $y(x_0) = \pm 1$. Clearly the constant functions $y(x) = \pm 1$ are two solutions: if we can plug back $y = \pm 1$ in the original equation we get that $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ and y is a constant function of x. From the preceding two cases we know that if $\frac{y(x)-1}{y(x)+1}$ is defined and not equal to zero for some value of x, then $\frac{y(x)-1}{y(x)+1}$ is defined and not equal to zero for all values of x. Therefore the present case yields only two solutions, the constant functions $y(x) = \pm 1$.

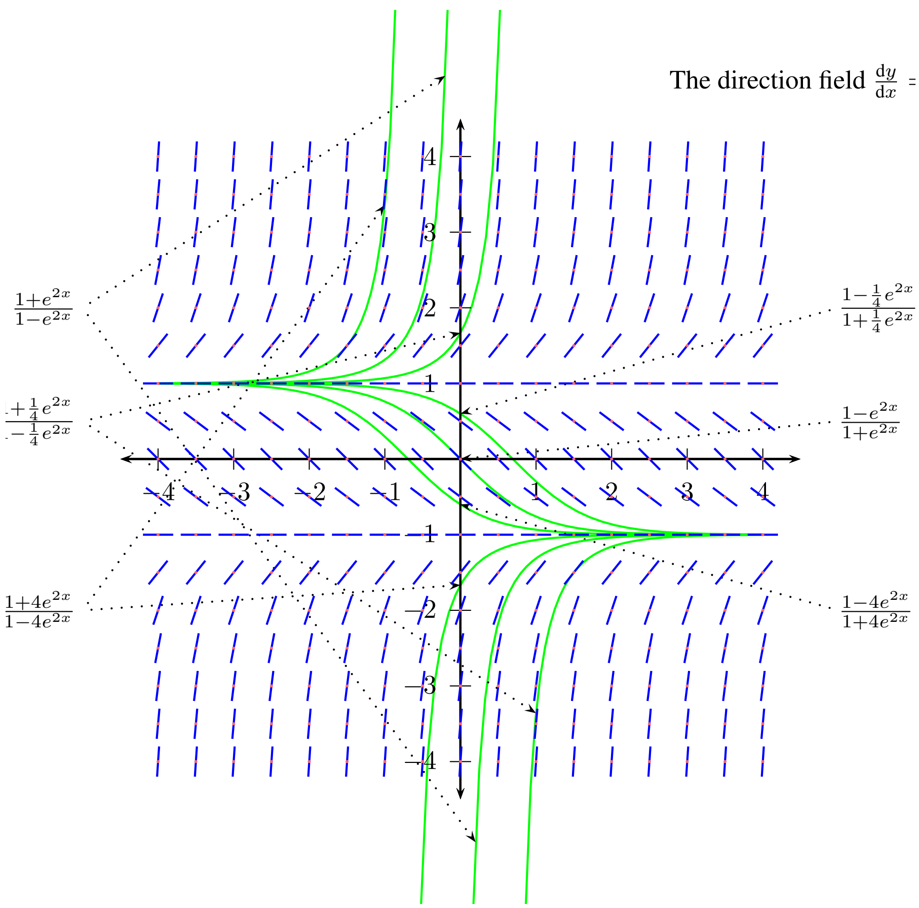
Our final answer is

$$y(x) = \frac{1 + De^{2x}}{1 - De^{2x}} \quad \text{or} \quad y(x) = -1,$$

where D is an arbitrary real number. Notice that in the above answer, we have combined Cases 1.1, 1.2 and the case $y(x) = 1$: by allowing D to be negative we included Case 1.2 and by allowing D to be zero we included the case $y(x) = 1$. Finally, we note that if we let $D \to \infty$, the solution $y(x) = \frac{1+De^{2x}}{1-De^{2x}}$ tends to the solution $y(x) = -1$ (for all values of x).

Solution plots.

We may plot solutions for a few values of D as follows. We overlay the solutions on top of the direction field of the differential equation. The picture tells us a lot about the properties of the solutions of the differential equations.



16.a.ii. From the computer generated picture above, we may visually estimate that $y(x) = \frac{1-4e^{2x}}{1+4e^{2x}}$ intersects the x-axis at $\left(0, -\frac{3}{5}\right)$. Furthermore, we may check directly that for

$$y(x) = \frac{1 - 4e^{2x}}{1 + 4e^{2x}}$$

we have $y(0) = \frac{1-4}{1+5} = -\frac{3}{5}$ and that is a solution to our problem (this however does not prove the solution is unique).

Alternatively, let us give an algebraic solution. As we are given that $y(0) = -\frac{3}{5}$ and so

$$\begin{aligned} -\frac{3}{5} &= y(0) = \frac{1 - De^{2\cdot 0}}{1 + De^{2\cdot 0}} = \frac{1 - D}{1 + D} \\ -\frac{3}{5}(1 + D) &= 1 - D \\ \frac{2}{5}D &= \frac{8}{5} \\ D &= 4 \quad , \end{aligned}$$

which is our final answer.

Solution. 16.c.

This is a concise solution written up in a form suitable for exam taking.

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= y^2(1+x)\\
\frac{\mathrm{d}y}{y^2} &= (1+x)\mathrm{d}x\\
\int \frac{\mathrm{d}y}{y^2} &= \int (1+x)\mathrm{d}x\\
-\frac{1}{y} &= x+\frac{x^2}{2}+C\\
-\frac{1}{3} &= 0+0+C\\
y &= -\frac{1}{\frac{x^2}{2}+x-\frac{1}{3}} = -\frac{3}{3x^2+6x-2} \quad .
\end{aligned}$$

Solution. 16.f.i and 16.f.ii

$$\begin{aligned}
y' &= x\tan y\\
\frac{y'}{\tan y} &= x\\
\frac{(\cos y)y'}{\sin y} &= x && \Big| \text{ Integrate from 0}\\
\int\limits_{t=0}^{t=x} \frac{\cos(y(t))}{\sin(y(t))}(y'\mathrm{d}t) &= \int\limits_{t=0}^{x} t\mathrm{d}t\\
\int\limits_{t=0}^{t=x} \frac{\cos(y(t))}{\sin(y(t))}\mathrm{d}(y(t)) &= \frac{x^2}{2} && \Big| \text{ Set } z=y(t)\\
\int\limits_{z=y(0)}^{z=y(x)} \frac{\cos z}{\sin z}\mathrm{d}z &= \frac{x^2}{2}\\
\int\limits_{z=y(0)}^{z=y(x)} \frac{\mathrm{d}(\sin z)}{\sin z} &= \frac{x^2}{2}\\
[\ln|\sin z|]_{y(0)}^{y} &= \frac{x^2}{2}\\
\ln|\sin y| - \ln|\sin(y(0))| &= \frac{x^2}{2}\\
\ln|\sin y| &= \frac{x^2}{2} + \ln|\sin(y(0))|\\
|\sin y| &= e^{\frac{x^2}{2}+\ln|\sin(y(0))|}\\
|\sin y| &= \begin{cases} e^{\frac{x^2}{2}+\ln\left|\sin\left(\arcsin\left(\frac{1}{e}\right)\right)\right|} & \text{for problem 16.f.i}\\ e^{\frac{x^2}{2}+\ln\left|\sin\left(\pi+\arcsin\left(\frac{1}{e}\right)\right)\right|} & \text{for problem 16.f.ii}\end{cases}\\
|\sin y| &= e^{\frac{x^2}{2}+\ln\left(\frac{1}{e}\right)}\\
|\sin y| &= e^{\frac{x^2}{2}-1} && \Big| \begin{array}{l} y(0)>0 \text{ for both problems}\\ \text{therefore } \sin y(0)>0\end{array}\\
\sin y &= e^{\frac{x^2}{2}-1} \quad .
\end{aligned}$$

From the elementary properties of the trigonometric functions, we know that $\sin y = \sin\alpha$ implies that either

- $y = \alpha + 2k\pi$, where k is an arbitrary integer or
- $y = (2k+1)\pi - \alpha$, where k is an arbitrary integer.

In other words, if we are given $\sin y$, we know y up to a choice of sign and a choice of an integer k. For our problem, this means that

$$y = \begin{cases} 2k\pi + \arcsin\left(e^{\frac{x^2}{2}-1}\right) & k - \text{integer} \\ \text{or} & \\ (2k+1)\pi - \arcsin\left(e^{\frac{x^2}{2}-1}\right) & k - \text{integer} \end{cases}$$

For problem 16.f.i, the only choice for k and sign which fits the initial condition $y(0) = \arcsin\left(\frac{1}{e}\right)$ is

$$y = \arcsin\left(e^{\frac{x^2}{2}-1}\right) \quad ,$$

which is our final answer.

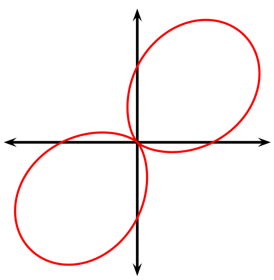
For problem 16.f.ii, the only choice for k and sign which fits the initial condition $y(0) = \pi + \arcsin\left(-\frac{1}{e}\right) = \pi - \arcsin\left(\frac{1}{e}\right)$ is

$$y = \pi - \arcsin\left(e^{\frac{x^2}{2}-1}\right) \quad ,$$

which is our final answer.

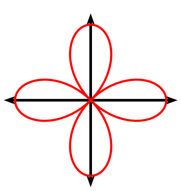
17. **This problem type will appear on the final as a bonus. We have not studied the material for this problem type.**

    (a) The curve given in polar coordinates by $r = 1 + \sin 2\theta$ is plotted below by computer. Find the area lying outside of this curve and inside of the circle $x^2 + y^2 = 1$.

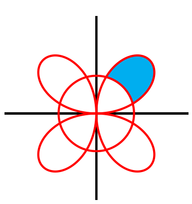
    answer: $a = 2 - \frac{\pi}{4}$

    (b) The curve given in polar coordinates by $r = \cos(2\theta)$ is plotted below by computer. Find the area lying inside the curve and outside of the circle $x^2 + y^2 = \frac{1}{4}$.

    answer: $\frac{\pi}{6} + \frac{\sqrt{3}}{4}$

    (c) Below is a computer generated plot of the curve $r = \sin(2\theta)$. Find the area locked inside one petal of the curve and outside of the circle $x^2 + y^2 = \dfrac{1}{4}$.

    answer: $\frac{\pi}{24} + \frac{\sqrt{3}}{16}$

Solution. 17.a. A computer generated plot of the two curves is included below. The circle $x^2 + y^2 = 1$ has one-to-one polar representation given by $r = 1, \theta \in [0, 2\pi)$. Except the origin, which is traversed four times by the curve $r = 1 + \sin(2\theta)$, the second curve is in a one-to-one correspondence with points in the r, θ-plane given by the equation $r = 1 + \sin(2\theta), \theta \in [0, 2\pi)$.

Since the two curves do not meet in the origin, we may conclude that the two curves may intersect only when their values for r and θ coincide. Therefore we have an intersection when

$$\begin{array}{rcll} 1+\sin(2\theta) & = & 1 & \\ \sin(2\theta) & = & 0 & \\ \theta & = & 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2} & \Big| \text{ because } \theta \in [0, 2\pi) \end{array}$$

Therefore the two curves meet in the points $(0,1)(-1,0)$ and $(0,-1),(1,0)$.

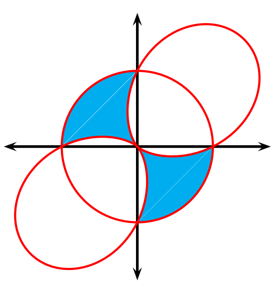
Denote the investigated region by A. From the computer-generated plot, it is clear that when a point has polar coordinates $\theta \in [\frac{\pi}{2}, \pi] \cup [\frac{3\pi}{2}, 2\pi]$, $r \in [1+\sin(2\theta), 1]$ it lies in A. Furthermore, the points r, θ lying in the above intervals are in one-to-one correspondence with the points in A.

Suppose we have a curve $r = f(\theta), \theta \in [a, b]$ for which no two points lie on the same ray from the origin. Recall from theory that the area swept by that curve is given by

$$\int\limits_a^b \frac{1}{2} f^2(\theta) \mathrm{d}\theta \quad .$$

Therefore the area a of A is computed via the integrals

$$\begin{array}{rcl|l} a & = & \displaystyle\int\limits_{\frac{\pi}{2}}^{\pi} \frac{1}{2}\left(\underbrace{1}_{\text{outer curve}}{}^2 - \left(\underbrace{1+\sin(2\theta)}_{\text{inner curve}}\right)^2\right) \mathrm{d}\theta + \int\limits_{\frac{3\pi}{2}}^{2\pi} \frac{1}{2}\left(1^2 - (1+\sin(2\theta))^2\right) \mathrm{d}\theta & \text{use the symmetry of } A \\ & = & \displaystyle\int\limits_{\frac{\pi}{2}}^{\pi} \left(1^2 - (1+\sin(2\theta))^2\right) \mathrm{d}\theta = \int\limits_{\frac{\pi}{2}}^{\pi} \left(-2\sin(2\theta) - \sin^2(2\theta)\right) \mathrm{d}\theta & \text{use } \sin^2 z = \frac{1-\cos(2z)}{2} \\ & = & \displaystyle\int\limits_{\frac{\pi}{2}}^{\pi} \left(-2\sin(2\theta) - \frac{1}{2} + \frac{1}{2}\cos(4\theta)\right) \mathrm{d}\theta = \left[\cos(2\theta) - \frac{1}{2}\theta - \frac{1}{8}\sin(4\theta)\right]_{\frac{\pi}{2}}^{\pi} & \\ & = & 2 - \frac{\pi}{4} \quad . & \end{array}$$



Solution. 17.b A computer generated plot of the figure is included below. The circle $x^2 + y^2 = \frac{1}{4}$ is centered at 0 and of radius $\frac{1}{2}$ and therefore can be parametrized in polar coordinates via $r = \frac{1}{2}, \theta \in [0, 2\pi]$.

Points with polar coordinates (r_1, θ_1) and (r_2, θ_2) coincide if one of the three holds:

- $r_1 = r_2 \neq 0$ and $\theta_1 = \theta_2 + 2k\pi, k \in \mathbb{Z}$,
- $r_1 = -r_2 \neq 0$ and $\theta_1 = \theta_2 + (2k+1)\pi, k \in \mathbb{Z}$,
- $r_1 = r_2 = 0$ and θ is arbitrary.

To find the intersection points of the two curves we have to explore each of the cases above. The third case is not possible as the circle does not pass through the origin. Suppose we are in the first case. Then the value of r (as a function of θ) is equal for the two curves. Thus the two curves intersect if

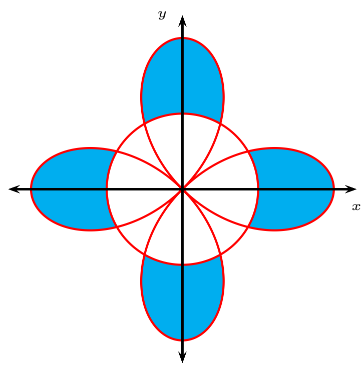
$$\begin{array}{rcl|l} r = \cos(2\theta) & = & \frac{1}{2} & \\ 2\theta & = & \pm\frac{\pi}{3} + 2k\pi & \text{where } k \in \mathbb{Z} \\ \theta & = & \pm\frac{\pi}{6} + k\pi & \text{where } k \in \mathbb{Z} \\ \theta & = & \frac{\pi}{6}, \frac{\pi}{6} + \pi, -\frac{\pi}{6} + \pi, -\frac{\pi}{6} + 2\pi & \text{all other values discarded as } \theta \in [0, 2\pi] \\ \theta & = & \frac{\pi}{6}, \frac{7\pi}{6}, \frac{5\pi}{6}, \frac{11\pi}{6} & \end{array}$$

This gives us only four intersection points, and the computer-generated plot shows eight. Therefore the second case must yield new intersection points: the two curves intersect also when

$$\begin{array}{rcll} r=\cos(2\theta) &=& -\frac{1}{2} & \\ 2\theta &=& \pm\frac{2\pi}{3}+2k\pi & \text{where } k\in\mathbb{Z} \\ \theta &=& \pm\frac{\pi}{3}+k\pi & \text{where } k\in\mathbb{Z} \\ \theta &=& \frac{\pi}{3}, \frac{\pi}{3}+\pi, \frac{-\pi}{3}+\pi, \frac{-\pi}{3}+2\pi & \text{all other values are discarded as } \theta\in[0,2\pi] \\ \theta &=& \frac{\pi}{3}, \frac{4\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{3} & . \end{array}$$

From the computer-generated plot below, we can see that the area we are looking for is 4 times the area locked between the two curves for $\theta\in\left[\frac{-\pi}{6},\frac{\pi}{6}\right]$. Therefore the area we are looking for is given by

$$4\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}}\frac{1}{2}\left(\cos^2(2\theta)-\left(\frac{1}{2}\right)^2\right)\mathrm{d}\theta \quad .$$



We leave the above integral to the reader.

Solution. 17.c. The circle $x^2+y^2=\frac{1}{4}$ is centered at 0 and of radius $\frac{1}{2}$ and therefore can be parametrized in polar coordinates via $r=\frac{1}{2}, \theta\in[0,2\pi)$.

Points with polar coordinates (r_1,θ_1) and (r_2,θ_2) coincide if one of the three holds:

- $r_1=r_2\neq 0$ and $\theta_1=\theta_2+2k\pi, k\in\mathbb{Z}$,
- $r_1=-r_2\neq 0$ and $\theta_1=\theta_2+(2k+1)\pi, k\in\mathbb{Z}$,
- $r_1=r_2=0$ and θ is arbitrary.

To find the intersection points of the two curves we have to explore each of the cases above. The third case is not possible as the circle does not pass through the origin. Suppose we are in the first case. Then the value of r (as a function of θ) is equal for the two curves. Thus the two curves intersect if

$$\begin{array}{rcll} r=\sin(2\theta) &=& \frac{1}{2} & \\ 2\theta &=& \frac{\pi}{6}+2k\pi \text{ or } \frac{5\pi}{6}+2k\pi & \text{where } k\in\mathbb{Z} \\ \theta &=& \frac{\pi}{12}+k\pi \text{ or } \frac{5\pi}{12} & \text{where } k\in\mathbb{Z} \\ \theta &=& \frac{\pi}{12}, \frac{13\pi}{12}, \frac{5\pi}{12}, \frac{17\pi}{12} & \text{other values discarded as } \theta\in[0,2\pi] \end{array}$$

This gives us only four intersection points, and the computer-generated plot shows eight. Therefore the second case must yield 4 new intersection points. However, from the figure we see there are only two intersection points that participate in the boundary of our area, and both of those were found above. Therefore we shall not find the remaining 4 intersections.

Both the areas locked by the petal and the area locked by the section of the circle are found by the formula for the area locked by a polar curve. Subtracting the two we get that the area we are looking for is:

$$
\begin{aligned}
\text{Area} &= \int_{\theta=\frac{\pi}{12}}^{\theta=\frac{5\pi}{12}} \frac{1}{2}\left(\sin^2(2\theta) - \left(\frac{1}{2}\right)^2\right) \mathrm{d}\theta \quad . \\
&= \frac{1}{2}\int_{\theta=\frac{\pi}{12}}^{\theta=\frac{5\pi}{12}} \left(\frac{1-\cos(4\theta)}{2} - \frac{1}{4}\right) \mathrm{d}\theta \\
&= \frac{1}{2}\left[\frac{1}{4}\theta - \frac{\sin(4\theta)}{8}\right]_{\theta=\frac{\pi}{12}}^{\theta=\frac{5\pi}{12}} \\
&= \tfrac{\pi}{24} + \tfrac{\sqrt{3}}{16} \quad .
\end{aligned}
$$