

Precalculus

Lecture 9

Laws of Sines and Cosines

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<https://github.com/tmilev/freecalculator>

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Outline

- 1 Law of sines
- 2 Law of cosines

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<https://github.com/tmilev/freecalc>

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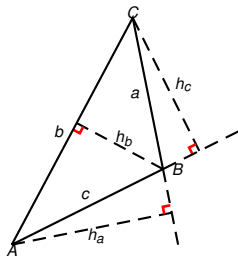
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Triangle area = $\frac{1}{2}$ base \cdot height

Let $\triangle ABC$ have side lengths a, b, c and height lengths h_a, h_b, h_c , as indicated - side a is opposite to vertex A and h_a starts at A , and so on.

Proposition (Triangle area)

$$\text{Area}(\triangle ABC) = \frac{1}{2} \text{height} \cdot \text{base} = \frac{1}{2} h_a a = \frac{1}{2} h_b b = \frac{1}{2} h_c c.$$



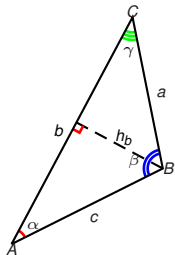
Triangle area from two sides and angle between them

Let $\triangle ABC$ have sides lengths a, b, c angles α, β, γ , as indicated: α is opposite to a , β is opposite to b , γ is opposite to c .

Proposition (\triangle area from two sides and angle between them)

The area of a triangle is half the product of the lengths of two of its sides times the sine of the angle between them. In other words,

$$\text{Area}(\triangle ABC) = \frac{ab \sin \gamma}{2} = \frac{bc \sin \alpha}{2} = \frac{ca \sin \beta}{2}$$



Proof.

$$\begin{aligned} \text{Area}(\triangle ABC) &= \frac{\text{base} \cdot \text{height}}{2} = \frac{bh_b}{2} \\ &= \frac{ba \sin \gamma}{2}. \end{aligned}$$

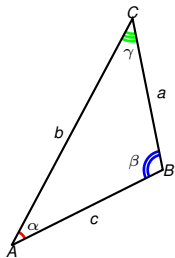
The proof of the other two cases is similar. □

Law of sines

Let $\triangle ABC$ have sides lengths a, b, c angles α, β, γ , as indicated: α is opposite to a , β is opposite to b , γ is opposite to c .

Proposition (Law of Sines)

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma}.$$

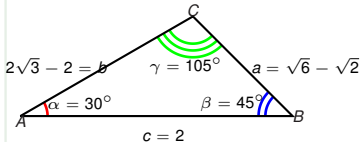


Proof.

$$\begin{aligned} \text{Area}(\triangle ABC) &= \frac{ab \sin \gamma}{2} = \frac{bc \sin \alpha}{2} & \left| \text{Div. by } \frac{b}{2} \right. \\ a \sin \gamma &= c \sin \alpha \\ \frac{a}{\sin \alpha} &= \frac{c}{\sin \gamma}. \end{aligned}$$

The remaining cases are similar. □

Example

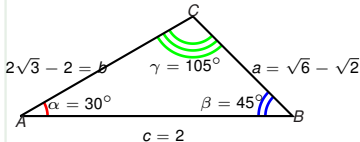


A triangle has a side of length 2cm ; the two angles adjacent to it are 30° and 45° .

- Find the other two sides of the triangle.
- Find the area of the triangle.

- Let the known side be $c = 2\text{cm}$.
- Let the known angles 30° , 45° be arranged as in the figure, and let the third angle be $\gamma = 180^\circ - 30^\circ - 45^\circ = 180^\circ - 75^\circ = 105^\circ$.
- Label the unknown sides a , b as indicated.

Example



A triangle has a side of length 2cm ; the two angles adjacent to it are 30° and 45° .

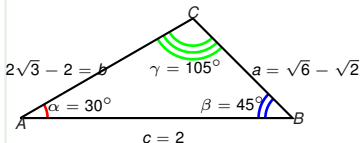
- Find the other two sides of the triangle.
- Find the area of the triangle.

$$\begin{aligned}\sin 105^\circ &= \sin(60^\circ + 45^\circ) = \sin 60^\circ \cos 45^\circ + \cos 60^\circ \sin 45^\circ \\ &= \frac{\sqrt{3}}{2} \frac{\sqrt{2}}{2} + \frac{1}{2} \frac{\sqrt{2}}{2} = \frac{\sqrt{6} + \sqrt{2}}{4}\end{aligned}$$

$$\frac{a}{\sin \alpha} = \frac{c}{\sin \gamma} \quad \text{Law of sines}$$

$$\begin{aligned}a &= \frac{c \sin \alpha}{\sin \gamma} = \frac{2 \sin 30^\circ}{\sin 105^\circ} = \frac{2 \cdot \frac{1}{2}}{\frac{\sqrt{6} + \sqrt{2}}{4}} = \frac{4(\sqrt{6} - \sqrt{2})}{(\sqrt{6} + \sqrt{2})(\sqrt{6} - \sqrt{2})} \\ &= \frac{4(\sqrt{6} - \sqrt{2})}{6 - 2} = \sqrt{6} - \sqrt{2}\end{aligned}$$

Example



A triangle has a side of length 2cm ; the two angles adjacent to it are 30° and 45° .

- Find the other two sides of the triangle.
- Find the area of the triangle.

| Law of sines

$$\frac{b}{\sin \beta} = \frac{c}{\sin \gamma}$$

$$b = \frac{c \sin \beta}{\sin \gamma} = \frac{2 \sin 45^\circ}{\sin 105^\circ} = \frac{\cancel{2} \frac{\sqrt{2}}{2}}{\frac{\sqrt{6} + \sqrt{2}}{4}} = \frac{4\sqrt{2}(\sqrt{6} - \sqrt{2})}{(\sqrt{6} + \sqrt{2})(\sqrt{6} - \sqrt{2})}$$

$$= \frac{\cancel{4}\sqrt{2}(\sqrt{6} - \sqrt{2})}{\cancel{4}} = 2\sqrt{3} - 2$$

$$\text{Area} = \frac{bc \sin \alpha}{2} = \frac{(2\sqrt{3} - 2)\cancel{2} \frac{1}{2}}{2} = \sqrt{3} - 1 \quad \text{cm}^2$$

Let $\triangle ABC$ have sides lengths a, b, c angles α, β, γ , as indicated.

Proposition (Law of Cosines)

$$c^2 = a^2 + b^2 - 2ab \cos \gamma$$

$$a^2 = b^2 + c^2 - 2bc \cos \alpha$$

$$b^2 = c^2 + a^2 - 2ca \cos \beta$$

Proof if $\gamma < 90^\circ$.

Drop a perpendicular h from B to AC .

$$|CD| = a \cos \gamma$$

$$h = a \sin \gamma$$

$$|AD| = b - |CD| = b - a \cos \gamma$$

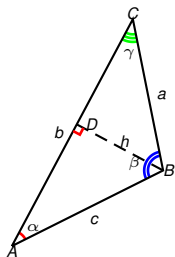
$$c^2 = |AD|^2 + h^2$$

$$= (b - a \cos \gamma)^2 + (a \sin \gamma)^2$$

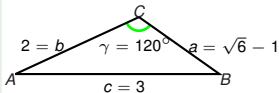
$$= b^2 - 2ab \cos \gamma + a^2 \cos^2 \gamma + a^2 \sin^2 \gamma$$

$$= b^2 - 2ab \cos \gamma + a^2.$$

Pyth. thm.
 $\triangle BDA$



Example



The longest side of a triangle has length 3 and the angle opposite to it is 120° . Another side of that triangle has length 2.

- Find the length of the third side.
- Find the area of the triangle.

$$a^2 + b^2 - 2ab \cos \gamma = c^2$$

$$a^2 + 2^2 - 2a \cdot 2 \cdot \cos 120^\circ = 3^2$$

$$a^2 - 4a \left(-\frac{1}{2} \right) - 5 = 0$$

$$a^2 + 2a - 5 = 0$$

$$a = \frac{-2 \pm \sqrt{2^2 - 4 \cdot (-5) \cdot 1}}{2 \cdot 1}$$

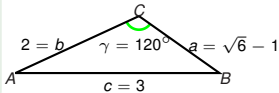
$$= \frac{-2 \pm \sqrt{24}}{2} = \frac{-2 + 2\sqrt{6}}{2}$$

$$= -1 + \sqrt{6}$$

Law of cosines
Solve for a :

$$a > 0$$

Example



The longest side of a triangle has length 3 and the angle opposite to it is 120° . Another side of that triangle has length 2.

- Find the length of the third side.
- Find the area of the triangle.

$$a^2 + b^2 - 2ab \cos \gamma = c^2$$

$$a^2 + 2^2 - 2a \cdot 2 \cdot \cos 120^\circ = 3^2$$

Law of cosines
Solve for a :

$$a = \frac{-2 \pm \sqrt{2^2 - 4 \cdot (-5) \cdot 1}}{2 \cdot 1}$$

$$= -1 + \sqrt{6}$$

$$\text{Area} = \frac{ab \sin \gamma}{2} = \frac{(\sqrt{6} - 1) \cancel{2} \sqrt{3}}{\cancel{2} 2}$$

$$= \frac{3\sqrt{2} - \sqrt{3}}{2}$$