Precalculus
Lecture 4
Complex Numbers

Todor Milev
https://github.com/tmilev/freecalc

2020

Outline

License to use and redistribute

Definition (Complex numbers)

The set of complex numbers $\mathbb{C}$ is defined as the set

$$\{a+bi | a, b - \text{real numbers}\},$$

where the number i is a number for which

$$i^2 = -1 \quad .$$

The number i is called the imaginary unit. By definition, $\sqrt{-1} = i$.

- Complex addition/subtraction

$$(a+bi) \pm (c+di) = (a \pm c) + (b \pm d)i \quad .$$

- Complex multiplication

$$\begin{aligned}(a+bi)(c+di) &= ac + adi + bci + bdi^2 = ac + adi + bci - bd \\ &= (ac - bd) + i(ad + bc)\end{aligned}$$

Let $u = 2 + 3i$, $v = 5 - 7i$.

Example (Addition)

$u + v = (2 + 3i) + (5 - 7i) = (2 + 5) + (3 - 7)i = 7 - 4i.$

Example (Subtraction)

$u - v = (2 + 3i) - (5 - 7i) = (2 - 5) + (3 - (-7))i = -3 + 10i.$

Example (Multiplication)

$$\begin{aligned} u \cdot v &= (2+3i)\cdot(5-7i) \\ &= 2\cdot 5 + 2\cdot(-7)i + 3i\cdot 5 + 3i(-7i) \\ &= 10 - 14i + 15i - 21i^2 \\ &= 10 + i - (-21) \\ &= 31 + i \end{aligned}$$

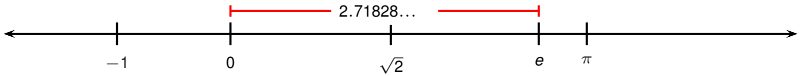
Example (Complex multiplication)

Multiply $u = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i$ by $v = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i$.

$$
\begin{aligned}
u \cdot v &= \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i \right) \cdot \left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i \right) \\
&= \frac{\sqrt{2}^2}{2^2} - \frac{\sqrt{2}^2}{2^2} i + \frac{\sqrt{2}^2}{2^2} i \cancel{\frac{\sqrt{2}^2}{2^2}} i + \cancel{\frac{\sqrt{2}^2}{2^2}} i - \frac{\sqrt{2}^2}{2^2} i^2 \\
&= \frac{2}{4} - \frac{2}{4}(-1) \\
&= \frac{1}{2} + \frac{1}{2} \\
&= 1
\end{aligned}
$$

Review of the basic types of numbers

- An integer, or whole number, is one of the numbers:
$$\dots, -2, -1, 0, 1, 2, \dots.$$
- A rational number is the quotient of two integers, for example:
$$\frac{1}{2}, \qquad \frac{2}{-3} = -\frac{2}{3}, \qquad \frac{8}{12} = \frac{4}{6} = \frac{2}{3}.$$
- A real number measures the location of a point on the real line:
$$\begin{array}{rcl} \sqrt{2} & = & 1.41421356237309504880168872420969 8\dots \\ \pi & = & 3.14159265358979323846264338327950 2\dots \\ e & = & 2.71828182845904523536028747135266 2\dots \\ -1 & & \end{array}$$



- A number is complex if it equals $a + bi$ with a, b- real, $\sqrt{-1} = i$:
$$2 + 3i, \qquad -i, \qquad 1 + \sqrt{2}i$$
- Geometric interpretation of complex numbers: beyond our scope.