Calculus II
Lecture 2

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Outline

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Antiderivatives

Definition (Antiderivative)

A function F is called an antiderivative of f on an interval I if $F'(x) = f(x)$ for all x in I.

Theorem

Let f be a continuous function on $[a, b]$. Then f is integrable over $[a, b]$.

In other words, $\int_a^b f(x)dx$ exists for any continuous (over $[a, b]$) function f.

Theorem (The Evaluation Theorem (FTC part 2))

If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\mathrm{d}x = F(b) - F(a),$$

where F is any antiderivative of f.

Indefinite Integrals

- The Evaluation Theorem establishes a connection between antiderivatives and definite integrals.
- It says that $\int_a^b f(x)\mathrm{d}x$ equals $F(b) - F(a)$, where F is an antiderivative of f.
- We need convenient notation for writing antiderivatives.
- This is what the indefinite integral is.

Definition (Indefinite Integral)

The indefinite integral of f is another way of saying the antiderivative of f, and is written $\int f(x)\mathrm{d}x$. In other words,

$$\int f(x)\mathrm{d}x = F(x) \qquad \text{means} \qquad F'(x) = f(x).$$

Example

$$\int x^4 \mathrm{d}x = \frac{x^5}{5} + C$$

because

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^5}{5} + C\right) = x^4.$$

- The indefinite integral represents a whole family of functions.
- Example: the general antiderivative of $\frac{1}{x}$ is

$$F(x) = \begin{cases} \ln|x| + C_1 & \text{if} \quad x > 0 \\ \ln|x| + C_2 & \text{if} \quad x < 0 \end{cases}$$

- We adopt the convention that the constant participating in an indefinite integral is only valid on one interval.
- $\int \frac{1}{x}\mathrm{d}x = \ln|x| + C$, and this is valid either on $(-\infty, 0)$ or $(0, \infty)$.

Differentials

- Recall $\Delta y, \Delta x$ stand for change of x, y. Recall: $\Delta y \approx \frac{\mathrm{d}y}{\mathrm{d}x}\Delta x$
- $\mathrm{d}y = \frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x = \mathrm{d}y$
- If we substitute Δy by the formal expression $\mathrm{d}y$ and Δx by the formal expression $\mathrm{d}x$, the expression $\mathrm{d}x$ appears to "cancel" to give a formal identity.
- Define the *differential* d and the *differential forms* $\mathrm{d}x$, $\mathrm{d}(f(x))$ by requesting that d and $\mathrm{d}x$ satisfy the transformation law

$$\mathrm{d}(f(x)) = f'(x)\mathrm{d}x$$

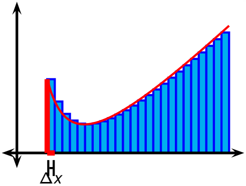
  for any differentiable function $f(x)$. In abbreviated notation:

$$\mathrm{d}f = f'\mathrm{d}x$$

  Expressions containing expression of the form d(*something*) are called differential forms.

- $\mathrm{d}f(x) = f'(x)\mathrm{d}x$.
- On the previous slide we stated the differential d and the differential forms $\mathrm{d}x, \mathrm{d}f(x)$ are formal expressions related by a transformation law.
- The precise definitions of differential forms and differentials are outside of the scope of Calculus I and II.
- Differential forms "encode" linear approximations which in turn "encode" "infinitesimal" lengths of segments.
- Courses such as "Integration and Manifolds" or "Differential geometry" usually give precise definitions and fill in the details.
- Nonetheless, what we studied is completely sufficient for practical purposes and carrying out computations.
- **Do not confuse differentials with derivatives.** The correct equality is this.

$$\cancel{\mathrm{d}f(x) = f'(x)} \qquad\qquad \mathrm{d}f(x) = f'(x)\mathrm{d}x$$



- $\int_a^b f(x)\mathrm{d}x$ is the definite integral of f.
- $\int f(x)\mathrm{d}x$ = corresponding anti-derivative.
- $\int$ stands for the limit of a Riemann sum (sum of approximating rectangles).

- $\mathrm{d}x$ "encodes" the base length of "infinitesimally small" approximating rectangle, $f(x)$ is the height.
- $f(x)\mathrm{d}x$ is a differential form as discussed already.
- We postponed a formal definition of differential form to another course, but we showed how to compute with those.
- This is consistent: integrals of equal differential forms are equal (follows from Net Change Theorem (subst. rule)).

- All rules for computing with derivatives have analogues for computing with differential forms.
- The rules for computing differential forms are a direct consequence of the corresponding derivative rules and the transformation law $\mathrm{d}(f(x)) = f'(x)\mathrm{d}x$.

Let c be a constant.

Differential rule	Derivative rule
$\mathrm{d}(fg) = g\mathrm{d}f + f\mathrm{d}g$	$(fg)' = f'g + fg'$
$\mathrm{d}c = 0$	$(c)' = 0$
$\mathrm{d}(cf) = c\,\mathrm{d}f$	$(cf)' = cf'$
$\mathrm{d}(f+g) = \mathrm{d}f + \mathrm{d}g$	$(f+g)' = f' + g'$
$\mathrm{d}f(g(x)) = f'(g(x))\mathrm{d}g(x)$	
$= f'(g(x))g'(x)\mathrm{d}x$	$(f(g(x)))' = f'(g(x))g'(x)$
$\mathrm{d}f(g) = f'(g)\mathrm{d}g$	
$\mathrm{d}x^n = nx^{n-1}\mathrm{d}x$	$(x^n)' = nx^{n-1}$
$\mathrm{d}e^x = e^x\mathrm{d}x$	$(e^x)' = e^x$
$\mathrm{d}\sin x = \cos x\mathrm{d}x$	$(\sin x)' = \cos x$
$\mathrm{d}\cos x = -\sin x\mathrm{d}x$	$(\cos x)' = -\sin x$
$\mathrm{d}\ln x = \frac{1}{x}\mathrm{d}x$	$(\ln x)' = \frac{1}{x}$

Let c be a constant.

Corresponding integration rules. Integration rules justified via the Fundamental Theorem of Calculus

Integration rule	Derivative rule
$\int \mathrm{d}(fg) = \int g\mathrm{d}f + \int f\mathrm{d}g$	$(fg)' = f'g + fg'$
$\int \mathrm{d}c = 0$	$(c)' = 0$
$\int \mathrm{d}(cf) = c\int \mathrm{d}f$	$(cf)' = cf'$
$\int \mathrm{d}(f+g) = \int \mathrm{d}f + \int \mathrm{d}g$	$(f+g)' = f' + g'$
$\int \mathrm{d}f(g(x)) = \int f'(g(x))\mathrm{d}g(x)$	
$= \int f'(g(x))g'(x)\mathrm{d}x$	$(f(g(x)))' = f'(g(x))g'(x)$
$\int \mathrm{d}f(g) = \int f'(g)\mathrm{d}g$	
$\mathrm{d}x^n = nx^{n-1}\mathrm{d}x$	$(x^n)' = nx^{n-1}$
$\mathrm{d}e^x = e^x\mathrm{d}x$	$(e^x)' = e^x$
$\mathrm{d}\sin x = \cos x\mathrm{d}x$	$(\sin x)' = \cos x$
$\mathrm{d}\cos x = -\sin x\mathrm{d}x$	$(\cos x)' = -\sin x$
$\mathrm{d}\ln x = \frac{1}{x}\mathrm{d}x$	$(\ln x)' = \frac{1}{x}$

We recall from previous slides that

$$\frac{\mathrm{d}}{\mathrm{d}x}(\ln|x|) = \frac{1}{x}.$$

This formula has a special application to integration:

Theorem (The Integral of $1/x$)

$$\int \frac{1}{x}\mathrm{d}x = \ln|x| + C.$$

This fills in the gap in the rule for integrating power functions:

$$\int x^n \mathrm{d}x = \frac{x^{n+1}}{n+1} + C, \qquad n \neq -1.$$

Now we know the formula for $n = -1$ too.