Precalculus
Lecture 3
Angle Sum Formulas

Todor Milev
https://github.com/tmilev/freecalc

2020

Outline

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Cofunction identities

Proposition (Cofunction identities)

$$
\begin{aligned}
\sin\left(\frac{\pi}{2}-\alpha\right) &= \cos\alpha & \sin\left(\frac{\pi}{2}+\alpha\right) &= \cos\alpha \\
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- The proof each formula is broken into 4 cases depending on which quadrant contains α.

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- This makes a total of 4 formulas $\times$4 cases per formula = 16 cases.

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- We show only a few of the cases.
- The proof provides intuition why the formulas are true.
- The Quadrant I part of the proof serves as a visual aid for memorization.
- There is an algebraically simpler (but theoretically advanced) way to prove the above identities through the angle sum f-las, derived in turn from Euler's formula (studied later/in another course).

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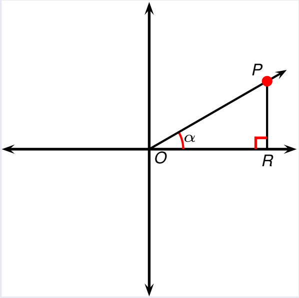
Part of Proof.



We are showing $\sin\left(\frac{\pi}{2}-\alpha\right) = \cos\alpha$ when α is in quadrant I.

□

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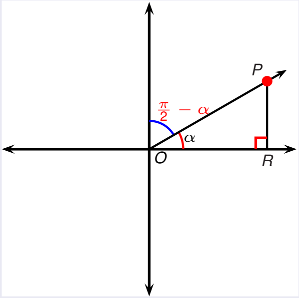

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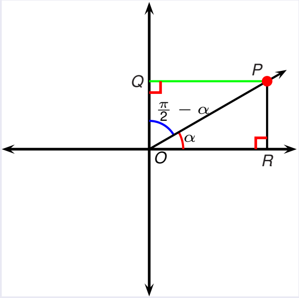

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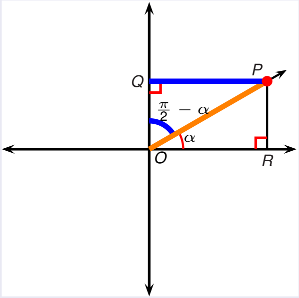

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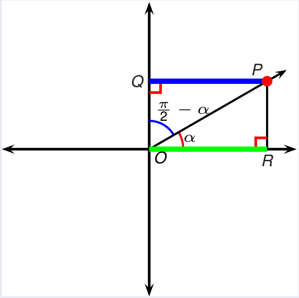

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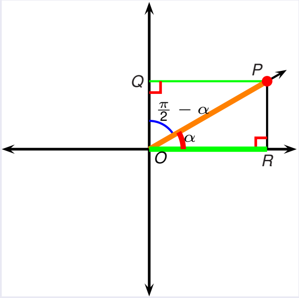

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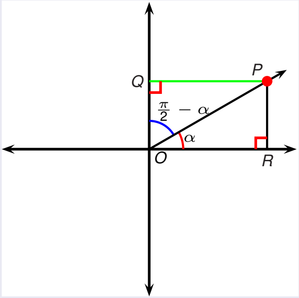

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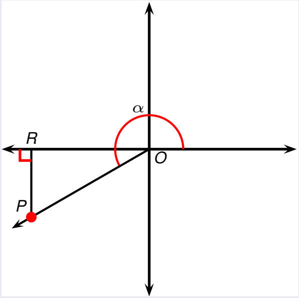

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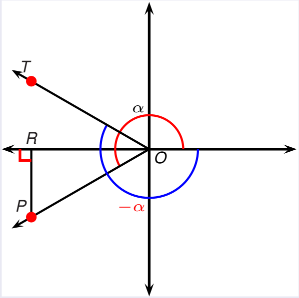

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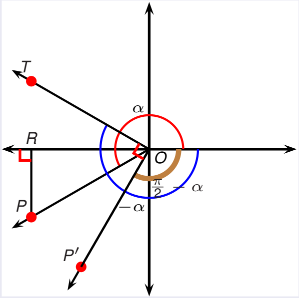

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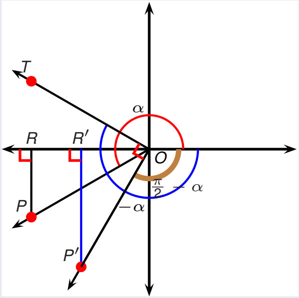

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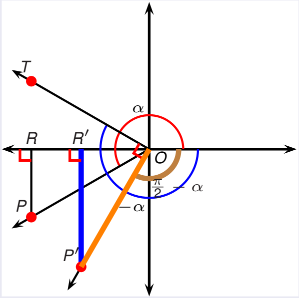

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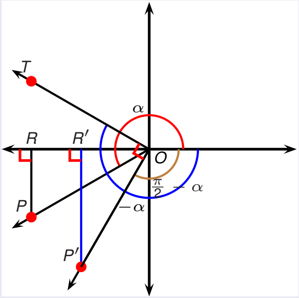

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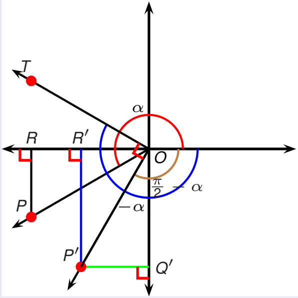

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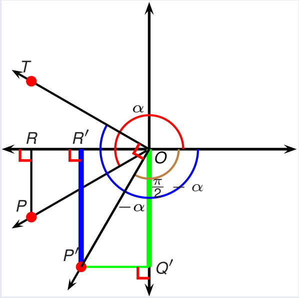

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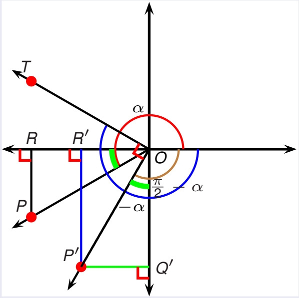

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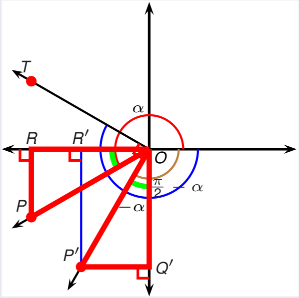

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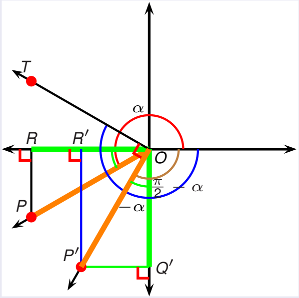

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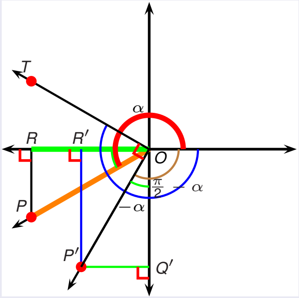

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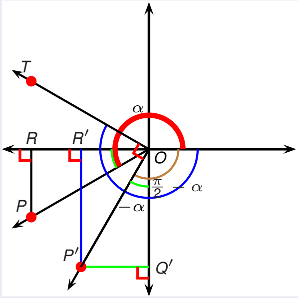

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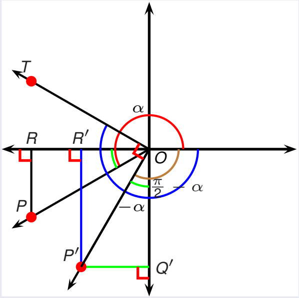

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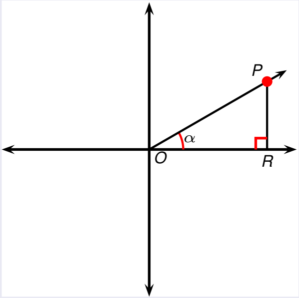

We show $\cos\left(\frac{\pi}{2}+\alpha\right) = -\sin\alpha$ when α is in Quadrant I.

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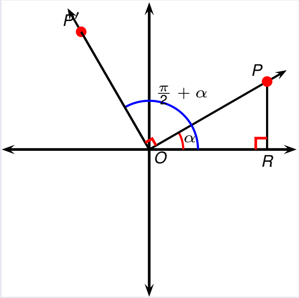
\cos\left(\frac{\pi}{2}-\alpha\right) &= \sin\alpha & \cos\left(\frac{\pi}{2}+\alpha\right) &= -\sin\alpha
\end{aligned}
$$

Part of Proof.



We show $\cos\left(\frac{\pi}{2}+\alpha\right) = -\sin\alpha$ when α is in Quadrant I.

$$\cos\left(\frac{\pi}{2}+\alpha\right) =$$

□

Cofunction identities

Proposition (Cofunction identities)

$$\begin{array}{rclrcl} \sin\left(\frac{\pi}{2}-\alpha\right) & = & \cos\alpha & \sin\left(\frac{\pi}{2}+\alpha\right) & = & \cos\alpha \\ \cos\left(\frac{\pi}{2}-\alpha\right) & = & \sin\alpha & \cos\left(\frac{\pi}{2}+\alpha\right) & = & -\sin\alpha \end{array}$$

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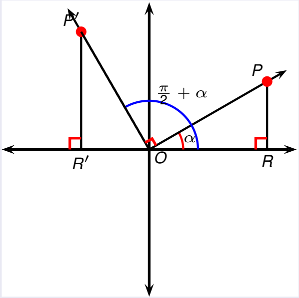

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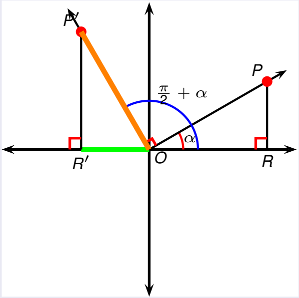

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$$\cos\left(\frac{\pi}{2}+\alpha\right) = -\frac{|OR'|}{|OP'|}$$

$\square$

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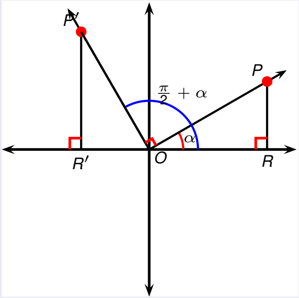

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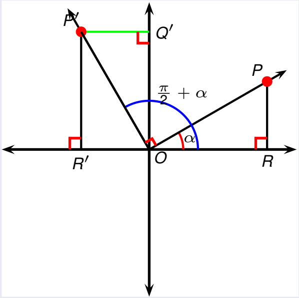

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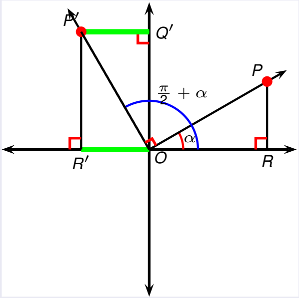

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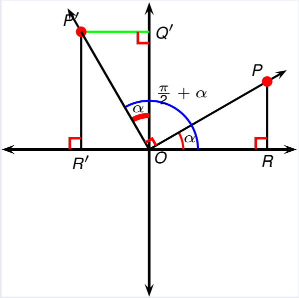

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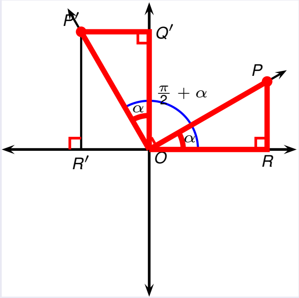

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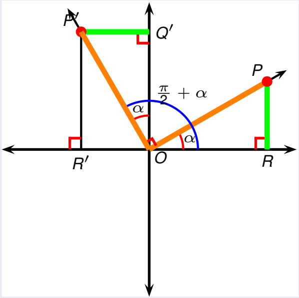

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$\Box$

Cofunction identities

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$$\begin{aligned} \sin\left(\frac{\pi}{2}-\alpha\right) &= \cos\alpha & \sin\left(\frac{\pi}{2}+\alpha\right) &= \cos\alpha \\ \cos\left(\frac{\pi}{2}-\alpha\right) &= \sin\alpha & \cos\left(\frac{\pi}{2}+\alpha\right) &= -\sin\alpha \end{aligned}$$

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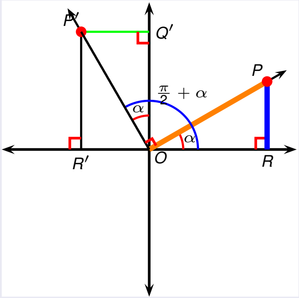

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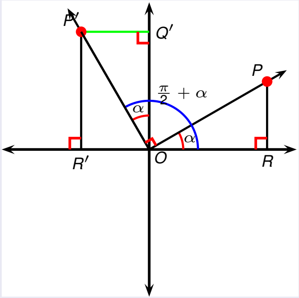

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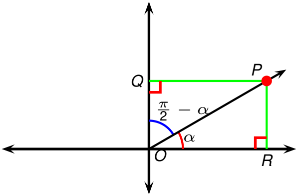
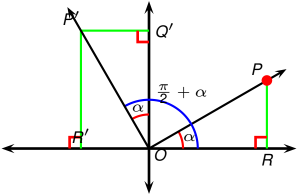
To memorize the cofunction identities it suffices to memorize the Quadrant I case via the two diagrams below.



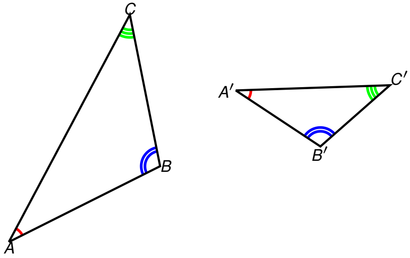
$$\begin{aligned} \sin\left(\tfrac{\pi}{2}-\alpha\right) &= \frac{|PQ|}{|OP|} \\ \cos\left(\tfrac{\pi}{2}-\alpha\right) &= \frac{|OQ|}{|OP|} \end{aligned}$$



$$\begin{aligned} \sin\left(\tfrac{\pi}{2}+\alpha\right) &= \frac{|OQ'|}{|OP|} \\ \cos\left(\tfrac{\pi}{2}+\alpha\right) &= -\frac{|PQ'|}{|OP'|} = \frac{|PR|}{|OP|} \end{aligned}$$

Definition (Similar triangles)

We say that a triangle $\triangle ABC$ is similar to a triangle $\triangle A'B'C'$ if the two triangles have equal angles.



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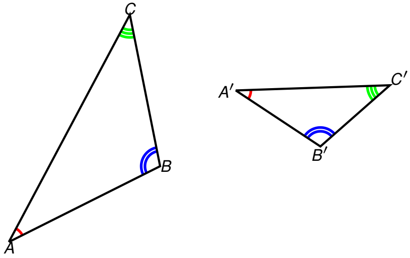
- The equal angles are assumed given in the same order for both triangles, that is, $\angle ABC = \angle A'B'C'$, $\angle BCA = \angle B'C'A'$, $\angle CAB = \angle C'A'B'$.



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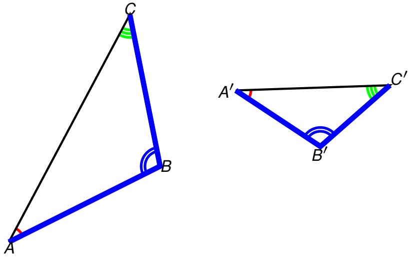

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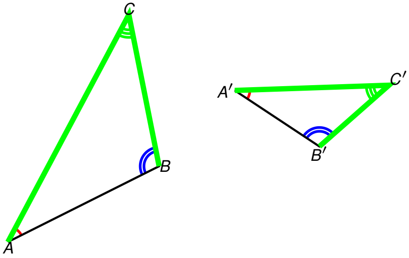

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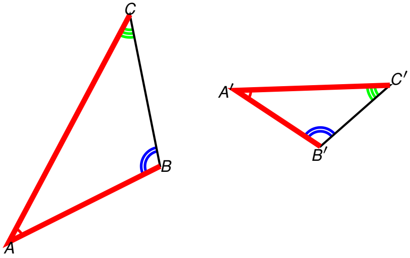
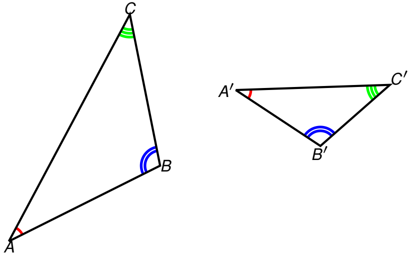


The following statement is proved in the subject of Euclidean (planar) geometry.

Theorem (Similar triangles have equal side ratios)

Let $\triangle ABC$ and $\triangle A'B'C'$ be two similar triangles. Then the ratios of the lengths of the sides of the two triangles are equal, that is

$$\frac{|AB|}{|BC|} = \frac{|A'B'|}{|B'C'|} \qquad \frac{|BC|}{|CA|} = \frac{|B'C'|}{|C'A'|} \qquad \frac{|CA|}{|AB|} = \frac{|C'A'|}{|A'B'|}$$



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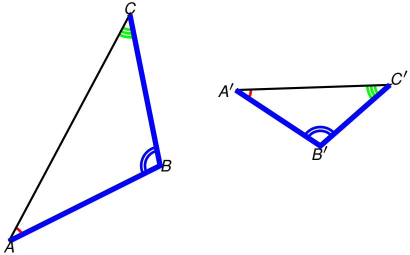

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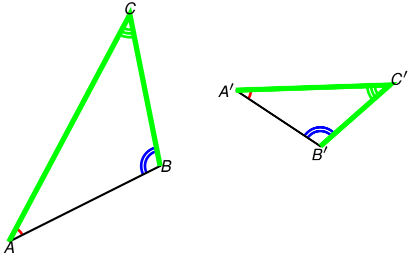

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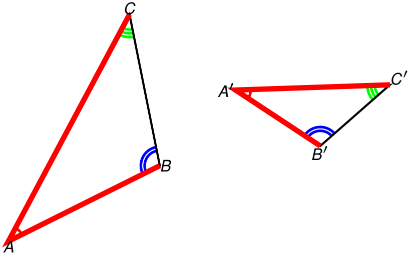

$\sin(\alpha+\beta), \cos(\alpha+\beta)$ via $\sin\alpha, \sin\beta, \cos\alpha, \cos\beta$

$\sin(\alpha+\beta) =$ **?**

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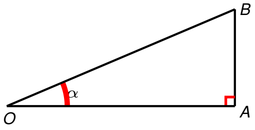

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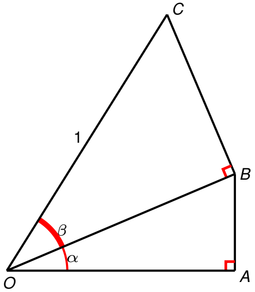

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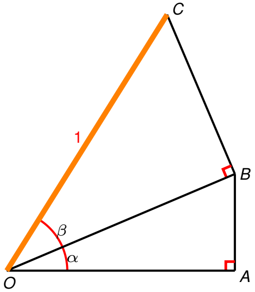

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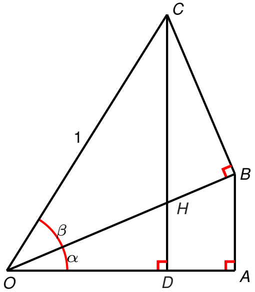

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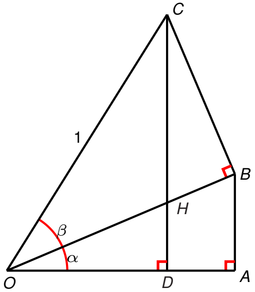

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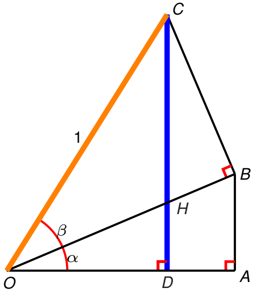

$$\sin(\alpha+\beta) = \frac{|CD|}{|OC|}$$

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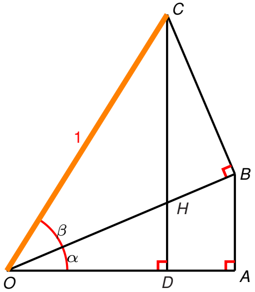

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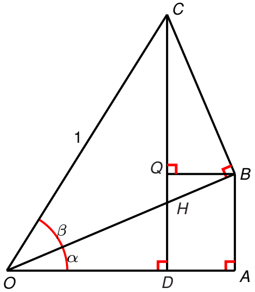

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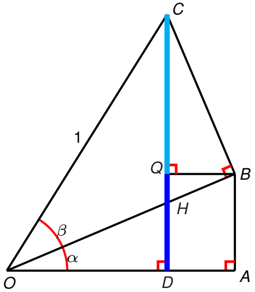

$$\sin(\alpha+\beta) = \frac{|CD|}{|OC|} = |CD|$$
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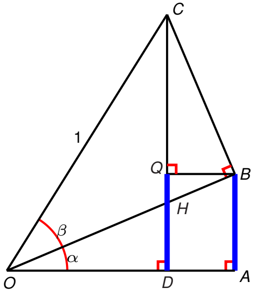

$|QD| = |BA|$ $\quad\Big|\ \square DABQ$

$$\sin(\alpha+\beta) = \frac{|CD|}{|OC|} = |CD|$$
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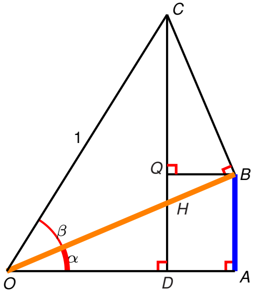

$$\begin{aligned}|QD| &= |BA| \\ &= \sin\alpha|OB|\end{aligned} \quad \Bigg| \begin{array}{l}\square DABQ \\ \triangle OAB\end{array}$$

$$\begin{aligned}\sin(\alpha+\beta) &= \frac{|CD|}{|OC|} = |CD| \\ &= |QD| + |CQ|\end{aligned}$$

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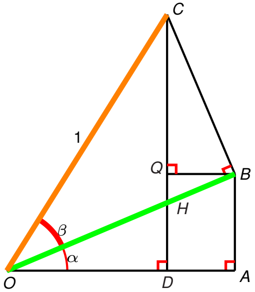
$\sin(\alpha+\beta), \cos(\alpha+\beta)$ via $\sin\alpha, \sin\beta, \cos\alpha, \cos\beta$

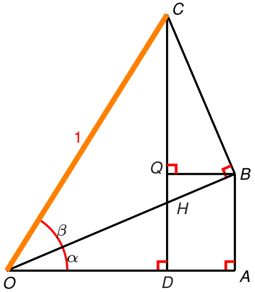


$$\begin{aligned} |QD| &= |BA| && \square DABQ \\ &= \sin\alpha |OB| && \triangle OAB \\ &= \sin\alpha \cos\beta |OC| && \triangle OBC \end{aligned}$$

$$\begin{aligned} \sin(\alpha+\beta) &= \frac{|CD|}{|OC|} = |CD| \\ &= |QD| + |CQ| \end{aligned}$$

$\cos(\alpha+\beta) =$ **?**

$\sin(\alpha+\beta)$, $\cos(\alpha+\beta)$ via $\sin\alpha$, $\sin\beta$, $\cos\alpha$, $\cos\beta$

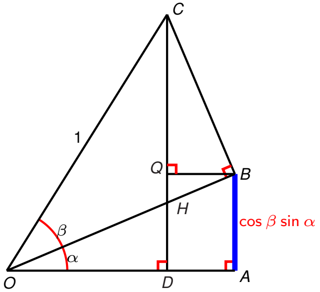


$$\begin{aligned}|QD| &= |BA| & &\square DABQ\\ &= \sin\alpha|OB| & &\triangle OAB\\ &= \sin\alpha\cos\beta{\color{red}|OC|} & &\triangle OBC\\ &= \sin\alpha\cos\beta\end{aligned}$$

$$\sin(\alpha+\beta) = \frac{|CD|}{|OC|} = |CD| = |QD| + |CQ|$$

$$\cos(\alpha+\beta) = \textbf{?}$$

$\sin(\alpha+\beta)$, $\cos(\alpha+\beta)$ via $\sin\alpha$, $\sin\beta$, $\cos\alpha$, $\cos\beta$



$$\begin{aligned}|QD| &= |BA| & \square DABQ\\ &= \sin\alpha|OB| & \triangle OAB\\ &= \sin\alpha\cos\beta|OC| & \triangle OBC\\ &= \sin\alpha\cos\beta\end{aligned}$$

$$\begin{aligned}\sin(\alpha+\beta) &= \frac{|CD|}{|OC|} = |CD|\\ &= |QD| + |CQ|\\ &= \sin\alpha\cos\beta + \textbf{?}\end{aligned}$$

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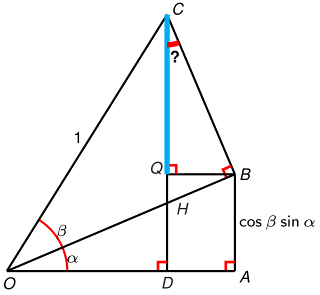

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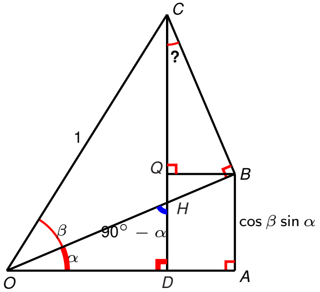
$\sin(\alpha+\beta)$, $\cos(\alpha+\beta)$ via $\sin\alpha$, $\sin\beta$, $\cos\alpha$, $\cos\beta$



$$\sin(\alpha+\beta) = \frac{|CD|}{|OC|} = |CD|$$
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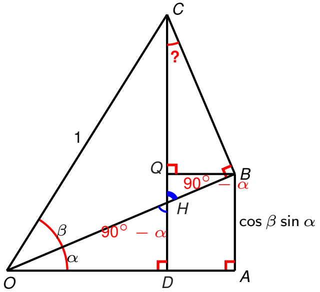
$$\cos(\alpha+\beta) = \textbf{?}$$

$$\begin{aligned} |QD| &= |BA| \\ &= \sin\alpha|OB| \\ &= \sin\alpha\cos\beta|OC| \\ &= \sin\alpha\cos\beta \end{aligned} \quad \left| \begin{aligned} &\square DABQ \\ &\triangle OAB \\ &\triangle OBC \end{aligned} \right.$$

$$|CQ| =$$

$\sin(\alpha+\beta)$, $\cos(\alpha+\beta)$ via $\sin\alpha$, $\sin\beta$, $\cos\alpha$, $\cos\beta$



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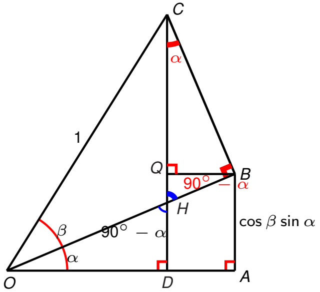

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$\lvert QD\rvert = \lvert BA\rvert$	$\square DABQ$
$= \sin\alpha\lvert OB\rvert$	$\triangle OAB$
$= \sin\alpha\cos\beta\lvert OC\rvert$	$\triangle OBC$
$= \sin\alpha\cos\beta$	

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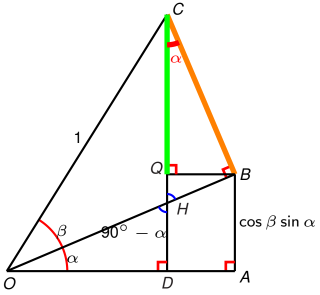

$$\begin{aligned} |QD| &= |BA| && \Box DABQ \\ &= \sin\alpha |OB| && \triangle OAB \\ &= \sin\alpha\cos\beta |OC| && \triangle OBC \\ &= \sin\alpha\cos\beta \\ |CQ| &= \cos\alpha |CB| && \triangle CQB \end{aligned}$$

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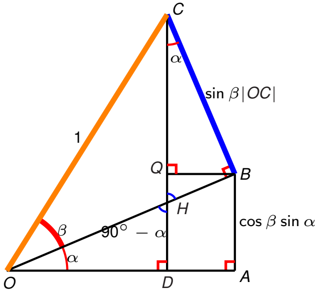

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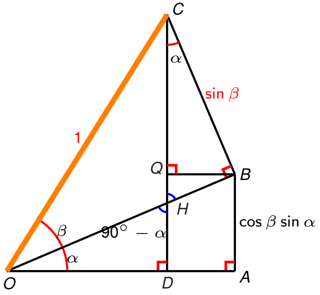

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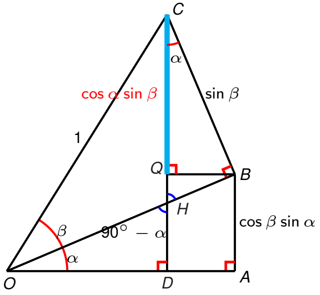

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$$\begin{aligned} {\color{red}|CQ|} &= \cos\alpha |CB| && \triangle CQB \\ &= \cos\alpha\sin\beta |OC| && \triangle OBC \\ &{\color{red}= \cos\alpha\sin\beta} \end{aligned}$$

$$\begin{aligned} \sin(\alpha+\beta) &= \frac{|CD|}{|OC|} = |CD| \\ &= |QD| + {\color{red}|CQ|} \\ &= \sin\alpha\cos\beta + {\color{red}\cos\alpha\sin\beta} \end{aligned}$$

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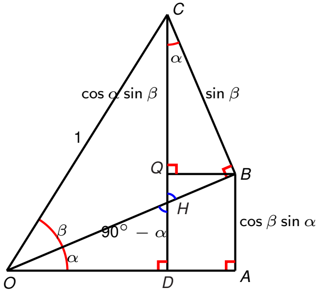

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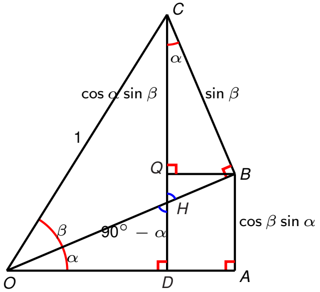

$$\sin(\alpha+\beta) = \frac{|CD|}{|OC|} = |CD| = |QD| + |CQ| = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$\cos(\alpha+\beta) = \frac{|OD|}{|OC|} = |OD| = |OA| - |DA| = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\begin{aligned} |QD| &= |BA| && \square DABQ \\ &= \sin\alpha|OB| && \triangle OAB \\ &= \sin\alpha\cos\beta|OC| && \triangle OBC \\ &= \sin\alpha\cos\beta \end{aligned}$$

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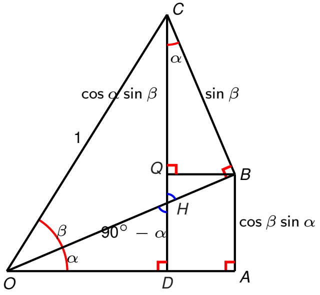
$\sin(\alpha+\beta), \cos(\alpha+\beta)$ via $\sin\alpha, \sin\beta, \cos\alpha, \cos\beta$



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$$\begin{aligned}
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&= \cos\alpha\sin\beta\\
|OA| &= \cos\alpha|OB| && \triangle OAB\\
&= \cos\alpha\cos\beta|OC| && \triangle OBC\\
&= \cos\alpha\cos\beta\\
|DA| &= |QB| && \square DABQ\\
&= \sin\alpha|CB| && \triangle CQB\\
&= \sin\alpha\sin\beta|OC| && \triangle OBC\\
&= \sin\alpha\sin\beta
\end{aligned}$$

Trig Functions of Sums and Differences of Angles

Theorem

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

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- We gave a geometric proof of the sum formulas when the two angles are acute and their sum is less than $\pi = 90^\circ$.

Trig Functions of Sums and Differences of Angles

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$$
\begin{aligned}
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$$

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- The theorem holds for all angles α, β without any restrictions.

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- There is a theoretically more advanced (but algebraically simpler) proof using Euler's formula (to be studied later/in another course).

Trig Functions of Sums and Differences of Angles

Theorem

$$\begin{aligned}
\sin(\alpha+\beta) &= \sin\alpha\cos\beta+\cos\alpha\sin\beta\\
\sin(\alpha-\beta) &= \sin\alpha\cos\beta-\cos\alpha\sin\beta\\
\cos(\alpha+\beta) &= \cos\alpha\cos\beta-\sin\alpha\sin\beta\\
\cos(\alpha-\beta) &= \cos\alpha\cos\beta+\sin\alpha\sin\beta
\end{aligned}$$

- We gave a geometric proof of the sum formulas when the two angles are acute and their sum is less than $\pi = 90^\circ$.
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- There is a theoretically more advanced (but algebraically simpler) proof using Euler's formula (to be studied later/in another course).
- The difference formulas are a consequence of the sum formulas and the fact that sin is an odd function and cos is even.

Trig Functions of Differences of Angles

Example

Prove the identities

$$\begin{aligned} \sin(\alpha-\beta) &= \sin\alpha\cos\beta-\cos\alpha\sin\beta\\ \cos(\alpha-\beta) &= \cos\alpha\cos\beta+\sin\alpha\sin\beta \end{aligned}$$

from the (already demonstrated) identities

$$\begin{aligned} \sin(\alpha+\beta) &= \sin\alpha\cos\beta+\cos\alpha\sin\beta\\ \cos(\alpha+\beta) &= \cos\alpha\cos\beta-\sin\alpha\sin\beta \end{aligned}$$

$$\begin{aligned} \sin(\alpha-\beta) &= \sin(\alpha+(-\beta))\\ &= \sin\alpha\cos(-\beta)+\cos\alpha\sin(-\beta) && \left| \begin{array}{l} \cos \text{ is even },\\ \sin \text{ is odd} \end{array}\right.\\ &= \sin\alpha\cos\beta-\cos\alpha\sin\beta\\ \cos(\alpha-\beta) &= \cos(\alpha+(-\beta))\\ &= \cos\alpha\cos(-\beta)-\sin\alpha\sin(-\beta) && \left| \begin{array}{l} \cos \text{ is even },\\ \sin \text{ is odd} \end{array}\right.\\ &= \cos\alpha\cos\beta+\cos\alpha\sin\beta \end{aligned}$$

Example

Find the exact value of the trigonometric function using radicals.

$\cos(105^\circ)$

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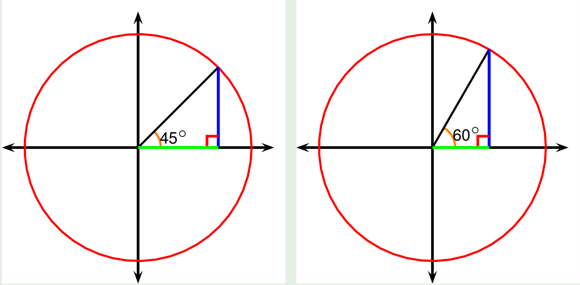
we know the trig f-ns of 45° and 60°



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$= \textbf{?}$

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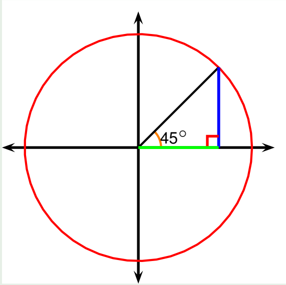
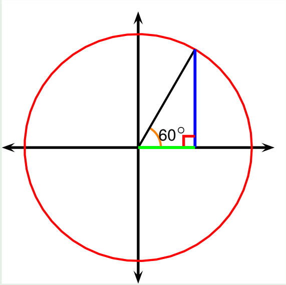
Angle sum f-la





Example

Find the exact value of the trigonometric function using radicals.

$$\cos(105^\circ) = \cos(45^\circ + 60^\circ)$$
$$= \cos(45^\circ)\cos(60^\circ) - \sin(45^\circ)\sin(60^\circ)$$

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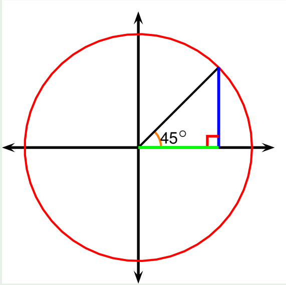
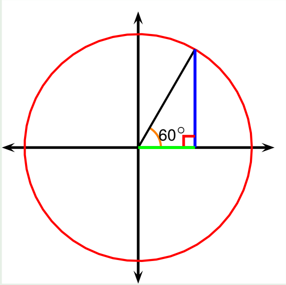
Angle sum f-la





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$$\begin{aligned}\cos(105^\circ) &= \cos(45^\circ + 60^\circ) \\ &= \cos(45^\circ)\cos(60^\circ) - \sin(45^\circ)\sin(60^\circ) \\ &= \mathbf{?} \quad \cdot \mathbf{?} - \mathbf{?} \quad \cdot \mathbf{?}\end{aligned}$$

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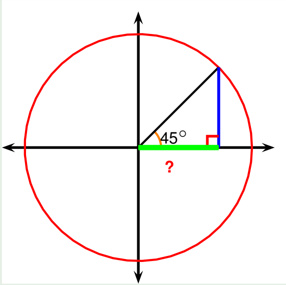
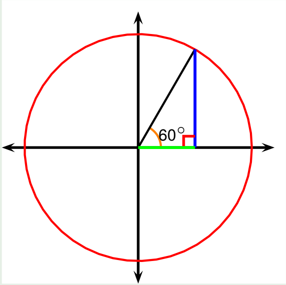
Angle sum f-la





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$$\begin{aligned} \cos(105^\circ) &= \cos(45^\circ + 60^\circ) \\ &= \cos(45^\circ)\cos(60^\circ) - \sin(45^\circ)\sin(60^\circ) \\ &= \frac{\sqrt{2}}{2} \cdot \textbf{?} - \textbf{?} \quad \cdot \textbf{?} \end{aligned}$$

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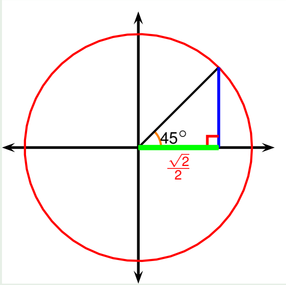
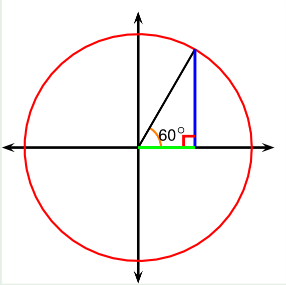
Angle sum f-la





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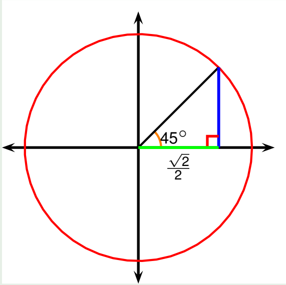

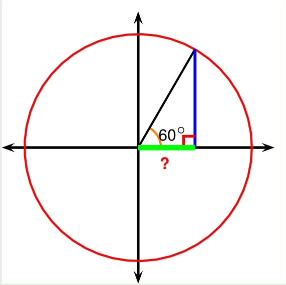

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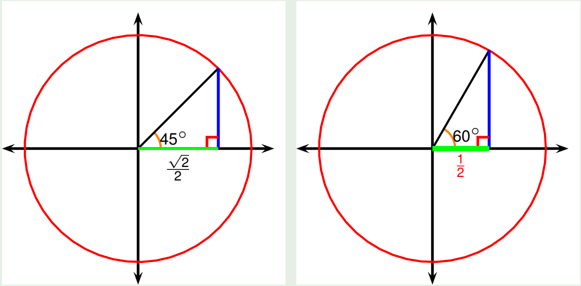
Angle sum f-la



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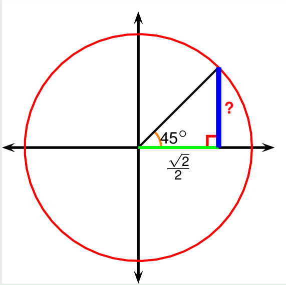
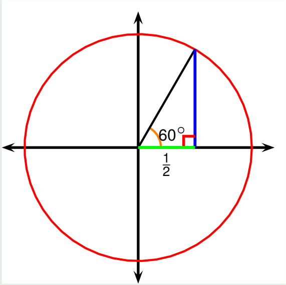
we know the trig f-ns of 45° and 60°

Angle sum f-la



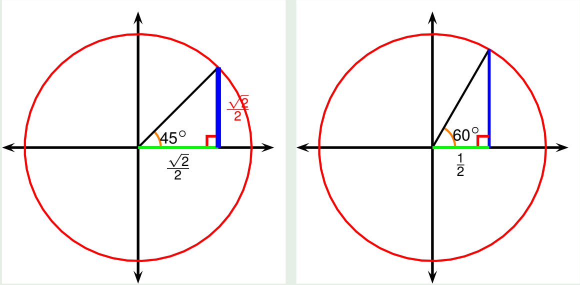


Example

Find the exact value of the trigonometric function using radicals.

$$\begin{aligned}\cos(105^\circ) &= \cos(45^\circ + 60^\circ) \\ &= \cos(45^\circ)\cos(60^\circ) - \sin(45^\circ)\sin(60^\circ) \\ &= \frac{\sqrt{2}}{2}\cdot\frac{1}{2} - \frac{\sqrt{2}}{2}\cdot \textbf{?}\end{aligned}$$

we know the trig f-ns of 45° and 60°
Angle sum f-la



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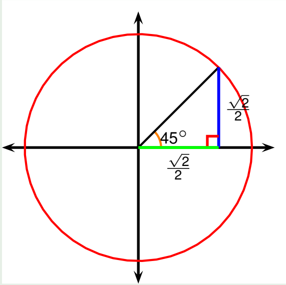
$$\cos(105^\circ) = \cos(45^\circ + 60^\circ)$$

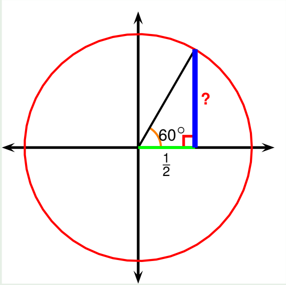
we know the trig f-ns of 45° and 60°

$$= \cos(45^\circ)\cos(60^\circ) - \sin(45^\circ)\sin(60^\circ)$$

Angle sum f-la

$$= \frac{\sqrt{2}}{2} \cdot \frac{1}{2} - \frac{\sqrt{2}}{2} \cdot \textbf{?}$$





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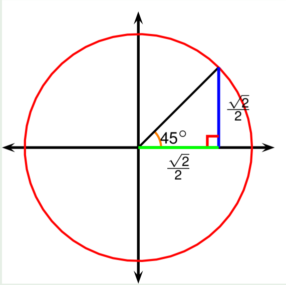
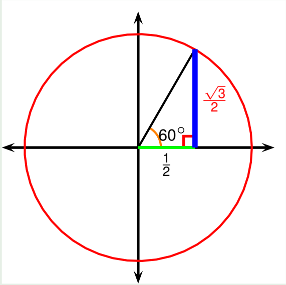
Angle sum f-la





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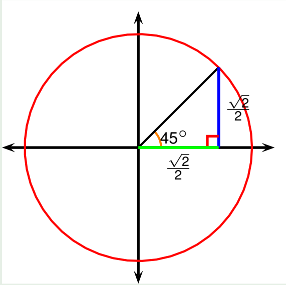
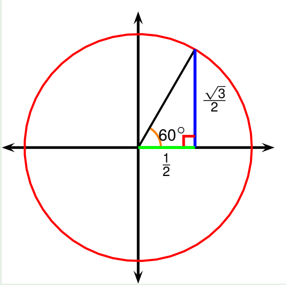
we know the trig f-ns of 45° and 60°

Angle sum f-la

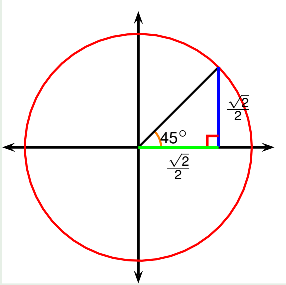


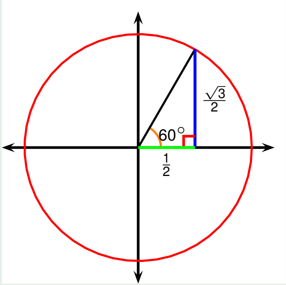


Example

Find the exact value of the trigonometric function using radicals.

$$
\begin{aligned}
\cos(105^\circ) &= \cos(45^\circ + 60^\circ) && \text{we know the trig f-ns of } 45^\circ \text{ and } 60^\circ \\
&= \cos(45^\circ)\cos(60^\circ) - \sin(45^\circ)\sin(60^\circ) && \text{Angle sum f-la} \\
&= \frac{\sqrt{2}}{2} \cdot \frac{1}{2} - \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} \\
&= \frac{\sqrt{2} - \sqrt{6}}{4}.
\end{aligned}
$$





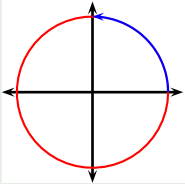
Example

Use the angle sum/difference formulas to simplify.

$$\cos\left(\frac{\pi}{2} - x\right)$$

Example

Use the angle sum/difference formulas to simplify.



$$\begin{aligned}\cos\left(\frac{\pi}{2}-x\right) &= \cos\left(\frac{\pi}{2}\right)\cos x+\sin\left(\frac{\pi}{2}\right)\sin x\\ &= 0\cdot\cos(x)+1\cdot\sin x\\ &= \sin x\end{aligned}$$

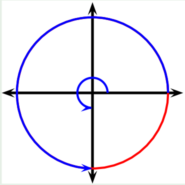
Example

Use the angle sum/difference formulas to simplify.

$$\cot\left(\frac{3\pi}{2} + x\right)$$

Example

Use the angle sum/difference formulas to simplify.



$$
\begin{aligned}
\cot\left(\frac{3\pi}{2}+x\right) &= \frac{\cos\left(\frac{3\pi}{2}+x\right)}{\sin\left(\frac{3\pi}{2}+x\right)}\\
&= \frac{\cos\left(\frac{3\pi}{2}\right)\cos x-\sin\left(\frac{3\pi}{2}\right)\sin x}{\sin\left(\frac{3\pi}{2}\right)\cos x+\cos\left(\frac{3\pi}{2}\right)\sin x}\\
&= \frac{0\cdot\cos x-(-1)\sin x}{(-1)\cos x+0\cdot\sin x}\\
&= \frac{\sin x}{-\cos x}=-\frac{\sin x}{\cos x}\\
&= -\tan x
\end{aligned}
$$

Example

Show that $\tan(\pi + x) = \tan x$ using the angle sum formulas.

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$$\tan(\pi + x) = \frac{\sin(\pi + x)}{\cos(\pi + x)}$$

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Show that $\tan(\pi + x) = \tan x$ using the angle sum formulas.

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Show that $\tan(\pi + x) = \tan x$ using the angle sum formulas.

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\tan(\pi + x) &= \frac{\sin(\pi + x)}{\cos(\pi + x)}\\
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&= \frac{-\sin x}{-\cos x} \\
&= \frac{\sin x}{\cos x} \\
&= \tan x,
\end{aligned}$$

as desired.

Proposition ($\tan, \cot$ are π-periodic)

The tangent and cotangent functions are π-periodic, in other words,

$$\begin{aligned} \tan(\theta + \pi) &= \tan\theta \\ \cot(\theta + \pi) &= \cot\theta \end{aligned}$$

Recall the angle sum formula $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$.

Example

Show that the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$ follows from the angle difference formula.

Recall the angle sum formula $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$.

Example

Show that the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$ follows from the angle difference formula.

$$\begin{aligned}
1 &= \cos 0 \\
&= \cos(\theta - \theta) \\
&= \cos\theta\cos\theta + \sin\theta\sin\theta \\
&= \cos^2\theta + \sin^2\theta,
\end{aligned}$$

as desired.

Example

Prove the angle sum formula $\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$.

$$\tan(\alpha+\beta) =$$

Example

Prove the angle sum formula $\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$.

$$\begin{aligned}
\tan(\alpha+\beta) &= \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}\\
&= \frac{(\sin\alpha\cos\beta+\cos\alpha\sin\beta)\frac{1}{\cos\alpha\cos\beta}}{(\cos\alpha\cos\beta-\sin\alpha\sin\beta)\frac{1}{\cos\alpha\cos\beta}}\\
&= \frac{\frac{\sin\alpha\cancel{\cos\beta}}{\cos\alpha\cancel{\cos\beta}}+\frac{\cancel{\cos\alpha}\sin\beta}{\cancel{\cos\alpha}\cos\beta}}{\frac{\cancel{\cos\alpha\cos\beta}}{\cancel{\cos\alpha\cos\beta}}-\frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}\\
&= \frac{\frac{\sin\alpha}{\cos\alpha}+\frac{\sin\beta}{\cos\beta}}{1-\frac{\sin\alpha}{\cos\alpha}\cdot\frac{\sin\beta}{\cos\beta}}\\
&= \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}
\end{aligned}$$

Double angle formulas

Proposition (Double angle formulas)

$$\begin{aligned}\sin(2\alpha) &= 2\sin\alpha\cos\alpha\\ \cos(2\alpha) &= \cos^2\alpha - \sin^2\alpha\\ &= 2\cos^2\alpha - 1\\ &= 1 - 2\sin^2\alpha\end{aligned}$$

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\cos(2\alpha) &= \cos^2\alpha - \sin^2\alpha \\
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\end{aligned}$$

- The double angle formulas play a special role in integration.

Example

Derive the double-angle formulas.

$$\sin(2\alpha) =$$

$$\cos(2\alpha) =$$

Example

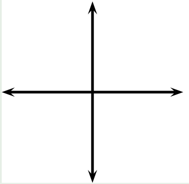
Derive the double-angle formulas.

$$\begin{aligned}
\sin(2\alpha) &= \sin(\alpha+\alpha)\\
&= \sin\alpha\cos\alpha + \cos\alpha\sin\alpha\\
&= 2\sin\alpha\cos\alpha\\
\cos(2\alpha) &= \cos(\alpha+\alpha)\\
&= \cos\alpha\cos\alpha - \sin\alpha\sin\alpha\\
&= \cos^2\alpha - \sin^2\alpha\\
&= \cos^2\alpha - (1-\cos^2\alpha)\\
&= 2\cos^2\alpha - 1\\
&= 1-\sin^2\alpha - \sin^2\alpha\\
&= 1-2\sin^2\alpha
\end{aligned}$$

Example

Using radicals, find the exact value of the trigonometric expression.

$$\cos 105^\circ$$



Recall the half angle formula $\cos\alpha = \pm\sqrt{\frac{1+\cos(2\alpha)}{2}}$.

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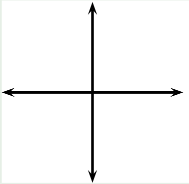

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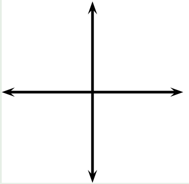
$$\cos 105^\circ = \pm\sqrt{\frac{1+\cos\left(2\cdot 105^\circ\right)}{2}}$$



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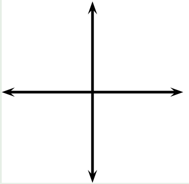
$$\cos 105^\circ = \pm\sqrt{\frac{1+\cos\left(2\cdot 105^\circ\right)}{2}} \qquad \Big| \quad \cos 105^\circ \mathbf{?}\ 0$$



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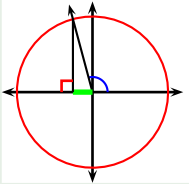
Example

Using radicals, find the exact value of the trigonometric expression.

$$\cos 105^\circ = \pm\sqrt{\frac{1+\cos\left(2\cdot 105^\circ\right)}{2}} \qquad \Big| \quad \cos 105^\circ < 0$$



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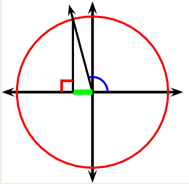

$$\begin{aligned}\cos 105^\circ &= \pm\sqrt{\frac{1+\cos(2\cdot 105^\circ)}{2}} \\ &= -\sqrt{\frac{1+\cos(210^\circ)}{2}}\end{aligned} \qquad \Bigg| \quad \cos 105^\circ < 0$$

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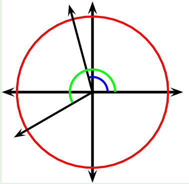

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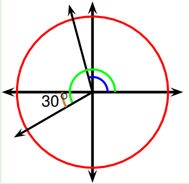

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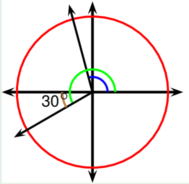

$$\begin{aligned}\cos 105^\circ &= \pm\sqrt{\frac{1+\cos(2\cdot 105^\circ)}{2}} \\ &= -\sqrt{\frac{1+\cos(210^\circ)}{2}} \\ &= -\sqrt{\frac{1-\cos(30^\circ)}{2}}\end{aligned}$$

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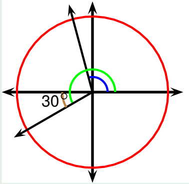

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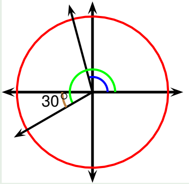

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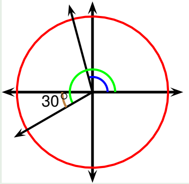
Using radicals, find the exact value of the trigonometric expression.



$$\begin{aligned}\cos 105^\circ &= \pm\sqrt{\frac{1+\cos(2\cdot 105^\circ)}{2}} & \Big|\ \cos 105^\circ < 0\\ &= -\sqrt{\frac{1+\cos(210^\circ)}{2}}\\ &= -\sqrt{\frac{1-\cos(30^\circ)}{2}}\\ &= -\sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}} = -\sqrt{\frac{2-\sqrt{3}}{2\cdot 2}}\end{aligned}$$

Recall the half angle formula $\cos\alpha = \pm\sqrt{\frac{1+\cos(2\alpha)}{2}}$.

Example

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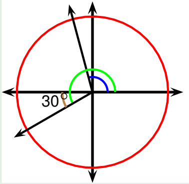

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\cos 105^\circ &= \pm\sqrt{\frac{1+\cos\left(2\cdot 105^\circ\right)}{2}} && \Big|\ \cos 105^\circ < 0\\
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&= -\sqrt{\frac{1-\cos\left(30^\circ\right)}{2}}\\
&= -\sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}} = -\sqrt{\frac{2-\sqrt{3}}{2\cdot 2}}\\
&= -\frac{\sqrt{2-\sqrt{3}}}{2}
\end{aligned}$$

Proposition (Power-Reducing Formulas)

$$\sin^2\alpha = \frac{1-\cos(2\alpha)}{2} \qquad \cos^2\alpha = \frac{1+\cos(2\alpha)}{2}$$

Proof.

$\square$

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$$\cos(2\alpha) = 1 - 2\sin^2\alpha$$

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$$\begin{aligned}\cos(2\alpha) &= 1 - 2\sin^2\alpha\\ 2\sin^2\alpha &= 1-\cos(2\alpha)\end{aligned}$$

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$$\begin{aligned} \cos(2\alpha) &= 1 - 2\sin^2\alpha \\ 2\sin^2\alpha &= 1 - \cos(2\alpha) \end{aligned}$$

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$$\begin{aligned}\cos(2\alpha) &= 1-2\sin^2\alpha \\ 2\sin^2\alpha &= 1-\cos(2\alpha) \\ \sin^2\alpha &= \frac{1-\cos(2\alpha)}{2}\end{aligned} \qquad \cos(2\alpha) = 2\cos^2\alpha - 1$$

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$$\begin{aligned}
\cos(2\alpha) &= 1 - 2\sin^2\alpha & \cos(2\alpha) &= 2\cos^2\alpha - 1\\
2\sin^2\alpha &= 1-\cos(2\alpha) & 2\cos^2\alpha - 1 &= \cos(2\alpha)\\
\sin^2\alpha &= \frac{1-\cos(2\alpha)}{2}
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Corollary

$$\sin\alpha = \pm\sqrt{\frac{1-\cos(2\alpha)}{2}} \qquad \cos\alpha = \pm\sqrt{\frac{1+\cos(2\alpha)}{2}}$$

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$$\sin\alpha = \pm\sqrt{\frac{1-\cos(2\alpha)}{2}} \qquad \cos\alpha = \pm\sqrt{\frac{1+\cos(2\alpha)}{2}}$$

Corollary (Half-Angle Formulas)

$$\sin\left(\frac{\beta}{2}\right) = \pm\sqrt{\frac{1-\cos\beta}{2}} \qquad \cos\left(\frac{\beta}{2}\right) = \pm\sqrt{\frac{1+\cos\beta}{2}}$$

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Proof.

$$\begin{aligned}
\cos(2\alpha) &= 1-2\sin^2\alpha & \cos(2\alpha) &= 2\cos^2\alpha - 1\\
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\sin^2\alpha &= \frac{1-\cos(2\alpha)}{2} & \cos^2\alpha &= \frac{1+\cos(2\alpha)}{2}
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- The power reducing formulas are used to express $\sin^k\alpha$ and $\cos^k\alpha$ via lower powers of the $\sin$ and $\cos$ functions (applied to angles other than α).

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- The power reducing formulas are used to express $\sin^k\alpha$ and $\cos^k\alpha$ via lower powers of the $\sin$ and $\cos$ functions (applied to angles other than α).
- This technique will play a key role in integration (studied later/in another course).

Example

Rewrite $\sin^4 \alpha$ in terms of first powers of the cosines and sines of multiples of the angle α.

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$$\sin^{4}\alpha = \left(\sin^{2}\alpha\right)^{2}$$

Recall the formulas: $\sin^2\beta = ?$, $\cos^2\beta = ?$.

Example

Rewrite $\sin^4\alpha$ in terms of first powers of the cosines and sines of multiples of the angle α.

$$\begin{aligned}\sin^4\alpha &= \left(\sin^2\alpha\right)^2\\ &= \left(?\qquad\right)^2\end{aligned}$$

Recall the formulas: $\sin^2\beta = \frac{1-\cos(2\beta)}{2}$, $\cos^2\beta = \textbf{?}$.

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$$\begin{aligned}\sin^4\alpha &= \left(\sin^2\alpha\right)^2\\ &= \left(\frac{1-\cos(2\alpha)}{2}\right)^2\end{aligned}$$

Recall the formulas: $\sin^2\beta = \frac{1-\cos(2\beta)}{2}$, $\cos^2\beta = \mathbf{?}$.

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$$\begin{aligned}\sin^4\alpha &= \left(\sin^2\alpha\right)^2\\ &= \left(\frac{1-\cos(2\alpha)}{2}\right)^2\\ &= \frac{1}{4}\left(\mathbf{?}\qquad\qquad\qquad\qquad\right)\end{aligned}$$

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$$\begin{aligned}
\sin^4\alpha &= \left(\sin^2\alpha\right)^2\\
&= \left(\frac{1-\cos(2\alpha)}{2}\right)^2\\
&= \frac{1}{4}\left(1-2\cos(2\alpha)+\cos^2(2\alpha)\right)
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&= \frac{1}{4}\left(1-2\cos(2\alpha)+\frac{\cos(2\cdot 2\alpha)}{2}+\frac{1}{2}\right) \\
&= \frac{1}{4}\left(\frac{3}{2}-2\cos(2\alpha)+\frac{\cos(4\alpha)}{2}\right)
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&= \frac{1}{4}\left(\frac{3}{2}-2\cos(2\alpha)+\frac{\cos(4\alpha)}{2}\right)\\
&= \frac{1}{8}\left(3-4\cos(2\alpha)+\cos(4\alpha)\right)
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