

Calculus I

Lecture 9

Product and Quotient Rule

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<https://github.com/tmilev/freecalc>

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Outline

1 Differentiation Formulas

- General Power Functions
- The Constant Multiple Rule
- The Sum and Difference Rules
- Derivatives of Exponential Functions

2 The Product and Quotient Rules

- The Product Rule
- The Quotient Rule

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Theorem (The Power Rule (General Version))

If n is any real number, then

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

Example (Power Rule, negative exponent)

Differentiate $y = \frac{1}{x}$.

$$y = x^{-1}.$$

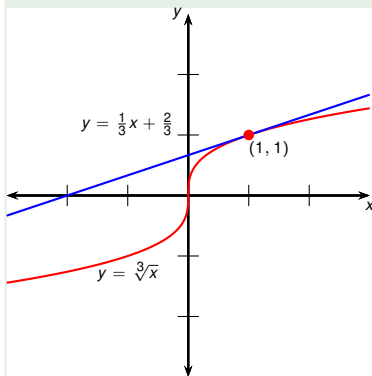
Power Rule: $\frac{dy}{dx} = (-1)x^{-2}$

$$= -\frac{1}{x^2}.$$

Example (Calculating the tangent line using the Power Rule)

Find an equation for the tangent line to the cubic $y = \sqrt[3]{x}$ at the point $P = (1, 1)$.

Here $a = 1$ and $f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$.



$$\begin{aligned}f'(x) &= \frac{1}{3}x^{\frac{1}{3}-1} \\&= \frac{1}{3}x^{-\frac{2}{3}} \\&= \frac{1}{3\sqrt[3]{x^2}}. \\f'(1) &= \frac{1}{3}.\end{aligned}$$

Point-slope form: $y - 1 = \frac{1}{3}(x - 1)$, or
 $y = \frac{1}{3}x + \frac{2}{3}$.

Example (Power Rule, fractional exponent)

Differentiate $y = \sqrt[6]{x^5}$.

$$y = x^{\frac{5}{6}}.$$

Power Rule: $\frac{dy}{dx} = \frac{5x^{-\frac{1}{6}}}{6}$
 $= \frac{5}{6\sqrt[6]{x}}.$

Theorem (The Constant Multiple Rule)

If c is a constant and f is a differentiable function, then

$$\frac{d}{dx}[cf(x)] = c \frac{d}{dx}f(x).$$

Proof.

Let $g(x) = cf(x)$.

$$\begin{aligned}\text{Then } g'(x) &= \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{cf(x+h) - cf(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{c(f(x+h) - f(x))}{h} \\ \text{Limit Law 3: } &= c \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= cf'(x).\end{aligned}$$



Example (Constant Multiple Rule, Power Rule)

Find the derivative of $y = \frac{2x^5}{7}$.

$$y = \left(\frac{2}{7}\right) (x^5).$$

$$\frac{dy}{dx} = \frac{d}{dx} \left[\left(\frac{2}{7}\right) (x^5) \right]$$

$$\begin{aligned}\text{Constant Multiple Rule: } &= \left(\frac{2}{7}\right) \frac{d}{dx} (x^5) \\ &= \left(\frac{2}{7}\right) (5x^4) \\ &= \frac{10x^4}{7}.\end{aligned}$$

Example (Constant Multiple Rule, Power Rule)

Find the derivative of $u = -x$.

$$u = (-1)(x).$$

$$\frac{du}{dx} = \frac{d}{dx} [(-1)(x)]$$

$$\begin{aligned}\text{Constant Multiple Rule: } &= (-1) \frac{d}{dx} (x) \\ &= (-1)(1) \\ &= -1.\end{aligned}$$

Example (Constant Multiple Rule, Power Rule, Negative Exponent)

Find the derivative of $t = \frac{2\pi}{x^4}$.

$$t = (2\pi) (x^{-4}) .$$

$$\frac{dt}{dx} = \frac{d}{dx} \left[(2\pi) (x^{-4}) \right]$$

$$\begin{aligned} \text{Constant Multiple Rule: } &= (2\pi) \frac{d}{dx} (x^{-4}) \\ &= (2\pi) (-4x^{-5}) \\ &= -\frac{8\pi}{x^5} . \end{aligned}$$

Theorem (The Sum Rule)

If f and g are both differentiable, then

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x).$$

Proof.

Let $F(x) = f(x) + g(x)$.

$$\begin{aligned}\text{Then } F'(x) &= \lim_{h \rightarrow 0} \frac{F(x+h) - F(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{[f(x+h) + g(x+h)] - [f(x) + g(x)]}{h} \\ &= \lim_{h \rightarrow 0} \left[\frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h} \right]\end{aligned}$$

$$\begin{aligned}\text{Limit Law 1: } &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \\ &= f'(x) + g'(x).\end{aligned}$$



The Sum Rule can be extended to any number of summands. For instance, using the theorem twice, we get

$$(f + g + h)' = [(f + g) + h]' = (f + g)' + h' = f' + g' + h'.$$

By writing $f - g$ as $f + (-1)g$ and applying the Sum Rule and the Constant Multiple Rule, we get

Theorem (The Difference Rule)

If f and g are both differentiable, then

$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x).$$

The Constant Multiple Rule, the Sum Rule, the Difference Rule, and the Power Rule can be combined to differentiate any polynomial.

Example (Derivative of a Polynomial)

$$\text{If } y = x^{16} + 2\sqrt{3}x^7 - 4x^3 + \frac{x}{8} - 5,$$

$$\begin{aligned}\text{Then } \frac{dy}{dx} &= \frac{d}{dx} \left(x^{16} + 2\sqrt{3}x^7 - 4x^3 + \frac{x}{8} - 5 \right) \\ &= \frac{d}{dx} (x^{16}) + \frac{d}{dx} (2\sqrt{3}x^7) - \frac{d}{dx} (4x^3) + \frac{d}{dx} \left(\frac{x}{8} \right) - \frac{d}{dx} (5) \\ &= \frac{d}{dx} (x^{16}) + 2\sqrt{3} \frac{d}{dx} (x^7) - 4 \frac{d}{dx} (x^3) + \frac{1}{8} \frac{d}{dx} (x) - \frac{d}{dx} (5) \\ &= (16x^{15}) + 2\sqrt{3} (7x^6) - 4 (3x^2) + \frac{1}{8} (1) - (0) \\ &= 16x^{15} + 14\sqrt{3}x^6 - 12x^2 + \frac{1}{8}.\end{aligned}$$

Example (Difference Rule, Negative Fractional Exponents)

Differentiate $v = \frac{3\sqrt{x} - \sqrt[3]{x}}{x}$.

$$v = 3\frac{\sqrt{x}}{x} - \frac{\sqrt[3]{x}}{x}$$

$$v = 3x^{-\frac{1}{2}} - x^{-\frac{2}{3}}.$$

Difference Rule: $\frac{dv}{dx} = \frac{d}{dx} \left(3x^{-\frac{1}{2}} \right) - \frac{d}{dx} \left(x^{-\frac{2}{3}} \right)$

Constant Multiple Rule: $= 3\frac{d}{dx} \left(x^{-\frac{1}{2}} \right) - \frac{d}{dx} \left(x^{-\frac{2}{3}} \right)$

Power Rule: $= 3 \left(-\frac{1}{2}x^{-\frac{3}{2}} \right) - \left(-\frac{2}{3}x^{-\frac{5}{3}} \right)$
 $= \frac{2}{3}x^{-\frac{5}{3}} - \frac{3}{2}x^{-\frac{3}{2}}.$

Derivatives of Exponential Functions

Compute the derivative of $f(x) = a^x$ using the definition:

$$\begin{aligned}f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{a^{x+h} - a^x}{h} \\&= \lim_{h \rightarrow 0} \frac{a^x a^h - a^x}{h} \\&= \lim_{h \rightarrow 0} \frac{a^x (a^h - 1)}{h} \\&= a^x \lim_{h \rightarrow 0} \frac{a^h - 1}{h} \\&= a^x \lim_{h \rightarrow 0} \frac{a^h - a^0}{h} \\&= a^x f'(0).\end{aligned}$$

We have shown that, if $f(x) = a^x$ is differentiable at 0, then it is differentiable everywhere, and

$$f'(x) = f'(0)a^x.$$

We leave the following theorem without proof.

Theorem

Let a be a positive number and let $f(x) = a^x$. Then the limit

$$f'(0) = \lim_{h \rightarrow 0} \frac{a^h - 1}{h}$$

exists.

We will later show that

$$f'(0) = \lim_{h \rightarrow 0} \frac{a^h - 1}{h} = \ln(a).$$

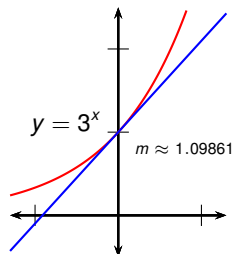
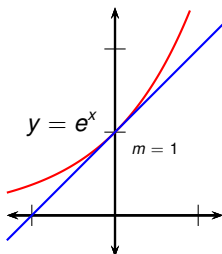
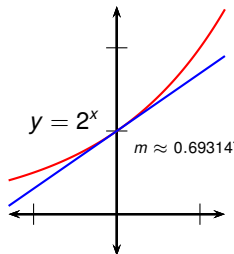
Here, \ln is the natural logarithm function.

If $f(x) = a^x$, then $f'(x) = f'(0)a^x$.

The formula above is simplest when $f'(0) = 1$. Since $\lim_{h \rightarrow 0} \frac{2^h - 1}{h} \approx 0.69$ and $\lim_{h \rightarrow 0} \frac{3^h - 1}{h} \approx 1.10$, we expect there is a number a between 2 and 3 such that $\lim_{h \rightarrow 0} \frac{a^h - 1}{h} = 1$.

Definition (e)

e is the number such that $\lim_{h \rightarrow 0} \frac{e^h - 1}{h} = 1$.



Definition (Natural Exponential Function)

e^x is called the natural exponential function. Its derivative is

$$\frac{d}{dx} (e^x) = e^x.$$

Example (Derivative of a Polynomial and the Natural Exponential Function)

Differentiate $y = e^x + x^7$.

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(e^x) + \frac{d}{dx}(x^7) \\ &= e^x + 7x^6.\end{aligned}$$

We need a formula for the derivative of the product of two functions. One might guess that the derivative of a product is the product of the derivatives; however, this is wrong.

Example (Not the Product Rule)

Let $f(x) = x$ and $g(x) = x^2$.

$$f'(x) = 1.$$

$$(fg)(x) = f(x)g(x) = x^3.$$

$$g'(x) = 2x.$$

$$(fg)'(x) = 3x^2.$$

$$f'(x)g'(x) = 2x.$$

Therefore

$$f'(x)g'(x) \neq (fg)'(x) \quad .$$

The correct formula is called the Product Rule.

Theorem (The Product Rule)

If f and g are both differentiable, then

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x).$$

Proof.

Let $F(x) = f(x)g(x)$. Then

$$\begin{aligned} F'(x) &= \lim_{h \rightarrow 0} \frac{F(x+h) - F(x)}{h} = \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h} \\ &= \lim_{h \rightarrow 0} \left(f(x+h) \frac{g(x+h) - g(x)}{h} + g(x) \frac{f(x+h) - f(x)}{h} \right) \\ &= \lim_{h \rightarrow 0} f(x+h) \cdot \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \\ &\quad + \lim_{h \rightarrow 0} g(x) \cdot \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = f(x)g'(x) + g(x)f'(x). \quad \square \end{aligned}$$

Example (Product Rule, polynomial times the Natural Exponential Function)

Differentiate $f(x) = x^3 e^x$.

$$\begin{aligned}\text{Product Rule: } f'(x) &= \frac{d}{dx} (x^3) (e^x) + (x^3) \frac{d}{dx} (e^x) \\ &= (3x^2) (e^x) + (x^3) (e^x) \\ &= e^x (x^3 + 3x^2).\end{aligned}$$

Theorem (The Quotient Rule)

If f and g are differentiable and $g(x) \neq 0$, then

$$\begin{array}{lcl} \frac{d}{dx} \left(\frac{f(x)}{g(x)} \right) & = & \frac{\frac{d}{dx} (f(x)) g(x) - f(x) \frac{d}{dx} (g(x))}{(g(x))^2} & \text{(Leibniz notation)} \\ \left(\frac{f(x)}{g(x)} \right)' & = & \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2} & \text{' notation} \\ \left(\frac{f}{g} \right)' & = & \frac{f'g - fg'}{g^2} & \text{abbreviated} \end{array}$$

- The proof of the Quotient Rule is similar to the proof of the Product Rule.
- There is an alternative algebraic proof via the Product Rule, the Power Rule and the (not yet studied) Chain Rule.

Example (Quotient Rule, rational function)

Differentiate $y = \frac{x^5 + 2x}{-x^6 + 2}$.

Quotient Rule:

$$\begin{aligned}\frac{dy}{dx} &= \frac{\frac{d}{dx}(x^5 + 2x)(-x^6 + 2) - (x^5 + 2x)\frac{d}{dx}(-x^6 + 2)}{(-x^6 + 2)^2} \\&= \frac{(5x^4 + 2)(-x^6 + 2) - (x^5 + 2x)(-6x^5)}{(-x^6 + 2)^2} \\&= \frac{(-5x^{10} - 2x^6 + 10x^4 + 4) - (-6x^{10} - 12x^6)}{(-x^6 + 2)^2} \\&= \frac{x^{10} + 10x^6 + 10x^4 + 4}{(-x^6 + 2)^2}.\end{aligned}$$