Calculus III
Lecture 12

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2020

Outline

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Minima, Maxima

Function $f\colon D \to \mathbb{R}$ defined on a region D in $\mathbb{R}^2$, we want to know:

- The largest and the smallest values of f attained on D, if any;
- The points where these extreme values are attained.

A point P_0 in D is a point of:

- absolute maximum, if $f(P) \leqslant f(P_0)$ for all P in D;
- absolute minimum, if $f(P) \geqslant f(P_0)$ for all P in D.

These notions are relative to the domain D. A point P_0 that is not an extreme point might become one if we focus only around that point.
A point P_0 in D is a point of:

- local maximum, if there exists an open disk $B = B_r(P_0)$ centered at P_0 such that $f(P) \leqslant f(P_0)$ for all P in $B \cap D$;
- local minimum, if there exists an open disk $B = B_r(P_0)$ centered at P_0 such that $f(P) \geqslant f(P_0)$ for all P in $B \cap D$.

How do we find points of extreme?

Critical Points

If $\mathbf{u} = (\nabla f)(P_0)$ exists and is non-zero, then

- f increases along $\mathbf{u}$;
- f decreases along $-\mathbf{u}$;

If we can move along $\pm\mathbf{u}$ and stay in D, then P_0 is not an extreme.
If:

- P_0 is a point of extreme (minimum or maximum);
- P_0 is an *interior point* of D, which means that there exists an open disk centered at P_0 and completely included in D;
- directional derivatives at P_0 exist in all directions

then $(\nabla f)(P_0) = \mathbf{0}$. In particular, $f_x(P_0) = f_y(P_0) = 0$.

Geometric Interpretation: At an interior point of extreme, the tangent plane to the graph surface is horizontal.

The converse is not true: if $f_x(P_0) = f_y(P_0) = 0$, then P_0 is not necessarily a point of extreme.

Where else can one find extreme points?

- At points P_0 where some directional derivatives do not exist (suffices that one of $f_x(P_0)$ or $f_y(P_0)$ does not exist.);
- At points P_0 in D that are not interior points of D.

Important concept: A point P in $\mathbb{R}^2$ is a *boundary point* for a region D if every open disk centered at P has points both in D and outside of D. Similar definition for $\mathbb{R}^3$, but replace open disk with open ball.
Examples:

- D=open unit disk $\Longrightarrow$set of boundary points = unit circle;
- D=closed unit disk $\Longrightarrow$set of boundary points = unit circle;

Notice that a boundary point may or may not be included in D.
Strategy for finding extreme points:

- Check the *critical points* of f:
  - Points P_0 for which $f_x(P_0)$ or $f_y(P_0)$ does not exist;
  - Points P_0 for which $f_x(P_0) = f_y(P_0) = 0$.
- Check boundary points included in the domain.

Example

Find the critical points of $f(x,y) = x^4 + y^4 - 4xy$ on $D = \mathbb{R}^2$.

- All points are interior; the function is differentiable everywhere.
- It remains to find the points (x,y) for which $f_x(x,y) = f_y(x,y) = 0$.

$$\left|\begin{array}{ll} f_x(x,y) & = 0 \\ f_y(x,y) & = 0 \end{array}\right. \iff \left|\begin{array}{ll} 4x^3 - 4y & = 0 \\ 4y^3 - 4x & = 0 \end{array}\right. \iff \left|\begin{array}{ll} x^3 & = y \\ y^3 & = x \end{array}\right.$$

- This system is a non-linear, haven't studied methods for those. This system can be solved using ad-hoc methods.
- There are three values of x that work:

$$\begin{aligned} x = 0 \implies y = 0 \implies &\text{ Point } (0,0) \\ x = 1 \implies y = 1 \implies &\text{ Point } (1,1) \\ x = -1 \implies y = -1 \implies &\text{ Point } (-1,-1) \end{aligned}$$

Typical mistake: $x^9 = x \iff x^8 = 1$.

Second Derivative Test

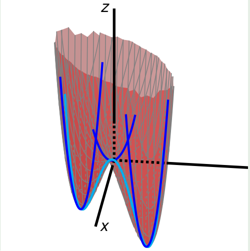
When is an interior critical point a pt. of min/max? Define the *Hessian matrix* H of f as follows. Denote by D the determinant of H.

$$H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} \qquad D = \det H = f_{xx} f_{yy} - f_{xy}^2$$

Test: Let $P(x_0, y_0)$ be an interior critical point of f and suppose that f has continuous second order derivatives around P.

- If $D(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) > 0$, then (x_0, y_0) is a local minimum. Example:**?** crit. pt. $(0,0)$ for $f(x,y) = x^2 + y^2$.
- If $D(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) < 0$, then (x_0, y_0) is a local maximum. Example: **?** crit. pt. $(0,0)$ for $f(x,y) = -x^2 - y^2$.
- If $D(x_0, y_0) < 0$, then (x_0, y_0) is neither a minimum nor a maximum. Such points are called *saddle points*. Example:**?** crit. pt. $(0,0)$ for $f(x,y) = x^2 - y^2$.
- If $D(x_0, y_0) = 0$, then the test is inconclusive. Examples: **?** $x^4 + y^4, -x^4 - y^4, x^4 - y^4$.

Example (Finding maxima, minima)



Find the local and global maxima and minima of $f(x, y) = x^4 + y^4 - 4xy$.

$$\begin{aligned} f_{xx} &= 12x^2 \\ f_{xy} &= -4 \\ f_{yy} &= 12y^2 \\ D = f_{xx}f_{yy} - f_{xy}^2 &= 144x^2y^2 - 16. \end{aligned}$$

The critical points were previously computed.

(x_0, y_0)	f_{xx}	f_{yy}	f_{xy}	D	Extremum ?
$(0,0)$	**?** 0	0	-4	$-16 < 0$	Saddle point
$(1,1)$	12	12	-4	$128 > 0$	Local min
$(-1,-1)$	12	12	-4	$128 > 0$	Local min

In this case it turns out that the two local minimum points are actually global minimum points, because
$f(x, y) = x^4 + y^4 - 4xy = (x^2 - 1)^2 + (y^2 - 1)^2 + 2(x - y)^2 - 2 \geq -2$.

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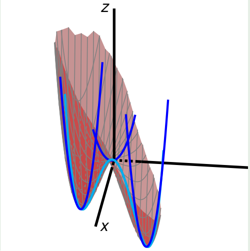

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Example

Let $P(2,1,0)$ and let $\mathcal{P}$ be the plane $3x+2y+z=6$. Find the shortest distance between P and a point on $\mathcal{P}$.

Let $Q(x,y,z)$ be a point on $\mathcal{P}$. We seek to minimize $d=\sqrt{(x-2)^2+(y-1)^2+z^2}$, equivalently to minimize $f=d^2$:

$$f(x,y)=(x-2)^2+(y-1)^2+z^2=(x-2)^2+(y-1)^2+(6-3x-2y)^2 \quad .$$

To find the critical points, solve the system:

$$\begin{aligned} 0=f_x(x,y) &= \mathbf{?}2(x-2)-6(6-3x-2y)=20x+12y-40\\ 0=f_y(x,y) &= \mathbf{?}2(y-1)-4(6-3x-2y)=12x+10y-26 \end{aligned}$$

From first equation $x=\frac{10-3y}{5}$. Substitute in the second eqn.: $\frac{14}{5}y-2=0$. Finally $x=\frac{11}{7}$, $y=\frac{5}{7}$. To find whether $\left(\frac{11}{7},\frac{5}{7}\right)$ is local extremum, compute Hessian: $H=\begin{pmatrix} f_{xx} & f_{xy}\\ f_{yx} & f_{yy}\end{pmatrix}=\mathbf{?}\begin{pmatrix} 20>0 & 12\\ 12 & 10\end{pmatrix}$.

$D=\det H=\mathbf{?}200-144=56>0$. Therefore we have a local **?** minimum at $x=\frac{11}{7}$, $y=\frac{5}{7}$, and the min. is: $f\left(\frac{11}{7},\frac{5}{7}\right)=\frac{\sqrt{14}}{7}$.

Extreme Value Theorem

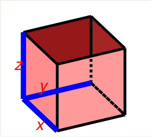
Global extreme points are guaranteed to exist if:

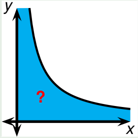
- $f\colon D \to \mathbb{R}$ is continuous, and
- the domain D has the following properties:
  - D is *bounded*: The points in D don't go farther than a certain fixed, finite distance from a fixed point.
  - D is *closed*: D contains all its boundary points.

The statement above is the **Extreme Value Theorem.**

- Why does D have to be bounded: to exclude $f\colon \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = x$;
- Why does D have to be closed: to exclude $f\colon \mathbb{R}^2 \setminus \{(0, 0)\} \to \mathbb{R}$, $f(x, y) = (x^2 + y^2)^{-1}$. In this situation the boundary of D is $\{(0, 0)\}$ and is not included in D, so D is not closed.

Example





Find the maximal volume of a box with no lid whose surface area is $10m^2$.
Let dimensions of box be x, y, z. We seek to maximize $V = xyz$.

Restrictions: $xy + 2(zx + zy) = 10$. $\Rightarrow z = \frac{10-xy}{2(x+y)}$. $\Rightarrow V = xy\frac{10-xy}{2(x+y)}$.
$(x, y) \in \mathcal{R} = \{(x, y) | xy \leq 10, x \geq 0, y \geq 0\}$. We're solving:

$$\begin{aligned} 0 = V_x &= \frac{-xy^3 - \frac{1}{2}x^2y^2 + 5y^2}{(x+y)^2} \\ 0 = V_y &= \frac{-yx^3 - \frac{1}{2}x^2y^2 + 5x^2}{(x+y)^2} \end{aligned}$$

We can assume $y \neq 0$, $x \neq 0$ (else the volume is zero). Then

$$\begin{aligned} 0 &= -xy - \frac{1}{2}x^2 + 5 \\ 0 &= -yx - \frac{1}{2}y^2 + 5 \end{aligned}$$

Therefore $x^2 = y^2$ and so $x = y$ (both quantities are positive).
Therefore $\frac{3}{2}x^2 = 5$, and so $x = \sqrt{\frac{10}{3}} = y$. By EVT max exists $\Rightarrow$ is achieved for $x = y = \sqrt{\frac{10}{3}}$. Max volume $V_{max} = \frac{5}{9}\sqrt{30}$.

Let S be a surface that has implicit equation $F(x, y, z) = 0$ for some differentiable function F. Let P be a point on the surface.

Theorem

Suppose $\nabla F(P) \neq \mathbf{0}$. Then $\nabla F(P)$ is perpendicular to the tangent vector at P of every differentiable curve lying in S passing through P.

Proof.

Suppose $\mathbf{r}(t) = (x(t), y(t), z(t))$ is a curve in S. Compute:

$$\begin{array}{rcll} F(x(t), y(t), z(t)) & = & 0 & \text{apply } \frac{\mathrm{d}}{\mathrm{d}t} \\ \left(F_x \frac{\mathrm{d}x}{\mathrm{d}t} + F_y \frac{\mathrm{d}y}{\mathrm{d}t} + F_z \frac{\mathrm{d}z}{\mathrm{d}t}\right)_{|x=x(t), y=y(t), z=z(t)} & = & 0 & \\ \mathbf{r}'(t) \cdot (F_x, F_y, F_z)_{|x=x(t), y=y(t), z=z(t)} & = & 0 & \\ \mathbf{r}'(t) \cdot (\nabla F)_{|x=x(t), y=y(t), z=z(t)} & = & 0 & \end{array}$$

$\square$

Definition (Tangent plane to level surface)

Suppose $\nabla F(P) \neq 0$. We define the tangent plane to the surface S at P to be the plane passing through P with normal vector $\nabla F(P)$.

Method of Lagrange Multipliers

Problem

Find the maximum of a function $G(x, y, z)$ subject to the variable restriction $F(x, y, z) = 0$.

- Let $S = \{(x, y, z) | F(x, y, z) = 0\}$.
- Suppose the max is achieved at $P(x_0, y_0, z_0)$. Let $\mathbf{r}(t) = (x(t), y(t), z(t))$ be a smooth curve on S such that $\mathbf{r}(0) = P$.
- Then $G(\mathbf{r}(t)) = G(x(t), y(t), z(t))$ has maximum at $t = 0$.

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0}(G(\mathbf{r}(t))) &= 0 \\ \left(\frac{\partial G}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial G}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial G}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t}\right)_{|t=0} &= 0 \\ \nabla G \cdot \mathbf{r}'(0) &= 0 \end{aligned}$$

- Therefore ∇G is $\perp$ to tangent at P of every curve in S through P.
- Therefore $\nabla F(P)$ and $\nabla G(P)$ are parallel, i.e., there exists λ s.t.: $(\nabla G)(P) = \lambda(\nabla F)(P)$.

Example

Find the maximum and the minimum values of $f(x, y) = xy$ on the region $D = \{(x, y) \mid |x| + |y| \leqslant 2\}$.
Region: closed square of vertices $(2, 0)$, $(0, 2)$, $(-2, 0)$, and $(0, -2)$.
The function is continuous and the domain is bounded and closed.
Extreme Value Theorem $\Longrightarrow$ f has global minimum and maximum points.
Strategy:

- Find critical points in the interior of the disk;
- Find extreme points on the boundary of the disk;
- Compare the values.

Since f is differentiable everywhere, the interior extreme points are among the solutions of the system

$$\left\{\begin{array}{rl} f_x(x, y) = & 0 \\ f_y(x, y) = & 0 \end{array}\right. \iff \left\{\begin{array}{rl} y = & 0 \\ x = & 0 \end{array}\right.$$

Find the maximum and the minimum values of $f(x, y) = xy$ on the region $D = \{(x, y) \mid |x| + |y| \leqslant 2\}$.
Extreme points on the boundary: check each of the four sides.
For the segment joining $(2, 0)$ with $(0, 2)$ we get:

$$\text{Find min/max of } f(x, y) = xy$$
$$\text{Subject to } g(x, y) = x + y - 2 = 0$$

The Lagrange function is

$$F(x, y, \lambda) = f(x, y) - \lambda g(x, y) = xy - \lambda(x + y - 2)$$

The critical points of F are the solutions of the system

$$\left\{\begin{array}{ll} F_x(x, y, \lambda) & = 0 \\ F_y(x, y, \lambda) & = 0 \\ F_\lambda(x, y, \lambda) & = 0 \end{array}\right. \iff \left\{\begin{array}{ll} y - \lambda & = 0 \\ x - \lambda & = 0 \\ x + y = 2 & = 0 \end{array}\right. \iff \left\{\begin{array}{ll} x & = 1 \\ y & = 1 \\ \lambda & = 1 \end{array}\right.$$

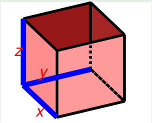
Gradient Analysis

- $(\nabla f)_{(1,1)} = \langle y, x\rangle|_{x=1,y=1} = \langle 1, 1\rangle$.
- If we move along the direction of the gradient at $(1, 1)$:
  - the value of the objective would increase;
  - the level curves of f we cross no longer intersect the constraint
  - those levels of f are unattainable on the constraint set $x + y = 2$.
- The point $(1, 1)$ corresponds to a local maxim.

Three more critical points on the boundary: $(-1, 1)$, $(-1, -1)$, $(1, -1)$. Compare the values at all points:

- the global maximum is 1, attained at $(1, 1)$ and $(-1, -1)$;
- the global minimum is -1, attained at $(1, -1)$ and $(-1, 1)$;
- the critical point $(0, 0)$ is a saddle point.

Example



Find the maximal volume of a box with no lid whose surface area is $10m^2$.
Let the three dimensions of the box be x, y, z.

We seek to maximize $V = xyz$ under the restriction

$$g(x,y,z) = xy + 2(zx + yz) - 10 = 0 \quad .$$

By the Lagrange multiplier method, we need to solve the system

$$(yz, zx, xy) = \nabla V = \lambda \nabla g = \lambda(2z + y, 2z + x, 2y + 2x) \quad .$$

In other words we are solving the following system.

$$\left| \begin{array}{rcl} yz &=& \lambda(2z+y) \\ xz &=& \lambda(2z+x) \\ xy &=& \lambda(2x+2y) \\ 10 &=& xy + 2(zx+yz) \end{array} \right. \Rightarrow \left| \begin{array}{rcl} xyz &=& \lambda(2z+y)x \\ xyz &=& \lambda(2z+x)y \\ xy &=& \lambda(2x+2y) \\ 10 &=& xy + 2(zx+yz) \end{array} \right. \quad .$$

Multiple Constraints

$$\begin{aligned} \text{Find } &\min / \max f(x, y, z) \\ \text{Subject to } &g(x, y, z) = 0 \\ &h(x, y, z) = 0 \end{aligned}$$

Each constraint defines a surface $\Longrightarrow$ their intersection defines a curve.
Condition: $(\nabla g)_P (\nabla h)_P$ are non-collinear for each intersection point P
The level surface of f through a point of extreme P_0 is tangent to the constraint curve, so $(\nabla f)(P_0)$ is perpendicular to the curve at P_0.
Constraint curve included in both surfaces $\Longrightarrow$
$(\nabla g)(P_0)$ and $(\nabla h)(P_0)$ are perpendicular to the curve $\Longrightarrow$
there exist constants λ and μ such that

$$(\nabla f)(P_0) = \lambda (\nabla g)(P_0) + \mu (\nabla h)(P_0) \ .$$

The Lagrange function is in this case

Example

Find the extreme points of $x + 2y$ on the intersection of the the cylinder $y^2 + z^2 = 5$ and the plane $x + y + z = 1$.

- Objective function: $f(x, y, z) = x + 2y$.
- Constraints: $g(x, y, z) = y^2 + z^2 - 5$ and $h(x, y, z) = x + y + z - 1$.
- Lagrange function:

$$F(x, y, z, \lambda, \mu) = x + 2y - \lambda(y^2 + z^2 - 5) - \mu(x + y + z - 1) \ .$$

- Critical points of F: $(1, \sqrt{5/2}, -\sqrt{5/2})$ and $(1, -\sqrt{5/2}, \sqrt{5/2})$
- Values of objective function at these points:

$$f(1, \sqrt{5/2}, -\sqrt{5/2}) = 1+2\sqrt{5/2}, \quad f(1, -\sqrt{5/2}, \sqrt{5/2}) = 1-2\sqrt{5/2}$$

- Constraint set is bounded and closed, function f is continuous $\Longrightarrow$ f attains its extreme on the constraint $\Longrightarrow$ $(1, -\sqrt{5/2}, \sqrt{5/2})$ corresponds to an absolute minimum and $(1, \sqrt{5/2}, -\sqrt{5/2})$ corresponds to an absolute maximum.
- The minimum value is $f(1, -\sqrt{5/2}, \sqrt{5/2}) = 1 - 2\sqrt{5/2}$ and the maximal value is $f(1, \sqrt{5/2}, -\sqrt{5/2}) = 1 + 2\sqrt{5/2}$