Precalculus
Lecture 17

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https://github.com/tmilev/freecalc

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Outline

1. Cartesian coordinate system
   - The Pythagorean Theorem, Euclidean Distance
   - Vectors
   - Segments, Midpoints

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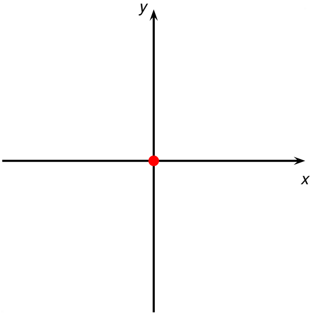
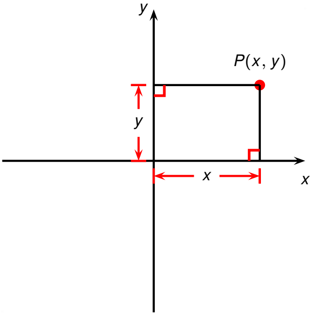
Rectangular/Cartesian Coordinates



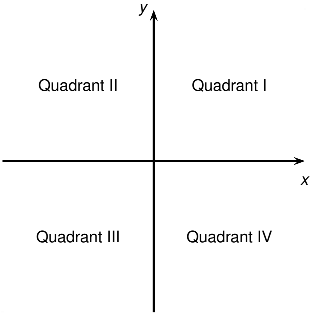
- The Cartesian (rectangular) coordinate system is a way to represent points on the plane.
- To introduce Cartesian coordinates, fix:
  - a point O (called the origin),
  - 2 pairwise perpendicular lines intersecting at the origin, called axes,
  - a direction in each of the coordinate axis.
- The axes are labeled as x-axis and y-axis.
- The x axis is drawn horizontal with direction pointing from left to right.
- The y axis is drawn vertical, pointing up.
- The Cartesian coordinate system is named after René Descartes (1596-1650) (Latinized name: Cartesius).

Rectangular/Cartesian Coordinates



- Let P -point. We assign to it a pair of numbers (x, y).
- Distinct points are assigned distinct pairs.
- $Q =$ base of perpendicular from P to x-axis.
- Define x as signed distance b-n O and Q.
- Take distance with $+$ sign if OQ points in direction of x-axis, $-$ sign else.
- To define y, do the same with the y axis.
- (x, y) = Cartesian coordinates of P.
- x is called the x-coordinate of P, y- the y coordinate.
- (x, y) = singed lengths of sides of the rectangle indicated in the picture.

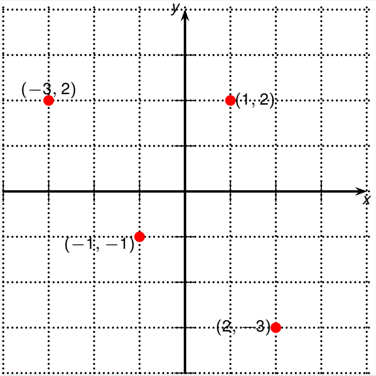
Rectangular/Cartesian Coordinates



Quadrant	(x, y)
I	$(+,+)$
II	$(-,+)$
III	$(-,-)$
IV	$(+,-)$

- The coordinate axes split the plane in 4 regions, called quadrants.
- The quadrants are labeled as indicated.
- For a point has coordinates (x, y), $x \neq 0$, $y \neq 0$, the signs of x and y are determined by the quadrant that contains the point.

Example



Plot the points and name the Quadrant that contains them

- $(1, 2)$. Quadrant I
- $(2, -3)$. Quadrant IV
- $(-3, 2)$. Quadrant II
- $(-1, -1)$. Quadrant III

- A triangle is a right-angled triangle if two of its sides are perpendicular.
- The two sides perpendicular to one another are called legs.
- The two legs form a right angle (90°).
- The side opposite to the right angle is called the hypothenuse.

Theorem

Let a, b be the lengths of the legs of a right-angled triangle and c the length of its hypotenuse. Then

$$a^2 + b^2 = c^2.$$

Theorem

Let (x_1, y_1) and (x_2, y_2) be two points in the plane. Then the distance d between the two points is given by

$$d = \sqrt{(x_1 - x_2)^2 - (y_1 - y_2)^2}$$

Example

Find the distance between $(-2, 3)$ and $(3, 5)$.
$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(-2-3)^2 + (3-5)^2} = \sqrt{(-5)^2 + (-2)^2} = \sqrt{25+4} = \sqrt{29} \approx 5.385$.

Example

Find the distances between the indicated points.

P	Q	distance
$(2, 3)$	$(3, 5)$	**?**
$(-2, -3)$	$(3, 5)$	**?**
$(-2, -3)$	$(3, -5)$	**?**
$(-2, 3)$	$(3, -5)$	**?**

Example

Do the points $(1, 2)$, $(2, 3)$, $(4, -1)$ form a right-angled triangle?

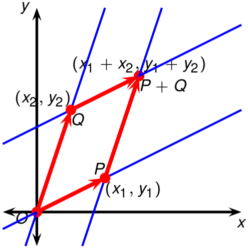
Example

Do the points $(1, 2)$, $(2, 4)$, $(3, 1)$ form a right-angled triangle?

Example

Do the indicated points form a right-angled triangle?

$(-1, -2)$	$(3, 5)$	$(6, -6)$	**?**
$(1, 2)$	$(3, 5)$	$(6, 6)$	**?**
$(0, 0)$	$(2, 3)$	$(3, -2)$	**?**
$(0, 0)$	$(2, 3)$	$(-2, 3)$	**?**

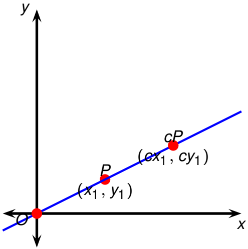


Definition ($+, \cdot$ in $\mathbb{R}^2$)

Let $\mathbf{u} = (x_1, y_1)$ and $\mathbf{v} = (x_2, y_2)$ be pairs of numbers and let c be a number. Define

$$\begin{aligned} \mathbf{u} + \mathbf{v} &= (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \\ c \cdot \mathbf{u} &= c \cdot (x_1, y_1) = (cx_1, cy_1). \end{aligned}$$

- Fix a Cartesian coordinate system in the plane.
- Let P, Q be points with respective coordinates (x_1, y_1) and (x_2, y_2); let O be the origin.
- $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ corresponds to a new point which we denote by $P + Q$.
- One can show: the line through O and P is parallel to the line through Q and $P + Q$.
- One can show: the line through O and Q is parallel to the line through P and $P + Q$.
- The points O, P, $P + Q$ and Q form a parallelogram.

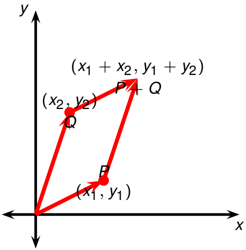


Definition ($+, \cdot$ in $\mathbb{R}^2$)

Let $\mathbf{u} = (x_1, y_1)$ and $\mathbf{v} = (x_2, y_2)$ be pairs of numbers and let c be a number. Define

$$\begin{aligned} \mathbf{u} + \mathbf{v} &= (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \\ c \cdot \mathbf{u} &= c \cdot (x_1, y_1) = (cx_1, cy_1). \end{aligned}$$

- Fix a Cartesian coordinate system in the plane.
- Let P, Q be points with respective coordinates (x_1, y_1) and (x_2, y_2); let O be the origin.
- $c(x_1, y_1) = (cx_1, cy_1)$ corresponds to a new point which we denote by cP.
- One can show O, P and cP lie on the same line.

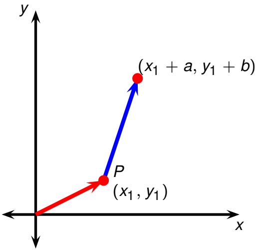


Definition ($+, \cdot$ in $\mathbb{R}^2$)

Let $\mathbf{u} = (x_1, y_1)$ and $\mathbf{v} = (x_2, y_2)$ be pairs of numbers and let c be a number. Define

$$\begin{aligned} \mathbf{u} + \mathbf{v} &= (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \\ c \cdot \mathbf{u} &= c \cdot (x_1, y_1) = (cx_1, cy_1). \end{aligned}$$

- Fix a Cartesian coordinate system in the plane.
- The correspondence between points in the plane and pairs of numbers depends on the choice of Cartesian coordinate system.
- If we change the coordinate system we change $+, \cdot$.
- The points in the plane, equipped with the operations $+, \cdot$ form a mathematical object which we call a vector space.



Definition ($+, \cdot$ in $\mathbb{R}^2$)

Let $\mathbf{u} = (x_1, y_1)$ and $\mathbf{v} = (x_2, y_2)$ be pairs of numbers and let c be a number. Define

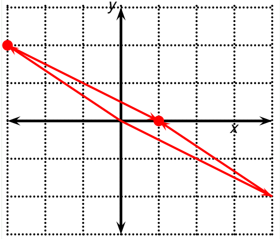
$$\begin{aligned} \mathbf{u} + \mathbf{v} &= (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \\ c \cdot \mathbf{u} &= c \cdot (x_1, y_1) = (cx_1, cy_1). \end{aligned}$$

Definition (Translation)

Let P with coordinates (x, y) be a point and let (a, b) be a pair of numbers. The point $P = (x, y) + (a, b) = (x + a, y + b)$ is called the translation (shift) of P a units right and b units up.

- We allow shifts by negative units.
- Translation down by b units we define to be translation up by $-b$ units.
- Translation left by a units we define to be translation right by $-a$ units.

Example



Translate $(-3, 2)$ 4 units right and 2 units down.

$$(-3, 2) + (4, -2) = (-3 + 4, 2 + (-2)) = (1, 0).$$

Example

Translate the point in the indicated way.

Point	Translation	result
$(2, 3)$	2 units left 3 units up	**?**
$(2, 1)$	2 units left -2 units down	**?**
$(-2, 1)$	-1 units right 2 units down	**?**
$(-2, 3)$	-1 units left 2 units up	**?**

Observation

The segment connecting P and Q consists of all points of the form

$$tP + (1-t)Q,$$

where t runs over all numbers in the interval $[0, 1]$*.*

- Let P have coordinates (x_1, y_1) and Q have coordinates (x_2, y_2).
- Then the segment between P and Q consists of the points with coordinates

$$t(x_1, y_1) + (1-t)(x_2, y_2).$$

Observation

The midpoint of the segment between P and Q is the point with $t = \frac{1}{2}$*.*

$$Midpoint(P, Q) = \frac{1}{2}P + \frac{1}{2}Q = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

Example

Let P have coordinates (x_1, y_1) and Q have coordinates (x_2, y_2). Let the midpoint of P and Q be R. Write the formula for the distance a between P and Q, and for the distance b between Q and R. Show that $b = \frac{1}{2}a$.

Example

Find the midpoint of the indicated pairs of points.

P	Q	midpoint
$(1,2)$	$(-1,-2)$	**?**
$(1,2)$	$(1,-2)$	**?**
$(-1,2)$	$(1,-2)$	**?**
$(-2,-3)$	$(3,2)$	**?**