

# Calculus I

## Lecture 20

### Areas, Integration

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<https://github.com/tmilev/freecalc>

2020

# Outline

## 1 Areas and Distances

- The Area Problem

## 2 The Definite Integral

- Review of the  $\sum$  notation
- Riemann sums, areas and integrals

## 3 The Definite Integral

- Evaluating Integrals with Riemann Sums
- Properties of the Definite Integral

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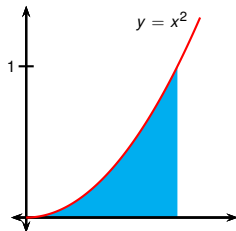
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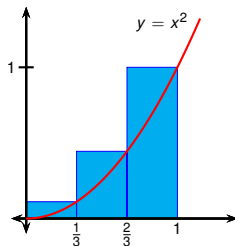
# The Area Problem

- How can we find the area under  $y = x^2$  between  $x = 0$  and  $x = 1$ ?



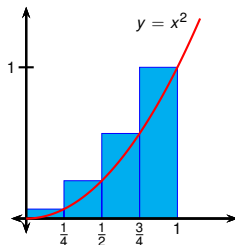
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- How can we find the area under  $y = x^2$  between  $x = 0$  and  $x = 1$ ?
- We can approximate it using rectangles.
- Divide  $[0, 1]$  into three strips of width  $\frac{1}{3}$ , and draw rectangles in those strips, the heights of which are the same as the height of the function at the right end of that strip.



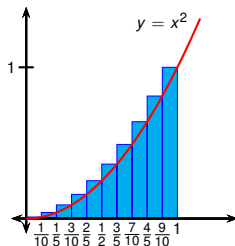
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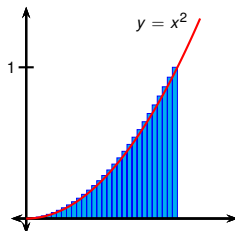
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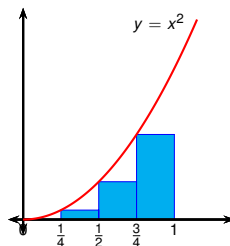
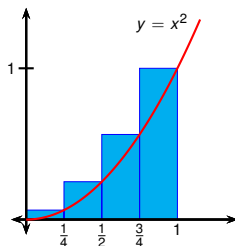
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# The Area Problem

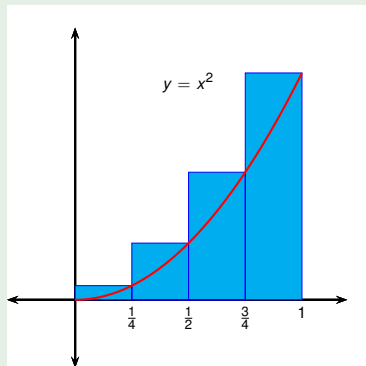
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- Divide  $[0, 1]$  into three strips of width  $\frac{1}{3}$ , and draw rectangles in those strips, the heights of which are the same as the height of the function at the right end of that strip.
- Four strips gives a better approximation.
- We could use the left endpoints to find the heights instead.



## Example

Find the sum of the areas of the four approximating rectangles obtained using right endpoints.

- Let  $R_4$  denote the sum of the areas of the rectangles.
- Each rectangle has width  $\frac{1}{4}$ .
- The heights are  $\left(\frac{1}{4}\right)^2$ ,  $\left(\frac{1}{2}\right)^2$ ,  $\left(\frac{3}{4}\right)^2$ , and  $1^2$ .

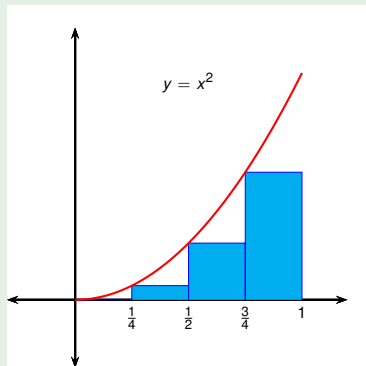


$$R_4 = \frac{1}{4} \cdot \left(\frac{1}{4}\right)^2 + \frac{1}{4} \cdot \left(\frac{1}{2}\right)^2 + \frac{1}{4} \cdot \left(\frac{3}{4}\right)^2 + \frac{1}{4} \cdot (1)^2 = \frac{15}{32} = 0.46875$$

## Example

Find the sum of the areas of the four approximating rectangles obtained using right endpoints.

- Let  $R_4$  denote the sum of the areas of the rectangles.
- Each rectangle has width  $\frac{1}{4}$ .
- The heights are  $\left(\frac{1}{4}\right)^2$ ,  $\left(\frac{1}{2}\right)^2$ ,  $\left(\frac{3}{4}\right)^2$ , and  $1^2$ .
- A similar calculation works for  $L_4$ , the sum of the areas of the left endpoint rectangles.



$$R_4 = \frac{1}{4} \cdot \left(\frac{1}{4}\right)^2 + \frac{1}{4} \cdot \left(\frac{1}{2}\right)^2 + \frac{1}{4} \cdot \left(\frac{3}{4}\right)^2 + \frac{1}{4} \cdot (1)^2 = \frac{15}{32} = 0.46875$$

$$L_4 = \frac{1}{4} \cdot (0)^2 + \frac{1}{4} \cdot \left(\frac{1}{4}\right)^2 + \frac{1}{4} \cdot \left(\frac{1}{2}\right)^2 + \frac{1}{4} \cdot \left(\frac{3}{4}\right)^2 = \frac{7}{32} = 0.21875$$

## Example

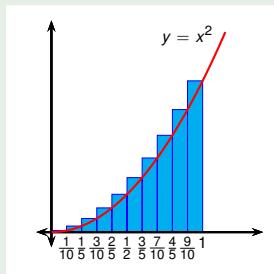
For the region  $S$  underneath the parabola  $y = x^2$  from 0 to 1, show that the area under the approximating rectangles approaches  $\frac{1}{3}$ , that is,

$$\lim_{n \rightarrow \infty} R_n = \frac{1}{3}.$$

- Each rectangle has width  $\frac{1}{n}$ .
- The heights are  $\left(\frac{1}{n}\right)^2, \left(\frac{2}{n}\right)^2, \dots, \left(\frac{n}{n}\right)^2$ .
- New formula:
- $1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$ .

$$R_n = \frac{1}{n} \left(\frac{1}{n}\right)^2 + \frac{1}{n} \left(\frac{2}{n}\right)^2 + \dots + \frac{1}{n} \left(\frac{n}{n}\right)^2 = \frac{1}{n^3} (1^2 + 2^2 + \dots + n^2)$$

$$\lim_{n \rightarrow \infty} R_n = \lim_{n \rightarrow \infty} \frac{1}{n^3} \frac{n(n+1)(2n+1)}{6} = \lim_{n \rightarrow \infty} \frac{1}{6} \left(1 + \frac{1}{n}\right) \left(2 + \frac{1}{n}\right) = \frac{1}{3}$$



## Example (The ... and $\sum$ notations for series)

Let  $A$  be the sum of the positive even integers between 2 and 124. Write  $A$  using the ... notation and using the  $\sum$  notation.

$$\begin{aligned} A &= 2 + 4 + 6 + \dots + 124 \\ &= 2 + 4 + 6 + \dots + 2n + \dots + 124 \\ &= 2 \cdot 1 + 2 \cdot 2 + 2 \cdot 3 + \dots + 2 \cdot n + \dots + 2 \cdot 62 \\ &= \sum_{n=1}^{62} 2n . \end{aligned}$$

- We aim to introduce the  $\sum$  notation for series via this example.
- The ... notation is informal but easier to read.
- If the ... are too ambiguous, we should include the general term.
- To make it clearer we should rewrite all elements in the pattern of the general term.
- If that is still ambiguous we should switch to the completely unambiguous  $\sum$  notation.

## Example (The ... and $\sum$ notations for series)

Let  $A$  be the sum of the positive even integers between 2 and 124. Write  $A$  using the ... notation and using the  $\sum$  notation.

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- The number  $n$  is the index (counter) of the sum.
- $\sum$  tells us to add several copies of the summed term, where in each term the index is replaced by a concrete value.
- The values taken by the index are determined by the boundaries of summation.
- The index varies over all integers starting with the lower boundary and ending with upper boundary.
- In programming, what objects are similar to  $\sum$ ?

## Example (The $\dots$ and $\sum$ notations for series)

Let  $A$  be the sum of the positive even integers between 2 and 124.  
Write  $A$  using the  $\dots$  notation and using the  $\sum$  notation.

$$\begin{aligned} A &= 2 + 4 + 6 + \dots + 124 \\ &= 2 + 4 + 6 + \dots + 2n + \dots + 124 \\ &= 2 \cdot 1 + 2 \cdot 2 + 2 \cdot 3 + \dots + 2 \cdot n + \dots + 2 \cdot 62 \\ &= \sum_{n=1}^{62} 2n . \end{aligned}$$

- To go from  $\sum$  to  $\dots$  notation: substitute few values for the index.  
Make sure to include the last value.
- To go from  $\dots$  to  $\sum$  notation:
  - figure out a pattern for the general term just as with sequences;
  - select first and last index so that your general term formula reproduces the first and last terms of the sequence.



## Example (The ... and $\sum$ notations for series)

Let  $A$  be the sum of the positive even integers between 2 and 124.  
Write  $A$  using the ... notation and using the  $\sum$  notation.

$$\begin{aligned} A &= 2 + 4 + 6 + \cdots + 124 \\ &= 2 + 4 + 6 + \cdots + 2n + \cdots + 124 \\ &= 2 \cdot 1 + 2 \cdot 2 + 2 \cdot 3 + \cdots + 2 \cdot n + \cdots + 2 \cdot 62 \\ &= \sum_{n=1}^{62} 2n . \end{aligned}$$

- Bear in mind the ... notation is informal.
  - There are infinitely many formulas that fit any single pattern.
  - Thus it is acceptable to use the ... notation only when we believe there is a single completely obvious pattern that will be recognized by every one.
  - The pattern should be obvious not only to us, but also to our potential readers.
  - If in doubt or seeking complete rigor we should use the  $\sum$  notation.

## Definition

**Sigma Notation:** The sum of  $n$  terms  $a_1, a_2, \dots, a_n$  is written as

$$\sum_{i=1}^n a_i = a_1 + a_2 + \cdots + a_n$$

where  $i$  is the *index of summation*,  $a_i$  is the  $i$ 'th term, and the *upper and lower bounds of summation* are  $n$  and 1 respectively.

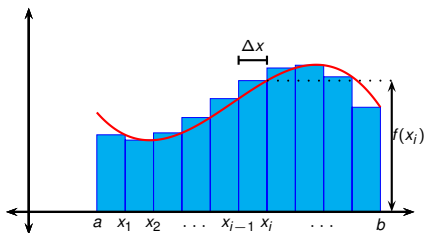
NOTE: The lower bound doesn't have to be 1. Any integer less than or equal to the upper bound is legitimate.

The index  $i$  may be replaced with another symbol, often  $j$  or  $k$ .

## Example

$$\sum_{j=3}^7 j^2 = 9 + 16 + 25 + 36 + 49$$

Estimate the area under  $y = f(x)$  between  $a$  and  $b$  using  $n$  strips.



- The right endpoints of the subintervals are

$$x_1 = a + \Delta x$$

$$x_2 = a + 2\Delta x$$

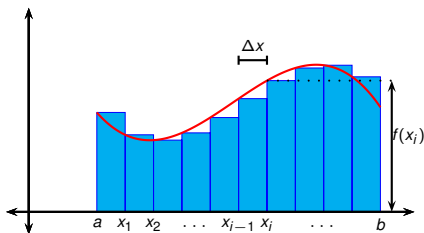
$$x_3 = a + 3\Delta x$$

$$\vdots$$

- The width of the interval is  $b - a$ .
- The width one strip is  $\Delta x = \frac{b-a}{n}$ .
- $[a, b]$  is divided into  $n$  subintervals:  $[x_0, x_1], [x_1, x_2], \dots, [x_{n-1}, x_n]$ , where  $x_0 = a$  and  $x_n = b$ .
- The height of the  $i$ th rectangle is  $f(x_i)$ .
- The area of the  $i$ th rectangle is  $f(x_i)\Delta x$ .

$$R_n = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + \dots + f(x_n)\Delta x$$

Estimate the area under  $y = f(x)$  between  $a$  and  $b$  using  $n$  strips.



- The **left** endpoints of the subintervals are

$$x_0 = a$$

$$x_1 = a + \Delta x$$

$$x_2 = a + 2\Delta x$$

$$\vdots$$

- The width of the interval is  $b - a$ .
- The width one strip is  $\Delta x = \frac{b-a}{n}$ .
- $[a, b]$  is divided into  $n$  subintervals:  $[x_0, x_1], [x_1, x_2], \dots, [x_{n-1}, x_n]$ , where  $x_0 = a$  and  $x_n = b$ .
- The height of the  $i$ th rectangle is  $f(x_{i-1})$ .
- The area of the  $i$ th rectangle is  $f(x_{i-1})\Delta x$ .

$$L_n = f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + \dots + f(x_{n-1})\Delta x$$

## Definition (Area Under a Curve)

Let  $f(x) > 0$ . The area of the region  $S$  that lies under  $y = f(x)$  is the limit of the sum of the areas of the approximating rectangles:

$$A = \lim_{n \rightarrow \infty} R_n = \lim_{n \rightarrow \infty} [f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x]$$

- This limit always exists if  $f$  is continuous.
- If  $f$  is continuous, we get the same limit if we use left endpoints:

$$A = \lim_{n \rightarrow \infty} L_n = \lim_{n \rightarrow \infty} [f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x]$$

- If  $f$  is continuous, we get the same limit if we use any number  $x_i^*$  in the interval  $[x_{i-1}, x_i]$ .  $x_i^*$  is called a sample point.

$$A = \lim_{n \rightarrow \infty} [f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x]$$

## Definition (Riemann Sum)

A Riemann sum is any sum of the form

$$f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x.$$

# The Definite Integral

## Definition (Definite Integral)

- Let  $f$  be a function defined for  $a \leq x \leq b$ .
- Divide the interval  $[a, b]$  into  $n$  subintervals of equal width  $\Delta x = (b - a)/n$  and set  $x_0 = a$ ,  $x_n = b$ .
- Let  $x_0, x_1, \dots, x_n$  be the endpoints of the subintervals.
- Let  $x_1^*, x_2^*, \dots, x_n^*$  be any sample points in these subintervals; that is,  $x_i^*$  is in  $[x_{i-1}, x_i]$ .

Suppose the limit  $\lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x$  exists and is independent of the choice of sample points  $x_i^*$ . Then we say that  $f$  is an integrable function. If  $f$  is integrable we call the limit the integral of  $f$  over  $[a, b]$  and write

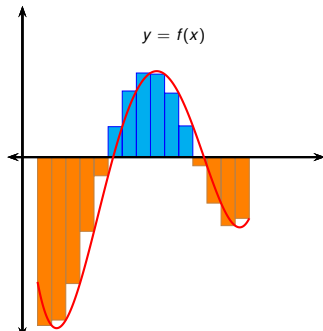
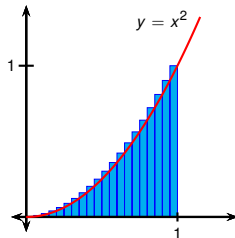
$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x \quad .$$

$$\int_a^b f(x)dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*)\Delta x,$$

- $\int$  is called the integration sign.
- $f(x)$  is called the integrand.
- $a$  and  $b$  are called the limits of integration.
- The definite integral is a number. It does not depend on  $x$ . We could use any variable instead of  $x$ .

$$\int_a^b f(x)dx = \int_a^b f(t)dt = \int_a^b f(r)dr = \int_a^b f(\theta)d\theta$$

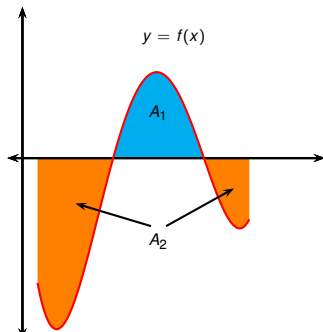
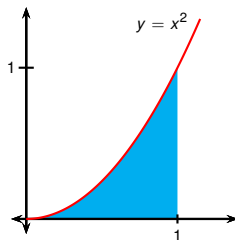
- We know already that if  $f(x)$  is always positive, then  $\int_a^b f(x)dx$  is the area under the curve.



- What if  $f(x)$  is sometimes negative?



- We know already that if  $f(x)$  is always positive, then  $\int_a^b f(x)dx$  is the area under the curve.



- What if  $f(x)$  is sometimes negative?
- Then  $\int_a^b f(x)dx = A_1 - A_2$ .
- $A_1$  is the area of the region above the x-axis and below the graph of  $f$ .
- $A_2$  is the area of the region below the x-axis and above the graph of  $f$ .

## Theorem

*Let  $f$  be a continuous function on  $[a, b]$ . Then  $f$  is integrable over  $[a, b]$ .*

- In particular the integral does not depend the choice of sampling points so long as the intervals containing them shrink.
- The proof of this theorem is not difficult, but is outside of the scope of Calculus I and II.
- The only “difficulty” in the proof stems from the fact that we have not rigorously constructed the real numbers.
- We already (silently) assumed a construction of the real numbers when we defined limits.
- Such a construction is also (silently) assumed in most regular high school mathematics courses.
- A student interested in a proof of the theorem should google “Darboux integral”.

The following **power sums** will be useful in what follows:

$$① \sum_{i=1}^n i = \frac{n(n+1)}{2}$$

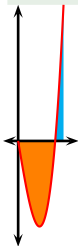
$$② \sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$③ \sum_{i=1}^n i^3 = \left[ \frac{n(n+1)}{2} \right]^2$$

## Example

Evaluate  $\int_0^3 (x^3 - 6x) dx$ .  $\Delta x = \frac{b-a}{n} = \frac{3}{n}$ .

$$\begin{aligned}
 \int_0^3 (x^3 - 6x) dx &= \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x = \lim_{n \rightarrow \infty} \sum_{i=1}^n f\left(\frac{3i}{n}\right) \frac{3}{n} \\
 &= \lim_{n \rightarrow \infty} \frac{3}{n} \sum_{i=1}^n \left[ \left(\frac{3i}{n}\right)^3 - 6 \left(\frac{3i}{n}\right) \right] = \lim_{n \rightarrow \infty} \frac{3}{n} \sum_{i=1}^n \left[ \frac{27}{n^3} i^3 - \frac{18}{n} i \right] \\
 &= \lim_{n \rightarrow \infty} \left[ \frac{81}{n^4} \sum_{i=1}^n i^3 - \frac{54}{n^2} \sum_{i=1}^n i \right] \\
 &= \lim_{n \rightarrow \infty} \left( \frac{81}{n^4} \left[ \frac{n(n+1)}{2} \right]^2 - \frac{54}{n^2} \frac{n(n+1)}{2} \right) \\
 &= \lim_{n \rightarrow \infty} \left[ \frac{81}{4} \left( 1 + \frac{1}{n} \right)^2 - 27 \left( 1 + \frac{1}{n} \right) \right] = \frac{81}{4} - 27 = -\frac{27}{4}
 \end{aligned}$$



# Properties of the Definite Integral

- So far when we have calculated  $\int_a^b f(x)dx$ , we have assumed that  $a < b$ .
- The definition as a limit of Riemann sums will still work even if we don't assume this.
- If we reverse  $a$  and  $b$ , then  $\Delta x$  changes from  $\frac{b-a}{n}$  to  $\frac{a-b}{n}$ .

$$\int_b^a f(x)dx = - \int_a^b f(x)dx$$

- If  $a = b$ , then  $\Delta x = 0$ .

$$\int_a^a f(x)dx = 0$$

## Properties of the Integral

- ①  $\int_a^b c dx = c(b - a)$ , where  $c$  is any constant.
- ②  $\int_a^b [f(x) + g(x)] dx = \int_a^b f(x) dx + \int_a^b g(x) dx$ .
- ③  $\int_a^b c f(x) dx = c \int_a^b f(x) dx$ , where  $c$  is any constant.
- ④  $\int_a^b [f(x) - g(x)] dx = \int_a^b f(x) dx - \int_a^b g(x) dx$ .

## Example

Use the properties of integrals to evaluate

$$\int_0^1 (4 + 3x^2)dx = \int_0^1 4dx + \int_0^1 3x^2dx \quad \text{Property 2}$$

$$= \int_0^1 4dx + 3 \int_0^1 x^2dx \quad \text{Property 3}$$

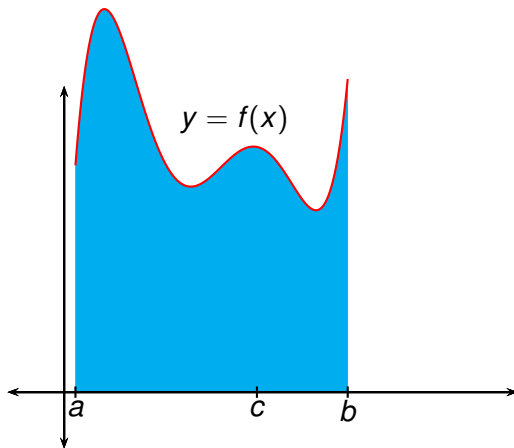
$$= 4(1 - 0) + 3 \int_0^1 x^2dx \quad \text{Property 1}$$

$$= 4 + 3 \cdot \frac{1}{3} \quad \text{From preceding lectures/slides}$$

$$= 5$$

# Properties of the Integral

$$⑤ \int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$





## Example

If it is known that  $\int_0^{10} f(x)dx = 17$  and  $\int_0^8 f(x)dx = 12$ , then find  $\int_8^{10} f(x)dx$ .

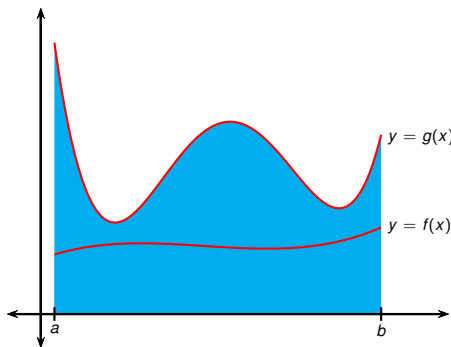
$$\begin{aligned}\int_0^8 f(x)dx + \int_8^{10} f(x)dx &= \int_0^{10} f(x)dx \\ \int_8^{10} f(x)dx &= \int_0^{10} f(x)dx - \int_0^8 f(x)dx \\ &= 17 - 12 \\ &= 5\end{aligned}$$

## Comparison Properties of the Integral

⑥ If  $f(x) \geq 0$  for all  $a \leq x \leq b$ , then  $\int_a^b f(x)dx \geq 0$ .

## Comparison Properties of the Integral

7 If  $f(x) \leq g(x)$  for all  $a \leq x \leq b$ , then  $\int_a^b f(x)dx \leq \int_a^b g(x)dx$ .



$$\int_a^b f(x)dx \leq \int_a^b g(x)dx$$

## Comparison Properties of the Integral

8 If  $m \leq f(x) \leq M$  for all  $a \leq x \leq b$ , then

$$m(b-a) \leq \int_a^b f(x) dx \leq M(b-a)$$

