Calculus I
Lecture 14
Logarithmic Differentiation

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Outline

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Derivatives of Logarithmic Functions

Theorem (The Derivative of $\log_a x$)

$$\frac{\mathrm{d}}{\mathrm{d}x}(\log_a x) = \frac{1}{x \ln a}.$$

Proof.

$\square$

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Example (Chain Rule)

Differentiate $f(x) = \log_3(5^x + 1)$.

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$$\begin{aligned} \text{Differentiate} \quad & f(x) = \log_3(5^x + 1). \\ \text{Let} \quad & h(x) = \textbf{?} \\ \text{Let} \quad & g(x) = \textbf{?} \\ \text{Then} \quad & f(x) = g(h(x)). \end{aligned}$$

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&= \left(\frac{1}{h(x)\ln 3}\right)(5^x \ln 5) \\
&= \frac{5^x \ln 5}{(5^x + 1)\ln 3}.
\end{aligned}
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Theorem (The Derivative of $\log_a x$)

$$\frac{\mathrm{d}}{\mathrm{d}x}(\log_a x) = \frac{1}{x \ln a}.$$

$\ln x = \log_e x$. Therefore when we set $a = e$ we get the derivative of $\ln x$:

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Differentiate $y = \ln(e^x \sec x)$.

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Example (Chain Rule, Natural Logarithm)

Differentiate $y = \ln(e^x \sec x)$.

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Then $\ln(\sec x) = \ln u$.

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&= 1 + \left(\frac{1}{u}\right)(\sec x \tan x) \\
&= 1 + \frac{1}{\cancel{\sec x}}\cancel{\sec x}\tan x \\
&= 1 + \tan x.
\end{aligned}
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Example

Find $f'(x)$ if $f(x) = \ln|x|$.

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$$= \frac{1}{x} \text{ if } x \neq 0.$$

Example

Differentiate $x^{\tan x}$, where $x > 0$.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{\tan x}\right)$$

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$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{\tan x}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\left(e^{\mathbf{?}}\quad\right)^{\tan x}\right)$$

Convert base to $e^?$

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Differentiate $x^{\tan x}$, where $x > 0$.

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$$= \frac{d}{dx}\left(e^{(\ln x)\tan x}\right)$$

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Differentiate $x^{\tan x}$, where $x > 0$.

$$\begin{aligned}\frac{d}{dx}\left(x^{\tan x}\right) &= \frac{d}{dx}\left(\left(e^{\ln x}\right)^{\tan x}\right) && \text{Convert base to } e^? \\ &= \frac{d}{dx}\left(e^{(\ln x)\tan x}\right) \\ &= \frac{d}{dx}\left(e^{u}\right) && \text{Set } (\ln x)\tan x = u\end{aligned}$$

Example

Differentiate $x^{\tan x}$, where $x > 0$.

$$\frac{d}{dx}\left(x^{\tan x}\right) = \frac{d}{dx}\left(\left(e^{\ln x}\right)^{\tan x}\right)$$

Convert base to $e^?$

$$= \frac{d}{dx}\left(e^{(\ln x)\tan x}\right)$$

$$= \frac{d}{dx}\left(e^{u}\right)$$

Set $(\ln x)\tan x = u$

$$= \frac{d}{du}\left(e^{u}\right)\frac{du}{dx}$$

Chain rule

Example

Differentiate $x^{\tan x}$, where $x > 0$.

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$$= \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{(\ln x)\tan x}\right)$$

$$= \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{u}\right) \qquad \text{Set } (\ln x)\tan x = u$$

$$= \frac{\mathrm{d}}{\mathrm{d}u}\left(e^{u}\right)\frac{\mathrm{d}u}{\mathrm{d}x} \qquad \text{Chain rule}$$

$$= \textbf{?}\ \frac{\mathrm{d}}{\mathrm{d}x}\left((\ln x)\tan x\right)$$

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Differentiate $x^{\tan x}$, where $x > 0$.

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\frac{d}{dx}\left(x^{\tan x}\right) &= \frac{d}{dx}\left(\left(e^{\ln x}\right)^{\tan x}\right) && \text{Convert base to } e^{?}\\
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&= \frac{d}{dx}\left(e^{u}\right) && \text{Set } (\ln x)\tan x = u\\
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&= \mathbf{?}\ \frac{d}{dx}\left((\ln x)\tan x\right)
\end{aligned}$$

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Differentiate $x^{\tan x}$, where $x > 0$.

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\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{\tan x}\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\left(e^{\ln x}\right)^{\tan x}\right) && \Big| \text{ Convert base to } e^? \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{(\ln x)\tan x}\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{u}\right) && \Big| \text{ Set } (\ln x)\tan x = u \\
&= \frac{\mathrm{d}}{\mathrm{d}u}\left(e^{u}\right)\frac{\mathrm{d}u}{\mathrm{d}x} && \Big| \text{ Chain rule} \\
&= e^{u}\frac{\mathrm{d}}{\mathrm{d}x}\left((\ln x)\tan x\right)
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&= \frac{d}{du}\left(e^{u}\right)\frac{du}{dx} && \text{Chain rule}\\
&= e^{u}\frac{d}{dx}\left((\ln x)\tan x\right)\\
&= e^{(\ln x)\tan x}\left((\ln x)'\tan x + (\ln x)(\tan x)'\right)
\end{aligned}$$

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&= e^{u}\frac{d}{dx}\left((\ln x)\tan x\right) \\
&= e^{(\ln x)\tan x}\left((\ln x)'\tan x + (\ln x)(\tan x)'\right) && \text{Prod. rule}
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&= e^{(\ln x)\tan x}\left((\ln x)'\tan x + (\ln x)(\tan x)'\right) && \text{Prod. rule}
\end{aligned}
$$

Example

Differentiate $x^{\tan x}$, where $x > 0$.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{\tan x}\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\left(e^{\ln x}\right)^{\tan x}\right) && \Big|\ \text{Convert base to } e^{?}\\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{(\ln x)\tan x}\right)\\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{u}\right) && \Big|\ \text{Set } (\ln x)\tan x = u\\
&= \frac{\mathrm{d}}{\mathrm{d}u}\left(e^{u}\right)\frac{\mathrm{d}u}{\mathrm{d}x} && \Big|\ \text{Chain rule}\\
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&= x^{\tan x}\left(\mathbf{?}\ \tan x + (\ln x)\mathbf{?}\qquad\right)
\end{aligned}
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&= x^{\tan x}\left(\mathbf{?}\ \tan x + (\ln x)\mathbf{?}\qquad\right)
\end{aligned}$$

Example

Differentiate $x^{\tan x}$, where $x > 0$.

$$\frac{d}{dx}\left(x^{\tan x}\right) = \frac{d}{dx}\left(\left(e^{\ln x}\right)^{\tan x}\right) \quad \Big| \text{ Convert base to } e^{?}$$

$$= \frac{d}{dx}\left(e^{(\ln x)\tan x}\right)$$

$$= \frac{d}{dx}\left(e^{u}\right) \quad \Big| \text{ Set } (\ln x)\tan x = u$$

$$= \frac{d}{du}\left(e^{u}\right)\frac{du}{dx} \quad \Big| \text{ Chain rule}$$

$$= e^{u}\frac{d}{dx}\left((\ln x)\tan x\right)$$

$$= e^{(\ln x)\tan x}\left((\ln x)'\tan x + (\ln x)(\tan x)'\right) \quad \Big| \text{ Prod. rule}$$

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&= x^{\tan x}\left(\frac{1}{x}\tan x + (\ln x)\,{\color{red}\mathbf{?}} \qquad\qquad \right)
\end{aligned}$$

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Differentiate $x^{\tan x}$, where $x > 0$.

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Example

Differentiate $(3x+1)^{\ln x}$, where $3x+1>0$.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left((3x+1)^{\ln x}\right)$$

Example

Differentiate $(3x+1)^{\ln x}$, where $3x+1>0$.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left((3x+1)^{\ln x}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\left(e^{\mathbf{?}} \qquad \right)^{\ln x}\right) \quad \Bigg| \quad \text{Convert base to } e^{?}$$

Example

Differentiate $(3x+1)^{\ln x}$, where $3x+1>0$.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left((3x+1)^{\ln x}\right)=\frac{\mathrm{d}}{\mathrm{d}x}\left(\left(e^{\ln(3x+1)}\right)^{\ln x}\right) \quad \Big| \text{ Convert base to } e^?$$

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$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left((3x+1)^{\ln x}\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\left(e^{\ln(3x+1)}\right)^{\ln x}\right) \\ &= \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{\ln(3x+1)\ln x}\right)\end{aligned}$$

Convert base to $e^{?}$

Example

Differentiate $(3x+1)^{\ln x}$, where $3x+1>0$.

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\begin{aligned}
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&= \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{\ln(3x+1)\ln x}\right) \\
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&= \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{u}\right) = \color{red}{\frac{\mathrm{d}}{\mathrm{d}u}\left(e^{u}\right)}\frac{\mathrm{d}u}{\mathrm{d}x} && \Big|\ \text{Set } \ln(3x+1)\ln x = u \\
&= \color{red}{e^{u}}\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln(3x+1)\ln x\right)
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$$= \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{\ln(3x+1)\ln x}\right)$$

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&= (3x+1)^{\ln x}\left(\mathbf{?} \qquad \ln x + \ln(3x+1)\mathbf{?}\right)
\end{aligned}$$

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Differentiate $(3x+1)^{\ln x}$, where $3x+1>0$.

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Differentiate $(3x+1)^{\ln x}$, where $3x+1>0$.

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Theorem

$$\frac{\mathrm{d}}{\mathrm{d}x}\left((a(x))^{b(x)}\right) = (a(x))^{b(x)}\left(\frac{a'(x)}{a(x)}b(x) + \ln(a(x))b'(x)\right), \quad a(x)>0$$

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Differentiate $(3x+1)^{\ln x}$, where $3x+1>0$.

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Differentiate $(3x+1)^{\ln x}$, where $3x+1>0$.

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&= \left(\frac{5}{3(x-1)} + 3\cot x - \frac{e^x}{2(e^x+1)}\right) \frac{(x-1)^{5/3}\sin^3 x}{\sqrt{e^x+1}}.
\end{aligned}$$

Steps in Logarithmic Differentiation

1. Take natural logarithms of both sides of an equation $y = f(x)$.
2. Use the properties of logarithms to simplify.
3. Differentiate implicitly with respect to x.
4. Solve the resulting equation for y'.

Note: If $f(x) < 0$, then we use $\ln|f(x)|$ instead as $\ln(f(x))$ is not defined. We computed the derivative of $\ln|f(x)|$ in the previous lecture.

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Differentiate $y = (3x+1)^{\ln x}$.

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Theorem (The Number e as a Limit)

$$e = \lim_{x\to 0}(1+x)^{\frac{1}{x}} = \lim_{y\to\infty}\left(1+\frac{1}{y}\right)^{y}.$$

Proof.

$\square$

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Proof.

Let $f(x) = \ln x$.

$\square$

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$$e = \lim_{x\to 0}(1+x)^{\frac{1}{x}} = \lim_{y\to\infty}\left(1+\frac{1}{y}\right)^{y}.$$

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