Precalculus
Lecture 21

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Outline

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Combinations of Functions

Two functions f and g can be combined to form new functions $f+g$, $f-g$, $f\cdot g$, and $\frac{f}{g}$:

$$\begin{array}{rcl|l}
(f+g)(x) & = & f(x)+g(x) & \\
(f-g)(x) & = & f(x)-g(x) & \\
(f\cdot g)(x) & = & f(x)\cdot g(x) & \\
\left(\frac{f}{g}\right)(x) & = & \frac{f(x)}{g(x)} & \text{for } g(x)\neq 0 \quad .
\end{array}$$

Let $\mathrm{Dom}(f)$ denote the domain of f. The function $f+g$ is defined only if both f and g are defined, and similarly for the others. Therefore

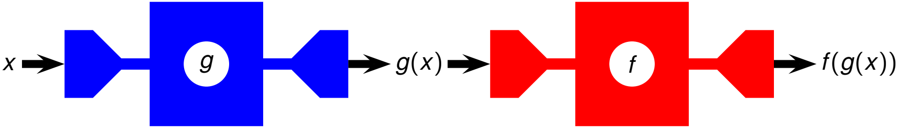
$$\begin{array}{rcl|l}
\mathrm{Dom}(f+g) & = & \mathrm{Dom}(f)\cap\mathrm{Dom}(g) & \cap \text{ stands for} \\
\mathrm{Dom}(f-g) & = & \mathrm{Dom}(f)\cap\mathrm{Dom}(g) & \text{set intersection} \\
\mathrm{Dom}(f\cdot g) & = & \mathrm{Dom}(f)\cap\mathrm{Dom}(g) & \\
\mathrm{Dom}\left(\frac{f}{g}\right) & = & \mathrm{Dom}(f)\cap\mathrm{Dom}(g)\cap\{x|g(x)\neq 0\} & \begin{array}{l}\text{right expr.}\\ \text{stands for set}\\ \text{where } g(x)\neq 0\end{array}
\end{array}$$

Definition (Composition of f and g)

If f and g are two functions, then the composition of f and g is written $f \circ g$ and is defined by the formula

$$(f \circ g)(x) = f(g(x)).$$

Imagine f and g as machines taking some input and producing some output. Then $f \circ g$ corresponds to attaching both machines end-to-end so that the output of g becomes the input of f.



The domain of $f \circ g$ is the set of all numbers x in the domain of g such that $g(x)$ is in the domain of f. If the domain of f is A and the domain of g is B, we write this as

$$\{x \in B |\ g(x) \in A\}.$$

Example

Find $f \circ g, g \circ f, g \circ g$ and their domains, where $f(x) = \sqrt{x}$ and $g(x) = \sqrt{3-x}$.

$$(f \circ g)(x) \quad = \quad f(g(x)) = f\left(\sqrt{3-x}\right) = \sqrt{\sqrt{3-x}} = \sqrt[4]{3-x}$$

Domain:

$$\begin{aligned} 3 - x &\geq 0 \\ -x &\geq -3 \\ x &\leq 3 \\ x &\in (-\infty, 3]. \end{aligned}$$

$$(g \circ f)(x) \quad = \quad g(f(x)) = g(\sqrt{x}) = \sqrt{3 - \sqrt{x}}$$

Domain :

$$\begin{aligned} x &\geq 0 \\ 3 - \sqrt{x} &\geq 0 \\ -\sqrt{x} &\geq -3 \\ \sqrt{x} &\leq 3 \\ x &\leq 9 \\ x &\in [0, 9] \end{aligned}$$

Example

Find $f \circ g, g \circ f, g \circ g$ and their domains, where $f(x) = \sqrt{x}$ and $g(x) = \sqrt{3-x}$.

$$
\begin{aligned}
(g \circ g)(x) &= g(g(x)) = g\left(\sqrt{3-x}\right) = \sqrt{3-\sqrt{3-x}} \\
\text{Domain :} & \\
3 - x &\geq 0 \\
-x &\geq -3 \\
x &\leq 3 \\
3 - \sqrt{3-x} &\geq 0 \\
-\sqrt{3-x} &\geq -3 \\
\sqrt{3-x} &\leq 3 \\
3 - x &\leq 9 \\
-x &\leq 6 \\
x &\geq -6 \\
x &\in [-6, 3].
\end{aligned}
$$

Example

Give simplified f-las for $f \circ g$, $f \circ f$, $g \circ f$, $g \circ g$. Find the implied domains.

$$f(x) = \frac{2x-1}{x+2} \qquad \Big| \quad x \neq -2$$

$$g(x) = \frac{2x+3}{5x-7} \qquad \Big| \quad x \neq \frac{7}{5}$$

$$(f \circ g)(x) = f(g(x)) = f\left(\frac{2x+3}{5x-7}\right) = \frac{2\left(\frac{2x+3}{5x-7}\right) - 1}{\frac{2x+3}{5x-7} + 2}$$

$$= \frac{\frac{2(2x+3)}{5x-7} - \frac{5x-7}{5x-7}}{\frac{2x+3}{5x-7} + \frac{2(5x-7)}{5x-7}} = \frac{\frac{4x+6-(5x-7)}{\cancel{5x-7}}}{\frac{2x+3+(10x-14)}{\cancel{5x-7}}} = \frac{-x+13}{12x-11} \qquad \Big| \quad x \neq \frac{11}{12}, \frac{7}{5}$$

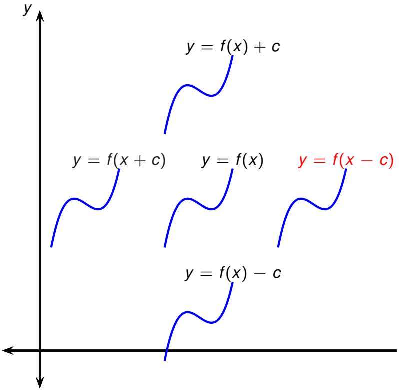
$$(f \circ f)(x) = f(f(x)) = f\left(\frac{2x-1}{x+2}\right) = \frac{2\left(\frac{2x-1}{x+2}\right) - 1}{\frac{2x-1}{x+2} + 2}$$

$$= \frac{3x-4}{4x+3} \qquad \Big| \quad x \neq -2, -\frac{3}{4}$$

$$(g \circ f)(x) = \frac{7x+4}{3x-19} \qquad \Big| \quad x \neq -2, \frac{19}{3}$$

$$(g \circ g)(x) = \frac{19x-15}{-25x+64} \qquad \Big| \quad x \neq \frac{7}{5}, \frac{64}{25}$$

Transformations of Functions

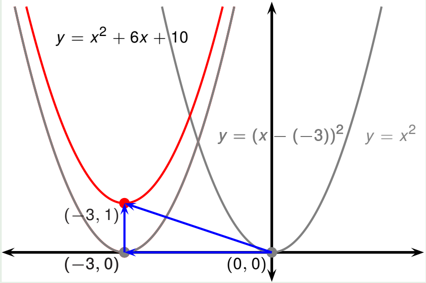


- What happens to the graph if we add/subtract a positive constant c in the equation of a function f?
- What happens if we add or subtract c from x before applying the function f?

$f(x) + c$	Shift the graph of $f(x)$ c units up.
$f(x) - c$	Shift the graph of $f(x)$ c units down.
$f(x - c)$	Shift the graph of $f(x)$ c units right .
$f(x + c)$	Shift the graph of $f(x)$ c units left.

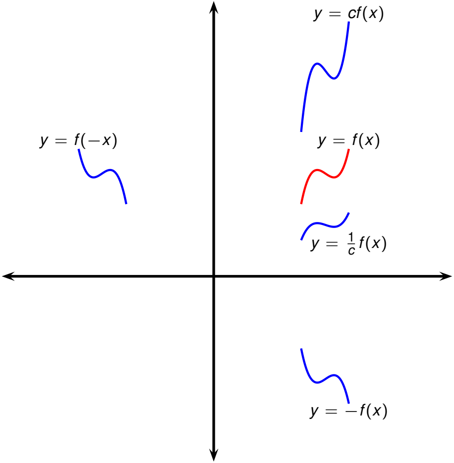
Example

Relative to the graph of $f(x) = x^2$, draw a graph of $f(x) = x^2 + 6x + 10$. Assume the graph of $f(x) = x^2$ given.



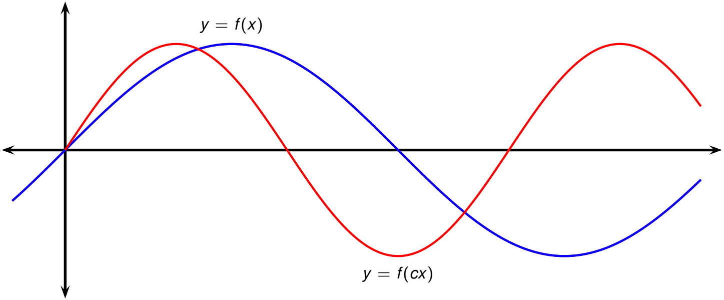
Complete the square:

$$
\begin{aligned}
f(x) &= x^2 + 6x + 10\\
&= \left(x^2 + 6x + 9\right) + 10 - 9\\
&= (x+3)^2 + 1\\
&= (x - (-3))^2 + 1
\end{aligned}
$$



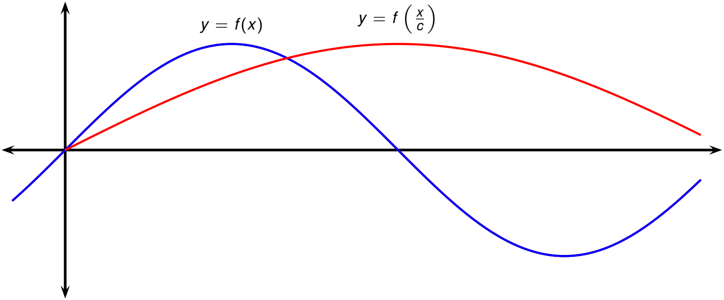
- What happens if we multiply or divide by a constant $c > 1$ in the equation of a function f?
- What happens if we multiply f by -1?
- What happens if we multiply x by -1 before applying f?

$cf(x)$	Stretch the graph of $f(x)$ vertically by a factor of c.
$\frac{1}{c}f(x)$	Compress the graph of $f(x)$ vertically by a factor of c.
$-f(x)$	Reflect the graph of $f(x)$ in the x-axis.
$f(-x)$	Reflect the graph of $f(x)$ in the y-axis.



- What happens if we multiply x by const. $c > 1$ before applying f?
- What happens if we divide x by const. $c > 1$ before applying f?

$f(cx)$	Compress the graph of $f(x)$ horizontally by a factor of c.
$f\left(\frac{1}{c}x\right)$	Stretch the graph of $f(x)$ horizontally by a factor of c.



- What happens if we multiply x by const. $c > 1$ before applying f?
- What happens if we divide x by const. $c > 1$ before applying f?

$f(cx)$	Compress the graph of $f(x)$ horizontally by a factor of c.
$f\left(\frac{1}{c}x\right)$	Stretch the graph of $f(x)$ horizontally by a factor of c.

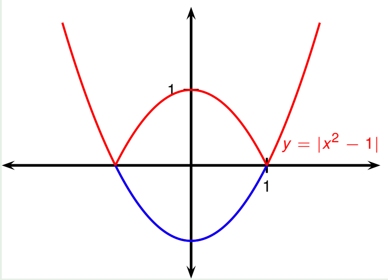
What happens when we take the absolute value of a function?

$$|f(x)| = \begin{cases} f(x) & \text{if} \quad f(x) \geq 0 \\ -f(x) & \text{if} \quad f(x) < 0 \end{cases}$$

This tells us how to draw the graph of $y = |f(x)|$: the part of the graph above the x-axis remains the same; the part below the x-axis is reflected about the x-axis.

Example

Draw the graph of the function $f(x) = |x^2 - 1|$.



- Draw the graph of $f(x) = x^2 - 1$.
- Identify the part(s) below the x-axis.
- Flip those parts over the x-axis.