Calculus II
Homework on Lecture 15

1. List the first 4 elements of the sequence.

   (a) $a_n = \dfrac{(-1)^n}{n}$.

   answer: $(a_1, a_2, a_3, a_4, a_5) = \left(-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}\right)$

   (b) $a_n = \dfrac{1}{n!}$.

   answer: $(a_1, a_2, a_3, a_4, a_5) = \left(1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}\right)$

   (c) $a_n = \cos(\pi n)$.

   answer: $(a_1, a_2, a_3, a_4, a_5) = (-1, 1, -1, 1)$

   (d) $a_n = \dfrac{(-1)^n}{2n+1}$.

   answer: $(a_1, a_2, a_3, a_4, a_5) = \left(-\frac{1}{3}, \frac{1}{5}, -\frac{1}{7}, \frac{1}{9}\right)$

   (e) $a_n = \dfrac{\sqrt{5}}{5}\left(\left(\dfrac{1+\sqrt{5}}{2}\right)^n - \left(\dfrac{1-\sqrt{5}}{2}\right)^n\right)$

   answer: $(a_1, a_2, a_3, a_4, a_5) = (1, 1, 2, 3)$

2. List the first 5 elements of the sequence.

   (a) $a_{n+1} = \dfrac{1}{2}\left(a_n + \dfrac{3}{a_n}\right)$, $a_1 = 1$.

   (b) $a_n = a_{n-1} + a_{n-2}$, $a_1 = 1$, $a_2 = 1$.

   (c) $a_n = \dfrac{\left(\frac{1}{2} - n\right)}{n} a_{n-1}$, $a_0 = 1$.

   (d) $a_n = a_{n-1} + 2n + 1$, $a_0 = 1$.

   (e) $a_n := \dfrac{1}{n} a_{n-1}$, $a_1 = 1$.

3. Give a simple sequence formula that matches the pattern below.

   (a) $\left(1, \dfrac{1}{3}, \dfrac{1}{5}, \dfrac{1}{7}, \dfrac{1}{9}, \dots\right)$.

   answer: $a_n = \frac{1}{2n-1}$

   (b) $\left(-1, \dfrac{1}{5}, -\dfrac{1}{25}, \dfrac{1}{125}, -\dfrac{1}{625}, \dfrac{1}{3125} \dots\right)$

   answer: $a_n = -\left(-\frac{1}{5}\right)^{n-1}$

   (c) $\left(-5, 2, -\dfrac{4}{5}, \dfrac{8}{25}, -\dfrac{16}{125}, \dfrac{32}{625}, \dots\right)$

   answer: $a_n = -5\left(-\frac{2}{5}\right)^{n-1}$

   (d) $(4, 7, 10, 13, 16, 19, \dots)$

   answer: $a_n = 3n + 1$

   (e) $\left(-2, \frac{3}{4}, -\frac{4}{9}, \frac{5}{16}, -\frac{6}{25}, \frac{7}{36}, \dots\right)$

   answer: $a_n = (-1)^n \left(\frac{n+1}{n^2}\right)$

   (f) $(0, -1, 0, 1, 0, -1, 0, 1, 0, -1, 0, 1, \dots)$

   answer: $a_n = \cos\left(n\frac{\pi}{2}\right)$

4. Determine if the sequence is convergent or divergent. If convergent, find the limit of the sequence.

   (a) $a_n = n$.

   answer: divergent

   (b) $a_n = 2^n$.

   answer: divergent

   (c) $a_n = 1.0001^n$.

   answer: divergent

   (d) $a_n = 0.999999^n$.

   answer: convergent, $\lim\limits_{n\to\infty} a_n = 0$

   (e) $a_n = n - \sqrt{n+1}\sqrt{n+2}$

   answer: convergent, $\lim_{n\to\infty} a_n = -\frac{3}{2}$

   (f) $a_n = \dfrac{\ln n}{n}$.

   answer: convergent, $\lim_{n\to\infty} a_n = 0$

   (g) $a_n = \dfrac{\ln n}{\sqrt[10]{n}}$.

   answer: convergent, $\lim_{n\to\infty} a_n = 0$

   (h) $a_n = \dfrac{1}{n}$.

   answer: convergent, $\lim_{n\to\infty} a_n = 0$

   (i) $a_n = \dfrac{1}{n!}$.

   answer: convergent, $\lim_{n\to\infty} a_n = 0$

   (j) $a_n = \dfrac{n^n}{n!}$.

   answer: divergent

   (k) $a_n = \cos n$.

   answer: divergent

   (l) $a_n = \cos\left(\dfrac{1}{n}\right)$

answer: convergent, $\lim_{n\to\infty} a_n = 1$

(m) $a_n = \left(\frac{n+1}{n}\right)^n$.

answer: convergent, $\lim_{n\to\infty} a_n = e$

(n) $a_n = \left(\frac{2n+1}{n}\right)^n$.

answer: divergent

(o) $a_n = \left(\frac{n+1}{n}\right)^{2n}$.

answer: convergent, $\lim_{n\to\infty} a_n = e^2$

(p) $a_n = \left(\frac{n+1}{2n}\right)^n$.

answer: convergent, $\lim_{n\to\infty} a_n = 0$

Solution. 4m.

Consider $f(x) = \left(\frac{x+1}{x}\right)^x$, where x is a positive number. We will now show that $\lim\limits_{x\to\infty} f(x)$ exists. Since the limit is of the form 1^∞, we will start by finding the limit of the logarithm $\ln(f(x))$. We will then exponentiate that limit to find the limit of $f(x)$.

$$
\begin{array}{rcll}
\lim\limits_{x\to\infty} \ln\left(\left(\frac{x+1}{x}\right)^x\right) & = & \lim\limits_{x\to\infty} x \ln\left(\frac{x+1}{x}\right) & \\
& = & \lim\limits_{x\to\infty} \frac{\ln\left(\frac{x+1}{x}\right)}{\frac{1}{x}} & \\
& = & \lim\limits_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} & \left| \begin{array}{l}\text{Form ``}\frac{0}{0}\text{''} \\ \text{L'Hospital rule}\end{array}\right. \\
& = & \lim\limits_{x\to\infty} \frac{\frac{1}{1+\frac{1}{x}}\left(1+\frac{1}{x}\right)'}{-\frac{1}{x^2}} & \\
& = & \lim\limits_{x\to\infty} \frac{\frac{1}{\left(1+\frac{1}{x}\right)}\cancel{\left(-\frac{1}{x^2}\right)}}{\cancel{\left(-\frac{1}{x^2}\right)}} & \\
& = & \lim\limits_{x\to\infty} \frac{1}{1+\frac{1}{x}} & \\
& = & 1 & \\
\lim\limits_{x\to\infty} \left(\frac{x+1}{x}\right)^x & = & \lim\limits_{x\to\infty} e^{\ln\left(\left(\frac{x+1}{x}\right)^x\right)} & \left| \text{The exponent is continuous} \right. \\
& = & e^{\lim\limits_{x\to\infty} \ln\left(\left(\frac{x+1}{x}\right)^x\right)} & \\
& = & e^1 & \left| \text{use preceding} \right. \\
& = & e & .
\end{array}
$$

Therefore $\lim\limits_{\substack{n\to\infty \\ n-\text{ integer}}} \left(\frac{n+1}{n}\right)^n = \lim\limits_{\substack{x\to\infty \\ x-\text{ real}}} \left(\frac{x+1}{x}\right)^x = e$ and the sequence converges (to e).

Solution. 4n.

This problem can be solved in fashion similar to Problem 4m. However there is a much simpler solution:

$$
\begin{array}{rcll}
\frac{2n+1}{n} & \geq & 2 & \left| \begin{array}{l} \text{for } n > 0 \\ \text{limits respect non-strict inequalities} \\ \lim\limits_{n\to\infty} 2^n \text{ computed in Problem 4b} \end{array} \right. \\
\lim\limits_{n\to\infty} \left(\frac{2n+1}{n}\right)^n & \geq & \lim\limits_{n\to\infty} 2^n & \\
\lim\limits_{n\to\infty} \left(\frac{2n+1}{n}\right)^n & = & \infty & .
\end{array}
$$