Calculus III
Lecture 12

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https://github.com/tmilev/freecalc

2020

Outline

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Minima, Maxima

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How do we find points of extreme?

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The converse is not true: if $f_x(P_0) = f_y(P_0) = 0$, then P_0 is not necessarily a point of extreme.

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Strategy for finding extreme points:

- Check the *critical points* of f:
  - Points P_0 for which $f_x(P_0)$ or $f_y(P_0)$ does not exist;
  - Points P_0 for which $f_x(P_0) = f_y(P_0) = 0$.
- Check boundary points included in the domain.

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- It remains to find the points (x,y) for which $f_x(x,y) = f_y(x,y) = 0$.

$$\left| \begin{array}{ll} f_x(x,y) & = 0 \\ f_y(x,y) & = 0 \end{array} \right. \iff \left| \begin{array}{ll} 4x^3 - 4y & = 0 \\ 4y^3 - 4x & = 0 \end{array} \right. \iff \left| \begin{array}{ll} x^3 & = y \\ y^3 & = x \end{array} \right.$$

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Typical mistake: $x^9 = x \iff x^8 = 1$.

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- If $D(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) > 0$, then (x_0, y_0) is a local minimum. Example: crit. pt. $(0, 0)$ for $f(x, y) = x^2 + y^2$.
- If $D(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) < 0$, then (x_0, y_0) is a local maximum. Example: crit. pt. $(0, 0)$ for $f(x, y) = -x^2 - y^2$.
- If $D(x_0, y_0) < 0$, then (x_0, y_0) is neither a minimum nor a maximum. Such points are called *saddle points*. Example: crit. pt. $(0, 0)$ for $f(x, y) = x^2 - y^2$.

Second Derivative Test

When is an interior critical point a pt. of min/max? Define the *Hessian matrix* H of f as follows. Denote by D the determinant of H.

$$H = \left(\begin{array}{cc} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{array}\right) \qquad D = \det H = f_{xx} f_{yy} - f_{xy}^2$$

Test: Let $P(x_0, y_0)$ be an interior critical point of f and suppose that f has continuous second order derivatives around P.

- If $D(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) > 0$, then (x_0, y_0) is a local minimum. Example: crit. pt. $(0,0)$ for $f(x,y) = x^2 + y^2$.
- If $D(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) < 0$, then (x_0, y_0) is a local maximum. Example: crit. pt. $(0,0)$ for $f(x,y) = -x^2 - y^2$.
- If $D(x_0, y_0) < 0$, then (x_0, y_0) is neither a minimum nor a maximum. Such points are called *saddle points*. Example: crit. pt. $(0,0)$ for $f(x,y) = x^2 - y^2$.
- If $D(x_0, y_0) = 0$, then the test is inconclusive.

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- If $D(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) > 0$, then (x_0, y_0) is a local minimum. Example: crit. pt. $(0,0)$ for $f(x,y) = x^2 + y^2$.
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Second Derivative Test

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Test: Let $P(x_0, y_0)$ be an interior critical point of f and suppose that f has continuous second order derivatives around P.

- If $D(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) > 0$, then (x_0, y_0) is a local minimum. Example: crit. pt. $(0, 0)$ for $f(x, y) = x^2 + y^2$.
- If $D(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) < 0$, then (x_0, y_0) is a local maximum. Example: crit. pt. $(0, 0)$ for $f(x, y) = -x^2 - y^2$.
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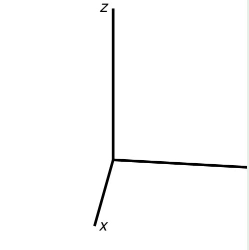
- If $D(x_0, y_0) = 0$, then the test is inconclusive. Examples: $x^4 + y^4$, $-x^4 - y^4$, $x^4 - y^4$.

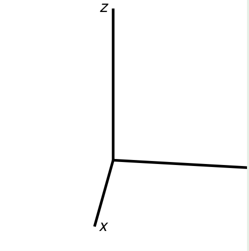
Example (Finding maxima, minima)



Find the local and global maxima and minima of $f(x, y) = x^4 + y^4 - 4xy$.

$$\begin{aligned} f_{xx} &= \\ f_{xy} &= \\ f_{yy} &= \\ D = f_{xx}f_{yy} - f_{xy}^2 &= \end{aligned}$$

Example (Finding maxima, minima)



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\end{aligned}
$$

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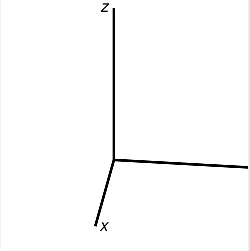

Find the local and global maxima and minima of $f(x, y) = x^4 + y^4 - 4xy$.

$$\begin{aligned} f_{xx} &= 12x^2 \\ f_{xy} &= \\ f_{yy} &= \\ D = f_{xx}f_{yy} - f_{xy}^2 &= \end{aligned}$$

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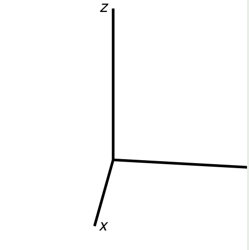

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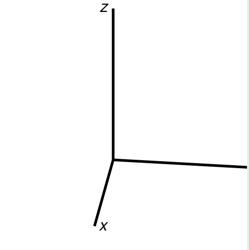

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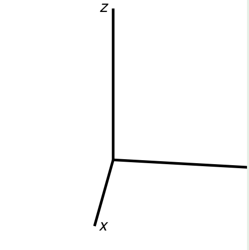

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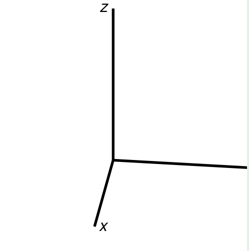

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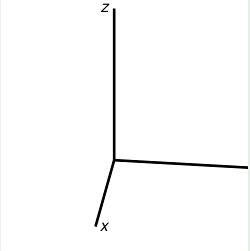

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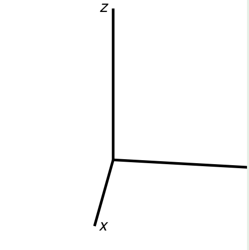

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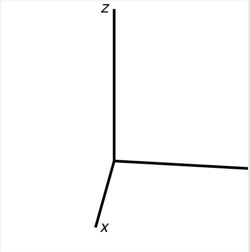

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The critical points were previously computed.

(x_0, y_0)	f_{xx}	f_{yy}	f_{xy}	D	Extremum ?
$(0,0)$					
$(1,1)$					
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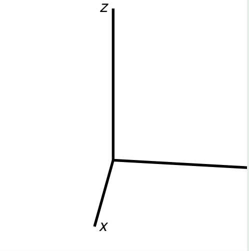

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$(1, 1)$					
$(-1, -1)$					

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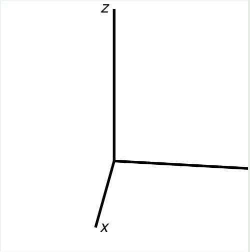

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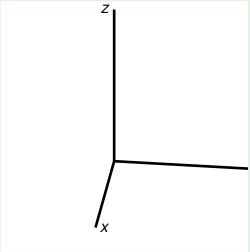

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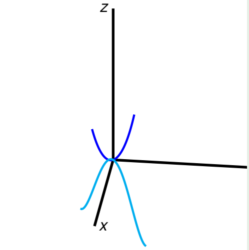

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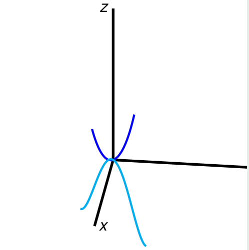

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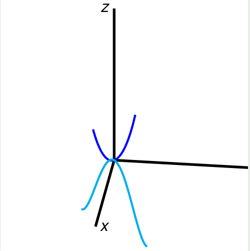

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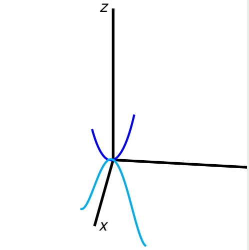

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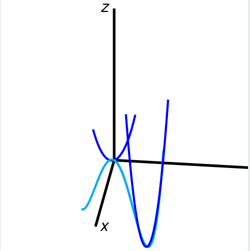

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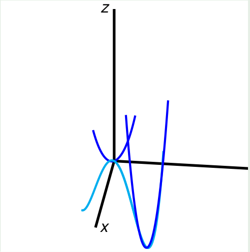

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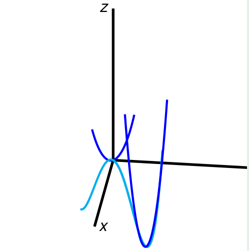

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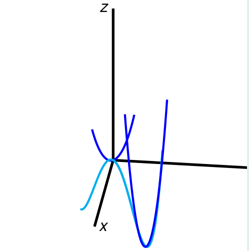

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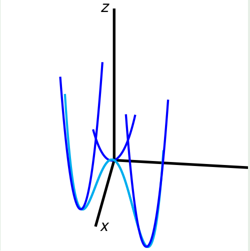

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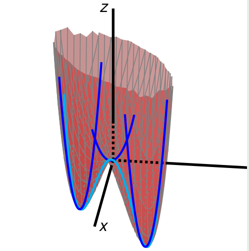

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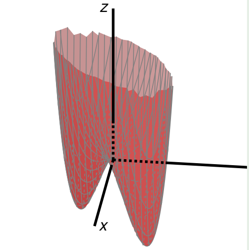

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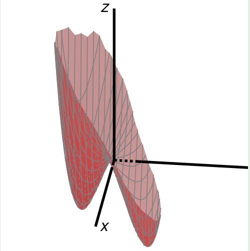

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$(0, 0)$	0	0	-4	$-16 < 0$	Saddle point
$(1, 1)$	12	12	-4	$128 > 0$	Local min
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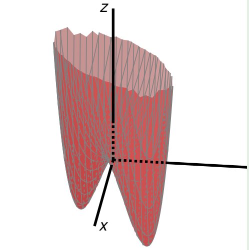
Example (Finding maxima, minima)



Find the local and global maxima and minima of $f(x, y) = x^4 + y^4 - 4xy$.

$$\begin{aligned} f_{xx} &= 12x^2 \\ f_{xy} &= -4 \\ f_{yy} &= 12y^2 \\ D = f_{xx}f_{yy} - f_{xy}^2 &= 144x^2y^2 - 16. \end{aligned}$$

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In this case it turns out that the two local minimum points are actually global minimum points, because
$f(x, y) = x^4 + y^4 - 4xy = (x^2 - 1)^2 + (y^2 - 1)^2 + 2(x - y)^2 - 2 \geq -2$.

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Extreme Value Theorem

Global extreme points are guaranteed to exist if:

- $f\colon D \to \mathbb{R}$ is continuous, and
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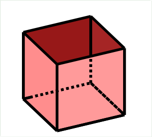

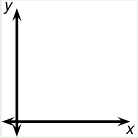

Find the maximal volume of a box with no lid whose surface area is $10m^2$.

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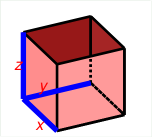

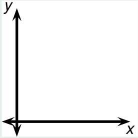

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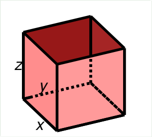

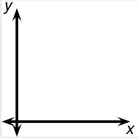

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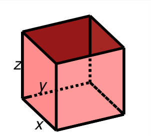

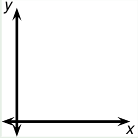

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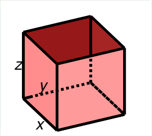

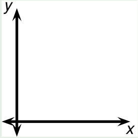

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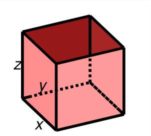

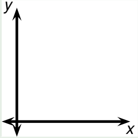

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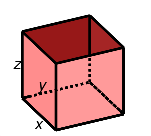

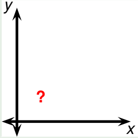

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Let dimensions of box be x, y, z. We seek to maximize $V = xyz$.

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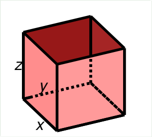

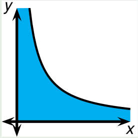

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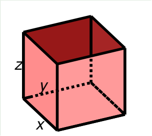

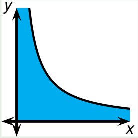

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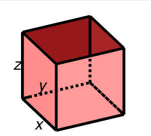

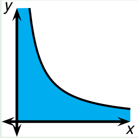

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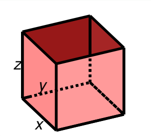

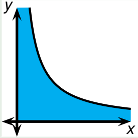

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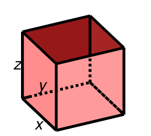

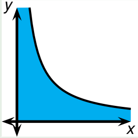

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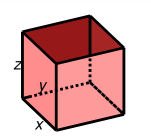

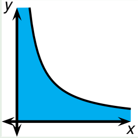

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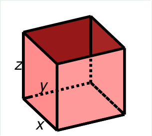

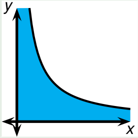

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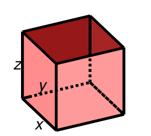

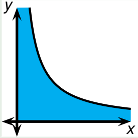

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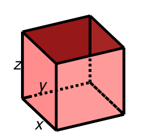

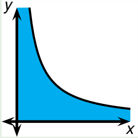

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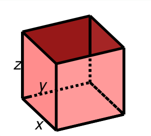

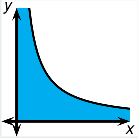

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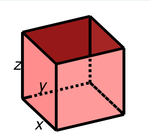

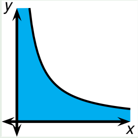

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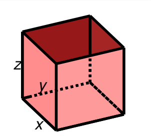

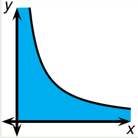

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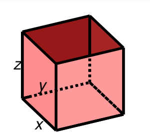

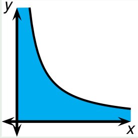

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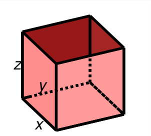

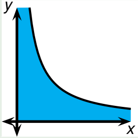

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$\square$

Let S be a surface that has implicit equation $F(x, y, z) = 0$ for some differentiable function F. Let P be a point on the surface.

Theorem

Suppose $\nabla F(P) \neq \mathbf{0}$. Then $\nabla F(P)$ is perpendicular to the tangent vector at P of every differentiable curve lying in S passing through P.

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\mathbf{r}'(t) \cdot (F_x, F_y, F_z)_{|x=x(t), y=y(t), z=z(t)} & = & 0 & \\
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Definition (Tangent plane to level surface)

Suppose $\nabla F(P)\neq 0$. We define the tangent plane to the surface S at P to be the plane passing through P with normal vector $\nabla F(P)$.

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Find the maximum of a function $G(x, y, z)$ subject to the variable restriction $F(x, y, z) = 0$.

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- Therefore ∇G is $\perp$ to tangent at P of every curve in S through P.
- Therefore $\nabla F(P)$ and $\nabla G(P)$ are parallel, i.e., there exists λ s.t.: $(\nabla G)(P)=\lambda(\nabla F)(P)$.

Example

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- Find critical points in the interior of the disk;
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Since f is differentiable everywhere, the interior extreme points are among the solutions of the system

$$\left\{\begin{array}{rl} f_x(x, y) = & 0 \\ f_y(x, y) = & 0 \end{array}\right. \iff \left\{\begin{array}{rl} y = & 0 \\ x = & 0 \end{array}\right.$$

Find the maximum and the minimum values of $f(x, y) = xy$ on the region $D = \{(x, y) \mid |x| + |y| \leqslant 2\}$.
Extreme points on the boundary: check each of the four sides.
For the segment joining $(2, 0)$ with $(0, 2)$ we get:

$$\begin{aligned} &\text{Find min/max of } f(x, y) = xy \\ &\qquad \text{Subject to } g(x, y) = x + y - 2 = 0 \end{aligned}$$

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$$F(x, y, \lambda) = f(x, y) - \lambda g(x, y) = xy - \lambda(x + y - 2)$$

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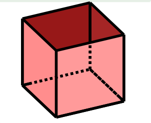
Three more critical points on the boundary: $(-1, 1)$, $(-1, -1)$, $(1, -1)$.
Compare the values at all points:

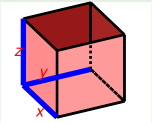
- the global maximum is 1, attained at $(1, 1)$ and $(-1, -1)$;
- the global minimum is -1, attained at $(1, -1)$ and $(-1, 1)$;
- the critical point $(0, 0)$ is a saddle point.

Example



Find the maximal volume of a box with no lid whose surface area is $10m^2$.

Example



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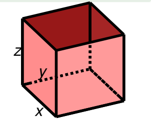

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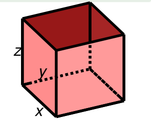

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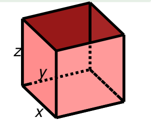

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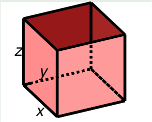

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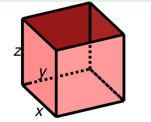

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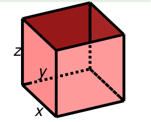

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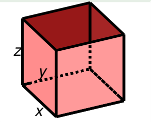

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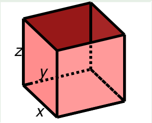

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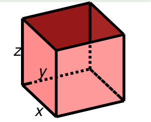

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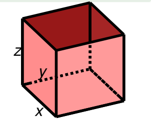

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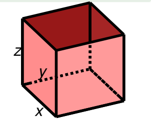

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$$\left| \begin{array}{rcl} yz &=& \lambda(2z + y) \\ xz &=& \lambda(2z + x) \\ xy &=& \lambda(2x + 2y) \\ 10 &=& xy + 2(zx + yz) \end{array} \right.$$

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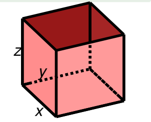

Find the maximal volume of a box with no lid whose surface area is $10m^2$.
Let the three dimensions of the box be x, y, z.

We seek to maximize $V = xyz$ under the restriction

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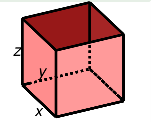

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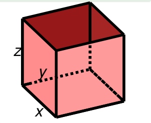

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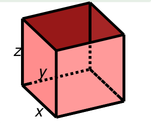

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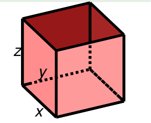

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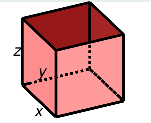

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From the first two equalities we get $2\lambda xz + \lambda xy = 2\lambda yz + \lambda xy$

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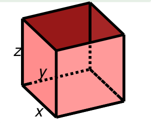

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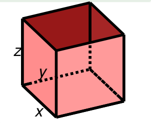

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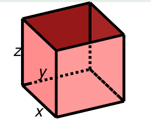

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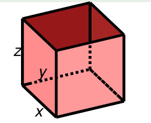
$$\lambda x z = \lambda y z \quad .$$

We have that $\lambda \neq 0, z \neq 0$ (else the volume would be zero). Therefore

$$x = y$$

Example



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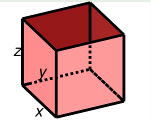

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We substitute $y = x$ in the third equality to get $x^2 = 4\lambda x$

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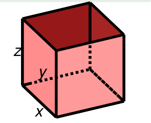

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We substitute $y = x$ in the third equality to get $x^2 = 4\lambda x$ and since $x \neq 0$ we get $\lambda = \frac{x}{4}$.

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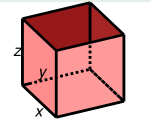

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We substitute $y = x$ in the third equality to get $x^2 = 4\lambda x$ and since $x \neq 0$ we get ${\color{red}\lambda = \frac{x}{4}}$. We substitute ${\color{red}\lambda = \frac{x}{4}}$ in the original second equality to get ${\color{red}zx = \frac{x}{4}(2z + x)}$.

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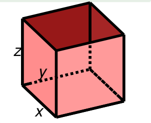

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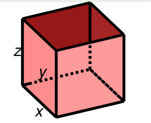

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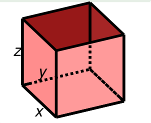

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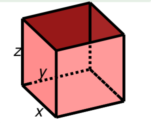

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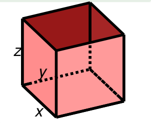

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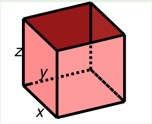

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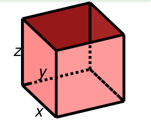
Thus $x = \frac{\sqrt{30}}{3}$

Example



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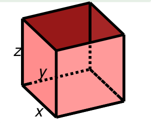

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Thus $x = \frac{\sqrt{30}}{3}$ and therefore $y = \frac{\sqrt{30}}{3}$, $z = \frac{\sqrt{30}}{6}$, our final answer.

Multiple Constraints

$$\begin{aligned} \text{Find } & \min / \max f(x, y, z) \\ \text{Subject to } & g(x, y, z) = 0 \\ & h(x, y, z) = 0 \end{aligned}$$

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Each constraint defines a surface $\implies$ their intersection defines a curve.

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Each constraint defines a surface $\implies$ their intersection defines a curve.
Condition: $(\nabla g)_P (\nabla h)_P$ are non-collinear for each intersection point P
The level surface of f through a point of extreme P_0 is tangent to the constraint curve, so $(\nabla f)(P_0)$ is perpendicular to the curve at P_0.
Constraint curve included in both surfaces $\implies$
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The Lagrange function is in this case

Example

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- Constraint set is bounded and closed, function f is continuous $\implies$ f attains its extreme on the constraint $\implies$ $(1, -\sqrt{5/2}, \sqrt{5/2})$ corresponds to an absolute minimum and $(1, \sqrt{5/2}, -\sqrt{5/2})$ corresponds to an absolute maximum.
- The minimum value is $f(1, -\sqrt{5/2}, \sqrt{5/2}) = 1 - 2\sqrt{5/2}$ and the maximal value is $f(1, \sqrt{5/2}, -\sqrt{5/2}) = 1 + 2\sqrt{5/2}$