Calculus II
Homework on Lecture 21

1. Plot the number z on the complex plane (you may use one drawing only for all the numbers). Find all real numbers φ and ρ for which $z = e^{\rho + i\varphi}$. Your answer may contain expressions of the form $\arcsin x$, $\arccos x$, $\arctan x$, $\ln x$, only if x is a real number.

   (a) $z = 1 + i\sqrt{3}$.

   answer: $z = e^{\ln 2 + i\left(\frac{\pi}{3} + 2k\pi\right)}, k \in \mathbb{Z}$

   (b) $z = -2 - 3i$.

   answer: $z = e^{\frac{1}{2}\ln(13) + i\left(\arctan\left(\frac{3}{2}\right) + 2k\pi\right)}, k \in \mathbb{Z}$

   (c) $z = 1 - i\sqrt{3}$.

   answer: $z = e^{\ln 2 + i\left(-\frac{\pi}{3} + 2k\pi\right)}, k \in \mathbb{Z}$

   (d) $z = 1 + i$.

   answer: $z = e^{\frac{1}{2}\ln 2 + i\left(\frac{\pi}{4} + 2k\pi\right)}, k \in \mathbb{Z}$

   (e) $z = -1 - i$.

   answer: $z = e^{\frac{1}{2}\ln 2 + i\left(\frac{5\pi}{4} + 2k\pi\right)}, k \in \mathbb{Z}$

   (f) $z = \frac{\sqrt{3} + i}{4}$.

   answer: $z = e^{-\ln 2 + i\left(\frac{\pi}{6} + 2k\pi\right)}, k \in \mathbb{Z}$

   (g) $z = -i$.

   answer: $z = e^{i\left(-\frac{\pi}{2} + 2k\pi\right)}, k \in \mathbb{Z}$

   (h) $z = 3 + 4i$.

   answer: $z = e^{\ln 5 + i\left(\arctan\left(\frac{4}{3}\right) + 2k\pi\right)}, k \in \mathbb{Z}$

Solution. 1.a.

Solution I. We have that

$$|z| = \sqrt{z\bar{z}} = \sqrt{\left(1 + i\sqrt{3}\right)\left(1 - i\sqrt{3}\right)} = \sqrt{1^2 + \sqrt{3}^2} = \sqrt{4} = 2 \quad .$$

Recall that $e^{\rho + i\varphi} = e^{\rho}(\cos\varphi + i\sin\varphi)$ and therefore

$$\begin{aligned}
\cos\varphi &= \frac{|z|\cos\varphi}{|z|} = \frac{\operatorname{Re} z}{|z|} = \frac{1}{2}\\
\sin\varphi &= \frac{|z|\sin\varphi}{|z|} = \frac{\operatorname{Im} z}{|z|} = \frac{\sqrt{3}}{2}\\
\tan\varphi &= \frac{\sin\varphi}{\cos\varphi} = \frac{\sqrt{3}}{3} \quad .
\end{aligned}$$

Therefore φ is of the form $\varphi = \arctan\left(\frac{\sqrt{3}}{3}\right) = \frac{\pi}{3} + k\pi$. However φ cannot be of the form $\frac{\pi}{3} + (2k+1)\pi$ because $\cos\left(\frac{\pi}{3} + (2k+1)\pi\right) = -\frac{1}{2}$. On the other hand, $\sin\left(\frac{\pi}{3} + 2k\pi\right) = \frac{\sqrt{3}}{2}$ and $\cos\left(\frac{\pi}{3} + 2k\pi\right) = \frac{1}{2}$). Therefore

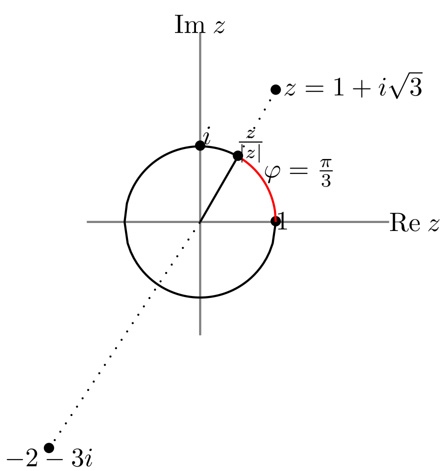
$$\varphi = \frac{\pi}{3} + 2k\pi, \qquad \text{for all } k \in \mathbb{Z}$$

(Recall that $\mathbb{Z}$ denotes the integers).

As studied in class $e^{\rho} = |z| = 2$, and therefore $\rho = \ln(e^{\rho}) = \ln|z| = \ln 2$. Therefore we get the answer

$$1 + i\sqrt{3} = e^{\ln 2 + i\left(\frac{\pi}{3} + 2k\pi\right)}$$

for all $k \in \mathbb{Z}$. To finish the task we need to plot the number z.



Solution II. We draw the number z as above. We compute that $\sin\varphi = \frac{\text{Im } z}{|z|} = \frac{\sqrt{3}}{2}$, $\cos\varphi = \frac{\text{Re } z}{|z|} = \frac{1}{2}$. Therefore we have that

$$1 + i\sqrt{3} = e^{\ln|1+i\sqrt{3}| + i\left(\frac{\pi}{3} + 2k\pi\right)} = e^{\ln 2 + i\left(\frac{\pi}{3} + 2k\pi\right)} \quad .$$

Solution. 1.b.
We draw the number as indicated on the figure. We compute that $\sin\varphi = -\frac{3}{\sqrt{13}}$, $\cos\varphi = -\frac{2}{\sqrt{13}}$, $\tan\varphi = \frac{3}{2}$. By the convention of our course, $\arctan\varphi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Therefore $\varphi = \left(\arctan\left(\frac{3}{2}\right) + \pi\right) + 2k\pi$ for all $k \in \mathbb{Z}$. Finally, we get

$$\begin{array}{rcl} -2 - 3i & = & e^{\ln|-2-3i| + i\left(\left(\arctan\left(\frac{3}{2}\right) + \pi\right) + 2k\pi\right)} = e^{\ln\sqrt{13} + i\left(\left(\arctan\left(\frac{3}{2}\right) + \pi\right) + 2k\pi\right)} \\ & = & e^{\frac{1}{2}\ln 13 + i\left(\left(\arctan\left(\frac{3}{2}\right) + \pi\right) + 2k\pi\right)} \quad . \end{array}$$

2. Carry out the operations. For some of the problems you may want to review the Newton Binomial formula.

   (a) $(5 + 3i)^2$.
   answer: $30i + 16$

   (b) $\dfrac{5 + 3i}{2 - 3i}$.
   answer: $\frac{21}{13}i + \frac{1}{13}$

   (c) $(5 + 3i)^{-2}$.
   answer: $-\frac{15}{578}i + \frac{4}{289}$

   (d) $(1 + i)^3$.
   answer: $2i - 2$

   (e) $(1 + i)^4$.
   answer: -4

   (f) $(1 + i)^5$.
   answer: $-4i - 4$

   (g) $(1 + i)^{-5}$.
   answer: $\frac{1}{8}i - \frac{1}{8}$

Solution. 2.f. By the Newton Binomial formula, we have that

$$(1 + i)^5 = 1 + 5i + 10i^2 + 10i^3 + 5i^4 + i^5 = 1 - 10 + 5 + i(5 - 10 + 1) = -4 - 4i.$$

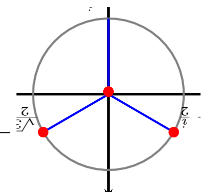
Solution. 2.g. Using the preceding example, we have that

$$(1 + i)^{-5} = \frac{1}{(1 + i)^5} = \frac{1}{-4 - 4i} = \frac{-4 + 4i}{(-4 - 4i)(-4 + 4i)} = \frac{-4 + 4i}{32} = -\frac{1}{8} + \frac{1}{8}i \quad .$$

3. Find all complex solutions of the equation. The answer key has not been proofread. Use with caution.

(a) $z^3 = i$.

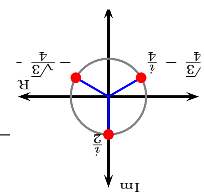
answer:



polar form	value
$\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)$	$-\frac{\sqrt{3}}{2} + \frac{i}{2}$
$\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)$	$\frac{\sqrt{3}}{2} + \frac{i}{2}$
$\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)$	$-i$

(b) $z^3 = -\dfrac{i}{8}$.

answer:



polar form	value
$\frac{1}{2}\left(\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)\right)$	$-\frac{\sqrt{3}}{4} - \frac{i}{4}$
$\frac{1}{2}\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$	$\frac{\sqrt{3}}{4} - \frac{i}{4}$
$\frac{1}{2}\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right)$	$\frac{i}{2}$

(c) $z^4 = -16$.

answer: $\pm\sqrt{2} \pm \sqrt{2}i$ (in all four combinations).

(d) $z^3 = -27$.

answer: $\frac{3}{2} + \frac{3\sqrt{3}}{2}i,\ \frac{3}{2} - \frac{3\sqrt{3}}{2}i,\ -3$.

(e) $z^8 = 1$.

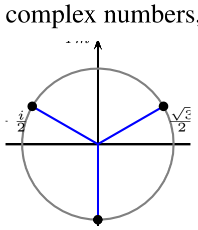
answer: $\pm\frac{\sqrt{2}}{2} \pm \frac{\sqrt{2}}{2}i$ (all four combinations), $\pm i$, ± 1 (total 8 values).

Solution. 3.a. Let $z = |z|(\cos\theta + i\sin\theta)$ be the polar form of $|z|$ for which $\theta \in (-\pi, \pi]$. We have $|z|^3 = |i| = 1$. Therefore $|z| = 1$.

We can write i in polar form as $i = \cos\left(\dfrac{\pi}{2}\right) + i\sin\left(\dfrac{\pi}{2}\right)$. Therefore

$$\begin{array}{rcll}
z^3 &=& i & \text{use de Moivre's formula} \\
|z|^3\left(\cos(3\theta) + i\sin(3\theta)\right) &=& \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) & \text{use } |z| = 1 \\
 & & & \quad\text{when sines and cosines} \\
\cos(3\theta) + i\sin(3\theta) &=& \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) & \quad\text{coincide the angles differ} \\
 & & & \quad\text{by even multiple of } \pi \\
3\theta &=& \frac{\pi}{2} + 2k\pi, & k - \text{integer} \\
\theta &=& \frac{\pi}{6} + k\frac{2\pi}{3} & \theta \in (-\pi, \pi] \Rightarrow k = -1, 0, \text{ or } 1 \\
\theta &=& -\frac{\pi}{2}, \frac{\pi}{6}, \text{ or } \frac{5\pi}{6} & .
\end{array}$$

To find out the values of z in non-polar form, we simply plot the numbers $z = (\cos\theta + i\sin\theta)$. The three complex solutions lie on a circle of radius 1; the numbers form an equilateral triangle, as shown on the picture. To find the actual values for these complex numbers, we use known values of the trigonometric functions. Our final answer is as follows.



polar form	value
$\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)$	$-\frac{\sqrt{3}}{2} + \frac{i}{2}$
$\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)$	$\frac{\sqrt{3}}{2} + \frac{i}{2}$
$\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)$	$-i$

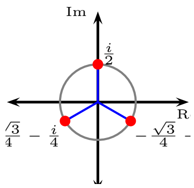
Solution. 3.b

Let $z = |z|(\cos\theta + i\sin\theta)$ be the polar form of $|z|$ for which $\theta \in (-\pi, \pi]$. We have $|z|^3 = \left|\frac{i}{8}\right| = \frac{1}{8}$. Since $|z|$ is a positive real number it follows that $|z| = \sqrt[3]{\dfrac{1}{8}} = \dfrac{1}{2}$.

We can write $-\frac{i}{8}$ in polar form as $-\frac{i}{8} = \frac{1}{8}\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right)$. Therefore

$$\begin{array}{rcl|l}
z^3 &=& \frac{-i}{8} & \text{use de Moivre's formula} \\
|z|^3\left(\cos(3\theta) + i\sin(3\theta)\right) &=& \frac{1}{8}\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right) & \text{use } |z| = \frac{1}{2} \\
& & & \quad\text{when sines and cosines} \\
\cos(3\theta) + i\sin(3\theta) &=& \cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right) & \quad\text{coincide the angles differ} \\
& & & \quad\text{by even multiple of } \pi \\
3\theta &=& -\frac{\pi}{2} + 2k\pi, & k - \text{integer} \\
\theta &=& -\frac{\pi}{6} + k\frac{2\pi}{3} & \theta \in (-\pi, \pi] \Rightarrow k = -1, 0, \text{ or } 1 \\
\theta &=& -\frac{5\pi}{6}, -\frac{\pi}{6}, \text{ or } \frac{\pi}{2} \quad .
\end{array}$$

To find out the values of z in non-polar form, we simply plot the numbers $z = \frac{1}{2}(\cos\theta + i\sin\theta)$. The three complex solutions lie on a circle of radius $\frac{1}{2}$; the numbers form an equilateral triangle, as shown on the picture. To find the actual values for these complex numbers, we use known values of the trigonometric functions. Our final answer is as follows.



polar form	value
$\frac{1}{2}\left(\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)\right)$	$-\frac{\sqrt{3}}{4} - \frac{i}{4}$
$\frac{1}{2}\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$	$\frac{\sqrt{3}}{4} - \frac{i}{4}$
$\frac{1}{2}\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right)$	$\frac{i}{2}$

4. Express the number in polar form and compute the indicated power. The answer key has not been proofread, use with caution.

   (a) $z = \sqrt{3} + i$, find z^3.

   answer: $z = \sqrt{3} + i = 2\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right)$, $z^3 = 8\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right) = 8i$.

   (b) $z = \sqrt{3}i - 1$, find z^{10}.

   answer: $z = 2\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right)$, $z^{10} = 2^{10}\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = -512 + 512\sqrt{3}i$.

   (c) $z = -1 - i$, find z^{21}.

   answer: $z = \sqrt{2}\left(\cos\left(\frac{5}{4}\pi\right) + \sin\left(\frac{5}{4}\pi\right)\right)$, $z^{21} = 1024 + 1024i$.

5. The de Moivre follows directly from Euler's formula and states that $(\cos(n\alpha) + i\sin(n\alpha)) = (\cos\alpha + i\sin\alpha)^n$. Expand the indicated expression and use it to express $\cos(n\alpha)$ and $\sin(n\alpha)$ via $\cos\alpha$ and $\sin\alpha$.

   You may want to use the Newton binomial formulas (derived, say, via Pascal's triangle). The formulas you may want to use are:

$$\begin{array}{rcl}
(a+b)^2 &=& a^2 + 2ab + b^2 \\
(a+b)^3 &=& a^3 + 3a^2b + 3ab^2 + b^3 \\
(a+b)^4 &=& a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 \quad .
\end{array}$$

   (a) Expand $(\cos\alpha + i\sin\alpha)^2$. Express $\cos(2\alpha)$ and $\sin(2\alpha)$ via $\cos\alpha$ and $\sin\alpha$.

   answer: $\begin{array}{rcl} \cos(2\alpha) &=& \cos^2\alpha - \sin^2\alpha \\ \sin(2\alpha) &=& 2\sin\alpha\cos\alpha . \end{array}$

   (b) Expand $(\cos\alpha + i\sin\alpha)^3$. Express $\cos(3\alpha)$ and $\sin(3\alpha)$ via $\cos\alpha$ and $\sin\alpha$.

   answer: $\begin{array}{rcl} \cos(3\alpha) &=& \cos^3\alpha - 3\cos\alpha\sin^2\alpha \\ \sin(3\alpha) &=& -\sin^3\alpha + 3\sin\alpha\cos^2\alpha . \end{array}$

   (c) Expand $(\cos\alpha + i\sin\alpha)^4$. Express $\cos(4\alpha)$ and $\sin(4\alpha)$ via $\cos\alpha$ and $\sin\alpha$.

   answer: $\begin{array}{rcl} \cos(4\alpha) &=& \cos^4\alpha - 6\cos^2\alpha\sin^2\alpha - \sin^4\alpha \\ \sin(4\alpha) &=& 4\sin\alpha\cos^3\alpha - 4\sin^3\alpha\cos\alpha . \end{array}$