Calculus I
Homework The Fundamental Theorem of Calculus
Lecture 23

1. Differentiate $f(x)$ using the Fundamental Theorem of Calculus part 1.

(a) $f(x) = \int\limits_1^x \sin\left(t^2\right) \mathrm{d}t$

answer: $\sin\left(x^2\right)$

(b) $f(x) = \int_1^x \left(t - \sqrt{t}\right) \mathrm{d}t.$

answer: $x - \sqrt{t}$

(c) $f(x) = \int\limits_x^1 (2+t^4)^5 \ \mathrm{d}t$

answer: $-\left(2+x^4\right)^5$

(d) $f(x) = \int\limits_0^{x^2} t^2 \mathrm{d}t.$

answer: $f'(x) = 2x^5$

(e) $f(x) = \int\limits_{\ln x}^{e^x} t^3 \mathrm{d}t.$

answer: $f'(x) = e^{4x} - \frac{x}{(\ln x)^3}$

(f) $f(x) = \int_1^x \left(\sqrt{t} - \sqrt[3]{t}\right) \mathrm{d}t.$

answer: $\sqrt{x} - \sqrt[3]{x}$

(g) $f(x) = \int_1^{\frac{1}{x+1}} \sin\left(t^2\right) \mathrm{d}t.$

answer:

(h) $f(x) = \int_1^{\frac{1}{x+1}} \cos\left(t^2\right) \mathrm{d}t.$

answer: $-\frac{1}{(x+1)^2} \cos\left(\frac{1}{(x+1)^2}\right)$

(i) $f(x) = \int_0^{x^3} \cos^2 t \ \mathrm{d}t$

answer: $3x^2 \cos^2\left(x^3\right)$

Solution. 1.b

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_1^x \left(t - \sqrt{t}\right) \mathrm{d}t\right) = x - \sqrt{x}. \qquad \Big| \text{ FTC, part 1}$$

Solution. 1.c We recall that the Fundamental Theorem of Calculus part 1 states that $\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_a^x h(t)dt\right) = h(x)$ where a is a constant. We can rewrite the integral so it has x as the upper limit:

$$f(x) = \int_x^1 (2+1^4)^5 \mathrm{d}t = -\int_1^x (2+1^4)^5 \mathrm{d}t \quad .$$

Therefore

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(-\int_1^x (2+t^4)^5 \mathrm{d}t\right) = -\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_1^x (2+t^4)^5 \mathrm{d}t\right) \overset{\text{FTC part 1}}{=} -(2+x^4)^5 \quad .$$

Solution. 1.e

$$\begin{aligned} f'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\int\limits_{\ln x}^{e^x} t^3 \mathrm{d}t\right) \\ &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\int\limits_{\ln x}^{0} t^3 \mathrm{d}t + \int\limits_0^{e^x} t^3 \mathrm{d}t\right) \\ &= \frac{\mathrm{d}}{\mathrm{d}x}\left(-\int\limits_0^{\ln x} t^3 \mathrm{d}t + \int\limits_0^{e^x} t^3 \mathrm{d}t\right). \end{aligned}$$

The Fundamental Theorem of Calculus part I states that for an arbitrary constant a, $\frac{\mathrm{d}}{\mathrm{d}u}\left(\int_a^u g(t)\mathrm{d}t\right) = g(u)$ (for a continuous g). We use this two compute the two derivatives:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(\int\limits_0^{\ln x} t^3\mathrm{d}t\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\int\limits_0^{u} t^3\mathrm{d}t\right) && \Big| \text{ Set } u = \ln x \\
&= u^3 \cdot \frac{\mathrm{d}u}{\mathrm{d}x} \\
&= \frac{(\ln x)^3}{x} \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\int\limits_0^{e^x} t^3\mathrm{d}t\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\int\limits_0^{w} t^3\mathrm{d}t\right) && \Big| \text{ Set } w = e^x \\
&= w^3 \cdot \frac{\mathrm{d}w}{\mathrm{d}x} \\
&= e^{3x}e^x = e^{4x}.
\end{aligned}$$

Finally, we combine the above computations to a single answer.

$$f'(x) = e^{4x} - \frac{(\ln x)^3}{x}.$$

Solution. 1.f

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_1^x \left(\sqrt{t} - \sqrt[3]{t}\right)\mathrm{d}t = \sqrt{x} - \sqrt[3]{x} \quad \Big| \text{ FTC part I}$$

Solution. 1.g

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\int_1^{\frac{1}{x+1}} \sin\left(t^2\right)\mathrm{d}t &= \frac{\mathrm{d}}{\mathrm{d}x}\int_1^{u} \sin(t^2)\mathrm{d}t && \Big| \; u = \tfrac{1}{x+1}, \text{ use FTC part I, chain rule} \\
&= \sin\left(u^2\right)\frac{\mathrm{d}u}{\mathrm{d}x} \\
&= \sin\left(\frac{1}{(x+1)^2}\right)\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x+1}\right) \\
&= \sin\left(\frac{1}{(x+1)^2}\right)\left(-\frac{1}{(x+1)^2}\right) \\
&= -\frac{1}{(x+1)^2}\sin\left(\frac{1}{(x+1)^2}\right)
\end{aligned}$$

Solution. 1.h

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_1^{\frac{1}{x+1}} \cos\left(t^2\right)\mathrm{d}t\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\int_1^{u} \cos\left(t^2\right)\mathrm{d}t\right) && \Big| \text{ Set } \tfrac{1}{x+1} = u \\
&= \cos\left(u^2\right)\frac{\mathrm{d}}{\mathrm{d}x}(u) && \Big| \text{ FTC part I and Chain Rule} \\
&= -\frac{1}{(x+1)^2}\cos\left(\frac{1}{(x+1)^2}\right)
\end{aligned}$$