Calculus III
Lecture 3

Todor Milev
https://github.com/tmilev/freecalc

2020

Outline

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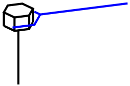
Torque

- If we tighten a bolt

Torque



- If we tighten a bolt using a wrench,

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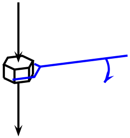

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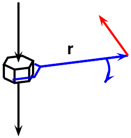

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- Let arm of the wrench: given by vector **r**.
- Suppose we are applying a force **F** at arm of the wrench. The force has three components:

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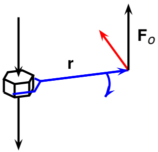

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  - component $\mathbf{F}_o$ orthogonal to the plane of rotation

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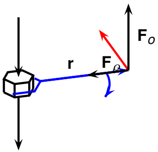

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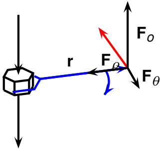

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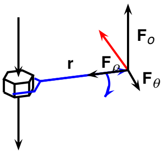

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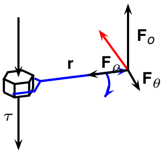

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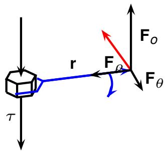

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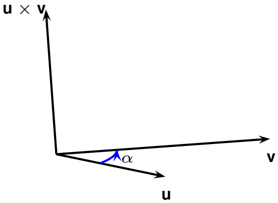
- Only $\mathbf{F}_\theta$ contributes to the bolt motion.
- The force of bolt motion τ is proportional to length of wrench.
- It turns out $\tau = \mathbf{r} \times (\mathbf{F}_\rho + \mathbf{F}_\theta)$, where $\times$ is the vector cross product.

The Cross Product $\times$



Definition (Cross product)

$\mathbf{u} \times \mathbf{v}$ is the vector uniquely determined by the following.

- If $\mathbf{u}$, $\mathbf{v}$ are non-zero and non-collinear.
  - $\mathbf{u} \times \mathbf{v}$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$.
  - The magnitude of $\mathbf{u} \times \mathbf{v}$ equals $|\mathbf{u}||\mathbf{orth_u v}| = |\mathbf{u}||\mathbf{v}| \sin \alpha$.
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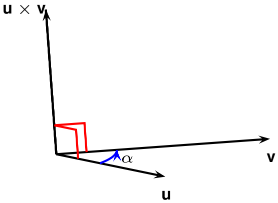

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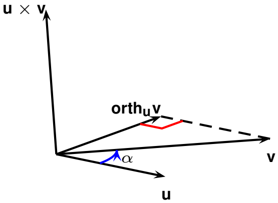

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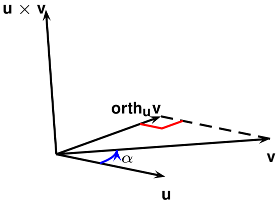

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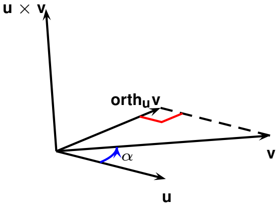

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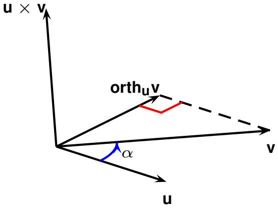

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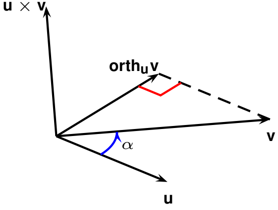

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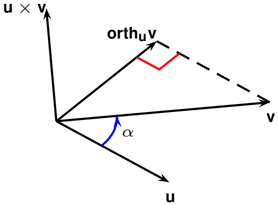

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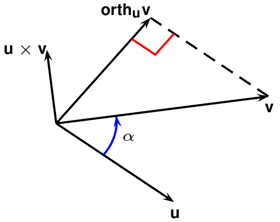

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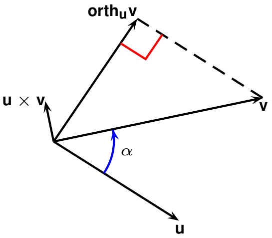

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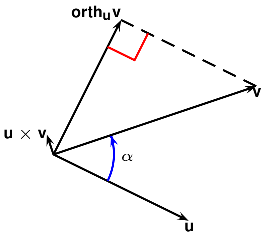

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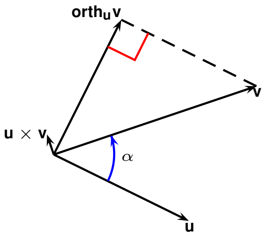

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There are a couple of hand rules to help figure out the direction of the cross product.

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  Indeed, that is because

$$|\mathbf{orth}_{\boldsymbol{u}}\mathbf{v}| = |\mathbf{v}| \sin \alpha \Longrightarrow |\mathbf{u} \times \mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}|\ \sin \alpha$$

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- Cross product is linear in each argument:

$$\mathbf{u} \times (a\mathbf{v} + b\mathbf{w}) = a\mathbf{u} \times \mathbf{v} + b\mathbf{u} \times \mathbf{w}$$

$$(a\mathbf{u} + b\mathbf{w}) \times \mathbf{v} = a\mathbf{u} \times \mathbf{v} + b\mathbf{w} \times \mathbf{v}$$

$\mathbf{orth_u}$ is a linear operator

Theorem

$\mathbf{orth_u}(\mathbf{v}_1 + \mathbf{v}_2) = \mathbf{orth_u}\mathbf{v}_1 + \mathbf{orth_u}\mathbf{v}_2$

Proof.

Geometric proof:

Algebraic proof:

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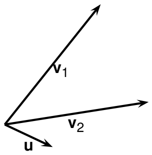
$$\mathbf{orth_u}(\mathbf{v}_1 + \mathbf{v}_2) = \mathbf{orth_u}\mathbf{v}_1 + \mathbf{orth_u}\mathbf{v}_2$$

Proof.

Geometric proof:



Algebraic proof:

$$\begin{aligned}\mathbf{orth_u}(\mathbf{v}_1 + \mathbf{v}_2) &= (\mathbf{v}_1 + \mathbf{v}_2) - \mathbf{proj_u}(\mathbf{v}_1 + \mathbf{v}_2)\\ &= (\mathbf{v}_1 + \mathbf{v}_2) - (\mathbf{proj_u}(\mathbf{v}_1) + \mathbf{proj_u}(\mathbf{v}_2))\\ &= (\mathbf{v}_1 - \mathbf{proj_u}(\mathbf{v}_1)) + (\mathbf{v}_2 - \mathbf{proj_u}(\mathbf{v}_2))\\ &= \mathbf{orth_u}\mathbf{v}_1 + \mathbf{orth_u}\mathbf{v}_2\end{aligned}$$

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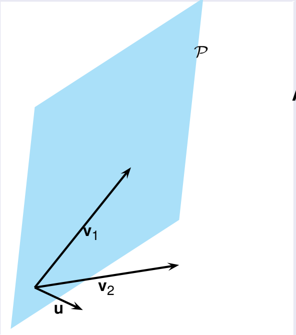

Let $\mathcal{P}$: plane $\perp$ $\mathbf{u}$.

Algebraic proof:

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\mathbf{orth_u}(\mathbf{v}_1 + \mathbf{v}_2) &= (\mathbf{v}_1 + \mathbf{v}_2) - \mathbf{proj_u}(\mathbf{v}_1 + \mathbf{v}_2) \\
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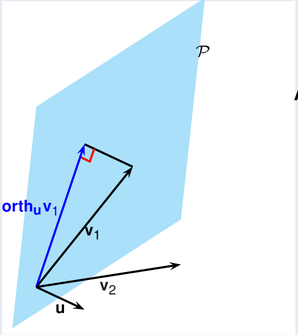

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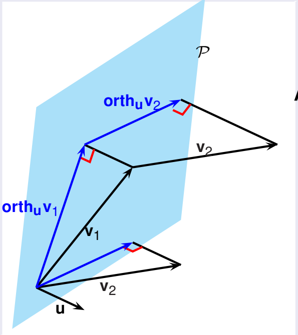

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\mathbf{orth_u}(\mathbf{v}_1 + \mathbf{v}_2) &= (\mathbf{v}_1 + \mathbf{v}_2) - \mathbf{proj_u}(\mathbf{v}_1 + \mathbf{v}_2) \\
&= (\mathbf{v}_1 + \mathbf{v}_2) - (\mathbf{proj_u}(\mathbf{v}_1) + \mathbf{proj_u}(\mathbf{v}_2)) \\
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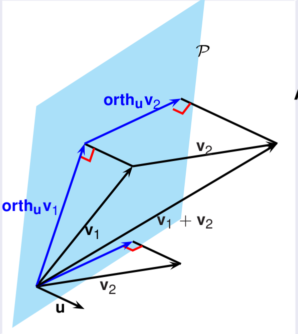

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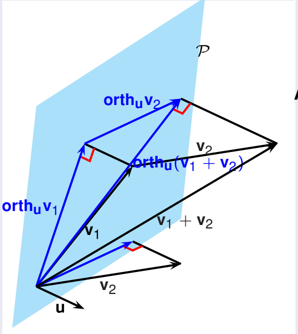

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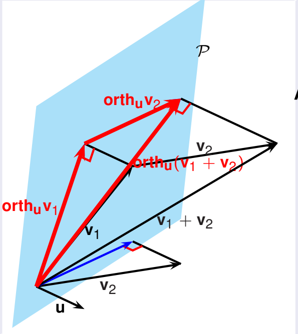

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Justification of Linearity of $\times$ Product

Theorem

$\mathbf{u} \times (\mathbf{v}_1 + \mathbf{v}_2) = \mathbf{u} \times \mathbf{v}_1 + \mathbf{u} \times \mathbf{v}_2.$

Geometric justification.

$\square$

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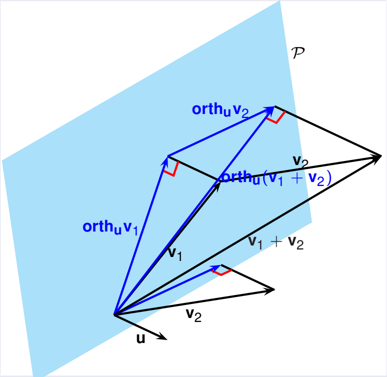

$$\begin{aligned} \mathbf{u} \times \mathbf{v}_1 &= \mathbf{u} \times \mathbf{orth_u v}_1 \\ \mathbf{u} \times \mathbf{v}_2 &= \mathbf{u} \times \mathbf{orth_u v}_2 \\ \mathbf{u} \times (\mathbf{v}_1 + \mathbf{v}_2) &= \mathbf{u} \times (\mathbf{orth_u}(\mathbf{v}_1 + \mathbf{v}_2)) \end{aligned}$$

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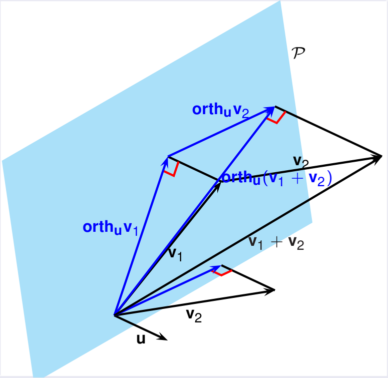

$$
\begin{aligned}
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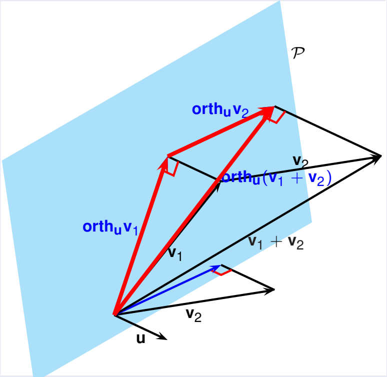

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$\Rightarrow$ suffices to prove theorem when $\mathbf{v}_1, \mathbf{v}_2 \perp \mathbf{u}$.

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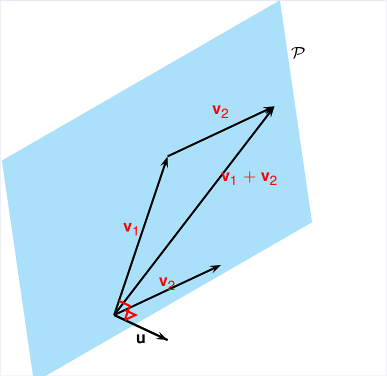

$$
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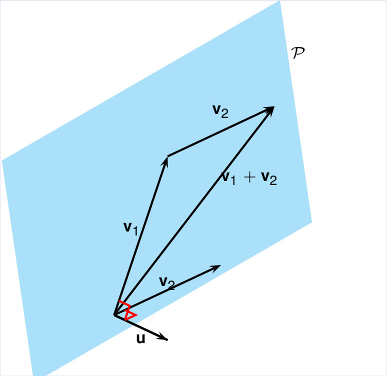

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$\Rightarrow$ suffices to prove theorem when $\mathbf{v}_1, \mathbf{v}_2 \perp \mathbf{u}$. Since $(a\mathbf{u}) \times \mathbf{v} = a(\mathbf{u} \times \mathbf{v}) \Rightarrow$ suffices to prove theorem when $|\mathbf{u}| = 1$.

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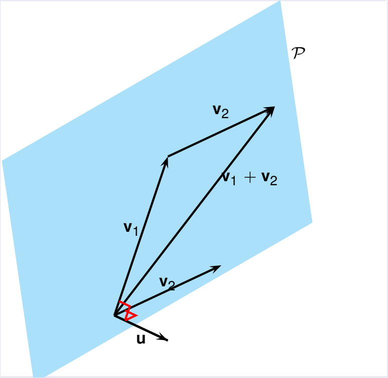

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$\mathbf{u} \times (\mathbf{v}_1 + \mathbf{v}_2) = \mathbf{u} \times \mathbf{v}_1 + \mathbf{u} \times \mathbf{v}_2.$

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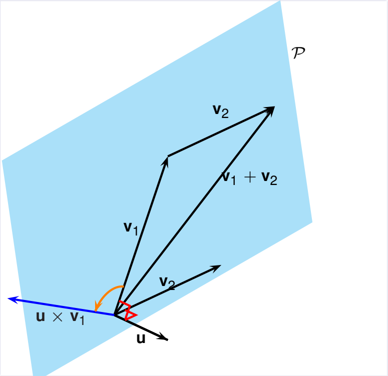

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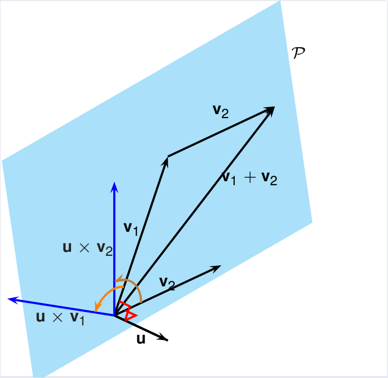

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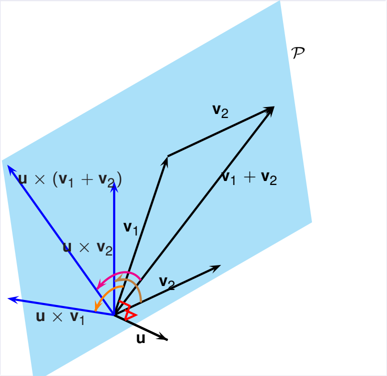

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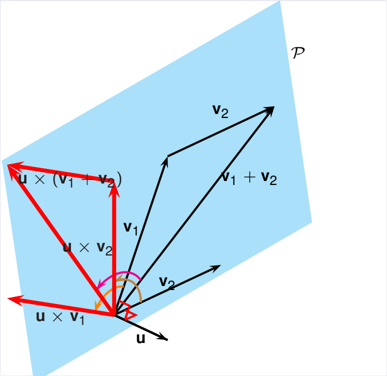

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When $|\mathbf{u}| = 1$, applying $\mathbf{u}\times$ rotates all vectors in the plane $\mathcal{P}$ at angle $\frac{\pi}{2}$. The statement of the theorem now follows from the fact that rotation preserves sums of vectors.

$\square$

Permutations and permutation signs

- Let $\sigma : \{1, 2, \dots n\} \to \{1, 2, \dots, n\}$ be one to one function.

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  - and so on, total: $n \cdot (n - 1) \cdots 1 = n!$ ways to make a permutation.

Sign of permutation

- Given two sequences of numbers, define them to be transpositions of one another if one is obtained from the other with a single swap of neighboring numbers.

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- If $sign(\sigma) = 1$, σ is called even, if $sign(\sigma) = -1$, σ is called odd.

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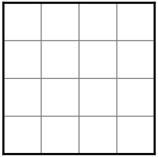

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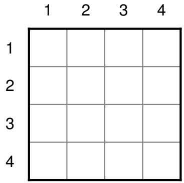

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1				
2				
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	1	2	3	4
1	?	?	?	?
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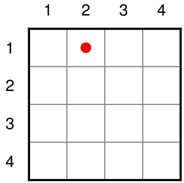

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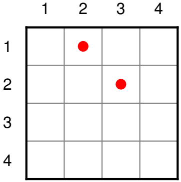

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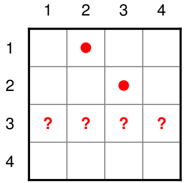

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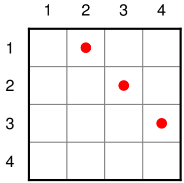

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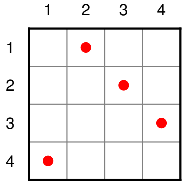

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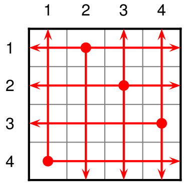

- $\sigma(k)$ are different $\Rightarrow$ rook placements are peaceful: rooks never hit one another. i.e., no two points lie on same column or row.

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n^{th} column

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- Non-square matrices: used & important but we discuss them elsewhere.

- The determinant $\det A$ of a square matrix A is a number written as:

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- For each rook placement we have a summand obtained by multiplying the numbers on which the rooks are standing.

- The determinant $\det A$ of a square matrix A is a number written as:

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & & & & \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{vmatrix}$$

- The formula for the determinant is:

$$\det A = \sum_{\text{all permutations } \sigma} a_{1\sigma(1)} a_{2\sigma(2)} \dots a_{n\sigma(n)} \mathit{sign}(\sigma) \quad .$$

- For every permutation σ we have one summand.
- Every pair $(k, \sigma(k))$ can be identified with a peaceful of a rook placement (as described in previous slides/lectures).
- For each rook placement we have a summand obtained by multiplying the numbers on which the rooks are standing.
- The sign of each summand is determined by the sign of the permutation.

2×2 determinants

$$\det \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} =$$

- We specialize the $n \times n$ determinant formula to the case $n = 2$.

2×2 determinants

$$\det \begin{vmatrix} \color{red}{\bullet} & a_{12} \\ a_{21} & \color{red}{\bullet} \end{vmatrix} =$$

- We specialize the $n \times n$ determinant formula to the case $n = 2$.
- There are two peaceful rook placements for a 2×2 chessboard.

2×2 determinants

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2×2 determinants

$$\det \begin{vmatrix} \bullet & a_{12} \\ a_{21} & \bullet \end{vmatrix} = a_{11}a_{22}$$

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- For each peaceful rook placement we got one summand.

2×2 determinants

$$\det \begin{vmatrix} a_{11} & \bullet \\ \bullet & a_{22} \end{vmatrix} = a_{11}a_{22} \qquad \color{red}{a_{12}a_{21}}$$

- We specialize the $n \times n$ determinant formula to the case $n = 2$.
- There are two peaceful rook placements for a 2×2 chessboard.
- For each peaceful rook placement we got one summand.

2×2 determinants

$$\det \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

- We specialize the $n \times n$ determinant formula to the case $n = 2$.
- There are two peaceful rook placements for a 2×2 chessboard.
- For each peaceful rook placement we got one summand.
- The permutation $(\sigma(1), \sigma(2)) = (2, 1)$ is odd, so one of the summands comes with negative sign.

3×3 determinants

$$\det \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =$$

- We specialize the $n \times n$ determinant formula to the case $n = 3$.

3×3 determinants

$$\det \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =$$

- We specialize the $n \times n$ determinant formula to the case $n = 3$.
- There are $6 = 3!$ peaceful rook placements for a 3×3 chessboard.

3×3 determinants

$$\det \begin{vmatrix} \bullet & a_{12} & a_{13} \\ a_{21} & \bullet & a_{23} \\ a_{31} & a_{32} & \bullet \end{vmatrix} = \quad a_{11}a_{22}a_{33}$$

- We specialize the $n \times n$ determinant formula to the case $n = 3$.
- There are $6 = 3!$ peaceful rook placements for a 3×3 chessboard.
- For each peaceful rook placement we got one summand.

3×3 determinants

$$\det \begin{vmatrix} a_{11} & \bullet & a_{13} \\ a_{21} & a_{22} & \bullet \\ \bullet & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31}$$

- We specialize the $n \times n$ determinant formula to the case $n = 3$.
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3×3 determinants

$$\det \begin{vmatrix} a_{11} & a_{12} & \bullet \\ \bullet & a_{22} & a_{23} \\ a_{31} & \bullet & a_{33} \end{vmatrix} = \begin{array}{l} a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} \\ + a_{13}a_{21}a_{32} \end{array}$$

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3×3 determinants

$$\det \begin{vmatrix} \color{red}{\bullet} & a_{12} & a_{13} \\ a_{21} & a_{22} & \color{red}{\bullet} \\ a_{31} & \color{red}{\bullet} & a_{33} \end{vmatrix} = \begin{array}{l} a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} \\ + a_{13}a_{21}a_{32} \\ - a_{13}a_{22}a_{31} \color{red}{- a_{11}a_{23}a_{32}} \end{array}$$

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3×3 determinants

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3×3 determinants

$$\det \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{array}{l} a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} \\ + a_{13}a_{21}a_{32} \\ - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} \\ - a_{12}a_{21}a_{33} \end{array}$$

- We specialize the $n \times n$ determinant formula to the case $n = 3$.
- There are $6 = 3!$ peaceful rook placements for a 3×3 chessboard.
- For each peaceful rook placement we got one summand.
- The rook placements along the down-right “broken” diagonals correspond to even permutations, and the rook placements along the right-up “broken” diagonals correspond to negative permutations.

Cross Product in Coordinates

- Let $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$: unit vectors along coordinate axes.
- We have that

$$\begin{array}{lll} \mathbf{i}\times\mathbf{i}=\mathbf{0}, & \mathbf{j}\times\mathbf{j}=\mathbf{0}, & \mathbf{k}\times\mathbf{k}=\mathbf{0} \\ \mathbf{i}\times\mathbf{j}=\mathbf{k}, & \mathbf{j}\times\mathbf{k}=\mathbf{i}, & \mathbf{k}\times\mathbf{i}=\mathbf{j} \\ \mathbf{j}\times\mathbf{i}=-\mathbf{k}, & \mathbf{k}\times\mathbf{j}=-\mathbf{i}, & \mathbf{i}\times\mathbf{k}=-\mathbf{j} \end{array}$$

- Let $\begin{array}{lllll} \mathbf{u} & = & u_1\mathbf{i}+u_2\mathbf{j}+u_3\mathbf{k} & = & (u_1,u_2,u_3) \\ \mathbf{v} & = & v_1\mathbf{i}+v_2\mathbf{j}+v_3\mathbf{k} & = & (v_1,v_2,v_3) \end{array}$.
-

$$\begin{array}{lll} \mathbf{u}\times\mathbf{v} & = & (u_1\mathbf{i}+u_2\mathbf{j}+u_3\mathbf{k})\times(v_1\mathbf{i}+v_2\mathbf{j}+v_3\mathbf{k}) \\ & = & (u_2v_3-u_3v_2)\mathbf{i}+(u_3v_1-u_1v_3)\mathbf{j}+(u_1v_2-u_2v_1)\mathbf{k} \\ \mathbf{u}\times\mathbf{v} & = & (u_1,u_2,u_3)\times(v_1,v_2,v_3)=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \end{array}$$

$$\mathbf{u} \times \mathbf{v} = (u_1, u_2, u_3) \times (v_1, v_2, v_3) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Example

Find $\mathbf{u} \times \mathbf{v}$, where $\mathbf{u} = (1, 2, 3)$ and $\mathbf{v} = (6, 5, 4)$.

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$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= (1, 2, 3) \times (6, 5, 4) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 6 & 5 & 4 \end{vmatrix} \\ &= \begin{vmatrix} 2 & 3 \\ 5 & 4 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 3 \\ 6 & 4 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 2 \\ 6 & 5 \end{vmatrix} \mathbf{k} \end{aligned}$$

$$\mathbf{u} \times \mathbf{v} = (u_1, u_2, u_3) \times (v_1, v_2, v_3) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

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$$\mathbf{u} \times \mathbf{v} = (u_1, u_2, u_3) \times (v_1, v_2, v_3) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

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Use $\times$ to find vector perpendicular to two given

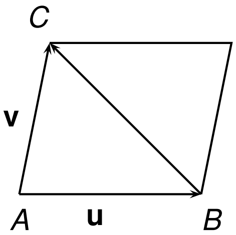
Recall $\mathbf{u} \times \mathbf{v}$ is perpendicular to $\mathbf{u}$ and $\mathbf{v}$.

Example

Find a vector $\mathbf{w}$ perpendicular to $\mathbf{u} = (1, 1, 0) = \mathbf{i} + \mathbf{j}$ and $\mathbf{v} = \mathbf{j} + \mathbf{k} = (0, 1, 1)$.

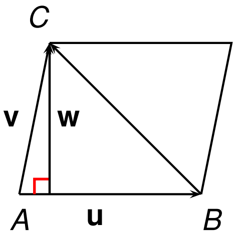
$$\begin{aligned} \mathbf{w} &= (\mathbf{i}+\mathbf{j}) \times (\mathbf{j}+\mathbf{k}) = \mathbf{i}\times\mathbf{j} + \mathbf{i}\times\mathbf{k} + \mathbf{j}\times\mathbf{j} + \mathbf{j}\times\mathbf{k} = \\ &= \mathbf{k} - \mathbf{j} + \mathbf{0} + \mathbf{i} = \mathbf{i} - \mathbf{j} + \mathbf{k} = (1, -1, 1) \ . \end{aligned}$$

Use $\times$ to find area of triangle in space



- A, B, C points in space, $\mathbf{u} = \mathbf{AB}$, $\mathbf{v} = \mathbf{AC}$.

Use $\times$ to find area of triangle in space



- A, B, C points in space, $\mathbf{u} = \mathbf{AB}$, $\mathbf{v} = \mathbf{AC}$.
- Then
  $|\mathbf{w}| = |\mathbf{orth_u v}| =$ distance from C to AB .

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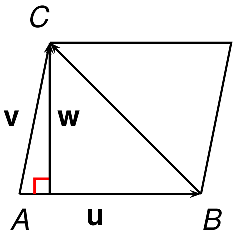

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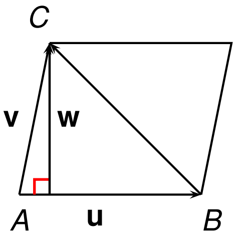
  $|\mathbf{w}| = |\mathbf{orth_u v}| =$ distance from C to AB .
- $|\mathbf{u} \times \mathbf{v}| = |\mathbf{orth_u v}|\,|\mathbf{u}| = 2\mathrm{area}(ABC) = \mathrm{area}(ABDC)$

Use $\times$ to find area of triangle in space



- A, B, C points in space, $\mathbf{u} = \mathbf{AB}$, $\mathbf{v} = \mathbf{AC}$.
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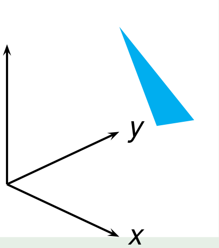
- $|\mathbf{u} \times \mathbf{v}| = |\mathbf{orth_u v}|\,|\mathbf{u}| = 2\mathrm{area}(ABC) = \mathrm{area}(ABDC)$
- $|\mathbf{u} \times \mathbf{v}|$ = Area of parallelogram on sides **u** and **v**.

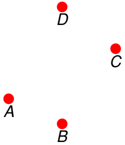
Example



Find the area of the triangle $A(1,2,3)$, $B(2,3,1)$, $C(3,1,2)$.

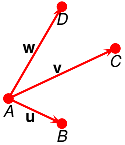
$$
\begin{aligned}
\mathrm{Area}(ABC) &= \frac{1}{2}|\mathbf{AB}\times\mathbf{AC}| = \frac{1}{2}|(1,1,-2)\times(2,-1,-1)| \\
&= \frac{1}{2}|(-3,-3,-3)| \\
&= \frac{3\sqrt{3}}{2}\,.
\end{aligned}
$$

Scalar Triple Product



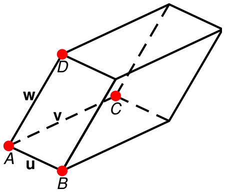
- A, B, C, D points in space;

Scalar Triple Product



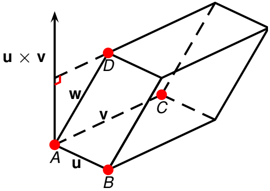
- A, B, C, D points in space;
- $\mathbf{u} = \mathbf{AB}$, $\mathbf{v} = \mathbf{AC}$, $\mathbf{w} = \mathbf{AD}$;

Scalar Triple Product



- A, B, C, D points in space;
- $\mathbf{u} = \mathbf{AB}$, $\mathbf{v} = \mathbf{AC}$, $\mathbf{w} = \mathbf{AD}$;
- $R = R(\mathbf{u}, \mathbf{v}, \mathbf{w})$: box on sides $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$.

Scalar Triple Product



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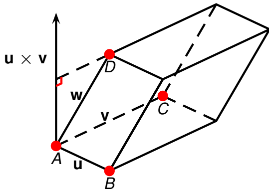
Scalar Triple Product



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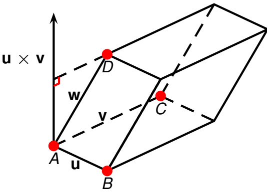
Definition

The quantity $\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})$ is called the scalar triple product of $\mathbf{w}, \mathbf{u}, \mathbf{v}$.

Scalar Triple Product



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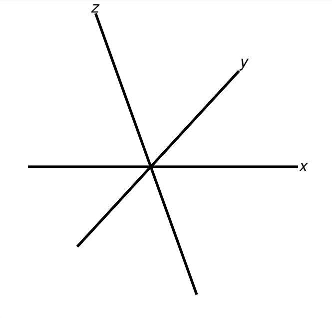
Definition

The quantity $\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})$ is called the scalar triple product of $\mathbf{w}, \mathbf{u}, \mathbf{v}$.

- If $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$, and $\mathbf{w} = (w_1, w_2, w_3)$, then

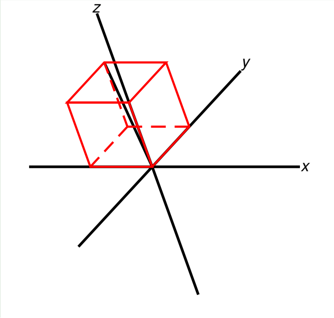
$$\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) = \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Example



Find the volume of the parallelepiped (slanted box) with vertex at the origin spanned by the vectors $(-1, 1, 1)$, $(1, -1, 1)$, $(1, 1, -1)$.

Example



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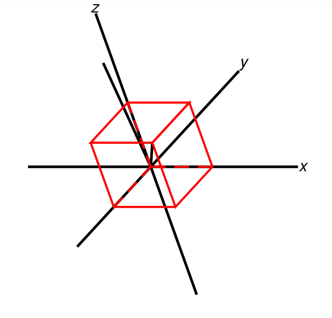

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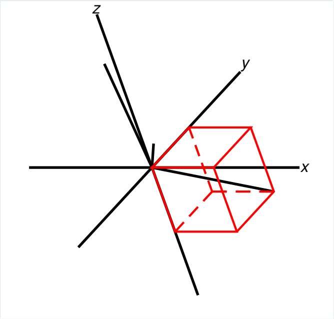

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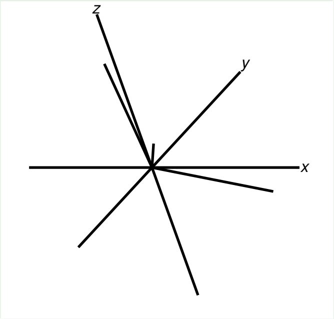

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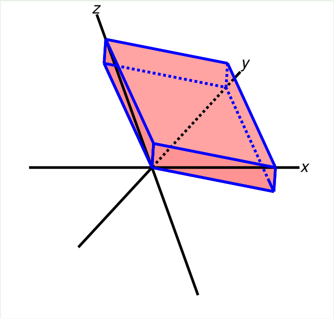
Example



Find the volume of the parallelepiped (slanted box) with vertex at the origin spanned by the vectors $(-1, 1, 1)$, $(1, -1, 1)$, $(1, 1, -1)$.

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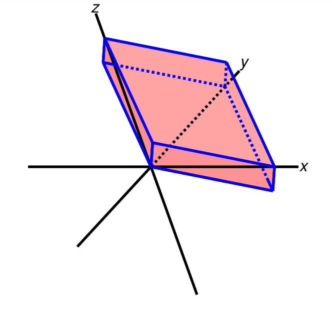

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$$\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = \textbf{?} \quad + \textbf{?} + \textbf{?} - \textbf{?} \quad - \textbf{?} \quad - \textbf{?}$$

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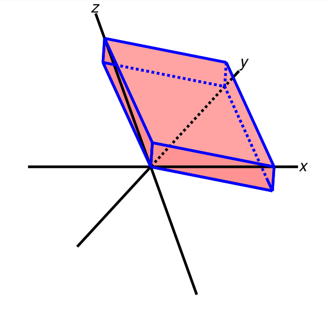

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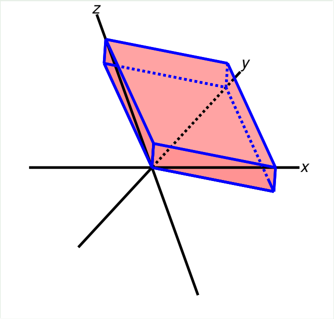

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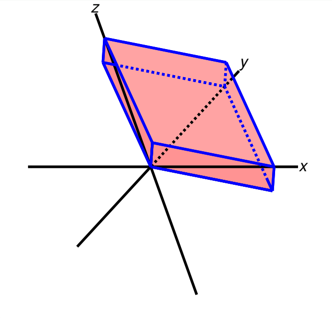

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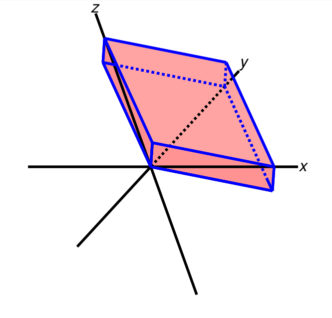

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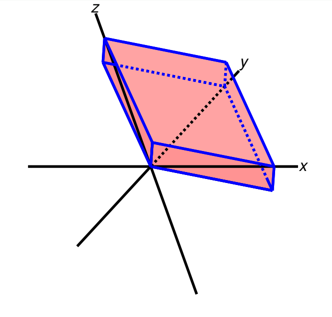

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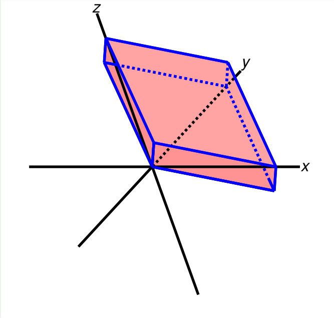

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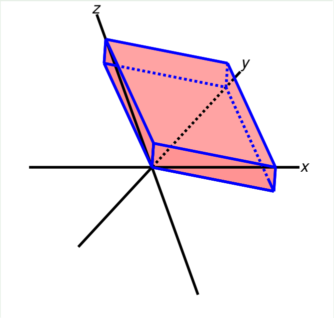

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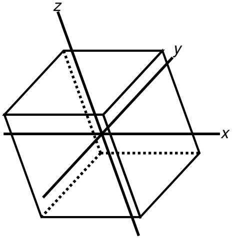
$$= 4$$

Example



Find the volume of the tetrahedron with vertices $(1, 1, 1)$, $(1, -1, -1)$, $(-1, 1, -1)$, $(-1, -1, 1)$.

$$\text{Vol(tetrahedron)} =$$

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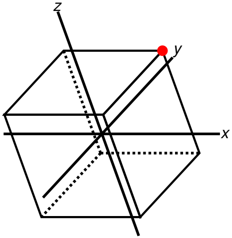

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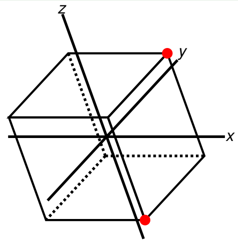

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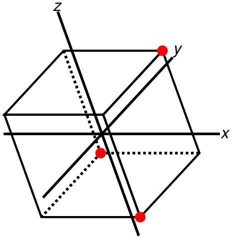

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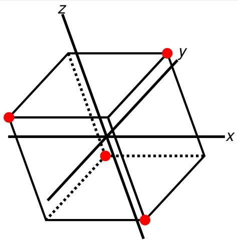

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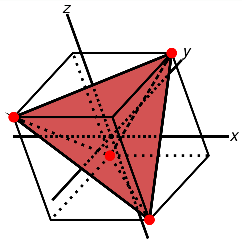

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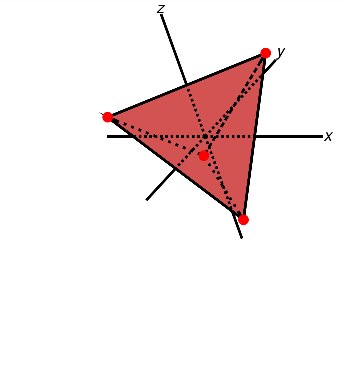

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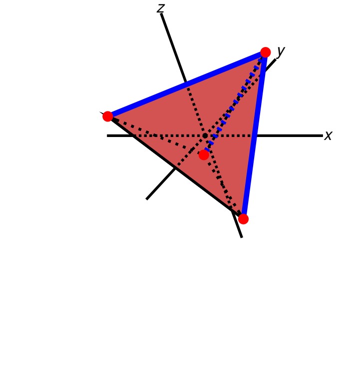

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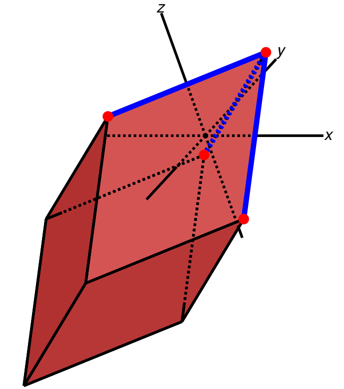

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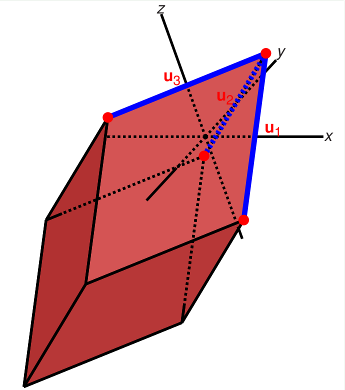

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$\mathbf{u}_1 = \mathbf{?}$

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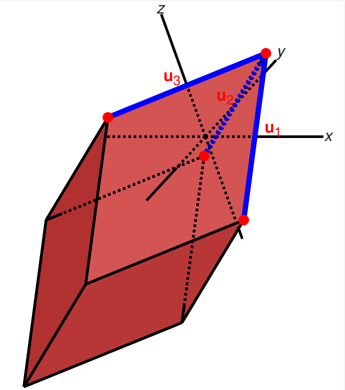

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$\mathbf{u}_1 = (1, -1, -1) - (1, 1, 1) = (0, -2, -2)$

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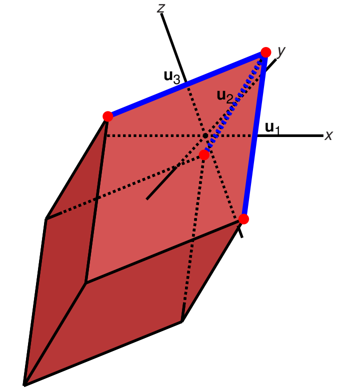

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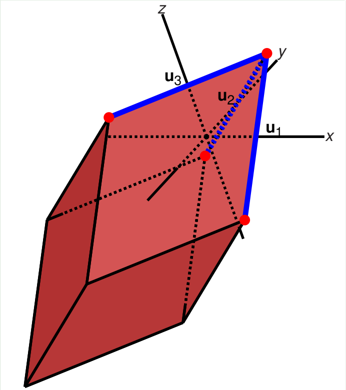

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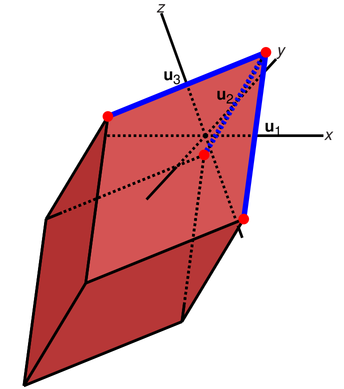

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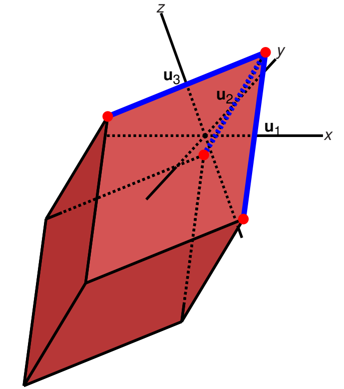

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Example

Do the points $(1, 2, 3)$, $(2, 3, 5)$, $(3, 5, 7)$, $(5, 7, 11)$ lie in one plane?

Example

Do the points $(1, -1, -1)$, $(-1, 1, -1)$, $(-1, -1, 1)$, $(1, 2, 3)$ lie in one plane?

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Definition

The frame $(\mathbf{u}, \mathbf{v}, \mathbf{w})$ is positively oriented if $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) > 0$.