

Calculus II

Lecture 10

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<https://github.com/tmilev/freecalc>

2020

Outline

1 Polar Coordinates

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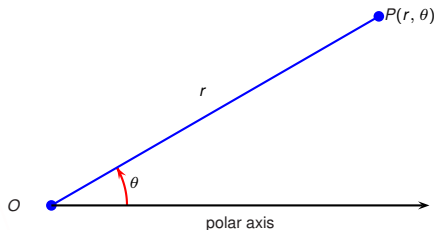
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Polar Coordinates

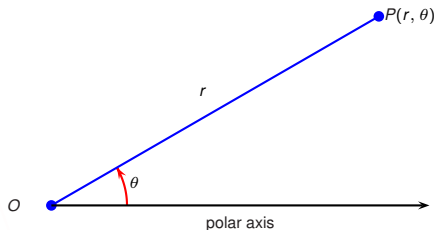
- The polar coordinate system is an alternative to the Cartesian coordinate system.
- Choose a point in the plane called O (the origin).
- Draw a ray starting at O . The ray is called the polar axis. This ray is usually drawn horizontally to the right.



- Let P be a point in the plane.
- Let θ denote the angle between the polar axis and the line OP .
- Let r denote the length of the segment OP .
- Then P is represented by the ordered pair (r, θ) .

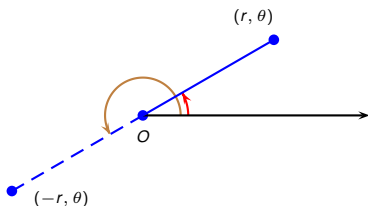
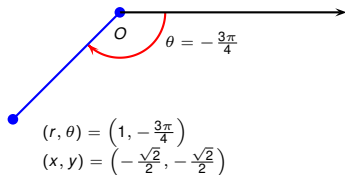
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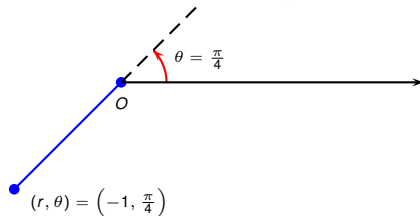
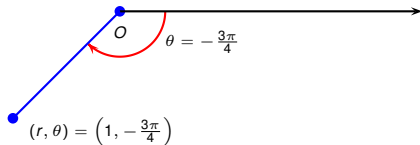
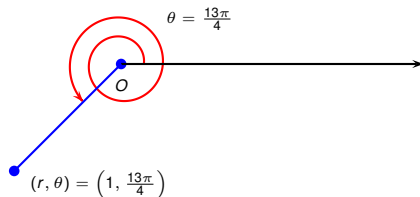
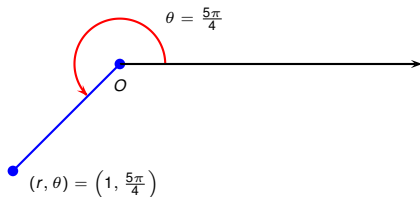


- The letters (x, y) imply Cartesian coordinates and the letters (r, θ) - polar. When we use other letters, it should be clear from context whether we mean Cartesian or polar coordinates. If not, one must request clarification.

- ❶ What if θ is negative?
- ❷ What if r is negative?
- ❸ What if r is 0?



- ❶ Positive angles θ are measured in the counterclockwise direction from O . Negative angles are measured in the clockwise direction.
- ❷ Points with polar coordinates $(-r, \theta)$ and (r, θ) lie on the same line through O and at the same distance from O , but on opposite sides.
- ❸ If $r = 0$, then $(0, \theta)$ represents O for all values of θ .



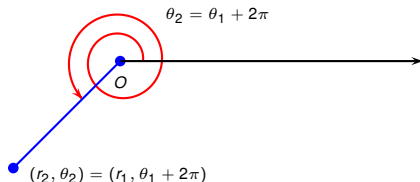
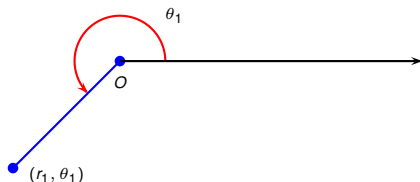
- There are many ways to represent the same point.
- We could use a negative θ .
- We could go around more than once.
- We could use a negative r .

- Let P_1 be point with polar coordinates (r_1, θ_1) .
- Let P_2 be point with polar coordinates (r_2, θ_2) .

Observation

P_1 coincides with P_2 if one of the three mutually exclusive possibilities holds:

- $r_1 = r_2 \neq 0$ and $\theta_2 = \theta_1 + 2k\pi, k \in \mathbb{Z}$,
- $r_1 = -r_2 \neq 0$ and $\theta_2 = \theta_1 + (2k + 1)\pi, k \in \mathbb{Z}$,
- $r_1 = r_2 = 0$ and θ is arbitrary.

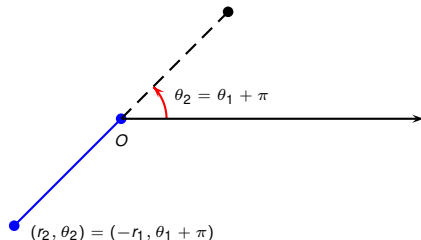
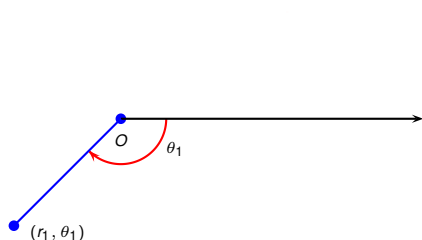


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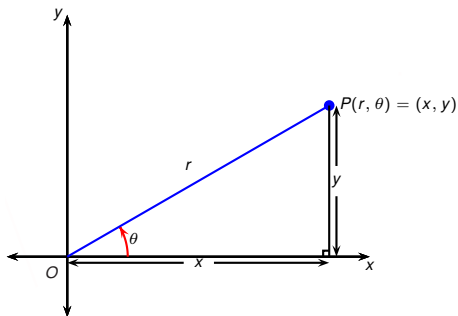
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- How do we go from polar coordinates to Cartesian coordinates?
- Suppose a point has polar coordinates (r, θ) and Cartesian coordinates (x, y) .
- How do we go from Cartesian coordinates to polar coordinates?



$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$\cos \theta = \frac{x}{r}$$

$$\sin \theta = \frac{y}{r}$$

$$r^2 = x^2 + y^2$$

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \arcsin\left(\frac{y}{r}\right) \quad \text{if } x > 0$$

$$= \arccos\left(\frac{x}{r}\right) \quad \text{if } y > 0$$

$$= \arctan\left(\frac{y}{x}\right) \quad \text{if } x > 0$$

Example

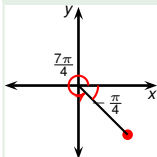
Convert the point $(2, \frac{\pi}{3})$ from polar to Cartesian coordinates.

$$x = r \cos \theta = 2 \cos \frac{\pi}{3} = 2 \left(\frac{1}{2} \right) = 1$$

$$y = r \sin \theta = 2 \sin \frac{\pi}{3} = 2 \left(\frac{\sqrt{3}}{2} \right) = \sqrt{3}$$

Therefore the point with polar coordinates $(2, \frac{\pi}{3})$ has Cartesian coordinates $(1, \sqrt{3})$.

Example



Represent the point with Cartesian coordinates $(1, -1)$ in terms of polar coordinates.

- Suppose r is positive.
- $\tan \theta = -1$ for $\theta = \frac{3\pi}{4}, \frac{7\pi}{4}$, and many other angles.
- $(1, -1)$ is in the fourth quadrant.
- Of the two values above, only $\theta = \frac{7\pi}{4}$ gives a point in the fourth quadrant.
- \Rightarrow one representation of $(1, -1)$ in polar coordinates is $(\sqrt{2}, \frac{7\pi}{4})$.
- $(\sqrt{2}, -\frac{\pi}{4})$ is another.

$$\begin{aligned} r &= \pm \sqrt{x^2 + y^2} \\ &= \sqrt{1^2 + (-1)^2} \\ &= \sqrt{2} \end{aligned}$$

$$\begin{aligned} \tan \theta &= \frac{y}{x} \\ &= -1 \end{aligned}$$