Calculus III
Lecture 5

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https://github.com/tmilev/freecalc

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Outline

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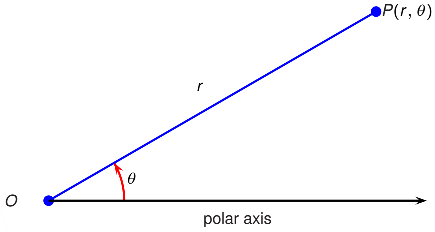
Polar Coordinates

- The polar coordinate system is an alternative to the Cartesian coordinate system.
- Choose a point in the plane called O (the origin).
- Draw a ray starting at O. The ray is called the polar axis. This ray is usually drawn horizontally to the right.



- Let P be a point in the plane.
- Let θ denote the angle between the polar axis and the line OP.
- Let r denote the length of the segment OP.
- Then P is represented by the ordered pair (r, θ).

Polar Coordinates

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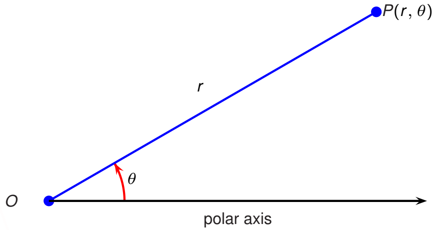
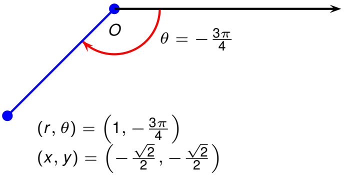
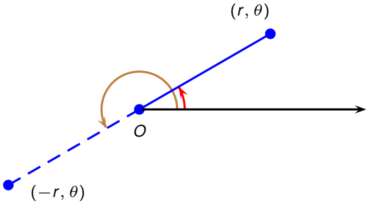


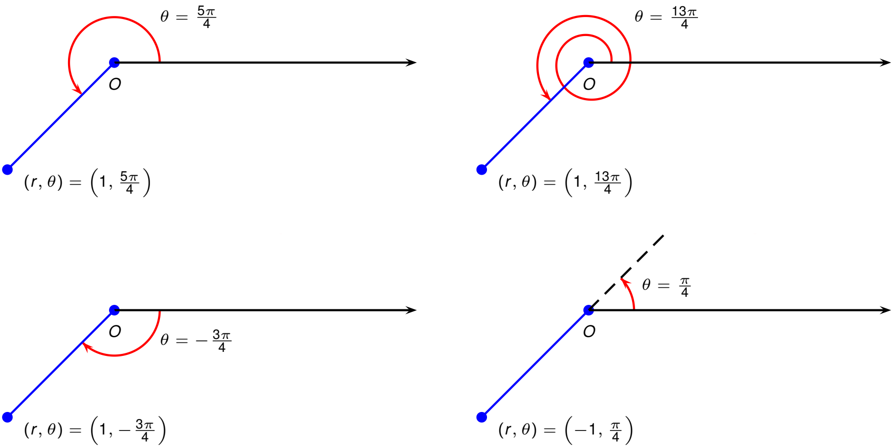
- The letters (x, y) imply Cartesian coordinates and the letters (r, θ)- polar. When we use other letters, it should be clear from context whether we mean Cartesian or polar coordinates. If not, one must request clarification.

1. What if θ is negative?
2. What if r is negative?
3. What if r is 0?





1. Positive angles θ are measured in the counterclockwise direction from O. Negative angles are measured in the clockwise direction.
2. Points with polar coordinates $(-r, \theta)$ and (r, θ) lie on the same line through O and at the same distance from O, but on opposite sides.
3. If $r = 0$, then $(0, \theta)$ represents O for all values of θ.



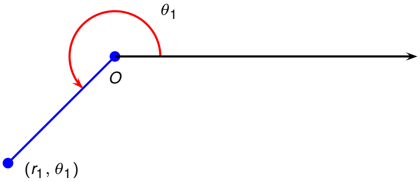
- There are many ways to represent the same point.
- We could use a negative θ.
- We could go around more than once.
- We could use a negative r.

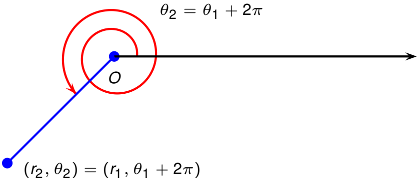
- Let P_1 be point with polar coordinates (r_1, θ_1).
- Let P_2 be point with polar coordinates (r_2, θ_2).

Observation

P_1 coincides with P_2 if one of the three mutually exclusive possibilities holds:

- $r_1 = r_2 \neq 0$ *and* $\theta_2 = \theta_1 + 2k\pi, k \in \mathbb{Z}$,
- $r_1 = -r_2 \neq 0$ *and* $\theta_2 = \theta_1 + (2k+1)\pi, k \in \mathbb{Z}$,
- $r_1 = r_2 = 0$ *and* θ *is arbitrary.*





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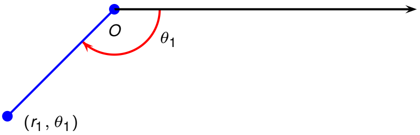
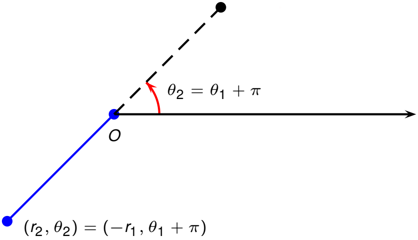
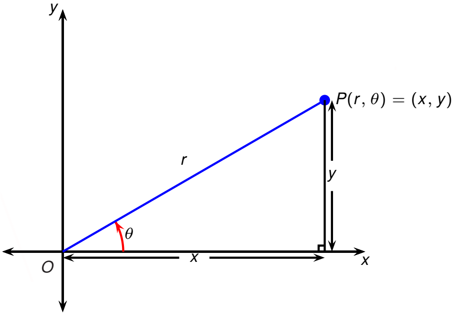




- How do we go from polar coordinates to Cartesian coordinates?
- Suppose a point has polar coordinates (r, θ) and Cartesian coordinates (x, y).
- How do we go from Cartesian coordinates to polar coordinates?



$$
\begin{array}{rcll}
x & = & r\cos\theta & \\
y & = & r\sin\theta & \\
\hline
\cos\theta & = & \frac{x}{r} & \\
\sin\theta & = & \frac{y}{r} & \\
r^2 & = & x^2 + y^2 & \\
\hline
r & = & \sqrt{x^2+y^2} & \\
\theta & = & \arcsin\left(\frac{y}{r}\right) & \text{if } x > 0 \\
 & = & \arccos\left(\frac{x}{r}\right) & \text{if } y > 0 \\
 & = & \arctan\left(\frac{y}{x}\right) & \text{if } x > 0
\end{array}
$$

Example

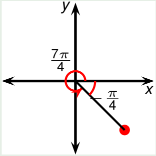
Convert the point $(2, \frac{\pi}{3})$ from polar to Cartesian coordinates.

$$x = r\cos\theta = 2\cos\frac{\pi}{3} = 2\left(\frac{1}{2}\right) = 1$$

$$y = r\sin\theta = 2\sin\frac{\pi}{3} = 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}$$

Therefore the point with polar coordinates $(2, \frac{\pi}{3})$ has Cartesian coordinates $(1, \sqrt{3})$.

Example

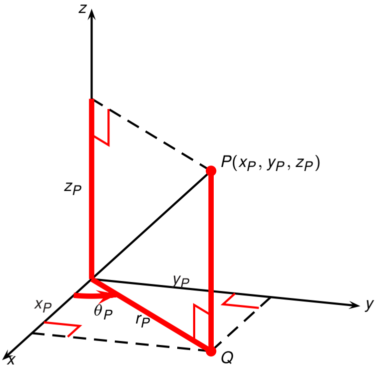


Represent the point with Cartesian coordinates $(1, -1)$ in terms of polar coordinates.

- Suppose r is positive.
- $\tan\theta = -1$ for $\theta = \frac{3\pi}{4}, \frac{7\pi}{4}$, and many other angles.
- $(1, -1)$ is in the fourth quadrant.
- Of the two values above, only $\theta = \frac{7\pi}{4}$ gives a point in the fourth quadrant.
- $\Rightarrow$ one representation of $(1, -1)$ in polar coordinates is $\left(\sqrt{2}, \frac{7\pi}{4}\right)$.
- $\left(\sqrt{2}, -\frac{\pi}{4}\right)$ is another.

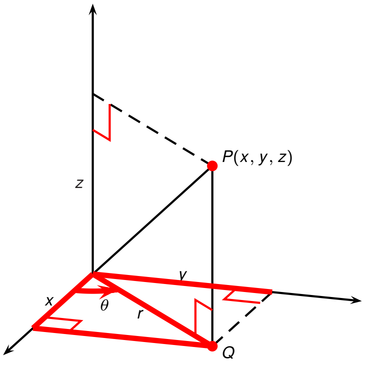
$$
\begin{aligned}
r &= \pm\sqrt{x^2 + y^2} \\
&= \sqrt{1^2 + (-1)^2} \\
&= \sqrt{2} \\
\tan\theta &= \frac{y}{x} \\
&= -1
\end{aligned}
$$

Cylindrical coordinates



- In Cartesian coordinates, a point P is given by triple (x_P, y_P, z_P).
- We introduce alternative cylindrical coordinates (r_P, θ_P, z_P).
- Cylindrical coordinates are obtained by "adding a z-coordinate" to the (2-dimensional) polar coordinates.
- More precisely, to P we assign triple (r_P, θ_P, z_P), where:
  - z_P equals the z-coordinate of P,
  - r_P is the distance $|OQ|$, where Q is the projection of P in the xy-plane and O-origin,
  - θ_P is an angle between the x-axis and **OQ**.

Cylindrical to and from cartesian coordinates



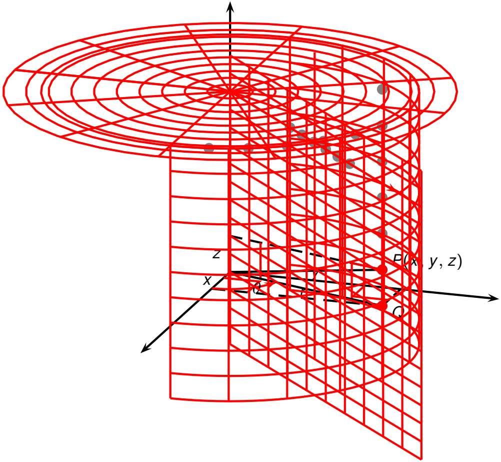
To transform cylindrical to rectangular coordinates:

$$
\begin{aligned}
x &= r\cos\theta \\
y &= r\sin\theta \\
z_{rectangular} &= z_{cylindrical}
\end{aligned}
$$

To transform rectangular to cylindrical:

$$
\begin{aligned}
r &= \sqrt{x^2+y^2} \\
\cos\theta &= \frac{x}{r} \\
\sin\theta &= \frac{y}{r} \\
\theta &= \arcsin\left(\frac{y}{r}\right) && \text{if } x>0 \\
&= \arccos\left(\frac{x}{r}\right) && \text{if } y>0 \\
&= \arctan\left(\frac{y}{x}\right) && \text{if } x>0 \\
z_{cylindirical} &= z_{rectangular}
\end{aligned}
$$

Constant Coordinate Sets



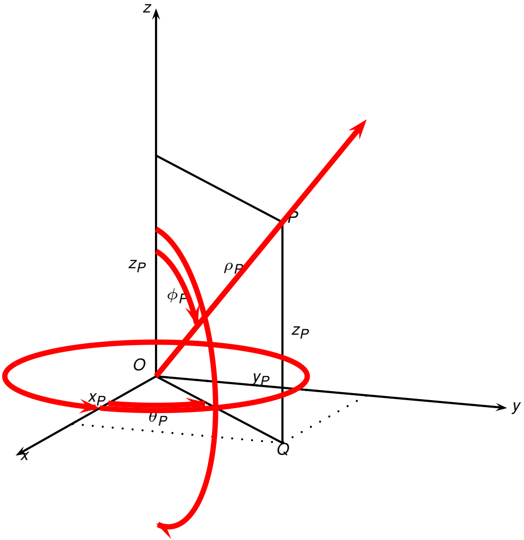
What curve is traced when:

- keep θ, z constant, let r vary: horizontal ray;
- keep r, z constant, let θ vary: horizontal circle;
- keep r, θ constant, let z vary: vertical line.

What surface is traced when:

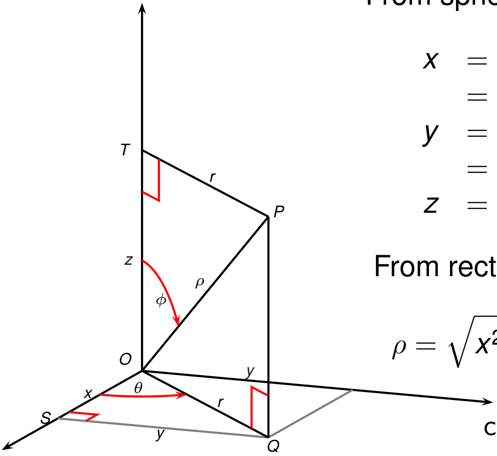
- keep r constant, let θ, z vary: vertical cylinder;
- keep θ constant, let r, z vary: vertical half plane;
- keep z constant, let r, θ vary: horizontal plane.

Spherical Coordinates



- In Cartesian coordinates, a point P is given by triple (x_P, y_P, z_P).
- We introduce alternative spherical coordinates $(\rho_P, \phi_P, \theta_P)$.
  - ρ_P: distance $|OP|$;
  - ϕ_P: angle Oz to OP;
  - θ_P: angle Ox to OP_{xy}.
- Coordinates range:
  - ρ: $[0, \infty)$;
  - ϕ: $[0, \pi]$;
  - θ: $[0, 2\pi)$.

Transition Spherical - Rectangular coordinates



From spherical to rectangular coords:

$$\begin{array}{rcl|l} x &=& r\cos\theta & \text{use } \triangle SQO \\ &=& \rho\sin\phi\cos\theta & \text{use } \triangle OPT \\ y &=& r\sin\theta & \\ &=& \rho\sin\phi\sin\theta & \\ z &=& \rho\cos\phi & \end{array}$$

From rectangular to spherical coords:

$$\rho = \sqrt{x^2+y^2+z^2} \quad , \quad r = \sqrt{x^2+y^2}$$

$$\cos\phi = \frac{z}{\rho} \quad , \quad \sin\phi = \frac{r}{\rho}$$

$$\cos\theta = \frac{x}{r} \quad , \quad \sin\theta = \frac{y}{r}$$

Constant Coordinate Sets

What curve is traced when:

- keep θ, ϕ constant, let ρ vary: ray through the origin;
- keep ρ, ϕ constant, let θ vary: circle parallel to the *xy*-plane, "parallel";
- keep ρ, θ constant, let ϕ vary: circle passing through *z* axis, "meridian".

What surface is traced when:

- keep ϕ constant, let θ, ρ vary: cone;
- keep θ constant, let ρ, ϕ vary: vertical half plane;

Polar curvilinear "boxes"

Polar "wedge":

$$C = \{P(r, \theta) \mid r_0 \leqslant r \leqslant r_0 + \Delta r, \theta_0 \leqslant \theta \leqslant \theta_0 + \Delta\theta\} \ .$$

Shape? Area = ...?

Cylindrical curvilinear “boxes”

Cylindrical “box”:

$$X = \{P(r, \theta, z) \mid 0 \leqslant r \leqslant r_0 \,, 0 \leqslant \theta \leqslant \theta_0 \,, 0 \leqslant z \leqslant z_0\}$$

Shape ? Volume = ...?

Spherical curvilinear "boxes"

- Cut off a rectangular box B in the ρ, ϕ, θ-coordinates. $B :=$
$$\left\{ (\rho, \phi, \theta) \middle| \begin{array}{ccccc} \rho_{min} & \leq & \rho & \leq & \rho_{max} \\ \phi_{min} & \leq & \phi & \leq & \phi_{max} \\ \theta_{min} & \leq & \theta & \leq & \theta_{max} \end{array} \right\}$$
- As (ρ, ϕ, θ) traverse B, the point $P(\rho, \phi, \theta)$ traverses curvilinear "box" Y:
$$Y = \{P(\rho, \phi, \theta) | (\rho, \phi, \theta) \in B\} .$$
- What is the shape of that curvilinear box?
- What is the volume?