Calculus I
Homework Volumes of Solids of Revolution
Lecture 25

1. (a) Consider the region bounded by the curves $y = 2x^2 - x + 1$ and $y = x^2 + 1$. What is the volume of the solid obtained by rotating this region about the line $x = 0$?

   answer: $\frac{2}{5}\pi$.

   (b) Consider the region bounded by the curves $y = 1 - x^2$ and $y = 0$. What is the volume of the solid obtained by rotating this region about the line $y = 0$?

   answer: $\frac{16\pi}{15}$

   (c) Consider the region bounded by the curves $y = x^2$ and $x = y^2$. What is the volume of the solid obtained by rotating this region about the line $x = 2$?

   answer: $\frac{31\pi}{30}$

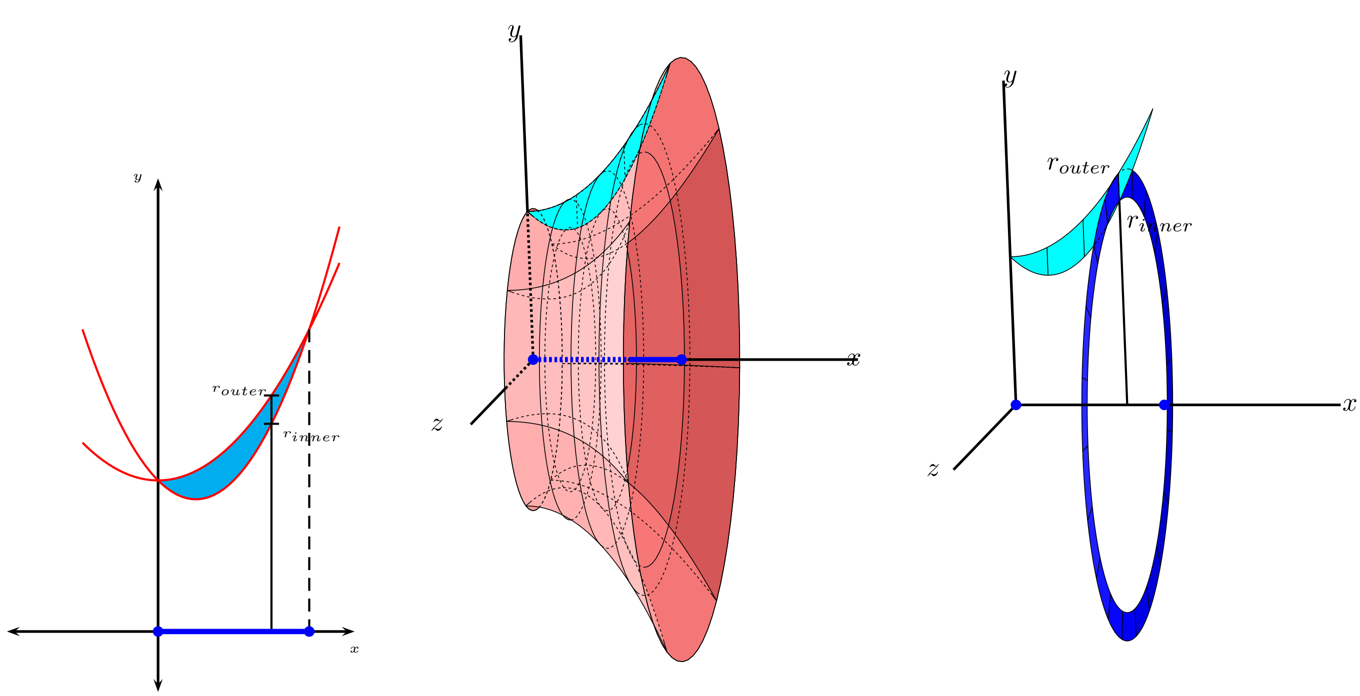
   (d) Set up BUT DO NOT EVALUATE an integral to calculate the volume of the solid obtained by rotating the region bounded by $y = -x^2 + 2$ and $y = 0$ about the given line.

   - The x axis.
   - The line $y = -3$.

   (e) Set up BUT DO NOT EVALUATE an integral to calculate the volume of the solid obtained by rotating the region bounded by $y = -x^2 + 1$ and $y = 0$ about the given line.

   - The x axis.
   - The line $y = -4$.

Solution. 1.a First, plot $y = 2x^2 - x + 1$ and $y = x^2 + 1$.



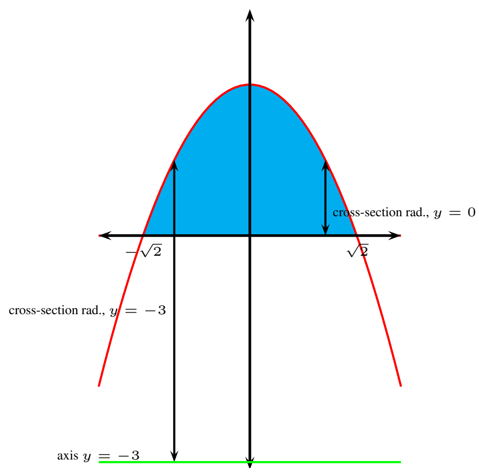
The two curves intersect when

$$\begin{aligned} 2x^2 - x + 1 &= x^2 + 1 \\ x^2 - x &= 0 \\ x(x-1) &= 0 \\ x = 0 \quad &\text{or} \quad x = 1. \end{aligned}$$

Therefore the two points of intersection have x-coordinates between $x = 0$ and $x = 1$. Therefore we need we need to integrate the volumes of washers with inner radii $r_{inner} = 2x^2 - x + 1$, outer radii $r_{outer} = x^2 + 1$ and infinitesimal heights $\mathrm{d}x$. The volume of an individual infinitesimal washer is then $\pi(r_{outer}^2 - r_{inner}^2)\mathrm{d}x$

$$\begin{aligned} V &= \int_0^1 \pi \left(\left(x^2+1\right)^2 - \left(2x^2 - x + 1\right)^2 \right) \mathrm{d}x \\ &= \pi \int_0^1 \left(-3x^4 + 4x^3 - 3x^2 + 2x\right) \mathrm{d}x \\ &= \pi \left[-\frac{3}{5}x^5 + x^4 - x^3 + x^2 \right]_0^1 \\ &= \frac{2}{5}\pi. \end{aligned}$$

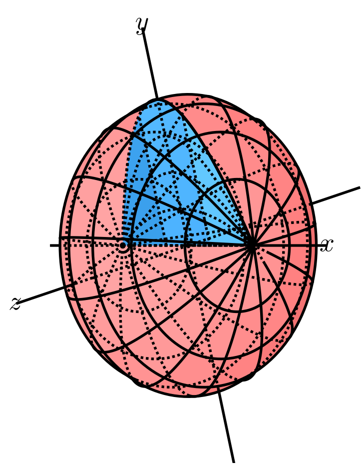
Solution. 1.d First, we plot the 2d region. The two curves intersect when $-x^2 + 2 = 0$, i.e., when $x = \pm\sqrt{2}$.



Rotation about $y = 0$**.**

Unless explicitly stated in the problem, a 3d plot of the solid is not required in the solution. Nevertheless generating such a plot helps to understand the problem.

To generate a 3d plot of the solid, we draw the circular cross-sections of the solid of revolution. By hand, this can be done roughly by drawing ovals (circles look like ovals when observed at an angle) centered at the axis about which we revolve the 2d-region. We include a computer-generated plot below; the plot's precision is above what is expected on an exam.

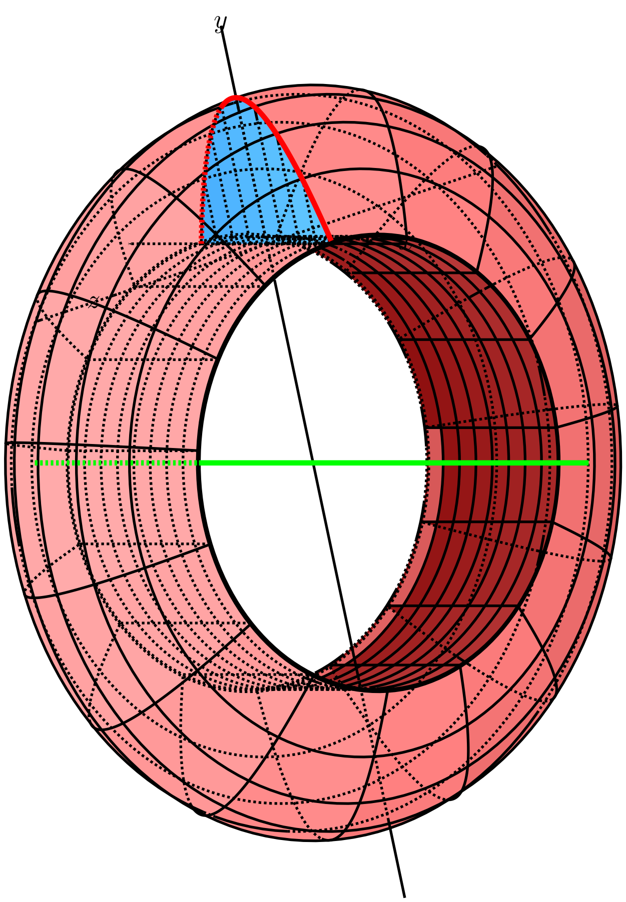


The volume of a solid (and in particular, of a solid of revolution) is computed by integrating the area $A(x) = \pi(\text{radius cross-section})^2 = \pi(-x^2+2)^2$ of the cross-section of the solid. Therefore the volume V equals

$$\begin{array}{rcll} V &=& \displaystyle\int_a^b A(x)\mathrm{d}x & \\ &=& \displaystyle\int_{-\sqrt{2}}^{\sqrt{2}} \pi(-x^2+2)^2 \mathrm{d}x & \\ &=& \pi\left[\frac{1}{5}x^5 - \frac{4}{3}x^3 + 4x\right]_{-\sqrt{2}}^{\sqrt{2}} & \text{step not required by problem} \\ &=& \pi\frac{64}{15}\sqrt{2} & \text{step not required by problem.} \end{array}$$

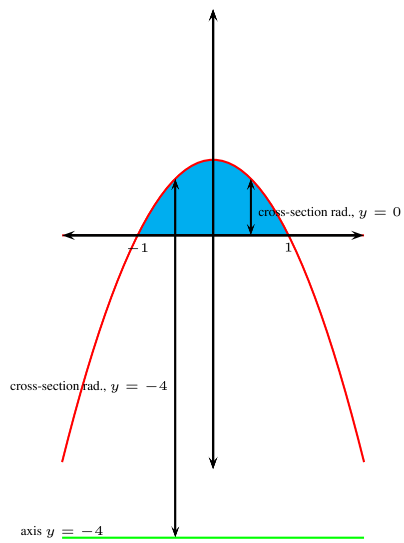
Rotation about $y = -3$. The cross-section of this solid of revolution is a washer with inner radius 3 and outer radius $-x^2+2-(-3) = 5-x^2$. Therefore the area of the cross-section is $\pi(5-x^2)^2 - \pi 3^2$ and the volume is computed via

$$\begin{array}{rcll} V &=& \displaystyle\int_a^b A(x)\mathrm{d}x & \\ &=& \displaystyle\int_{-\sqrt{2}}^{\sqrt{2}} \pi\left((5-x^2)^2 - 3^2\right) \mathrm{d}x & \\ &=& \pi\left[\frac{1}{5}x^5 - \frac{10}{3}x^3 + 16x\right]_{-\sqrt{2}}^{\sqrt{2}} & \text{step not required by problem} \\ &=& \pi\frac{304}{15}\sqrt{2} & \text{step not required by problem.} \end{array}$$

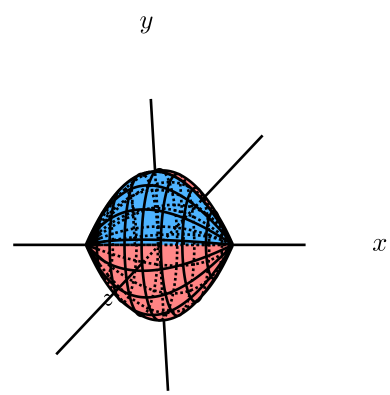


Solution. 1.e

First, we plot the 2d region. The two curves intersect when $-x^2+1 = 0$, i.e., when $x = \pm 1$.



Rotation about $y = 0$**.**

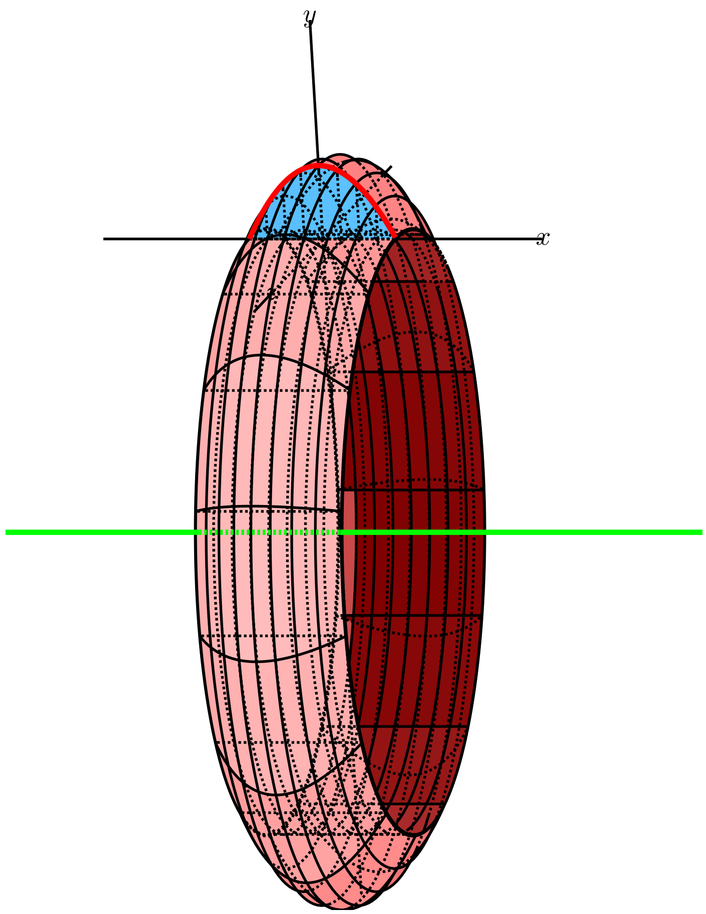


The volume of a solid (and in particular, of a solid of revolution) is computed by integrating the area $A(x) = \pi(\text{radius cross-section})^2 = \pi(-x^2+1)^2$ of the cross-section of the solid. Therefore the volume V equals

$$\begin{array}{rcll} V &=& \int_a^b A(x)\mathrm{d}x & \\ &=& \int_{-1}^1 \pi(-x^2+1)^2\mathrm{d}x & \\ &=& \pi\left[\frac{1}{5}x^5 - \frac{2}{3}x^3 + x\right]_{-1}^1 & \text{step not required by problem} \\ &=& \pi\frac{16}{15} & \text{step not required by problem.} \end{array}$$

Rotation about $y = -4$**.** The cross-section of this solid of revolution is a washer with inner radius 4 and outer radius $-x^2+1-(-4) = 5-x^2$. Therefore the area of the cross-section is $\pi(5-x^2)^2 - \pi 4^2$ and the volume is computed via

$$\begin{array}{rcll} V &=& \int_a^b A(x)\mathrm{d}x & \\ &=& \int_{-1}^1 \pi\left((5-x^2)^2 - 4^2\right)\mathrm{d}x & \\ &=& \pi\left[\frac{1}{5}x^5 - \frac{10}{3}x^3 + 9x\right]_{-1}^1 & \text{step not required by problem} \\ &=& \frac{176}{15}\pi & \text{step not required by problem.} \end{array}$$



2. (a) Consider the region bounded by the curves $y = \sqrt{x}$, $x = 0$, $y = 2$. Use the method of cylindrical shells to find the volume of the solid obtained by rotating this region about the x-axis.

answer: 8π

(b) Consider the region bounded by the curves $y = x^2$ and $y = 2 - x^2$. Use the method of cylindrical shells to find the volume of the solid obtained by rotating this region about the line $x = 1$.

answer: $16\frac{\pi}{3}$