Calculus III
Lecture 6

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https://github.com/tmilev/freecalc

2020

Outline

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Definition ($\mathbb{R}^2$)

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$(0, 5, -2, 4, 0) \in$
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Parametric Equations of a Line Segment

- Recall parametric vector equation of line L:

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\mathbf{r} &= \mathbf{r}_0 + t\mathbf{u}, && t \text{ real number.}\\
\mathbf{r} &= \mathbf{r}_0 + t(\mathbf{r}_1 - \mathbf{r}_0), && t \text{ real number.}\\
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- Parametric scalar equations:

$$\left| \begin{array}{l} x = x_0 + tu_1 \\ y = y_0 + tu_2 \\ z = z_0 + tu_3 \end{array} \right. \Leftrightarrow \left| \begin{array}{l} x = x_0 + t(x_1 - x_0) \\ y = y_0 + t(y_1 - y_0) \\ z = z_0 + t(z_1 - z_0) \end{array} \right. \Leftrightarrow \left| \begin{array}{l} x = (1-t)x_0 + tx_1 \\ y = (1-t)y_0 + ty_1 \\ z = (1-t)z_0 + tz_1 \end{array} \right.$$

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- Segment with endpoints $P_0(\mathbf{r}_0)$ and $P_1(\mathbf{r}_1)$:

$$\mathbf{r} = (1-t)\mathbf{r}_0 + t\mathbf{r}_1, \quad 0 \leqslant t \leqslant 1$$

Example

Parametrize

- the line L passing through $P_0(1, 2, 3)$ and $P_1(5, 2, 1)$;
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Parametric vectorial equation of segment S:

$$\mathbf{r} = t(1,2,3) + (1-t)(5,2,1) \quad t \in [0,1] \quad .$$

Parametrized Curves

- A curve parametrization is a function $\mathbf{r} : [a, b] \to \mathbb{R}^2$ $\mathbf{r} : [a, b] \to \mathbb{R}^3$, or $\mathbf{r} : [a, b] \to \mathbb{R}^n$ in general.

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- $x, y, z\colon [a, b] \to \mathbb{R}$, coordinate functions.

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Describe the curve:

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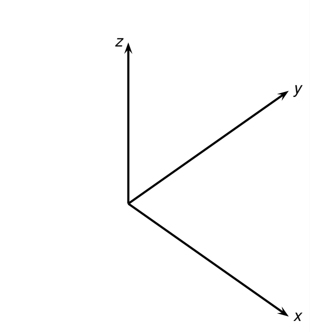

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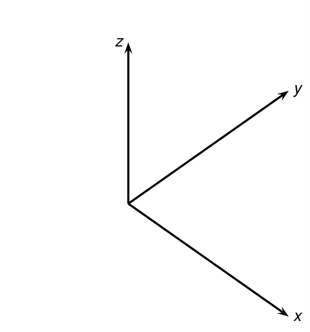

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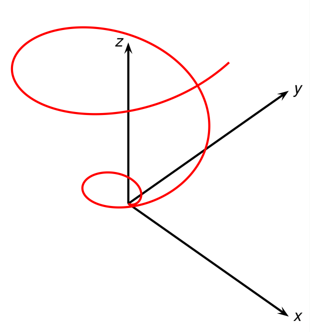

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- "Tornado".

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if by selecting that $t \neq a$ be close enough to a we can guarantee that $\mathbf{r}(t)$ is as close to $\mathbf{u}$ as we want.

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- Then

$$\lim_{t\to a} \mathbf{r}(t) = \mathbf{u} \Longleftrightarrow \left| \begin{array}{l} \lim\limits_{t\to a} x(t) = u_1 \\ \lim\limits_{t\to a} y(t) = u_2 \\ \lim\limits_{t\to a} z(t) = u_3 \end{array} \right. .$$

Continuity

Definition

Suppose

- $\mathbf{r}$ is defined at t_0

Observation

$\mathbf{r}(t) = (x(t), y(t), z(t))$ is continuous at $t_0 \iff x(t), y(t), z(t)$ are all continuous at t_0.

Continuity

Definition

Suppose

- $\mathbf{r}$ is defined at t_0
- $\lim_{t \to t_0} \mathbf{r}(t)$ exists.

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$\mathbf{r}(t) = (x(t), y(t), z(t))$ *is continuous at* $t_0 \iff x(t), y(t), z(t)$ *are all continuous at* t_0*.*

Continuity

Definition

Suppose

- $\mathbf{r}$ is defined at t_0
- $\lim_{t\to t_0} \mathbf{r}(t)$ exists.

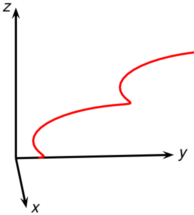
Then we say that $\mathbf{r}\colon [a,b] \to \mathbb{R}^3$ is continuous at t_0 if

$$\lim_{t\to t_0} \mathbf{r}(t) = \mathbf{r}(t_0) \quad .$$

Observation

$\mathbf{r}(t) = (x(t), y(t), z(t))$ *is continuous at* $t_0 \iff x(t), y(t), z(t)$ *are all continuous at* t_0*.*

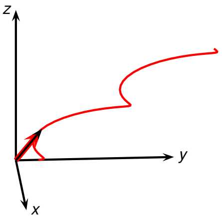
Derivatives



$$\begin{aligned}\mathbf{f}\colon [a,b] &\to \mathbb{R}^3\\ \mathbf{f}(t) &= (x(t), y(t), z(t))\end{aligned}$$

- **Average Velocity** $= \dfrac{\textbf{change in position}}{\text{change in time}} = \dfrac{\mathbf{f}(t)-\mathbf{f}(t_0)}{t-t_0}.$

Derivatives



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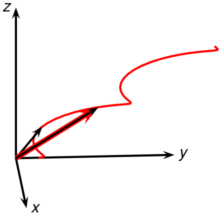

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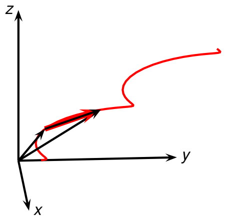

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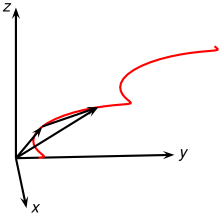

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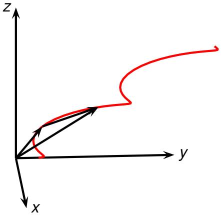

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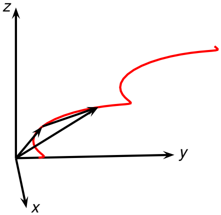

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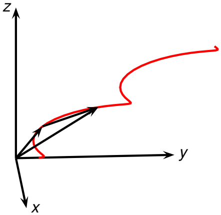
Derivatives



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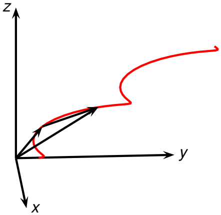

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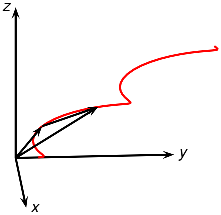
- $\mathbf{f}(t)$ vector $\Longrightarrow$ $\mathbf{f}'(t)$ vector.

Derivatives



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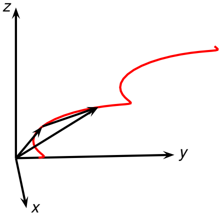

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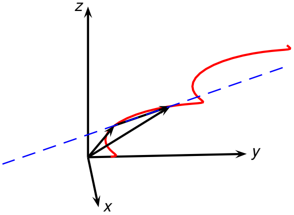
- $\mathbf{f}(t)$ vector $\Longrightarrow$ $\mathbf{f}'(t)$ vector.
- Higher order derivatives: $\mathbf{f}'(t)$, $\mathbf{f}''(t) = (\mathbf{f}'(t))'$ (acceleration), ...

Tangent Lines



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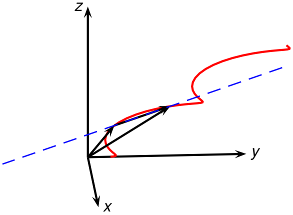
Tangent Lines



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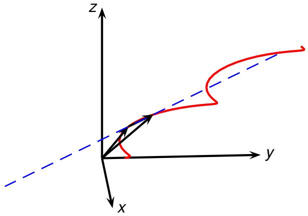

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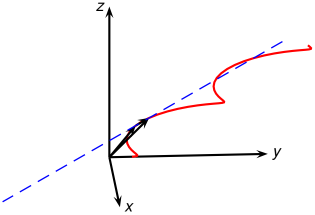

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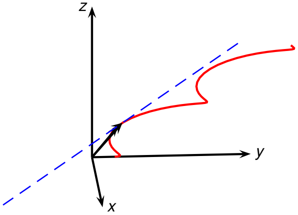

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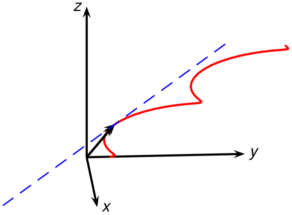

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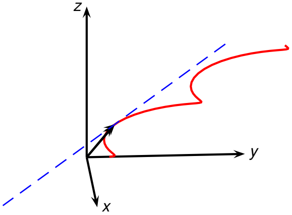

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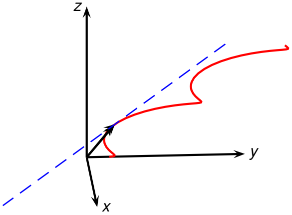
- $\mathbf{f}'(t_0)$: direction of tangent line through $\mathbf{f}(t_0)$.
- Tangent equation at $\mathbf{f}(t_0)$:

$$\mathbf{r}(t) = \mathbf{f}(t_0) + t\mathbf{f}'(t_0).$$

Tangent Lines



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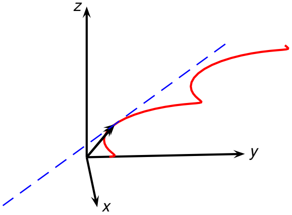
Tangent Lines



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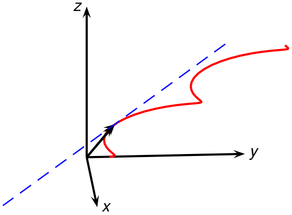
Tangent Lines



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- Differentials:
$$d\mathbf{f} = \mathbf{f}' dt = (x', y', z') dt.$$

Example

Let $\mathbf{r}(t)$ be the coordinate curves for the spherical coordinates, i.e., let

$$\begin{array}{rcl} \mathbf{e}_\rho(t) & = & (t\sin\phi\cos\theta, t\sin\phi\sin\theta, t\cos\phi) \\ \mathbf{e}_\phi(t) & = & (\rho\sin t\cos\theta, \rho\sin t\sin\theta, \rho\cos t) \\ \mathbf{e}_\theta(t) & = & (\rho\sin\phi\cos t, \rho\sin\phi\sin t, \rho\cos\phi) \end{array}$$

where ρ, ϕ, θ are regarded as constants and t as the curve parameter. Find $\mathbf{e}'_\rho(t)$, $\mathbf{e}'_\phi(t)$, $\mathbf{e}'_\theta(t)$. Compute $(\mathbf{e}'_\rho(\rho) \times \mathbf{e}'_\theta(\theta)) \cdot \mathbf{e}'_\phi(\phi)$.

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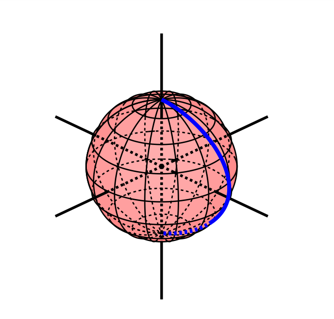
- What can we say about constant acceleration?

$$\mathbf{r}\cdot\mathbf{r}' \equiv 0 \implies [\mathbf{r}\cdot\mathbf{r}']' = 0 \iff \mathbf{r}'\cdot\mathbf{r}' + \mathbf{r}\cdot\mathbf{r}'' = 0 \implies \mathbf{r}\cdot\mathbf{r}'' = -|\mathbf{r}'|^2 \leq 0$$

Acceleration vector $\mathbf{r}''$ points inside the sphere.

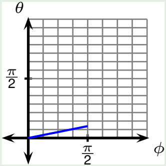
Example



Compute the acceleration vector when traversing the loxodrome curve below.

$$\begin{aligned} x &= \rho \sin(at) \cos(bt) \\ y &= \rho \sin(at) \sin(bt) \\ z &= \rho \cos(at) \end{aligned}$$

Spherical coordinates:



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$\mathbf{r}\colon [a, b] \to \mathbb{R}^3$

- Division $a = t_0 < t_1 < t_2 < \cdots < t_k < t_{k+1} < \cdots < t_n = b$

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- Result: a vector.

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Parabola in the plane determined by $\mathbf{v}_0$ and $\mathbf{k}$.

Arclength

- $\mathbf{r}\colon [a, b] \to \mathbb{R}^3$: piecewise smooth function
- Distance traveled $=$ Speed $\cdot$ Time

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- Length of parametrized curve:

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Arclength Function

- Fix $t = a$ as starting point. For $t \geqslant a$, let
- $L(t) =$ distance traveled between a and t = length of the piece of the curve corresponding to values of the parameter between a and t.

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- The function $L\colon [a, b] \to \mathbb{R}$ is called the *arclength function*.

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Let $\mathbf{r}(t) = (\cos t, \sin t, t)$

- Do you know the name of this curve?
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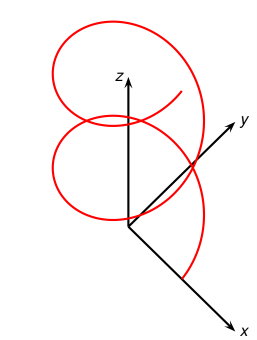

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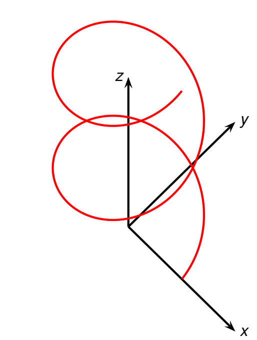

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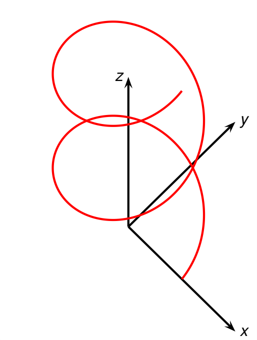

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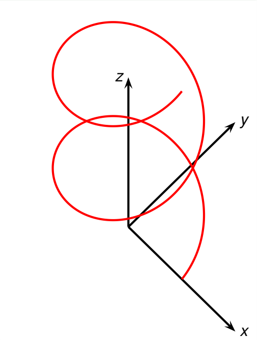

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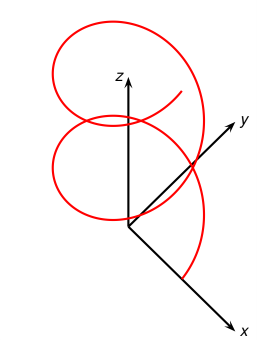

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Parametrization by Arclength

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$$\mathbf{p}(s) = \left(\cos\left(\frac{s}{\sqrt{2}}\right), \sin\left(\frac{s}{\sqrt{2}}\right), \left(\frac{s}{\sqrt{2}}\right)\right)$$

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- $\mathbf{r} = \mathbf{r}(t)$: smooth parametrization, not necessary by arclength.

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{|\mathbf{r}''(t) \times \mathbf{r}'(t)|}{|\mathbf{r}'(t)|^3} \ .$$

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| \Rightarrow \kappa = \frac{|\mathbf{r}''(t)\times\mathbf{r}'(t)|}{|\mathbf{r}'(t)|^3}$$

$s(t) = \int_{t_0}^{t} |\mathbf{r}'(x)|dx$-curve (arc)length function.

$$|\mathbf{v}|' = \left(\sqrt{\mathbf{v}\cdot\mathbf{v}}\right)' = \frac{(\mathbf{v}\cdot\mathbf{v})'}{2\sqrt{\mathbf{v}\cdot\mathbf{v}}} = \frac{2\mathbf{v}'\cdot\mathbf{v}}{2|\mathbf{v}|} = \frac{\mathbf{v}'\cdot\mathbf{v}}{|\mathbf{v}|}. \quad \frac{ds}{dt} = |\mathbf{r}'(t)| \quad \frac{d\mathbf{T}}{ds}\frac{ds}{dt} = \frac{d\mathbf{T}}{dt}$$

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \left|\frac{\frac{d\mathbf{T}}{dt}}{\frac{ds}{dt}}\right| = \left|\frac{\mathbf{T}'(t)}{\mathbf{r}'(t)}\right| = \frac{\left|\left(\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}\right)'\right|}{|\mathbf{r}'(t)|} = \frac{\frac{\left|\mathbf{r}''|\mathbf{r}'|-\mathbf{r}'(|\mathbf{r}'|)'\right|}{|\mathbf{r}'|^2}}{|\mathbf{r}'|}$$

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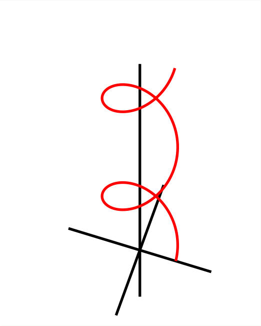
Example

Compute the curvature of $\mathbf{r}(t) = (\cos t, \sin t, t)$.



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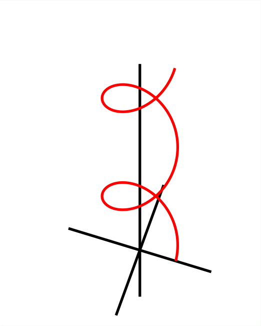

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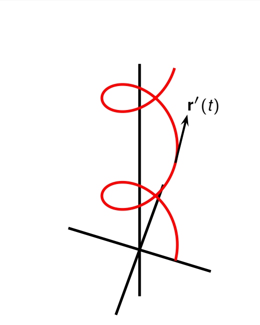

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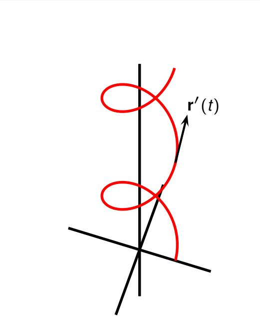

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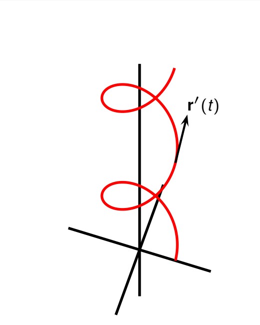

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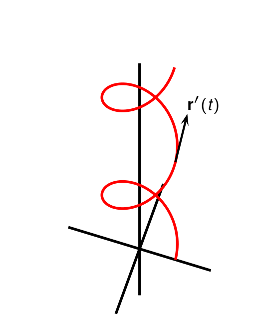

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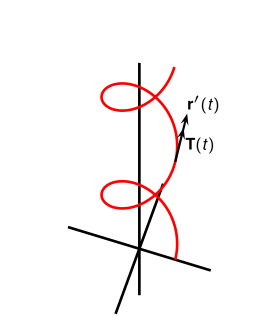

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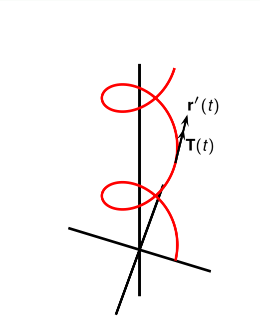

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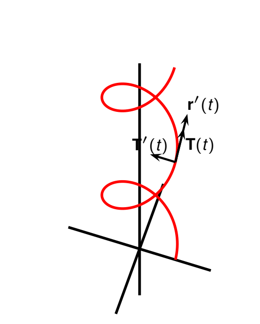

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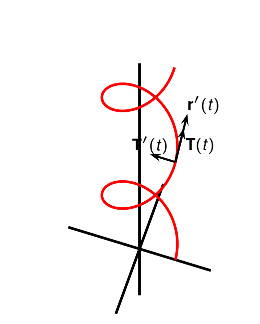

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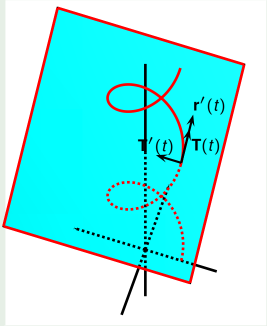

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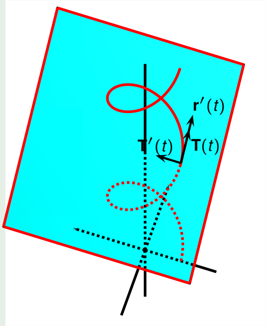

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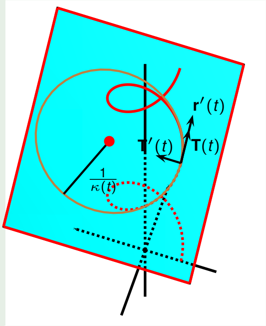
.



Notation: **T**- unit tangent vector, **r**- position vector, κ-curvature.

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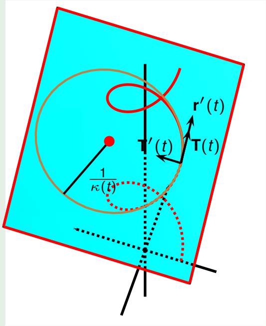

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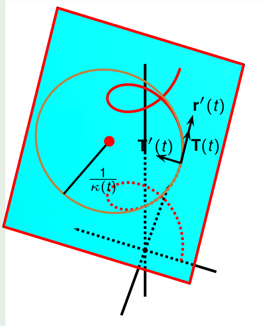

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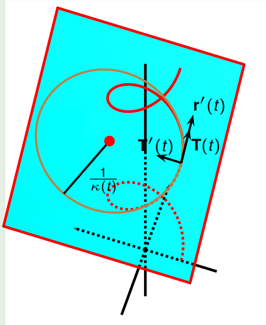

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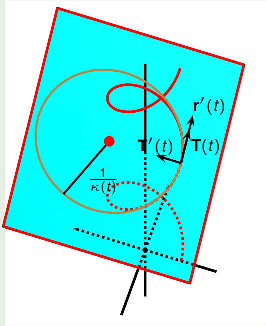

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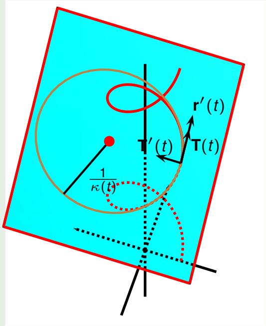

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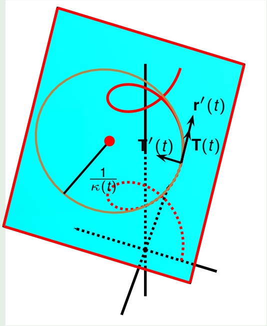

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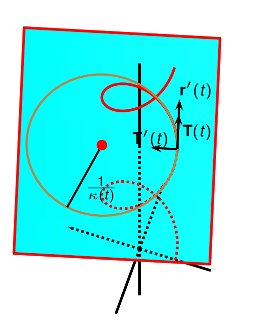

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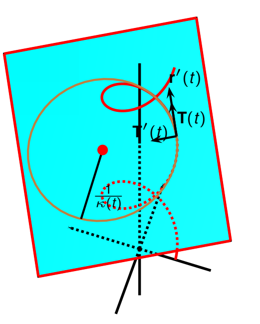

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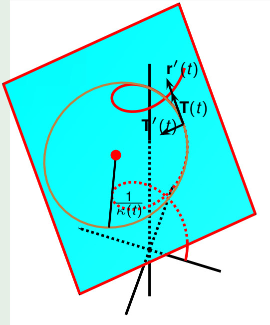

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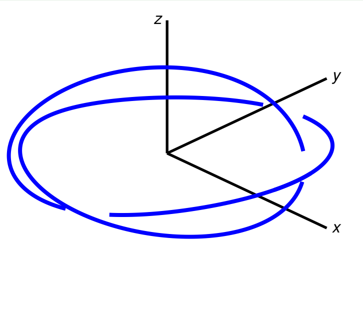
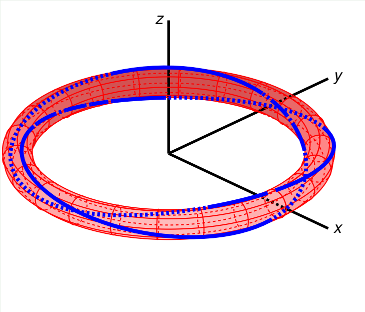


Compute the curvature of the (torus) trefoil curve

$$
\begin{aligned}
x &= (R + r\sin(3t))\cos(2t)\\
y &= (R + r\sin(3t))\sin(2t)\\
z &= r\cos(3t)
\end{aligned}
$$

Example



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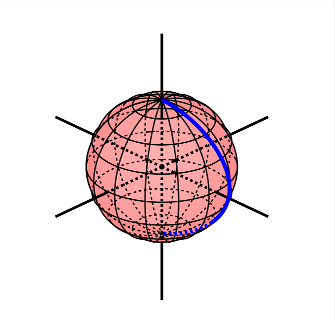
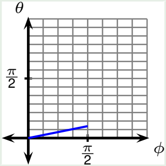


Compute the curvature of the loxodrome

$$\begin{aligned} x &= \rho \sin(at)\cos(bt) \\ y &= \rho \sin(at)\sin(bt) \\ z &= \rho \cos(at). \end{aligned}$$

Spherical coordinates:



$x = \rho \sin\phi \cos\theta$

$y = \rho \sin\phi \sin\theta$

$z = \rho \cos\phi$

Components of Acceleration

- Object moves through space, $\mathbf{r} = \mathbf{r}(t)$ position vector at time t;
- Velocity vector $\mathbf{v}(t) = \mathbf{r}'(t)$;
- Tangent direction: $\mathbf{T}(t) = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|}$;
- Speed is $v(t) = |\mathbf{v}(t)|$;
- Acceleration $\mathbf{a}(t) = \mathbf{r}''(t)$;
- Tangential component $\mathbf{a}_T(t)$:

$$\mathbf{a}_T(t) = \mathrm{proj}_{\mathbf{T}(t)}\mathbf{a}(t) = \frac{\mathbf{a}\cdot\mathbf{T}}{|\mathbf{T}|}\mathbf{T} = \frac{\mathbf{v}'\cdot\mathbf{v}}{|\mathbf{v}|}\mathbf{T} = |\mathbf{v}|'\mathbf{T} = v'\mathbf{T} \ ,$$

$$a_T(t) = |\mathbf{a}_T(t)| = |v'(t)| \ .$$

- Normal component $\mathbf{a}_N(t) = \mathrm{orth}_{\mathbf{T}(t)}\mathbf{a}(t)$,

$$a_N(t) = |\mathbf{a}_N(t)| = |\mathrm{orth}_{\mathbf{T}}\mathbf{a}| = |\mathbf{a}\times\mathbf{T}| = \frac{|\mathbf{r}''\times\mathbf{r}'|}{|\mathbf{r}'|} = \kappa|\mathbf{r}'|^2 = \kappa(t)v^2(t) \ .$$