Calculus I
Homework Limits involving ∞
Lecture 5

1. Show the following limits do not exist and compute whether they evaluate to ∞, $-\infty$, or neither.

   (a) $\displaystyle\lim_{x\to 3^+} \frac{x^2+x-1}{x^2-2x-3}$. answer: ∞.

   (b) $\displaystyle\lim_{x\to 3^-} \frac{x^2+x-1}{x^2-2x-3}$. answer: $-\infty$.

   (c) $\displaystyle\lim_{x\to 1^+} \frac{x^2+1}{\sqrt{x^2+3}-2}$. answer: ∞.

   (d) $\displaystyle\lim_{x\to 1^-} \frac{x^2+1}{\sqrt{x^2+3}-2}$. answer: $-\infty$.

   (e) $\displaystyle\lim_{x\to 2^+} \frac{\sqrt{x^3-8}}{-x^2+x+2}$. answer: $-\infty$.

   (f) $\displaystyle\lim_{x\to -1^+} \frac{\sqrt[3]{x^2+2x+1}}{x^2-2x-3}$. answer: $-\infty$.

2. Find the limit or show that it does not exist. If the limit does not exist, indicate whether it is $\pm\infty$, or neither. The answer key has not been proofread, use with caution.

   (a) $\displaystyle\lim_{x\to\infty} \frac{x-2}{2x+1}$. answer: $\frac{1}{2}$

   (b) $\displaystyle\lim_{x\to\infty} \frac{1-x^2}{x^3-x-1}$. answer: 0

   (c) $\displaystyle\lim_{x\to-\infty} \frac{x-2}{x^2+5}$. answer: 0

   (d) $\displaystyle\lim_{x\to-\infty} \frac{3x^3+2}{2x^3-4x+5}$. answer: $\frac{3}{2}$

   (e) $\displaystyle\lim_{x\to\infty} \frac{\sqrt{x}+x^2}{\sqrt{x}-x^2}$. answer: -1

   (f) $\displaystyle\lim_{x\to\infty} \frac{3-x\sqrt{x}}{2x^{\frac{3}{2}}-2}$. answer: $-\frac{1}{2}$

   (g) $\displaystyle\lim_{x\to\infty} \frac{(2x^2+3)^2}{(x-1)^2(x^2+1)}$. answer: 4

   (h) $\displaystyle\lim_{x\to\infty} \frac{x^2-3}{\sqrt{x^4+3}}$. answer: 1

   (i) $\displaystyle\lim_{x\to-\infty} \frac{\sqrt{x^2+1}}{x+1}$. answer: -1

   (j) $\displaystyle\lim_{x\to\infty} \frac{\sqrt{16x^6-3x}}{x^3+2}$. answer: 4

   (k) $\displaystyle\lim_{x\to-\infty} \frac{\sqrt{16x^6-3x}}{x^3+2}$. answer: -4

   (l) $\displaystyle\lim_{x\to\infty} \frac{\sqrt{3x^2+2x+1}}{x+1}$. answer: $\sqrt{3}$

   (m) $\displaystyle\lim_{x\to\infty} \sqrt{4x^2+x}-2x$. answer: $\frac{1}{4}$

   (n) $\displaystyle\lim_{x\to-\infty} x+\sqrt{x^2+3x}$. answer: $-\frac{3}{2}$

   (o) $\displaystyle\lim_{x\to\infty} \sqrt{x^2+2x}-\sqrt{x^2-2x}$. answer: 2.

   (p) $\displaystyle\lim_{x\to-\infty} \sqrt{x^2+x}-\sqrt{x^2-x}$. answer: -1.

   (q) $\displaystyle\lim_{x\to\infty} \sqrt{x^2+ax}-\sqrt{x^2+bx}$. answer: $\frac{a-b}{2}$

   (r) $\displaystyle\lim_{x\to\infty} \cos x$. answer: DNE

   (s) $\displaystyle\lim_{x\to\infty} \frac{x^4+x}{x^3-x+2}$. answer: ∞

   (t) $\displaystyle\lim_{x\to\infty} \sqrt{x^2+1}$. answer: ∞

   (u) $\displaystyle\lim_{x\to-\infty} (x^4+x^5)$. answer: $-\infty$

   (v) $\displaystyle\lim_{x\to-\infty} \frac{\sqrt{1+x^6}}{1+x^2}$. answer: ∞

   (w) $\displaystyle\lim_{x\to\infty} (x-\sqrt{x})$. answer: ∞

   (x) $\displaystyle\lim_{x\to\infty} (x^2-x^3)$. answer: $-\infty$

   (y) $\displaystyle\lim_{x\to\infty} x\sin x$. answer: DNE

   (z) $\displaystyle\lim_{x\to\infty} \sqrt{x}\sin x$. answer: DNE

Solution. 2.d.

$$\begin{array}{rcll}
\displaystyle\lim_{x\to-\infty} \frac{3x^3+2}{2x^3-4x+5} & = & \displaystyle\lim_{x\to-\infty} \frac{\left(3x^3+2\right)\frac{1}{x^3}}{\left(2x^3-4x+5\right)\frac{1}{x^3}} & \Bigg| \text{ Divide top and bottom} \\
& = & \displaystyle\lim_{x\to-\infty} \frac{3+\frac{2}{x^3}}{2-\frac{4}{x^2}+\frac{5}{x^3}} & \Bigg| \text{ by highest term in denominator} \\
& = & \displaystyle\frac{3+0}{2-0+0} = \frac{3}{2}. &
\end{array}$$

Solution. 2.i

$$\lim_{x\to-\infty}\frac{\sqrt{x^2+1}}{x+1} = \lim_{x\to-\infty}\frac{\frac{1}{x}\sqrt{x^2+1}}{\frac{1}{x}(x+1)} = \lim_{x\to-\infty}\frac{-\frac{1}{\sqrt{x^2}}\sqrt{x^2+1}}{\frac{1}{x}(x+1)} \qquad \Big|\ x=-\sqrt{x^2},\ \text{whenever } x<0$$

$$= \lim_{x\to-\infty}\frac{-\sqrt{\frac{x^2+1}{x^2}}}{1+\frac{1}{x}} = \lim_{x\to-\infty}\frac{-\sqrt{1+\underbrace{\frac{1}{x^2}}_{\to 0}}}{1+\underbrace{\frac{1}{x}}_{x\to 0}}$$

$$= 1.$$

Solution. 2.k.

$$\begin{aligned}
\lim_{x\to-\infty}\frac{\sqrt{16x^6-3x}}{x^3+2} &= \lim_{x\to-\infty}\frac{\sqrt{x^6\left(16-\frac{3}{x^5}\right)}}{x^3+2}\\
&= \lim_{x\to-\infty}\frac{\sqrt{x^6}\sqrt{\left(16-\frac{3}{x^5}\right)}}{x^3+2} && \Big|\ \sqrt{x^6}=-x^3 \text{ because } x<0 \text{ as } x\to-\infty\\
&= \lim_{x\to-\infty}\frac{-x^3\sqrt{\left(16-\frac{3}{x^5}\right)}}{x^3+2} && \Big|\ \text{Divide by highest order term in denominator}\\
&= \lim_{x\to-\infty}\frac{-x^3\sqrt{\left(16-\frac{3}{x^5}\right)}}{x^3+2}\\
&= \lim_{x\to-\infty}\frac{-\cancel{x^3}\sqrt{\left(16-\frac{3}{x^5}\right)}\frac{1}{\cancel{x^3}}}{\left(x^3+2\right)\frac{1}{x^3}}\\
&= \lim_{x\to-\infty}\frac{-\sqrt{\left(16-\underbrace{\frac{3}{x^5}}_{\to 0}\right)}}{1+\underbrace{\frac{2}{x^3}}_{\to 0}}\\
&= \frac{-\sqrt{16}}{1}=-4.
\end{aligned}$$

Solution. 2.l

$$\begin{aligned}
\lim_{x\to\infty}\frac{\sqrt{3x^2+2x+1}}{x+1} &= \lim_{x\to\infty}\frac{\frac{1}{x}\sqrt{3x^2+2x+1}}{\frac{1}{x}(x+1)}\\
&= \lim_{x\to\infty}\frac{\sqrt{\frac{3x^2+2x+1}{x^2}}}{\left(1+\frac{1}{x}\right)}\\
&= \lim_{x\to\infty}\frac{\sqrt{3+\frac{2}{x}+\frac{1}{x^2}}}{\left(1+\frac{1}{x}\right)}\\
&= \frac{\sqrt{3+0+0}}{1+0}\\
&= \sqrt{3}.
\end{aligned}$$

Solution. 2.p.

$$\begin{aligned}
\lim_{x\to-\infty}\sqrt{x^2+x}-\sqrt{x^2-x} &= \lim_{x\to-\infty}\left(\sqrt{x^2+x}-\sqrt{x^2-x}\right)\frac{\left(\sqrt{x^2+x}+\sqrt{x^2-x}\right)}{\left(\sqrt{x^2+x}+\sqrt{x^2-x}\right)}\\
&= \lim_{x\to-\infty}\frac{x^2+x-\left(x^2-x\right)}{\sqrt{x^2+x}+\sqrt{x^2-x}} = \lim_{x\to-\infty}\frac{2x\frac{1}{x}}{\left(\sqrt{x^2+x}+\sqrt{x^2-x}\right)\frac{1}{x}}\\
&= \lim_{x\to-\infty}\frac{2}{\frac{\sqrt{x^2+x}}{x}+\frac{\sqrt{x^2-x}}{x}} = \lim_{x\to-\infty}\frac{2}{-\sqrt{\frac{x^2+x}{x^2}}-\sqrt{\frac{x^2-x}{x^2}}}\\
&= \lim_{x\to-\infty}\frac{2}{-\sqrt{1+\frac{1}{x}}-\sqrt{1-\frac{1}{x}}} = \frac{2}{-\sqrt{1+0}-\sqrt{1-0}}=-1.
\end{aligned}$$

The sign highlighted in red arises from the fact that, for negative x, we have that $x=-\sqrt{x^2}$.

3. Find the horizontal and vertical asymptotes of the graph of the function. If a graphing device is available, check your work by plotting the function.

(a) $y = \dfrac{2x}{\sqrt{x^2+x+3}-3}$.

answer: Vertical: $x = 2$, $x = -3$, horizontal: $y = 2$, $y = -2$

(b) $y = \dfrac{3x^2}{\sqrt{x^2+2x+10}-5}$.

answer: Vertical: $x = 3$, $x = -5$, horizontal: none.

(c) $y = \dfrac{3x+1}{x-2}$.

answer: vertical: $x = 2$, horizontal: $y = 3$

(d) $y = \dfrac{x^2-1}{2x^2-3x-2}$.

answer: vertical: $x = 2$, $x = -\frac{1}{2}$, horizontal: $y = \frac{1}{2}$

(e) $y = \dfrac{2x^2-2x-1}{x^2+x-2}$.

answer: vertical: $x = 1$, $x = -2$, horizontal: $y = 2$

(f) $f(x) = \dfrac{-5x^2-3x+5}{x^2-2x-3}$

answer: vertical: $x = -1$, $x = 3$, horizontal: $y = -5$

(g) $y = \dfrac{1+x^4}{x^2-x^4}$.

answer: vertical: $x = 0$, $x = 1$, $x = -1$, horizontal: $y = -1$

(h) $y = \dfrac{x^3-x}{x^2-7x+6}$.

answer: vertical: $x = 6$, no horizontal asymptote

(i) $y = \dfrac{x-9}{\sqrt{4x^2+3x+3}}$.

answer: no vertical asymptote, horizontal: $y = \pm\frac{1}{2}$

(j) $y = \dfrac{\sqrt{x^2+1}-x}{x}$.

answer: vertical: $x = 0$, horizontal: $y = 0$, $y = -2$

(k) $f(x) = \dfrac{x}{\sqrt{x^2+3}-2x}$

answer: vertical: $x = 1$, horizontal: $y = -\frac{1}{3}$, $y = -1$

Solution. 3.a **Vertical asymptotes.** A function $f(x)$ has a vertical asymptote at $x = a$ if $\lim\limits_{x\to a} f(x) = \pm\infty$.

The function is algebraic, and therefore has a finite limit at every point it is defined (i.e., no asymptote). Therefore the function can have vertical asymptotes only for those x for which $f(x)$ is not defined. The function is not defined for $\sqrt{x^2+x+3}-3 = 0$, which has two solutions, $x = 2$ and $x = -3$. These are precisely the vertical asymptotes: indeed,

$$\lim_{x\to 2^+} \frac{2x}{\sqrt{x^2+x+3}-3} = \infty \qquad \lim_{x\to 2^-} \frac{2x}{\sqrt{x^2+x+3}-3} = -\infty$$

and

$$\lim_{x\to -3^+} \frac{2x}{\sqrt{x^2+x+3}-3} = \infty \qquad \lim_{x\to -3^-} \frac{2x}{\sqrt{x^2+x+3}-3} = -\infty$$

Horizontal asymptotes. A function $f(x)$ has a horizontal asymptote if $\lim\limits_{x\to\pm\infty} f(x)$ exists. If that limit exists, and is some number, say, N, then $y = N$ is the equation of the corresponding asymptote.

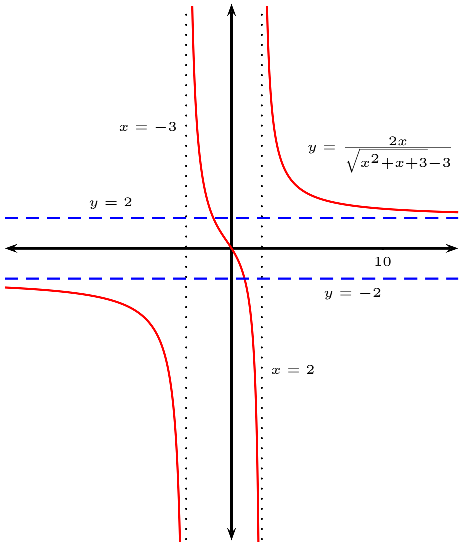
Consider the limit $x \to -\infty$. We have that

$$\begin{aligned}
\lim_{x\to-\infty} \frac{2x}{\sqrt{x^2+3x+3}-3} &= \lim_{x\to-\infty} \frac{2}{\frac{\sqrt{x^2+x+3}}{x} - \frac{3}{x}} \\
&= \lim_{x\to-\infty} \frac{2}{-\sqrt{\frac{x^2+3x+3}{x^2}} - \frac{3}{x}} && \left|\ \frac{1}{x} = -\sqrt{\frac{1}{x^2}} \text{ when } x < 0\right. \\
&= \lim_{x\to-\infty} \frac{2}{-\sqrt{1+\frac{3}{x}+\frac{3}{x^2}} - \frac{3}{x}} \\
&= \frac{\lim\limits_{x\to-\infty} 2}{-\sqrt{\lim\limits_{x\to-\infty} 1 + \lim\limits_{x\to-\infty}\frac{3}{x} + \lim\limits_{x\to-\infty}\frac{3}{x^2}} - \lim\limits_{x\to-\infty}\frac{3}{x}} \\
&= \frac{2}{-\sqrt{1+0+0}-0} \\
&= -2 \quad .
\end{aligned}$$

Therefore $y = -2$ is a horizontal asymptote.

The case $x \to \infty$, is handled similarly and yields that $y = 2$ is a horizontal asymptote.

A computer generated graph confirms our computations.



Solution. 3.d

Vertical asymptotes. A function $f(x)$ has a vertical asymptote at $x = a$ if $\lim\limits_{x\to a} f(x) = \pm\infty$.

The function is algebraic, and therefore has a finite limit at every point it is defined (i.e., no asymptote). Therefore the function can have vertical asymptotes only for those x for which $f(x)$ is not defined. The function is not defined for $2x^2 - 3x - 2 = 0$, which has two solutions, $x = 2$ and $x = -\frac{1}{2}$. These are precisely the vertical asymptotes: indeed,

$$\begin{array}{rcll|l}
\lim\limits_{x\to 2^+} \dfrac{x^2-1}{2x^2-3x-2} & = & \lim\limits_{x\to 2^+} \dfrac{x^2-1}{2(x-2)\left(x+\frac{1}{2}\right)} & = \infty & \text{Limit of form } \frac{(+)}{(+)(+)} \\
\lim\limits_{x\to 2^-} \dfrac{x^2-1}{2x^2-3x-2} & = & \lim\limits_{x\to 2^-} \dfrac{x^2-1}{2(x-2)\left(x+\frac{1}{2}\right)} & = -\infty & \text{Limit of form } \frac{(+)}{(-)(+)}
\end{array}$$

and

$$\begin{array}{rcll|l}
\lim\limits_{x\to -\frac{1}{2}^+} \dfrac{x^2-1}{2x^2-3x-2} & = & \lim\limits_{x\to -\frac{1}{2}^+} \dfrac{x^2-1}{2(x-2)\left(x+\frac{1}{2}\right)} & = \infty & \text{Limit of form } \frac{(-)}{(+)(-)} \\
\lim\limits_{x\to -\frac{1}{2}^-} \dfrac{x^2-1}{2x^2-3x-2} & = & \lim\limits_{x\to -\frac{1}{2}^-} \dfrac{x^2-1}{2(x-2)\left(x+\frac{1}{2}\right)} & = -\infty & \text{Limit of form } \frac{(-)}{(-)(-)}
\end{array}$$

Horizontal asymptotes. A function $f(x)$ has a horizontal asymptote if $\lim\limits_{x\to\pm\infty} f(x)$ exists. If that limit exists, and is some number, say, N, then $y = N$ is the equation of the corresponding asymptote.

We have that

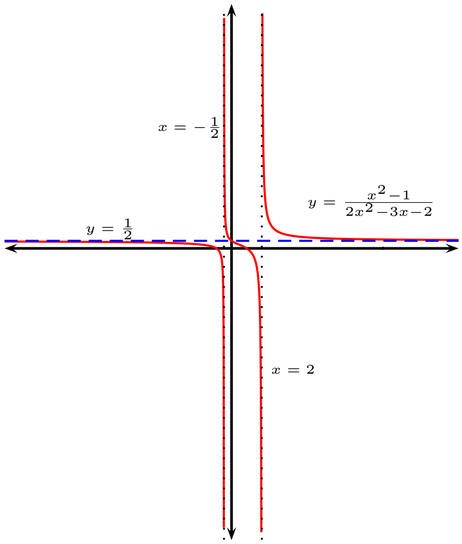
$$\begin{array}{rcl|l}
\lim\limits_{x\to\infty} \dfrac{x^2-1}{2x^2-3x-2} & = & \lim\limits_{x\to\infty} \dfrac{\left(x^2-1\right)\frac{1}{x^2}}{\left(2x^2-3x-2\right)\frac{1}{x^2}} & \text{Divide by highest term in den.} \\
& = & \lim\limits_{x\to\infty} \dfrac{1-\frac{1}{x^2}}{2-\frac{3}{x}-\frac{2}{x^2}} & \\
& = & \dfrac{\lim\limits_{x\to\infty} 1 - \lim\limits_{x\to\infty} \frac{1}{x^2}}{\lim\limits_{x\to\infty} 2 - \lim\limits_{x\to\infty} \frac{3}{x} - \lim\limits_{x\to\infty} \frac{2}{x^2}} & \text{Step may be skipped} \\
& = & \dfrac{1-0}{2-0-0} & \\
& = & \dfrac{1}{2} &
\end{array}$$

A similar computation shows that

$$\lim_{x\to-\infty} \frac{x^2-1}{2x^2-3x-2} = \frac{1}{2}$$

Therefore $y = \frac{1}{2}$ is the only horizontal asymptote, valid in both directions ($x \to \pm\infty$).

A computer generated graph confirms our computations.



Solution. 3.f

Vertical asymptotes. The function is rational, and therefore has a finite limit (and therefore no vertical asymptote) at every point it its domain. The function is not defined for $x^2 - 2x - 3 = 0$, which has two solutions, $x = -1$ and $x = 3$. These are precisely the vertical asymptotes: indeed,

$$\begin{array}{rcll} \lim\limits_{x \to -1^+} \dfrac{-5x^2 - 3x + 5}{x^2 - 2x - 3} & = & \lim\limits_{x \to -1^+} \dfrac{-5x^2 - 3x + 5}{(x+1)(x-3)} = -\infty & \Big| \text{ Limit of form } \frac{(+)}{(+)(-)} \\ \lim\limits_{x \to -1^-} \dfrac{-5x^2 - 3x + 5}{x^2 - 2x - 3} & = & \lim\limits_{x \to -1^-} \dfrac{-5x^2 - 3x + 5}{(x+1)(x-3)} = \infty & \Big| \text{ Limit of form } \frac{(+)}{(-)(-)} \end{array}$$

and

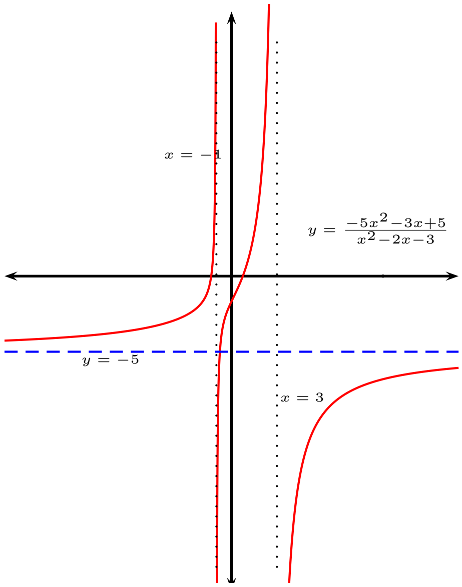
$$\begin{array}{rcll} \lim\limits_{x \to 3^+} \dfrac{-5x^2 - 3x + 5}{x^2 - 2x - 3} & = & \lim\limits_{x \to 3^+} \dfrac{-5x^2 - 3x + 5}{(x+1)(x-3)} = -\infty & \Big| \text{ Limit of form } \frac{(-)}{(+)(+)} \\ \lim\limits_{x \to 3^-} \dfrac{-5x^2 - 3x + 5}{x^2 - 2x - 3} & = & \lim\limits_{x \to 3^-} \dfrac{-5x^2 - 3x + 5}{(x+1)(x-3)} = \infty & \Big| \text{ Limit of form } \frac{(-)}{(+)(-)} \end{array}$$

Horizontal asymptotes.

$$\begin{array}{rcll} \lim\limits_{x \to \pm\infty} \dfrac{-5x^2 - 3x + 5}{x^2 - 2x - 3} & = & \lim\limits_{x \to \pm\infty} \dfrac{\left(-5x^2 - 3x + 5\right) \frac{1}{x^2}}{\left(x^2 - 2x - 3\right) \frac{1}{x^2}} & \Big| \text{ Divide by highest term in den.} \\ & = & \lim\limits_{x \to \pm\infty} \dfrac{-5 - \frac{3}{x} + \frac{5}{x^2}}{1 - \frac{2}{x} - \frac{3}{x^2}} & \\ & = & \dfrac{-\lim\limits_{x \to \pm\infty} 5 - \lim\limits_{x \to \pm\infty} \frac{3}{x} + \lim\limits_{x \to \pm\infty} \frac{5}{x^2}}{\lim\limits_{x \to \pm\infty} 1 - \lim\limits_{x \to \pm\infty} \frac{2}{x} - \lim\limits_{x \to \pm\infty} \frac{3}{x^2}} & \Big| \text{ Step may be skipped} \\ & = & \dfrac{-5 - 0 + 0}{1 - 0 - 0} & \\ & = & -5. & \end{array}$$

Therefore $y = -5$ is the only horizontal asymptote, valid in both directions ($x \to \pm\infty$).

A computer generated graph confirms our computations.



Solution. 3.k

Vertical asymptotes. A function $f(x)$ has a vertical asymptote at $x=a$ if $\lim\limits_{x\to a} f(x)=\pm\infty$.

The function is algebraic, and therefore has a finite limit at every point it is defined (i.e., no asymptote). Therefore the function can have vertical asymptotes only for those x for which $f(x)$ is not defined. The function is not defined for

$$\begin{array}{rcll}
\sqrt{x^2+3}-2x &=& 0 & \\
\sqrt{x^2+3} &=& 2x & \left| \begin{array}{l}\text{square both sides}\\ \text{may introduce extraneous solutions}\end{array}\right. \\
x^2+3 &=& 4x^2 & \\
3x^2-3 &=& 0 & \\
3(x-1)(x+1) &=& 0 & \\
x=1 & \text{or} & \cancel{x=-1} & \\
 & & x=-1 \text{ is extraneous:} & \\
 & & \sqrt{(-1)^2+3}-(-1)2=4\neq 0 &
\end{array}$$

$x=-1$ is indeed a vertical asymptote:

$$\lim_{x\to 1^+}\frac{x}{\sqrt{x^2+3}-2x}=\infty \qquad \lim_{x\to 1^-}\frac{x}{\sqrt{x^2+3}-2x}=-\infty.$$

Horizontal asymptotes.

$$
\begin{aligned}
\lim_{x\to-\infty}\frac{x}{\sqrt{x^2+3}-2x} &= \lim_{x\to-\infty}\frac{1}{\frac{\sqrt{x^2+3}}{x}-2} \\
&= \lim_{x\to-\infty}\frac{1}{-\sqrt{\frac{x^2+3}{x^2}}-2} && \Big|\ \frac{1}{x}=-\sqrt{\frac{1}{x^2}}\text{ when } x<0 \\
&= \lim_{x\to-\infty}\frac{1}{-\sqrt{1+\frac{3}{x^2}}-2} \\
&= \frac{1}{-\sqrt{1+0}-2} \\
&= -\frac{1}{3}. \\
\lim_{x\to-\infty}\frac{x}{\sqrt{x^2+3}-2x} &= \lim_{x\to\infty}\frac{1}{\frac{\sqrt{x^2+3}}{x}-2} \\
&= \lim_{x\to\infty}\frac{1}{\sqrt{\frac{x^2+3}{x^2}}-2} && \Big|\ \frac{1}{x}=\sqrt{\frac{1}{x^2}}\text{ when } x>0 \\
&= \lim_{x\to\infty}\frac{1}{\sqrt{1+\frac{3}{x^2}}-2} \\
&= \frac{1}{\sqrt{1+0}-2} \\
&= -1.
\end{aligned}
$$

Therefore $y=-\frac{1}{3}$ and $y=-1$ are the two horizontal asymptotes.

A computer generated graph confirms our computations.