Precalculus
Lecture 14
Graphing Equations

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https://github.com/tmilev/freecalc

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Outline

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- Let $F(x, y)$ and $G(x, y)$ be arbitrary functions in the variables x, y.
- An equation in x, y is an expression of the form

$$F(x, y) = G(x, y).$$

- An ordered pair of numbers (a, b) is a solution of the equation if the number $F(a, b)$ equals the number $G(a, b)$.

Definition

The graph of the equation $F(x, y) = G(x, y)$ is the set of all solutions (a, b) of the equation.

- Two equations are equivalent if they have the same graphs (set of solutions).
- If we set $H(x, y) = F(x, y) - G(x, y)$, we transform an arbitrary equation to an equivalent equation of the form:

$$H(x, y) = 0.$$

Example

Determine which of the following points

- $(-3, -5)$
- $(3, 5)$

is a solution to the equation

$$7x - 4y = -1.$$

- For $x = -3$, $y = -5$, we have:

  $$7x - 4y = 7(-3) - 4(-5) = -21 + 20 = -1,$$

  so $(-3, -5)$ is a solution.
- For $x = 3$, $y = 5$, we have:

  $$(7x - 4y) = 7(3) - 4(5) = 21 - 20 = 1 \neq -1,$$

  so $(3, 5)$ is **not** a solution.

Example

Determine which of the points $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$, $\left(\frac{\sqrt{3}}{2}, \frac{\sqrt{2}}{4}\right)$ is a solution to the equation

$$x^2 + 2y^2 = 1.$$

- For $x = \frac{\sqrt{3}}{2}$, $y = \frac{1}{2}$, we have:

$$x^2 + 2y^2 = \left(\frac{\sqrt{3}}{2}\right)^2 + 2\left(\frac{1}{2}\right)^2 = \frac{3}{4} + 2 \cdot \frac{1}{4} = \frac{5}{4} \neq 1,$$

so $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ is **not** a solution.

- For $x = \frac{\sqrt{3}}{2}$, $y = \frac{\sqrt{2}}{4}$, we have:

$$x^2 + 2y = \left(\frac{\sqrt{3}}{2}\right)^2 + 2\left(\frac{\sqrt{2}}{4}\right)^2 = \frac{3}{4} + \cancel{2} \cdot \frac{\cancel{2}}{4^{\cancel{2}}} = 1,$$

so $\left(\frac{\sqrt{3}}{2}, \frac{\sqrt{2}}{4}\right)$ is a solution.

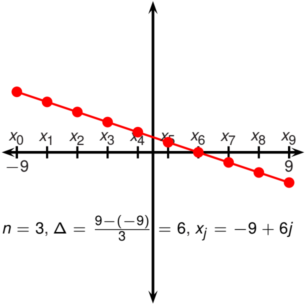
Question

Can we plot the graph of an arbitrary equation, $F(x, y) = G(x, y)$?

- For sufficiently well behaved equations the answer is yes.
- We illustrate one computer algorithm for doing this.
- The theory behind plotting arbitrary equations, even when they are well behaved, is well beyond the scope of the present course.
- In particular, while computer algorithms plot graphs of well-behaved equations relatively well, it is not clear why those algorithms work.
- When, using algebra, we can express one variable in terms of the other, it is easy to produce the graph of the equation.

Question

Can we plot the graph of an equation of the form $y = f(x)$ or $x = h(y)$ (for continuous h, f)? Yes.



Demonstration of the algorithm for $y = -\frac{1}{3}x + 1$ from $x = -9$ to $x = 9$.

- Suppose $y = f(x)$.
- To plot the graph from $x = a$ to $x = b$, select $n + 1$ points $x_0, x_1, \dots, x_n$ on $[a, b]$.
    - Usually we choose the points to be evenly spaced with $x_0 = a$ and $x_n = b$.
    - $n + 1$ points split $[a, b]$ into n intervals.
    - Each interval has length $\Delta = \frac{b-a}{n}$.
    - The formula for the j^{th} point is then $x_j = a + \Delta \cdot j$.
- The points $(x_0, f(x_0)), \dots, (x_n, f(x_n))$ all lie on the graph.
- Connect them with straight lines.
- Repeat for increasing number of segments n.

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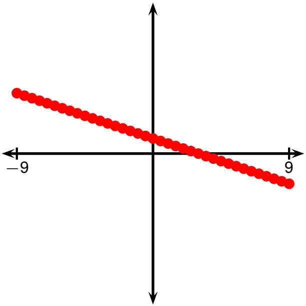

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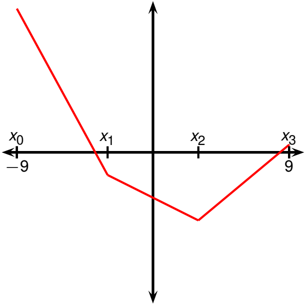

Demonstration of the algorithm for $y = \frac{1}{9}x^2 - \frac{1}{2}x + 4$ from $x = -9$ to $x = 9$.

- Suppose $y = f(x)$.
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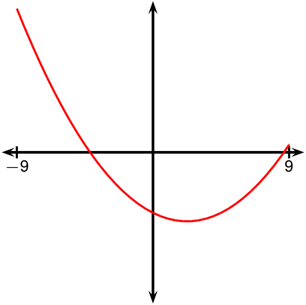

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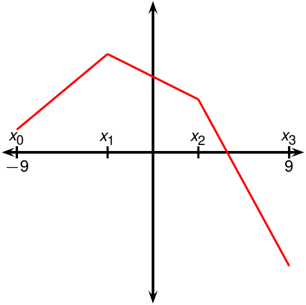

Demonstration of the algorithm for $y = -\frac{1}{9}x^2 - \frac{1}{2}x + 6$ from $x = -9$ to $x = 9$.

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- To plot the graph from $x = a$ to $x = b$, select $n + 1$ points $x_0, x_1, \ldots, x_n$ on $[a, b]$.
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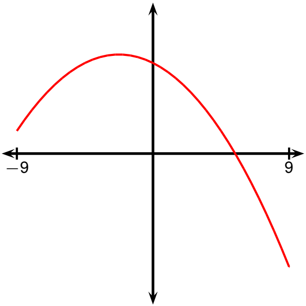

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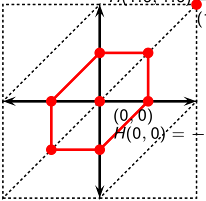
You will not be tested on the material in the following slide.

(Elementary Computer algorithm for sketching graphs)

Let H-continuous; is there simple algorithm to sketch $H(x, y) = 0$? Yes.



We illustrate the algorithm for:

$$x^2 + 2y^2 = 1$$
$$x^2 + 2y^2 - 1 = 0$$
Set $H(x, y) = x^2 + 2y^2 - 1$

- Elementary algorithm: fix large rectangle.
- Split the grid in triangular mesh. One strategy to do that is shown.
- For each triangle:
  - Fix two corners $P(x_P, y_P)$ and $Q(x_Q, y_Q)$.
  - If $H(x_P, y_P)$ and $H(x_Q, y_Q)$ have different sign then H must become zero somewhere on the segment between P and Q.
  - Select a point between P and Q and "guess" that H is zero there.
    - In our implementation, we select the midpoint (i.e., $\frac{1}{2}P + \frac{1}{2}Q$).
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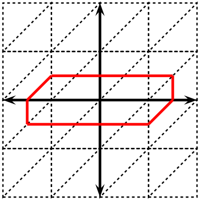

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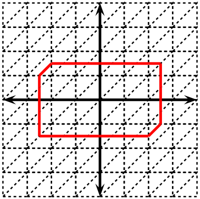

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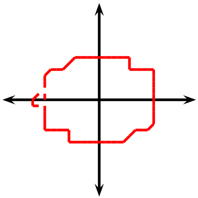

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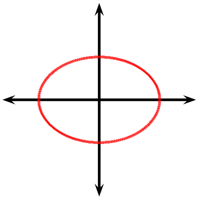

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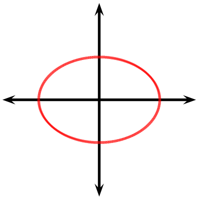

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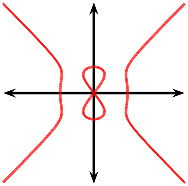

Illustrate the algorithm for:

$y^2(y^2 - 3) = x^2(x^2 - 5)$

$$H(x, y) = y^2(y^2 - 3) - x^2(x^2 - 5)$$

- Elementary algorithm: fix large rectangle.
- Split the grid in triangular mesh. One strategy to do that is shown.
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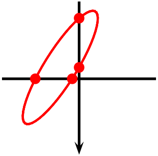
Definition

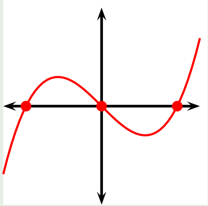
Points on the graph of $F(x, y) = G(x, y)$ for which

- $x = 0$ are called y-intercepts (those lie on the y axis)
- $y = 0$ are called x-intercept (those lie on the x axis).



- To find the x intercepts set $y = 0$ and solve for x.
- To find the y intercepts set $x = 0$ and solve for y.

Example



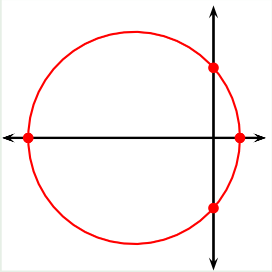
Find the x and y intercepts of the graph of the equation $y = x^3 - x$.

To find the y intercept, set $x = 0$ to get $y = 0$. To find the x intercepts, set $y = 0$ and solve

$$\begin{aligned} x^3 - x &= 0 \\ x(x^2 - 1) &= 0 \\ x(x-1)(x+1) &= 0 \\ x = 0 \text{ or } x = 1 & \quad \text{or } x = -1. \end{aligned}$$

The x-intercepts are: $(-1, 0), (0, 0), (1, 0)$, the y-intercept is $(0, 0)$.

Example



Answer: the y-intercepts are: $\left(0, \sqrt{\frac{7}{4}}\right)$, $\left(0, -\sqrt{\frac{7}{4}}\right)$; the x intercepts are: $(\frac{1}{2}, 0))$ and $(-\frac{7}{2}, 0)$

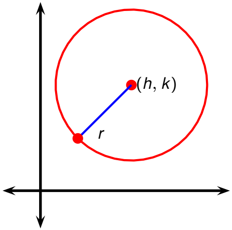
Find the x and y intercepts of the graph of the equation $x^2 + 3x + y^2 = \frac{7}{4}$.
To find the y intercept, set $x = 0$ and solve:

$$y^2 = \frac{7}{4} \Rightarrow y = \pm\sqrt{\frac{7}{4}}$$

To find the x intercepts, set $y = 0$ and solve:

$$\begin{aligned} x^2 + 3x &= \frac{7}{4} \\ x^2 + 3x - \frac{7}{4} &= 0 \\ 4x^2 + 12x - 7 &= 0 \\ (2x - 1)(2x + 7) &= 0 \\ 2x - 1 = 0 \quad &\text{or} \quad 2x + 7 = 0 \\ x = \frac{1}{2} \quad &\text{or} \quad x = -\frac{7}{2} \end{aligned}$$

- A graph is symmetric with respect to the x axis for each (x, y) lying on the graph $(x, -y)$ also lies on the graph.
- A graph is symmetric with respect to the y axis for each (x, y) lying on the graph $(-x, y)$ also lies on the graph.
- A graph is symmetric with respect to the origin if for each (x, y) lying on the graph $(-x, -y)$ also lies on the graph.



Observation

The graph of the equation

$$(x-h)^2+(y-k)^2=r^2$$

is a circle with radius r and center (h, k).

Definition (Completing the square)

Let $a \neq 0$. To *complete the square* means to carry out the following algebraic manipulation.

$$\begin{aligned}
ax^2+bx+c &= a\left(x^2+\frac{b}{a}x\right)+c\\
&= a\left(x^2+2\cdot\frac{b}{2a}x\right)+c\\
&= a\left(x^2+2\frac{b}{2a}x+\left(\frac{b}{2a}\right)^2-\left(\frac{b}{2a}\right)^2\right)+c\\
&= a\left(\left(x+\frac{b}{2a}\right)^2-\frac{b^2}{4a^2}\right)+c\\
&= a\left(x+\frac{b}{2a}\right)^2-\cancel{a}\cdot\frac{b^2}{4a^{\cancel{2}}}+c\\
&= a\left(x+\frac{b}{2a}\right)^2+c-\frac{b^2}{4a}.
\end{aligned}$$

Add & subtract $\left(\frac{b}{2a}\right)^2$ use $(A+B)^2 = A^2+2AB+B^2$

Example (Completing the square)

Complete the square.

$$
\begin{aligned}
3x^2 - 5x + 1 &= 3\left(x^2 - \frac{5}{3}x\right) + 1 \\
&= 3\left(x^2 - 2\cdot\frac{5}{2\cdot 3}x\right) + 1 \\
&= 3\left(x^2 - 2\cdot\frac{5}{6}x + \left(\frac{5}{6}\right)^2 - \left(\frac{5}{6}\right)^2\right) + 1 \\
&= 3\left(\left(x - \frac{5}{6}\right)^2 - \frac{25}{36}\right) + 1 \\
&= 3\left(x - \frac{5}{6}\right)^2 - \frac{25}{12} + 1 \\
&= 3\left(x - \frac{5}{6}\right)^2 - \frac{13}{12}.
\end{aligned}
$$

Example

Show that the graph of the given equation is a circle. Find the center and radius of the circle.

- $x^2 + 2x + y^2 = 1$.
- $x^2 + x + 2y^2 + y = 1 + y^2$.
- $x^2 = 3x - y^2 - 2y$.
- $3x^2 + y = -3y^2$.
- $2x^2 + y = \frac{1}{2}x - 2y^2$.

Example

Find an equation of a circle with center $(2, 3)$ and passing through the point $(-1, 1)$.