Calculus III
Lecture 3

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https://github.com/tmilev/freecalc

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Outline

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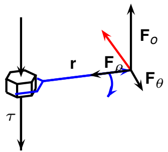
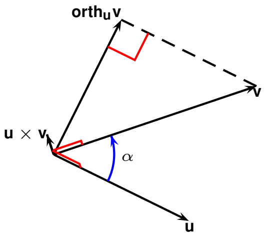
Torque



- If we tighten a bolt using a wrench, it moves in direction perpendicular to the motion of the wrench.
- Let arm of the wrench: given by vector **r**.
- Suppose we are applying a force **F** at arm of the wrench. The force has three components:
  - component $\mathbf{F}_o$ orthogonal to the plane of rotation
  - component $\mathbf{F}_\rho$ in the plane of rotation towards/away from the center
  - component $\mathbf{F}_\theta$ tangent to the motion of the wrench.
- Only $\mathbf{F}_\theta$ contributes to the bolt motion.
- The force of bolt motion τ is proportional to length of wrench.
- It turns out $\tau = \mathbf{r} \times (\mathbf{F}_\rho + \mathbf{F}_\theta)$, where $\times$ is the vector cross product.

The Cross Product $\times$



Definition (Cross product)

$\mathbf{u} \times \mathbf{v}$ is the vector uniquely determined by the following.

- If $\mathbf{u}$, $\mathbf{v}$ are non-zero and non-collinear.
  - $\mathbf{u} \times \mathbf{v}$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$.
  - The magnitude of $\mathbf{u} \times \mathbf{v}$ equals $|\mathbf{u}||\mathbf{orth_u v}| = |\mathbf{u}||\mathbf{v}| \sin \alpha$.
  - The direction of $\mathbf{u} \times \mathbf{v}$ is such that when viewed from the tip of $\mathbf{u} \times \mathbf{v}$, $\mathbf{v}$ is counter-clockwise from $\mathbf{u}$.
- If $\mathbf{u}$, $\mathbf{v}$ are colinear or zero then $\mathbf{u} \times \mathbf{v} = \mathbf{0}$.

There are a couple of hand rules to help figure out the direction of the cross product.

Properties of Cross Product

Let **u**, **v** non-zero vectors, $\alpha = \angle(\mathbf{u}, \mathbf{v})$.

- $|\mathbf{v} \times \mathbf{u}| = |\mathbf{u} \times \mathbf{v}|$.
  Indeed, that is because
$$|\mathbf{orth}_{\mathbf{u}}\mathbf{v}| = |\mathbf{v}| \sin\alpha \Longrightarrow |\mathbf{u} \times \mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}|\, \sin\alpha$$
- Cross product is anti-symmetric:
$$\mathbf{v} \times \mathbf{u} = -\mathbf{u} \times \mathbf{v}\ .$$
- Cross product is linear in each argument:
$$\mathbf{u} \times (a\mathbf{v} + b\mathbf{w}) = a\mathbf{u} \times \mathbf{v} + b\mathbf{u} \times \mathbf{w}$$
$$(a\mathbf{u} + b\mathbf{w}) \times \mathbf{v} = a\mathbf{u} \times \mathbf{v} + b\mathbf{w} \times \mathbf{v}$$

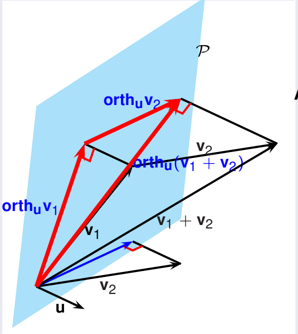
$\mathbf{orth_u}$ is a linear operator

Theorem

$$\mathbf{orth_u}(\mathbf{v}_1 + \mathbf{v}_2) = \mathbf{orth_u}\mathbf{v}_1 + \mathbf{orth_u}\mathbf{v}_2$$

Proof.

Geometric proof:



Algebraic proof:

$$\begin{aligned}
\mathbf{orth_u}(\mathbf{v}_1 + \mathbf{v}_2) &= (\mathbf{v}_1 + \mathbf{v}_2) - \mathbf{proj_u}(\mathbf{v}_1 + \mathbf{v}_2)\\
&= (\mathbf{v}_1 + \mathbf{v}_2) - (\mathbf{proj_u}(\mathbf{v}_1) + \mathbf{proj_u}(\mathbf{v}_2))\\
&= (\mathbf{v}_1 - \mathbf{proj_u}(\mathbf{v}_1)) + (\mathbf{v}_2 - \mathbf{proj_u}(\mathbf{v}_2))\\
&= \mathbf{orth_u}\mathbf{v}_1 + \mathbf{orth_u}\mathbf{v}_2
\end{aligned}$$

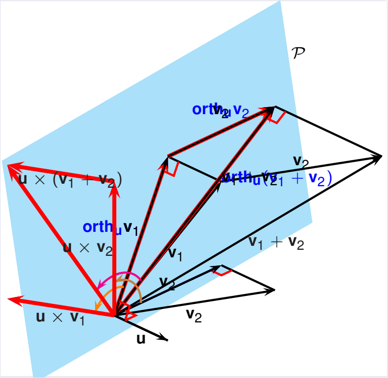
Let $\mathcal{P}$: plane $\perp$ $\mathbf{u}$.

Justification of Linearity of $\times$ Product

Theorem

$\mathbf{u} \times (\mathbf{v}_1 + \mathbf{v}_2) = \mathbf{u} \times \mathbf{v}_1 + \mathbf{u} \times \mathbf{v}_2.$

Geometric justification.



$$\begin{aligned}
\mathbf{u} \times \mathbf{v}_1 &= \mathbf{u} \times \mathbf{orth}_{\mathbf{u}} \mathbf{v}_1 \\
\mathbf{u} \times \mathbf{v}_2 &= \mathbf{u} \times \mathbf{orth}_{\mathbf{u}} \mathbf{v}_2 \\
\mathbf{u} \times (\mathbf{v}_1 + \mathbf{v}_2) &= \mathbf{u} \times (\mathbf{orth}_{\mathbf{u}} (\mathbf{v}_1 + \mathbf{v}_2)) \\
&= \mathbf{u} \times (\mathbf{orth}_{\mathbf{u}} (\mathbf{v}_1) + \mathbf{orth}_{\mathbf{u}} (\mathbf{v}_2))
\end{aligned}$$

$\Rightarrow$ suffices to prove theorem when $\mathbf{v}_1, \mathbf{v}_2 \perp \mathbf{u}$.
Since $(a\mathbf{u}) \times \mathbf{v} = a(\mathbf{u} \times \mathbf{v}) \Rightarrow$ suffices to prove theorem when $|\mathbf{u}| = 1$.
When $|\mathbf{u}| = 1$, applying $\mathbf{u}\times$ rotates all vectors in the plane $\mathcal{P}$ at angle $\frac{\pi}{2}$. The statement of the theorem now follows from the fact that rotation preserves sums of vectors. $\square$

Permutations and permutation signs

- Let $\sigma : \{1, 2, \dots n\} \to \{1, 2, \dots, n\}$ be one to one function.
- Since σ - one to one, $(\sigma(1), \sigma(2), \dots, \sigma(n))$ have no repetition.

Definition

A one-to-one function from the set $\{1, 2, \dots, n\}$ to itself is called a permutation ("shuffling").

- There are $n!$ different permutations:
  - there are n ways to select $\sigma(1)$,
  - $n - 1$ ways to select $\sigma(2)$ (one number is already taken),
  - and so on, total: $n \cdot (n-1) \cdots 1 = n!$ ways to make a permutation.

Sign of permutation

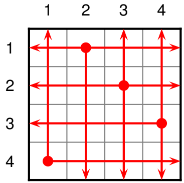
- Given two sequences of numbers, define them to be transpositions of one another if one is obtained from the other with a single swap of neighboring numbers.
- $(2,3,4,1)$ and $(2,4,3,1)$ are transpositions of one another. $(2,3,4,1)$ and $(1,3,4,2)$ are **not** transpositions of one another.
- Write the numbers $(\sigma(1),\sigma(2),\dots,\sigma(n))$ in a sequence.
- Using transpositions, get from $(\sigma(1),\sigma(2),\dots,\sigma(n))$ to the properly ordered sequence $1,2,\dots,n$.
- Number of transpositions used varies depending how we do it, but parity (even-ness) of # of transpositions is always the same.
- If $sign(\sigma)=1$, σ is called even, if $sign(\sigma)=-1$, σ is called odd.

Definition

If we can get from $(\sigma(1),\sigma(2),\dots,\sigma(n))$ to $(1,2,\dots,n)$ with even # of transpositions, define $sign(\sigma)$ to be 1, else define $sign(\sigma)$ to be -1.

- To each permutation σ, assign n pairs of numbers $(1, \sigma(1))$, $(2, \sigma(2))$,... $(n, \sigma(n))$.
- Consider a $n \times n$ chess board. Interpret pair $(k, \sigma(k))$ as $(row, column)$-coordinates in the board.
- For each pair $(k, \sigma(k))$, place a rook on the board.
- Example: $\begin{array}{rcl}\sigma(1) &=& 2\\ \sigma(2) &=& 3\\ \sigma(3) &=& 4\\ \sigma(4) &=& 1\end{array}$, pairs: $\begin{array}{rcl}(1,\sigma(1)) &=& (1,2)\\ (2,\sigma(2)) &=& (2,3)\\ (3,\sigma(3)) &=& (3,4)\\ (4,\sigma(4)) &=& (4,1)\end{array}$.

  Corresponding peaceful rook placement:



- $\sigma(k)$ are different $\Rightarrow$ rook placements are peaceful: rooks never hit one another. i.e., no two points lie on same column or row.

Square matrices

- Let A be $n \times n$ (square) table of numbers.
- Technical term: A is a (square) *matrix*.
- Matrices are often denoted by surrounding with ()-parenthesis:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & & & & \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{pmatrix}.$$

First row Second row n^{th} row First column Second column n^{th} column

- Most common convention for matrix notation:
  - $(i,j)^{th}$ entry of a matrix = denoted by letter with indices i,j, such as a_{ij}
  - no comma between indices i,j in a_{ij}
  - first index stands for row, second - for column.
- Non-square matrices: used & important but we discuss them

- The determinant $\det A$ of a square matrix A is a number written as:

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & & & & \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{vmatrix}$$

- The formula for the determinant is:

$$\det A = \sum_{\text{all permutations } \sigma} a_{1\sigma(1)} a_{2\sigma(2)} \dots a_{n\sigma(n)} sign(\sigma) \quad .$$

- For every permutation σ we have one summand.
- Every pair $(k, \sigma(k))$ can be identified with a peaceful of a rook placement (as described in previous slides/lectures).
- For each rook placement we have a summand obtained by multiplying the numbers on which the rooks are standing.
- The sign of each summand is determined by the sign of the permutation.

2×2 determinants

$$\det \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

- We specialize the $n \times n$ determinant formula to the case $n = 2$.
- There are two peaceful rook placements for a 2×2 chessboard.
- For each peaceful rook placement we got one summand.
- The permutation $(\sigma(1), \sigma(2)) = (2, 1)$ is odd, so one of the summands comes with negative sign.

3×3 determinants

$$\det \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{array}{l} a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} \\ + a_{13}a_{21}a_{32} \\ - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} \\ - a_{12}a_{21}a_{33} \end{array}$$

- We specialize the $n \times n$ determinant formula to the case $n = 3$.
- There are $6 = 3!$ peaceful rook placements for a 3×3 chessboard.
- For each peaceful rook placement we got one summand.
- The rook placements along the down-right “broken” diagonals correspond to even permutations, and the rook placements along the right-up “broken” diagonals correspond to negative permutations.

Cross Product in Coordinates

- Let **i**, **j**, **k**: unit vectors along coordinate axes.
- We have that

$$\begin{array}{lll} \mathbf{i}\times\mathbf{i}=\mathbf{0}, & \mathbf{j}\times\mathbf{j}=\mathbf{0}, & \mathbf{k}\times\mathbf{k}=\mathbf{0}\\ \mathbf{i}\times\mathbf{j}=\mathbf{k}, & \mathbf{j}\times\mathbf{k}=\mathbf{i}, & \mathbf{k}\times\mathbf{i}=\mathbf{j}\\ \mathbf{j}\times\mathbf{i}=-\mathbf{k}, & \mathbf{k}\times\mathbf{j}=-\mathbf{i}, & \mathbf{i}\times\mathbf{k}=-\mathbf{j} \end{array}$$

- Let $\begin{array}{lllll} \mathbf{u} & = & u_1\mathbf{i}+u_2\mathbf{j}+u_3\mathbf{k} & = & (u_1,u_2,u_3)\\ \mathbf{v} & = & v_1\mathbf{i}+v_2\mathbf{j}+v_3\mathbf{k} & = & (v_1,v_2,v_3) \end{array}$.
-

$$\begin{array}{lll} \mathbf{u}\times\mathbf{v} & = & (u_1\mathbf{i}+u_2\mathbf{j}+u_3\mathbf{k})\times(v_1\mathbf{i}+v_2\mathbf{j}+v_3\mathbf{k})\\ & = & (u_2v_3-u_3v_2)\mathbf{i}+(u_3v_1-u_1v_3)\mathbf{j}+(u_1v_2-u_2v_1)\mathbf{k}\\ \mathbf{u}\times\mathbf{v} & = & (u_1,u_2,u_3)\times(v_1,v_2,v_3)=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\ u_1&u_2&u_3\\ v_1&v_2&v_3\end{vmatrix} \end{array}$$

$$\mathbf{u}\times\mathbf{v} = (u_1, u_2, u_3)\times(v_1, v_2, v_3) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Example

Find $\mathbf{u}\times\mathbf{v}$, where $\mathbf{u} = (1, 2, 3)$ and $\mathbf{v} = (6, 5, 4)$.

$$\begin{aligned}
\mathbf{u}\times\mathbf{v} &= (1, 2, 3)\times(6, 5, 4) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 6 & 5 & 4 \end{vmatrix} \\
&= \begin{vmatrix} 2 & 3 \\ 5 & 4 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 3 \\ 6 & 4 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 2 \\ 6 & 5 \end{vmatrix}\mathbf{k} \\
&= (2\cdot 4 - 3\cdot 5)\mathbf{i} - (1\cdot 4 - 3\cdot 6)\mathbf{j} + (1\cdot 5 - 2\cdot 6)\mathbf{k} \\
&= -7\mathbf{i} + 14\mathbf{j} - 7\mathbf{k} = (-7, 14, -7)\ .
\end{aligned}$$

Use $\times$ to find vector perpendicular to two given

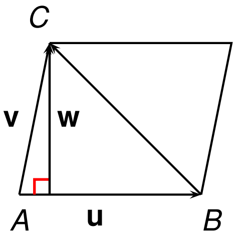
Recall $\mathbf{u} \times \mathbf{v}$ is perpendicular to $\mathbf{u}$ and $\mathbf{v}$.

Example

Find a vector $\mathbf{w}$ perpendicular to $\mathbf{u} = (1, 1, 0) = \mathbf{i} + \mathbf{j}$ and $\mathbf{v} = \mathbf{j} + \mathbf{k} = (0, 1, 1)$.

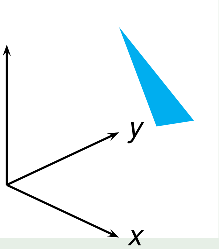
$$\begin{aligned} \mathbf{w} &= (\mathbf{i}+\mathbf{j}) \times (\mathbf{j}+\mathbf{k}) = \mathbf{i} \times \mathbf{j} + \mathbf{i} \times \mathbf{k} + \mathbf{j} \times \mathbf{j} + \mathbf{j} \times \mathbf{k} = \\ &= \mathbf{k} - \mathbf{j} + \mathbf{0} + \mathbf{i} = \mathbf{i} - \mathbf{j} + \mathbf{k} = (1, -1, 1) \ . \end{aligned}$$

Use $\times$ to find area of triangle in space



- A, B, C points in space, $\mathbf{u} = \mathbf{AB}$, $\mathbf{v} = \mathbf{AC}$.
- Then
  $|\mathbf{w}| = |\mathbf{orth_u v}| =$ distance from C to AB .
- $|\mathbf{u} \times \mathbf{v}| = |\mathbf{orth_u v}|\,|\mathbf{u}| = 2\text{area}(ABC) = \text{area}(ABDC)$
- $|\mathbf{u} \times \mathbf{v}|$ = Area of parallelogram on sides $\mathbf{u}$ and $\mathbf{v}$.

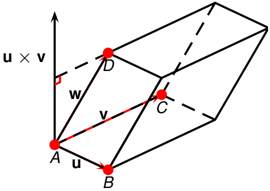
Example



Find the area of the triangle $A(1,2,3)$, $B(2,3,1)$, $C(3,1,2)$.

$$\begin{aligned}\mathrm{Area}(ABC) &= \frac{1}{2}|\mathbf{AB}\times\mathbf{AC}| = \frac{1}{2}|(1,1,-2)\times(2,-1,-1)| \\ &= \frac{1}{2}|(-3,-3,-3)| \\ &= \frac{3\sqrt{3}}{2}\ .\end{aligned}$$

Scalar Triple Product



- A, B, C, D points in space;
- $\mathbf{u} = \mathbf{AB}$, $\mathbf{v} = \mathbf{AC}$, $\mathbf{w} = \mathbf{AD}$;
- $R = R(\mathbf{u}, \mathbf{v}, \mathbf{w})$: box on sides $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$.
- $\mathrm{Vol}(R) = |\mathbf{u} \times \mathbf{v}||\mathbf{r}| = |\mathbf{u} \times \mathbf{v}|\,|\mathbf{proj}_{\boldsymbol{u}\times\boldsymbol{v}}\mathbf{w}| = |\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})|$.

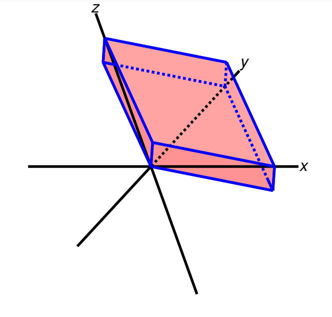
Definition

The quantity $\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})$ is called the scalar triple product of $\mathbf{w}, \mathbf{u}, \mathbf{v}$.

- If $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$, and $\mathbf{w} = (w_1, w_2, w_3)$, then

$$\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) = \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

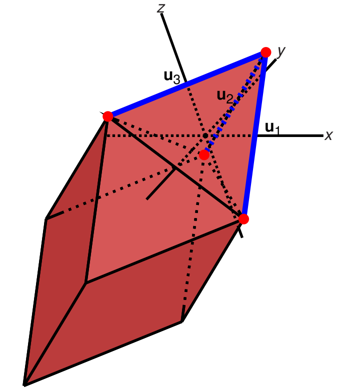
Example



Find the volume of the parallelepiped (slanted box) with vertex at the origin spanned by the vectors $(-1,1,1)$, $(1,-1,1)$, $(1,1,-1)$.

$$\begin{aligned}\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} &= -1+1+1-(-1)-(-1)-(-1) \\ &= 4\end{aligned}$$

Example



Find the volume of the tetrahedron with vertices $(1,1,1)$, $(1,-1,-1)$, $(-1,1,-1)$, $(-1,-1,1)$.

$\mathbf{u}_1 = (1,-1,-1) - (1,1,1) = (0,-2,-2)$
$\mathbf{u}_2 = (-1,1,-1) - (1,1,1) = (-2,0,-2)$
$\mathbf{u}_3 = (-1,-1,1) - (1,1,1) = (-2,-2,0)$

$$\begin{aligned}\mathrm{Vol(tetrahedron)} &= \frac{1}{6}\mathrm{Vol(Box\ generated\ by\ any\ 3\ edges)}\\ &= \frac{1}{6}\left|\det\begin{pmatrix} 0 & -2 & -2\\ -2 & 0 & -2\\ -2 & -2 & 0\end{pmatrix}\right| = \frac{1}{6}|-16| = \frac{8}{3}.\end{aligned}$$

Example

Do the points $(1, 2, 3)$, $(2, 3, 5)$, $(3, 5, 7)$, $(5, 7, 11)$ lie in one plane?

Example

Do the points $(1, -1, -1)$, $(-1, 1, -1)$, $(-1, -1, 1)$, $(1, 2, 3)$ lie in one plane?

Orientations of Space

- The following are equivalent:
  - every vector in space can be decomposed along **u**, **v**, **w**;
  - the box $R(\mathbf{u}, \mathbf{v}, \mathbf{w})$ is non-degenerate;
  - $Vol(R(\mathbf{u}, \mathbf{v}, \mathbf{w})) \neq 0$;
  - $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) \neq 0$.
- If any of the above is valid: $(\mathbf{u}, \mathbf{v}, \mathbf{w})$ is a frame.
- Rectangular coordinate system $\to$ fundamental frame $(\mathbf{u}, \mathbf{v}, \mathbf{w})$
- The hand rules for determining directions of cross products $(\mathbf{w} = \mathbf{u} \times \mathbf{v})$ are consistent with this coordinate system if and only if

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) > 0$$

Definition

The frame $(\mathbf{u}, \mathbf{v}, \mathbf{w})$ is positively oriented if $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) > 0$.