Calculus III
Lecture 11

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https://github.com/tmilev/freecalc

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Outline

1. Surfaces
   - Quadric Surfaces
2. Tangent Planes

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- A surface S can be given in a number of ways.
  - Implicit form, as a *level surface*:
  $$F(x,y,z)=0$$
  .
  - Explicit form, as a *parametric surface*:
  $$\left|\begin{array}{rcl} x & = & g(u,v) \\ y & = & h(u,v) \\ z & = & f(u,v) \end{array}\right.$$
  - A partial case of the parametric surface is the *graph surface*:
  $$z=f(x,y)\Longrightarrow \left|\begin{array}{rcl} x & = & u \\ y & = & v \\ z & = & f(u,v) \end{array}\right.$$
- A graph surface $z=f(x,y)$ can be represented as a level surface:
$$z=f(x,y)\Longleftrightarrow F(x,y,z)=0 \quad \text{for} \quad F(x,y,z)=z-f(x,y)\ .$$

Quadratic surfaces

- The level sets for second degree polynomial functions

  $$f(x,y,z) = Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J.$$

  are called quadratic surfaces.
- At least one of the second-degree terms is required to be non-zero.
- The coefficients are allowed to be zero.

Canonical forms of quadratic surfaces.

Through rigid motions (translations and rotations) a quadratic surface can be reduced to one of the two canonical forms.

- $$Ax^2 + By^2 + Cz^2 + D = 0,$$

  $(A, B, C) \neq (0, 0, 0)$. These quadratics posses central symmetry: if (x, y, z) belongs to the surface, so does $(-x, -y, -z)$.

- $$Ax^2 + By^2 + Iz = 0,$$

  $(A, B) \neq (0, 0)$, $I \neq 0$. These quadratics do not posses central symmetry.

The canonical forms above are in addition split into sub-forms depending on the sign of A, B, C, D, I.

$Ax^2 + By^2 + Cz^2 + D = 0$, $A > 0, B > 0, C > 0, D > 0$

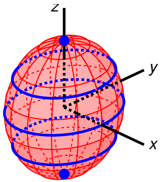
Consider the surface $\mathcal{C} = \{(x, y, z) | Ax^2 + By^2 + Cz^2 + D = 0\}$.

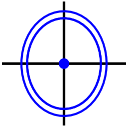
- Let $A > 0, B > 0, C > 0, D > 0$.
- Then the surface is the empty set.
- Example:

$$x^2 + 2y^2 + 3z^2 + 4 = 0.$$

$Ax^2 + By^2 + Cz^2 + D = 0$, $A > 0$, $B > 0$, $C > 0$, $D > 0$

Consider the surface $\mathcal{C} = \{(x, y, z) | Ax^2 + By^2 + Cz^2 + D = 0\}$.

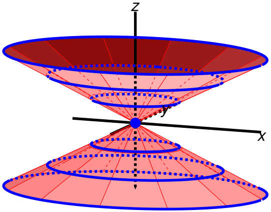


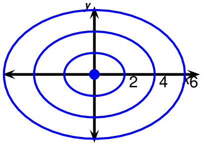


$$\begin{aligned} z &= -3\,-2\,-1 0 1 2 \\ \tfrac{1}{2}x^2 + \tfrac{1}{3}y^2 &= 1 - \frac{z^2}{4} \\ &= -\tfrac{5}{4} 0 \tfrac{3}{4} 1 \tfrac{3}{4} 0 \end{aligned}$$

- Let $A > 0$, $B > 0$, $C > 0$, $D < 0$. Rescale so $D = -1$.
- Set $A = \frac{1}{a^2}$, $B = \frac{1}{b^2}$, $C = \frac{1}{c^2}$.
- Surface becomes: $\left\{ (x, y, z) | \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1 \right\}$.
- We illustrate the theory on the example: $\frac{1}{2}x^2 + \frac{1}{3}y^2 + \frac{z^2}{4} = 1$.
- Rewrite: $\frac{1}{2}x^2 + \frac{1}{3}y^2 = 1 - \frac{z^2}{4}$.
- The level curves are:
  - None for $z < 2$ and $z > 2$.
  - Two points for $z = \pm 2$.
  - Ellipses for $z \in (-2, 2)$.
- Figure is called ellipsoid.

$Ax^2 + By^2 + Cz^2 + D = 0, A > 0, B > 0, C < 0, D = 0$





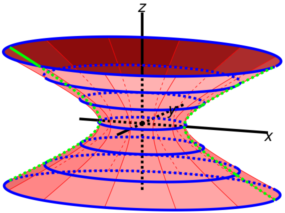
$$\begin{aligned} z &= \pm 3 \\ \left(\frac{x}{2}\right)^2 + \left(\frac{y}{\sqrt{2}}\right)^2 &= \\ \Rightarrow (x, y, z) &\in \text{ellipse} \end{aligned}$$

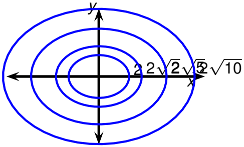
- Consider the surface $\mathcal{C} = \left\{(x, y, z) | \frac{x^2}{4} + \frac{y^2}{2} = z^2\right\}$
- The level curves $z = const$ are:
  - A point for $z = 0$.
  - Ellipses for $z \neq 0$.
- For $y = 0$:

$$\begin{aligned} \left(\frac{x}{2}\right)^2 &= z^2 \\ \frac{x}{2} &= \pm z \\ x &= \pm 2z \end{aligned}$$

- $\Rightarrow$ the ellipses are stacked along lines .
- $\Rightarrow$ The figure is a ("two-piece") cone.

$Ax^2 + By^2 + Cz^2 + D = 0, A > 0, B > 0, C < 0, D < 0$





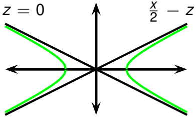
$$z = \pm 3$$

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{\sqrt{2}}\right)^2 = 10 = 1 + (\pm 3)^2$$

$$\Rightarrow (x, y, z) \in \text{ellipse}$$

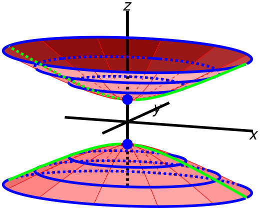
- Consider the surface $\mathcal{C} = \left\{(x, y, z) | \frac{x^2}{4} + \frac{y^2}{2} = z^2 + 1\right\}$
- The level curves $z = const$ are: Ellipses for all z. For $y = 0$:

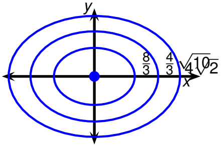
$$\begin{aligned} \left(\frac{x}{2}\right)^2 &= z^2 + 1 \\ \left(\frac{x}{2}\right)^2 - z^2 &= 1 \\ \left(\frac{x}{2} - z\right)\left(\frac{x}{2} + z\right) &= 1 \\ \left(\frac{x}{2} - z\right) &= \frac{1}{\left(\frac{x}{2} + z\right)} \end{aligned}$$



- $\Rightarrow$ ellipses: stacked along hyperbolas.
- Figure called: one-sheet hyperboloid.

$Ax^2 + By^2 + Cz^2 + D = 0, A > 0, B > 0, C < 0, D > 0$





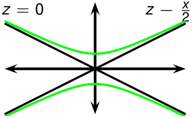
$$z = \pm 3$$

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{\sqrt{2}}\right)^2 = 8 = (\pm 3)^2 - 1$$

$$\Rightarrow (x, y, z) \in \text{ellipse}$$

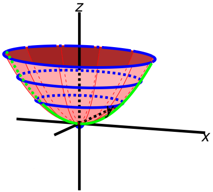
- Consider the surface $\mathcal{C} = \left\{ (x, y, z) | \frac{x^2}{4} + \frac{y^2}{2} = z^2 - 1 \right\}$
- When $z = const$: Two pts. for $z = \pm 1$. Ellipses for $|z| > 1$. When

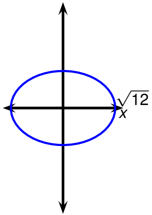
$$y = 0: \quad \begin{array}{rcl} \left(\frac{x}{2}\right)^2 &=& z^2 - 1 \\ z^2 - \left(\frac{x}{2}\right)^2 &=& 1 \\ \left(z - \frac{x}{2}\right)\left(\frac{x}{2} + z\right) &=& 1 \\ \left(z - \frac{x}{2}\right) &=& \frac{1}{\left(\frac{x}{2} + z\right)} \end{array}$$



- $\Rightarrow$ellipses: stacked along hyperbolas.
- Figure called: two-sheet hyperboloid.

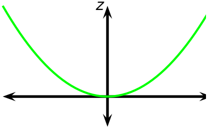
$Ax^2 + By^2 + Iz = 0$, $A > 0$, $B > 0$, $I \neq 0$





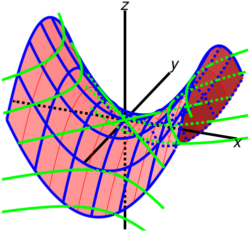
$$z=3$$
$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{\sqrt{2}}\right)^2 = 3$$
$$\Rightarrow (x, y, z) \in \text{ellipse}$$

- Consider the surface $\mathcal{C} = \left\{(x, y, z) \middle| \frac{x^2}{4} + \frac{y^2}{2} = z\right\}$
- The level curves $z = const$ are: a point for $z = 0$; ellipses for $z > 0$. For $y = 0$: $\left(\frac{x}{2}\right)^2 = z$



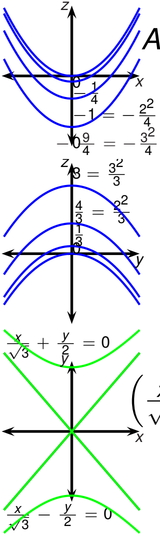
- $\Rightarrow$ ellipses: stacked along parabola.
- Surface name: paraboloid. If $A \neq B$: elliptic paraboloid.
- What happens if we decrease B?

$Ax^2 + By^2 + Iz = 0$, $A > 0$, $B < 0$, $C = 0$, $I \neq 0$



Surface: $\mathcal{C} = \left\{(x, y, z) | \frac{x^2}{3} - \frac{y^2}{4} = z\right\}$.

- Name: hyperbolic paraboloid.
- What happens if $|B|$ decreases?



$A = \frac{1}{3}$, $|B| = \frac{1}{4} = 0$, $I = -1$.

Set: $y =$

$\frac{x^2}{3} - 0 = z$ | parab

Set: $x =$

$-\frac{y^2}{4} = z$ | parab

Set: $z = 210 - 1 - 2$

$\frac{x^2}{3} - \frac{y^2}{4} = 210 - 1 - 2$

$\left(\frac{x}{\sqrt{3}} - \frac{y}{2}\right)\left(\frac{x}{\sqrt{3}} + \frac{y}{2}\right) = 210 - 1 - 2$

$\left(\frac{x}{\sqrt{3}} - \frac{y}{2}\right) = \frac{210-1-2}{\left(\frac{x}{\sqrt{3}} + \frac{y}{2}\right)}$ | hyperb

Summary: surfaces of form $Ax^2 + By^2 + Cz^2 + D = 0$

A	B	C	D	$x = x_0$	$y = y_0$	$z = z_0$	Example	Name
> 0	> 0	> 0	> 0	empty	empty	empty	$x^2 + 2y^2 + 3z^2 + 4 = 0$	empty
> 0	> 0	> 0	$= 0$					
> 0	> 0	> 0	< 0	ellipse	ellipse	ellipse	$x^2 + 2y^2 + 3z^2 - 4 = 0$	Ellipsoid
> 0	> 0	$= 0$	> 0					
> 0	> 0	$= 0$	$= 0$					
> 0	> 0	$= 0$	< 0					
> 0	> 0	< 0	> 0					
> 0	> 0	< 0	$= 0$					
> 0	> 0	< 0	< 0					
> 0	$= 0$	$= 0$	> 0					
> 0	$= 0$	$= 0$	$= 0$					
> 0	$= 0$	$= 0$	< 0					

Fill in the rest of the table.

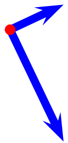
Quadratics $Ax^2 + By^2 + Iz = 0$ (no central symmetry)

$$Ax^2 + By^2 + Iz = 0$$

A	B	$x = x_0$	$y = y_0$	$z = z_0$	Example	Name
> 0	> 0	parabola	parabola	ellipse, point, or empty	$x^2 + 2y^2 + 3z = 0$	Elliptic paraboloid
> 0	$= 0$					
> 0	< 0					

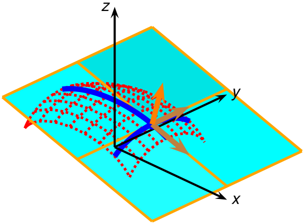
Fill in the rest of the table.

Tangent Plane



- Consider a surface S in space and a point P on the surface.
- How should we define the notion of "a plane tangent to S at P" so that it matches our geometric intuition?
- Intuitively, it should include all tangents at P to curves passing through P and contained in the surface.
- Therefore it should be the plane
  - passing through P;
  - parallel to the directions of all tangent vectors of curves passing through P and contained in the surface.

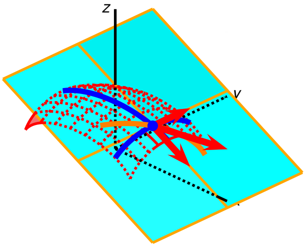
Tangent Plane to a Graph Surface



- Graph surface $z = f(x, y)$, point $P(x_0, y_0, z_0)$ on the surface.
- Call $\mathbf{p}(x)$ the curve given by $f(x, y)$ by keeping $y = y_0$ constant; call $\mathbf{q}(y)$ the curve given by $f(x, y)$ by keeping $x = x_0$ constant.
- $$\begin{array}{rclcrcl} \mathbf{p}(x) &=& (x, y_0, f(x, y_0)) & & \mathbf{p}'(x_0) &=& (1, 0, f_x(x_0, y_0)) \\ \mathbf{q}(y) &=& (x_0, y, f(x_0, y)) & & \mathbf{q}'(y_0) &=& (0, 1, f_y(x_0, y_0)) \end{array} .$$
- Normal to tangent plane at P: $\mathbf{n} = \mathbf{p}'(x_0) \times \mathbf{q}'(y_0) = (1, 0, f_x(x_0, y_0)) \times (0, 1, f_y(x_0, y_0)) = (-f_x(x_0, y_0), -f_y(x_0, y_0), 1)$.
- Equation of tangent plane at $P(x_0, y_0, f(x_0, y_0))$: $\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$
  $-f_x(x_0, y_0)(x - x_0) - f_y(x_0, y_0)(y - y_0) + (z - f(x_0, y_0)) = 0$
  $$\boxed{z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)} .$$

- Let $z = f(x, y)$ be a function and let (x_0, y_0) be a point such that f is differentiable in a small disk near (x_0, y_0).
- The graph of $f(x, y)$ is the surface $S = \{(x, y, f(x, y))\}$.
- Let (x, y) be a point in the domain of f. By the vertical line test there is exactly one point in S whose first two coordinates are x, y. This is the point $(x, y, f(x, y))$.
- Let $\mathbf{q}(t) = (x(t), y(t), z(t))$ be a curve lying in S such that $x(0) = x_0$ and $y(0) = y_0$.
- By the preceding remarks it follows that $z(t) = f(x(t), y(t))$.
- Then the tangent vector of $\mathbf{q}(t)$ at $t = 0$ is

$$
\begin{aligned}
\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t}_{|t=0} &= \left(\frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t}, \frac{\mathrm{d}}{\mathrm{d}t}\left(f(x(t), y(t))\right)\right)_{|t=0} \\
&= \left(\frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t}, \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}\right)_{|t=0} \\
&= \frac{\mathrm{d}x}{\mathrm{d}t}_{|t=0}\left(1, 0, \frac{\partial f}{\partial x}\right)_{|t=0} + \frac{\mathrm{d}y}{\mathrm{d}t}_{|t=0}\left(0, 1, \frac{\partial f}{\partial y}\right)_{|t=0}
\end{aligned}
$$



- $\mathbf{q}(t) = (x(t), y(t), z(t))$ - smooth curve in $S = \{(x, y, f(x, y))\}$,
- (x_0, y_0, z_0)- point in S, $\mathbf{q}(0) = (x(0), y(0), z(0)) = (x_0, y_0, z_0)$.
- $$\frac{d\mathbf{q}}{dt}_{|t=0} = \frac{dx}{dt}_{|t=0}\left(1, 0, \frac{\partial f}{\partial x}\right)_{|t=0} + \frac{dy}{dt}_{|t=0}\left(0, 1, \frac{\partial f}{\partial y}\right)_{|t=0}.$$

- Recall the tangent (space) space plane at (x_0, y_0, z_0) was defined as the (space) space passing through (x_0, y_0, z_0) and spanned by the tangents of all curves lying in the surface and passing through (x_0, y_0, z_0).

Corollary (Justification of tangent plane definition)

The tangent vector to any curve passing through (x_0, y_0, z_0) is a linear combination of the vectors $\left(1, 0, \frac{\partial f}{\partial x}\right)$ and $\left(0, 1, \frac{\partial f}{\partial y}\right)$.

Question

Can a given level surface be represented as a graph surface?

- Globally, level surfaces are cannot be represented as graph surfaces.
- Example: $x^2 + y^2 + z^2 = 1$: can't solve for z globally. Informally, $z = \pm\sqrt{1 - x^2 - y^2}$, however this is not a function if we can't decide on choice of $+$ or $-$.
- Locally, with additional requirements, a level surface can be represented as graph surfaces.
  - Near $P(0, 0, 1)$, the surface is the graph surface of $z = \sqrt{1 - x^2 - y^2}$.
  - Near $P(1, 0, 0)$, the surface is not a graph surface w.r.t. to z.

- Let $F(x, y, z)$- function, let $P(x_0, y_0, z_0)$ - point in the domain of F.
- Let $F(x_0, y_0, z_0) = k$.
- We say the level surface is a graph surface around P if there is a function $z = f(x, y)$ such that:
  - f is defined on an open disk D around (x_0, y_0);
  - $f(x_0, y_0) = z_0$;
  - $F(x, y, f(x, y)) = 0$ for all (x, y) in the disk D.
- If the level surface is a graph surface, we say that the equation $F(x, y, z) = k$ implicitly defines $z = f(x, y)$ satisfying the condition $f(x_0, y_0) = z_0$.