Calculus II
Homework on Lecture 7

1. Find a linear substitution (via completing the square) to transform the radical to a multiple of an expression of the form $\sqrt{u^2+1}$, $\sqrt{u^2-1}$ or $\sqrt{1-u^2}$.
   (a) $\sqrt{x^2+x+1}$.
   (b) $\sqrt{-2x^2+x+1}$.

1 Trig or Euler substitution, solutions use trig sub

1.1 Case 1: $\sqrt{x^2+1}$

2. Compute the integral.
   (a) $\int \frac{\sqrt{1+x^2}}{x^2}\mathrm{d}x$.

1.2 Case 2: $\sqrt{1-x^2}$

3. Compute the integral using a trigonometric substitution.
   (a) $\int \frac{\sqrt{9-x^2}}{x^2}\mathrm{d}x$.

2 Trig or Euler substitution, solutions use Euler sub

2.1 Case 1: $\sqrt{x^2+1}$

4. Compute the integral.
   (a) $\int \sqrt{x^2+1}\mathrm{d}x$
   (b) $\int \sqrt{x^2+2}\mathrm{d}x$
   (c) $\int \sqrt{x^2+x+1}\mathrm{d}x$
   (d) $\int \sqrt{(2x^2+2x+1)}\mathrm{d}x$
   (e) $\int \sqrt{(3x^2+2x+1)}\mathrm{d}x$
   (f) $\int \frac{\sqrt{x^2+1}}{x+1}\mathrm{d}x$
5. **This problem will not be quizzed.** Let $b^2-4ac<0$ and $a>0$ be (real) numbers. Show that
   $\int \sqrt{(ax^2+bx+c)}\mathrm{d}x = \frac{\sqrt{a}D}{2}\left(\ln\left(\sqrt{\left(\frac{2xa+b}{2\sqrt{D}a}\right)^2+1}+\frac{2xa+b}{2\sqrt{D}a}\right)+\frac{2xa+b}{2\sqrt{D}a}\sqrt{\left(\frac{2xa+b}{2\sqrt{D}a}\right)^2+1}\right)+C,$
   where $D=\dfrac{4ac-b^2}{4a^2}$.

2.2 Case 2: $\sqrt{1-x^2}$

6. Integrate
   (a) $\int \sqrt{1-x^2}\mathrm{d}x$
   (b) $\int \sqrt{2-x^2}\mathrm{d}x$
   (c) $\int \sqrt{-x^2+x+1}\mathrm{d}x$
   (d) $\int \sqrt{2-x-x^2}\mathrm{d}x$
   (e) $\int \frac{\sqrt{1-x^2}}{1+x}\mathrm{d}x$
   (f) $\int \frac{\sqrt{1-x^2}}{2+x}\mathrm{d}x$

2.3 Case 3: $\sqrt{x^2-1}$

7. Integrate
   (a) $\int \sqrt{x^2-1}\mathrm{d}x$
   (b) $\int \sqrt{x^2-2}\mathrm{d}x$
   (c) $\int \sqrt{2x^2+x-1}\mathrm{d}x$
   (d) $\int \sqrt{x^2+x-1}\mathrm{d}x$

3 Theory through problems (Optional homework, will not be quizzed, will not be tested)

3.1 Case 1: $\sqrt{x^2+1}$

3.1.1 $x = \cot\theta$

8. (a) Express $x, \mathrm{d}x$ and $\sqrt{x^2+1}$ via θ and $\mathrm{d}\theta$ for the trigonometric substitution $x = \cot\theta$, $\theta \in (0, \pi)$.
   (b) Express $x, \mathrm{d}x$ and $\sqrt{x^2+1}$ via t and $\mathrm{d}t$ for the Euler substitution $x = \cot(2\arctan t)$, $t > 0$. Express t via x.
9. Let the variables x and t be related via $\sqrt{x^2+1} = x + t$.
   (a) Express x via t.
   (b) Express $\sqrt{x^2+1}$ via t alone.
   (c) Express $\mathrm{d}x$ via t and $\mathrm{d}t$.

3.1.2 $x = \tan\theta$

10. (a) Express $x, \mathrm{d}x$ and $\sqrt{x^2+1}$ via θ and $\mathrm{d}\theta$ for the trigonometric substitution $x = \tan\theta$, $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.
    (b) Express $x, \mathrm{d}x$ and $\sqrt{x^2+1}$ via t and $\mathrm{d}t$ for the Euler substitution $x = \tan(2\arctan t)$, $t \in (-1, 1)$. Express t via x.
11. Let the variables x and t be related via $\sqrt{x^2+1} = \frac{x}{t} - 1$.
    (a) Express x via t.
    (b) Express $\sqrt{x^2+1}$ via t alone.
    (c) Express $\mathrm{d}x$ via t and $\mathrm{d}t$.

3.2 Case 2: $\sqrt{1-x^2}$

3.2.1 $x = \cos\theta$

12. (a) Express $x, \mathrm{d}x$ and $\sqrt{1-x^2}$ via θ and $\mathrm{d}\theta$ for the trigonometric substitution $x = \cos\theta$, $\theta \in [0, \pi]$.
    (b) Express $x, \mathrm{d}x$ and $\sqrt{1-x^2}$ via t and $\mathrm{d}t$ for the Euler substitution $x = \cos(2\arctan t)$, $t \geq 0$. Express t via x.
13. Let the variables x and t be related via $\sqrt{-x^2+1} = (1-x)t$.
    (a) Express x via t.
    (b) Express $\sqrt{-x^2+1}$ via t alone.
    (c) Express $\mathrm{d}x$ via t and $\mathrm{d}t$.

3.2.2 $x = \sin\theta$

14. (a) Express $x, \mathrm{d}x$ and $\sqrt{1-x^2}$ via θ and $\mathrm{d}\theta$ for the trigonometric substitution $x = \sin\theta$, $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.
    (b) Express $x, \mathrm{d}x$ and $\sqrt{1-x^2}$ via t and $\mathrm{d}t$ for the Euler substitution $x = \sin(2\arctan t)$, $t \in [-1, 1]$. Express t via x.
15. Let the variables x and t be related via $\sqrt{-x^2+1} = 1 - xt$.
    (a) Express x via t.
    (b) Express $\sqrt{-x^2+1}$ via t alone.
    (c) Express $\mathrm{d}x$ via t and $\mathrm{d}t$.

3.3 Case 3: $\sqrt{x^2-1}$

3.3.1 $x = \sec\theta$

16. (a) Express $x, \mathrm{d}x$ and $\sqrt{x^2-1}$ via θ and $\mathrm{d}\theta$ for the trigonometric substitution $x = \csc\theta$, $\theta \in \left[0, \frac{\pi}{2}\right] \cup \left[\pi, \frac{3\pi}{2}\right)$.
    (b) Express $x, \mathrm{d}x$ and $\sqrt{1-x^2}$ via t and $\mathrm{d}t$ for the Euler substitution $x = \sec(2\arctan t)$, $t \in (-\infty, -1) \cup [1, 0)$. Express t via x.
17. Let the variables x and t be related via $\sqrt{x^2-1} = (x+1)t$.
    (a) Express x via t.
    (b) Express $\sqrt{x^2-1}$ via t alone.
    (c) Express $\mathrm{d}x$ via t and $\mathrm{d}t$.

3.3.2 $x = \csc\theta$

18. (a) Express $x, \mathrm{d}x$ and $\sqrt{1-x^2}$ via θ and $\mathrm{d}\theta$ for the trigonometric substitution $x = \csc\theta$, $\theta \in \left[0, \frac{\pi}{2}\right] \cup \left[\pi, \frac{3\pi}{2}\right)$.
    (b) Express $x, \mathrm{d}x$ and $\sqrt{1-x^2}$ via t and $\mathrm{d}t$ for the Euler substitution $x = \csc(2\arctan t)$, $t \in (-\infty, -1) \cup [0, 1)$. Express t via x.
19. Let the variables x and t be related via $\sqrt{x^2-1} = \frac{1}{t} - x$.
    (a) Express x via t.
    (b) Express $\sqrt{x^2-1}$ via t alone.
    (c) Express $\mathrm{d}x$ via t and $\mathrm{d}t$.