Precalculus
Lecture 16
Factoring Polynomials

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Outline

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Recall that $i^2 = -1$, $\sqrt{-1} = i$.

Example (Polynomial factorizations)

$$
\begin{aligned}
2x^2 + 3x - 5 &= (2x+5)(x-1) = 2\left(x - \left(-\tfrac{5}{2}\right)\right)(x-1) \\
x^2 + 1 &= x^2 - (-1) = x^2 - i^2 = (x+i)(x-i) \\
x^4 - 1 &= (x^2-1)(x^2+1) = (x-1)(x+1)(x^2+1) \\
&= (x-1)(x+1)(x-i)(x+i) \\
x^4 + 1 &= \left(x^2 - \sqrt{2}x + 1\right)\left(x^2 + \sqrt{2}x + 1\right) \\
&= \left(x - \left(\tfrac{\sqrt{2}}{2} + \tfrac{\sqrt{2}}{2}i\right)\right)\left(x - \left(\tfrac{\sqrt{2}}{2} - \tfrac{\sqrt{2}}{2}i\right)\right) \\
&\quad \left(x - \left(-\tfrac{\sqrt{2}}{2} + \tfrac{\sqrt{2}}{2}i\right)\right)\left(x - \left(-\tfrac{\sqrt{2}}{2} - \tfrac{\sqrt{2}}{2}i\right)\right)
\end{aligned}
$$

Theorem (The Fundamental Theorem of Algebra)

Every polynomial can be factored into product of linear terms

$$p(x) = a_0x^n + a_1x^{n-1} + \dots + a_n = a_0(x - x_1)\dots(x - x_n),$$

where $x_1, \dots, x_n$ are the (not necessarily different) roots of $p(x)$.

- Every pol. of deg. n can be factored as product of n linear factors.
- $x_1, \dots, x_n$ may be complex numbers. Reminder: complex numbers are of the form $p + qi$, where $i^2 = -1$ and $\sqrt{-1} = i$.
- While we can find $x_1, \dots x_n$ with arbitrary precision, there may not exist a formula involving radicals for computing each $x_1, \dots, x_n$.

Corollary

Every real polynomial can be factored into a product of real linear terms and real quadratic terms with no real roots, i.e., factors of form

- $(x - r)$, *where r is real and*
- $ax^2 + bx + c$ *with $b^2 - 4ac < 0$ where a, b, c are real.*

$$a_0x^n + a_1x^{n-1} + \dots + a_n = a_0(x-x_1)\dots(x-x_n)$$
$=$prod. real quadratics no roots & lin. terms.

Example

$$2x^2+3x-5=(2x+5)(x-1)=2\left(x-\left(-\tfrac{5}{2}\right)\right)(x-1)$$ real roots

$$x^2+1=x^2-(-1)=x^2-i^2=(x-(-i))(x-i)$$ complex roots

$$x^4-1=(x^2-1)(x^2+1)=(x-1)(x+1)(x^2+1)$$
$$=(x-1)(x-(-1))(x-i)(x-(-i))$$ mixed roots

$$x^4+1=(x^2-\sqrt{2}x+1)(x^2+\sqrt{2}x+1)$$
$$=\left(x-\left(\tfrac{\sqrt{2}}{2}+\tfrac{\sqrt{2}}{2}i\right)\right)\left(x-\left(\tfrac{\sqrt{2}}{2}-\tfrac{\sqrt{2}}{2}i\right)\right)$$
$$\left(x-\left(-\tfrac{\sqrt{2}}{2}+\tfrac{\sqrt{2}}{2}i\right)\right)\left(x-\left(-\tfrac{\sqrt{2}}{2}-\tfrac{\sqrt{2}}{2}i\right)\right)$$ complex roots

Factoring polynomials in practice

- In theory every polynomial can be factored.

$$a_0x^n + a_1x^{n-1} + \cdots + a_n = a_0(x - x_1)\dots(x - x_n)$$

- Theory guarantees numerical approximations for roots $x_1, \dots, x_n$.
- Can we find algebraic formulas for $x_1, \dots, x_n$?
- No, if using finitely many operations $+, -, *, /, \sqrt[n]{}$.
- First (advanced) proof by Norwegian Niels Henrik Abel(1824) based on work of Italian Paolo Ruffini(1799).
- Yes, with extra operations. Difficult: google Galois Theory to get started.

What does factorization mean?

- Based on context, "to factor a polynomial" means one of:
  - Factor the polynomial over the rational numbers. Use integers/quotients, but no $\sqrt[n]{\ }$).
  - Factor the polynomial over the real numbers. Use radicals and/or numerical approximations, no use of $i = \sqrt{-1}$.
  - Fully factor the polynomial using complex numbers.

These poly's are equal	Type of factorization
$x^4 + 1$	factored over rationals
$(x^2 - \sqrt{2}x + 1)(x^2 + \sqrt{2}x + 1)$	factored over the reals
$\left(x - \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)\right)\left(x - \left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)\right)$ $\left(x - \left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right)\right)\left(x - \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right)\right)$	full complex factorization

Factorization over the rationals

- Suppose we want to factor a polynomial using only rational numbers (no $\sqrt[n]{}$ or numerical approximations).
- No guarantee to get:

$$a_0x^n + a_1x^{n-1} + \dots + a_n = a_0(x - x_1)\dots(x - x_n)$$

- A factorization using rationals may have arbitrarily large factors.
- Efficient algorithms for factoring using rationals exist.
  - Kronecker algorithm (German Leopold Kronecker (1823-1891)).
  - Methods based on finite fields.
  - Lenstra-Lenstra-Lovász algorithm (Dutch, Dutch, Hungarian mathematicians, all contemporary).
- Above methods require computer; no rational roots assumption.
- If we assume rational roots there are practical algorithms by hand.
- We study those for cubics with the aid of scientific calculator.

Example (Polynomial long division)

Divide with quotient and remainder $x^3 + 2x^2 + 1$ by $x - 1$.

$$
\begin{array}{rrl}
\textbf{Quotient:} & & x^2 + 3x + 3 \\
x - 1 & & \overline{)\, x^3 + 2x^2 \qquad + 1} \\
- & & x^3 - x^2 \\
\hline
 & & \quad 3x^2 \qquad + 1 \\
- & & \quad 3x^2 - 3x \\
\hline
 & & \qquad\quad 3x + 1 \\
- & & \qquad\quad 3x - 3 \\
\hline
\textbf{Remainder:} & & \qquad\qquad 4
\end{array}
$$

$$
\begin{array}{rcl}
\textbf{(Dividend)} & = & \textbf{(Quotient)} \cdot \textbf{(Divisor)} + \textbf{(Remainder)} \\
(x^3 + 2x^2 + 1) & = & (x^2 + 3x + 3) \cdot (x - 1) + 4
\end{array}
$$

Example

Demonstrate that $6x^3 - 19x^2 + 17x - 3$ is divisible by $2x - 3$ using polynomial long division. Use your work to factor the cubic. Solve the equation $6x^3 - 19x^2 + 17x - 3 = 0$.

$$
\begin{array}{rr}
\textbf{Quotient:} & 3x^2 - 5x + 1 \\
2x - 3 & \overline{\big)\, 6x^3 - 19x^2 + 17x - 3} \\
- & 6x^3 - 9x^2 \qquad\qquad \\
\hline
 & -10x^2 + 17x - 3 \\
- & -10x^2 + 15x \quad\;\; \\
\hline
 & 2x - 3 \\
- & 2x - 3 \\
\hline
\textbf{Remainder:} & 0
\end{array}
$$

$$\textbf{(Dividend)} = \textbf{(Quotient)} \cdot \textbf{(Divisor)} + \textbf{(Remainder)}$$

$$(6x^3 - 19x^2 + 17x - 3) = (3x^2 - 5x + 1) \cdot (2x - 3)$$

Example

Demonstrate that $6x^3 - 19x^2 + 17x - 3$ is divisible by $2x - 3$ using polynomial long division. Use your work to factor the cubic. Solve the equation $6x^3 - 19x^2 + 17x - 3 = 0$.

$$\begin{aligned}(6x^3 - 19x^2 + 17x - 3) &= (3x^2 - 5x + 1) \cdot (2x - 3) \\ &= 3\left(x - \left(\frac{5+\sqrt{13}}{6}\right)\right)\left(x - \left(\frac{5-\sqrt{13}}{6}\right)\right)(2x - 3)\end{aligned}$$

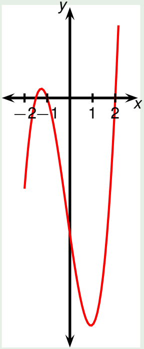
No easy factorization of quadratic, so use formula:

$x_1, x_2 = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{-(-5)\pm\sqrt{(-5)^2-4\cdot 3\cdot 1}}{2\cdot 3} = \frac{5\pm\sqrt{13}}{6}$ We are ready to solve the equation.

$$\begin{aligned}6x^3 - 19x^2 + 17x - 3 &= 0 \\ 3\left(x - \left(\frac{5+\sqrt{13}}{6}\right)\right)\left(x - \left(\frac{5-\sqrt{13}}{6}\right)\right)(2x - 3) &= 0\end{aligned}$$

$$\begin{aligned} 2x - 3 &= 0 \quad \textbf{or} \quad x = \left(\frac{5+\sqrt{13}}{6}\right) \quad \textbf{or} \quad x = \left(\frac{5-\sqrt{13}}{6}\right) \\ x &= \frac{3}{2}\end{aligned}$$

Example



Plot the left hand side of the equation with a graphing calculator. Solve the equation.

$$\begin{aligned} 2x^3 + x^2 - 7x - 6 &= 0 \\ (2x+3)(x+1)(x-2) &= 0 \end{aligned}$$

$$x = -\frac{3}{2} \quad \textbf{or} \quad x = -1 \quad \textbf{or} \quad x = 2$$

Make sure to practice with the graphing calculator you will on your exam(s). The graph appears to intersect the x axis at: $-1.5, -1, 2$. The left hand side should factor as:

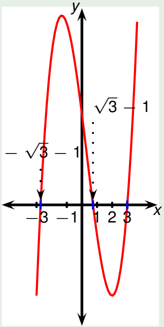
$$2(x - (-1.5))(x - (-1))(x - 2) = (2x+3)(x+1)(x-2)$$

$$= (2x^2 + 5x + 3)(x - 2) = (2x^3 + 5x^2 + 3x) - (4x^2 + 10x + 6)$$

$$= 2x^3 + x^2 - 7x - 6$$

Check work to make sure we guessed the roots correctly.

Example



Plot the left hand side of the equation with a graphing calculator. Solve the equation.

$$\begin{aligned} x^3 - x^2 - 8x + 6 &= 0 \\ (x-3)(x^2+2x-2) &= 0 \end{aligned}$$

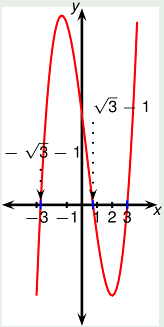
The graph appears to intersect the x axis at: $-\sqrt{3}-1$, $\sqrt{3}-1, 3$. What are the two roots besides 3?

Quotient: $x^2 + 2x - 2$

$$\begin{array}{r|l} x-3 & x^3 - x^2 - 8x + 6 \\ - & x^3 - 3x^2 \\ \hline & \quad 2x^2 - 8x + 6 \\ - & \quad 2x^2 - 6x \\ \hline & \qquad -2x + 6 \\ - & \qquad -2x + 6 \\ \hline & \qquad\quad 0 \end{array}$$

Remainder: 0

Example



Plot the left hand side of the equation with a graphing calculator. Solve the equation.

$$\begin{aligned} x^3 - x^2 - 8x + 6 &= 0 \\ (x-3)(x^2+2x-2) &= 0 \end{aligned}$$

$$x - 3 = 0 \quad \textbf{or} \quad x = \frac{-2 \pm \sqrt{(2)^2 - 4\cdot 1\cdot(-2)}}{2\cdot 1}$$

$$x = 3 \qquad x = \frac{-2 \pm \sqrt{12}}{2}$$

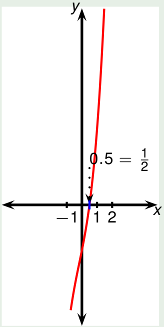
$$x = \frac{-2 \pm 2\sqrt{3}}{2} = -1 \pm \sqrt{3}.$$

The graph appears to intersect the x axis at: $-\sqrt{3}-1, \sqrt{3}-1, 3$. What are the two roots besides 3?

Final answer:

$x = 3$ **or** $x = -1-\sqrt{3}$ **or** $x = -1+\sqrt{3}$.

Example



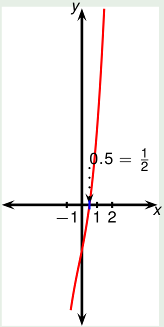
Plot the left hand side of the equation with a graphing calculator. Find all real solutions of the equation.

$$\begin{aligned} 2x^3 + x^2 + 5x - 3 &= 0 \\ \left(x - \tfrac{1}{2}\right)\left(2x^2 + 2x + 6\right) + 0 &= 0 \end{aligned}$$

We see only one root, $x = 0.5 = \frac{1}{2}$. Is our guess correct? Is there another root (far away from 0)? Factor:

$$\begin{array}{rr} \textbf{Quotient:} & 2x^2 + 2x + 6 \\ x - \frac{1}{2} & \overline{\big)\, 2x^3 + x^2 + 5x - 3} \\ - & 2x^3 - x^2 \\ \hline & 2x^2 + 5x - 3 \\ - & 2x^2 - x \\ \hline & 6x - 3 \\ - & 6x - 3 \\ \hline \textbf{Remainder:} & 0 \end{array}$$

Example



Plot the left hand side of the equation with a graphing calculator. Find all real solutions of the equation.

$$\begin{aligned} 2x^3 + x^2 + 5x - 3 &= 0 \\ \left(x - \tfrac{1}{2}\right)\left(2x^2 + 2x + 6\right) + 0 &= 0 \end{aligned}$$

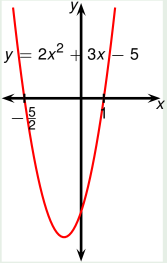
$$x - \frac{1}{2} = 0 \quad \textbf{or} \quad x = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 2 \cdot 6}}{2 \cdot 2}$$

$$x = \frac{1}{2} \qquad\qquad x = \frac{-2 \pm \sqrt{-44}}{2 \cdot 2}$$

no real solution

We see only one root, $x = 0.5 = \frac{1}{2}$. Is our guess correct? Is there another root (far away from 0)?

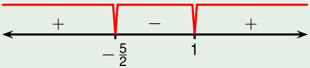
Example



Solve the inequality.

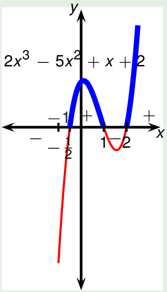
$$\begin{aligned} 2x^2 + 3x - 5 &\geq 0 \\ (2x+5)(x-1) &\geq 0 \\ x \in \left(-\infty, -\frac{5}{2}\right] \cup [1, \infty) \end{aligned}$$

Left hand side vanishes when $x = -\frac{5}{2}$ and when $x = 1$. The two roots split the real line into three intervals: $\left(-\infty, -\frac{5}{2}\right), \left(-\frac{5}{2}, 1\right), (1, \infty)$.



Interval	Factor signs	Final sign	Sample pt	Value at sample pt
$\left(-\infty, -\frac{5}{2}\right)$	$(-)(-)$	$+$	-100	$f(-100) > 0$
$\left(-\frac{5}{2}, 1\right)$	$(+)(-)$	$-$	0	$f(0) = -5 < 0$
$(1, \infty)$	$(+)(+)$	$+$	100	$f(100) > 0$

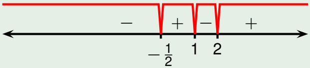
Example



Plot the function $2x^3 - 5x^2 + x + 2$. Solve the inequality.

$$\begin{aligned} 2x^3 - 5x^2 + x + 2 &> 0 \\ 2\left(x - \left(-\tfrac{1}{2}\right)\right)(x-1)(x-2) &> 0 \\ x \in \left(-\tfrac{1}{2}, 1\right) \cup (2, \infty) \end{aligned}$$

Left hand side vanishes when $x = -\frac{1}{2}$, when $x = 1$ and when $x = 2$. The two roots split the real line into four intervals: $\left(-\infty, -\frac{1}{2}\right), \left(-\frac{1}{2}, 1\right), (1, 2), (2, \infty)$.



Interval	Factor signs	Final sign from plot
$\left(-\infty, -\frac{1}{2}\right)$	$(-)(-)(-)$	$-$
$\left(-\frac{1}{2}, 1\right)$	$(+)(-)(-)$	$+$
$(1, 2)$	$(+)(+)(-)$	$-$
$(2, \infty)$	$(+)(+)(+)$	$+$