Calculus III
Homework on Lecture 11

1. Determine the type of the quadratic surface given by the equation. The answer key has not been proofread, use with extreme caution.

   (a) $x^2 + y^2 + z^2 + x + 2y + 3z = 0.$ answer: sphere (also ellipsoid)

   (b) $x^2 + 2y^2 + z^2 + x + 2y + 3z = 0.$ answer: (circular) ellipsoid

   (c) $x^2 + 2y^2 + 3z^2 + x + 2y + 3z = 0.$ answer: ellipsoid

   (d) $z^2 + 2y^2 - 3x^2 + x + y + 1 = 0.$ answer: (elliptic) hyperboloid two sheets

   (e) $z^2 - y^2 + \frac{1}{4}x^2 + x - y + 1 = 0.$ answer: (elliptic) hyperboloid two sheets

   (f) $x^2 + y^2 - \frac{1}{4}z^2 + x - y + 5 = 0.$ answer: (circular) hyperboloid two sheets

   (g) $\frac{1}{4}x^2 - y^2 + z^2 - x + 1 = 0$ answer: (elliptic) cone

   (h) $-\frac{1}{4}x^2 + y^2 + z^2 - x - 1 = 0$ answer: (circular) cone

   (i) $xy + z^2 + 1 = 0$. Hint: write $x = \frac{1}{\sqrt{2}}(u + v)$, $y = \frac{1}{\sqrt{2}}(u - v)$ for some new variables u, v. Solve the problem in the z, u, v -coordinates. Argue that the (axes of the) u, v, z-coordinate system can be obtained from the x, y, z-coordinate system via rotation. answer: (circular) hyperboloid one sheet

   (j) $x^2 + 2y^2 + z = 0.$ answer: (elliptic) paraboloid

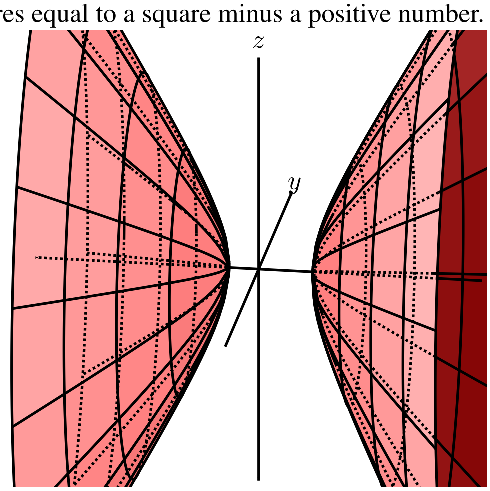
   (k) $x^2 + y^2 + 2xy + z = 0.$ answer: cylindrical paraboloid

   (l) $x^2 - y^2 + 2x + z = 0.$ answer: parabolic hyperboloid

   Solution. 1d We have that

$$
\begin{array}{rcl}
z^2 + 2y^2 - 3x^2 + x + y + 1 & = & 0 \\
z^2 + 2\left(y + \frac{1}{4}\right)^2 - 3\left(x - \frac{1}{6}\right)^2 - \frac{1}{8} + \frac{1}{12} + 1 & = & 0 \\
z^2 + 2\left(y + \frac{1}{4}\right)^2 & = & 3\left(x - \frac{1}{6}\right)^2 - \frac{23}{24}
\end{array}
$$

   This figure is given by sum of two squares equal to a square minus a positive number. That makes is a hyperboloid two sheet, as explained in the theoretical discussions.



2. Find an equation of the tangent plane to the surface at the given point. The surface is given via an implicit equation.

   (a) The sphere $x^2 + y^2 + z^2 = 1$ at $(x, y, z) = \left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$.

   answer: tangent plane: $x + y + z = \sqrt{3}$

(b) The two-sheet hyperboloid $x^2 + y^2 - z^2 = -3$ at $(x, y, z) = (2, 3, 4)$.

answer: tangent plane: $2x + 3y - 4z = -3$

(c) The ellipsoid $x^2 + 2y^2 + 3z^2 = 20$ at $(x, y, z) = (3, 2, 1)$.

answer: tangent plane: $3x + 4y + 3z = 20$

Solution. 2.b As studied, a normal to the tangent plane to a surface with implicit equation $f = 0$ is given by ∇f. Since the tangent plane passes through $(2, 3, 4)$, this determines the tangent plane.

$$
\begin{array}{rcll}
f = x^2 + y^2 - z^2 - 3 & = & 0 & \mid \text{equation of the surface} \\
\nabla f & = & (2x, 2y, -2z) & \\
\nabla f_{|(x,y,z)=(2,3,4)} & = & (4, 6, -8) & \\
\nabla f_{|(x,y,z)=(2,3,4)} \cdot (x-2, y-3, z-4) & = & 0 & \mid \text{equation of plane} \\
4(x-2) + 6(y-3) - 8(z-4) & = & 0 & \\
2x + 3y - 4z & = & -3 & \mid \text{final answer in simplified form.}
\end{array}
$$

Find the equation of the tangent plane to the graph of the function at the indicated point.

3. (a) $z = x^2 - y^2$, at the point $(1, 1, 0)$.

answer:

(b) $z = e^{-x^2-y^2}$, at the point $(0, 0, 1)$

answer: $z = 1$

(c) $z = e^{x^2-y^2}$, at the point $(1, -1, 1)$.

answer: $2x + 2y - z = -1$

(d) $z = \sqrt{3 - x^2 - y^2}$, at the point $(1, 1, 1)$.

answer: