Calculus I
Lecture 21
The Fundamental Theorem of Calculus
Part I

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Outline

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Antiderivatives

Definition (Antiderivative)

A function F is called an antiderivative of f on an interval I if $F'(x) = f(x)$ for all x in I.

Example

- Let $f(x) = x^2$.
- Use the Power Rule to find an antiderivative of f:
- If $F(x) = \frac{1}{3}x^3$, then $F'(x) = x^2 = f(x)$.
- Is this the only one?
- No. If $G(x) = \frac{1}{3}x^3 + 1$, then $G'(x) = x^2 = f(x)$.
- $\frac{1}{3}x^3 + 2$ will also work.
- Any function of the form $H(x) = \frac{1}{3}x^3 + C$, where C is a constant, is an antiderivative of f.

Theorem

If F is an antiderivative of f on an interval I, then an arbitrary antiderivative of f on I is of the form

$$F(x) + C$$

where C is an arbitrary constant.

Example

Find all antiderivatives of each of the following functions.

$$f(x) = \sin x$$

- If $F(x) = -\cos x$, then $F'(x) = \sin x$.
- Therefore antiderivative is of the form $G(x) = -\cos x + C$.

$$f(x) = x^n, n \geq 0$$

- If $F(x) = \frac{x^{n+1}}{n+1}$, then $F'(x) = x^n$.
- Therefore any antiderivative is of the form $G(x) = \frac{x^{n+1}}{n+1} + C$.

Example

Find the most general antiderivative of $f(x) = \frac{1}{x}$.

- If $F(x) = \ln|x|$, then $F'(x) = \frac{1}{x}$.
- This is valid for any interval on which $\frac{1}{x}$ is defined.
- $\frac{1}{x}$ is defined everywhere except at 0.
- The most general answer needs two different constants, one for $(-\infty, 0)$ and one for $(0, \infty)$.

$$G(x) = \begin{cases} \ln|x| + C_1 & \text{if} \quad x > 0 \\ \ln|x| + C_2 & \text{if} \quad x < 0 \end{cases}$$

Every differentiation formula gives rise to an antidifferentiation formula. Suppose $F' = f$ and $G' = g$.

Function	Particular Antiderivative
$cf(x)$	$cF(x)$
$f(x) + g(x)$	$F(x) + G(x)$
$x^n (n \neq -1)$	$\frac{x^{n+1}}{n+1}$
$\frac{1}{x}$	$\ln\|x\|$
e^x	e^x
$\cos x$	$\sin x$
$\sin x$	$-\cos x$
$\sec^2 x$	$\tan x$
$\sec x \tan x$	$\sec x$

Example

Find all functions g such that

$$g'(x) = 4\sin x + \frac{2x^5 - \sqrt{x}}{x}.$$

Rewrite:

$$g'(x) = 4\sin x + 2\frac{x^5}{x} - \frac{\sqrt{x}}{x} = 4\sin x + 2x^4 - \frac{1}{\sqrt{x}}$$

Find the antiderivative:

$$g'(x) = 4\sin x + 2x^4 - \frac{1}{\sqrt{x}}$$

$$g(x) = 4(-\cos x) + 2\frac{x^5}{5} - \frac{x^{1/2}}{\frac{1}{2}} + C$$

$$= -4\cos x + \frac{2}{5}x^5 - 2\sqrt{x} + C$$

Example

Find f if $f'(x) = \frac{1}{x\sqrt{x}}$ for $x > 0$, and $f(1) = 1$.

$$f'(x) = \frac{1}{x\sqrt{x}} = x^{-3/2}$$

$$f(x) = \frac{x^{-1/2}}{-\frac{1}{2}} + C$$

$$= -\frac{2}{\sqrt{x}} + C$$

To find C, use the fact that $f(1) = 1$.

$$f(1) = 1$$

$$-\frac{2}{\sqrt{1}} + C = 1$$

$$C = 3$$

Therefore

$$f(x) = -\frac{2}{\sqrt{x}} + 3.$$

Theorem

Let f be a continuous function on $[a, b]$. Then f is integrable over $[a, b]$.

In other words, $\int_a^b f(x)dx$ exists for any continuous (over $[a, b]$) function f.

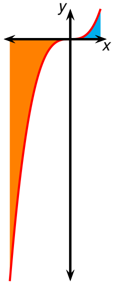
Theorem (The Evaluation Theorem (FTC part 2))

If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\mathrm{d}x = F(b) - F(a),$$

where F is any antiderivative of f.

Example



Evaluate the integral $\int_{-2}^{1} x^3 \, \mathrm{d}x$.

- x^3 is continuous on $[-2, 1]$ (in fact, it's continuous everywhere).
- An antiderivative is $F(x) = \frac{1}{4}x^4$.

$$\int_{-2}^{1} x^3 \, \mathrm{d}x = F(1) - F(-2) = \frac{1}{4}(1)^4 - \frac{1}{4}(-2)^4 = \frac{1}{4} - \frac{16}{4} = -\frac{15}{4}$$

We often use the notation

$$F(x)]_a^b = F(b) - F(a)$$

or

$$[F(x)]_a^b = F(b) - F(a)$$

Therefore we can write

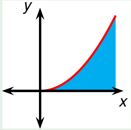
$$\int_a^b f(x)\mathrm{d}x = F(x)]_a^b$$

or

$$\int_a^b f(x)\mathrm{d}x = [F(x)]_a^b$$

Example

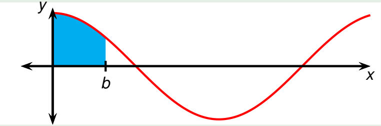
Find the area under the parabola $y = x^2$ from 0 to 1.



- x^2 is continuous on $[0, 1]$ (in fact, it's continuous everywhere).
- An antiderivative of x^2 is $\frac{1}{3}x^3$.

$$\int_0^1 x^2 \, dx = \left[\frac{1}{3}x^3\right]_0^1 = \frac{1}{3}(1)^3 - \frac{1}{3}(0)^3 = \frac{1}{3}$$

Example



Find the area under the cosine curve from 0 to b, where $0 \leq b \leq \frac{\pi}{2}$.

- $\cos x$ is continuous on $[0, \frac{\pi}{2}]$ (in fact, it's continuous everywhere).
- An antiderivative of $\cos x$ is $\sin x$.

$$\int_0^b \cos x \, dx = [\sin x]_0^b = \sin(b) - \sin(0) = \sin b$$

Indefinite Integrals

- The Evaluation Theorem establishes a connection between antiderivatives and definite integrals.
- It says that $\int_a^b f(x)\mathrm{d}x$ equals $F(b) - F(a)$, where F is an antiderivative of f.
- We need convenient notation for writing antiderivatives.
- This is what the indefinite integral is.

Definition (Indefinite Integral)

The indefinite integral of f is another way of saying the antiderivative of f, and is written $\int f(x)\mathrm{d}x$. In other words,

$$\int f(x)\mathrm{d}x = F(x) \qquad \text{means} \qquad F'(x) = f(x).$$

Example

$$\int x^4 \mathrm{d}x = \frac{x^5}{5} + C$$

because

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^5}{5} + C\right) = x^4.$$

- The indefinite integral represents a whole family of functions.
- Example: the general antiderivative of $\frac{1}{x}$ is
$$F(x) = \begin{cases} \ln|x| + C_1 & \text{if} \quad x > 0 \\ \ln|x| + C_2 & \text{if} \quad x < 0 \end{cases}$$
- We adopt the convention that the constant participating in an indefinite integral is only valid on one interval.
- $\int \frac{1}{x}\mathrm{d}x = \ln|x| + C$, and this is valid either on $(-\infty, 0)$ or $(0, \infty)$.

Example

Find the indefinite integral.

$$\begin{aligned}\int (8x^3 - 3\sec^2 x)\mathrm{d}x &= 8\int x^3\mathrm{d}x - 3\int \sec^2 x\mathrm{d}x\\ &= 8\frac{x^4}{4} - 3\tan x + C\\ &= 2x^4 - 3\tan x + C\end{aligned}$$

Example

Find the general indefinite integral.

$$\begin{aligned}\int \frac{\cos\theta}{\sin^2\theta}\mathrm{d}\theta &= \int \left(\frac{1}{\sin\theta}\right)\left(\frac{\cos\theta}{\sin\theta}\right)\mathrm{d}\theta \\ &= \int \csc\theta\,\cot\theta\mathrm{d}\theta \\ &= -\csc\theta + C\end{aligned}$$

Example

$$\begin{aligned}
\int_0^3 (x^3 - 6x)\mathrm{d}x &= \left[\int (x^3 - 6x)\mathrm{d}x\right]_0^3 \\
&= \left[\int x^3\mathrm{d}x - 6\int x\mathrm{d}x\right]_0^3 \\
&= \left[\frac{x^4}{4} - 6\,\frac{x^2}{2}\right]_0^3 \\
&= \left(\frac{1}{4}\cdot 3^4 - 3\cdot 3^2\right) - \left(\frac{1}{4}\cdot 0^4 - 3\cdot 0^2\right) \\
&= \frac{81}{4} - 27 - 0 + 0 = -\frac{27}{4}.
\end{aligned}$$

Example

Evaluate: $\displaystyle\int_1^9 \frac{2t^3 + t^2\sqrt{t} - 1}{t^2} \mathrm{d}t$

$$= \int_1^9 \left(2t + t^{\frac{1}{2}} - t^{-2}\right) \mathrm{d}t = \left[\int (2t + t^{\frac{1}{2}} - t^{-2}) \mathrm{d}t\right]_1^9$$

$$= \left[\int 2t\mathrm{d}t + \int t^{\frac{1}{2}}\mathrm{d}t - \int t^{-2}\mathrm{d}t\right]_1^9$$

$$= \left[t^2 + \frac{t^{\frac{3}{2}}}{\frac{3}{2}} - \frac{t^{-1}}{-1}\right]_1^9 = \left[t^2 + \frac{2}{3}t^{\frac{3}{2}} + \frac{1}{t}\right]_1^9$$

$$= \left(9^2 + \frac{2}{3} \cdot 9^{\frac{3}{2}} + \frac{1}{9}\right) - \left(1^2 + \frac{2}{3} \cdot 1^{\frac{3}{2}} + \frac{1}{1}\right)$$

$$= 81 + 18 + \frac{1}{9} - 1 - \frac{2}{3} - 1 = \frac{868}{9}.$$