Calculus II
Lecture 21

Todor Milev
https://github.com/tmilev/freecalc

2020

Outline

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Definition (Complex numbers)

The set of complex numbers $\mathbb{C}$ is defined as the set

$$\{a + bi | a, b - \text{real numbers}\},$$

where the number i is a number for which

$$i^2 = -1 \quad .$$

The number i is called the imaginary unit.

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$$(a + bi) \pm (c + di) = (a \pm c) + (b \pm d)i \quad .$$

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$$(a + bi) \pm (c + di) = (a \pm c) + (b \pm d)i \quad .$$

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$$(a + \color{red}{bi}\color{black})(c + \color{red}{di}\color{black}) \quad = \quad ac + adi + bci + \color{red}{bdi^2}$$

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$$\begin{aligned}(a + bi)(c + di) &= ac + adi + bci + bdi^2 = ac + adi + bci - bd \\ &= (ac - bd) + i(ad + bc)\end{aligned}$$

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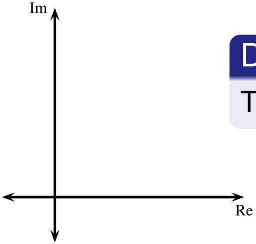

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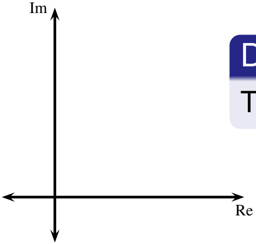
The complex numbers are the set $\{x + yi | x, y \in \mathbb{R}\}$.



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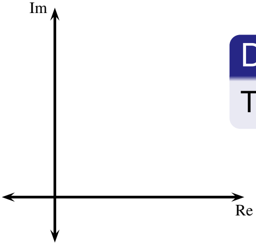

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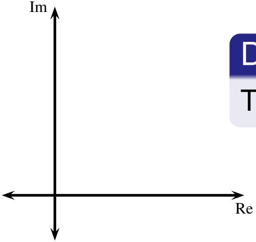

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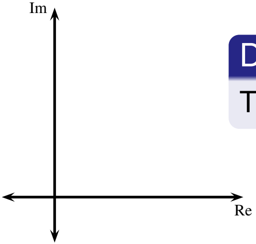

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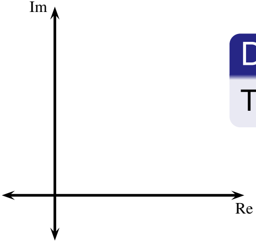

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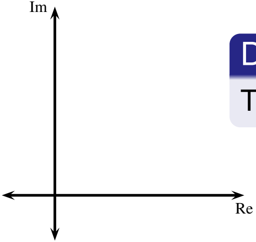

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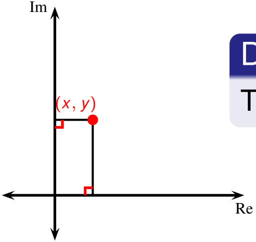

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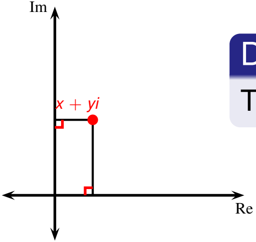

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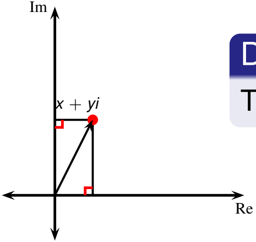

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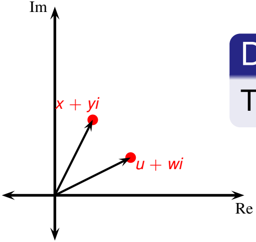

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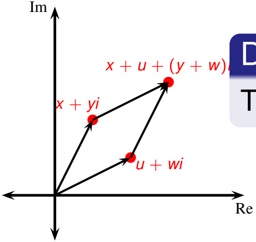

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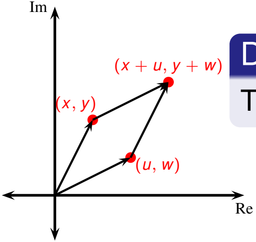

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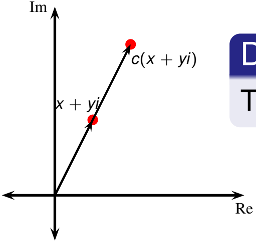

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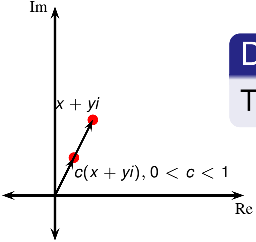

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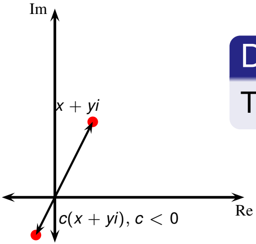

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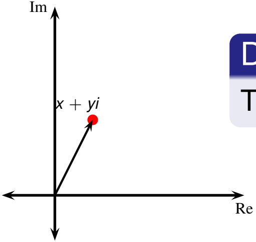

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- The addition of complex numbers corresponds to vector addition.
- Multiplication by a real number c corresponds to vector scalar multiplication by c (scaling).
- The space the complex numbers is referred to as the complex plane (sometimes alternatively called the complex line).

Let $u = 2 + 3i$, $v = 5 - 7i$.

Example (Addition)

$u + v =$.

Example (Subtraction)

$u - v =$.

Example (Multiplication)

$u \cdot v \quad =$

Let $u = 2 + 3i$, $v = 5 - 7i$.

Example (Addition)

$u + v = (2 + 3i) + (5 - 7i) = \mathbf{?}$.

Example (Subtraction)

$u - v =$.

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$u \cdot v =$

Let $u = 2 + 3i$, $v = 5 - 7i$.

Example (Addition)

$u + v = (2 + 3i) + (5 - 7i) = (2 + 5) + (3 - 7)i = 7 - 4i.$

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$u - v = (2 + 3i) - (5 - 7i) = (2 - 5) + (3 - (-7))i = -3 + 10i.$

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$u - v = (2 + 3i) - (5 - 7i) = (2 - 5) + (3 - (-7))i = -3 + 10i.$

Example (Multiplication)

$$
\begin{aligned}
u \cdot v &= (2 + 3i) \cdot (5 - 7i) \\
&= 2 \cdot 5 + 2 \cdot (-7)i + 3i \cdot 5 + 3i(-7i) \\
&= 10 - 14i + 15i - 21i^2
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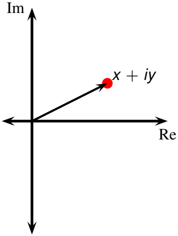

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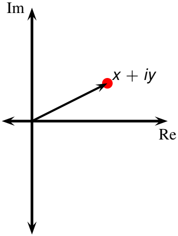

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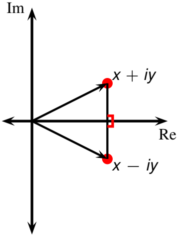

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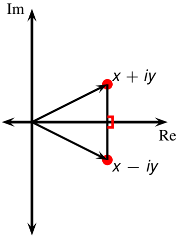

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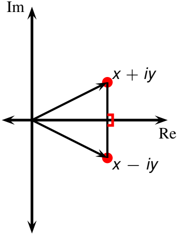

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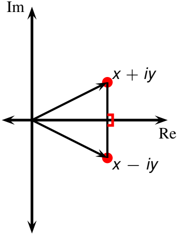

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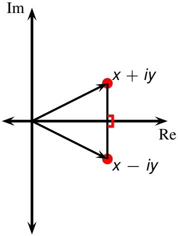

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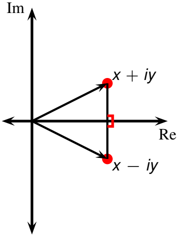

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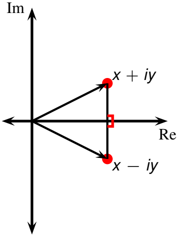

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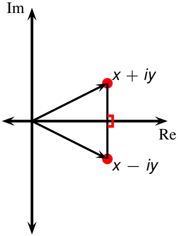

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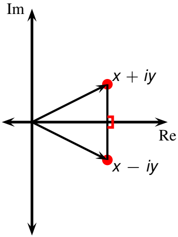

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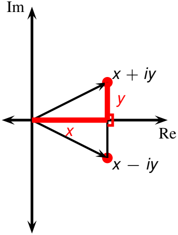

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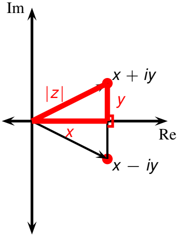

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Theorem (Conjugation preserves $\cdot$)

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Corollary

$\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$, $w \neq 0$.

Let $u = 2 + 3i$, $v = 5 - 7i$.

Example (Division)

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Multiply and divide
by complex conjugate
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$$\begin{aligned} \frac{u}{v} &= \frac{2+3i}{5-7i} \\ &= \frac{(2+3i)}{(5-7i)} \frac{(5+7i)}{(5+7i)} \end{aligned}$$

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Multiply and divide by complex conjugate of denominator

Let $u = a + bi$, $v = c + di$, $v \neq 0$.

Example (Complex number division)

$$
\begin{aligned}
\frac{u}{v} &= \frac{a+bi}{c+di} \\
&= \frac{(a+bi)}{(c+di)}\frac{(c-di)}{(c-di)} \\
&= \frac{(a+bi)(c-di)}{c^2-(di)^2} \\
&= \frac{ac-adi+cbi-bdi^2}{c^2+d^2} \\
&= \frac{ac+bd+(bc-ad)i}{c^2+d^2} \\
&= \frac{ac+bd}{c^2+d^2}+\frac{(bc-ad)}{c^2+d^2}i
\end{aligned}
$$

Multiply and divide by complex conjugate of denominator

Definition (Complex number division)

The quotient $\frac{u}{v}$, $v \neq 0$ is defined via the formula above.

Theorem

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Let $e(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$, $z \in \mathbb{C}$. *Then* $e(z)e(w) = e(z+w)$.

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Proof.

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$$\begin{aligned} e(z)e(w) &= \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{m=0}^{\infty} \frac{w^m}{m!} = \sum_{s=0}^{\infty} \sum_{k=0}^{s} \frac{z^k w^{s-k}}{k!(s-k)!} \\ &= \sum_{s=0}^{\infty} \sum_{k=0}^{s} \frac{z^k w^{s-k}}{s!} \frac{s!}{k!(s-k)!} = \sum_{s=0}^{\infty} \frac{(z+w)^s}{s!} = e(z+w). \end{aligned}$$

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\end{aligned}
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Lemma (Newton Binomial formula)

$(z+w)^s = \sum_{k=0}^{s} z^k w^{s-k} \frac{s!}{k!(s-k)!}$.

Definition (Real exponent, Definition I)

Let $z \in \mathbb{R}$. The real exponent e^z is defined as $\lim\limits_{\substack{p \to x \\ p \text{ is rational}}} e^p$.

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Sketch of Proof. Definition I implies Definition II over $\mathbb{R}$.

Under Definition I the Maclaurin series of e^z was computed to be $\sum\limits_{n=0}^{\infty} \frac{z^n}{n!}$.

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Euler's Formula

Theorem (Euler's Formula)

$$e^{ix} = \cos x + i \sin x,$$

where $e \approx 2.71828$ is Euler's/Napier's constant .

Proof.

Recall $n! = 1 \cdot 2 \cdot 3 \cdot \dots \cdot (n-1) \cdot n$. Borrow from Calc II the f-las:

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$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + \frac{(-1)^n x^{2n}}{(2n)!} + \dots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots$$

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\sin x & = & & x & & -\frac{x^3}{3!} & & +\frac{x^5}{5!} & -\dots \\
\cos x & = & 1 & & -\frac{x^2}{2!} & & +\frac{x^4}{4!} & & +\dots \\
\hline
e^z & = & 1 & +z & +\frac{z^2}{2!} & +\frac{z^3}{3!} & +\frac{z^4}{4!} & +\frac{z^5}{5!} & +\dots
\end{array}$$

Rearrange.

$\square$

Euler's Formula

Theorem (Euler's Formula)

$$e^{ix} = \cos x + i \sin x,$$

where $e \approx 2.71828$ is Euler's/Napier's constant .

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Rearrange. Plug-in $z = ix$.

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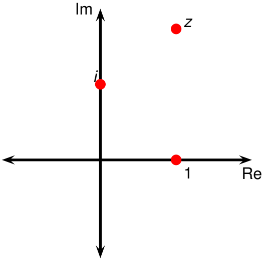

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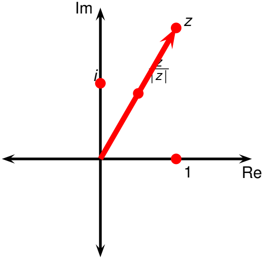

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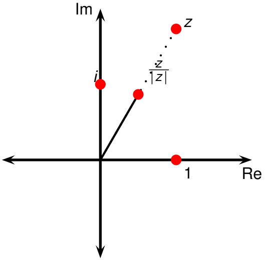

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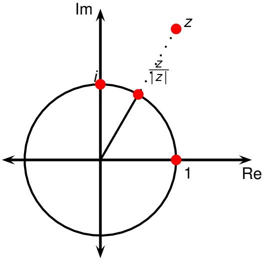

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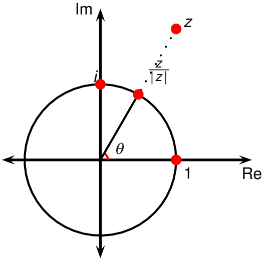

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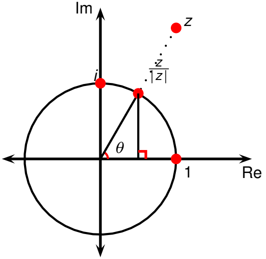

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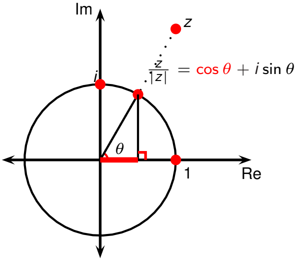

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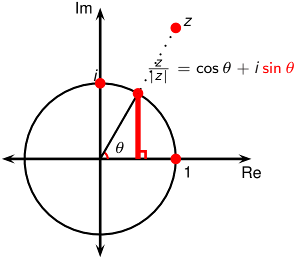

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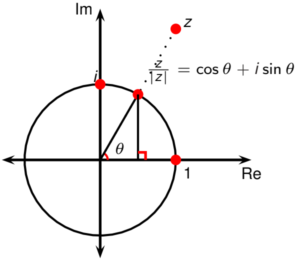

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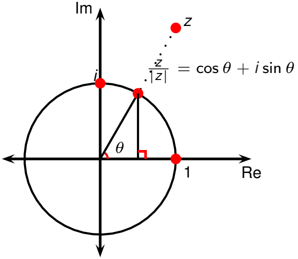

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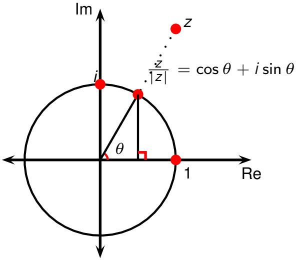

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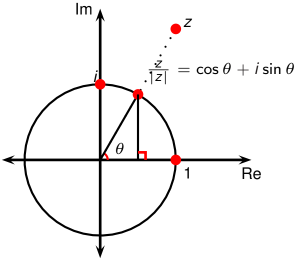

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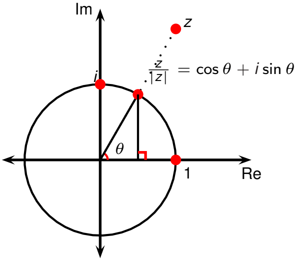

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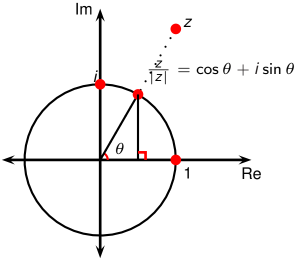

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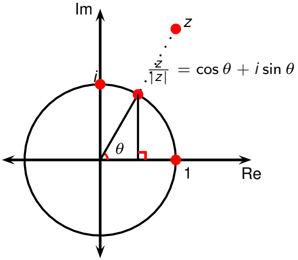

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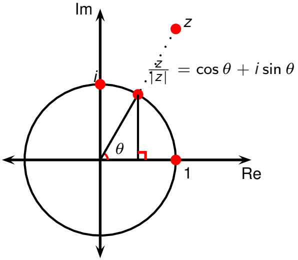

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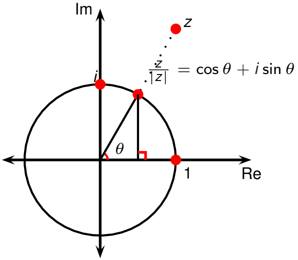

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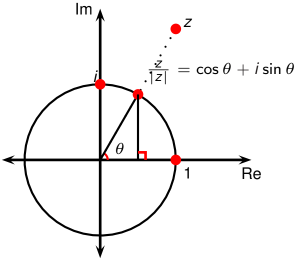

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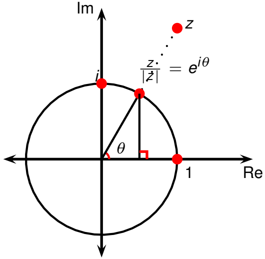

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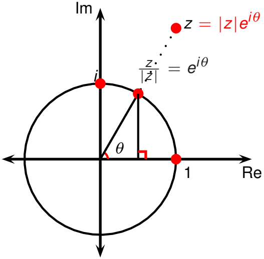

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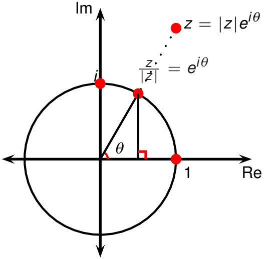

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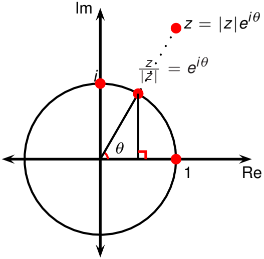

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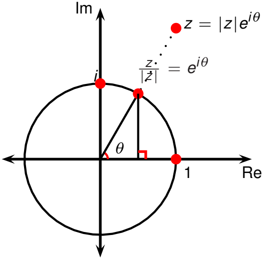

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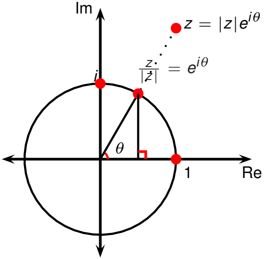

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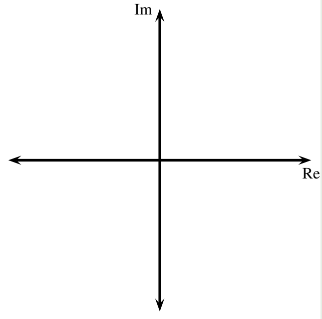
- One should never write $\theta = \arg z$ without clarifying the choice of argument.

Example

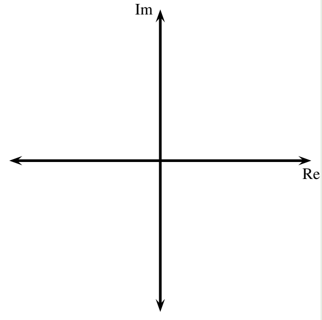


Plot the number z. Write z in polar form, using the principal value of the argument of z (polar angle).

Recall that θ is the principal argument $\Rightarrow$ $\theta \in (-\pi, \pi]$.

z	$\lvert z\rvert$	θ	$\lvert z\rvert(\cos\theta + i\sin\theta)$
1	1	0	$\cos 0 + i \sin 0$
i	1	$\frac{\pi}{2}$	$\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)$
-1	1	π	$\cos\pi + i\sin\pi$
$-i$	1	$-\frac{\pi}{2}$	$\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)$

Example



Plot the number z. Write z in polar form, using the principal value of the argument of z (polar angle).
Recall that θ is the principal argument $\Rightarrow$ $\theta \in (-\pi, \pi]$.

z	$\lvert z\rvert$	θ	$\lvert z\rvert(\cos\theta + i\sin\theta)$
$\frac{1}{2}+\frac{\sqrt{3}}{2}i$	1	$\frac{\pi}{3}$	$\cos\left(\frac{\pi}{3}\right)+i\sin\left(\frac{\pi}{3}\right)$
$1+i$	2	$\frac{\pi}{4}$	$2\left(\cos\left(\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{4}\right)\right)$
$1-i$	2	$-\frac{\pi}{4}$	$2\left(\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right)$
$-\sqrt{3}-i$	2	$-\frac{2\pi}{3}$	$2\left(\cos\left(\frac{2\pi}{3}\right)+i\sin\left(\frac{2\pi}{3}\right)\right)$
$\frac{3}{5}+\frac{4}{5}i$	5	$\arctan\left(\frac{4}{3}\right)$	$5\left(\cos\left(\arctan\left(\frac{4}{3}\right)\right)+i\sin\left(\arctan\left(\frac{4}{3}\right)\right)\right)$

Definition (Real exponent)

Let $\rho \in \mathbb{R}$. The real exponent e^ρ is defined as $\lim\limits_{\substack{p \to \rho \\ p \text{ is rational}}} e^p$.

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Let $\rho, \theta \in \mathbb{R}$. Define the complex exponent $e^{\rho+i\theta}$ via
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Proof of (a).

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- The trig. f-las used above need separate (relatively long) proof.

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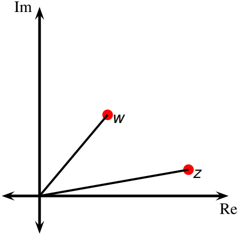

- Let $z, w \neq 0$

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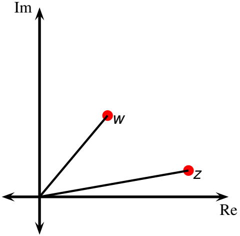

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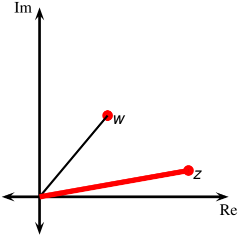

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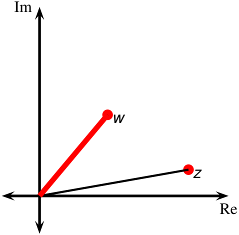

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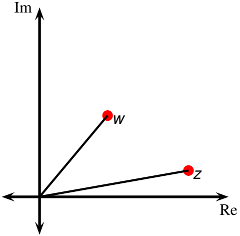

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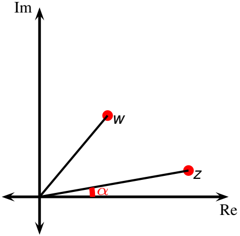
Geometric interpretation of complex multiplication



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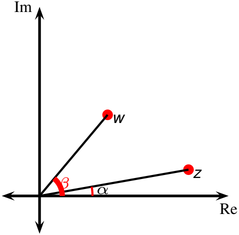

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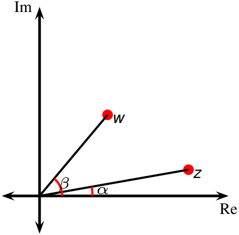

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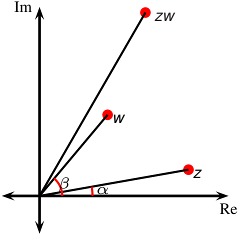

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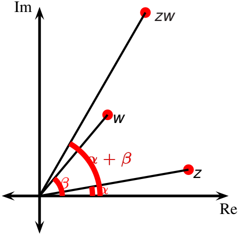

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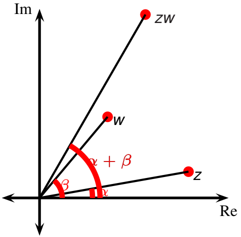

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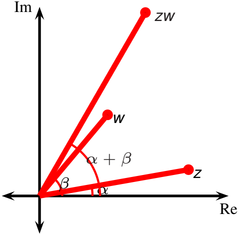

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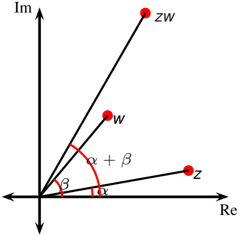

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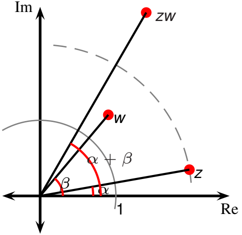

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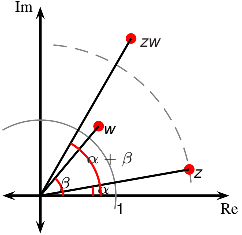

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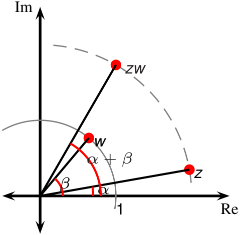

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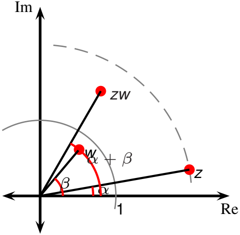

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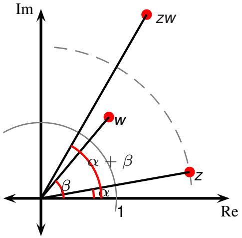

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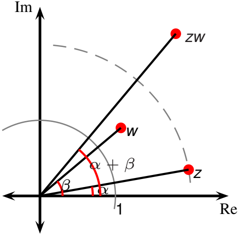

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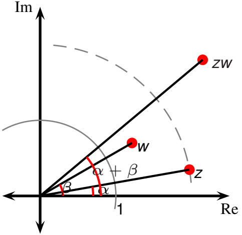

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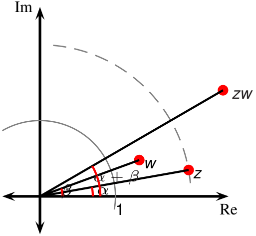

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$(\cos\alpha + i\sin\alpha)^n = \cos(n\alpha) + i\sin(n\alpha).$

Proof.

$\square$

The formula is named after the French mathematician A. de Moivre (1667-1754).

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Polar form $z = |z|(\cos\theta + i\sin\theta)$.

Example

Compute $\left(\sqrt{3} + i\right)^{2014}$ and its polar form.

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Example

Compute $\left(\sqrt{3}+i\right)^{2014}$ and its polar form.

Write $\sqrt{3}+i$ in polar form: $\sqrt{3}+i = \mathbf{?}\left(\cos\left(\mathbf{?}\right) + i\sin\left(\mathbf{?}\right)\right)$.

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Polar form $z = |z|(\cos\theta + i\sin\theta)$.

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Compute $\left(\sqrt{3}+i\right)^{2014}$ and its polar form.

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&= 2^{2013}\left(1-\sqrt{3}i\right).
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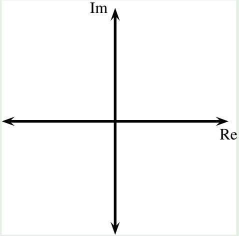
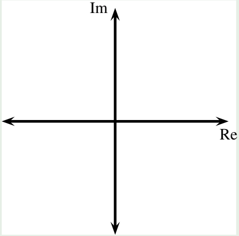
$$

Example



Find all complex solutions of the equation $z^4 = 1$.

Example



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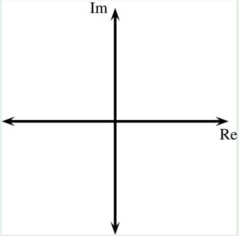

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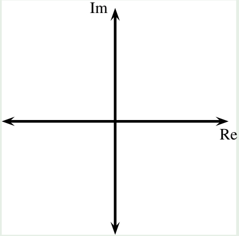

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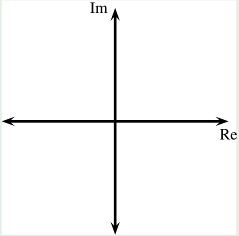

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By de Moivre's equality $z^4 = \cos(4\theta) + i\sin(4\theta) = 1$.

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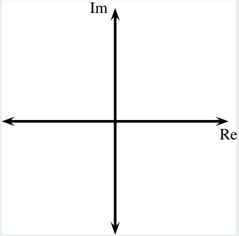

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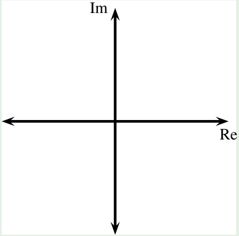

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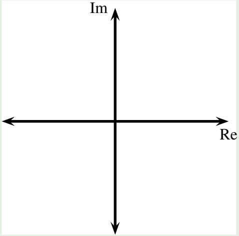

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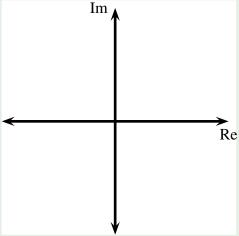

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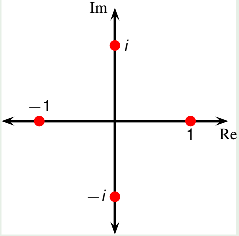

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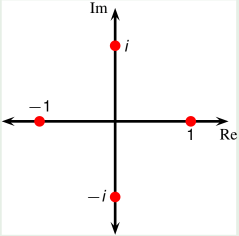

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Therefore the equation $z^4 = 1$ has 4 roots given by

$$\begin{aligned} z &= \cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right) = -i \\ z &= \cos 0 + i\sin 0 = 1 \\ z &= \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) = i \\ z &= \cos\pi + i\sin\pi = -1 \quad . \end{aligned}$$

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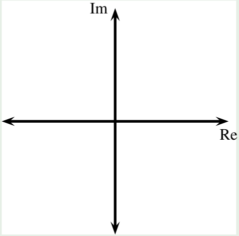

Find all complex numbers z such that $z^3 = i$.

$$\begin{array}{rcl|l} i &=& (\cos(\theta) + i\sin(\theta))^3 & \text{de Moivre} \\ \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) &=& \cos(3\theta) + i\sin(3\theta) & \text{Polar form } i \\ 3\theta &=& \frac{\pi}{2} + 2k\pi & k \text{ any integer} \\ \theta &=& \frac{\pi}{6} + \frac{2k}{3}\pi & k \text{ any integer} \end{array}$$

Values of θ that differ by even multiple of π produce the same value for $z \Rightarrow$ restrict our attention to $\theta \in (-\pi, \pi]$, i.e. $k = 0, 1, -1 \Rightarrow$ $\theta = \frac{\pi}{6}, \frac{5\pi}{6}, -\frac{\pi}{2}$. Our final answer is **to be continued.**

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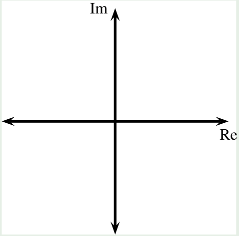

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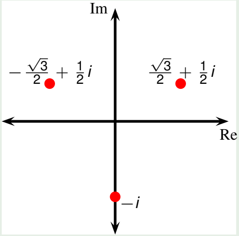

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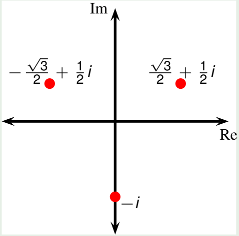

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