Calculus II
Lecture 11

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https://github.com/tmilev/freecalc

2020

Outline

1. Curves
   - The Cycloid
   - Polar Curves

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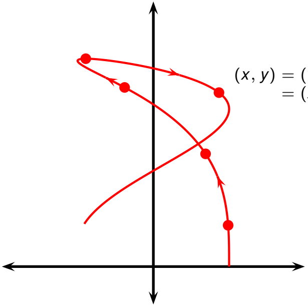
Curves Defined by Parametric Equations



- Suppose a particle moves along the curve in the picture.
- The x-coordinate and y-coordinate of the particle are some functions of the time t.
- We can write $x = f(t)$ and $y = g(t)$.
- Less formally, we may directly write $(x, y) = (x(t), y(t))$.
- We say that the equations $\left| \begin{array}{rcl} x & = & f(t) \\ y & = & g(t) \end{array} \right.$ are parametric equations of a parametric curve.
- Note that the curve can't be written as $y = f(x)$: it fails the vertical line test.

Definition (Curve in *n*-dimensional space)

We define an arbitrary n-tuple of functions $f_1, \dots, f_n$ on $[a, b]$ to be a *parametric curve* (or simply *curve*). If C is a curve, we write C as:

$$C : \left| \begin{array}{rcl} x_1 &=& f_1(t) \\ x_2 &=& f_2(t) \\ & \vdots & \\ x_n &=& f_n(t) \end{array} \right., t \in [a, b]$$

where $x_1, \dots, x_n$ are the labels of the n-dimensional coordinate system.

Curves in 2- and 3-dimensional space will be of special interest:

A curve in dimension 2 is given by:

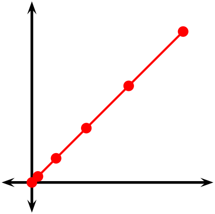
$$C : \left| \begin{array}{rcl} x &=& f(t) \\ y &=& g(t) \end{array} \right., t \in [a, b] \quad .$$

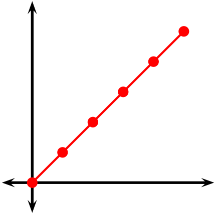
A curve in dimension 3 is given by:

$$C : \left| \begin{array}{rcl} x &=& f(t) \\ y &=& g(t) \\ z &=& h(t) \end{array} \right., t \in [a, b] \quad .$$

Consider the two parametric curves:

$$\gamma_1 : \left| \begin{array}{lcl} x &=& t^2 \\ y &=& t^2 \end{array} \right. , t \in [0, 1] \qquad \gamma_2 : \left| \begin{array}{lcl} x &=& t \\ y &=& t \end{array} \right. , t \in [0, 1]$$





Plug in $t = 0$, $t = 0.2$, $t = 0.4$, $t = 0.6$, $t = 0.8$, $t = 1$.

Question

Are the above curves different?

To answer this question we need a definition.

Recall a parametric curve C was defined as the data

$$C : \left| \begin{array}{lcl} x_1 & = & f_1(t) \\ x_2 & = & f_2(t) \\ & \vdots & \\ x_n & = & f_n(t) \end{array} \right. , t \in [a, b]$$

Definition

A *curve image* (or simply a curve) is any set of points that arises by traversing some continuous curve. In other words, a curve image is any set that can be written in the form

$$\{(f_1(t), \dots, f_n(t)) \mid t \in [a, b]\} \quad ,$$

for some continuous functions $f_1, \dots, f_n$.

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If we don't require that the functions be continuous, every set of points will be a curve and the definition would be pointless.

Recall a parametric curve C was defined as the data

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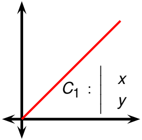
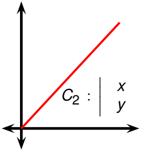
Informally, a curve image “remembers” only the points lying on the curve but forgets the “speed” with which each point was visited and “how many times” each point was visited.



$$C_1 : \left| \begin{array}{rcl} x & = & t^2 \\ y & = & t^2 \end{array} \right. , t \in [0, 1]$$



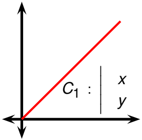
$$C_2 : \left| \begin{array}{rcl} x & = & t \\ y & = & t \end{array} \right. , t \in [0, 1]$$

Question

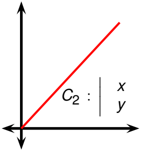
~~Are the above curves different?~~

Are the above parametric curves different? Yes.
Are the above curve images different? No.

- As parametric curves, C_1 and C_2 are different: C_1, C_2 are given by different functions.
- As curve images, C_1, C_2 coincide.
- The original question is incorrectly posed: the word “curve” does not have a mathematical definition without the words “parametric” or “image” attached to it.



$$C_1 : \left| \begin{array}{ccc} x & = & t^2 \\ y & = & t^2 \end{array} \right. , t \in [0, 1]$$



$$C_2 : \left| \begin{array}{ccc} x & = & t \\ y & = & t \end{array} \right. , t \in [0, 1]$$

Question

~~Are the above curves different?~~

Are the above parametric curves different? Yes.
Are the above curve images different? No.

- Nonetheless we sometimes use the word “curve” informally, without specifying “parametric curve” or “curve image”.
- In this case, whether we mean “parametric curve” or “curve image” should be clear from the context. If not, we are using mathematical language incorrectly.

Graphs of functions as curve images

- Consider a graph of a function given by

$$y = f(x)$$

- Write $x = t$. Then $y = f(x) = f(t)$, so we get the system

$$C : \left| \begin{array}{lcl} x & = & t \\ y & = & f(t) \end{array} \right. , t \in [a, b]$$

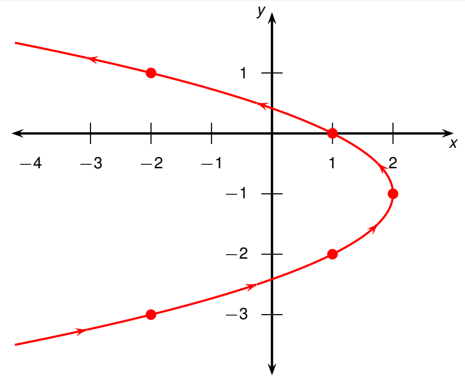
Observation

The graph of an arbitrary function can be written as the image of a curve C using the above transformation.

Example

Sketch and identify the curve image defined by the equations

$$\left|\begin{array}{rcl} x &=& -t^2+2 \\ y &=& t-1 \end{array}\right.$$

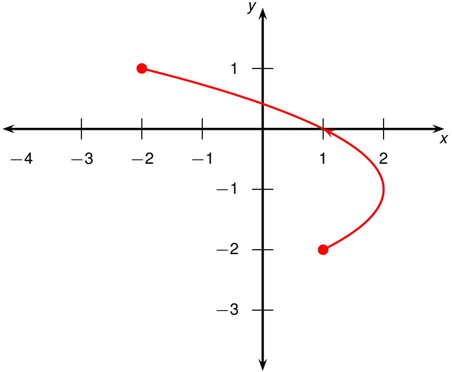


t	x	y
-2	-2	-3
-1	1	-2
0	2	-1
1	1	0
2	-2	1

Eliminate t: from second equation we have $t = y + 1$ and therefore:

$$\begin{array}{rcl} x &=& -t^2+2 \\ &=& -(y+1)^2+2 \\ &=& -y^2-2y+1 \end{array}$$

Thus our curve image is a parabola, as expected.



$$\left| \begin{array}{rcl} x & = & -t^2+2 \\ y & = & t-1 \end{array} \right., -1 \leq t \leq 2$$

- There was no restriction placed on t in the last example.
- In such a case we assume $t \in (-\infty, \infty)$, i.e., t runs over all real numbers.
- In general we are expected to specify the interval in which t lies.
- For example, if we restrict the previous example to $t \in [-1, 2]$, we get the part of the parabola that begins at $(1, -2)$ and ends at $(-2, 1)$.
- We say that $(1, -2)$ is the initial point and $(-2, 1)$ is the terminal point of the curve.

Implicit vs Explicit (Parametric) Curve Equations

- Consider the parametric curve

$$\left| \begin{array}{lll} x & = & -t^2+2 \\ y & = & t-1 \end{array} \right. .$$

- As we saw in preceding slides/lectures, all points (x, y) on the image of this curve satisfy the equation

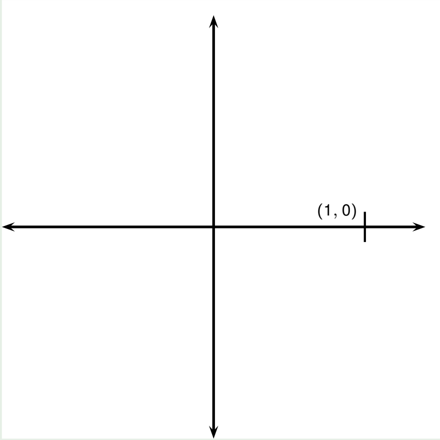
$$x+(y+1)^2-2=0$$

- Equations of the first form are called explicit (parametric) curve equations.
- Equations of the second form are called implicit equations of the curve image.
- Explicit (parametric) curve equations have the advantage that it is easy to generate points on the curve.
- Implicit curve equations have the advantage that it is easy to check whether a point is on the curve.

Example

Sketch and identify the curve defined by the parametric equations

$$x = \cos t, \qquad y = \sin t.$$



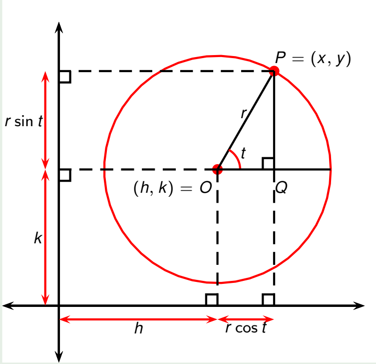
t	x	y
0	1	0
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{2}$	0	1
π	-1	0
$\frac{3\pi}{2}$	0	-1
2π	1	0

$$x^2 + y^2 = \cos^2 t + \sin^2 t = 1$$

Therefore (x, y) travels on the unit circle $x^2 + y^2 = 1$.

Example

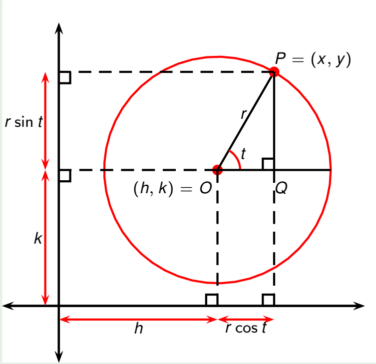
Find parametric equations for the circle with center (h, k) and radius r.



- Let O be the center of the circle with coordinates (h, k).
- Let P be a point on the circle with coordinates (x, y).
- Let t, Q be as indicated on the figure.
- Then $|OQ| = r \cos t$.
- $|PQ| = r \sin t$.
- Then the coordinates of P are $(h + r \cos t, k + r \sin t)$.
- In this way we get the parametric equations $\left| \begin{array}{l} x = h + r \cos t \\ y = k + r \sin t \end{array} \right., t \in [0, 2\pi]$

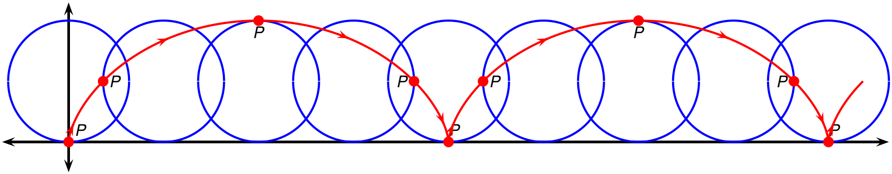
Example

Find parametric equations for the circle with center (h, k) and radius r.



- Alternative solution: $x = \cos t$, $y = \sin t$ are parametric equations of the unit circle.
- Multiply by r to scale the circle to have radius r: $x = r\cos t, \quad y = r \sin t$.
- Add h to x and k to y to translate the circle h units to the left and k units up: $x = h + r\cos t, \quad y = k + r\sin t$

The Cycloid

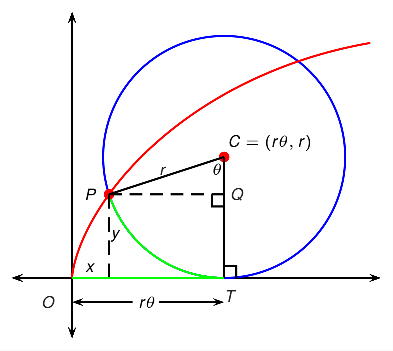


Definition (Cycloid)

The curve traced out by a point P on the circumference of a circle as the circle rolls along a straight line is called a cycloid.

Example

Find parametric equations of a cycloid made using a circle with radius r that rolls along the x-axis such that P hits the origin.



- We choose our parameter to be θ, the angle of rotation of the circle.
- How far has the circle moved if it has rolled through θ radians?

$$|OT| = \operatorname{arc} PT = r\theta$$

- Then the center is $C = (r\theta, r)$.
- Let the coordinates of P be (x, y).

$$\begin{aligned} x &= |OT| - |PQ| &= r\theta - r\sin\theta \\ y &= |CT| - |CQ| &= r - r\cos\theta \end{aligned}$$

Therefore the equations are

$$x = r(\theta - \sin\theta), \qquad y = r(1 - \cos\theta), \qquad \theta \in \mathbb{R}$$

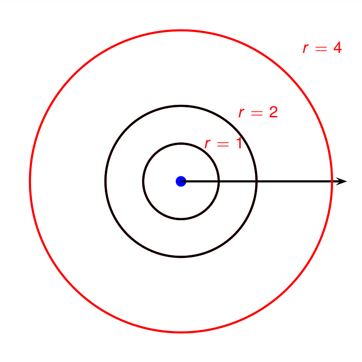
- Recall polar coordinates:

$$\left| \begin{array}{lcl} x & = & r\cos\theta \\ y & = & r\sin\theta \end{array} \right.$$

- A curve in polar coordinates is given by specifying explicit or implicit equations in polar coordinates.

Example

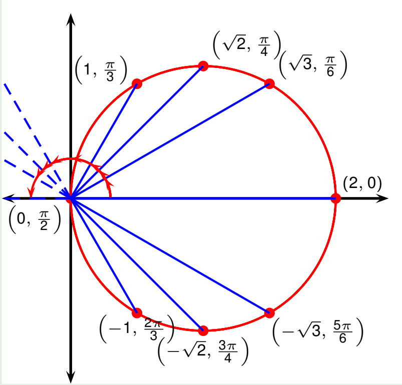
What curve is represented by the polar equation $r = 2$?



- The equation describes all points that are 2 units away from O.
- This is the circle with center O and radius 2.
- The equation $r = 1$ describes the unit circle.
- The equation $r = 4$ describes the circle with center O and radius 4.

Example

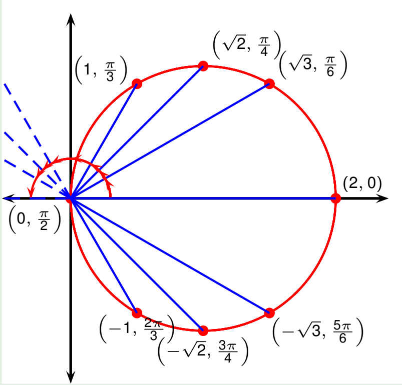
1. Sketch the curve with polar equation $r = 2\cos\theta$.
2. Find a Cartesian equation for this curve.



θ	r
0	2
$\pi/6$	$\sqrt{3}$
$\pi/4$	$\sqrt{2}$
$\pi/3$	1
$\pi/2$	0
$2\pi/3$	-1
$3\pi/4$	$-\sqrt{2}$
$5\pi/6$	$-\sqrt{3}$
π	-2

Example

1. Sketch the curve with polar equation $r = 2\cos\theta$.
2. Find a Cartesian equation for this curve.

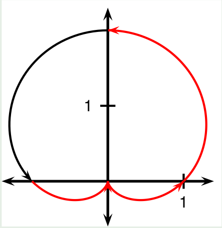


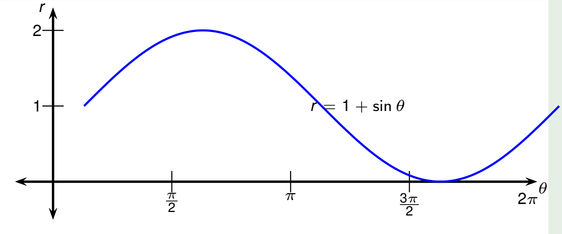
- $x = r\cos\theta$.
- $\cos\theta = x/r$.
- $r = 2\cos\theta = 2x/r$.
- $2x = r^2 = x^2 + y^2$.
- $x^2 + y^2 - 2x = 0$.
- Complete the square:

$$\begin{aligned}(x^2 - 2x + 1) + y^2 &= 0 + 1\\ (x-1)^2 + y^2 &= 1\end{aligned}$$

Example (Cardioid)

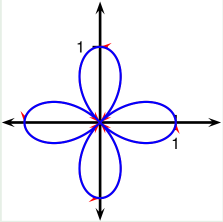
Sketch the curve $r = 1 + \sin\theta$.

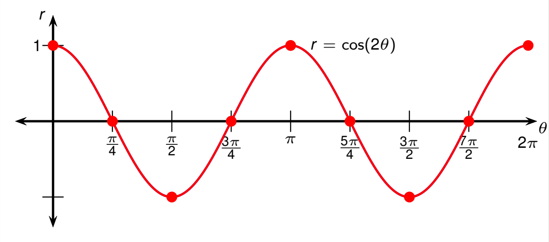




Example

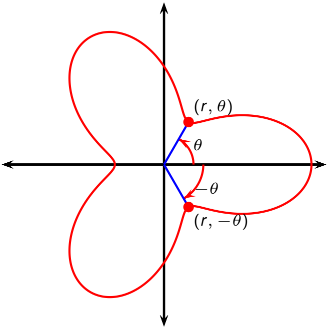
Sketch the curve $r = \cos(2\theta)$.





Symmetry

- If the polar equation is unchanged when θ is replaced by $-\theta$, the curve is symmetric about the polar axis.
- If the equation is unchanged when θ is replaced by $\pi + \theta$, the curve is symmetric under rotation about the pole.
- If the equation is unchanged when θ is replaced by $\pi - \theta$, the curve is symmetric about the vertical line $\theta = \frac{\pi}{2}$.



Symmetry

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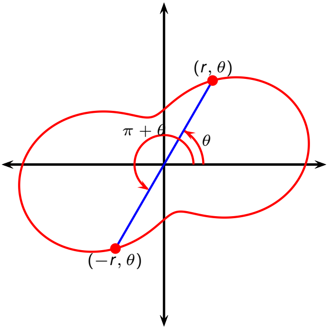
- If the equation is unchanged when θ is replaced by $\pi-\theta$, the curve is symmetric about the vertical line $\theta=\frac{\pi}{2}$.



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