

Master Problem Sheet
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Calculus III
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1 Distances and coordinates

Problem 1. Find the distance between the points. The answer key has not been proofread, use with caution.

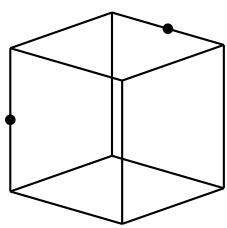
1. $(2, 3, 5)$ and $(3, 5, 7)$.

answer: 3
2. $(1, 1, 1)$ and $(0, 0, -1)$.

answer: $\sqrt{6}$
3. A vertex of a cube with edge 2cm and the midpoint of one of the three opposing sides.

answer: $\sqrt{6}$
4. Consider a cube with edge 2cm. Consider two edges that do not have a common point and are not parallel. Find the distance between the midpoints of those two edges.

answer: $\sqrt{6}$



Problem 2. Show that the equation is an equation of a sphere. Determine the center of the sphere and its radius. The answer key has not been proofread, use with caution.

1. $x^2 + y^2 + z^2 - 2x + 3y + 5z = 0$

answer: Sphere with center $(1, -\frac{3}{2}, -\frac{5}{2})$ and radius $\frac{\sqrt{38}}{2}$
2. $x^2 + y^2 + z^2 - x - 2y - 3z = 0$

answer: Sphere with center $(\frac{1}{2}, 1, \frac{3}{2})$ and radius $\frac{\sqrt{14}}{2}$
3. $\frac{1}{2}((x - y)^2 + (x + y)^2) + z^2 + 2z = 0$

answer: Sphere with center $(0, 0, -1)$ and radius 1

2 Vectors

2.1 Vector basics

Problem 3. Carry out the indicated operations between the indicated vectors.

$$\begin{aligned}\mathbf{u} &= (-1, 2, 3) \\ \mathbf{v} &= (2, -3, -5) \\ \mathbf{w} &= (3, 5, -7).\end{aligned}$$

1. $-\mathbf{u}$

ANSWER: $(1, -2, -3)$

2. $\mathbf{u} + \mathbf{v}$

ANSWER: $(1, -1, -2)$

3. $\mathbf{u} - 2\mathbf{w}$

ANSWER: $(-7, -8, 17)$

4. $-3\mathbf{w} + \frac{\mathbf{v}}{2}$

ANSWER: $(-\frac{23}{2}, -\frac{23}{2}, \frac{23}{2})$

5. $\frac{\mathbf{w} + 2\mathbf{u} + 3\mathbf{v}}{6}$

ANSWER: $(\frac{3}{8}, 0, \frac{9}{2})$

6. $\mathbf{u} + \mathbf{w} - (2\mathbf{v} + 3\mathbf{u})$

ANSWER: $(1, 7, -3)$

2.2 Dot product

Problem 4. Compute the dot product.

1. $\mathbf{u} = (2, -3, 5), \mathbf{v} = (-3, 5, 7).$

ANSWER: $-6 - 15 + 35 = 14$

2. $\mathbf{u} = (\frac{1}{2}, \frac{1}{3}, \frac{1}{4}), \mathbf{v} = (\frac{1}{3}, \frac{1}{4}, \frac{1}{5}).$

ANSWER: $\frac{10}{3}$

Problem 5. Determine if the vectors are orthogonal.

1. $\mathbf{u} = (1, 2, 3), \mathbf{v} = (-1, 2, -1).$

ANSWER: $\mathbf{u} \perp \mathbf{v}$

2. $\mathbf{u} = (1, 0, 1), \mathbf{v} = (-1, 1, 1).$

ANSWER: $\mathbf{u} \perp \mathbf{v}$

3. $\mathbf{u} = (-1, 0, 1), \mathbf{v} = (-1, 1, 1).$

ANSWER: $\mathbf{u} \not\perp \mathbf{v}$

Problem 6. Find the angles between the vectors. You may use a calculator to get a numerical approximation.

1. $\mathbf{u} = (1, 2, 3), \mathbf{v} = (3, 1, 2).$

ANSWER: $\arccos\left(\frac{14}{11}\right) \approx 0.666946$

2. $\mathbf{u} = (-1, -1, -1), \mathbf{v} = (0, 0, 1)$

ANSWER: $\arccos\left(-\frac{\sqrt{3}}{3}\right) \approx 2.186276$

Problem 7. A tetrahedron is a pyramid whose base is a triangle. The 8 points $(1, 1, 1), (-1, 1, 1), (1, -1, 1), (-1, -1, 1), (1, 1, -1), (-1, 1, -1), (1, -1, -1), (-1, -1, -1)$ (all possible sign combinations) give the vertices of a cube with edge 2 units.

1. Find 4 vertices of the cube so they form a regular tetrahedron, i.e., 4 points that are not in the same plane and such that the distance between any two is equal.

2. Form two vectors, \mathbf{u} and \mathbf{v} , by connecting the origin with any two of the 4 points you found.

3. Find the angle between \mathbf{u} and \mathbf{v} .

4. What is the angle between the two bonds of hydrogen atoms in the methane molecule CH_4 ?

ANSWER: $\arccos\left(-\frac{1}{3}\right) \approx 109.471207^\circ$

Problem 8. Find the

- Scalar projection $\text{comp}_{\mathbf{v}} \mathbf{u}$ of \mathbf{u} onto \mathbf{v} .
- The vector projection $\text{proj}_{\mathbf{v}} \mathbf{u}$ of \mathbf{u} along \mathbf{v} .
- The component $\text{orth}_{\mathbf{v}} \mathbf{u}$ of \mathbf{u} orthogonal of \mathbf{v} .

The answer key has not been proofread, use with caution.

1. $\mathbf{v} = (2, 3, 5), \mathbf{u} = (3, 5, 7).$

ANSWER: $\text{comp}_{\mathbf{v}} \mathbf{u} = \frac{61}{88}, \text{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{61}{88}, \frac{61}{88}, \frac{61}{88}\right), \text{orth}_{\mathbf{v}} \mathbf{u} = \left(\frac{61}{88}, \frac{61}{88}, \frac{61}{88}\right)$

2. $\mathbf{v} = (5, 1, -3), \mathbf{u} = (2, 3, 5).$

ANSWER: $\text{comp}_{\mathbf{v}} \mathbf{u} = \frac{95}{161}, \text{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{95}{161}, \frac{95}{161}, -\frac{95}{161}\right), \text{orth}_{\mathbf{v}} \mathbf{u} = \left(\frac{95}{161}, \frac{95}{161}, -\frac{95}{161}\right)$

2.3 Cross product

Problem 9. Find the area of the triangle $\triangle ABC$. The answer key has not been proofread, use with caution.

1. $A(1, 0, 0), B(0, 1, 0), C(0, 0, 1).$

ANSWER: $\frac{\sqrt{2}}{2}$

2. $A(1, -1, 0), B(0, 1, -1), C(-1, 0, 1).$

ANSWER: $\frac{\sqrt{3}}{2}$

3. $A(1, 2, 3), B(5, 7, 11), C(13, 17, 19).$

ANSWER: $4\sqrt{11}$

Problem 10. Find a vector orthogonal to the two given vectors. The answer key has not been proofread, use with caution.

1. $\mathbf{u} = (2, 3, 5), \mathbf{v} = (3, 5, 7).$

ANSWER: $\mathbf{n} = (-4, 1, 1)$

2. $\mathbf{u} = (2, -5, -3), \mathbf{v} = (3, 5, 7).$

ANSWER: $\mathbf{n} = (-20, -23, 25)$

Problem 11. Let the 4 vertices of a tetrahedron be O, A, B, C . Let $\mathbf{v}_1 = \mathbf{OA}, \mathbf{v}_2 = \mathbf{OB}, \mathbf{v}_3 = \mathbf{OC}$ (the vectors given by the edges of the tetrahedron that pass through O). It can be shown that the volume of the tetrahedron equals $\frac{1}{6} = \frac{1}{6}$ of the volume of the slanted box spanned by $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. Using that information find the volumes of the following tetrahedra.

1. The volume of the tetrahedron with vertices $(1, 1, 1), (1, -1, -1), (-1, 1, -1), (-1, -1, 1).$

ANSWER: $\frac{3}{8}$

2. The volume of the tetrahedron with vertices $(1, 2, 3), (2, 3, 5), (3, 5, 7), (5, 7, 13).$

ANSWER: $\frac{3}{4}$

3. The volume of the tetrahedron with vertices $A(1, 2, 2), B(1, 3, 3), C(1, 0, 2), D(-2, 3, 2).$

Solution. 11.3

$$\begin{aligned} \mathbf{AB} &= (1, 3, 3) - (1, 2, 2) = (0, 1, 1) \\ \mathbf{AC} &= (1, 0, 2) - (1, 2, 2) = (0, -2, 0) \\ \mathbf{AD} &= (-2, 3, 2) - (1, 2, 2) = (-3, 1, 0) \\ \text{Vol}(ABCD) &= \left| \frac{1}{6} \det \begin{pmatrix} 0 & 1 & 1 \\ 0 & -2 & 0 \\ -3 & 1 & 0 \end{pmatrix} \right| \\ &= 1. \end{aligned}$$

Problem 12. Do the points $(1, 2, 3), (2, 3, 5), (3, 5, 7), (5, 7, 11)$ lie in one plane?

ANSWER: YES.

Problem 13. Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be arbitrary vectors. Show that the Jacobi identity for the cross product holds, i.e., show that

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) + \mathbf{v} \times (\mathbf{w} \times \mathbf{u}) + \mathbf{w} \times (\mathbf{u} \times \mathbf{v}) = \mathbf{0}.$$

3 Lines, planes, points and relationships between them

3.1 Lines

3.1.1 Line from point and direction

Problem 14. Write vectorial and scalar equations of the line L passing through the given point and with the given direction.

1. $P_0 = (1, 2, 3)$, $\mathbf{u} = (-3, -2, -1)$.

$$\text{ANSWER: } L : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t \begin{pmatrix} -3 \\ -2 \\ -1 \end{pmatrix}, t \in \mathbb{R}$$

2. $P_0 = (3, 5, 7)$, $\mathbf{u} = (2, 3, 4)$.

$$\text{ANSWER: } L : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix} + t \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}, t \in \mathbb{R}$$

3.1.2 Line from two points

Problem 15. Write vectorial and scalar equations of the line passing L through the given points.

1. $(2, 3, 5)$ and $(3, 5, 7)$.

$$\text{ANSWER: } L : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \text{ or } L : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix} + t \begin{pmatrix} -1 \\ -2 \\ -2 \end{pmatrix}, t \in \mathbb{R}$$

2. $(-1, -1, 1)$ and $(-1, 1, -1)$.

$$\text{ANSWER: } L : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \text{ or } L : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} + t \begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix}, t \in \mathbb{R}$$

Problem 16. We recall that the 8 points $(1, 1, 1)$, $(-1, 1, 1)$, $(1, -1, 1)$, $(-1, -1, 1)$, $(1, 1, -1)$, $(-1, 1, -1)$, $(1, -1, -1)$, $(-1, -1, -1)$ (all possible sign combinations) give the vertices of a cube with edge 2 units.

Find equations for all lines connecting two vertices in the cube above that pass through the origin (how many connecting two vertices of a cube are there? How many of them are edges?).

ANSWER: There are 4 such edges. See the solution below for their equations.

Solution. 16. A cube has a total of 8 vertices. A line is given by two (distinct) points, therefore there are $\binom{8}{2} = \frac{8 \cdot 7}{2} = 28$ total lines connecting two distinct vertices of a cube. Of those 28 lines are cube edges, $12 = 6 \cdot 2$ are diagonals of cube faces, and 4 are inner diagonals. All four inner diagonals contain the origin. A justification for this can undoubtedly be given by writing all 28 line equations. However, the origin is in the center of the cube, and we know from our every-day geometric intuition that only the inner diagonals contain the center of a cube; we give no further justification.

The 4 inner diagonals of the cube, call them L_1, L_2, L_3, L_4 pass through the points

$$\begin{aligned} (1, 1, 1), (-1, -1, -1) &\in L_1 \\ (1, 1, -1), (-1, -1, 1) &\in L_2 \\ (1, -1, 1), (-1, 1, -1) &\in L_3 \\ (-1, 1, 1), (1, -1, -1) &\in L_4 \end{aligned}$$

Therefore equations for these lines are given by:

$$\begin{aligned} L_1 : & t(1, 1, 1) \\ L_2 : & t(1, 1, -1) \\ L_3 : & t(1, -1, 1) \\ L_4 : & t(-1, 1, 1) \end{aligned}$$

3.2 Planes

3.2.1 Plane from point and normal

Problem 17. Find an equation of the plane \mathcal{P} passing through the given point and with the given normal. Find parametric vectorial equations of the plane.

1. $P_0(1, 2, 3)$, $\mathbf{n} = (4, 5, 6)$.

$$\text{ANSWER: } \mathcal{P} : 4x + 5y + 6z = 32, \mathcal{P} : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t \begin{pmatrix} 5 \\ -4 \\ 0 \end{pmatrix} + s \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix}$$

2. $P_0(2, 3, 5)$, $\mathbf{n} = (-3, -5, -7)$.

3. $P_0(1, 1, 1)$, $\mathbf{n} = (1, 1, 1)$.

Solution. 17.1 As studied, an equation passing through $(1, 2, 3)$ and with normal $(4, 5, 6)$ has equation:

$$\begin{aligned}(x, y, z) \cdot (4, 5, 6) &= (4, 5, 6) \cdot (1, 2, 3) \\ 4x + 5y + 6z &= 23\end{aligned}$$

To find parametric equations of the plane, we need to find two directions, \mathbf{u} , \mathbf{v} , that can be added to the base point to obtain all points in the plane. This means that a direction vector \mathbf{u} has to be perpendicular to \mathbf{n} . Equivalently, a direction vector \mathbf{u} lies in the plane passing through the origin and orthogonal to \mathbf{n} . This means $\mathbf{u}(u_1, u_2, u_3)$ satisfies the equation:

$$\begin{aligned}\mathbf{u} \cdot \mathbf{n} &= 0 \\ 4u_1 + 5u_2 + 6u_3 &= 0.\end{aligned}\tag{1}$$

There are infinitely many solutions to that equation - in fact, for each point in the plane passing through the origin and orthogonal to \mathbf{n} there is one solution. However, we only need to find two such non-colinear solutions, and declare them to be our vectors \mathbf{u} and \mathbf{v} . It is very easy to do that: if we set u_1 and u_2 to be arbitrary, then u_3 can always be chosen so as to make the equation above hold. There are a number of accepted ways to choose u_1 and u_2 in a not-so-arbitrary fashion. For reasons outside of the scope of this homework, such ways to choose u_1 and u_2 may be preferable to the choosing at random. Our scheme for choosing a vector \mathbf{u} will be to choose $u_1 = 1$ and $u_2 = 0$ (or the other way round for \mathbf{v}), and then to rescale the resulting vector so all coordinates are integers and the first non-zero coordinate is positive. In other words, we select \mathbf{u} to be proportional to $(1, 0, -\frac{4}{6})$, and \mathbf{v} to be proportional to $(0, 1, -\frac{5}{6})$, i.e., we select

$$\begin{aligned}\mathbf{u} &= (3, 0, -2) \\ \mathbf{v} &= (0, 6, -5)\end{aligned}.$$

Finally we get that a parametric equation of the plane is given by:

$$(1, 2, 3) + s(3, 0, -2) + t(0, 6, -5) \quad .\tag{2}$$

The above equations are not unique; our problem has many correct answers.

How do we check if two plane parametrizations are equivalent? Equivalently, how do we check that equation (2) gives a plane that coincides with the plane in given in (1)? Here's what we need to do to make sure our answer is correct (we leave the justification for that to the reader):

- Check that our \mathbf{u} , \mathbf{v} are orthogonal to \mathbf{n} .
- Check that our \mathbf{u} , \mathbf{v} are not proportional to one another.
- Check that the base point of our equation is in the original plane.

3.2.2 Plane from point and two directions

Problem 18. Find an equation of plane \mathcal{P} passing through the point and parallel to the given directions.

1. $P_0(1, 2, 3)$, $\mathbf{u} = (2, 3, 5)$, $\mathbf{v} = (3, 5, 7)$.

2. $P_0(1, 1, 1)$, $\mathbf{u} = (1, -1, 0)$, $\mathbf{v} = (0, 1, -1)$.

3.2.3 Plane from three points

Problem 19. Find an equation of the plane \mathcal{P} passing through the given points.

1. $P_0(2, 3, 5)$, $P_1(3, 5, 7)$, $P_2(5, 7, 11)$.

2. $P_0(1, 1, 1)$, $P_1(1, -1, -1)$, $P_2(-1, -1, 1)$.

3.3 Distances between points, lines, planes

3.3.1 Distance between line and point

Problem 20. Find the distance between the line and the point.

1. The line passing through $P_0(1, 1, 1)$ and $P_1(-1, -1, -1)$ and the point $Q(1, 0, 0)$.
2. The line passing through $P_0(-2, 3, -5)$ and $P_1(3, 4, 5)$ and the point $Q(2, -2, 2)$.

ANSWER: $\frac{3}{\sqrt{6}}$

ANSWER: $\frac{\sqrt{57610}}{42}$

3.3.2 Distance between plane and point

Problem 21. Find the distance between the plane and the point.

1. The plane passing through $P_0(1, 2, 3)$, $P_1(2, 3, 5)$ and $P_2(3, 5, 7)$ and the point $Q(2, -2, 2)$.
2. The plane passing through $P_0(1, 2, 3)$, $P_1(2, 3, 5)$ and $P_2(3, 5, 7)$ and the point $Q(5, 7, 11)$.
3. The plane passing through the points $P_0(1, 1, 1)$, $P_1(1, -1, -1)$, $P_2(-1, -1, 1)$ and the point $Q(-1, 1, -1)$.

ANSWER: $\frac{2}{3}\sqrt{6}$

ANSWER: 0

ANSWER: $\frac{3}{4}\sqrt{3}$

3.3.3 Distance between lines

Problem 22. Recall that a regular tetrahedron can be realized using 4 vertices of a cube.

Find the distance between two opposite edges of a regular tetrahedron inscribed in a $2 \times 2 \times 2$ cm cube.

ANSWER: 2 cm

Problem 23. Find the distance between the lines.

1. The line passing through $Q_0(1, 2, 3)$ and $Q_1(6, 5, 4)$ and the line passing through $P_0(1, 3, 5)$ and $P_1(2, 4, 6)$.
2. The line passing through $Q_0(1, 2, 3)$ and $Q_1(2, 3, 5)$ and the line passing through $P_0(3, 5, 7)$ and $P_1(5, 7, 11)$.

ANSWER: 0

ANSWER: $\frac{9}{\sqrt{30}}$

3. The line passing through $Q_0(1, 1, 1)$ and $Q_1(-1, -1, -1)$ and the line passing through $P_0(1, -1, -1)$ and $P_1(-1, 1, -1)$.

ANSWER: $\frac{3}{\sqrt{6}}$

4. The line passing through $(1, 3, 4)$ and $(2, 3, 1)$ and the line passing through $(1, 2, 2)$ and $(0, 2, 5)$.

ANSWER: $\frac{9}{\sqrt{35}}$

5. The line passing through $(1, 3, 4)$ and $(2, 3, 1)$ and the line passing through $(1, 2, 2)$ and $(0, 2, 4)$.

ANSWER: 1

Solution. 23.1 We need to first establish whether the two lines are parallel. Let \mathbf{u} be the direction vector of the first line given by

$$\mathbf{u} = \mathbf{Q}_0\mathbf{Q}_1 = (6, 5, 4) - (1, 2, 3) = (5, 3, 1)$$

and let \mathbf{v} be the direction vector of the second line given by

$$\mathbf{v} = \mathbf{P}_0\mathbf{P}_1 = (2, 4, 6) - (1, 3, 5) = (1, 1, 1).$$

Now it is straightforward to see that the two lines are not parallel - indeed, one immediately sees that $\mathbf{u} = (5, 3, 1)$ is not a scalar multiple of $\mathbf{v} = (1, 1, 1)$. Since the two lines are not parallel, the two direction vectors determine a plane through the origin whose normal vector is given by

$$\mathbf{n} = \mathbf{u} \times \mathbf{v} = (5, 3, 1) \times (1, 1, 1) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & 3 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 2\mathbf{i} - 4\mathbf{j} + 2\mathbf{k} = (2, -4, 2).$$

We note that if the vectors \mathbf{u}, \mathbf{v} were parallel, then the cross product above would have been zero. Now the distance between the two lines is obtained by taking an arbitrary vector with tail on one line and head on the other, and computing the length of its projection onto \mathbf{n} . We use the vector $\mathbf{r} = \mathbf{Q}_0\mathbf{P}_0$. Then the distance d between the two lines is given by:

$$d = \frac{|\mathbf{r} \cdot \mathbf{n}|}{|\mathbf{n}|} = \frac{|((1, 3, 5) - (1, 2, 3)) \cdot \mathbf{n}|}{|\mathbf{n}|} = \frac{|(0, 1, 2) \cdot (2, -4, 2)|}{|\mathbf{n}|} = 0.$$

Therefore the distance between the two lines is zero. This completes our solution.

We note that since the distance between the lines is zero, they must intersect. As a consistency check for our work, let us verify that the two lines do indeed intersect. The first line is parametrized by $(1, 2, 3) + t(5, 3, 1)$ i.e., has parametric equations

$$\begin{cases} x = 1 + 5t \\ y = 2 + 3t \\ z = 3 + t \end{cases}.$$

Similarly, the second line is given by the equations

$$\begin{cases} x = 1 + s \\ y = 3 + s \\ z = 5 + s \end{cases}.$$

Therefore to find an intersection of the two lines, we need to solve the system

$$\begin{cases} 1 + 5t = 1 + s \\ 2 + 3t = 3 + s \\ 3 + t = 5 + s \end{cases}.$$

From the first equality we get that $s = 5t$. We substitute that into the second equality to get that $t = -\frac{1}{2}$. Therefore the intersection of the two lines is the point

$$(1, 2, 3) - \frac{1}{2}(5, 3, 1) = \left(-\frac{3}{2}, \frac{1}{2}, \frac{5}{2}\right) = (1, 3, 5) - \frac{5}{2}(1, 1, 1) ;$$

all our error checks have been successful.

Solution. 23.4 We present a solution in a concise form suitable for exam taking.

Let L_1, L_2 be the two lines.

$$\begin{aligned} \mathbf{u} &= (2, 3, 1) - (1, 3, 4) = (1, 0, -3) & \left| \begin{array}{l} \text{direction vector } L_1 \\ \text{direction vector } L_2 \end{array} \right. \\ \mathbf{v} &= (0, 2, 5) - (1, 2, 2) = (-1, 0, 3) = -\mathbf{u} \\ &\text{Therefore } L_1 \parallel L_2 \\ \mathbf{r} &= (2, 3, 1) - (0, 2, 5) = (2, 1, -4) & \left| \begin{array}{l} \text{arbitrary vector connecting } L_1, L_2 \end{array} \right. \\ L_1 \parallel L_2 \Rightarrow \\ \text{dist}(L_1, L_2) &= |\text{orth}_{\mathbf{u}} \mathbf{r}| \\ &= |\mathbf{r} - \text{proj}_{\mathbf{u}} \mathbf{r}| \\ &= \left| \mathbf{r} - \frac{\mathbf{r} \cdot \mathbf{u}}{|\mathbf{u}|^2} \mathbf{u} \right| \\ &= \left| (2, 1, -4) - \frac{(2, 1, -4) \cdot (1, 0, -3)}{1^2 + 0^2 + (-3)^2} (1, 0, -3) \right| \\ &= \left| \left(\frac{3}{5}, 1, \frac{1}{5}\right) \right| \\ &= \sqrt{\left(\frac{3}{5}\right)^2 + 1^2 + \left(\frac{1}{5}\right)^2} \\ &= \frac{\sqrt{35}}{5}. \end{aligned}$$

Solution. 23.5 We present a solution in a concise form suitable for exam taking.

$$\begin{aligned} \mathbf{u} &= (2, 3, 1) - (1, 3, 4) = (1, 0, -3) & \left| \begin{array}{l} \text{direction vector } L_1 \\ \text{direction vector } L_2 \end{array} \right. \\ \mathbf{v} &= (0, 2, 4) - (1, 2, 2) = (-1, 0, 2) = -\mathbf{u} \\ \mathbf{r} &= (1, 3, 4) - (1, 2, 2) = (0, 1, 2) & \left| \begin{array}{l} \text{arbitrary vector connecting } L_1, L_2 \\ \neq 0 \Rightarrow L_1 \nparallel L_2 \end{array} \right. \\ \mathbf{n} &= \mathbf{u} \times \mathbf{v} = (0, 1, 0) \\ L_1 \nparallel L_2 \Rightarrow \\ \text{dist}(L_1, L_2) &= |\text{proj}_{\mathbf{n}} \mathbf{r}| \\ &= \left| \mathbf{r} \cdot \frac{\mathbf{n}}{|\mathbf{n}|} \right| \\ &= |(0, 1, 2) \cdot (0, 1, 0)| \\ &= 1. \end{aligned}$$

3.4 Angles (in space)

3.4.1 Angles between lines

Problem 24. Recall that a regular tetrahedron can be realized using 4 vertices of a cube.

1. In a regular tetrahedron, find the angle between two edges that share a common vertex.

009 = $\frac{\pi}{2}$: ANSWER

2. In a regular tetrahedron, find the angle between two edges that share a common vertex.

006 = $\frac{\pi}{2}$: ANSWER

3.4.2 Angle between plane and line

Problem 25. Find the angle between the line and the plane.

1. The line passing through $(-1, -1, -1)$ and $(1, 1, 1)$ and the plane with equation $z = -1$.

$$\text{ANSWER: } \arcsin \left(\frac{\sqrt{3}}{3} \right) \approx 0.61548 \approx 35.2644^\circ$$

2. The line passing through $(2, 3, 5)$ and $(3, 5, 7)$ and the plane passing through $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$.

$$\text{ANSWER: } \arcsin \left(\frac{\sqrt{3}}{3} \right) \approx 1.295 \approx 74.21^\circ$$

Problem 26. Recall that a regular tetrahedron can be realized using 4 vertices of a cube. Find the angle between an edge of a regular tetrahedron and one of the two sides of the tetrahedron not containing the edge.

$$\text{ANSWER: } \arcsin \left(\frac{\sqrt{6}}{3} \right) \approx 0.955 \approx 54.736^\circ$$

3.4.3 Angles between planes

Problem 27. Recall that a regular tetrahedron can be realized using 4 vertices of a cube.

Find the angle between two faces of a regular tetrahedron.

$$\text{ANSWER: } \arccos \left(\frac{1}{3} \right) \approx 1.230959 \approx 70.528755^\circ$$

4 Polar, Cylindrical and Spherical coordinates

4.1 Polar coordinates

Problem 28. Find polar equations of the line given below.

1. The line $x + y = 1$.

$$\text{ANSWER: } r = \frac{1}{\cos \theta + \sin \theta}$$

2. The line $x + \sqrt{3}y = 2$.

$$\text{ANSWER: } r = \frac{2}{\cos \theta + \sqrt{3} \sin \theta}$$

3. The line passing through $(3, 5)$ and $(5, 7)$.

$$\text{ANSWER: } r = -\frac{2}{\cos \theta + \sin \theta}$$

4. The line passing through $(2, 3)$ and $(-3, -2)$.

$$\text{ANSWER: } r = \frac{1}{\cos \theta - \sin \theta}$$

Solution. 28.2

Polar coordinates are given by

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases}$$

All we need to do to obtain polar equations for our line is substitute the above expressions in the equation for the line.

$$r \cos \theta + \sqrt{3}r \sin \theta = 2.$$

This is a perfectly good answer, but we can transform the equation to make it look more compact:

$$\begin{aligned} r \cos \theta + \sqrt{3}r \sin \theta &= 2 \\ r \underbrace{\frac{1}{2} \cos \theta}_{=\cos(\frac{\pi}{3})} + r \underbrace{\frac{\sqrt{3}}{2} \sin \theta}_{=\sin(\frac{\pi}{3})} &= 1 \\ r \cos \left(-\frac{\pi}{3} \right) \cos \theta - \sin \left(-\frac{\pi}{3} \right) \sin \theta &= 1 \\ r \cos \left(\theta - \frac{\pi}{3} \right) &= 1 \\ r &= \frac{1}{\cos \left(\theta - \frac{\pi}{3} \right)} \\ &= \sec \left(\theta - \frac{\pi}{3} \right). \end{aligned} \quad \left| \text{ use } \cos(a+b) = \cos a \cos b - \sin a \sin b \right.$$

Problem 29. Find polar equations of the circle given below.

1. The circle given by $(x - 1)^2 + y^2 = 1$.

2. The circle given by $x^2 + x + y^2 = 1$.

3. The circle with center $(1, 2)$ and radius 3.

4. The circle with center $(2, 3)$ and radius 4.

answer: $r = 2 \cos \theta$

answer: $r^2 = 1 - \cos \alpha + r \cos \alpha = 0$

answer: $r^2 = 4r \sin \theta - 2r \cos \theta - 4 = 0$

answer: $r^2 = 6r \sin \theta - 4r \cos \theta - 3 = 0$

4.2 Cylindrical coordinates - basics

Problem 30. Find an equation of the plane in cylindrical coordinates.

1. The plane given by $x + y + z = 1$.

2. The plane given by $2x + 3y - 5z = 0$.

3. The plane passing through $(-1, 1, 1)$, $(1, 1, -1)$ and $(1, -1, 1)$.

4. The plane passing through $(2, 3, 5)$, $(3, 5, 2)$ and $(5, 2, 3)$.

Problem 31. Find an equation of the sphere in cylindrical coordinates.

1. The unit sphere.

2. The sphere with equation $x^2 + x + y^2 + 2y + z^2 + 3z = 0$.

3. The sphere with center $(1, 2, 3)$ and radius 5.

4.3 Spherical coordinates - basics

Problem 32. Find an equation of the plane in spherical coordinates.

1. The plane given by $x + y + z = 1$.

2. The plane given by $2x + 3y - 5z = 0$.

3. The plane passing through $(-1, 1, 1)$, $(1, 1, -1)$ and $(1, -1, 1)$.

4. The plane passing through $(2, 3, 5)$, $(3, 5, 2)$ and $(5, 2, 3)$.

Problem 33. Find an equation of the sphere in spherical coordinates.

1. The unit sphere.

2. The sphere with equation $x^2 + x + y^2 + 2y + z^2 + 3z = 0$.

3. The sphere with center $(1, 2, 3)$ and radius 5.

5 Quadratic surfaces

Problem 34. This problem is in preparation, please skip it. Match the surface graph to its mathematical name and to its equation.

1.

Problem 35. Determine the type of the quadratic surface given by the equation. The answer key has not been proofread, use with extreme caution.

1. $x^2 + y^2 + z^2 + x + 2y + 3z = 0$.

answer: sphere (also ellipsoid)

2. $x^2 + 2y^2 + z^2 + x + 2y + 3z = 0$.

answer: (circular) ellipsoid

3. $x^2 + 2y^2 + 3z^2 + x + 2y + 3z = 0$.

answer: ellipsoid

4. $z^2 + 2y^2 - 3x^2 + x + y + 1 = 0$.

answer: (elliptic) hyperboloid two sheets

5. $z^2 - y^2 + \frac{1}{4}x^2 + x - y + 1 = 0$.

answer: (elliptic) hyperboloid two sheets

6. $x^2 + y^2 - \frac{1}{4}z^2 + x - y + 5 = 0$.

answer: (circular) hyperboloid two sheets

$$7. \frac{1}{4}x^2 - y^2 + z^2 - x + 1 = 0$$

answer: (elliptic) cone

$$8. -\frac{1}{4}x^2 + y^2 + z^2 - x - 1 = 0$$

answer: (circular) cone

9. $xy + z^2 + 1 = 0$. *Hint: write $x = \frac{1}{\sqrt{2}}(u + v)$, $y = \frac{1}{\sqrt{2}}(u - v)$ for some new variables u, v . Solve the problem in the z, u, v -coordinates. Argue that the (axes of the) u, v, z -coordinate system can be obtained from the x, y, z -coordinate system via rotation.*

answer: (circular) hyperboloid one sheet

$$10. x^2 + 2y^2 + z = 0.$$

answer: (elliptic) paraboloid

$$11. x^2 + y^2 + 2xy + z = 0.$$

answer: cylindrical paraboloid

$$12. x^2 - y^2 + 2x + z = 0.$$

answer: parabolic hyperboloid

Solution. 4 We have that

$$\begin{aligned} z^2 + 2y^2 - 3x^2 + x + y + 1 &= 0 \\ z^2 + 2\left(y + \frac{1}{4}\right)^2 - 3\left(x - \frac{1}{6}\right)^2 - \frac{1}{8} + \frac{1}{12} + 1 &= 0 \\ z^2 + 2\left(y + \frac{1}{4}\right)^2 &= 3\left(x - \frac{1}{6}\right)^2 - \frac{23}{24} \end{aligned}$$

This figure is given by sum of two squares equal to a square minus a positive number. That makes is a hyperboloid two sheet, as explained in the theoretical discussions.

6 Curves in space

6.1 Curvature

Problem 36. Compute the tangent vector, the normal vector and the curvature at each point on the curve.

1. The equator of the unit sphere $\mathbf{r}(t) = (\cos t, \sin t, 0)$.
2. The equator of the unit sphere $\mathbf{r}(t) = (\cos t, \sin t, 0)$.
3. The loxodromic curve $\mathbf{r}(t) = (\cos(10t)\sin(t), \sin(10t)\sin(t), \cos(t))$. A loxodromic curve is a curve obtained by taking a straight line in the $\{(\rho, \phi, \theta) | \rho = \text{const}\}$ -plane in spherical coordinates and mapping it into the x, y, z -coordinates. Loxodromic curves were used in navigation: maintaining a course on a loxodromic curve requires only keeping a constant angle with the north direction (which one approximately obtained via compass).
4. The ellipse $\mathbf{r}(t) = (a \cos t, b \sin t)$. Where is the curvature largest? Where smallest? Can you answer without computation, and does your answer match your computation?
5. The loxodromic meridian $\mathbf{r}(t) = (\sin(at) \cos(bt), \sin(at) \sin(bt), \rho \cos(at))$.
6. The trefoil (torus) knot $\mathbf{r} = ((R + r \sin(3t)) \cos(2t), (R + r \sin(3t)) \sin(2t), r \cos(3t))$.
7. The torus curve $\mathbf{r} = ((R + r \sin(20t)) \cos(t), (R + r \sin(20t)) \sin(2t), r \cos(20t))$.
8. The helix $\mathbf{r} = (\cos t, \sin t, t)$.
9. The cone curve $\mathbf{r} = (t \cos t, t \sin t, -t)$.

Solution. 36.9

$$\begin{aligned}
\mathbf{r}(t) &= (t \cos t, t \sin t, -t) \\
\mathbf{r}'(t) &= (\cos t - t \sin t, \sin t + t \cos t, -1) \\
\mathbf{r}''(t) &= (-t \cos t - 2 \sin t, -t \sin t + 2 \cos t, 0) \\
\mathbf{r}'(t) \times \mathbf{r}''(t) &= (-t \sin t + 2 \cos t, t \cos t + 2 \sin t, t^2 \sin^2 t + t^2 \cos^2 t + 2 \sin^2 t + 2 \cos^2 t) \\
&= (-t \sin t + 2 \cos t, t \cos t + 2 \sin t, t^2 + 2) \\
|\mathbf{r}'(t)| &= \sqrt{(\cos t - t \sin t)^2 + (\sin t + t \cos t)^2 + (-1)^2} \\
&= \sqrt{\cos^2 t - 2t \cos t \sin t + t^2 \sin^2 t + \sin^2 t + 2t \sin t \cos t + t^2 \cos^2 t} \\
&= \sqrt{2 + t^2} \\
|\mathbf{r}'(t) \times \mathbf{r}''(t)| &= \sqrt{(-t \sin t + 2 \cos t)^2 + (t \cos t + 2 \sin t)^2 + (t^2 + 2)^2} \\
&= \sqrt{t^2 \sin^2 t + t^2 \cos^2 t + 4 \sin^2 t + 4 \cos^2 t + t^4 + 4t^2 + 4} \\
&= \sqrt{t^4 + 5t^2 + 8} \\
\kappa &= \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} \\
&= \sqrt{\frac{t^4 + 5t^2 + 8}{(2 + t^2)^3}}.
\end{aligned}$$

6.2 Curve length

Problem 37. Find the length of the curve.

1. The helix $\mathbf{r} = (\cos t, \sin t, t)$, $t \in [0, 2\pi]$.
2. The cone curve $\mathbf{r} = (t \cos t, t \sin t, t)$, $t \in [0, 2\pi]$.
3. The paraboloid curve $\mathbf{r} = (t \cos t, t \sin t, t^2)$, $t \in [0, 2\pi]$.

Solution. ??

$$\begin{aligned}
\mathbf{r}(t) &= (t \cos t, t \sin t, -t) \\
\mathbf{r}'(t) &= (\cos t - t \sin t, \sin t + t \cos t, -1) \\
|\mathbf{r}'(t)| &= \sqrt{(\cos t - t \sin t)^2 + (\sin t + t \cos t)^2 + (-1)^2} \\
&= \sqrt{\cos^2 t - 2t \cos t \sin t + t^2 \sin^2 t + \sin^2 t + 2t \sin t \cos t + t^2 \cos^2 t} \\
&= \sqrt{2 + t^2} \\
\text{length} &= \int_{t=0}^{t=2\pi} \sqrt{2 + t^2} dt && \left| \begin{array}{l} \text{Set } t = \sqrt{2} \tan \theta \end{array} \right. \\
&= \int_{t=0}^{t=2\pi} \sqrt{2 \sec^2 \theta} \sqrt{2} \sec^2 \theta d\theta \\
&= 2 \int_{t=0}^{t=2\pi} \sec^3 \theta d\theta && \left| \begin{array}{l} \text{Integral studied} \end{array} \right. \\
&= [\sec \theta \tan \theta + \ln(\sec \theta + \tan \theta)]_{t=0}^{t=2\pi} \\
&= \left[\sqrt{1 + \frac{t^2}{2}} \frac{t}{\sqrt{2}} + \ln \left(\sqrt{1 + \frac{t^2}{2}} + \frac{t}{\sqrt{2}} \right) \right]_{t=0}^{t=2\pi} \\
&= \sqrt{4\pi^2 + 2\pi} + \ln \left(\sqrt{1 + 2\pi^2} + \pi\sqrt{2} \right) \\
&\approx 22.429913.
\end{aligned}$$

Problem 38. Write down the integral expressing the length of the curve. Please do not try to solve the integrals by hand. Optionally, for this exercise only, you may type the integrals in an on-line computer algebra system and see what you get.

1. The loxodromic curve $\mathbf{r}(t) = (\cos(10t) \sin(t), \sin(10t) \sin(t), \cos(t))$ from $t = 0$ to $t = t_0$.
2. The ellipse $\mathbf{r}(t) = (a \cos t, b \sin t)$ from $t = 0$ to $t = t_0$.
3. The trefoil (torus) knot $\mathbf{r} = ((3 + \sin(3t)) \cos(2t), (3 + \sin(3t)) \sin(2t), \cos(3t))$ from $t = 0$ to $t = 2\pi$.
4. The torus curve $\mathbf{r} = ((R + r \sin(20t)) \cos(t), (R + r \sin(20t)) \sin(t), r \cos(20t))$ from $t = 0$ to $t = 2\pi$.

7 Multivariable limits

Problem 39. Find the limit or show that it does not exist. The answer key has not been proofread, use with caution.

1. $\lim_{(x,y) \rightarrow (0,0)} \frac{y^4}{x^4 + 2y^2}.$

ANSWER: 0

2. $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 + (\ln(1+y))^2}{x^2 + y^2}.$

ANSWER: the limit does not exist.

3. $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 \ln(1+y)}{x^2 + y^2}.$

ANSWER: 0

4. $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y^4}{x^4 + y^8}.$

ANSWER: the limit does not exist.

Solution. 39.1 Limits respect inequalities, therefore

$$0 \leq \lim_{(x,y) \rightarrow 0} \frac{y^4}{x^4 + 2y^2} \leq \lim_{(x,y) \rightarrow 0} \frac{y^4}{2y^2} = \lim_{(x,y) \rightarrow 0} \frac{1}{2} y^2 = 0.$$

Therefore $\lim_{(x,y) \rightarrow 0} \frac{y^4}{x^4 + 2y^2} = 0.$

8 Tangent planes

Problem 40. Find the equation of the tangent plane to the graph of the function at the indicated point.

1. $z = x^2 - y^2$, at the point $(1, 1, 0).$

ANSWER:

2. $z = e^{-x^2 - y^2}$, at the point $(0, 0, 1)$

ANSWER: $z = 1$

3. $z = e^{x^2 - y^2}$, at the point $(1, -1, 1).$

ANSWER: $2x + 2y + z = -1$

4. $z = \sqrt{3 - x^2 - y^2}$, at the point $(1, 1, 1).$

ANSWER:

Problem 41. Find an equation of the tangent plane to the surface at the given point. The surface is given via an implicit equation.

1. The sphere $x^2 + y^2 + z^2 = 1$ at $(x, y, z) = (\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}).$

ANSWER: tangent plane: $x + y + z = \sqrt{3}$

2. The two-sheet hyperboloid $x^2 + y^2 - z^2 = -3$ at $(x, y, z) = (2, 3, 4).$

ANSWER: tangent plane: $2x + 3y - 4z = -3$

3. The ellipsoid $x^2 + 2y^2 + 3z^2 = 20$ at $(x, y, z) = (3, 2, 1).$

ANSWER: tangent plane: $3x + 4y + 3z = 20$

Solution. 41.2 As studied, a normal to the tangent plane to a surface with implicit equation $f = 0$ is given by ∇f . Since the tangent plane passes through $(2, 3, 4)$, this determines the tangent plane.

$f = x^2 + y^2 - z^2 - 3$	$= 0$	equation of the surface
∇f	$= (2x, 2y, -2z)$	
$\nabla f _{(x,y,z)=(2,3,4)}$	$= (4, 6, -8)$	
$\nabla f _{(x,y,z)=(2,3,4)} \cdot (x - 2, y - 3, z - 4)$	$= 0$	equation of plane
$4(x - 2) + 6(y - 3) - 8(z - 4)$	$= 0$	
$2x + 3y - 4z$	$= -3$	final answer in simplified form.

9 Partial derivatives

Problem 42. Compute the indicated partial derivatives. Answer key has not been proofread, use with caution.

$$1. \frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}, r = \sqrt{x^2 + y^2}.$$

$$\frac{\partial}{\partial x} = \frac{z^{\frac{1}{2}} + z^{\frac{1}{2}}}{\frac{1}{2}} = \frac{r}{x}, \frac{\partial}{\partial y} = \frac{z^{\frac{1}{2}} + z^{\frac{1}{2}}}{\frac{1}{2}} = \frac{r}{y}$$

$$2. \frac{\partial^2 r}{\partial x^2}, \frac{\partial^2 r}{\partial y^2}, \frac{\partial^2 r}{\partial y \partial x}, r = \sqrt{x^2 + y^2}.$$

$$\frac{\partial^2}{\partial x^2} = \frac{x}{x^2} \frac{\partial}{\partial x} = \frac{1}{x^2}, \frac{\partial^2}{\partial y^2} = \frac{y}{y^2} \frac{\partial}{\partial y} = \frac{1}{y^2}, \frac{\partial^2}{\partial y \partial x} = \frac{1}{xy}$$

$$3. \frac{\partial \theta}{\partial x}, \frac{\partial \theta}{\partial y}, \theta = \arctan\left(\frac{y}{x}\right).$$

$$\frac{\partial \theta}{\partial x} = \frac{r}{x} = \frac{1}{x}, \frac{\partial \theta}{\partial y} = \frac{r}{y} = \frac{1}{y}$$

$$4. \frac{\partial^2 \theta}{\partial x^2}, \frac{\partial^2 \theta}{\partial y \partial x}, \frac{\partial^2 \theta}{\partial y^2}, \theta = \arctan\left(\frac{y}{x}\right).$$

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{x}{x^2} \frac{\partial}{\partial x} = \frac{1}{x^2}, \frac{\partial^2 \theta}{\partial y \partial x} = \frac{1}{xy}, \frac{\partial^2 \theta}{\partial y^2} = \frac{1}{y^2}$$

Solution. 42.3

$$\begin{aligned} \frac{\partial}{\partial x} \left(\arctan\left(\frac{y}{x}\right) \right) &= \frac{\frac{\partial}{\partial x} \left(\frac{y}{x} \right)}{1 + \left(\frac{y}{x} \right)^2} = \frac{-\frac{y}{x^2}}{1 + \frac{y^2}{x^2}} = \frac{-y}{x^2 + y^2} \\ \frac{\partial}{\partial y} \left(\arctan\left(\frac{y}{x}\right) \right) &= \frac{\frac{\partial}{\partial y} \left(\frac{y}{x} \right)}{1 + \left(\frac{y}{x} \right)^2} = \frac{\frac{1}{x}}{1 + \frac{y^2}{x^2}} = \frac{x}{x^2 + y^2} \end{aligned}$$

9.1 Variable changes in differential operators

Problem 43. 1. Let the variables b, c, x_1, x_2 be related via $b = -x_1 - x_2$ and $c = x_1 x_2$.

(a) Express the differential operators $\frac{\partial}{\partial c}$ and $\frac{\partial}{\partial b}$ via $\frac{\partial}{\partial x_1}$ and $\frac{\partial}{\partial x_2}$.

(b) Express the differential operators $\frac{\partial}{\partial x_1}$ and $\frac{\partial}{\partial x_2}$ via $\frac{\partial}{\partial c}$ and $\frac{\partial}{\partial b}$.

2. Let x, y, z and ρ, ϕ, θ be related via the usual spherical coordinates equations i.e., $x =$

(a) Express the differential operators $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ via $\frac{\partial}{\partial \rho}, \frac{\partial}{\partial \phi}, \frac{\partial}{\partial \theta}$.

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial}{\partial \rho} \cos \phi \cos \theta + \frac{\partial}{\partial \phi} \frac{\cos \phi \sin \theta}{\rho} + \frac{\partial}{\partial \theta} \frac{\cos \phi \cos \theta}{\rho} \\ \frac{\partial}{\partial y} &= \frac{\partial}{\partial \rho} \cos \phi \sin \theta + \frac{\partial}{\partial \phi} \frac{\cos \phi \cos \theta}{\rho} + \frac{\partial}{\partial \theta} \frac{\cos \phi \sin \theta}{\rho} \\ \frac{\partial}{\partial z} &= \frac{\partial}{\partial \rho} \sin \phi + \frac{\partial}{\partial \phi} \frac{\sin \phi}{\rho} \end{aligned}$$

(b) Express the differential operators $\frac{\partial}{\partial \rho}, \frac{\partial}{\partial \phi}, \frac{\partial}{\partial \theta}$ via $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$.

$$\begin{aligned} \frac{\partial}{\partial \rho} &= \cos \phi \cos \theta \frac{\partial}{\partial x} + \cos \phi \sin \theta \frac{\partial}{\partial y} + \sin \phi \frac{\partial}{\partial z} \\ \frac{\partial}{\partial \phi} &= -\sin \phi \cos \theta \frac{\partial}{\partial x} - \sin \phi \sin \theta \frac{\partial}{\partial y} + \cos \phi \frac{\partial}{\partial z} \\ \frac{\partial}{\partial \theta} &= -\cos \phi \sin \theta \frac{\partial}{\partial x} + \cos \phi \cos \theta \frac{\partial}{\partial y} \end{aligned}$$

(c) Express the Laplace differential operator $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ via $\frac{\partial}{\partial \rho}, \frac{\partial}{\partial \phi}, \frac{\partial}{\partial \theta}$ (in other words, write the 3 dimensional Laplace operator in spherical coordinates).

$$\begin{aligned} &= \frac{1}{\rho^2} \left(\frac{\partial}{\partial \rho} \left(\rho^2 \frac{\partial}{\partial \rho} \right) + \frac{1}{\sin \phi} \frac{\partial}{\partial \phi} \left(\sin \phi \frac{\partial}{\partial \phi} \right) + \frac{1}{\sin^2 \phi} \frac{\partial^2}{\partial \theta^2} \right) \\ &= \frac{1}{\rho^2} \left(\frac{\partial}{\partial \rho} \left(\rho^2 \frac{\partial}{\partial \rho} \right) + \frac{1}{\sin \phi} \frac{\partial}{\partial \phi} \left(\sin \phi \frac{\partial}{\partial \phi} \right) + \frac{1}{\sin^2 \phi} \frac{\partial^2}{\partial \theta^2} \right) \end{aligned}$$

Solution. 2a To be written.

10 Multivariable optimization (min/max)

Problem 44. Using the second derivative test, find the local minima and maxima as well as the saddle points of the function.

$$1. f(x, y) = 1 + x^3 + y^3 - 3xy.$$

$$2. f(x, y) = x^3 y + x^2 - 27y.$$

$$3. f(x, y) = e^{2y-x^2-y^2}.$$

$$4. f(x, y) = e^x \sin y.$$

$$5. f(x, y) = x^2 + y^2 + \frac{1}{x^2 y^2}.$$

6. $f(x, y) = x^2 + x^2y + y^3 - 4y$.

(x, y)	critical point type
$(0, -2\frac{\sqrt{3}}{3})$	local maximum
$(\frac{\sqrt{3}}{3}, 0)$	local minimum
$(1, -1)$	saddle
$(-1, -1)$	saddle

Solution. 6 The critical points of f are given by:

$$\begin{aligned}\frac{\partial f}{\partial x} &= 0 &= 2xy + 2x \\ \frac{\partial f}{\partial y} &= 0 &= 3y^2 + x^2 - 4.\end{aligned}$$

The first equality implies $x(y + 1) = 0$, and we have two cases: $x = 0$ and $y = -1$.

Case 1. $x = 0$. We substitute in second equality and solve:

$$\begin{aligned}3y^2 - 4 &= 0 \\ y^2 &= \frac{4}{3} \\ y &= \pm 2\frac{\sqrt{3}}{3}.\end{aligned}$$

Case 1 provides us with two critical points, $(x, y) = (0, 2\frac{\sqrt{3}}{3})$ and $(x, y) = (0, -2\frac{\sqrt{3}}{3})$.

Case 2. $x \neq 0$. It follows that $y = -1$. We substitute in the second equality and solve:

$$\begin{aligned}3 + x^2 - 4 &= 0 \\ x^2 &= 1 \\ x &= \pm 1.\end{aligned}$$

Case 2 provides us with two additional critical points, $(x, y) = (1, -1)$ and $(x, y) = (-1, -1)$.

The Hessian matrix of f and its determinant are:

$$H = \begin{pmatrix} 2y + 2 & 2x \\ 2x & 6y \end{pmatrix} \quad \det H = 12y(y + 1) - 4x^2.$$

At $(x, y) = (0, 2\frac{\sqrt{3}}{3})$, $\det H = 8\sqrt{3} + 16 > 0$, and $\frac{\partial f}{\partial x^2} > 0$ so f has a local minimum at that point. At $(x, y) = (0, -2\frac{\sqrt{3}}{3})$, we have $\det H = -8\sqrt{3} + 16 > 0$. We further have $\frac{\partial f}{\partial x^2} = 2(\frac{2}{\sqrt{3}} - 1) < 0$ so f has a local maximum at that point. Finally at $(x, y) = (\pm 1, -1)$, we have $\det H = -4 < 0$ and so both points are saddle points of f .

Our final answer is as follows.

(x, y)	critical point type
$(0, -2\frac{\sqrt{3}}{3})$	local maximum
$(0, 2\frac{\sqrt{3}}{3})$	local minimum
$(-1, -1)$	saddle
$(1, -1)$	saddle

10.1 Lagrange multipliers

Problem 45. Find the maximum of the function subject to the given restriction, or show the maximum does not exist.

The problems don't have an answer key yet. If you think that a problem is incorrectly posed, make a clean argument why that is the case.

1. $f(x, y) = x^2 + 2y^2, xy = 1$.

2. $f(x, y) = 4x + 5y, x^2 + y^2 = 13$.

3. $f(x, y) = x^2y, x^2 + 2y^2 = 1$.

4. $f(x, y) = e^{xy}, x^3 + y^3 = 2$.

$$e^x = (1 + \frac{x}{n})^n f = x^n u^n f \cdot (1 + \frac{1}{n}) = (n + x) u^n f$$

5. $f(x, y) = x + 3y + 5z, x^2 + y^2 + z^2 = 35$.

6. $f(x, y) = x - z, x^2 + 3y^2 + z^2 = 1$.

7. $f(x, y) = xyz, x^2 + 3y^2 + 5z^2 = 8.$
8. $f(x, y) = x^2y^2z^2, x^2 + y^2 + z^2 = 1.$
9. $f(x, y) = x^2 + y^2 + z^2, x^4 + y^4 + z^4 = 1.$
10. $f(x, y) = x^4 + y^4 + z^4, x^2 + y^2 + z^2 = 1.$
11. $f(x_1, \dots, x_n) = x_1 + \dots + x_n, x_1^2 + \dots + x_n^2 = 1.$
12. Find the local extrema of $f(x, y) = y + x$ when x, y satisfy the restriction $y^2 + y + x^2 + x = 1.$

maximum	$\left(\frac{\sqrt{2}-1}{2}, \frac{\sqrt{2}-1}{2} \right)$
minimum	$\left(\frac{\sqrt{2}+1}{2}, \frac{\sqrt{2}+1}{2} \right)$
critical point type	(x, y)

Solution. 45.4 The restriction is $g(x, y) = x^3 + y^3 - 2 = 0$. We use the method of Lagrange multipliers. We have that $\nabla f = (e^{x+y}, e^{x+y})$ and $\nabla g = (3x^2, 3y^2)$. We have a local extremum when $\lambda \nabla f = \nabla g$, i.e., when

$$\begin{aligned}\lambda e^{x+y} &= 3x^2 \\ \lambda e^{x+y} &= 3y^2 \\ x^3 + y^3 &= 2\end{aligned}$$

The first two equations imply $y^2 = x^2$ which implies $y = \pm x$.

Case 1. Suppose $y = -x$. Then the last equation $x^3 + y^3 = 2$ reduces to $0 = 2$, which has no solutions; this case yields no candidates for maxima and minima.

Case 2. Suppose $y = x$. We substitute into the third equation and solve:

$$\begin{aligned}2x^3 &= 2 \\ x^3 - 1 &= 0 \\ (x-1)(x^2+x+1) &= 0 \quad | \quad x^2+x+1 \neq 0 \text{ for all real } x \\ x &= 1\end{aligned}$$

Therefore $x = 1, y = 1$ is the only critical point obtained by the method of Lagrange multipliers. To find out whether the critical point is a maximum or minimum, we can rewrite our restriction as $y(x) = \sqrt[3]{2-x^3}$ and so $f(x, y(x)) = e^{x+\sqrt[3]{2-x^3}}$. Since the exponent is an increasing function, $e^{x+\sqrt[3]{2-x^3}}$ has extrema if and only if the function $x + \sqrt[3]{2-x^3}$ has the same type of extrema. $x + \sqrt[3]{2-x^3}$ has second derivative $-2x^4(-x^3+2)^{-\frac{2}{3}} - 2x(-x^3+2)^{-\frac{2}{3}}$, which evaluates to -4 when $x = 1$. Therefore by the single-variable second derivative criterion $f(x, y(x)) = e^{x+\sqrt[3]{2-x^3}}$ has a local maximum and so the critical point is a local maximum.

We point out that via the equality $f(x, y(x)) = e^{x+\sqrt[3]{2-x^3}}$ this problem can be solved without using Lagrange multipliers, however the computations would be longer.

Solution. 45.12 The restriction is $g(x, y) = y^2 + y + x^2 + x - 1 = 0$. We use the method of Lagrange multipliers. We have that $\nabla f = (1, 1)$ and $\nabla g = (2y+1, 2x+1)$. We have a local extremum when $\lambda \nabla f = \nabla g$, i.e., when

$$\begin{aligned}\lambda &= (2y+1) \\ \lambda &= (2x+1) \\ y^2 + y + x^2 + x - 1 &= 0\end{aligned}$$

The first two equations imply $y = x$. We substitute that into the last equation to get that $2x^2 + 2x - 1 = 0$. The solutions to the latter are $x = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 2 \cdot (-1)}}{4} = \frac{-1 \pm \sqrt{3}}{2}$. The only restriction on the points (x, y) is that they lie on the curve $y^2 + y + x^2 + x = 1$ (a circle). A circle is a bounded and closed set. We recall that a set in space is bounded if it is contained in a ball (with finite radius) and a set in space is closed if it contains all of its boundary points. Therefore f must attain both its minimum and its maximum on it. Therefore the two critical points are maximum and minimum of f . Substitution of our answer in f shows that f attains its minimum at

$(x, y) = \left(\frac{-1-\sqrt{3}}{2}, \frac{-1-\sqrt{3}}{2} \right)$ and its maximum at $(x, y) = \left(\frac{-1+\sqrt{3}}{2}, \frac{-1+\sqrt{3}}{2} \right)$. Our final answer is below.

(x, y)	max or min
$\left(\frac{-1-\sqrt{3}}{2}, \frac{-1-\sqrt{3}}{2} \right)$	minimum
$\left(\frac{-1+\sqrt{3}}{2}, \frac{-1+\sqrt{3}}{2} \right)$	maximum

11 Double integrals

Problem 46. Evaluate the double integral.

1. $\iint_D x^3 y^2 dx dy$, $D = \{(x, y) | 0 \leq x \leq 2, -x \leq y \leq x\}$.
2. $\iint_D \frac{4y}{x^3 + 2} dx dy$, $D = \{(x, y) | 1 \leq x \leq 2, 0 \leq y \leq 2x\}$.
3. $\iint_D \frac{2y}{x^2 + 1} dx dy$, $D = \{(x, y) | 0 \leq x \leq 1, 0 \leq y \leq \sqrt{x}\}$.
4. $\iint_D e^{y^2} dx dy$, $D = \{(x, y) | 0 \leq y \leq 1, 0 \leq x \leq y\}$.
5. $\iint_D x \cos y dx dy$, D bounded by $y = 0$, $y = x^2$, $x = 1$.
6. $\iint_D (x + y) dx dy$, D bounded by $y = \sqrt{x}$ and $y = x^2$.
7. $\iint_D y^3 dx dy$, D - triangle with vertices $(0, 2)$, $(1, 1)$, $(3, 2)$.
8. $\iint_D xy^2 dx dy$, D enclosed by $x = 0$ and $x^2 + y^2 = 1$.
9. $\iint_D (2x - y) dx dy$, D bounded by circle with radius 2 centered at the origin.
10. $\iint_D 2xy dx dy$, D - triangular region with vertices $(0, 0)$, $(1, 2)$, $(0, 3)$.

Problem 47. Evaluate the double integral. The answer key has not been proofread, use with caution.

1. $\iint_{\mathcal{R}} xy dx dy$ where \mathcal{R} is bounded by the curves $x = 3$, $x + 1 = y^2$, $x = y^2 + 2y + 3$.

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2. $\iint_{\mathcal{R}} xy dx dy$.

where \mathcal{R} is the region enclosed by $y = x^2 + 1$ and $y = 2x^2 - x - 1$.

Solution. 47.1

We start by plotting the region. $x = y^2 - 1$ is a parabola symmetric across the x axis; $x = y^2 + 2y + 3$ is a parabola with vertex at $x = 3$, $y = -2$. The two parabolas intersect when

$$\begin{aligned} x = y^2 - 1 &= y^2 + 2y + 3 \\ 2y + 4 &= 0 \\ y &= -2 \\ x &= y^2 - 1 = (-2)^2 - 1 = 3, \end{aligned}$$

i.e., when $(x, y) = (3, -2)$. The line $x = 3$ intersects $x + 1 = y^2$ when $y^2 = 3 + 1 = 4$, i.e., when $(x, y) = (3, \pm 2)$. The line $x = 3$ intersects $x = y^2 + 2y + 3$ when $3 = y^2 + 2y + 3$. This implies $y(y + 2) = 0$ and finally we conclude the intersections of $x = 3$ with $x = y^2 + 2y + 3$ are $(x, y) = (3, 0)$ and $(x, y) = (3, 2)$. We can conclude that there three curves are plotted as indicated in the figure. Of the 8 regions bounded by the curves only two are bounded, and only one of them is bounded by all three curves. Since no further instruction is given in the problem, we assume that the intended region is the one bounded by all three curves, i.e., the region indicated in the figure above (the other bounded region can be enclosed using two of the curves only). Let Q_1 and Q_2 be the regions indicated in the figure above. Those two regions are curvilinear trapezoids with vertical bases. Consider the region Q_1 . Fix the y coordinate of a point in Q_1 . The figure shows that, for that fixed value of y , x varies between $y^2 - 1$ and 3. For Q_2 , it similarly follows that, for a fixed

y , x varies between $y^2 - 1$ and $y^2 + 2y + 3$. The points in Q_1 have y coordinates in the range $y \in [0, 2]$, and similarly, in Q_2 we have that the y coordinate varies in the range $y \in [-2, 0]$. Thus our regions are parametrized as

$$\begin{aligned} Q_1 &= \{(x, y) | 0 \leq y \leq 2, y^2 - 1 \leq x \leq 3\} \\ Q_2 &= \{(x, y) | -2 \leq y \leq 0, y^2 - 1 \leq x \leq y^2 + 2y + 3\}. \end{aligned}$$

Finally our integral becomes

$$\begin{aligned} \iint_{\mathcal{R}} xy dx dy &= \iint_{Q_1} xy dx dy + \iint_{Q_2} xy dx dy \\ &= \int_{y=0}^{y=2} \left(\int_{x=y^2-1}^{x=3} xy dx \right) dy + \int_{y=-2}^{y=0} \left(\int_{x=y^2-1}^{x=y^2+2y+3} xy dx \right) dy \\ &= \int_{y=0}^{y=2} \left[\frac{x^2 y}{2} \right]_{x=y^2-1}^{x=3} dy + \int_{y=-2}^{y=0} \left[\frac{x^2 y}{2} \right]_{x=y^2-1}^{x=y^2+2y+3} dy \\ &= \int_{y=0}^{y=2} \left(-\frac{1}{2}y(y^2-1)^2 + \frac{9}{2}y \right) dy + \int_{y=-2}^{y=0} \left(\frac{1}{2}y(y^2+2y+3)^2 - \frac{1}{2}y(y^2-1)^2 \right) dy \\ &= \int_{y=0}^{y=2} \left(-\frac{1}{2}y^5 + y^3 + 4y \right) dy + \int_{y=-2}^{y=0} (2y^4 + 6y^3 + 6y^2 + 4y) dy \\ &= \left[-\frac{1}{12}y^6 + \frac{1}{4}y^4 + 2y^2 \right]_{y=0}^{y=2} + \left[\frac{2}{5}y^5 + \frac{3}{2}y^4 + 2y^3 + 2y^2 \right]_{y=-2}^{y=0} \\ &= \frac{20}{3} - \frac{16}{5} = \frac{52}{15} \end{aligned}$$

Solution. 47.2

We start by plotting the region. $y = x^2 + 1$ is a parabola symmetric across the y axis; $y = 2x^2 - x - 1$ is a parabola with vertex at $x = \frac{1}{4}$, $y = -\frac{9}{8}$. The two parabolas intersect when

$$\begin{aligned} x^2 + 1 &= 2x^2 - x - 1 \\ x^2 - x - 2 &= 0 \\ x &= -1 \text{ or } 2 \end{aligned}$$

Thus the region looks as plotted below.

Therefore our region \mathcal{R} is parametrized as

$$\mathcal{R} = \{(x, y) | 2x^2 - x - 1 \leq x^2 + 1\}.$$

This is a single curvilinear trapezoid. We can integrate directly as follows.

$$\begin{aligned} \iint_{\mathcal{R}} xy dx dy &= \int_{x=-1}^{x=2} \int_{y=2x^2-x-1}^{y=x^2+1} xy dy dx \\ &= \int_{x=-1}^{x=2} \left[\frac{1}{2}xy^2 \right]_{y=2x^2-x-1}^{y=x^2+1} dx \\ &= \int_{x=-1}^{x=2} \left(-\frac{3}{2}x^5 + 2x^4 + \frac{5}{2}x^3 - x^2 \right) dx \\ &= \left[-\frac{1}{4}x^6 + \frac{2}{5}x^5 + \frac{5}{8}x^4 - \frac{1}{3}x^3 \right]_{x=-1}^{x=2} \\ &= \frac{153}{40} \end{aligned}$$

11.1 Double integrals solved via changing integration order

Problem 48. *Integrate.*

1. $\int_{y=0}^{y=\sqrt{\pi}} \int_{x=y}^{x=\sqrt{\pi}} \cos(x^2) dx dy.$
2. $\int_{y=0}^{y=1} \int_{x=\sqrt[5]{y}}^{x=\sqrt[5]{y}} e^{-x^3} dx dy.$

Solution. 1 The issue with this integral is that we cannot integrate $\cos(x^2)$ with respect to x using (finitely many) elementary functions and their compositions. However, this expression is easy to integrate with respect to y . Therefore changing the order of integration (using Fubini's Theorem) could possibly help. Let the region of integration be \mathcal{R} . Then

$$\mathcal{R} = \{(x, y) | 0 \leq y \leq \sqrt{\pi}, y \leq x \leq \sqrt{\pi}\}.$$

We plot the region to find it is the triangle indicated in the figure below. When we fix the value of x , y varies between 0 and x . Therefore we can re-parametrize \mathcal{R} via vertical slices:

$$\mathcal{R} = \{(x, y) | 0 \leq x \leq \sqrt{\pi}, 0 \leq y \leq x\}.$$

By Fubini's theorem, the iterated integral equals the double integral, which in turn can be evaluated using the iterated integral using the second parametrization of \mathcal{R} .

$$\begin{aligned} \int_{y=0}^{y=\sqrt{\pi}} \int_{x=y}^{x=\sqrt{\pi}} \cos(x^2) \, dx \, dy &= \iint_{\mathcal{R}} \cos(x^2) \, dx \, dy && \left| \begin{array}{l} \text{By Fubini's Theorem} \\ \text{again by Fubini's Theorem} \end{array} \right. \\ &= \int_{x=0}^{\sqrt{\pi}} \int_{y=0}^{y=x} \cos(x^2) \, dy \, dx \\ &= \int_{x=0}^{\sqrt{\pi}} [y \cos(x^2)]_{y=0}^{y=x} \, dx \\ &= \int_{x=0}^{\sqrt{\pi}} x \cos(x^2) \, dx \\ &= \int_{x=0}^{\sqrt{\pi}} \cos(x^2) \frac{1}{2} d(x^2) \\ &= \frac{1}{2} [\sin(x^2)]_{x=0}^{x=\sqrt{\pi}} \\ &= \frac{1}{2} (\sin \pi - \sin 0) = 0. \end{aligned}$$

Solution. 2 This problem exploits the same idea as Problem 1 - that sometimes changing the order of integration is helpful for the algebraic manipulations.

Let the region of integration be \mathcal{R} . We have

$$\mathcal{R} = \{(x, y) | \sqrt[5]{y} \leq x \leq \sqrt[5]{y}, 0 \leq y \leq 1\}.$$

\mathcal{R} can be plotted as follows.

Therefore \mathcal{R} can be reparametrized as follows.

$$\mathcal{R} = \{(x, y) | x^5 \leq y \leq x^2, 0 \leq x \leq 1\}.$$

By Fubini's theorem, the iterated integral equals the double integral, which in turn can be evaluated using the iterated integral with respect to the second parametrization of \mathcal{R} .

$$\begin{aligned} \int_{y=0}^{y=1} \int_{x=\sqrt[5]{y}}^{x=\sqrt[5]{y}} e^{-x^3} \, dx \, dy &= \iint_{\mathcal{R}} e^{-x^3} \, dx \, dy && \left| \begin{array}{l} \text{By Fubini's Theorem} \\ \text{again by Fubini's Theorem} \end{array} \right. \\ &= \int_{x=0}^1 \int_{y=x^5}^{y=x^2} e^{-x^3} \, dy \, dx \\ &= \int_{x=0}^1 [ye^{-x^3}]_{y=x^5}^{y=x^2} \, dx \\ &= \int_{x=0}^1 x^2(1-x^3)e^{-x^3} \, dx \\ &= \int_{x=0}^1 (1-x^3)e^{-x^3} \frac{1}{3} d(x^3) && \left| \begin{array}{l} \text{Set } z = x^3 \end{array} \right. \\ &= \frac{1}{3} \int_{z=0}^{z=1} (1-z)e^{-z} \, dz \\ &= \frac{1}{3} [ze^{-z}]_{z=0}^{z=1} \\ &= \frac{1}{3} e^{-1} = \frac{1}{3e}. \end{aligned}$$

12 Triple Integrals

Problem 49. Set up an integral of the given function over the given region in space. Integrate.

1. $f(x, y, z) = x + y + z$, over the region \mathbb{R}^3 bounded by $x + 2y + z = 2$, $x = 2y$, $x = 0$, $z = 0$.
2. $f(x, y, z) = x + y + z$, over the region \mathbb{R}^3 bounded by $x + 3y + z = 2$, $x = 3y$, $x = 0$, $z = 0$.

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Solution. 2 First we need to plot the four planes enclosing our region \mathcal{R} . A region of non-zero volume in dimension n is bounded by at least $n + 1$ planes; since our region is given by 4 planes, our region, if bounded, must be a tetrahedron (the only type of figure in dimension 3 with 4 sides is a tetrahedron). Since we have the equations of our planes, we know their normal vectors. Therefore we can quickly plot those planes as indicated in the figure. We see that the corresponding figure is indeed a tetrahedron. A tetrahedron is given by its four vertices; let us compute those. Each of the vertices lies on three of the four planes, so it is determined by solving the system given by those four planes. The points and the systems used to find them are indicated in the figure.

Next, we need to parametrize the tetrahedron \mathcal{R} . There is a simple way to parametrize a tetrahedron by using one of its vertices and the three edges adjacent to that vertex. The integral resulting from this parametrization would require the multivariable substitution rule. Rather than doing that, let us solve the problem directly by more elementary means. We proceed to parametrize the tetrahedron directly by the variables x, y, z .

We start our parametrization by selecting one of the variables x, y, z and letting it vary over an interval. We choose the variable z (the other two choices would also work, we leave those for exercise). Since no restrictions are placed on the variables x, y , the possible z coordinates of all points in the tetrahedron are between 0 and 2, as evident from the figure. Now fix a value $z = \text{const}$ between 0 and 2. The points on the tetrahedron with the fixed z -coordinate are obtained by intersecting the tetrahedron with the plane $z = \text{const}$. The intersection is a triangle; let its vertices be A, B, C as labeled in the figure. Inside the plane $z = \text{const}$, the triangle ABC has sides given by the lines $y = 3x$ and $x + 3y + z = 2$, or what is the same, $y = \frac{2-z-x}{3}$. The coordinates of C are solutions to the system

$$\begin{cases} 3y = x \\ x + 3y + z = 2 \end{cases} \Rightarrow (x, y) = \left(\frac{2-z}{2}, \frac{2-z}{6} \right).$$

The y coordinate of B is obtained by setting $x = 0$ in $y = \frac{2-z-x}{3}$, in other words, the coordinates of B are $(0, \frac{2-z}{3})$. We have two variants -fix an interval for x next, or fix an interval for y next.

Variant I. Fix a value for x . The points of the triangle ABC have x coordinates between 0 and the x -coordinate of the point C , in other words, $x \in [0, \frac{2-z}{2}]$. Finally, for the fixed value for x , the variable y varies between $\frac{x}{3}$ and $\frac{2-z-x}{3}$. Our integral becomes

$$\begin{aligned} \iiint_{\mathcal{R}} (x + y + z) dx dy dz &= \int_{z=0}^{z=2} \int_{x=0}^{x=\frac{2-z}{2}} \int_{y=\frac{x}{3}}^{y=\frac{2-z-x}{3}} (x + y + z) dy dx dz \\ &= \int_{z=0}^{z=2} \int_{x=0}^{x=\frac{2-z}{2}} \left[xy + \frac{y^2}{2} + zy \right]_{y=\frac{x}{3}}^{y=\frac{2-z-x}{3}} dx dz \\ &= \int_{z=0}^{z=2} \int_{x=0}^{x=\frac{2-z}{2}} \left(-\frac{5}{18}z^2 - \frac{2}{3}x^2 - \frac{8}{9}xz + \frac{4}{9}z + \frac{4}{9}x + \frac{2}{9} \right) dx dz \\ &= \int_{z=0}^{z=2} \left[-\frac{4}{9}x^2z - \frac{5}{18}xz^2 - \frac{2}{9}x^3 + \frac{2}{9}x^2 + \frac{4}{9}zx + \frac{2}{9}x \right]_{x=0}^{x=\frac{2-z}{2}} dz \\ &= \int_{z=0}^{z=2} \left(\frac{1}{18}z^3 - \frac{1}{6}z^2 + \frac{2}{9} \right) dz \\ &= \left[\frac{1}{72}z^4 - \frac{1}{18}z^3 + \frac{2}{9}z \right]_{z=0}^{z=2} \\ &= \frac{2}{9}. \end{aligned}$$

Variant II. Fix a value for y . The points of the triangle ABC have y -coordinates between 0 and $\frac{2-z}{3}$. For the fixed value of y , the variable x varies between 0 and either $3y$ or $\frac{2-z-x}{3}$ - whichever is smaller. In other words, $x \in [0, \min\{3y, 2 - z - 3y\}]$. To avoid the use of the minimum function, we need to subdivide our triangle ABC into two triangles: $y \in [0, \frac{2-z}{6}]$, $x \in [0, 3y]$ and

$y \in \left[\frac{2-z}{6}, \frac{2-z}{3}\right]$, $x \in [0, 2 - z - 3y]$. Our integral becomes

$$\begin{aligned}
\iiint_{\mathcal{R}} (x + y + z) dx dy dz &= \int_{z=0}^{z=2} \int_{y=0}^{y=\frac{2-z}{6}} \int_{x=0}^{x=3y} (x + y + z) dx dy dz \\
&+ \int_{z=0}^{z=2} \int_{y=\frac{2-z}{6}}^{y=\frac{2-z}{3}} \int_{x=2-z-3y}^{x=0} (x + y + z) dx dy dz \\
&= \int_{z=0}^{z=2} \int_{y=0}^{y=\frac{2-z}{6}} \left[\frac{1}{2}x^2 + zx + yx \right]_{x=0}^{x=3y} dy dz \\
&+ \int_{z=0}^{z=2} \int_{y=\frac{2-z}{6}}^{y=\frac{2-z}{3}} \left[\frac{1}{2}x^2 + zx + yx \right]_{x=2-z-3y}^{x=0} dy dz \\
&= \int_{z=0}^{z=2} \int_{y=0}^{y=\frac{2-z}{6}} \left(\frac{15}{2}y^2 + 3zy \right) dy dz \\
&+ \int_{z=0}^{z=2} \int_{y=\frac{2-z}{6}}^{y=\frac{2-z}{3}} \left(-\frac{1}{2}z^2 + \frac{3}{2}y^2 - yz - 4y + 2 \right) dy dz \\
&= \int_{z=0}^{z=2} \left[\frac{3}{2}zy^2 + \frac{5}{2}y^3 \right]_{y=0}^{y=\frac{2-z}{6}} dz \\
&+ \int_{z=0}^{z=2} \left[-\frac{1}{2}z^2y - \frac{1}{2}y^2z + \frac{1}{2}y^3 - 2y^2 + 2y \right]_{y=\frac{2-z}{6}}^{y=\frac{2-z}{3}} dz \\
&= \int_{z=0}^{z=2} \left(\frac{1}{18}z^3 - \frac{1}{6}z^2 + \frac{2}{9} \right) dz \\
&= \frac{2}{9}.
\end{aligned}$$

The fact that the two variants produce the same answer provides an error check for our computations.

13 Variable Changes in Multivariable Integrals

Problem 50. *Problem 50.5 is of higher difficulty than the problem you will get on the exam.*

- Write the Jacobian matrix of the indicated variable change.
- Set up an integral expressing the volume of the region using the indicated variable change and the multivariable integral substitution rule.
- Integrate to find the volume of the region.

1. Spherical coordinates; use to find the volume of a ball of radius R .

2. Spherical coordinates; use to find the volume of a curvilinear spherical box, given in spherical coordinates by $\rho_{\min} \leq \rho \leq \rho_{\max}$, $\phi_{\min} \leq \phi \leq \phi_{\max}$, $\theta_{\min} \leq \theta \leq \theta_{\max}$.

3. Ellipsoidal coordinates: $\mathbf{f} : \begin{cases} x = a\rho \sin \phi \cos \theta \\ y = b\rho \sin \phi \sin \theta \\ z = c\rho \cos \phi \end{cases}$; use to find the volume of an ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, $a, b, c > 0$.

4. Variable change: $T : \begin{cases} x = (R + \rho \cos \theta) \cos \phi \\ y = (R + \rho \cos \theta) \sin \phi \\ z = \rho \sin \theta \end{cases}$; use to find the volume of a torus with major radius R and minor radius r , i.e., the figure given by $\rho \in [0, r]$, $\phi \in [0, 2\pi]$, $\theta \in [0, 2\pi]$.

5. Variable change: $\begin{cases} x = (2 + \rho \cos \theta) \cos \phi \\ y = (2 + \rho \cos \theta) \sin \phi \\ z = \rho \sin \theta + \frac{\phi}{3} \end{cases}$, use to find the volume of the horn given by $\theta \in [0, 2\pi]$, $\phi \in [0, 3\pi]$, $\rho \in \left[0, \frac{\phi}{9}\right]$.

6. Variable change: $\begin{cases} x = (2 + \phi/3 + \rho \cos \theta) \cos \phi \\ y = (2 + \phi/3 + \rho \cos \theta) \sin \phi \\ z = \rho \sin \theta + \frac{\phi}{3} \end{cases}$, use to find the volume of the horn given by $\theta \in [0, 2\pi]$, $\phi \in [0, 3\pi]$, $\rho \in \left[0, \frac{\phi}{9}\right]$.

Solution. 50.5 This solution is only partial.

Let \mathbf{f} be the map given by the variable change:

$$\begin{cases} x = (2 + \rho \cos \theta) \cos \phi \\ y = (2 + \rho \cos \theta) \sin \phi \\ z = \rho \sin \theta + \frac{\phi}{3} \end{cases}$$

$$J_{\mathbf{f}} = \begin{pmatrix} \frac{dx}{d\rho} & \frac{dx}{d\theta} & \frac{dx}{d\phi} \\ \frac{dy}{d\rho} & \frac{dy}{d\theta} & \frac{dy}{d\phi} \\ \frac{dz}{d\rho} & \frac{dz}{d\theta} & \frac{dz}{d\phi} \end{pmatrix} = \begin{pmatrix} \cos \phi \cos \theta & -\rho \cos \theta \sin \phi - 2 \sin \phi & -\rho \cos \phi \sin \theta \\ \cos \theta \sin \phi & \rho \cos \phi \cos \theta + 2 \cos \phi & -\rho \sin \phi \sin \theta \\ \sin \theta & 1/3 & \rho \cos \theta \end{pmatrix}$$

$$\begin{aligned} \text{Then } \det J_{\mathbf{f}} &= \cos^2 \phi \cos \theta \sin^2 \theta + \rho^2 \cos \theta \sin^2 \phi \sin^2 \theta \\ &\quad + \rho^2 \cos^3 \theta \sin^2 \phi + \rho^2 \cos^2 \phi \cos^3 \theta + 2\rho \sin^2 \phi \sin^2 \theta \\ &\quad + 2\rho \cos^2 \theta \sin^2 \phi + 2\rho \cos^2 \phi \sin^2 \theta + 2\rho \cos^2 \phi \cos^2 \theta \\ &= \rho^2 \cos \theta + 2\rho \end{aligned}$$

The rest of the problem we leave to the student.

Solution. 50.5 This solution is only partial.

Let \mathbf{f} be the map given by the variable change:

$$\begin{cases} x = (2 + \phi/3\rho \cos \theta) \cos \phi \\ y = (2 + \phi/3\rho \cos \theta) \sin \phi \\ z = \rho \sin \theta + \frac{\phi}{3} \end{cases}$$

$$J_{\mathbf{f}} = \begin{pmatrix} \frac{dx}{d\rho} & \frac{dx}{d\theta} & \frac{dx}{d\phi} \\ \frac{dy}{d\rho} & \frac{dy}{d\theta} & \frac{dy}{d\phi} \\ \frac{dz}{d\rho} & \frac{dz}{d\theta} & \frac{dz}{d\phi} \end{pmatrix} = \begin{pmatrix} \cos \phi \cos \theta & -\rho \cos \theta \sin \phi - 1/3\phi \sin \phi - 2 \sin \phi + 1/3 \cos \phi & -\rho \cos \phi \sin \theta \\ \cos \theta \sin \phi & \rho \cos \phi \cos \theta + 1/3\phi \cos \phi + 1/3 \sin \phi + 2 \cos \phi & -\rho \sin \phi \sin \theta \\ \sin \theta & 1/3 & \rho \cos \theta \end{pmatrix}$$

$$\begin{aligned} \text{Then } \det J_{\mathbf{f}} &= \rho^2 \cos^2 \phi \cos \theta \sin^2 \theta + \rho^2 \cos \theta \sin^2 \phi \sin^2 \theta + \rho^2 \cos^3 \theta \sin^2 \phi + \rho^2 \cos^2 \phi \cos^3 \theta \\ &\quad + \frac{1}{3}\phi \rho \sin^2 \phi \sin^2 \theta + \frac{1}{3}\phi \rho \cos^2 \theta \sin^2 \phi + \frac{1}{3}\phi \rho \cos^2 \phi \sin^2 \theta + \frac{1}{3}\phi \rho \cos^2 \phi \cos^2 \theta \\ &\quad + 2\rho \sin^2 \phi \sin^2 \theta + 2\rho \cos^2 \theta \sin^2 \phi + 2\rho \cos^2 \phi \sin^2 \theta + 2\rho \cos^2 \phi \cos^2 \theta \\ &= \rho^2 \cos \theta + \frac{\phi \rho}{3} + 2\rho \end{aligned}$$

The rest of the problem we leave to the student.

13.1 Integrals in Polar, Cylindrical and Spherical coordinates

13.1.1 Polar coordinates

Problem 51. Compute the double integral. The integrals are set up to be easy in polar coordinates.

1. $\iint_{\mathcal{S}} (x+y) dx dy$, where \mathcal{S} is the region left of the y -axis and between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.
2. $\iint_{\mathcal{S}} (x+y) dx dy$, where \mathcal{S} is the sector region in the first quadrant locked between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$, and the lines $-\sqrt{3}x + y = 0$, $x - \sqrt{3}y = 0$.

13.1.2 Cylindrical coordinates

Problem 52. Compute the triple integral. The integrals are set up to be easy in cylindrical coordinates.

1. $\iiint_{\mathcal{S}} \sqrt{x^2 + y^2} dx dy dz$, where \mathcal{S} is the solid conical body with vertical axis along the z axis, pointing upwards, of height 1 and with circular base of radius 1 lying on the xy -plane.

13.1.3 Spherical coordinates

Problem 53. 1. Find the centroid of a semi-ball of radius R whose base is a circle in the x, y -plane.

2. Integrate $\iiint_{\mathcal{S}} z dx dy dz$, where \mathcal{S} is the semi-ball from the previous point.

14 2D Field Potential

Problem 54. Show the field \mathbf{F} has scalar potential f ; find f . Compute the indicated line integral.

We recall that f is scalar potential for \mathbf{F} if $\mathbf{F} = \nabla f = (f_x, f_y)$.

1. $\mathbf{F} = (3 + 2xy)\mathbf{i} + (x^2 - 3y^2)\mathbf{j}$. Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is any curve from $(1, 0)$ to $(0, 1)$.
2. $\mathbf{F} = (e^{xy}y + 2x)\mathbf{i} + (e^{xy}x + 1)\mathbf{j}$. Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is any curve from $(1, 0)$ to $(0, 1)$.

15 Green's Theorem

15.1 Line Integrals via Greens Theorem

Problem 55. Compute the line integral.

1. $\int_C \left(-y^3 + e^{x^3 + \arctan x} \right) dx + \left(x^3 + \arctan \left(e^{y^2} + y \right) \right) dy$, where C is the positively oriented boundary of the unit disk.
2. $\int_C (\sin(\cos x) + y)dx + (\cos(\sin y) + 2x)dy$, where C is the oriented curve consisting of the four segments connecting $(0, -1)$, $(1, 0)$, $(0, 1)$, $(-1, 0)$, $(0, -1)$ in this order.

16 Surface Integrals

16.1 Surface area

Problem 56. 1. Derive the formula for the surface area of the sphere with radius R .

2. Find the surface area of a torus with major radius R and minor radius r , $r < R$.

16.2 Flux

Problem 57. Find the flux of the field \mathbf{F} through the surface \mathcal{S} .

1. $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, \mathcal{S} is the hemisphere, oriented upwards, with radius 1, whose axis is the z -axis and whose base lies in the x, y -plane.
2. $\mathbf{F} = x^2\mathbf{i}$, where \mathcal{S} is the part of the paraboloid with equation $z = 1 - x^2 - y^2$ that lies above the x, y -plane. \mathcal{S} is oriented upwards.

16.3 Divergence Theorem

Problem 58. Use the divergence theorem to compute the flux of \mathbf{F} through the surface \mathcal{S} .

1. $\mathbf{F} = \mathbf{k}$, \mathcal{S} is the hemisphere, oriented upwards, with radius 1, whose axis is the z -axis and whose base lies in the x, y -plane.
2. $\mathbf{F} = (x^2 + y^2)\mathbf{k}$, where \mathcal{S} is the part of the paraboloid with equation $z = 1 - x^2 - y^2$ that lies above the x, y -plane. \mathcal{S} is oriented upwards.