# Calculus I
# Lecture 8
# Derivatives

Todor Milev
https://github.com/tmilev/freecalc

2020

# Outline

1. Tangents

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1. Tangents
2. Derivatives
   - Other Notations
   - The Derivative as a Function
   - Velocities
   - Differentiability
   - How Can a Function Fail to be Differentiable?
   - Higher Derivatives

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3. Differentiation Formulas
   - Power Functions

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3. Differentiation Formulas
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4. Balls, spheres, circles, disks and differentiation

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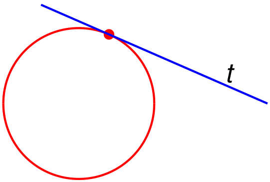


- A tangent is a line that touches a curve.
- Moreover, a tangent should have the same “direction” as the curve at the point of contact.
- For a circle, a tangent is a line that intersects the circle at exactly one point.

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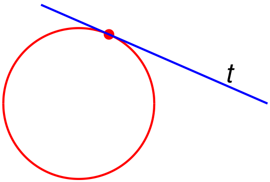


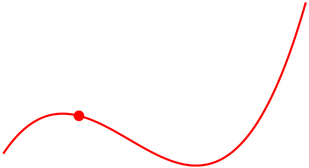

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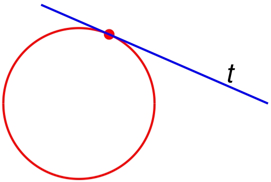


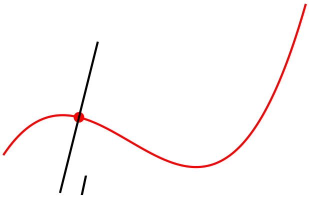


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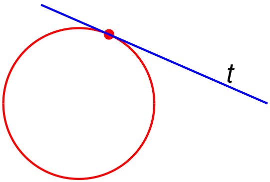


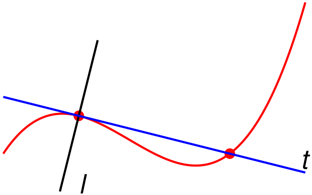


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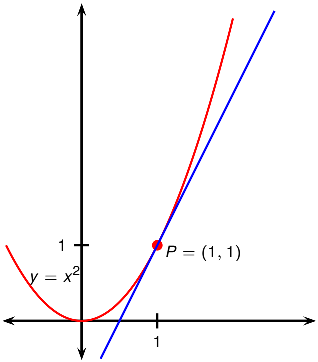
- The line $l$ intersects the curve at exactly one point, but it doesn't look like a tangent.
- The line $t$ does look like a tangent, but it intersects the curve at two points.




| $x$ | $m_{PQ}$ | $x$ | $m_{PQ}$ |
|---|---|---|---|
| 2 | | 0 | |
| 1.5 | | 0.5 | |
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| 1.1 | | 0.9 | |
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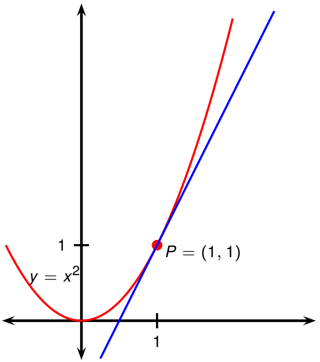


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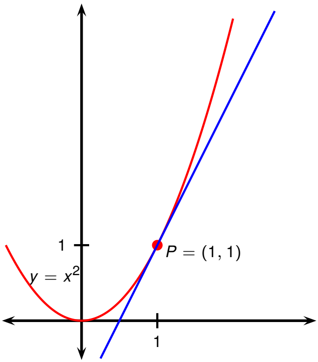


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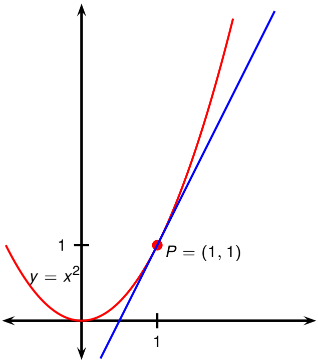


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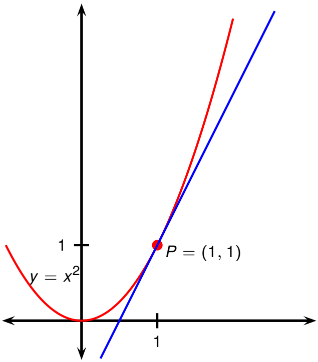


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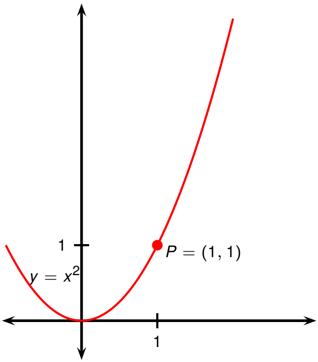


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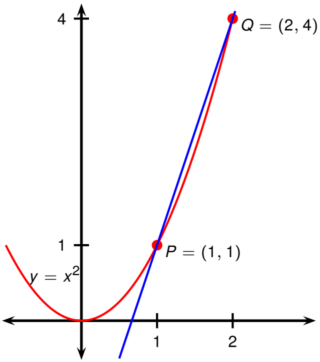


| $x$ | $m_{PQ}$ | $x$ | $m_{PQ}$ |
|---|---|---|---|
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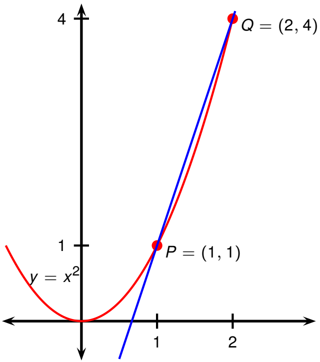


| $x$ | $m_{PQ}$ | $x$ | $m_{PQ}$ |
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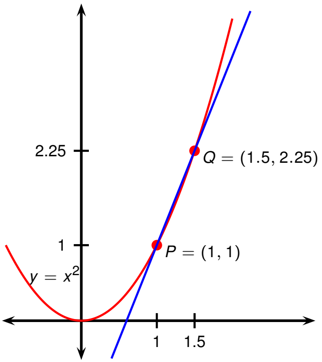


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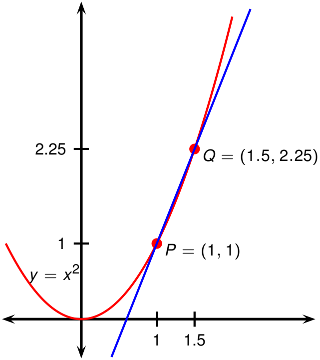


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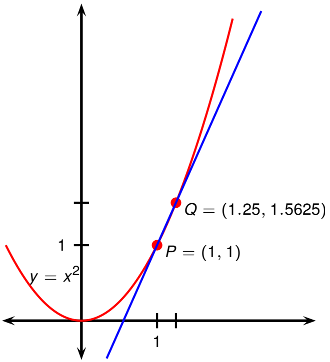


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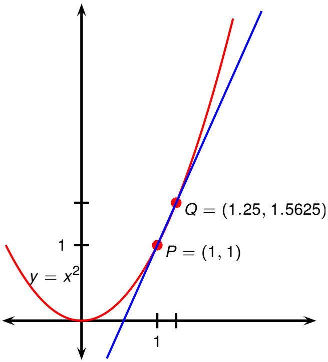


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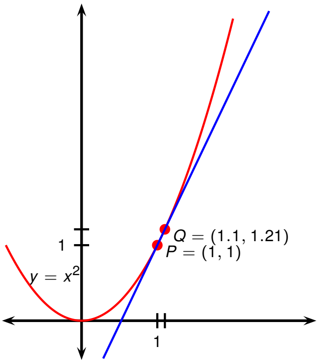


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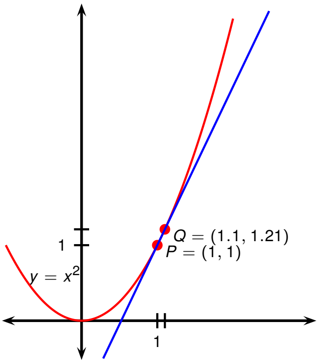


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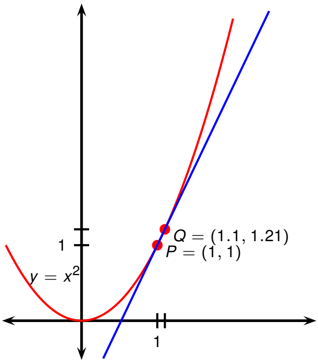


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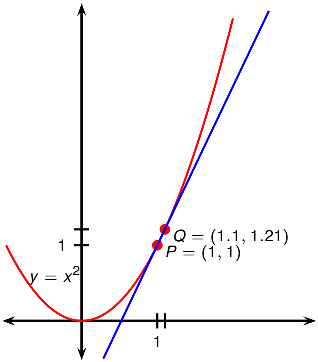


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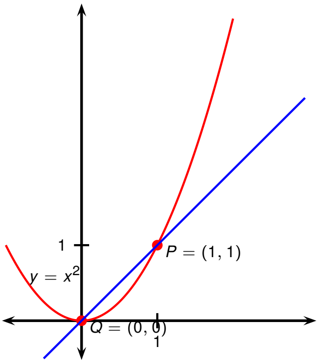


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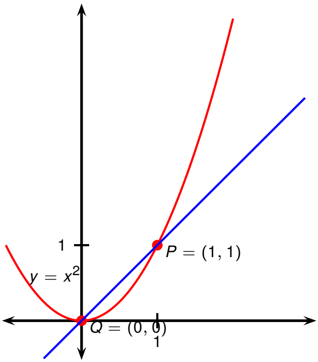


| $x$ | $m_{PQ}$ | $x$ | $m_{PQ}$ |
|---|---|---|---|
| 2 | 3 | 0 | 1 |
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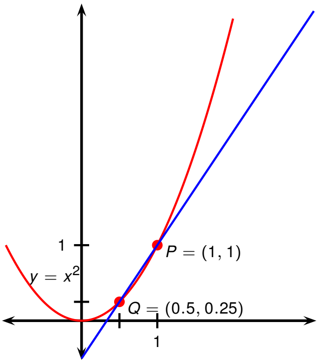


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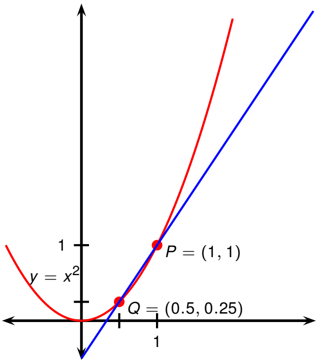


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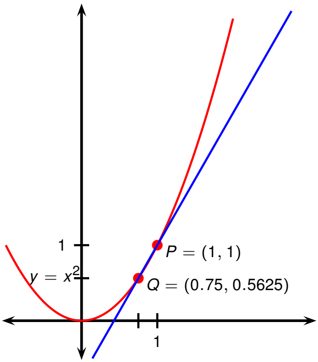


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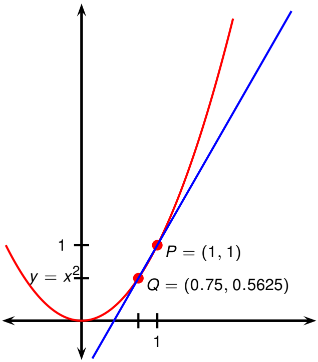


| $x$ | $m_{PQ}$ | $x$ | $m_{PQ}$ |
|---|---|---|---|
| 2 | 3 | 0 | 1 |
| 1.5 | 2.5 | 0.5 | 1.5 |
| 1.25 | 2.25 | 0.75 | 1.75 |
| 1.1 | 2.1 | 0.9 | |
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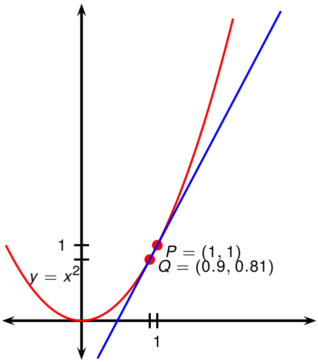


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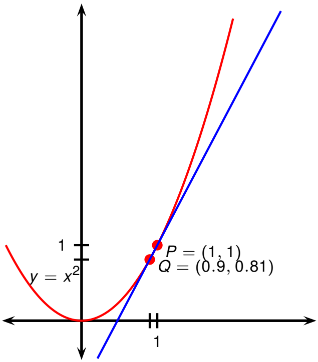


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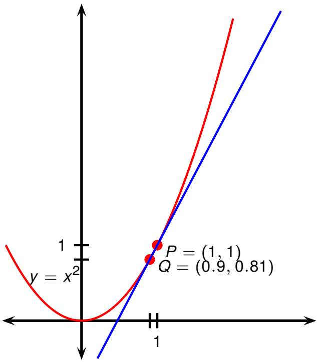


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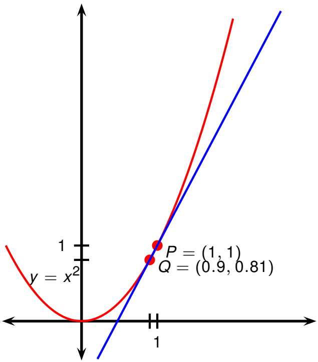


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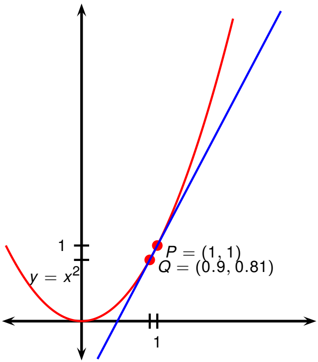


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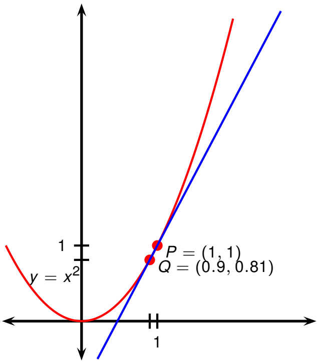


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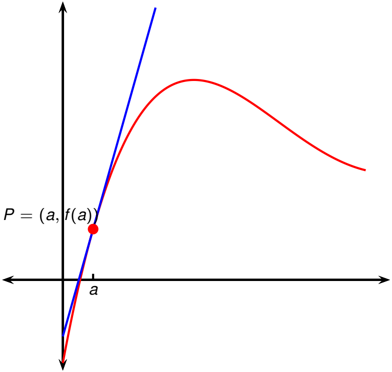
- This suggests the slope of the tangent should be 2.

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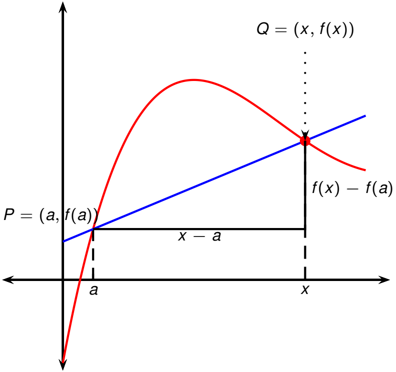


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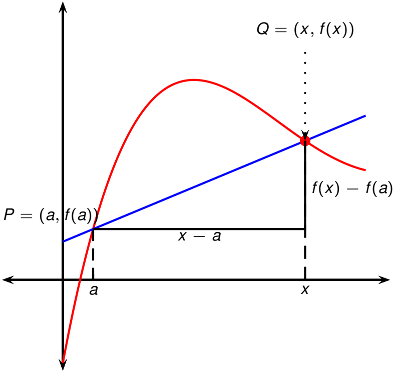


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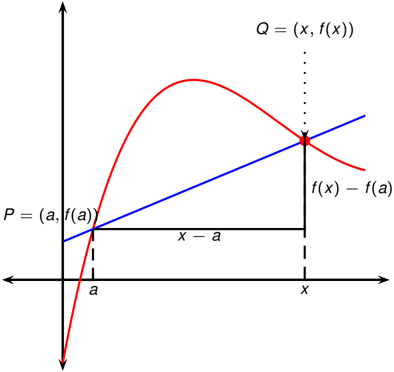


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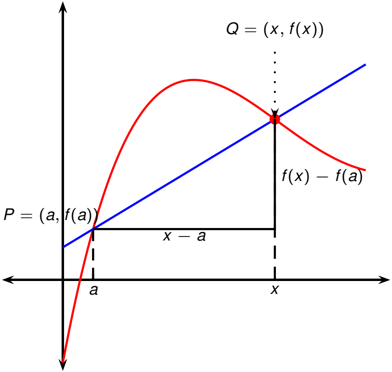


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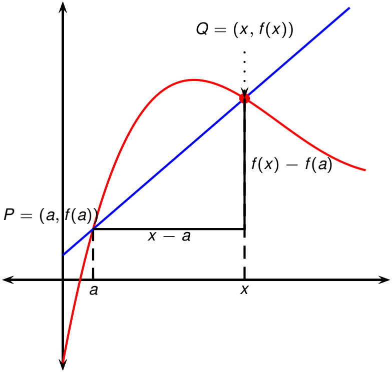


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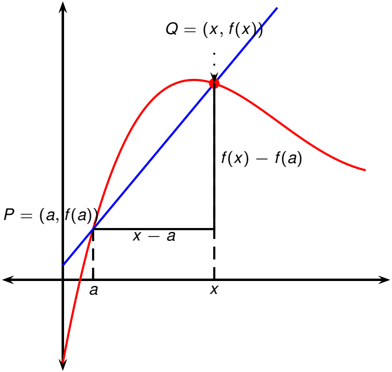


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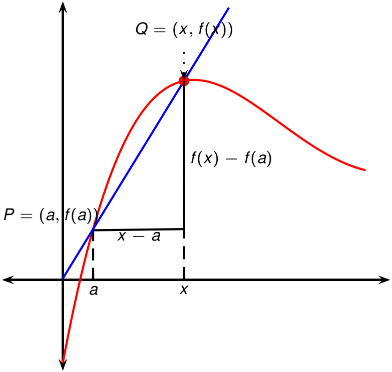


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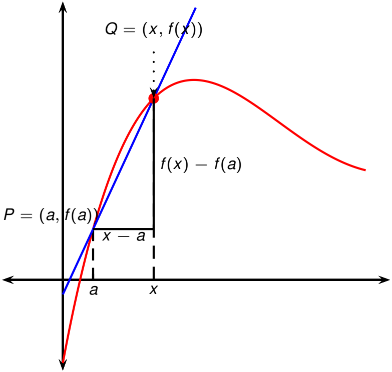


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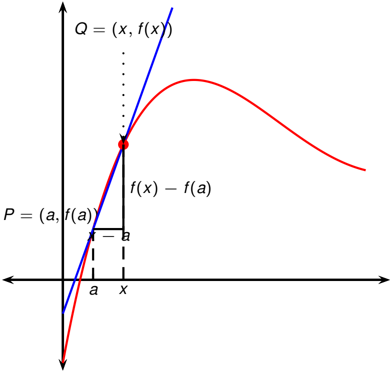


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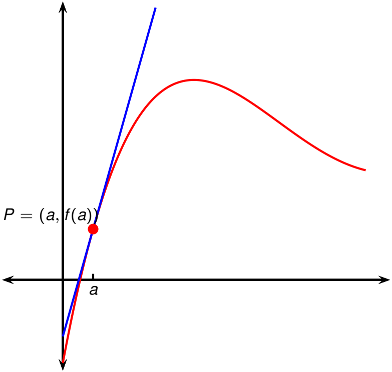


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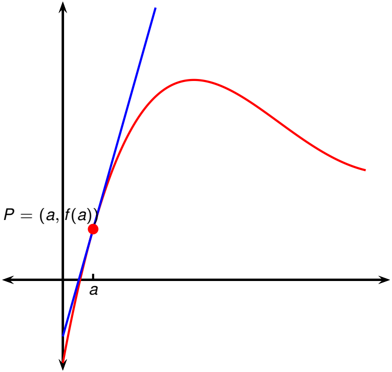
Let $P = (a, f(a))$. Suppose the limit $m = \lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$ exists. Define the tangent to $y = f(x)$ at $P$ to be the line passing through $P$ with slope $m$

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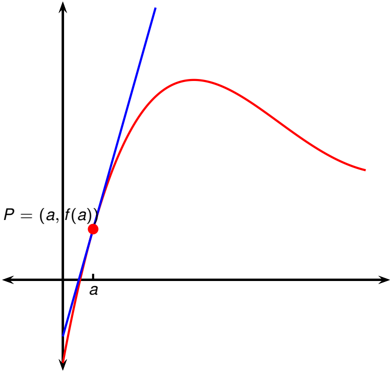


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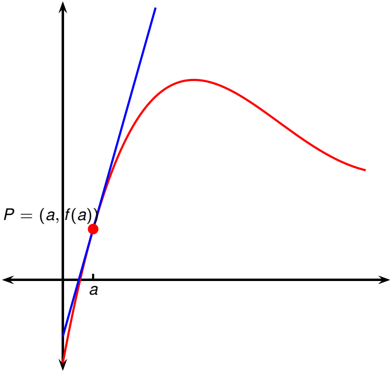


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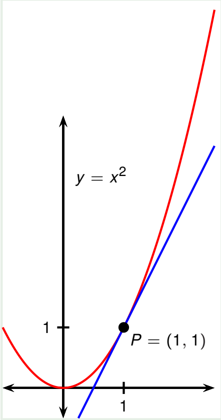
**Note.** Even if the limit does not exist a reasonable notion of a tangent line may still exist.

## Example

Find an equation for the tangent line to the parabola $y = x^2$ at the point $P = (1, 1)$.



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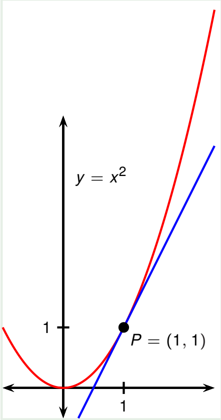


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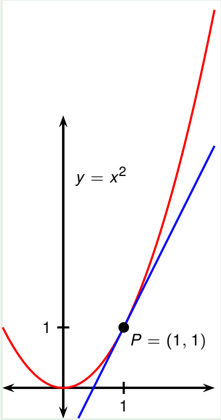


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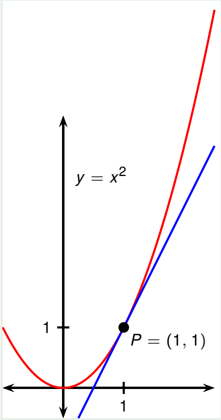


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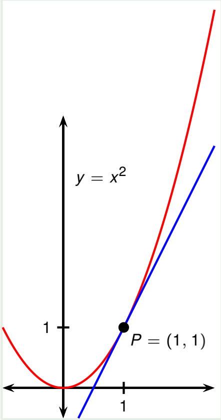


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$$\begin{aligned} m &= \lim_{x\to 1} \frac{f(x) - f(1)}{x - 1} \\ &= \lim_{x\to 1} \frac{x^2 - 1}{x - 1} \\ &= \lim_{x\to 1} \frac{(\textbf{?} \quad)(\textbf{?} \quad)}{x - 1} \end{aligned}$$

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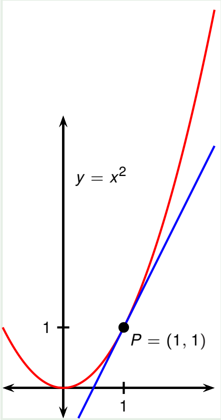


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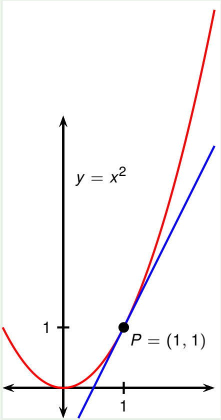


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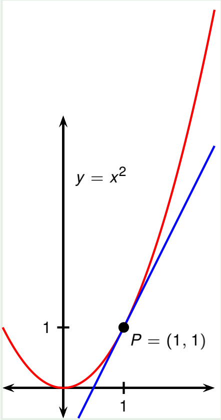


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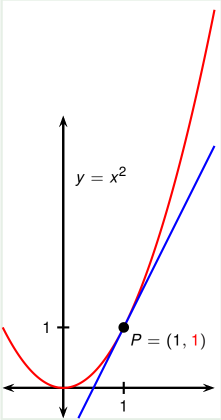


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Point-slope form: $y - 1 = 2(x - 1)$

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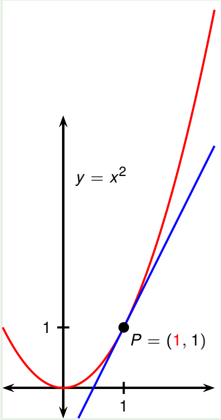


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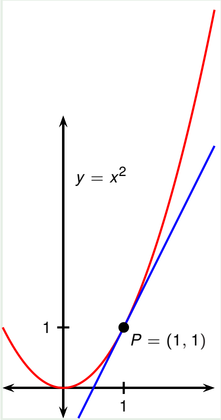


Here $a = 1$ and $f(x) = x^2$.

$$\begin{aligned} m &= \lim_{x\to 1} \frac{f(x) - f(1)}{x - 1} \\ &= \lim_{x\to 1} \frac{x^2 - 1}{x - 1} \\ &= \lim_{x\to 1} \frac{\cancel{(x - 1)}(x + 1)}{\cancel{x - 1}} \\ &= \lim_{x\to 1} (x + 1) = 1 + 1 = 2 \end{aligned}$$

Point-slope form: $y - 1 = 2(x - 1)$

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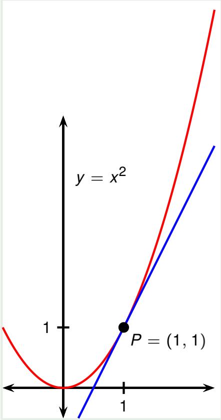


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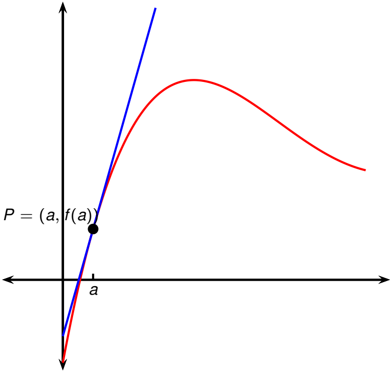
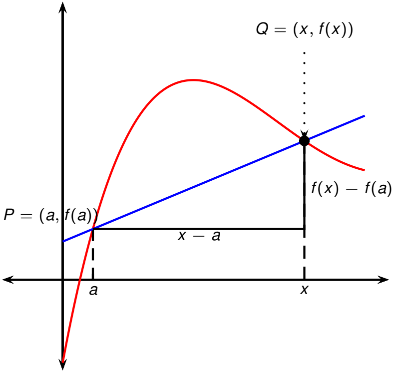
Point-slope form: $y - 1 = 2(x - 1)$, or finally $y = 2x - 1$.




- There is an equivalent expression for the slope of the tangent.

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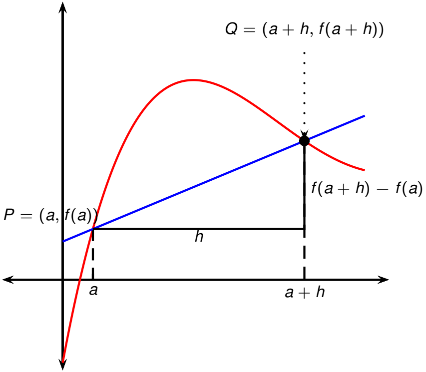


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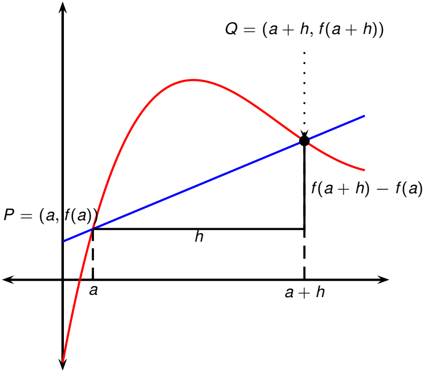


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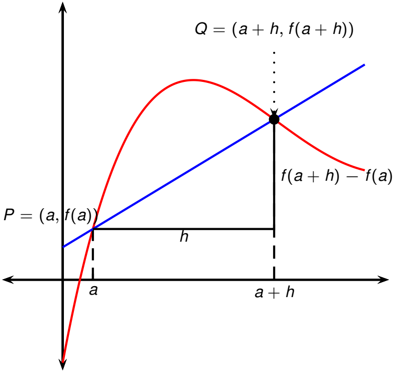


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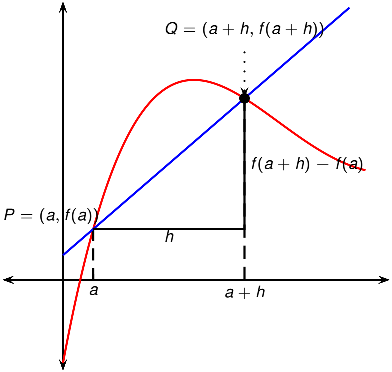


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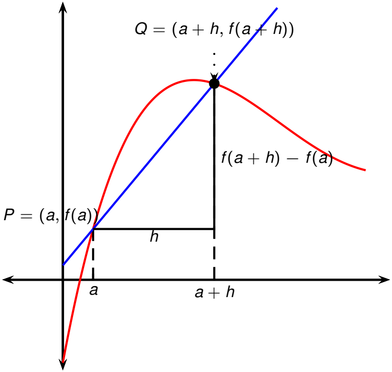


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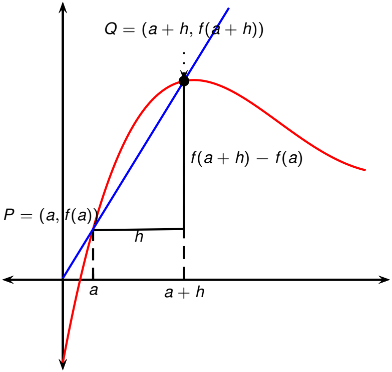


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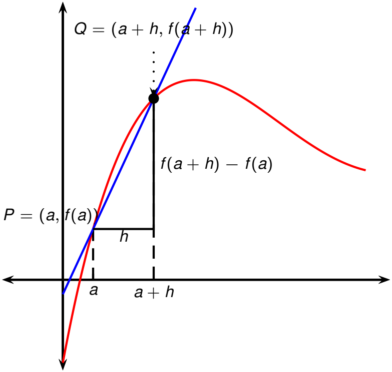


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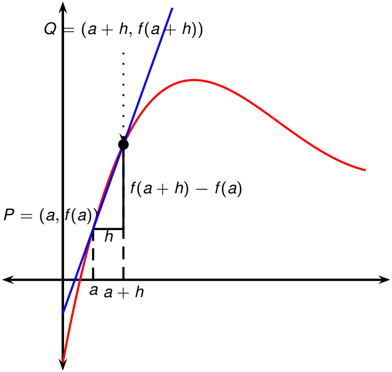


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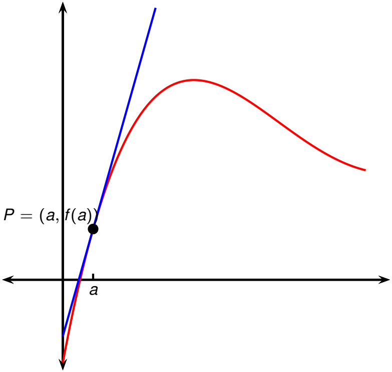


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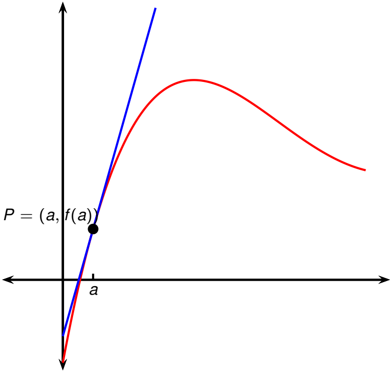


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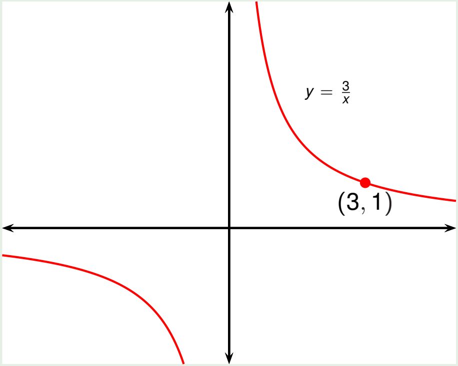
Tangent slope - equivalent expression:

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

## Example

Find an equation for the tangent line to the hyperbola $y = \frac{3}{x}$ at the point $(3, 1)$.



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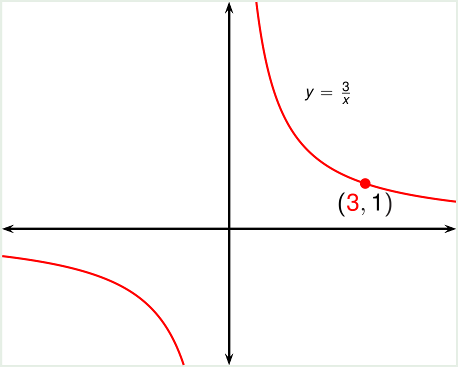


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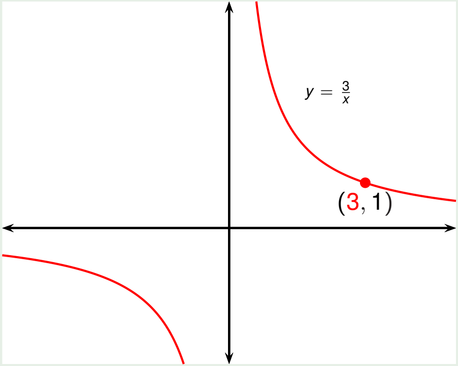


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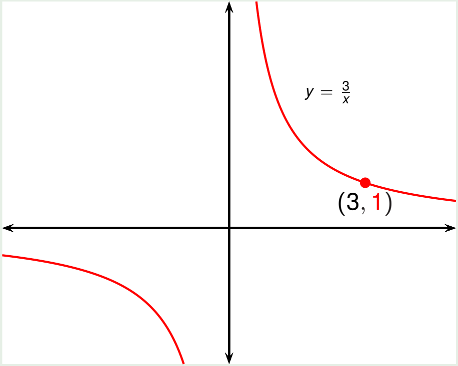


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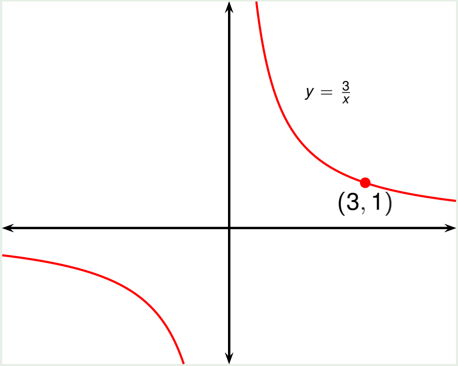


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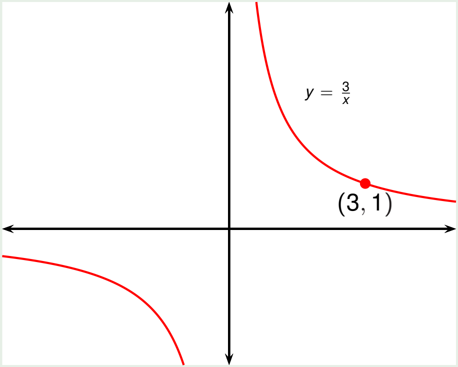


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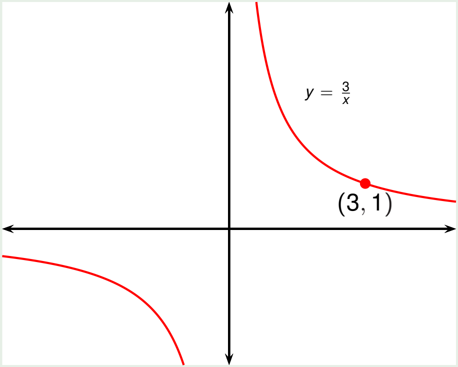


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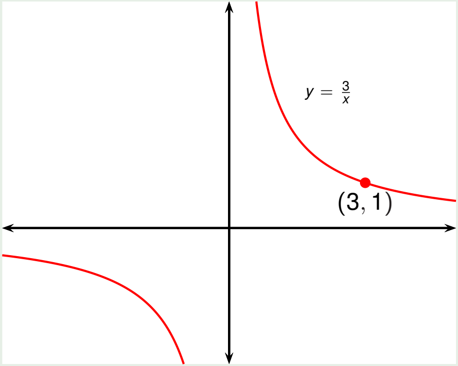


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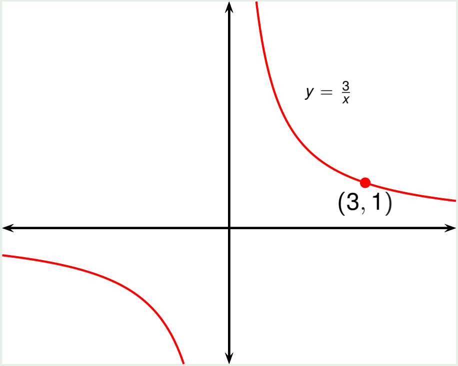


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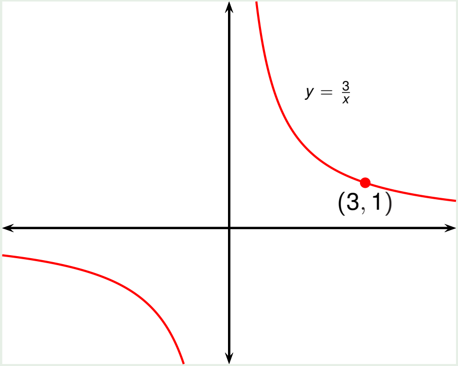


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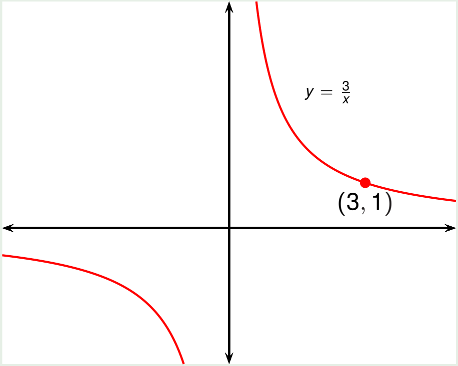


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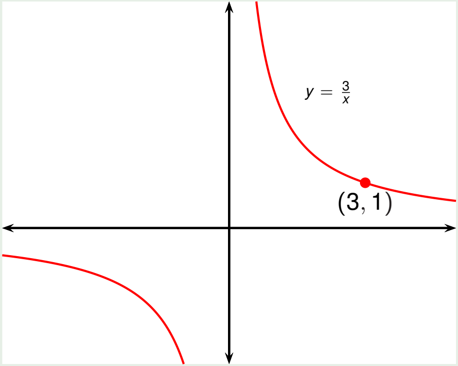


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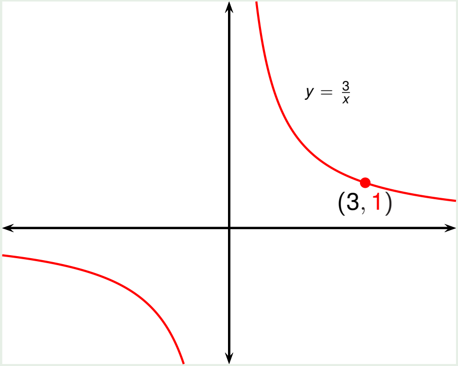


Point-slope form: $y - 1 = -\frac{1}{3}(x - 3)$

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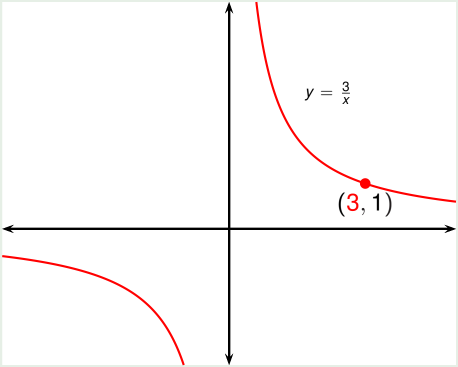


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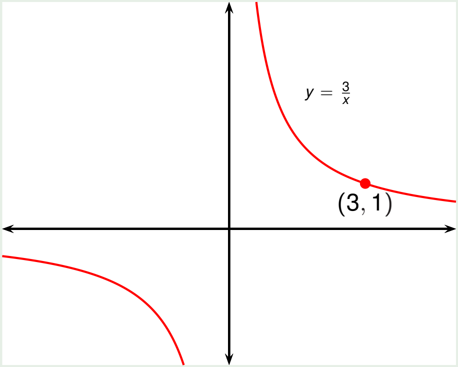


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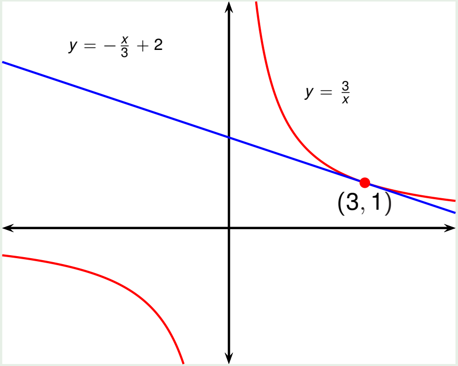


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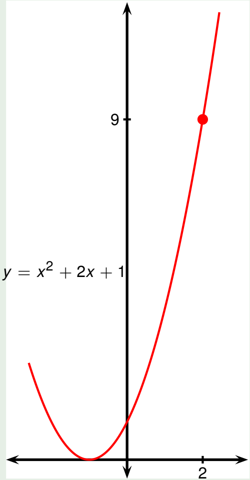
## Example (Tangent line to a polynomial)

Find an equation for the tangent line to the parabola $y = x^2 + 2x + 1$ at the point $P = (2, 9)$.



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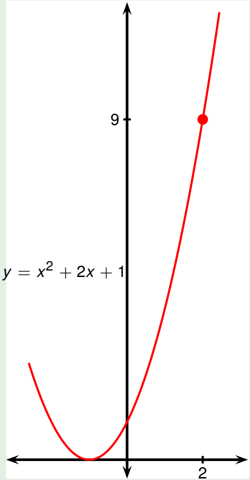


Here $a =$ **?** and $f(x) = x^2 + 2x + 1$.

$$m \quad \lim_{x \to ?} \frac{f(x) - f(?)}{x - ?}$$

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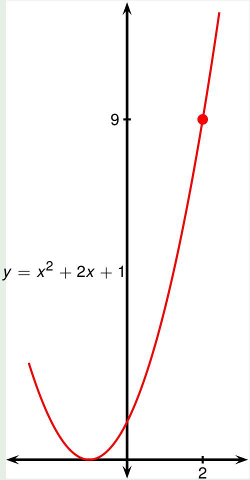


Here $a = 2$ and $f(x) = x^2 + 2x + 1$.

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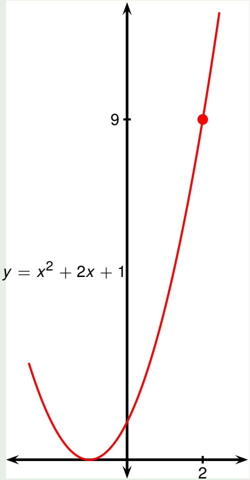


Here $a = 2$ and $f(x) = x^2 + 2x + 1$.

$$m = \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2}$$

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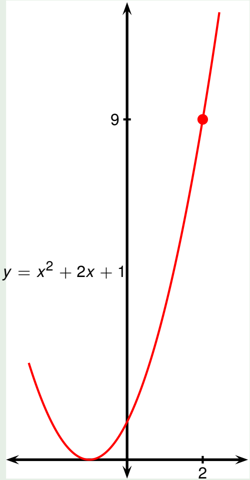


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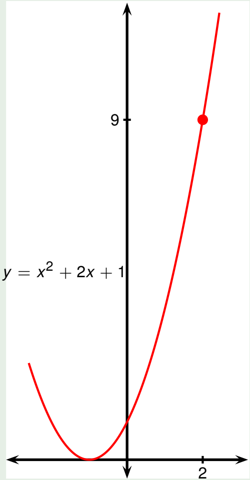


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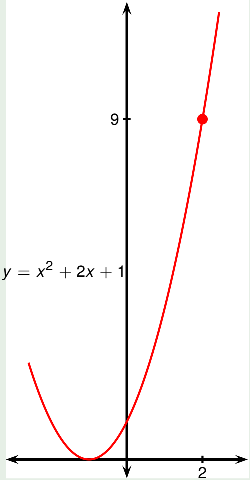


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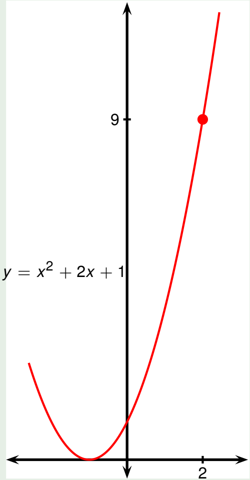


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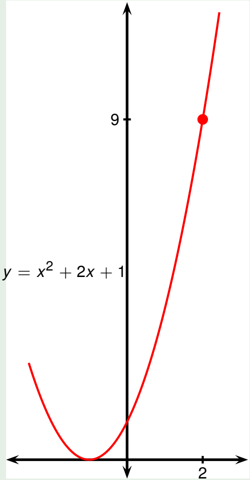


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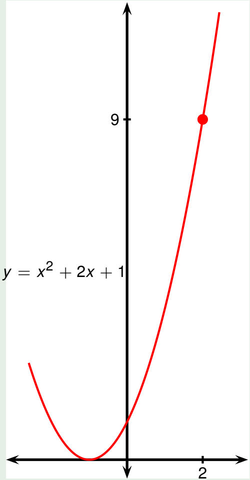


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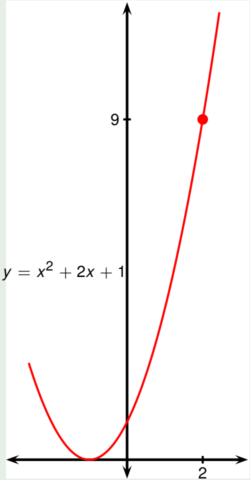


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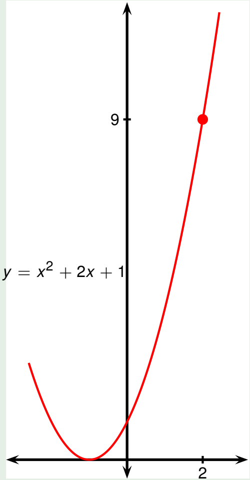


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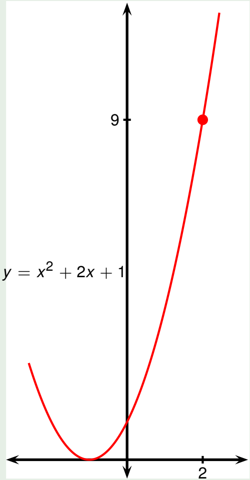


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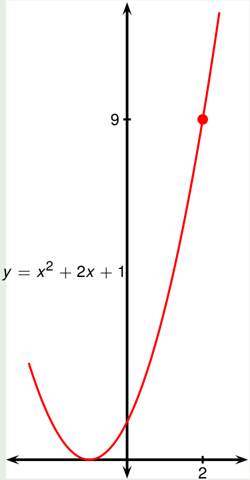


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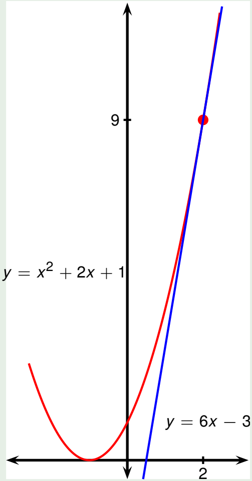


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The derivative of a function $f$ at a number $a$, denoted by $f'(a)$, is

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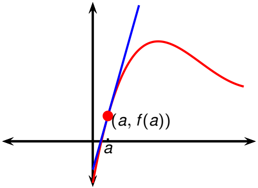

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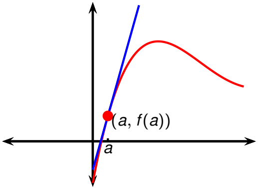


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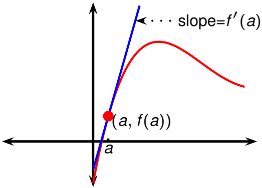


- The two alternative formulas result in equivalent definitions.
- Equivalent formulation. The derivative $f'(a)$ is the slope of the tangent line to $y = f(x)$ at $(a, f(a))$, provided that tangent line exists and is non-vertical.

## Example

Find the derivative of the function $f(x) = x^2 - 8x + 9$ at the number $a$.

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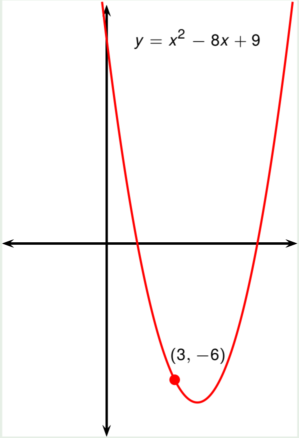

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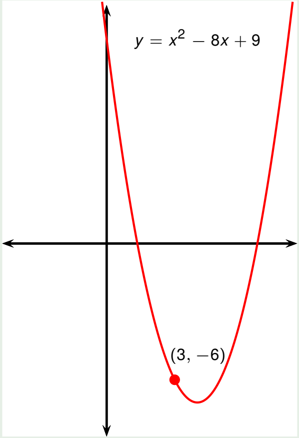


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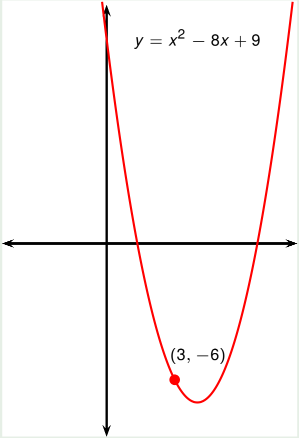


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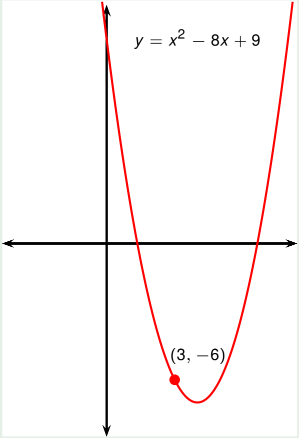


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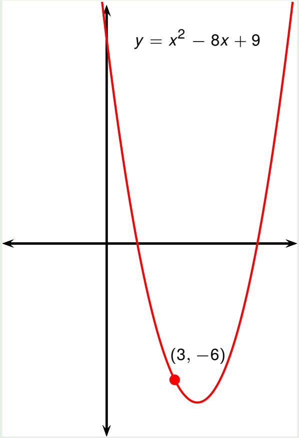


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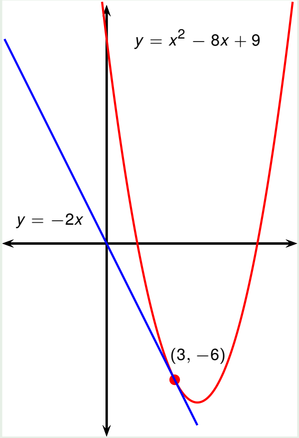


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- Therefore $f'(3) = 2 \cdot 3 - 8 = -2$.
- Point-slope form: $y - (-6) = -2(x - 3)$.
- Slope $y$-intercept form: $y = -2x$.

# Other Notations for Derivative

If $y = f(x)$ is a function, there are many ways to write its derivative.

$$f'(x) = y' = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}f(x) = Df(x) = D_x f(x)$$

- $\frac{\mathrm{d}}{\mathrm{d}x}$ are called differentiation operators because they indicate the operation of differentiation, which is the process of calculating the derivative.
- $\mathrm{d}y/\mathrm{d}x$ is called Leibniz notation; it means the same as $f'(x)$.
- If we want to indicate the value of the derivative $\mathrm{d}y/\mathrm{d}x$ in Leibniz notation at a point $a$, we write

$$\frac{\mathrm{d}y}{\mathrm{d}x}_{|x=a} \qquad \text{or} \qquad \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_a \qquad \text{or} \qquad \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right]_a$$

# The Derivative as a Function

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

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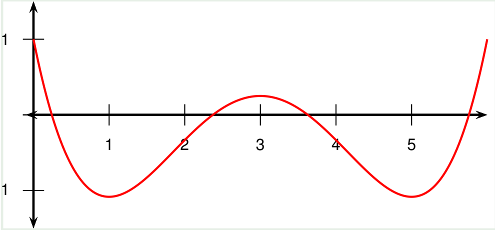
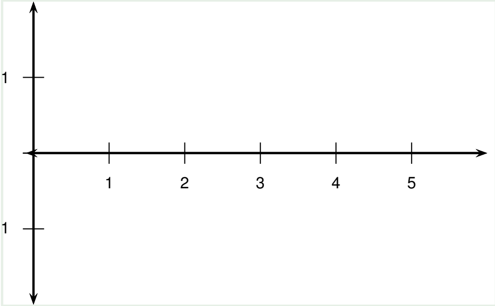
- The domain of $f'$ may be smaller than the domain of $f$.

## Example

The graph of a function $f$ appears below. Use it to sketch the graph of the derivative $f'$.



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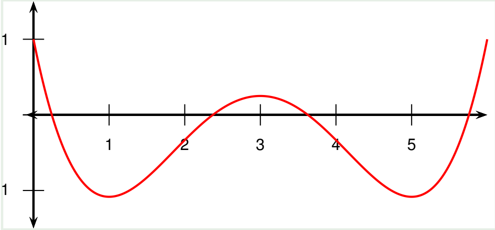


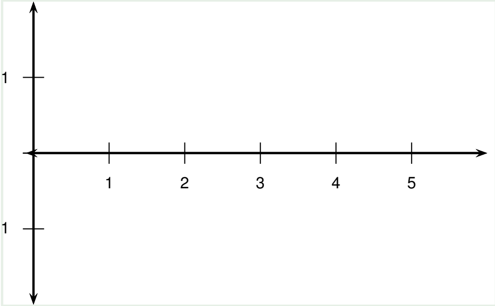


- Find the points where the tangent is horizontal ($m = 0$).

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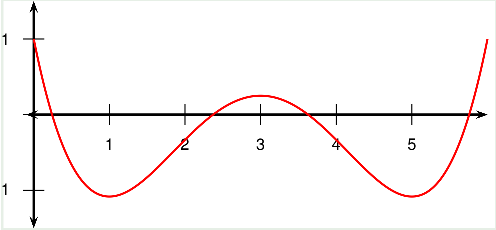


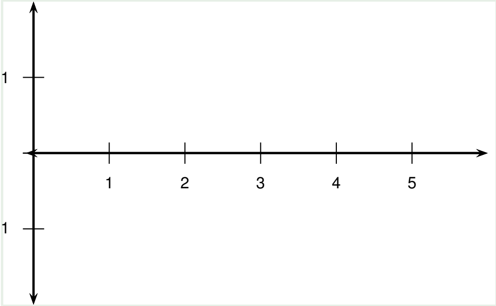


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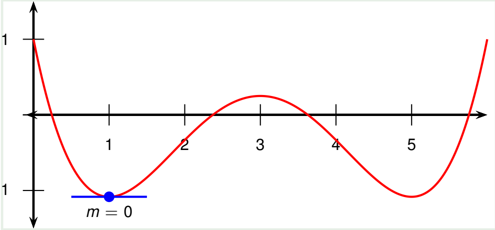


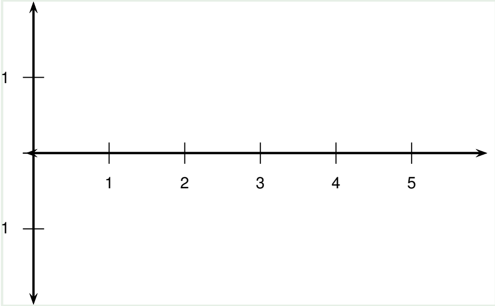


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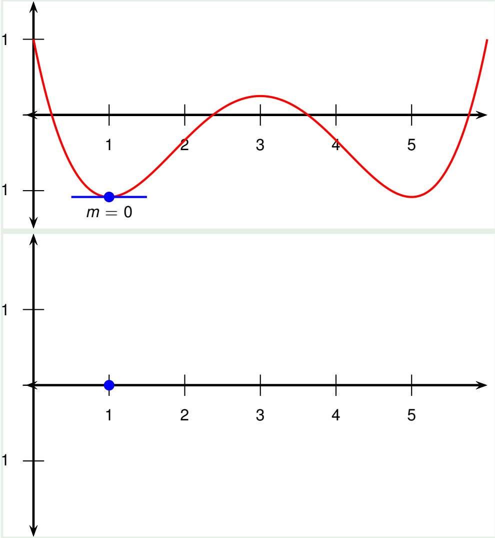


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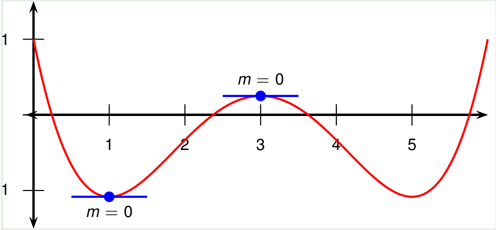


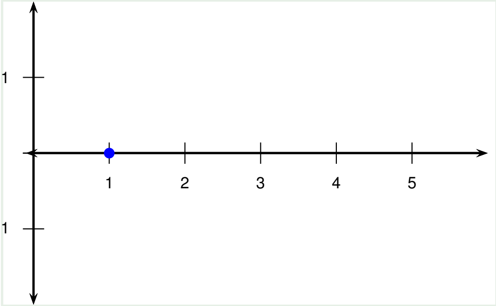


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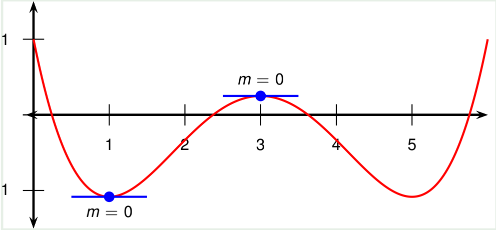


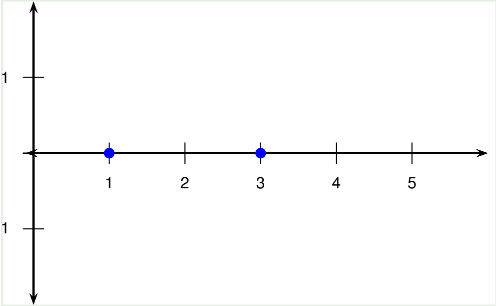


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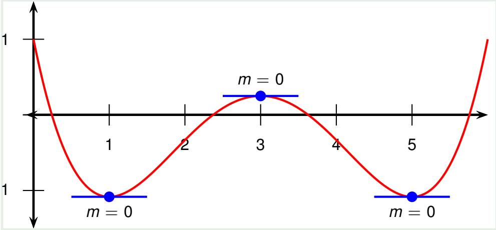


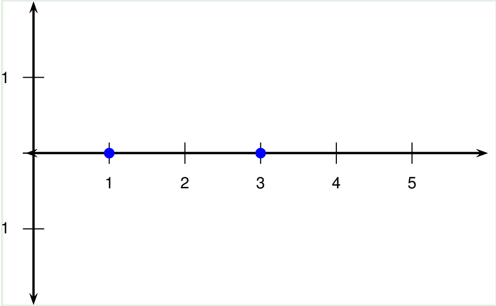


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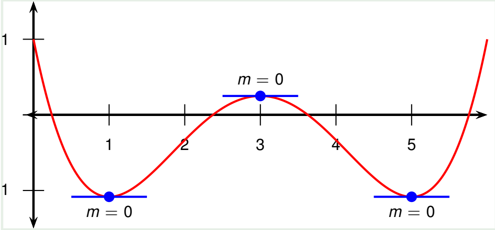


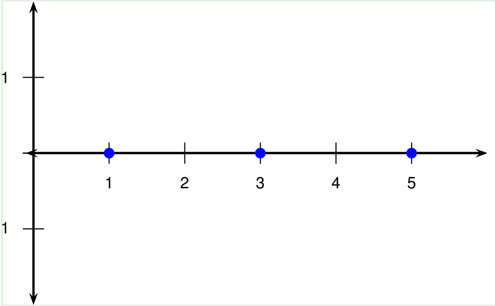


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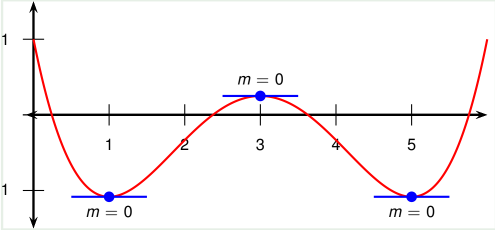


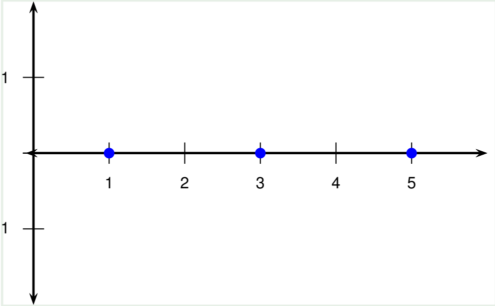


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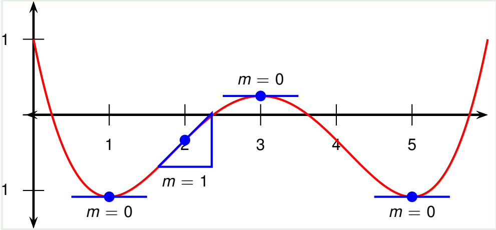


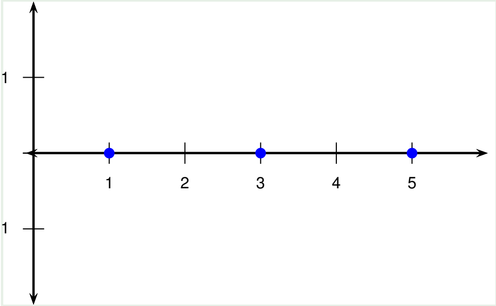


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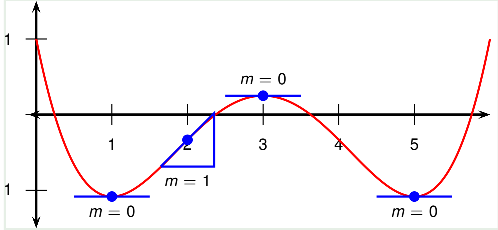


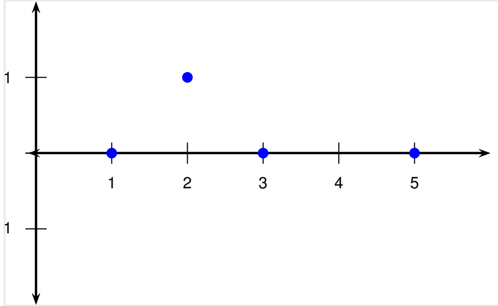


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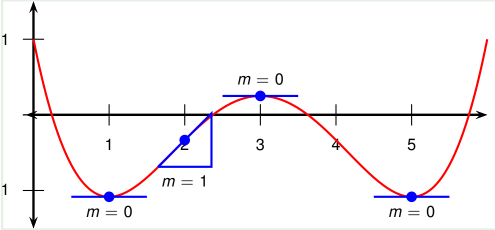


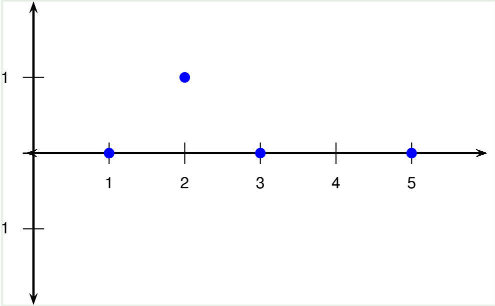


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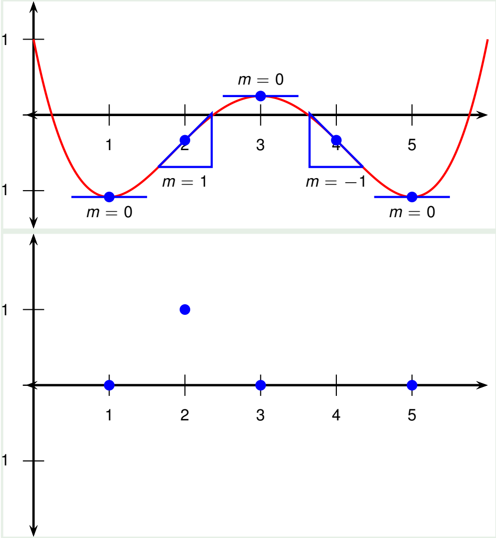


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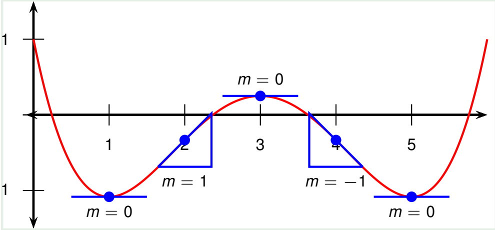


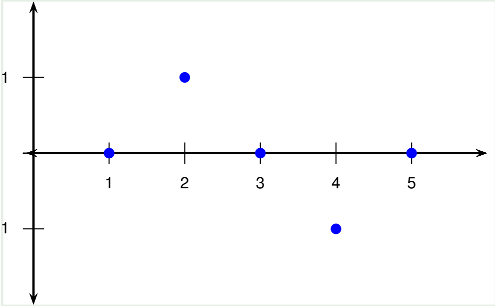


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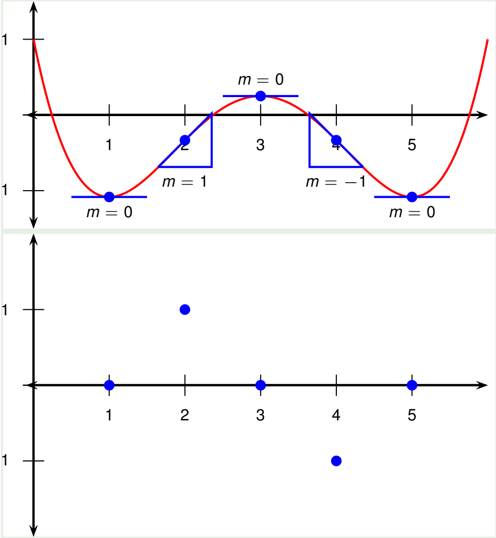


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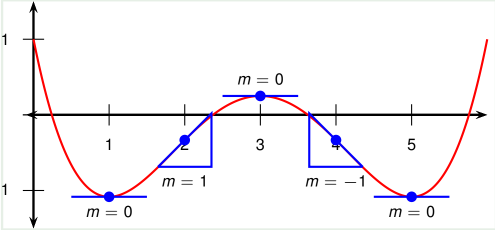


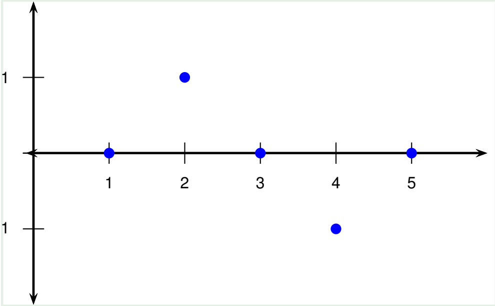


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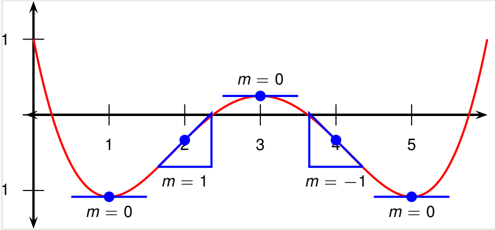


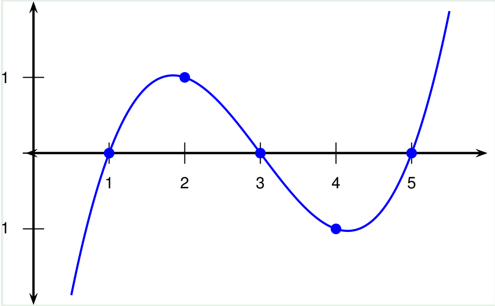


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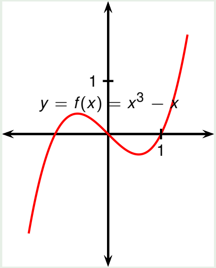
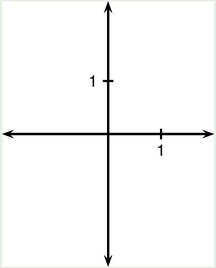
If $f(x) = x^3 - x$, find formula for $f'(x)$.



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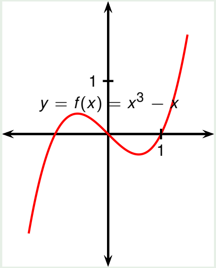


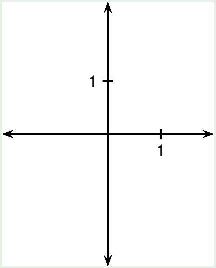


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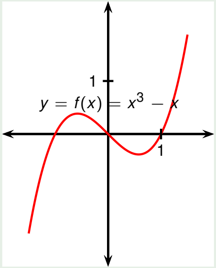


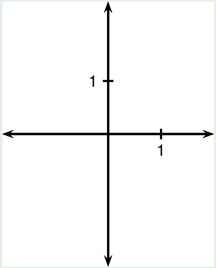


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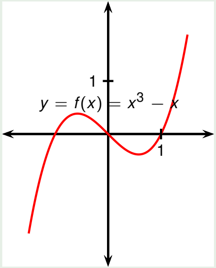


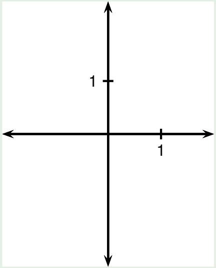


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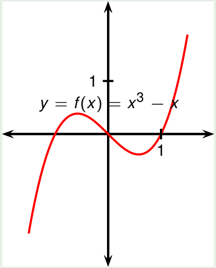


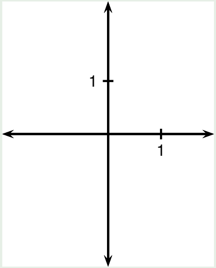


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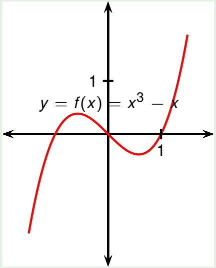


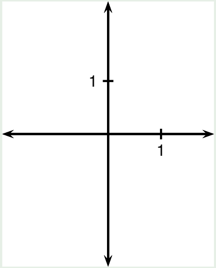


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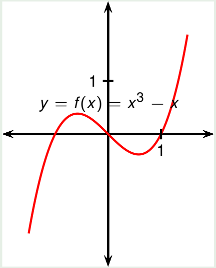


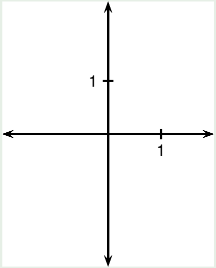


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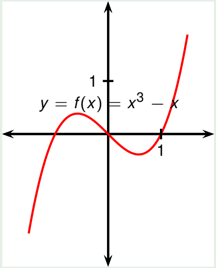


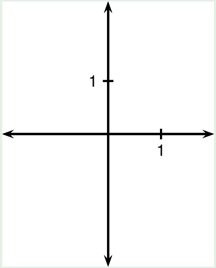


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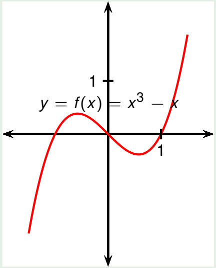


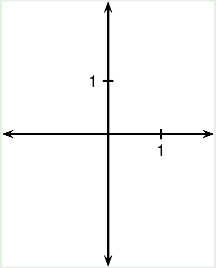


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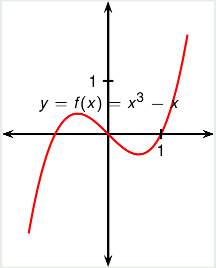


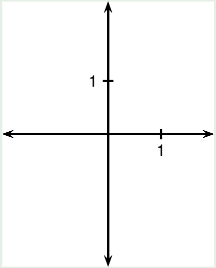


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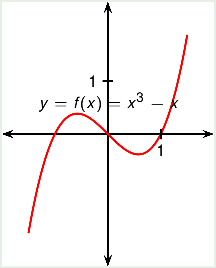


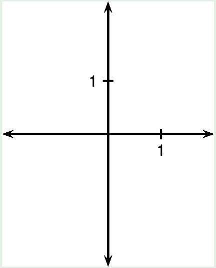


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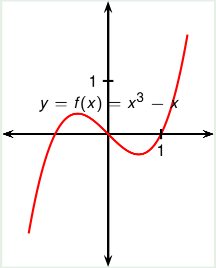


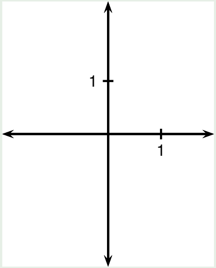


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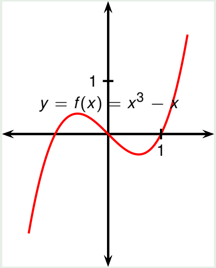


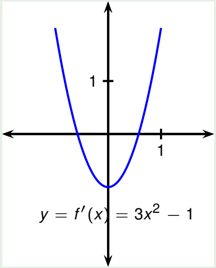


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# Velocities

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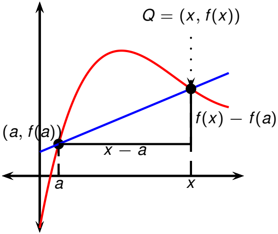


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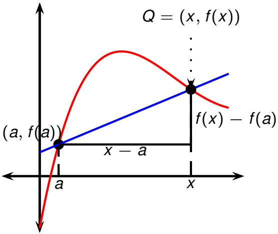


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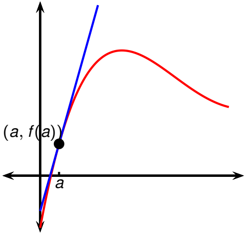


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Therefore the velocity after 5s is $v(5) = 9.8(5) = 49$m/s.

**Definition (Differentiable at a point)**

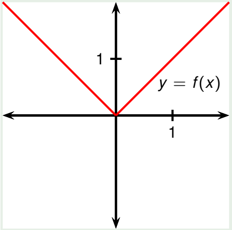
A function $f$ is differentiable at $a$ if $f'(a)$ exists.

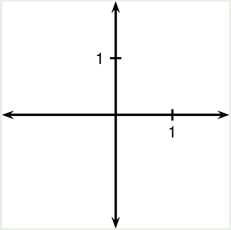
**Definition (Differentiable on an interval)**

A function $f$ is differentiable on an open interval $(a, b)$ (allowing $a = -\infty, b = \infty$) if it is differentiable at every number in the interval.

## Example

Where is the function $f(x) = |x|$ differentiable?



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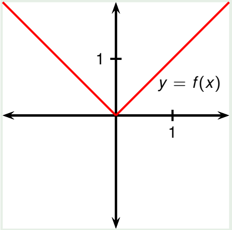


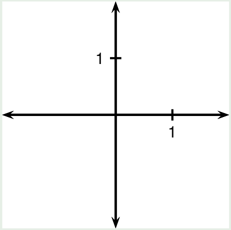


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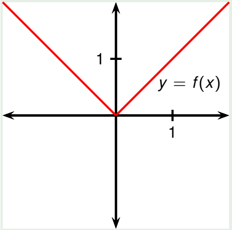


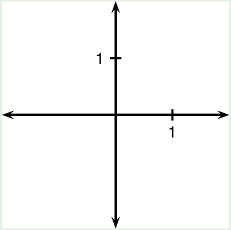


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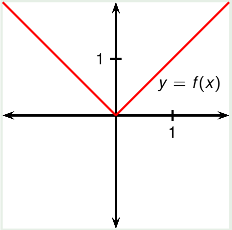


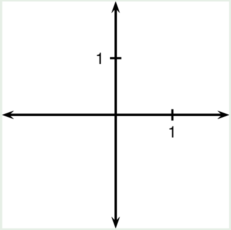


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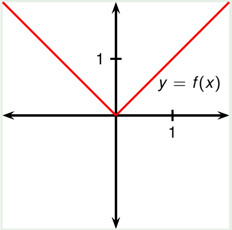


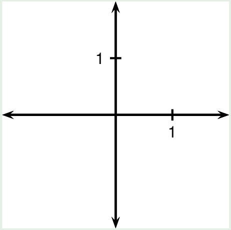


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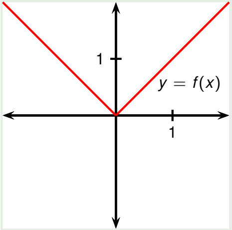


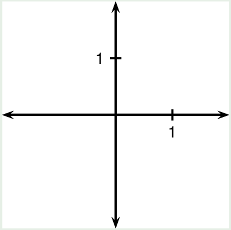


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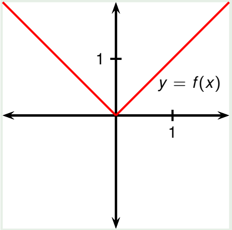


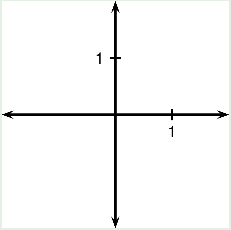


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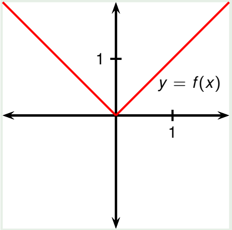


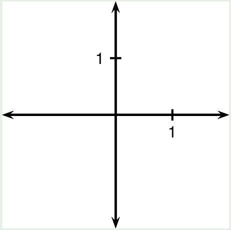


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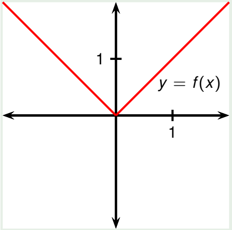


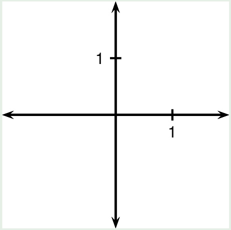


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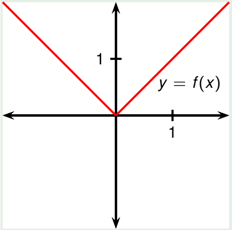


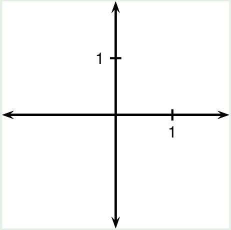


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Therefore $f$ is differentiable for any $x > 0$.

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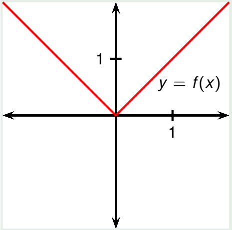


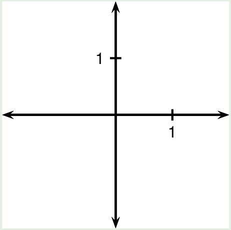


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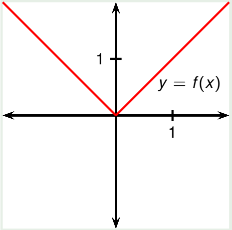


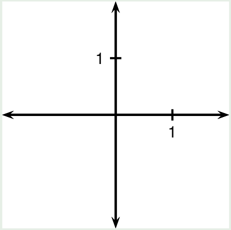


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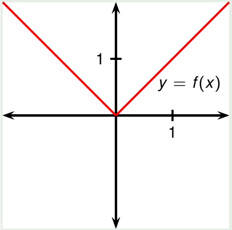


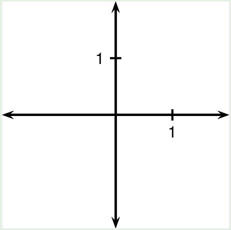


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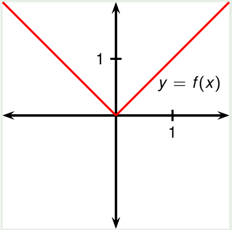


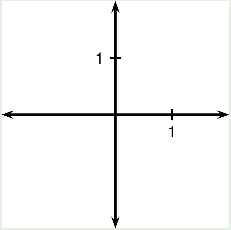


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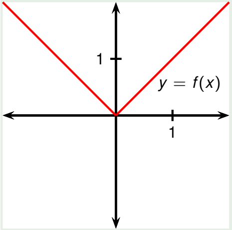


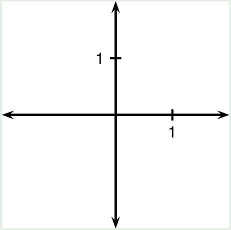


- Suppose $x < 0$.
- Then $|x| = -x$.
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$$f'(x) = \lim_{h \to 0} \frac{|x + h| - |x|}{h}$$

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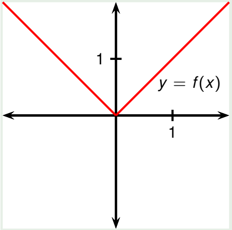


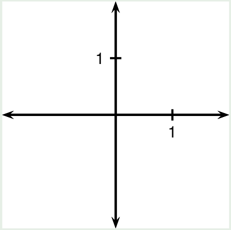


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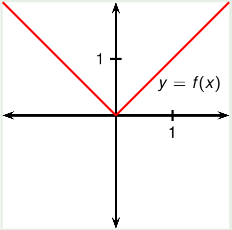


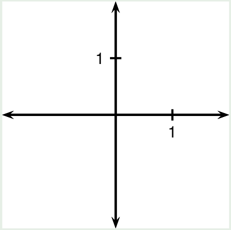


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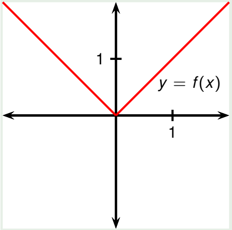


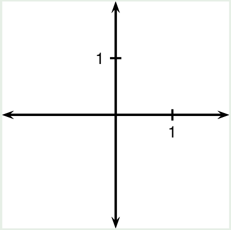


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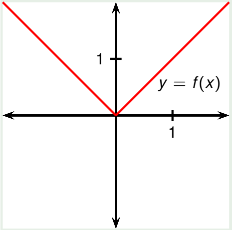


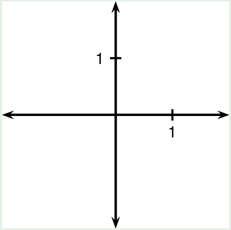


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Therefore $f$ is differentiable for any $x < 0$.

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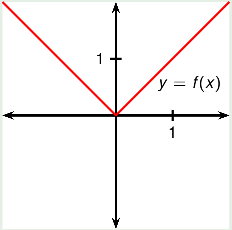


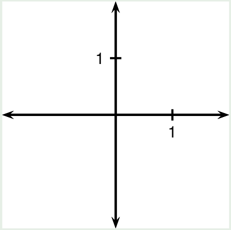


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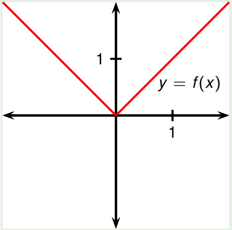


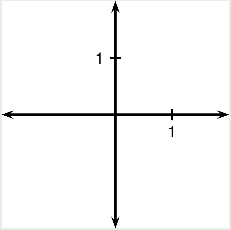


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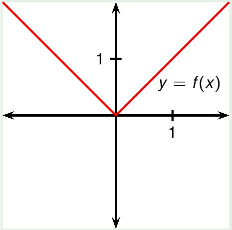


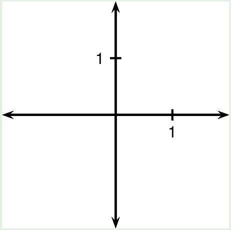


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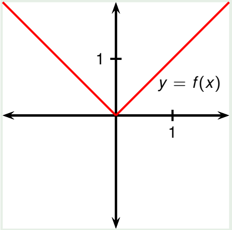


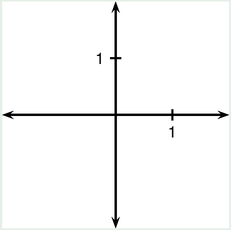


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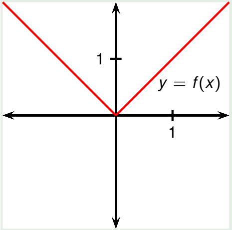


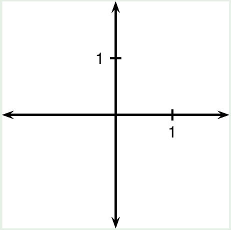


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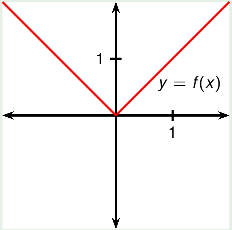


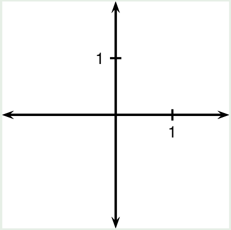


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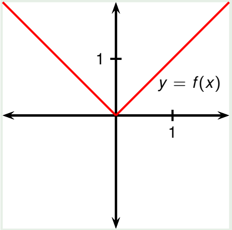


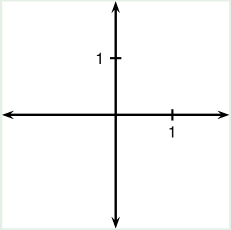


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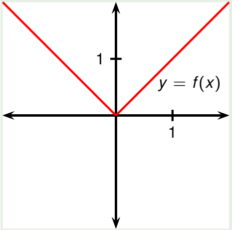


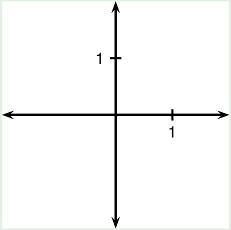


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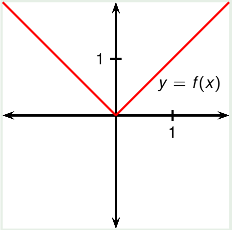


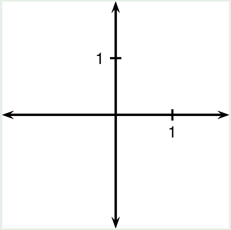


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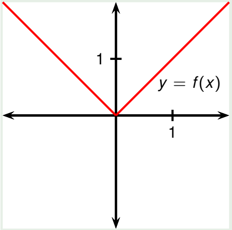


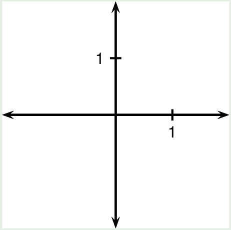


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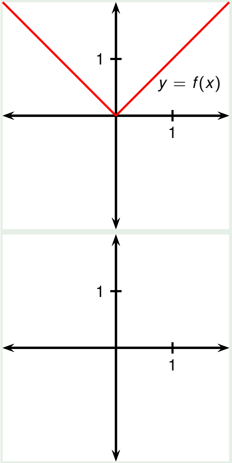


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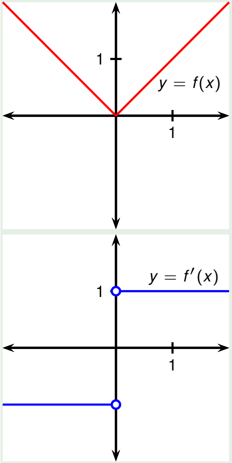


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$$f'(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

## Theorem (Differentiability Implies Continuity)

*If $f$ is differentiable at $a$, then $f$ is continuous at $a$.*

## Proof.

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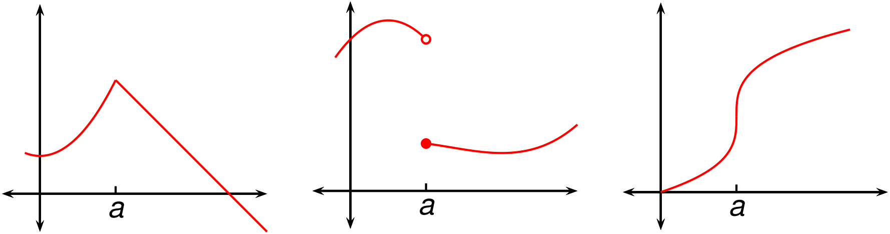
Therefore $f$ is continuous at $a$. $\square$

# How Can a Function Fail to be Differentiable?



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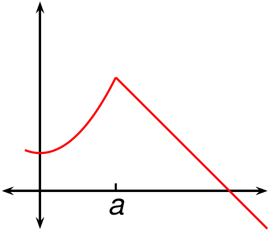
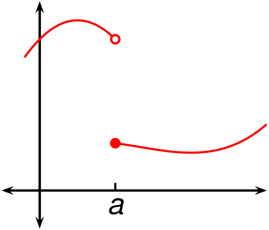
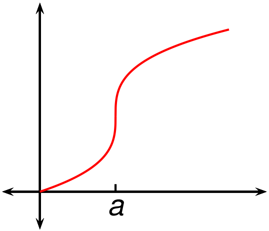



corner



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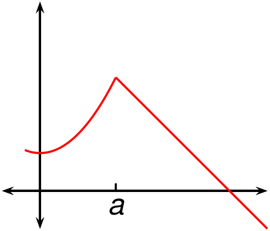


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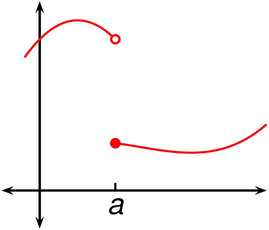
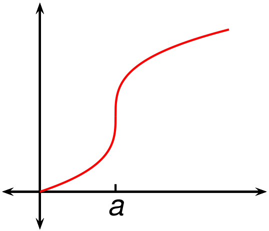



discontinuity



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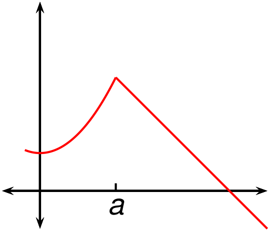


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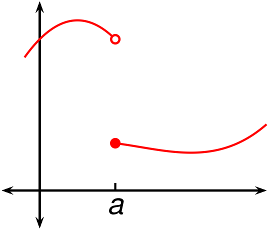


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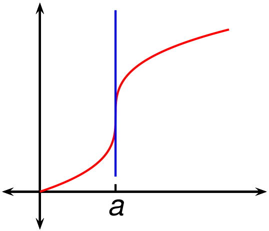


vertical tangent

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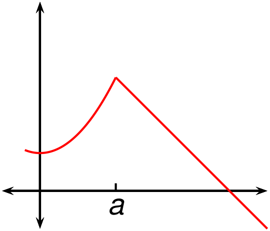


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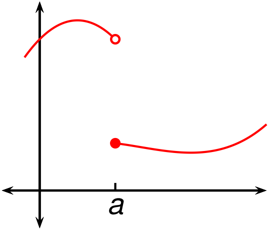


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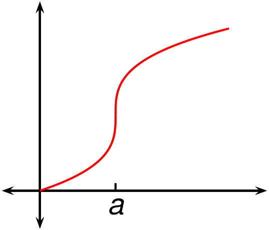


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| first derivative | $f'(x)$ | $y'$ | $\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\mathrm{d}f}{\mathrm{d}x}(x)$ | $\frac{\mathrm{d}y}{\mathrm{d}x}$ |

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**Note:** Do not confuse the superscript in the notation for $n^{th}$ derivative with exponent. The parenthesis indicate we mean derivatives rather than exponents.

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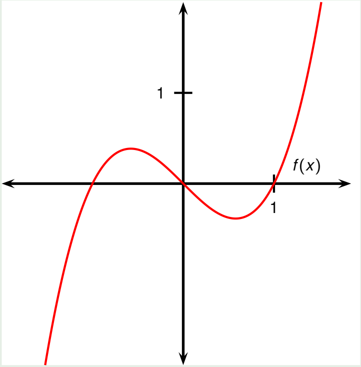
If $f(x) = x^3 - x$, find $f''(x)$.



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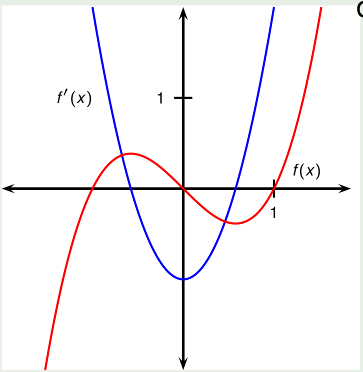


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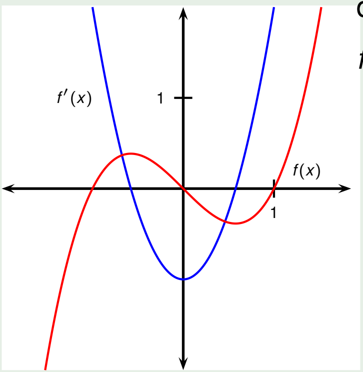


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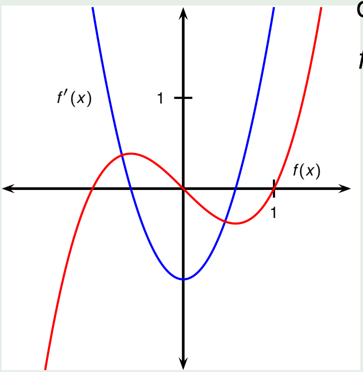


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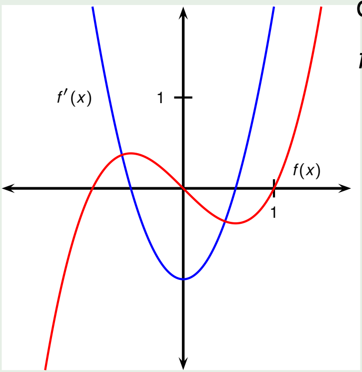


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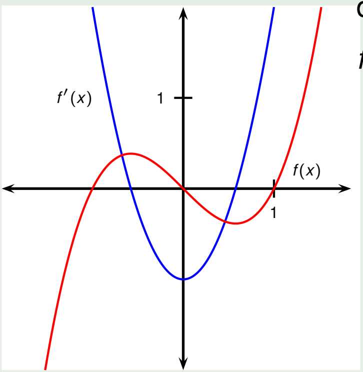


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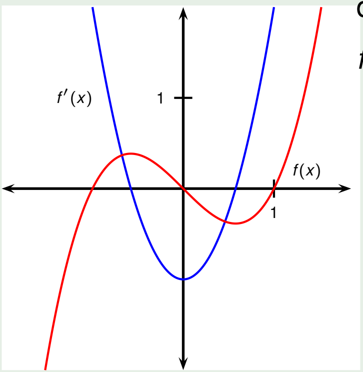


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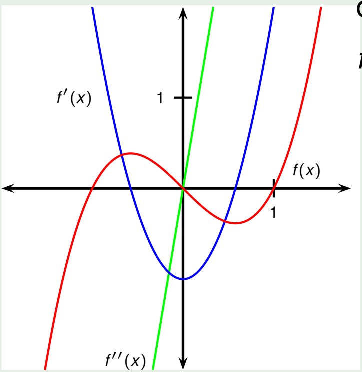


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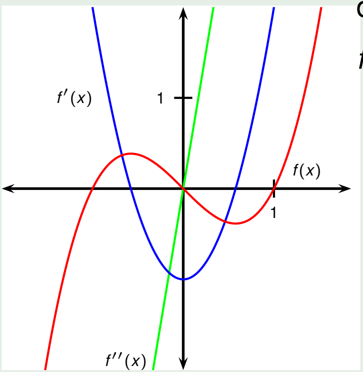


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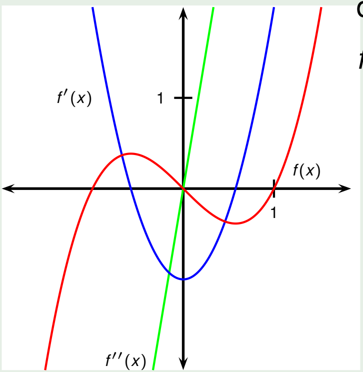


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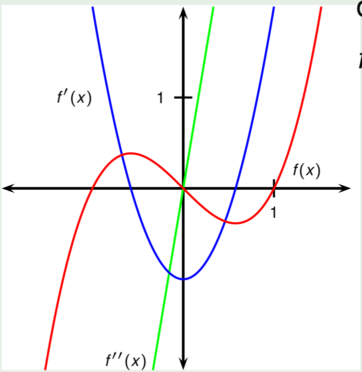


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**Theorem (Derivative of a Constant Function)**

$$\frac{\mathrm{d}}{\mathrm{d}x}(c) = 0$$

# Power Functions

Now consider functions of the form $f(x) = x^n$, where $n$ is a positive integer. For $f(x) = x$, the graph is the line $y = x$, which has slope 1. So

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What about $n = 2$ and $n = 3$?

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}(x^2)\\
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## Theorem (The Power Rule)

*If $n$ is a positive integer, then* $$\frac{\mathrm{d}}{\mathrm{d}x}(x^n) = nx^{n-1}.$$

## Proof.

$\square$

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Use this formula (which you can verify):

$$x^n - a^n = (x-a)(x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1}).$$

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&= a^{n-1} + a^{n-2}a + \dots + aa^{n-2} + a^{n-1} = na^{n-1}.
\end{aligned}$$

□

## Example (Power Rule)

If $f(x) = x^5$,
Then $f'(x) =$

If $y = x^{1000}$,
Then $y' =$

If $u = t^{22}$,
Then $\frac{\mathrm{d}u}{\mathrm{d}t} =$

$\frac{\mathrm{d}}{\mathrm{d}r}(r^3) =$

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If $f(x) = x^5$,

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You will not be tested on the material in the following slide.

# The Relation between Ball Volume and Surface Area

There is a relationship between the surface area and the volume of a ball (in any dimension).

| Dimension | Set of pts. at dist. $\leq r$ from origin | Inside measure name | Measure f-la | Boundary name | Boundary measure formula | Derivative of inside measure |
|---|---|---|---|---|---|---|
| 3 | | | | | | |
| 2 | | | | | | |
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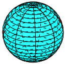
| 3 |  ball | | | | | |
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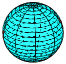
| 3 |  ball | **?** | | | | |
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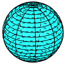
| 3 |  ball | volume | | | | |
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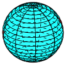
| 3 |  ball | volume | **?** | | | |
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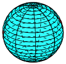
| 3 |  ball | volume | $\frac{4}{3}\pi r^3$ | | | |
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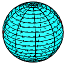
| 3 |  ball | volume | $\frac{4}{3}\pi r^3$ | | | |
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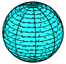
| 3 |  ball | volume | $\frac{4}{3}\pi r^3$ |  sphere | | |
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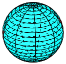
| 3 |  ball | volume | $\frac{4}{3}\pi r^3$ |  sphere | **?** | |
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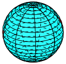
| 3 |  ball | volume | $\frac{4}{3}\pi r^3$ |  sphere | $4\pi r^2$ | |
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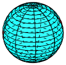
| 3 |  ball | volume | $\frac{4}{3}\pi r^3$ |  sphere | $4\pi r^2$ | $\frac{d}{dr}\left(\frac{4}{3}\pi r^3\right) = \mathbf{?}$ |
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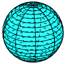
| 3 |  ball | volume | $\frac{4}{3}\pi r^3$ |  sphere | $4\pi r^2$ | $\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2$ |
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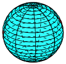
| 3 |  ball | volume | $\frac{4}{3}\pi r^3$ |  sphere | $4\pi r^2$ | $\frac{d}{dr}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2$ |
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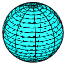
| 3 |  ball | volume | $\frac{4}{3}\pi r^3$ |  sphere | $4\pi r^2$ | $\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2$ |
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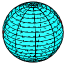
| 3 |  ball | volume | $\frac{4}{3}\pi r^3$ |  sphere | $4\pi r^2$ | $\frac{d}{dr}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2$ |
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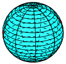
| 3 |  ball | volume | $\frac{4}{3}\pi r^3$ |  sphere | $4\pi r^2$ | $\frac{d}{dr}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2$ |
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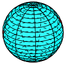
| 3 |  ball | volume | $\frac{4}{3}\pi r^3$ |  sphere | $4\pi r^2$ | $\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2$ |
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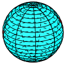
| 3 |  ball | volume | $\frac{4}{3}\pi r^3$ |  sphere | $4\pi r^2$ | $\frac{d}{dr}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2$ |
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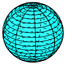
| 3 |  ball | volume | $\frac{4}{3}\pi r^3$ |  sphere | $4\pi r^2$ | $\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2$ |
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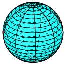
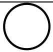
| 3 |  ball | volume | $\frac{4}{3}\pi r^3$ |  sphere | $4\pi r^2$ | $\frac{d}{dr}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2$ |
| 2 |   disk, circle | circle area | $\pi r^2$ |  circle (circumference) | | |
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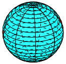
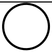
| 3 |  ball | volume | $\frac{4}{3}\pi r^3$ |  sphere | $4\pi r^2$ | $\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2$ |
| 2 |  disk, circle | circle area | $\pi r^2$ |  circle (circumference) | **?** | |
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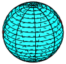
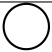
| 3 |  ball | volume | $\frac{4}{3}\pi r^3$ |  sphere | $4\pi r^2$ | $\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2$ |
| 2 |   disk, circle | circle area | $\pi r^2$ |  circle (circumference) | $2\pi r$ | |
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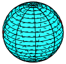
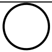
| 3 |  ball | volume | $\frac{4}{3}\pi r^3$ |  sphere | $4\pi r^2$ | $\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2$ |
| 2 |   disk, circle | circle area | $\pi r^2$ |  circle (circumference) | $2\pi r$ | $\frac{\mathrm{d}}{\mathrm{d}r}\left(\pi r^2\right) = $ **?** |
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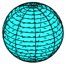
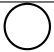
| 3 |  ball | volume | $\frac{4}{3}\pi r^3$ |  sphere | $4\pi r^2$ | $\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2$ |
| 2 |  disk, circle | circle area | $\pi r^2$ |  circle (circumference) | $2\pi r$ | $\frac{\mathrm{d}}{\mathrm{d}r}\left(\pi r^2\right) = 2\pi r$ |
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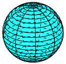
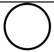
| 3 |  ball | volume | $\frac{4}{3}\pi r^3$ |  sphere | $4\pi r^2$ | $\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2$ |
| 2 |  disk, circle | circle area | $\pi r^2$ |  circle (circumference) | $2\pi r$ | $\frac{\mathrm{d}}{\mathrm{d}r}\left(\pi r^2\right) = 2\pi r$ |
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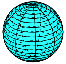
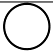
| 3 |  ball | volume | $\frac{4}{3}\pi r^3$ |  sphere | $4\pi r^2$ | $\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2$ |
| 2 |  disk, circle | circle area | $\pi r^2$ |  circle (circumference) | $2\pi r$ | $\frac{\mathrm{d}}{\mathrm{d}r}\left(\pi r^2\right) = 2\pi r$ |
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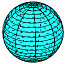
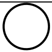
| 3 |  ball | volume | $\frac{4}{3}\pi r^3$ |  sphere | $4\pi r^2$ | $\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2$ |
| 2 |  disk, circle | circle area | $\pi r^2$ |  circle (circumference) | $2\pi r$ | $\frac{\mathrm{d}}{\mathrm{d}r}\left(\pi r^2\right) = 2\pi r$ |
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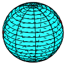
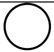
| 3 |  ball | volume | $\frac{4}{3}\pi r^3$ |  sphere | $4\pi r^2$ | $\frac{d}{dr}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2$ |
| 2 |  disk, circle | circle area | $\pi r^2$ |  circle (circumference) | $2\pi r$ | $\frac{d}{dr}\left(\pi r^2\right) = 2\pi r$ |
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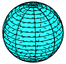
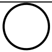
| 3 |  ball | volume | $\frac{4}{3}\pi r^3$ |  sphere | $4\pi r^2$ | $\frac{d}{dr}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2$ |
| 2 |  disk, circle | circle area | $\pi r^2$ |  circle (circumference) | $2\pi r$ | $\frac{d}{dr}\left(\pi r^2\right) = 2\pi r$ |
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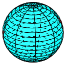
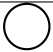
| 3 |  ball | volume | $\frac{4}{3}\pi r^3$ |  sphere | $4\pi r^2$ | $\frac{d}{dr}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2$ |
| 2 |  disk, circle | circle area | $\pi r^2$ |  circle (circumference) | $2\pi r$ | $\frac{d}{dr}\left(\pi r^2\right) = 2\pi r$ |
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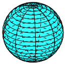
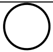
| 3 |  ball | volume | $\frac{4}{3}\pi r^3$ |  sphere | $4\pi r^2$ | $\frac{d}{dr}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2$ |
| 2 |   disk, circle | circle area | $\pi r^2$ |  circle (circumference) | $2\pi r$ | $\frac{d}{dr}\left(\pi r^2\right) = 2\pi r$ |
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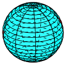
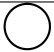
| 3 |  ball | volume | $\frac{4}{3}\pi r^3$ |  sphere | $4\pi r^2$ | $\frac{d}{dr}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2$ |
| 2 |   disk, circle | circle area | $\pi r^2$ |  circle (circumference) | $2\pi r$ | $\frac{d}{dr}\left(\pi r^2\right) = 2\pi r$ |
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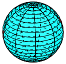
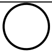
| 3 |  ball | volume | $\frac{4}{3}\pi r^3$ |  sphere | $4\pi r^2$ | $\frac{d}{dr}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2$ |
| 2 |  disk, circle | circle area | $\pi r^2$ |  circle (circumference) | $2\pi r$ | $\frac{d}{dr}\left(\pi r^2\right) = 2\pi r$ |
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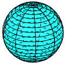
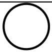
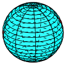
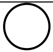
| 3 |  ball | volume | $\frac{4}{3}\pi r^3$ |  sphere | $4\pi r^2$ | $\frac{d}{dr}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2$ |
| 2 |   disk, circle | circle area | $\pi r^2$ |  circle (circumference) | $2\pi r$ | $\frac{d}{dr}\left(\pi r^2\right) = 2\pi r$ |
| 1 |   interval | length | $2r$ |  endpts. | 2 | |

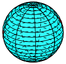
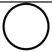
# The Relation between Ball Volume and Surface Area

There is a relationship between the surface area and the volume of a ball (in any dimension).

| Dimension | Set of pts. at dist. $\leq r$ from origin | Inside measure name | Measure f-la | Boundary name | Boundary measure formula | Derivative of inside measure |
|---|---|---|---|---|---|---|
| 3 |  ball | volume | $\frac{4}{3}\pi r^3$ |  sphere | $4\pi r^2$ | $\frac{d}{dr}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2$ |
| 2 |  disk, circle | circle area | $\pi r^2$ |  circle (circumference) | $2\pi r$ | $\frac{d}{dr}\left(\pi r^2\right) = 2\pi r$ |
| 1 |  interval | length | $2r$ |  endpts. | 2 | $\frac{d}{dr}(2r) =$ **?** |

# The Relation between Ball Volume and Surface Area

There is a relationship between the surface area and the volume of a ball (in any dimension).

| Dimension | Set of pts. at dist. $\leq r$ from origin | Inside measure name | Measure f-la | Boundary name | Boundary measure formula | Derivative of inside measure |
|---|---|---|---|---|---|---|
| 3 |  ball | volume | $\frac{4}{3}\pi r^3$ |  sphere | $4\pi r^2$ | $\frac{d}{dr}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2$ |
| 2 |   disk, circle | circle area | $\pi r^2$ |  circle (circumference) | $2\pi r$ | $\frac{d}{dr}\left(\pi r^2\right) = 2\pi r$ |
| 1 |   interval | length | $2r$ |  endpts. | $2$ | $\frac{d}{dr}(2r) = 2$ |