Calculus II
Homework on Lecture 7

1. Find a linear substitution (via completing the square) to transform the radical to a multiple of an expression of the form $\sqrt{u^2+1}$, $\sqrt{u^2-1}$ or $\sqrt{1-u^2}$.

   (a) $\sqrt{x^2+x+1}$.

   (b) $\sqrt{-2x^2+x+1}$.

   Solution. 1.a

$$
\begin{aligned}
\sqrt{x^2+x+1} &= \sqrt{x^2+2\frac{1}{2}x+\frac{1}{4}-\frac{1}{4}+1}\\
&= \sqrt{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\\
&= \sqrt{\frac{3}{4}\left(\frac{4}{3}\left(x+\frac{1}{2}\right)^2+1\right)}\\
&= \frac{\sqrt{3}}{2}\sqrt{\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right)^2+1}\\
&= \frac{\sqrt{3}}{2}\sqrt{u^2+1},
\end{aligned}
$$

   where $u=\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)=\frac{2\sqrt{3}}{3}x+\frac{\sqrt{3}}{3}$.

   Solution. 1.b

$$
\begin{aligned}
\sqrt{-2x^2+x+1} &= \sqrt{-2\left(x^2-\frac{1}{2}x-\frac{1}{2}\right)}\\
&= \sqrt{-2\left(x^2-2\frac{1}{4}x+\frac{1}{16}-\frac{1}{16}-\frac{1}{2}\right)}\\
&= \sqrt{-2\left(\left(x-\frac{1}{16}\right)^2-\frac{9}{16}\right)}\\
&= \sqrt{\frac{9}{8}\left(-\frac{16}{9}\left(x-\frac{1}{16}\right)^2+1\right)}\\
&= \frac{3}{\sqrt{8}}\sqrt{-\left(\frac{4}{3}\left(x-\frac{1}{16}\right)\right)^2+1}\\
&= \frac{3}{\sqrt{8}}\sqrt{-u^2+1}
\end{aligned}
$$

   where $u=\frac{4}{3}\left(x-\frac{1}{16}\right)=\frac{4}{3}x-\frac{1}{12}$.

1 Trig or Euler substitution, solutions use trig sub

1.1 Case 1: $\sqrt{x^2+1}$

2. Compute the integral.

(a) $\displaystyle\int \frac{\sqrt{1+x^2}}{x^2}\mathrm{d}x.$

answer: $\ln\left(\sqrt{1+x^2}+x\right)-\frac{\sqrt{1+x^2}}{x}+C$

Solution. 2.a

Variant I. In this variant, we use the trigonometric substitution $x=\tan\theta$ and then solve the integral using a few algebraic tricks.

$$\begin{aligned}
\int \frac{\sqrt{1+x^2}}{x^2}\mathrm{d}x &= \int \frac{\sqrt{1+\tan^2\theta}}{\tan^2\theta}\mathrm{d}(\tan\theta) \\
&= \int \frac{|\sec\theta|}{\tan^2\theta}\sec^2\theta\mathrm{d}\theta \\
&= \int \frac{\cos^2\theta}{\cos^3\theta\sin^2\theta}\mathrm{d}\theta \\
&= \int \frac{\cos\theta}{\cos^2\theta\sin^2\theta}\mathrm{d}\theta
\end{aligned}$$

Set
$x=\tan\theta$
$\theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$
$|\sec\theta|=\sec\theta$
for $\theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

$$= \int \frac{\mathrm{d}(\sin\theta)}{(1-\sin^2\theta)\sin^2\theta}$$

Set
$u=\sin\theta$
for $\theta\in\left(0,\frac{\pi}{2}\right)$
$u=\sqrt{1-\cos^2\theta}$
$u=\sqrt{1-\frac{1}{\sec^2\theta}}$
$u=\sqrt{1-\frac{1}{1+\tan^2\theta}}$
$u=\sqrt{\frac{\tan^2\theta}{1+\tan^2\theta}}$
$u=\frac{\tan\theta}{\sqrt{1+\tan^2\theta}}$
$u=\frac{x}{\sqrt{1+x^2}}$

$$\begin{aligned}
&= \int \frac{\mathrm{d}u}{(1-u^2)u^2} \\
&= \int \frac{\mathrm{d}u}{(1-u)u^2(u+1)} && \text{use part. frac.} \\
&= \int \left(\frac{\frac{1}{2}}{u+1}+\frac{-\frac{1}{2}}{u-1}+\frac{1}{u^2}\right)\mathrm{d}u \\
&= -\frac{1}{2}\ln|u-1|+\frac{1}{2}\ln(u+1)-u^{-1}+C \\
&= -\frac{1}{2}\ln(1-u)+\frac{1}{2}\ln(u+1)-u^{-1}+C && u=\tfrac{x}{\sqrt{1+x^2}}<1 \\
&= \frac{1}{2}\ln\left(\frac{1+u}{1-u}\right)-u^{-1}+C \\
&= \frac{1}{2}\ln\left(\frac{(1+u)}{(1-u)}\cdot\frac{(1+u)}{(1+u)}\right)-u^{-1}+C \\
&= \frac{1}{2}\ln\left(\frac{(1+u)^2}{1-u^2}\right)-u^{-1}+C && \text{use } u=\tfrac{x}{\sqrt{1+x^2}} \\
&= \frac{1}{2}\ln\left(\frac{(1+u)^2}{\frac{1}{1+x^2}}\right)-\frac{\sqrt{1+x^2}}{x}+C \\
&= \frac{1}{2}\ln\left(\left((1+u)\sqrt{1+x^2}\right)^2\right)-\frac{\sqrt{1+x^2}}{x}+C \\
&= \ln\left(\sqrt{1+x^2}+x\right)-\frac{\sqrt{1+x^2}}{x}+C \quad .
\end{aligned}$$

Variant II. In this variant, we use directly the Euler substitution

$$\begin{aligned}
x &= \cot(2\arctan t)\\
&= \tfrac{1}{2}\left(\tfrac{1}{t}-t\right)\\
\mathrm{d}x &= -\tfrac{1}{2}\left(\tfrac{1}{t^2}+1\right)\mathrm{d}t\\
\sqrt{1+x^2} &= \tfrac{1}{2}\left(\tfrac{1}{t}+t\right)\\
t &= \sqrt{x^2+1}-x\\
\tfrac{1}{t} &= \sqrt{x^2+1}+x \quad .
\end{aligned}$$

$$\begin{aligned}
\int \frac{\sqrt{1+x^2}}{x^2}\mathrm{d}x &= \int \frac{\frac{1}{2}\left(\frac{1}{t}+t\right)}{\frac{1}{4}\left(\frac{1}{t}-t\right)^2}\left(-\frac{1}{2}\right)\left(\frac{1}{t^2}+1\right)\mathrm{d}t\\
&= \int \frac{-t^4-2t^2-1}{(t-1)^2t(t+1)^2}\mathrm{d}t && \Big| \text{ Part. frac}\\
&= \int\left(-\frac{1}{t}+\frac{1}{(t+1)^2}-\frac{1}{(t-1)^2}\right)\mathrm{d}t\\
&= -\ln t-\frac{1}{t+1}+\frac{1}{t-1}+C\\
&= \ln\left(\frac{1}{t}\right)+\frac{2}{t^2-1}+C\\
&= \ln\left(\sqrt{1+x^2}+x\right)+\frac{1}{t\frac{1}{2}\left(t-\frac{1}{t}\right)}+C\\
&= \ln\left(\sqrt{1+x^2}+x\right)-\frac{1}{t}\cdot\frac{1}{\frac{1}{2}\left(\frac{1}{t}-t\right)}+C\\
&= \ln\left(\sqrt{1+x^2}+x\right)-\left(\sqrt{x^2+1}+x\right)\cdot\frac{1}{x}+C\\
&= \ln\left(\sqrt{1+x^2}+x\right)-\frac{\sqrt{x^2+1}}{x}-1+C \quad .
\end{aligned}$$

1.2 Case 2: $\sqrt{1-x^2}$

3. Compute the integral using a trigonometric substitution.

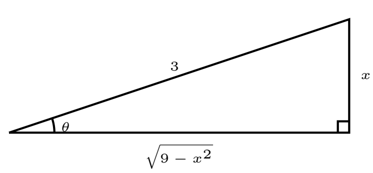
   (a) $\displaystyle\int \frac{\sqrt{9-x^2}}{x^2}\mathrm{d}x$.

answer: $-\frac{\sqrt{9-x^2}}{x}-\arcsin\left(\frac{x}{3}\right)+C$

Solution. 3.a

$$\begin{aligned}
\int \frac{\sqrt{9-x^2}}{x^2}\mathrm{d}x &= \int \frac{3\sqrt{\cos^2\theta}}{9\sin^2\theta}(3\cos\theta)\mathrm{d}\theta && \Big| \text{ Set } x=3\sin\theta \text{ for } \theta\in\left[\tfrac{\pi}{2},0\right)\cup\left(0,\tfrac{\pi}{2}\right]\\
&&& \ \ \mathrm{d}x = 3\cos\theta\mathrm{d}\theta\\
&= 9\int \frac{|\cos\theta|}{\sin^2\theta}\cos\theta\mathrm{d}\theta && \ \ \text{For } \theta\in\left[\tfrac{\pi}{2},0\right)\cup\left(0,\tfrac{\pi}{2}\right] \text{ we have} |\cos\theta|=\cos\theta\\
&= \int \cot^2\theta\mathrm{d}\theta\\
&= \int(\csc^2\theta-1)\mathrm{d}\theta\\
&= -\cot\theta-\theta+C\\
&= -\frac{\sqrt{9-x^2}}{x}-\arcsin\left(\frac{x}{3}\right)+C,
\end{aligned}$$

where we expressed $\cot\theta$ via $\sin\theta$ by considering the following triangle.



2 Trig or Euler substitution, solutions use Euler sub

2.1 Case 1: $\sqrt{x^2+1}$

4. Compute the integral.

(a) $\displaystyle\int \sqrt{x^2+1}\mathrm{d}x$

answer: $\frac{1}{2}x\sqrt{x^2+1}+\frac{1}{2}\ln\left(\sqrt{x^2+1}+x\right)+C$

(b) $\displaystyle\int \sqrt{x^2+2}\mathrm{d}x$

answer: $\ln\left(\sqrt{\frac{1}{2}x^2+1}+\frac{\sqrt{2}}{2}x\right)+\frac{\sqrt{2}}{2}x\sqrt{\frac{1}{2}x^2+1}+C$

(c) $\displaystyle\int \sqrt{x^2+x+1}\mathrm{d}x$

answer: $\frac{3}{4}\left(\frac{1}{2}\ln\left(\sqrt{\frac{4}{3}(x+\frac{1}{2})^2+1}+\frac{2}{3}\sqrt{3}\left(x+\frac{1}{2}\right)\right)+\frac{\sqrt{3}}{3}\left(x+\frac{1}{2}\right)\sqrt{\frac{4}{3}(x+\frac{1}{2})^2+1}\right)+C$

(d) $\displaystyle\int \sqrt{(2x^2+2x+1)}\mathrm{d}x$

answer: $\frac{\sqrt{2}}{4}\left(\frac{1}{2}(2x+1)\sqrt{(2x+1)^2+1}+\frac{1}{2}\ln\left(\sqrt{(2x+1)^2+1}+2x+1\right)\right)+C$

(e) $\displaystyle\int \sqrt{(3x^2+2x+1)}\mathrm{d}x$

answer: $\frac{2}{9}\sqrt{3}\left(\frac{1}{2}\ln\left(\sqrt{\frac{9}{2}(x+\frac{1}{3})^2+1}+\frac{3}{2}\sqrt{2}\left(x+\frac{1}{3}\right)\right)\right.$
$\left.+\frac{3}{4}\sqrt{2}\left(x+\frac{1}{3}\right)\sqrt{\frac{9}{2}(x+\frac{1}{3})^2+1}\right)+C$

(f) $\displaystyle\int \frac{\sqrt{x^2+1}}{x+1}\mathrm{d}x$

answer:
$-\sqrt{2}\ln\left(\sqrt{x^2+1}-x+\sqrt{2}-1\right)$
$+\sqrt{2}\ln\left(\sqrt{x^2+1}-x-\sqrt{2}-1\right)$
$+\ln\left(\sqrt{x^2+1}-x\right)$
$+\sqrt{x^2+1}$

Solution. 4.a.

This problem can be solved both via the Euler substitution and by transforming to a trigonometric integral and solving the trigonometric integral on its own. We present both variants.

Variant I. We recall the Euler substitution for $\sqrt{x^2+1}$ given in (2):

$$\begin{array}{rcl} x &=& \frac{1}{2}\left(\frac{1}{t}-t\right)\\ \sqrt{x^2+1} &=& \frac{1}{2}\left(\frac{1}{t}+t\right)\\ \mathrm{d}x &=& -\frac{1}{2}\left(\frac{1}{t^2}+1\right)\mathrm{d}t\\ t &=& \sqrt{x^2+1}-x \quad . \end{array}$$

Therefore

$$\begin{array}{rll} \displaystyle\int\sqrt{(x^2+1)}\mathrm{d}x &= \displaystyle -\int\frac{1}{4}\left(\frac{1}{t}+t\right)\left(\frac{1}{t^2}+1\right)\mathrm{d}t & \\ &= \displaystyle -\frac{1}{4}\int\left(\frac{1}{t^3}+2\frac{1}{t}+t\right)\mathrm{d}t & \\ &= \displaystyle -\frac{1}{4}\left(-\frac{t^{-2}}{2}+2\ln|t|+\frac{t^2}{2}\right)+C & \\ &= \displaystyle \frac{1}{8}\left(t^{-2}-t^2\right)-\frac{1}{2}\ln|t|+C & \left| \begin{array}{l} a^2-b^2=\\ (a-b)(a+b)\end{array}\right. \\ &= \displaystyle \frac{1}{2}\left(\underbrace{\frac{1}{2}\left(t^{-1}-t\right)}_{=x}\right)\left(\underbrace{\frac{1}{2}\left(t^{-1}+t\right)}_{=\sqrt{x^2+1}}\right)-\frac{1}{2}\ln|t|+C & \\ &= \displaystyle \frac{1}{2}x\sqrt{x^2+1}-\frac{1}{2}\ln\left|\sqrt{x^2+1}-x\right|+C & \left|\text{ See below}\right. \\ &= \displaystyle \frac{1}{2}x\sqrt{x^2+1}+\frac{1}{2}\ln\left(\sqrt{x^2+1}+x\right)+C \quad . & \end{array}$$

Our problem is solved.

A few comments are in order. In the above expression we would have obtained a perfectly good answer if we plugged in $t=\sqrt{x^2+1}-x$ into the fourth line, however our answer would look much more complicated. Indeed, had we not used the formula $a^2-b^2=(a-b)(a+b)$ in the fourth line, the term $t^{-2}-t^2$ would be equal to $\frac{1}{(\sqrt{x^2+1}-x)^2}-(\sqrt{x^2+1}-x)^2$. In turn, the term $\frac{1}{(\sqrt{x^2+1}-x)^2} - (\sqrt{x^2+1}-x)^2$ can be simplified to $4x\sqrt{x^2+1}$ as follows. We carry out the simplifications to illustrate some of the algebraic issues arising when dealing with integrals of radicals.

$$\begin{aligned}
t^{-2} - t^2 &= \frac{1}{(\sqrt{x^2+1}-x)^2} - (\sqrt{x^2+1}-x)^2 \\
&= \frac{(\sqrt{x^2+1}+x)^2}{(\sqrt{x^2+1}-x)^2(\sqrt{x^2+1}+x)^2} \\
&\quad -(\sqrt{x^2+1}-x)^2 \\
&= \underbrace{\frac{(\sqrt{x^2+1}+x)^2}{((\sqrt{x^2+1})^2-x^2)^2}}_{=1} - (\sqrt{x^2+1}-x)^2 \\
&= 4x\sqrt{x^2+1} \quad .
\end{aligned}$$

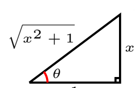
Of course, the above computations are unnecessary if we use the formula $a^2-b^2 = (a-b)(a+b)$ as done in the original solution. We note that in the last transformation we transformed $\ln\left|\sqrt{x^2+1}-x\right|$ to $\ln\left(\sqrt{x^2+1}-x\right)$ because the quantity $\sqrt{x^2+1}-x$ is always positive. The proof of that fact we leave for the reader's exercise.

Finally, we note that as a last simplification to our solution, we used the transformation $\ln|t| = \ln\left(\sqrt{x^2+1}-x\right) = -\ln|\frac{1}{t}| = -\ln\left(\sqrt{x^2+1}+x\right)$. This is seen as follows.

$$\begin{array}{rcll}
\ln|t| &=& -\ln\left|\dfrac{1}{t}\right| & \\
&=& -\ln\left(\dfrac{1}{\sqrt{x^2+1}-x}\right) & \Big| \text{ rationalize} \\
&=& -\ln\left(\dfrac{\left(\sqrt{x^2+1}+x\right)}{\left(\sqrt{x^2+1}-x\right)\left(\sqrt{x^2+1}+x\right)}\right) & \\
&=& -\ln\left(\dfrac{\sqrt{x^2+1}+x}{x^2+1-x^2}\right) & \\
&=& -\ln\left(\sqrt{x^2+1}+x\right) \quad . &
\end{array}$$

Variant II. In this variant we transform to a trigonometric integral and solve it using ad-hoc methods. We recall that if we decided to solve the trigonometric integral using the standard substitution $\theta = 2\arctan t$, we would arrive at the Euler substitution given in Variant I.

$$\begin{array}{rll}
\displaystyle\int \sqrt{x^2+1}\mathrm{d}x &= \displaystyle\int \sqrt{\tan^2\theta+1}\ \mathrm{d}(\tan\theta) & \Bigg| \begin{array}{l} \text{Set} \\ x=\tan\theta \\ \theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right) \end{array} \\
&= \displaystyle\int \sqrt{sec^2\theta}\sec^2\theta\mathrm{d}\theta & \Big| \sec\theta > 0 \\
&= \displaystyle\int \sec^3\theta\mathrm{d}\theta & \Big| \text{Problem solved already} \\
&= \dfrac{1}{2}\left(\tan\theta\sec\theta + \ln|\sec\theta+\tan\theta|\right) + C & \Bigg| \begin{array}{l} \sec\theta = \sqrt{x^2+1} \\ \tan\theta = x \end{array} \\
&= \dfrac{1}{2}\left(x\sqrt{x^2+1} + \ln\left(\sqrt{x^2+1}+x\right)\right) + C &
\end{array}$$



Solution. 4.d

$$
\begin{aligned}
\int \sqrt{(2x^2+2x+1)}\mathrm{d}x &= \int \sqrt{2}\sqrt{\left(\left(x+\frac{1}{2}\right)^2+\frac{1}{4}\right)}\mathrm{d}x && \text{complete square}\\
&= \sqrt{2}\int \sqrt{\frac{1}{4}\left(4\left(x+\frac{1}{2}\right)^2+1\right)}\mathrm{d}x\\
&= \frac{\sqrt{2}}{2}\int \sqrt{\left(4\left(x+\frac{1}{2}\right)^2+1\right)}\mathrm{d}x\\
&= \frac{\sqrt{2}}{2}\int \sqrt{\left((2x+1)^2+1\right)}\frac{1}{2}\mathrm{d}\left(2x+1\right) && \text{Set } u=2x+1\\
& && \text{Euler subst.:}\\
& && u=\tfrac{1}{2}\left(\tfrac{1}{t}-t\right),\\
& && t>0\\
&= \frac{\sqrt{2}}{4}\int \sqrt{(u^2+1)}\mathrm{d}u && \mathrm{d}u=-\tfrac{1}{2}\left(\tfrac{1}{t^2}+1\right)\mathrm{d}t\\
& && \sqrt{u^2+1}=\tfrac{1}{2}\left(\tfrac{1}{t}+t\right)\\
& && t=\sqrt{u^2+1}-u\\
&= -\frac{\sqrt{2}}{16}\int \left(\frac{1}{t}+t\right)\left(\frac{1}{t^2}+1\right)\mathrm{d}t\\
&= -\frac{\sqrt{2}}{16}\int \left(t^{-3}+2t^{-1}+t\right)\mathrm{d}t\\
&= -\frac{\sqrt{2}}{16}\left(-\frac{t^{-2}}{2}+2\ln|t|+\frac{t^2}{2}\right)+C && \text{simplify as in Problem 4.a}\\
&= \frac{\sqrt{2}}{8}\left(u\sqrt{u^2+1}+\ln\left(\sqrt{u^2+1}+u\right)\right)+C\\
&= \frac{\sqrt{2}}{8}\left((2x+1)\sqrt{(2x+1)^2+1}\right.\\
&\qquad \left.+\ln\left(\sqrt{(2x+1)^2+1}+2x+1\right)\right)+C.
\end{aligned}
$$

Solution. 4.f

$$\begin{aligned}
\int \frac{\sqrt{x^2+1}}{x+1}\mathrm{d}x =& \int \frac{\frac{1}{2}\left(\frac{1}{t}+t\right)}{\frac{1}{2}\left(\frac{1}{t}-t\right)+1}\mathrm{d}\left(\frac{1}{2}\left(\frac{1}{t}-t\right)\right) && \left| \begin{array}{r} \text{Euler sub:} \\ x = \frac{1}{2}\left(\frac{1}{t}-t\right) \\ \sqrt{x^2+1} = \frac{1}{2}\left(\frac{1}{t}+t\right) \end{array}\right. \\
=& \int \left(\frac{1+t^2}{1-t^2+2t}\right)\frac{1}{2}\left(-t^{-2}-1\right)\mathrm{d}t \\
=& \int \frac{1}{2}\frac{\left(1+t^2\right)\left(-t^{-2}-1\right)}{1-t^2+2t}\mathrm{d}t \\
=& \frac{1}{2}\int \frac{t^4+2t^2+1}{t^4-2t^3-t^2}\mathrm{d}t && \left| \begin{array}{l} \text{pol. long div.} \\ \\ \text{part. fractions} \end{array}\right. \\
=& \frac{1}{2}\int \left(1+\frac{2t^3+3t^2+1}{t^2\left(t^2-2t-1\right)}\right)\mathrm{d}t \\
=& \frac{1}{2}\int \left(1+\frac{2\sqrt{2}}{t-\sqrt{2}-1}+\frac{-2\sqrt{2}}{t+\sqrt{2}-1}+\frac{2}{t}+\frac{-1}{t^2}\right)\mathrm{d}t \\
=& -\sqrt{2}\ln\left|t+\sqrt{2}-1\right|+\sqrt{2}\ln\left|t-\sqrt{2}-1\right| \\
&+\frac{1}{2}t^{-1}+\ln|t|+\frac{1}{2}t+C && \left| \; t=\sqrt{x^2+1}-x \right. \\
=& -\sqrt{2}\ln\left(\sqrt{x^2+1}-x+\sqrt{2}-1\right) \\
&+\sqrt{2}\ln\left(\sqrt{x^2+1}-x-\sqrt{2}-1\right) \\
&+\ln\left(\sqrt{x^2+1}-x\right) \\
&+\frac{1}{2}\left(\sqrt{x^2+1}-x\right)^{-1}+\frac{1}{2}\sqrt{x^2+1}-\frac{1}{2}x+C && \left| \begin{array}{l} \text{Last 3 terms} \\ \text{simplify} \end{array}\right. \\
=& -\sqrt{2}\ln\left(\sqrt{x^2+1}-x+\sqrt{2}-1\right) \\
&+\sqrt{2}\ln\left(\sqrt{x^2+1}-x-\sqrt{2}-1\right) \\
&+\ln\left(\sqrt{x^2+1}-x\right) \\
&+\sqrt{x^2+1}+C \quad .
\end{aligned}$$

5. **This problem will not be quizzed.** Let $b^2-4ac<0$ and $a>0$ be (real) numbers. Show that

$$\int \sqrt{(ax^2+bx+c)}\mathrm{d}x = \frac{\sqrt{a}D}{2}\left(\ln\left(\sqrt{\left(\frac{2xa+b}{2\sqrt{Da}}\right)^2+1}+\frac{2xa+b}{2\sqrt{Da}}\right)+\frac{2xa+b}{2\sqrt{Da}}\sqrt{\left(\frac{2xa+b}{2\sqrt{Da}}\right)^2+1}\right)+C,$$

where $D=\dfrac{4ac-b^2}{4a^2}$.

2.2 Case 2: $\sqrt{1-x^2}$

6. Integrate

(a) $\displaystyle\int \sqrt{1-x^2}\mathrm{d}x$

answer:

(b) $\displaystyle\int \sqrt{2-x^2}\mathrm{d}x$

(c) $\displaystyle\int \sqrt{-x^2+x+1}\mathrm{d}x$

(d) $\displaystyle\int \sqrt{2-x-x^2}\mathrm{d}x$

(e) $\displaystyle\int \frac{\sqrt{1-x^2}}{1+x}\mathrm{d}x$

(f) $\displaystyle\int \frac{\sqrt{1-x^2}}{2+x}\mathrm{d}x$

Solution. 6.a

Variant I. This integral can quickly be solved using a trig substitution. The Euler substitution results in a slightly longer solution, shown in the next solution variant.

$$
\begin{aligned}
\int \sqrt{1-x^2}\mathrm{d}x &= \int \sqrt{1-\cos^2\theta}\mathrm{d}(\cos\theta) && \text{Set } x=\cos\theta, \theta\in[0,\pi]\\
&= \int \sqrt{\sin^2\theta}(-\sin\theta)\mathrm{d}\theta && \theta\in[0,\pi]\Rightarrow \sin\theta\geq 0\\
&= -\int \sin^2\theta\mathrm{d}\theta && \sin^2\theta=\tfrac{1-\cos(2\theta)}{2}\\
&= -\int \frac{1-\cos(2\theta)}{2}\mathrm{d}\theta\\
&= -\frac{\theta}{2}+\frac{\sin(2\theta)}{4}+C\\
&= -\frac{\theta}{2}+\frac{2\sin\theta\cos\theta}{4}+C && \begin{array}{r}x{=}\cos\theta\\ \theta{=}\arccos x\\ \sin\theta{=}\sin\left(\arccos x\right)\\ {=}\sqrt{1-x^2}\end{array}\\
&= -\frac{\arccos x}{2}+\frac{x\sqrt{1-x^2}}{2}+C\\
&= \frac{\arcsin x}{2}+\frac{x\sqrt{1-x^2}}{2}+K \quad ,
\end{aligned}
$$

where for the last equality we recall that the derivative of $\arcsin x$ is minus the derivative of $\arccos x$.

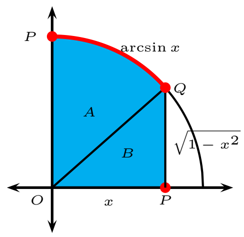
Variant II. We show how to do this integral via the Euler substitution $x = \cos(2\arctan t)$.

$$
\begin{aligned}
\int \sqrt{1-x^2}\mathrm{d}x &= \int \sqrt{1-\cos^2\theta}\mathrm{d}(\cos\theta) && \begin{array}{r}\text{Set}\\ x{=}\cos(2\arctan t)\\ \tfrac{1}{2}\arccos x{=}\arctan t\\ x{=}\tfrac{1-t^2}{1+t^2}\\ {=}\tfrac{2}{1+t^2}-1\\ \sqrt{1-x^2}{=}\tfrac{2t}{1+t^2}\end{array}\\
&= \int \frac{2t}{1+t^2}\mathrm{d}\left(\frac{1-t^2}{1+t^2}\right)\\
&= \int \frac{2t}{1+t^2}\left(\frac{-4t}{\left(1+t^2\right)^2}\right)\mathrm{d}t && \begin{array}{l}\text{Integral rational}\\ \text{function}\\ \text{we skip details}\end{array}\\
&= \frac{-t}{t^2+1}+\frac{2t}{\left(t^2+1\right)^2}\\
&\quad -\arctan t+C\\
&= -\frac{1}{2}\sqrt{1-x^2}+\frac{\sqrt{1-x^2}}{t^2+1}\\
&\quad -\arctan t+C\\
&= \frac{1}{2}\sqrt{1-x^2}\left(\frac{2}{t^2+1}-1\right)\\
&\quad -\arctan t+C\\
&= \frac{x\sqrt{1-x^2}}{2}-\frac{1}{2}\arccos x+C\\
&= \frac{x\sqrt{1-x^2}}{2}+\frac{1}{2}\arcsin x+K,
\end{aligned}
$$

where for the very last equality we used the fact that the derivatives of $\arcsin x$ and $\arccos x$ are negatives of one another.

Variant III. We show how to do this integral geometrically, provided already know the area of a sector of circle. Of course, here we assume we have already derived the formula for an area of a circle. We warn the reader that most methods for deriving the formula of a sector area rely on integrals, so it is possible we are making a circular reasoning argument. Since we already did the integral purely algebraically in the preceding solution variants, we can safely ignore the danger of the aforementioned circular reasoning argument. In other words, the present solution Variant is a geometric interpretation of the problem which relies on the formula for sector area of a circle (which we assumed proved elsewhere, possibly using similar integration techniques to the ones presented in Variant I and II).

By the Fundamental Theorem of Calculus, the indefinite integral measures up to a constant the area locked under the graph of $\sqrt{1-x^2}$. This graph is a part of a circle. Therefore, up to a constant, $\int\sqrt{1-t^2}\mathrm{d}t$ equals $\int_0^x\sqrt{1-t^2}\mathrm{d}t$. In turn $\int_0^x\sqrt{1-t^2}\mathrm{d}t$ is given by the area highlighted in the picture below.



$$\begin{aligned}
\text{Area}(A) &= \frac{\text{length}\left(\widehat{PQ}\right)}{2\pi}\pi = \frac{\text{length}\left(\widehat{PQ}\right)}{2} = \frac{\arcsin x}{2}\\
\text{Area}(B) &= \text{Area}(\triangle OPQ) = \frac{x\sqrt{1-x^2}}{2}\\
\int_0^x \sqrt{1-t^2}\mathrm{d}t &= \text{Area}(A) + \text{Area}(B)\\
&= \frac{\arcsin x}{2} + \frac{x\sqrt{1-x^2}}{2}\\
&\Rightarrow\\
\int \sqrt{1-x^2}\mathrm{d}x &= \frac{\arcsin x}{2} + \frac{x\sqrt{1-x^2}}{2} + C \quad .
\end{aligned}$$

Solution. 6.e In this problem solution we use the standard Euler substitution $x = \cos(2\arctan t)$. We recall from (6) that

$$\begin{aligned}
x &= \cos(2\arctan t) = \frac{1-t^2}{1+t^2}\\
\arccos(x) &= 2\arctan t\\
\mathrm{d}x &= -\frac{4t}{(1+t^2)^2}\mathrm{d}t\\
\sqrt{1-x^2} &= \sin(2\arctan t) = \frac{2t}{1+t^2}\\
t &= \frac{\sqrt{1-x^2}}{x+1} \quad .
\end{aligned}$$

$$\begin{aligned}
\int \frac{\sqrt{1-x^2}}{1+x}\mathrm{d}x &= \int t\left(-\frac{4t}{(1+t^2)^2}\right)\mathrm{d}t && \left|\ \begin{array}{l}\text{Set } x = \frac{1-t^2}{1+t^2}\\ \text{Use f-las above}\end{array}\right.\\
&= -4\int \frac{t^2}{(1+t^2)^2}\mathrm{d}t\\
&= -4\int \frac{1+t^2-1}{(1+t^2)^2}\mathrm{d}t\\
&= -4\int \left(\frac{1}{1+t^2} - \frac{1}{(1+t^2)^2}\right)\mathrm{d}t\\
&= -4\left(\arctan t - \frac{1}{2}\left(\arctan t + \frac{t}{1+t^2}\right)\right) + C\\
&= -2\left(\arctan t - \frac{t}{1+t^2}\right) + C\\
&= -2\left(\arctan\left(\frac{\sqrt{1-x^2}}{1+x}\right) - \frac{1}{2}\sqrt{1-x^2}\right) + C\\
&= -2\arctan t + \sqrt{1-x^2} + C && \left|\ \text{Use f-las above}\right.\\
&= -\arccos x + \sqrt{1-x^2} + C\\
&= \arcsin x + \sqrt{1-x^2} + K \quad .
\end{aligned}$$

We have included the last equality to remind the student that derivatives of $\arcsin(x)$ and $\arccos x$ are negatives of one another.

2.3 Case 3: $\sqrt{x^2-1}$

7. Integrate

(a) $\int \sqrt{x^2-1}\mathrm{d}x$

(b) $\int \sqrt{x^2-2}\mathrm{d}x$

(c) $\int \sqrt{2x^2+x-1}\mathrm{d}x$

(d) $\int \sqrt{x^2+x-1}\mathrm{d}x$

3 Theory through problems (Optional homework, will not be quizzed, will not be tested)

3.1 Case 1: $\sqrt{x^2+1}$

3.1.1 $x = \cot\theta$

8. (a) Express $x, \mathrm{d}x$ and $\sqrt{x^2+1}$ via θ and $\mathrm{d}\theta$ for the trigonometric substitution $x = \cot\theta$, $\theta \in (0, \pi)$.
   (b) Express $x, \mathrm{d}x$ and $\sqrt{x^2+1}$ via t and $\mathrm{d}t$ for the Euler substitution $x = \cot(2\arctan t)$, $t > 0$. Express t via x.

Solution. 8.a The trigonometric substitution $x = \cot\theta$ is given by

$$\begin{array}{rcll}
\sqrt{x^2+1} &=& \sqrt{\cot^2\theta+1} & \\
&=& \sqrt{\dfrac{\cos^2\theta}{\sin^2\theta}+1} & \\
&=& \sqrt{\dfrac{\cos^2\theta+\sin^2\theta}{\sin^2\theta}} & \\
&=& \sqrt{\dfrac{1}{\sin^2\theta}} & \left| \begin{array}{l}\text{when } \theta\in(0,\pi) \text{ we have} \\ \sin\theta\geq 0 \text{ and so } \sqrt{\sin^2\theta} = \sin\theta\end{array}\right. \\
&=& \dfrac{1}{\sin\theta} = \csc\theta & .
\end{array}$$

The differential $\mathrm{d}x$ can be expressed via $\mathrm{d}\theta$ from $x = \cot\theta$. The substitution $x = \cot\theta$ can be now summarized as:

$$\begin{array}{rcl}
x &=& \cot\theta \\
\sqrt{x^2+1} &=& \dfrac{1}{\sin\theta} = \csc\theta \\
\mathrm{d}x &=& -\dfrac{\mathrm{d}\theta}{\sin^2\theta} = -\csc^2\theta\mathrm{d}\theta \\
\theta &=& \operatorname{arccot} x \quad .
\end{array}$$

Solution. 8.b We recall that the substitution $\theta = 2\arctan t$ transforms a trigonometric integral into an integral of a rational function. We now apply the substitution $\theta = 2\arctan t$ after the substitution $x = \cot\theta$:

$$\begin{array}{rcll}
x &=& \cot\theta & \left| \text{use } \theta = 2\arctan t \right. \\
&=& \cot(2\arctan t) & \left| \text{use } \cot 2z = \dfrac{\cos(2z)}{\sin(2z)} = \dfrac{1-\tan^2 z}{2\tan z} \right. \\
&=& \dfrac{1-\tan^2(\arctan t)}{2\tan(\arctan t)} & \\
&=& \dfrac{1-t^2}{2t} & \\
&=& \dfrac{1}{2}\left(\dfrac{1}{t}-t\right) & .
\end{array}$$

We can furthermore compute

$$\begin{array}{rcll}
\sqrt{x^2+1} &=& \sqrt{\dfrac{1}{4}\left(\dfrac{1}{t}-t\right)^2+1} & \\
&=& \dfrac{1}{2}\sqrt{\left(\dfrac{1}{t}+t\right)^2} & \left| \sqrt{\left(\dfrac{1}{t}+t\right)^2} = \dfrac{1}{t}+t \text{ because } t>0 \right. \\
&=& \dfrac{1}{2}\left(\dfrac{1}{t}+t\right) & .
\end{array} \qquad (1)$$

The differential dx can via dx as follows.

$$\mathrm{d}x = \mathrm{d}\left(\frac{1}{2}\left(\frac{1}{t}-t\right)\right) = -\frac{1}{2}\left(\frac{1}{t^2}-1\right) \quad .$$

Finally, we can subtract $x = \frac{1}{2}\left(\frac{1}{t}-t\right)$ from $\sqrt{x^2+1} = \frac{1}{2}\left(\frac{1}{t}+t\right)$ to get that

$$t = \sqrt{x^2+1} - x \quad .$$

The Euler substitution $x = \cot\theta = \cot(\arctan 2t)$ can be now summarized as:

$$\begin{array}{rcl} x &=& \frac{1}{2}\left(\frac{1}{t}-t\right) \\ \sqrt{x^2+1} &=& \frac{1}{2}\left(\frac{1}{t}+t\right) \\ \mathrm{d}x &=& -\frac{1}{2}\left(\frac{1}{t^2}+1\right)\mathrm{d}t \\ t &=& \sqrt{x^2+1}-x \quad . \end{array} \qquad (2)$$

9. Let the variables x and t be related via $\sqrt{x^2+1} = x+t$.

   (a) Express x via t.

   (b) Express $\sqrt{x^2+1}$ via t alone.

   (c) Express dx via t and dt.

Solution. 9.a.

$$\begin{array}{rcll} \sqrt{x^2+1} &=& x+t & \big| \text{ square both sides} \\ x^2+1 &=& x^2+2xt+t^2 & \\ -2xt &=& t^2-1 & \\ x &=& \frac{1}{2}\left(\frac{1}{t}-t\right) \quad . & \end{array}$$

Solution. 9.b.

Use Problem 9.a to get:

$$\sqrt{x^2+1} = x+t = \frac{1}{2}\left(\frac{1}{t}-t\right)+t = \frac{1}{2}\left(\frac{1}{t}+t\right) \quad .$$

3.1.2 $x = \tan\theta$

10. (a) Express x, dx and $\sqrt{x^2+1}$ via θ and dθ for the trigonometric substitution $x = \tan\theta$, $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

    (b) Express x, dx and $\sqrt{x^2+1}$ via t and dt for the Euler substitution $x = \tan(2\arctan t)$, $t \in (-1, 1)$. Express t via x.

Solution. 10.a The trigonometric substitution $x = \tan\theta$ is given by

$$\begin{array}{rcll} \sqrt{x^2+1} &=& \sqrt{\tan^2\theta+1} & \\ &=& \sqrt{\frac{\sin^2\theta}{\cos^2\theta}+1} & \\ &=& \sqrt{\frac{\sin^2\theta+\cos^2\theta}{\cos^2\theta}} & \\ &=& \sqrt{\frac{1}{\cos^2\theta}} & \Bigg| \begin{array}{l}\text{when } \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \text{ we have} \\ \cos\theta > 0 \text{ and so } \sqrt{\cos^2\theta} = \cos\theta\end{array} \\ &=& \frac{1}{\cos\theta} = \sec\theta \quad . & \end{array}$$

The differential $\mathrm{d}x$ can be expressed via $\mathrm{d}\theta$ from $x = \tan\theta$. The substitution $x = \tan\theta$ can be now summarized as:

$$\begin{array}{rcl} x &=& \tan\theta \\ \sqrt{x^2+1} &=& \dfrac{1}{\cos\theta} = \sec\theta \\ \mathrm{d}x &=& \dfrac{\mathrm{d}\theta}{\cos^2\theta} = \sec^2\theta \mathrm{d}\theta \\ \theta &=& \arctan x \quad . \end{array}$$

Solution. 10.b We recall that the substitution $\theta = 2\arctan t$ transforms a trigonometric integral into an integral of a rational function. We now apply the substitution $\theta = 2\arctan t$ after the substitution $x = \tan\theta$:

$$\begin{array}{rcll} x &=& \tan\theta & \Big| \text{ use } \theta = 2\arctan t \\ &=& \tan(2\arctan t) & \Big| \text{ use: } \tan 2z = \dfrac{\sin(2z)}{\cos(2z)} = \dfrac{2\tan z}{1-\tan^2 z} \\ &=& \dfrac{2\tan(\arctan t)}{1-\tan^2(\arctan t)} & \\ &=& \dfrac{2t}{1-t^2} \quad . & \end{array}$$

We can furthermore compute

$$\begin{array}{rcll} \sqrt{x^2+1} &=& \sqrt{\left(\dfrac{2t}{1-t^2}\right)^2+1} & \\ &=& \sqrt{\dfrac{4t^2+(1-t^2)^2}{(1-t^2)^2}} & \\ &=& \sqrt{\dfrac{(1+t^2)^2}{(1-t^2)^2}} & \Big| \sqrt{(1-t^2)^2} = 1-t^2 \text{ because } |t|<1 \\ &=& \dfrac{1+t^2}{1-t^2} & \\ &=& \dfrac{2-(1-t^2)}{1-t^2} & \\ &=& -1+\dfrac{2}{1-t^2} \quad . & \end{array} \tag{3}$$

From $\sqrt{x^2+1} = -1 + \dfrac{2}{1-t^2}$ and $x = \dfrac{2t}{1-t^2}$ we can express t via x:

$$\begin{array}{rcll} \sqrt{x^2+1} &=& -1+\dfrac{2}{1-t^2} & \\ &=& -1+\dfrac{1}{t}\left(\dfrac{2t}{1-t^2}\right) & \Big| \text{ use } x = \dfrac{2t}{1-t^2} \\ &=& -1+\dfrac{x}{t} & \\ 1+\sqrt{x^2+1} &=& \dfrac{x}{t} & \\ t &=& \dfrac{x}{1+\sqrt{x^2+1}} & \\ &=& \dfrac{x}{1+\sqrt{x^2+1}}\left(\dfrac{1-\sqrt{x^2+1}}{1-\sqrt{x^2+1}}\right) & \\ &=& \dfrac{x(1-\sqrt{x^2+1})}{1-x^2-1} & \\ &=& \dfrac{\sqrt{x^2+1}-1}{x} \quad . & \end{array}$$

The differential $\mathrm{d}x$ can expressed via $\mathrm{d}t$ from $x = 1 + \dfrac{2}{t^2-1}$. The Euler substitution $x = \tan\theta = \tan(2\arctan t)$ can now be summarized as follows.

$$\begin{array}{rcl} x &=& \dfrac{2t}{1-t^2} \\ \sqrt{x^2+1} &=& -1+\dfrac{2}{1-t^2} \\ \mathrm{d}x &=& \dfrac{2(1+t^2)}{(1-t^2)^2}\mathrm{d}t \\ t &=& \dfrac{\sqrt{x^2+1}-1}{x} \quad . \end{array} \tag{4}$$

11. Let the variables x and t be related via $\sqrt{x^2+1} = \frac{x}{t} - 1$.

   (a) Express x via t.

   (b) Express $\sqrt{x^2+1}$ via t alone.

   (c) Express $\mathrm{d}x$ via t and $\mathrm{d}t$.

3.2 Case 2: $\sqrt{1-x^2}$

3.2.1 $x = \cos\theta$

12. (a) Express $x, \mathrm{d}x$ and $\sqrt{1-x^2}$ via θ and $\mathrm{d}\theta$ for the trigonometric substitution $x = \cos\theta$, $\theta \in [0, \pi]$.

   (b) Express $x, \mathrm{d}x$ and $\sqrt{1-x^2}$ via t and $\mathrm{d}t$ for the Euler substitution $x = \cos(2\arctan t)$, $t \geq 0$. Express t via x.

Solution. 12.a The trigonometric substitution $x = \cos\theta$ is given by

$$\begin{array}{rcll}
\sqrt{-x^2+1} & = & \sqrt{1-\cos^2\theta} & \\
& = & \sqrt{\sin^2\theta} & \left| \begin{array}{l} \text{when } \theta \in [0,\pi] \text{ we have} \\ \sin\theta \geq 0 \text{ and so } \sqrt{\sin^2\theta} = \sin\theta \end{array} \right. \\
& = & \sin\theta \quad . &
\end{array}$$

The differential $\mathrm{d}x$ can be expressed via $\mathrm{d}\theta$ from $x = \cos\theta$. The substitution $x = \cos\theta$ can be now summarized as:

$$\begin{array}{rcl}
x & = & \cos\theta \\
\sqrt{-x^2+1} & = & \sin\theta \\
\mathrm{d}x & = & -\sin\theta \mathrm{d}\theta \\
\theta & = & \arccos x \quad .
\end{array}$$

Solution. 12.b We recall that the substitution $\theta = 2\arctan t$ transforms a trigonometric integral into an integral of a rational function. We now apply the substitution $2\arctan t$ after the substitution $x = \cos\theta$:

$$\begin{array}{rcll}
x & = & \cos\theta & \left| \text{use } \theta = 2\arctan t \right. \\
& = & \cos(2\arctan t) & \left| \text{use } \cos(2z) = \dfrac{1-\tan^2 z}{1+\tan^2 z} \right. \\
& = & \dfrac{1-\tan^2(\arctan t)}{1+\tan^2(\arctan t)} & \\
& = & \dfrac{1-t^2}{1+t^2} \quad . &
\end{array}$$

We can furthermore compute

$$\begin{array}{rcll}
\sqrt{-x^2+1} & = & \sqrt{1-\left(\dfrac{1-t^2}{1+t^2}\right)^2} & \\
& = & \sqrt{\dfrac{(1+t^2)^2-(1-t^2)^2}{(1+t^2)^2}} & \\
& = & \sqrt{\dfrac{4t^2}{(1+t^2)^2}} & \left| \sqrt{4t^2} = 2t \text{ because } t \geq 0 \right. \\
& = & \dfrac{2t}{1+t^2} \quad . &
\end{array} \qquad (5)$$

The differential $\mathrm{d}x$ can be computed from $x = \frac{1-t^2}{1+t^2}$. Finally, we can express t via x with a little algebra:

$$
\begin{aligned}
x &= \frac{1-t^2}{1+t^2} \\
(1+t^2)x &= 1-t^2 \\
t^2(x+1) &= 1-x \\
t^2 &= \frac{1-x}{1+x} \\
t &= \sqrt{\frac{1-x}{1+x}} && \Big| \text{ here we use } t>0 \\
t &= \frac{\sqrt{1-x}}{\sqrt{1+x}}\frac{\sqrt{1+x}}{\sqrt{1+x}} \\
t &= \frac{\sqrt{-x^2+1}}{x+1} \quad .
\end{aligned}
$$

The Euler substitution $x = \cos(2\arctan t)$ can be now summarized as:

$$
\begin{aligned}
x &= \frac{1-t^2}{1+t^2} \\
\sqrt{-x^2+1} &= \frac{2t}{1+t^2} \\
\mathrm{d}x &= -\frac{4t}{(t^2+1)^2}\mathrm{d}t \\
t &= \frac{\sqrt{-x^2+1}}{x+1} \quad .
\end{aligned}
\tag{6}
$$

13. Let the variables x and t be related via $\sqrt{-x^2+1} = (1-x)t$.

    (a) Express x via t.

    (b) Express $\sqrt{-x^2+1}$ via t alone.

    (c) Express $\mathrm{d}x$ via t and $\mathrm{d}t$.

Solution. 13.a.

$$
\begin{aligned}
\sqrt{-x^2+1} &= (1-x)t && \Big| \text{ square both sides} \\
(1-x)(1+x) &= (1-x)^2t^2 && \Big| \text{ divide by } (1-x) \\
1+x &= (1-x)t^2 \\
x(1+t^2) &= t^2-1 \\
x &= \frac{t^2-1}{t^2+1} = 1-\frac{2}{t^2+1} \quad .
\end{aligned}
$$

Solution. 13.b.

Use Problem 13.a to get

$$
\sqrt{-x^2+1} = (1-x)t = \left(1-\left(1-\frac{2t}{t^2+1}\right)\right)t = \frac{2t}{t^2+1} \quad .
$$

3.2.2 $x = \sin\theta$

14. (a) Express $x, \mathrm{d}x$ and $\sqrt{1-x^2}$ via θ and $\mathrm{d}\theta$ for the trigonometric substitution $x = \sin\theta$, $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

    (b) Express $x, \mathrm{d}x$ and $\sqrt{1-x^2}$ via t and $\mathrm{d}t$ for the Euler substitution $x = \sin(2\arctan t)$, $t \in [-1, 1]$. Express t via x.

Solution. 14.a The trigonometric substitution $x = \sin\theta$ is given by

$$
\begin{aligned}
\sqrt{-x^2+1} &= \sqrt{1-\sin^2\theta} \\
&= \sqrt{\cos^2\theta} && \Big| \text{ when } \theta \in \left[-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right] \text{ we have } \cos\theta \geq 0 \text{ and so } \sqrt{\cos^2\theta} = \cos\theta \\
&= \cos\theta \quad .
\end{aligned}
$$

The differential dx can be expressed via dθ from $x = \sin\theta$. The substitution $x = \sin\theta$ can be now summarized as:

$$\begin{array}{rcl} x &=& \sin\theta \\ \sqrt{-x^2+1} &=& \sin\theta \\ \mathrm{d}x &=& \cos\theta \mathrm{d}\theta \\ \theta &=& \arcsin x \quad . \end{array}$$

Solution. 14.b We recall that the substitution $\theta = 2\arctan t$ transforms a trigonometric integral into an integral of a rational function. We now apply the substitution $2\arctan t$ after the substitution $x = \sin\theta$:

$$\begin{array}{rcll} x &=& \sin\theta & \Big| \text{ use } \theta = 2\arctan t \\ &=& \sin(2\arctan t) & \Big| \text{ use } \sin(2z) = \dfrac{2\tan z}{1+\tan^2 z} \\ &=& \dfrac{2\tan(\arctan t)}{1+\tan^2(\arctan t)} & \\ &=& \dfrac{2t}{1+t^2} \quad . & \end{array}$$

We can furthermore compute

$$\begin{array}{rcll} \sqrt{-x^2+1} &=& \sqrt{1-\left(\dfrac{2t}{1+t^2}\right)^2} & \\ &=& \sqrt{\dfrac{(1+t^2)^2-4t^2}{(1+t^2)^2}} & \\ &=& \sqrt{\dfrac{(1-t^2)^2}{(1+t^2)^2}} & \Big| \sqrt{(1-t^2)^2} = 1-t^2 \text{ because } |t| \leq 1 \\ &=& \dfrac{1-t^2}{1+t^2} & \\ &=& \dfrac{2-(1+t^2)}{1+t^2} & \\ &=& -1+\dfrac{2}{1+t^2} \quad . & \end{array} \tag{7}$$

The differential dx can be computed from $x = \frac{2t}{1+t^2}$. Finally, we can express t via x with a little algebra:

$$\begin{array}{rcll} \sqrt{-x^2+1} &=& -1+\dfrac{2}{1+t^2} & \\ &=& -1+\dfrac{1}{t}\left(\dfrac{2t}{1+t^2}\right) & \Big| \text{ use } x = \frac{2t}{1+t^2} \\ &=& -1+\dfrac{x}{t} & \Big| +1 \text{ to both sides} \\ \dfrac{x}{t} &=& 1+\sqrt{-x^2+1} & \\ t &=& \dfrac{x}{1+\sqrt{-x^2+1}} & \\ &=& \dfrac{x}{(1+\sqrt{-x^2+1})}\dfrac{(1-\sqrt{-x^2+1})}{(1-\sqrt{-x^2+1})} & \\ &=& \dfrac{1-\sqrt{-x^2+1}}{x} \quad . & \end{array}$$

The Euler substitution $x = \sin(2\arctan t)$ can be now summarized as:

$$\begin{array}{rcl} x &=& \dfrac{2t}{1+t^2} \\ \sqrt{-x^2+1} &=& -1+\dfrac{2}{1+t^2} \\ \mathrm{d}x &=& 2\left(\dfrac{1-t^2}{(1+t^2)^2}\right)\mathrm{d}t \\ t &=& \dfrac{1-\sqrt{-x^2+1}}{x} \quad . \end{array}$$

15. Let the variables x and t be related via $\sqrt{-x^2+1} = 1 - xt$.

    (a) Express x via t.

    (b) Express $\sqrt{-x^2+1}$ via t alone.

    (c) Express $\mathrm{d}x$ via t and $\mathrm{d}t$.

3.3 Case 3: $\sqrt{x^2-1}$

3.3.1 $x = \sec\theta$

16. (a) Express $x, \mathrm{d}x$ and $\sqrt{x^2-1}$ via θ and $\mathrm{d}\theta$ for the trigonometric substitution $x = \csc\theta$, $\theta \in \left[0, \frac{\pi}{2}\right] \cup \left[\pi, \frac{3\pi}{2}\right)$.

    (b) Express $x, \mathrm{d}x$ and $\sqrt{1-x^2}$ via t and $\mathrm{d}t$ for the Euler substitution $x = \sec(2\arctan t)$, $t \in (-\infty, -1) \cup [1, 0)$. Express t via x.

Solution. 16.a The trigonometric substitution $x = \sec\theta$ is given by

$$\begin{array}{rcll}
\sqrt{x^2-1} & = & \sqrt{\sec^2\theta - 1} = \sqrt{\dfrac{1}{\cos^2\theta} - 1} & \\
& = & \sqrt{\dfrac{\sin^2\theta}{\cos^2\theta}} = \sqrt{\tan^2\theta} & \left|\begin{array}{l}\text{when } \theta \in \theta \in \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right) \text{ we have} \\ \tan\theta \geq 0 \text{ and so } \sqrt{\tan^2\theta} = \tan\theta\end{array}\right. \\
& = & \tan\theta \quad . &
\end{array}$$

The differential $\mathrm{d}x$ can be expressed via $\mathrm{d}\theta$ from $x = \sec\theta$. The substitution $x = \sec\theta$ can be now summarized as:

$$\begin{array}{rcl}
x & = & \sec\theta = \dfrac{1}{\cos\theta} \\
\sqrt{x^2-1} & = & \tan\theta \\
\mathrm{d}x & = & \dfrac{\sin\theta}{\cos^2\theta}\mathrm{d}\theta = \sec\theta\tan\theta\mathrm{d}\theta \\
\theta & = & \operatorname{arcsec} x \quad .
\end{array}$$

Solution. 16.b We recall that the substitution $\theta = 2\arctan t$ transforms a trigonometric integral into an integral of a rational function. We now apply the substitution $2\arctan t$ after the substitution $x = \sec\theta$:

$$\begin{array}{rcll}
x & = & \sec\theta = \dfrac{1}{\cos\theta} & \left|\begin{array}{l}\text{use } \theta = 2\arctan t \\ \text{use } \cos(2z) = \dfrac{1-\tan^2 z}{1+\tan^2 z}\end{array}\right. \\
& = & \dfrac{1}{\cos(2\arctan t)} & \\
& = & \dfrac{1+\tan^2(\arctan t)}{1-\tan^2(\arctan t)} & \\
& = & \dfrac{1+t^2}{1-t^2} & \\
& = & -1 + \dfrac{2}{1-t^2} \quad . &
\end{array}$$

We can furthermore compute

$$\begin{array}{rcll}
\sqrt{x^2-1} & = & \sqrt{\left(\dfrac{1+t^2}{1-t^2}\right)^2 - 1} & \\
& = & \sqrt{\dfrac{(1+t^2)^2 - (1-t^2)^2}{(1-t^2)^2}} & \\
& = & \sqrt{\dfrac{4t^2}{(1-t^2)^2}} & \left|\begin{array}{l} t \text{ and } 1-t^2 \text{ have the same} \\ \text{sign for } t \in (-\infty, -1) \cup [0, 1)\end{array}\right. \\
& = & \dfrac{2t}{1-t^2} \quad . &
\end{array} \qquad (8)$$

The differential dx can be computed from $x = \frac{1+t^2}{1-t^2}$. Finally, we can express t via x with a little algebra:

$$\begin{array}{rcll}
x &=& \dfrac{1+t^2}{1-t^2} & \\
(1-t^2)x &=& 1+t^2 & \\
(1+x)t^2 &=& x-1 & \\
t^2 &=& \dfrac{x-1}{x+1} & \\
t &=& \begin{cases} \sqrt{\frac{x-1}{x+1}} & x>1 \\ -\sqrt{\frac{x-1}{x+1}} & x<-1 \end{cases} & \left| \begin{array}{l}\text{because when } x<-1, \\ \text{we have } t\in(-\infty,-1] \end{array}\right. \\
t &=& \begin{cases} \frac{\sqrt{x^2-1}}{x+1} & x>1 \\ -\frac{\sqrt{x^2-1}}{x+1} & x<-1 \end{cases} & .
\end{array}$$

The Euler substitution $x = \sec(2\arctan t)$ can be now summarized as:

$$\begin{array}{rcl}
x &=& \dfrac{1+t^2}{1-t^2} \\
\sqrt{x^2-1} &=& \dfrac{2t}{1-t^2} \\
\mathrm{d}x &=& \dfrac{4t}{(1-t^2)^2}\mathrm{d}t \\
t &=& \pm\dfrac{\sqrt{x^2-1}}{x+1} \quad .
\end{array}$$

17. Let the variables x and t be related via $\sqrt{x^2-1} = (x+1)t$.

    (a) Express x via t.

    (b) Express $\sqrt{x^2-1}$ via t alone.

    (c) Express dx via t and dt.

Solution. 17.a.

$$\begin{array}{rcll}
\sqrt{x^2-1} &=& (x+1)t & \left| \text{square both sides} \right. \\
(x-1)(x+1) &=& (x+1)^2t^2 & \left| \text{divide by } (x+1) \right. \\
x-1 &=& (x+1)t^2 & \\
x(1-t^2) &=& 1+t^2 & \\
x &=& \dfrac{1+t^2}{1-t^2} = -1 + \dfrac{2}{1-t^2} &
\end{array}$$

Solution. 17.b.

We use Problem 17.a to get

$$\sqrt{x^2-1} = (x+1)t = \left(-1 + \frac{2}{1-t^2} + 1\right)t = \frac{2t}{1-t^2}$$

3.3.2 $x = \csc\theta$

18. (a) Express $x, \mathrm{d}x$ and $\sqrt{1-x^2}$ via θ and dθ for the trigonometric substitution $x = \csc\theta$, $\theta \in \left[0, \frac{\pi}{2}\right] \cup \left[\pi, \frac{3\pi}{2}\right)$.

    (b) Express $x, \mathrm{d}x$ and $\sqrt{1-x^2}$ via t and dt for the Euler substitution $x = \csc(2\arctan t)$, $t \in (-\infty,-1) \cup [0,1)$. Express t via x.

Solution. 18.a The trigonometric substitution $x = \csc\theta$ is given by

$$\begin{array}{rcll} \sqrt{x^2-1} &=& \sqrt{\dfrac{1}{\sin^2\theta}-1} & \\ &=& \sqrt{\dfrac{\cos^2\theta}{\sin^2\theta}} = \sqrt{\cot^2\theta} & \Bigg| \begin{array}{l}\text{when } \theta\in\theta\in\left[0,\frac{\pi}{2}\right)\cup\left[\pi,\frac{3\pi}{2}\right) \text{ we have} \\ \cot\theta\geq 0 \text{ and so } \sqrt{\cot^2\theta}=\tan\theta\end{array} \\ &=& \cot\theta \quad . & \end{array}$$

The differential dx can be expressed via dθ from $x = \csc\theta$. The substitution $x = \csc\theta$ can be now summarized as:

$$\begin{array}{rcl} x &=& \csc\theta \\ \sqrt{x^2-1} &=& \cot\theta \\ \mathrm{d}x &=& -\dfrac{\cos\theta}{\sin^2\theta}\mathrm{d}\theta = -\csc\theta\cot\theta\mathrm{d}\theta \\ \theta &=& \csc^{-1}x \quad . \end{array}$$

Solution. 18.b We recall that the substitution $\theta = 2\arctan t$ transforms a trigonometric integral into an integral of a rational function. We now apply the substitution $2\arctan t$ after the substitution $x = \csc\theta$:

$$\begin{array}{rcll} x &=& \csc\theta = \dfrac{1}{\sin\theta} & \Big| \text{use } \theta = 2\arctan t \\ &=& \dfrac{1}{\sin(2\arctan t)} & \Big| \text{use } \sin(2z) = \dfrac{2\tan z}{1+\tan^2 z} \\ &=& \dfrac{1+\tan^2(\arctan t)}{2\tan(\arctan t)} & \\ &=& \dfrac{1+t^2}{2t} & \\ &=& \dfrac{1}{2}\left(\dfrac{1}{t}+t\right) \quad . & \end{array}$$

We can furthermore compute

$$\begin{array}{rcll} \sqrt{x^2-1} &=& \sqrt{\left(\dfrac{1+t^2}{2t}\right)^2-1} & \\ &=& \sqrt{\dfrac{(1+t^2)^2-4t^2}{4t^2}} & \\ &=& \sqrt{\dfrac{(1-t^2)^2}{4t^2}} & \Big| \dfrac{1-t^2}{2t} > 0 \text{ when } t\in(-\infty,-1)\cup[0,1) \\ &=& \dfrac{1-t^2}{2t} & \\ &=& \dfrac{1}{2}\left(\dfrac{1}{t}-t\right) \quad . & \end{array} \tag{9}$$

The differential dx can be computed from $x = \frac{1}{2}\left(\frac{1}{t}-t\right)$. Finally, we can express t via x with a little algebra:

$$\begin{array}{rcll} \sqrt{x^2-1} &=& \dfrac{1-t^2}{2t} & \\ \sqrt{x^2-1} &=& \dfrac{2-(1+t^2)}{2t} & \Big| \text{use } x = \dfrac{1+t^2}{2t} \\ \sqrt{x^2-1} &=& \dfrac{1}{t}-x & \\ \dfrac{1}{t} &=& \sqrt{x^2-1}+x & \\ t &=& \dfrac{1}{\sqrt{x^2-1}+x} = \dfrac{1}{(\sqrt{x^2-1}+x)}\dfrac{(-\sqrt{x^2-1}+x)}{(-\sqrt{x^2-1}+x)} & \\ t &=& x-\sqrt{x^2-1} & \end{array}$$

The Euler substitution $x = \cos(2 \arctan t)$ can be now summarized as:

$$
\begin{array}{rcl}
x & = & \frac{1}{2}\left(\frac{1}{t} + t\right) \\
\sqrt{-x^2+1} & = & \frac{1}{2}\left(\frac{1}{t} - t\right) \\
\mathrm{d}x & = & -\frac{1}{2}\left(\frac{1}{t^2} + 1\right)\mathrm{d}t \\
t & = & x - \sqrt{x^2 - 1} \quad .
\end{array}
$$

19. Let the variables x and t be related via $\sqrt{x^2 - 1} = \frac{1}{t} - x$.

    (a) Express x via t.

    (b) Express $\sqrt{x^2 - 1}$ via t alone.

    (c) Express $\mathrm{d}x$ via t and $\mathrm{d}t$.