Calculus III
Lecture 7

Todor Milev
https://github.com/tmilev/freecalc

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Outline

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- So far, the functions we studied had one dimensional (scalar) input.
- Most mathematical models deal with phenomena where the output depends on several variables.
- Variables may be “dependent” or independent - issue dealt with in the subject of probabilities/statistics.
- We need to build and use functions with multidimensional input.
- Such input is typically represented as a bundle of scalar variables.

Describing multivariable functions

- When doing mathematical modeling, there are several ways to define a function of several variables.
- Usually: start with *verbal* description, then give specific meanings to our input and output variables.
- We explain by examples.

Verbal description examples

- The apparent temperature, W, felt on exposed skin depends on several factors, including the actual temperature, T, the wind speed, v, and the humidity. The *wind chill temperature* is a mathematical model for W under the assumption that the humidity is 0 and that the only factors influencing W are T and v:

$$W = W(T, v) .$$

  The domain of the function W consists of all reasonable pairs (T, v).
- The Cobb-Douglas production function models the production output, P, under the assumption that the only factors are the amount of labor, L, and the amount of capital, K:

$$P = P(L, K) .$$

Verbal description examples

- The magnitude G of the attraction force between two mass points depends on the masses m and M of the bodies and the distance d between them:
$$G = G(m, M, d) \quad .$$
- A set (ρ, ϕ, θ) of spherical coordinates determines the rectangular coordinates (x, y, z) of a point. In this case, both the input and the output are multidimensional:
$$(x, y, z) = \mathbf{F}(\rho, \theta, \phi) \ ,$$

- The wind velocity $\mathbf{v}$ at a point P depends on the position $\mathbf{r}$ of P,

$$\mathbf{v} = \mathbf{V}(\mathbf{r}) \ .$$

  In this case both the input and the output are vectors.
- The electric force on a charge q displaced by $\mathbf{r}$ from a charge Q depends on the two charges, the displacement, and the medium in which the charges are placed:

$$\mathbf{E} = \mathbf{E}(q, Q, \mathbf{r}) \ .$$

  Note that in this case the output data is a vector and the input data is a mix of scalars and vectors.

Numerical description

- Verbal description is essential for understanding.
- However this does not include quantitative or visual information.
- A *numerical* description gives output data for a relevant set of input data.
- This facilitates construction/study of a mathematical model.
- Numerical description is typically given by table.
- This table contains numerical data collected through experiments at selected input levels.

Example: Describing Function Via Numerical Data

- the following is Wind Chill Chart provided by NOOA. The table entries indicate the temperature felt on exposed skin under the corresponding wind speed and temperature.

	Temperature °F																	
Wind (mph)	40	35	30	25	20	15	10	5	0	-5	-10	-15	-20	-25	-30	-35	-40	-45
5	36	31	25	19	13	7	1	-5	-11	-16	-22	-28	-34	-40	-46	-52	-57	-63
10	34	27	21	15	9	3	-4	-10	-16	-22	-28	-35	-41	-47	-53	-59	-66	-72
15	32	25	19	13	6	0	-7	-13	-19	-26	-32	-39	-45	-51	-58	-64	-71	-77
20	30	24	17	11	4	-2	-9	-15	-22	-29	-35	-42	-48	-55	-61	-68	-74	-81
25	29	23	16	9	3	-4	-11	-17	-24	-31	-37	-44	-51	-58	-64	-71	-78	-84
30	28	22	15	8	1	-5	-12	-19	-26	-33	-39	-46	-53	-60	-67	-73	-80	-87
35	28	21	14	7	0	-7	-14	-21	-27	-34	-41	-48	-55	-62	-69	-76	-82	-89
40	27	20	13	6	-1	-8	-15	-22	-29	-36	-43	-50	-57	-64	-71	-78	-84	-91
45	26	19	12	5	-2	-9	-16	-23	-30	-37	-44	-51	-58	-65	-72	-79	-86	-93
50	26	19	12	4	-3	-10	-17	-24	-31	-38	-45	-52	-60	-67	-74	-81	-88	-95
55	25	18	11	4	-3	-11	-18	-25	-32	-39	-46	-54	-61	-68	-75	-82	-89	-97
60	25	17	10	3	-4	-11	-19	-26	-33	-40	-48	-55	-62	-69	-76	-84	-91	-98

- Another example is the Income Tax Table. Explain what the input and output variables are in that case.

Analytical description of multivarible function

- Numerical data has output data for selected inputs only.
- If output is not tabulated for given input we need to approximate.
- This is done by inter/extrapolation from given information.
- It would be better to have procedure to determine output from any reasonable input.
- This would be an *analytical* description of the function.
- By analytical description we mean giving a procedure to compute the value of the function:
  - via formula or
  - via another algorithmic procedure.

From numerical to analytical description

- One way is to try to guess a formula that fits approximately the input data.
- For wind chill, one such formula is:

$$W(T, v) = 35.74 + 0.6215T - 35.75v^{0.16} + 0.4275Tv^{0.16}$$

  with W and T in Fahrenheit and v in *mph*.
- A proper mathematical model requires that we
  - compute unknown output for some input and
  - make a new measurement and compare with the model's output to see if model gives correct prediction.
- Constructing mathematical models to fit numerical data (approximately) is the subject of "Approximation theory".
- Mathematicians dealing with "approximation theory" are often called "applied mathematicians".
- The above terms are not precisely defined and not fully agreed upon.

- For the Cobb-Douglas production function: economic analysis motivates properties such function should have.
- One formula (model) with these properties is:

$$P(K, L) = cL^aK^{1-a}\ ;$$

  where a is a parameter between 0 and 1.
- While the function P depends on three variables a, L, and K, we treat them differently: we consider a to be a parameter of the model; once we decide on the value of a, we treat it as a constant.
- The transition formulas from spherical to rectangular coordinates are a derived via geometric reasoning.

- An important class of functions of several variables is the class of polynomial functions.
- Polynomial functions in n variables are obtained using n variables, the constants and three easiest arithmetic operations - $+, -, \cdot$.
- Polynomials of degree one in two and three variables are parametrized by:

$$\begin{aligned} f(x,y) &= ax + by + c \\ g(x,y,z) &= ax + by + cz + d \quad . \end{aligned}$$

- Polynomials of degree two in two and three variables are parametrized by:

$$\begin{aligned} f(x,y) &= a_{11}x^2 + a_{12}xy + a_{22}y^2 + a_1x + a_2y + a_0 \\ g(x,y,z) &= a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + a_{12}xy + a_{13}xz + a_{23}yz + \\ &\quad + a_1x + a_2y + a_3z + a_0 \end{aligned}$$

where the a_{ij}'s are real numbers.

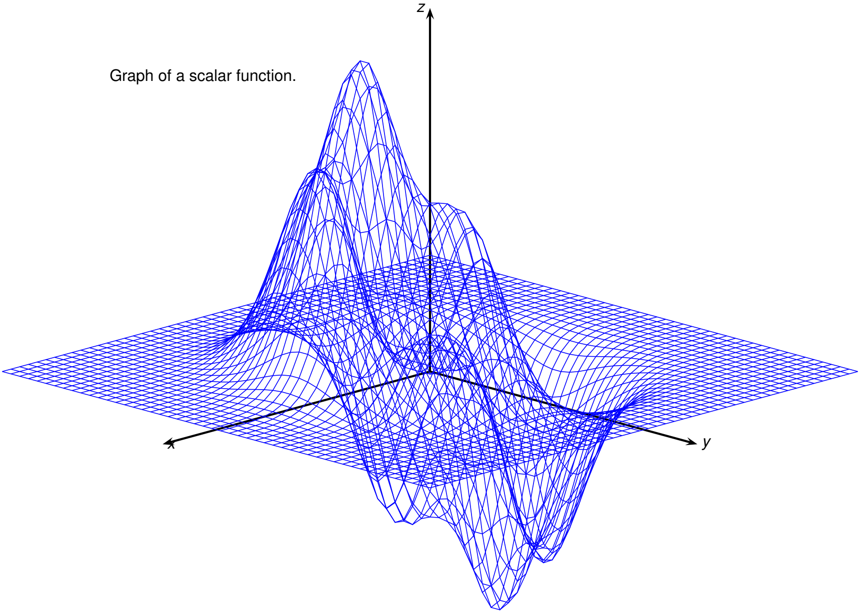
- The formula for electric force is given by laws of physics: the magnitude of the force is directly proportional to the charges q, Q, and inversely proportional to the square of the distance between them. The force acts along the line joining the two points, attracts q to Q if the charges have different sign and rejects q from Q if the charges have the same sign. The mathematical translation is

$$\mathbf{E}(q, Q, \mathbf{r}, \epsilon) = \frac{\epsilon q Q}{|\mathbf{r}|^3}\mathbf{r} \; ,$$

where ϵ is a proportionality constant, depending on the medium the charges are placed in.

- An analytical description is technically best, but not easy to interpret.
- If output is a scalar, where does the function attain its extreme values (maxima, minima)?
- How do values change for nearby points - are they decreasing, increasing, how fast?
- We will learn to decode this information from the analytical descriptions.
- Even so, “a picture is worth a thousand words”.

Graph of a scalar function.



Graph of a function

- For one variable function, $y = f(x)$, the graph of f is a set of points in $\mathbb{R}^2$: the set of points (x, y) such that $y = f(x)$.
- Example: if $f(x) = x^2$, then $(3, 9)$ is on the graph, because $9 = 3^2$, but $(2, 5)$ is not because $5 \neq 2^2$.
- We can extend this graphical representation for functions with two dimensional input and one dimensional (scalar) output.
- The *graph* of the function $f\colon D \to \mathbb{R}$, where D is a region in $\mathbb{R}^2$, is the set of points $P(x, y, z)$ in $\mathbb{R}^3$ whose coordinates satisfy the condition
$$z = f(x, y) \quad .$$
- For example, the graph of $f(x, y) = 2x - y + 3$ is the set
$$\{(x, y, z) \,|\, z = 2x - y + 3\} \implies \text{plane } 2x - y - z + 3 = 0 \ .$$

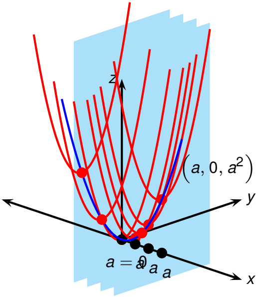
A motivating example

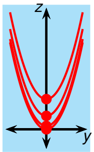
$g(x, y) = x^2 + 2y^2$

- What does the graph Γ of g look like?
- Γ = points in $\mathbb{R}^3$ such that $z = x^2 + 2y^2$. The set is not a plane: what does it look like?
- To answer look at sections. Use imaginary CT scan to cut the graph; assemble resulting sections into a graph.

A motivating example

$g(x, y) = x^2 + 2y^2$



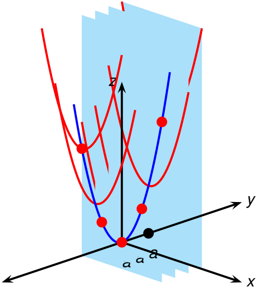


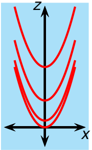
The plane $x = a$.

- Cut by vertical planes $x = a$, a-constant, parallel to the $Oyz-$plane.
- In other words, treat x as constant and study the f-n $y \to z = a^2 + 2y^2 = g(a, y)$.
- The cross-sections are the curves: $\{(a, y, z) \quad \text{where } z = a^2 + 2y^2\}$
- These are parabolas lying inside the plane $x = a$ with vertices at $(a, 0, a^2)$.
- As a moves away from 0, the parabola vertex rises.
- The vertices traverse the curve given by $\{(a, 0, a^2)\}$.

A motivating example

$g(x, y) = x^2 + 2y^2$



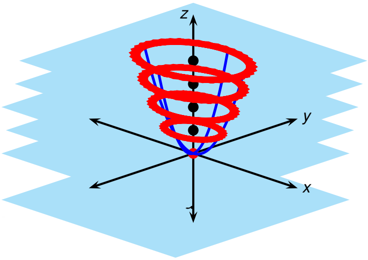


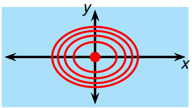
The plane $x = a$.

- Similarly, cut by vertical planes $y = a$, i.e., planes parallel to the Oxz plane.
- In other words, treat y as constant and study the f-n $z = g(x, a) = x^2 + 2a^2$.
- The cross-sections are the curves $\{(x, a, z) \text{ where } z = x^2 + 2a^2\}$. These are parabolas lying inside the plane $y = a$.
- $\Rightarrow$ the vertical sections along both the x and y axes are parabolas.
- The vertices are rising as we move away from the origin.

A motivating example

$g(x, y) = x^2 + 2y^2$





$x^2 + 2y^2 = a.$

- For horizontal sections keep constant the output variable, $z = a$.
- When we intersect with $z = a$ we get the curve with equations $x^2 + 2y^2 = a$, $z = a$.
- For $a < 0$ intersection is empty.
- For $a = 0$ intersection is $(0, 0, 0)$.
- For $a > 0$ intersection is an ellipse.
- Figure is called ellipsoidal paraboloid.

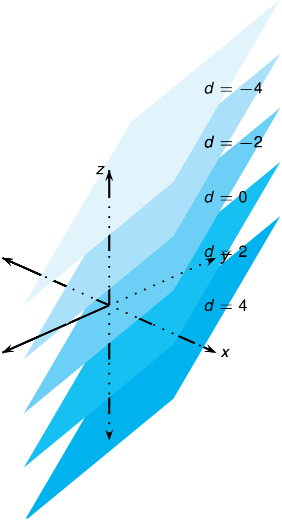
Definition

The sets $\{(x, y, a) | g(x, y) = a\}$ are called level curves of the function g.

- You should be familiarized with level curves if you have ever seen a topographic map or from weather reports on the tv.
- What are the functions in those cases?

- Previously we considered functions $z = g(x, y)$ with scalar output and two dimensional input.
- The graphs of such functions live in $\mathbb{R}^3 = \mathbb{R}^{2+1}$.
- 2 dimensions were used to represent the input.
- 1 dimension was used to represent the output.
- To represent functions with 3 dimensional input (3 variables) and scalar output: need $3 + 1 = 4$ dimensions.
- That's difficult for eyes used to visualizing physical 3d- space.
- Instead: label the level sets of the function with color or other means to indicate value.
- In this way we represent the f-n graphically using dimension equal to the number of input variables.

Example



- Let $f(x, y, z) = x + y - z$.
- The graph consists of the quadruples (x, y, z, w) in $\mathbb{R}^4$ such that $w = x + y - z$. Can't plot that graphically (yet).
- However, can represent with labeled level sets.
- The level set $f(x, y, z) = d$ is the surface $x + y - z = d$ in $\mathbb{R}^3$, and that surface is a plane.
- For varying values of d we plot the level set. $f(x, y, z) = d$. Darker color = larger d.

To understand surfaces in space we need the following.

Remark

The level set $f(x, y, z) = 0$ for the function

$$f(x, y, z) = ax + by - z + d$$

is the same as the graph of the function $g(x, y) = ax + by + c$.

- Graph surfaces can always be represented as level surfaces
- The converse is not true: level surfaces can't always be represented as graph surfaces.
- Example: a sphere centered at the origin is the level surface of $f(x, y, z) = x^2 + y^2 + z^2$ but it "fails the vertical line test in all directions", so it cannot be globally represented as a graph surface, no matter how we change the coordinate system.
- We'll show that under reasonable assumptions, level surfaces can *locally* be described as graph surfaces.

Vector fields

- *Vector fields* are functions with multidimensional input and output.
- Input is point in space; output is a vector, which we plot as a vector with a tail at the input point.
- Examples
  - Velocity of fluid/air at given point;
  - Electric force per unit of charge;
  - Gravitational field;

Coordinate representation of vector fields

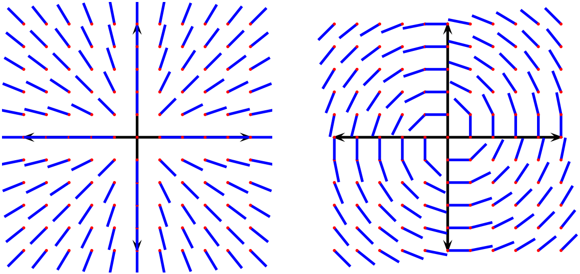
- In rectangular coordinates a vector field **F** can be decomposed along the fundamental directions:

$$\mathbf{F}(x,y,z) = F_1(x,y,z)\mathbf{i} + F_2(x,y,z)\mathbf{j} + F_3(x,y,z)\mathbf{k}\ .$$

- For regions in the plane 2-dim vector fields are defined in a similar fashion: as function from subsets of $\mathbb{R}^2$ to $\mathbb{R}$:

$$\mathbf{F}(x,y) = F_1(x,y)\mathbf{i} + F_2(x,y)\mathbf{j}$$

- Example: define the vector field $\mathbf{e}_r$ on $\mathbb{R}^2 \setminus \{(0,0)\}$ via
  $\mathbf{e}_r = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j} = \frac{x}{r}\,\mathbf{i} + \frac{y}{r}\,\mathbf{j} = \frac{x}{\sqrt{x^2+y^2}}\,\mathbf{i} + \frac{y}{\sqrt{x^2+y^2}}\,\mathbf{j}$
- Similarly define the vector field $\mathbf{e}_\theta$ by:
  $\mathbf{e}_\theta = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j} = -\frac{y}{r}\,\mathbf{i} + \frac{x}{r}\,\mathbf{j} = -\frac{y}{\sqrt{x^2+y^2}}\,\mathbf{i} + \frac{x}{\sqrt{x^2+y^2}}\,\mathbf{j}$.



- From the picture it is evident what trajectory would be followed by an object that "flows along the vector field".
- By "flowing" we mean an object whose velocity at each point is given by the value of the field.

Similar to decomposition in rectangular coordinates we can decompose a vector field along fundamental vectors corresponding to other coordinate systems. Things are a bit trickier, since the fundamental vectors change from point to point.
In particular, a planar vector field can be written in terms of $\mathbf{e}_r$ and $\mathbf{e}_\theta$:

$$\mathbf{X}(r,\theta) = X_1(r,\theta)\mathbf{e}_r + X_2(r,\theta)\mathbf{e}_\theta \ .$$

For example, if $X(P) = \mathbf{i}$, then

$$X(r,\theta) = \cos\theta\,\mathbf{e}_r - \sin\theta\,\mathbf{e}_\theta \ .$$