Calculus II
Homework on Lecture 1

1. Let $x \in (0,1)$. Express the following using x and $\sqrt{1-x^2}$.

   (a) $\sin(\arcsin(x))$.

   answer: x

   (b) $\sin(2\arcsin(x))$.

   answer: $2x\sqrt{1-x^2}$

   (c) $\sin(3\arcsin(x))$.

   answer: $-4x^3+3x$

   (d) $\sin(\arccos(x))$.

   answer: $\sqrt{1-x^2}$

   (e) $\sin(2\arccos(x))$.

   answer: $2x\sqrt{1-x^2}$

   (f) $\sin(3\arccos(x))$.

   answer: $(4x^2-1)\sqrt{1-x^2}$ $= -4\left(\sqrt{1-x^2}\right)^3 + 3\sqrt{1-x^2}$

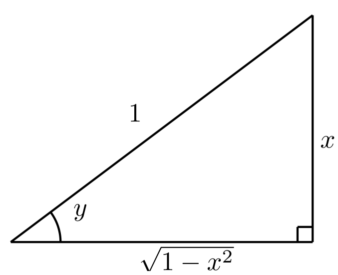
   (g) $\cos(2\arcsin(x))$.

   answer: $1-2x^2$

   (h) $\cos(3\arccos(x))$.

   answer: $4x^3-3x$

Solution. 1.b. Let $y = \arcsin x$. Then $\sin y = x$, and we can draw a right triangle with opposite side length x and hypotenuse length 1 to find the other trigonometric ratios of y.



Then $\cos y = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$. Now we use the double angle formula to find $\sin(2\arcsin x)$.

$$\begin{aligned}\sin(2\arcsin x) &= \sin(2y)\\ &= 2\sin y\cos y\\ &= 2x\sqrt{1-x^2}.\end{aligned}$$

Solution. 1.c. Use the result of Problem 1.b. This also requires the addition formula for sine:

$$\sin(A+B) = \sin A\cos B + \sin B\cos A,$$

and the double angle formula for cosine:

$$\cos(2y) = \cos^2 y - \sin^2 y.$$

$$
\begin{aligned}
\sin(3\arcsin x) &= \sin(3y)\\
&= \sin(2y+y)\\
&= \sin(2y)\cos y + \sin y\cos(2y) && \text{Use addition formula}\\
&= (2\sin y\cos y)\cos y + \sin y(\cos^2 y - \sin^2 y) && \text{Use double angle formulas}\\
&= 2\sin y\cos^2 y + \sin y\cos^2 y - \sin^3 y\\
&= 3\sin y\cos^2 y - \sin^3 y\\
&= 3\sin y(1-\sin^2 y) - \sin^3 y\\
&= 3x(1-x^2) - x^3\\
&= 3x - 4x^3.
\end{aligned}
$$

The solution is complete. A careful look at the solution above reveals a strategy useful for problems similar to this one.

(a) Identify the inverse trigonometric expression- $\arcsin x, \arccos x, \arctan x, \dots$. In the present problem that was $y = \arcsin x$.

(b) The problem is therefore a trigonometric function of y.

(c) Using trig identities and algebra, rewrite the problem as a trigonometric expression involving only the trig function that transforms y to x. In the present problem we rewrote everything using $\sin y$.

(d) Use the fact that $\sin(\arcsin x) = x$, $\cos(\arccos x) = x, \dots$, etc. to simplify.

Solution. 1.f We use the same strategy outlined in the end of the solution of Problem 1.c. Set $y = \arccos x$ and so $\cos(y) = x$. Therefore:

$$
\begin{aligned}
\sin(3y) &= \sin(2y+y)\\
&= \sin(2y)\cos y + \sin y\cos(2y)\\
&= 2\sin y\cos y\cos y + \sin y(2\cos^2 y - 1)\\
&= 2\sin y\cos^2 y + \sin y(2\cos^2 y - 1)\\
&= \sin y(4\cos^2 y - 1) && \text{use } \begin{array}{rcl}\cos y &=& x\\ \sin y &=& \sqrt{1-x^2}\end{array}\\
&= \sqrt{1-x^2}(4x^2-1) \quad .
\end{aligned}
$$

2. Express as the following as an algebraic expression of x. In other words, "get rid" of the trigonometric and inverse trigonometric expressions.

(a) $\cos^2(\arctan x)$. answer: $\frac{1}{1+x^2}$

(b) $-\sin^2(\operatorname{arccot} x)$. answer: $-\frac{1}{1+x^2}$

(c) $\dfrac{1}{\cos(\arcsin x)}$. answer: $\frac{1}{\sqrt{1-x^2}}$

(d) $-\dfrac{1}{\sin(\arccos x)}$. answer: $-\frac{1}{\sqrt{1-x^2}}$

Solution. 2.b. We follow the strategy outlined in the end of the solution of Problem 1.c. We set $y = \operatorname{arccot} x$. Then we need to express $-\sin^2 y$ via $\cot y$. That is a matter of algebra:

$$
\begin{aligned}
-\sin^2(\operatorname{arccot} x) &= -\sin^2 y && \text{Set } y = \operatorname{arccot} x\\
&= -\frac{\sin^2 y}{\sin^2 y + \cos^2 y} && \text{use } \sin^2 y + \cos^2 y = 1\\
&= -\frac{1}{\frac{\sin^2 y + \cos^2 y}{\sin^2 y}}\\
&= -\frac{1}{1+\cot^2 y} && \text{Substitute back } \cot y = x\\
&= -\frac{1}{1+x^2} \quad .
\end{aligned}
$$

3. Rewrite as a rational function of t. This problem will be later used to derive the Euler substitutions (an important technique for integrating).

(a) $\cos(2\arctan t)$.

answer: $\frac{1-t^2}{1+t^2}$.

(b) $\sin(2\arctan t)$.

answer: $\frac{2t}{1+t^2}$

(c) $\tan(2\arctan t)$.

answer: $\frac{2t}{1-t^2}$

(d) $\cot(2\arctan t)$.

answer: $\frac{1}{2}\left(\frac{1}{t}-t\right)$.

(e) $\csc(2\arctan t)$.

answer: $\frac{1}{2}\left(t+\frac{1}{t}\right)$

(f) $\sec(2\arctan t)$.

answer: $\frac{1+t^2}{1-t^2}$

(g) $\cos(2\,\mathrm{arccot}\, t)$.

answer: $\frac{t^2-1}{t^2+1}$

(h) $\sin(2\,\mathrm{arccot}\, t)$.

answer: $\frac{2t}{t^2+1}$

(i) $\tan(2\,\mathrm{arccot}\, t)$.

answer: $\frac{2t}{t^2-1}$

(j) $\cot(2\,\mathrm{arccot}\, t)$.

answer: $\frac{1}{2}\left(t-\frac{1}{t}\right)$

(k) $\csc(2\,\mathrm{arccot}\, t)$.

answer: $\frac{1}{2}\left(t+\frac{1}{t}\right)$

(l) $\sec(2\,\mathrm{arccot}\, t)$.

answer: $\frac{t^2+1}{t^2-1}$

Solution. 3.a Set $z = \arctan t$, and so $\tan z = t$. Then

$$\begin{array}{rcll}
\cos(2\arctan t) &=& \cos(2z) & \\
&=& \dfrac{\cos(2z)}{1} & \text{use double angle formulas and } 1=\sin^2 z+\cos^2 z \\
&=& \dfrac{\cos^2 z - \sin^2 z}{\cos^2 z + \sin^2 z} & \text{divide top and bottom by } \cos^2 z \\
&=& \dfrac{(\cos^2 z - \sin^2 z)\frac{1}{\cos^2 z}}{(\sin^2 z + \cos^2 z)\frac{1}{\cos^2 z}} & \\
&=& \dfrac{1-\tan^2 z}{1+\tan^2 z} & \\
&=& \dfrac{1-t^2}{1+t^2} \quad . &
\end{array}$$

Solution. 3.d Set $z = \arctan t$, and so $\tan z = t$. Then

$$\begin{array}{rcll}
\cot(2\arctan t) &=& \cot(2z) & \\
&=& \dfrac{\cos(2z)}{\sin(2z)} & \text{use double angle formulas} \\
&=& \dfrac{\cos^2 z - \sin^2 z}{2\sin z\cos z} & \\
&=& \dfrac{1-\tan^2 z}{2\tan z} & \\
&=& \dfrac{1-t^2}{2t} & \\
&=& \dfrac{1}{2}\left(\dfrac{1}{t}-t\right) \quad . &
\end{array}$$

4. Compute the derivative (derive the formula).

(a) $(\arctan x)'$. answer: $\frac{1}{1+x^2}$

(b) $(\mathrm{arccot}\, x)'$. answer: $-\frac{1}{1+x^2}$

(c) $(\arcsin x)'$. answer: $\frac{1}{\sqrt{1-x^2}}$

(d) $(\arccos x)'$. answer: $-\frac{1}{\sqrt{1-x^2}}$

(e) Let arcsec denote the inverse of the secant function. Compute $(\mathrm{arcsec} x)'$. answer: $\frac{1}{x\sqrt{x^2-1}}$

5. (a) Let $a+b \neq k\pi$, $a \neq k\pi + \frac{\pi}{2}$ and $b \neq k\pi + \frac{\pi}{2}$ for any $k \in \mathbb{Z}$ (integers). Prove that

$$\frac{\tan a + \tan b}{1 - \tan a \tan b} = \tan(a+b) \quad .$$

(b) Let x and y be real. Prove that, for $xy \neq 1$, we have

$$\arctan x + \arctan y = \arctan\left(\frac{x+y}{1-xy}\right)$$

if the left hand side lies between $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Solution. 5.a We start by recalling the formulas

$$\begin{array}{rcl} \cos(a+b) & = & \cos a \cos b - \sin a \sin b \\ \sin(a+b) & = & \sin a \cos b + \sin b \cos a \end{array} \quad .$$

These formulas have been previously studied; alternatively they follow from Euler's formula and the computation

$$\begin{array}{rcl} \cos(a+b) + i\sin(a+b) & = & e^{i(a+b)} = e^{ia}e^{ib} = (\cos a + i \sin a)(\cos b + i \sin b) \\ & = & \cos a \cos b - \sin a \sin b + i(\sin a \cos b + \sin b \cos a) \end{array} \quad .$$

Now 5.a is done via a straightforward computation:

$$\begin{array}{rcl} \tan(a+b) & = & \dfrac{\sin(a+b)}{\cos(a+b)} = \dfrac{\sin a \cos b + \sin b \cos a}{\cos a \cos b - \sin a \sin b} = \dfrac{(\sin a \cos b + \sin b \cos a)\frac{1}{\cos a \cos b}}{(\cos a \cos b - \sin a \sin b)\frac{1}{\cos a \cos b}} \\ & = & \dfrac{\tan a + \tan b}{1 - \tan a \tan b} \quad . \end{array} \tag{1}$$

5.b is a consequence of 5.a. Let $a = \arctan x$, $b = \arctan y$. Then (1) becomes

$$\tan(\arctan x + \arctan y) = \frac{\tan(\arctan x) + \tan(\arctan y)}{1 - \tan(\arctan x)\tan(\arctan y)} = \frac{x+y}{1-xy} \quad ,$$

where we use the fact that $\tan(\arctan w) = w$ for all w. We recall that $\arctan(\tan z) = z$ whenever $z \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Now take $\arctan$ on both sides of the above equality to obtain

$$\arctan x + \arctan y = \arctan\left(\frac{x+y}{1-xy}\right) \quad .$$