Calculus II
Lecture (not covered in class)

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Outline

1. Exponential Functions and logarithms, Review
2. Derivatives of Exponential Functions
   - Natural Exponent
3. A More Advanced Approach to Exponents
   - Derivative using series definition
4. Logarithmic Functions, Review
   - Natural Logarithms
5. Derivatives of Logarithms, Review
   - The Natural Logarithm
   - The Number e as a Limit
   - Derivatives of Exponents with Arbitrary Base
   - Derivatives of Arbitrary Exponents with Arbitrary Base

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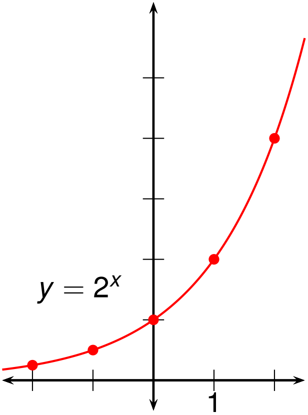
Exponential Functions

The function $f(x) = 2^x$ is called an exponential function because the variable x is the exponent.

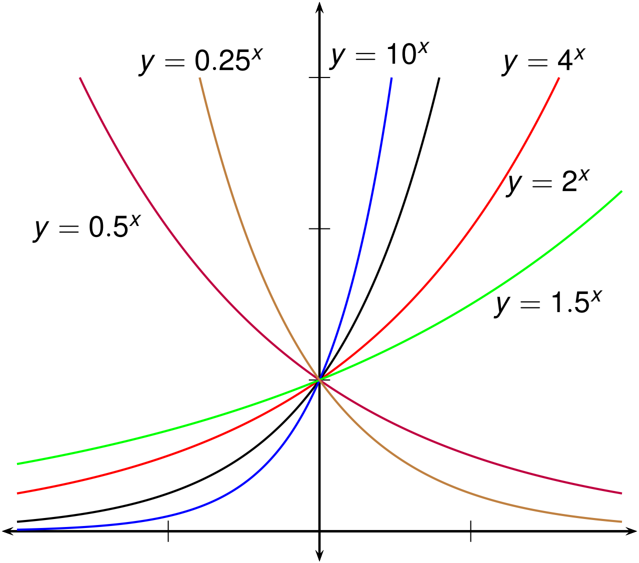


x	y
2	4
1	2
0	1
-1	$\frac{1}{2}$
-2	$\frac{1}{4}$

(Exponential Function Terminology)

An exponential function is a function of the form $f(x) = a^x$, where a is a positive constant.

Graphs of various exponential functions.



Derivatives of Exponential Functions

Compute the derivative of $f(x) = a^x$ using the definition:

$$\begin{aligned}
f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h\to 0} \frac{a^{x+h} - a^x}{h} \\
&= \lim_{h\to 0} \frac{a^x a^h - a^x}{h} \\
&= \lim_{h\to 0} \frac{a^x \left(a^h - 1\right)}{h} \\
&= a^x \lim_{h\to 0} \frac{a^h - 1}{h} \\
&= a^x \lim_{h\to 0} \frac{a^h - a^0}{h} \\
&= a^x f'(0).
\end{aligned}$$

We have shown that, if $f(x) = a^x$ is differentiable at 0, then it is differentiable everywhere, and

$$f'(x) = f'(0)a^x.$$

We leave the following theorem without proof.

Theorem

Let a be a positive number and let $f(x) = a^x$. Then the limit

$$f'(0) = \lim_{h \to 0} \frac{a^h - 1}{h}$$

exists.

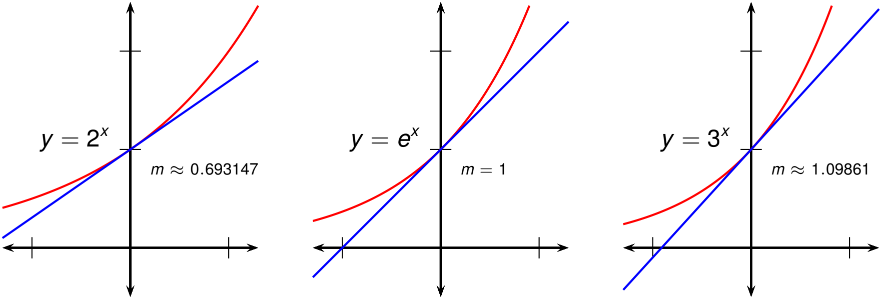
We will later show that

$$f'(0) = \lim_{h \to 0} \frac{a^h - 1}{h} = \ln(a).$$

Here, $\ln$ is the natural logarithm function.

The Natural Exponential Function

- One base for an exponential function is especially useful.
- It has a special property: its tangent line at $x = 0$ has slope $m = 1$.
- We call this number e, known as Euler's number or Napier's constant.
- e is a number between 2 and 3.
- In fact, $e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots \approx 2.71828$.



Definition (Natural Exponential Function)

e^x is called the natural exponential function. Its derivative is

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(e^x\right) = e^x.$$

Exponents overview

- For integer x, we know how to compute a^x as a function of a.
- How do we compute $f(x) = a^x$ when x is not an integer?
- We need to go back to the definition of a^x (for x non-integer).
- In what follows we give/recall an elementary way to define exponent.
- Then we give an alternative second definition.
- The second definition will be studied in sufficient depth only much later.
- The two definitions are equivalent: if we choose one definition the other becomes a theorem and the other way round.
- Choosing one definition makes some statements easier to prove and others more difficult.
- We shall discuss pros and cons of the two. In a nutshell:
  - the first elementary definition is easier to motivate;
  - the second alternative definition is easier to compute with.

Exponent definition using limits (approach I)

- For integer p we know to compute a^p.
- Therefore for integer q we know to compute $a^{\frac{1}{q}} = \sqrt[q]{a} = \max\{x|$ for which $x^q \leq a\}$.
- Therefore we know to compute $a^{\frac{p}{q}}$ for all rational $\frac{p}{q}$.
- We can then define
$$a^x = \lim_{\substack{y \to x \\ y\text{-rational}}} a^y$$
For example, a^π would be defined as the limit of the sequence $a^{3.14}, a^{3.141}, a^{3.1415}, \dots$.
- Cons: not computationally effective; not how computers compute.
- Pros: for non-integer x and y, it is very easy to prove that $a^{x+y} = a^x a^y$ - this follows from the definition of limit above.
- This is the definition assumed in many elementary courses.

Exponent definition using series (approach II)

- The following formula (studied much later) can be used as alternative definition.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots$$

Here $n! = 1 \cdot 2 \cdot 3 \cdot \dots \cdot (n-1) \cdot n$ and is read "n factorial".

- For $|x| < 1$ define

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + \frac{(-1)^{n+1} x^n}{n} + \dots$$

Infinite sum studied much later.

- For arbitrary $a > 0$ define a^x as $a^x = e^{x \ln a}$.
- Cons: more difficult to prove $e^{x+y} = e^x e^y$ and $e^{\ln(1+x)} = 1 + x$, proof done later.
- Pros: this is how e^x and a^x are actually computed (by modern computers and by humans in the past).

Example

Derive the exponent rule $(e^x)' = e^x$ using the Calc II formula below, the infinite (both sides uniformly convergent) sum rule $(f_1 + f_2 + f_3 + \dots)' = f_1' + f_2' + f_3' + \dots$ and the power rule $(x^n)' = nx^{n-1}$.

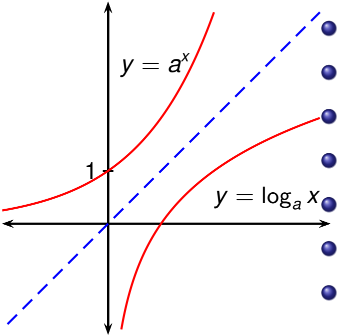
$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots,$$

where $n! = 1 \cdot 2 \cdot 3 \cdot \dots \cdot n$. We have that $\frac{n}{n!} = \frac{n}{1 \cdot 2 \cdot \dots \cdot (n-1) \cdot n} = \frac{1}{1 \cdot 2 \cdot \dots \cdot (n-1)} = \frac{1}{(n-1)!}$.

$$\begin{aligned}
(e^x)' &= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right)' \\
&= (1)' + (x)' + \frac{(x^2)'}{2!} + \frac{(x^3)'}{3!} + \dots + \frac{(x^n)'}{n!} + \dots \\
&= 0 + 1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \dots + \frac{nx^{n-1}}{n!} + \dots \\
&= \quad 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^{n-1}}{(n-1)!} + \dots = e^x
\end{aligned}$$

as desired.

Logarithmic Functions



- Suppose $a > 0$, $a \neq 1$.
- Let $f(x) = a^x$.
- Then f is either increasing or decreasing.
- Therefore f is one-to-one.
- Therefore f has an inverse function, f^{-1}.
- The graph shows $y = a^x$ for $a > 1$.
- The graph of $y = \log_a x$ is the reflection of this in the line $y = x$.

Definition ($\log_a x$)

The inverse function of $f(x) = a^x$ is called the logarithmic function with base a, and is written $\log_a x$. It is defined by the formula

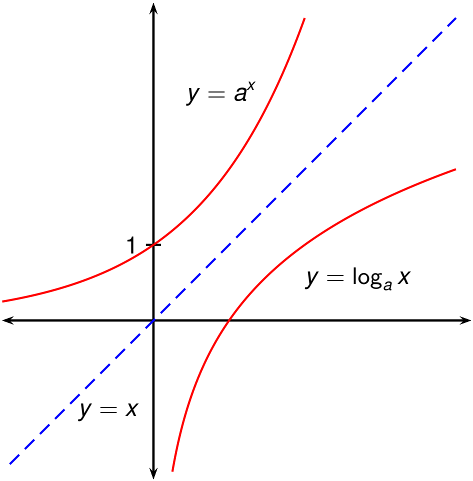
$$\log_a x = y \quad \Leftrightarrow \quad a^y = x.$$

If $x > 0$, then $\log_a x$ is the exponent to which the base a must be raised to give x.

Example

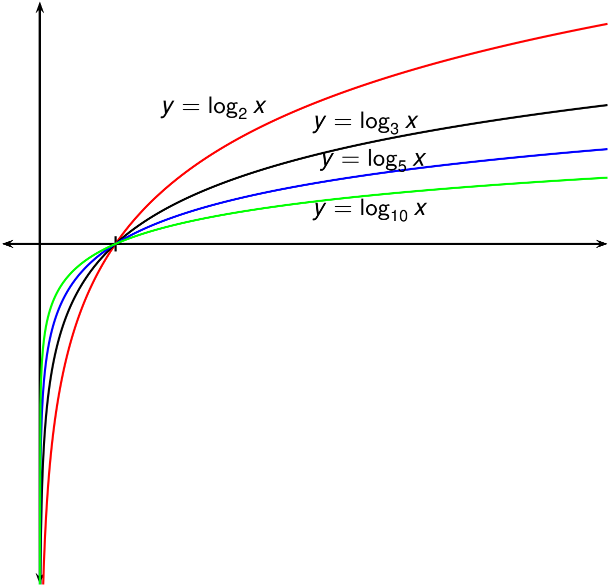
Evaluate:

1. $\log_3 81 = 4$ because $3^4 = 81$.
2. $\log_{25} 5 = \frac{1}{2}$ because $25^{\frac{1}{2}} = \sqrt{25} = 5$.
3. $\log_{10} 0.001 = -3$ because $10^{-3} = 0.001$.



- Suppose $a > 1$.
- Domain of a^x: $\mathbb{R}$.
- Range of a^x: $(0, \infty)$.
- Domain of $\log_a x$: $(0, \infty)$.
- Range of $\log_a x$: $\mathbb{R}$.
- $\log_a(a^x) = x$ for $x \in \mathbb{R}$.
- $a^{\log_a x} = x$ for $x > 0$.

Graphs of various logarithmic functions with $a > 1$

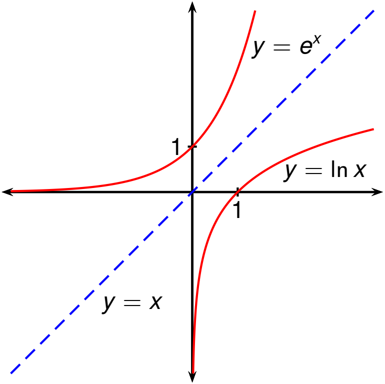


Natural Logarithms

Definition ($\ln x$)

The logarithm with base e is called the natural logarithm, and has a special notation:

$$\log_e x = \ln x.$$



- $\ln x = y \quad \Leftrightarrow \quad e^y = x$.
- $\ln(e^x) = x$ for $x \in \mathbb{R}$.
- $e^{\ln x} = x$ for $x > 0$.

Theorem (Properties of Logarithmic Functions)

If $a > 1$, the function $f(x) = \log_a x$ is a one-to-one, continuous, increasing function with domain $(0, \infty)$ and range $\mathbb{R}$. If $x, y, a, b > 0$ and r is any real number, then

1. $\log_a(xy) = \log_a x + \log_a y$.
2. $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$.
3. $\log_a(x^r) = r \log_a x$.
4. $\log_a(x) = \log_b x \log_a b = \frac{\log_b x}{\log_b a} = \frac{\ln x}{\ln a}$.

The Derivative of the Natural Logarithm

Theorem (The Derivative of $\ln x$)

$$\frac{\mathrm{d}}{\mathrm{d}x}(\ln x) = \frac{1}{x}.$$

Proof.

- Let $y = \ln x$.
- Then $e^y = x$.
- Differentiate this implicitly with respect to x:
- $e^y \frac{\mathrm{d}y}{\mathrm{d}x} = 1$.
- Rearrange:
- $\frac{\mathrm{d}}{\mathrm{d}x}(\ln x) = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{e^y} = \frac{1}{e^{\ln x}} = \frac{1}{x}$.

$\square$

Example (Chain Rule,)

$$\begin{aligned}
\text{Differentiate}\quad & y = \ln(x^3+1).\\
\text{Let}\quad & u = x^3+1.\\
\text{Then}\quad & y = \ln u.\\
\text{Chain Rule:}\quad & \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}\\
& = \left(\frac{1}{u}\right)\left(3x^2\right)\\
& = \frac{3x^2}{x^3+1}.
\end{aligned}$$

Theorem (The Number e as a Limit)

$$e = \lim_{x\to 0}(1+x)^{\frac{1}{x}} = \lim_{y\to\infty}\left(1+\frac{1}{y}\right)^{y}.$$

Proof.

Let $f(x) = \ln x$. Then $f'(x) = \frac{1}{x}$, so $f'(1) = 1$.

$$\begin{aligned}1 = f'(1) &= \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = \lim_{x\to 0}\frac{f(1+x)-f(1)}{x}\\ &= \lim_{x\to 0}\frac{\ln(1+x)-\ln(1)}{x} = \lim_{x\to 0}\frac{1}{x}\ln(1+x)\\ &= \lim_{x\to 0}\ln(1+x)^{\frac{1}{x}}.\end{aligned}$$

Then use the fact that the exponential function is continuous:

$$e = e^1 = e^{\lim\limits_{x\to 0}\ln(1+x)^{\frac{1}{x}}} = \lim_{x\to 0} e^{\ln(1+x)^{\frac{1}{x}}} = \lim_{x\to 0}(1+x)^{\frac{1}{x}}. \qquad \square$$

Example

Compute

$$
\begin{array}{rcll}
\displaystyle\lim_{x\to\infty}\left(\frac{x+3}{x}\right)^x &=& \displaystyle\lim_{x\to\infty}\left(1+\frac{3}{x}\right)^x & \\
&=& \displaystyle\lim_{x\to\infty}\left(1+\frac{1}{\frac{x}{3}}\right)^{3\frac{x}{3}} & \left|\ \text{Set } \frac{x}{3}=y\right. \\
&=& \displaystyle\lim_{\substack{x\to\infty\\ \frac{x}{3}=y\to\infty}}\left(1+\frac{1}{y}\right)^{3y} & \\
&=& \displaystyle\lim_{y\to\infty}\left(\left(1+\frac{1}{y}\right)^{y}\right)^{3}=e^3 & .
\end{array}
$$

Example

Compute

$$
\begin{aligned}
& \lim_{x\to\infty}\left(\frac{x}{x-2}\right)^{2x+2} \\
= & \lim_{x\to\infty}\left(\frac{x-2+2}{x-2}\right)^{2x+2} = \lim_{x\to\infty}\left(1+\frac{2}{x-2}\right)^{2x+2} \\
= & \lim_{x\to\infty}\left(1+\frac{1}{\frac{x-2}{2}}\right)^{2(x-2+2)+2} \\
= & \lim_{x\to\infty}\left(1+\frac{1}{\frac{x-2}{2}}\right)^{4\frac{x-2}{2}+6} = \lim_{\substack{\frac{x-2}{2}=y\\ y\to\infty}}\left(1+\frac{1}{y}\right)^{4y+6} \qquad \Bigg| \text{ Set } y=\frac{x-2}{2} \\
= & \lim_{y\to\infty}\left(\left(1+\frac{1}{y}\right)^{y}\right)^{4}\lim_{y\to\infty}\left(1+\frac{1}{y}\right)^{6} \\
= & e^4\cdot(1+0)^6 = e^4 \quad .
\end{aligned}
$$

Theorem (The Derivative of a^x)

$$\frac{\mathrm{d}}{\mathrm{d}x}(a^x) = a^x \ln a.$$

Proof.

Use the fact that $a = e^{\ln a}$.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(a^x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left((e^{\ln a})^x\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{x \ln a}\right) \\
&= e^{x\ln a}\frac{\mathrm{d}}{\mathrm{d}x}(x \ln a) \\
&= (e^{\ln a})^x \ln a \\
&= a^x(\ln a).
\end{aligned}$$

□

Example (Chain Rule)

$$
\begin{aligned}
\text{Differentiate} \quad y &= 10^{x^2}. \\
\text{Let} \quad u &= x^2. \\
\text{Then} \quad y &= 10^u. \\
\text{Chain Rule:} \quad \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} \\
&= \left(10^u(\ln 10)\right)(2x) \\
&= (2\ln 10)x10^{x^2}.
\end{aligned}
$$

Example

Compute $\frac{\mathrm{d}}{\mathrm{d}x}\left((\tan x)^{\frac{1}{x}}\right)$, where $x \in (0, \frac{\pi}{2})$.

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left((\tan x)^{\frac{1}{x}}\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\left(e^{\ln \tan x}\right)^{\frac{1}{x}}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{\frac{1}{x}\ln \tan x}\right)\\
&= e^{\frac{1}{x}\ln(\tan x)}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x}\ln(\tan x)\right)\\
&= (\tan x)^{\frac{1}{x}}\left(-\frac{1}{x^2}\ln(\tan x) + \frac{1}{x}\frac{(\tan x)'}{\tan x}\right)\\
&= (\tan x)^{\frac{1}{x}}\left(-\frac{1}{x^2}\ln(\tan x) + \frac{1}{x}\frac{\frac{1}{\cos^2(x)}}{\frac{\sin x}{\cos x}}\right)\\
&= (\tan x)^{\frac{1}{x}}\left(-\frac{1}{x^2}\ln(\tan x) + \frac{1}{x}\frac{1}{\sin x \cos x}\right)
\end{aligned}
$$

Example

Suppose $g(x)$ and $f(x)$ are differentiable functions and suppose $g(x) > 0$. Prove that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(g(x)^{f(x)}\right) = g(x)^{f(x)}\left(f'(x)\ln(g(x)) + f(x)\frac{g'(x)}{g(x)}\right) \quad .$$

Proof.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(g(x)^{f(x)}\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\left(e^{\ln g(x)}\right)^{f(x)}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{f(x)\ln g(x)}\right) \\
&= e^{f(x)\ln g(x)}\frac{\mathrm{d}}{\mathrm{d}x}(f(x)\ln g(x)) \\
&= g(x)^{f(x)}\left(f'(x)\ln(g(x)) + f(x)\frac{g'(x)}{g(x)}\right) \quad ,
\end{aligned}$$

as desired. $\square$