Calculus I
Lecture 16
Optimization in One Variable

Todor Milev
https://github.com/tmilev/freecalc

2020

Outline

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Fermat's Theorem suggests that we should look at three types of points to find local maxima and minima:

1. Points c for which $f'(c) = 0$.
2. Points c for which $f'(c)$ doesn't exist.
3. Points c at ends of intervals where f is defined. Here, we need also that f be defined at c.

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Fermat's Theorem says that if f has a local maximum or minimum at c, and c is not an endpoint, then c is a critical number for f.

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- f isn't defined at $-\frac{2}{3}$. Therefore the critical numbers are 0 and $\frac{2}{3}$.

The Closed Interval Method

We know from the Extreme Value Theorem that a continuous function attains its maximum and minimum on a closed interval $[a, b]$. The maximum might occur at an endpoint. The minimum might occur at an endpoint.

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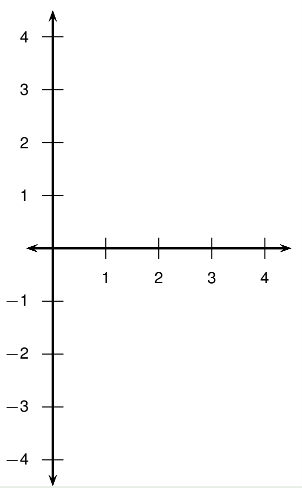
To find the maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

1. Find the values of f at the critical numbers of f in $[a, b]$.
   - Find the values c with $f'(c) = 0$.
   - Find the values c where f' does not exist.
2. Find the values of f at the endpoints a and b.
3. The maximum of f is maximum of the preceding values; the minimum value is the minimum.

Example

Find the maximum and minimum values of the function
$f(x) = -x^3 + 2x^2 + 4x - 5$ on the interval $[1, 3]$.



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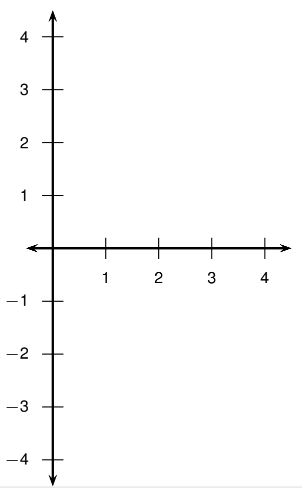

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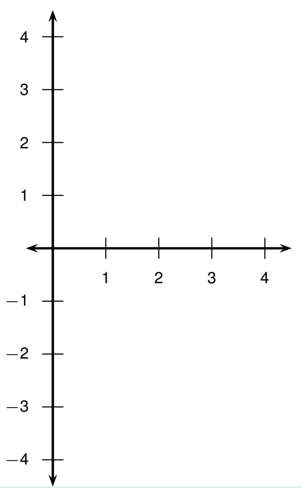

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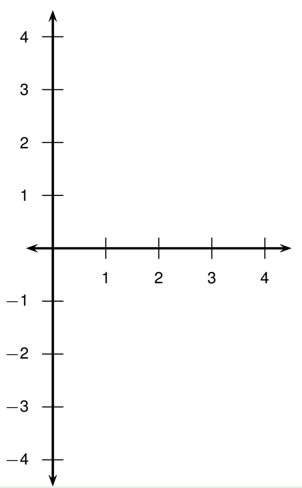

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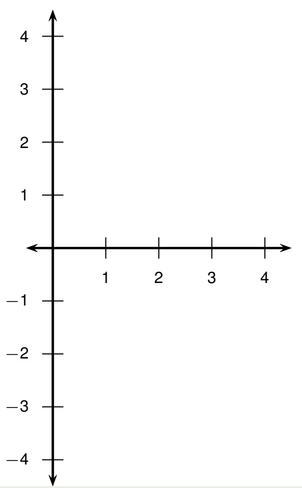

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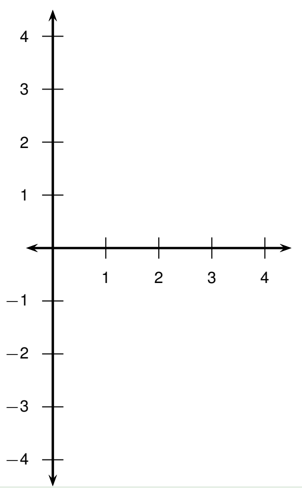

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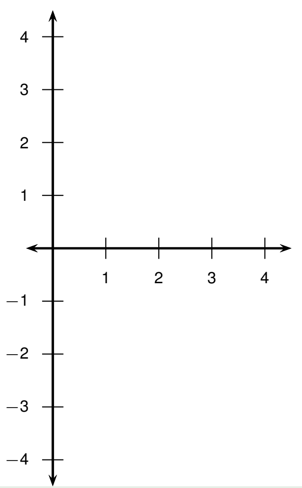

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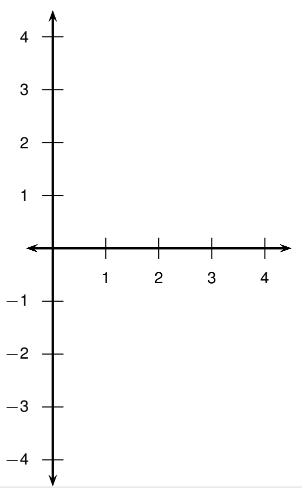

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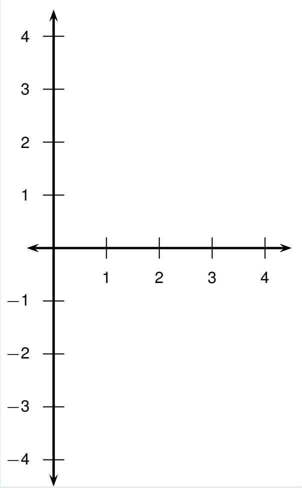

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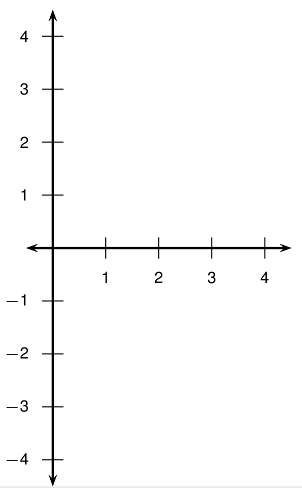

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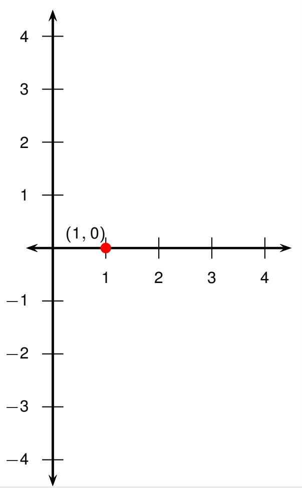

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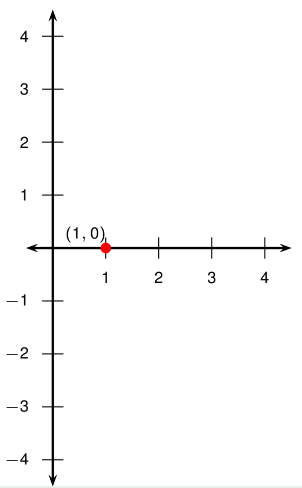

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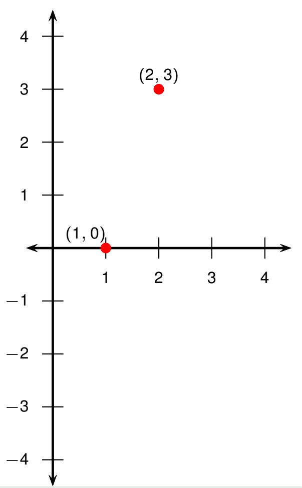

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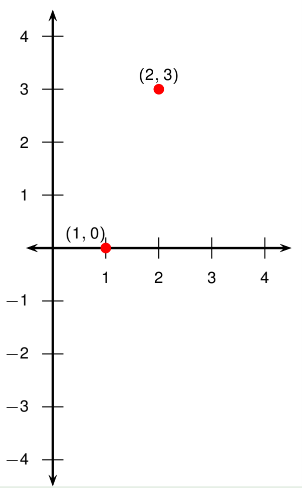

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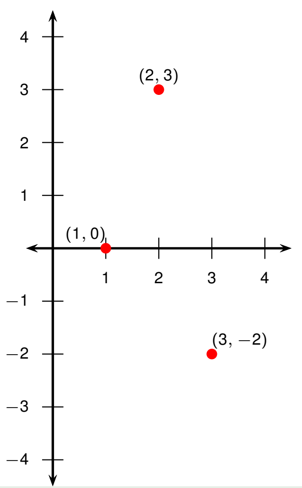

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1. The critical numbers of f in $[a, b]$.
2. The endpoints a and b.

x	$f(x)$
1	0
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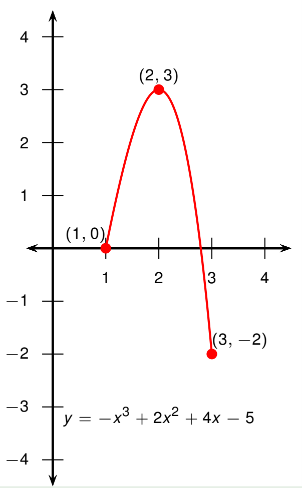
3	−2

Example

Find the maximum and minimum values of the function $f(x) = -x^3 + 2x^2 + 4x - 5$ on the interval $[1, 3]$.



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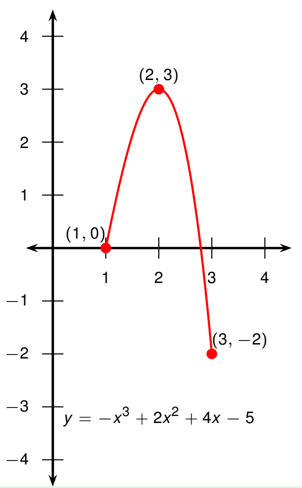

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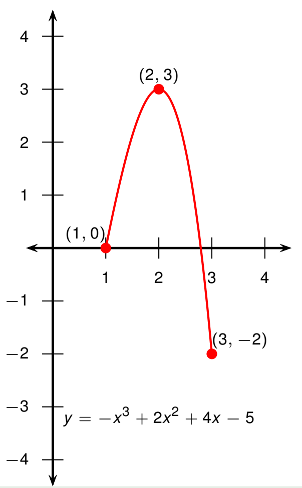

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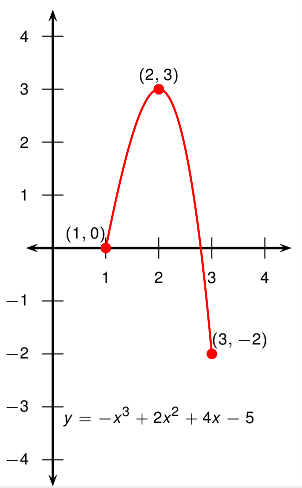

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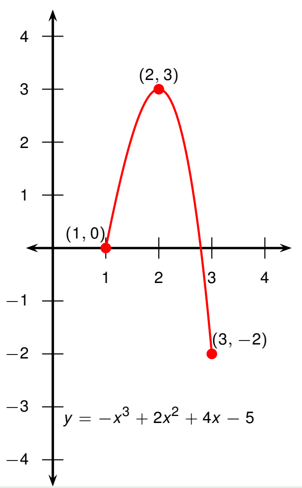

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The problem of finding minimum/maximum of a differentiable one-variable function often arises in practice.

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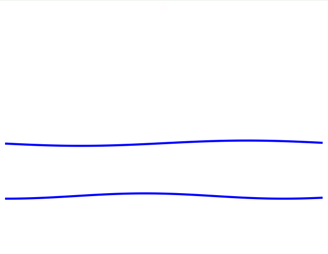
3. Use the closed interval method to find the maximum/minimum value of the desired quantity.

Example

A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. He doesn't need to put fencing along the river. What are the dimensions of the field with the largest area?



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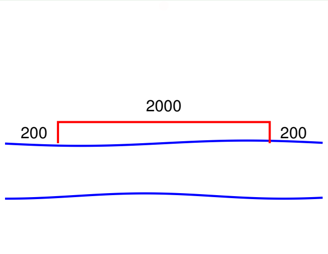

$\text{Area} = 200 \cdot 2000 = 400,000 \text{ft}^2$

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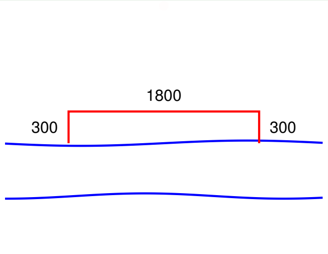

Area $= 300 \cdot 1800 = 540,000\text{ft}^2$

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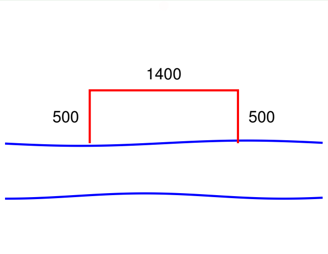

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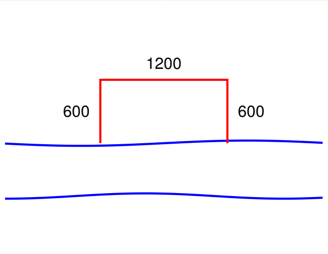

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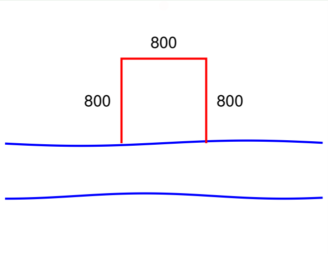

Area $= 800 \cdot 800 = 640,000\text{ft}^2$

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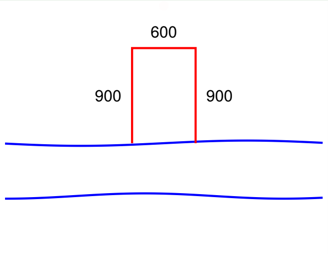

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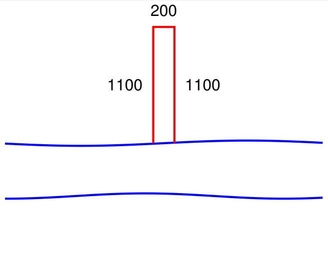

$\text{Area} = 1100 \cdot 200 = 220,000 \text{ft}^2$

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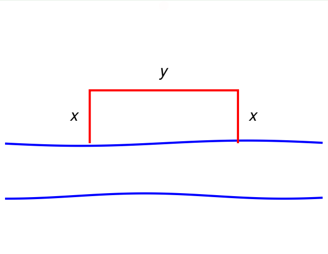

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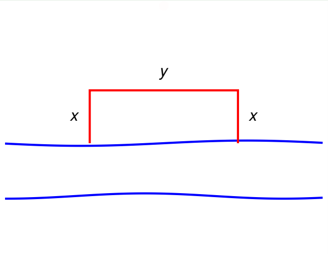

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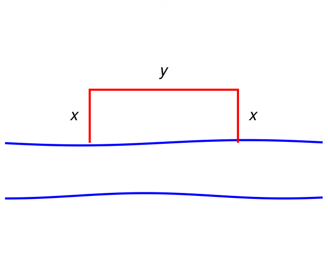

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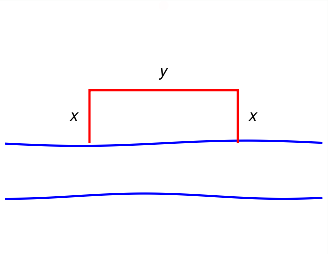

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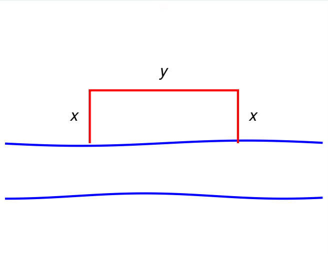

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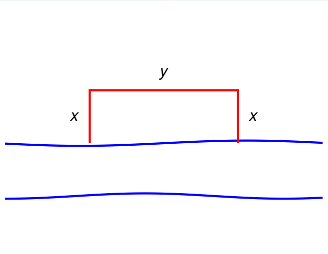

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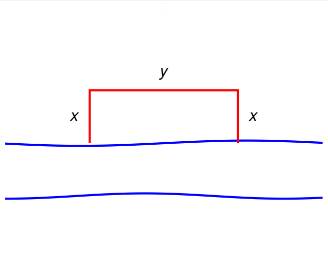

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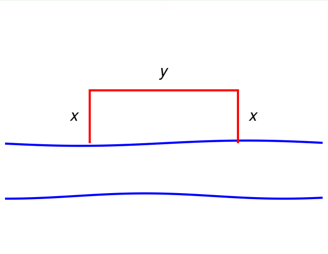

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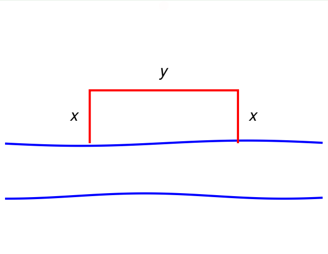

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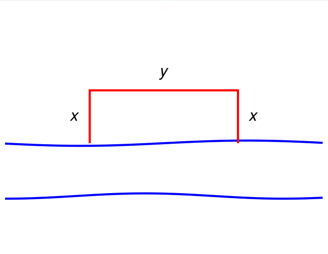

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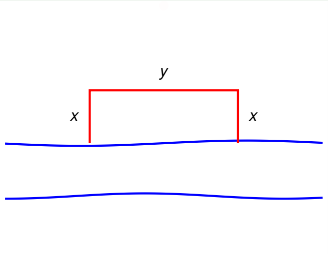

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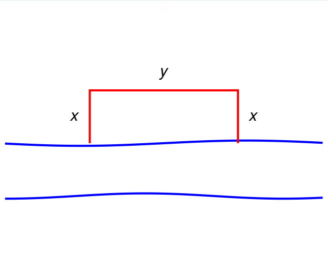

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x	$A(x)$
0	
600	
1200	

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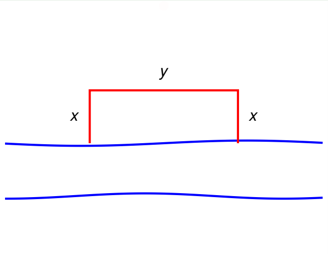

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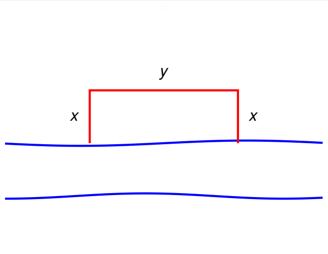

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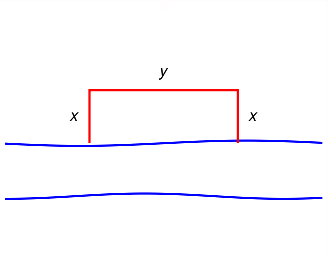

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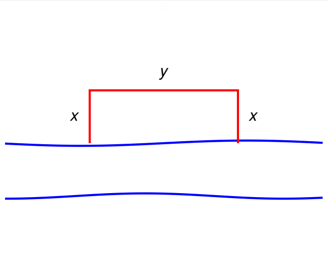

Area $= A = xy$

x	$A(x)$
0	0
600	720,000
1200	

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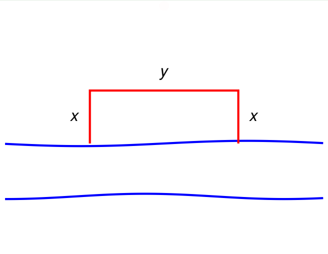
Maximize the function $A(x)$:

$$A'(x) = 2400 - 4x$$

Critical number: $x = 600$.

Example

A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. He doesn't need to put fencing along the river. What are the dimensions of the field with the largest area?



Area $= A = xy$

x	$A(x)$
0	0
600	$720,000$
1200	**?**

Let x and y denote the depth and width of the rectangle (in feet). Let A be its area.

$$\begin{aligned} 2x + y &= 2400 \\ y &= 2400 - 2x \\ A &= xy = x(2400 - 2x) \\ &= 2400x - 2x^2 \end{aligned}$$

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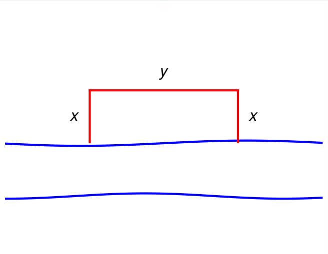

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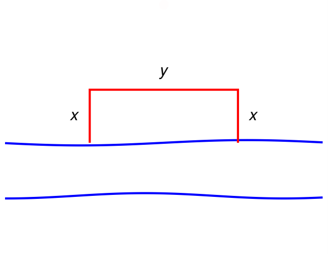

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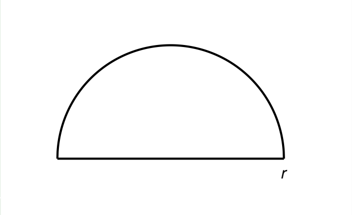
Therefore the maximum area occurs when $x = 600$ft and $y = 1200$ft.

Example

Find the largest possible area of a rectangle inscribed in a semicircle of radius r.



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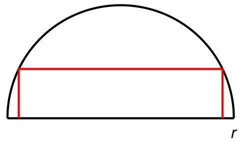

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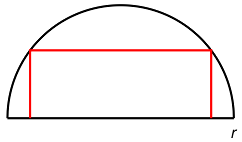

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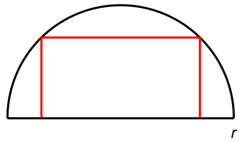

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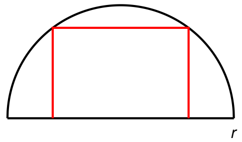

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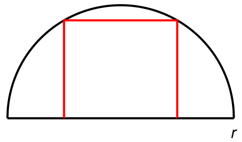

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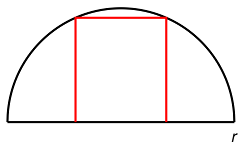

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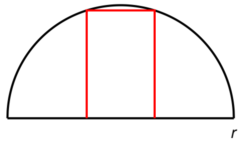

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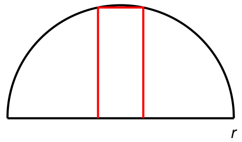

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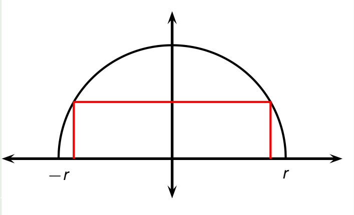

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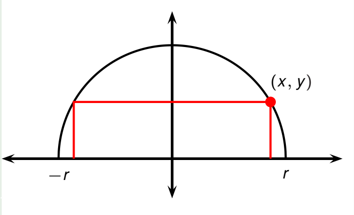

Let the semicircle have center at the origin. Let (x, y) -coord. of top right corner of rectangle.

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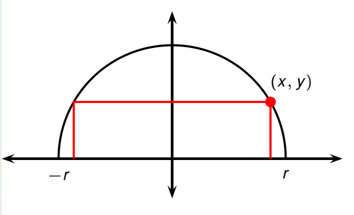

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$A =$ base $\cdot$ height

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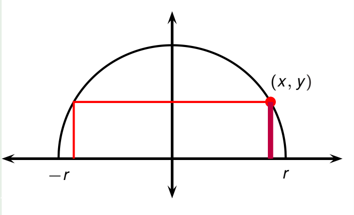

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$= \mathbf{?} \quad \cdot \mathbf{?}$

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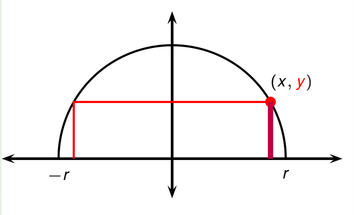

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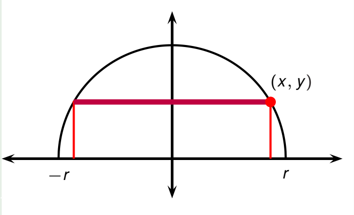

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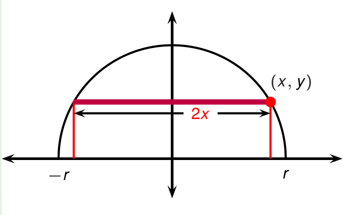

Let the semicircle have center at the origin. Let (x, y) -coord. of top right corner of rectangle. Let A be its area.

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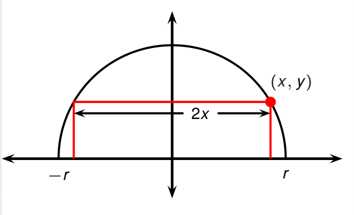

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A=base $\cdot$ height

$= 2x \cdot y = 2x \cdot$ **?**

To eliminate y, use that (x, y) lies on the semicircle.

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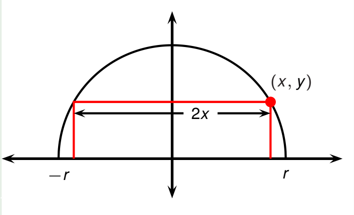

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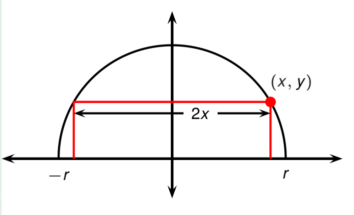

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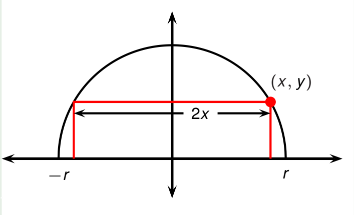

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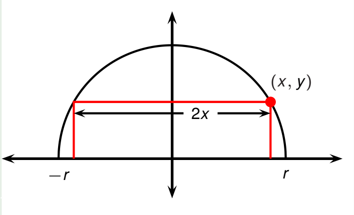

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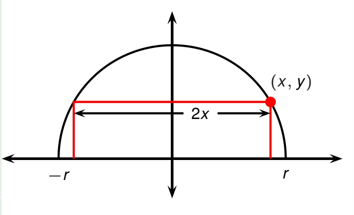

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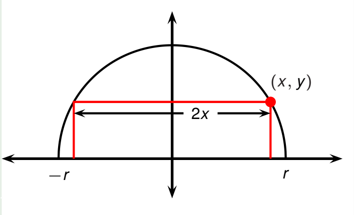

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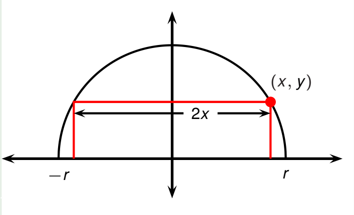

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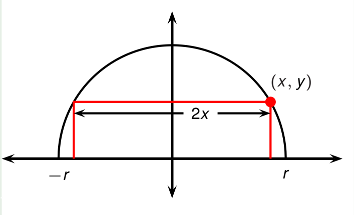

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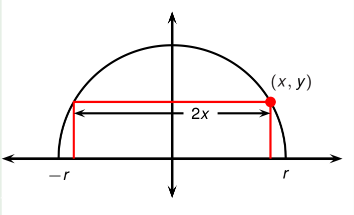

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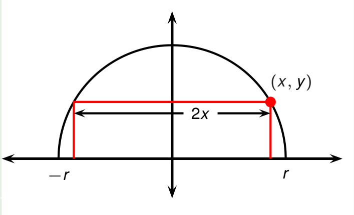

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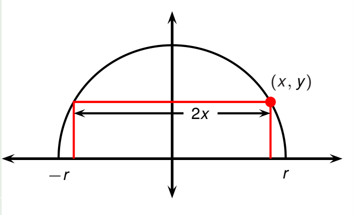

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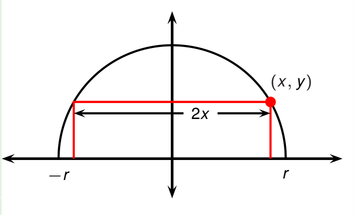

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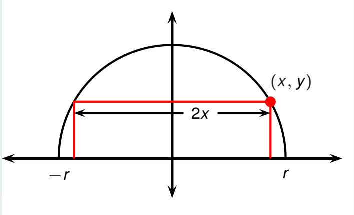

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Critical numbers: $x = \mathbf{?}$.

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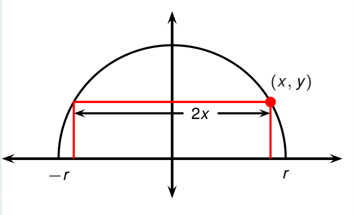

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Critical numbers: $x = \frac{r}{\sqrt{2}}$ and r.

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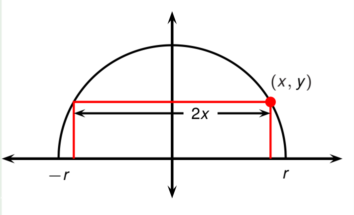

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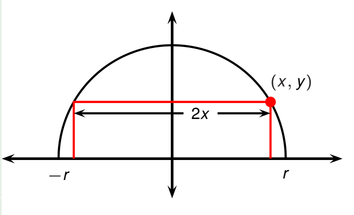
Critical numbers: $x = \frac{r}{\sqrt{2}}$ and r.

Example

Find the largest possible area of a rectangle inscribed in a semicircle of radius r.



To eliminate y, use that (x, y) lies on the semicircle.

$$y^2 = r^2 - x^2$$
$$y = \sqrt{r^2 - x^2}$$

Let the semicircle have center at the origin. Let (x, y) -coord. of top right corner of rectangle. Let A be its area.

$$A = \text{base} \cdot \text{height}$$
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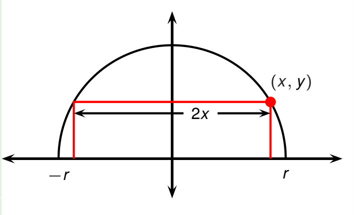

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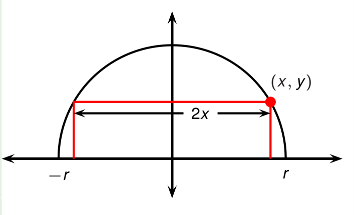

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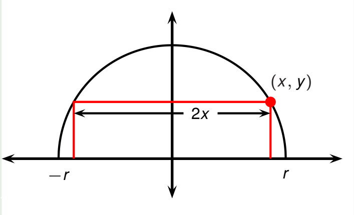

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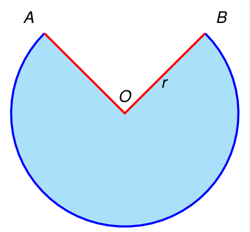
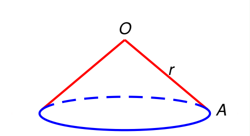
Example

A cone is folded from a wedge-shaped profile of radius r. Find the maximal possible volume V of such a cone.





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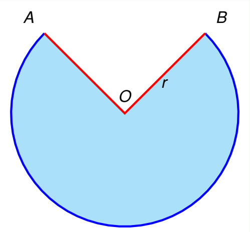

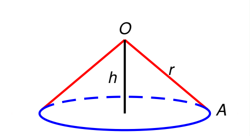

Set h - cone height,

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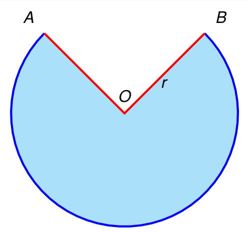

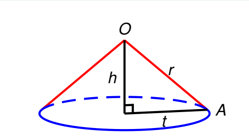

Set h - cone height, t - cone radius.

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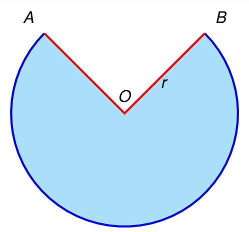

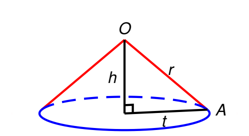

Set h - cone height, t - cone radius. Then

$V =$.

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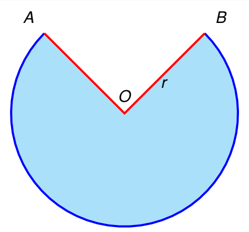

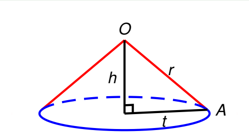

Set h - cone height, t - cone radius. Then $V = \frac{1}{3}(\text{area cone base})h$.

Example

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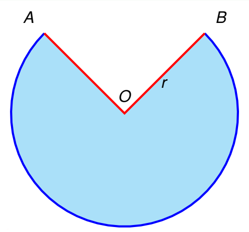

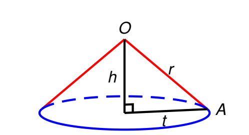

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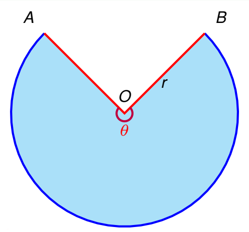

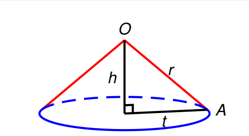

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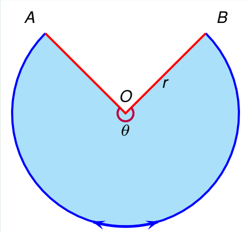

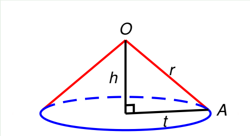

Set h - cone height, t - cone radius. Then $V = \frac{1}{3}(\text{area cone base})h = \frac{1}{3}\pi t^2 h$. Let θ - angle of the wedge. Then arc$AB =$

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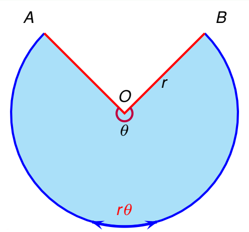

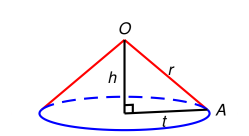

Set h - cone height, t - cone radius. Then $V = \frac{1}{3}(\text{area cone base})h = \frac{1}{3}\pi t^2 h$. Let θ - angle of the wedge. Then $\text{arc}AB = r\theta$

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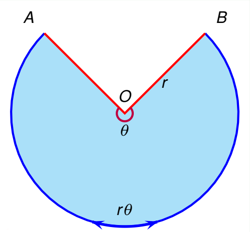

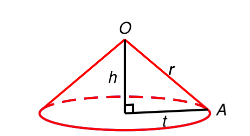

Set h - cone height, t - cone radius. Then $V = \frac{1}{3}(\text{area cone base})h = \frac{1}{3}\pi t^2 h$. Let θ - angle of the wedge. Then arc$AB = r\theta$ = perimeter cone base =

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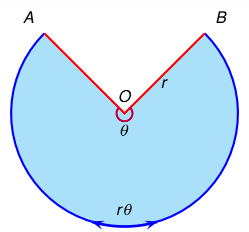

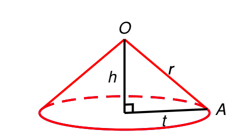

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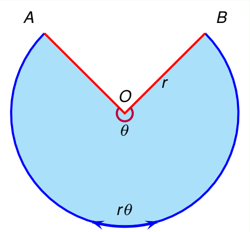

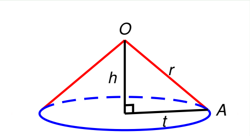

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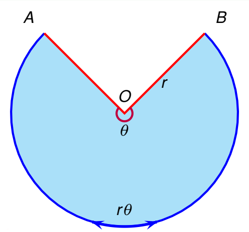

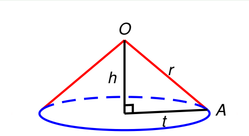

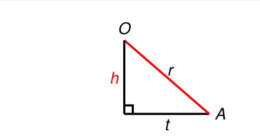

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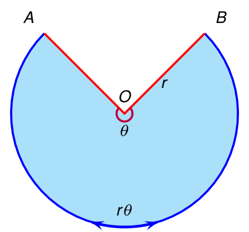

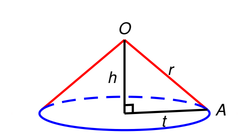

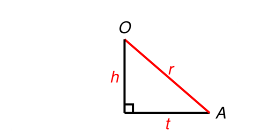

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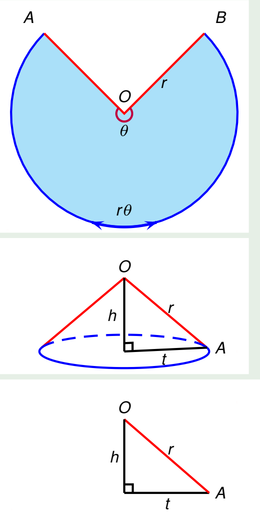

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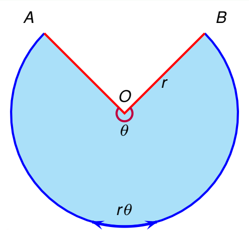

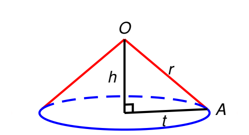

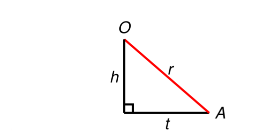

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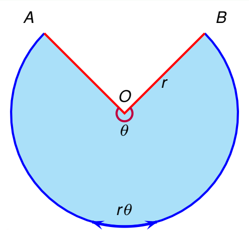

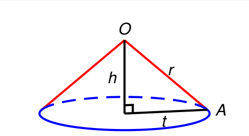

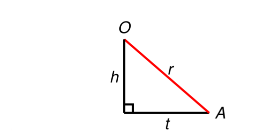

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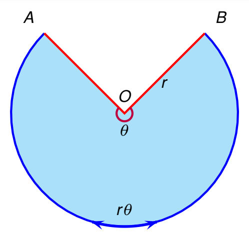

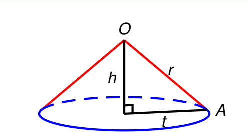

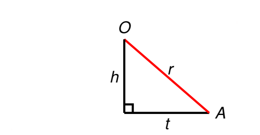

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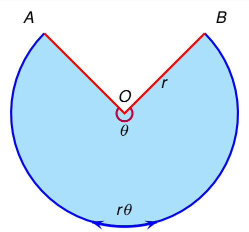

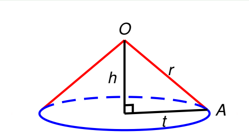

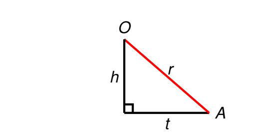

Set h - cone height, t - cone radius. Then $V = \frac{1}{3}(\text{area cone base})h = \frac{1}{3}\pi t^2 h$. Let θ - angle of the wedge. Then arc$AB = r\theta$ = perimeter cone base = $2\pi t$. Therefore $t = \frac{r\theta}{2\pi}$. Then

$$h = \sqrt{r^2 - t^2} = \sqrt{r^2 - \left(\frac{r\theta}{2\pi}\right)^2} = \frac{r}{2\pi}\sqrt{4\pi^2 - \theta^2},$$

and therefore

$$V = \frac{1}{3}\pi t^2 h = \frac{1}{3}\pi\left(\frac{r\theta}{2\pi}\right)^2 \frac{r}{2\pi}\sqrt{4\pi^2 - \theta^2}$$

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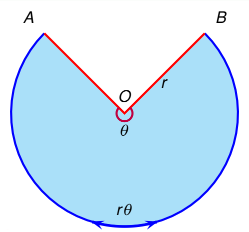

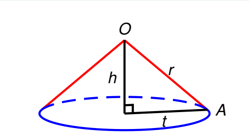

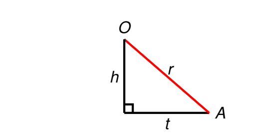

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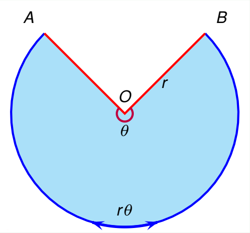
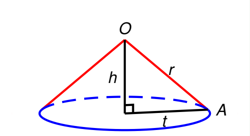
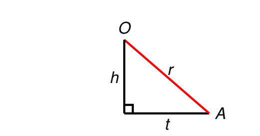
$$\begin{aligned} V &= \frac{1}{3}\pi t^2 h = \frac{1}{3}\pi\left(\frac{r\theta}{2\pi}\right)^2 \frac{r}{2\pi}\sqrt{4\pi^2 - \theta^2} \\ &= \frac{r^3}{24\pi^2}\theta^2\sqrt{4\pi^2 - \theta^2} \quad . \end{aligned}$$

Example

A cone is folded from a wedge-shaped profile of radius r. Find the maximal possible volume V of such a cone.







We reduced the problem to: find the maximum of

$$V = \frac{r^3}{24\pi^2}\theta^2\sqrt{4\pi^2 - \theta^2},$$

as function of θ (using the closed interval method).

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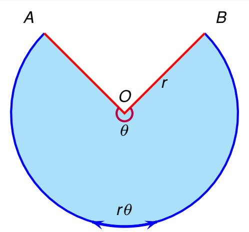

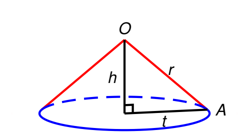

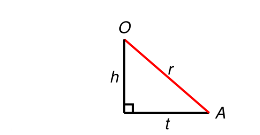

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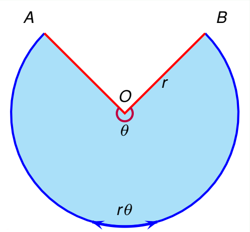

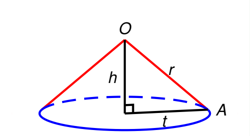

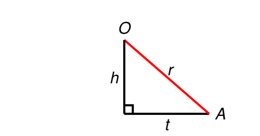

We reduced the problem to: find the maximum of

$$V = \frac{r^3}{24\pi^2}\theta^2\sqrt{4\pi^2-\theta^2}, \qquad 0 \leq \theta \leq 2\pi$$

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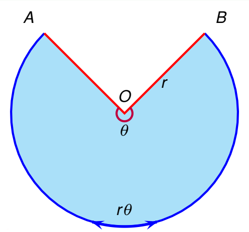

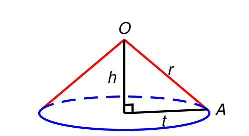

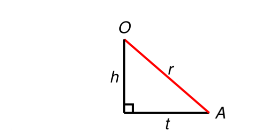

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$$V = \frac{r^3}{24\pi^2}\theta^2\sqrt{4\pi^2-\theta^2}, \qquad 0 \leq \theta \leq 2\pi$$

as function of θ (using the closed interval method). We need to find the critical points of V, i.e., the values of θ for which $\frac{\mathrm{d}V}{\mathrm{d}\theta} = 0$ and the values of θ for which $\frac{\mathrm{d}V}{\mathrm{d}\theta}$ is not defined.

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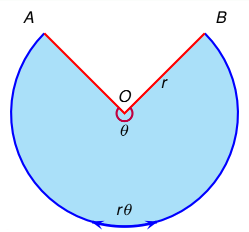

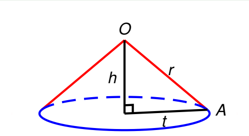

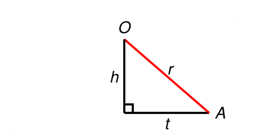

$$V = \frac{r^3}{24\pi^2}\theta^2\sqrt{4\pi^2-\theta^2}, \qquad 0 \leq \theta \leq 2\pi$$

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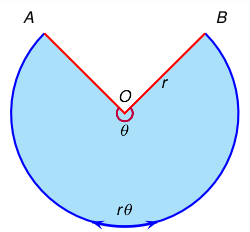

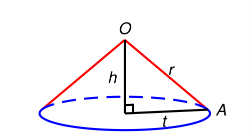

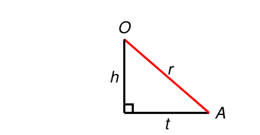

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$$\frac{dV}{d\theta} =$$

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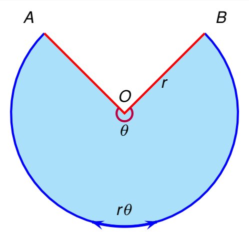

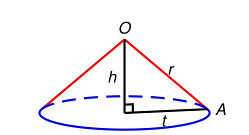

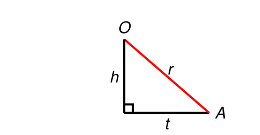

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\begin{aligned}
V &= \frac{r^3}{24\pi^2}\theta^2\sqrt{4\pi^2-\theta^2}, \qquad 0\leq\theta\leq 2\pi\\
\frac{\mathrm{d}V}{\mathrm{d}\theta} &= \left(\frac{r^3}{24\pi^2}\right)\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\theta^2\right)\sqrt{4\pi^2-\theta^2}\\
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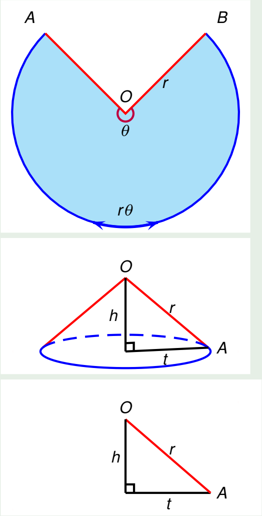

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&= \left(\frac{r^3}{24\pi^2}\right)(\quad)\sqrt{4\pi^2-\theta^2}\\
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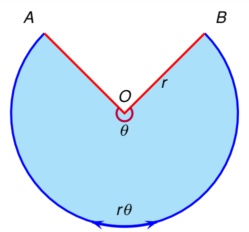

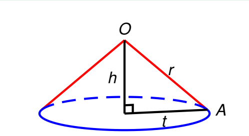

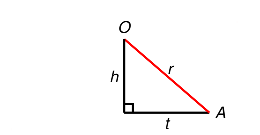

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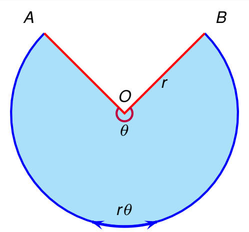

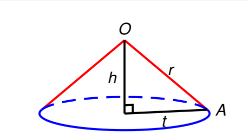

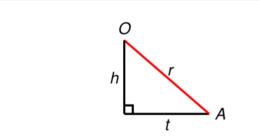

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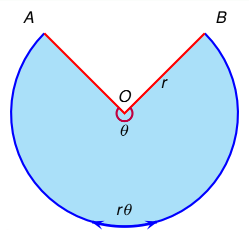

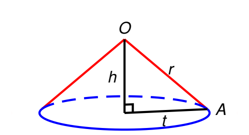

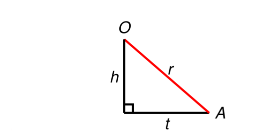

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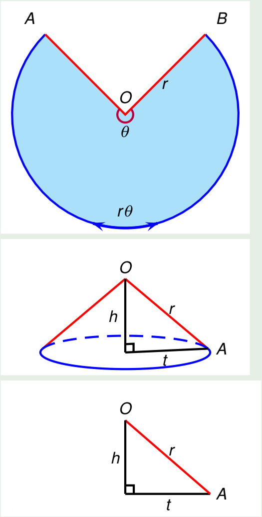

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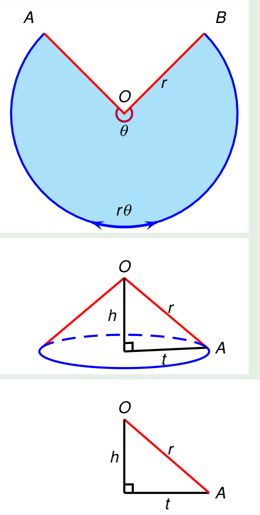

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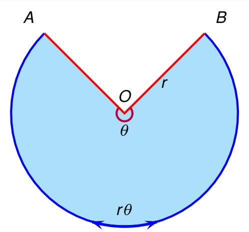

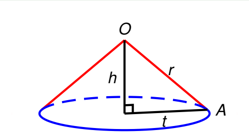

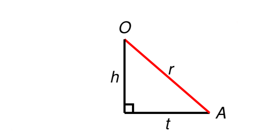

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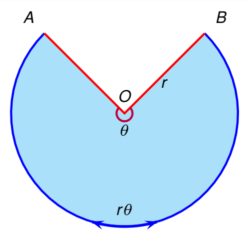

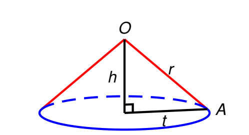

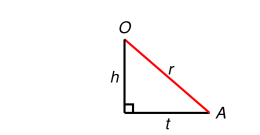

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&= \left(\frac{r^3}{24\pi^2}\right)(2\theta)\sqrt{4\pi^2-\theta^2}\\
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&= \left(\frac{r^3}{24\pi^2}\right)\frac{2\theta(4\pi^2-\theta^2)-\theta^3}{\sqrt{4\pi^2-\theta^2}}\\
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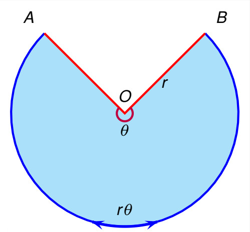

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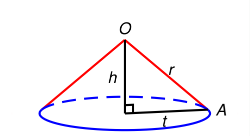

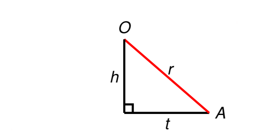

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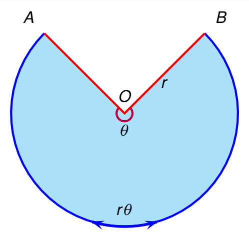

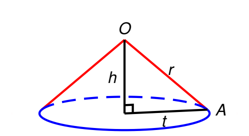

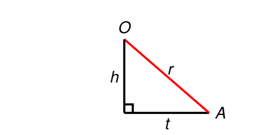

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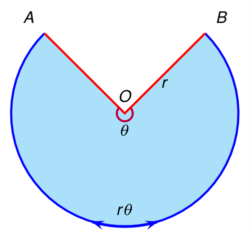

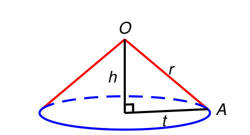

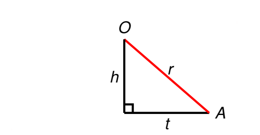

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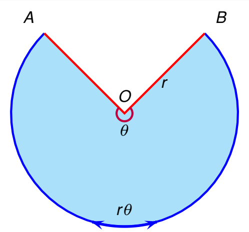

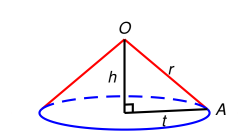

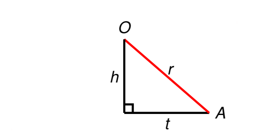

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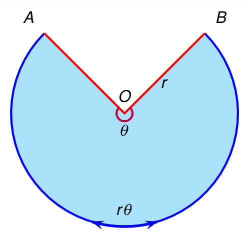

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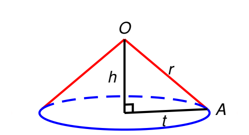

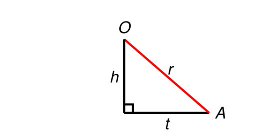

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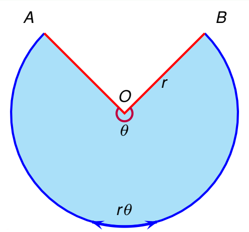

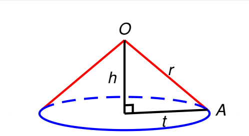

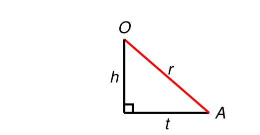

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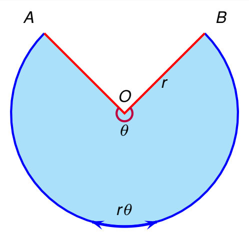

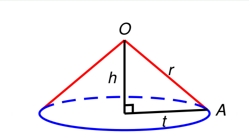

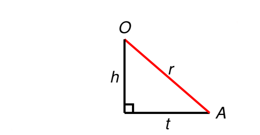

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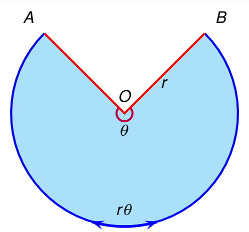

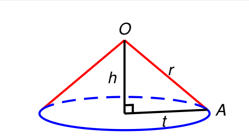

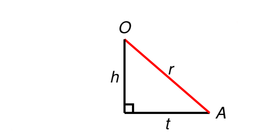

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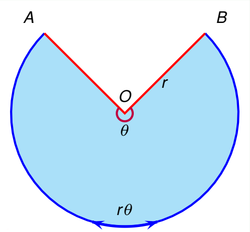

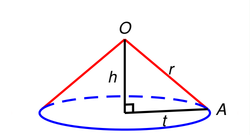

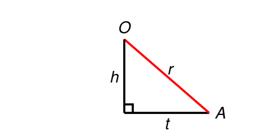

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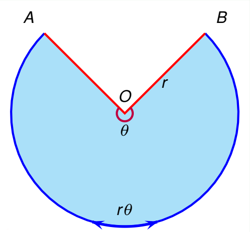

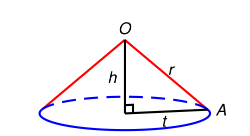

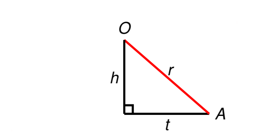

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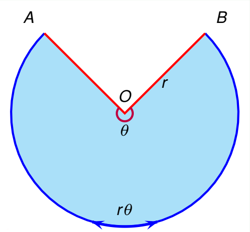

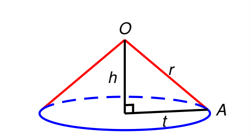

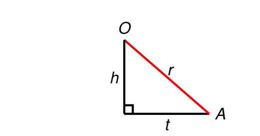

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Finally, the answer to the problem is

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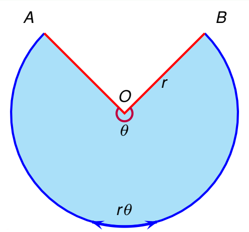

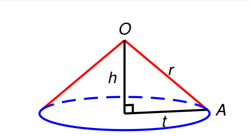

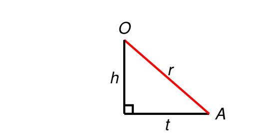

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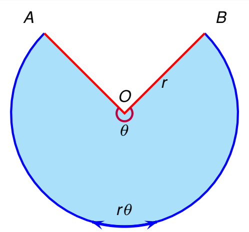

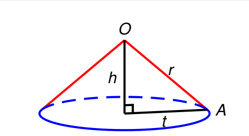

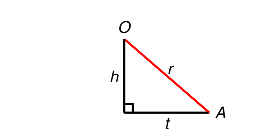

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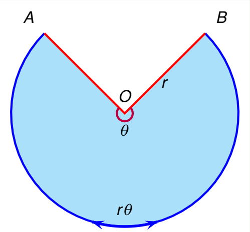

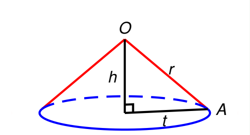

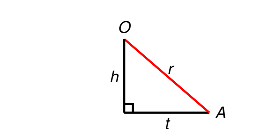

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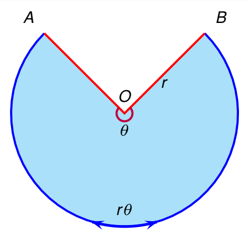
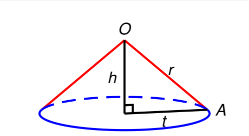
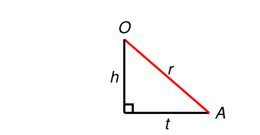






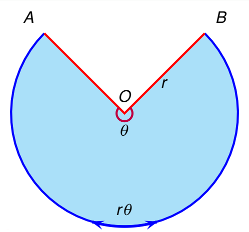
$$V(\theta) = \frac{r^3}{24\pi^2}\theta^2\sqrt{4\pi^2-\theta^2}$$

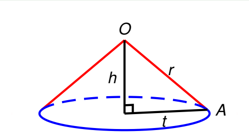
Finally, the answer to the problem is

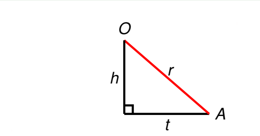
$$\begin{aligned} V_{max} &= V\left(\sqrt{\frac{8}{3}}\pi\right) \\ &= \frac{r^3}{24\pi^2}\left(\sqrt{\frac{8}{3}}\pi\right)^2\sqrt{4\pi^2-\left(\sqrt{\frac{8}{3}}\pi\right)^2} \\ &= \frac{r^3}{9}\pi\sqrt{4-\frac{8}{3}} \\ &= \pi\frac{r^3}{9}\sqrt{\frac{4}{3}} \end{aligned}$$

Example

A cone is folded from a wedge-shaped profile of radius r. Find the maximal possible volume V of such a cone.







$$V(\theta) = \frac{r^3}{24\pi^2}\theta^2\sqrt{4\pi^2-\theta^2}$$

Finally, the answer to the problem is

$$\begin{aligned} V_{max} &= V\left(\sqrt{\frac{8}{3}}\pi\right) \\ &= \frac{r^3}{24\pi^2}\left(\sqrt{\frac{8}{3}}\pi\right)^2\sqrt{4\pi^2-\left(\sqrt{\frac{8}{3}}\pi\right)^2} \\ &= \frac{r^3}{9}\pi\sqrt{4-\frac{8}{3}} \\ &= \pi\frac{r^3}{9}\sqrt{\frac{4}{3}} = \frac{2\pi r^3}{9\sqrt{3}} \end{aligned}$$