Calculus III
Lecture 8

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Outline

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Multivariable Limits

- Let $f\colon D \to \mathbb{R}$, where D a subset of the plane.
- Let P_0 be point in plane such that:
  - f is defined arbitrarily close to P_0;
  - f is not necessarily defined at P_0.
- For example,

$$\begin{aligned} f\colon \mathbb{R}^2 \setminus \{P_0(0,0)\} &\to \mathbb{R} \\ f(x,y) &= \frac{x^2 y}{x^2+y^2} \end{aligned}$$

  - is defined arbitrarily close to $(0,0)$;
  - and is not defined at $P_0(0,0)$.
- Question: What happens to $f(Q)$ as Q gets closer to P_0?

Numerical Exploration of Limits - Example

Example

$$f(x,y) = \frac{x^2 y}{x^2+y^2}$$

$$f(Q) \to ? \text{ as } Q \to P_0(0,0)$$

Numerical approach:

$$\begin{array}{r|rcl} Q_1(0.1, 0.1) & f(Q_1) & = & f(0.1, 0.1) \simeq 0.05 \\ Q_2(0.01, -0.02) & f(Q_2) & = & f(0.01, -0.02) \simeq -0.004 \\ Q_3(-0.003, 0.001) & f(Q_3) & = & f(-0.003, 0.001) \simeq 0.0009 \end{array}$$

Numerical data suggests $f(Q)$ approaches 0 as $Q \to P_0(0,0)$.

Definition of Multivariable Limit

- Let $f\colon D \to \mathbb{R}$, with D a subset of the plane.
- Let P_0 be a point in the plane such that:
  - f is defined arbitrarily close to P_0;
  - f is not necessarily defined at P_0.

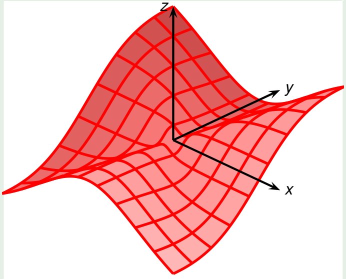
Definition

L is the limit of f at P_0 if we can keep the values of $f(Q)$ as close to L as we want by keeping Q close enough to P_0, but not equal to P_0. We write:

$$\begin{aligned} L &= \lim_{Q\to P_0} f(Q) \qquad\qquad \text{or} \\ L &= \lim_{(x,y)\to(x_0,y_0)} f(x,y) \end{aligned}$$

- As usual, we extend the definition to allow $L=\infty$. By convention, if $M > N$, we say that M is closer to ∞ than N.
- If the limit L exists, it is unique.

Example



$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2}$$

- f is not defined at $P_0(0,0)$;
- Even if it were, the actual value might be different from the limit.

Polar coordinates: $x = r\cos\theta$, $y = r\sin\theta$. We have: $(x,y)\to(0,0)$ if an only if $r\to 0$ in polar coordinates.

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2} &= \lim_{r\to 0} \frac{\cancel{r^2}\cos^2\theta r\sin\theta}{\cancel{r^2}}\\ &= \lim_{r\to 0} r\cos^2\theta\sin\theta\\ &= 0\end{aligned}$$

For the last equality, we use the squeeze theorem:
$0 = \lim_{r\to 0} -r \leq \lim_{r\to 0} r\cos^2\theta\sin\theta \leq \lim_{r\to 0} r = 0.$

If $\mathbf{r} = (a, b)$ is a vector, by $f(\mathbf{r})$ we understand $f(a, b)$ (i.e., define $f(\mathbf{r})$ via the vector-point identification).

Definition

- Let $f : D \to \mathbb{R}$, where D is a region in the plane;
- let f be defined near P with position vector $\mathbf{r}$; f is not necessarily defined at P;
- let $\mathbf{u}$ be an arbitrary vector.

We say that the one-variable limit

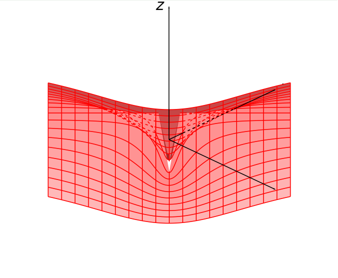
$$\lim_{t \to 0} f(\mathbf{r} + t\mathbf{u})$$

is the limit of f along the direction $\mathbf{u}$.

Theorem

If the limit $\lim\limits_{Q \to P} f(Q)$ exists, then every directional limit $\lim\limits_{t \to 0} f(\mathbf{r} + t\mathbf{u})$ exists and all directional limits are equal.

Example (Limit may fail to exist)



$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

Let $\mathbf{u} = (1, m)$. Directional limit along $\mathbf{u}$:

$$\begin{aligned}\lim_{t\to 0} f(t\mathbf{u}) &= \lim_{t\to 0} f(t(1,m)) = \lim_{t\to 0} f(t, tm) \\ &= \lim_{t\to 0} \frac{mt^2}{t^2+m^2t^2} \\ &= \frac{m}{1+m^2} \quad .\end{aligned}$$

Directional limit depends on $m \Rightarrow$ directional limit is not the same for all values of $\mathbf{u} \Rightarrow$ the multivariable limit does not exist.

If we'd used polar coordinates, we would had obtained:

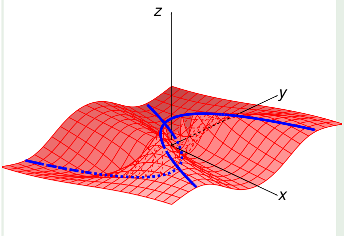
$$\frac{xy}{x^2+y^2} = \cos\theta\sin\theta$$

This expression depends only on θ; as $r \to 0$ permits arbitrary behavior of θ, we'd had guessed correctly that the limit doesn't exist.

Side Limits and Directional Limits

- Directional limits in dimension 1 are equal to the left or right hand limits (depending on the direction of the vector).
- Directional limits are therefore the natural analog of 1-dim side limits.
- Similarities b-n side and directional limits.
  - Single variable functions: limit exists $\implies$ side limits exist, have the same value.
  - Multivariable functions: limit exists $\implies$ directional limits exist, have the same value.
  - Singe variable functions: side limits are different $\implies$ limit does not exist.
  - Multivariable functions: directional limits have different values $\implies$ limit does not exist.
- Differences b-n side and directional limits.
  - Single variable functions: Side limits are equal $\implies$ limit exists.
  - Multivariable functions: even if all directional limits have the same value the limit does not necessarily exist.

Example (All directional limits exist, limit doesn't)



$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$$

Along $y = mx$:

$$\frac{xy^2}{x^2+y^4} = \frac{m^2x^3}{x^2+m^4x^4} = \frac{m^2x}{1+m^2x^4} \to 0$$

as $x \to 0$.

For direction $x = 0$, $y = m$: directional limit is again 0. $\Rightarrow$ all directional limits exist and equal 0. However, along $x = y^2$:

$$\frac{xy^2}{x^2+y^4} = \frac{y^4}{y^4+y^4} = \frac{1}{2} \to \frac{1}{2} \text{ as } x \to 0$$

Therefore $\lim\limits_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$ does not exist.

Limits along paths

Definition

- Let $f : D \to \mathbb{R}$, where D is a region in the plane;
- let f be defined near point P with position vector $\mathbf{p}$.
- Let $\mathbf{r}(t) = (x(t), y(t))$, $t \in I$ be a continuous path such that:
  - 0 is in I, $\mathbf{r}(0) = \mathbf{p}$;
  - $\mathbf{r}$ is continuous at 0;
  - $\mathbf{r}(t)$ lies in D for $t \neq 0$

We say that the one-variable limit

$$\lim_{t \to 0} f(\mathbf{r}(t))$$

is the limit of f along the path $\mathbf{r}(t)$.

Theorem

If the limit $\lim_{Q \to P} f(Q)$ *exists, then every path limit exists and all path limits are equal.*

- If we pick our path to be of the form $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{u}$, we see that the directional limit is a special case of the path limit.

Continuity

Definition (Continuous at a point)

We say that f is continuous at (x_0, y_0) if $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$.

Definition (Continuity)

We say that f is a continuous function if it is continuous at all points where it is defined.

- Polynomial functions are continuous.
- Sum, difference, product of continuous functions are continuous.
- If defined, quotients of continuous are continuous.
- Powers, exponentials of continuous functions are continuous.
- Compositions of continuous functions are continuous.
- A function fails to be continuous if:
  - (Removable discontinuity) limit exists but is different from f-n value;
  - (Essential discontinuity) the limit does not exist.

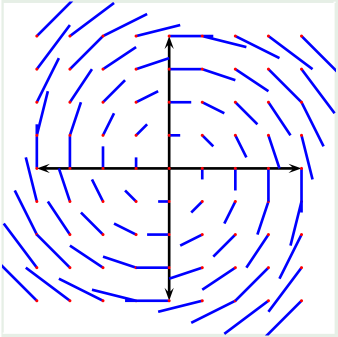
Continuity of vector fields

- Recall that a vector field is a function

$$\begin{array}{rcl} \mathbf{F}\colon D & \to & \mathbb{R}^2 \\ \mathbf{F}(x,y) & = & F_1(x,y)\mathbf{i} + F_2(x,y)\mathbf{j} \end{array}$$

- F_1, F_2 are two-variable functions with scalar output.
- We have already defined the notion of continuity of F_1 and F_2.
- We define **F** to be continuous if F_1 F_2 are continuous.

Example (continuous vector field)



Discuss the continuity of the following vector field.

$$\mathbf{F}(x, y) = \frac{y}{3}\mathbf{i} - \frac{x}{3}\mathbf{j} \quad .$$

Then

$$F_1(x, y) = \frac{y}{3} \quad , \quad F_2(x, y) = -\frac{x}{3}$$

We have that $\frac{y}{3}, -\frac{x}{3}$ are two-variable polynomials $\Longrightarrow$ F_1 and F_2 are continuous $\Longrightarrow$ the vector field is continuous.