Calculus III
Lecture 17

Todor Milev
https://github.com/tmilev/freecalc

2020

Outline

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Overview of Integrals Covered so Far

- We have so far studied integrals over:
  - intervals on a line;
  - planar regions in the plane;
  - solid regions in space.

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- Can we make sense of an integral over region that has lower dimension then the ambient space?
- We can for arbitrary k-dimensional surface in n dimensional space. We will only consider the examples of
  - a curve (1D region) embedded in a plane (2D)
  - a curve (1D region) embedded in space (3D)
  - a surface (2D region) embedded in space (3D).

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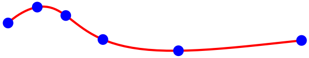

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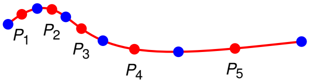

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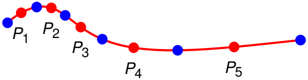

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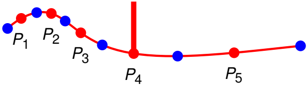

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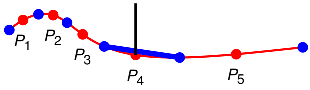

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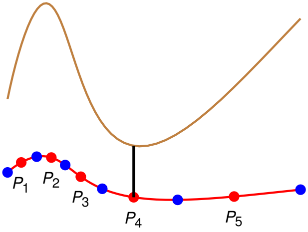

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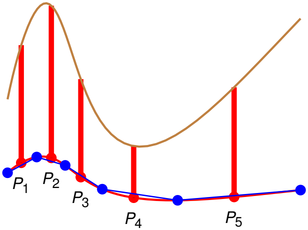

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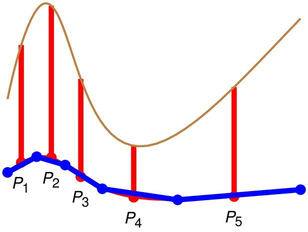

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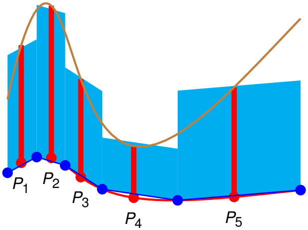

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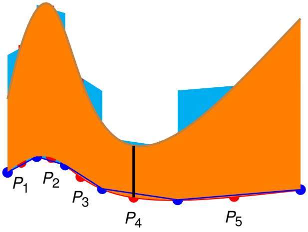

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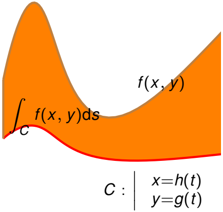

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Suppose the limit

$$\lim_{\max_k(\text{segment length})\to 0} \sum_{k=1}^{N} f(P_k) \cdot \text{length}(D_k)$$

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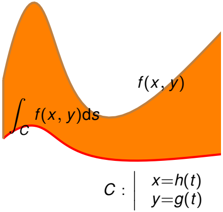

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The line integral is guaranteed to exists if f is a continuous functions or is bounded and continuous except at a finite number of points.

Parametrizations and Computations

Let $\mathbf{r}\colon [a,b] \to C$ be a regular, piecewise smooth parametrization of C. Then $\int_C f(x,y)\mathrm{d}s$ is computed as follows.

$$\begin{aligned} \mathrm{d}s &= |\mathbf{r}'(t)|\mathrm{d}t \\ \int_{(x,y)\in C} f(x,y)\mathrm{d}s &= \int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)|\mathrm{d}t. \end{aligned}$$

The result is independent of the parametrization of C we use.

$$\mathbf{r}\colon [a,b] \to \mathbb{R}^2 \quad , \quad \mathbf{r}(t) = (x(t), y(t))$$

$$\mathrm{d}s = |\mathbf{r}'(t)|\mathrm{d}t = \sqrt{(x'(t))^2 + (y'(t))^2}\,\mathrm{d}t$$

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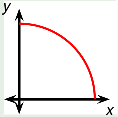

Compute $\int_C x^2 y \mathrm{d}s$, where C is the first quadrant part of the circle of radius 2 centered at origin and $\mathrm{d}s$ is the arclength form.

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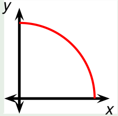

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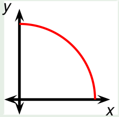

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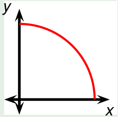

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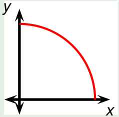

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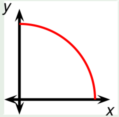

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- Line integral = work done by force $\mathbf{F}$ on particle moving along C.
- Line integral across C = flux across a membrane: $\mathbf{F} \cdot \mathbf{N}$ is the normal component of $\mathbf{F}$.

Line Integral Computations

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$\mathbf{r}\colon [a, b] \to C$: regular parametrization compatible with the orientation.

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$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \mathbf{F}\cdot\mathbf{T}\,ds$$

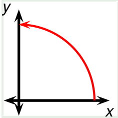
$\mathbf{r}\colon [a,b] \to C$: regular parametrization compatible with the orientation. Recall that

$$\mathbf{T} = \frac{1}{|\mathbf{r}'(t)|}\mathbf{r}'(t), \quad ds = |\mathbf{r}'(t)|\,dt \ \Rightarrow \mathbf{T}\,ds = \mathbf{r}'(t)\,dt = d\mathbf{r}\ .$$

Let $\mathbf{F}$ be given by $\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$. Then we can compute the line integral as follows.

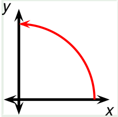
$$\begin{aligned}
\mathbf{r}'(t) &= (x'(t), y'(t)) = x'(t)\mathbf{i} + y'(t)\mathbf{j}\\
\mathbf{F}\cdot d\mathbf{r} &= \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt = \big(P(x(t),y(t))x'(t) + Q(x(t),y(t))y'(t)\big)\,dt\\
\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_{t=a}^{t=b} \big(P(x(t),y(t))x'(t) + Q(x(t),y(t))y'(t)\big)\,dt
\end{aligned}$$

Example



Find the work done by the force $\mathbf{F} = (x, -y) = x\,\mathbf{i} - y\,\mathbf{j}$ on a particle moving from $(1, 0)$ to $(0, 1)$ along the quarter of the unit circle contained in the first quadrant.

Example



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A parametrization of C compatible with the given orientation is $\mathbf{r}(t) = (\cos t, \sin t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}, t \in \left[0, \frac{\pi}{2}\right]$.

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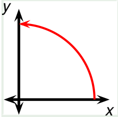

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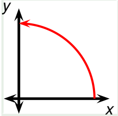
$$\mathbf{F} = (\cos t)\mathbf{i} - (\sin t)\mathbf{j}$$

Example



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$$\begin{array}{rcl} \mathbf{F} & = & (\cos t)\mathbf{i} - (\sin t)\mathbf{j} \\ \mathbf{r}'(t) & = & -(\sin t)\mathbf{i} + (\cos t)\mathbf{j} \end{array}$$

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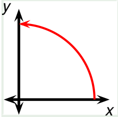

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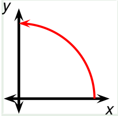

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\mathbf{F}\cdot\mathbf{r}'(t) &= -2\sin t\cos t \\
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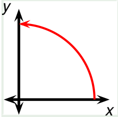

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&= \int_{t=0}^{t=\frac{\pi}{2}} -2 \sin t \cos t \,\mathrm{d}t = \left[\cos^2 t\right]_{t=0}^{t=\frac{\pi}{2}} = -1 \ .
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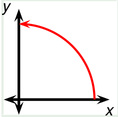

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What if the parametrization is not compatible with the orientation?

Differential 1-Forms

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Definition (Differential 1-form)

An expression of the type

$$\omega = P(x, y)\,\mathrm{d}x + Q(x, y)\,\mathrm{d}y \qquad \Big|\ \text{(in 2D)}$$

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Integrals of 1-Forms

Let $\omega = P(x, y)\,\mathrm{d}x + Q(x, y)\,\mathrm{d}y$ be a 1-form, let C be an oriented curve. Let $\mathbf{r}\colon [a, b] \to C$, $\mathbf{r}(t) = (x(t), y(t))$ be an orientation-compatible parametrization.

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P(x, y)\mathrm{d}x + Q(x, y)\mathrm{d}y &= \left(P(x(t), y(t))x'(t) + Q(x(t), y(t))y'(t)\right)\mathrm{d}t \\
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\end{aligned}$$

If we re-parametrize the curve, the substitution rule and the multivariable chain rule imply that the integral doesn't change.

The Notation $\oint$: Closed Path Integrals

- Suppose that the curve image C, parametrized by $\mathbf{r}(t) : (x(t), y(t)), t \in [a, b]$ is a closed curve.

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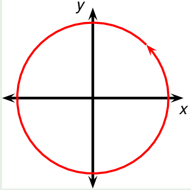
- The notation is mostly useful when we are integrating an closed 1-form. (Definition of closed form is/will be studied separately).

Example



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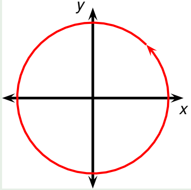

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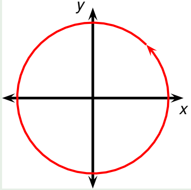

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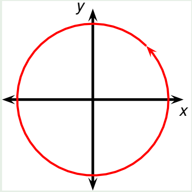

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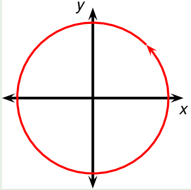

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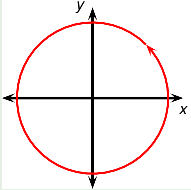

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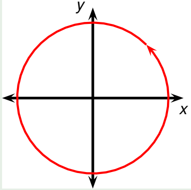

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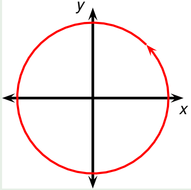

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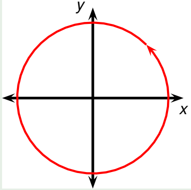

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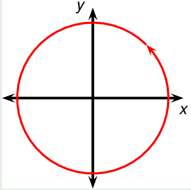

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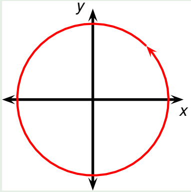

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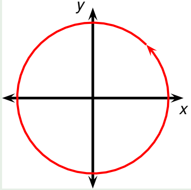

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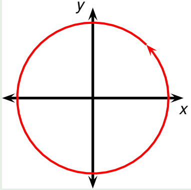

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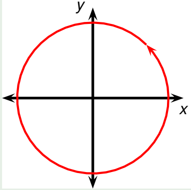

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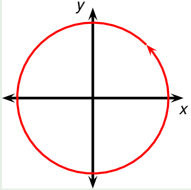

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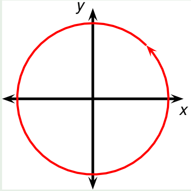

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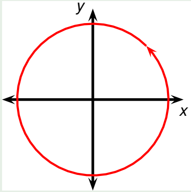

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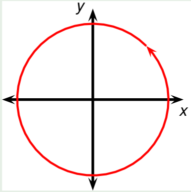

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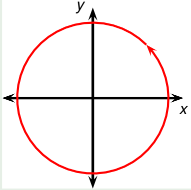

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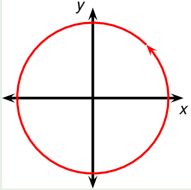

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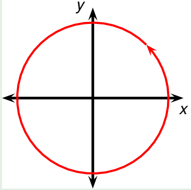

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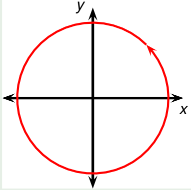

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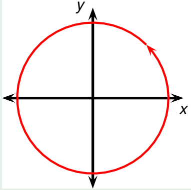

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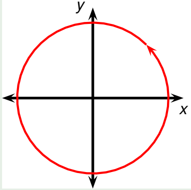

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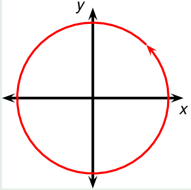

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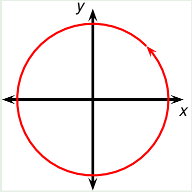
$$\oint_C \frac{x}{x^2+y^2}\mathrm{d}x + \frac{y}{x^2+y^2}\mathrm{d}y$$

Parametrize: $C: \left|\begin{array}{rcl} x &=& R\cos t \\ y &=& R\sin t \\ \mathrm{d}x &=& (-R\sin t)\mathrm{d}t \\ \mathrm{d}y &=& (R\cos t)\mathrm{d}t \end{array}\right., 0 \leq t \leq 2\pi.$

$$\begin{aligned}\frac{x}{x^2+y^2}\mathrm{d}x + \frac{y}{x^2+y^2}\mathrm{d}y &= \frac{R\cos t(-R\sin t\mathrm{d}t)}{R^2} + \frac{R\sin t(R\cos t\mathrm{d}t)}{R^2} \\ &= 0\end{aligned}$$

Example



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$$= 0$$

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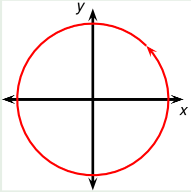
1-Forms in Polar Coordinates

Example



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1-Forms in Polar Coordinates

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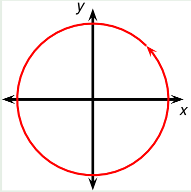

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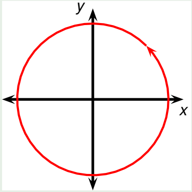
In polar coord.: $\left| \begin{array}{lcl} x &=& r\cos\theta \\ y &=& r\sin\theta \end{array} \right. \Rightarrow \left| \begin{array}{l} \mathrm{d}x = \textbf{?} \\ \mathrm{d}y = \textbf{?} \end{array} \right.$.

1-Forms in Polar Coordinates

Example



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1-Forms in Polar Coordinates

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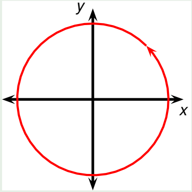

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1-Forms in Polar Coordinates

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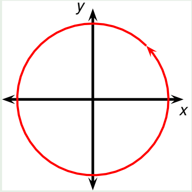

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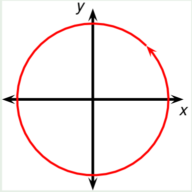

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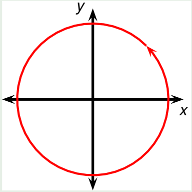

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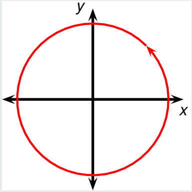
1-Forms in Polar Coordinates

Example



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1-Forms in Polar Coordinates

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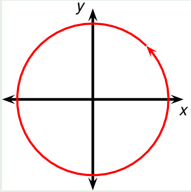

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1-Forms in Polar Coordinates

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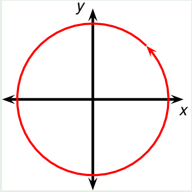

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1-Forms in Polar Coordinates

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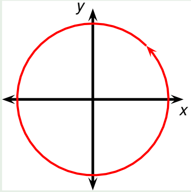

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1-Forms in Polar Coordinates

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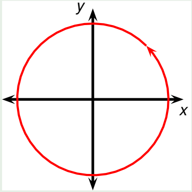

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$$-\frac{y}{x^2+y^2}\mathrm{d}x + \frac{x}{x^2+y^2}\mathrm{d}y \quad = \quad \mathrm{d}\theta$$

1-Forms in Polar Coordinates

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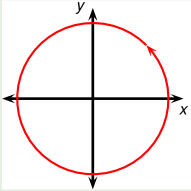

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1-Forms in Polar Coordinates

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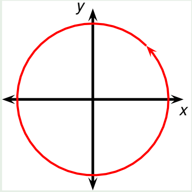

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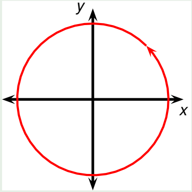

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1-Forms in Polar Coordinates

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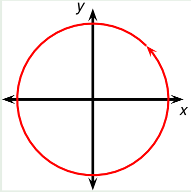

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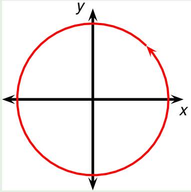

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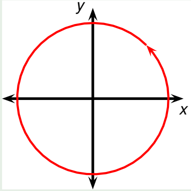

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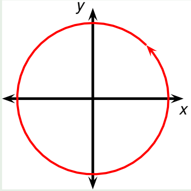

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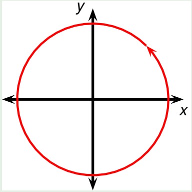
1-Forms in Polar Coordinates

Example



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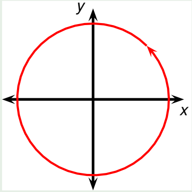

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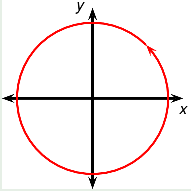

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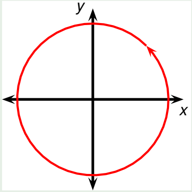

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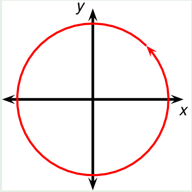

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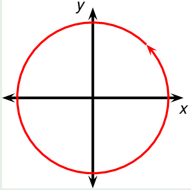

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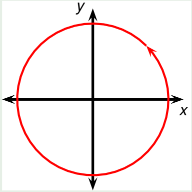

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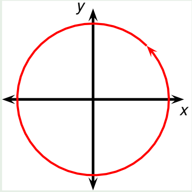

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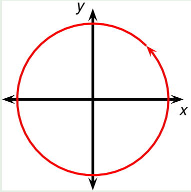

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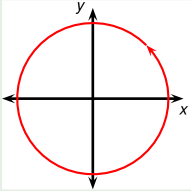

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1-Forms in Polar Coordinates

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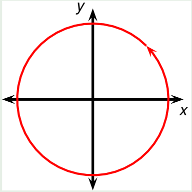

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1-Forms in Polar Coordinates

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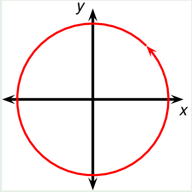

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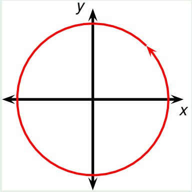

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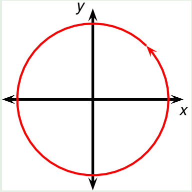

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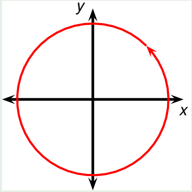

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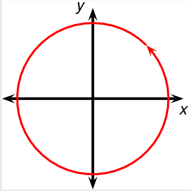

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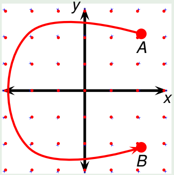
Example (Work Done by Point Mass Gravity Field)



Let **F** be the vector field

$$\mathbf{F}(\mathbf{v}) = -\frac{1}{|\mathbf{v}|^3}\mathbf{v} \quad ,$$

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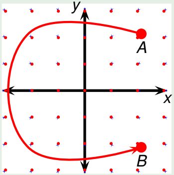

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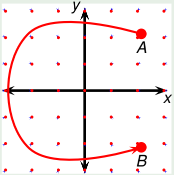

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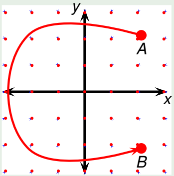

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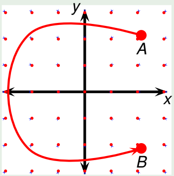

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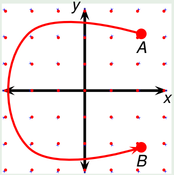

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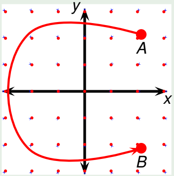
Example (Work Done by Point Mass Gravity Field)



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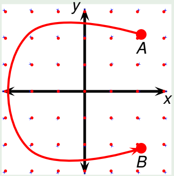

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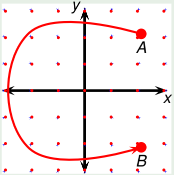

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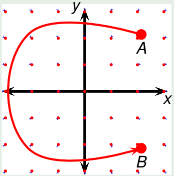

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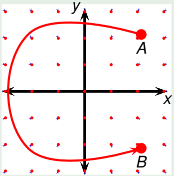

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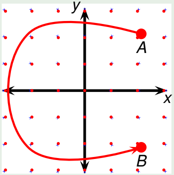

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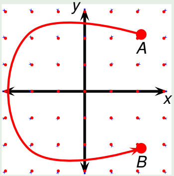

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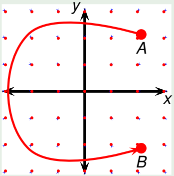

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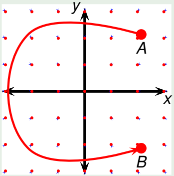

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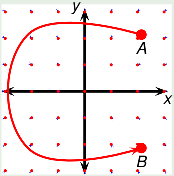

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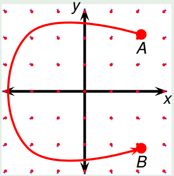

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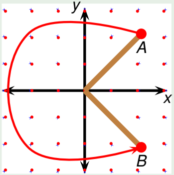

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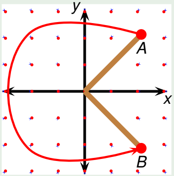

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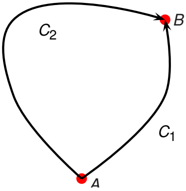
In this example, we established that the line integral depends only on the endpoints A and B but not on the connecting path.

Conservative Fields



Definition

A vector field $\mathbf{F}$ is called *conservative* if for any two points A and B and any two paths C_1 and C_2 from A to B we have $\displaystyle\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$.

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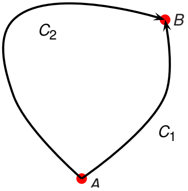

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A vector field $\mathbf{F}$ *is conservative if and only if every point A every path C starting and ending at A we have* $\displaystyle\oint_C \mathbf{F}\cdot d\mathbf{r} = 0$.

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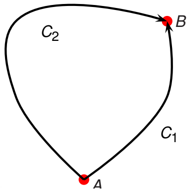

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The path $C = C_1 \cup (-C_2)$ starts and ends at A

□

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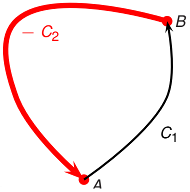

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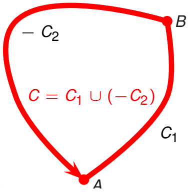

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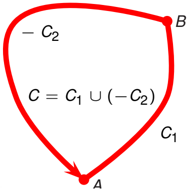

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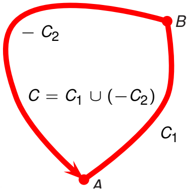

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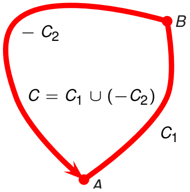

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A vector field $\mathbf{F}$ is called *conservative* if for any two points A and B and any two paths C_1 and C_2 from A to B we have $\displaystyle\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$.

Lemma (alternative definition)

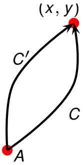
A vector field $\mathbf{F}$ *is conservative if and only if every point A every path C starting and ending at A we have* $\displaystyle\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$.

Proof.

The path $C = C_1 \cup (-C_2)$ starts and ends at A and therefore

$$0 = \oint_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} - \int_{C_2} \mathbf{F} \cdot d\mathbf{r} \; . \Leftrightarrow \int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}. \qquad \square$$

Conservative Field $\Rightarrow$ Gradient Field



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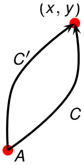

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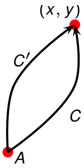

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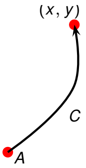

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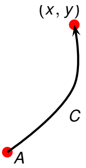

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Theorem

$$\mathbf{F} = \nabla f.$$

Proof.

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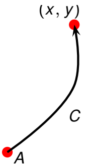

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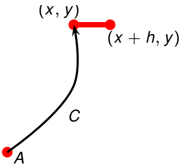

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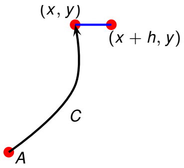

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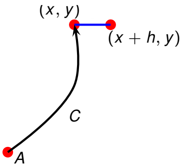

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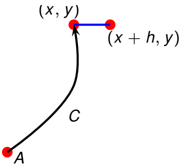

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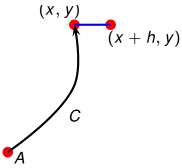

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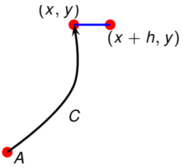

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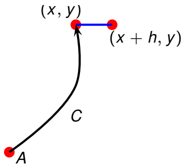

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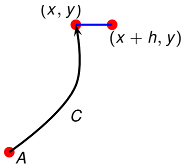

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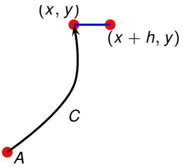

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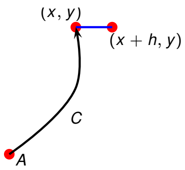

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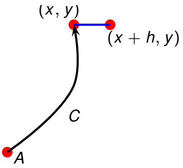

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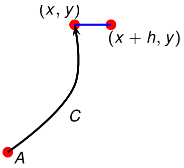

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where the last equality is the single-variable Fundamental Theorem of Calculus. $\square$

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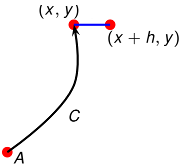

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where the last equality is the single-variable Fundamental Theorem of Calculus. Similarly it follows that $\frac{\partial}{\partial y}(f) = Q(x, y)$. $\square$

Gradient Field $\Rightarrow$ Conservative Field

Theorem (Fundamental Theorem of Calculus for Line Integrals)

$$\int_C (\nabla f)\cdot \mathbf{dr} = f(B) - f(A)\ ,\ \textit{for every smooth curve } C \textit{ from } A \textit{ to } B.$$

Proof.

$$\begin{aligned} \int_C (\nabla f)\cdot \mathbf{dr} &= \int_C f_x \mathrm{d}x + f_y \mathrm{d}y = \int_C (f_x x'(t) + f_y y'(t))\,\mathrm{d}t \\ &= \int_a^b \frac{\mathrm{d}}{\mathrm{d}t}(f(\mathbf{r}(t))\mathrm{d}t = f(B) - f(A). \end{aligned}$$

$\square$

Definition

If $\mathbf{F} = \nabla f$ then f is called *scalar potential* of $\mathbf{F}$; $\mathbf{F}$ is called *gradient field*.

Let $\mathbf{F} = \nabla f$ be gradient field. For a curve C joining points A and B

$$\int_C \mathbf{F}\cdot \mathbf{dr} = \int_C (\nabla f)\cdot \mathbf{dr} = f(B) - f(A)$$

depends only on A and B, but not on C $\Rightarrow$ $\mathbf{F}$ is conservative.

A Criterion for Conservative (Gradient) Fields

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Proposition

If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ is a gradient field, then $P_y = Q_x$.

A Criterion for Conservative (Gradient) Fields

- Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ be a smooth conservative (gradient) field.
- Then for some f, $\mathbf{F} = \nabla f$, hence

$$P = f_x, \qquad Q = f_y \ .$$

- Since mixed partial derivatives are equal, it follows that

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If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ is a gradient field, then $P_y = Q_x$.

- A similar consideration in 3 dimensions shows the following.

Proposition

If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ is a gradient field, then

$$P_y = Q_x, \quad P_z = R_x, \quad Q_z = R_y \ .$$

Simply Connected Regions

If $P_y(x, y) \neq Q_x(x, y)$, then **F** is not a gradient field.

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If $P_y(x, y) \neq Q_x(x, y)$, then **F** is not a gradient field.
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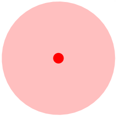

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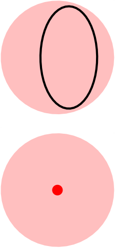

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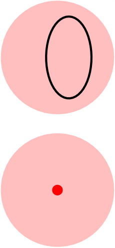

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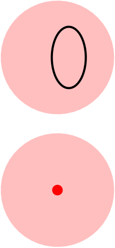

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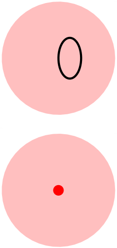

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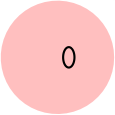

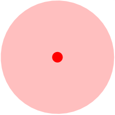

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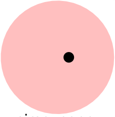
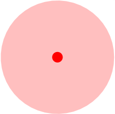


simp. conn.



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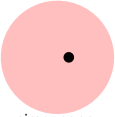

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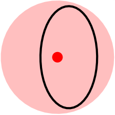

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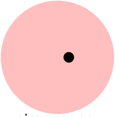

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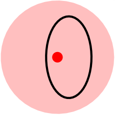

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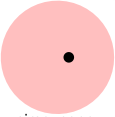
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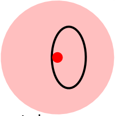
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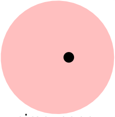

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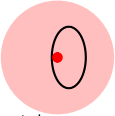

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Theorem

Let $\mathbf{F} = P\,\mathbf{i} + Q\mathbf{j}$ *and* $P_y = Q_x$. *Suppose* **F** *is defined over a simply connected open set. Then* **F** *is a gradient field.*

Example

Show the field $\mathbf{F} = (3 + 2xy)\mathbf{i} + (x^2 - 3y^2)\mathbf{j}$ is gradient and find a scalar potential. Compute $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$, where C is any curve from $(1, 0)$ to $(0, 1)$.

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$\mathbb{R}^2$ – simp. conn.
$\Rightarrow$ potential exists

Example

Show the field $\mathbf{F} = (3 + 2xy)\mathbf{i} + (x^2 - 3y^2)\mathbf{j}$ is gradient and find a scalar potential. Compute $\int_C \mathbf{F} \cdot \mathbf{dr}$, where C is any curve from $(1, 0)$ to $(0, 1)$.

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f &= \int P \mathrm{d}x
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&= (-1+K) - (3+K) = -4 \quad .
\end{aligned}
$$

Exact 1-Forms

Let $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ be a vector field and $\omega = \mathbf{F} \cdot \mathrm{d}\mathbf{r} = P\mathrm{d}x + Q\mathrm{d}y$ be the corresponding 1-form.

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1-forms that are (total) differentials of functions are called *exact*.

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Theorem (Net Change Theorem for Line Integrals)

If C is a curve from A and B, then $\displaystyle\int_C \mathrm{d}f = \int_C (\nabla f) \cdot \mathrm{d}\mathbf{r} = f(B) - f(A)$.