Calculus II
Lecture 15

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Outline

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- We are interested to study sequences such as:

$$
\begin{array}{l}
(a_1, a_2, a_3 \dots) \\
(1, 2, 3, \dots) \\
(1, 3, 5, 7, \dots) \\
(1, -1, 1, -1, \dots) \quad .
\end{array}
$$

Definition (Most general form)

A sequence is an ordered collection of objects in which repetitions are allowed.

- The definition is too general for our purposes.
- For example it allows sequences indexed by the real numbers.
- We give a less general (but possibly easier to understand) definition shortly.
- Our less general definition will cover all uses of sequences in the present course/lectures.
- We start by a few examples.

Example (Sequence notation)

- Consider the sequence

$$(2, 4, 6, 8, \ldots).$$

- That appears to be the sequence of all positive even integers.
- We can express this sequence more compactly using the notation

$$a_n = 2n,$$

where a_n denotes the nth term.

$$\begin{aligned} a_1 &= 2 \cdot 1 = 2 \\ a_2 &= 2 \cdot 2 = 4 \\ a_3 &= 2 \cdot 3 = 6 \\ a_4 &= 2 \cdot 4 = 8 \\ &\vdots \end{aligned}$$

Example

The sequence
$$(-1, 1, -1, 1, -1, 1, \dots)$$
can be written as $b_n = (-1)^n$.

Example

The sequence
$$(1, 2, 4, 8, 16, \dots)$$
can be written as $c_n = 2^{n-1}$.

Example

The sequence
$$\left(\frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, -\frac{1}{16}, \dots\right)$$
can be written as $d_n = -\left(-\frac{1}{2}\right)^n$.

Definition (Sequence indexed by the integers)

A sequence is a list of numbers indexed by consecutive integers bounded below and written in a definite order

$$(a_1, a_2, a_3, a_4, \dots, a_n, \dots) \quad .$$

- In our course/lectures we assume all sequences are indexed by consecutive integers.
- Unless stated/implied otherwise
  - We assume the first index is 1.
  - Under above assumption a_1 is called the first term, a_2 is called the second term, and a_n is the n^{th} term.
- We often denote the sequence of elements $(a_1, a_2, \dots)$ by

$$\{a_n\} \quad \text{and more precisely} \quad \{a_n\}_{n=1}^{\infty}$$

or by

$$(a_n) \quad \text{and more precisely} \quad (a_n)_{n=1}^{\infty}$$

- The use of $\{\}$ versus () differs between authors and instructors.

Definition (Sequence indexed by the integers)

A sequence is a list of numbers indexed by consecutive integers bounded below and written in a definite order

$$(a_1, a_2, a_3, a_4, \dots, a_n, \dots) \quad .$$

- To indicate a sequence labeled so the first index is not 1 write:

$$\begin{array}{rcl} (a_n)_{n=0}^{\infty} & \text{for} & (a_0, a_1, a_2, \dots) \\ (a_n)_{n=2}^{\infty} & \text{for} & (a_2, a_3, a_4, \dots) \\ (a_n)_{n=-1}^{\infty} & \text{for} & (a_{-1}, a_0, a_1, \dots) \end{array}$$

Definition

A sequence is finite if it has a finite number of elements.

- To indicate a sequence is finite either write all elements of the sequence or use indices as shown below.

$$\begin{array}{rcl} & & (1, 2, 3, 4, 5, 6, 7, 8, 9, 10) \\ (a_n)_{n=1}^{5} & \text{for} & (a_1, a_2, a_3, a_4, a_5) \end{array}$$

Defining sequences

Question

How can we define a sequence of numbers $(a_1, a_2, a_3 \dots, a_n, \dots)$*?*

- A sequence can be interpreted as a function that takes as arguments a subset of the integers.
- Since functions can be defined in exotic and indirect ways, so can sequences.
- We will focus on the three most frequently used ways to define sequences:
  - by specifying a formula for the n^{th} term;
  - by recursion;
  - by specifying a property of integers and constructing a sequence of all integers with that property.

Sequences via formulas

- Sequences can be defined by presenting a formula to obtain the n^{th} term a_n as a function of the index n.
- Another frequently used notation: include the f-la in parenthesis and indicate the index ranges by super- and subscripts.
- There is a third informal but frequently used notation: list few terms of the sequence and let the reader guess the formula.

Example

$a_n = \frac{n}{n+1}$	$\left(\frac{n}{n+1}\right)_{n=1}^{\infty}$	$\left(\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots\right)$
$a_n = \frac{(-1)^n(n+1)}{3^n}$	$\left(\frac{(-1)^n(n+1)}{3^n}\right)_{n=1}^{\infty}$	$\left(\frac{-2}{3}, \frac{3}{9}, \frac{-4}{27}, \frac{5}{81}, \dots\right)$
$a_n = \sqrt{n-3}, n \geq 3$	$\left(\sqrt{n-3}\right)_{n=3}^{\infty}$	$\left(0, 1, \sqrt{2}, \sqrt{3}, \dots\right)$
$a_n = \cos\left(\frac{n\pi}{6}\right), n \geq 0$	$\left(\cos\frac{n\pi}{6}\right)_{n=0}^{\infty}$	$\left(1, \frac{\sqrt{3}}{2}, \frac{1}{2}, 0, \dots\right)$

Example (Sequences via formulas: find sequence terms)

Find the first five terms of each of the following sequences.

1. $a_n = 3 \cdot 2^{-n}$
2. $b_n = 1$
3. $c_n = -3(n-1) + 5$
4. $d_n = n^2 + 1$

Example (Sequences via f-las: guess f-la from terms)

Find a formula for the general term a_n of the sequence

$$\left(0, \frac{1}{4}, -\frac{2}{8}, \frac{3}{16}, -\frac{4}{32}, \frac{5}{64}, \dots\right)$$

$$a_1 = 0,\ a_2 = \frac{1}{4},\ a_3 = -\frac{2}{8},\ a_4 = \frac{3}{16},\ a_5 = -\frac{4}{32},\ a_6 = \frac{5}{64},$$

- The numerators start at 0 and go up by one with each term.
- The n^{th} term has numerator $n - 1$.
- The denominators start at 2 and double with each term.
- The n^{th} term has denominator 2^n.
- The signs of the terms alternate between positive and negative.
- We take this into account by multiplying by $(-1)^n$.

$$a_n = (-1)^n \frac{n-1}{2^n}$$

Example (Sequences via f-las: guess f-la from terms)

Find a formula for the *n*th term of each of the following sequences.

1. $a_n = 2 \cdot \left(\frac{1}{4}\right)^{n-1}$

$$\left(2, \frac{1}{2}, \frac{1}{8}, \frac{1}{32}, \frac{1}{128}, \dots\right)$$

2. $b_n = (-1)^n n^2$

$$-1, 4, -9, 16, -25, \dots$$

3. $c_n = -1 + 6(n-1)$

$$-1, 5, 11, 17, 23, \dots$$

Warning about implied sequence formulas

- We found the sequence $\left(0, \frac{1}{4}, -\frac{2}{8}, \frac{3}{16}, -\frac{4}{32}, \frac{5}{64}, \dots\right)$ can be given by: $a_n = (-1)^n \frac{n-1}{2^n}$
- For any finite number of terms we can produce infinitely many different formulas that match them - but disagree on the terms after.
- For example the sequence above can also be obtained by:

$$a_n = \frac{27}{512}n^5 - \frac{477}{512}n^4 + \frac{3159}{512}n^3 - \frac{9651}{512}n^2 + \frac{6643}{256}n - \frac{793}{64}$$

  and that produces $a_7 = \frac{363}{32}$.
- Bear in mind that using implied sequence formulas is **informal**.
  - It is acceptable to use the implied sequence notation only when we believe there is a single completely obvious pattern that will be recognized by every one.
  - The pattern should be obvious not only to us, but also to our potential readers.
  - If in doubt we should switch to a more rigorous notation.

Sequences via recursion

- Sequences can be defined by recursive formulas.
- A sequence formula is recursive if it expresses the term a_n via the preceding terms $a_1, a_2, \dots, a_{n-1}$, rather than directly as a function of n.

Example (Defining sequences by recursion)

Define recursively the Fibonacci sequence $(f_n)_{n=1}^{\infty}$ by requesting that

$$f_1 = 1 \qquad f_2 = 1 \qquad f_n = f_{n-1} + f_{n-2}, \quad n \geq 3.$$

The first few terms are

$$1, 1,$$

- In fact the Fibonacci sequence can be described by a formula, but it is not very simple: $a_n = \frac{\sqrt{5}}{5}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$.

Sequences via inclusion criterion

- A sequence can also be given by specifying a criterion to check whether a number should be included in the sequence or not.

Example (Defining sequence by criterion)

Define $(p_n)_{n=1}^{\infty}$ as the sequence of all primes.

$$(2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, \dots)$$

- We know how to check whether a number is prime.
- For example, a crude test for whether a number is prime is to check whether it is divisible by all positive numbers smaller than it.
- Our sequence is well defined; we could generate it, say, by computer.
- However, we have given no closed or even recursive formula to generate the entire sequence.

Sequences defined indirectly

- We note that in addition to the illustrated ways to define sequences, we are also free to use for the task any well-posed statement.
- Such ways to define a sequence may be very indirect or obscure and we will not use them in our course.
- We hint the challenges that can arise by using arbitrary (but well-posed) definitions on a few examples.

Example

1. Let a_n be the n^{th} digit in the decimal expansion of the number e. The first few terms of (a_n):

$$2, 7, 1, 8, 2, 8, 1, 8, 2, 8, 4, 5, \dots$$

2. Consider the sequence (p_n), where p_n is the population of the world as of January 1 of year n.

Definition (Arithmetic sequence)

An arithmetic sequence is one in which successive terms differ by a constant number. This constant is called the difference of the arithmetic sequence.

Example (Which are arithmetic?)

1,	2,	3,	4,	5,	$\dots$	is arithmetic with difference 1.
23,	16,	9,	2,	-5,	$\dots$	is arithmetic with difference -7.
8,	9,	12,	17,	24,	$\dots$	is not arithmetic.
						$(9-8=1$ but $12-9=3.)$

Example (Which are arithmetic?)

Sequence	Arithmetic?	Difference	First term	nth term
$1, -1, 1, -1, \ldots$				
$\frac{1}{6}, \frac{1}{2}, \frac{5}{6}, \frac{7}{6}, \frac{3}{2}, \ldots$				
$2, 2, 2, 2, \ldots$				

If an arithmetic sequence has difference d, then the nth term has formula

$$a_n = a_1 + d(n-1),$$

where a_1 is the first term.

Definition (Geometric sequence)

A geometric sequence is one in which each term is obtained by multiplying the previous one by the same constant. This constant is called the ratio of the geometric sequence.

Example (Which are geometric?)

$$\begin{array}{rrrrrll}
2, & 4, & 8, & 16, & 32, & \dots & \text{is geometric with ratio 2.}\\
1, & -3, & 9, & -27, & 81, & \dots & \text{is geometric with ratio } -3.\\
-42, & -14, & -21, & 31, & -22, & \dots & \text{is not geometric.}\\
 & & & & & & \left(\frac{-14}{-42} = \frac{1}{3} \text{ but } \frac{-21}{-14} = \frac{3}{2}.\right)
\end{array}$$

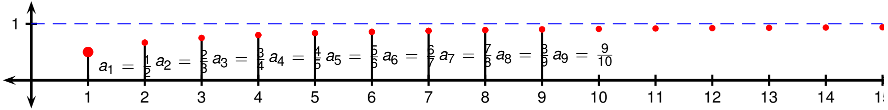
Example (Arithmetic and geometric)

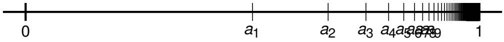
Sequence	Arithmetic/ geometric	Diff.	Ratio	a_1	a_n
$\frac{2}{3}, \frac{4}{9}, \frac{8}{27}, \frac{16}{81}, \dots$					
$7, 3, -1, -5, \dots$					
$4, 4, 4, 4, \dots$					
$\pi, -\pi^2, \pi^3, -\pi^4, \dots$					
$1, 1, 2, 2, 3, 3, \dots$					

If a geometric sequence has ratio r, then the nth term has formula

$$a_n = a_1 r^{n-1}.$$

where a_1 is the first term.





- The sequence $a_n = \frac{n}{n+1}$ can be plotted on a number line or using Cartesian coordinates.
- From the pictures, the terms in the sequence appear to approach 1 as n gets larger.
- $1 - \frac{n}{n+1} = \frac{1}{n+1}$.
- This can be made arbitrarily small by choosing n large enough.
- We express this by writing $\lim\limits_{n\to\infty} \frac{n}{n+1} = 1$.

Definition (Limit of a Sequence)

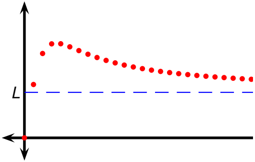
A sequence $\{a_n\}$ has the limit L, and we write

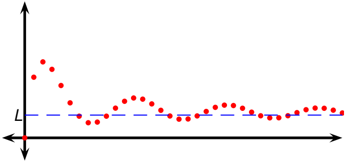
$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L \text{ as } n \to \infty$$

if we can make a_n as close to L as we like by taking n large enough.

Definition (Convergent)

A sequence that has a limit is called convergent. A sequence that has no limit is called divergent.

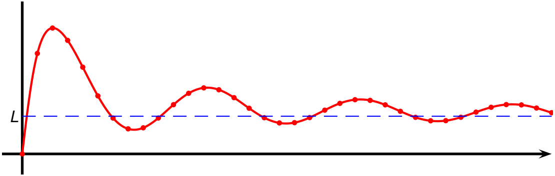




If you compare the definition of the limit of a sequence with the definition of the infinite limit of a function, you'll see that the only difference between

$$\lim_{n\to\infty} a_n = L \qquad \text{and} \qquad \lim_{x\to\infty} f(x) = L$$

is that n is required to be an integer.



Theorem

If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ for all integers n, then $\lim_{n\to\infty} a_n = L$.

Example

Find $\lim_{n\to\infty} \frac{n}{n+1}$.
Divide numerator and denominator by the highest power of n, and use the limit laws:

$$\begin{aligned}
\lim_{n\to\infty} \frac{n}{n+1} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} &= \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} \\
&= \frac{\lim\limits_{n\to\infty} 1}{\lim\limits_{n\to\infty} 1 + \lim\limits_{n\to\infty} \frac{1}{n}} \\
&= \frac{1}{1+0} \\
&= 1
\end{aligned}$$

Just like for functions, there is a notion of sequences tending to infinity:
If a_n grows large as n becomes large, we write $\lim_{n\to\infty} a_n = \infty$.
You can probably guess what $\lim_{n\to\infty} a_n = -\infty$ means.

The Limit Laws for continuous functions also hold for sequences:
If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then

1. $\lim\limits_{n\to\infty}(a_n + b_n) = \lim\limits_{n\to\infty} a_n + \lim\limits_{n\to\infty} b_n$
2. $\lim\limits_{n\to\infty}(a_n - b_n) = \lim\limits_{n\to\infty} a_n - \lim\limits_{n\to\infty} b_n$
3. $\lim\limits_{n\to\infty} ca_n = c \lim\limits_{n\to\infty} a_n$
4. $\lim\limits_{n\to\infty}(a_n b_n) = \lim\limits_{n\to\infty} a_n \cdot \lim\limits_{n\to\infty} b_n$
5. $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}$ if $\lim_{n\to\infty} b_n \neq 0$
6. $\lim\limits_{n\to\infty} a_n^p = \left[\lim\limits_{n\to\infty} a_n\right]^p$ if $p > 0$ and $a_n > 0$.

Example

Calculate $\lim_{n\to\infty} \frac{\ln n}{n}$.

- Both $\ln n$ and n go to ∞ as n gets bigger.
- We can't use L'Hospital's Rule directly, because L'Hospital's Rule is for functions.
- Define $f(x) = \frac{\ln x}{x}$. Now use L'Hospital's Rule:

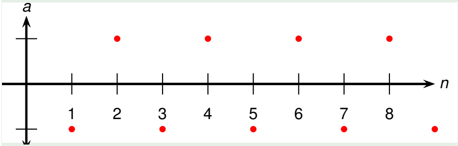
$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0$$

- Therefore

$$\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{x\to\infty} f(x) = 0$$

Example

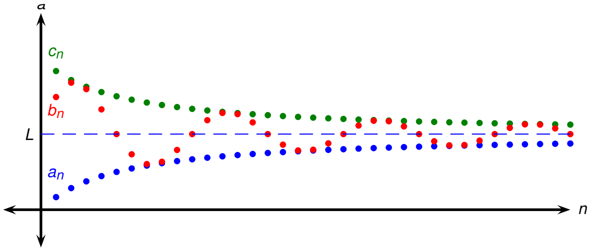
Is the sequence $a_n = (-1)^n$ convergent or divergent?



- The terms oscillate between -1 and 1 infinitely many times.
- Therefore the sequence doesn't approach any number.
- $\{a_n\}$ is divergent.

Theorem (The Squeeze Theorem for Sequences)

If $a_n \leq b_n \leq c_n$ *for* $n \geq n_0$ *and* $\lim\limits_{n\to\infty} a_n = L = \lim\limits_{n\to\infty} c_n$, *then* $\lim\limits_{n\to\infty} b_n = L$.



Corollary (to the squeeze theorem)

If $\lim_{n\to\infty} |a_n| = 0$, *then* $\lim_{n\to\infty} a_n = 0$.

Theorem (Squeeze theorem for functions at ∞)

If $f(x) \leq g(x) \leq h(x)$ *and* $\lim_{x\to\infty} f(x) = L = \lim_{x\to\infty} h(x)$, *then* $\lim_{x\to\infty} g(x) = L$.

Example

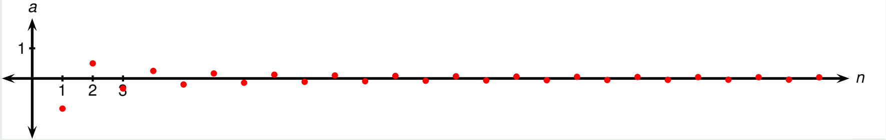
Is $a_n = \frac{(-1)^n}{n}$ convergent or divergent?

$$\lim_{n\to\infty} \left| \frac{(-1)^n}{n} \right| = \lim_{n\to\infty} \frac{1}{n} = 0$$

Therefore, by the corollary to the Squeeze Theorem,

$$\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$$

Therefore $\left\{ \frac{(-1)^n}{n} \right\}$ is convergent.



Theorem

If $\lim_{n\to\infty} a_n = L$ *and the function* f *is continuous at* L*, then*

$$\lim_{n\to\infty} f(a_n) = f(L)$$

Example

Find $\lim_{n\to\infty} \sin(\pi/n)$.
Sine is continuous at 0.

$$\begin{aligned} & \lim_{n\to\infty} \sin(\pi/n) \\ = & \sin\left(\lim_{n\to\infty}(\pi/n)\right) \\ = & \sin 0 \\ = & 0 \end{aligned}$$

Find $\lim_{n\to\infty} \cos(\pi/n)$.
Cosine is continuous at 0.

$$\begin{aligned} & \lim_{n\to\infty} \cos(\pi/n) \\ = & \cos\left(\lim_{n\to\infty}(\pi/n)\right) \\ = & \cos 0 \\ = & 1 \end{aligned}$$

Example

Discuss the convergence of the sequence $a_n = \frac{n!}{n^n}$, where $n! = 1 \cdot 2 \cdot 3 \cdot \dots \cdot n$.

- Both the top and the bottom go to infinity as $n \to \infty$.
- We can't use L'Hospital's Rule, because we have no function corresponding to $n!$ ($x!$ isn't defined if x isn't an integer).

$$a_1 = 1 \qquad a_2 = \frac{1 \cdot 2}{2 \cdot 2} \qquad a_3 = \frac{1 \cdot 2 \cdot 3}{3 \cdot 3 \cdot 3}$$

$$\begin{aligned} a_n &= \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot \dots \cdot n}{n \cdot n \cdot n \cdot n \cdot \dots \cdot n} \\ &= \frac{1}{n} \left(\frac{2 \cdot 3 \cdot 4 \cdot \dots \cdot n}{n \cdot n \cdot n \cdot \dots \cdot n} \right) \end{aligned}$$

- $\frac{2}{n} \leq 1$, $\frac{3}{n} \leq 1$, $\frac{4}{n} \leq 1$,… $\frac{n}{n} \leq 1$. Therefore $0 \leq a_n \leq \frac{1}{n}$.
- Since $\frac{1}{n} \to 0$ as $n \to \infty$, by the Squeeze Theorem $a_n \to 0$ as $n \to \infty$.

Example

For what values of r is the sequence $\{r^n\}$ convergent?

Consider the exponential function $y = r^x$.

$$\lim_{x\to\infty} r^x = \begin{cases} \infty & \text{if} \quad r > 1 \\ 0 & \text{if} \quad 0 < r < 1 \end{cases}$$

Therefore

$$\lim_{n\to\infty} r^n = \begin{cases} \infty & \text{if} \quad r > 1 \\ 0 & \text{if} \quad 0 < r < 1 \end{cases}$$

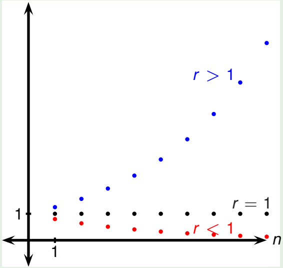
Also, $\lim\limits_{n\to\infty} 1^n = 1$ and $\lim\limits_{n\to\infty} 0^n = 0$.

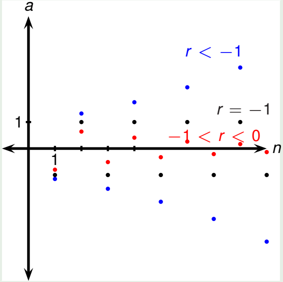
If $-1 < r < 0$, then $0 < |r| < 1$, and

$$\lim_{n\to\infty} |r^n| = \lim_{n\to\infty} |r|^n = 0$$

Therefore $\lim\limits_{n\to\infty} r^n = 0$.

If $r \leq -1$, then r^n diverges. In particular, $(-1)^n$ diverges.





This theorem summarizes the results of the previous example.

Theorem (Convergence of Geometric Sequences)

The sequence $\{r^n\}$ is convergent if $-1 < r \leq 1$ and divergent otherwise.

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \textit{if} \quad -1 < r < 1 \\ 1 & \textit{if} \quad r = 1 \end{cases}$$

Definition (Increasing and Decreasing)

A sequence $\{a_n\}$ is called increasing if $a_n < a_{n+1}$ for all $n \geq 1$. In other words, $\{a_n\}$ is increasing if $a_1 < a_2 < a_3 < \cdots$.
A sequence $\{a_n\}$ is called decreasing if $a_n > a_{n+1}$ for all $n \geq 1$. In other words, $\{a_n\}$ is decreasing if $a_1 > a_2 > a_3 > \cdots$.
A sequence is called monotonic if it is either increasing or decreasing.

Example

The sequence $\left\{\frac{1}{2n+1}\right\}$ is decreasing because

$$a_n = \frac{1}{2n+1} \qquad a_{n+1} = \frac{1}{2(n+1)+1} = \frac{1}{2n+3}$$

and

$$\frac{1}{2n+1} > \frac{1}{2n+3}$$

because the denominator of the latter is bigger.

Definition (Bounded Sequence)

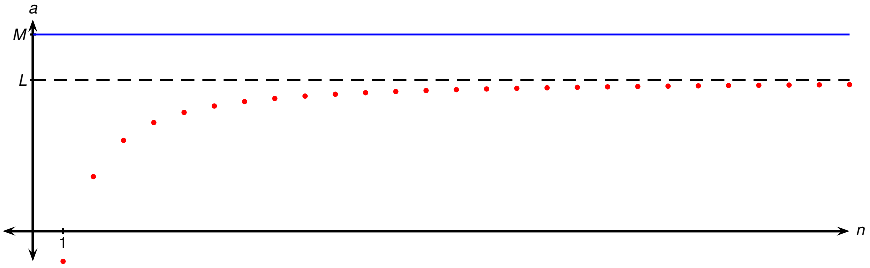
A sequence $\{a_n\}$ is called bounded above if there exists a number M such that

$$a_n < M \qquad \text{for all} \qquad n \geq 1.$$

It is called bounded below if there exists a number M such that

$$a_n > M \qquad \text{for all} \qquad n \geq 1.$$

A bounded sequence is a sequence that is bounded below and above.



Theorem (Monotonic Sequence Theorem)

Every bounded, monotonic sequence is convergent.