Calculus I
Lecture 14
Logarithmic Differentiation

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Outline

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Derivatives of Logarithmic Functions

Theorem (The Derivative of $\log_a x$)

$$\frac{\mathrm{d}}{\mathrm{d}x}(\log_a x) = \frac{1}{x \ln a}.$$

Proof.

$$\begin{aligned} \text{Let } y &= \log_a x. \\ \text{Then } a^y &= x. \\ \text{Differentiate implicitly:} \quad a^y (\ln a) y' &= 1 \\ y' &= \frac{1}{a^y \ln a} \\ &= \frac{1}{x \ln a}. \end{aligned}$$

$\square$

Example (Chain Rule)

Differentiate $f(x) = \log_3(5^x + 1)$.

Let $h(x) =$

Let $g(x) = \log_3 x$.

Theorem (The Derivative of $\log_a x$)

$$\frac{\mathrm{d}}{\mathrm{d}x}(\log_a x) = \frac{1}{x \ln a}.$$

$\ln x = \log_e x$. Therefore when we set $a = e$ we get the derivative of $\ln x$:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\ln x) &= \frac{1}{x \ln e}\\ &= \frac{1}{x(1)}\\ &= \frac{1}{x}.\end{aligned}$$

Theorem (The Derivative of $\ln x$)

$$\frac{\mathrm{d}}{\mathrm{d}x}(\ln x) = \frac{1}{x}.$$

Example (Chain Rule, Natural Logarithm)

$$\text{Differentiate} \quad y = \ln(e^x \sec x).$$

$$y = \ln e^x + \ln(\sec x)$$

$$= x + \ln(\sec x).$$

$$\frac{dy}{dx} = \quad + \frac{d}{dx}(\ln(\sec x))$$

$$\text{Let } u =$$

$$\text{Then } \ln(\sec x) =$$

$$\text{Chain Rule:} \quad \frac{dy}{dx} = 1 + \frac{d}{du}(\quad)\frac{du}{dx}$$

$$= 1 + \left(\quad \right)(\qquad)$$

$$=$$

$$=$$

Example

Find $f'(x)$ if $f(x) = \ln|x|$.

$$f(x) = \begin{cases} \ln x & \text{if } x > 0 \\ \ln(-x) & \text{if } x < 0 \end{cases} .$$

$$\begin{aligned} f'(x) &= \begin{cases} \frac{1}{x} & \text{if } x > 0 \\ \frac{1}{-x}(-1) & \text{if } x < 0 \end{cases} \\ &= \begin{cases} \frac{1}{x} & \text{if } x > 0 \\ \frac{1}{x} & \text{if } x < 0 \end{cases} \\ &= \frac{1}{x} \text{ if } x \neq 0. \end{aligned}$$

Example

Differentiate $x^{\tan x}$, where $x > 0$.

$$\begin{aligned}
\frac{d}{dx}\left(x^{\tan x}\right) &= \frac{d}{dx}\left(\left(e^{\ln x}\right)^{\tan x}\right) && \text{Convert base to } e^? \\
&= \frac{d}{dx}\left(e^{(\ln x)\tan x}\right) \\
&= \frac{d}{dx}\left(e^{u}\right) && \text{Set } (\ln x)\tan x = u \\
&= \frac{d}{du}\left(e^{u}\right)\frac{du}{dx} && \text{Chain rule} \\
&= e^{u}\frac{d}{dx}\left((\ln x)\tan x\right) \\
&= e^{(\ln x)\tan x}\left((\ln x)'\tan x + (\ln x)(\tan x)'\right) && \text{Prod. rule} \\
&= x^{\tan x}\left(\frac{1}{x}\tan x + (\ln x)\sec^2 x\right)
\end{aligned}$$

Example

Differentiate $(3x+1)^{\ln x}$, where $3x+1>0$.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left((3x+1)^{\ln x}\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\left(e^{\ln(3x+1)}\right)^{\ln x}\right) && \Big|\ \text{Convert base to } e^?\\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{\ln(3x+1)\ln x}\right)\\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{u}\right) = \frac{\mathrm{d}}{\mathrm{d}u}\left(e^{u}\right)\frac{\mathrm{d}u}{\mathrm{d}x} && \Big|\ \text{Set } \ln(3x+1)\ln x = u\\
&= e^{u}\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln(3x+1)\ln x\right)\\
&= e^{\ln(3x+1)\ln x}\left((\ln(3x+1))'\ln x + \ln(3x+1)(\ln x)'\right)\\
&= (3x+1)^{\ln x}\left(\frac{(3x+1)'}{3x+1}\ln x + \ln(3x+1)\frac{1}{x}\right)\\
&= (3x+1)^{\ln x}\left(\frac{3\ln x}{3x+1} + \ln(3x+1)\frac{1}{x}\right)
\end{aligned}$$

Example

Differentiate $(3x+1)^{\ln x}$, where $3x+1>0$.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left((3x+1)^{\ln x}\right) = (3x+1)^{\ln x}\left(\frac{3\ln x}{3x+1} + \ln(3x+1)\frac{1}{x}\right)$$

Theorem

$$\frac{\mathrm{d}}{\mathrm{d}x}\left((a(x))^{b(x)}\right) = (a(x))^{b(x)}\left(\frac{a'(x)}{a(x)}b(x) + \ln(a(x))b'(x)\right), \quad a(x)>0$$

Example (Logarithmic Differentiation)

Differentiate $y = \frac{(x-1)^{5/3}\sin^3 x}{\sqrt{e^x+1}}$.

Take the natural logarithm of both sides:

$$\ln y = \ln \frac{(x-1)^{5/3}\sin^3 x}{\sqrt{e^x+1}}$$

$$\ln y = (5/3)\ln(x-1) + 3\ln(\sin x) - (1/2)\ln(e^x+1).$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\ln y) = \frac{\mathrm{d}}{\mathrm{d}x}\left((5/3)\ln(x-1) + 3\ln(\sin x) - (1/2)\ln(e^x+1)\right)$$

$$\frac{1}{y}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \left(\frac{5}{3}\left(\frac{1}{x-1}\right)\right) + \left(\frac{3\cos x}{\sin x}\right) - \left(\frac{1}{2}\left(\frac{e^x}{e^x+1}\right)\right)$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \left(\frac{5}{3(x-1)} + 3\cot x - \frac{e^x}{2(e^x+1)}\right) y$$

$$= \left(\frac{5}{3(x-1)} + 3\cot x - \frac{e^x}{2(e^x+1)}\right) \frac{(x-1)^{5/3}\sin^3 x}{\sqrt{e^x+1}}.$$

Steps in Logarithmic Differentiation

1. Take natural logarithms of both sides of an equation $y = f(x)$.
2. Use the properties of logarithms to simplify.
3. Differentiate implicitly with respect to x.
4. Solve the resulting equation for y'.

Note: If $f(x) < 0$, then we use $\ln|f(x)|$ instead as $\ln(f(x))$ is not defined. We computed the derivative of $\ln|f(x)|$ in the previous lecture.

Example (Variable base and exponent)

Differentiate $y = (3x+1)^{\ln x}$.

Take logarithms of both sides:

$$\begin{aligned}
\ln y &= \ln(3x+1)^{\ln x}\\
\ln y &= \ln x \ln(3x+1).
\end{aligned}$$

Differentiate implicitly with respect to x:

$$\begin{aligned}
\frac{1}{y}y' &= (\ln x)\frac{\mathrm{d}}{\mathrm{d}x}(\ln(3x+1)) + (\ln(3x+1))\frac{\mathrm{d}}{\mathrm{d}x}(\ln x)\\
\frac{1}{y}y' &= (\ln x)\left(\frac{1}{3x+1}\cdot 3\right) + (\ln(3x+1))\left(\frac{1}{x}\right)\\
y' &= y\left(\frac{3\ln x}{3x+1} + \frac{\ln(3x+1)}{x}\right)\\
&= (3x+1)^{\ln x}\left(\frac{3\ln x}{3x+1} + \frac{\ln(3x+1)}{x}\right).
\end{aligned}$$

Example (Variable base and exponent)

Differentiate $y = x^{\tan x}$.

Take logarithms of both sides:

$$\begin{aligned}\ln y &= \ln x^{\tan x}\\ \ln y &= \tan x \ln x.\end{aligned}$$

Differentiate implicitly with respect to x:

$$\begin{aligned}\frac{1}{y}y' &= (\tan x)\frac{\mathrm{d}}{\mathrm{d}x}(\ln x) + (\ln x)\frac{\mathrm{d}}{\mathrm{d}x}(\tan x)\\ \frac{1}{y}y' &= (\tan x)\left(\frac{1}{x}\right) + (\ln x)\left(\sec^2 x\right)\\ y' &= y\left(\frac{\tan x}{x} + (\ln x)\sec^2 x\right)\\ &= x^{\tan x}\left(\frac{\tan x}{x} + (\ln x)\sec^2 x\right).\end{aligned}$$

Theorem (The Number e as a Limit)

$$e = \lim_{x\to 0}(1+x)^{\frac{1}{x}} = \lim_{y\to\infty}\left(1+\frac{1}{y}\right)^{y}.$$

Proof.

Let $f(x) = \ln x$. Then $f'(x) = \frac{1}{x}$, so $f'(1) = 1$.

$$\begin{aligned} 1 = f'(1) &= \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = \lim_{x\to 0}\frac{f(1+x)-f(1)}{x} \\ &= \lim_{x\to 0}\frac{\ln(1+x)-\ln(1)}{x} = \lim_{x\to 0}\frac{1}{x}\ln(1+x) \\ &= \lim_{x\to 0}\ln(1+x)^{\frac{1}{x}}. \end{aligned}$$

Then use the fact that the exponential function is continuous:

$$e = e^1 = e^{\lim\limits_{x\to 0}\ln(1+x)^{\frac{1}{x}}} = \lim_{x\to 0} e^{\ln(1+x)^{\frac{1}{x}}} = \lim_{x\to 0}(1+x)^{\frac{1}{x}}. \qquad \square$$