Calculus I
Homework Implicit Differentiation
Lecture 13

1. Express $\frac{\mathrm{d}y}{\mathrm{d}x}$ as a function of x and y by implicit differentiation. The answer key has not been proofread, use with caution.

(a) $x^3 + y^3 = 1$.

answer: $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x^2}{y^2}$

(b) $2\sqrt{x} + \sqrt{y} = 3$.

answer: $\frac{\mathrm{d}y}{\mathrm{d}x} = -2\sqrt{\frac{y}{x}}$

(c) $x^2 + xy - y^2 = 4$.

answer: $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-2x-y}{x-2y}$

(d) $2x^3 + x^2y - xy^3 = 2$.

answer: $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^3-6x^2-2xy}{-3xy^2+x^2}$

(e) $x^4(x + y) = y^2(3x - y)$.

answer: $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-5x^4-4x^3y+3y^2}{x^4-6xy-3y^2}$

(f) $y^5 + x^2y^3 = 1 + x^4y$.

answer: $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4x^3y-2xy^3}{5y^4+3x^2y^2-x^4}$

(g) $y \cos x = x^2 + y^2$.

answer: $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y\sin x+2x}{\cos x-2y}$

(h) $\cos(xy) = 1 + \sin y$.

answer: $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{y\sin(xy)}{\cos y+x\sin(xy)}$

(i) $4 \cos x \sin y = 1$.

answer: $\frac{\mathrm{d}y}{\mathrm{d}x} = \tan x \tan y$

(j) $y \sin\left(x^2\right) = x \sin\left(y^2\right)$.

answer: $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-2xy\cos\left(x^2\right)+\sin\left(y^2\right)}{-2xy\cos\left(y^2\right)+\sin\left(x^2\right)}$

(k) $\tan\left(\frac{x}{y}\right) = x + y$.

answer: $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-y^2+y\sec^2\left(\frac{y}{x}\right)}{y^2+x\sec^2\left(\frac{x}{y}\right)}$

(l) $\sqrt{x + y} = 1 + x^2y^2$.

answer: $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-\frac{1}{2}(y+x)^{-\frac{1}{2}}+8xy^2}{\frac{1}{2}(y+x)^{-\frac{1}{2}}-8x^2y}$

(m) $\sqrt{xy} = 1 + x^2y$.

answer: $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2xy-\frac{y}{2\sqrt{xy}}}{\frac{x}{2\sqrt{xy}}-x^2} = \frac{4xyx^2y+4xy-y}{-2x^4y-2x^2+x}$

(n) $x \sin y + y \sin x = 1$.

answer: $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-y\cos x-\sin y}{x\cos y+\sin x}$

(o) $y \cos x = 1 + \sin(xy)$.

answer: $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\cos(xy)y+y\sin x}{-\cos(xy)x+\cos x}$

(p) $\tan(x - y) = \frac{y}{1 + x^2}$.

answer: $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-(\sec(-y+x))^2x^4-2(\sec(-y+x))^2x^2-(\sec(-y+x))^2-2yx}{-(\sec(-y+x))^2x^4-2(\sec(-y+x))^2x^2-(\sec(-y+x))^2-x^2-1}$

(q) $x^4(x + y) = y^2(3x - y)$.

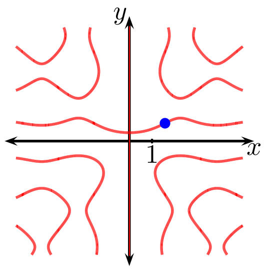
answer: $\frac{dy}{dx} = \frac{-4x^3y+3y^2-5x^4}{3y^2+x^4-6yx}$

(r) $2x^3 + x^2y - xy^3 = 2$.

answer: $\frac{dy}{dx} = \frac{y^3-6x^2-2xy}{-3xy^2+x^2}$

2. Verify that the coordinates of the given point satisfy the given equation. Use implicit differentiation to find an equation of the tangent line to the curve at the given point.

(a) $y\sin(2x) = x\cos(2y)$, $\left(\frac{\pi}{2}, \frac{\pi}{4}\right)$.

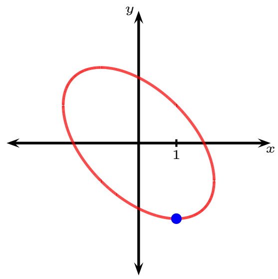


answer: $y = \frac{1}{2}x$

(b) $\sin(x+y) = 2x - 2y$, (π, π) .

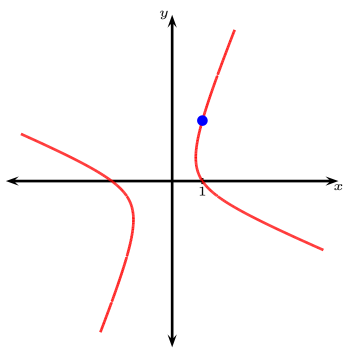
answer: $\frac{1}{3}x + \frac{2}{3}\pi$

(c) $x^2 + xy + y^2 = 3$, $(1, -2)$ (ellipse).



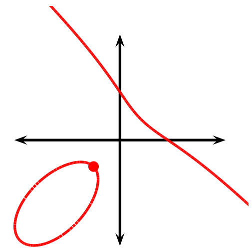
answer: $y = -2$

(d) $x^2 + 2xy - y^2 + x = 2$, $(1, 2)$ (hyperbola).



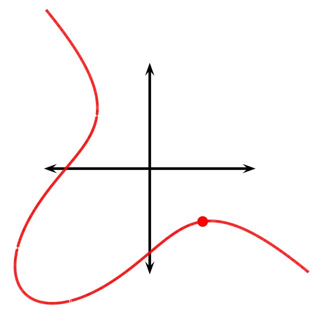
answer: $y = \frac{7}{2}x - \frac{3}{2}$

(e) $y^3 + x^3 + 4xy = \dfrac{3}{4}$, $\left(-\frac{1}{2}, -\frac{1}{2}\right)$



answer: $y = -x - 1$

(f) $y^3 + x^3 + 4xy = -4$, $(1, -1)$



answer:

(g) $x^2 + y^2 = (2x^2 + 2y^2 - x)^2$, $(0, \frac{1}{2})$.

answer: $y = x + \frac{1}{2}$

(h) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 4$, $(-3\sqrt{3}, 1)$.

answer: $y = \frac{1}{\sqrt{3}}x + 4$

(i) $2(x^2 + y^2)^2 = 25(x^2 - y^2)$, $(3, 1)$.

answer: $y = -\frac{9}{13}x + \frac{40}{13}$

(j) $y^2(y^2 - 4) = x^2(x^2 - 5)$, $(0, -2)$.

answer: $y = -2$

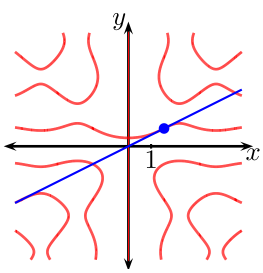
(k) $x^{\frac{4}{3}} + y^{\frac{4}{3}} = 10$ at $(-3\sqrt{3}, 1)$.

answer: $y = \sqrt{3}x + 10$

(l) $x^2y^3 + x^3 - y^2 = 1$ at $(1, 1)$.

answer: $y = -5x + 6$

Solution. 2.a



First we verify that the point $(x, y) = \left(\dfrac{\pi}{2}, \dfrac{\pi}{4}\right)$ indeed satisfies the given equation:

$$\begin{array}{rcll} y\sin(2x)_{|x=\frac{\pi}{2}, y=\frac{\pi}{4}} &=& \dfrac{\pi}{4}\sin\pi \quad = \quad 0 & \Big| \text{ left hand side} \\ x\cos(2y)_{|x=\frac{\pi}{2}, y=\frac{\pi}{4}} &=& \dfrac{\pi}{2}\cos\left(\dfrac{\pi}{2}\right) \quad = \quad 0 & \Big| \text{ right hand side} \end{array}$$

so the two sides of the equation are equal (both to 0) when $x=\frac{\pi}{2}$ and $y=\frac{\pi}{4}$.

Since we are looking an equation of the tangent line, we need to find $\frac{\mathrm{d}y}{\mathrm{d}x}_{|x=\frac{\pi}{2},y=\frac{\pi}{4}}$ - that is, the derivative of y at the point $x=\frac{\pi}{2}$, $y=\frac{\pi}{4}$. To do so we use implicit differentiation.

$$\begin{array}{rcll}
y\sin(2x) &=& x\cos(2y) & \Big|\ \frac{\mathrm{d}}{\mathrm{d}x}\\
\frac{\mathrm{d}y}{\mathrm{d}x}\sin(2x)+y\frac{\mathrm{d}}{\mathrm{d}x}(\sin(2x)) &=& \cos(2y)+x\frac{\mathrm{d}}{\mathrm{d}x}(\cos(2y)) & \\
\frac{\mathrm{d}y}{\mathrm{d}x}\sin(2x)+2y\cos(2x) &=& \cos(2y)-2x\sin(2y)\frac{\mathrm{d}y}{\mathrm{d}x} & \\
\frac{\mathrm{d}y}{\mathrm{d}x}(\sin(2x)+2x\sin(2y)) &=& \cos(2y)-2y\cos(2x) & \Big|\ \text{Set } x=\frac{\pi}{2}, y=\frac{\pi}{4}\\
\frac{\mathrm{d}y}{\mathrm{d}x}_{|x=\frac{\pi}{2},y=\frac{\pi}{4}}\left(\sin\pi+\pi\sin\left(\frac{\pi}{2}\right)\right) &=& \cos\left(\frac{\pi}{2}\right)-\frac{\pi}{2}\cos\pi & \\
\pi\frac{\mathrm{d}y}{\mathrm{d}x}_{|x=\frac{\pi}{2},y=\frac{\pi}{4}} &=& -\frac{\pi}{2}\cos\pi & \\
\frac{\mathrm{d}y}{\mathrm{d}x}_{|x=\frac{\pi}{2},y=\frac{\pi}{4}} &=& \frac{1}{2}. &
\end{array}$$

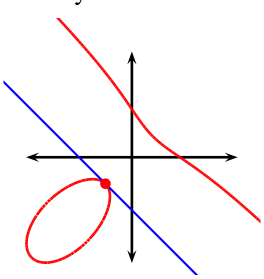
Therefore the equation of the line through $x=\frac{\pi}{2}, y=\frac{\pi}{4}$ is

$$\begin{array}{rcl}
y-\frac{\pi}{4} &=& \frac{1}{2}\left(x-\frac{\pi}{2}\right)\\
y &=& \frac{1}{2}x.
\end{array}$$

Solution. 2.e

$$\begin{array}{rcll}
y^3+x^3+4xy &=& \frac{3}{4} & \Big|\ \text{apply } \frac{\mathrm{d}}{\mathrm{d}x}\\
3y^2\frac{\mathrm{d}y}{\mathrm{d}x}+3x^2+4\left(y+x\frac{\mathrm{d}y}{\mathrm{d}x}\right) &=& 0 & \\
\frac{\mathrm{d}y}{\mathrm{d}x}\left(3y^2+4x\right) &=& -3x^2-4y & \\
\frac{\mathrm{d}y}{\mathrm{d}x} &=& \frac{-3x^2-4y}{3y^2+4x} & \Big|\ \text{substitute } x=-\frac{1}{2}, y=-\frac{1}{2}\\
\frac{\mathrm{d}y}{\mathrm{d}x}_{|x=-\frac{1}{2},y=-\frac{1}{2}} &=& \frac{-3\left(-\frac{1}{2}\right)^2-4\left(-\frac{1}{2}\right)}{3\left(-\frac{1}{2}\right)^2+4\left(-\frac{1}{2}\right)} & \\
\frac{\mathrm{d}y}{\mathrm{d}x}_{|x=-\frac{1}{2},y=-\frac{1}{2}} &=& -1 &
\end{array}$$

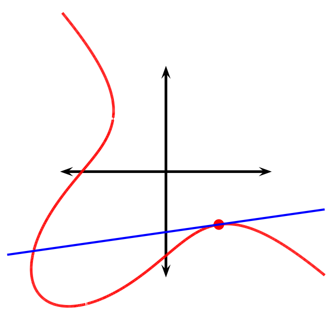
Therefore the equation of the tangent is $y-\left(-\frac{1}{2}\right)=-(x-\left(-\frac{1}{2}\right))$ which simplifies to $y=-x-1$. A computer-generated plot visually confirms our computations.



Solution. 2.f

$$\begin{aligned}
y^3 + x^3 + 4xy &= -4 && \Big|\ \text{apply } \tfrac{\mathrm{d}}{\mathrm{d}x}\\
3y^2 \frac{\mathrm{d}y}{\mathrm{d}x} + 3x^2 + 4\left(y + x\frac{\mathrm{d}y}{\mathrm{d}x}\right) &= 0\\
\frac{\mathrm{d}y}{\mathrm{d}x}\left(3y^2 + 4x\right) &= -3x^2 - 4y\\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{-3x^2 - 4y}{3y^2 + 4x} && \Big|\ \text{substitute } x = 1, y = -1\\
\frac{\mathrm{d}y}{\mathrm{d}x}_{|x=1,y=-1} &= \frac{-3(1)^2 - 4(-1)}{3(-1)^2 + 4(1)}\\
\frac{\mathrm{d}y}{\mathrm{d}x}_{|x=-\frac{1}{2},y=-\frac{1}{2}} &= \frac{1}{7}
\end{aligned}$$

Therefore the equation of the tangent is $y - (-1) = \frac{1}{7}(x-1)$ which simplifies to $y = \frac{x}{7} - \frac{8}{7}$. A computer-generated plot visually confirms our computations.



3. (a) If V is the volume of a cube with edge length x and the cube expands as time passes, find $\frac{dV}{dt}$ in terms of $\frac{dx}{dt}$.

answer: $3x^2\frac{dx}{dt}$

(b) Each side of a square is increasing at a rate of 1cm/s. At what rate is the area of the square increasing when the area of the square is 9 cm^2?

answer: $6\mathrm{cm}^2/\mathrm{s}$

(c) The radius of a ball is increasing at a constant rate of 5mm/s. At a point in time we measure the radius of the ball to be 5cm.
- How fast is the volume increasing at the time of our observation?
- How fast is the surface area increasing at the time of our observation?

(d) At a point in time, we measure the surface area of a ball to be increasing at a rate of $5\mathrm{cm}^2/\mathrm{s}$ and the radius of the ball to be 5cm.
- How fast is the volume increasing at the time of our observation?
- How fast is the radius increasing at the time of our observation?

(e) At a point in time, we measure the volume of a ball to be increasing at a rate of $5\mathrm{cm}^3/\mathrm{s}$ and the radius of the ball to be 5cm.
- How fast is the radius increasing at the time of our observation?
- How fast is the surface area increasing at the time of our observation?

(f) A 5m long ladder is leaning on a vertical wall. The base of the ladder is 4m away from the wall. At the moment of observation, the top end of the ladder is sliding downwards along the wall at a speed of 10 cm/s.
- At the time of observation, how fast is the bottom end of the ladder sliding away from the wall
- At the time of observation, how fast is the midpoint of the ladder moving away from the wall?
- At the time of observation, how fast is the midpoint of the ladder moving downwards towards the ground?

(g) A street light is mounted at the top of a 4m tall pole. A woman 160 cm tall walks away from the pole at a speed of 5km/h along a straight path. How fast is the tip of her shadow moving when she is 10m from the pole?

(h) A ship is pulled into a dock. On the dock, the rope is pulled by a pulley that is 1m lower than the mooring point on the ship. If the rope is pulled in at a constant rate of 10cm/s, how fast is the mooring point approaching the dock when it is 10m from the dock?

(i) A Ferris wheel with a radius of 10m is rotating at a rate of one revolution every 2 minutes. How fast is a riding rising when his seat is 16 m above ground level?

(j) The minute hand on a watch is 10mm long and the hour hand is 5mm long. How fast is the distance between the tips of the hands changing at two o'clock?

answer: (not proofread) $-\frac{55}{42}\sqrt{21}\pi$ mm/h

Solution. 3.a. The volume V is given by $V = x^3$, therefore

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(x^3\right) = 3x^2\frac{\mathrm{d}x}{\mathrm{d}t}.$$

Solution. 3.b The area A of the square is given by $A = x^2$, therefore

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(x^2\right) = 2x\frac{\mathrm{d}x}{\mathrm{d}t}.$$

When $A = 9\mathrm{cm}^2$, $x = 3\mathrm{cm}$ $(= \sqrt{9\mathrm{cm}^2})$, and so $\frac{\mathrm{d}A}{\mathrm{d}t}_{|x=3, \frac{\mathrm{d}x}{\mathrm{d}t}=1} = 2 \cdot 3\mathrm{cm} \cdot 1\mathrm{cm/s} = 6\mathrm{cm}^2/\mathrm{s}$.

Solution. 3.d Let S denote the surface area and r the radius of the ball. Then $S = 4\pi r^2$. Let the point of time be t_0. We are given that $\frac{\mathrm{d}S}{\mathrm{d}t}_{|t=t_0} = 5\mathrm{cm}^2/\mathrm{s}$. On the other hand,

$$\begin{array}{rcl} \frac{\mathrm{d}S}{\mathrm{d}t} & = & \frac{\mathrm{d}}{\mathrm{d}t}\left(4\pi r^2\right) = 8\pi r\frac{\mathrm{d}r}{\mathrm{d}t} \\ \frac{1}{8\pi}\frac{\mathrm{d}S}{\mathrm{d}t} & & \end{array}$$

Solution. 3.j Let the angle between the two arrows be θ. The cosine law states that for a triangle with angle θ and sides a, b, c we have that $c^2 = a^2 + b^2 - 2ab\cos\theta$ (where c is the length of the side opposite to the angle θ).

Then by the cosine law, the distance between the tips of the two hands is 1

$$y = \sqrt{5^2 + 10^2 + 2 \cdot 5 \cdot 10\cos\theta} = \sqrt{125 + 100\cos\theta}.$$

The short hand makes 1 full revolution every 12 hours, and the long hand makes 1 full revolution every 1 hour. Therefore the angle θ measured from the small hand to the long hand changes at the (constant) rate of $\frac{11}{12}$ revolutions per hour, or what is the same, at the rate $\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{11}{12}(2\pi) = \frac{11}{6}\pi$.

The problem asks us to compute $\frac{dy}{dt}$ at two o'clock, i.e., at $t = 2$. This is a straightforward computation using the chain rule:

$$\begin{array}{rcll} \frac{\mathrm{d}y}{\mathrm{d}t} & = & \frac{1}{2}\frac{-100\sin\theta}{\sqrt{125+100\cos\theta}}\frac{\mathrm{d}\theta}{\mathrm{d}t} & \Big| \text{at 2 o'clock } \theta = \frac{\pi}{3} \text{ and } \frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{11}{6}\pi \\ \frac{\mathrm{d}y}{\mathrm{d}t}_{|t=2} & = & \frac{1}{2}\frac{-100\sin\frac{\pi}{3}}{\sqrt{125+100\cos\frac{\pi}{3}}}\frac{11\pi}{6} = -\frac{55}{42}\sqrt{21}\pi & . \end{array}$$

The measurement unit of speed is mm/hour, so the distance is changing at the rate of $-\frac{55}{42}\sqrt{21}$ mm/hour.