Precalculus
Lecture 15

Todor Milev
https://github.com/tmilev/freecalc

2020

Outline

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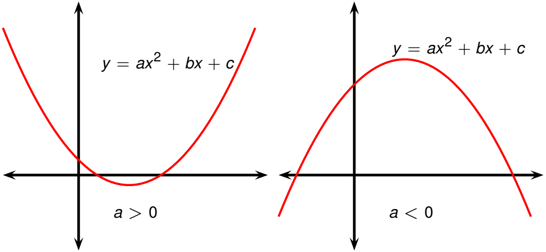
Definition

Let a, b, c be real numbers with $a \neq 0$. The function

$$f(x) = ax^2 + bx + c$$

is called a *quadratic function*.

- The graph of a quadratic function is called a parabola.



Example (Completing the square)

Complete the square.

$$\begin{aligned}
3x^2 - 5x + 1 &= 3\left(x^2 - \frac{5}{3}x\right) + 1 \\
&= 3\left(x^2 - 2\cdot\frac{5}{2\cdot 3}x\right) + 1 \\
&= 3\left(x^2 - 2\cdot\frac{5}{6}x + \left(\frac{5}{6}\right)^2 - \left(\frac{5}{6}\right)^2\right) + 1 \\
&= 3\left(\left(x - \frac{5}{6}\right)^2 - \frac{25}{36}\right) + 1 \\
&= 3\left(x - \frac{5}{6}\right)^2 - \frac{25}{12} + 1 \\
&= 3\left(x - \frac{5}{6}\right)^2 - \frac{13}{12}.
\end{aligned}$$

Definition (Completing the square)

Let $a \neq 0$. To *complete the square* means to carry out the following algebraic manipulation.

$$\begin{aligned}
ax^2 + bx + c &= a\left(x^2 + \frac{b}{a}x\right) + c \\
&= a\left(x^2 + 2\cdot\frac{b}{2a}x\right) + c \\
&= a\left(x^2 + 2\frac{b}{2a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right) + c \\
&= a\left(\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2}\right) + c \\
&= a\left(x + \frac{b}{2a}\right)^2 - \cancel{a}\cdot\frac{b^2}{4a^{\cancel{2}}} + c \\
&= a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}.
\end{aligned}$$

Add & subtract $\left(\frac{b}{2a}\right)^2$ use $(A+B)^2 = A^2 + 2AB + B^2$

Definition (Discriminant of quadratic function)

The quantity $D = b^2 - 4ac$ is called the *discriminant* of the quadratic function $ax^2 + bx + c$.

Let $a \neq 0$ and let $f(x) = ax^2 + bx + c$. Then we have the equality

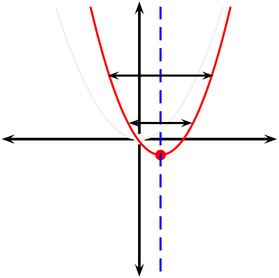
$$\begin{aligned} f(x) &= a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a} && \Big| \text{ complete the square} \\ &= a\left(x - \left(-\frac{b}{2a}\right)\right)^2 - \frac{b^2 - 4ac}{4a} \\ &= a\left(x - \left(-\frac{b}{2a}\right)\right)^2 - \frac{D}{4a}. \end{aligned}$$

Definition

The expression $f(x) = a(x - h)^2 + k$, where $h = -\frac{b}{2a}$ and $k = -\frac{D}{4a} = -\frac{b^2 - 4ac}{4a}$ is called the standard form of $ax^2 + bx + c$.

Definition

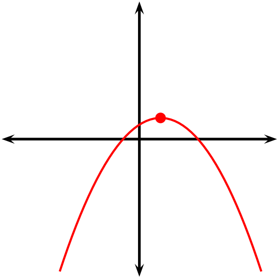
The expression $f(x) = a(x-h)^2 + k$, where $h = -\dfrac{b}{2a}$ and $k = -\dfrac{D}{4a} = -\dfrac{b^2-4ac}{4a}$ is called the standard form of ax^2+bx+c.



- The graph of $y = x^2$ is a parabola; its shape is assumed known.
- The standard form shows how the graph of an arbitrary quadratic is obtained from the graph of $y = x^2$:
  - ax^2 stretches $y = x^2$ by factor of a and possibly reflects across the x axis.
  - $a(x-h)^2$ shifts $y = ax^2$ by h units right.
  - $a(x-h)^2 + k$ shifts $y = a(x-h)^2$ by k units up.

Definition

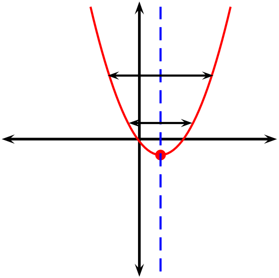
The expression $f(x) = a(x-h)^2 + k$, where $h = -\dfrac{b}{2a}$ and $k = -\dfrac{D}{4a} = -\dfrac{b^2-4ac}{4a}$ is called the standard form of ax^2+bx+c.



- The graph of a quadratic function is a parabola.
- When $a > 0$ the parabola opens upwards.
- When $a < 0$ the parabola opens downwards.
- When $|a|$ increases, the parabola becomes steeper.
- The point $(h,k) = \left(-\frac{b}{2a}, -\frac{D}{4a}\right)$ is called the vertex of the parabola.
- The parabola is symmetric with respect to the line $x = h = -\frac{b}{2a}$, i.e., the vertical line through its vertex.

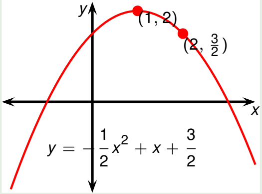
Definition

The expression $f(x) = a(x-h)^2 + k$, where $h = -\dfrac{b}{2a}$ and $k = -\dfrac{D}{4a} = -\dfrac{b^2-4ac}{4a}$ is called the standard form of ax^2+bx+c.



- When we change h and k we move the vertex of the parabola without change in steepness.
- Therefore when we change b and c we move the vertex of the parabola without change in steepness.

Example



Write an equation of a parabola with vertex at $(1, 2)$ that passes through the point $\left(2, \frac{3}{2}\right)$.

$$\begin{aligned}
a(x-h)^2 + k &= y && \text{Standard form}\\
a(x-1)^2 + 2 &= y && \text{Vertex at } (1,2)\\
a(2-1)^2 + 2 &= \frac{3}{2} && \text{Passes through } \left(2, \tfrac{2}{3}\right)\\
a &= \frac{3}{2} - 2 = -\frac{1}{2} &&\\
y &= -\frac{1}{2}(x-1)^2 + 2 && \text{Final answer}\\
y &= -\frac{1}{2}x^2 + x + \frac{3}{2} && \text{Alternative answer}
\end{aligned}$$

Problem (Quadratic equation formula)

Solve the general quadratic equation

$$
\begin{array}{rcll}
ax^2 + bx + c &=& 0 & \Big| \text{ complete the square} \\
a\left(x+\frac{b}{2a}\right)^2 - \frac{D}{4a} &=& 0 & \Big| \text{ where } D = b^2 - 4ac \\
a\left(\left(x+\frac{b}{2a}\right)^2 - \frac{D}{4a^2}\right) &=& 0 & \\
a\left(\left(x+\frac{b}{2a}\right)^2 - \left(\frac{\sqrt{D}}{2a}\right)^2\right) &=& 0 & \\
a\left(x+\frac{b}{2a}-\frac{\sqrt{D}}{2a}\right)\left(x+\frac{b}{2a}+\frac{\sqrt{D}}{2a}\right) &=& 0 & \Big| \text{ use } A^2 - B^2 = (A-B)(A+B)
\end{array}
$$

$$
x+\frac{b}{2a}-\frac{\sqrt{D}}{2a} = 0 \quad \text{or} \quad x+\frac{b}{2a}+\frac{\sqrt{D}}{2a} = 0
$$

$$
x = \frac{-b+\sqrt{D}}{2a} \quad \text{or} \quad x = \frac{-b-\sqrt{D}}{2a}.
$$

Theorem

The solutions of the quadratic equation

$$ax^2 + bx + c = 0$$

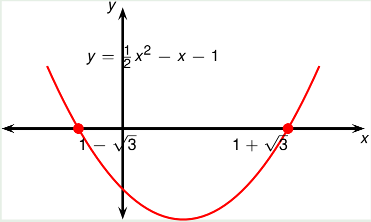
are given by:

$$x = x_1 = \frac{-b + \sqrt{D}}{2a} \quad or \quad x = x_2 = \frac{-b - \sqrt{D}}{2a} ,$$

where $D = b^2 - 4ac$*, or equivalently by:*

$$x_1, x_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

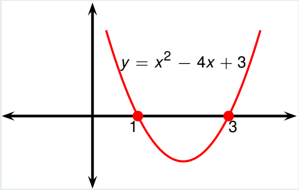
Example



Find the x-intercepts of $\frac{x^2}{2} - x - 1$.

$$
\begin{aligned}
x_1, x_2 &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \cdot \frac{1}{2} \cdot (-1)}}{\cancel{2} \cdot \frac{1}{\cancel{2}}} \\
&= 1 \pm \sqrt{3}
\end{aligned}
$$

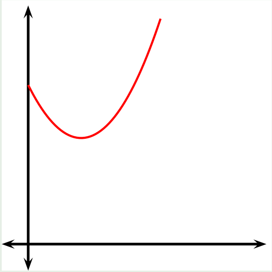
Example



Find the x-intercepts of $x^2 - 4x + 3$.

$$
\begin{aligned}
x_1, x_2 &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot 1 \cdot 3}}{2 \cdot 1} \\
&= \frac{4 \pm \sqrt{4}}{2} \\
&= \frac{4 \pm 2}{2} \\
&= \begin{cases} \frac{4+2}{2} = \frac{6}{2} = 3 \\ \frac{4-2}{2} = \frac{2}{2} = 1 \end{cases}
\end{aligned}
$$

Example



Find the x-intercepts of $x^2 - 2x + 3$.

$$
\begin{aligned}
x_1, x_2 &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \cdot 1 \cdot (3)}}{2 \cdot 1} \\
&= \frac{2 \pm \sqrt{-8}}{2} \\
&\text{no real solutions} \\
&\text{no } x - \text{intercepts}
\end{aligned}
$$

Proposition

Let $ax^2 + bx + c$, $a \neq 0$ be a quadratic with discriminant $D = b^2 - 4ac$ and roots x_1 and x_2. Then $D = a^2 (x_1 - x_2)^2$.

Proof.

$$
\begin{aligned}
a^2(x_1 - x_2)^2 &= a^2 \left(\frac{\cancel{-b} + \sqrt{D}}{2a} - \frac{\cancel{-b} - \sqrt{D}}{2a} \right)^2 \\
&= a^2 \left(\frac{\cancel{2}\sqrt{D}}{\cancel{2}a} \right)^2 \\
&= \cancel{a^2} \frac{D}{\cancel{a^2}} \\
&= D, \quad \text{as desired.}
\end{aligned}
$$

□

- Discriminant is zero $\Leftrightarrow$ the quadratic has non-distinct roots, hence the discriminant discriminates between the two roots.

Proposition (Vieta's formulas)

Let ax^2+bx+c be a quadratic functions with zeros x_1 and x_2. Then:

$$\begin{aligned} a(x-x_1)(x-x_2) &= ax^2+bx+c \\ ax^2-a(x_2+x_1)x+ax_1x_2 &= ax^2+bx+c \\ x_1x_2 &= \frac{c}{a} \\ x_1+x_2 &= -\frac{b}{a} \end{aligned}$$

The last two formulas are called Vieta's formulas (after François Viète (1540-1603), Latinized name: Franciscus Vieta).

Theorem

The quadratic $ax^2 + bx + c$ factors as follows.

$$ax^2 + bx + c = a(x - x_1)(x - x_2),$$

where x_1 and x_2 are the roots of the quadratic, given by:

$$x_1, x_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$ax^2 + bx + c = a(x - x_1)(x - x_2), \text{ where } x_1, x_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Example

Factor the polynomial. If possible, guess the factorization.

$$\begin{aligned} 3x^2 + 8x - 11 &= (3x + 11)(x + \ -1) \\ &= 3\left(x - \left(-\tfrac{11}{3}\right)\right)(x - 1) \end{aligned}$$

- If there is a factorization using integers, it should be of the form

$$\begin{aligned} 3x^2 + 8x - 11 &= (3x + p)(x + q) \\ &= 3x^2 + 3xq + px + pq \\ &= 3x^2 + x(3q + p) + pq \end{aligned}$$

(Vieta's formulas) This means that :

$$\begin{aligned} 8 &= 3q + p \\ -11 &= pq \end{aligned}$$

p, q must be divisors of 11: $\pm 1, \pm 11$

$$\begin{aligned} p &= 11 \\ q &= -1 \end{aligned}$$

$$ax^2+bx+c = a\left(x-x_1\right)\left(x-x_2\right), \text{ where } x_1,x_2 = \frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

Example

Factor the polynomial. If possible, guess the factorization.

$$\begin{aligned} 3x^2+8x-11 &= (3x+11)(x+\ -1) \\ &= 3\left(x-\left(-\tfrac{11}{3}\right)\right)\left(x-1\right) \end{aligned}$$

- What if we can't guess the factorization?
- Use the formulas for x_1, x_2.

$$\begin{aligned} x_1,x_2 &= \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{-8\pm\sqrt{8^2-4\cdot 3\cdot(-11)}}{2\cdot 3} \\ &= \frac{-8\pm\sqrt{64+132}}{6} = \frac{-8\pm\sqrt{196}}{6} \\ &= \frac{-8\pm 14}{6} = \begin{cases} \dfrac{-8+14}{6} = \dfrac{6}{6} = 1 \\ \dfrac{-8-14}{6} = -\dfrac{22}{6} = -\dfrac{11}{3} \end{cases} \end{aligned}$$

Proposition (Vieta's formulas)

Let $ax^2 + bx + c$ be a quadratic functions with zeros x_1 and x_2. Then:

$$\begin{aligned} a(x - x_1)(x - x_2) &= ax^2 + bx + c \\ ax^2 - a(x_2 + x_1)x + ax_1x_2 &= ax^2 + bx + c \\ x_1x_2 &= \frac{c}{a} \\ x_1 + x_2 &= -\frac{b}{a} \end{aligned}$$

The last two formulas are called Vieta's formulas (after François Viète (1540-1603), Latinized name: Franciscus Vieta).

$$ax^2 + bx + c = a(x - x_1)(x - x_2)$$

$$\begin{aligned} x_1 + x_2 &= -\frac{b}{a} \\ x_1 x_2 &= \frac{c}{a} \end{aligned} \quad \Bigg| \text{ Vieta's formulas}$$

Example

Factor the quadratic.

$$x^2 + 5x + 6 = (x + 2)(x + 3)$$

- The product of the two roots: $x_1 x_2 = 6$.
- The divisors of 6 are $\pm 1, \pm 2, \pm 3, \pm 6$.
- Therefore the pair x_1, x_2 is $\pm 1, \pm 6$ or $\pm 2, \pm 3$.
- The sum of the two roots: $x_1 + x_2 = -5$

$$ax^2 + bx + c = a\left(x - x_1\right)\left(x - x_2\right),$$

$$\begin{aligned} x_1 x_2 &= \frac{c}{a} \\ x_1 + x_2 &= -\frac{b}{a} \end{aligned}$$

Example

Factor the quadratic.

$$x^2 + 3x + 1 = \left(x \quad - \left(\tfrac{-3+\sqrt{5}}{2}\right)\right)\left(x \quad - \left(\tfrac{-3-\sqrt{5}}{2}\right)\right)$$

- The product of the two roots: $x_1 x_2 = 1$.
- Integer options: $x_1 = 1, x_2 = 1$ and $x_1 = -1, x_2 = -1$.
- $\begin{aligned} (x-1)(x-1) &= (x-1)^2 = x^2 - 2x + 1 \\ (x+1)(x+1) &= (x+1)^2 = x^2 + 2x + 1 \end{aligned}$ both don't work.
- $\Rightarrow$ No easy factorization; must use quadratic formula.

$$\begin{aligned} x_1, x_2 &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-3 \pm \sqrt{3^2 - 4 \cdot 1 \cdot 1}}{2 \cdot 1} \\ &= \frac{-3 \pm \sqrt{5}}{2} \end{aligned}$$

$$ax^2+bx+c = a(x-x_1)(x-x_2),$$

$$\begin{aligned} x_1x_2 &= \frac{c}{a} \\ x_1+x_2 &= -\frac{b}{a} \end{aligned}$$

Example

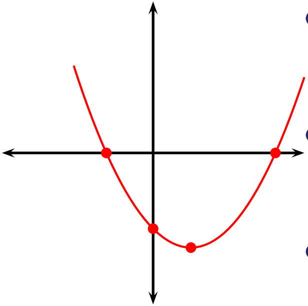
Factor the quadratic, using complex numbers if needed.

$$x^2+x+1 = \left(x \quad - \left(\frac{-1+\sqrt{3}i}{2}\right)\right)\left(x \quad - \left(\frac{-1-\sqrt{3}i}{2}\right)\right)$$

- The product of the two roots: $x_1x_2 = 1$.
- Integer options: $x_1 = 1, x_2 = 1$ and $x_1 = -1, x_2 = -1$.
- $\begin{aligned} (x-1)(x-1) &= (x-1)^2 = x^2-2x+1 \\ (x+1)(x+1) &= (x+1)^2 = x^2+2x+1 \end{aligned}$ both don't work.
- $\Rightarrow$ No easy factorization; must use quadratic formula.

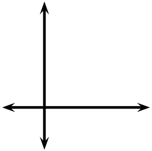
$$\begin{aligned} x_1, x_2 &= \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{-1\pm\sqrt{1^2-4\cdot 1\cdot 1}}{2\cdot 1} \\ &= \frac{-1\pm\sqrt{-3}}{2} = \frac{-1\pm\sqrt{3}i}{2} \end{aligned}$$

To plot a parabola by hand roughly, we need to do the following.



- Find the vertex of the parabola.
- Find the y intercept.
- Find the x intercept(s) if any.
- Select (or re-select) axes scale so all important points found in the preceding items fit in the plot.
- Plot the parabola freehand, making sure that the parabola passes through all special points you found in the preceding items.
- If $a > 0$ your parabola should open upwards, if $a < 0$ your parabola should open downwards.
- For $|a| > 1$ we should aim to draw the graph steeper than $a = x^2$, for $|a| < 1$ we should aim to draw the graph flatter than $a = x^2$.

Example



Vertex at: $\left(\frac{21}{4}, \frac{171}{8}\right)$
y-intercept at $y = 3$
x-intercepts at
$x = \frac{21 - 3\sqrt{57}}{4}$,
$x = \frac{21 + 3\sqrt{57}}{4}$.

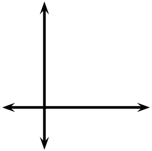
Plot roughly by hand the graph of
$f(x) = -\frac{2}{3}x^2 + 7x + 3$.

- The vertex of the parabola is given by:

$$
\begin{aligned}
x &= -\frac{b}{2a} = -\frac{7}{2\left(-\frac{2}{3}\right)} = \frac{21}{4}\\
y &= f\left(-\frac{b}{2a}\right) = -\frac{D}{4a} = -\frac{\left(b^2 - 4ac\right)}{4a}\\
&= -\frac{7^2 - 4\left(-\frac{2}{\cancel{3}}\right)\cancel{3}}{4\left(-\frac{2}{3}\right)} = \frac{49 + 8}{\frac{8}{3}}\\
&= \frac{3 \cdot 57}{8} = \frac{171}{8}.
\end{aligned}
$$

- The y-intercept is $f(0) = 3$.

Example



Vertex at: $\left(\frac{21}{4}, \frac{171}{8}\right)$
y-intercept at $y = 3$
x-intercepts at
$x = \frac{21-3\sqrt{57}}{4}$,
$x = \frac{21+3\sqrt{57}}{4}$.

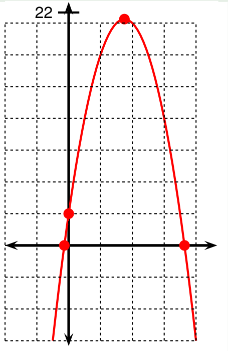
Plot roughly by hand the graph of
$f(x) = -\frac{2}{3}x^2 + 7x + 3$.

- The x intercepts are given by the solutions of

$$
\begin{aligned}
-\frac{2}{3}x^2 + 7x + 3 &= 0 \qquad \Big| \cdot 3\\
-2x^2 + 21x + 9 &= 0
\end{aligned}
$$

$$
\begin{aligned}
x &= \frac{-b\pm\sqrt{b^2-4ac}}{2a}\\
&= \frac{-21\pm\sqrt{21^2-4\cdot(-2)\cdot 9}}{2\cdot(-2)}\\
&= \frac{-21\pm\sqrt{441+72}}{-4}\\
&= \frac{21\mp\sqrt{513}}{4}\\
&= \frac{21\mp\sqrt{9\cdot 57}}{4}\\
&= \frac{21\mp\sqrt{9}\sqrt{57}}{4}\\
&= \frac{21\mp 3\sqrt{57}}{4}
\end{aligned}
$$

Example

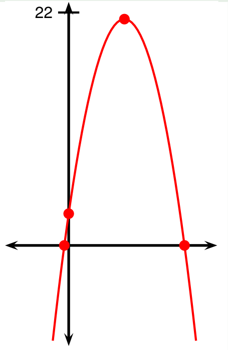


Vertex at: $\left(\frac{21}{4}, \frac{171}{8}\right)$
y-intercept at $y = 3$
x-intercepts at
$x = \frac{21-3\sqrt{57}}{4}$,
$x = \frac{21+3\sqrt{57}}{4}$.

Plot roughly by hand the graph of $f(x) = -\frac{2}{3}x^2 + 7x + 3$.

- Select scale to fit the picture:
  - $\frac{21}{4}$ is close to $\frac{20}{4} = 5$.
  - $\frac{171}{8}$ is between the integers 21 and 22.
  - $\frac{21+3\sqrt{57}}{4}$ is close to $\frac{21+3\sqrt{64}}{4} = \frac{21+24}{4} = \frac{45}{4}$ which is close to $\frac{44}{4} = 11$.
  - $\frac{21-3\sqrt{57}}{4}$ is close to $\frac{21-3\sqrt{64}}{4} = \frac{21-24}{4} = -\frac{3}{4}$ which is close to -1.
  - The parabola vertex is less than 22 units high and the parabola opens downwards.
  - Axes height of 22 units appears reasonable.
  - A grid of width 3 units appears reasonable.
  - Plot all relevant points.
  - Finally “connect the dots with a freehand drawing”.

Example



Vertex at: $\left(\frac{21}{4}, \frac{171}{8}\right)$
y-intercept at $y = 3$
x-intercepts at
$x = \frac{21-3\sqrt{57}}{4}$,
$x = \frac{21+3\sqrt{57}}{4}$.

Plot roughly by hand the graph of
$f(x) = -\frac{2}{3}x^2 + 7x + 3$.

- Select scale to fit the picture:
  - $\frac{21}{4}$ is close to $\frac{20}{4} = 5$.
  - $\frac{171}{8}$ is between the integers 21 and 22.
  - $\frac{21+3\sqrt{57}}{4}$ is close to $\frac{21+3\sqrt{64}}{4} = \frac{21+24}{4} = \frac{45}{4}$ which is close to $\frac{44}{4} = 11$.
  - $\frac{21-3\sqrt{57}}{4}$ is close to $\frac{21-3\sqrt{64}}{4} = \frac{21-24}{4} = -\frac{3}{4}$ which is close to -1.
  - The parabola vertex is less than 22 units high and the parabola opens downwards.
  - Axes height of 22 units appears reasonable.
  - A grid of width 3 units appears reasonable.
  - Plot all relevant points.
  - Finally “connect the dots with a freehand drawing”.

Maximum or minimum value of a quadratic function

- Let $f(x) = ax^2 + bx + c$ - quadratic ($a \neq 0$).
- Let D be the discriminant $D = b^2 - 4ac$.

$$f(x) = a\left(x - \left(-\frac{b}{2a}\right)\right)^2 - \frac{D}{4a} \quad \Big| \text{ complete the square}$$

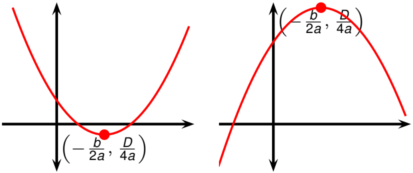
- Therefore if $a > 0$ then $f(x) = a(\text{square}) - \dfrac{D}{4a} \geq -\dfrac{D}{4a}$.
- Similarly if $a < 0$ then $f(x) = a(\text{square}) - \dfrac{D}{4a} \leq -\dfrac{D}{4a}$.

Recall $f(x) = ax^2 + bx + c = a\left(x - \left(-\frac{b}{2a}\right)\right)^2 - \frac{D}{4a}$.

Proposition

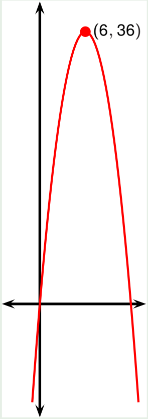
Let $f(x) = ax^2 + bx + c$, $a \neq 0$ and let $D = b^2 - 4ac$.

- *If $a > 0$ then $f(x)$ has no maximum and has minimum at $x = -\frac{b}{2a}$.*
- *If $a < 0$ then $f(x)$ has no minimum and has maximum at $x = -\frac{b}{2a}$.*
- *In both cases, the extremal value (either maximum or minimum) is $f\left(-\frac{b}{2a}\right) = -\frac{b^2-4ac}{4a} = -\frac{D}{4a}$.*



Example

Let x, z be two numbers that add to 12. Choose x and z so that the product $x \cdot z$ is maximal.



$$\begin{aligned} x + z &= 12 \\ z &= 12 - x \end{aligned}$$

Maximizing:

$$\begin{aligned} xz &= x(12 - x) \\ &= -x^2 + 12x \end{aligned}$$

Parabola opens down $\Rightarrow$ has maximum, attained at:

$$\begin{aligned} x &= -\frac{b}{2a} \\ &= -\frac{12}{-2} = 6 \\ z &= 12 - x = 12 - 6 = 6 \\ \text{Max. product} &= xz = 6 \cdot 6 = 36. \end{aligned}$$