Calculus I
Homework Substitution Rule
Lecture 22

1. Evaluate the indefinite integral. The answer key has not been proofread, use with caution.

(a) $\displaystyle\int (1+3x)^9 \mathrm{d}x$.

answer: $\frac{(1+3x)^{10}}{30}+C$.

(b) $\displaystyle\int \left(\sqrt{2x+1}\right) \mathrm{d}x$.

answer: $\frac{(2x+1)^{\frac{3}{2}}}{3}+C$

(c) $\displaystyle\int (3x+2)^{2.4} \mathrm{d}x$.

answer: $\frac{(3x+2)^{3.4}}{10.2}+C$

(d) $\displaystyle\int (x-1)\sqrt{2x-x^2} \mathrm{d}x$.

answer: $-\frac{\left(2x-x^2\right)^{\frac{3}{2}}}{3}+C$

(e) $\displaystyle\int x\sqrt{1-x^2} \mathrm{d}x$.

answer: $-\frac{\left(1-x^2\right)^{\frac{3}{2}}}{3}+C$

(f) $\displaystyle\int \frac{1+x^2}{\sqrt{3x+x^3}} \mathrm{d}x$.

answer: $\frac{2}{3}\left(3x+x^3\right)^{\frac{1}{2}}+C$

(g) $\displaystyle\int (x^2+1)(x^3+3x)^5 \mathrm{d}x$.

answer: $\frac{\left(x^3+3x\right)^6}{18}+C$

(h) $\displaystyle\int \frac{x^2}{\sqrt[3]{1+x^3}} \mathrm{d}x$. answer: $\frac{1}{2}\left(1+x^3\right)^{\frac{2}{3}}+C$

(i) $\displaystyle\int x^2\left(\sqrt{1+x}\right) \mathrm{d}x$.

answer: $\frac{2}{7}(1+x)^{\frac{7}{2}}-\frac{4}{5}(1+x)^{\frac{5}{2}}+\frac{2}{3}(1+x)^{\frac{3}{2}}+C$

(j) $\displaystyle\int x(2x+5)^{2014} \mathrm{d}x$.

answer: $\frac{1}{8064}(2x+5)^{2016}-\frac{1}{1612}(2x+5)^{2015}+C$

(k) $\displaystyle\int x^3\left(\sqrt{x^2+1}\right) \mathrm{d}x$.

answer: $\frac{1}{5}\left(x^2+1\right)^{\frac{5}{2}}-\frac{1}{3}\left(x^2+1\right)^{\frac{3}{2}}+C$

(l) $\displaystyle\int \sqrt{x}\sin\left(2+x^{\frac{3}{2}}\right) \mathrm{d}x$.

answer: $-\frac{2}{3}\cos\left(2+x^{\frac{3}{2}}\right)+C$

(m) $\displaystyle\int \frac{\cos\left(\frac{\pi}{x}\right)}{x^2} \mathrm{d}x$.

answer: $-\frac{\sin\left(\frac{\pi}{x}\right)}{\pi}+C$

(n) $\displaystyle\int \csc^2(2t) \mathrm{d}t$.

answer: $-\frac{1}{2}\cot(2t)+C$

(o) $\displaystyle\int \sec(5t)\tan(5t) \mathrm{d}t$.

answer: $\frac{1}{5}\sec(5t)+C$

(p) $\displaystyle\int \frac{\cos t}{\sin t} \mathrm{d}t$.

answer: $\ln|\sin t|+C$

(q) $\displaystyle\int \tan t \mathrm{d}t$.

answer: $-\ln|\cos t|+C$

(r) $\displaystyle\int \cot(2t) \mathrm{d}t$.

answer: $\frac{1}{2}\ln|\sin(2t)|+C$

(s) $\displaystyle\int \frac{\sin\sqrt{t}}{\sqrt{t}} \mathrm{d}t$.

answer: $-2\cos\left(\sqrt{t}\right)+C$

(t) $\displaystyle\int \sec^2 t\tan^3 t \mathrm{d}t$.

answer: $\frac{\tan^4 t}{4}+C$

(u) $\displaystyle\int \cos^4 t\sin t \mathrm{d}t$.

answer: $-\frac{\cos^5 t}{5}$

(v) $\displaystyle\int \frac{\mathrm{d}t}{\cos^2 t\sqrt{1+\tan t}}$.

answer: $2\sqrt{1+\tan t}+C$

(w) $\displaystyle\int \sqrt{\cot t}\csc^2 t \mathrm{d}t$.

answer: $-\frac{2}{3}\cot^{\frac{3}{2}} t+C$

(x) $\displaystyle\int \sin t\sec^2(\cos t) \mathrm{d}t$.

answer: $-\tan(\cos t)+C$

(y) $\displaystyle\int \sec^3 t\tan t \mathrm{d}t$.

answer: $\frac{\sec^3 t}{3}+C$

(z) $\displaystyle\int t\sin\left(t^2\right) \mathrm{d}t$.

answer: $-\frac{1}{2}\cos\left(t^2\right)+C$

Solution. 1.a We present two solution variants. The variants are equivalent. The only difference between them is that they use two interchangeable notations for differentials. Both variants are acceptable both when taking tests and writing scientific texts.

Variant I

$$\begin{aligned}\int(1+3x)^9\mathrm{d}x &= \int(1+3x)^9\frac{\mathrm{d}(3x)}{3}\\ &= \int u^9\frac{\mathrm{d}u}{3}\\ &= \frac{1}{3}\int u^9\mathrm{d}u\\ &= \frac{1}{30}u^{10}+C=\frac{(1+3x)^{10}}{30}+C.\end{aligned}$$

Set
$$\begin{aligned}u &= 1+3x\\ \mathrm{d}u &= 3\mathrm{d}x\\ \mathrm{d}x &= \tfrac{1}{3}\mathrm{d}u\end{aligned}$$

Variant II This variant is equivalent to the previous but uses the differential notation.

$$\begin{aligned}
\int (1+3x)^9 \mathrm{d}x &= \int (1+3x)^9 \frac{\mathrm{d}(3x)}{3} && \text{differentials are linear: } \mathrm{d}(3x) = (3x)'\mathrm{d}x = 3\mathrm{d}x \\
&= \int (1+3x)^9 \frac{\mathrm{d}(1+3x)}{3} && \text{differentials don't change when we add constants} \\
&= \frac{1}{3}\int u^9 \mathrm{d}u && \text{Set } u = 1+3x \\
&= \tfrac{1}{30}u^{10} + C = \tfrac{(1+3x)^{10}}{30} + C.
\end{aligned}$$

2. **Since we haven't studied** $\arctan$ **yet, please ignore problem 2.u. You can solve the problem using the formula** $\int \frac{1}{1+x^2}\mathrm{d}x = \arctan x + C$**. The function** $\arctan x$ **is the arctangent function (the inverse function to the tangent function).** Evaluate the integral. The answer key has not been proofread, use with caution.

(a) $\displaystyle\int \frac{\mathrm{d}x}{3x+5}$.

answer: $\frac{1}{3}\ln|3x+5| + C$

(b) $\displaystyle\int \frac{\mathrm{d}x}{2-3x}$.

answer: $-\frac{1}{3}\ln|2-3x| + C$

(c) $\displaystyle\int e^x\cos(e^x)\mathrm{d}x$.

answer: $\sin(e^x) + C$

(d) $\displaystyle\int \frac{(\ln x)^3}{x}\mathrm{d}x$.

answer: $\frac{(\ln x)^4}{4} + C$

(e) $\displaystyle\int e^x\left(\sqrt{e^x+1}\right)\mathrm{d}x$

answer: $\frac{2}{3}(e^x+1)^{\frac{3}{2}} + C$

(f) $\displaystyle\int e^x\sqrt{1-e^x}\mathrm{d}x$.

answer: $-\frac{2}{3}(1-e^x)^{\frac{3}{2}} + C$

(g) $\displaystyle\int e^{\sin t}\cos t\mathrm{d}t$.

answer: $e^{\sin t} + C$

(h) $\displaystyle\int e^{\cot x}\csc^2 x\mathrm{d}x$.

answer: $-e^{\cot x} + C$

(i) $\displaystyle\int \frac{x}{1+x^2}\mathrm{d}x$.

answer: $\frac{1}{2}\ln\left(x^2+1\right) + C$

(j) $\displaystyle\int \frac{x}{2+3x^2}\mathrm{d}x$.

answer: $\frac{1}{6}\ln\left(x^2+\frac{2}{3}\right) + C$

(k) $\displaystyle\int \frac{x}{\sqrt{1-x^2}}\mathrm{d}x$.

answer: $-\sqrt{1-x^2} + C$

(l) $\displaystyle\int \frac{\cos(\ln x)}{x}\mathrm{d}x$.

answer: $\sin(\ln x) + C$

(m) $\displaystyle\int \frac{\sin(\ln x)}{x}\mathrm{d}x$.

answer: $-\cos(\ln x) + C$

(n) $\displaystyle\int \frac{\sin(2x)}{2+\cos^2 x}\mathrm{d}x$.

answer: $-\ln(2+\cos^2 x) + C$

(o) $\displaystyle\int \frac{\cos x}{\sin x}\mathrm{d}x$

answer: $\ln|\sin x| + C$

(p) $\displaystyle\int \cot x\mathrm{d}x$.

answer: $\ln|\sin x| + C$

(q) $\displaystyle\int \cot\left(\frac{x}{2}\right)\mathrm{d}x$

answer: $2\ln\left|\sin\left(\frac{x}{2}\right)\right| + C$

(r) $\displaystyle\int \tan(2x)\mathrm{d}x$.

answer: $-\frac{1}{2}\ln|\cos(2x)| + C$

(s) $\displaystyle\int \frac{x^4+3x}{x^2}\mathrm{d}x$

answer: $\frac{x^3}{3} + 3\ln|x| + C$

(t) $\displaystyle\int x^2 e^{x^3}\mathrm{d}x$

answer: $\frac{1}{3}e^{x^3} + C$

(u) $\displaystyle\int \frac{\arctan x}{1+x^2}\mathrm{d}x$.

answer: $\frac{(\arctan x)^2}{2} + C$

Solution. 2.e.

$$\begin{aligned}
\int e^x\sqrt{e^x+1}\,\mathrm{d}x &= \int \sqrt{e^x+1}\,\mathrm{d}\left(e^x\right) \\
&= \int \sqrt{e^x+1}\,\mathrm{d}\left(e^x+1\right) && \text{Set } u = e^x+1 \\
&= \int \sqrt{u}\,\mathrm{d}u \\
&= \frac{2}{3}u^{\frac{3}{2}} + C \\
&= \frac{2}{3}\left(e^x+1\right)^{\frac{3}{2}} + C
\end{aligned}$$

Solution. 2.n

$$\begin{array}{rcl|l}
\displaystyle\int \frac{\sin(2x)}{2+\cos^2 x}\mathrm{d}x &=& \displaystyle\int \frac{2\cos x \sin x \mathrm{d}x}{2+\cos^2 x} & \text{use } \sin(2x) = 2\sin x\cos x \\
&=& \displaystyle\int \frac{2\cos x \mathrm{d}(-\cos x)}{2+\cos^2 x} & \text{use } \mathrm{d}(\cos x) = -\sin x \mathrm{d}x \\
&=& \displaystyle -\int \frac{2u\mathrm{d}(u)}{2+u^2} & \text{set } u = \cos x \\
&=& \displaystyle -\int \frac{\mathrm{d}\left(2+u^2\right)}{2+u^2} & \text{use } \mathrm{d}(u^2+2) = 2u\mathrm{d}u \\
&=& \displaystyle -\int \frac{\mathrm{d}z}{z} & \text{set } z = 2+u^2 \\
&=& -\ln|z| + C & \text{Substitute back } z = u^2+2 \\
& & & \quad u^2+2 \text{ is positive} \\
&=& -\ln(u^2+2) + C & \quad \Rightarrow \text{omit the abs. value} \\
& & & \quad \text{Substitute back } u = \cos x \\
&=& -\ln(\cos^2 x + 2) + C. &
\end{array}$$

Solution. 2.m

$$\begin{array}{rcl|l}
\displaystyle\int \frac{\sin(\ln x)}{x}\mathrm{d}x &=& \displaystyle\int \sin(\ln x)\mathrm{d}\left(\ln x\right) & u = \ln x \\
&=& \displaystyle\int \sin u \mathrm{d}u & \\
&=& -\cos u + C & \\
&=& -\cos(\ln x) + C &
\end{array}$$

3. Evaluate the definite integral. The answer key has not been proofread, use with caution.

   (a) $\displaystyle\int_{e}^{e^3} \frac{\mathrm{d}x}{x\sqrt[3]{\ln x}}.$

   answer: $\frac{3}{2}\left(\sqrt[3]{9}-1\right)$

   (b) $\displaystyle\int_{0}^{1} xe^{-x^2}\mathrm{d}x.$

   answer: $\frac{1-e^{-1}}{2}$

   (c) $\displaystyle\int_{0}^{1} \frac{e^x+1}{e^x+x}\mathrm{d}x.$

   answer: $\ln(e+1)$

   (d) $\displaystyle\int_{1}^{2} \frac{x}{2x^2+1}\mathrm{d}x.$

   answer: $\frac{1}{4}\ln 3$

   (e) $\displaystyle\int_{-3}^{-2} \frac{x}{1-x^2}\mathrm{d}x.$

   answer: $\left[-\frac{1}{2}\ln\left|1-x^2\right|\right]_{-3}^{-2} = \frac{1}{2}\ln\left(\frac{8}{3}\right)$

   (f) $\displaystyle\int_{-3}^{-2} \frac{3x}{2-x^2}\mathrm{d}x.$

   answer: $\left[-\frac{3}{2}\ln\left|2-x^2\right|\right]_{-3}^{-2} = \frac{3}{2}\ln\frac{7}{2}$

   (g) $\displaystyle\int_{0}^{\frac{1}{4}} \frac{x}{\sqrt{1-3x^2}}\mathrm{d}x.$

   answer: $\frac{1}{3}\left(1-\sqrt{\frac{13}{16}}\right)$

Solution. 3.d

$$\begin{aligned}\int\limits_1^2 \frac{x}{2x^2+1}\mathrm{d}x &= \int\limits_{x=1}^{x=2} \frac{\frac{1}{4}\mathrm{d}(2x^2)}{2x^2+1} = \frac{1}{4}\int\limits_{x=1}^{x=2} \frac{\mathrm{d}(2x^2+1)}{2x^2+1} \quad \Bigg| \text{ Set } u = 2x^2+1 \\ &= \frac{1}{4}\int\limits_{\substack{x=1\\u=3}}^{\substack{x=2\\u=9}} \frac{\mathrm{d}u}{u} = \frac{1}{4}\left[\ln u\right]_3^9 = \frac{1}{4}\left(\ln 9 - \ln 3\right) = \frac{\ln 3}{4}.\end{aligned}$$

Solution. 3.e

$$\begin{aligned}\int_{-3}^{-2} \frac{x}{1-x^2}\mathrm{d}x &= \int\limits_{\substack{x = -3\\ u = -8}}^{\substack{x = -2\\ u = -3}} \frac{1}{u}\left(-\frac{1}{2}\mathrm{d}u\right) \quad \left|\ \begin{aligned} u &= 1-x^2 \\ \mathrm{d}u &= -2x\mathrm{d}x \\ x\mathrm{d}x &= -\tfrac{1}{2}\mathrm{d}u \end{aligned}\right. \\ &= -\frac{1}{2}\left[\ln|u|\right]_{-8}^{-3} \\ &= -\frac{1}{2}\left(\ln|3| - \ln|8|\right) \\ &= \frac{\ln\left|\frac{8}{3}\right|}{2}\end{aligned}$$

Solution. 3.f

$$\begin{aligned}\int_{-3}^{-2} \frac{3x}{2-x^2}\mathrm{d}x &= \int_{-3}^{-2} \frac{3\frac{\mathrm{d}(x^2)}{2}}{2-x^2}\mathrm{d}x \\ &= \frac{3}{2}\int_{-3}^{-2} \frac{-\mathrm{d}\left(-x^2\right)}{2-x^2} \\ &= -\frac{3}{2}\int_{-3}^{-2} \frac{\mathrm{d}\left(-x^2\right)}{2-x^2} \\ &= -\frac{3}{2}\int_{x=-3}^{x=-2} \frac{\mathrm{d}\left(2-x^2\right)}{2-x^2} \quad \Big| \text{ Set } 2-x^2 = u \\ &= -\frac{3}{2}\int_{x=-3,u=-7}^{x=-2,u=-2} \frac{\mathrm{d}u}{u} \\ &= -\frac{3}{2}\left[\ln|u|\right]_{-7}^{-2} \\ &= -\frac{3}{2}(\ln 2 - \ln 7) \\ &= \frac{3}{2}\ln\left(\frac{7}{2}\right).\end{aligned}$$