Calculus II
Homework on Lecture 17

1. Find whether the series is convergent or divergent using an appropriate test. Some of the problems require the alternating series test. The test states the following.

   Alternating series test. Suppose $b_n \searrow 0$. Then $\sum (-1)^n b_n$ is convergent.

   Here, $b_n \searrow 0$ means the following.

   - The sequence of numbers b_n is decreasing.
   - The sequence decreases to 0, that is,
   $$\lim_{n\to\infty} b_n = 0 \quad .$$

   (a) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \ln n.$

   answer: diverges, basic divergence test

   (b) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}.$

   answer: converges, alternating series test

   (c) $\displaystyle\sum_{n=2}^{\infty} \frac{n}{\ln n}$

   answer: diverges, basic divergence test

   (d) $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n}$

   answer: converges, alternating series test

Solution. 1.a. $\lim_{n\to\infty} (-1)^n \ln n$ does not exist and therefore the sum is not convergent.

Solution. 1.b. For $n > 2$, we have that $\ln n$ is a positive increasing function and therefore $\frac{1}{\ln n}$ is a decreasing positive function. Furthermore $\displaystyle\lim_{n\to\infty} \frac{1}{\ln n} = 0$. Therefore the series is convergent by the alternating series test.

2. (The last problem will NOT appear on the quiz) Use the integral test, the comparison test or the limit comparison test to determine whether the series is convergent or divergent. Justify your answer.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n+1}.$

   answer: divergent

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n^2+n^3}.$

   answer: convergent, compare to $\sum_{n=1}^{\infty} \frac{1}{2n^2}$

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+3}{3n^5+n}$

   answer: convergent, can use limit comparison test

   (d) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{3^n+5}.$

   answer: convergent, compare to $\sum_{n=0}^{\infty} \frac{1}{3^n}$

   (e) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln n}$

   answer: divergent, integral test

   (f) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(2n+1)\ln(n)}.$

   answer: divergent

   (g) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

   answer: convergent, can use integral test

   (h) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(2n+1)(\ln(n))^2}.$

   answer: convergent

   (i) Determine all values of p, q r for which the series
   $$\sum_{n=30}^{\infty} \frac{1}{n^p(\ln n)^q(\ln(\ln n))^r}$$
   is convergent.

Solution. 2.e. The function $\frac{1}{x\ln x}$ is decreasing, as for $x > 2$, it is the quotient of 1 by increasing positive functions. $\frac{1}{x\ln x}$ tends to 0 as $x \to \infty$, and therefore the integral criterion implies that $\sum_{n=2}^{\infty} \frac{1}{n\ln n}$ is convergent/divergent if and only if $\int_2^{\infty} \frac{1}{x\ln x}\mathrm{d}x$ is.

$$
\begin{aligned}
\int_2^{\infty} \frac{1}{x\ln x}\mathrm{d}x &= \lim_{t\to\infty}\int_2^t \frac{1}{x\ln x}\mathrm{d}x\\
&= \lim_{t\to\infty}\int_2^t \frac{1}{\ln x}\mathrm{d}(\ln x)\\
&= \lim_{t\to\infty}\int_2^t \mathrm{d}(\ln(\ln x))\\
&= \lim_{t\to\infty} [\ln(\ln x)]_{x=2}^{x=t}\\
&= \lim_{t\to\infty} (\ln(\ln t)-\ln(\ln 2))\\
&= \infty.
\end{aligned}
$$

The integral is divergent (and diverges to $+\infty$) and therefore, by the integral criterion, so is the sum.

Solution. 2.f The integral criterion appears to be of little help: the improper integral $\int \frac{1}{(2x+1)\ln x}\mathrm{d}x$ cannot be integrated algebraically with any of the techniques we have studied so far. Therefore it makes sense to try to solve this problem using a comparison test.

We present two solution variants. In Variant I we use the limit-comparison test. This is an easier (but slightly longer) solution. In Variant II we use the comparison test - this solution is harder as it requires algebraic intuition to select a series to compare to.

Variant I. This variant uses the limit comparison test.

The "dominant term"[1] of the denominator of $\frac{1}{(2n+1)\ln n} = \frac{1}{2n\ln n+\ln n}$ is $2n\ln n$. Therefore it makes sense to compare - or limit-compare - with $\frac{1}{n\ln n}$.

We will use the Limit Comparison Test for the series $\sum\limits_{n=2}^{\infty} a_n = \sum\limits_{n=2}^{\infty}\frac{1}{(2n+1)\ln n}$ and $\sum\limits_{n=2}^{\infty} b_n = \sum\limits_{n=2}^{\infty}\frac{1}{n\ln n}$. Both a_n and b_n are positive (for $n>2$) and therefore the Limit Comparison Test applies.

$$
\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\frac{1}{(2n+1)\ln n}}{\frac{1}{n\ln n}} = \lim_{n\to\infty}\frac{n}{2n+1} = \lim_{n\to\infty}\frac{1}{2+\frac{1}{n}} = \frac{1}{2}.
$$

Since $\lim\limits_{n\to\infty}\frac{a_n}{b_n} = \frac{1}{2}\neq 0$, the Limit Comparison Test implies that the series $\sum\limits_{n=2}^{\infty} a_n$ has same convergence/divergence properties as the series $\sum\limits_{n=2}^{\infty} b_n$. In Problem 2.e we demonstrated that the series $\sum\limits_{n=2}^{\infty} b_n$ is divergent; therefore the series $\sum\limits_{n=2}^{\infty} a_n = \sum\limits_{n=2}^{\infty}\frac{1}{(2n+1)\ln n}$ is divergent as well.

Variant II. This variant uses directly the comparison test. It is slightly shorter than the preceding variant but requires more intuition.

Let $a_n = \frac{1}{(2n+1)\ln n}$. Consider the series $\sum\limits_{n=2}^{\infty} b_n$ for $b_n = \frac{1}{3n\ln n}$. We have that

$$
\begin{array}{rcl|l}
3n & \geq & 2n+1 & \text{for } n\geq 1\\
\frac{1}{3n} & \leq & \frac{1}{2n+1} \quad . & \text{Inverting positive quantities reverses inequalities}
\end{array}
$$

Therefore $b_n \geq a_n$. In Problem 2.e we illustrated (using the integral test) that $\sum\limits_{n=2}^{\infty}(3b_n)$ is divergent and therefore so is its constant multiple $\sum\limits_{n=2}^{\infty} b_n$. Therefore $\sum\limits_{n=2}^{\infty}\frac{1}{(2n+1)\ln n}$ is divergent by the comparison test.

[1] since we do not speak of rational functions, here the expression "dominant term" is used informally