Calculus I
Lecture 20
Areas, Integration

Todor Milev
https://github.com/tmilev/freecalc

2020

Outline

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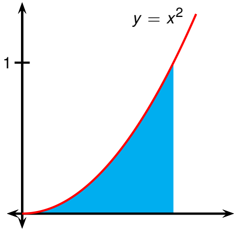
The Area Problem

- How can we find the area under $y = x^2$ between $x = 0$ and $x = 1$?



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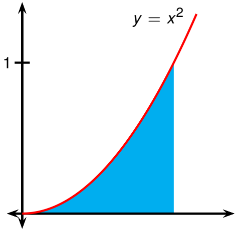

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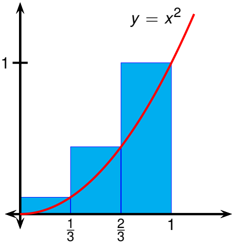

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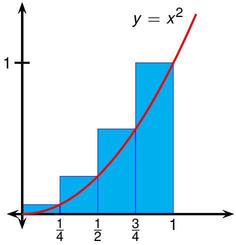

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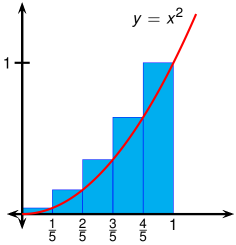

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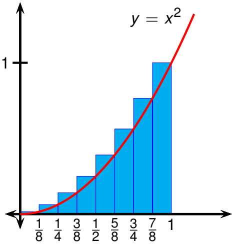

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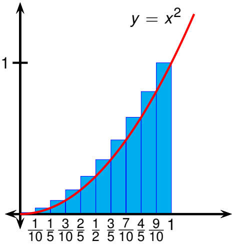

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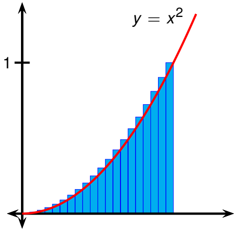

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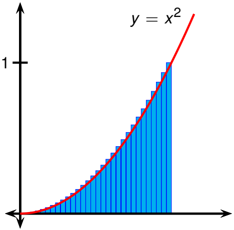

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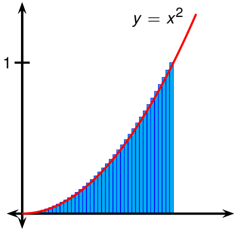

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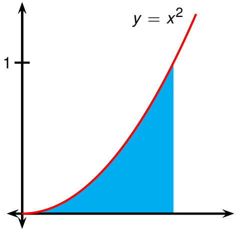

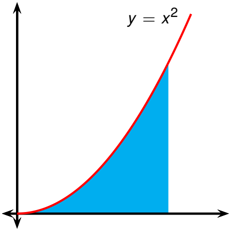

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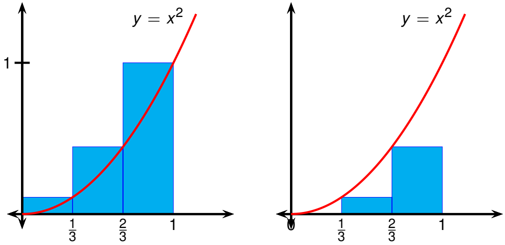

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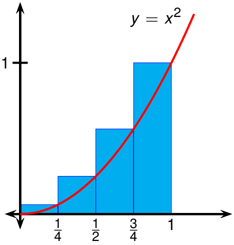

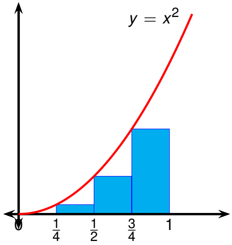

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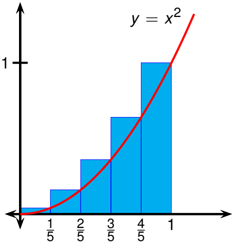

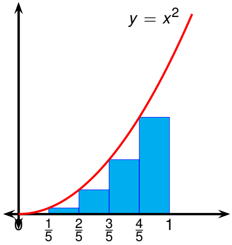

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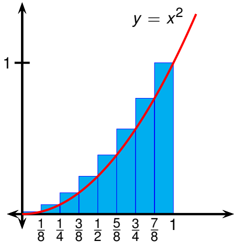

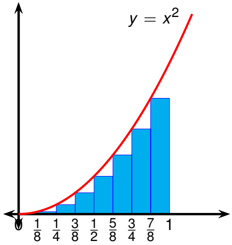

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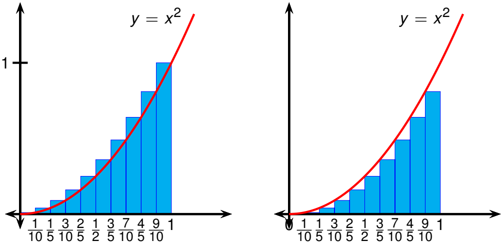

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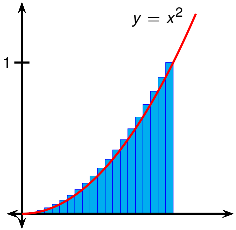

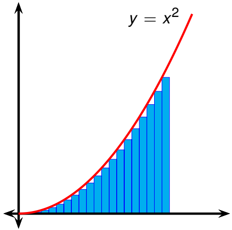

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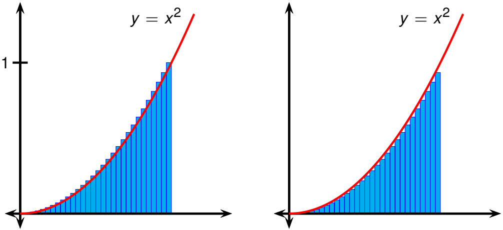

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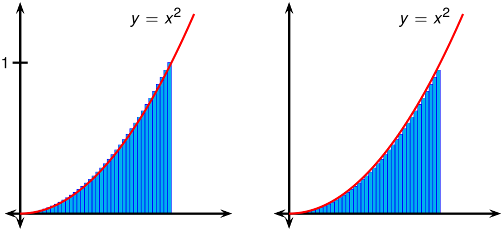
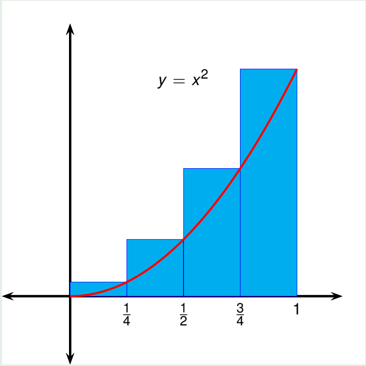


Example

Find the sum of the areas of the four approximating rectangles obtained using right endpoints.



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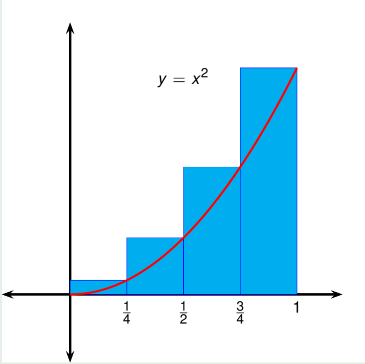
- Let R_4 denote the sum of the areas of the rectangles.



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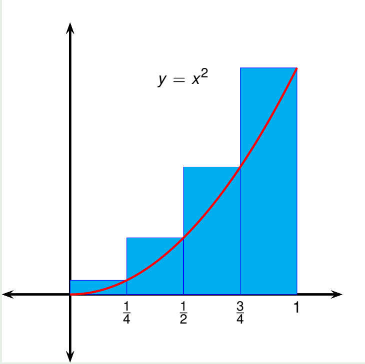
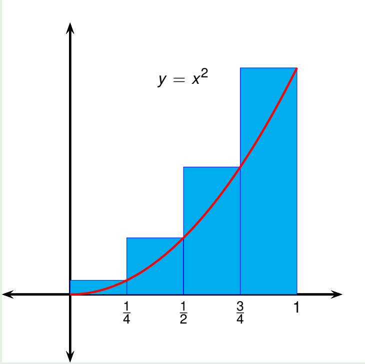
- Let R_4 denote the sum of the areas of the rectangles.
- Each rectangle has width **?**.



Example

Find the sum of the areas of the four approximating rectangles obtained using right endpoints.

- Let R_4 denote the sum of the areas of the rectangles.
- Each rectangle has width $\frac{1}{4}$.



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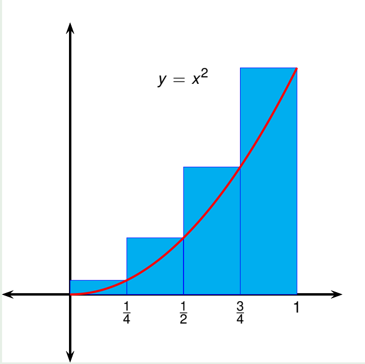

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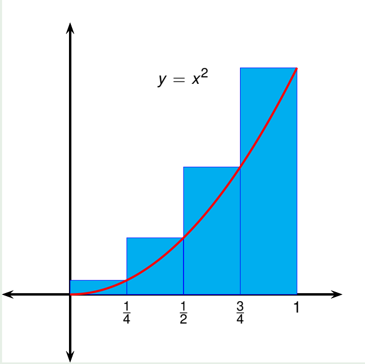

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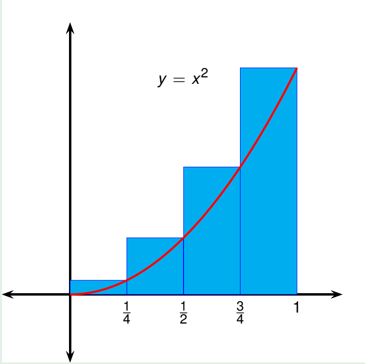

$$R_4 = \frac{1}{4} \cdot \left(\frac{1}{4}\right)^2 + \frac{1}{4} \cdot \left(\frac{1}{2}\right)^2 + \frac{1}{4} \cdot \left(\frac{3}{4}\right)^2 + \frac{1}{4} \cdot (1)^2$$

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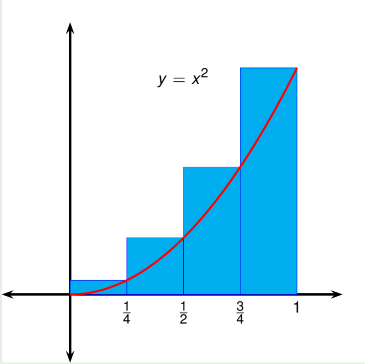

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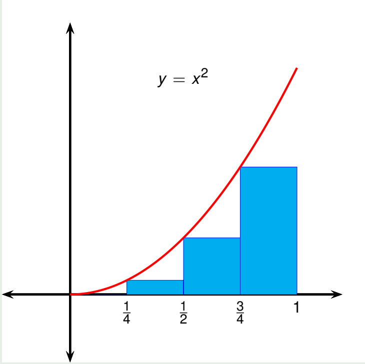

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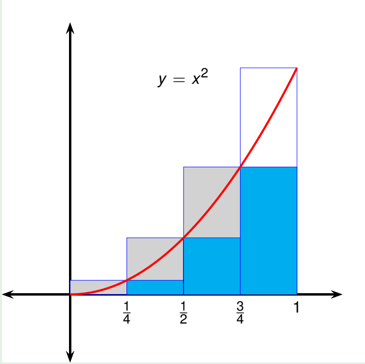

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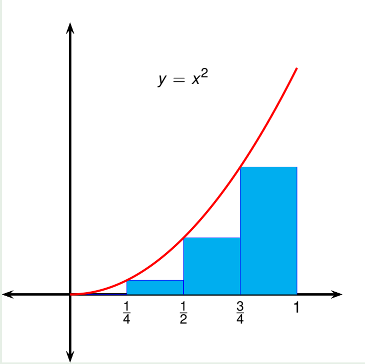

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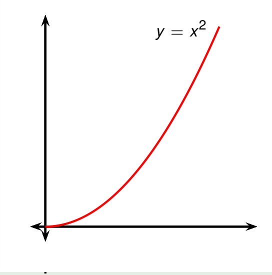
Example

For the region S underneath the parabola $y = x^2$ from 0 to 1, show that the area under the approximating rectangles approaches $\frac{1}{3}$, that is,

$$\lim_{n\to\infty} R_n = \frac{1}{3}.$$



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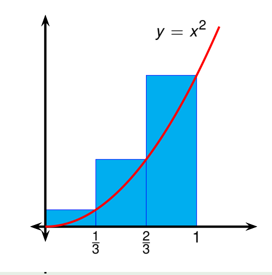

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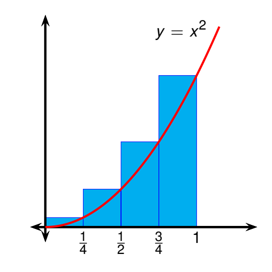

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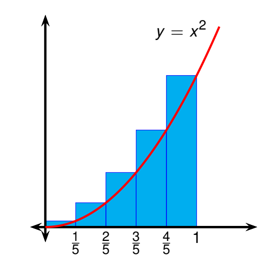

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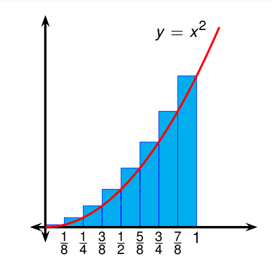

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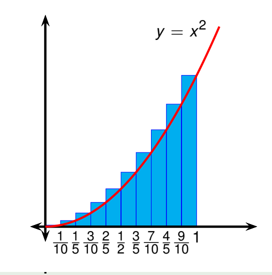

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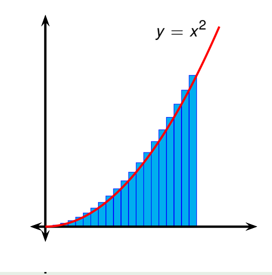

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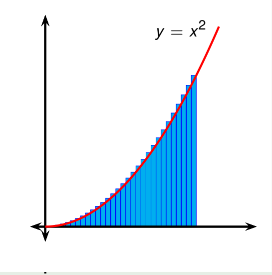

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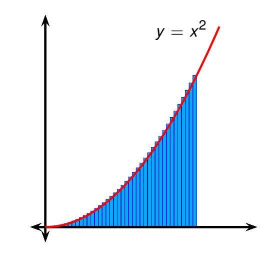

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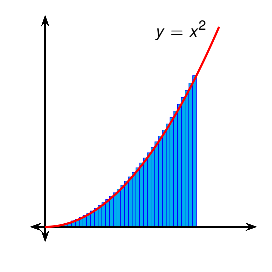

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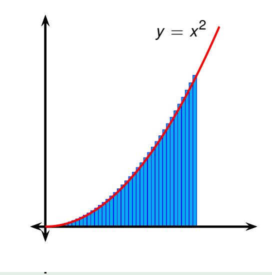

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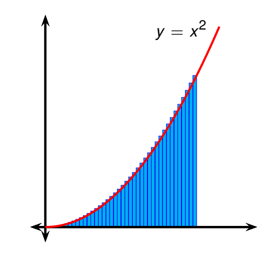

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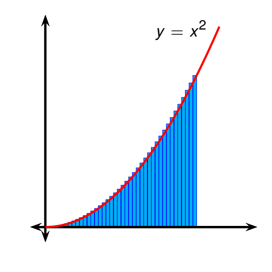

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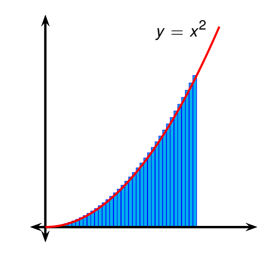

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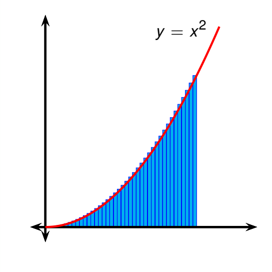

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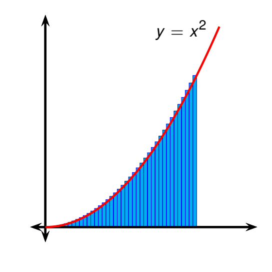

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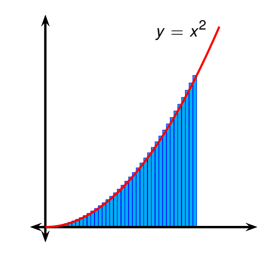

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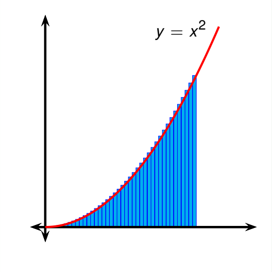

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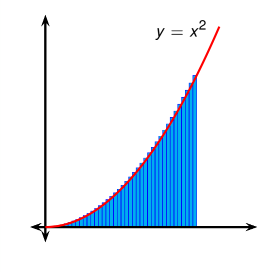

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Example (The ... and $\sum$ notations for series)

Let A be the sum of the positive even integers between 2 and 124.

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  - If in doubt or seeking complete rigor we should use the $\sum$ notation.

Definition

Sigma Notation: The sum of n terms $a_1, a_2, \ldots, a_n$ is written as

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$$

where i is the *index of summation*, a_i is the *i'th term*, and the *upper and lower bounds of summation* are n and 1 respectively.

NOTE: The lower bound doesn't have to be 1. Any integer less than or equal to the upper bound is legitimate.
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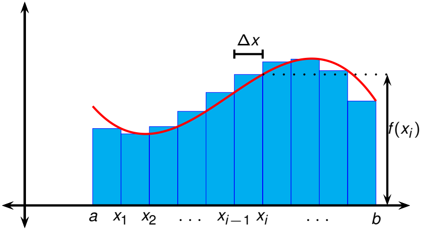
Example

$$\sum_{j=3}^{7} j^2 = 9 + 16 + 25 + 36 + 49$$

Estimate the area under $y = f(x)$ between a and b using n strips.



- The right endpoints of the subintervals are

$$\begin{aligned} x_1 &= a + \Delta x \\ x_2 &= a + 2\Delta x \\ x_3 &= a + 3\Delta x \\ &\vdots \end{aligned}$$

- The width of the interval is $b - a$.
- The width one strip is $\Delta x = \frac{b-a}{n}$.
- $[a, b]$ is divided into n subintervals: $[x_0, x_1], [x_1, x_2], \dots, [x_{n-1}, x_n]$, where $x_0 = a$ and $x_n = b$.
- The height of the ith rectangle is $f(x_i)$.
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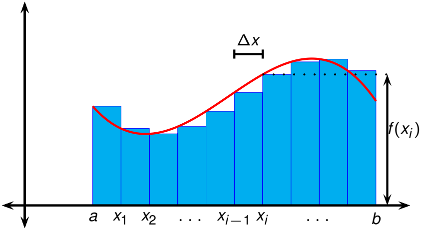

- The left endpoints of the subintervals are

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Let $f(x) > 0$. The area of the region S that lies under $y = f(x)$ is the limit of the sum of the areas of the approximating rectangles:

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Definition (Riemann Sum)

A Riemann sum is any sum of the form

$$f(x_1^*)\Delta x + f(x_2^*)\Delta x + \dots + f(x_n^*)\Delta x.$$

The Definite Integral

Definition (Definite Integral)

- Let f be a function defined for $a \leq x \leq b$.
- Divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$ nd set $x_0 = a$, $x_n = b$.
- Let $x_0, x_1, \dots, x_n$ be the endpoints of the subintervals.
- Let $x_1^*, x_2^*, \dots, x_n^*$ be any sample points in these subintervals; that is, x_i^* is in $[x_{i-1}, x_i]$.

Suppose the limit $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} f(x_i^*)\Delta x$ exists and is independent of the choice of sample points x_i^*. Then we say that f is an integrable function. If f is integrable we call the limit the integral of f over $[a, b]$ and write

$$\int_a^b f(x)\mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x \quad .$$

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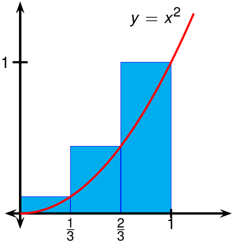
- a and b are called the limits of integration.
- The definite integral is a number. It does not depend on x. We could use any variable instead of x.

$$\int_a^b f(x)\mathrm{d}x = \int_a^b f(t)\mathrm{d}t = \int_a^b f(r)\mathrm{d}r = \int_a^b f(\theta)\mathrm{d}\theta$$

- We know already that if $f(x)$ is always positive, then $\int_a^b f(x)dx$ is the area under the curve.



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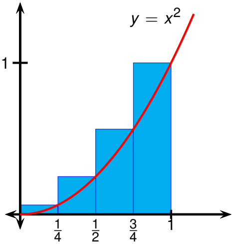

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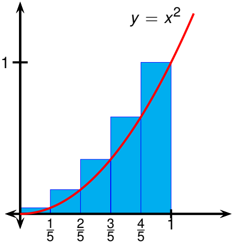

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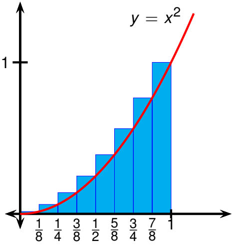

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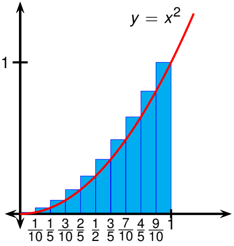

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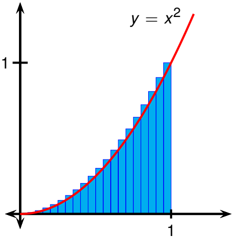

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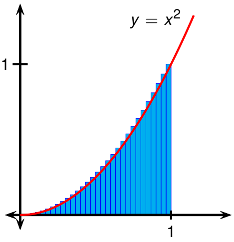

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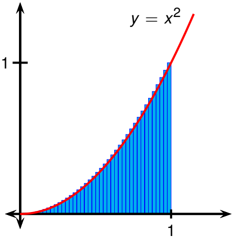

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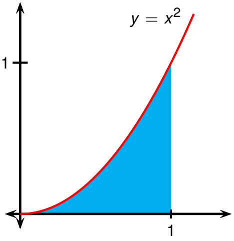

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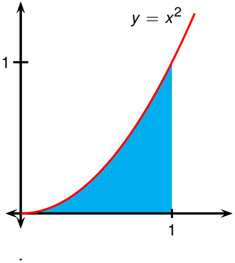

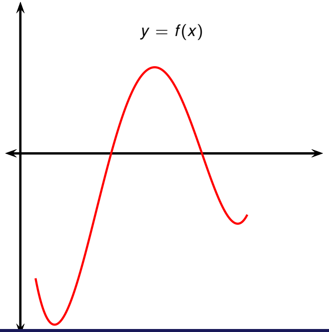

- What if $f(x)$ is sometimes negative?

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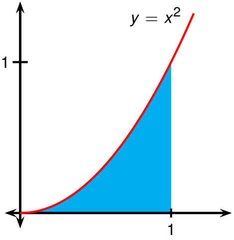

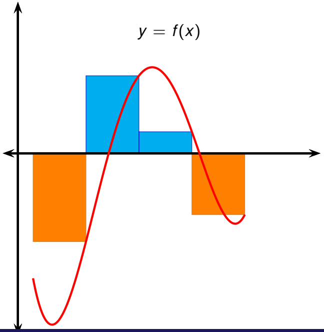

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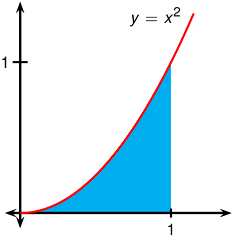

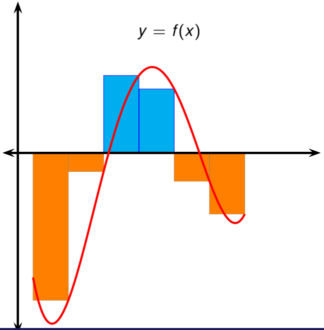

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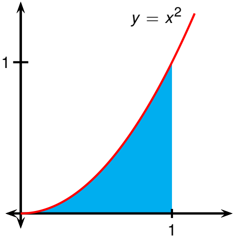

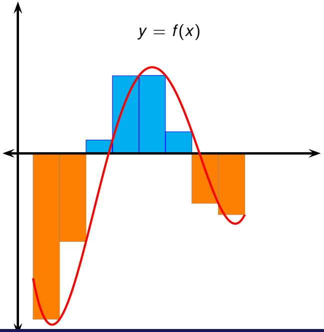

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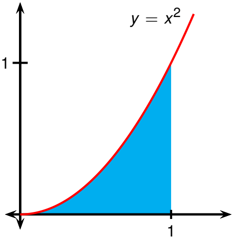

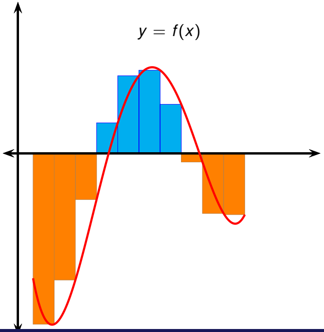

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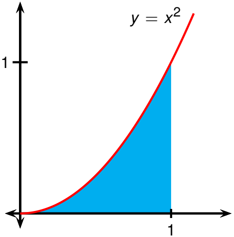

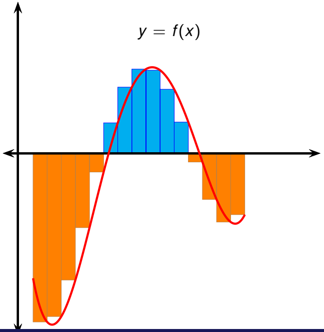

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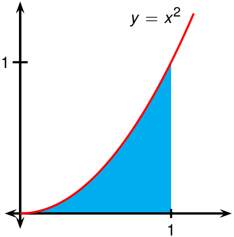

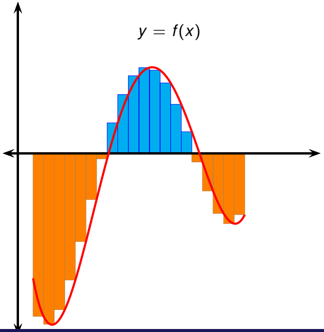

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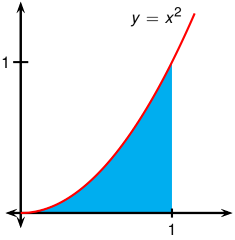

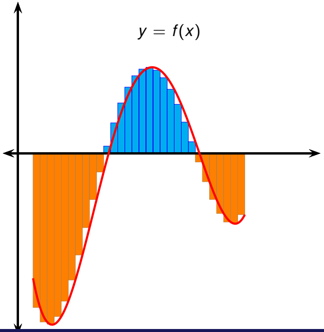

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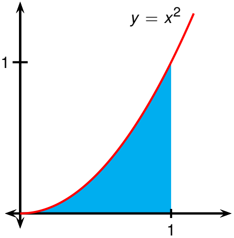

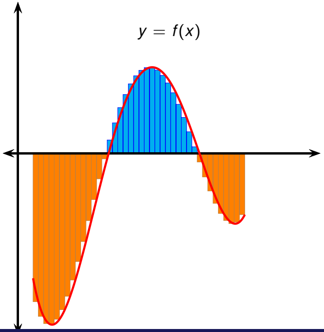

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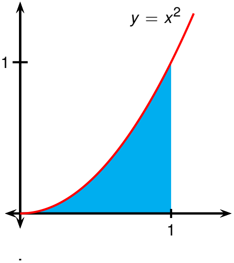

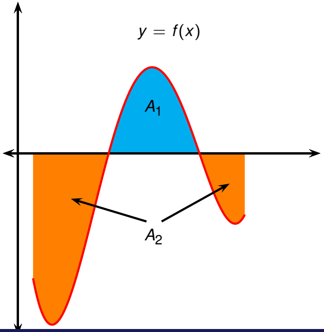

- What if $f(x)$ is sometimes negative?
- Then $\int_a^b f(x)\mathrm{d}x = A_1 - A_2$.
- A_1 is the area of the region above the x-axis and below the graph of f.
- A_2 is the area of the region below the x-axis and above the graph of f.

Theorem

Let f be a continuous function on $[a, b]$. Then f is integrable over $[a, b]$.

- In particular the integral does not depend the choice of sampling points so long as the intervals containing them shrink.
- The proof of this theorem is not difficult, but is outside of the scope of Calculus I and II.
- The only “difficulty” in the proof stems from the fact that we have not rigorously constructed the real numbers.
- We already (silently) assumed a construction of the real numbers when we defined limits.
- Such a construction is also (silently) assumed in most regular high school mathematics courses.
- A student interested in a proof of the theorem should google “Darboux integral”.

The following **power sums** will be useful in what follows:

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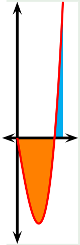
Example

Evaluate $\int_0^3 (x^3 - 6x)\mathrm{d}x$.



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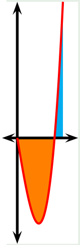

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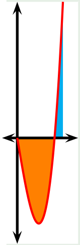

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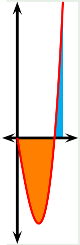

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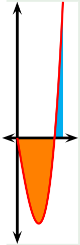

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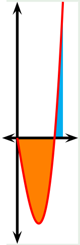

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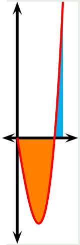

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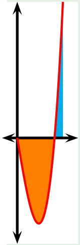

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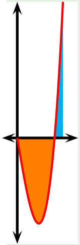

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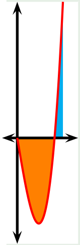

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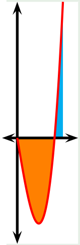

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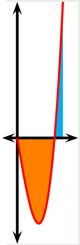

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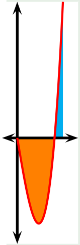

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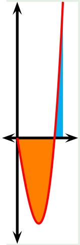

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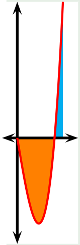

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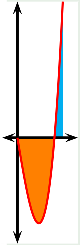

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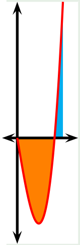

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\end{aligned}
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Properties of the Definite Integral

- So far when we have calculated $\int_a^b f(x)\mathrm{d}x$, we have assumed that $a < b$.
- The definition as a limit of Riemann sums will still work even if we don't assume this.

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- If we reverse a and b, then Δx changes from $\frac{b-a}{n}$ to $\frac{a-b}{n}$.

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1. $\int_a^b c\mathrm{d}x = c(b-a)$, where c is any constant.
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Example

Use the properties of integrals to evaluate

$$\int_0^1 (4 + 3x^2)\mathrm{d}x$$

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$$\int_0^1 (4+3x^2)\mathrm{d}x = \int_0^1 4\mathrm{d}x + \int_0^1 3x^2\mathrm{d}x \qquad \text{Property}$$

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$$\begin{aligned}\int_0^1 (4+3x^2)\mathrm{d}x &= \int_0^1 4\mathrm{d}x + \int_0^1 3x^2\mathrm{d}x && \text{Property 2}\\ &= \int_0^1 4\mathrm{d}x + 3\int_0^1 x^2\mathrm{d}x && \text{Property 3}\end{aligned}$$

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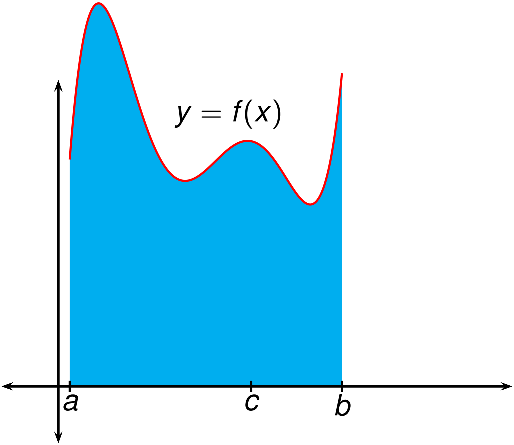
\end{aligned}
$$

Properties of the Integral

5. $$\int_a^b f(x)\mathrm{d}x = \int_a^c f(x)\mathrm{d}x + \int_c^b f(x)\mathrm{d}x$$



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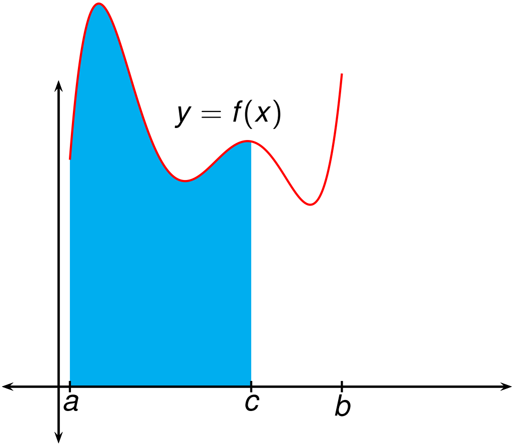

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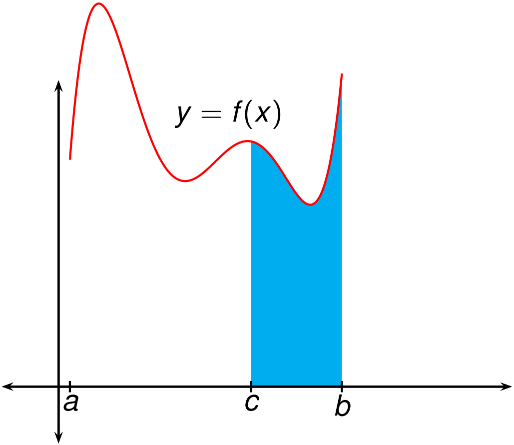

Example

If it is known that $\int_0^{10} f(x)\mathrm{d}x = 17$ and $\int_0^{8} f(x)\mathrm{d}x = 12$, then find $\int_8^{10} f(x)\mathrm{d}x$.

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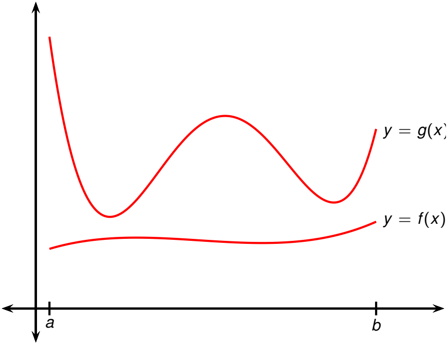
Comparison Properties of the Integral

6. If $f(x) \geq 0$ for all $a \leq x \leq b$, then $\displaystyle\int_a^b f(x)\mathrm{d}x \geq 0$.

Comparison Properties of the Integral

7. If $f(x) \leq g(x)$ for all $a \leq x \leq b$, then $\int_a^b f(x)\mathrm{d}x \leq \int_a^b g(x)\mathrm{d}x$.



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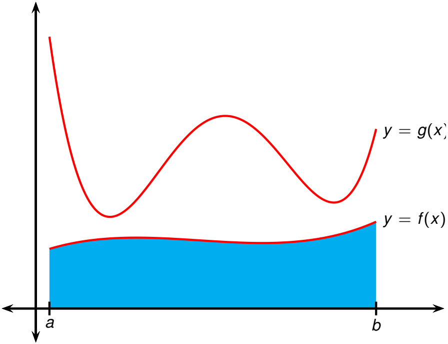

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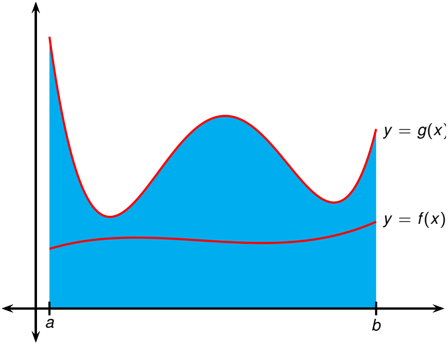

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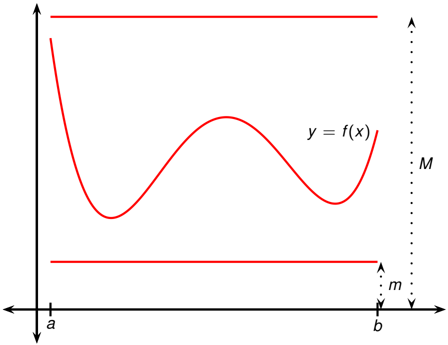
8. If $m \leq f(x) \leq M$ for all $a \leq x \leq b$, then

$$m(b-a) \leq \int_a^b f(x)\mathrm{d}x \leq M(b-a)$$



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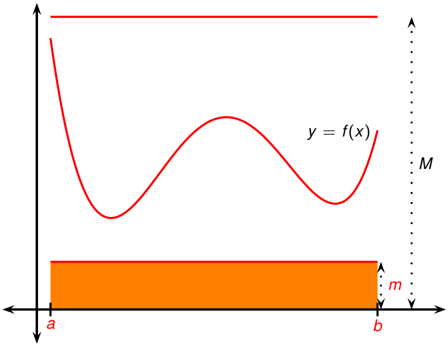

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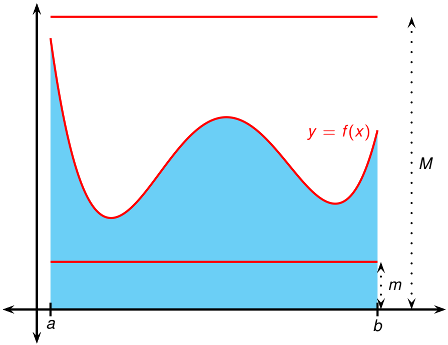

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