Calculus I
Lecture 11
The Chain Rule

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https://github.com/tmilev/freecalc

2020

Outline

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The Chain Rule

- What is the derivative of $f(x) = \sqrt{x^2 + 1}$?

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- $u = h(x) = \textbf{?} \qquad .$
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- It would be nice if we could find the derivative of f in terms of the derivatives of y and u.
- It turns out that the derivative of the composition $g \circ h$ is the product of the derivative of g and the derivative of h.
- This important fact is called the Chain Rule.

The Chain Rule

Let g and h be functions. Recall that the composite function $f = g \circ h$ is defined via $f(x) = g(h(x))$.

Theorem

Let h be differentiable at x and let g be a differentiable at $h(x)$. Then the composite function $f = g \circ h$ is differentiable at x and f' is given by the product

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The last equality uses the Leibniz notation (due to G. Leibniz (1646-1716)).

Chain rule notations

- As we saw, the chain rule can be written using a number of notations:

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(g(h(x)))' & = & g'(h(x)) \cdot h'(x) & & \text{(notation 1)} \\
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- Most authors tend to prefer one of these notations over the others.
- In order to exercise ourselves we shall use all three notations throughout our course.
- There are additional notations (not covered here) used in practice.
- Whenever in doubt about derivative notation, if possible, request clarification.

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Example (Chain Rule, Notation 1)

Differentiate $f(x) = \sqrt{x^2+1}.$

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\text{Differentiate} & f(x) &=& \sqrt{x^2+1}. \\
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\begin{array}{rrcl}
\text{Differentiate} & f(x) & = & \sqrt{x^2+1}. \\
\text{Let} & h(x) & = & x^2+1. \\
\text{Let} & g(u) & = & \sqrt{u}. \\
\text{Then} & f(x) & = & g(h(x)). \\
\text{Chain Rule:} & f'(x) & = & g'(h(x))h'(x) \\
 & & = & \left(\frac{1}{2\sqrt{h(x)}}\right)(2x) \\
 & & = & \frac{x}{\sqrt{x^2+1}}.
\end{array}
$$

$$\begin{array}{rcll} (g(h(x)))' & = & g'(h(x)) \cdot h'(x) & & \text{(notation 1)} \\ (g(u))' & = & g'(u)u' & \text{where } u = h(x) & \text{(notation 2)} \\ \dfrac{\mathrm{d}y}{\mathrm{d}x} & = & \dfrac{\mathrm{d}y}{\mathrm{d}u}\dfrac{\mathrm{d}u}{\mathrm{d}x} & \text{where } y = g(u) & \text{(notation 3)} \end{array} .$$

Example (Chain Rule, Notation 2)

Differentiate $\quad f(x) \quad = \quad \sqrt{x^2+1}.$

$$\begin{array}{rcll} (g(h(x)))' &=& g'(h(x))\cdot h'(x) & & \text{(notation 1)}\\ (g(u))' &=& g'(u)u' & \text{where } u=h(x) & \text{(notation 2)}\\ \dfrac{dy}{dx} &=& \dfrac{dy}{du}\dfrac{du}{dx} & \text{where } y=g(u) & \text{(notation 3)} \end{array} .$$

Example (Chain Rule, Notation 2)

$$\begin{array}{rrcl} \text{Differentiate} & f(x) &=& \sqrt{x^2+1}.\\ \text{Let} & u &=& \textbf{?}\\ \text{Let} & g(u) &=& \textbf{?}\\ \text{Then} & f(x) &=& g(u). \end{array}$$

$$\begin{array}{rcll} (g(h(x)))' & = & g'(h(x)) \cdot h'(x) & & \text{(notation 1)} \\ (g(u))' & = & g'(u)u' & \text{where } u = h(x) & \text{(notation 2)} \\ \frac{dy}{dx} & = & \frac{dy}{du}\frac{du}{dx} & \text{where } y = g(u) & \text{(notation 3)} \end{array}.$$

Example (Chain Rule, Notation 2)

$$\begin{array}{rrcl} \text{Differentiate} & f(x) & = & \sqrt{x^2+1}. \\ \text{Let} & u & = & x^2+1. \\ \text{Let} & g(u) & = & \textbf{?} \\ \text{Then} & f(x) & = & g(u). \end{array}$$

$$
\begin{array}{rcll}
(g(h(x)))' & = & g'(h(x)) \cdot h'(x) & & \text{(notation 1)} \\
(g(u))' & = & g'(u)u' & \text{where } u = h(x) & \text{(notation 2)} \\
\frac{\mathrm{d}y}{\mathrm{d}x} & = & \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} & \text{where } y = g(u) & \text{(notation 3)}
\end{array} .
$$

Example (Chain Rule, Notation 2)

$$
\begin{array}{rrcl}
\text{Differentiate} & f(x) & = & \sqrt{x^2+1}. \\
\text{Let} & u & = & x^2+1. \\
\text{Let} & g(u) & = & \textbf{?} \\
\text{Then} & f(x) & = & g(u).
\end{array}
$$

$$
\begin{array}{rcll}
(g(h(x)))' & = & g'(h(x)) \cdot h'(x) & & \text{(notation 1)} \\
{\color{red}(g(u))'} & {\color{red}=} & {\color{red}g'(u)u'} & \text{where } u = h(x) & \text{(notation 2)} \\
\dfrac{\mathrm{d}y}{\mathrm{d}x} & = & \dfrac{\mathrm{d}y}{\mathrm{d}u}\dfrac{\mathrm{d}u}{\mathrm{d}x} & \text{where } y = g(u) & \text{(notation 3)}
\end{array} .
$$

Example (Chain Rule, Notation 2)

$$
\begin{array}{rrcl}
\text{Differentiate} & f(x) & = & \sqrt{x^2+1}. \\
\text{Let} & u & = & x^2+1. \\
\text{Let} & {\color{red}g(u)} & {\color{red}=} & {\color{red}\sqrt{u}.} \\
\text{Then} & f(x) & = & g(u).
\end{array}
$$

$$
\begin{array}{rcll}
(g(h(x)))' &=& g'(h(x)) \cdot h'(x) & & \text{(notation 1)}\\
(g(u))' &=& g'(u)u' & \text{where } u = h(x) & \text{(notation 2)}\\
\frac{\mathrm{d}y}{\mathrm{d}x} &=& \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} & \text{where } y = g(u) & \text{(notation 3)}
\end{array} .
$$

Example (Chain Rule, Notation 2)

$$
\begin{array}{rrcl}
\text{Differentiate} & f(x) &=& \sqrt{x^2+1}.\\
\text{Let} & u &=& x^2+1.\\
\text{Let} & g(u) &=& \sqrt{u}.\\
\text{Then} & f(x) &=& g(u).\\
\text{Chain Rule:} & f'(x) &=& g'(u)u'
\end{array}
$$

$$\begin{array}{rcll} (g(h(x)))' & = & g'(h(x))\cdot h'(x) & & \text{(notation 1)} \\ (g(u))' & = & g'(u)u' & \text{where } u=h(x) & \text{(notation 2)} \\ \dfrac{dy}{dx} & = & \dfrac{dy}{du}\dfrac{du}{dx} & \text{where } y=g(u) & \text{(notation 3)} \end{array} .$$

Example (Chain Rule, Notation 2)

$$\begin{array}{rrcl} \text{Differentiate} & f(x) & = & \sqrt{x^2+1}. \\ \text{Let} & u & = & x^2+1. \\ \text{Let} & g(u) & = & \sqrt{u}. \\ \text{Then} & f(x) & = & g(u). \\ \text{Chain Rule:} & f'(x) & = & g'(u)u' \\ & & = & \left(\mathbf{?}\quad\right)(\mathbf{?}\ ) \end{array}$$

$$\begin{aligned}
(g(h(x)))' &= g'(h(x)) \cdot h'(x) && && \text{(notation 1)}\\
(g(u))' &= g'(u)u' && \text{where } u = h(x) && \text{(notation 2)}\\
\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} && \text{where } y = g(u) && \text{(notation 3)}
\end{aligned}.$$

Example (Chain Rule, Notation 2)

$$\begin{array}{rrcl}
\text{Differentiate} & f(x) & = & \sqrt{x^2+1}.\\
\text{Let} & u & = & x^2+1.\\
\text{Let} & g(u) & = & \sqrt{u}.\\
\text{Then} & f(x) & = & g(u).\\
\text{Chain Rule:} & f'(x) & = & g'(u)u'\\
 & & = & \left(\frac{1}{2\sqrt{u}}\right)(\textbf{?}\quad)
\end{array}$$

$$\begin{aligned}
(g(h(x)))' &= g'(h(x)) \cdot h'(x) && && \text{(notation 1)}\\
(g(u))' &= g'(u)u' && \text{where } u = h(x) && \text{(notation 2)}\\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} && \text{where } y = g(u) && \text{(notation 3)}
\end{aligned} \quad .$$

Example (Chain Rule, Notation 2)

$$\begin{array}{rrcl}
\text{Differentiate} & f(x) & = & \sqrt{x^2+1}.\\
\text{Let} & u & = & x^2+1.\\
\text{Let} & g(u) & = & \sqrt{u}.\\
\text{Then} & f(x) & = & g(u).\\
\text{Chain Rule:} & f'(x) & = & g'(u)u'\\
 & & = & \left(\frac{1}{2\sqrt{u}}\right) (\textbf{?}\quad)
\end{array}$$

$$
\begin{aligned}
(g(h(x)))' &= g'(h(x)) \cdot h'(x) && && \text{(notation 1)} \\
(g(u))' &= g'(u)u' && \text{where } u = h(x) && \text{(notation 2)} \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} && \text{where } y = g(u) && \text{(notation 3)}
\end{aligned} .
$$

Example (Chain Rule, Notation 2)

$$
\begin{array}{rrcl}
\text{Differentiate} & f(x) & = & \sqrt{x^2+1}. \\
\text{Let} & u & = & x^2+1. \\
\text{Let} & g(u) & = & \sqrt{u}. \\
\text{Then} & f(x) & = & g(u). \\
\text{Chain Rule:} & f'(x) & = & g'(u)u' \\
 & & = & \left(\frac{1}{2\sqrt{u}}\right)(2x)
\end{array}
$$

$$
\begin{aligned}
(g(h(x)))' &= g'(h(x)) \cdot h'(x) && && \text{(notation 1)} \\
(g(u))' &= g'(u)u' && \text{where } u = h(x) && \text{(notation 2)} \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} && \text{where } y = g(u) && \text{(notation 3)}
\end{aligned} .
$$

Example (Chain Rule, Notation 2)

$$
\begin{array}{rrcl}
\text{Differentiate} & f(x) & = & \sqrt{x^2+1}. \\
\text{Let} & u & = & x^2+1. \\
\text{Let} & g(u) & = & \sqrt{u}. \\
\text{Then} & f(x) & = & g(u). \\
\text{Chain Rule:} & f'(x) & = & g'(u)u' \\
 & & = & \left(\frac{1}{2\sqrt{u}}\right)(2x) \\
 & & = & \frac{x}{\sqrt{x^2+1}}.
\end{array}
$$

$$\begin{array}{rcll} (g(h(x)))' &=& g'(h(x))\cdot h'(x) & & \text{(notation 1)}\\ (g(u))' &=& g'(u)u' & \text{where } u = h(x) & \text{(notation 2)}\\ \frac{\mathrm{d}y}{\mathrm{d}x} &=& \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} & \text{where } y = g(u) & \text{(notation 3)} \end{array} .$$

Example (Chain Rule, Notation 3)

Differentiate $y = \sqrt{x^2+1}.$

$$
\begin{array}{rcll}
(g(h(x)))' & = & g'(h(x)) \cdot h'(x) & & \text{(notation 1)} \\
(g(u))' & = & g'(u)u' & \text{where } u = h(x) & \text{(notation 2)} \\
\frac{\mathrm{d}y}{\mathrm{d}x} & = & \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} & \text{where } y = g(u) & \text{(notation 3)} \quad .
\end{array}
$$

Example (Chain Rule, Notation 3)

$$
\begin{array}{rcll}
\text{Differentiate} & y & = & \sqrt{x^2+1}. \\
\text{Let} & u & = & \mathbf{?} \\
\text{Then} & y & = & \mathbf{?}
\end{array}
$$

$$\begin{aligned}
(g(h(x)))' &= g'(h(x)) \cdot h'(x) & & \text{(notation 1)}\\
(g(u))' &= g'(u)u' & \text{where } u = h(x) & \text{(notation 2)}\\
\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} & \text{where } y = g(u) & \text{(notation 3)} \quad .
\end{aligned}$$

Example (Chain Rule, Notation 3)

$$\begin{aligned}
\text{Differentiate} \quad y &= \sqrt{x^2+1}.\\
\text{Let} \quad u &= x^2+1.\\
\text{Then} \quad y &= \textbf{?}
\end{aligned}$$

$$
\begin{array}{rclll}
(g(h(x)))' & = & g'(h(x)) \cdot h'(x) & & \text{(notation 1)} \\
(g(u))' & = & g'(u)u' & \text{where } u = h(x) & \text{(notation 2)} \\
\frac{\mathrm{d}y}{\mathrm{d}x} & = & \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} & \text{where } y = g(u) & \text{(notation 3)}
\end{array} .
$$

Example (Chain Rule, Notation 3)

$$
\begin{array}{rrcl}
\text{Differentiate} & y & = & \sqrt{x^2+1}. \\
\text{Let} & u & = & x^2+1. \\
\text{Then} & y & = & \textbf{?}
\end{array}
$$

$$
\begin{array}{rcll}
(g(h(x)))' & = & g'(h(x)) \cdot h'(x) & & \text{(notation 1)} \\
(g(u))' & = & g'(u)u' & \text{where } u = h(x) & \text{(notation 2)} \\
\frac{\mathrm{d}y}{\mathrm{d}x} & = & \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} & \text{where } y = g(u) & \text{(notation 3)}
\end{array} .
$$

Example (Chain Rule, Notation 3)

$$
\begin{array}{rrcl}
\text{Differentiate} & y & = & \sqrt{x^2+1}. \\
\text{Let} & u & = & x^2+1. \\
\text{Then} & y & = & \sqrt{u}.
\end{array}
$$

$$
\begin{aligned}
(g(h(x)))' &= g'(h(x)) \cdot h'(x) & & \text{(notation 1)} \\
(g(u))' &= g'(u)u' & \text{where } u = h(x) & \text{(notation 2)} \\
\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} & \text{where } y = g(u) & \text{(notation 3)} \quad .
\end{aligned}
$$

Example (Chain Rule, Notation 3)

$$
\begin{aligned}
\text{Differentiate} \quad y &= \sqrt{x^2+1}. \\
\text{Let} \quad u &= x^2+1. \\
\text{Then} \quad y &= \sqrt{u}. \\
\text{Chain Rule:} \quad \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx}
\end{aligned}
$$

$$\begin{array}{rcll} (g(h(x)))' & = & g'(h(x)) \cdot h'(x) & & \text{(notation 1)} \\ (g(u))' & = & g'(u)u' & \text{where } u = h(x) & \text{(notation 2)} \\ \frac{dy}{dx} & = & \frac{dy}{du}\frac{du}{dx} & \text{where } y = g(u) & \text{(notation 3)} \end{array} .$$

Example (Chain Rule, Notation 3)

$$\begin{array}{rrcl} \text{Differentiate} & y & = & \sqrt{x^2+1}. \\ \text{Let} & u & = & x^2+1. \\ \text{Then} & y & = & \sqrt{u}. \\ \text{Chain Rule:} & \frac{dy}{dx} & = & \frac{dy}{du}\frac{du}{dx} \\ & & = & \left(\mathbf{?} \quad \right)(\mathbf{?} \quad) \end{array}$$

$$
\begin{aligned}
(g(h(x)))' &= g'(h(x)) \cdot h'(x) && \text{(notation 1)}\\
(g(u))' &= g'(u)u' \quad \text{where } u = h(x) && \text{(notation 2)}\\
\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \quad \text{where } y = g(u) && \text{(notation 3)} \quad .
\end{aligned}
$$

Example (Chain Rule, Notation 3)

$$
\begin{aligned}
\text{Differentiate} \quad y &= \sqrt{x^2+1}.\\
\text{Let} \quad u &= x^2+1.\\
\text{Then} \quad y &= \sqrt{u}.\\
\text{Chain Rule:} \quad \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx}\\
&= \left(\frac{1}{2\sqrt{u}}\right)(\textbf{?} \quad)
\end{aligned}
$$

$$
\begin{array}{rcll}
(g(h(x)))' & = & g'(h(x)) \cdot h'(x) & & \text{(notation 1)} \\
(g(u))' & = & g'(u)u' & \text{where } u = h(x) & \text{(notation 2)} \\
\frac{\mathrm{d}y}{\mathrm{d}x} & = & \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} & \text{where } y = g(u) & \text{(notation 3)}
\end{array} .
$$

Example (Chain Rule, Notation 3)

$$
\begin{array}{rrcl}
\text{Differentiate} & y & = & \sqrt{x^2+1}. \\
\text{Let} & u & = & x^2+1. \\
\text{Then} & y & = & \sqrt{u}. \\
\text{Chain Rule:} & \frac{\mathrm{d}y}{\mathrm{d}x} & = & \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} \\
& & = & \left(\frac{1}{2\sqrt{u}}\right) (\textbf{?} \quad)
\end{array}
$$

$$\begin{aligned}
(g(h(x)))' &= g'(h(x)) \cdot h'(x) && && \text{(notation 1)}\\
(g(u))' &= g'(u)u' && \text{where } u = h(x) && \text{(notation 2)}\\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} && \text{where } y = g(u) && \text{(notation 3)} \quad .
\end{aligned}$$

Example (Chain Rule, Notation 3)

$$\begin{aligned}
\text{Differentiate} \quad y &= \sqrt{x^2+1}.\\
\text{Let} \quad u &= x^2+1.\\
\text{Then} \quad y &= \sqrt{u}.\\
\text{Chain Rule:} \quad \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}\\
&= \left(\frac{1}{2\sqrt{u}}\right)(2x)
\end{aligned}$$

$$\begin{aligned}
(g(h(x)))' &= g'(h(x)) \cdot h'(x) && && \text{(notation 1)}\\
(g(u))' &= g'(u)u' && \text{where } u = h(x) && \text{(notation 2)}\\
\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} && \text{where } y = g(u) && \text{(notation 3)}
\end{aligned}\quad.$$

Example (Chain Rule, Notation 3)

$$\begin{aligned}
\text{Differentiate} \quad y &= \sqrt{x^2+1}.\\
\text{Let} \quad u &= x^2+1.\\
\text{Then} \quad y &= \sqrt{u}.\\
\text{Chain Rule:} \quad \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx}\\
&= \left(\frac{1}{2\sqrt{u}}\right)(2x)\\
&= \frac{x}{\sqrt{x^2+1}}.
\end{aligned}$$

$$
\begin{array}{rcll}
(g(h(x)))' & = & g'(h(x)) \cdot h'(x) & & \text{(notation 1)} \\
(g(u))' & = & g'(u)u' & \text{where } u = h(x) & \text{(notation 2)} \\
\frac{\mathrm{d}y}{\mathrm{d}x} & = & \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} & \text{where } y = g(u) & \text{(notation 3)}
\end{array} .
$$

Example (Chain Rule, Notation 1, square root of a trigonometric function)

$$\text{Differentiate} \quad f(x) \quad = \quad \sqrt{\sin x + 2}.$$

$$\begin{aligned}
(g(h(x)))' &= g'(h(x)) \cdot h'(x) & & \text{(notation 1)}\\
(g(u))' &= g'(u)u' & \text{where } u = h(x) & \text{(notation 2)}\\
\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} & \text{where } y = g(u) & \text{(notation 3)} \quad .
\end{aligned}$$

Example (Chain Rule, Notation 1, square root of a trigonometric function)

$$\begin{aligned}
\text{Differentiate} \quad f(x) &= \sqrt{\sin x + 2}.\\
\text{Let} \quad h(x) &= \textbf{?}\\
\text{Let} \quad g(u) &= \textbf{?}\\
\text{Then} \quad f(x) &= g(h(x)).
\end{aligned}$$

$$\begin{aligned} (g(h(x)))' &= g'(h(x)) \cdot h'(x) && && \text{(notation 1)} \\ (g(u))' &= g'(u)u' && \text{where } u = h(x) && \text{(notation 2)} \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} && \text{where } y = g(u) && \text{(notation 3)} \end{aligned} .$$

Example (Chain Rule, Notation 1, square root of a trigonometric function)

$$\begin{array}{rrcl} \text{Differentiate} & f(x) & = & \sqrt{\sin x + 2}. \\ \text{Let} & h(x) & = & \sin x + 2. \\ \text{Let} & g(u) & = & \mathbf{?} \\ \text{Then} & f(x) & = & g(h(x)). \end{array}$$

$$
\begin{array}{rcll}
(g(h(x)))' &=& g'(h(x)) \cdot h'(x) & \text{(notation 1)}\\
(g(u))' &=& g'(u)u' \quad \text{where } u = h(x) & \text{(notation 2)}\\
\frac{dy}{dx} &=& \frac{dy}{du}\frac{du}{dx} \quad \text{where } y = g(u) & \text{(notation 3)}
\end{array} .
$$

Example (Chain Rule, Notation 1, square root of a trigonometric function)

$$
\begin{array}{rrcl}
\text{Differentiate} & f(x) &=& \sqrt{\sin x + 2}.\\
\text{Let} & h(x) &=& \sin x + 2.\\
\text{Let} & g(u) &=& \textbf{?}\\
\text{Then} & f(x) &=& g(h(x)).
\end{array}
$$

$$\begin{array}{rcll} (g(h(x)))' &=& g'(h(x))\cdot h'(x) & & \text{(notation 1)}\\ (g(u))' &=& g'(u)u' & \text{where } u=h(x) & \text{(notation 2)}\\ \dfrac{dy}{dx} &=& \dfrac{dy}{du}\dfrac{du}{dx} & \text{where } y=g(u) & \text{(notation 3)} \end{array} .$$

Example (Chain Rule, Notation 1, square root of a trigonometric function)

$$\begin{array}{rrcl} \text{Differentiate} & f(x) &=& \sqrt{\sin x+2}.\\ \text{Let} & h(x) &=& \sin x+2.\\ \text{Let} & g(u) &=& \sqrt{u}.\\ \text{Then} & f(x) &=& g(h(x)). \end{array}$$

$$\begin{array}{rcll} (g(h(x)))' &=& g'(h(x)) \cdot h'(x) & & \text{(notation 1)} \\ (g(u))' &=& g'(u)u' & \text{where } u = h(x) & \text{(notation 2)} \\ \dfrac{\mathrm{d}y}{\mathrm{d}x} &=& \dfrac{\mathrm{d}y}{\mathrm{d}u}\dfrac{\mathrm{d}u}{\mathrm{d}x} & \text{where } y = g(u) & \text{(notation 3)} \end{array} .$$

Example (Chain Rule, Notation 1, square root of a trigonometric function)

$$\begin{array}{rrcl} \text{Differentiate} & f(x) &=& \sqrt{\sin x + 2}. \\ \text{Let} & h(x) &=& \sin x + 2. \\ \text{Let} & g(u) &=& \sqrt{u}. \\ \text{Then} & f(x) &=& g(h(x)). \\ \text{Chain Rule:} & f'(x) &=& g'(h(x))h'(x) \end{array}$$

$$
\begin{aligned}
(g(h(x)))' &= g'(h(x)) \cdot h'(x) && && \text{(notation 1)}\\
(g(u))' &= g'(u)u' && \text{where } u = h(x) && \text{(notation 2)}\\
\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} && \text{where } y = g(u) && \text{(notation 3)}
\end{aligned}
.
$$

Example (Chain Rule, Notation 1, square root of a trigonometric function)

$$
\begin{aligned}
\text{Differentiate} \quad f(x) &= \sqrt{\sin x + 2}.\\
\text{Let} \quad h(x) &= \sin x + 2.\\
\text{Let} \quad g(u) &= \sqrt{u}.\\
\text{Then} \quad f(x) &= g(h(x)).\\
\text{Chain Rule:} \quad f'(x) &= g'(h(x))h'(x)\\
&= \left(\textbf{?} \qquad \right) (\textbf{?} \qquad)
\end{aligned}
$$

$$\begin{array}{rcll} (g(h(x)))' &=& g'(h(x))\cdot h'(x) & & \text{(notation 1)}\\ (g(u))' &=& g'(u)u' & \text{where } u=h(x) & \text{(notation 2)}\\ \frac{\mathrm{d}y}{\mathrm{d}x} &=& \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} & \text{where } y=g(u) & \text{(notation 3)} \end{array}.$$

Example (Chain Rule, Notation 1, square root of a trigonometric function)

$$\begin{array}{rrcl} \text{Differentiate} & f(x) &=& \sqrt{\sin x+2}.\\ \text{Let} & h(x) &=& \sin x+2.\\ \text{Let} & g(u) &=& \sqrt{u}.\\ \text{Then} & f(x) &=& g(h(x)).\\ \text{Chain Rule:} & f'(x) &=& g'(h(x))h'(x)\\ & &=& \left(\frac{1}{2\sqrt{h(x)}}\right)(\textbf{?}\quad) \end{array}$$

$$
\begin{aligned}
(g(h(x)))' &= g'(h(x)) \cdot h'(x) && && \text{(notation 1)}\\
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\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} && \text{where } y = g(u) && \text{(notation 3)} \quad .
\end{aligned}
$$

Example (Chain Rule, Notation 1, square root of a trigonometric function)

$$
\begin{array}{rrcl}
\text{Differentiate} & f(x) & = & \sqrt{\sin x + 2}.\\
\text{Let} & h(x) & = & \sin x + 2.\\
\text{Let} & g(u) & = & \sqrt{u}.\\
\text{Then} & f(x) & = & g(h(x)).\\
\text{Chain Rule:} & f'(x) & = & g'(h(x))h'(x)\\
 & & = & \left(\frac{1}{2\sqrt{h(x)}}\right) (? \quad)
\end{array}
$$

$$
\begin{array}{rcll}
(g(h(x)))' & = & g'(h(x)) \cdot h'(x) & & \text{(notation 1)} \\
(g(u))' & = & g'(u)u' & \text{where } u = h(x) & \text{(notation 2)} \\
\frac{\mathrm{d}y}{\mathrm{d}x} & = & \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} & \text{where } y = g(u) & \text{(notation 3)}
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$$

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$$\begin{array}{rcll} (g(h(x)))' &=& g'(h(x)) \cdot h'(x) & & \text{(notation 1)} \\ (g(u))' &=& g'(u)u' & \text{where } u = h(x) & \text{(notation 2)} \\ \frac{dy}{dx} &=& \frac{dy}{du}\frac{du}{dx} & \text{where } y = g(u) & \text{(notation 3)} \end{array} .$$

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$$
\begin{aligned}
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\end{aligned}
$$

Example (Chain Rule, Notation 2)

Differentiate $f(x) = \cos\left(x^3\right)$.

$$
\begin{array}{rclrr}
(g(h(x)))' & = & g'(h(x)) \cdot h'(x) & & \text{(notation 1)} \\
(g(u))' & = & g'(u)u' & \text{where } u = h(x) & \text{(notation 2)} \\
\dfrac{dy}{dx} & = & \dfrac{dy}{du}\dfrac{du}{dx} & \text{where } y = g(u) & \text{(notation 3)}
\end{array} .
$$

Example (Chain Rule, Notation 2)

$$
\begin{array}{rrcl}
\text{Differentiate} & f(x) & = & \cos\left(x^3\right). \\
\text{Let} & u & = & ? \\
\text{Let} & g(u) & = & ? \\
\text{Then} & f(x) & = & g(u).
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$$

$$\begin{array}{rcll} (g(h(x)))' &=& g'(h(x)) \cdot h'(x) & & \text{(notation 1)} \\ (g(u))' &=& g'(u)u' & \text{where } u = h(x) & \text{(notation 2)} \\ \dfrac{dy}{dx} &=& \dfrac{dy}{du}\dfrac{du}{dx} & \text{where } y = g(u) & \text{(notation 3)} \end{array} .$$

Example (Chain Rule, Notation 2)

$$\begin{array}{rrcl} \text{Differentiate} & f(x) &=& \cos\left(x^3\right). \\ \text{Let} & u &=& x^3. \\ \text{Let} & g(u) &=& \textbf{?} \\ \text{Then} & f(x) &=& g(u). \end{array}$$

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\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} && \text{where } y = g(u) && \text{(notation 3)}
\end{aligned} .$$

Example (Chain Rule, Notation 2)

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\end{array}$$

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Example (Chain Rule, Notation 2)

$$\begin{aligned} \text{Differentiate} \quad f(x) &= \cos\left(x^3\right). \\ \text{Let} \quad u &= x^3. \\ \text{Let} \quad g(u) &= \cos u. \\ \text{Then} \quad f(x) &= g(u). \\ \text{Chain Rule:} \quad f'(x) &= g'(u)u' \\ &= (\textbf{?} \qquad) \left(\textbf{?} \quad\right) \end{aligned}$$

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\begin{array}{rcll}
(g(h(x)))' & = & g'(h(x)) \cdot h'(x) & & \text{(notation 1)} \\
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\end{array} .
$$

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\text{Chain Rule:} & f'(x) & = & g'(u)u' \\
 & & = & (-\sin u)\left(\textbf{?} \quad \right)
\end{array}
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\text{Then} & f(x) & = & g(u). \\
\text{Chain Rule:} & f'(x) & = & g'(u)u' \\
 & & = & (-\sin u)\left(3x^2\right)
\end{array}
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Example (Chain Rule, Notation 2)

Differentiate $f(x) = \cos^3 x$.

$$\begin{array}{rcll} (g(h(x)))' &=& g'(h(x))\cdot h'(x) & & \text{(notation 1)}\\ {\color{red}(g(u))'} &{\color{red}=}& {\color{red}g'(u)u'} & \text{where } u=h(x) & \text{(notation 2)}\\ \dfrac{dy}{dx} &=& \dfrac{dy}{du}\dfrac{du}{dx} & \text{where } y=g(u) & \text{(notation 3)} \end{array} .$$

Example (Chain Rule, Notation 2)

$$\begin{array}{rrcl} \text{Differentiate} & f(x) &=& \cos^3 x.\\ \text{Let} & {\color{red}u} &{\color{red}=}& {\color{red}\textbf{?}}\\ \text{Let} & g(u) &=& \textbf{?}\\ \text{Then} & f(x) &=& g(u). \end{array}$$

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\text{Then} & f(x) &=& g(u). \\
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& &=& \left(\textbf{?} \quad \right) (\textbf{?} \qquad)
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\begin{array}{rrcl}
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\text{Let} & g(u) & = & u^3. \\
\text{Then} & f(x) & = & g(u). \\
\text{Chain Rule:} & f'(x) & = & g'(u){\color{red}u'} \\
 & & = & \left(3u^2\right) {\color{red}(\mathbf{?} \qquad)}
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\begin{array}{rcll}
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\end{array}
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$$\begin{array}{rcll} (g(h(x)))' &=& g'(h(x)) \cdot h'(x) & & \text{(notation 1)} \\ (g(u))' &=& g'(u)u' & \text{where } u = h(x) & \text{(notation 2)} \\ \dfrac{\mathrm{d}y}{\mathrm{d}x} &=& \dfrac{\mathrm{d}y}{\mathrm{d}u}\dfrac{\mathrm{d}u}{\mathrm{d}x} & \text{where } y = g(u) & \text{(notation 3)} \end{array} .$$

Example (Chain Rule, Notation 2)

$$\begin{array}{rrcl} \text{Differentiate} & f(x) &=& \cos^3 x. \\ \text{Let} & u &=& \cos x. \\ \text{Let} & g(u) &=& u^3. \\ \text{Then} & f(x) &=& g(u). \\ \text{Chain Rule:} & f'(x) &=& g'(u)u' \\ & &=& \left(3u^2\right)(-\sin x) \\ & &=& -3\sin x(\cos x)^2. \end{array}$$

- In the example $y = \cos^3 x$, the outer function was a power function: $y = u^3$.

- In the example $y = \cos^3 x$, the outer function was a power function: $y = u^3$.
- The derivative was $\frac{\mathrm{d}y}{\mathrm{d}x} = 3u^2 \frac{\mathrm{d}u}{\mathrm{d}x} = (3\cos^2 x)(-\sin x)$.

- In the example $y = \cos^3 x$, the outer function was a power function: $y = u^3$.
- The derivative was $\frac{dy}{dx} = 3u^2 \frac{du}{dx} = (3\cos^2 x)(-\sin x)$.
- We can generalize this:

Observation (The Power Rule Combined with the Chain Rule)

If n is any real number and $u = h(x)$ is differentiable, then

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$$

Alternatively,

$$\frac{d}{dx}[h(x)]^n = n[h(x)]^{n-1} \cdot h'(x)$$

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$$

$$\begin{array}{rcll} (g(h(x)))' &=& g'(h(x)) \cdot h'(x) & & \text{(notation 1)} \\ (g(u))' &=& g'(u)u' & \text{where } u = h(x) & \text{(notation 2)} \\ \frac{dy}{dx} &=& \frac{dy}{du}\frac{du}{dx} & \text{where } y = g(u) & \text{(notation 3)} \end{array} .$$

Example (Chain Rule, Notation 3, Power Rule)

Differentiate $y = (x^3 - 1)^{100}$.

$$\frac{\mathrm{d}}{\mathrm{d}x}(u^n) = nu^{n-1}\frac{\mathrm{d}u}{\mathrm{d}x}$$

$$\begin{array}{rcll}
(g(h(x)))' & = & g'(h(x)) \cdot h'(x) & & \text{(notation 1)} \\
(g(u))' & = & g'(u)u' & \text{where } u = h(x) & \text{(notation 2)} \\
\frac{\mathrm{d}y}{\mathrm{d}x} & = & \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} & \text{where } y = g(u) & \text{(notation 3)}
\end{array} .$$

Example (Chain Rule, Notation 3, Power Rule)

$$\begin{array}{rrcl}
\text{Differentiate} & y & = & (x^3 - 1)^{100}. \\
\text{Let} & u & = & \mathbf{?} \\
\text{Then} & y & = & \mathbf{?}
\end{array}$$

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$$

$$\begin{array}{rcll}
(g(h(x)))' & = & g'(h(x)) \cdot h'(x) & & \text{(notation 1)} \\
(g(u))' & = & g'(u)u' & \text{where } u = h(x) & \text{(notation 2)} \\
\frac{dy}{dx} & = & \frac{dy}{du}\frac{du}{dx} & \text{where } y = g(u) & \text{(notation 3)}
\end{array}.$$

Example (Chain Rule, Notation 3, Power Rule)

$$\begin{array}{rrcl}
\text{Differentiate} & y & = & (x^3 - 1)^{100}. \\
\text{Let} & u & = & x^3 - 1. \\
\text{Then} & y & = & \textbf{?}
\end{array}$$

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$$

$$\begin{array}{rcll} (g(h(x)))' &=& g'(h(x)) \cdot h'(x) & \text{(notation 1)} \\ (g(u))' &=& g'(u)u' \quad \text{where } u = h(x) & \text{(notation 2)} \\ \frac{dy}{dx} &=& \frac{dy}{du}\frac{du}{dx} \quad \text{where } y = g(u) & \text{(notation 3)} \end{array}.$$

Example (Chain Rule, Notation 3, Power Rule)

$$\begin{array}{rrcl} \text{Differentiate} & y &=& (x^3-1)^{100}. \\ \text{Let} & u &=& x^3-1. \\ \text{Then} & y &=& \textbf{?} \end{array}$$

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Differentiate $f(x) = \frac{1}{\sqrt[3]{x^2+x+1}}$.

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Example

Find the derivative of $y = (2x + 1)^5 (x^3 - x + 1)^4$.

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Find the derivative of $y = (2x+1)^5(x^3-x+1)^4$.

Product Rule:

$$y' = \frac{d}{dx}\left((2x+1)^5\right)(x^3-x+1)^4 + (2x+1)^5\frac{d}{dx}\left((x^3-x+1)^4\right)$$

Chain Rule:

$$= \left(\textbf{?} \qquad\qquad\qquad\qquad \right)(x^3-x+1)^4$$

$$+ (2x+1)^5\left(\textbf{?} \qquad\qquad\qquad\qquad \right)$$

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$$= \left(5(2x+1)^4\frac{d}{dx}(2x+1)\right)(x^3-x+1)^4$$
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$$\begin{aligned} &= \left(5(2x+1)^4\frac{\mathrm{d}}{\mathrm{d}x}(2x+1)\right)(x^3-x+1)^4 \\ &\quad + (2x+1)^5\left(4(x^3-x+1)^3\frac{\mathrm{d}}{\mathrm{d}x}(x^3-x+1)\right) \\ &= 5(2x+1)^4\,(\mathbf{?})\,(x^3-x+1)^4 + 4(2x+1)^5(x^3-x+1)^3(3x^2-1) \end{aligned}$$

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Common factor $2(2x+1)^4(x^3-x+1)^3$:

$$= 2(2x+1)^4(x^3-x+1)^3(2(2x+1)(3x^2-1)+5(x^3-x+1)))$$

Example

Find the derivative of $y=(2x+1)^5(x^3-x+1)^4$.

Product Rule:

$$y' = \frac{d}{dx}\left((2x+1)^5\right)(x^3-x+1)^4+(2x+1)^5\frac{d}{dx}\left((x^3-x+1)^4\right)$$

Chain Rule:

$$= \left(5(2x+1)^4\frac{d}{dx}(2x+1)\right)(x^3-x+1)^4 + (2x+1)^5\left(4(x^3-x+1)^3\frac{d}{dx}(x^3-x+1)\right)$$

$$= 5(2x+1)^4(2)(x^3-x+1)^4+4(2x+1)^5(x^3-x+1)^3(3x^2-1)$$

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$$= 5(2x+1)^4(2)(x^3-x+1)^4 + 4(2x+1)^5(x^3-x+1)^3(3x^2-1)$$

Common factor $2(2x+1)^4(x^3-x+1)^3$:

$$= 2(2x+1)^4(x^3-x+1)^3(2(2x+1)(3x^2-1)+5(x^3-x+1)))$$

$$= 2(2x+1)^4(x^3-x+1)^3(17x^3+6x^2-9x+3)$$

Example (Chain Rule, general exponential function)

Differentiate $y = 2^x$.

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Let $u = x \ln 2.$

Then $y = e^u.$

Example (Chain Rule, general exponential function)

$$\begin{aligned}
\text{Differentiate} \quad y &= 2^x. \\
y &= \left(e^{\ln 2}\right)^x \\
y &= e^{x \ln 2}. \\
\text{Let} \quad u &= x \ln 2. \\
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\end{aligned}$$

$$\text{Chain Rule:} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}$$

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\text{Then} \quad y &= e^u. \\
\text{Chain Rule:} \quad \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} \\
&= (\mathbf{?} \quad)(\mathbf{?} \qquad)
\end{aligned}
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Chain Rule: $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}$

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Differentiate $y = 2^x$.

$$y = \left(e^{\ln 2}\right)^x$$

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Let $u = x \ln 2$.

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Chain Rule: $$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

$$= (e^u)(\ln 2)$$

$$= \left(e^{(\mathbf{?} \quad)}\right)(\ln 2)$$

Example (Chain Rule, general exponential function)

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\text{Differentiate}\quad y &= 2^x.\\
y &= \left(e^{\ln 2}\right)^x\\
y &= e^{x\ln 2}.\\
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\text{Then}\quad y &= e^u.\\
\text{Chain Rule:}\quad \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx}\\
&= (e^u)(\ln 2)\\
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Example (Chain Rule, general exponential function)

$$\begin{aligned}
\text{Differentiate} \quad y &= a^x. \\
y &= \left(e^{\ln a}\right)^x \\
y &= e^{x \ln a}. \\
\text{Let} \quad u &= x \ln a. \\
\text{Then} \quad y &= e^u.
\end{aligned}$$

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\text{Chain Rule:} \quad \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} \\
&= (e^u)(\ln a) \\
&= \left(e^{(x \ln a)}\right)(\ln a) \\
&= \left(e^{\ln a}\right)^x (\ln a) \\
&= a^x \ln a.
\end{aligned}$$

Theorem (The Derivative of a^x)

$$\frac{\mathrm{d}}{\mathrm{d}x}(a^x) = a^x \ln a.$$

- We can add more “links” when we use the Chain Rule.

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Example (Using the Chain Rule twice)

Differentiate: $y = \sin \sqrt{10^x + 1}$.

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Differentiate: $y = \sin\sqrt{10^x + 1}$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\sin\sqrt{10^x + 1}\right)$$

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Differentiate: $y = \sin\sqrt{10^x + 1}$.

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Chain Rule: $$= \left(\textbf{?} \qquad\qquad\qquad\right)\frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{10^x+1}\right)$$

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Differentiate: $y = \sin\sqrt{10^x + 1}$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\sin\sqrt{10^x+1}\right)$$

Chain Rule: $$= \left(\cos\sqrt{10^x+1}\right)\frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{10^x+1}\right)$$

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Chain Rule: $$= \left(\cos\sqrt{10^x + 1}\right)\frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{10^x + 1}\right)$$

Chain Rule: $$= \left(\cos\sqrt{10^x + 1}\right)\left(\frac{1}{2\sqrt{10^x + 1}}\right)\frac{\mathrm{d}}{\mathrm{d}x}\left(10^x + 1\right)$$

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Differentiate: $y = \sin\sqrt{10^x+1}$.

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$$= \left(\cos\sqrt{10^x+1}\right)\left(\frac{1}{2\sqrt{10^x+1}}\right)(10^x\ln 10)$$

$$= \frac{(\ln 10)10^x\cos\sqrt{10^x+1}}{2\sqrt{10^x+1}}.$$

Example (Using the Chain Rule twice)

Differentiate: $y = e^{\tan(\pi x)}$.

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$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{\tan(\pi x)}\right)$$

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Differentiate: $y = e^{\tan(\pi x)}$.

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Differentiate: $y = e^{\tan(\pi x)}$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{\tan(\pi x)}\right)$$

Chain Rule: $$= \left(e^{\tan(\pi x)}\right)\frac{\mathrm{d}}{\mathrm{d}x}\left(\tan\left(\pi x\right)\right)$$

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Differentiate: $y = e^{\tan(\pi x)}$.

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{\tan(\pi x)}\right)$$

Chain Rule: $$= \left(e^{\tan(\pi x)}\right) \frac{d}{dx}\left(\tan(\pi x)\right)$$

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Differentiate: $y = e^{\tan(\pi x)}$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{\tan(\pi x)}\right)$$

Chain Rule: $= \left(e^{\tan(\pi x)}\right) \frac{\mathrm{d}}{\mathrm{d}x}\left(\tan(\pi x)\right)$

Chain Rule: $= \left(e^{\tan(\pi x)}\right) \left(\sec^2(\pi x)\right) \frac{\mathrm{d}}{\mathrm{d}x}(\pi x)$

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Differentiate: $y = e^{\tan(\pi x)}$.

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Chain Rule: $$= \left(e^{\tan(\pi x)}\right)\left(\sec^2(\pi x)\right)\frac{\mathrm{d}}{\mathrm{d}x}(\pi x)$$

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Chain Rule: $$= \left(e^{\tan(\pi x)}\right)\left(\sec^2(\pi x)\right)\frac{\mathrm{d}}{\mathrm{d}x}(\pi x)$$

$$= \left(e^{\tan(\pi x)}\right)\left(\sec^2(\pi x)\right)(\pi)$$

$$= \pi e^{\tan(\pi x)}\sec^2(\pi x).$$

Theorem (Chain rule)

Let g-differentiable at neighborhood of a, f-diff. at neighb. of $g(a)$.

$$(f(g(x)))'_{|x=a} = f'(g(a))g'(a)$$

Proof with additional assumptions -motivation for actual proof.

□

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g-diff. near a, f-diff. near $g(a)$ $\Rightarrow (f(g(a)))' = f'(g(a))g'(a)$.

Proof.

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g-diff. near a, f-diff. near $g(a) \Rightarrow (f(g(a)))' = f'(g(a))g'(a)$.

Proof.

Define $Q(y) = \begin{cases} \frac{f(y)-f(g(a))}{y-g(a)}, & y \neq g(a) \\ f'(g(a)), & y = g(a) \end{cases}$. $Q(g(x))$ - defined for all x near a. Therefore $f'(g(a)) = \lim_{y\to a} Q(y) = \lim_{x\to a} Q(g(x))$.

$$\begin{aligned} Q(g(x))\frac{g(x)-g(a)}{x-a} &= \begin{cases} \frac{(f(g(x))-f(g(a)))}{(g(x)-g(a))}\frac{(g(x)-g(a))}{x-a}, & g(x) \neq g(a) \\ f'(g(a))\frac{g(a)-g(a)}{x-a} = 0, & g(x) = g(a) \end{cases} \\ &= \frac{f(g(x))-f(a)}{x-a}. \end{aligned}$$

$$\begin{aligned} (f\circ g)'(a) &= \lim_{x\to a} \frac{f(g(x)) - f(g(a))}{x-a} = \lim_{x\to a} Q(g(x))\frac{g(x)-g(a)}{x-a} \\ &= \lim_{x\to a} Q(g(x)) \lim_{x\to a} \frac{g(x)-g(a)}{x-a} = f'(g(a))g'(a) \quad . \end{aligned}$$

$\square$

Theorem (Chain rule)

g-diff. near a, f-diff. near $g(a)$ $\Rightarrow (f(g(a)))' = f'(g(a))g'(a)$.

Proof.

Define $Q(y) = \begin{cases} \frac{f(y)-f(g(a))}{y-g(a)}, & y \neq g(a) \\ f'(g(a)), & y = g(a) \end{cases}$. $Q(g(x))$ - defined for all x near a. Therefore $f'(g(a)) = \lim_{y\to a} Q(y) = \lim_{x\to a} Q(g(x))$.

$$\begin{aligned} Q(g(x))\frac{g(x)-g(a)}{x-a} &= \begin{cases} \frac{(f(g(x))-f(g(a)))}{(g(x)-g(a))}\frac{(g(x)-g(a))}{x-a}, & g(x) \neq g(a) \\ f'(g(a))\frac{g(a)-g(a)}{x-a} = 0, & g(x) = g(a) \end{cases} \\ &= \frac{f(g(x))-f(a)}{x-a}. \end{aligned}$$

$$\begin{aligned} (f\circ g)'(a) &= \lim_{x\to a}\frac{f(g(x))-f(g(a))}{x-a} = \lim_{x\to a} Q(g(x))\frac{g(x)-g(a)}{x-a} \\ &= \lim_{x\to a} Q(g(x)) \lim_{x\to a}\frac{g(x)-g(a)}{x-a} = f'(g(a))g'(a) \quad . \end{aligned}$$

□