# Calculus II
## Lecture (not covered in class)

Todor Milev
https://github.com/tmilev/freecalc

2020

# Outline

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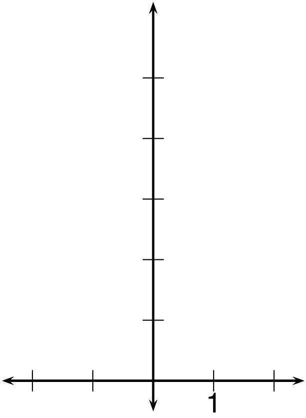
# Exponential Functions

The function $f(x) = 2^x$ is called an exponential function because the variable $x$ is the exponent.

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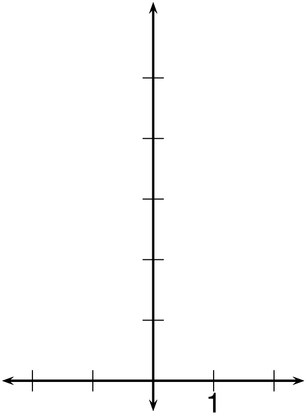


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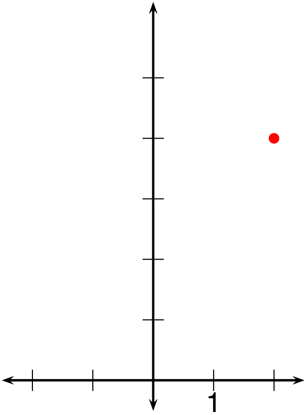


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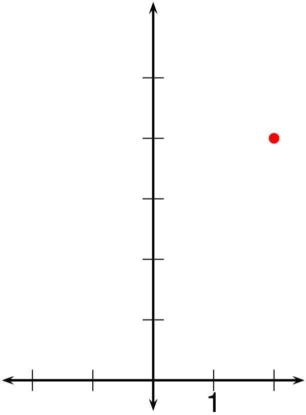


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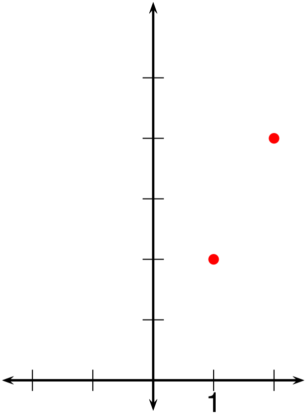


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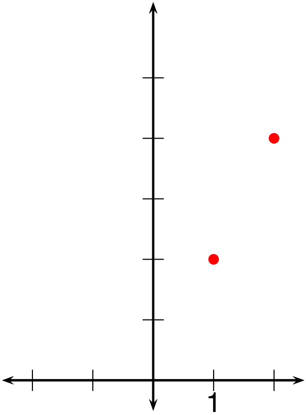


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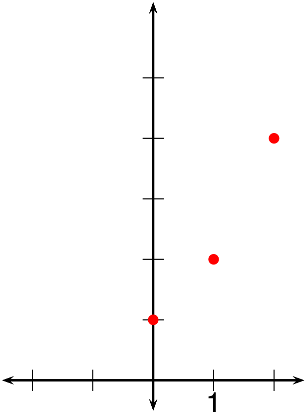


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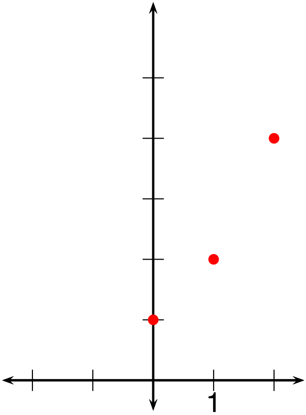


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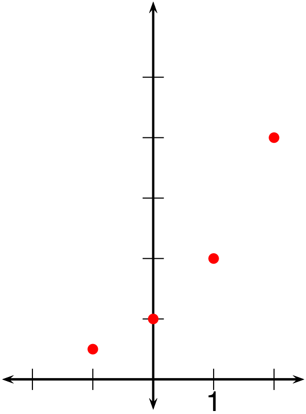


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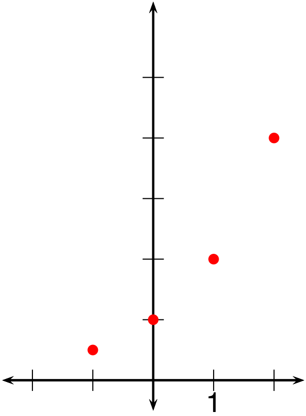


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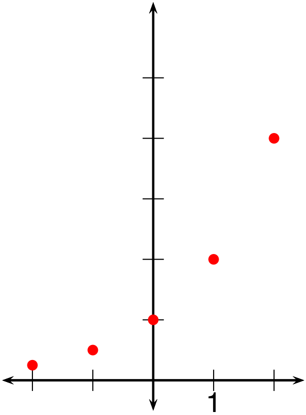


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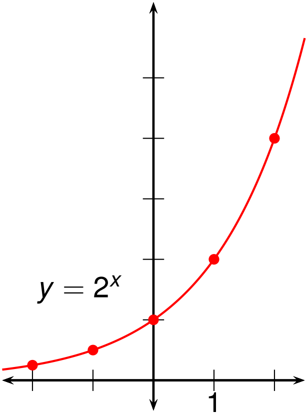


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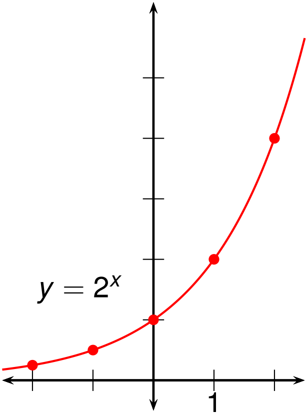


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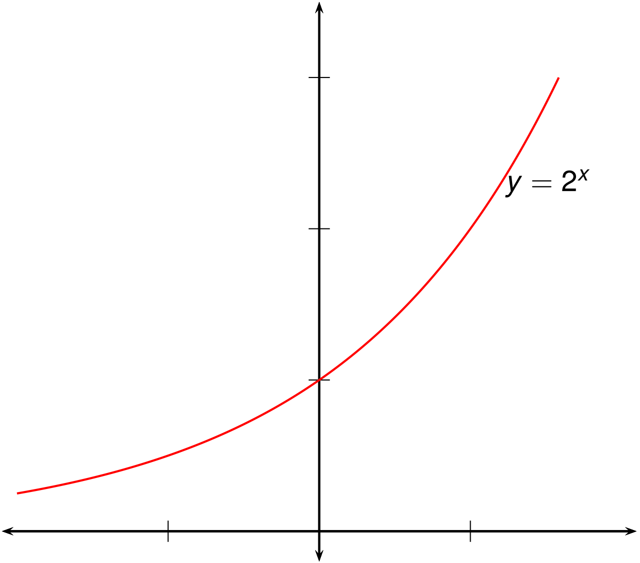
**(Exponential Function Terminology)**

*An exponential function is a function of the form $f(x) = a^x$, where $a$ is a positive constant.*

# Graphs of various exponential functions.



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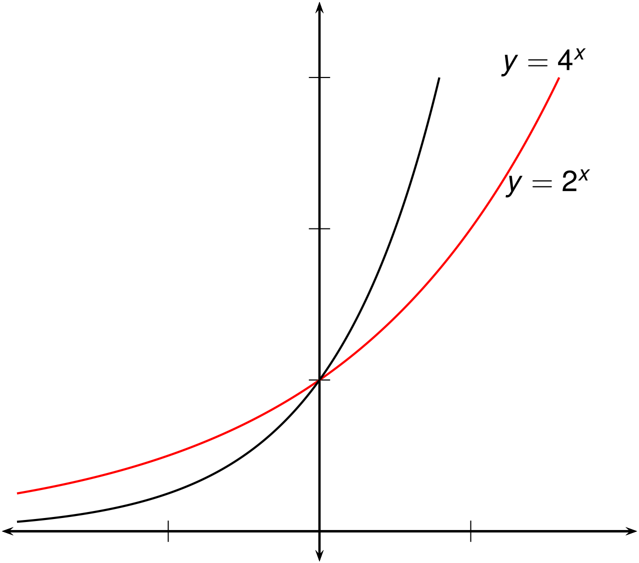

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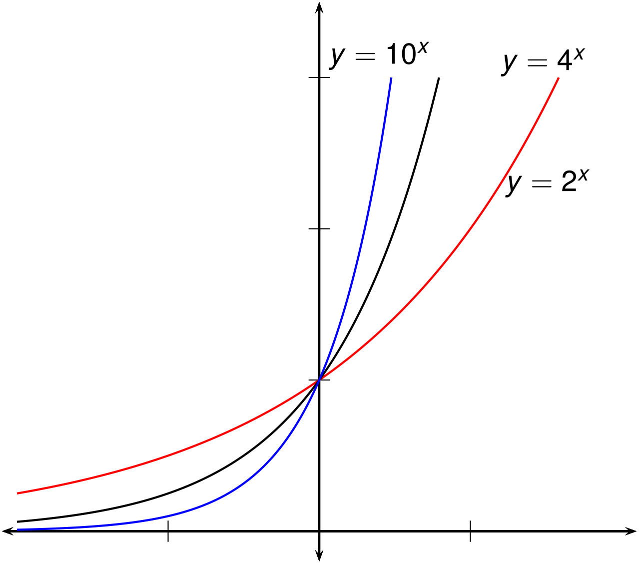

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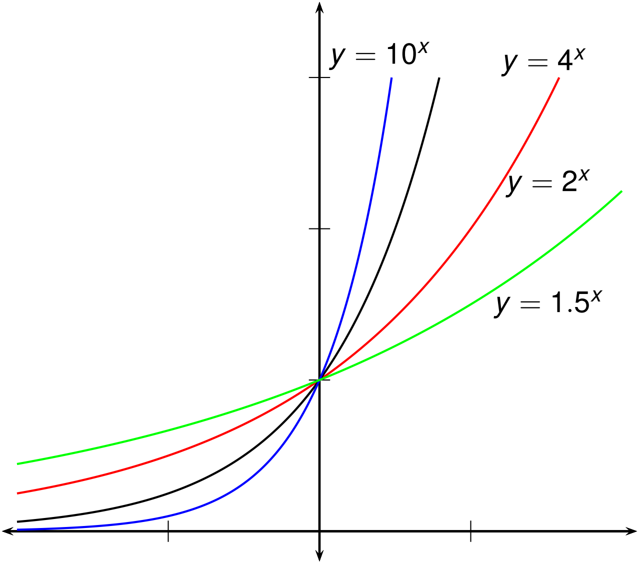

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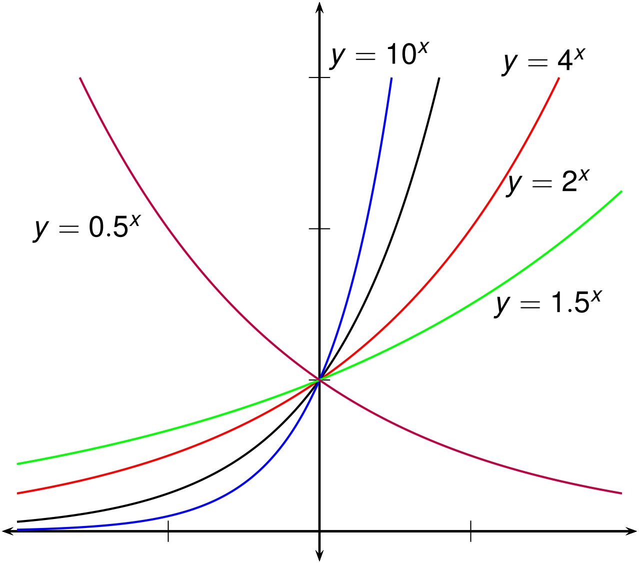

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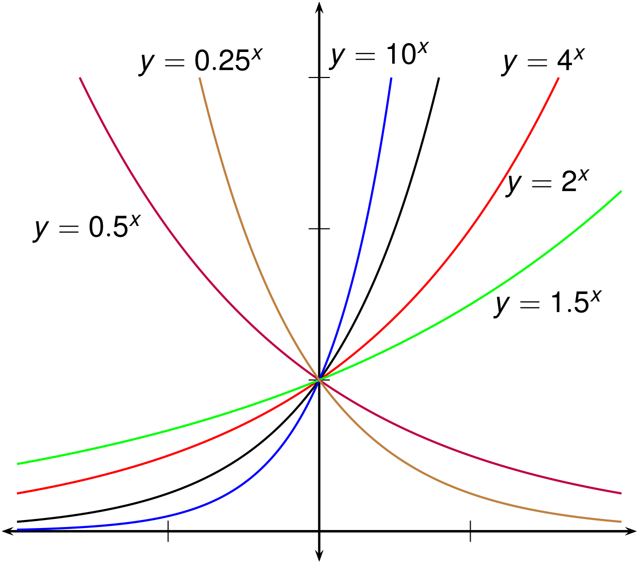

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&= a^x f'(0).
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We leave the following theorem without proof.

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*Let a be a positive number and let $f(x) = a^x$. Then the limit*

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*exists.*

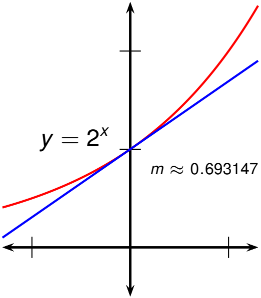
We will later show that

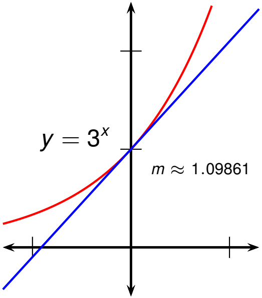
$$f'(0) = \lim_{h \to 0} \frac{a^h - 1}{h} = \ln(a).$$

Here, $\ln$ is the natural logarithm function.

# The Natural Exponential Function

- One base for an exponential function is especially useful.



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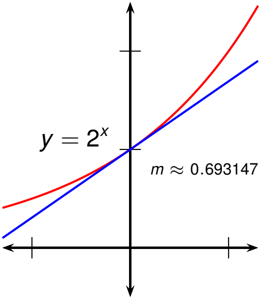
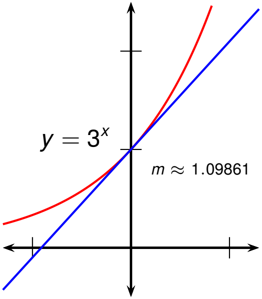
- It has a special property: its tangent line at $x = 0$ has slope $m = 1$.



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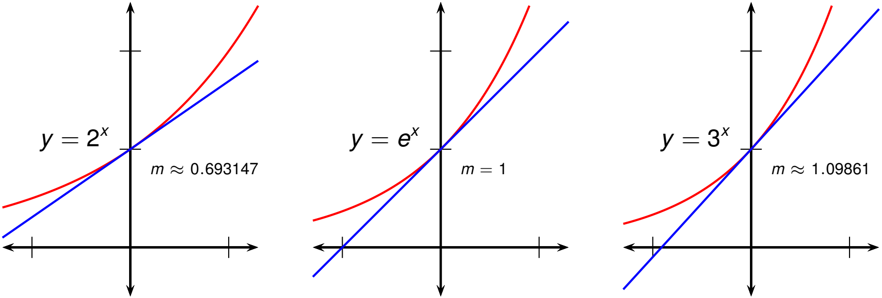
- We call this number $e$, known as Euler's number or Napier's constant.



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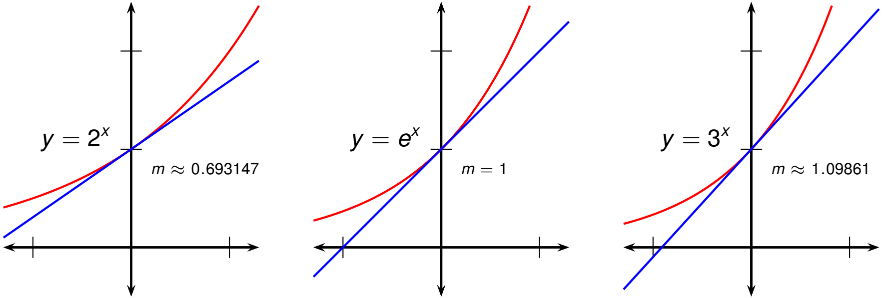
- $e$ is a number between 2 and 3.



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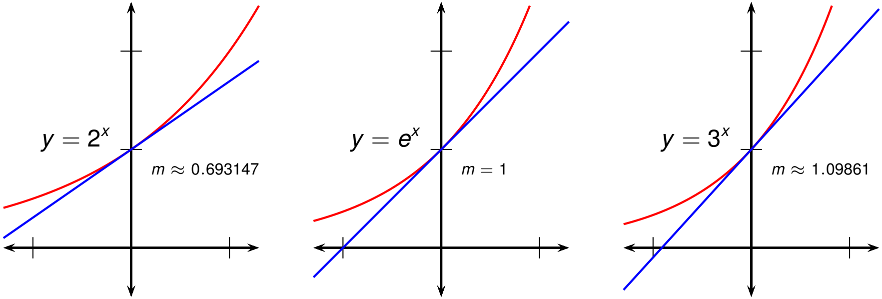
- $e$ is a number between 2 and 3.
- In fact, $e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots \approx 2.71828$.



**Definition (Natural Exponential Function)**

$e^x$ is called the natural exponential function. Its derivative is

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(e^x\right) = e^x.$$

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- Cons: not computationally effective; not how computers compute.

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- Cons: not computationally effective; not how computers compute.
- Pros: for non-integer $x$ and $y$, it is very easy to prove that $a^{x+y} = a^x a^y$ - this follows from the definition of limit above.

# Exponent definition using limits (approach I)

- For integer $p$ we know to compute $a^p$.
- Therefore for integer $q$ we know to compute $a^{\frac{1}{q}} = \sqrt[q]{a} = \max\{x |$ for which $x^q \leq a\}$.
- Therefore we know to compute $a^{\frac{p}{q}}$ for all rational $\frac{p}{q}$.
- We can then define
$$a^x = \lim_{\substack{y \to x \\ y\text{-rational}}} a^y$$
For example, $a^\pi$ would be defined as the limit of the sequence $a^{3.14}, a^{3.141}, a^{3.1415}, \dots$.
- Cons: not computationally effective; not how computers compute.
- Pros: for non-integer $x$ and $y$, it is very easy to prove that $a^{x+y} = a^x a^y$ - this follows from the definition of limit above.
- This is the definition assumed in many elementary courses.

# Exponent definition using series (approach II)

- The following formula (studied much later) can be used as alternative definition.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots$$

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- For $|x| < 1$ define

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + \frac{(-1)^{n+1} x^n}{n} + \dots$$

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- Cons: more difficult to prove $e^{x+y} = e^x e^y$ and $e^{\ln(1+x)} = 1 + x$, proof done later.
- Pros: this is how $e^x$ and $a^x$ are actually computed (by modern computers and by humans in the past).

## Example

Derive the exponent rule $(e^x)' = e^x$

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$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots,$$

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$$(e^x)' = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right)'$$

## Example

Derive the exponent rule $(e^x)' = e^x$ using the Calc II formula below, the infinite (both sides uniformly convergent) sum rule
$(f_1 + f_2 + f_3 + \dots)' = f_1' + f_2' + f_3' + \dots$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots,$$

where $n! = 1 \cdot 2 \cdot 3 \cdot \dots \cdot n$.

$$\begin{aligned} (e^x)' &= \left(1 + x + \tfrac{x^2}{2!} + \tfrac{x^3}{3!} + \dots\right)' \\ &= (1)' + (x)' + \frac{\left(x^2\right)'}{2!} + \frac{\left(x^3\right)'}{3!} + \dots + \frac{(x^n)'}{n!} + \dots \end{aligned}$$

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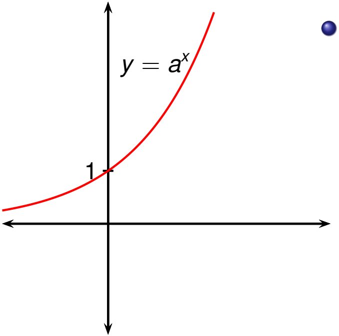
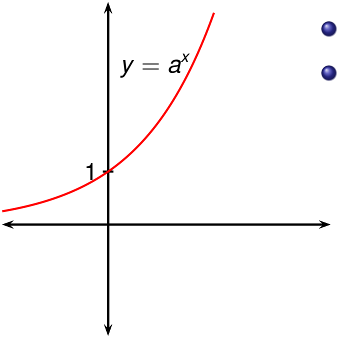
as desired.

# Logarithmic Functions




- Suppose $a > 0$, $a \neq 1$.

# Logarithmic Functions




- Suppose $a > 0$, $a \neq 1$.
- Let $f(x) = a^x$.

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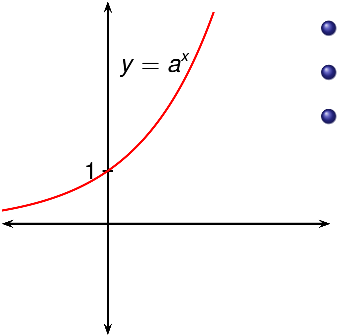


- Suppose $a > 0$, $a \neq 1$.
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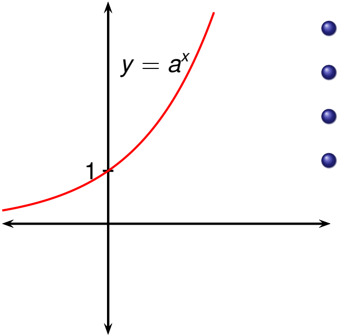


- Suppose $a > 0$, $a \neq 1$.
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- Therefore $f$ is one-to-one.

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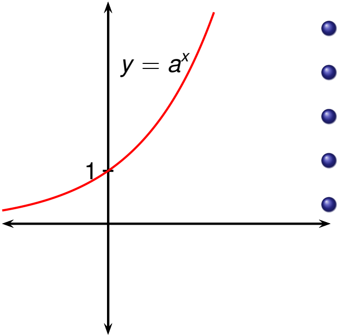


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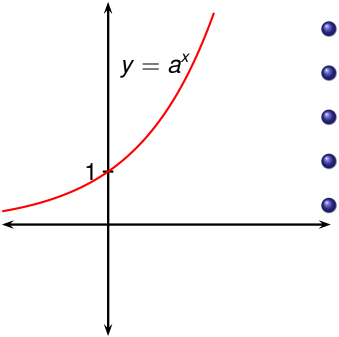


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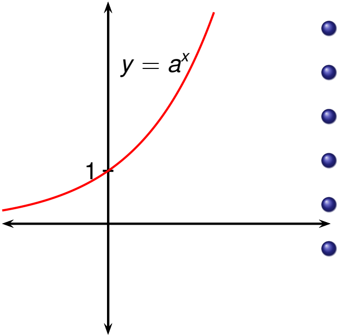
## Definition ($\log_a x$)

The inverse function of $f(x) = a^x$ is called the logarithmic function with base $a$, and is written $\log_a x$. It is defined by the formula

$$\log_a x = y \qquad \Leftrightarrow \qquad a^y = x.$$

# Logarithmic Functions

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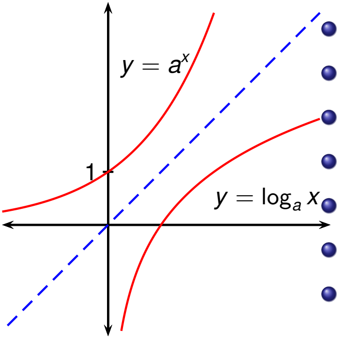


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- The graph of $y = \log_a x$ is the reflection of this in the line $y = x$.

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If $x > 0$, then $\log_a x$ is the exponent to which the base $a$ must be raised to give $x$.

## Example

Evaluate:

1. $\log_3 81 =$
2. $\log_{25} 5 =$
3. $\log_{10} 0.001 =$

If $x > 0$, then $\log_a x$ is the exponent to which the base $a$ must be raised to give $x$.

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## Example

Evaluate:

1. $\log_3 81 = 4$ because $3^4 = 81$.
2. $\log_{25} 5 = \textbf{?}$
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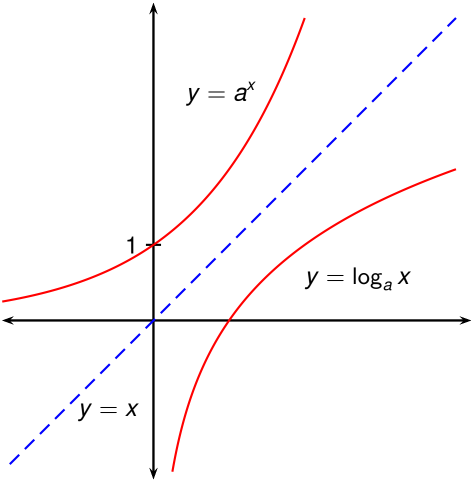


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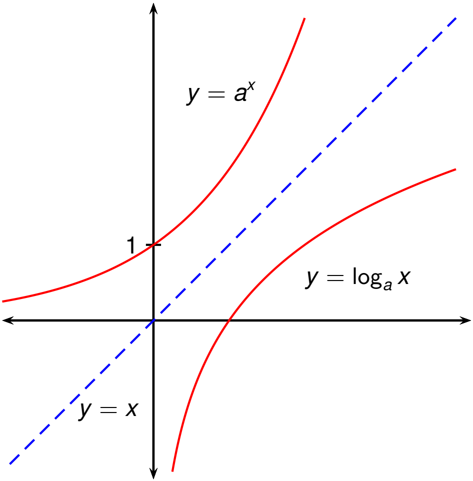


- Suppose $a > 1$.
- Domain of $a^x$: **?**
- Range of $a^x$: **?**
- Domain of $\log_a x$: **?**
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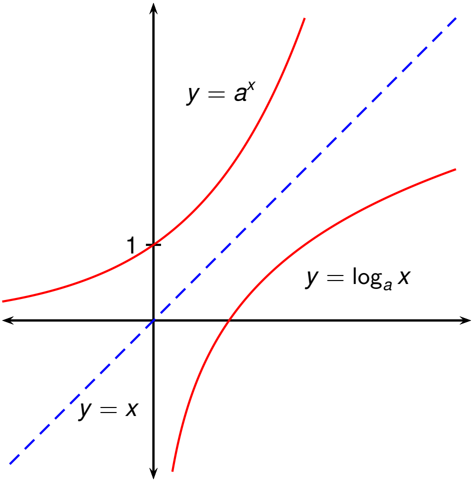


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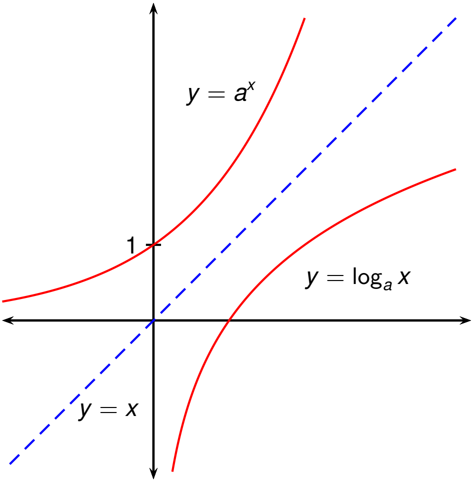


- Suppose $a > 1$.
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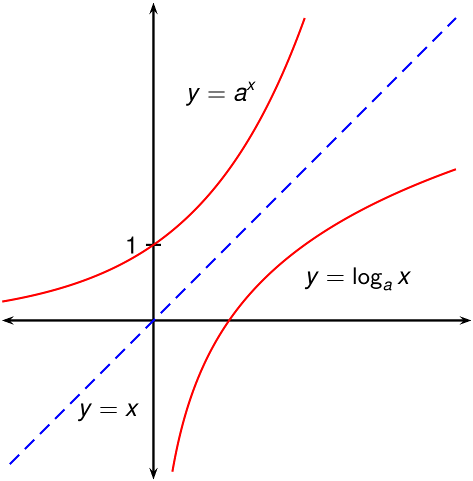


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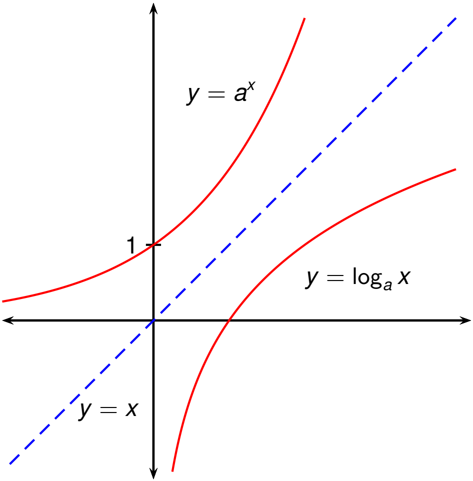


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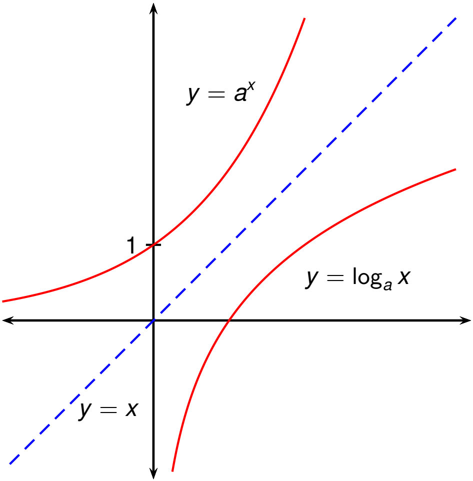


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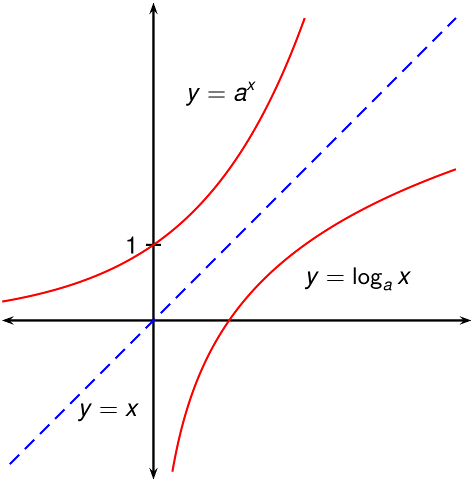


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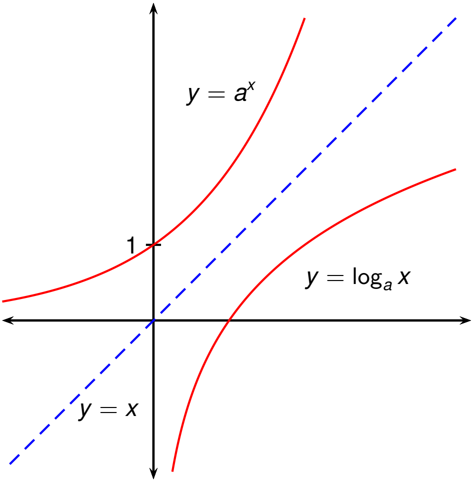


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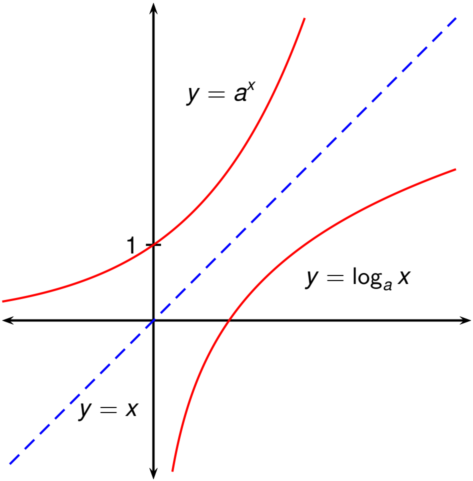


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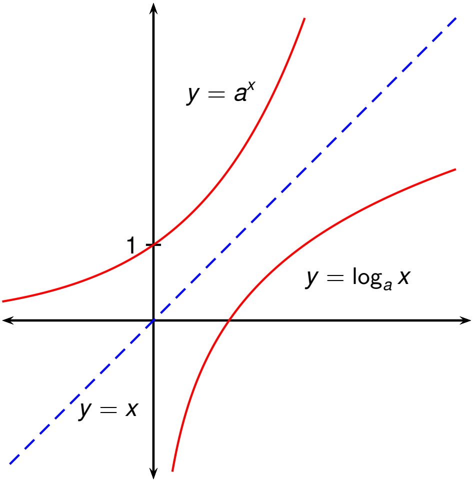


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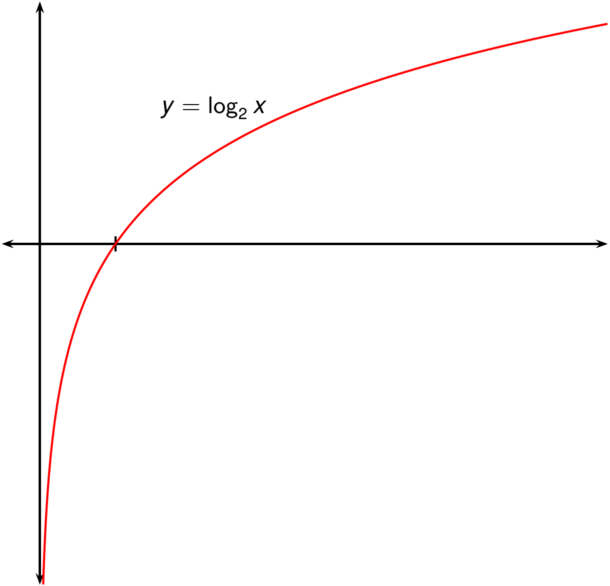
# Graphs of various logarithmic functions with $a > 1$



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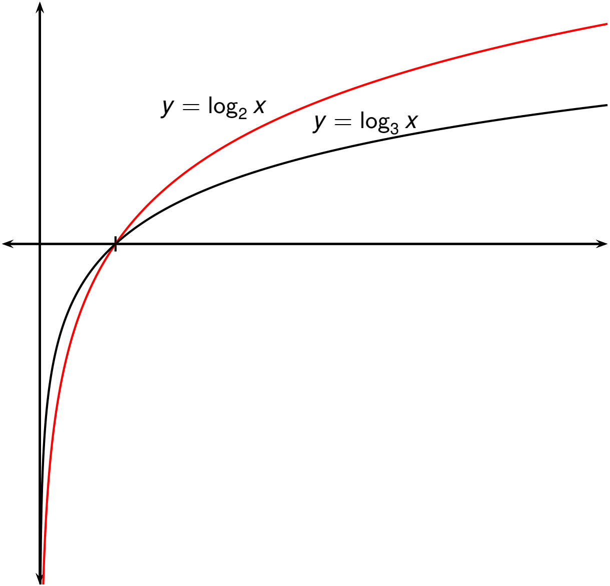

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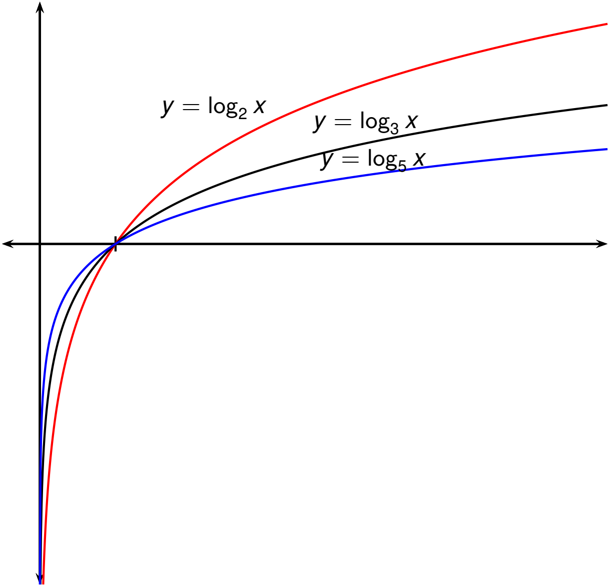

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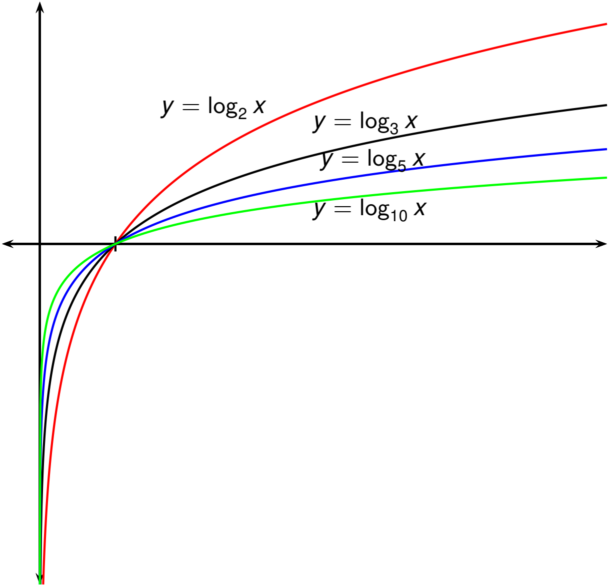
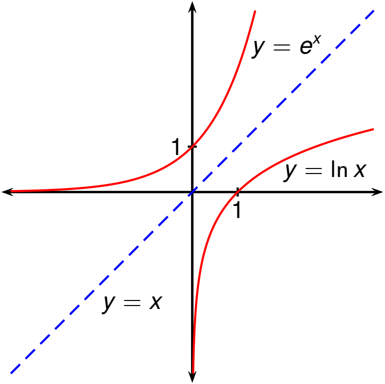


# Natural Logarithms

## Definition ($\ln x$)

The logarithm with base $e$ is called the natural logarithm, and has a special notation:

$$\log_e x = \ln x.$$



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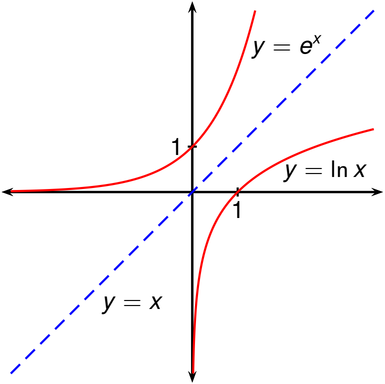


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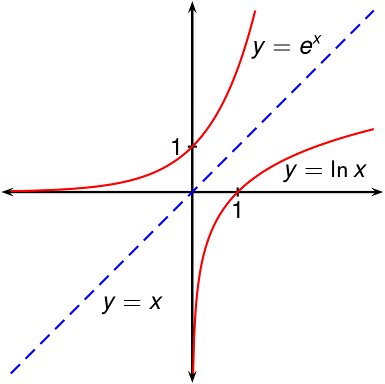


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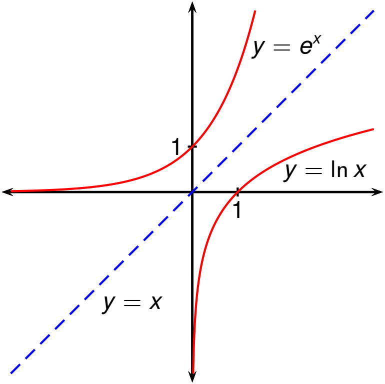


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## Theorem (Properties of Logarithmic Functions)

*If $a > 1$, the function $f(x) = \log_a x$ is a one-to-one, continuous, increasing function with domain $(0, \infty)$ and range $\mathbb{R}$. If $x, y, a, b > 0$ and $r$ is any real number, then*

1. $\log_a(xy) = \log_a x + \log_a y$.
2. $\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$.
3. $\log_a(x^r) = r \log_a x$.
4. $\log_a(x) = \log_b x \log_a b = \frac{\log_b x}{\log_b a} = \frac{\ln x}{\ln a}$.

# The Derivative of the Natural Logarithm

## Theorem (The Derivative of $\ln x$)

$$\frac{\mathrm{d}}{\mathrm{d}x}(\ln x) = \frac{1}{x}.$$

## Proof.

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## Example (Chain Rule, )

Differentiate $\quad y = \ln(x^3 + 1).$

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## Theorem (The Number $e$ as a Limit)

$$e = \lim_{x\to 0}(1+x)^{\frac{1}{x}} = \lim_{y\to\infty}\left(1+\frac{1}{y}\right)^{y}.$$

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## Example

Compute

$$\lim_{x\to\infty}\left(\frac{x+3}{x}\right)^x$$

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\lim_{x\to\infty}\left(\frac{x+3}{x}\right)^{x} &= \lim_{x\to\infty}\left(1+\frac{3}{x}\right)^{x} \\
&= \lim_{x\to\infty}\left(1+\frac{1}{\frac{x}{3}}\right)^{3\frac{x}{3}} && \Big| \text{ Set } \tfrac{x}{3}=y
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## Example

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Set $\frac{x}{3}=y$

## Example

Compute

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\lim_{x\to\infty}\left(\frac{x+3}{x}\right)^x &= \lim_{x\to\infty}\left(1+\frac{3}{x}\right)^x \\
&= \lim_{x\to\infty}\left(1+\frac{1}{\frac{x}{3}}\right)^{3\frac{x}{3}} \\
&= \lim_{\substack{x\to\infty\\ \frac{x}{3}=y\to\infty}}\left(1+\frac{1}{y}\right)^{3y}
\end{aligned}$$

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Compute

$$
\begin{aligned}
\lim_{x\to\infty}\left(\frac{x+3}{x}\right)^x &= \lim_{x\to\infty}\left(1+\frac{3}{x}\right)^x \\
&= \lim_{x\to\infty}\left(1+\frac{1}{\frac{x}{3}}\right)^{3\frac{x}{3}} && \Big|\ \text{Set } \tfrac{x}{3}=y \\
&= \lim_{\substack{x\to\infty \\ \frac{x}{3}=y\to\infty}}\left(1+\frac{1}{y}\right)^{\color{red}{3y}} \\
&= \lim_{y\to\infty}\left(\left(1+\frac{1}{y}\right)^{\color{red}{y}}\right)^{\color{red}{3}}
\end{aligned}
$$

## Example

Compute

$$\begin{aligned}
\lim_{x\to\infty}\left(\frac{x+3}{x}\right)^{x} &= \lim_{x\to\infty}\left(1+\frac{3}{x}\right)^{x} \\
&= \lim_{x\to\infty}\left(1+\frac{1}{\frac{x}{3}}\right)^{3\frac{x}{3}} && \Bigg|\ \text{Set } \tfrac{x}{3}=y \\
&= \lim_{\substack{x\to\infty\\ \frac{x}{3}=y\to\infty}}\left(1+\frac{1}{y}\right)^{3y} \\
&= \lim_{y\to\infty}\left(\left(1+\frac{1}{y}\right)^{y}\right)^{3} =
\end{aligned}$$

## Example

Compute

$$\begin{aligned}
\lim_{x\to\infty}\left(\frac{x+3}{x}\right)^x &= \lim_{x\to\infty}\left(1+\frac{3}{x}\right)^x \\
&= \lim_{x\to\infty}\left(1+\frac{1}{\frac{x}{3}}\right)^{3\frac{x}{3}} && \left|\ \text{Set } \tfrac{x}{3}=y\right. \\
&= \lim_{\substack{x\to\infty \\ \frac{x}{3}=y\to\infty}}\left(1+\frac{1}{y}\right)^{3y} \\
&= \lim_{y\to\infty}\left(\left(1+\frac{1}{y}\right)^y\right)^3 = e^3 \quad .
\end{aligned}$$

## Example

Compute

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\lim_{x\to\infty}\left(\frac{x+3}{x}\right)^x &= \lim_{x\to\infty}\left(1+\frac{3}{x}\right)^x \\
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&= \lim_{y\to\infty}\left(\left(1+\frac{1}{y}\right)^y\right)^3 = e^3 \quad .
\end{aligned}$$

## Example

Compute

$$\lim_{x\to\infty}\left(\frac{x}{x-2}\right)^{2x+2}$$

## Example

Compute

$$
\begin{aligned}
& \lim_{x\to\infty} \left( \frac{x}{x-2} \right)^{2x+2} \\
= & \lim_{x\to\infty} \left( \frac{x-2+2}{x-2} \right)^{2x+2}
\end{aligned}
$$

## Example

Compute

$$
\begin{aligned}
& \lim_{x\to\infty}\left(\frac{x}{x-2}\right)^{2x+2} \\
= & \lim_{x\to\infty}\left(\frac{x-2+2}{x-2}\right)^{2x+2} = \lim_{x\to\infty}\left(1+\frac{2}{x-2}\right)^{2x+2}
\end{aligned}
$$

## Example

Compute

$$\begin{aligned}
& \lim_{x\to\infty}\left(\frac{x}{x-2}\right)^{2x+2} \\
= & \lim_{x\to\infty}\left(\frac{x-2+2}{x-2}\right)^{2x+2} = \lim_{x\to\infty}\left(1+\frac{2}{x-2}\right)^{2x+2} \\
= & \lim_{x\to\infty}\left(1+\frac{1}{\frac{x-2}{2}}\right)^{2(x-2+2)+2}
\end{aligned}$$

## Example

Compute

$$
\begin{aligned}
& \lim_{x\to\infty}\left(\frac{x}{x-2}\right)^{2x+2} \\
= & \lim_{x\to\infty}\left(\frac{x-2+2}{x-2}\right)^{2x+2} = \lim_{x\to\infty}\left(1+\frac{2}{x-2}\right)^{2x+2} \\
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= & \lim_{x\to\infty}\left(\frac{x-2+2}{x-2}\right)^{2x+2} = \lim_{x\to\infty}\left(1+\frac{2}{x-2}\right)^{2x+2} \\
= & \lim_{x\to\infty}\left(1+\frac{1}{\frac{x-2}{2}}\right)^{\color{red}2(x-2+2)+2} \\
= & \lim_{x\to\infty}\left(1+\frac{1}{\frac{x-2}{2}}\right)^{\color{red}4\frac{x-2}{2}+6}
\end{aligned}
$$

## Example

Compute

$$
\begin{aligned}
& \lim_{x\to\infty}\left(\frac{x}{x-2}\right)^{2x+2} \\
= & \lim_{x\to\infty}\left(\frac{x-2+2}{x-2}\right)^{2x+2} = \lim_{x\to\infty}\left(1+\frac{2}{x-2}\right)^{2x+2} \\
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= & \lim_{x\to\infty}\left(1+\frac{1}{\frac{x-2}{2}}\right)^{4\frac{x-2}{2}+6}
\end{aligned}
$$

Set $y = \frac{x-2}{2}$

## Example

Compute

$$
\begin{aligned}
& \lim_{x\to\infty}\left(\frac{x}{x-2}\right)^{2x+2} \\
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= & \lim_{x\to\infty}\left(1+\frac{1}{\frac{x-2}{2}}\right)^{2(x-2+2)+2} \\
= & \lim_{x\to\infty}\left(1+\frac{1}{\frac{x-2}{2}}\right)^{4\frac{x-2}{2}+6} = \lim_{\substack{\frac{x-2}{2}=y \\ y\to\infty}}\left(1+\frac{1}{y}\right)^{4y+6}
\end{aligned}
$$

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## Example

Compute

$$
\begin{aligned}
& \lim_{x\to\infty}\left(\frac{x}{x-2}\right)^{2x+2} \\
= & \lim_{x\to\infty}\left(\frac{x-2+2}{x-2}\right)^{2x+2} = \lim_{x\to\infty}\left(1+\frac{2}{x-2}\right)^{2x+2} \\
= & \lim_{x\to\infty}\left(1+\frac{1}{\frac{x-2}{2}}\right)^{2(x-2+2)+2} \\
= & \lim_{x\to\infty}\left(1+\frac{1}{\frac{x-2}{2}}\right)^{4\frac{x-2}{2}+6} = \lim_{\substack{\frac{x-2}{2}=y\\ y\to\infty}}\left(1+\frac{1}{y}\right)^{4y+6} \\
= & \lim_{y\to\infty}\left(\left(1+\frac{1}{y}\right)^{y}\right)^{4}\lim_{y\to\infty}\left(1+\frac{1}{y}\right)^{6}
\end{aligned}
$$

Set $y=\frac{x-2}{2}$

## Example

Compute

$$\lim_{x\to\infty}\left(\frac{x}{x-2}\right)^{2x+2}$$

$$= \lim_{x\to\infty}\left(\frac{x-2+2}{x-2}\right)^{2x+2} = \lim_{x\to\infty}\left(1+\frac{2}{x-2}\right)^{2x+2}$$

$$= \lim_{x\to\infty}\left(1+\frac{1}{\frac{x-2}{2}}\right)^{2(x-2+2)+2}$$

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Set $y = \frac{x-2}{2}$

$$= \lim_{y\to\infty}\left(\left(1+\frac{1}{y}\right)^{y}\right)^{4}\lim_{y\to\infty}\left(1+\frac{1}{y}\right)^{6}$$

## Example

Compute

$$
\begin{array}{rl}
& \displaystyle\lim_{x\to\infty}\left(\frac{x}{x-2}\right)^{2x+2} \\
= & \displaystyle\lim_{x\to\infty}\left(\frac{x-2+2}{x-2}\right)^{2x+2} = \lim_{x\to\infty}\left(1+\frac{2}{x-2}\right)^{2x+2} \\
= & \displaystyle\lim_{x\to\infty}\left(1+\frac{1}{\frac{x-2}{2}}\right)^{2(x-2+2)+2} \\
= & \displaystyle\lim_{x\to\infty}\left(1+\frac{1}{\frac{x-2}{2}}\right)^{4\frac{x-2}{2}+6} = \lim_{\substack{\frac{x-2}{2}=y\\ y\to\infty}}\left(1+\frac{1}{y}\right)^{4y+6} \\
= & \displaystyle\lim_{y\to\infty}\left(\left(1+\frac{1}{y}\right)^{y}\right)^{4}\lim_{y\to\infty}\left(1+\frac{1}{y}\right)^{6} \\
= & e^4\cdot(1+0)^6
\end{array}
\qquad \left|\ \text{Set } y=\frac{x-2}{2}\right.
$$

## Example

Compute

$$
\begin{aligned}
& \lim_{x\to\infty}\left(\frac{x}{x-2}\right)^{2x+2} \\
= & \lim_{x\to\infty}\left(\frac{x-2+2}{x-2}\right)^{2x+2} = \lim_{x\to\infty}\left(1+\frac{2}{x-2}\right)^{2x+2} \\
= & \lim_{x\to\infty}\left(1+\frac{1}{\frac{x-2}{2}}\right)^{2(x-2+2)+2} \\
= & \lim_{x\to\infty}\left(1+\frac{1}{\frac{x-2}{2}}\right)^{4\frac{x-2}{2}+6} = \lim_{\substack{\frac{x-2}{2}=y\\ y\to\infty}}\left(1+\frac{1}{y}\right)^{4y+6} \qquad \Bigg| \text{ Set } y=\frac{x-2}{2} \\
= & \lim_{y\to\infty}\left(\left(1+\frac{1}{y}\right)^{y}\right)^{4}\lim_{y\to\infty}\left(1+\frac{1}{y}\right)^{6} \\
= & e^4\cdot(1+0)^6
\end{aligned}
$$

## Example

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= & \lim_{x\to\infty}\left(1+\frac{1}{\frac{x-2}{2}}\right)^{4\frac{x-2}{2}+6} = \lim_{\substack{\frac{x-2}{2}=y\\ y\to\infty}}\left(1+\frac{1}{y}\right)^{4y+6} \qquad \Bigg| \text{ Set } y=\frac{x-2}{2} \\
= & \lim_{y\to\infty}\left(\left(1+\frac{1}{y}\right)^{y}\right)^{4} \lim_{y\to\infty}\left(1+\frac{1}{y}\right)^{6} \\
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\end{aligned}$$

## Example

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=&\lim_{x\to\infty}\left(1+\frac{1}{\frac{x-2}{2}}\right)^{2(x-2+2)+2}\\
=&\lim_{x\to\infty}\left(1+\frac{1}{\frac{x-2}{2}}\right)^{4\frac{x-2}{2}+6} = \lim_{\substack{\frac{x-2}{2}=y\\ y\to\infty}}\left(1+\frac{1}{y}\right)^{4y+6} \qquad \left|\ \text{Set } y=\frac{x-2}{2}\right.\\
=&\lim_{y\to\infty}\left(\left(1+\frac{1}{y}\right)^{y}\right)^{4}\lim_{y\to\infty}\left(1+\frac{1}{y}\right)^{6}\\
=&\ e^4\cdot(1+0)^6 = e^4 \quad .
\end{aligned}$$

## Theorem (The Derivative of $a^x$)

$$\frac{\mathrm{d}}{\mathrm{d}x}(a^x) = a^x \ln a.$$

## Proof.

$\square$

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Use the fact that $a = e^{\ln a}$.

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□

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Use the fact that $a = e^{\ln a}$.

$$\frac{\mathrm{d}}{\mathrm{d}x}(a^x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\left(e^{\ln a}\right)^x\right)$$

□

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Use the fact that $a = e^{\ln a}$.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(a^x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left((e^{\ln a})^x\right)\\ &= \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{x\ln a}\right)\end{aligned}$$

$\square$

## Theorem (The Derivative of $a^x$)

$$\frac{d}{dx}(a^x) = a^x \ln a.$$

## Proof.

Use the fact that $a = e^{\ln a}$.

$$\begin{aligned}\frac{d}{dx}(a^x) &= \frac{d}{dx}\left((e^{\ln a})^x\right)\\ &= \frac{d}{dx}\left(e^{x\ln a}\right)\\ &= e^{x\ln a}\frac{d}{dx}(x\ln a)\end{aligned}$$

$\square$

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$\square$

## Theorem (The Derivative of $a^x$)

$$\frac{\mathrm{d}}{\mathrm{d}x}(a^x) = a^x \ln a.$$

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Use the fact that $a = e^{\ln a}$.

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\frac{\mathrm{d}}{\mathrm{d}x}(a^x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left((e^{\ln a})^x\right)\\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{x\ln a}\right)\\
&= e^{x\ln a}\frac{\mathrm{d}}{\mathrm{d}x}(x\ln a)\\
&= (e^{\ln a})^x \ln a\\
&= a^x(\ln a).
\end{aligned}$$

□

## Example (Chain Rule)

Differentiate $y = 10^{x^2}$.

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Let $u = \textbf{?}$

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## Example (Chain Rule)

$$
\begin{aligned}
\text{Differentiate} \quad & y = 10^{x^2}. \\
\text{Let} \quad & u = x^2. \\
\text{Then} \quad & y = 10^u.
\end{aligned}
$$

## Example (Chain Rule)

$$\text{Differentiate} \quad y = 10^{x^2}.$$

$$\text{Let} \quad u = x^2.$$

$$\text{Then} \quad y = 10^u.$$

$$\text{Chain Rule:} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}$$

## Example (Chain Rule)

Differentiate $y = 10^{x^2}$.

Let $u = x^2$.

Then $y = 10^u$.

Chain Rule: $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$

$= (\mathbf{?} \qquad\qquad )\,(\quad)$

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$= (10^u(\ln 10))\,(\quad)$

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## Example (Chain Rule)

$$
\begin{aligned}
\text{Differentiate}\quad & y = 10^{x^2}.\\
\text{Let}\quad & u = x^2.\\
\text{Then}\quad & y = 10^u.\\
\text{Chain Rule:}\quad & \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}\\
& \quad = \left(10^u(\ln 10)\right)(2x)
\end{aligned}
$$

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$$
\begin{aligned}
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\text{Let} \quad & u = x^2. \\
\text{Then} \quad & y = 10^u. \\
\text{Chain Rule:} \quad & \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \\
& = \left(10^u (\ln 10)\right)(2x) \\
& = (2 \ln 10) x 10^{x^2}.
\end{aligned}
$$

## Example

Compute $\frac{\mathrm{d}}{\mathrm{d}x}\left((\tan x)^{\frac{1}{x}}\right)$, where $x \in (0, \frac{\pi}{2})$.

## Example

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Compute $\frac{\mathrm{d}}{\mathrm{d}x}\left((\tan x)^{\frac{1}{x}}\right)$, where $x \in (0, \frac{\pi}{2})$.

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Compute $\frac{\mathrm{d}}{\mathrm{d}x}\left((\tan x)^{\frac{1}{x}}\right)$, where $x \in (0, \frac{\pi}{2})$.

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## Example

Compute $\frac{\mathrm{d}}{\mathrm{d}x}\left((\tan x)^{\frac{1}{x}}\right)$, where $x \in (0, \frac{\pi}{2})$.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left((\tan x)^{\frac{1}{x}}\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\left(e^{\ln \tan x}\right)^{\frac{1}{x}}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{\frac{1}{x}\ln \tan x}\right)\\
&= e^{\frac{1}{x}\ln(\tan x)}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x}\ln(\tan x)\right)
\end{aligned}$$

## Example

Compute $\frac{d}{dx}\left((\tan x)^{\frac{1}{x}}\right)$, where $x \in (0, \frac{\pi}{2})$.

$$\begin{aligned}
\frac{d}{dx}\left((\tan x)^{\frac{1}{x}}\right) &= \frac{d}{dx}\left(\left(e^{\ln \tan x}\right)^{\frac{1}{x}}\right) = \frac{d}{dx}\left(e^{\frac{1}{x}\ln \tan x}\right) \\
&= e^{\frac{1}{x}\ln(\tan x)}\frac{d}{dx}\left(\frac{1}{x}\ln(\tan x)\right) \\
&= (\tan x)^{\frac{1}{x}}\left( \qquad \right)
\end{aligned}$$

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\frac{\mathrm{d}}{\mathrm{d}x}\left((\tan x)^{\frac{1}{x}}\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\left(e^{\ln\tan x}\right)^{\frac{1}{x}}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{\frac{1}{x}\ln\tan x}\right) \\
&= e^{\frac{1}{x}\ln(\tan x)}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x}\ln(\tan x)\right) \\
&= (\tan x)^{\frac{1}{x}}\left( \qquad \qquad \qquad \qquad \right)
\end{aligned}$$

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Compute $\frac{d}{dx}\left((\tan x)^{\frac{1}{x}}\right)$, where $x \in (0, \frac{\pi}{2})$.

$$\begin{aligned}
\frac{d}{dx}\left((\tan x)^{\frac{1}{x}}\right) &= \frac{d}{dx}\left(\left(e^{\ln\tan x}\right)^{\frac{1}{x}}\right) = \frac{d}{dx}\left(e^{\frac{1}{x}\ln\tan x}\right)\\
&= e^{\frac{1}{x}\ln(\tan x)}\frac{d}{dx}\left(\frac{1}{x}\ln(\tan x)\right)\\
&= (\tan x)^{\frac{1}{x}}\left(-\frac{1}{x^2}\ln(\tan x)+\frac{1}{x}\frac{(\tan x)'}{\tan x}\right)
\end{aligned}$$

## Example

Compute $\frac{\mathrm{d}}{\mathrm{d}x}\left((\tan x)^{\frac{1}{x}}\right)$, where $x \in (0, \frac{\pi}{2})$.

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left((\tan x)^{\frac{1}{x}}\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\left(e^{\ln \tan x}\right)^{\frac{1}{x}}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{\frac{1}{x}\ln \tan x}\right)\\
&= e^{\frac{1}{x}\ln(\tan x)}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x}\ln(\tan x)\right)\\
&= (\tan x)^{\frac{1}{x}}\left(-\frac{1}{x^2}\ln(\tan x) + \frac{1}{x}\frac{(\tan x)'}{\tan x}\right)\\
&= (\tan x)^{\frac{1}{x}}\left(-\frac{1}{x^2}\ln(\tan x) + \frac{1}{x}\frac{\frac{1}{\cos^2(x)}}{\frac{\sin x}{\cos x}}\right)
\end{aligned}
$$

## Example

Compute $\frac{d}{dx}\left((\tan x)^{\frac{1}{x}}\right)$, where $x \in (0, \frac{\pi}{2})$.

$$
\begin{aligned}
\frac{d}{dx}\left((\tan x)^{\frac{1}{x}}\right) &= \frac{d}{dx}\left(\left(e^{\ln\tan x}\right)^{\frac{1}{x}}\right) = \frac{d}{dx}\left(e^{\frac{1}{x}\ln\tan x}\right)\\
&= e^{\frac{1}{x}\ln(\tan x)}\frac{d}{dx}\left(\frac{1}{x}\ln(\tan x)\right)\\
&= (\tan x)^{\frac{1}{x}}\left(-\frac{1}{x^2}\ln(\tan x)+\frac{1}{x}\frac{(\tan x)'}{\tan x}\right)\\
&= (\tan x)^{\frac{1}{x}}\left(-\frac{1}{x^2}\ln(\tan x)+\frac{1}{x}\frac{\frac{1}{\cos^2(x)}}{\frac{\sin x}{\cos x}}\right)\\
&= (\tan x)^{\frac{1}{x}}\left(-\frac{1}{x^2}\ln(\tan x)+\frac{1}{x}\frac{1}{\sin x\cos x}\right)
\end{aligned}
$$

## Example

Suppose $g(x)$ and $f(x)$ are differentiable functions and suppose $g(x) > 0$. Prove that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(g(x)^{f(x)}\right) = g(x)^{f(x)}\left(f'(x)\ln(g(x)) + f(x)\frac{g'(x)}{g(x)}\right) \quad .$$

## Proof.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(g(x)^{f(x)}\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\left(e^{\ln g(x)}\right)^{f(x)}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{f(x)\ln g(x)}\right) \\
&= e^{f(x)\ln g(x)}\frac{\mathrm{d}}{\mathrm{d}x}(f(x)\ln g(x)) \\
&= g(x)^{f(x)}\left(f'(x)\ln(g(x)) + f(x)\frac{g'(x)}{g(x)}\right) \quad ,
\end{aligned}$$

as desired. □