Calculus II
Lecture 20

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Outline

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Modeling with Differential Equations

- When modeling real-world problems, we often have a relationship between an unknown function and some of its derivatives.
- Such a relationship is called a differential equation.
- It is not always possible to find an explicit solution to a differential equation, but sometimes a graphical or approximate answer can be good enough for applications.

Models of Population Growth

- One model for population growth assumes that the population grows at a rate proportional to its size.
- In other words, if a certain number of bacteria produce a certain number of offspring in a certain time, then ten times that many bacteria produce ten times that many offspring in the same time.
- This is plausible when the population has unlimited food and environment and no restrictions on its size.
- Name the variables:

  $t =$ *time*

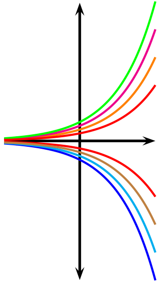
  $P =$ *the number of individuals in the population*

- The rate of growth is $\mathrm{d}P/\mathrm{d}t$.
- Then "rate of growth proportional to population size" means

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP$$

where k is the proportionality constant.

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP$$



- This is a differential equation.
- Exponential functions satisfy this condition.
- Let $P(t) = Ce^{kt}$ (C is a constant). Then
$$\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(Ce^{kt}) = Cke^{kt} = kCe^{kt} = kP(t)$$
- Therefore any function of the form $P(t) = Ce^{kt}$ satisfies the equation. We will see later that there is no other solution.
- Letting C vary over the real numbers gives a family of solutions.
- Since populations are non-negative, only solutions with $C > 0$ are relevant.

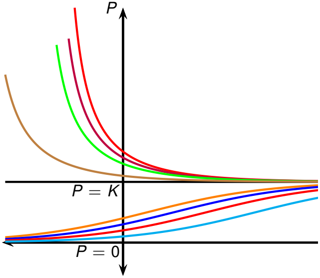
- This model works well under ideal conditions.
- In real life, most populations are constrained by the environment, the amount of food, etc.
- Many populations start by increasing exponentially, but then level off when they approach some upper bound, called the carrying capacity K.
- To take this into account, make two assumptions:
    - $\frac{\mathrm{d}P}{\mathrm{d}t} \approx kP$ if P is small (Initially, the growth rate is proportional to P).
    - $\frac{\mathrm{d}P}{\mathrm{d}t} < 0$ if $P > K$ (P decreases if it ever exceeds K).
- Here is an expression that takes both assumptions into account:

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1 - \frac{P}{K}\right)$$

- This is called the logistic differential equation.

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1 - \frac{P}{K}\right)$$

- What do the solutions look like?
- $P = 0$ and $P = K$ are special solutions, called equilibrium solutions.
- If $P > K$, then $1 - P/K < 0$, so $\mathrm{d}P/\mathrm{d}t < 0$, and P decreases.
- If $P < K$, then $1 - P/K > 0$, so $\mathrm{d}P/\mathrm{d}t > 0$, and P increases.
- As $P \to K$, $1 - P/K \to 0$, so $\mathrm{d}P/\mathrm{d}t \to 0$ and P levels off.



A Model for the Motion of a Spring

- Suppose we have an object with mass m attached to a spring.
- Hooke's Law: if the spring is stretched or compressed x units from its natural length, then it exerts a force that is proportional to x.
- Force equals mass times acceleration.
- Acceleration is the second derivative of displacement with respect to time.

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -kx$$

- This is called a second-order differential equation because it involves second derivatives.
- Sine and cosine functions are solutions.

General Differential Equations

Definition (Differential Equation)

A differential equation is an equation that contains an unknown function and some of its derivatives.

Definition (Order of a Differential Equation)

The order of a differential equation is the highest derivative that appears in it.

Definition (Solution)

A function f is called a solution of a differential equation if the equation is satisfied when f and its derivatives are plugged in.

Definition (To Solve a Differential Equation)

When we are asked to solve a differential equation we are expected to find all possible solutions.

Example

Show that every member of the family of functions

$$y = \frac{1 + ce^t}{1 - ce^t}$$

is a solution of the differential equation $y' = \frac{1}{2}(y^2 - 1)$.

$$\begin{aligned}
\mathrm{LHS} &= \frac{(1 - ce^t)(ce^t) - (1 + ce^t)(-ce^t)}{(1 - ce^t)^2} \\
&= \frac{ce^t - c^2e^{2t} + ce^t + c^2e^{2t}}{(1 - ce^t)^2} = \frac{2ce^t}{(1 - ce^t)^2}
\end{aligned}$$

$$\begin{aligned}
\mathrm{RHS} &= \frac{1}{2}\left[\left(\frac{1 + ce^t}{1 - ce^t}\right)^2 - 1\right] = \frac{1}{2}\left[\frac{(1 + ce^t)^2 - (1 - ce^t)^2}{(1 - ce^t)^2}\right] \\
&= \frac{1}{2}\left[\frac{1 + 2ce^t + c^2e^{2t} - 1 + 2ce^t - c^2e^{2t}}{(1 - ce^t)^2}\right] \\
&= \frac{1}{2}\frac{4ce^t}{(1 - ce^t)^2} = \frac{2ce^t}{(1 - ce^t)^2} = \mathrm{LHS}
\end{aligned}$$

- Often we don't want to find all solutions (the general solution).
- Instead, we only want to find a single solution that satisfies some additional requirement.
- Often that requirement has the form $y(t_0) = y_0$.
- This is called an initial condition.
- This type of problem is called an initial value problem.

Example

Find a solution of the differential equation $y' = \frac{1}{2}(y^2 - 1)$ that satisfies the initial condition $y(0) = 2$.
Substitute $t = 0$ and $y = 2$ into the formula

$$y = \frac{1 + ce^t}{1 - ce^t}$$

from Example 1.

$$\begin{aligned} 2 &= \frac{1 + ce^0}{1 - ce^0} = \frac{1 + c}{1 - c} \\ 2(1 - c) &= 1 + c \\ 2 - 2c &= 1 + c \\ c &= 1/3 \end{aligned}$$

Therefore the solution to the initial-value problem is

$$y = \frac{1 + \frac{1}{3}e^t}{1 - \frac{1}{3}e^t} = \frac{3 + e^t}{3 - e^t}.$$

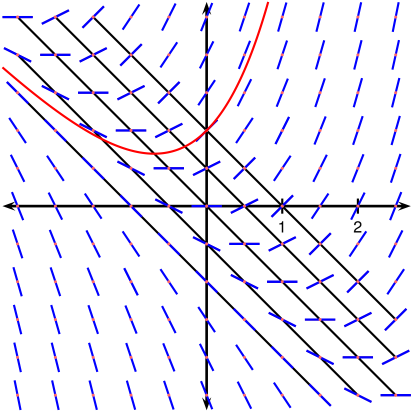
Direction Fields and Euler's Method

- Often we don't know how to find explicit solutions to a differential equation.
- Nevertheless, we can learn a lot about the solutions using:
  - A graphical approach (direction fields)
  - A numerical approach (Euler's method)
- Today we will discuss direction fields, but not Euler's method.

Direction Fields

- How do we sketch the graph of the solution to $y' = x + y$ that satisfies the initial condition $y(0) = 1$?
- Make a table of values of y'.

Point	y'
$(1,0)$	1
$(-1,0)$	-1
$(0,1)$	1
$(0,-1)$	-1
$(0,0)$	0
$(1,1)$	2
$(1,-1)$	0
$(-1,1)$	0
$(-1,-1)$	-2



Line	y'
$y=-x$	0
$y=-x+\frac{1}{2}$	$\frac{1}{2}$
$y=-x+1$	1
$y=-x-\frac{1}{2}$	$-\frac{1}{2}$
$y=-x-1$	-1

Separable Equations

In this section, we will discuss a type of differential equation, called a separable equation, for which it is possible to find an explicit solution.

Definition (Separable Equation)

A separable equation is a first-order equation in which the expression for $\mathrm{d}y/\mathrm{d}x$ can be factored as a function of x times a function of y. In other words,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = g(x)f(y).$$

Let $f(y) = 1/h(y)$. Then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{g(x)}{h(y)}.$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{g(x)}{h(y)}.$$

- To solve, write this in differential form:
$$h(y)\mathrm{d}y = g(x)\mathrm{d}x$$
- Now integrate:
$$\int h(y)\mathrm{d}y = \int g(x)\mathrm{d}x$$
- This defines y implicitly as a function of x.
- Sometimes we might be able to solve explicitly for y in terms of x.

Why does this process yield a function that satisfies the original differential equation? Suppose that $\int h(y)\mathrm{d}y = \int g(x)\mathrm{d}x$. Then we will use the Chain Rule to show that y satisfies the original equation.

$$
\begin{aligned}
\int h(y)\mathrm{d}y &= \int g(x)\mathrm{d}x \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\int h(y)\mathrm{d}y\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\int g(x)\mathrm{d}x\right) \\
\frac{\mathrm{d}}{\mathrm{d}y}\left(\int h(y)\mathrm{d}y\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\int g(x)\mathrm{d}x\right) \\
h(y)\frac{\mathrm{d}y}{\mathrm{d}x} &= g(x) \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{g(x)}{h(y)}
\end{aligned}
$$

Example

Solve the differential equation $\frac{dy}{dx} = \frac{x^2}{y^2}$, and find the solution that satisfies the initial condition $y(0) = 2$.

$$\begin{aligned} y^2 dy &= x^2 dx \\ \int y^2 dy &= \int x^2 dx \\ \frac{y^3}{3} &= \frac{x^3}{3} + C \\ y &= \sqrt[3]{x^3 + 3C} \\ y &= \sqrt[3]{x^3 + K} \end{aligned}$$

To find the solution satisfying the initial condition, set $2 = y(0) = \sqrt[3]{0^3 + K} = \sqrt[3]{K}$. Then $\sqrt[3]{K} = 2$, so $K = 8$.

$$y = \sqrt[3]{x^3 + 8}.$$

Example

Solve the equation $y' = x^2 y$.

$$
\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= x^2 y \\
\frac{1}{y}\mathrm{d}y &= x^2 \mathrm{d}x \qquad y \neq 0 \\
\int \frac{1}{y}\mathrm{d}y &= \int x^2 \mathrm{d}x \\
\ln|y| &= \frac{1}{3}x^3 + C \\
e^{\ln|y|} &= e^{x^3/3+C} \\
|y| &= e^C e^{x^3/3} \\
y &= \pm e^C e^{x^3/3}
\end{aligned}
$$

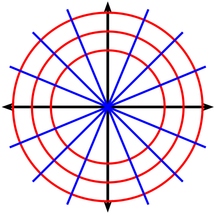
The function $y = 0$ satisfies the equation. General solution:

$$y = Ae^{x^3/3}.$$

Orthogonal Trajectories

Definition (Orthogonal Trajectory)

An orthogonal trajectory to a family of curves is a curve that intersects each curve of the family orthogonally (that is, at right angles).

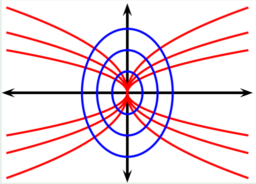


Each member of the family $y = mx$ of straight lines passing through the origin is an orthogonal trajectory to the family $x^2 + y^2 = r^2$ of circles centered at the origin.

Example

Find the orthogonal trajectories of the family $x = ky^2$, where k is an arbitrary constant. Differentiate implicitly:

$$\begin{aligned} x &= ky^2 \\ 1 &= 2ky\frac{dy}{dx} \\ 1 &= 2\left(\frac{x}{y^2}\right)y\frac{dy}{dx} \\ \frac{dy}{dx} &= \frac{y}{2x} \end{aligned}$$



An orthogonal trajectory will have a slope that is the negative reciprocal of the slope of the curve.

$$\begin{aligned} \frac{dy}{dx} &= -\frac{2x}{y} \\ \int y dy &= -\int 2x dx \\ \frac{y^2}{2} &= -x^2 + C \\ x^2 + \frac{y^2}{2} &= C \end{aligned}$$

The ellipses $x^2 + \frac{y^2}{2} = C$ are all orthogonal trajectories to $x = ky^2$.

Mixing Problems

- Typical mixing problems involve:
- A tank of fixed capacity.
- A completely mixed solution of some substance in the tank.
- A solution of a certain concentration enters the tank at a fixed rate.
- In the tank, the solution immediately becomes completely stirred.
- The mixture leaves at the other end at a fixed rate (possibly a different rate).
- Let $y(t)$ denote the amount of substance in the tank at time t.
- Then $y'(t)$ denotes the rate at which the substance is being added minus the rate at which it is being removed.
- This often gives a differential equation.

Example

A tank contains 20 kg of salt dissolved in 5000 L of water. Brine that contains 0.03 kg of salt per liter of water enters the tank at a rate of 25 L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate. How much salt is in the tank after half an hour?

- Let $y(t)$ denote the amount of salt (in kg) after t minutes.
- Given: $y(0) = 20$. We want to know: $y(30)$.

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}t} &= \text{(rate in)} - \text{(rate out)} = 0.75 - \frac{y(t)}{200} = \frac{150 - y(t)}{200} \\
\text{rate in} &= \text{(concentration in)(rate of volume in)} \\
&= \left(0.03\frac{\mathrm{kg}}{\mathrm{L}}\right)\left(25\frac{\mathrm{L}}{\mathrm{min}}\right) = 0.75\frac{\mathrm{kg}}{\mathrm{min}} \\
\text{rate out} &= \text{(concentration out)(rate of volume out)} \\
&= \left(\frac{y(t)}{5000}\frac{\mathrm{kg}}{\mathrm{L}}\right)\left(25\frac{\mathrm{L}}{\mathrm{min}}\right) = \frac{y(t)}{200}\frac{\mathrm{kg}}{\mathrm{min}}
\end{aligned}$$

Example (Example 6, p. 621)

A tank contains 20 kg of salt dissolved in 5000 L of water. Brine that contains 0.03 kg of salt per liter of water enters the tank at a rate of 25 L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate. How much salt is in the tank after half an hour?

$$
\begin{aligned}
\frac{dy}{dt} &= \frac{150 - y(t)}{200}\\
\int \frac{dy}{150 - y} &= \int \frac{dt}{200}\\
-\ln|150 - y| &= t/200 + C \qquad y(0) = 20, \text{ so } C = -\ln 130\\
-\ln|150 - y| &= t/200 - \ln 130\\
|150 - y| &= 130e^{-t/200}\\
& y < 150 = (0.03)(5000), \text{ so } |150 - y| = 150 - y\\
y &= 150 - 130e^{-t/200}\\
y(30) &= 150 - 130e^{-30/200} \approx 38.1\text{kg}
\end{aligned}
$$

The Law of Natural Growth

- Recall that differential equations could be used to model population growth.
- The Law of Natural Growth works in ideal cases, where populations are unconstrained by lack of food, or the environment.
- Let $P(t)$ be the population at time t.
- Then the Law of Natural Growth says:

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP$$

- The constant k is sometimes called the relative growth rate.

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP$$

This is a separable equation, so we can solve it.

$$\begin{aligned}
\int \frac{\mathrm{d}P}{P} &= \int k\mathrm{d}t \\
\ln|P| &= kt + C \\
|P| &= e^C e^{kt} \\
P &= \pm e^C e^{kt}
\end{aligned}$$

- Let $A = \pm e^C$. Then the solution is $P = Ae^{kt}$.
- $A = \pm e^C$ can be any positive or negative number.
- The function $P = 0$ is also a solution, so A can be any number.
- $P(0) = Ae^{k\cdot 0} = A$.

The solution to the initial value problem

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP, \qquad P(0) = P_0$$

$$\text{is} \qquad P(t) = P_0 e^{kt}.$$

The Logistic Model

- The Logistic Model works in cases when the population is constrained by its environment.
- Let $P(t)$ be the population at time t.
- Then the Logistic Equation is:

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1 - \frac{P}{K}\right)$$

- The constant K is called the carrying capacity. It represents how many individuals the environment can sustain in the long run.

$$
\begin{aligned}
\frac{\mathrm{d}P}{\mathrm{d}t} &= kP\left(1-\frac{P}{K}\right)\\
\int \frac{1}{P(1-P/K)}\mathrm{d}P &= \int k\mathrm{d}t\\
\int \frac{K}{P(K-P)}\mathrm{d}P &= \int k\mathrm{d}t\\
\int \left(\frac{1}{P}+\frac{1}{K-P}\right)\mathrm{d}P &= \int k\mathrm{d}t\\
\ln|P| - \ln|K-P| &= kt + C\\
\ln\left|\frac{K-P}{P}\right| &= -kt - C\\
\frac{K-P}{P} &= \pm e^{-C}e^{-kt} = Ae^{-kt}\\
K &= P(1+Ae^{-kt})\\
P &= \frac{K}{1+Ae^{-kt}}
\end{aligned}
$$

Plug in $P(0) = P_0$:

$$\frac{K-P_0}{P_0} = Ae^{-k\cdot 0} = A.$$

The solution to the initial value problem

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1 - \frac{P}{K}\right), \qquad P(0) = P_0$$

is

$$P = \frac{K}{1 + Ae^{-kt}}, \qquad A = \frac{K - P_0}{P_0}.$$

Example

Write the solution of the initial value problem

$$\frac{\mathrm{d}P}{\mathrm{d}t} = 0.08P\left(1 - \frac{P}{1000}\right), \qquad P(0) = 100$$

and use it to find when the population reaches 900.

$$P(t) = \frac{1000}{1 + Ae^{-0.08t}}, \qquad A = \frac{1000 - 100}{100} = 9$$

$$\text{Therefore} \qquad P(t) = \frac{1000}{1 + 9e^{-0.08t}}.$$

$$\text{Set } P(t) = 900: \qquad \begin{aligned} \frac{1000}{1 + 9e^{-0.08t}} &= 900 \\ 1 + 9e^{-0.08t} &= 1000/900 \\ e^{-0.08t} &= \frac{1000/900 - 1}{9} = \frac{1}{81} \\ -0.08t &= -\ln 81 \\ t &= \frac{\ln 81}{0.08} \approx 54.9 \end{aligned}$$