Precalculus
Homework Lecture 15

1. Use the definition of a logarithm to evaluate each of the following without using a calculator. The answer key has not been proofread, use with caution.

   (a) $\log_2 16$.

   answer: 4

   (b) $\log_3 \left(\frac{1}{9}\right)$.

   answer: -2

   (c) $\log_{10} 1000$.

   answer: 3

   (d) $\log_6 36^{-\frac{2}{3}}$.

   answer: $-\frac{4}{3}$

   (e) $\log_2(8\sqrt{2})$.

   answer: $\frac{7}{2}$

   (f) $\log_{\frac{1}{2}}(4)$.

   answer: -2

   (g) $\log_{\frac{1}{9}}(\sqrt{3})$.

   answer: $-\frac{1}{4}$

2. Find the exact value of each expression.

   (a) $\log_5 125$.

   answer: 3

   (b) $\log_3 \frac{1}{27}$.

   answer: -3

   (c) $\ln \left(\frac{1}{e}\right)$.

   answer: -1

   (d) $\log_{10} \sqrt{10}$.

   answer: $\frac{1}{2}$

   (e) $e^{\ln 4.5}$.

   answer: 4.5

   (f) $\log_{10} 0.0001$.

   answer: 4

   (g) $\log_{1.5} 2.25$.

   answer: 2

   (h) $\log_5 4 - \log_5 500$.

   answer: -3

   (i) $\log_2 6 - \log_2 15 + \log_2 20$.

   answer: 3

   (j) $\log_3 100 - \log_3 18 - \log_3 50$.

   answer: -2

   (k) $e^{-2\ln 5}$.

   answer: $\frac{1}{25}$

   (l) $\ln\left(\ln e^{e^{10}}\right)$.

   answer: 10

   (m) $\log_7 \left(\frac{49^x}{343^y}\right)$

   answer: $2x - 3y$

   Solution. 2.m.

$$
\begin{aligned}
\log_7\left(\tfrac{49^x}{343^y}\right) &= \log_7 49^x - \log_7 343^y \\
&= x\log_7 49 - y\log_7 343 \\
\text{However } 49 = 7^2 \text{ and } 343 = 7^3, \text{ therefore } \log_7\left(\tfrac{49^x}{343^y}\right) &= 2x - 3y.
\end{aligned}
$$

3. Using only the $\ln$ operation of your calculator compute the indicated logarithm. Confirm your computation numerically by exponentiation.

   (a) $\log_5(13)$.

   answer: $\frac{\ln 13}{\ln 5} \approx 1.593693$

   (b) $\log_{12}(9)$.

   answer: $\frac{\ln 9}{\ln 12} \approx 0.884228$

   (c) $\log_{13}(101)$.

   answer: $\frac{\ln 101}{\ln 13} \approx 1.799303$

   (d) $\log_{10}(2015)$.

   answer: $\frac{\ln 2015}{\ln 10} \approx 3.304275$

Solution.

$$\log_5(13) = \frac{\ln 13}{\ln 5} \approx \frac{2.564949357}{1.609437912} \approx 1.593693.$$

As a check of our computations, we compute by calculator: $13 = 5^{\log_5 13} \approx 5^{1.593693} \approx 13.000007508$, and our computations check out.

4. Express each of the following as a single logarithm. If possible, compute the logarithm without using a calculator. The answer key has not been proofread, use with caution.

   (a) $\ln 4 + \ln 6 - \ln 5$.

   answer: $\ln\left(\frac{24}{5}\right)$

   (b) $2\ln 2 - 3\ln 3 + 4\ln 4$.

   answer: $\ln\left(\frac{1024}{27}\right)$

   (c) $\ln 36 - 2\ln 3 - 3\ln 2$.

   answer: $-\ln 2 = \ln\left(\frac{1}{2}\right)$

   (d) $\log_2(24) - \log_4 9$.

   answer: 3

   (e) $\log_7(24) + \log_{\frac{1}{7}} 3 - \log_{49}(64)$.

   answer: 0

   (f) $\log_3(24) + \log_3\left(\frac{3}{8}\right)$.

   answer: 2

Solution. 4.b.

$$\begin{aligned} 2\ln 2 - 3\ln 3 + 4\ln 4 &= \ln 2^2 - \ln 3^3 + \ln 4^4 \\ &= \ln 4 - \ln 27 + \ln 256 \\ &= \ln\left(\frac{4}{27}\right) + \ln 256 \\ &= \ln\left(\frac{4\cdot 256}{27}\right) \\ &= \ln\left(\frac{1024}{27}\right). \end{aligned}$$

$\frac{1024}{27}$ is not a rational power of e, therefore $\ln\left(\frac{1024}{27}\right)$ is not a rational number and there are no further simplifications of the answer (except possibly a numerical approximation with a calculator or equivalent).

Solution. 4.e

$$\begin{aligned} \log_7(24) + \log_{\frac{1}{7}}(3) - \log_{49}(64) &= \log_7(24) + \frac{\log_7(3)}{\log_7\left(\frac{1}{7}\right)} - \frac{\log_7(64)}{\log_7(49)} && \text{common base} \\ &= \log_7(24) + \frac{\log_7(3)}{-1} - \frac{\log_7(64)}{2} && \text{simplify logarithms} \\ &= \log_7(24) - \log_7(3) - \frac{1}{2}\log_7(64) \\ &= \log_7\left(\frac{24}{3}\right) - \log_7\left(64^{\frac{1}{2}}\right) && \text{rule: } \log_a x - \log_a y = \log_a\left(\frac{x}{y}\right) \\ & && \text{rule: } \log_a x^r = r\log_a x \\ &= \log_7(8) - \log_7\left(\sqrt{64}\right) \\ &= \log_7 8 - \log_7 8 = 0 && \text{alternatively:} \\ &= \log_7\left(\frac{8}{8}\right) \\ &= \log_7(1) \\ &= 0. \end{aligned}$$

5. Demonstrate the identity(s).

   (a) $-\ln\left(\sqrt{1+x^2}-x\right) = \ln\left(x+\sqrt{1+x^2}\right)$

6. Solve each equation for x. If available, use a calculator to give an ($\approx$) answer in decimal notation. If available, use a calculator to verify your approximate solutions.

   (a) $e^{7-4x} = 7$.

   answer: $\frac{7-\ln 7}{4} \approx 1.263522$

   (b) $\ln(2x-9) = 2$.

   answer: $\frac{e^2+9}{2} \approx 8.194528$

   (c) $\ln(x^2-2) = 3$.

   answer: $\pm\sqrt{e^3+2} \approx \pm 4.699525$

   (d) $2^{x-3} = 5$.

   answer: $\log_2 5 + 3 = \frac{\ln 5}{\ln 2} + 3 \approx 5.321928$

   (e) $\ln x + \ln(x-1) = 1$.

   answer: $\frac{1}{2}\left(1+\sqrt{1+4e}\right) \approx 2.223$

   (f) $e^{2x+1} = t$.

   answer: $\frac{\ln t - 1}{2}$

   (g) $\log_2(mx) = c$.

   answer: $\frac{2^c}{m}$

   (h) $e - e^{-2x} = 1$.

   answer: $-\frac{1}{2}\ln(e-1) \approx -0.271$

   (i) $8(1+e^{-x})^{-1} = 3$.

   answer: $-\ln\frac{5}{3} = \ln\frac{3}{5} \approx -0.510826$

   (j) $\ln(\ln x) = 1$.

   answer: $e^e \approx 15.154$

   (k) $e^{e^x} = 10$.

   answer: $\ln(\ln 10) \approx 0.834$

   (l) $\ln(2x+1) = 3 - \ln x$.

   answer: $\frac{-1+\sqrt{1+8e^3}}{4} \approx 2.928878$

   (m) $e^{2x} - 4e^x + 3 = 0$.

   answer: $x = \ln 3 \approx 1.098612, \ , x = 0$

   (n) $e^{4x} + 3e^{2x} - 4 = 0$.

   answer: $x = 0$

   (o) $e^{2x} - e^x - 6 = 0$.

   answer: $x = \ln 3$

   (p) $4^{3x} - 2^{3x+2} - 5 = 0$.

   answer: $x = \frac{\log_2 5}{3}$

   (q) $3 \cdot 2^x + 2\left(\frac{1}{2}\right)^{x-1} - 7 = 0$.

   answer: $x = 0$ or $2 - \log_2 3 = 2 - \frac{\ln 3}{\ln 2}$

Solution. 6.d

$$\begin{array}{rcl|l} 2^{x-3} &=& 5 & \text{take } \log_2 \\ x-3 &=& \log_2(5) & \text{add 3 to both sides} \\ x &=& \log_2(5)+3 & \text{answer is complete} \\ &=& \dfrac{\ln 5}{\ln 2}+3 & \text{optional step: convert to } \ln \\ &\approx& 5.321928095 & \text{calculator} \end{array}$$

Solution. 6.h

$$\begin{array}{rcll} e - e^{-2x} &=& 1 & \\ e^{-2x} &=& e-1 & \big|\ \text{apply } \ln \\ \ln e^{-2x} &=& \ln(e-1) & \\ -2x &=& \ln(e-1) & \\ x &=& -\dfrac{1}{2}\ln(e-1) & \\ &\approx& -0.270662427 & \big|\ \text{calculator} \end{array}$$

Solution. 6.e

$$\begin{array}{rcl} \ln x + \ln(x-1) &=& 1 \\ \ln\left(x^2-x\right) &=& 1 \\ e^{\ln(x^2-x)} &=& e^1 \\ x^2-x &=& e \\ x^2-x-e &=& 0 \\ \text{Quadratic formula:}\quad x &=& \dfrac{-(-1)\pm\sqrt{(-1)^2-4(1)(-e)}}{2(1)} \\ &=& \dfrac{1\pm\sqrt{1+4e}}{2}. \end{array}$$

However $\frac{1-\sqrt{1+4e}}{2}$ is negative, so $\ln\left(\frac{1-\sqrt{1+4e}}{2}\right)$ is undefined. Hence the only solution is $x = \frac{1+\sqrt{1+4e}}{2} \approx 2.2229$.

Solution. 6.p

$$
\begin{aligned}
4^{3x} - 2^{3x+2} - 5 &= 0 \\
4^{3x} - 4 \cdot 2^{3x} - 5 &= 0 \\
u^2 - 4u - 5 &= 0 \\
(u-5)(u+1) &= 0 \\
u = 5 \quad &\text{or} \quad u = -1 \\
2^{3x} = 5 \quad & \quad 2^{3x} = -1 \\
3x = \log_2(5) \quad & \quad \text{no real solution} \\
x = \frac{\log_2 5}{3} & \\
\text{Calculator: } x \approx 0.773976 &
\end{aligned}
$$

$$
\left|\ \text{Set}\ \begin{aligned} 2^{3x} &= u \\ 4^{3x} &= u^2 \end{aligned}\right.
$$

Solution. 6.q

$$
\begin{aligned}
3 \cdot 2^x + 2\left(\frac{1}{2}\right)^{x-1} - 7 &= 0 \\
3 \cdot 2^x + 2\left(\frac{1}{2}\right)^{x}\left(\frac{1}{2}\right)^{-1} - 7 &= 0 \\
3 \cdot 2^x + 4\left(\frac{1}{2}\right)^{x} - 7 &= 0 \\
3u + \frac{4}{u} - 7 &= 0 \\
3u^2 - 7u + 4 &= 0 \\
(u-1)(3u-4) &= 0 \\
u = 1 \quad &\text{or} \quad 3u - 4 = 0 \\
2^x = 1 \quad & \quad u = \frac{4}{3} \\
x = 0 \quad & \quad 2^x = \frac{4}{3} \\
& \quad x = \log_2 \frac{4}{3} = \log_2 4 - \log_2 3 \\
& \quad x = 2 - \log_2 3 \\
\text{Calculator:} \quad & \quad x \approx 0.415037
\end{aligned}
$$

$$
\left|\ \begin{aligned} &\text{Set } 2^x = u \\ &\text{Multiply by } u \end{aligned}\right.
$$