Calculus II
Lecture 5

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https://github.com/tmilev/freecalc

2020

Outline

1. Integration of Rational Functions
   - Partial fractions

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From building blocks to all rational functions: example

- We know how to solve $\displaystyle\int \frac{2}{x-1}\mathrm{d}x$ and $\displaystyle\int \frac{1}{x+2}\mathrm{d}x$.
- Consider the difference

$$\frac{2}{x-1} - \frac{1}{x+2} =$$

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$$\frac{2}{x-1} - \frac{1}{x+2} = \frac{2(x+2)-(x-1)}{(x-1)(x+2)} =$$

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$$\frac{2}{x-1}-\frac{1}{x+2}=\frac{2(x+2)-(x-1)}{(x-1)(x+2)}=\frac{x+5}{x^2+x-2} \quad .$$

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- From (linear substitutions of) basic building blocks we constructed a larger example, which we can therefore solve.
- We now learn how to do the reverse procedure: given a rational function, split it into “partial fractions”.

Partial fractions definition

Definition

A partial fraction is rational function of one of the 2 forms below.

- $\frac{A}{(ax+b)^n}$, $n \geq 1$.
- $\frac{Ax+B}{(ax^2+bx+c)^n}$, where $b^2 - 4ac < 0$ and $n \geq 1$.

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Theorem

Every rational function can be written as a sum of a polynomial and partial fractions.

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Theorem

Every rational function can be written as a sum of a polynomial and partial fractions.

- We already learned know how to integrate all partial fractions (using linear substitutions and building blocks I, II and III).
- Thus, if we can produce the partial fractions whose existence is promised by the theorem, we can integrate all rational functions.

Review of polynomial notation

- Recall that a rational function is a function of the form

$$f(x) = \frac{P(x)}{Q(x)}$$

where P and $Q \neq 0$ are polynomials.

Review of polynomial notation

- Recall that a rational function is a function of the form

$$f(x) = \frac{P(x)}{Q(x)}$$

  where P and $Q \neq 0$ are polynomials.
- Recall that the degree of P is the highest power of x in P that has a non-zero coefficient.

Ensure denominator degree $>$ numerator degree

- To decompose $\frac{P(x)}{Q(x)}$ in partial fractions we ensure first the degree of the numerator is smaller than the degree of the denominator.

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- We recall that to divide the dividend $P(x)$ by the divisor $Q(x)$ to get quotient $S(x)$ with remainder $R(x)$ means to find polynomials $S(x), R(x)$ such that $\deg R < \deg Q$ and

$$P(x) = S(x)Q(x) + R(x)$$

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$$\begin{array}{rcll} P(x) & = & S(x)Q(x) + R(x) & \Big| \text{ divide by } Q(x) \\ \dfrac{P(x)}{Q(x)} & = & \dfrac{S(x)Q(x)}{Q(x)} + \dfrac{R(x)}{Q(x)} & \end{array}$$

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\dfrac{P(x)}{Q(x)} & = & \dfrac{S(x)\cancel{Q(x)}}{\cancel{Q(x)}} + \dfrac{R(x)}{Q(x)} & \\
\dfrac{P(x)}{Q(x)} & = & S(x) + \dfrac{R(x)}{Q(x)} &
\end{array}
$$

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\end{array}
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- The above transforms $\frac{P(x)}{Q(x)}$ to a polynomial plus a fraction in which the numerator has degree smaller than the denominator.

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\end{array}
$$

- The above transforms $\frac{P(x)}{Q(x)}$ to a polynomial plus a fraction in which the numerator has degree smaller than the denominator.
- The polynomials $Q(x)$ and $S(x)$ are computed via polynomial long division. We recall the procedure through examples.

Example

Find $\int \frac{x^3+x}{x-1} \mathrm{d}x$.

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$$x-1\overline{\big)\,x^3 \qquad + \quad x}$$

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Divide x^3 by x

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Find $\int \frac{x^3+x}{x-1}\mathrm{d}x$.

$$\begin{array}{r|l} & x^2 \\ \hline x-1 & x^3 \quad + \quad x \end{array}$$

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Example

Find $\int \frac{x^3+x}{x-1} dx$.

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Multiply x^2 by $x-1$

Example

Find $\int \frac{x^3+x}{x-1}\mathrm{d}x$.

$$
\begin{array}{r l}
 & x^2 \\
x-1 \big) & \overline{x^3 \qquad + \; x} \\
 & \underline{x^3 - x^2}
\end{array}
$$

Multiply x^2 by $x - 1$

Example

Find $\int \frac{x^3+x}{x-1} \mathrm{d}x$.

$$
\begin{array}{r|l}
 & x^2 \\
\cline{2-2}
x-1 & x^3 \qquad + \ x \\
 & \underline{x^3 - x^2}
\end{array}
$$

Subtract $x^3 - x^2$ from x^3

Example

Find $\int \frac{x^3+x}{x-1} dx$.

$$
\begin{array}{r|l}
 & x^2 \\
x-1 & x^3 \qquad + \; x \\
 & x^3 - x^2 \\
\hline
 & \quad\;\; x^2
\end{array}
$$

Subtract $x^3 - x^2$ from x^3

Example

Find $\int \frac{x^3+x}{x-1} \mathrm{d}x$.

$$
\begin{array}{r|llll}
 & x^2 & & & \\
\hline
x-1 & x^3 & & + & x \\
 & x^3 & - x^2 & & \\
\cline{2-3}
 & & x^2 & + & x \\
\end{array}
$$

Bring down the x

Example

Find $\int \frac{x^3+x}{x-1} dx$.

$$
\begin{array}{r|llll}
 & x^2 & & & \\
\cline{2-5}
x-1 & x^3 & & + & x \\
 & x^3 & - x^2 & & \\
\cline{2-3}
 & & x^2 & + & x \\
\end{array}
$$

Divide x^2 by x

Example

Find $\int \frac{x^3+x}{x-1} dx$.

$$
\begin{array}{r|llll}
 & x^2 & + x & & \\
\hline
x-1 & x^3 & & + & x \\
 & x^3 - x^2 & & & \\
\hline
 & & x^2 & + & x \\
\end{array}
$$

Divide x^2 by x

Example

Find $\int \frac{x^3+x}{x-1} dx$.

$$
\begin{array}{r|l}
 & x^2 + x \\
\hline
x-1 & x^3 \quad\quad + \; x \\
 & x^3 - x^2 \\
\hline
 & \quad\; x^2 + \; x \\
\end{array}
$$

Multiply x by $x-1$

Example

Find $\int \frac{x^3+x}{x-1}\mathrm{d}x$.

$$
\begin{array}{r|l}
 & x^2 + x \\
\hline
x-1 & x^3 \qquad\quad + \ x \\
 & \underline{x^3 - x^2} \\
 & \qquad\; x^2 + \ x \\
 & \qquad\; \underline{x^2 - \ x}
\end{array}
$$

Multiply x by $x-1$

Example

Find $\int \frac{x^3+x}{x-1} \mathrm{d}x$.

$$
\begin{array}{r|l}
 & x^2 + x \\
\hline
x-1 & x^3 \quad\quad + \; x \\
 & x^3 - x^2 \\
\hline
 & \quad\quad x^2 + \; x \\
 & \quad\quad x^2 - \; x \\
\hline
\end{array}
$$

Subtract $x^2 - x$ from $x^2 + x$

Example

Find $\int \frac{x^3+x}{x-1} \mathrm{d}x$.

$$
\begin{array}{r|l}
 & x^2 + x \\
\hline
x-1 & x^3 \quad\quad + x \\
 & x^3 - x^2 \\
\hline
 & \quad\quad x^2 + x \\
 & \quad\quad x^2 - x \\
\hline
 & \quad\quad\quad 2x
\end{array}
$$

Subtract $x^2 - x$ from $x^2 + x$

Example

Find $\int \frac{x^3+x}{x-1} \mathrm{d}x$.

$$
\begin{array}{r@{}l}
 & x^2 + x \\
x-1 \big) & x^3 \quad\quad + x \\
 & x^3 - x^2 \\ \hline
 & \quad\;\; x^2 + x \\
 & \quad\;\; x^2 - x \\ \hline
 & \quad\quad\quad\;\; 2x
\end{array}
$$

Divide $2x$ by x

Example

Find $\int \frac{x^3+x}{x-1}\mathrm{d}x$.

$$
\begin{array}{r|l}
 & x^2 + x + 2 \\
\hline
x-1 & x^3 \quad\quad + x \\
 & \underline{x^3 - x^2} \\
 & \quad\quad x^2 + x \\
 & \quad\quad \underline{x^2 - x} \\
 & \quad\quad\quad\quad 2x
\end{array}
$$

Divide $2x$ by x

Example

Find $\int \frac{x^3+x}{x-1} dx$.

$$
\begin{array}{r|rrrr}
 & x^2 & + x & + 2 & \\
\hline
x-1 & x^3 & & + x & \\
 & x^3 - x^2 & & & \\
\hline
 & & x^2 & + x & \\
 & & x^2 & - x & \\
\hline
 & & & 2x &
\end{array}
$$

Multiply 2 by $x - 1$

Example

Find $\int \frac{x^3+x}{x-1} dx$.

$$
\begin{array}{r|lllll}
\multicolumn{1}{r}{} & x^2 & + & x & + & 2 \\
\cline{2-6}
x-1 & x^3 & & & + & x \\
 & x^3 & - & x^2 & & \\
\cline{2-4}
 & & & x^2 & + & x \\
 & & & x^2 & - & x \\
\cline{4-6}
 & & & & & 2x \\
 & & & & & 2x - 2 \\
\end{array}
$$

Multiply 2 by $x - 1$

Example

Find $\int \frac{x^3+x}{x-1}\mathrm{d}x$.

$$
\begin{array}{rl}
 & x^2 + x + 2 \\
x-1 \big) & \overline{x^3 \qquad\quad + x} \\
 & \underline{x^3 - x^2} \\
 & \qquad x^2 + x \\
 & \qquad \underline{x^2 - x} \\
 & \qquad\qquad 2x \\
 & \qquad\qquad \underline{2x - 2}
\end{array}
$$

Subtract $2x - 2$ from $2x$

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Find $\int \frac{x^3+x}{x-1} \mathrm{d}x$.

$$
\begin{array}{r|l}
\multicolumn{1}{r}{} & \quad x^2 + x + 2 \\ \cline{2-2}
x-1 & x^3 \qquad\quad + x \\
 & \underline{x^3 - x^2} \\
 & \qquad x^2 + x \\
 & \qquad \underline{x^2 - x} \\
 & \qquad\qquad 2x \\
 & \qquad\qquad \underline{2x - 2} \\
 & \qquad\qquad\qquad 2
\end{array}
$$

Subtract $2x-2$ from $2x$

Example

Find $\int \frac{x^3+x}{x-1} \mathrm{d}x$.

$$
\begin{array}{r|rrrr}
 & x^2 & +\ x & +\ 2 & \\
\cline{2-4}
x-1 & x^3 & & +\ x & \\
 & x^3 & -\ x^2 & & \\
\cline{2-3}
 & & x^2 & +\ x & \\
 & & x^2 & -\ x & \\
\cline{3-4}
 & & & 2x & \\
 & & & 2x & -\ 2 \\
\cline{4-5}
 & & & & 2
\end{array}
$$

$$
\begin{aligned}
&\int \frac{x^3+x}{x-1} \mathrm{d}x \\
= \quad &\int \left(x^2 + x + 2 + \frac{2}{x-1} \right) \mathrm{d}x
\end{aligned}
$$

Example

Find $\int \frac{x^3+x}{x-1}\mathrm{d}x$.

$$
\begin{array}{r|l}
 & x^2 + x + 2 \\
\hline
x-1 & x^3 + x \\
 & x^3 - x^2 \\
\hline
 & x^2 + x \\
 & x^2 - x \\
\hline
 & 2x \\
 & 2x - 2 \\
\hline
 & 2
\end{array}
$$

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Example

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$$
\begin{array}{r}
x^2 + x + 2 \\
x-1 \overline{) x^3 + x} \\
\underline{x^3 - x^2} \\
x^2 + x \\
\underline{x^2 - x} \\
2x \\
\underline{2x - 2} \\
2
\end{array}
$$

$$
\int \frac{x^3+x}{x-1} \mathrm{d}x = \int \left(x^2 + x + 2 + \frac{2}{x-1} \right) \mathrm{d}x
$$

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Find $\int \frac{x^3+x}{x-1}\mathrm{d}x$.

$$
\begin{array}{r|l}
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\hline
x-1 & x^3 \qquad + x \\
 & x^3 - x^2 \\
\hline
 & \qquad x^2 + x \\
 & \qquad x^2 - x \\
\hline
 & \qquad\qquad 2x \\
 & \qquad\qquad 2x - 2 \\
\hline
 & \qquad\qquad\qquad 2
\end{array}
$$

$$
\begin{aligned}
& \int \frac{x^3+x}{x-1}\mathrm{d}x \\
= \quad & \int \left(x^2 + x + 2 + \frac{2}{x-1}\right)\mathrm{d}x \\
= \quad & \frac{x^3}{3} + \frac{x^2}{2} + 2x \\
& \qquad + 2\ln|x-1| + C
\end{aligned}
$$

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- Factoring of $Q(x)$ can always be done in quadratic and linear terms as asserted in the following.

Corollary (Corollary to the Fundamental Theorem of Algebra)

Let $Q(x)$ be a polynomial (with real coefficients). Then $Q(x)$ can be factored as a product of terms of the form $(ax+b)^n$ (powers of linear terms) and product of terms of the form $(ax^2+bx+c)^n$ with $b^2-4ac<0$ (powers of quadratic terms).

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- The above result is a corollary to the Fundamental Theorem of Algebra.

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- The above result is a corollary to the Fundamental Theorem of Algebra. We state the Fundamental Theorem of algebra without proving it.

Theorem (The Fundamental Theorem of Algebra)

Every polynomial has at least one complex root.

- Let $\frac{R(x)}{Q(x)}$ be a rational function with $\deg Q > \deg R$.

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- Suppose $Q(x)$ factors into factors of the form

$$(ax+b)^N \qquad \text{and} \qquad (ax^2+bx+c)^M.$$

- Then we can split $\frac{R(x)}{Q(x)}$ into sum of partial fractions of the form

$$\frac{A_i}{(ax+b)^i}, \text{ with } i \leq N \qquad \text{or} \qquad \frac{B_j x + C_j}{(ax^2+bx+c)^j}, \text{ with } j \leq M,$$

- Let $\frac{R(x)}{Q(x)}$ be a rational function with $\deg Q > \deg R$.
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- Thus the total number of constants used equals the degree of Q.
- The difficulty of finding the constants A_i, B_j, C_j increases as the number of distinct factors increases, as well as when the exponents of those factors increase.

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- Suppose $Q(x)$ is a product of distinct linear factors:

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- We show how to find $A_1, A_2, \cdots, A_k$ on examples.

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Find $\int \frac{x^2+2x-1}{2x^3+3x^2-2x} \mathrm{d}x$.

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- In a similar fashion we add more partial fractions to account for all other terms of the form $(a_sx + b_s)^t$.

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- Then the partial fraction decomposition should include summands of the form

$$\frac{A_1 x + B_1}{ax^2 + bx + c} + \frac{A_2 x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_r x + B_r}{(ax^2 + bx + c)^r}$$

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For example of this size it makes sense to use a computer algebra system; one such system easily produces the decomposition:

$$= \frac{-1}{x} + \frac{\frac{1}{8}}{x-1} + \frac{-x-1}{(x^2+x+1)} + \frac{\frac{15}{8}x-\frac{1}{8}}{(x^2+1)} + \frac{\frac{3}{4}x+\frac{3}{4}}{(x^2+1)^2} + \frac{-\frac{x}{2}+\frac{1}{2}}{(x^2+1)^3}.$$