Master Problem Sheet
Calculus I

This master problem sheet contains all freecalc problems on the topics studied in Calculus I. For a list of contributors/authors of the freecalc project (and in particular, the present problem collection) see file contributors.tex.

Contents

1 Functions, Basic Facts

1.1 Understanding function notation

Problem 1. *Evaluate the difference quotient and simplify your answer.*

1. $\dfrac{f(2+h)-f(2)}{h}$, *where* $f(x)=x^2-x-1$.

   answer: $h+3$

2. $\dfrac{f(a+h)-f(a)}{h}$, *where* $f(x)=x^2$.

   answer: $h+2a$

3. $\dfrac{f(a+h)-f(a)}{h}$, *where* $f(x)=x^3$.

   answer: h^2+3a^2+3ah

4. $\dfrac{f(a+h)-f(a)}{h}$, *where* $f(x)=x^4$.

   answer: $6a^2h+4ah^2+h^3+4a^3$

5. $\dfrac{f(x)-f(a)}{x-a}$, *where* $f(x)=\frac{1}{x}$.

   answer: $-\frac{1}{ax}$.

6. $\dfrac{f(x)-f(1)}{x-1}$, *where* $f(x)=\frac{x-1}{x+1}$.

   answer: $\frac{1}{x+1}$.

1.2 Domains and ranges

Problem 2. *Find the implied domain of the function.*

1. $f(x) = \dfrac{x+4}{x^2-4}$.

answer: $x \neq \pm 2$, *alternatively:* $x \in (-\infty, -2) \cup (-2, 2) \cup (2, \infty)$

2. $f(x) = \dfrac{2x^3-5}{x^2+5x+6}$.

answer: $x \neq -2, -3$, *alternatively:* $x \in (-\infty, -3) \cup (-3, -2) \cup (-2, \infty)$

3. $f(t) = \sqrt[3]{3t-1}$.

answer: $x \in \mathbb{R}$ *(the domain is all real numbers)*

4. $g(t) = \sqrt{5-t} - \sqrt{1+t}$.

answer: $x \in [-1, 5]$.

5. $h(x) = \dfrac{1}{\sqrt[6]{x^2-7x}}$.

answer: $x \in (-\infty, 0) \cup (7, \infty)$.

6. $f(u) = \dfrac{u+1}{1+\frac{1}{u+1}}$.

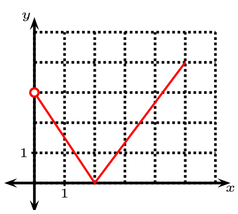
answer: $u \neq -1, -2$ *or* $u \in (-\infty, -2) \cup (-2, -1) \cup (-1, \infty)$..

7. $F(x) = \sqrt{10 - \sqrt{x}}$.

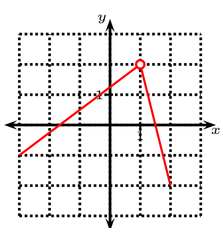
answer: $x \in [0, 100]$

1.3 Piecewise Defined Functions

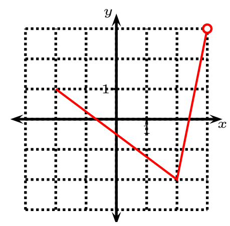
Problem 3. *Write down a formula for a function whose graphs is given below. The graphs are up to scale. Please note that there is more than one way to write down a correct answer.*

1. 

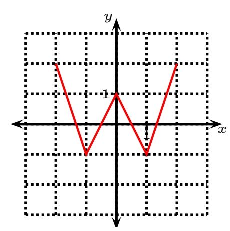
answer: $y = \begin{cases} -\frac{3}{2}x + 3 & \text{if } 0 < x \leq 2 \\ \frac{3}{4}x - \frac{3}{2} & \text{if } 2 < x \leq 6 \end{cases}$

2. 

answer: $y = \begin{cases} \frac{3}{4}x + \frac{5}{4} & \text{if } -3 \leq x < 1 \\ -4x + 6 & \text{if } 1 < x \leq 2 \end{cases}$

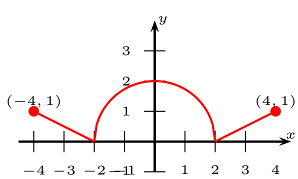
3. 

answer: $y = \begin{cases} -\frac{3}{4}x - \frac{1}{2} & \text{if } -2 \leq x < 2 \\ 5x - 12 & \text{if } 2 \leq x < 3 \end{cases}$

4. 

answer: $y = \begin{cases} -3x - 4 & \text{if } -2 \leq x < -1 \\ 2x + 1 & \text{if } -1 \leq x < 0 \\ -2x + 1 & \text{if } 0 \leq x < 1 \\ 3x - 4 & \text{if } 1 \leq x \leq 2 \end{cases}$

Problem 4. *Write down formulas for function whose graphs are as follows. The graphs are up to scale. All arcs are parts of circles.*

1. 

Problem 5. *Plot the piecewise defined functions by hand. Compare your answer to the plot of a computer algebra system.*

1. $G(x) = \frac{x+|x|}{2x}$.
2. $g(x) = |x| - x$.
3. $f(x) = \begin{cases} x & x \leq 1 \\ x^2 & x \geq 1 \end{cases}$.

1.4 Function composition

Problem 6. *Find the functions $f \circ g$, $g \circ f$, $f \circ f$ and $g \circ g$ and their implied domains. The answer key has not been proofread, use with caution.*

1. $f(x) = x^2 + 1$, $g(x) = x + 1$.

answer: *in some order:* $(x+1)^2 + 1$, $(x)^2 + 2$, $((x)^2 + 1)^2 + 1$, $x + 2$
Domain, *all 4 cases:* $x \in \mathbb{R}$ *(all reals)*

2. $f(x) = \sqrt{x+1}$, $g(x) = x + 1$.

answer: *Domain of* $f \circ g$ *is* $x \geq -2$. *Domain of* $g \circ f$ *is* $x \geq -1$. *Domain of* $f \circ f$ *is* $x \geq -1$. *Domain of* $g \circ g$ *is all reals* $(x \in \mathbb{R})$. *in some order:* $\sqrt{2+x}$, $1+\sqrt{1+x}$, $\sqrt{1+\sqrt{1+x}}$, $2+x$

3. $f(x) = 2x$, $g(x) = \tan x$.

In this subproblem, you are not required to find the domain.

answer: *Domain* $f \circ f$*: all reals* $(x \in \mathbb{R})$. *Domain* $g \circ f$*:* $x \neq (2k+1)\frac{\pi}{2}$ *for all* $k \in \mathbb{Z}$. *Domain* $f \circ g$*:* $x \neq (4k+1)\frac{\pi}{4}, x \neq (4k+3)\frac{\pi}{4}$ *for all* $k \in \mathbb{Z}$. *Domain* $g \circ g$*:* $x \neq (2k+1)\frac{\pi}{2}$ *and* $x \neq k\pi + \arctan\left(\frac{\pi}{2}\right)$ *for all* $k \in \mathbb{Z}$. *in some order:* $2\tan x$, $\tan(2x)$, $4x$, $\tan(\tan x)$

4. $f(x) = \dfrac{x+1}{x-1}$, $g(x) = \dfrac{x-1}{x+1}$.

answer: *Domain* $f \circ f$*:* $x \neq 1$. *Domain* $g \circ g$*:* $x \neq 0, x \neq -1$. *Domain* $f \circ g$*:* $x \neq -1$. *Domain* $g \circ f$*:* $x \neq 0, x \neq 1$. *in some order:* $-x$, $\frac{1}{x}$, x, $-\frac{1}{x}$

Problem 7. *Compute the composite functions* $(f \circ g)(x)$, $(g \circ f)(x)$. *Simplify your answer to a single fraction. Find the domain of the composite function.*

1. $f(x) = \dfrac{x+2}{x-2}, g(x) = \dfrac{x-1}{x+2}$.

answer: $(f \circ g)(x) = \frac{3+3x}{-5-x}$, $x \neq -2, -5$; $(g \circ f)(x) = \frac{4}{-2+3x}$, $x \neq 2, \frac{2}{3}$

2. $f(x) = \dfrac{x+1}{3x-2}, g(x) = \dfrac{x-2}{x-1}$.

answer: $(f \circ g)(x) = \frac{-3+2x}{-4+x}$, $x \neq 4, 1$; $(g \circ f)(x) = \frac{5-5x}{3-2x}$, $x \neq \frac{2}{3}, \frac{3}{2}$

3. $f(x) = \dfrac{2x+1}{3x-1}, g(x) = \dfrac{x-2}{2x-1}$.

answer: $(f \circ g)(x) = \frac{-5+4x}{-5+x}$, $x \neq 5, \frac{1}{2}$; $(g \circ f)(x) = \frac{3-4x}{3+x}$, $x \neq -3, \frac{1}{3}$

4. $f(x) = \dfrac{x+1}{x-2}, g(x) = \dfrac{x+2}{2x-1}$.

answer: $(f \circ g)(x) = \frac{1+3x}{4-3x}$, $x \neq \frac{4}{3}, \frac{1}{2}$; $(g \circ f)(x) = \frac{-3+3x}{4+x}$, $x \neq -4, 2$

5. $f(x) = \dfrac{5x+1}{4x-1}, g(x) = \dfrac{4x-1}{3x+1}$.

answer: $(f \circ g)(x) = \frac{-4+23x}{-5+13x}$, $x \neq \frac{5}{13}, -\frac{1}{3}$; $(g \circ f)(x) = \frac{5+16x}{2+19x}$, $x \neq -\frac{2}{19}, \frac{1}{4}$

6. $f(x) = \dfrac{3x-5}{x-2}$, $g(x) = \dfrac{x-2}{x-4}$.

answer: $(f \circ g)(x) = \frac{-2x+14}{-x+6}$, $x \neq 6, 4$; $(g \circ f)(x) = \frac{x-1}{-x+3}$, $x \neq 3, 2$

7. $f(x) = \dfrac{x-3}{x+2}$, $g(y) = \dfrac{y+3}{y-4}$.

answer: $(f \circ g)(x) = \frac{-2x+15}{3x-5}$, $x \neq \frac{5}{3}, 4$; $(g \circ f)(x) = \frac{4x+3}{-3x-11}$, $x \neq -\frac{11}{3}, -2$

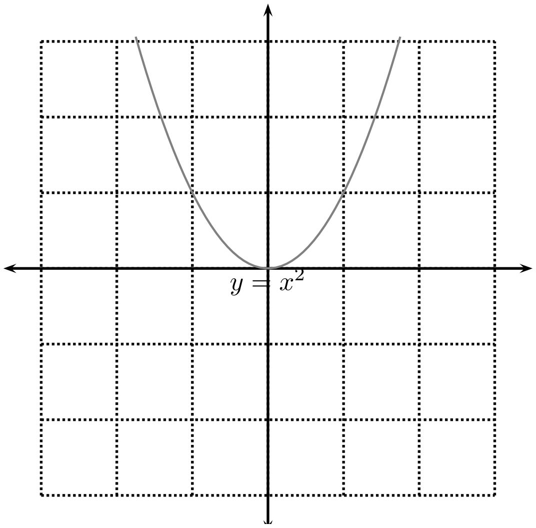
1.5 Linear Transformations and Graphs of Functions

Problem 8. *Graph the functions roughly by hand, by applying consecutively the transformations learned in class. Each consequtive graph is a transformations of the preceding one. Compare your answer with the graph produced by a graphic calculator.*

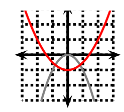
1. $y = \dfrac{1}{x}$.
2. $y = \dfrac{1}{x+1}$.
3. $y = \dfrac{1}{2x+1}$.
4. $y = \dfrac{3}{2x+1}$.
5. $y = \dfrac{3+x}{2x+1}$.
6. $y = \left|\dfrac{3+x}{2x+1}\right|$.

Problem 9. *Sketch by hand approximately the given function. The function is obtained by transforming linearly the graph of a known function. The known function has been sketched for you by computer.*

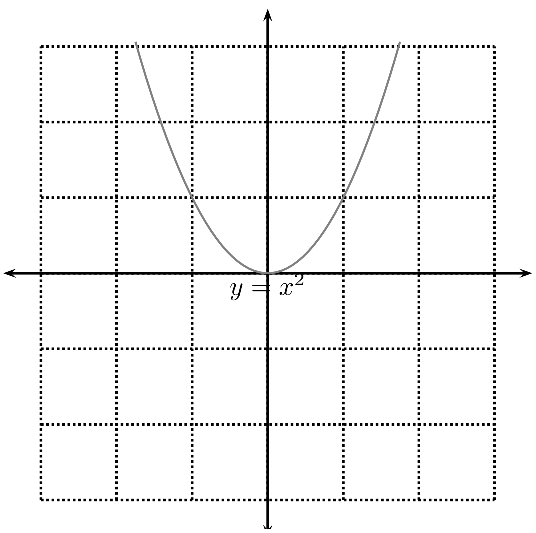
1. $f(x) = -\frac{1}{2}x^2 + 1.$



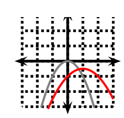
answer:



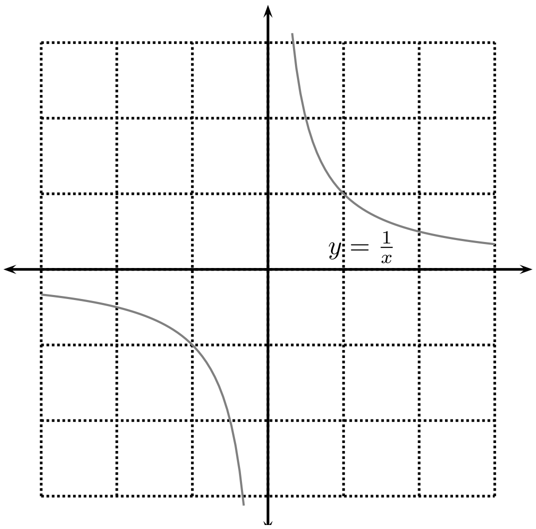
2. $f(x) = \frac{1}{2}x^2 + x - 1.$

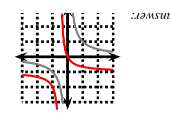


answer:

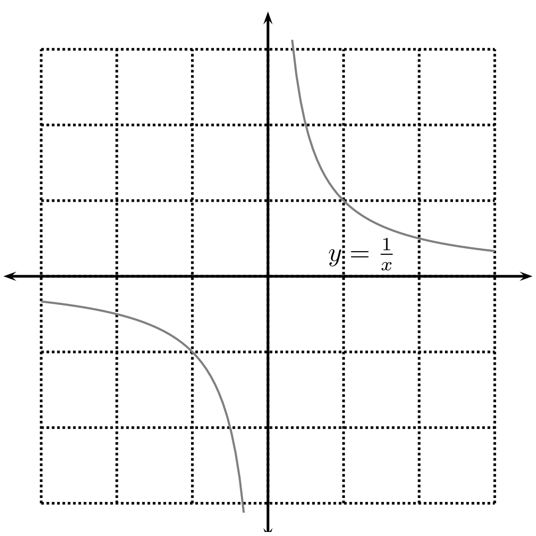


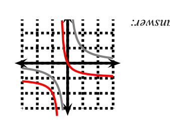
3. $f(x) = \dfrac{1}{2x-1} + 1.$



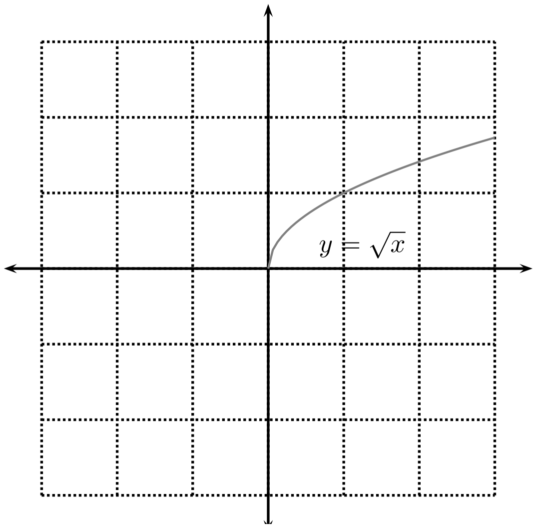


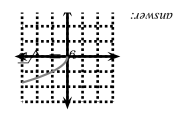
4. $f(x) = \dfrac{\frac{1}{2}x + \frac{1}{4}}{x - \frac{1}{2}} + \dfrac{1}{2}.$



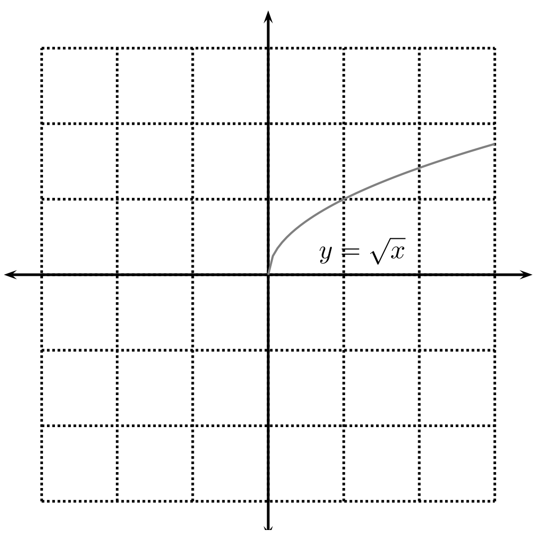


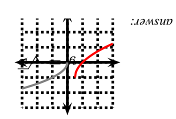
5. $f(x) = -\sqrt{2x-1} - 1$





6. $f(x) = -\sqrt{-2x-1} + 1$





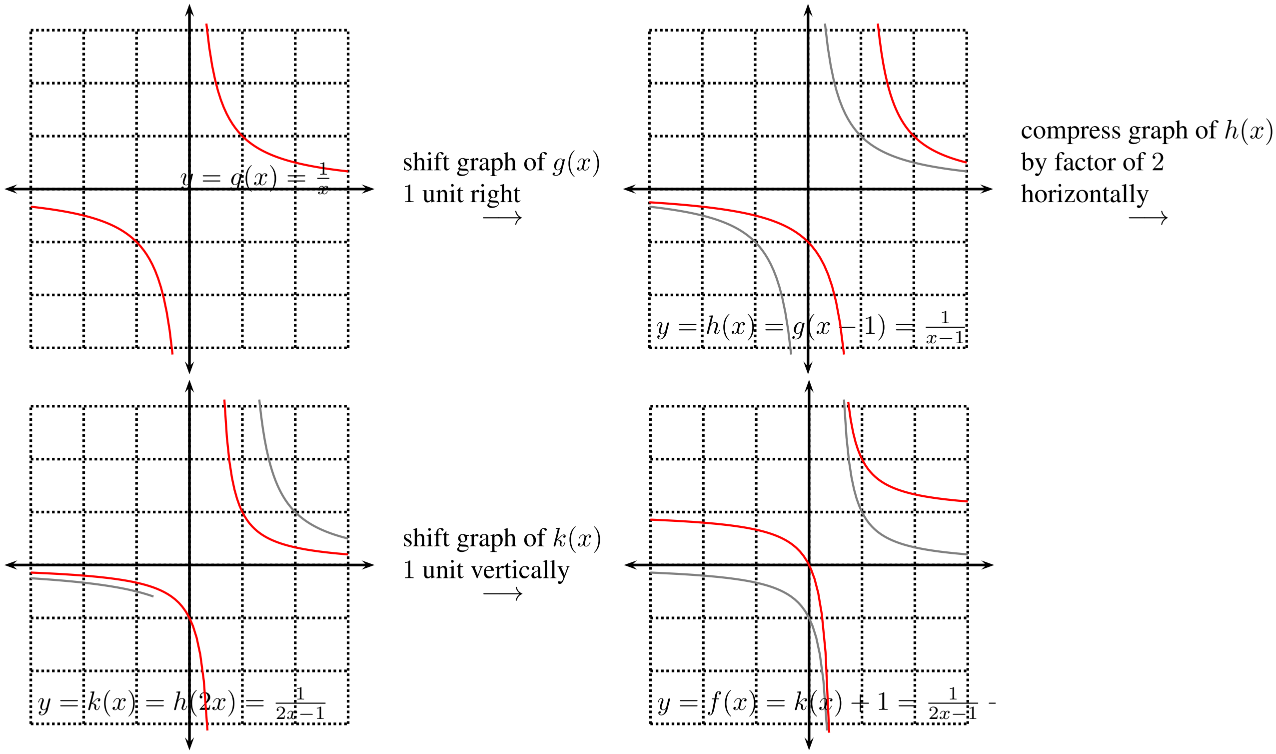
Solution. 9.3.

We are asked to plot $f(x) = \dfrac{1}{2x-1} + 1$ by linearly transforming the graph of $g(x) = \frac{1}{x}$ to the graph of $f(x)$. To do that we have to compose g with a sequence of linear transformations to obtain $f(x)$. There are two natural ways to do that; we show both by presenting two different solutions.

Solution I. We show how to get from $g(x) = \frac{1}{x}$ to $f(x)$ by composing g with a sequence of linear transformations.

$$\begin{array}{r|rcl} g(x) & & = & \frac{1}{x} \\ \text{Define } h(x) \text{ via:} & h(x) & = & g(x+1) = \dfrac{1}{x-1} \\ \text{Define } k(x) \text{ via:} & k(x) & = & h(2x) = \dfrac{1}{2x-1} \\ \text{Therefore} & f(x) & = & k(x) + 1 = \dfrac{1}{2x-1} + 1 \end{array}$$

We plot consecutively the functions $g(x)$, $h(x)$, $k(x)$ and $f(x)$. We start from the given graph of $g(x)$.



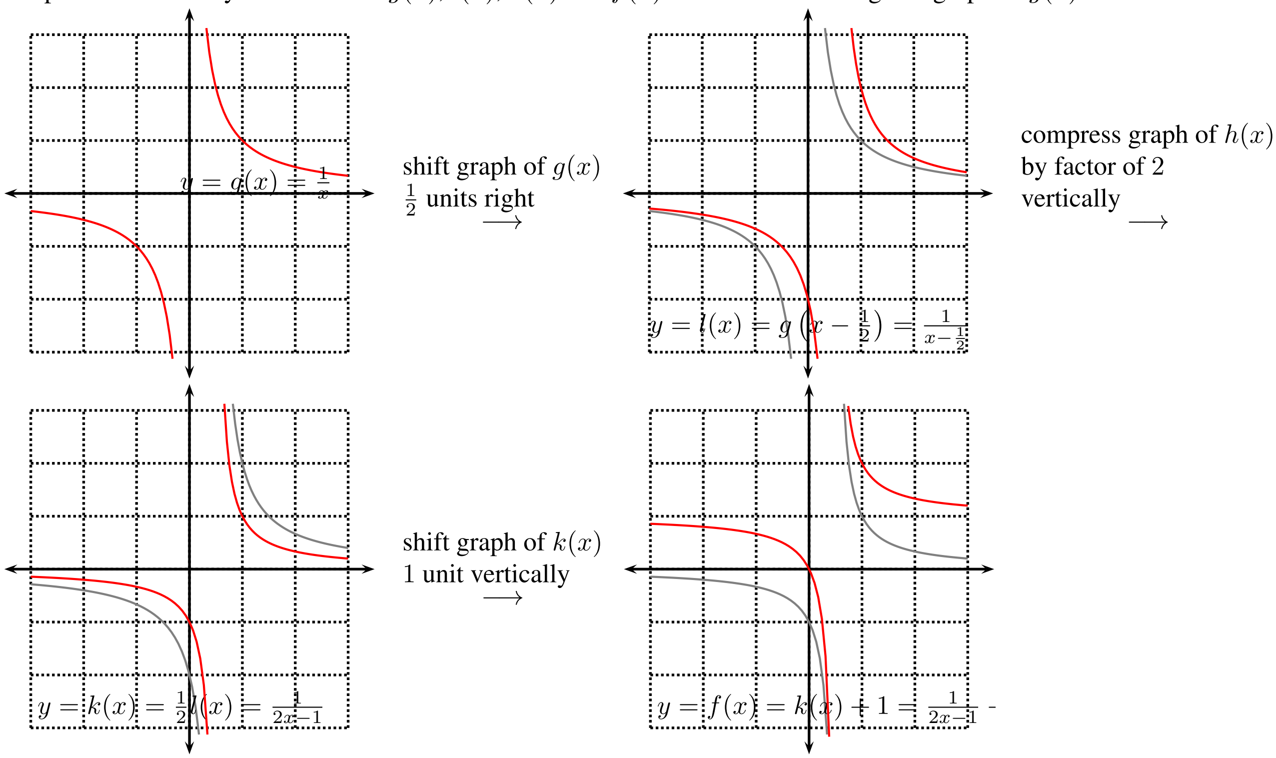
Solution II. In the previous solution we used horizontal stretch to transform the graph of $h(x)$ to the graph of $k(x) = h(2x)$. Algebra suggests a second way to transform the graph of $g(x)$ to the graph of $f(x)$, this time using a vertical stretch. Indeed, we have the equality

$$f(x) = \frac{1}{2x-1} + 1 = \frac{1}{2} \cdot \frac{1}{x - \frac{1}{2}} + 1.$$

Therefore we can carry out the sequence of transformations shown below.

$$\begin{array}{r|rcl}
g(x) & & = & \frac{1}{x} \\
\text{Define } l(x) \text{ via:} & l(x) & = & g\left(x - \frac{1}{2}\right) = \frac{1}{x - \frac{1}{2}} \\
\text{Define } k(x) \text{ via:} & k(x) & = & \frac{1}{2}h(x) = \frac{1}{2} \cdot \frac{1}{\left(x - \frac{1}{2}\right)} = \frac{1}{(2x-1)} \\
\text{Therefore} & f(x) & = & k(x) + 1 = \frac{1}{2x-1} + 1
\end{array}$$

We plot consecutively the functions $g(x)$, $l(x)$, $k(x)$ and $f(x)$. We start from the given graph of $g(x)$.



2 Trigonometry

2.1 Angle conversion

Problem 10. *Convert from degrees to radians.*

1. 15°.
   answer: $\frac{\pi}{12} \approx 0.261799388$
2. 30°.
   answer: $\frac{\pi}{6} \approx 0.523598776$
3. 36°.
   answer: $\frac{\pi}{5} \approx 0.628318531$
4. 45°.
   answer: $\frac{\pi}{4} \approx 0.785398163$
5. 60°.
   answer: $\frac{\pi}{3} \approx 1.047197551$
6. 75°.
   answer: $\frac{5\pi}{12} \approx 1.308997$
7. 90°.
   answer: $\frac{\pi}{2}$
8. 120°.
   answer: $\frac{2\pi}{3}$
9. 135°.
   answer: $\frac{3\pi}{4}$
10. 150°.
   answer: $\frac{5\pi}{6}$
11. 180°.
   answer: π
12. 225°.
   answer: $\frac{5\pi}{4}$
13. 270°.
   answer: $\frac{3\pi}{2}$
14. 305°.
   answer: $\frac{61\pi}{36} \approx 5.323254$
15. 360°.
   answer: 2π
16. 405°.
   answer: $\frac{9\pi}{4}$
17. 1200°.
   answer: $\frac{20\pi}{3}$
18. -900°.
   answer: -5π
19. -2014°.
   answer: $-\frac{1007}{90}\pi \approx -35.150931$

Problem 11. *Convert from radians to degrees. The answer key has not been proofread, use with caution.*

1. 4π.
   answer: 720°
2. $-\frac{7}{6}\pi$.
   answer: -210°
3. $\frac{7}{12}\pi$.
   answer: 105°
4. $\frac{4}{3}\pi$.
   answer: 240°
5. $-\frac{3}{8}\pi$.
   answer: -67.5°
6. 2014π.
   answer: 362520°

7. 5.

answer: $\left(\frac{900}{\pi}\right)^\circ \approx 286^\circ$

8. -2014.

answer: -362520°

2.2 Trigonometry identities

Problem 12. *Prove the trigonometry identities.*

1. $\sin\theta\cot\theta = \cos\theta$.
2. $(\sin\theta+\cos\theta)^2 = 1+\sin(2\theta)$.
3. $\sec\theta - \cos\theta = \tan\theta\sin\theta$.
4. $\tan^2\theta - \sin^2\theta = \tan^2\theta\sin^2\theta$.
5. $\cot^2\theta + \sec^2\theta = \tan^2\theta + \csc^2\theta$.
6. $2\csc(2\theta) = \sec\theta\csc\theta$.
7. $\tan(2\theta) = \dfrac{2\tan\theta}{1-\tan^2\theta}$.
8. $\dfrac{1}{1-\sin\theta} + \dfrac{1}{1+\sin\theta} = 2\sec^2\theta$.
9. $\tan\alpha + \tan\beta = \dfrac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta}$.
10. $\tan(\alpha+\beta) = \dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$.
11. $\sin(3\theta)+\sin\theta = 2\sin(2\theta)\cos\theta$.
12. $\cos(3\theta) = 4\cos^3\theta - 3\cos\theta$.
13. $1+\tan^2\theta = \sec^2\theta$.
14. $1+\csc^2\theta = \cot^2\theta$.
15. $2\cos^2(2x) = 2\sin^4\theta + 2\cos^4\theta - \sin^2(2\theta)$.
16. $\dfrac{1+\tan\left(\frac{\theta}{2}\right)}{1-\tan\left(\frac{\theta}{2}\right)} = \tan\theta + \sec\theta$.

2.3 Trigonometry equations

Problem 13. *Find all values of x in the interval $[0, 2\pi]$ that satisfy the equation.*

1. $2\cos x - 1 = 0$.

   answer: $x = \frac{\pi}{3}$ or $x = \frac{5\pi}{3}$

2. $\sin(2x) = \cos x$.

   answer: $x = \frac{\pi}{2}$, $x = \frac{3\pi}{2}$, $x = \frac{\pi}{6}$, or $x = \frac{5\pi}{6}$

3. $\sqrt{3}\sin x = \sin(2x)$.

   answer: $x = \frac{\pi}{6}, \frac{11\pi}{6}, 0, \pi, 2\pi$

4. $2\sin^2 x = 1$.

   answer: $x = \frac{\pi}{4}$, $x = \frac{3\pi}{4}$, $x = \frac{5\pi}{4}$, or $x = \frac{7\pi}{4}$

5. $2+\cos(2x) = 3\cos x$.

   answer: $x = 0$, $x = 2\pi$, $x = \frac{\pi}{3}$, or $x = \frac{5\pi}{3}$

6. $2\cos x + \sin(2x) = 0$.

   answer: $x = \frac{\pi}{2}$, $x = \frac{3\pi}{2}$

7. $2\cos^2 x - (1+\sqrt{2})\cos x + \dfrac{\sqrt{2}}{2} = 0$.

   answer: $x = \frac{\pi}{4}, \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{4}$

8. $|\tan x| = 1$.

   answer: $x = \frac{\pi}{4}$, $x = \frac{3\pi}{4}$, $x = \frac{5\pi}{4}$, or $x = \frac{7\pi}{4}$

9. $3\cot^2 x = 1$.

   answer: $x = \frac{\pi}{3}$, $x = \frac{2\pi}{3}$, $x = \frac{4\pi}{3}$, or $x = \frac{5\pi}{3}$

10. $\sin x = \tan x$.

    answer: $x = 0$, $x = \pi$, or $x = 2\pi$

Solution. 13.7 Set $\cos x = u$. Then

$$2\cos^2 x - (1+\sqrt{2})\cos x + \frac{\sqrt{2}}{2} = 0$$

becomes

$$2u^2 - (1+\sqrt{2})u + \frac{\sqrt{2}}{2} = 0.$$

This is a quadratic equation in u and therefore has solutions

$$\begin{aligned} u_1, u_2 &= \frac{1+\sqrt{2} \pm \sqrt{(1+\sqrt{2})^2 - 4\sqrt{2}}}{4} \\ &= \frac{1+\sqrt{2} \pm \sqrt{1-2\sqrt{2}+2}}{4} \\ &= \frac{1+\sqrt{2} \pm \sqrt{(1-\sqrt{2})^2}}{4} \\ &= \frac{1+\sqrt{2} \pm (1-\sqrt{2})}{4} = \begin{cases} \frac{1}{2} & \text{or} \\ \frac{\sqrt{2}}{2} \end{cases} \end{aligned}$$

Therefore $u = \cos x = \frac{1}{2}$ or $u = \cos x = \frac{\sqrt{2}}{2}$, and, as x is in the interval $[0, 2\pi]$, we get $x = \frac{\pi}{3}, \frac{5\pi}{3}$ (for $\cos x = \frac{1}{2}$) or $x = \frac{\pi}{4}, \frac{7\pi}{4}$ (for $\cos x = \frac{\sqrt{2}}{2}$).

3 Limits and Continuity

3.1 Limits as x tends to a number

Problem 14. *Evaluate the limit if it exists.*

1. $\displaystyle\lim_{x\to 2} \frac{x^2-5x+6}{x-2}$.

   answer: -1

2. $\displaystyle\lim_{x\to 3} \frac{x^2-3x}{x^2-2x-3}$.

   answer: $\frac{3}{4}$

3. $\displaystyle\lim_{x\to -2} \frac{2x^2+x-6}{x^2-4}$

   answer: $\frac{7}{4}$

4. $\displaystyle\lim_{x\to 2} \frac{x^2-5x-6}{x-2}$.

   answer: DNE

5. $\displaystyle\lim_{x\to -1} \frac{x^2-3x}{x^2-2x-3}$.

   answer: DNE

6. $\displaystyle\lim_{x\to -2} \frac{x^2-4}{2x^2+5x+2}$.

   answer: $\frac{4}{3}$

7. $\displaystyle\lim_{x\to -1} \frac{2x^2+3x+1}{3x^2-2x-5}$.

   answer: $\frac{1}{8}$

8. $\displaystyle\lim_{x\to -4} \frac{x^2+7x+12}{x^2+6x+8}$.

   answer: $\frac{1}{2}$

9. $\displaystyle\lim_{h\to 0} \frac{(-3+h)^2-9}{h}$.

   answer: -6

10. $\displaystyle\lim_{h\to 0} \frac{(-2+h)^3+8}{h}$.

    answer: 12

11. $\displaystyle\lim_{x\to -3} \frac{x+3}{x^3+27}$.

    answer: $\frac{1}{27}$

12. $\displaystyle\lim_{x\to 1} \frac{x^4-1}{x^3-1}$.

    answer: $\frac{4}{3}$

13. $\displaystyle\lim_{h\to 0} \frac{\sqrt{4+h}-2}{h}$.

    answer: $\frac{1}{4}$

14. $\displaystyle\lim_{x\to 3} \frac{\sqrt{5x+1}-4}{x-3}$.

    answer: $\frac{5}{8}$

15. $\displaystyle\lim_{x\to -3} \frac{\sqrt{x^2+16}-5}{x+3}$.

    answer: $-\frac{3}{5}$

16. $\displaystyle\lim_{x\to -3} \frac{\frac{1}{3}+\frac{1}{x}}{3+x}$.

    answer: $-\frac{1}{9}$

17. $\displaystyle\lim_{x\to -2} \frac{x^2+4x+4}{x^4-16}$.

    answer: 0

18. $\displaystyle\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x}$.

    answer: 1

19. $\displaystyle\lim_{x\to 0} \left(\frac{1}{x}-\frac{1}{x^2+x}\right)$.

    answer: 1

20. $\displaystyle\lim_{x\to 9} \frac{3-\sqrt{x}}{9x-x^2}$.

    answer: $\frac{1}{54}$

21. $\displaystyle\lim_{h\to 0} \frac{(2+h)^{-1}-2^{-1}}{h}$.

    answer: $-\frac{1}{4}$

22. $\displaystyle\lim_{x\to 0} \left(\frac{1}{x\sqrt{1+x}}-\frac{1}{x}\right)$.

    answer: $-\frac{1}{2}$

23. $\displaystyle\lim_{h\to 0} \frac{(x+h)^3-x^3}{h}$.

    answer: $3x^2$

24. $\displaystyle\lim_{h\to 0} \frac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h}$.

    answer: $-\frac{2}{x^3}$

25. $\displaystyle\lim_{h\to 0} \frac{\frac{1}{(2+h)^2}-\frac{1}{4}}{h}$.

    answer: $-\frac{1}{4}$

26. $\displaystyle\lim_{h\to 0} \frac{\frac{1}{(1+h)^2}-1}{h}$.

    answer: -2

Solution. 14.1

$$\begin{aligned}
\lim_{x\to 2}\frac{x^2-5x+6}{x-2} &= \lim_{x\to 2}\frac{(x-3)\cancel{(x-2)}}{\cancel{x-2}} && \Big| \text{ factor and cancel}\\
&= 2-3=-1
\end{aligned}$$

Solution. 14.3

$$\begin{aligned}
\lim_{x\to -2}\frac{2x^2+x-6}{x^2-4} &= \lim_{x\to -2}\frac{(2x-3)\cancel{(x+2)}}{(x-2)\cancel{(x+2)}} && \Big| \text{ factor and cancel}\\
&= \frac{(2(-2)-3)}{-2-2} && \Big| \text{ substitute}\\
&= \frac{7}{4}
\end{aligned}$$

Solution. 14.6

$$\begin{aligned}
\lim_{x\to 2}\frac{x^2-4}{2x^2+5x+2} &= \lim_{x\to -2}\frac{(x-2)\cancel{(x+2)}}{(2x+1)\cancel{(x+2)}} && \Big| \text{ factor and cancel}\\
&= \frac{(-2)-2}{2(-2)+1}=\frac{4}{3}.
\end{aligned}$$

Solution. 14.7

$$\begin{aligned}
\lim_{x\to -1}\frac{2x^2+3x+1}{3x^2-2x-5} &= \lim_{x\to -1}\frac{(2x+1)\cancel{(x+1)}}{(3x-5)\cancel{(x+1)}} && \Big| \text{ factor and cancel}\\
&= \frac{2(-1)+1}{3(-1)-5}=\frac{1}{8}.
\end{aligned}$$

Solution. 14.8.

$$\begin{aligned}
\lim_{x\to -4}\frac{x^2+7x+12}{x^2+6x+8} &= \lim_{x\to -4}\frac{(x+3)\cancel{(x+4)}}{(x+2)\cancel{(x+4)}} && \Big| \text{ factor}\\
&= \frac{-4+3}{-4+2}=-\frac{1}{2}.
\end{aligned}$$

Solution. 14.24

$$\begin{aligned}
\lim_{h\to 0}\frac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h} &= \lim_{h\to 0}\frac{x^2-(x+h)^2}{hx^2(x+h)^2}=\lim_{h\to 0}\frac{x^2-(x^2+2xh+h^2)}{hx^2(x+h)^2}\\
&= \lim_{h\to 0}\frac{\cancel{h}(-2x+h)}{\cancel{h}x^2(x+h)^2}=\frac{-2x+0}{x^2(x+0)^2}=-\frac{2}{x^3}.
\end{aligned}$$

Solution. 14.25.

Variant I.

$$\begin{aligned}
\lim_{h\to 0}\frac{\frac{1}{(2+h)^2}-\frac{1}{4}}{h} &= \lim_{h\to 0}\frac{\frac{4-(2+h)^2}{4(2+h)^2}}{h}\\
&= \lim_{h\to 0}\frac{4-(4+4h+h^2)}{4h(2+h)^2}\\
&= \lim_{h\to 0}\frac{-4h-h^2}{4h(2+h)^2}\\
&= \lim_{h\to 0}\frac{\cancel{h}(-4-h)}{4\cancel{h}(2+h)^2} && \Big| \text{ substitute } h=0\\
&= \frac{-4-0}{4(2+0)^2}\\
&= -\frac{1}{4}
\end{aligned}$$

Variant II.

$$\begin{aligned}
\lim_{h\to 0}\frac{\frac{1}{(2+h)^2}-\frac{1}{4}}{h} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x^2}\right)_{|x=2}\\
&= \left(\frac{-2}{x^3}\right)_{|x=2}\\
&= -\frac{1}{4}
\end{aligned}$$

Solution. 14.26.

Variant I.

$$\begin{aligned}
\lim_{h\to 0}\frac{\frac{1}{(1+h)^2}-1}{h} &= \lim_{h\to 0}\frac{\frac{1-(1+h)^2}{(1+h)^2}}{h}\\
&= \lim_{h\to 0}\frac{1-(1+2h+h^2)}{h(1+h)^2}\\
&= \lim_{h\to 0}\frac{-2h-h^2}{h(1+h)^2}\\
&= \lim_{h\to 0}\frac{\cancel{h}(-2-h)}{\cancel{h}(1+h)^2} && \Big|\ \text{substitute } h=0\\
&= \frac{-2-0}{(1+0)^2}\\
&= -2.
\end{aligned}$$

Variant II.

$$\begin{aligned}
\lim_{h\to 0}\frac{\frac{1}{(1+h)^2}-1}{h} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x^2}\right)_{|x=1} && \Big|\ \text{derivative definition}\\
&= \left(\frac{-2}{x^3}\right)_{|x=1}\\
&= -2.
\end{aligned}$$

Problem 15. *Evaluate the limit if it exists.*

1. $\displaystyle\lim_{x\to 1}\frac{3x^2+4x-7}{x^3-x}$ *answer: 5.*
2. $\displaystyle\lim_{x\to -1}\frac{2x^2-3x-5}{x^3+1}$ *answer: $-\frac{7}{3}$.*

Problem 16. *Evaluate the limits. Justify your computations.*

1. $\displaystyle\lim_{x\to 2}2x^2-3x-6.$ *answer: -4*
2. $\displaystyle\lim_{x\to -1}\frac{x^4-x}{x^2+2x+3}.$ *answer: 1*
3. $\displaystyle\lim_{x\to -1}\frac{1}{x^2-3x+2}.$ *answer: $\frac{1}{6}$*
4. $\displaystyle\lim_{x\to -2}\sqrt{x^4+16}.$ *answer: $\sqrt{32}$*
5. $\displaystyle\lim_{x\to 8}(1+\sqrt[3]{x})(2-x).$ *answer: -18*

3.2 Limits involving ∞

3.2.1 Limits as $x\to\pm\infty$

Problem 17. *Find the limit or show that it does not exist. If the limit does not exist, indicate whether it is $\pm\infty$, or neither. The answer key has not been proofread, use with caution.*

1. $\displaystyle\lim_{x\to\infty}\frac{x-2}{2x+1}.$ *answer: $\frac{1}{2}$*
2. $\displaystyle\lim_{x\to\infty}\frac{1-x^2}{x^3-x-1}.$ *answer: 0*
3. $\displaystyle\lim_{x\to-\infty}\frac{x-2}{x^2+5}.$ *answer: 0*
4. $\displaystyle\lim_{x\to-\infty}\frac{3x^3+2}{2x^3-4x+5}.$ *answer: $\frac{3}{2}$*
5. $\displaystyle\lim_{x\to\infty}\frac{\sqrt{x}+x^2}{\sqrt{x}-x^2}.$ *answer: -1*
6. $\displaystyle\lim_{x\to\infty}\frac{3-x\sqrt{x}}{2x^{\frac{3}{2}}-2}.$ *answer: $-\frac{1}{2}$*
7. $\displaystyle\lim_{x\to\infty}\frac{(2x^2+3)^2}{(x-1)^2(x^2+1)}.$ *answer: 4*
8. $\displaystyle\lim_{x\to\infty}\frac{x^2-3}{\sqrt{x^4+3}}.$ *answer: 1*
9. $\displaystyle\lim_{x\to-\infty}\frac{\sqrt{x^2+1}}{x+1}.$ *answer: -1*
10. $\displaystyle\lim_{x\to\infty}\frac{\sqrt{16x^6-3x}}{x^3+2}.$ *answer: 4*
11. $\displaystyle\lim_{x\to-\infty}\frac{\sqrt{16x^6-3x}}{x^3+2}.$ *answer: -4*
12. $\displaystyle\lim_{x\to\infty}\frac{\sqrt{3x^2+2x+1}}{x+1}.$ *answer: $\sqrt{3}$*
13. $\displaystyle\lim_{x\to\infty}\sqrt{4x^2+x}-2x.$ *answer: $\frac{1}{4}$*
14. $\displaystyle\lim_{x\to-\infty}x+\sqrt{x^2+3x}.$ *answer: $-\frac{3}{2}$*
15. $\displaystyle\lim_{x\to\infty}\sqrt{x^2+2x}-\sqrt{x^2-2x}.$ *answer: 2.*
16. $\displaystyle\lim_{x\to-\infty}\sqrt{x^2+x}-\sqrt{x^2-x}.$ *answer: -1.*
17. $\displaystyle\lim_{x\to\infty}\sqrt{x^2+ax}-\sqrt{x^2+bx}.$ *answer: $\frac{a-b}{2}$*
18. $\displaystyle\lim_{x\to\infty}\cos x.$ *answer: DNE*
19. $\displaystyle\lim_{x\to\infty}\frac{x^4+x}{x^3-x+2}.$ *answer: ∞*

20. $\lim\limits_{x\to\infty}\sqrt{x^2+1}$.

answer: ∞

21. $\lim\limits_{x\to-\infty}(x^4+x^5)$.

answer: $-\infty$

22. $\lim\limits_{x\to-\infty}\dfrac{\sqrt{1+x^6}}{1+x^2}$.

answer: ∞

23. $\lim\limits_{x\to\infty}(x-\sqrt{x})$.

answer: ∞

24. $\lim\limits_{x\to\infty}(x^2-x^3)$.

answer: $-\infty$

25. $\lim\limits_{x\to\infty} x\sin x$.

answer: DNE

26. $\lim\limits_{x\to\infty}\sqrt{x}\sin x$.

answer: DNE

Solution. 17.4.

$$\begin{aligned}
\lim_{x\to-\infty}\frac{3x^3+2}{2x^3-4x+5} &= \lim_{x\to-\infty}\frac{\left(3x^3+2\right)\frac{1}{x^3}}{\left(2x^3-4x+5\right)\frac{1}{x^3}} \\
&= \lim_{x\to-\infty}\frac{3+\frac{2}{x^3}}{2-\frac{4}{x^2}+\frac{5}{x^3}} \\
&= \frac{3+0}{2-0+0}=\frac{3}{2}.
\end{aligned}$$

Divide top and bottom by highest term in denominator

Solution. 17.9

$$\begin{aligned}
\lim_{x\to-\infty}\frac{\sqrt{x^2+1}}{x+1} &= \lim_{x\to-\infty}\frac{\frac{1}{x}\sqrt{x^2+1}}{\frac{1}{x}(x+1)} = \lim_{x\to-\infty}\frac{-\frac{1}{\sqrt{x^2}}\sqrt{x^2+1}}{\frac{1}{x}(x+1)} \\
&= \lim_{x\to-\infty}\frac{-\sqrt{\frac{x^2+1}{x^2}}}{1+\frac{1}{x}} = \lim_{x\to-\infty}\frac{-\sqrt{1+\underbrace{\frac{1}{x^2}}_{\to 0}}}{1+\underbrace{\frac{1}{x}}_{x\to 0}} \\
&= 1.
\end{aligned}$$

$x=-\sqrt{x^2}$, whenever $x<0$

Solution. 17.11.

$$\begin{aligned}
\lim_{x\to-\infty}\frac{\sqrt{16x^6-3x}}{x^3+2} &= \lim_{x\to-\infty}\frac{\sqrt{x^6\left(16-\frac{3}{x^5}\right)}}{x^3+2} \\
&= \lim_{x\to-\infty}\frac{\sqrt{x^6}\sqrt{\left(16-\frac{3}{x^5}\right)}}{x^3+2} \\
&= \lim_{x\to-\infty}\frac{-x^3\sqrt{\left(16-\frac{3}{x^5}\right)}}{x^3+2} \\
&= \lim_{x\to-\infty}\frac{-x^3\sqrt{\left(16-\frac{3}{x^5}\right)}}{x^3+2} \\
&= \lim_{x\to-\infty}\frac{-\cancel{x^3}\sqrt{\left(16-\frac{3}{x^5}\right)}\frac{1}{\cancel{x^3}}}{\left(x^3+2\right)\frac{1}{x^3}} \\
&= \lim_{x\to-\infty}\frac{-\sqrt{\left(16-\underbrace{\frac{3}{x^5}}_{\to 0}\right)}}{1+\underbrace{\frac{2}{x^3}}_{\to 0}} \\
&= \frac{-\sqrt{16}}{1}=-4.
\end{aligned}$$

$\sqrt{x^6}=-x^3$ because $x<0$ as $x\to-\infty$

Divide by highest order term in denominator

Solution. 17.12

$$\begin{aligned}
\lim_{x\to\infty}\frac{\sqrt{3x^2+2x+1}}{x+1} &= \lim_{x\to\infty}\frac{\frac{1}{x}\sqrt{3x^2+2x+1}}{\frac{1}{x}(x+1)}\\
&= \lim_{x\to\infty}\frac{\sqrt{\frac{3x^2+2x+1}{x^2}}}{\left(1+\frac{1}{x}\right)}\\
&= \lim_{x\to\infty}\frac{\sqrt{3+\frac{2}{x}+\frac{1}{x^2}}}{\left(1+\frac{1}{x}\right)}\\
&= \frac{\sqrt{3+0+0}}{1+0}\\
&= \sqrt{3}.
\end{aligned}$$

Solution. 17.16.

$$\begin{aligned}
\lim_{x\to-\infty}\sqrt{x^2+x}-\sqrt{x^2-x} &= \lim_{x\to-\infty}\left(\sqrt{x^2+x}-\sqrt{x^2-x}\right)\frac{\left(\sqrt{x^2+x}+\sqrt{x^2-x}\right)}{\left(\sqrt{x^2+x}+\sqrt{x^2-x}\right)}\\
&= \lim_{x\to-\infty}\frac{x^2+x-(x^2-x)}{\sqrt{x^2+x}+\sqrt{x^2-x}} = \lim_{x\to-\infty}\frac{2x\frac{1}{x}}{\left(\sqrt{x^2+x}+\sqrt{x^2-x}\right)\frac{1}{x}}\\
&= \lim_{x\to-\infty}\frac{2}{\frac{\sqrt{x^2+x}}{x}+\frac{\sqrt{x^2-x}}{x}} = \lim_{x\to-\infty}\frac{2}{-\sqrt{\frac{x^2+x}{x^2}}-\sqrt{\frac{x^2-x}{x^2}}}\\
&= \lim_{x\to-\infty}\frac{2}{-\sqrt{1+\frac{1}{x}}-\sqrt{1-\frac{1}{x}}} = \frac{2}{-\sqrt{1+0}-\sqrt{1-0}} = -1.
\end{aligned}$$

The sign highlighted in red arises from the fact that, for negative x, we have that $x=-\sqrt{x^2}$.

3.2.2 Limits involving vertical asymptote

Problem 18. *Show the following limits do not exist and compute whether they evaluate to ∞, $-\infty$, or neither.*

1. $\displaystyle\lim_{x\to3^+}\frac{x^2+x-1}{x^2-2x-3}$. answer: ∞.
2. $\displaystyle\lim_{x\to3^-}\frac{x^2+x-1}{x^2-2x-3}$. answer: $-\infty$.
3. $\displaystyle\lim_{x\to1^+}\frac{x^2+1}{\sqrt{x^2+3}-2}$. answer: ∞.
4. $\displaystyle\lim_{x\to1^-}\frac{x^2+1}{\sqrt{x^2+3}-2}$. answer: $-\infty$.
5. $\displaystyle\lim_{x\to2^+}\frac{\sqrt{x^3-8}}{-x^2+x+2}$. answer: $-\infty$.
6. $\displaystyle\lim_{x\to-1^+}\frac{\sqrt[3]{x^2+2x+1}}{x^2-2x-3}$. answer: $-\infty$.

Problem 19. *Evaluate the limit if it exists.*

1. $\displaystyle\lim_{x\to3^+}\frac{\sqrt{\frac{x^2}{9}-1}}{2x^2-3x-9}$. answer: ∞.
2. $\displaystyle\lim_{x\to-2^-}\frac{\sqrt{\frac{x^2}{4}-1}}{2x^2+3x-2}$. answer: ∞.

Solution. 19.1. We have that

$$\begin{aligned}
\lim_{x\to3^+}\frac{\sqrt{\frac{x^2}{9}-1}}{2x^2-3x-9} &= \lim_{x\to3^+}\frac{\sqrt{(\frac{x}{3}-1)(\frac{x}{3}+1)}}{2(x+\frac{3}{2})(x-3)} = \lim_{x\to3^+}\frac{\left(\frac{1}{3}(x-3)(\frac{x}{3}+1)\right)^{\frac{1}{2}}}{2(x+\frac{3}{2})(x-3)}\\
&= \lim_{x\to3^+}\frac{\sqrt{\frac{1}{3}\left(\frac{x}{3}+1\right)}}{2\left(x+\frac{3}{2}\right)(x-3)^{\frac{1}{2}}} = \lim_{x\to3^+}\frac{\sqrt{\underbrace{\frac{1}{3}\left(\frac{x}{3}+1\right)}_{\to\frac{2}{3}}}}{\underbrace{2\left(x+\frac{3}{2}\right)}_{\to9}\underbrace{(x-3)^{\frac{1}{2}}}_{\to0^+}} = \infty,
\end{aligned}$$

where the latest term is $+\infty$ because it is of the form $\frac{(+)}{(+)(+)}$.

3.2.3 Find the Horizontal and Vertical Asymptotes

Problem 20. *Find the horizontal and vertical asymptotes of the graph of the function. If a graphing device is available, check your work by plotting the function.*

1. $y = \dfrac{2x}{\sqrt{x^2+x+3}-3}$.

answer: Vertical: $x=2$, $x=-3$, horizontal: $y=2$, $y=-2$

2. $y = \dfrac{3x^2}{\sqrt{x^2+2x+10}-5}$.

answer: Vertical: $x=3$, $x=-5$, horizontal: none.

3. $y = \dfrac{3x+1}{x-2}$.

answer: vertical: $x=2$, horizontal: $y=3$

4. $y = \dfrac{x^2-1}{2x^2-3x-2}$.

answer: vertical: $x=2$, $x=-\frac{1}{2}$, horizontal: $y=\frac{1}{2}$

5. $y = \dfrac{2x^2-2x-1}{x^2+x-2}$.

answer: vertical: $x=1$, $x=-2$, horizontal: $y=2$

6. $f(x) = \dfrac{-5x^2-3x+5}{x^2-2x-3}$

answer: vertical: $x=-1$, $x=3$, horizontal: $y=-5$

7. $y = \dfrac{1+x^4}{x^2-x^4}$.

answer: vertical: $x=0$, $x=1$, $x=-1$, horizontal: $y=-1$

8. $y = \dfrac{x^3-x}{x^2-7x+6}$.

answer: vertical: $x=6$, no horizontal asymptote

9. $y = \dfrac{x-9}{\sqrt{4x^2+3x+3}}$.

answer: no vertical asymptote, horizontal: $y=\pm\frac{1}{2}$

10. $y = \dfrac{\sqrt{x^2+1}-x}{x}$.

answer: vertical: $x=0$, horizontal: $y=0$, $y=-2$

11. $f(x) = \dfrac{x}{\sqrt{x^2+3}-2x}$

answer: vertical: $x=1$, horizontal: $y=-\frac{1}{3}$, $y=-1$

Solution. 20.1 **Vertical asymptotes.** A function $f(x)$ has a vertical asymptote at $x=a$ if $\lim\limits_{x\to a} f(x) = \pm\infty$.

The function is algebraic, and therefore has a finite limit at every point it is defined (i.e., no asymptote). Therefore the function can have vertical asymptotes only for those x for which $f(x)$ is not defined. The function is not defined for $\sqrt{x^2+x+3}-3=0$, which has two solutions, $x=2$ and $x=-3$. These are precisely the vertical asymptotes: indeed,

$$\lim_{x\to 2^+}\frac{2x}{\sqrt{x^2+x+3}-3} = \infty \qquad \lim_{x\to 2^-}\frac{2x}{\sqrt{x^2+x+3}-3} = -\infty$$

and

$$\lim_{x\to -3^+}\frac{2x}{\sqrt{x^2+x+3}-3} = \infty \qquad \lim_{x\to -3^-}\frac{2x}{\sqrt{x^2+x+3}-3} = -\infty$$

Horizontal asymptotes. A function $f(x)$ has a horizontal asymptote if $\lim\limits_{x\to\pm\infty} f(x)$ exists. If that limit exists, and is some number, say, N, then $y=N$ is the equation of the corresponding asymptote.

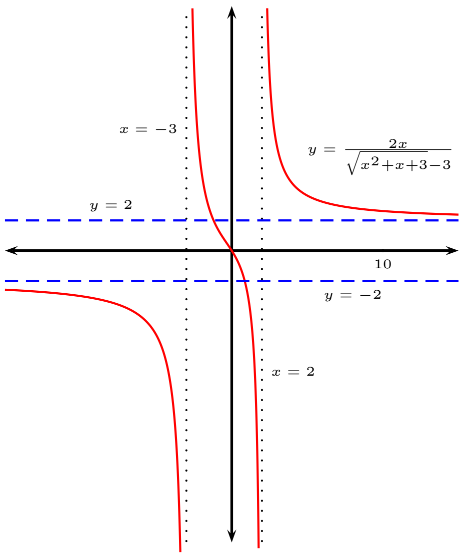
Consider the limit $x\to-\infty$. We have that

$$\begin{aligned}
\lim_{x\to-\infty}\frac{2x}{\sqrt{x^2+3x+3}-3} &= \lim_{x\to-\infty}\frac{2}{\frac{\sqrt{x^2+x+3}}{x}-\frac{3}{x}} \\
&= \lim_{x\to-\infty}\frac{2}{-\sqrt{\frac{x^2+3x+3}{x^2}}-\frac{3}{x}} && \Big|\ \frac{1}{x} = -\sqrt{\frac{1}{x^2}} \text{ when } x<0 \\
&= \lim_{x\to-\infty}\frac{2}{-\sqrt{1+\frac{3}{x}+\frac{3}{x^2}}-\frac{3}{x}} \\
&= \frac{\lim\limits_{x\to-\infty} 2}{-\sqrt{\lim\limits_{x\to-\infty}1+\lim\limits_{x\to-\infty}\frac{3}{x}+\lim\limits_{x\to-\infty}\frac{3}{x^2}}-\lim\limits_{x\to-\infty}\frac{3}{x}} \\
&= \frac{2}{-\sqrt{1+0+0}-0} \\
&= -2 \quad .
\end{aligned}$$

Therefore $y=-2$ is a horizontal asymptote.

The case $x\to\infty$, is handled similarly and yields that $y=2$ is a horizontal asymptote.

A computer generated graph confirms our computations.



Solution. 20.4

Vertical asymptotes. A function $f(x)$ has a vertical asymptote at $x = a$ if $\lim\limits_{x\to a} f(x) = \pm\infty$.

The function is algebraic, and therefore has a finite limit at every point it is defined (i.e., no asymptote). Therefore the function can have vertical asymptotes only for those x for which $f(x)$ is not defined. The function is not defined for $2x^2 - 3x - 2 = 0$, which has two solutions, $x = 2$ and $x = -\frac{1}{2}$. These are precisely the vertical asymptotes: indeed,

$$\begin{array}{rcll} \lim\limits_{x\to 2^+} \dfrac{x^2-1}{2x^2-3x-2} & = & \lim\limits_{x\to 2^+} \dfrac{x^2-1}{2(x-2)\left(x+\frac{1}{2}\right)} = \infty & \Big| \text{ Limit of form } \frac{(+)}{(+)(+)} \\ \lim\limits_{x\to 2^-} \dfrac{x^2-1}{2x^2-3x-2} & = & \lim\limits_{x\to 2^-} \dfrac{x^2-1}{2(x-2)\left(x+\frac{1}{2}\right)} = -\infty & \Big| \text{ Limit of form } \frac{(+)}{(-)(+)} \end{array}$$

and

$$\begin{array}{rcll} \lim\limits_{x\to -\frac{1}{2}^+} \dfrac{x^2-1}{2x^2-3x-2} & = & \lim\limits_{x\to -\frac{1}{2}^+} \dfrac{x^2-1}{2(x-2)\left(x+\frac{1}{2}\right)} = \infty & \Big| \text{ Limit of form } \frac{(-)}{(+)(-)} \\ \lim\limits_{x\to -\frac{1}{2}^-} \dfrac{x^2-1}{2x^2-3x-2} & = & \lim\limits_{x\to -\frac{1}{2}^-} \dfrac{x^2-1}{2(x-2)\left(x+\frac{1}{2}\right)} = -\infty & \Big| \text{ Limit of form } \frac{(-)}{(-)(-)} \end{array}$$

Horizontal asymptotes. A function $f(x)$ has a horizontal asymptote if $\lim\limits_{x\to\pm\infty} f(x)$ exists. If that limit exists, and is some number, say, N, then $y = N$ is the equation of the corresponding asymptote.

We have that

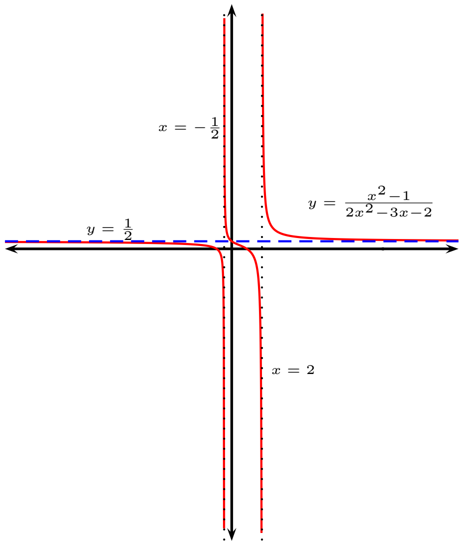
$$\begin{array}{rcll} \lim\limits_{x\to\infty} \dfrac{x^2-1}{2x^2-3x-2} & = & \lim\limits_{x\to\infty} \dfrac{\left(x^2-1\right)\frac{1}{x^2}}{\left(2x^2-3x-2\right)\frac{1}{x^2}} & \Big| \text{ Divide by highest term in den.} \\ & = & \lim\limits_{x\to\infty} \dfrac{1-\frac{1}{x^2}}{2-\frac{3}{x}-\frac{2}{x^2}} & \\ & = & \dfrac{\lim\limits_{x\to\infty} 1 - \lim\limits_{x\to\infty} \frac{1}{x^2}}{\lim\limits_{x\to\infty} 2 - \lim\limits_{x\to\infty} \frac{3}{x} - \lim\limits_{x\to\infty} \frac{2}{x^2}} & \Big| \text{ Step may be skipped} \\ & = & \dfrac{1-0}{2-0-0} & \\ & = & \dfrac{1}{2} & \end{array}$$

A similar computation shows that

$$\lim_{x\to -\infty} \frac{x^2-1}{2x^2-3x-2} = \frac{1}{2}$$

Therefore $y = \frac{1}{2}$ is the only horizontal asymptote, valid in both directions ($x \to \pm\infty$).

A computer generated graph confirms our computations.



Solution. 20.6

Vertical asymptotes. The function is rational, and therefore has a finite limit (and therefore no vertical asymptote) at every point it its domain. The function is not defined for $x^2-2x-3=0$, which has two solutions, $x=-1$ and $x=3$. These are precisely the vertical asymptotes: indeed,

$$\begin{array}{rcll}\lim\limits_{x\to-1^+}\dfrac{-5x^2-3x+5}{x^2-2x-3} &=& \lim\limits_{x\to-1^+}\dfrac{-5x^2-3x+5}{(x+1)(x-3)}=-\infty & \Bigg| \text{ Limit of form } \frac{(+)}{(+)(-)}\\ \lim\limits_{x\to-1^-}\dfrac{-5x^2-3x+5}{x^2-2x-3} &=& \lim\limits_{x\to-1^-}\dfrac{-5x^2-3x+5}{(x+1)(x-3)}=\infty & \Bigg| \text{ Limit of form } \frac{(+)}{(-)(-)}\end{array}$$

and

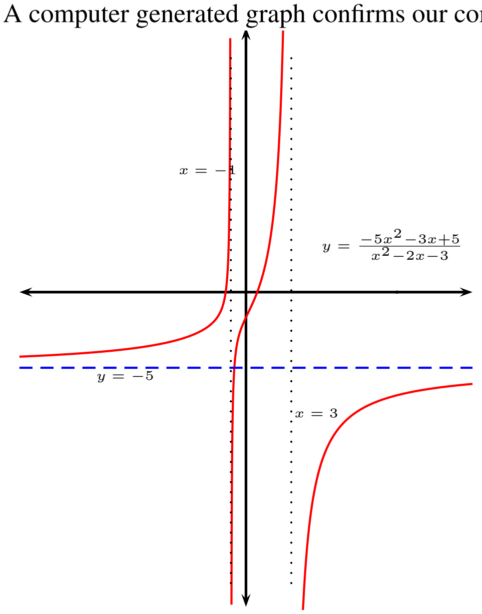
$$\begin{array}{rcll}\lim\limits_{x\to3^+}\dfrac{-5x^2-3x+5}{x^2-2x-3} &=& \lim\limits_{x\to3^+}\dfrac{-5x^2-3x+5}{(x+1)(x-3)}=-\infty & \Bigg| \text{ Limit of form } \frac{(-)}{(+)(+)}\\ \lim\limits_{x\to3^-}\dfrac{-5x^2-3x+5}{x^2-2x-3} &=& \lim\limits_{x\to3^-}\dfrac{-5x^2-3x+5}{(x+1)(x-3)}=\infty & \Bigg| \text{ Limit of form } \frac{(-)}{(+)(-)}\end{array}$$

Horizontal asymptotes.

$$\begin{array}{rcll}\lim\limits_{x\to\pm\infty}\dfrac{-5x^2-3x+5}{x^2-2x-3} &=& \lim\limits_{x\to\pm\infty}\dfrac{\left(-5x^2-3x+5\right)\frac{1}{x^2}}{\left(x^2-2x-3\right)\frac{1}{x^2}} & \Bigg| \text{ Divide by highest term in den.}\\ &=& \lim\limits_{x\to\pm\infty}\dfrac{-5-\frac{3}{x}+\frac{5}{x^2}}{1-\frac{2}{x}-\frac{3}{x^2}} & \\ &=& \dfrac{-\lim\limits_{x\to\pm\infty}5-\lim\limits_{x\to\pm\infty}\frac{3}{x}+\lim\limits_{x\to\pm\infty}\frac{5}{x^2}}{\lim\limits_{x\to\pm\infty}1-\lim\limits_{x\to\pm\infty}\frac{2}{x}-\lim\limits_{x\to\pm\infty}\frac{3}{x^2}} & \Bigg| \text{ Step may be skipped}\\ &=& \dfrac{-5-0+0}{1-0-0} & \\ &=& -5. & \end{array}$$

Therefore $y=-5$ is the only horizontal asymptote, valid in both directions ($x\to\pm\infty$).

A computer generated graph confirms our computations.



Solution. 20.11

Vertical asymptotes. A function $f(x)$ has a vertical asymptote at $x = a$ if $\lim_{x\to a} f(x) = \pm\infty$.

The function is algebraic, and therefore has a finite limit at every point it is defined (i.e., no asymptote). Therefore the function can have vertical asymptotes only for those x for which $f(x)$ is not defined. The function is not defined for

$$
\begin{array}{rcll}
\sqrt{x^2+3}-2x &=& 0 & \\
\sqrt{x^2+3} &=& 2x & \left|\begin{array}{l}\text{square both sides}\\ \text{may introduce extraneous solutions}\end{array}\right. \\
x^2+3 &=& 4x^2 & \\
3x^2-3 &=& 0 & \\
3(x-1)(x+1) &=& 0 & \\
x=1 & \text{or} & \cancel{x=-1} & \\
 & & x=-1 \text{ is extraneous:} & \\
 & & \sqrt{(-1)^2+3}-(-1)2 = 4 \neq 0 &
\end{array}
$$

$x = -1$ is indeed a vertical asymptote:

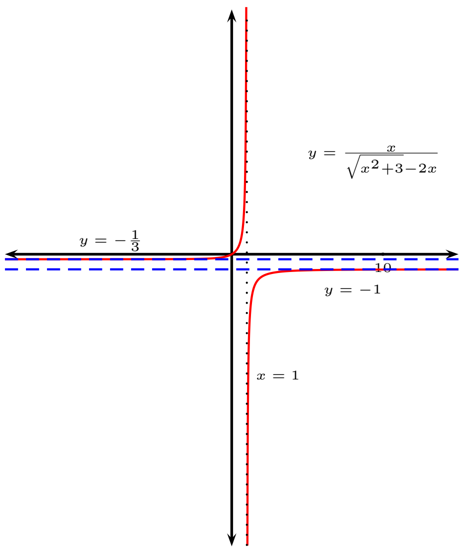
$$\lim_{x\to 1^+} \frac{x}{\sqrt{x^2+3}-2x} = \infty \qquad \lim_{x\to 1^-} \frac{x}{\sqrt{x^2+3}-2x} = -\infty.$$

Horizontal asymptotes.

$$
\begin{array}{rcll}
\lim\limits_{x\to-\infty} \dfrac{x}{\sqrt{x^2+3}-2x} &=& \lim\limits_{x\to-\infty} \dfrac{1}{\frac{\sqrt{x^2+3}}{x}-2} & \\
&=& \lim\limits_{x\to-\infty} \dfrac{1}{-\sqrt{\frac{x^2+3}{x^2}}-2} & \left| \frac{1}{x} = -\sqrt{\frac{1}{x^2}} \text{ when } x<0 \right. \\
&=& \lim\limits_{x\to-\infty} \dfrac{1}{-\sqrt{1+\frac{3}{x^2}}-2} & \\
&=& \dfrac{1}{-\sqrt{1+0}-2} & \\
&=& -\dfrac{1}{3}. & \\
\lim\limits_{x\to-\infty} \dfrac{x}{\sqrt{x^2+3}-2x} &=& \lim\limits_{x\to\infty} \dfrac{1}{\frac{\sqrt{x^2+3}}{x}-2} & \\
&=& \lim\limits_{x\to\infty} \dfrac{1}{\sqrt{\frac{x^2+3}{x^2}}-2} & \left| \frac{1}{x} = \sqrt{\frac{1}{x^2}} \text{ when } x>0 \right. \\
&=& \lim\limits_{x\to\infty} \dfrac{1}{\sqrt{1+\frac{3}{x^2}}-2} & \\
&=& \dfrac{1}{\sqrt{1+0}-2} & \\
&=& -1. &
\end{array}
$$

Therefore $y = -\frac{1}{3}$ and $y = -1$ are the two horizontal asymptotes.

A computer generated graph confirms our computations.



3.3 Limits - All Cases - Problem Collection

Problem 21. *Find the following limits, or show that they do not exist:*

1. $\lim\limits_{x\to 2} \dfrac{x^2-4}{x^2-x-2}$ — *answer:* $\frac{4}{3}$
2. $\lim\limits_{x\to -\infty} \dfrac{5x^3+x-1}{2x^3-7}$ — *answer:* $\frac{5}{2}$
3. $\lim\limits_{x\to 1^+} \dfrac{x-3}{x-1}$ — *answer:* $-\infty$
4. $\lim\limits_{h\to 0} \dfrac{2(x+h)^3-2x^3}{h}$ — *answer:* $6x^2$
5. $\lim\limits_{x\to \infty} \dfrac{\sqrt{9x^2-2}}{x+4}$ — *answer:* 3
6. $\lim\limits_{x\to -1} \dfrac{2x+3}{x+1}$ — *answer:* Does not exist

3.4 Continuity

3.4.1 Continuity to evaluate limits

Problem 22. *Use continuity to evaluate the limits. The answer key has not been proofread, use with caution.*

- $\lim\limits_{x\to \frac{\pi}{4}} x\tan x.$

answer: $\frac{\pi}{4}$

- $\lim\limits_{x\to 0} \dfrac{1}{1-\sqrt{3+\cos x}}$

answer: -1

- $\lim\limits_{x\to 0} \tan(x+\sin x)$

answer: 0

- $\lim\limits_{x\to \pi} \cos(\ln x \sin x)$

answer: 1

3.4.2 Conceptual problems

Problem 23. *For which values of x is f continuous?*

- $f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$
- $f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ x & \text{if } x \text{ is irrational} \end{cases}$

Problem 24. *Show that $f(x)$ is continuous at all irrational points and discontinuous at all rational ones.*

$$f(x)=\begin{cases}\frac{1}{q^2} & \text{if } x \text{ is rational and } x=\frac{p}{q}\\ 0 & \text{if } x \text{ is irrational}\end{cases}$$

where in the first item p, q are relatively prime integers (i.e., integers without a common divisor).

3.4.3 Continuity and Piecewise Defined Functions

Problem 25. *Find the (implied) domain of $f(x)$. Extend the definition of f at $x = 3$ to make f continuous at 3.*

1. $f(x) = \dfrac{x^2 - x - 6}{x - 3}$.
2. $f(x) = \dfrac{x^3 - 27}{x^2 - 9}$.

answer: *Implied domain:* $x \in (-\infty, 3) \cup (3, \infty)$. *Extend* $f(x)$ *to* $f(x) = x + 2$.

answer: *Implied domain:* $x \in (-\infty, -3) \cup (-3, 3) \cup (3, \infty)$. *Extend* $f(x)$ *to* $f(x) = \frac{x^2+3x+9}{x+3}$ *with domain* $x \in (-\infty, -3) \cup (-3, \infty)$.

Problem 26. *Find the numbers x for which f is discontinuous. At which of these numbers is f continuous from the right, from the left, or neither?*

1. $f(x)=\begin{cases} 2+x^2 & \text{if } x\leq 0\\ -2x & \text{if } 0<x\leq 2\\ -x^2 & \text{if } x>2\end{cases}$.
2. $f(x)=\begin{cases} x+1 & \text{if } x\leq 1\\ \frac{1}{x} & \text{if } 1<x<2\\ \sqrt{x-2} & \text{if } x\geq 2\end{cases}$.
3. $f(x)=\begin{cases} x+2 & \text{if } x< 0\\ 2x^2 & \text{if } 0\leq x\leq 1\\ 2-x & \text{if } x>1\end{cases}$.

Problem 27. *Find the values of a and b that make f continuous everywhere.*

1. $f(x)=\begin{cases} 1 & \text{if } x< 0\\ ax+b & \text{if } 0\leq x<1\\ 2x & \text{if } x\geq 1\end{cases}$.
2. $f(x)=\begin{cases} \dfrac{x^2-1}{x-1} & \text{if } x< 1\\ ax^2-bx+3 & \text{if } 1\leq x<3\\ 2x-a+b & \text{if } x\geq 3\end{cases}$.

3.5 Intermediate Value Theorem

Problem 28. *Use the Intermediate Value Theorem to show that there is a real number solution of the given equation in the specified interval.*

1. $x^5 + x - 3 = 0$ *where* $x \in (1, 2)$.
2. $\sqrt[4]{x} = 1 - x$ *where* $x \in \mathbb{R}$ *(i.e., x is an arbitrary real number).*
3. $\cos x = 2x$, *where* $x \in (0, 1)$.
4. $\sin x = x^2 - x - 1$, *where* $x \in \mathbb{R}$ *(i.e., x is an arbitrary real number).*
5. $\cos x = x^4$, *where* $x \in \mathbb{R}$ *(i.e., x is an arbitrary real number).*
6. $x^5 - x^2 + x + 3 = 0$, *where* $x \in \mathbb{R}$.

Problem 29.

1. (a) *Solve the equation* $x^2 + 13x + 41 = 1$.
   (b) *Use the intermediate value theorem to prove that the equation* $x^2 + 13x + 41 = \sin x$ *has at least two solutions, lying between the two solutions to 29.1.a.*
2. (a) *Solve the equation* $x^2 - 15x + 55 = 1$.
   (b) *Use the intermediate value theorem to prove that the equation* $x^2 - 15x + 55 = \cos x$ *has at least two solutions, lying between the two solutions to the equation in the preceding item.*

Solution. 29.1.a.

$$\begin{array}{rcl} x^2+13x+41 &=& 1 \\ x^2+13x+40 &=& 0 \\ (x+5)(x+8) &=& 0 \end{array} .$$

eqnarray Therefore the two solutions are $x_1 = -5$ and $x_2 = -8$.
29.1.b. Consider the function

$$f(x) = x^2 + 13x + 41 - \sin x \quad .$$

Our strategy for proving $f(x) = 0$ has a solution consists in finding a number a such that $f(a) < 0$ and a number b such that $f(b) > 0$, and then using the Intermediate Value Theorem (IVT) with $N = 0$.

Let

$$g(x) = x^2 + 13x + 41,$$

and so $f(x) = g(x) - \sin x$. We have no techniques for evaluating $\sin x$ without calculator, but we do have all knowledge necessary to evaluate $g(x)$. Indeed, from high school we know that the lowest point of the parabola $g(x)$ is located at $x = -\frac{13}{2} = -6.5$. Then $g(-6.5) = -1.25$. Therefore

$$f(-6.5) = g(-6.5) - \sin(-6.5) = g(-6.5) + \sin(6.5) = -1.25 + \sin 6.5 \leq -0.25,$$

where for the very last inequality we use the fact that $\sin 6.5 < 1$ (remember $\sin t \leq 1$ for all real values of t).

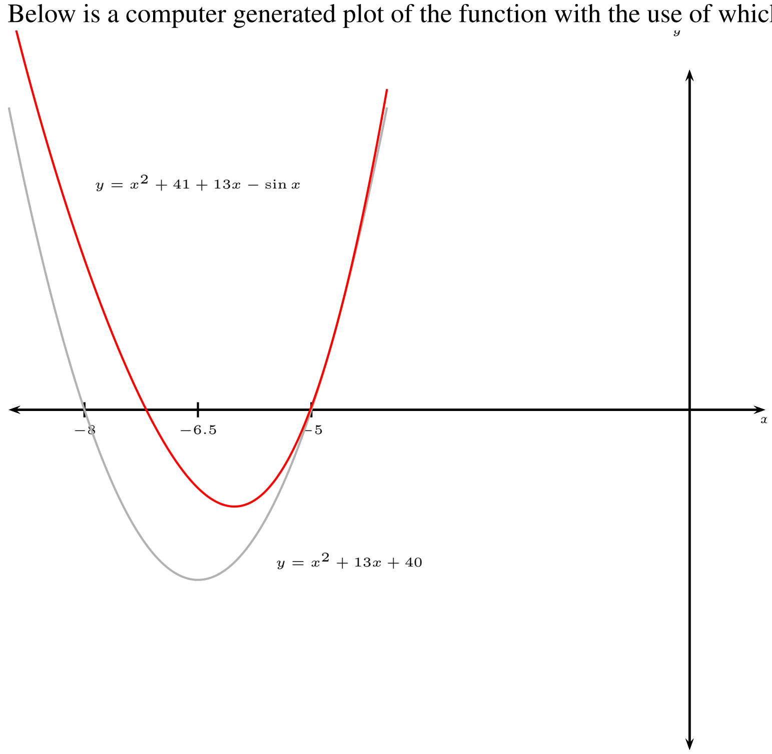
On the other hand,

$$f(-5) = g(-5) - \sin(-5) = 1 + \sin 5 > 0$$

as $\sin 5 > -1$ (remember $\sin t \geq -1$ for all real values of t). Therefore $f(-5) > 0$ and $f(-6.5) < 0$ and by the Intermediate Value Theorem (IVT) $f(x) = 0$ has a solution in the interval $x \in (-6.5, -5)$.

Proving $f(x) = 0$ has a solution in the interval $x \in (-8, -6.5)$ is similar and we leave it to the student.

Below is a computer generated plot of the function with the use of which we can visually verify our answer.



4 Inverse Functions

4.1 Problems Using Rational Functions Only

Problem 30. *Find the inverse function. You are asked to do the algebra only; you are not asked to determine the domain or range of the function or its inverse.*

1. $f(x) = 3x^2 + 4x - 7$, *where* $x \geq -\frac{2}{3}$.

answer: $f^{-1}(x) = -\frac{2}{3} + \frac{\sqrt{25+3x}}{3}, \quad x \geq -\frac{25}{3}$

2. $f(x) = 2x^2 + 3x - 5$, *where* $x \geq -\frac{3}{4}$.

answer: $f^{-1}(x) = -\frac{3}{4} + \frac{\sqrt{49+8x}}{4}$, $x \geq -\frac{49}{8}$

3. $f(x) = \dfrac{2x+5}{x-4}$, *where* $x \neq 4$.

answer: $f^{-1}(x) = \frac{4x+5}{x-2}$, $x \neq 2$

4. $f(x) = \dfrac{3x+5}{2x-4}$, *where* $x \neq 2$.

answer: $f^{-1}(x) = \dfrac{4x+5}{2x-3}$, $x \neq \dfrac{3}{2}$

5. $f(x) = \dfrac{5x+6}{4x+5}$, *where* $x \neq -\frac{5}{4}$.

answer: $f^{-1}(x) = \frac{-5x+6}{4x-5}$, $x \neq \frac{5}{4}$

6. $f(x) = \dfrac{2x-3}{-3x+4}$, *where* $x \neq \frac{4}{3}$..

answer: $f^{-1}(x) = \frac{4x+3}{3x+2}$, $x \neq -\frac{2}{3}$

Solution. 30.4 This is a concise solution written in form suitable for test taking.

$$\begin{array}{rrcl}
& y &=& \dfrac{3x+5}{2x-4} \\
& y(2x-4) &=& 3x+5 \\
& 2xy-4y &=& 3x+5 \\
& 2xy-3x &=& 4y+5 \\
& x(2y-3) &=& 4y+5 \\
& x &=& \dfrac{4y+5}{2y-3} \\
\text{Therefore} & f^{-1}(y) &=& \dfrac{5+4y}{2y-3} \\
& f^{-1}(x) &=& \dfrac{5+4x}{2x-3}.
\end{array}$$

Solution. 30.5. Set $f(x) = y$. Then

$$\begin{array}{rcl}
y &=& \dfrac{5x+6}{4x+5} \\
y(4x+5) &=& 5x+6 \\
x(4y-5) &=& -5y+6 \\
x &=& \dfrac{-5y+6}{4y-5}.
\end{array}$$

Therefore the function $x = g(y) = \dfrac{-5y+6}{4y-5}$ is the inverse of $f(x)$. We write $g = f^{-1}$. The function $g = f^{-1}$ is defined for $y \neq \dfrac{5}{4}$. For our final answer we relabel the argument of g to x:

$$g(x) = f^{-1}(x) = \frac{-5x+6}{4x-5} \quad .$$

Let us check our work. In order for f and g to be inverses, we need that $g(f(x))$ be equal to x.

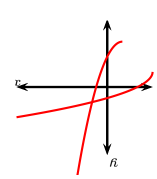
$$g(f(x)) = \frac{-5f(x)+6}{4f(x)-5} = \frac{-5\frac{(5x+6)}{4x+5}+6}{4\frac{(5x+6)}{4x+5}-5} = \frac{-5(5x+6)+6(4x+5)}{4(5x+6)-5(4x+5)} = \frac{-x}{-1} = x \quad ,$$

as expected.

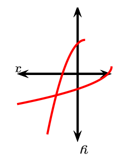
Problem 31. *Find the inverse function f^{-1}. Plot roughly by hand $y = f(x)$. Using the plot of $y = f(x)$, plot roughly by hand $f^{-1}(x)$. Indicate the relationship between the graph of $f(x)$ and $f^{-1}(x)$.*

1. $f(x) = x^2 + 2x - 2, \qquad x \geq -1.$

answer: $f^{-1}(x) = \sqrt{x+3} - 1$



2. $f(x) = x^2 + x - 2, \qquad x \geq -\frac{1}{2}.$



answer: $f^{-1}(x) = \frac{\sqrt{4x+9}-1}{2}$

4.2 Problems Involving Exponents, Logarithms

Problem 32. *Find the inverse function and its domain.*

1. $y = \ln(x + 3)$.

   answer: $f^{-1}(x) = e^x - 3$

2. $y = 4\ln\,(x - 3) - 4$.

   answer: $f^{-1}(x) = e^{\frac{x+4}{4}x} + 3$

3. $y = 2\ln\,(-2x + 4) + 1$

   answer: $f^{-1}(x) = -\frac{1}{2}e^{\frac{1}{2}x - \frac{1}{2}} + 2$

4. $f(x) = e^{x^3}$.

   answer: $f^{-1}(x) = \sqrt[3]{\ln x}, \quad x > 0$

5. $y = (\ln x)^2$, $x \geq 1$.

   answer: $f^{-1}(x) = e^{\sqrt{x}}, \quad x \geq 0$

6. $y = \dfrac{e^x}{1 + 2e^x}$.

   answer: $f^{-1}(x) = \ln\left(\frac{x}{1-2x}\right), \quad x \in \left(0, \frac{1}{2}\right)$

7. $f(x) = 2^{2x} + 2^x - 2$.

   answer: $f^{-1}(x) = \log_2 \frac{-1+\sqrt{9+4x}}{2}, \quad x \geq -2$

Solution. 32.1

$$\begin{aligned} y &= \ln(x+3) \\ e^y &= e^{\ln(x+3)} \\ e^y &= x + 3 \\ e^y - 3 &= x \\ \text{Therefore} \quad f^{-1}(y) &= e^y - 3. \end{aligned}$$

The domain of e^y is all real numbers, so the domain of f^{-1} is all real numbers.

Solution. 32.2

$$\begin{array}{rcll} 4\ln(x-3) - 4 &=& y & \\ 4\ln(x-3) &=& y + 4 & \\ \ln(x-3) &=& \dfrac{y+4}{4} & \Big| \text{ exponentiate} \\ e^{\ln(x-3)} &=& e^{\frac{y+4}{4}} & \\ x - 3 &=& e^{\frac{y+4}{4}} & \\ f^{-1}(y) = x &=& e^{\frac{y+4}{4}} + 3 & \\ f^{-1}(x) &=& e^{\frac{x+4}{4}} + 3 & \Big| \text{ relabel.} \end{array}$$

The domain of f^{-1} is all real numbers (no restrictions on the domain).

Solution. 32.5

$$\begin{array}{rcll} y &=& (\ln x)^2 & \Big| \text{ take } \sqrt{\ } \text{ on both sides, } y \geq 0 \\ \sqrt{y} &=& \ln x & \Big| \text{ exponentiate} \\ e^{\sqrt{y}} &=& e^{\ln x} = x & \\ f^{-1}(y) &=& e^{\sqrt{y}} & \\ f^{-1}(x) &=& e^{\sqrt{x}} & \end{array}$$

Solution. 32.6

$$\begin{aligned}
y &= \frac{e^x}{1+2e^x} \\
y(1+2e^x) &= e^x \\
y &= e^x(1-2y) \\
\frac{y}{1-2y} &= e^x \\
\ln\frac{y}{1-2y} &= \ln e^x \\
\ln\frac{y}{1-2y} &= x \\
\text{Therefore}\quad f^{-1}(y) &= \ln\frac{y}{1-2y}.
\end{aligned}$$

The natural logarithm function is only defined for positive input values. Therefore the domain is the set of all y for which

$$\frac{y}{1-2y} > 0.$$

This inequality holds if the numerator and denominator are both positive or both negative. This happens if either

1. $y > 0$ and $y < \frac{1}{2}$, or
2. $y < 0$ and $y > \frac{1}{2}$.

The latter option is impossible, so the domain is $\{y \in \mathbb{R} \mid 0 < y < \frac{1}{2}\}$.

5 Logarithms and Exponent Basics

5.1 Exponents Basics

Problem 33. *Express each of the following as a single power.*

1. $\dfrac{2^5 \cdot 2^7}{2\sqrt{2}}$

answer: $2^{10.5} = 2^{\frac{21}{2}}$

2. $\dfrac{3^2 \cdot 3^{-1}}{3^3 \cdot \sqrt{3^3}}$

answer: $3^{-\frac{7}{2}}$

3. $\dfrac{\pi^3}{\pi^{-1}\sqrt{\pi^5}}$

answer: $\pi^{\frac{3}{2}}$

Solution. 33.2.

$$\begin{aligned}
\frac{3^2 \cdot 3^{-1}}{3^3 \cdot \sqrt{3^3}} &= \frac{3^2 \cdot 3^{-1}}{3^3 \cdot (3^3)^{\frac{1}{2}}} \\
&= \frac{3^2 \cdot 3^{-1}}{3^3 \cdot 3^{\frac{3}{2}}} \\
&= \frac{3^{2-1}}{3^{3+\frac{3}{2}}} \\
&= \frac{3^1}{3^{\frac{9}{2}}} \\
&= 3^{1-\frac{9}{2}} \\
&= 3^{-\frac{7}{2}}.
\end{aligned}$$

5.2 Logarithm Basics

Problem 34. *Use the definition of a logarithm to evaluate each of the following without using a calculator. The answer key has not been proofread, use with caution.*

1. $\log_2 16$.

   answer: 4

2. $\log_3 \left(\dfrac{1}{9}\right)$.

   answer: -2

3. $\log_{10} 1000$.

   answer: 3

4. $\log_6 36^{-\frac{2}{3}}$.

   answer: $-\frac{4}{3}$

5. $\log_2(8\sqrt{2})$.

   answer: $\frac{7}{2}$

6. $\log_{\frac{1}{2}}(4)$.

   answer: -2

7. $\log_{\frac{1}{9}}(\sqrt{3})$.

   answer: $-\frac{1}{4}$

Solution. 36.13.

$$\begin{array}{rcl} \log_7\left(\frac{49^x}{343^y}\right) & = & \log_7 49^x - \log_7 343^y \\ & = & x\log_7 49 - y\log_7 343 \\ \text{However } 49 = 7^2 \text{ and } 343 = 7^3, \text{ therefore } \log_7\left(\frac{49^x}{343^y}\right) & = & 2x - 3y. \end{array}$$

Problem 35. *Express each of the following as a single logarithm. If possible, compute the logarithm without using a calculator. The answer key has not been proofread, use with caution.*

1. $\ln 4 + \ln 6 - \ln 5$.

   answer: $\ln\left(\frac{24}{5}\right)$

2. $2\ln 2 - 3\ln 3 + 4\ln 4$.

   answer: $\ln\left(\frac{1024}{27}\right)$

3. $\ln 36 - 2\ln 3 - 3\ln 2$.

   answer: $-\ln 2 = \ln\left(\frac{1}{2}\right)$

4. $\log_2(24) - \log_4 9$.

   answer: 3

5. $\log_7(24) + \log_{\frac{1}{7}} 3 - \log_{49}(64)$.

   answer: 0

6. $\log_3(24) + \log_3\left(\frac{3}{8}\right)$.

   answer: 2

Solution. 35.2.

$$\begin{aligned} 2\ln 2 - 3\ln 3 + 4\ln 4 &= \ln 2^2 - \ln 3^3 + \ln 4^4 \\ &= \ln 4 - \ln 27 + \ln 256 \\ &= \ln\left(\frac{4}{27}\right) + \ln 256 \\ &= \ln\left(\frac{4\cdot 256}{27}\right) \\ &= \ln\left(\frac{1024}{27}\right). \end{aligned}$$

$\frac{1024}{27}$ is not a rational power of e, therefore $\ln\left(\frac{1024}{27}\right)$ is not a rational number and there are no further simplifications of the answer (except possibly a numerical approximation with a calculator or equivalent).

Solution. 35.5

$$\begin{aligned}
\log_7(24)+\log_{\frac{1}{7}}(3)-\log_{49}(64) &= \log_7(24)+\frac{\log_7(3)}{\log_7\left(\frac{1}{7}\right)}-\frac{\log_7(64)}{\log_7(49)} && \text{common base}\\
&= \log_7(24)+\frac{\log_7(3)}{-1}-\frac{\log_7(64)}{2} && \text{simplify logarithms}\\
&= \log_7(24)-\log_7(3)-\frac{1}{2}\log_7(64)\\
&= \log_7\left(\frac{24}{3}\right)-\log_7\left(64^{\frac{1}{2}}\right) && \text{rule: } \log_a x-\log_a y=\log_a\left(\tfrac{x}{y}\right)\\
& && \text{rule: } \log_a x^r = r\log_a x\\
&= \log_7(8)-\log_7\left(\sqrt{64}\right)\\
&= \log_7 8-\log_7 8 = 0 && \text{alternatively:}\\
&= \log_7\left(\frac{8}{8}\right)\\
&= \log_7(1)\\
&= 0.
\end{aligned}$$

Problem 36. *Find the exact value of each expression.*

1. $\log_5 125$.

   answer: 3

2. $\log_3 \frac{1}{27}$.

   answer: -3

3. $\ln\left(\frac{1}{e}\right)$.

   answer: -1

4. $\log_{10}\sqrt{10}$.

   answer: $\frac{1}{2}$

5. $e^{\ln 4.5}$.

   answer: 4.5

6. $\log_{10} 0.0001$.

   answer: 4

7. $\log_{1.5} 2.25$.

   answer: 2

8. $\log_5 4-\log_5 500$.

   answer: -3

9. $\log_2 6-\log_2 15+\log_2 20$.

   answer: 3

10. $\log_3 100-\log_3 18-\log_3 50$.

    answer: -2

11. $e^{-2\ln 5}$.

    answer: $\frac{1}{25}$

12. $\ln\left(\ln e^{e^{10}}\right)$.

    answer: 10

13. $\log_7\left(\frac{49^x}{343^y}\right)$

    answer: $2x-3y$

Solution. 36.13.

$$\begin{aligned}
\log_7\left(\tfrac{49^x}{343^y}\right) &= \log_7 49^x-\log_7 343^y\\
&= x\log_7 49-y\log_7 343\\
\text{However } 49=7^2 \text{ and } 343=7^3,\ \text{therefore } \log_7\left(\tfrac{49^x}{343^y}\right) &= 2x-3y.
\end{aligned}$$

5.3 Some Problems Involving Logarithms

Problem 37. *Solve each equation for x. If available, use a calculator to give an ($\approx$) answer in decimal notation. If available, use a calculator to verify your approximate solutions.*

1. $e^{7-4x} = 7.$

answer: $\frac{7-\ln 7}{4} \approx 1.263522$

2. $\ln(2x-9) = 2.$

answer: $\frac{e^2+9}{2} \approx 8.194528$

3. $\ln(x^2-2) = 3.$

answer: $\pm\sqrt{e^3+2} \approx \pm 4.69525$

4. $2^{x-3} = 5.$

answer: $\log_2 5 + 3 = \frac{\ln 5}{\ln 2} + 3 \approx 5.321928$

5. $\ln x + \ln(x-1) = 1.$

answer: $\frac{1}{2}\left(1+\sqrt{1+4e}\right) \approx 2.223$

6. $e^{2x+1} = t.$

answer: $\frac{\ln t - 1}{2}$

7. $\log_2(mx) = c.$

answer: $\frac{2^c}{m}$

8. $e - e^{-2x} = 1.$

answer: $-\frac{1}{2}\ln(e-1) \approx -0.271$

9. $8(1+e^{-x})^{-1} = 3.$

answer: $-\ln\frac{5}{3} = \ln\frac{3}{5} \approx -0.510826$

10. $\ln(\ln x) = 1.$

answer: $e^e \approx 15.154$

11. $e^{e^x} = 10.$

answer: $\ln(\ln 10) \approx 0.834$

12. $\ln(2x+1) = 3 - \ln x.$

answer: $\frac{-1+\sqrt{1+8e^3}}{4} \approx 2.92878$

13. $e^{2x} - 4e^x + 3 = 0.$

answer: $x = \ln 3 \approx 1.098612$, $x = 0$

14. $e^{4x} + 3e^{2x} - 4 = 0.$

answer: $x = 0$

15. $e^{2x} - e^x - 6 = 0.$

answer: $x = \ln 3$

16. $4^{3x} - 2^{3x+2} - 5 = 0.$

answer: $x = \frac{\log_2 5}{3}$

17. $3 \cdot 2^x + 2\left(\frac{1}{2}\right)^{x-1} - 7 = 0.$

answer: $x = 0$ *or* $2 - \log_2 3 = 2 - \frac{\ln 3}{\ln 2}$

Solution. 37.4

$$\begin{array}{rcl|l} 2^{x-3} &=& 5 & \text{take } \log_2 \\ x-3 &=& \log_2(5) & \text{add 3 to both sides} \\ x &=& \log_2(5)+3 & \text{answer is complete} \\ &=& \dfrac{\ln 5}{\ln 2}+3 & \text{optional step: convert to } \ln \\ &\approx& 5.321928095 & \text{calculator} \end{array}$$

Solution. 37.8

$$\begin{array}{rcll} e-e^{-2x} &=& 1 & \\ e^{-2x} &=& e-1 & \mid \text{apply } \ln \\ \ln e^{-2x} &=& \ln(e-1) & \\ -2x &=& \ln(e-1) & \\ x &=& -\dfrac{1}{2}\ln(e-1) & \\ &\approx& -0.270662427 & \mid \text{calculator} \end{array}$$

Solution. 37.5

$$\begin{array}{rcl} \ln x + \ln(x-1) &=& 1 \\ \ln\left(x^2-x\right) &=& 1 \\ e^{\ln(x^2-x)} &=& e^1 \\ x^2-x &=& e \\ x^2-x-e &=& 0 \\ \text{Quadratic formula:} \quad x &=& \dfrac{-(-1)\pm\sqrt{(-1)^2-4(1)(-e)}}{2(1)} \\ &=& \dfrac{1\pm\sqrt{1+4e}}{2}. \end{array}$$

However $\frac{1-\sqrt{1+4e}}{2}$ is negative, so $\ln\left(\frac{1-\sqrt{1+4e}}{2}\right)$ is undefined. Hence the only solution is $x = \frac{1+\sqrt{1+4e}}{2} \approx 2.2229.$

Solution. 37.16

$$\begin{aligned}
4^{3x}-2^{3x+2}-5 &= 0\\
4^{3x}-4\cdot 2^{3x}-5 &= 0 && \left|\ \text{Set}\ \begin{aligned} 2^{3x} &= u\\ 4^{3x} &= u^2\end{aligned}\right.\\
u^2-4u-5 &= 0\\
(u-5)(u+1) &= 0\\
u=5 \quad &\text{or} \quad u=-1\\
2^{3x}=5 \quad & \quad\ 2^{3x}=-1\\
3x=\log_2(5) \quad & \quad\ \text{no real solution}\\
x=\frac{\log_2 5}{3}\\
\text{Calculator: } x\approx 0.773976
\end{aligned}$$

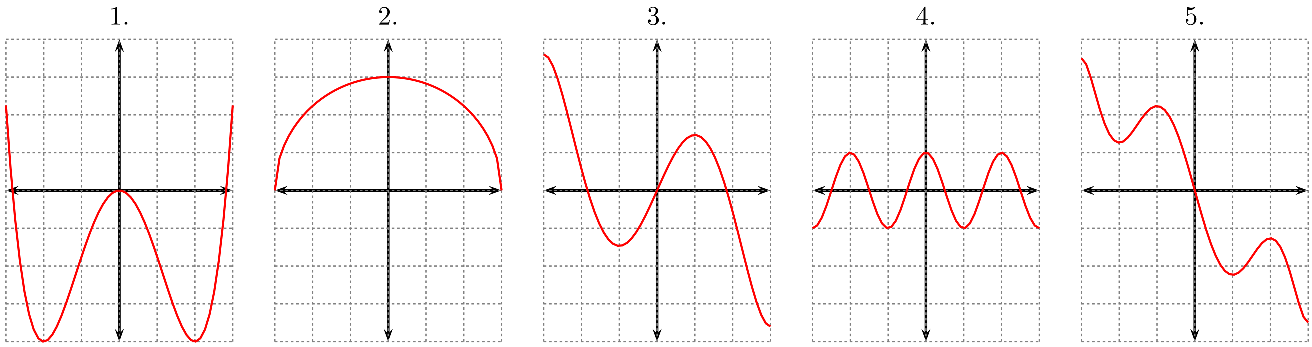
Solution. 37.17

$$\begin{aligned}
3\cdot 2^x+2\left(\frac{1}{2}\right)^{x-1}-7 &= 0\\
3\cdot 2^x+2\left(\frac{1}{2}\right)^{x}\left(\frac{1}{2}\right)^{-1}-7 &= 0\\
3\cdot 2^x+4\left(\frac{1}{2}\right)^{x}-7 &= 0 && \big|\ \text{Set } 2^x=u\\
3u+\frac{4}{u}-7 &= 0 && \big|\ \text{Multiply by } u\\
3u^2-7u+4 &= 0\\
(u-1)(3u-4) &= 0\\
u=1 \quad &\text{or} \quad 3u-4=0\\
2^x=1 \quad & \quad\ u=\frac{4}{3}\\
x=0 \quad & \quad\ 2^x=\frac{4}{3}\\
& \quad\ x=\log_2\frac{4}{3}=\log_2 4-\log_2 3\\
& \quad\ x=2-\log_2 3\\
\text{Calculator:} \quad & \quad\ x\approx 0.415037
\end{aligned}$$

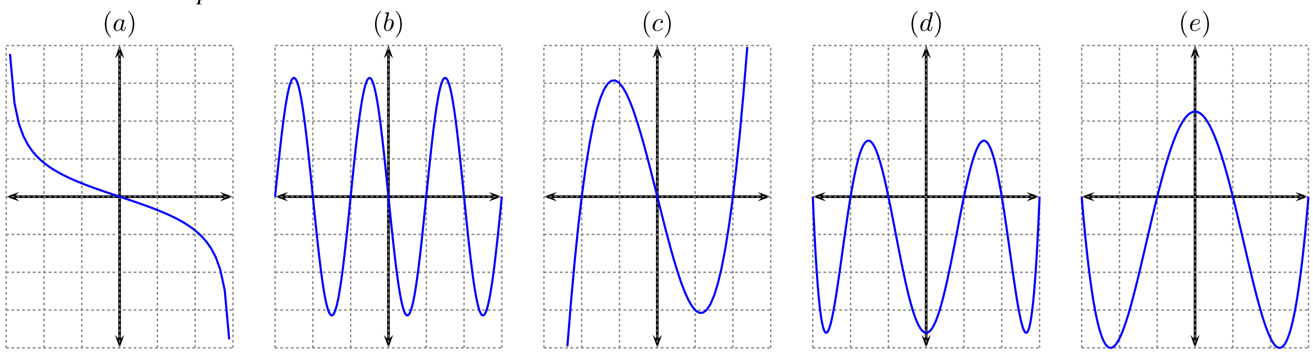
6 Derivatives

6.1 Derivatives and Function Graphs: basics

Problem 38.
Match each of the following function plots:



to their derivative plots:

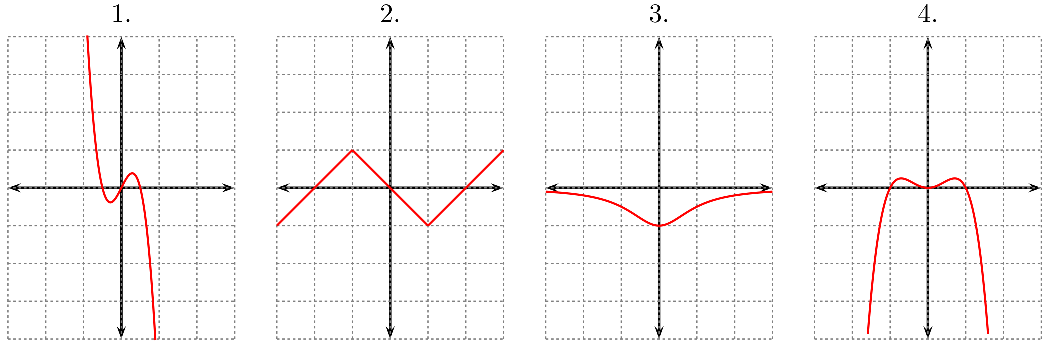


Problem 39. Solution. 38

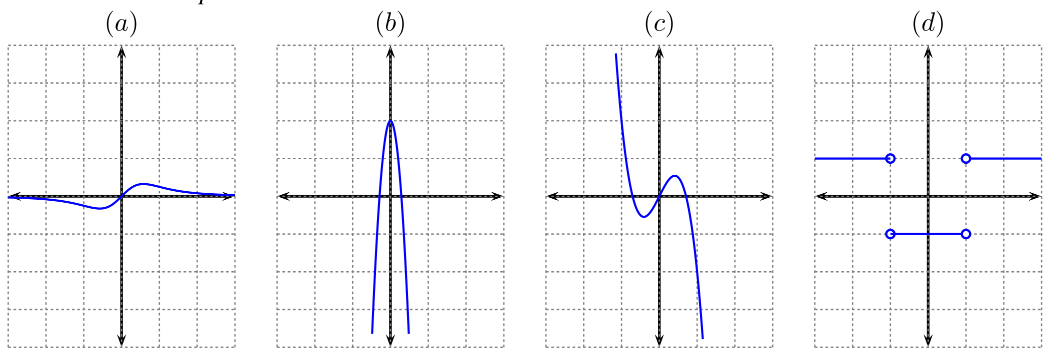
(1) matches (c) because (1) has three local extrema and (c) is the only derivative graph with three zeros.
(2) matches (a) because (2) has one local extrema and (a) is the only derivative graph with one zero.
(3) matches (e) because (3) has four local extrema (± 1 and ± 3) and (e) is the only derivative graph with four zeros.
(4) matches (b) because (4) has seven local extrema and (b) is the only derivative graph with seven zeros.
(5) matches (d) because (5) has six local extrema and (d) is the only derivative graph with six zeros.

Problem 40.

Match each of the following function plots:



to their derivative plots:



Give reasons for your choices. Can you guess formulas that would give a similar (or precisely the same) graph, and confirm visually your guess using a graphing device?

6.2 Product and Quotient Rules

Problem 41. *Compute the derivative.*

1. $f(x) = 2^{2015}$.

   answer: 0

2. $f(x) = \pi^{2015}$.

   answer: 0

3. $f(x) = 2 - \frac{2}{3}x$.

   answer: $-\frac{2}{3}$

4. $f(x) = \frac{3}{4}x^8$.

   answer: $6x^7$

5. $f(x) = x^3 - 4x + 6$.

   answer: $-4 + 3x^2$

6. $f(t) = \frac{1}{2}t^6 - 3t^4 + t$.

   answer: $3t^5 - 12t^3 + 1$

7. $g(x) = x^2(1 - 2x)$.

   answer: $2x - 6x^2$

8. $h(x) = (x - 2)(2x + 3)$.

   answer: $4x - 1$

9. $f(x) = 2x^{-\frac{3}{4}}$.

   answer: $-\frac{3}{2}x^{-\frac{7}{4}}$

10. $f(x) = cx^{-6}$.

    answer: $-6cx^{-7}$

11. $A(x) = -\frac{12}{x^5}$.

    answer: $60x^{-6}$

Solution. 41.7 Approach 1. Uncover the parenthesis, and then differentiate:

$$\left(x^2(1-2x)\right)' = \left(x^2 - 2x^3\right)' = 2x - 6x^2$$

Approach 2. Use first the product rule and then simplify:

$$\begin{aligned}\left(x^2(1-2x)\right)' &= (x^2)'(1-2x) + x^2(1-2x)' \\ &= 2x(1-2x) + x^2(-2) \\ &= 2x - 4x^2 - 2x^2 \\ &= 2x - 6x^2.\end{aligned}$$

Of course, both approaches lead to the same answer.

Problem 42. *Compute the derivative.*

1. $y = x^{\frac{5}{3}} - x^{\frac{2}{3}}$.

answer: $\frac{5}{3}x^{\frac{2}{3}} - 2/3x^{-\frac{1}{3}}$

2. $f(x) = \sqrt{x} - x$.

answer: $-1 + \frac{1}{2}x^{-\frac{1}{2}}$

3. $y = \sqrt{x}(x-1)$.

answer: $\frac{3}{2}x^{\frac{1}{2}} - \frac{1}{2}x^{-\frac{1}{2}}$

4. $f(x) = (2x+1)^2$.

answer: $4 + 8x$

5. $f(x) = 4\pi x^2$.

answer: $8\pi x$

6. $y = \dfrac{x^2+4x+3}{\sqrt{x}}$.

answer: $2x^{-\frac{1}{2}} + \frac{3}{2}x^{\frac{1}{2}} - \frac{3}{2}x^{-\frac{3}{2}}$

7. $y = \dfrac{\sqrt{x}+x}{x^2}$.

answer: $-x^{-2} - \frac{3}{2}x^{-\frac{5}{2}}$

8. $f(x) = \left(x + x^{-1}\right)^3$.

answer: $3x^2 + 3 - 3x^{-2} - 3x^{-4}$

9. $f(x) = \sqrt{2}x + \sqrt{5x}$.

answer: $\sqrt{2} + \frac{\sqrt{5}}{2}x^{-\frac{1}{2}} = \sqrt{2} + \frac{\sqrt{5}}{2\sqrt{x}}$

10. $y = \sqrt[5]{x} + 4\sqrt{x^5}$.

answer: $10x^{\frac{3}{2}} + \frac{1}{5}x^{-\frac{4}{5}}$

11. $y = \left(\sqrt{x} + \dfrac{1}{\sqrt[3]{x}}\right)^2$.

answer: $1 + \frac{1}{3}x^{-\frac{5}{6}} - \frac{2}{3}x^{-\frac{5}{3}}$

12. $f(x) = (1+2x^2)(x-x^2)$.

answer: $1 - 2x + 6x^2 - 8x^3$

13. $f(x) = \dfrac{x^4 - 5x^3 + \sqrt{x}}{x^2}$.

answer: $-5 + 2x - \frac{3}{2}x^{-\frac{5}{2}}$

14. $f(x) = (2x^3+3)(x^4-2x)$.

answer: $-6 - 4x^3 + 14x^6$

15. $f(x) = (1+x+x^2)(2-x^4)$.

answer: $2 + 4x - 4x^3 - 5x^4 - 6x^5$

16. $g(y) = \left(\dfrac{1}{y^2} - \dfrac{3}{y^4}\right)(y + 5y^3)$.

answer: $5 + 9y^{-4} + 14y^{-2}$

17. $f(x) = (x^3 - 2x)(x^{-4} + x^{-2})$.

answer: $1 + 6x^{-4} + x^{-2}$

18. $f(x) = \dfrac{1+2x}{3-4x}$.

answer: $10(3-4x)^{-2}$

Solution. 42.11

$$
\begin{aligned}
\left(\left(\sqrt{x} + \frac{1}{\sqrt[3]{x}}\right)^2\right)' &= \left(\left(x^{\frac{1}{2}} + x^{-\frac{1}{3}}\right)^2\right)' \\
&= \left(\left(x^{\frac{1}{2}}\right)^2 + 2x^{\frac{1}{2}}x^{-\frac{1}{3}} + \left(x^{-\frac{1}{3}}\right)^2\right)' \\
&= \left(x + 2x^{\frac{1}{6}} + x^{-\frac{2}{3}}\right)' \\
&= 1 + 2\cdot\frac{1}{6}x^{\frac{1}{6}-1} + \left(-\frac{2}{3}\right)x^{-\frac{2}{3}-1} \\
&= 1 + \frac{1}{3}x^{-\frac{5}{6}} - \frac{2}{3}x^{-\frac{5}{3}}.
\end{aligned}
$$

Problem 43. *Compute the derivative (with respect to the implied variable).*

1. $f(x) = \dfrac{x-3}{x+3}$.

answer: $6(x+3)^{-2}$

2. $y = \dfrac{x^3}{1-x^2}$.

answer: $\frac{3x^2-x^4}{(1-x^2)^2}$

3. $y = \dfrac{x+1}{x^3+x-2}$.

answer: $\frac{-3-3x^2-2x^3}{(-2+x+x^3)^2}$

4. $y = \dfrac{x-1}{x^3+x-2}$.

answer: $\frac{-2x-1}{\left(x^2+x+2\right)^2}$

5. $f(x) = \dfrac{x+1}{x^3+1}$.

answer: $\frac{-2x+1}{(x^2-x+2)^2}$

6. $y = \dfrac{x^3 - 2x\sqrt{x}}{x}$.

answer: $2x - x^{-\frac{1}{2}}$

7. $y = \dfrac{t}{(t-1)^2}$.

answer: $-\frac{t+1}{(t-1)^3}$

8. $y = \dfrac{t^2+2}{t^4-3t^2+1}$.

answer: $\frac{14t-8t^3-2t^5}{(1-3t^2+t^4)^2}$

9. $g(t) = \dfrac{t-\sqrt{t}}{t^{\frac{1}{3}}}$.

answer: $-\frac{1}{6}t^{-\frac{5}{6}} + \frac{2}{3}t^{-\frac{1}{3}}$

10. $y = ax^2 + bx + c$.

answer: $b + 2ax$

11. $y = A + \dfrac{B}{x} + \dfrac{C}{x^2}$.

answer: $\frac{-Bx-2C}{x^3}$

12. $f(t) = \dfrac{2t}{2+\sqrt{t}}$.

answer: $\frac{4+t^{\frac{1}{2}}}{\left(2+t^{\frac{1}{2}}\right)^2}$

13. $y = \dfrac{cx}{1+cx}$.

answer: $c(1+cx)^{-2}$

14. $y = \sqrt[3]{t}(t^2+t+t^{-1})$.

answer: $-\frac{2}{3}t^{-\frac{5}{3}} + \frac{4}{3}t^{\frac{1}{3}} + \frac{7}{3}t^{\frac{4}{3}}$

15. $y = \dfrac{u^6 - 2u^3 + 5}{u^2}$.

answer: $-2 - 10u^{-3} + 4u^3$

16. $f(x) = \dfrac{ax+b}{cx+d}$.

answer: $\frac{ad-bc}{(cx+d)^2}$

17. $f(x) = \dfrac{1+x}{1+\frac{2}{x}}$.

answer: $\frac{x^2+4x+2}{(2+x)^2}$

18. $f(x) = \dfrac{1+x}{1+\frac{3}{x}}$.

answer: $\frac{x^2+6x+3}{(3+x)^2}$

19. $f(x) = \dfrac{x}{x+\frac{c}{x}}$.

answer: $\frac{2xc}{(c+x^2)^2}$

Solution. 43.7 This can be differentiated more efficiently using the chain rule, however let us show how the problem can be solved directly using the quotient rule.

$$
\begin{aligned}
\left(\frac{t}{(t-1)^2}\right)' &= \frac{(t)'(t-1)^2 - t\left((t-1)^2\right)'}{(t-1)^4} \\
&= \frac{(t-1)^2 - t\left(t^2-2t+1\right)'}{(t-1)^4} \\
&= \frac{(t-1)^2 - t\,(2t-2)}{(t-1)^4} \\
&= \frac{\cancel{(t-1)}\left((t-1)-2t\right)}{(t-1)^{\cancel{4}\,3}} \\
&= \frac{-t-1}{(t-1)^3} \\
&= -\frac{t+1}{(t-1)^3}
\end{aligned}
$$

Solution. 43.5

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x+1}{x^3+1}\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\cancel{x+1}}{\cancel{(x+1)}(x^2-x+1)}\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x^2-x+1}\right)
\end{aligned}
$$

Variant I: use quotient rule.

$$
\begin{aligned}
&= \frac{\frac{\mathrm{d}}{\mathrm{d}x}(1)\cdot(x^2-x+1) - 1\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2-x+1\right)}{\left(x^2-x+1\right)^2} \\
&= \frac{-2x+1}{\left(x^2-x+1\right)^2}
\end{aligned}
$$

Variant I: use chain rule.

$$
\begin{aligned}
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(\left(x^2-x+1\right)^{-1}\right) \\
&= -(x^2-x+1)^{-2}\frac{\mathrm{d}}{\mathrm{d}x}(x^2-x+1) \\
&= -(x^2-x+1)^{-2}(2x-1) \\
&= \frac{-2x+1}{\left(x^2-x+1\right)^2}.
\end{aligned}
$$

6.3 Basic Trigonometric Derivatives

Problem 44. *Compute the derivative.*

1. $f(x) = 2x^3 - 3\cos x.$

   answer: $6x^2 + 3\sin x$

2. $f(x) = \sqrt{x}\cos x.$

   answer: $-x^{\frac{1}{2}}\sin x + \frac{1}{2}x^{-\frac{1}{2}}\cos x$

3. $f(x) = \sin x + \dfrac{1}{3}\cot x.$

   answer: $\frac{-\frac{1}{3}+\cos x\sin^2 x}{\sin^2 x} = \cos x - \frac{1}{3}\csc^2 x$

4. $y = 2\sec x - \csc x.$

   answer: $\frac{\cos^3 x+2\sin^3 x}{(\cos x\sin x)^2}$

5. $y = \dfrac{1+\sin^2\theta}{\cos^3\theta}.$

   answer: $y' = \frac{2\sin\theta\cos^2\theta + 3\sin^3\theta + 3\sin\theta}{\cos^4\theta}$

6. $g(t) = 4\sec t + \tan t - \csc t + 3\cot t.$

   answer: $4\sec t\tan t + \sec^2 t + \csc t\cot t - 3\csc^2 t$

7. $y = c\cos t + t^2\sin t.$

   answer: $-c\sin t + 2t\sin t + t^2\cos t$

8. $y = u(a\cos u + b\cot u).$

   answer: $\frac{-au\sin^3 u+a\cos u\sin^2 u-bu+b\cos u\sin u}{\sin^2 u}$

9. $y = \dfrac{x}{2-\tan x}.$

   answer: $\frac{x-\cos x\sin x+2\cos^2 x}{(2\cos x-\sin x)^2}$

10. $y = \sin\theta\cos\theta.$

    answer: $\cos(2\theta) = \cos^2\theta - \sin^2\theta$

11. $f(\theta) = \dfrac{\sec\theta}{1+\sec\theta}.$

    answer: $\frac{\sin\theta}{(1+\cos\theta)^2}$

12. $y = \dfrac{\cos x}{1-\sin x}.$

    answer: $\frac{1}{1-\sin x}$

13. $y = \dfrac{t\sin t}{1+t}.$

    answer: $\frac{\sin t+t\cos t+t^2\cos t}{(1+t)^2}$

14. $y = \dfrac{1-\sec x}{\tan x}.$

    answer: $\frac{\cos x-1}{\sin^2 x}$

15. $h(\theta) = \theta\csc\theta - \cot\theta.$

    answer: $\frac{1+\sin\theta-\theta\cos\theta}{\sin^2\theta}$

16. $y = x^2\sin x\tan x.$

    answer: $\frac{2x\cos x\sin^2 x+2x^2\sin x\cos^2 x+x^2\sin^3(x)}{\cos^2 x}$

Problem 45. *Differentiate.*

1. $\tan x.$

   answer: $\sec^2 x$

2. $\cot x.$

   answer: $-\csc^2 x$

3. $\sec x.$

   answer: $\sec x\tan x = \frac{\sin x}{\cos^2 x}$

4. $\csc x.$

   answer: $-\csc x\cot x = -\frac{\cos x}{\sin^2 x}$

5. $\sec x\tan x.$

   answer: $\sec x\tan^2 x + \sec^3 x$

6. $\sec x + \tan x.$

   answer: $\sec x(\tan x + \sec x)$

7. $\sec^2 x.$

   answer: $2\tan x\sec^2 x$

8. $\csc^2 x.$

   answer: $-2\cot x\csc^2 x$

9. $f(x) = (\sec x)e^x.$

   answer: $\sec x\tan x e^x + \sec x e^x = (\tan x + 1)(\sec x)e^x$

10. $f(x) = (\tan x)e^x.$

    answer: $\sec^2 x e^x + \tan x e^x$

11. $\dfrac{\sin x}{x}.$

    answer: $\frac{x\cos x-\sin x}{x^2}$

12. $\dfrac{\sin x}{e^x}.$

    answer: $\frac{\cos x-\sin x}{e^x}$

13. $x(\cos x)e^x.$

    answer: $e^x(x\cos x - x\sin x + \cos x)$

14. $\dfrac{e^x}{\tan x}.$

    answer: $e^x\left(\cot x - \csc^2 x\right)$

15. $\dfrac{e^x}{\sec x} + \sec x.$

    answer: $e^x(\cos x - \sin x) + \sec x\tan x$

6.4 Natural Exponent Derivatives

Differentiate each function.

1. $f(x) = \dfrac{e^x}{1+2e^x}$.
2. $r(t) = Ae^{-kt^2}$, where A and k are unknown constants.

   Solution.

$$
\begin{aligned}
& & r &= Ae^{-kt^2}. \\
&\text{Let} & u &= -kt^2. \\
&\text{Then} & r &= Ae^u. \\
&\text{Chain Rule} & \frac{\mathrm{d}r}{\mathrm{d}t} &= \frac{\mathrm{d}r}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}t} \\
& & &= (Ae^u)(-2kt) \\
& & &= -2Akte^{-kt^2}.
\end{aligned}
$$

3. $y = \dfrac{e^x}{2}(\sin x + \cos x)$.

6.5 The Chain Rule

Problem 46. *Compute the derivative using the chain rule.*

1. $f(x) = \sqrt{1+x^2}$

answer: $x(x^2+1)^{-\frac{1}{2}}$

2. $f(x) = \sqrt{3x^2 - x + 2}$.

answer: $\frac{6x-1}{2\sqrt{3x^2-x+2}}$

3. $f(x) = \dfrac{x}{\sqrt{1+\frac{2}{x^2}}}$.

answer: $\frac{\pm x^2}{\sqrt{x^2+2}}$

4. $f(x) = \sqrt{1-\sqrt{x}}$.

answer: $-\frac{1}{4}x^{-\frac{1}{2}}\left(-\sqrt{x}+1\right)^{-\frac{1}{2}}$

5. $y = (\cos x)^{\frac{1}{2}}$

answer: $\frac{1}{2}x^{-\frac{1}{2}}\cos\left(\sqrt{x}\right) = \frac{\cos(\sqrt{x})}{2\sqrt{x}}$

6. $f(x) = \sin^3 x$.

answer: $3\cos x\sin^2 x$

7. $y = (1+\cos x)^2$.

answer: $-2\cos x\sin x - 2\sin x = -\sin(2x) - 2\sin x$

8. $f(x) = \dfrac{1}{\sin^3 x}$.

answer: $-\frac{3\cos x}{\sin^4 x}$

9. $f(x) = \sqrt[3]{4+3\tan x}$.

answer: $(4+3\tan x)^{-\frac{2}{3}}\sec^2 x$

10. $f(x) = (\cos x + 3\sin x)^4$.

answer: $4(\cos x + 3\sin x)^3(3\cos x - \sin x)$

11. $y = \sin\left(\sqrt{x}\right)$

answer: $\frac{\cos\sqrt{x}}{2\sqrt{x}}$

12. $y = \cos(4x)$

answer: $-4\sin(4x)$

13. $\sec^2(3x^2)$.

answer: $12\frac{x\sin\left(3x^2\right)}{\left(\cos\left(3x^2\right)\right)^3}$

14. $\csc^2(3x^2)$.

answer: $-12\frac{x\cos\left(3x^2\right)}{\left(\sin\left(3x^2\right)\right)^3}$

15. e^{2x}.

answer: $2e^{2x}$

16. e^{-x^2}

answer: $-2xe^{-x^2}$

17. $e^{\sqrt{x}}$

answer: $\frac{1}{2\sqrt{x}}e^{\sqrt{x}}$

18. $f(x) = e^{-\frac{1}{x}}$.

answer: $\frac{e^{-\frac{1}{x}}}{x^2}$

19. 5^x.

answer: $(\ln 5)5^x$

20. e^{2^x}.

answer: $e^{2^x} 2^x (\ln 2)$

21. 2^{3^x}.

answer: $2^{3^x} 3^x (\ln 2)(\ln 3)$

22. 3^{2^x}.

answer: $3^{2^x} 2^x (\ln 2)(\ln 3)$

23. $y = \sqrt{\sec(4x)}$

answer: $2\sqrt{\sec(4x)}\tan(4x) = 2(\sec(4x))^{\frac{3}{2}}\sin(4x)$.

24. $y = x^2 \tan(5x)$

answer: $2x\tan(5x) - 5x^2\sec^2(5x)$

25. $y = \dfrac{1+\sin\left(x^2\right)}{1+\cos\left(x^2\right)}$.

answer: $\dfrac{2x\left(1+\cos\left(x^2\right)+\sin\left(x^2\right)\right)}{\left(1+\cos\left(x^2\right)\right)^2}$

Solution. 46.2

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{3x^2-x+2}\right) = \frac{(3x^2-x+2)'}{2\sqrt{3x^2-x+2}} = \frac{6x-1}{2\sqrt{3x^2-x+2}}.$$

Solution. 46.3

$$\begin{aligned}
\left(\frac{x}{\sqrt{1+\frac{2}{x^2}}}\right)' &= \frac{\sqrt{1+\frac{2}{x^2}} - x\left(\sqrt{1+\frac{2}{x^2}}\right)'}{1+\frac{2}{x^2}} = \frac{\sqrt{1+\frac{2}{x^2}} - x\frac{\frac{1}{2}}{\sqrt{1+\frac{2}{x^2}}}\left(\frac{2}{x^2}\right)'}{1+\frac{2}{x^2}} \\
&= \frac{\sqrt{1+\frac{2}{x^2}} + \frac{2}{x^2\sqrt{1+\frac{2}{x^2}}}}{1+\frac{2}{x^2}} = \frac{x^2\left(1+\frac{2}{x^2}\right)+2}{x^2\left(1+\frac{2}{x^2}\right)^{\frac{3}{2}}} = \frac{x^2+4}{x^2\left(1+\frac{2}{x^2}\right)^{\frac{3}{2}}}
\end{aligned}$$

Please note that this problem can be solved also by applying the transformation

$$\frac{x}{\sqrt{1+\frac{2}{x^2}}} = \frac{x}{\sqrt{\frac{x^2+2}{x^2}}} = \frac{x}{\frac{1}{\pm x}\sqrt{x^2+2}} = \frac{\pm x^2}{\sqrt{x^2+2}}$$

before differentiating, however one must not forget the $\pm$ sign arising from $\sqrt{x^2} = \pm x$. Our original approach resulted in more algebra, but did not have the disadvantage of dealing with the $\pm$ sign.

Solution. 46.4

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{1-\sqrt{x}}\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\left(1-x^{\frac{1}{2}}\right)^{\frac{1}{2}}\right) && \text{chain rule} \\
&= \frac{1}{2}\left(1-x^{\frac{1}{2}}\right)^{-\frac{1}{2}}\frac{\mathrm{d}}{\mathrm{d}x}\left(1-x^{\frac{1}{2}}\right) \\
&= -\frac{1}{4}x^{-\frac{1}{2}}\left(1-x^{\frac{1}{2}}\right)^{-\frac{1}{2}}
\end{aligned}$$

Solution. 46.5

$$\begin{aligned}
\text{Let}\quad & u = \cos x. \\
\text{Then}\quad & y = u^{\frac{1}{2}}. \\
\text{Chain Rule:}\quad & \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} \\
& \qquad = \left(\frac{1}{2}u^{-\frac{1}{2}}\right)(-\sin x) \\
& \qquad = -\frac{1}{2}\sin x(\cos x)^{-\frac{1}{2}}.
\end{aligned}$$

Solution. 46.7

$$\begin{aligned}
\text{Let} \quad & u = 1+\cos x. \\
\text{Then} \quad & y = u^2. \\
\text{Chain Rule:} \quad & \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} \\
& = (2u)(-\sin x) \\
& = -2\sin x(1+\cos x) \\
& = -2\sin x - 2\sin x\cos x \\
& = -2\sin x - \sin(2x). \quad \text{(This last step is optional.)}
\end{aligned}$$

Solution. 46.11

$$\begin{aligned}
\text{Let} \quad & u = \sqrt{x}. \\
\text{Then} \quad & y = \sin u. \\
\text{Chain Rule:} \quad & \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} \\
& = (\cos u)\left(\frac{1}{2}u^{-\frac{1}{2}}\right) \\
& = \frac{\cos\left(\sqrt{x}\right)}{2\sqrt{x}}.
\end{aligned}$$

Solution. 46.18

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}x}\left(e^{-\frac{1}{x}}\right) &= e^{-\frac{1}{x}}\tfrac{\mathrm{d}}{\mathrm{d}x}\left(-\tfrac{1}{x}\right) && \Big| \text{ chain rule} \\
&= -e^{-\frac{1}{x}}\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{-1}\right) \\
&= x^{-2}e^{-\frac{1}{x}} \\
&= \frac{e^{-\frac{1}{x}}}{x^2}
\end{aligned}$$

Solution. 46.23

$$\begin{aligned}
\text{Chain Rule:} \quad & \frac{\mathrm{d}y}{\mathrm{d}x} = \left(\frac{1}{2}(\sec(4x))^{-\frac{1}{2}}\right)\frac{\mathrm{d}}{\mathrm{d}x}(\sec(4x)) \\
\text{Chain Rule:} \quad & \frac{\mathrm{d}y}{\mathrm{d}x} = \left(\frac{1}{2\sqrt{\sec(4x)}}\right)(\sec(4x)\tan(4x))\frac{\mathrm{d}}{\mathrm{d}x}(4x) \\
& = \left(\frac{1}{2\sqrt{\sec(4x)}}\right)(\sec(4x)\tan(4x))(4) \\
& = \frac{2\sec(4x)\tan(4x)}{\sqrt{\sec(4x)}}
\end{aligned}$$

There are many ways to simplify this answer, including both of the following.

$$\begin{aligned}
& = 2\sqrt{\sec(4x)}\tan(4x). \\
& = 2(\sec(4x))^{\frac{3}{2}}\sin(4x).
\end{aligned}$$

Solution. 46.24

$$\text{Product Rule:} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = (x^2)\frac{\mathrm{d}}{\mathrm{d}x}(\tan(5x)) + (\tan(5x))\frac{\mathrm{d}}{\mathrm{d}x}(x^2)$$

Use the Chain Rule to differentiate $\tan(5x)$ in the first term.

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= (x^2)(-5\sec^2(5x) + (\tan(5x))(2x) \\ &= 2x\tan(5x) - 5x^2\sec^2(5x).\end{aligned}$$

Solution. 46.25

$$\text{Quotient Rule:} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\left(1+\cos\left(x^2\right)\right)\frac{\mathrm{d}}{\mathrm{d}y}\left(1+\sin\left(x^2\right)\right) - \left(1+\sin\left(x^2\right)\right)\frac{\mathrm{d}}{\mathrm{d}x}\left(1+\cos\left(x^2\right)\right)}{\left(1+\cos\left(x^2\right)\right)^2}$$

By the Chain Rule, $\frac{\mathrm{d}}{\mathrm{d}x}\left(1+\cos\left(x^2\right)\right) = -2x\sin\left(x^2\right)$ and $\frac{\mathrm{d}}{\mathrm{d}x}\left(1+\sin\left(x^2\right)\right) = 2x\cos\left(x^2\right)$.

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\left(1+\cos\left(x^2\right)\right)\left(2x\cos\left(x^2\right)\right) - \left(1+\sin\left(x^2\right)\right)\left(-2x\sin\left(x^2\right)\right)}{\left(1+\cos\left(x^2\right)\right)^2} \\ &= \frac{2x\cos\left(x^2\right) + 2x\cos^2\left(x^2\right) + 2x\sin\left(x^2\right) + 2x\sin^2\left(x^2\right)}{\left(1+\cos\left(x^2\right)\right)^2} \\ &= \frac{2x\left(\cos^2\left(x^2\right) + \sin^2\left(x^2\right)\right) + 2x\left(\cos\left(x^2\right) + \sin\left(x^2\right)\right)}{\left(1+\cos\left(x^2\right)\right)^2}\end{aligned}$$

By the Pythagorean Identity, $\cos^2\left(x^2\right) + \sin^2\left(x^2\right) = 1$.

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{2x + 2x\left(\cos\left(x^2\right) + \sin\left(x^2\right)\right)}{\left(1+\cos\left(x^2\right)\right)^2} \\ &= \frac{2x\left(1+\cos\left(x^2\right) + \sin\left(x^2\right)\right)}{\left(1+\cos\left(x^2\right)\right)^2}.\end{aligned}$$

Problem 47. *Differentiate. The answer key has not been proofread, use with caution.*

1. $f(x) = \sin(-5x)$.

   answer: $-5\cos(-5x) = -5\cos(5x)$

2. $f(x) = \cot(2x)$.

   answer: $-2\csc^2(2x)$

3. $f(x) = e^{-3x}$.

   answer: $-3e^{-3x}$

4. $f(x) = e^{\frac{1}{x}}$.

   answer: $-\frac{e^{\frac{1}{x}}}{x^2}$

5. $f(x) = e^{\sqrt{x}}$.

   answer: $\frac{1}{2\sqrt{x}}e^{\sqrt{x}}$

6. $f(x) = \ln(1+x)$

   answer: $\frac{1}{1+x}$

7. $f(x) = \ln(1+x^3)$

   answer: $\frac{3x^2}{1+x^3}$

8. $f(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$

   answer: $\frac{1}{1-x^2}$

Problem 48. *Compute the derivative.*

1. $f(x) = (x^4 + 3x^2 - 2)^5$.

   answer: $(30x + 20x^3)\left(-2 + 3x^2 + x^4\right)^4$

2. $f(x) = (4x - x^2)^{100}$.

   answer: $(-200x + 400)\left(4x - x^2\right)^{99}$

3. $f(x) = (2x-3)^4(x^2+x+1)^5$.

   answer: $\left(-7 - 12x + 28x^2\right)(-3+2x)^3\left(1 + x + x^2\right)^4$

4. $f(x) = (x^2+1)^3(x^2+2)^6$.

   answer: $\left(24x + 18x^3\right)\left(1 + x^2\right)^2\left(2 + x^2\right)^5$

5. $f(x) = (3x-1)^4(2x+1)^{-3}$.

   answer: $(3x-1)^3\frac{6x+18}{(2x+1)^4}$

6. $f(x) = \frac{1}{1+x^2}$.

   answer: $\frac{-2x}{(1+x^2)^2}$

7. $f(x) = \left(\frac{x^2+1}{x^2-1}\right)^3$.

   answer: $\frac{-12x}{\left(x^2-1\right)^2}\left(\frac{x^2+1}{x^2-1}\right)^2$

8. $f(x) = (x+1)^{\frac{2}{3}}(2x^2-1)^3$.

   answer: $\left(\frac{40}{3}x^2 + 12x - \frac{2}{3}\right)\left(2x^2-1\right)^2(x+1)^{-\frac{1}{3}}$

9. $f(x) = \sqrt{1-2x}$.

answer: $-(1-2x)^{-\frac{1}{2}}$

10. $f(x) = \sqrt{\dfrac{x^2+1}{x^2+4}}$.

answer: $\frac{3x}{\left(x^2+4\right)^2} \left(\frac{x^2+1}{x^2+4}\right)^{-\frac{1}{2}}$

11. $f(x) = 3\cot(2x)$.

answer: $\frac{-6}{(\sin(2x))^2}$

12. $f(x) = \dfrac{1}{(1+\sec x)^2}$.

answer: $\frac{-2\cos(x)\sin(x)}{(1+\cos(x))^3} = \frac{-\sin(2x)}{(1+\cos(x))^3}$

13. $f(x) = \sqrt[3]{1+\tan x}$.

answer: $\frac{1}{3}(1+\tan x)^{-\frac{2}{3}} \sec^2 x$

14. $f(x) = \cos(2+x^3)$.

answer: $-3x^2 \sin\left(2+x^3\right)$

15. $f(x) = \cos\left(\dfrac{1}{x}\right)\sin(x^2)$.

answer: $x^{-2}\sin\left(x^{-1}\right)\sin\left(x^2\right) + 2x\cos\left(x^{-1}\right)\cos\left(x^2\right)$

16. $f(x) = x\sec(kx)$.

answer: $\frac{\cos(kx)+kx\sin(kx)}{(\cos(kx))^2}$

Problem 49. *Differentiate.*

1. $f(x) = \sin(\tan(2x))$.

answer: $2\sec^2(2x)\cos(\tan 2x)$

2. $f(x) = \sec^2(mx)$.

answer: $\frac{2m\sin(mx)}{(\cos(mx))^3}$

3. $f(x) = \sec^2 x + \tan^2 x$.

answer: $\frac{4\sin x}{\cos^3 x}$

4. $f(x) = x\sin\left(\dfrac{1}{x}\right)$.

answer: $-x^{-1}\cos\left(x^{-1}\right) + \sin\left(x^{-1}\right)$

5. $f(x) = \left(\dfrac{1-\cos(2x)}{1+\cos(2x)}\right)^4$.

answer: $\frac{16\sin(2x)}{(\cos(2x)+1)^2}\left(\frac{-\cos(2x)+1}{\cos(2x)+1}\right)^3$

6. $f(x) = \sqrt{\dfrac{x}{x^2+4}}$.

answer: $\frac{-\frac{1}{2}x^2+2}{(x^2+4)^2}\left(\frac{x}{x^2+4}\right)^{-\frac{1}{2}}$

7. $f(t) = \cot^2(\sin t)$.

answer: $\frac{-2\cos t\cos(\sin t)}{\sin^3(\sin t)}$

8. $f(x) = \left(ax + \sqrt{x^2+b^2}\right)^{-2}$.

answer: $\frac{-2x\left(x^2+b^2\right)^{-\frac{1}{2}} - 2a}{\left(\left(x^2+b^2\right)^{\frac{1}{2}} + ax\right)^3}$

9. $f(x) = \left(x^2 + (1-3x)^5\right)^3$.

answer: $\left(-45(-3x+1)^4 + 6x\right)\left((-3x+1)^5 + x^2\right)^2$

Using computer algebra:

$(-3645x^4 + 4860x^3 - 2430x^2 + 546x - 45)\left((-3x+1)^5 + x^2\right)^2$

Using computer algebra full expansion:

$-215233605x^{14} + 1004423490x^{13} - 2176250895x^{12}$
$+2903793624x^{11} - 2666357595x^{10} + 1782098820x^9$
$-893713176x^8 + 341444160x^7 - 99805041x^6$
$+22199676x^5 - 3697470x^4 + 447132x^3$
$-37125x^2 + 1896x - 45$

10. $f(x) = \sin(\sin(\sin x))$.

answer: $\cos x\cos(\sin x)\cos(\sin(\sin x))$

11. $f(x) = \sqrt{x+\sqrt{x}}$.

answer: $\left(\frac{1}{2} + \frac{1}{4}x^{-\frac{1}{2}}\right)\left(x^{\frac{1}{2}} + x\right)^{-\frac{1}{2}}$

12. $f(x) = \sqrt{x+\sqrt{x+\sqrt{x}}}$.

answer: $\frac{1}{2}\left(\left(x^{\frac{1}{2}}+x\right)^{\frac{1}{2}} + x\right)^{-\frac{1}{2}}\left(\frac{1}{2}\left(x^{\frac{1}{2}}+x\right)^{-\frac{1}{2}}\left(\frac{1}{2}x^{-\frac{1}{2}}+1\right)+1\right)$

13. $f(x) = (2r\sin(rx)+n)^p$.

answer: $pr(2r\sin(rx)+n)^{p-1}\cos(rx)$

14. $f(x) = \cos^4(\sin^3 x)$.

answer: $-12\cos x\sin^2 x\sin\left(\sin^3 x\right)\cos^3\left(\sin^3 x\right)$

15. $f(x) = \cos\sqrt{\sin(\tan(\pi x))}$.

answer: $\frac{-\frac{1}{2}\pi\cos(\tan(\pi x))\sin\left(\sqrt{\sin(\tan(\pi x))}\right)}{\sqrt{\sin(\tan(\pi x))}\cos^2(\pi x)}$

16. $f(x) = \left(x + (x+\sin^2 x)^3\right)^4$.

answer: $4((\sin^2 x + x)^3 + x)^3(3(\sin^2 x + x)^2(2\sin x\cos x + 1) + 1)$

Problem 50. *Compute the second derivative.*

1. $f(x) = \sin(-5x)$.

answer: $25\sin(5x)$

2. $f(x) = \cot(2x)$.

answer: $8\cot(2x)\csc^2(2x) = \frac{8\cos(2x)}{\sin^3(2x)}$

3. $f(x) = e^{-3x}$.

answer: $9e^{-3x}$

4. $f(x) = e^{\frac{1}{x}}$.

answer: $2e^{x^{-1}}x^{-3} + e^{x^{-1}}x^{-4}$

5. $f(x) = e^{\sqrt{x}}$.

answer: $e^{x^{\frac{1}{2}}}x^{-\frac{3}{2}} + \frac{1}{4}e^{x^{\frac{1}{2}}}x^{-1}$

6. $f(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

answer: $\frac{-8\left(-e^{-x}+e^x\right)}{\left(e^{-x}+e^x\right)^3}$

7. $f(x) = \dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)$

answer: $\frac{1}{2}\left(-\frac{1}{(x+1)^2} + \frac{1}{(-x+1)^2}\right) = \frac{2x}{\left(1-x^2\right)^2}$

6.6 Problem Collection All Techniques

Problem 51. *Find the derivative of the following functions.*

1. $\dfrac{\sin x}{x^2}$

   answer: $\dfrac{x\cos x - 2\sin x}{x^3}$

2. $e^{\sqrt{x^2+1}}$

   answer: $e^{\sqrt{x^2+1}} \cdot \frac{1}{2}\left(x^2+1\right)^{-\frac{1}{2}} \cdot 2x = \dfrac{xe^{\sqrt{x^2+1}}}{\sqrt{x^2+1}}$

3. $\ln\left(x - \dfrac{1}{x}\right)$

   answer: $\dfrac{1}{x-\frac{1}{x}} \cdot \left(1 + \dfrac{1}{x^2}\right)$

4. $\sqrt[3]{x}\ln x$

   answer: $\dfrac{1}{3}\dfrac{1}{\sqrt[3]{x^2}}\ln x + \dfrac{1}{\sqrt[3]{x^2}}$

5. $\cos(e^x)$

   answer: $-\sin\left(e^x\right)\cdot e^x$

6. $\sin^3(2x)$

   answer: $3\sin^2(2x)\cdot\cos(2x)\cdot 2 = 6\sin^2(2x)\cos(2x)$

7. *Find* y' *if* $2x^2 + x + xy = 1$.

   answer: $y' = \dfrac{-4x-1-y}{x} = \dfrac{-2x^2-1}{x^2}$

8. *Find* y' *if* $x\sin y + y\sin x = 4$.

   answer: $y' = \dfrac{-\sin y - y\cos x}{\sin x + x\cos y}$

6.7 Implicit Differentiation

Problem 52. *Express* $\frac{\mathrm{d}y}{\mathrm{d}x}$ *as a function of* x *and* y *by implicit differentiation. The answer key has not been proofread, use with caution.*

1. $x^3 + y^3 = 1$.

   answer: $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x^2}{y^2}$

2. $2\sqrt{x} + \sqrt{y} = 3$.

   answer: $\frac{\mathrm{d}y}{\mathrm{d}x} = -2\sqrt{\frac{y}{x}}$

3. $x^2 + xy - y^2 = 4$.

   answer: $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-2x-y}{x-2y}$

4. $2x^3 + x^2y - xy^3 = 2$.

   answer: $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^3-6x^2-2xy}{-3xy^2+x^2}$

5. $x^4(x+y) = y^2(3x-y)$.

   answer: $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-5x^4-4x^3y+3y^2}{x^4-6xy-3y^2}$

6. $y^5 + x^2y^3 = 1 + x^4y$.

   answer: $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4x^3y-2xy^3}{5y^4+3x^2y^2-x^4}$

7. $y\cos x = x^2 + y^2$.

   answer: $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y\sin x+2x}{\cos x-2y}$

8. $\cos(xy) = 1 + \sin y$.

   answer: $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{y\sin(xy)}{\cos y+x\sin(xy)}$

9. $4\cos x\sin y = 1$.

   answer: $\frac{\mathrm{d}y}{\mathrm{d}x} = \tan x\tan y$

10. $y\sin\left(x^2\right) = x\sin\left(y^2\right)$.

    answer: $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-2xy\cos\left(x^2\right)+\sin\left(y^2\right)}{-2xy\cos\left(y^2\right)+\sin\left(x^2\right)}$

11. $\tan\left(\dfrac{x}{y}\right) = x + y$.

    answer: $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-y^2+y\sec^2\left(\frac{y}{x}\right)}{y^2+x\sec^2\left(\frac{x}{y}\right)}$

12. $\sqrt{x+y} = 1 + x^2y^2$.

    answer: $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-\frac{1}{2}(y+x)^{-\frac{1}{2}}+8xy^2}{\frac{1}{2}(y+x)^{-\frac{1}{2}}-8x^2y}$

13. $\sqrt{xy} = 1 + x^2y$.

    answer:
    $$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2xy-\frac{y}{2\sqrt{xy}}}{\frac{x}{2\sqrt{xy}}-x^2} = \frac{4xyx^2y+4xy-y}{-2x^4y-2x^2+x}$$

14. $x\sin y + y\sin x = 1$.

    answer: $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-y\cos x-\sin y}{x\cos y+\sin x}$

15. $y\cos x = 1 + \sin(xy)$.

    answer: $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\cos(xy)y+y\sin x}{-\cos(xy)x+\cos x}$

16. $\tan(x-y) = \dfrac{y}{1+x^2}$.

    answer: $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-(\sec(-y+x))^2x^4-2(\sec(-y+x))^2x^2-(\sec(-y+x))^2-2yx}{-(\sec(-y+x))^2x^4-2(\sec(-y+x))^2x^2-(\sec(-y+x))^2-x^2-1}$

17. $x^4(x+y) = y^2(3x-y)$.

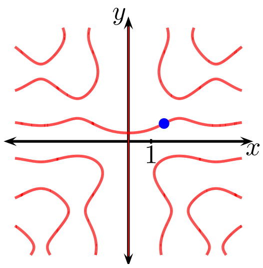
    answer: $\frac{dy}{dx} = \frac{-4x^3y+3y^2-5x^4}{3y^2+x^4-6yx}$

18. $2x^3 + x^2y - xy^3 = 2$.

    answer: $\frac{dy}{dx} = \frac{y^3-6x^2-2xy}{-3xy^2+x^2}$

Problem 53. *Verify that the coordinates of the given point satisfy the given equation. Use implicit differentiation to find an equation of the tangent line to the curve at the given point.*

1. $y \sin(2x) = x\cos(2y)$, $\left(\frac{\pi}{2}, \frac{\pi}{4}\right)$.

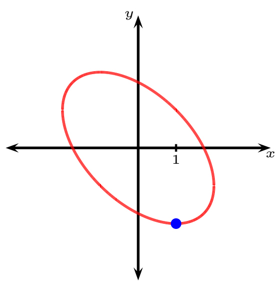


answer: $y = \frac{1}{2}x$

2. $\sin(x+y) = 2x - 2y$, (π, π) .

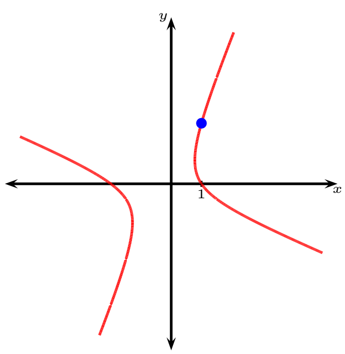
answer: $y = \frac{1}{3}x + \frac{2}{3}\pi$

3. $x^2 + xy + y^2 = 3$, $(1, -2)$ *(ellipse).*



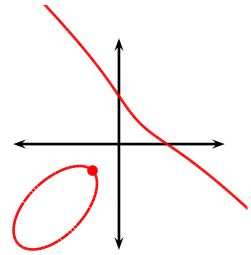
answer: $y = -2$

4. $x^2 + 2xy - y^2 + x = 2$, $(1, 2)$ *(hyperbola).*



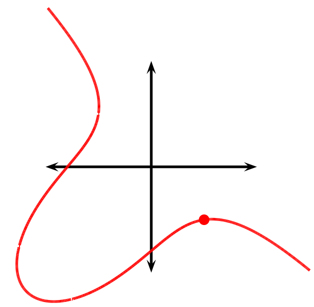
answer: $y = \frac{7}{2}x - \frac{3}{2}$

5. $y^3 + x^3 + 4xy = \frac{3}{4}$, $\left(-\frac{1}{2}, -\frac{1}{2}\right)$



answer: $y = -x - 1$

6. $y^3 + x^3 + 4xy = -4$, $(1, -1)$



answer:

7. $x^2 + y^2 = (2x^2 + 2y^2 - x)^2$, $(0, \frac{1}{2})$.

answer: $y = x + \frac{1}{2}$

8. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 4$, $(-3\sqrt{3}, 1)$.

answer: $y = \frac{1}{\sqrt{3}}x + 4$

9. $2(x^2 + y^2)^2 = 25(x^2 - y^2)$, $(3, 1)$.

answer: $y = -\frac{9}{13}x + \frac{40}{13}$

10. $y^2(y^2 - 4) = x^2(x^2 - 5)$, $(0, -2)$.

answer: $y = -2$

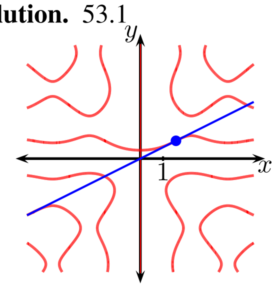
11. $x^{\frac{4}{3}} + y^{\frac{4}{3}} = 10$ *at* $(-3\sqrt{3}, 1)$.

answer: $y = \sqrt{3}x + 10$

12. $x^2y^3 + x^3 - y^2 = 1$ *at* $(1, 1)$.

answer: $y = -5x + 6$

Solution. 53.1



First we verify that the point $(x, y) = \left(\frac{\pi}{2}, \frac{\pi}{4}\right)$ indeed satisfies the given equation:

$$\begin{array}{rcll|l} y\sin(2x)_{|x=\frac{\pi}{2}, y=\frac{\pi}{4}} &=& \frac{\pi}{4}\sin\pi &=& 0 & \text{left hand side} \\ x\cos(2y)_{|x=\frac{\pi}{2}, y=\frac{\pi}{4}} &=& \frac{\pi}{2}\cos\left(\frac{\pi}{2}\right) &=& 0 & \text{right hand side} \end{array}$$

so the two sides of the equation are equal (both to 0) when $x=\frac{\pi}{2}$ and $y=\frac{\pi}{4}$.

Since we are looking an equation of the tangent line, we need to find $\frac{\mathrm{d}y}{\mathrm{d}x}_{|x=\frac{\pi}{2},y=\frac{\pi}{4}}$ - that is, the derivative of y at the point $x=\frac{\pi}{2}$, $y=\frac{\pi}{4}$. To do so we use implicit differentiation.

$$\begin{array}{rcll}
y\sin(2x) &=& x\cos(2y) & \Big|\ \frac{\mathrm{d}}{\mathrm{d}x}\\
\frac{\mathrm{d}y}{\mathrm{d}x}\sin(2x)+y\frac{\mathrm{d}}{\mathrm{d}x}(\sin(2x)) &=& \cos(2y)+x\frac{\mathrm{d}}{\mathrm{d}x}(\cos(2y)) & \\
\frac{\mathrm{d}y}{\mathrm{d}x}\sin(2x)+2y\cos(2x) &=& \cos(2y)-2x\sin(2y)\frac{\mathrm{d}y}{\mathrm{d}x} & \\
\frac{\mathrm{d}y}{\mathrm{d}x}(\sin(2x)+2x\sin(2y)) &=& \cos(2y)-2y\cos(2x) & \Big|\ \text{Set } x=\frac{\pi}{2}, y=\frac{\pi}{4}\\
\frac{\mathrm{d}y}{\mathrm{d}x}_{|x=\frac{\pi}{2},y=\frac{\pi}{4}}\left(\sin\pi+\pi\sin\left(\frac{\pi}{2}\right)\right) &=& \cos\left(\frac{\pi}{2}\right)-\frac{\pi}{2}\cos\pi & \\
\pi\frac{\mathrm{d}y}{\mathrm{d}x}_{|x=\frac{\pi}{2},y=\frac{\pi}{4}} &=& -\frac{\pi}{2}\cos\pi & \\
\frac{\mathrm{d}y}{\mathrm{d}x}_{|x=\frac{\pi}{2},y=\frac{\pi}{4}} &=& \frac{1}{2}. &
\end{array}$$

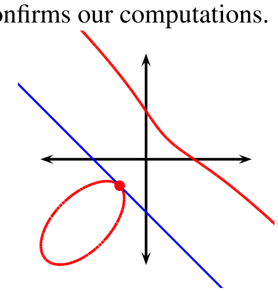
Therefore the equation of the line through $x=\frac{\pi}{2}, y=\frac{\pi}{4}$ is

$$\begin{array}{rcl}
y-\frac{\pi}{4} &=& \frac{1}{2}\left(x-\frac{\pi}{2}\right)\\
y &=& \frac{1}{2}x.
\end{array}$$

Solution. 53.5

$$\begin{array}{rcll}
y^3+x^3+4xy &=& \frac{3}{4} & \Big|\ \text{apply } \frac{\mathrm{d}}{\mathrm{d}x}\\
3y^2\frac{\mathrm{d}y}{\mathrm{d}x}+3x^2+4\left(y+x\frac{\mathrm{d}y}{\mathrm{d}x}\right) &=& 0 & \\
\frac{\mathrm{d}y}{\mathrm{d}x}\left(3y^2+4x\right) &=& -3x^2-4y & \\
\frac{\mathrm{d}y}{\mathrm{d}x} &=& \frac{-3x^2-4y}{3y^2+4x} & \Big|\ \text{substitute } x=-\frac{1}{2}, y=-\frac{1}{2}\\
\frac{\mathrm{d}y}{\mathrm{d}x}_{|x=-\frac{1}{2},y=-\frac{1}{2}} &=& \frac{-3\left(-\frac{1}{2}\right)^2-4\left(-\frac{1}{2}\right)}{3\left(-\frac{1}{2}\right)^2+4\left(-\frac{1}{2}\right)} & \\
\frac{\mathrm{d}y}{\mathrm{d}x}_{|x=-\frac{1}{2},y=-\frac{1}{2}} &=& -1 &
\end{array}$$

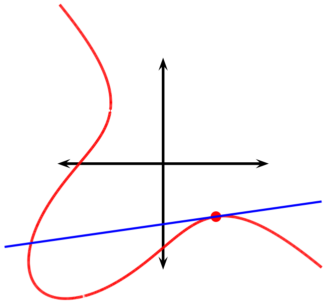
Therefore the equation of the tangent is $y-\left(-\frac{1}{2}\right)=-\left(x-\left(-\frac{1}{2}\right)\right)$ which simplifies to $y=-x-1$. A computer-generated plot visually confirms our computations.



Solution. 53.6

$$\begin{array}{rcll}
y^3+x^3+4xy &=& -4 & \Big|\ \text{apply } \frac{\mathrm{d}}{\mathrm{d}x}\\
3y^2\frac{\mathrm{d}y}{\mathrm{d}x}+3x^2+4\left(y+x\frac{\mathrm{d}y}{\mathrm{d}x}\right) &=& 0 & \\
\frac{\mathrm{d}y}{\mathrm{d}x}\left(3y^2+4x\right) &=& -3x^2-4y & \\
\frac{\mathrm{d}y}{\mathrm{d}x} &=& \frac{-3x^2-4y}{3y^2+4x} & \Big|\ \text{substitute } x=1, y=-1\\
\frac{\mathrm{d}y}{\mathrm{d}x}_{|x=1,y=-1} &=& \frac{-3(1)^2-4(-1)}{3(-1)^2+4(1)} & \\
\frac{\mathrm{d}y}{\mathrm{d}x}_{|x=-\frac{1}{2},y=-\frac{1}{2}} &=& \frac{1}{7} &
\end{array}$$

Therefore the equation of the tangent is $y-(-1)=\frac{1}{7}(x-1)$ which simplifies to $y=\frac{x}{7}-\frac{8}{7}$. A computer-generated plot visually confirms our computations.



6.8 Implicit Differentiation and Inverse Trigonometric Functions

Problem 54. *The variables x and y are related by*

$$x\arctan y+y\arctan x=\frac{\pi}{2}.$$

1. *Show that $(1,1)$ is on the graph of this relation.*

   Solution.

$$\begin{aligned}\mathrm{LS}&=1\cdot\arctan 1+1\cdot\arctan 1\\&=1\cdot\frac{\pi}{4}+1\cdot\frac{\pi}{4}\\&=\frac{\pi}{2}.\end{aligned}\qquad\qquad \mathrm{RS}=\frac{\pi}{2}.$$

   The fact that the left side equals the right side when we plug in $x=1$ and $y=1$ means that the point $(1,1)$ is on the graph of the relation.

2. *Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y.*

   Solution. Differentiate implicitly.

$$\begin{aligned}\left((x)\frac{\mathrm{d}}{\mathrm{d}x}(\arctan y)+(\arctan y)\frac{\mathrm{d}}{\mathrm{d}x}(x)\right)+\left((y)\frac{\mathrm{d}}{\mathrm{d}x}(\arctan x)+(\arctan x)\frac{\mathrm{d}}{\mathrm{d}x}(y)\right)&=0\\ x\cdot\frac{1}{1+y^2}\cdot\frac{\mathrm{d}y}{\mathrm{d}x}+(\arctan y)1+y\cdot\frac{1}{1+x^2}+(\arctan x)\frac{\mathrm{d}y}{\mathrm{d}x}&=0\\ \frac{x}{1+y^2}\frac{\mathrm{d}y}{\mathrm{d}x}+\arctan y+\frac{y}{1+x^2}+\frac{\mathrm{d}y}{\mathrm{d}x}\arctan x&=0.\end{aligned}$$

   Rearrange to isolate $\frac{\mathrm{d}y}{\mathrm{d}x}$ on one side.

$$\begin{aligned}\frac{x}{1+y^2}\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{\mathrm{d}y}{\mathrm{d}x}\arctan x&=-\frac{y}{1+x^2}-\arctan y\\ \left(\frac{x}{1+y^2}+\arctan x\right)\frac{\mathrm{d}y}{\mathrm{d}x}&=-\left(\frac{y}{1+x^2}+\arctan y\right)\\ \frac{\mathrm{d}y}{\mathrm{d}x}&=-\frac{\frac{y}{1+x^2}+\arctan y}{\frac{x}{1+y^2}+\arctan x}.\end{aligned}$$

3. *Find the equation of the tangent to the graph at $(1,1)$.*

   Solution. To find the slope of the tangent, plug in $x=1,y=1$ to the formula for $\frac{\mathrm{d}y}{\mathrm{d}x}$.

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{\frac{1}{1+1^2} + \arctan 1}{\frac{1}{1+1^2} + \arctan 1} \\ &= -1.\end{aligned}$$

Now use the point $(1, 1)$ to find an equation for the tangent line.

$$\begin{aligned} y - 1 &= (-1)(x - 1) \\ y &= -x + 2.\end{aligned}$$

Problem 55. *The variables x and y are related by*

$$x^2y + xy^2 + \arcsin x = \frac{\pi}{6}.$$

1. *Find all points on the graph of this relation for which $x = \frac{1}{2}$.*

   Solution. Set $x = \frac{1}{2}$ and solve for y.

$$\begin{aligned}\left(\frac{1}{2}\right)^2 y + \frac{1}{2}y^2 + \arcsin\frac{1}{2} &= \frac{\pi}{6} \\ \frac{1}{4}y + \frac{1}{2}y^2 + \frac{\pi}{6} &= \frac{\pi}{6} \\ \frac{1}{4}y + \frac{1}{2}y^2 &= 0 \\ \frac{1}{4}y(1 + 2y) &= 0,\end{aligned}$$

   so $y = 0$ or $y = -\frac{1}{2}$. Therefore $(\frac{1}{2}, 0)$ and $(\frac{1}{2}, -\frac{1}{2})$ are the points on the graph of the relation for which $x = \frac{1}{2}$.

2. *Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y.*

   Solution. Differentiate implicitly.

$$\begin{aligned}\left((x^2)\frac{\mathrm{d}}{\mathrm{d}x}(y) + (y)\frac{\mathrm{d}}{\mathrm{d}x}(x^2)\right) + \left((x)\frac{\mathrm{d}}{\mathrm{d}x}(y^2) + (y^2)\frac{\mathrm{d}}{\mathrm{d}x}(x)\right) + \frac{1}{\sqrt{1-x^2}} &= 0 \\ x^2\frac{\mathrm{d}y}{\mathrm{d}x} + y(2x) + x(2y)\frac{\mathrm{d}y}{\mathrm{d}x} + y^2 + \frac{1}{\sqrt{1-x^2}} &= 0 \\ x^2\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy + 2xy\frac{\mathrm{d}y}{\mathrm{d}x} + y^2 + \frac{1}{\sqrt{1-x^2}} &= 0.\end{aligned}$$

   Rearrange to isolate $\frac{\mathrm{d}y}{\mathrm{d}x}$ on one side.

$$\begin{aligned}x^2\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy\frac{\mathrm{d}y}{\mathrm{d}x} &= -y^2 - 2xy - \frac{1}{\sqrt{1-x^2}} \\ (x^2 + 2xy)\frac{\mathrm{d}y}{\mathrm{d}x} &= -\left(y^2 + 2xy + \frac{1}{\sqrt{1-x^2}}\right) \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{y^2 + 2xy + \frac{1}{\sqrt{1-x^2}}}{x^2 + 2xy}.\end{aligned}$$

3. *Find the equation of the tangent to the graph at each of the points you found in the first part.*

Solution. To find the slope of the tangent at $(\frac{1}{2}, 0)$, plug in $x = \frac{1}{2}, y = 0$ to the formula for $\frac{\mathrm{d}y}{\mathrm{d}x}$.

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{(0)^2 + 2(\frac{1}{2})(0) + \frac{1}{\sqrt{1-(\frac{1}{2})^2}}}{(\frac{1}{2})^2 + 2(\frac{1}{2})(0)} \\
&= -\frac{0 + 0 + \frac{1}{\sqrt{\frac{3}{4}}}}{\frac{1}{4} + 0} \\
&= -\frac{\frac{1}{\frac{\sqrt{3}}{2}}}{\frac{1}{4}} \\
&= -\frac{2}{\sqrt{3}} \cdot \frac{4}{1} \\
&= -\frac{8}{\sqrt{3}}.
\end{aligned}$$

Now use the point $(\frac{1}{2}, 0)$ to find an equation for the tangent line.

$$\begin{aligned}
y - 0 &= -\frac{8}{\sqrt{3}}\left(x - \frac{1}{2}\right) \\
y &= -\frac{8}{\sqrt{3}}x + \frac{4}{\sqrt{3}}.
\end{aligned}$$

This is the equation for one of the tangent lines.
To find the slope of the tangent at $\left(\frac{1}{2}, -\frac{1}{2}\right)$, plug in $x = \frac{1}{2}, y = -\frac{1}{2}$ to the formula for $\frac{\mathrm{d}y}{\mathrm{d}x}$.

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{(-\frac{1}{2})^2 + 2(\frac{1}{2})(-\frac{1}{2}) + \frac{1}{\sqrt{1-(\frac{1}{2})^2}}}{(\frac{1}{2})^2 + 2(\frac{1}{2})(-\frac{1}{2})} \\
&= -\frac{\frac{1}{4} - \frac{1}{2} + \frac{1}{\sqrt{\frac{3}{4}}}}{\frac{1}{4} - \frac{1}{2}} \\
&= -\frac{-\frac{1}{4} + \frac{2}{\sqrt{3}}}{-\frac{1}{4}} \\
&= \frac{-\frac{1}{4} + \frac{2}{\sqrt{3}}}{\frac{1}{4}} \\
&= 4\left(-\frac{1}{4} + \frac{2}{\sqrt{3}}\right) \\
&= -1 + \frac{8}{\sqrt{3}}.
\end{aligned}$$

Now use the point $(\frac{1}{2}, -\frac{1}{2})$ to find an equation for the tangent line.

$$\begin{aligned}
y - \left(-\frac{1}{2}\right) &= \left(-1 + \frac{8}{\sqrt{3}}\right)\left(x - \frac{1}{2}\right) \\
y &= \left(-1 + \frac{8}{\sqrt{3}}\right)x + \frac{1}{2} - \frac{4}{\sqrt{3}} - \frac{1}{2} \\
y &= \left(-1 + \frac{8}{\sqrt{3}}\right)x - \frac{4}{\sqrt{3}},
\end{aligned}$$

and this is the equation of the other tangent line.

6.9 Derivative of non-Constant Exponent with non-Constant Base

Problem 56. *Differentiate*

1. $x^{\sin x}$. *answer:* $x^{\sin x}\left(\frac{\sin x}{x} + \cos x \ln x\right)$
2. $x^{\tan x}$. *answer:* $x^{\tan x}\left(\frac{\tan x}{x} + (\ln x)\sec^2 x\right)$

Solution. 56.1. $\left(x^{\sin x}\right)' = \left(e^{(\ln x)\sin x}\right)' = e^{(\ln x)\sin x}(\ln x \sin x)' = x^{\sin x}\left(\frac{\sin x}{x} + \ln x \cos x\right)$.

Problem 57. *Differentiate.*

1. 10^{x^3}. *answer:* $3(\ln 10)x^2(10)^{x^3}$
2. $2^{\tan x}$. *answer:* $(\ln 2)2^{\tan x}\sec^2 x$
3. x^x. *answer:* $x^x(\log(x) + 1)$
4. x^{x^x}. *answer:* $(\ln(x))^2 x^{x^x+x} + x^{x^x+x-1} + (\ln x)x^{x^x+x}$
5. $(\sin x)^{\cos x}$. *answer:* $\frac{-\ln(\sin x)(\sin x)^{\cos x+2}+(\sin x)^{\cos x}\cos^2 x}{\sin x}$
6. $(\ln x)^{\ln x}$. *answer:* $\ln(\ln(x))(\ln(x))^{\ln(x)}x^{-1} + x^{-1}(\ln(x))^{\ln(x)}$

6.10 Related Rates

Problem 58. 1. *A spherical soap bubble is slowly shrinking. If its surface area is decreasing at a rate of 50 square millimeters per second, how quickly is the radius decreasing when the surface area is 1000 square millimeters?*

answer: $\frac{dr}{dt} = -\frac{50}{8\pi\sqrt{\frac{250}{\pi}}} = -\frac{\sqrt{10}}{8\sqrt{\pi}}$.

2. *A car drives along an elliptical track. The track can be modeled by the equation $x^2 + 5y^2 = 14$, where x and y are measured in kilometres of distance from the center of the track. As the car passes the point $(3, 1)$, the x-coordinate is increasing at a rate of* 1.5 *km/min. How quickly is the y-coordinate changing at that point?*

answer: $\frac{dy}{dt} = -\frac{9}{10}$ *km/min.*

3. *Gravel is being dumped from a conveyor belt at a constant rate of* 500 *litres per minute. The gravel pile forms a cone with circular base, the diameter of which remains equal to the hight of the cone at any given moment. Use related rates to approximate how fast is the height of the pile increasing when it is 2 meters tall?*

answer: $\frac{dh}{dt} = \frac{1}{2\pi}$ *m/min.*

Solution. 58.1 Let t denote time. Let R be the radius of the sphere. Let S be the surface area of the sphere. We are given that

$$\frac{\mathrm{d}S}{\mathrm{d}t} = -50 \quad ,$$

where the sign is negative because the bubble is decreasing. We are asked to find

$$\frac{\mathrm{d}R}{\mathrm{d}t} = ?$$

R and S are related via

$$\begin{array}{rcll} S &=& 4\pi R^2 & \Big| \text{ differentiate with respect to time} \\ \frac{\mathrm{d}S}{\mathrm{d}t} &=& 4\pi * 2R * \frac{\mathrm{d}R}{\mathrm{d}t} & \\ \frac{\mathrm{d}R}{\mathrm{d}t} &=& \frac{1}{8\pi R}\frac{\mathrm{d}S}{\mathrm{d}t} & . \end{array}$$

When $S = 1000$, we can find the corresponding value of R:

$$\begin{array}{rcl} 1000 &=& 4\pi R^2 \\ R^2 &=& \frac{250}{\pi} \\ R &=& \sqrt{\frac{250}{\pi}} \quad . \end{array}$$

Finally, we can compute:

$$\frac{\mathrm{d}R}{\mathrm{d}t}_{|S=1000} = \frac{1}{8\pi \underbrace{\sqrt{\frac{250}{\pi}}}_{=R \text{ when } S=1000}} * \underbrace{(-50)}_{=\frac{\mathrm{d}S}{\mathrm{d}t}} = -\frac{\sqrt{10\pi}}{8\pi} \quad .$$

Solution. 58.2. Let t denote time, and let t_0 be the point in time in which the measurements take place. We are given that $\frac{\mathrm{d}x}{\mathrm{d}t}_{|t=t_0} = 1.5km/min$, $x_{|t=t_0} = 3$, $y_{|t=t_0} = 1$. The problem asks us to find $\frac{\mathrm{d}y}{\mathrm{d}t}_{|t=t_0}$.

Compute:

$$\begin{array}{rcll} x^2+5y^2 &=& 14 & \Big| \text{ apply } \dfrac{\mathrm{d}}{\mathrm{d}t} \\ 2x\dfrac{\mathrm{d}x}{\mathrm{d}t}+10y\dfrac{\mathrm{d}y}{\mathrm{d}t} &=& 0 & \Big| \text{ fix time } t=t_0 \\ 2\cdot 3\cdot 1.5+10\cdot 1\cdot \dfrac{\mathrm{d}y}{\mathrm{d}t}_{|t=t_0} &=& 0 & \\ 10\dfrac{\mathrm{d}y}{\mathrm{d}t}_{|t=t_0} &=& -9 & \\ \dfrac{\mathrm{d}y}{\mathrm{d}t}_{|t=t_0} &=& -\dfrac{9}{10} & . \end{array}$$

The measurement unit of $\mathrm{d}y$ is km, the measurement unit of t is minutes, therefore the answer is $-\frac{9}{10}$km/min.

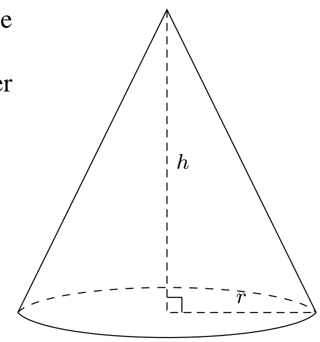
Solution. 58.3.

At time t (minutes) let h be the height of the pile (m), let r be the radius of the base (m), and let V be the volume of the pile (m^3).

We are given that $\frac{dV}{dt} = 500$L/min $= \frac{1}{2}$ m^3/min. We are also given the diameter is equal to the height, so $2r = h$.

We are asked to find $\frac{dh}{dt}$ when $h = 2$.

The formula for the volume of a cone is



$$V = \frac{1}{3}\pi r^2 h.$$

Since $r = \frac{h}{2}$ we obtain

$$V = \frac{1}{3}\pi\frac{h^2}{4}h = \frac{1}{12}\pi h^3$$

Differentiating with respect to t then gives

$$\frac{dV}{dt} = \frac{1}{4}\pi h^2\frac{dh}{dt} \quad \Rightarrow \quad \frac{dh}{dt} = \frac{4}{h^2\pi}\frac{dV}{dt}$$

Substituting $\frac{dV}{dt} = \frac{1}{2}$m^3/min, and $h = 2$ we obtain

$$\frac{dh}{dt} = \frac{4}{4\pi}\cdot\frac{1}{2} = \frac{1}{2\pi} \text{ m/min}$$

Problem 59. 1. *If V is the volume of a cube with edge length x and the cube expands as time passes, find $\frac{dV}{dt}$ in terms of $\frac{dx}{dt}$.*

answer: $3x^2\frac{dx}{dt}$

2. *Each side of a square is increasing at a rate of* 1*cm/s. At what rate is the area of the square increasing when the area of the square is 9 cm^2?*

answer: 6cm$^2/s$

3. *The radius of a ball is increasing at a constant rate of* 5*mm/s. At a point in time we measure the radius of the ball to be* 5*cm.*
   - *How fast is the volume increasing at the time of our observation?*
   - *How fast is the surface area increasing at the time of our observation?*

4. *At a point in time, we measure the surface area of a ball to be increasing at a rate of* 5*cm$^2/s$ and the radius of the ball to be* 5*cm.*
   - *How fast is the volume increasing at the time of our observation?*
   - *How fast is the radius increasing at the time of our observation?*

5. *At a point in time, we measure the volume of a ball to be increasing at a rate of* 5*cm$^3/s$ and the radius of the ball to be* 5*cm.*
   - *How fast is the radius increasing at the time of our observation?*
   - *How fast is the surface area increasing at the time of our observation?*

6. *A 5m long ladder is leaning on a vertical wall. The base of the ladder is 4m away from the wall. At the moment of observation, the top end of the ladder is sliding downwards along the wall at a speed of 10 cm/s.*
   - *At the time of observation, how fast is the bottom end of the ladder sliding away from the wall*

- *At the time of observation, how fast is the midpoint of the ladder moving away from the wall?*
- *At the time of observation, how fast is the midpoint of the ladder moving downwards towards the ground?*

7. *A street light is mounted at the top of a 4m tall pole. A woman 160 cm tall walks away from the pole at a speed of 5km/h along a straight path. How fast is the tip of her shadow moving when she is 10m from the pole?*
8. *A ship is pulled into a dock. On the dock, the rope is pulled by a pulley that is 1m lower than the mooring point on the ship. If the rope is pulled in at a constant rate of 10cm/s, how fast is the mooring point approaching the dock when it is 10m from the dock?*
9. *A Ferris wheel with a radius of 10m is rotating at a rate of one revolution every 2 minutes. How fast is a riding rising when his seat is 16 m above ground level?*
10. *The minute hand on a watch is 10mm long and the hour hand is 5mm long. How fast is the distance between the tips of the hands changing at two o'clock?*

answer: (not proofread) $-\frac{55}{42}\sqrt{21}\pi$ *mm/h*

Solution. 59.1. The volume V is given by $V = x^3$, therefore

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(x^3\right) = 3x^2\frac{\mathrm{d}x}{\mathrm{d}t}.$$

Solution. 59.2 The area A of the square is given by $A = x^2$, therefore

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(x^2\right) = 2x\frac{\mathrm{d}x}{\mathrm{d}t}.$$

When $A = 9\mathrm{cm}^2$, $x = 3\mathrm{cm}$ $(= \sqrt{9\mathrm{cm}^2})$, and so $\frac{\mathrm{d}A}{\mathrm{d}t}_{|x=3, \frac{\mathrm{d}x}{\mathrm{d}t}=1} = 2 \cdot 3\mathrm{cm} \cdot 1\mathrm{cm/s} = 6\mathrm{cm}^2/\mathrm{s}$.

Solution. 59.4 Let S denote the surface area and r the radius of the ball. Then $S = 4\pi r^2$. Let the point of time be t_0. We are given that $\frac{\mathrm{d}S}{\mathrm{d}t}_{|t=t_0} = 5\mathrm{cm}^2/\mathrm{s}$. On the other hand,

$$\begin{array}{rcl} \frac{\mathrm{d}S}{\mathrm{d}t} & = & \frac{\mathrm{d}}{\mathrm{d}t}\left(4\pi r^2\right) = 8\pi r\frac{\mathrm{d}r}{\mathrm{d}t} \\ \frac{1}{8\pi}\frac{\mathrm{d}S}{\mathrm{d}t} & & \end{array}$$

Solution. 59.10 Let the angle between the two arrows be θ. The cosine law states that for a triangle with angle θ and sides a, b, c we have that $c^2 = a^2 + b^2 - 2ab\cos\theta$ (where c is the length of the side opposite to the angle θ).

Then by the cosine law, the distance between the tips of the two hands is 1

$$y = \sqrt{5^2 + 10^2 + 2 \cdot 5 \cdot 10\cos\theta} = \sqrt{125 + 100\cos\theta}.$$

The short hand makes 1 full revolution every 12 hours, and the long hand makes 1 full revolution every 1 hour. Therefore the angle θ measured from the small hand to the long hand changes at the (constant) rate of $\frac{11}{12}$ revolutions per hour, or what is the same, at the rate $\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{11}{12}(2\pi) = \frac{11}{6}\pi$.

The problem asks us to compute $\frac{dy}{dt}$ at two o'clock, i.e., at $t = 2$. This is a straightforward computation using the chain rule:

$$\begin{array}{rcll} \frac{\mathrm{d}y}{\mathrm{d}t} & = & \frac{1}{2}\frac{-100\sin\theta}{\sqrt{125+100\cos\theta}}\frac{\mathrm{d}\theta}{\mathrm{d}t} & \Big| \text{ at 2 o'clock } \theta = \frac{\pi}{3} \text{ and } \frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{11}{6}\pi \\ \frac{\mathrm{d}y}{\mathrm{d}t}_{|t=2} & = & \frac{1}{2}\frac{-100\sin\frac{\pi}{3}}{\sqrt{125+100\cos\frac{\pi}{3}}}\frac{11\pi}{6} = -\frac{55}{42}\sqrt{21}\pi & . \end{array}$$

The measurement unit of speed is mm/hour, so the distance is changing at the rate of $-\frac{55}{42}\sqrt{21}$ mm/hour.

7 Graphical Behavior of Functions

7.1 Mean Value Theorem

Problem 60. *Recall that given a function f satisfying certain conditions, the Mean value Theorem relates the slope of a secant line of the graph of f to the slope of a tangent line to f at some intermediate point(s) c.*

Give the precise statement of the Mean Value Theorem.

For the given functions and the given intervals, find the point(s) c that satisfies the conclusion of the Mean Value Theorem.

The answer key has not been proofread, use with caution.

- $f(x) = x^2$, *for the interval* $[1, 3]$.

answer: $c = 2$

- $f(x) = x^3 - 6x^2 + 11x - 6$, *for the interval* $[1, 3]$.

answer: $c = 2 + \frac{1}{3}\sqrt{3}, 2 - \frac{1}{3}\sqrt{3}$

- $f(x) = \ln x$, *for the interval* $[1, e]$.

answer: $c = e - 1$

- $f(x) = \cos x$, *for the interval* $[0, \pi]$.

answer: $c = \frac{\pi}{2}$

Problem 61. 1. *We have that f is continuous in $[0, 10]$ and differentiable in $(0, 10)$. If $f(0) = 0$, $f(10) = 10$ and $|f'(x)| \leq 1$, how large can $f(x)$ be for x in the interval $[0, 10]$?*

answer: 10

2. *We have that f is continuous in $[0, 10]$ and differentiable in $(0, 10)$. If $f(0) = -3$, $f(2) = -13$ and $|f'(x)| \leq 5$, what is the smallest possible value of $f(1)$?*

answer: -8

Problem 62. *Use the Intermediate Value theorem and the Mean Value Theorem/Rolle's Theorem to prove that the function has* ***exactly one*** *real root.*

1. $f(x) = x^3 + 4x + 7$.
2. $f(x) = x^3 + x^2 + x + 1$.
3. $f(x) = \cos^3\left(\frac{x}{3}\right) + \sin x - 3x$.

Solution. 62.1. $f(-2) = -9$ and $f(1) = 12$. Since $f(x)$ is continuous and has both negative and positive outputs, it must have a zero. In other words, for some c between -2 and 1, $f(c) = 0$. If there were solutions $x = a$ and $x = b$, then we would have $f(a) = f(b)$, and Rolle's Theorem would guarantee that for some x-value, $f'(x) = 0$. However, $f'(x) = 3x^2 + 4$, which always positive and therefore is never 0. Therefore there cannot be 2 or more solutions.

The above can be stated informally as follows. Note that $f'(x) = 3x^2 + 4$, which is always positive. Therefore, the graph of f is increasing from left to right. So once the graph crosses the x-axis, it can never turn around and cross again, so there can only be a single zero (that is, a single solution to $f(x) = 0$).

Solution. 62.3. $f(5) = \cos^3\left(\frac{5}{3}\right) + \sin 5 - 15 \leq 2 - 15 = -13 < 0$ (because $\cos a, \sin b \in [-1, 1]$ for arbitrary a, b). Similarly $f(-5) = \cos^3\left(-\frac{5}{3}\right) + \sin(-5) + 15 \geq 15 - 2 > 0$. Therefore by the Intermediate Value Theorem $f(x) = 0$ has at least one solution in the interval $[-5, 5]$.

Suppose on the contrary to what we are trying to prove, $f(x) = 0$ has two or more solutions; call the first 2 solutions a, b. That means that $f(a) = f(b) = 0$, so by the Mean value theorem, there exists a $c \in (a, b)$ such that $f'(c) = (f(a) - f(b))/(a - b) = (0 - 0)/(a - b) = 0$. On the other hand we may compute:

$$f'(x) = -3 + \cos x - \cos^2\left(\frac{x}{3}\right)\sin\left(\frac{x}{3}\right) \leq -1 < 0,$$

where the first inequality follows from the fact that $\sin x, \cos x \in [-1, 1]$. So we got that $f'(c) = 0$ for some c but at the same time $f'(x) < 0$ for all x, which is a contradiction. Therefore $f(x) = 0$ has exactly one solution.

7.2 Maxima, Minima

7.2.1 Closed Interval method

Problem 63. *Find the maximum and minimum values of f on the given interval and the values of x for which they are attained.*

1. $f(x) = 9 + 3x - x^2$, $x \in [0, 4]$.

answer: $f_{max} = f\left(\frac{3}{2}\right) = \frac{45}{4}$, $f_{min} = f(4) = 5$

2. $f(x) = 5 + 4x - 2x^3$, $x \in [-1, 1]$.

answer: $f_{max} = f\left(\frac{\sqrt{6}}{3}\right) = \frac{8}{9}\sqrt{6} + 5$, $f_{min} = f\left(-\frac{\sqrt{6}}{3}\right) = 5 - \frac{8}{9}\sqrt{6}$

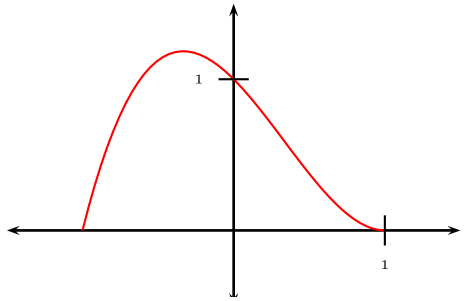
3. $f(x) = 2x^3 - x^2 - 20x + 1$, $x \in [-4, 3]$.

answer: $f_{max} = f\left(-\frac{5}{3}\right) = \frac{602}{27}$, $f_{min} = f(-4) = -63$

4. $f(x) = 3x^4 - 4x^3 - 12x^2 + 1$, $x \in [-2, 3]$.

answer: $f_{max} = f(-2) = 33$, $f_{min} = f(2) = -31$

5. $f(x) = x^3 - x^2 - x + 1$, $x \in [-1, 1]$.



6. $f(x) = x^3 - x + 1$, $x \in [-2, 1]$.

answer: $f_{max} = f\left(-\frac{\sqrt{3}}{3}\right) = \frac{2}{9}\sqrt{3} + 1$, $f_{min} = f(-2) = -5$

7. $f(x) = (x^2 - 1)^3$, $x \in [-1, 2]$.

answer: $f_{max} = f(2) = 27$, $f_{min} = f(0) = -1$

8. $f(x) = x + \dfrac{1}{x}$, $x \in [0.2, 4]$.

answer: $f_{max} = f(0.2) = \frac{26}{5} = 5.2$, $f_{min} = f(1) = 2$

9. $f(x) = \dfrac{x}{x^2 - x + 1}$, $x \in [0, 3]$.

answer: $f_{max} = f(1) = 1$, $f_{min} = f(-1) = -\frac{1}{3}$

10. $f(t) = t\sqrt{4 - t^2}$, $x \in [-1, 2]$.

answer: $f_{max} = f\left(\sqrt{2}\right) = 2$, $f_{min} = f\left(-\sqrt{2}\right) = -2$

11. $f(t) = \sqrt[3]{t}(8 - t)$, $x \in [0, 8]$.

answer: $f_{max} = f(2) = 6\sqrt[3]{2}$, $f_{min} = f(0) = f(8) = 0$

12. $f(t) = 2\cos t + \sin(2t)$, $x \in [0, \frac{\pi}{2}]$.

answer: $f_{max} = f\left(\frac{\pi}{6}\right) = \frac{3}{2}\sqrt{3}$, $f_{min} = f\left(\frac{\pi}{2}\right) = 0$

13. $f(t) = t + \cot\left(\dfrac{t}{2}\right)$, $x \in [\frac{\pi}{4}, \frac{7\pi}{4}]$.

answer: $f_{max} = f\left(\frac{3\pi}{2}\right) = \frac{3\pi}{2} - 1$, $f_{min} = f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} + 1$

14. $f(t) = t + \cot\left(\dfrac{t}{2}\right)$, $x \in [\frac{\pi}{4}, \frac{7\pi}{4}]$.

15. $f(x) = xe^{3x}$, $x \in \left[-3, \frac{1}{6}\right]$.

answer: $f_{max} = f\left(\frac{1}{6}\right) = \frac{e^{\frac{1}{2}}}{6} \approx 0.274787$, $f_{min} = f\left(-\frac{1}{3}\right) = -\frac{1}{3e} \approx -0.122626$

16. $f(x) = (x - 2)(x + 1)e^x$, $x \in [-5, 2]$.

answer: $f_{min} = f\left(\frac{\sqrt{13}}{2} - \frac{1}{2}\right) = \left(-\sqrt{13} + 2\right)e^{\left(\frac{\sqrt{13}}{2} - \frac{1}{2}\right)} \approx -5.907619$; $f_{max} = f\left(-\frac{\sqrt{13}}{2} - \frac{1}{2}\right) = \left(\sqrt{13} + 2\right)e^{\left(-\frac{\sqrt{13}}{2} - \frac{1}{2}\right)} \approx 0.560448$

17. $f(x) = (x + 1)e^{-x^2}$, $x \in [-3, 3]$.

answer: $f_{min} = f\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}\right) = \left(-\frac{\sqrt{3}}{2} + \frac{1}{2}\right)e^{-\frac{\sqrt{3}}{2} - 1} \approx -0.056638$; $f_{max} = f\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right) = \left(\frac{\sqrt{3}}{2} + \frac{1}{2}\right)e^{\frac{\sqrt{3}}{2} - 1} \approx 1.194743$

18. $f(x) = xe^{2x}$, $x \in \left[-2, \frac{1}{2}\right]$.

answer: $f_{max} = f\left(\frac{1}{2}\right) = \frac{e}{2}$, $f_{min} = f\left(-\frac{1}{2}\right) = -\frac{1}{2e}$

Problem 64. *A particle moves in such a way that, after t seconds, it is $s(t) = \ln\left(2 - t + t^2\right)$ m to the right of the origin.*

1. *What is the closest it comes to the origin?*

Solution.

$$
\begin{aligned}
s'(t) &= \frac{-1 + 2t}{2 - t + t^2}.\\
\text{Set}\quad s'(t) &= 0.\\
\frac{-1 + 2t}{2 - t + t^2} &= 0\\
-1 + 2t &= 0\\
t &= \frac{1}{2}.
\end{aligned}
$$

Therefore the position function has a critical number at $t = \frac{1}{2}$. The parabola $2 - t + t^2$ has a global minimum at $t = \frac{1}{2}$, and the natural logarithm function is an increasing function, so $\ln\left(2 - t + t^2\right)$ also has a global minimum at $t = \frac{1}{2}$. The minimum value is $s(\frac{1}{2}) = \ln\left(2 - \frac{1}{2} + \left(\frac{1}{2}\right)^2\right) = \ln\left(\frac{7}{4}\right) \approx 0.5596$ m.

2. *What is its acceleration when it is closest to the origin?*

Solution.

$$
\begin{aligned}
v(t) = s'(t) &= \frac{-1+2t}{2-t+t^2}.\\
a(t) = s''(t) &= \frac{(2-t+t^2)(2)-(-1+2t)(-1+2t)}{(2-t+t^2)^2}\\
&= \frac{(4-2t+2t^2)-(1-4t+4t^2)}{(2-t+t^2)^2}\\
&= \frac{3+2t-2t^2}{(2-t+t^2)^2}.\\
\text{Plug in } t=\frac{1}{2}: \quad a\left(\frac{1}{2}\right) &= \frac{3+2(\frac{1}{2})-2(\frac{1}{2})^2}{\left(\frac{7}{4}\right)^2}\\
&= \frac{3+1-\frac{1}{2}}{\frac{49}{16}}\\
&= \frac{\frac{7}{2}}{\frac{49}{16}}\\
&= \frac{8}{7}.
\end{aligned}
$$

Therefore the particle is accelerating at a rate of $\frac{8}{7} \approx 1.1429$ m/s^2 when it is closest to the origin.

3. *For which values of t is the position function $s(t)$ defined?*

   Solution. The natural logarithm function is defined for all positive input values. The formula $y = 2-t+t^2$ is always positive. To see this, note that its graph is a parabola with discriminant $b^2-4ac = (-1)^2-4(1)(2) = -7$, which is negative. This means that the graph of $y = 2-t+t^2$ never touches the x-axis, and hence it is either always positive or always negative. Since $y(0) = 2$ is positive, this means all values are positive. Therefore $\ln(2-t+t^2)$ is defined for all input values t.

7.2.2 Derivative tests

Problem 65. *Find the critical points of the function. Identify whether those are local maxima, minima, or neither. The answer key has not been proofread, use with caution.*

1. $f(x) = \dfrac{x}{1+x^2}$.

   answer: $x = 1$, local & global max, $x = -1$, local and global min

2. $f(x) = x^3-x^2-x-1$.

   answer: $x = -\frac{1}{3}$, local max, $x = 1$, local min

3. $f(x) = 2x^3-x^2-20x+1$.

   answer: $x = -\frac{5}{3}$, local max $x = 2$, local min

4. $f(x) = x + \dfrac{1}{x}$.

   answer: $x = -1$, local max, $x = 1$, local min

5. $f(x) = \dfrac{x-\frac{1}{2}}{x^2-2x+\frac{7}{4}}$.

   answer: $x = -\frac{1}{2}$, local and global min, $x = \frac{3}{2}$, local and global max

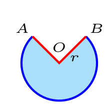
7.2.3 Optimization

Problem 66. 1. *Find the dimensions of a rectangle with area 1000 m^2 whose perimeter is as small as possible.*

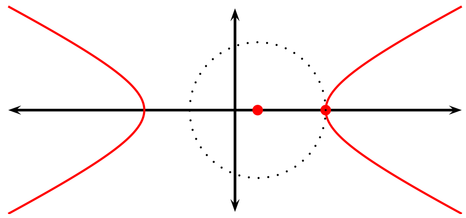
2. *A box with an open top is to be constructed from a square piece of cardboard, 1m wide, by cutting out a square from each of the four corners and bending up the sides. Find the largest volume that such a box can have.*

3. *A right circular cylinder is inscribed in a sphere of radius r. Find the largest possible volume of such a cylinder.*

4. *A wedge of radius 2 (depicted below) is folded into a cone cup. The volume varies depending on the angle of the wedge. Find the maximal possible volume of the cone cup and the angle of the wedge for which this maximal volume is achieved.*

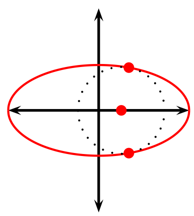


Problem 67. 1. *What is the x-coordinate of the point on the hyperbola $x^2 - 4y^2 = 16$ that is closest to the point $(1, 0)$?*



answer: $x = 4$

2. *What is the x-coordinate of the point on the ellipse $x^2 + 4y^2 = 16$ closest to the point $(1, 0)$?*



answer: $x = \frac{4}{3}$

3. *A rectangular box with a square base is being built out of sheet metal. 2 pieces of sheet will be used for the bottom of the box, and a single piece of sheet metal for the 4 sides and the top of the box. What is the largest possible volume of the resulting box that can be obtained with $36m^2$ of metal sheet?*

answer: 12 cubic meters.

4. *Recall that the volume of a cylinder is computed as the product of the area of its base by its height. Recall also that the surface area of the wall of a cylinder is given by multiplying the perimeter of the base by the height of the cylinder.*

   A cylindrical container with an open top is being built from metal sheet. The total surface area of metal used must equal $10m^2$. Let r denote the radius of the base of the cylinder, and h its height. How should one choose h and r so as to get the maximal possible container volume? What will the resulting container volume be?

Solution. 67.1

The distance function between an arbitrary point (x, y) and the point $(1, 0)$ is $d = \sqrt{(x-1)^2 + (y-0)^2}$. On the other hand, when the point (x, y) lies on the hyperbola we have $y^2 = \frac{x^2-16}{4}$. In this way, the problem becomes that of minimizing the distance function

$$dist(x) = \sqrt{(x-1)^2 + y^2} = \sqrt{(x-1)^2 + \frac{x^2-16}{4}} \quad .$$

This is a standard optimization problem: we need to find the critical endpoints, i.e., the points where $dist' = 0$. As the square root function is an increasing function, the function $\sqrt{(x-1)^2 + \frac{x^2-16}{4}}$ achieves its minimum when the function

$$l = dist^2 = (x-1)^2 + \frac{x^2-16}{4}$$

does. l is a quadratic function of x and we can directly determine its minimimum via elementary methods. Alternatively, we find the critical points of l:

$$\begin{array}{rcl} l' &=& 0 \\ 2(x-1) + \frac{x}{2} &=& 0 \\ \frac{5}{2}x - 2 &=& 0 \\ x &=& \frac{4}{5} \quad . \end{array}$$

On the other hand, $x^2 = 16 + 4y^2$ and therefore $|x| \geq \sqrt{16} = 4$. Therefore $x \in (-\infty, -4] \cup [4, \infty)$. As $x = \frac{4}{5}$ is outside of the allowed range, it follows that our function either attains its minimum at one of the endpoints ± 4 or the function has no minimum at all. It is clear however that as x tends to ∞, so does $dist$. Therefore $dist$ attains its minimum for $x = 4$ or -4 and $y = \pm\sqrt{(\pm 4)^2 - 16} = 0$. Direct check shows that $dist_{|x=4} = \sqrt{(4-1)^2 + \frac{4^2-16}{4}} = 3$ and $dist_{|x=-4} = \sqrt{(-4-1)^2 + \frac{4^2-16}{4}} = 5$ so our function $dist$ has a minimal value of 3 achieved when $x = 4$, which is our final answer. Notice that this answer can be immediately given without computation by looking at the figure drawn for 67.1. Indeed, it is clear that there are no points from the hyperbola lying inside the dotted circle centered at $(1, 0)$. Therefore the point where this circle touches the hyperbola must have the shortest distance to the center of the circle.

Solution. 67.3 Let B denote the area of the base of the box, equal to the area of the top. Let W denote the area of the four walls of the box (the four walls are all equal because the base of the box is a square). Then the surface area S of material used will be

$$S = \underbrace{2B}_{\text{two pieces for the bottom}} + \underbrace{4W}_{\text{4 walls}} + \underbrace{B}_{\text{the box lid}} = 3B + 4W \quad .$$

Let x denote the length of the side of the square base and let y denote the height of the box. Then

$$B = x^2$$

and

$$W = xy \quad .$$

As the surface area S is fixed to be 36 square meters, we have that

$$S = 3B + 4W = 36 = 3x^2 + 4xy \quad .$$

As y is positive, the above formula shows that $3x^2 \leq 36$ and so $x \leq \sqrt{12}$. Let us now express y in terms of x:

$$\begin{array}{rcl} 3x^2 + 4xy &=& 36 \\ 4xy &=& 36 - 3x^2 \\ y &=& \dfrac{36 - 3x^2}{4x} \quad . \end{array}$$

The problem asks us to maximize the volume V of the box. The volume of the box equals the area of the base times the height of the box:

$$V = B \cdot y = yx^2 = \frac{(36 - 3x^2)}{4x} x^2 = \frac{36x - 3x^3}{4} \quad .$$

As x is non-negative, it follows that the domain for x is:

$$x \in [0, \sqrt{12}] \quad .$$

To maximize the volume we find the critical points, i.e., the values of x for which V' vanishes:

$$\begin{array}{rcl} 0 &=& V' = \left(\dfrac{36x - 3x^3}{4} \right)' \\ 0 &=& \dfrac{36 - 9x^2}{4} \\ 9x^2 &=& 36 \\ x^2 &=& 4 \\ x &=& \pm 2 \end{array}$$

As x measures length, $x = -2$ is not possible (outside of the domain for x). Therefore the only critical point is $x = 2$. Direct check shows that at the endpoints $x = 0$ and $x = \sqrt{12}$, we have that $V = 0$. Therefore the maximal volume is achieved when $x = 2$:

$$V_{max} = V_{|x=2} = \frac{36(2) - 3(2)^3}{4} = 12 \quad .$$

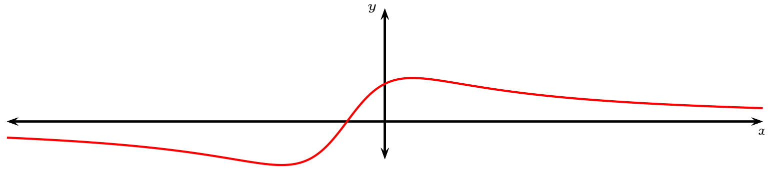
7.3 Function Graph Sketching

Problem 68. *Find the*

- *the implied domain of f,*
- *x and y intercepts of f,*
- *horizontal and vertical asymptotes,*
- *intervals of increase and decrease,*
- *local and global minima, maxima,*
- *intervals of concavity,*
- *points of inflection.*

Label all relevant points on the graph. Show all of your computations.

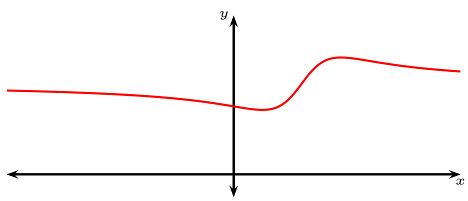
1. $f(x) = \dfrac{x + \frac{1}{2}}{x^2 + x + 1}$



answer:
y-intercept: $\frac{1}{2}$. x-intercept: $-\frac{1}{2}$
Horizontal asymptote: $y = 0$, vertical: none
local and global min at $x = \frac{-1-\sqrt{3}}{2}$, local and global max at $x = \frac{-1+\sqrt{3}}{2}$
Intervals of decrease: $\left(-\infty, \frac{-1-\sqrt{3}}{2}\right) \cup \left(\frac{-1+\sqrt{3}}{2}, \infty\right)$, intervals of decrease $\left(\frac{-1-\sqrt{3}}{2}, \frac{-1+\sqrt{3}}{2}\right)$
Concave down on $(-\infty, -2) \cup \left(-\frac{1}{2}, 1\right)$, concave up on $\left(-2, -\frac{1}{2}\right) \cup (1, \infty)$
Inflection points at: $x = -2, x = -\frac{1}{2}, x = 1$

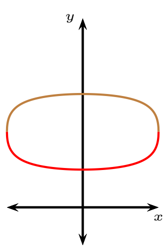
2. $f(x) = \dfrac{2x^2 - 5x + \frac{9}{2}}{x^2 - 3x + 3}$. ***For this problem, indicate only the x-coordinates of the local maxima/minima and inflection points; you do not need to compute the y-coordinates of those points.***

Computation shows that $f'(x) = \dfrac{-x^2 + 3x - \frac{3}{2}}{(x^2 - 3x + 3)^2}$ *and that* $f''(x) = \dfrac{(2x-3)x(x-3)}{(x^2 - 3x + 3)^3}$*; you may use those computations without further justification.*



answer:
y-intercept: $\frac{3}{2}$
horizontal asymptote: $y = 2$, vertical: none
increasing on $\left(\frac{3-\sqrt{3}}{2}, \frac{3+\sqrt{3}}{2}\right)$, decreasing on $\left(-\infty, \frac{3-\sqrt{3}}{2}\right) \cup \left(\frac{3+\sqrt{3}}{2}, \infty\right)$
local and global min at $x = \frac{3-\sqrt{3}}{2}$, local and global max at $x = \frac{3+\sqrt{3}}{2}$
concave up on $\left(0, \frac{3}{2}\right) \cup (3, \infty)$, concave down $(-\infty, 0) \cup \left(\frac{3}{2}, 3\right)$
inflection points at $x = 0$, $x = \frac{3}{2}$, $x = 3$

3. $f(x) = \dfrac{2\sqrt{-x^2+1}+1}{\sqrt{-x^2+1}+1}$, $f(x) = \dfrac{1}{\sqrt{-x^2+1}+1}$

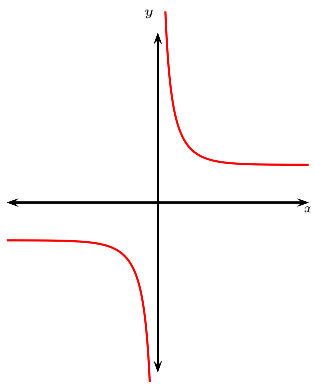


The two functions are plotted simultaneously in the x, y-plane. Indicate which part of the graph is the graph of which function.

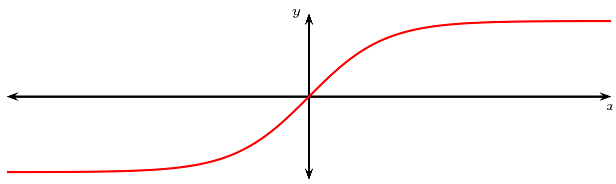
answer:
For $f(x) = \frac{2\sqrt{-x^2+1}+1}{\sqrt{-x^2+1}+1}$:
y-intercept: $x = \frac{3}{2}$, no x intercept
no asymptotes
increasing on $[-1, 0]$, decreasing on $[0, 1]$
global and local max at $x = 0$, global and local min at $x = \pm 1$.
concave down on $[-1, 1]$
no inflection points

answer:
For $f(x) = \frac{1}{\sqrt{-x^2+1}+1}$:
y-intercept: $x = \frac{1}{2}$, no x intercept
no asymptotes
decreasing on $[-1, 0]$, increasing on $[0, 1]$
global and local min at $x = 0$, global and local max at $x = \pm 1$.
concave up on $[-1, 1]$
no inflection points

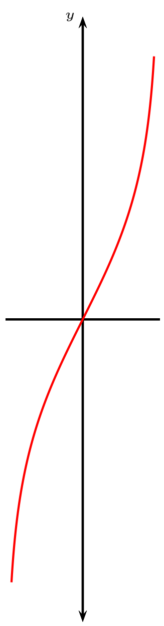
4. $f(x) = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$



5. $f(x) = \dfrac{-e^{-x} + e^x}{e^{-x} + e^x}$

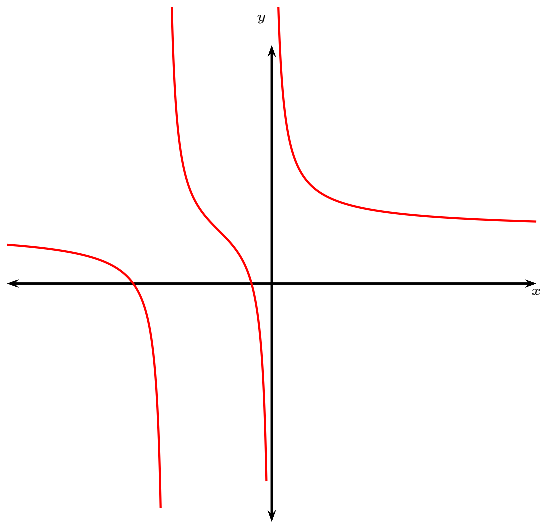


6. $f(x) = \ln\left(\dfrac{x+1}{-x+1}\right)$

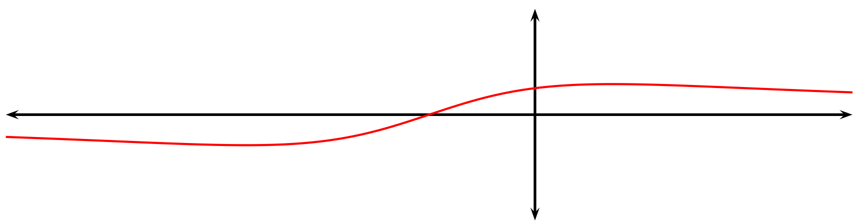


7. $f(x) = \dfrac{x^2 + 3x + 1}{x^2 + 2x}$. ***For this problem, indicate only the x-coordinates of the local maxima/minima and inflection points; you do not need to compute the y-coordinates of those points.***

Computation shows that $f'(x) = \dfrac{-x^2 - 2x - 2}{(x^2 + 2x)^2}$ *and that* $f''(x) = \dfrac{2x^3 + 6x^2 + 12x + 8}{(x^2 + 2x)^3} = \dfrac{(x+1)\left(2x^2 + 4x + 8\right)}{(x^2 + 2x)^3}$*; you may* *use those computations without further justification.*



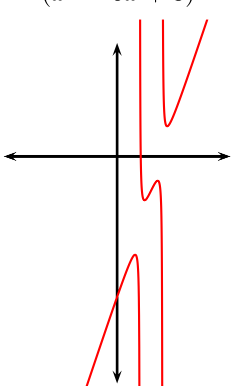
answer: *y-intercept: none, x-intercepts:* $\frac{-3\pm\sqrt{5}}{2}$
horizontal asymptote: $y = 1$, *vertical:* $x = -2$ *and* $x = 0$
always decreasing
no local/global minima/maxima
concave down on $(-\infty, -2) \cup (-1, 0)$, *concave up on* $(-2, -1) \cup (0, \infty)$
inflection point at $x = -1$



8. $f(x) = \dfrac{x+1}{x^2+2x+4}$

answer: *y-intercept:* $\frac{1}{4}$, *x-intercept:* -1
horizontal asymptote: $y = 0$, *vertical: none*
increasing on $\left(-1-\sqrt{3}, -1+\sqrt{3}\right)$, *decreasing on* $\left(-\infty, -1-\sqrt{3}\right) \cup \left(-1+\sqrt{3}, \infty\right)$
local and global min at $x = -1-\sqrt{3}$, *local and global max at* $x = -1+\sqrt{3}$
concave up on $(-4, -1) \cup (2, \infty)$, *concave down* $(-\infty, -4) \cup (-1, 2)$
inflection points at $x = -4$, $x = -1$, $x = 2$

9. $f(x) = \dfrac{3x^3 - 30x^2 + 97x - 99}{x^2 - 6x + 8}$. ***For this problem, do not find the x-intercepts of the function. Indicate only the x-coordinates of the local maxima/minima and inflection points; you do not need to compute the y-coordinates of those points.*** *Computation shows that* $f'(x) = \dfrac{3x^4 - 36x^3 + 155x^2 - 282x + 182}{\left(x^2-6x+8\right)^2} = \dfrac{\left(x^2-6x+7\right)\left(3x^2-18x+26\right)}{\left(x^2-6x+8\right)^2}$ *and that* $f''(x) = \dfrac{2x^3 - 18x^2 + 60x - 72}{\left(x^2-6x+8\right)^3} = \dfrac{(x-3)\left(2x^2-12x+24\right)}{\left(x^2-6x+8\right)^3}$*; you may use those computations without further justification.*



answer: *y-intercept* $\frac{-99}{8}$, *x-intercept: not requested*
horizontal asymptote: none, vertical: $x = 2$ *and* $x = 4$
increasing on $\left(-\infty, -\sqrt{2}+3\right) \cup \left(-\frac{\sqrt{3}}{3}+3, \frac{\sqrt{3}}{3}+3\right) \cup \left(\sqrt{2}+3, \infty\right)$, *decreasing on the complement intervals*
local maxima at $x = -\sqrt{2}+3$, $x = \frac{\sqrt{3}}{3}+3$
local minima at $x = -\frac{\sqrt{3}}{3}+3$, $x = \sqrt{2}+3$
concave up on $((2, 3) \cup (4, \infty))$
concave down on $(-\infty, 2) \cup (3, 4)$
inflection point at $x = 3$

Solution. 68.2

Domain. We have that f is not defined only when we have division by zero, i.e., if x^2-3x+3 equals zero. However, the roots of x^2-3x+3 are not real numbers: they are $\frac{3\pm\sqrt{3^2-4\cdot 3}}{2}=\frac{3\pm\sqrt{-3}}{2}$, and therefore x^2-3x+3 cannot equal zero (for real x). Alternatively, completing the square shows that the denominator is always positive:

$$x^2-3x+3=x^2-2\cdot\frac{3}{2}x+\frac{9}{4}-\frac{9}{4}+3=\left(x-\frac{3}{2}\right)^2+\frac{3}{4}>0$$

Therefore the domain of f is all real numbers.

x, y-intercepts. The y-intercept of f equals by definition $f(0)=\dfrac{2\cdot 0^2-5\cdot 0+\frac{9}{2}}{0^2-3\cdot 0+3}=\dfrac{\frac{9}{2}}{3}=\dfrac{3}{2}$. The x intercept of f is those values of x for which $f(x)=0$. The graph of f shows no such x, and that is confirmed by solving the equation $f(x)=0$:

$$\begin{array}{rcll} f(x) &=& 0 & \\ \dfrac{2x^2-5x+\frac{9}{2}}{x^2-3x+3} &=& 0 & \Big| \text{ Mult. by } x^2-3x+3 \\ 2x^2-5x+\dfrac{9}{2} &=& 0 & \\ x_1,x_2 &=& \dfrac{5\pm\sqrt{25-4\cdot 2\cdot\frac{9}{2}}}{4}=\dfrac{5\pm\sqrt{-9}}{4} & , \end{array}$$

so there are no real solutions (the number $\sqrt{-9}$ is not real).

Asymptotes. Since f is defined for all real numbers, its graph has no vertical asymptotes. To find the horizontal asymptote(s), we need to compute the limits $\lim\limits_{x\to\infty}f(x)$ and $\lim\limits_{x\to-\infty}f(x)$. The two limits are equal, as the direct computation below shows:

$$\begin{array}{rcll} \lim\limits_{x\to\pm\infty}\dfrac{2x^2-5x+\frac{9}{2}}{x^2-3x+3} &=& \lim\limits_{x\to\pm\infty}\dfrac{\left(2x^2-5x+\frac{9}{2}\right)\frac{1}{x^2}}{\left(x^2-3x+3\right)\frac{1}{x^2}} & \Big| \begin{array}{l}\text{Divide by leading}\\ \text{monomial in denominator}\end{array} \\ &=& \lim\limits_{x\to\pm\infty}\dfrac{2-\frac{5}{x}+\frac{9}{2x^2}}{1-\frac{3}{x}+\frac{3}{x^2}} & \\ &=& \dfrac{2-0+0}{1-0+0} & \\ &=& 2 & \end{array}$$

Therefore the graph of $f(x)$ has a single horizontal asymptote at $y=2$.

Intervals of increase and decrease. The intervals of increase and decrease of f are governed by the sign of f'. We compute:

$$\begin{array}{rcl} f'(x) &=& \left(\dfrac{2x^2-5x+\frac{9}{2}}{x^2-3x+3}\right)' \\ &=& \dfrac{\left(2x^2-5x+\frac{9}{2}\right)'\left(x^2-3x+3\right)-\left(2x^2-5x+\frac{9}{2}\right)\left(x^2-3x+3\right)'}{\left(x^2-3x+3\right)^2} \\ &=& \dfrac{-x^2+3x-\frac{3}{2}}{\left(x^2-3x+3\right)^2} \end{array}$$

As the denominator is a square, the sign of f' is governed by the sign of $-x^2+3x-\frac{3}{2}$. To find where $-x^2+3x-\frac{3}{2}$ changes sign, we compute the zeroes of this expression:

$$\begin{array}{rcll} -x^2+3x-\dfrac{3}{2} &=& 0 & \Big| \text{ Mult. by } -2 \\ 2x^2-6x+3 &=& 0 & \\ x_1,x_2 &=& \dfrac{6\pm\sqrt{36-24}}{4}=\dfrac{6\pm\sqrt{12}}{4} & \\ x_1,x_2 &=& \dfrac{3\pm\sqrt{3}}{2} & \end{array}$$

Therefore the quadratic $-x^2+3x-\frac{3}{2}$ factors as

$$-(x-x_1)(x-x_2)=-\left(x-\left(\frac{3-\sqrt{3}}{2}\right)\right)\left(x-\left(\frac{3+\sqrt{3}}{2}\right)\right) \tag{1}$$

The points x_1,x_2 split the real line into three intervals: $\left(-\infty,\frac{3-\sqrt{3}}{2}\right)$, $\left(\frac{3-\sqrt{3}}{2},\frac{3+\sqrt{3}}{2}\right)$ and $\left(\frac{3+\sqrt{3}}{2},\infty\right)$, and each of the factors of (1) has constant sign inside each of the intervals. If we choose x to be a very negative number, it follows that $-(x-x_1)(x-x_2)$

is a negative, and therefore $f'(x)$ is negative for $x \in (-\infty, \frac{3-\sqrt{3}}{2})$. For $x \in (\frac{3-\sqrt{3}}{2}, \frac{3+\sqrt{3}}{2})$, exactly one factor of f' changes sign and therefore $f'(x)$ is positive in that interval; finally only one factor of $f'(x)$ changes sign in the last interval so $f'(x)$ is negative on $(\frac{3+\sqrt{3}}{2}, \infty)$.

Our computations can be summarized in the following table.

Interval	$f'(x)$	$f(x)$
$\left(-\infty, \frac{3-\sqrt{3}}{2}\right)$	$-$	$\searrow$
$\left(\frac{3-\sqrt{3}}{2}, \frac{3+\sqrt{3}}{2}\right)$	$+$	$\nearrow$
$\left(\frac{3+\sqrt{3}}{2}, \infty\right)$	$-$	$\searrow$

Local and global minima and maxima. The table above shows that $f(x)$ changes from decreasing to increasing at $x = x_1 = \frac{3-\sqrt{3}}{2}$ and therefore f has a local minimum at that point. The table also shows that $f(x)$ changes from increasing to decreasing at $x = x_2 = \frac{3+\sqrt{3}}{2}$ and therefore f has a local maximum at that point. The so found local maximum and local minimum turn out to be global: there are two things to consider here. First, no other finite point is critical and thus cannot be maximum or minimum - however this leaves out the possibility of a maximum/minimum "at infinity". This possibility can be quickly ruled out by looking at the graph of f. To do so via algebra, compute first $f(x_1)$ and $f(x_2)$:

$$\begin{aligned} f(x_1) = f\left(\frac{3-\sqrt{3}}{2}\right) &= \frac{2\left(\frac{3-\sqrt{3}}{2}\right)^2 - 5\left(\frac{3-\sqrt{3}}{2}\right) + \frac{9}{2}}{\left(\frac{3-\sqrt{3}}{2}\right)^2 - 3\left(\frac{3-\sqrt{3}}{2}\right) + 3} = 2 - \frac{\sqrt{3}}{3} \\ f(x_2) = f\left(\frac{3+\sqrt{3}}{2}\right) &= \frac{2\left(\frac{3+\sqrt{3}}{2}\right)^2 - 5\left(\frac{3+\sqrt{3}}{2}\right) + \frac{9}{2}}{\left(\frac{3+\sqrt{3}}{2}\right)^2 - 3\left(\frac{3+\sqrt{3}}{2}\right) + 3} = 2 + \frac{\sqrt{3}}{3} \quad . \end{aligned}$$

On the other hand, while computing the horizontal asymptotes, we established that $\lim_{x\to\pm\infty} f(x) = 2$. This implies that all x sufficiently far away from $x = 0$, we have that $f(x)$ is close to 2. Therefore $f(x)$ is larger than $f(x_1)$ and smaller than $f(x_2)$ for all sufficiently far away from $x = 0$. This rules out the possibility for a maximum or a minimum "at infinity", as claimed above.

Intervals of concavity. The intervals of concavity of f are governed by the sign of f''. The second derivative of f is:

$$\begin{aligned} f''(x) &= (f'(x))' = \left(\frac{-x^2+3x-\frac{3}{2}}{(x^2-3x+3)^2}\right)' \\ &= \left(-x^2+3x-\frac{3}{2}\right)'\left(\frac{1}{(x^2-3x+3)^2}\right) + \left(-x^2+3x-\frac{3}{2}\right)\left(\frac{1}{(x^2-3x+3)^2}\right)' && \text{second differentiation: chain rule} \\ &= (-2x+3)\left(\frac{1}{(x^2-3x+3)^2}\right) + \left(-x^2+3x-\frac{3}{2}\right)(-2)\frac{(x^2-3x+3)'}{(x^2-3x+3)^3} \\ &= (-2x+3)\left(\frac{1}{(x^2-3x+3)^2}\right) + \left(2x^2-6x+3\right)\frac{(2x-3)}{(x^2-3x+3)^3} && \text{factor out } \tfrac{(2x-3)}{(x^2-3x+3)^2} \\ &= \frac{(2x-3)}{(x^2-3x+3)^2}\left(-1+\frac{(2x^2-6x+3)}{(x^2-3x+3)}\right) \\ &= \frac{(2x-3)}{(x^2-3x+3)^2}\left(\frac{-\left(x^2-3x+3\right)+(2x^2-6x+3)}{(x^2-3x+3)}\right) \\ &= \frac{(2x-3)(x^2-3x)}{(x^2-3x+3)^3} \\ &= \frac{(2x-3)x(x-3)}{(x^2-3x+3)^3} \end{aligned}$$

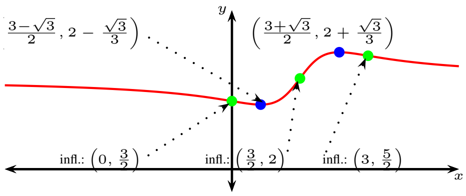
When computing the domain of f, we established that the denominator of the above expression is always positive. Therefore $f''(x)$ changes sign when the terms in the numerator change sign, namely, at $x = 0$, $x = \frac{3}{2}$ and $x = 3$.

Our computations can be summarized in the following table. In the table, we use the $\cup$ symbol to denote that the function is concave up in the indicated interval, and $\cap$ to denote that the function is concave down.

Interval	$f''(x)$	$f(x)$
$(-\infty, 0)$	$-$	$\cap$
$(0, \frac{3}{2})$	$+$	$\cup$
$(\frac{3}{2}, 3)$	$-$	$\cap$
$(3, \infty)$	$+$	$\cup$

Points of inflection. The preceding table shows that $f''(x)$ changes sign at $0, \frac{3}{2}, 3$ and therefore the points of inflection are located at $x = 0, x = \frac{3}{2}$ and $x = 3$, i.e., the points of inflection are $(0, f(0)) = \left(0, \frac{3}{2}\right)$, $\left(\frac{3}{2}, f\left(\frac{3}{2}\right)\right) = \left(\frac{3}{2}, 2\right)$, $(3, f(3)) = \left(3, \frac{5}{2}\right)$.

We can command our graphing device to use the so computed information to label the graph of the function. Finally, we can confirm visually that our function does indeed behave in accordance with our computations.



Solution. 68.8

This problem is very similar to Problem 68.2. We recommend to the student to solve the problem first "with closed textbook" and only then to compare with the present solution.

Domain. As f is a quotient of two polynomials (rational function), its implied domain is all x except those for which we get division by zero for f. Consequently the domain of f is all x for which $x^2+2x+4=0$. However, the polynomial x^2+2x+4 has no real roots - its roots are $\dfrac{-2\pm\sqrt{4-16}}{2}=-1\pm\sqrt{-3}$, and therefore the domain of f is all real numbers. Alternatively, we can complete the square: $x^2+2x+4=(x+1)^2+3$ and so x^2+2x+4 is positive for all values of x.

x, y-intercepts. The y-intercept of f equals by definition $f(0)=\dfrac{0+1}{0^2+2\cdot 0+4}=\dfrac{1}{4}$. The x intercept of f is those values of x for which $f(x)=0$. We compute

$$\begin{array}{rcl} f(x) &=& 0\\ \dfrac{x+1}{x^2+2x+4} &=& 0\\ x+1 &=& 0\\ x &=& -1 \quad , \end{array}$$

and the x-intercept of f is $x=-1$.

Asymptotes. The line $x=a$ is a vertical asymptote when $\lim\limits_{x\to a^\pm} f(x)=\pm\infty$; as f is defined for all real numbers, this implies that there are no vertical asymptotes.

The line $y=L$ is a horizontal asymptote if $\lim\limits_{x\to\pm\infty} f(x)$ exists and equals L. We compute:

$$\lim_{x\to\infty} f(x)=\lim_{x\to\infty}\frac{(x+1)\frac{1}{x^2}}{(x^2+2x+4)\frac{1}{x^2}}=\lim_{x\to\infty}\frac{\frac{1}{x}+\frac{1}{x^2}}{1+\frac{2}{x}+\frac{4}{x^2}}=\frac{0+0}{1+0+0}=0$$

Therefore $y=0$ is a horizontal asymptote for f. An analogous computation shows that $\lim\limits_{x\to\pm\infty} f(x)=0$ and so $y=0$ is the only horizontal asymptote of f.

Intervals of increase and decrease. The intervals of increase and decrease of f are governed by the sign of f'. We compute:

$$\begin{array}{rcl|l} f'(x) &=& \left(\dfrac{x+1}{x^2+2x+4}\right)' & \text{qutotient rule}\\ &=& \dfrac{(x+1)'\left(x^2+2x+4\right)-(x+1)\left(x^2+2x+4\right)'}{\left(x^2+2x+4\right)^2} &\\ &=& \dfrac{x^2+2x+4-(x+1)(2x+2)}{\left(x^2+2x+4\right)^2} &\\ &=& \dfrac{x^2+2x+4-\left(2x^2+4x+2\right)}{\left(x^2+2x+4\right)^2} &\\ &=& \dfrac{-x^2-2x+2}{\left(x^2+2x+4\right)^2} & \end{array}$$

As x^2+2x+4 is positive, the sign of f' is governed by the sign of $-x^2+2x+2$. To find out where $-x^2+2x+2$ changes sign, we compute the zeroes of this expression:

$$\begin{array}{rcl|l} -x^2-2x+2 &=& 0 &\\ x^2+2x-2 &=& 0 & \text{use the quadratic formula}\\ x_1,x_2 &=& -1\pm\sqrt{3} \quad . & \end{array}$$

Therefore the quadratic $-x^2+2x+2$ factors as

$$-(x-x_1)(x-x_2)=-\left(x-\left(-1-\sqrt{3}\right)\right)\left(x-\left(-1+\sqrt{3}\right)\right) \qquad (2)$$

The points x_1, x_2 split the real line into three intervals: $(-\infty, -1-\sqrt{3})$, $(-1-\sqrt{3}, -1+\sqrt{3})$ and $(-1+\sqrt{3}, \infty)$, and each of the factors of (2) has constant sign inside each of the intervals. If we choose x to be a very negative number, it follows that $-(x-x_1)(x-x_2)$ is a negative, and therefore $f'(x)$ is negative for $x \in (-\infty, -1-\sqrt{3})$. For $x \in (-1-\sqrt{3}, -1+\sqrt{3})$, exactly one factor of f' changes sign and therefore $f'(x)$ is positive in that interval; finally only one factor of $f'(x)$ changes sign in the last interval so $f'(x)$ is negative on $(-1+\sqrt{3}, \infty)$.

Our computations can be summarized in the following table.

Interval	$f'(x)$	$f(x)$
$(-\infty, -1-\sqrt{3})$	$-$	$\searrow$
$(-1-\sqrt{3}, -1+\sqrt{3})$	$+$	$\nearrow$
$(-1+\sqrt{3}, \infty)$	$-$	$\searrow$

Local and global minima and maxima. The table above shows that $f(x)$ changes from decreasing to increasing at $x = x_1 = -1-\sqrt{3}$ and therefore f has a local minimum at that point. The table also shows that $f(x)$ changes from increasing to decreasing at $x = x_2 = -1+\sqrt{3}$ and therefore f has a local maximum at that point. The so found local maximum and local minimum turn out to be global: indeed, no other finite point is critical and thus cannot be maximum or minimum; on the other hand $\lim\limits_{x\to\pm\infty} f(x) = 1$ and this implies that all x sufficiently far away from $x = 0$ have that $f(x)$ is close to 0, and therefore $f(x)$ is larger than $f(x_1)$ and smaller than $f(x_2)$ for all x.

Intervals of concavity. The intervals of concavity of f are governed by the sign of f''. The second derivative of f is:

$$
\begin{aligned}
f''(x) &= (f'(x))' = \left(\frac{-x^2-2x+2}{(x^2+2x+4)^2}\right)' \\
&= (-x^2-2x+2)'\left(\frac{1}{(x^2+2x+4)^2}\right) + (-x^2-2x+2)\left(\frac{1}{(x^2+2x+4)^2}\right)' && \text{use chain rule for second differentiation} \\
&= (-2x-2)\left(\frac{1}{(x^2+2x+4)^2}\right) + (-x^2-2x+2)(-2)\frac{(x^2+2x+4)'}{(x^2+2x+4)^3} \\
&= -(2x+2)\left(\frac{1}{(x^2+2x+4)^2}\right) + (2x^2+4x-4)\frac{(2x+2)}{(x^2+2x+4)^3} && \text{factor out } \tfrac{(2x+2)}{(x^2+2x+4)^2} \\
&= \frac{(2x+2)}{(x^2+2x+4)^2}\left(-1+\frac{(2x^2+4x-4)}{(x^2+2x+4)}\right) \\
&= \frac{(2x+2)}{(x^2+2x+4)^2}\left(\frac{-(x^2+2x+4)+(2x^2+4x-4)}{(x^2+2x+4)}\right) \\
&= \frac{(2x+2)(x^2+2x-8)}{(x^2+2x+4)^3} && \text{factor } (x^2+2x-8) \\
&= \frac{(2x+2)(x+4)(x-2)}{(x^2+2x+4)^3}
\end{aligned}
$$

As we previously established, the denominator of the above expression is always positive. Therefore the expression above changes sign when the terms in the numerator change sign, namely, at $x=-1$, $x=-4$ and $x=2$.

Our computations can be summarized in the following table.

Interval	$f''(x)$	$f(x)$
$(-\infty, -4)$	$-$	$\cap$
$(-4, -1)$	$+$	$\cup$
$(-1, 2)$	$-$	$\cap$
$(2, \infty)$	$+$	$\cup$

Points of inflection. The preceding table shows that $f''(x)$ changes sign at $-4, -1, 2$ and therefore the points of inflection are located at $x=-4, x=-1$ and $x=2$, i.e., the points of inflection are $\left(-4, -\frac{1}{4}\right)$, $(-1, 0)$, $\left(2, \frac{1}{4}\right)$.

Problem 69. 1. *Sketch the graph of $y = x^4 - 8x^2 + 8$ by determining the intervals of increase and decrease, finding the local mins and maxes, determining where the graph is concave up and concave down, and plotting a few key points.*

answer: *Check your graph with a calculator or online graphing program. Local max at 0, local mins at 2 and -2. Concave down between $-\sqrt{4/3}$ and $\sqrt{4/3}$, and concave up otherwise.*

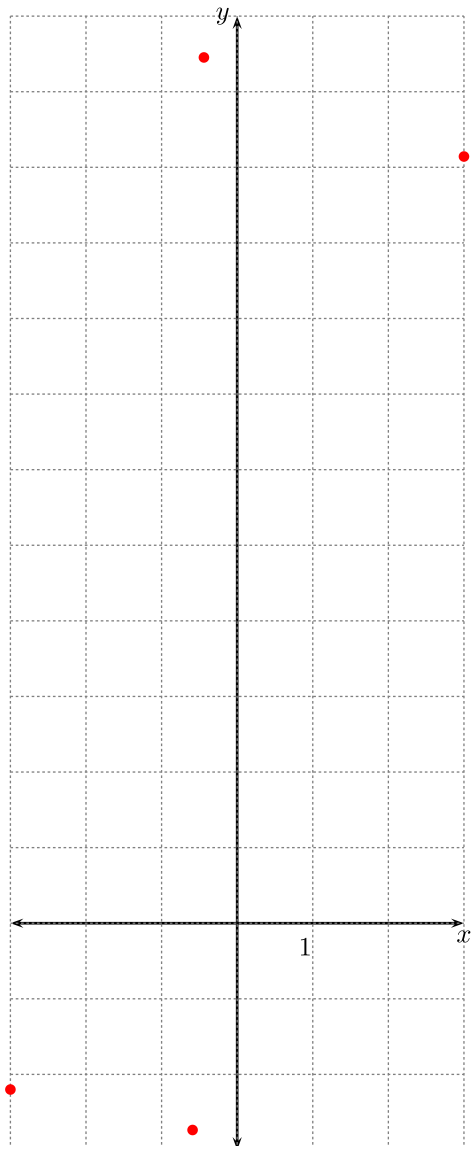
2. *Sketch the graph of $y = \frac{x-1}{x^2-9}$ by graphing any vertical and horizontal asymptotes, finding the x- and y-intercepts, and then sketching a graph that fits this information.*

answer: *Check your graph with a calculator or online graphing program. Vertical asymptotes at $x = 3$ and $x = -3$. Horizontal asymptote at $y = 0$. y-intercept of $\frac{1}{9}$; x-intercept of 1.*

3. *Consider the function $f(x) = \dfrac{4x^2+10x+5}{2x+1}$. Computation shows that $f'(x) = \dfrac{8x^2+8x}{(2x+1)^2}$ and $f''(x) = \dfrac{8}{(2x+1)^3}$.*

- *Find the intervals of increase and intervals of decrease of f.*
- *Find the local maxima and minima of f.*
- *Find where the function is concave up and where it is concave down.*
- *Sketch the function $f(x)$ roughly by hand. Make sure that your plot matches your computations from the preceding parts of the problem.*

  You may use the provided grid and coordinate system. From the previous page, we recall that $f(x) = \dfrac{4x^2+10x+5}{2x+1}$, $f'(x) = \dfrac{8x^2+8x}{(2x+1)^2}$ and $f''(x) = \dfrac{8}{(2x+1)^3}$.
  The 4 points plotted on the grid are known to lie on the curve.



4. *Consider the function $f(x) = \dfrac{2x^2-4x+2}{x^2-2x}$.*

- *Find the vertical asymptotes of f.* ***For this particular sub-question, and for this sub-question alone, no justification is required (just write the answer).***
- *Computation shows that $f'(x) = \dfrac{-4x+4}{(x^2-2x)^2}$. Find the intervals of increase and decrease of f.*
- *Find the local maxima and minima of f.*
- *Computation shows that $f''(x) = \dfrac{12x^2-24x+16}{(x^2-2x)^3}$. Find where the function is concave up and where it is concave down.*

- *Sketch the function $f(x)$ roughly by hand. Make sure that your plot matches your computations from the preceding parts of the problem.*

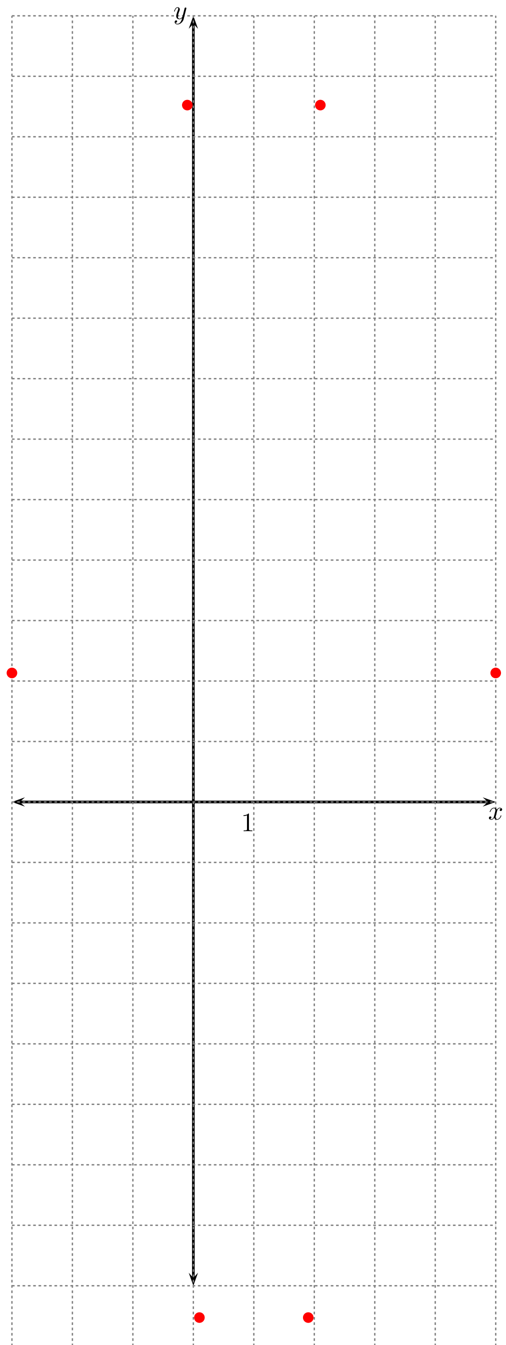
  You may use the provided grid and coordinate system. We recall that

  $$f(x) = \frac{2x^2 - 4x + 2}{x^2 - 2x},$$

  $$f'(x) = \frac{-4x + 4}{(x^2 - 2x)^2},$$

  $$f''(x) = \frac{12x^2 - 24x + 16}{(x^2 - 2x)^3}.$$

  The points plotted below are known to lie on the curve.



8 Linearizations and Differentials

Problem 70.

1. *Find the linearization of $f(x) = \sqrt{x}$ at $a = 100$ and use it to approximate $\sqrt{99.8}$.*

answer: $L(x) = 10 + 0.05(x - 100)$. *Therefore* $\sqrt{99.8} \approx L(99.8) = 9.99$.

2. *Find the linearization of* $f(x) = \sqrt{8+x}$ *at* $a = 1$ *and use it to approximate* $\sqrt{9.02}$.

answer: $f(x) \approx 3 + \frac{1}{6}(x-1) = \frac{1}{6}x + \frac{17}{6}$. *Therefore* $\sqrt{9.02} \approx \frac{901}{300} \approx 3.003333$

3. *Find the linearization of* $f(x) = \sqrt[3]{8+x}$ *at* $a = 0$ *and use it to approximate* $\sqrt[3]{7.97}$.

answer: $\sqrt[3]{8+x} \approx \frac{1}{12}x + 2$. *Therefore* $\sqrt[3]{7.97} \simeq \frac{799}{400} = 1.9975$

4. *Find the linearization of* $f(x) = \ln x$ *at* $a = 1$ *and use it to approximate* $\ln 1.01$.

answer: $f(x) \approx f(1) + f'(1)(x-1) = x - 1, \ln 1.01 \approx 0.01$.

5. *Use a linear approximation to estimate* $(1.001)^9$.

answer: $(1.001)^9 \approx 1.009$.

6. *Use a linear approximation to estimate* $(0.9999)^{2014}$.

answer: $(0.9999)^{2014} \approx 0.7986$.

Solution. 70.6 Let $f(x) = x^{2014}$. We are looking to approximate $(0.9999)^{2014} = f(0.9999)$. As $f(1) = 1^{2014} = 1$ is easy to compute, is makes sense to use linear approximation at $a = 1$ to approximate $(0.9999)^{2014}$. We have that

$$f'(x) = 2014x^{2013} \quad .$$

Therefore the linear approximation of $f(x) = x^{2014}$ at $a = 1$ is:

$$f(x) \approx f(1) + f'(1)(x-1) = 1^{2014} + 2014 \cdot 1^{2013}(x-1) = 1 + 2014(x-1) = 2014x - 2013 \quad .$$

Therefore

$$f(0.9999) \approx 2014 \cdot 0.9999 - 2013 = 1 \cdot 0.9999 + 2013(0.9999 - 1) = 0.9999 - 2013 \cdot 0.0001 = 0.9999 - 0.2013 = 0.7986$$

A computation with computer shows that $0.999^{2014} = 0.817577\ldots$. While our approximation of 0.7986 is less than perfect, it is within the same order of magnitude. We study techniques for estimating errors in linear approximations later.

9 Integration Basics

9.1 Riemann Sums

Problem 71. *Estimate the integral using a Riemann sum using the indicated sample points and interval length.*

1. $\displaystyle\int_0^4 \left(\sqrt{8x+1}\right) \mathrm{d}x$. *Use four intervals of equal width, choose the sample point to be the left endpoint of each interval.*

answer: $\Delta x = 1$ *and* $f(x) = \sqrt{8x+1}$. *Thus* $\int_0^4 f(x)\mathrm{d}x \approx 9 + \sqrt{17}$.

2. $\displaystyle\int_0^6 \frac{1}{x^2+1}\mathrm{d}x$. *Use three intervals of equal width, choose the sample point to be the left endpoint.*

answer: $\Delta x = 2$ *and* $f(x) = \frac{1}{x^2+1}$. *Thus* $\int_0^6 f(x)\mathrm{d}x \approx \frac{214}{85}$.

3. $\displaystyle\int_{-3.5}^{-0.5} \frac{\mathrm{d}x}{x^2+1}$. *Use three intervals of equal width, choose the sample point to be the midpoint of each interval.*

answer: $\Delta x = 1$ *and* $f(x) = \frac{1}{x^2+1}$. *Thus* $\displaystyle\int_{-3.5}^{-0.5} f(x)\mathrm{d}x \approx \Delta x\left(f(-3) + f(-2) + f(-1)\right) = \frac{4}{5} = 0.8$.

4. $\displaystyle\int_0^2 \frac{\mathrm{d}x}{1+x+x^3}$. *Use* $\Delta x = \frac{1}{2}$ *and right endpoint sampling points.*

answer: $\frac{1}{2}\left(\frac{8}{13} + \frac{1}{3} + \frac{8}{47} + \frac{1}{11}\right) = \frac{12197}{20163} \approx 0.604920$

5. $\displaystyle\int_{-2}^0 \frac{\mathrm{d}x}{1+x+x^2}$. *Use* $\Delta x = \frac{2}{3}$ *and left endpoint sampling points.*

answer: $\frac{2}{3}\left(\frac{1}{3} + \frac{9}{13} + \frac{9}{7}\right) = \frac{1262}{819} \approx 1.540904$

6. $\displaystyle\int_0^2 \frac{\mathrm{d}x}{1+x^3}$. *Use four intervals of equal width, choose the sample point to be the left endpoint of each interval.*

answer: $\Delta x = 0.5$ *and* $f(x) = \frac{1}{1+x^3}$. *Thus* $\displaystyle\int_0^2 f(x)\mathrm{d}x \approx \Delta x \left(f(0) + f(1) + f\left(\frac{1}{2}\right) + f\left(\frac{3}{2}\right)\right) = \frac{1649}{1260} \approx 1.30873.$

7. $\displaystyle\int_{-2}^0 \frac{\mathrm{d}x}{x^4+1}$. *Use four intervals of equal width, choose the sample point to be the right endpoint.*

answer: $\Delta x = 0.5$ *and* $f(x) = \frac{1}{1+x^3}$. *Thus* $\displaystyle\int_0^2 f(x)\mathrm{d}x \approx \Delta x \left(f\left(-\frac{3}{2}\right) + f(-1) + f\left(-\frac{1}{2}\right) + f(0)\right) = \frac{8595}{6596} \approx 1.303062.$

8. $\displaystyle\int_{-1}^0 \frac{1}{3x^2+1}\mathrm{d}x$. *Use* 3 ***intervals*** *of equal width, choose the sampling points to be the* ***left endpoints*** *of each interval. Simplify your answer to a rational number (single fraction of two integers).*

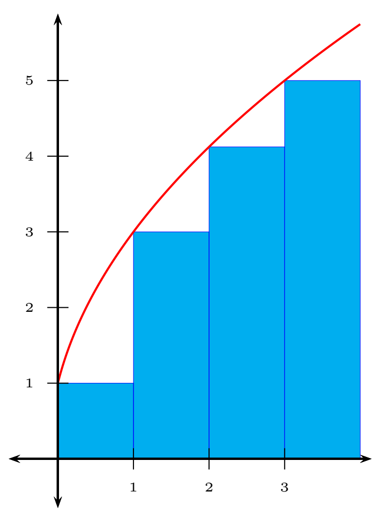
answer: $\Delta x = \frac{1}{3}$ *and* $f(x) = \frac{1}{3x^2+1}$. *Thus* $\displaystyle\int_{-1}^0 f(x)\mathrm{d}x$ *is approximated by* $\Delta x \left(f(-1) + f\left(-\frac{2}{3}\right) + f\left(-\frac{1}{3}\right)\right) = \frac{10}{21}.$

Solution. 71.1. The interval $[0, 4]$ is subdivided into $n = 4$ intervals, therefore the length of each is $\Delta x = 1$. The intervals are therefore

$$[0, 1], [1, 2], [2, 3], [3, 4] \quad .$$

The problem asks us to use the left endpoints of each interval as sampling points. Therefore our sampling points are $0, 1, 2, 3$. Therefore the Riemann sum we are looking for is

$$\Delta x \left(f(0) + f(1) + f(2) + f(3)\right) = 1 \cdot \left(\sqrt{8 \cdot 0 + 1} + \sqrt{8 \cdot 1 + 1} + \sqrt{8 \cdot 2 + 1} + \sqrt{8 \cdot 3 + 1}\right) = 9 + \sqrt{17} \approx 13.1231$$

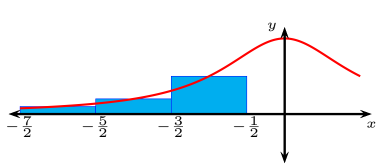


Solution. 71.3. The interval $[-3.5, -0.5]$ is subdivided into $n = 3$ intervals, therefore the length of each is $\Delta x = \frac{-0.5-(-3.5)}{3} = \frac{3}{3} = 1$. The intervals are therefore

$$[-3.5, -2.5], [-2.5, -1.5], [-1.5, -0.5] \quad .$$

The problem asks us to use the midpoint of each interval as a sampling point. Therefore our sampling points are $-3, -2, -1$. Therefore the Riemann sum we are looking for is

$$\Delta x \left(f(-3) + f(-2) + f(-1)\right) = 1 \cdot \left(\frac{1}{10} + \frac{1}{5} + \frac{1}{2}\right) = 0.8 \quad .$$



Solution. 71.8

$\Delta x = \frac{1}{3}$ and $f(x) = \frac{1}{3x^2+1}$. Thus $\int\limits_{-1}^{0} f(x)\mathrm{d}x$ is approximated by $\Delta x \left(f(-1) + f\left(-\frac{2}{3}\right) + f\left(-\frac{1}{3}\right)\right) = \frac{10}{21}$.

9.2 Antiderivatives

Problem 72. *Find all antiderivatives of the functions.*

1. $f(x) = \sqrt{3} + \pi^2$.

   answer: $x\left(\pi^2 + \sqrt{3}\right) + C$

2. $f(x) = x - 5$.

   answer: $\frac{x^2}{2} - 5x + C$

3. $f(x) = x^2 - 2x + 6$.

   answer: $\frac{x^3}{3} - x^2 + 6x + C$

4. $f(x) = \dfrac{x(x+1)}{2}$.

   answer: $\frac{1}{6}x^3 + \frac{1}{4}x^2 + C$

5. $f(x) = x(x+1)(2x+1)$.

   answer: $\frac{1}{2}x^4 + x^3 + \frac{1}{2}x^2 + C$

6. $f(x) = 7x^{\frac{2}{7}} + x^{-\frac{4}{7}}$.

   answer: $\frac{49}{9}x^{\frac{9}{7}} + \frac{7}{3}x^{\frac{3}{7}} + C$

7. $f(x) = x^{2.4} - 2x^{\sqrt{3}-1}$.

   answer: $\frac{5}{17}x^{\frac{17}{5}} - \frac{2\sqrt{3}x^{\sqrt{3}}}{3} + C$

8. $f(x) = \dfrac{8}{x^7}$.

   answer: $-\frac{4}{3}x^{-6} + C$

9. $f(x) = \dfrac{x+1}{x^3}$.

   answer: $-x^{-1} - \frac{1}{2}x^{-2} + C$

10. $f(x) = \dfrac{1}{x}$.

    answer: $\ln|x| + C$

11. $f(x) = \dfrac{x^2+1}{x}$.

    answer: $\frac{1}{2}x^2 + \ln|x| + C$

12. $f(x) = \dfrac{5 - 4x^3 + 2x^6}{x^4}$.

    answer: $\frac{2}{3}x^3 - \frac{5}{3}x^{-3} - 4\ln|x| + C$

13. $g(x) = \dfrac{1 + \sqrt{x} + x}{\sqrt{x^3}}$.

    answer: $2x^{\frac{1}{2}} - 2x^{-\frac{1}{2}} + \ln|x| + C$

14. $f(t) = 3\sin t - 4\cos t$.

    answer: $-3\cos t - 4\sin t + C$

15. $f(\theta) = \sec^2\theta$.

    answer: $\tan\theta + C$

16. $f(\theta) = \csc^2\theta$.

    answer: $-\cot\theta + C$

17. $f(t) = \sec t \tan t + \csc t \cot t$.

    answer: $\sec t - \csc t + C$

18. $f(x) = \dfrac{2 + x\cos x}{x}$.

    answer: $2\ln|x| + \sin x$

Problem 73.

1. *Find $f(x)$ if $f'(x) = 3 + \frac{1}{x}$ and $f(1) = 2$.*

   answer: $f(x) = 3x + \ln|x| - 1$

2. *Find $f(x)$ if $f'(x) = x - \sin x$ and $f(0) = 7$.*

   answer: $f(x) = \frac{x^2}{2} + \cos x + 6$

Problem 74. *Verify by differentiation that the formula is correct.*

1. $\displaystyle\int \sqrt{1+x^2}\mathrm{d}x = \frac{1}{2}\left(x\sqrt{1+x^2} + \ln\left(x + \sqrt{1+x^2}\right) + C\right)$.

2. $\displaystyle\int \sin^2 x\mathrm{d}x = -\frac{1}{4}\sin(2x) + \frac{1}{2}x + C$.

3. $\displaystyle\int \sin^3 x\mathrm{d}x = \frac{1}{3}\cos^3 x - \cos x + C$.

4. $\displaystyle\int \frac{x}{\sqrt{1+x}}\mathrm{d}x = \frac{2}{3}(x-2)\sqrt{1+x} + C$

9.3 Basic Definite Integrals

Problem 75. *Evaluate the integral (definite or indefinite).*

1. $\displaystyle\int\limits_{-2}^{3} \left(x^2 - 1\right)\mathrm{d}x$.

   answer: $\left[\frac{1}{3}x^3 - x\right]_{-2}^{3} = \frac{20}{3}$

2. $\displaystyle\int\limits_{1}^{2} \left(4x^3 + 3x^2 + 2x + 1\right)\mathrm{d}x$.

   answer: $\left[x^4 + x^3 + x^2 + x\right]_1^2 = 26$

3. $\displaystyle\int\limits_{0}^{2} (x-1)(x^2+1)\mathrm{d}x$.

   answer: $\left[\frac{1}{4}x^4 - \frac{1}{3}x^3 + \frac{1}{2}x^2 - x\right]_0^2 = \frac{4}{3}$

4. $\displaystyle\int\limits_{-1}^{1} \left(\frac{x(x+1)}{2}\right)^2\mathrm{d}x$.

   answer: $\left[\frac{1}{20}x^5 + \frac{1}{8}x^4 + \frac{1}{12}x^3\right]_{-1}^{1} = \frac{4}{15}$

5. $\displaystyle\int\limits_{0}^{1} (1+x^2)^3 dx$.

   answer: $\left[\frac{1}{7}x^7 + \frac{3}{5}x^5 + x^3 + x\right]_0^1 = \frac{96}{35}$

6. $\displaystyle\int\limits_{1}^{2} \left(\frac{1}{x} - \frac{4}{x^2}\right)\mathrm{d}x$.

   answer: $\left[4x^{-1} + \ln x\right]_1^2 = \ln 2 - 2$

7. $\int\limits_1^4 \sqrt{x}(1+x)\mathrm{d}x.$

answer: $\left[\frac{2}{5}x^{\frac{5}{2}} + \frac{2}{3}x^{\frac{3}{2}}\right]_1^4 = \frac{256}{15}$

8. $\int\limits_1^4 \sqrt{\frac{6}{x}}\mathrm{d}x.$

answer: $\left[2\sqrt{6}\sqrt{x}\right]_1^4 = 2\sqrt{6}$

9. $\int\limits_1^4 \frac{\frac{1}{\sqrt{x}}+1+x}{\sqrt{x}}\mathrm{d}x.$

answer: $\left[\frac{2}{3}x^{\frac{3}{2}} + 2\sqrt{x} + \ln|x|\right]_1^4 = \ln(4) + \frac{20}{3}$

10. $\int\limits_1^8 \frac{1+x}{\sqrt[3]{x}}\mathrm{d}x.$

answer: $\left[\frac{3}{5}x^{\frac{5}{3}} + \frac{3}{2}x^{\frac{2}{3}}\right]_1^8 = \frac{231}{10}$

11. $\int\limits_1^{64} \frac{\frac{1}{\sqrt[3]{x}}+\sqrt[3]{x}}{\sqrt{x}}\mathrm{d}x.$

answer: $\left[\frac{6}{5}x^{\frac{5}{6}} + 6x^{\frac{1}{6}}\right]_1^{64} = \frac{216}{5}$

12. $\int\limits_0^1 \left(\sqrt[5]{x^6}+\sqrt[6]{x^5}\right)\mathrm{d}x.$

answer: $\left[\frac{5}{11}x^{\frac{11}{5}} + \frac{6}{11}x^{\frac{11}{6}}\right]_0^1 = 1$

13. $\int\limits_1^2 \left(x+\frac{1}{x}\right)^2\mathrm{d}x.$

answer: $\left[frac13x^3 - x^{-1} + 2x\right]_1^2 = \frac{29}{6}$

14. $\int\limits_1^2 \left(x+\frac{1}{x}\right)^3\mathrm{d}x.$

answer: $\left[\frac{1}{4}x^4 + \frac{3}{2}x^2 - \frac{1}{2}x^{-2} + 3\ln x\right]_1^2 = \frac{69}{8} + 3\ln 2$

15. $\int\limits_1^2 \left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2\mathrm{d}x.$

answer: $\left[\frac{1}{2}x^2 + \ln x + 2x\right]_1^2 = \frac{7}{2} + \ln(2)$

16. $\int\limits_1^2 \left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^3\mathrm{d}x.$

answer: $\left[\frac{2}{5}x^{\frac{5}{2}} + 2x^{\frac{3}{2}} + 6\sqrt{x} - 2x^{-\frac{1}{2}}\right]_1^2 = \frac{53}{5}\sqrt{2} - \frac{32}{5}$

17. $\int\limits_0^2 |x-1|\mathrm{d}x.$

answer: 1

18. $\int\limits_0^1 \left|x-\frac{1}{2}\right|\mathrm{d}x.$

answer: $\frac{1}{4}$

19. $\int\limits_{-1}^1 (x-3|x|)\mathrm{d}x.$

answer: -3

20. $\int\limits_{\frac{\pi}{4}}^{\frac{\pi}{2}} \csc^2\theta\mathrm{d}\theta.$

answer: [] =

21. $\int\limits_0^{\frac{\pi}{4}} \frac{1-\cos^2\theta}{\cos^2\theta}\mathrm{d}\theta.$

answer: [] =

22. $\int\limits_0^{\frac{\pi}{4}} \frac{\sin^2\theta}{\cos^2\theta}\mathrm{d}\theta.$

answer: [] =

23. $\int\limits_0^{\frac{\pi}{4}} \tan^2\theta\mathrm{d}\theta.$

answer: [] =

24. $\int\limits_0^{\frac{\pi}{3}} \frac{\sin\theta+\sin\theta\tan^2\theta}{\sec^2\theta}\mathrm{d}\theta.$

answer: [] =

25. $\int\limits_0^{\pi} (\sin\theta-\cos\theta)\mathrm{d}\theta.$

answer: [] =

26. $\int\limits_0^{\pi} |\sin x|\mathrm{d}x.$

Solution. 75.18

$$
\begin{aligned}
\int\limits_0^1 \left|x-\frac{1}{2}\right|\mathrm{d}x &= \int\limits_0^{\frac{1}{2}} \left|x-\frac{1}{2}\right|\mathrm{d}x + \int\limits_{\frac{1}{2}}^1 \left|x-\frac{1}{2}\right|\mathrm{d}x \\
&= \int\limits_0^{\frac{1}{2}} \left(\frac{1}{2}-x\right)\mathrm{d}x + \int\limits_{\frac{1}{2}}^1 \left(x-\frac{1}{2}\right)\mathrm{d}x \\
&= \left[-\frac{x^2}{2}+\frac{x}{2}\right]_0^{\frac{1}{2}} + \left[\frac{x^2}{2}-\frac{x}{2}\right]_{\frac{1}{2}}^1 \\
&= \left(-\frac{1}{8}+\frac{1}{4}\right) + \left(\frac{1}{2}-\frac{1}{2}-\left(\frac{1}{8}-\frac{1}{4}\right)\right) \\
&= \frac{1}{4}
\end{aligned}
$$

$$
\begin{aligned}
\left|x-\tfrac{1}{2}\right| &= \tfrac{1}{2}-x \text{ when } x \leq \tfrac{1}{2} \\
\left|x-\tfrac{1}{2}\right| &= x-\tfrac{1}{2} \text{ when } x \geq \tfrac{1}{2}
\end{aligned}
$$

Problem 76. *Integrate (definite or indefinite).*

1. $\int\limits_1^8 \frac{t-t^{\frac{1}{3}}+2}{t^{\frac{4}{3}}}\mathrm{d}t \quad .$

answer: $-\ln 8 + \frac{15}{2}$

2. $\int\limits_1^4 \left(x+\sqrt{x}\right)^2 dx \quad .$

answer: $\frac{533}{10}$

3. $\displaystyle\int \frac{\sqrt[3]{x} - x^{\frac{1}{2}} + 1}{x} \mathrm{d}x.$

answer: $-2\sqrt{x} + 3x^{\frac{1}{3}} + \ln|x| + C$

4. $\displaystyle\int \frac{\sqrt[3]{x} - 1}{x} \mathrm{d}x.$

answer: $3x^{\frac{1}{3}} - \ln|x| + C$

9.4 Fundamental Theorem of Calculus Part I

Problem 77. *Differentiate $f(x)$ using the Fundamental Theorem of Calculus part 1.*

1. $f(x) = \displaystyle\int_1^x \sin\left(t^2\right) \mathrm{d}t$

   answer: $\sin\left(x^2\right)$

2. $f(x) = \displaystyle\int_1^x \left(t - \sqrt{t}\right) \mathrm{d}t.$

   answer: $x - \sqrt{t}$

3. $f(x) = \displaystyle\int_x^1 (2 + t^4)^5 \, \mathrm{d}t$

   answer: $-\left(2 + x^4\right)^5$

4. $f(x) = \displaystyle\int_0^{x^2} t^2 \mathrm{d}t.$

   answer: $f'(x) = 2x^5$

5. $f(x) = \displaystyle\int_{\ln x}^{e^x} t^3 \mathrm{d}t.$

   answer: $f'(x) = e^{4x} - \frac{(\ln x)^3}{x}$

6. $f(x) = \displaystyle\int_1^x \left(\sqrt{t} - \sqrt[3]{t}\right) \mathrm{d}t.$

   answer: $\sqrt{x} - \sqrt[3]{x}$

7. $f(x) = \displaystyle\int_1^{\frac{1}{x+1}} \sin\left(t^2\right) \mathrm{d}t.$

   answer:

8. $f(x) = \displaystyle\int_1^{\frac{1}{x+1}} \cos\left(t^2\right) \mathrm{d}t.$

   answer: $-\frac{1}{(x+1)^2} \cos\left(\frac{1}{(x+1)^2}\right)$

9. $f(x) = \displaystyle\int_0^{x^3} \cos^2 t \, \mathrm{d}t$

   answer: $3x^2 \cos^2\left(x^3\right)$

Solution. 77.2

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_1^x \left(t - \sqrt{t}\right) \mathrm{d}t\right) = x - \sqrt{x}. \quad \Big| \text{ FTC, part 1}$$

Solution. 77.3 We recall that the Fundamental Theorem of Calculus part 1 states that $\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_a^x h(t)dt\right) = h(x)$ where a is a constant. We can rewrite the integral so it has x as the upper limit:

$$f(x) = \int_x^1 (2 + 1^4)^5 \mathrm{d}t = -\int_1^x (2 + 1^4)^5 \mathrm{d}t \quad .$$

Therefore

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(-\int_1^x (2 + t^4)^5 \mathrm{d}t\right) = -\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_1^x (2 + t^4)^5 \mathrm{d}t\right) \overset{\text{FTC part 1}}{=} -(2 + x^4)^5 \quad .$$

Solution. 77.5

$$\begin{aligned}
f'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\int_{\ln x}^{e^x} t^3 \mathrm{d}t\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(\int_{\ln x}^{0} t^3 \mathrm{d}t + \int_0^{e^x} t^3 \mathrm{d}t\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(-\int_0^{\ln x} t^3 \mathrm{d}t + \int_0^{e^x} t^3 \mathrm{d}t\right).
\end{aligned}$$

The Fundamental Theorem of Calculus part I states that for an arbitrary constant a, $\frac{\mathrm{d}}{\mathrm{d}u}\left(\int_a^u g(t)\mathrm{d}t\right) = g(u)$ (for a continuous g). We use this two compute the two derivatives:

$$\begin{array}{rcll}\frac{\mathrm{d}}{\mathrm{d}x}\left(\int\limits_0^{\ln x} t^3\mathrm{d}t\right) &=& \frac{\mathrm{d}}{\mathrm{d}x}\left(\int\limits_0^{u} t^3\mathrm{d}t\right) & \Big| \text{Set } u=\ln x\\ &=& u^3\cdot\frac{\mathrm{d}u}{\mathrm{d}x} \\ &=& \frac{(\ln x)^3}{x}\\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\int\limits_0^{e^x} t^3\mathrm{d}t\right) &=& \frac{\mathrm{d}}{\mathrm{d}x}\left(\int\limits_0^{w} t^3\mathrm{d}t\right) & \Big| \text{Set } w=e^x\\ &=& w^3\cdot\frac{\mathrm{d}w}{\mathrm{d}x}\\ &=& e^{3x}e^x=e^{4x}.\end{array}$$

Finally, we combine the above computations to a single answer.

$$f'(x) = e^{4x} - \frac{(\ln x)^3}{x}.$$

Solution. 77.6

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_1^x\left(\sqrt{t}-\sqrt[3]{t}\right)\mathrm{d}t = \sqrt{x}-\sqrt[3]{x} \quad \Big| \text{FTC part I}$$

Solution. 77.7

$$\begin{array}{rcll}\frac{\mathrm{d}}{\mathrm{d}x}\int_1^{\frac{1}{x+1}}\sin\left(t^2\right)\mathrm{d}t &=& \frac{\mathrm{d}}{\mathrm{d}x}\int_1^u \sin(t^2)\mathrm{d}t & \Big| u=\frac{1}{x+1}, \text{ use FTC part I, chain rule}\\ &=& \sin\left(u^2\right)\frac{\mathrm{d}u}{\mathrm{d}x}\\ &=& \sin\left(\frac{1}{(x+1)^2}\right)\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x+1}\right)\\ &=& \sin\left(\frac{1}{(x+1)^2}\right)\left(-\frac{1}{(x+1)^2}\right)\\ &=& -\frac{1}{(x+1)^2}\sin\left(\frac{1}{(x+1)^2}\right)\end{array}$$

Solution. 77.8

$$\begin{array}{rcll}\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_1^{\frac{1}{x+1}}\cos\left(t^2\right)\mathrm{d}t\right) &=& \frac{\mathrm{d}}{\mathrm{d}x}\left(\int_1^u\cos\left(t^2\right)\mathrm{d}t\right) & \Big| \text{Set } \frac{1}{x+1}=u\\ &=& \cos\left(u^2\right)\frac{\mathrm{d}}{\mathrm{d}x}(u) & \Big| \text{FTC part I and Chain Rule}\\ &=& -\frac{1}{(x+1)^2}\cos\left(\frac{1}{(x+1)^2}\right)\end{array}$$

9.5 Integration with The Substitution Rule

9.5.1 Substitution in Indefinite Integrals

Problem 78. *Evaluate the indefinite integral. The answer key has not been proofread, use with caution.*

1. $\int (1+3x)^9 \mathrm{d}x.$

answer: $\frac{(1+3x)^{10}}{30} + C$

2. $\int \left(\sqrt{2x+1}\right) \mathrm{d}x.$

answer: $\frac{(2x+1)^{\frac{3}{2}}}{3} + C$

3. $\int (3x+2)^{2.4} \mathrm{d}x.$

answer: $\frac{(3x+2)^{3.4}}{10.2} + C$

4. $\int (x-1)\sqrt{2x-x^2} \mathrm{d}x.$

answer: $-\frac{\left(2x-x^2\right)^{\frac{3}{2}}}{3} + C$

5. $\int x\sqrt{1-x^2} \mathrm{d}x.$

answer: $-\frac{\left(1-x^2\right)^{\frac{3}{2}}}{3} + C$

6. $\int \frac{1+x^2}{\sqrt{3x+x^3}} \mathrm{d}x.$

answer: $\frac{2}{3}\left(3x+x^3\right)^{\frac{1}{2}} + C$

7. $\int (x^2+1)(x^3+3x)^5 \mathrm{d}x.$

answer: $\frac{\left(x^3+3x\right)^6}{18} + C$

8. $\int \frac{x^2}{\sqrt[3]{1+x^3}} \mathrm{d}x.$ *answer:* $\frac{1}{2}\left(1+x^3\right)^{\frac{2}{3}} + C$

9. $\int x^2\left(\sqrt{1+x}\right) \mathrm{d}x.$

answer: $\frac{2}{7}(x-1)^{\frac{7}{2}} - \frac{4}{5}(x-1)^{\frac{5}{2}} + \frac{2}{3}(x-1)^{\frac{3}{2}} + C$

10. $\int x(2x+5)^{2014} \mathrm{d}x.$

answer: $\frac{1}{8064}(2x+5)^{2016} - \frac{1}{1612}(2x+5)^{2015} + C$

11. $\int x^3\left(\sqrt{x^2+1}\right) \mathrm{d}x.$

answer: $\frac{1}{5}\left(x^2+1\right)^{\frac{5}{2}} - \frac{1}{3}\left(x^2+1\right)^{\frac{3}{2}} + C$

12. $\int \sqrt{x}\sin\left(2+x^{\frac{3}{2}}\right) \mathrm{d}x.$

answer: $-\frac{2}{3}\cos\left(2+x^{\frac{3}{2}}\right) + C$

13. $\int \frac{\cos\left(\frac{\pi}{x}\right)}{x^2} \mathrm{d}x.$

answer: $-\frac{\sin\left(\frac{\pi}{x}\right)}{\pi} + C$

14. $\int \csc^2(2t) \mathrm{d}t.$

answer: $-\frac{1}{2}\cot(2t) + C$

15. $\int \sec(5t)\tan(5t) \mathrm{d}t.$

answer: $\frac{1}{5}\sec(5t) + C$

16. $\int \frac{\cos t}{\sin t} \mathrm{d}t.$

answer: $\ln|\sin t| + C$

17. $\int \tan t \mathrm{d}t.$

answer: $-\ln|\cos t| + C$

18. $\int \cot(2t) \mathrm{d}t.$

answer: $\frac{1}{2}\ln|\sin(2t)| + C$

19. $\int \frac{\sin\sqrt{t}}{\sqrt{t}} \mathrm{d}t.$

answer: $-2\cos\left(\sqrt{t}\right) + C$

20. $\int \sec^2 t\tan^3 t \mathrm{d}t.$

answer: $\frac{\tan^4 t}{4} + C$

21. $\int \cos^4 t\sin t \mathrm{d}t.$

answer: $-\frac{\cos^5 t}{5}$

22. $\int \frac{\mathrm{d}t}{\cos^2 t\sqrt{1+\tan t}}.$

answer: $2\sqrt{1+\tan t} + C$

23. $\int \sqrt{\cot t}\csc^2 t \mathrm{d}t.$

answer: $-\frac{2}{3}\cot^{\frac{3}{2}} t + C$

24. $\int \sin t\sec^2(\cos t) \mathrm{d}t.$

answer: $-\tan(\cos t) + C$

25. $\int \sec^3 t\tan t \mathrm{d}t.$

answer: $\frac{\sec^3 t}{3} + C$

26. $\int t\sin\left(t^2\right) \mathrm{d}t.$

answer: $-\frac{1}{2}\cos\left(t^2\right) + C$

Solution. 78.1 We present two solution variants. The variants are equivalent. The only difference between them is that they use two interchangeable notations for differentials. Both variants are acceptable both when taking tests and writing scientific texts.

Variant I

$$\begin{array}{rcl|l} \int (1+3x)^9 \mathrm{d}x & = & \int (1+3x)^9 \frac{\mathrm{d}(3x)}{3} & \begin{array}{rcl} \multicolumn{3}{l}{\text{Set}} \\ u & = & 1+3x \\ \mathrm{d}u & = & 3\mathrm{d}x \\ \mathrm{d}x & = & \frac{1}{3}\mathrm{d}u \end{array} \\ & = & \int u^9 \frac{\mathrm{d}u}{3} & \\ & = & \frac{1}{3}\int u^9 \mathrm{d}u & \\ & = & \frac{1}{30}u^{10} + C = \frac{(1+3x)^{10}}{30} + C. & \end{array}$$

Variant II This variant is equivalent to the previous but uses the differential notation.

$$\begin{array}{rcl|l} \int (1+3x)^9 \mathrm{d}x & = & \int (1+3x)^9 \frac{\mathrm{d}(3x)}{3} & \text{differentials are linear: } \mathrm{d}(3x) = (3x)'\mathrm{d}x = 3\mathrm{d}x \\ & = & \int (1+3x)^9 \frac{\mathrm{d}(1+3x)}{3} & \text{differentials don't change when we add constants} \\ & = & \frac{1}{3}\int u^9 \mathrm{d}u & \text{Set } u = 1+3x \\ & = & \frac{1}{30}u^{10} + C = \frac{(1+3x)^{10}}{30} + C. & \end{array}$$

Problem 79. *Evaluate the integral. The answer key has not been proofread, use with caution.*

1. $\int \frac{dx}{3x+5}$.

answer: $\frac{1}{3}\ln|3x+5|+C$

2. $\int \frac{dx}{2-3x}$.

answer: $-\frac{1}{3}\ln|2-3x|+C$

3. $\int e^x \cos(e^x) dx$.

answer: $\sin(e^x)+C$

4. $\int \frac{(\ln x)^3}{x} dx$.

answer: $\frac{(\ln x)^4}{4}+C$

5. $\int e^x \left(\sqrt{e^x+1}\right) dx$

answer: $\frac{2}{3}(e^x+1)^{\frac{3}{2}}+C$

6. $\int e^x \sqrt{1-e^x} dx$.

answer: $-\frac{2}{3}(1-e^x)^{\frac{3}{2}}+C$

7. $\int e^{\sin t}\cos t dt$.

answer: $e^{\sin t}+C$

8. $\int e^{\cot x}\csc^2 x dx$.

answer: $-e^{\cot x}+C$

9. $\int \frac{x}{1+x^2} dx$.

answer: $\frac{1}{2}\ln\left(x^2+1\right)+C$

10. $\int \frac{x}{2+3x^2} dx$.

answer: $\frac{1}{6}\ln\left(x^2+\frac{2}{3}\right)+C$

11. $\int \frac{x}{\sqrt{1-x^2}} dx$.

answer: $-\sqrt{1-x^2}+C$

12. $\int \frac{\cos(\ln x)}{x} dx$.

answer: $\sin(\ln x)+C$

13. $\int \frac{\sin(\ln x)}{x} dx$.

answer: $-\cos(\ln x)+C$

14. $\int \frac{\sin(2x)}{2+\cos^2 x} dx$.

answer: $-\ln(2+\cos^2 x)+C$

15. $\int \frac{\cos x}{\sin x} dx$

answer: $\ln|\sin x|+C$

16. $\int \cot x dx$.

answer: $\ln|\sin x|+C$

17. $\int \cot\left(\frac{x}{2}\right) dx$

answer: $2\ln\left|\sin\left(\frac{x}{2}\right)\right|+C$

18. $\int \tan(2x) dx$.

answer: $-\frac{1}{2}\ln|\cos(2x)|+C$

19. $\int \frac{x^4+3x}{x^2} dx$

answer: $\frac{x^3}{3}+3\ln|x|+C$

20. $\int x^2 e^{x^3} dx$

answer: $\frac{1}{3}e^{x^3}+C$

21. $\int \frac{\arctan x}{1+x^2} dx$.

answer: $\frac{(\arctan x)^2}{2}+C$

Solution. 79.5.

$$
\begin{array}{rcll}
\int e^x\sqrt{e^x+1}\, dx & = & \int \sqrt{e^x+1}\, d\left(e^x\right) & \\
& = & \int \sqrt{e^x+1}\, d\left(e^x+1\right) & \Big| \text{ Set } u=e^x+1 \\
& = & \int \sqrt{u}\, du & \\
& = & \frac{2}{3}u^{\frac{3}{2}}+C & \\
& = & \frac{2}{3}\left(e^x+1\right)^{\frac{3}{2}}+C &
\end{array}
$$

Solution. 79.14

$$
\begin{array}{rcl|l}
\int \frac{\sin(2x)}{2+\cos^2 x}dx & = & \int \frac{2\cos x\sin x dx}{2+\cos^2 x} & \text{use } \sin(2x)=2\sin x\cos x \\
& = & \int \frac{2\cos x d(-\cos x)}{2+\cos^2 x} & \text{use } d(\cos x)=-\sin x dx \\
& = & -\int \frac{2u d(u)}{2+u^2} & \text{set } u=\cos x \\
& = & -\int \frac{d\left(2+u^2\right)}{2+u^2} & \text{use } d(u^2+2)=2u du \\
& = & -\int \frac{dz}{z} & \text{set } z=2+u^2 \\
& = & -\ln|z|+C & \text{Substitute back } z=u^2+2 \\
& & & \quad u^2+2 \text{ is positive} \\
& = & -\ln(u^2+2)+C & \quad \Rightarrow \text{omit the abs. value} \\
& & & \quad \text{Substitute back } u=\cos x \\
& = & -\ln(\cos^2 x+2)+C. &
\end{array}
$$

Solution. 79.13

$$\begin{aligned}\int \frac{\sin(\ln x)}{x}\mathrm{d}x &= \int \sin(\ln x)\mathrm{d}\left(\ln x\right) && \Big|\ u=\ln x\\ &= \int \sin u \mathrm{d}u\\ &= -\cos u + C\\ &= -\cos(\ln x)+C\end{aligned}$$

9.5.2 Substitution in Definite Integrals

Problem 80. *Evaluate the definite integral. The answer key has not been proofread, use with caution.*

1. $\displaystyle\int_{e}^{e^3} \frac{\mathrm{d}x}{x\sqrt[3]{\ln x}}$.

   answer: $\frac{3}{2}\left(\sqrt[3]{9}-1\right)$

2. $\displaystyle\int_0^1 xe^{-x^2}\mathrm{d}x$.

   answer: $\frac{1-e^{-1}}{2}$

3. $\displaystyle\int_0^1 \frac{e^x+1}{e^x+x}\mathrm{d}x$.

   answer: $\ln(e+1)$

4. $\displaystyle\int_1^2 \frac{x}{2x^2+1}\mathrm{d}x$.

   answer: $\frac{1}{4}\ln 3$

5. $\displaystyle\int_{-3}^{-2} \frac{x}{1-x^2}\mathrm{d}x$.

   answer: $\left[-\frac{1}{2}\ln\left|1-x^2\right|\right]_{-3}^{-2}=\frac{1}{2}\ln\left(\frac{8}{3}\right)$

6. $\displaystyle\int_{-3}^{-2} \frac{3x}{2-x^2}\mathrm{d}x$.

   answer: $\left[-\frac{3}{2}\ln\left|2-x^2\right|\right]_{-3}^{-2}=\frac{3}{2}\ln\frac{7}{2}$

7. $\displaystyle\int_0^{\frac{1}{4}} \frac{x}{\sqrt{1-3x^2}}\mathrm{d}x$.

   answer: $\frac{1}{3}\left(1-\sqrt{\frac{13}{16}}\right)$

Solution. 80.4

$$\begin{aligned}\int_1^2 \frac{x}{2x^2+1}\mathrm{d}x &= \int_{x=1}^{x=2} \frac{\frac{1}{4}\mathrm{d}(2x^2)}{2x^2+1} = \frac{1}{4}\int_{x=1}^{x=2}\frac{\mathrm{d}(2x^2+1)}{2x^2+1} && \Big|\ \text{Set } u=2x^2+1\\ &= \frac{1}{4}\int_{\substack{x=1\\u=3}}^{\substack{x=2\\u=9}} \frac{\mathrm{d}u}{u} = \frac{1}{4}\left[\ln u\right]_3^9 = \frac{1}{4}\left(\ln 9-\ln 3\right)=\frac{\ln 3}{4}.\end{aligned}$$

Solution. 80.5

$$
\int_{-3}^{-2} \frac{x}{1-x^2}\mathrm{d}x = \int_{\substack{x=-3\\u=-8}}^{\substack{x=-2\\u=-3}} \frac{1}{u}\left(-\frac{1}{2}\mathrm{d}u\right) \qquad \left|\ \begin{array}{rcl} u &=& 1-x^2\\ \mathrm{d}u &=& -2x\mathrm{d}x\\ x\mathrm{d}x &=& -\frac{1}{2}\mathrm{d}u \end{array}\right.
$$

$$
\begin{aligned}
&= -\frac{1}{2}\left[\ln|u|\right]_{-8}^{-3}\\
&= -\frac{1}{2}\left(\ln|3|-\ln|8|\right)\\
&= \frac{\ln\left|\frac{8}{3}\right|}{2}
\end{aligned}
$$

Solution. 80.6

$$
\begin{aligned}
\int_{-3}^{-2}\frac{3x}{2-x^2}\mathrm{d}x &= \int_{-3}^{-2}\frac{3\frac{\mathrm{d}\left(x^2\right)}{2}}{2-x^2}\mathrm{d}x\\
&= \frac{3}{2}\int_{-3}^{-2}\frac{-\mathrm{d}\left(-x^2\right)}{2-x^2}\\
&= -\frac{3}{2}\int_{-3}^{-2}\frac{\mathrm{d}\left(-x^2\right)}{2-x^2}\\
&= -\frac{3}{2}\int_{x=-3}^{x=-2}\frac{\mathrm{d}\left(2-x^2\right)}{2-x^2} && \left|\ \text{Set } 2-x^2=u\right.\\
&= -\frac{3}{2}\int_{x=-3,u=-7}^{x=-2,u=-2}\frac{\mathrm{d}u}{u}\\
&= -\frac{3}{2}\left[\ln|u|\right]_{-7}^{-2}\\
&= -\frac{3}{2}(\ln 2-\ln 7)\\
&= \frac{3}{2}\ln\left(\frac{7}{2}\right).
\end{aligned}
$$

10 First Applications of Integration

10.1 Area Between Curves

Problem 81. 1. *Find the area of the region bounded by the curves $y = 2x^2$ and $y = 4 + x^2$.*

answer: $\frac{32}{3}$

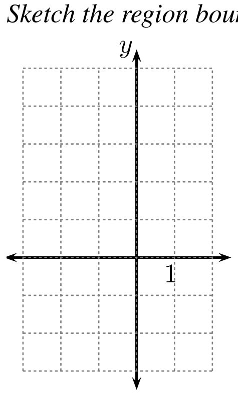
2. *Find the area of the region bounded by the curves $x = 4 - y^2$ and $y = 2 - x$.*

answer: $\frac{9}{2}$

3. *Find the area of the region bounded by the curves $y = x^2$ and $y = 2x^2 + x - 2$.*

answer: $\frac{9}{2}$

4. • *Sketch the region bounded by the curves $y = x^2$ and $y = 2x^2 + x - 2$.*

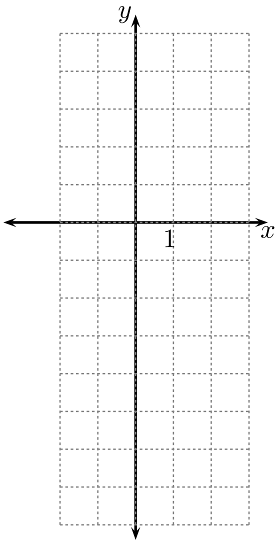


- *Find the area of the region.*

answer: $\frac{9}{2}$

5.

- *Sketch the region bounded by the curves $y = -x^2 + 2x - 1$ and $y = -2x^2 + 3x + 1$. Make sure to indicate the points where the curves intersect.*



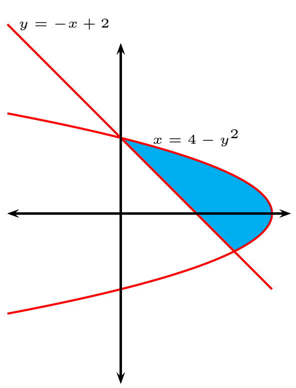
- *Find the area of the region.*

Solution. 81.2. $x = 4 - y^2$ is a parabola (here we consider x as a function of y). $y = -x + 2$ implies that $x = 2 - y$ and so the two curves intersect when

$$\begin{array}{rcl} 4 - y^2 &=& 2 - y \\ -y^2 + y + 2 &=& 0 \\ -(y+1)(y-2) &=& 0 \\ y &=& -1 \text{ or } 2 \end{array} .$$

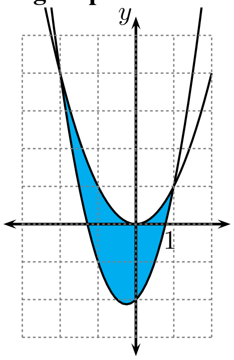
As $x = 2 - y$, this implies that $x = 0$ when $y = 2$ and $x = 3$ when $y = -1$, or in other words the points of intersection are $(0, 2)$ and $(3, -1)$. Therefore we the region is the one plotted below. Integrating with respect to y, we get that the area is

$$\begin{array}{rcl} A &=& \displaystyle\int_{-1}^{2} \left|4 - x^2 - (-x+2)\right| \mathrm{d}y = \int_{-1}^{2} \left(-y^2 + y + 2\right) \mathrm{d}y \\ &=& \displaystyle\left[-\frac{y^3}{3} + \frac{y^2}{2} + 2y\right]_{-1}^{2} = -\frac{8}{3} + 2 + 4 - \left(-\frac{(-1)^3}{3} + \frac{(-1)^2}{2} - 2\right) \\ &=& \dfrac{9}{2} \end{array} .$$



Solution. 81.4

Region plot.



The intersection between the two parabolas are found via

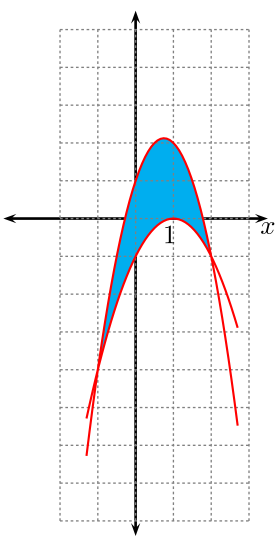
$$\begin{array}{rcl} x^2 &=& 2x^2+x-2 \\ x^2+x-2 &=& 0 \\ (x-1)(x+2) &=& 0 \\ x=1 & & x=-2 \\ y=1 & & y=4. \end{array}$$

Area of the region.

$$\begin{array}{rcl} A &=& \displaystyle\int_{-2}^{1} \left|x^2-(2x^2+x-2)\right| \mathrm{d}x \qquad \Big| \; x^2 > (2x^2+x-2) \text{ for } x\in[-2,1] \text{ (from plot)} \\ &=& \displaystyle\int_{-2}^{1} \left(x^2-(2x^2+x-2)\right) \mathrm{d}x \\ &=& \left[-\frac{1}{3}x^3-\frac{1}{2}x^2+2x\right]_{-2}^{1} \\ &=& \frac{9}{2}. \end{array}$$

Solution. 81.5

Region plot.



The intersections between the two parabolas are found via

$$\begin{array}{rcl} -2x^2+3x+1 &=& -x^2+2x-1 \\ -x^2+x+2 &=& 0 \\ -(x+1)(x-2) &=& 0 \\ x=-1 & \text{or} & x=2 \\ y=-4 & & y=-1. \end{array}$$

Area of the region.

$$
\begin{aligned}
A &= \int_{-1}^{2} \left|-2x^2+3x+1-(-x^2+2x-1)\right| \mathrm{d}x & \Bigg| \quad & -2x^2+3x+1 > -x^2+2x-1 \\
& & & \text{for } x \in [-1,2] \text{ (from plot)} \\
&= \int_{-1}^{2} \left(-2x^2+3x+1-(-x^2+2x-1)\right) \mathrm{d}x \\
&= \int_{-1}^{2} \left(-x^2+x+2\right) \mathrm{d}x \\
&= \left[-\frac{1}{3}x^3+\frac{1}{2}x^2+2x\right]_{-1}^{2} \\
&= \left(-\frac{1}{3}2^3+\frac{1}{2}2^2+2\cdot 2\right)-\left(-\frac{1}{3}(-1)^3+\frac{1}{2}(-1)^2+2(-1)\right) \\
&= \frac{9}{2}.
\end{aligned}
$$

10.2 Volumes of Solids of Revolution

10.2.1 Problems Geared towards the Washer Method

Problem 82. 1. *Consider the region bounded by the curves $y = 2x^2 - x + 1$ and $y = x^2 + 1$. What is the volume of the solid obtained by rotating this region about the line $x = 0$?*

answer: $\frac{2}{5}\pi$.

2. *Consider the region bounded by the curves $y = 1 - x^2$ and $y = 0$. What is the volume of the solid obtained by rotating this region about the line $y = 0$?*

answer: $\frac{16\pi}{15}$

3. *Consider the region bounded by the curves $y = x^2$ and $x = y^2$. What is the volume of the solid obtained by rotating this region about the line $x = 2$?*

answer: $\frac{31\pi}{30}$

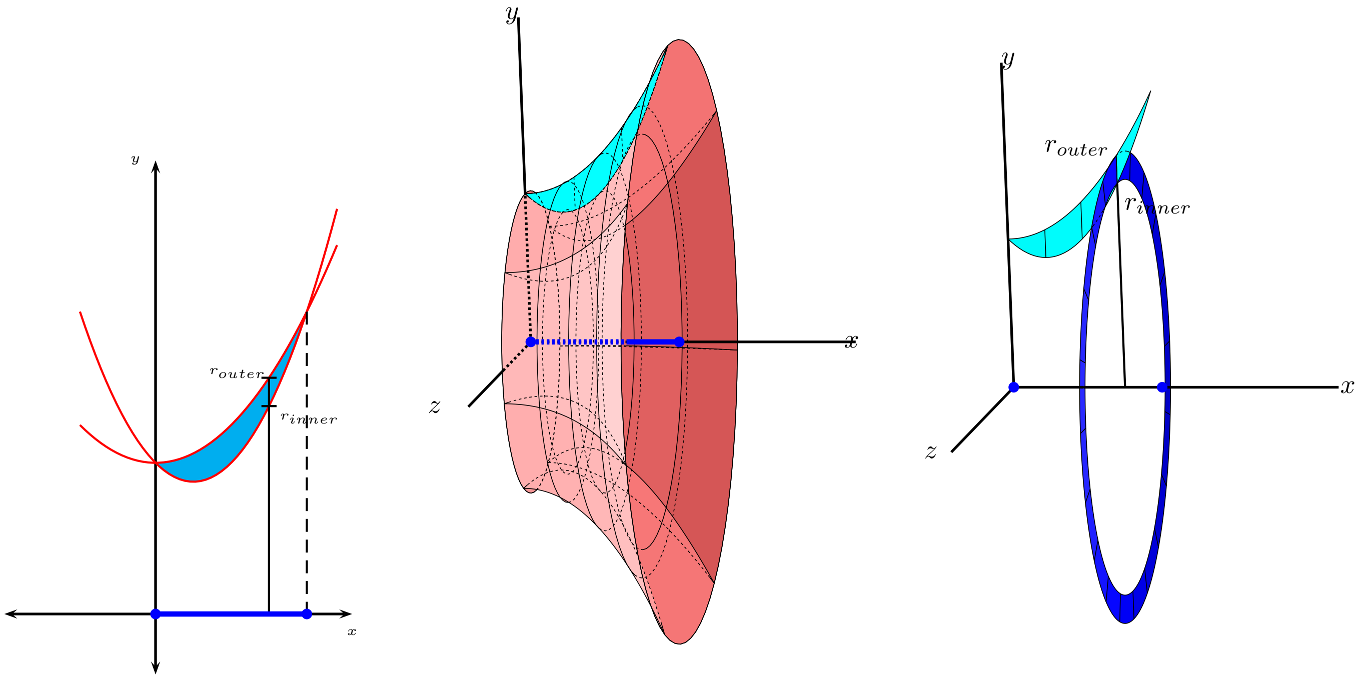
4. *Set up* BUT DO NOT EVALUATE *an integral to calculate the volume of the solid obtained by rotating the region bounded by $y = -x^2 + 2$ and $y = 0$ about the given line.*

   - *The x axis.*
   - *The line $y = -3$.*

5. *Set up* BUT DO NOT EVALUATE *an integral to calculate the volume of the solid obtained by rotating the region bounded by $y = -x^2 + 1$ and $y = 0$ about the given line.*

   - *The x axis.*
   - *The line $y = -4$.*

Solution. 82.1 First, plot $y = 2x^2 - x + 1$ and $y = x^2 + 1$.



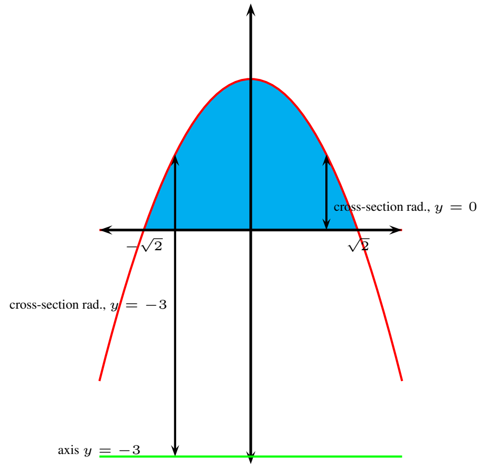
The two curves intersect when

$$\begin{aligned} 2x^2 - x + 1 &= x^2 + 1 \\ x^2 - x &= 0 \\ x(x-1) &= 0 \\ x = 0 &\text{ or } x = 1. \end{aligned}$$

Therefore the two points of intersection have x-coordinates between $x = 0$ and $x = 1$. Therefore we need we need to integrate the volumes of washers with inner radii $r_{inner} = 2x^2 - x + 1$, outer radii $r_{outer} = x^2 + 1$ and infinitesimal heights $\mathrm{d}x$. The volume of an individual infinitesimal washer is then $\pi(r_{outer}^2 - r_{inner}^2)\mathrm{d}x$

$$\begin{aligned} V &= \int_0^1 \pi \left(\left(x^2+1\right)^2 - \left(2x^2 - x + 1\right)^2 \right) \mathrm{d}x \\ &= \pi \int_0^1 \left(-3x^4 + 4x^3 - 3x^2 + 2x\right) \mathrm{d}x \\ &= \pi \left[-\frac{3}{5}x^5 + x^4 - x^3 + x^2 \right]_0^1 \\ &= \frac{2}{5}\pi. \end{aligned}$$

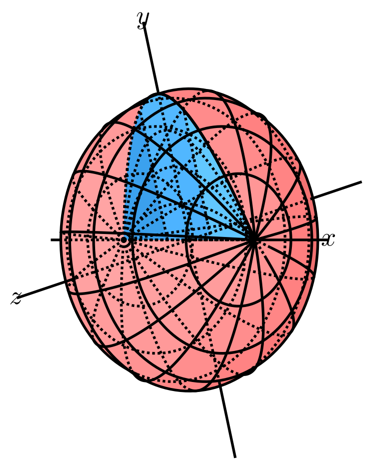
Solution. 82.4 First, we plot the 2d region. The two curves intersect when $-x^2 + 2 = 0$, i.e., when $x = \pm\sqrt{2}$.



Rotation about $y = 0$**.**

Unless explicitly stated in the problem, a 3d plot of the solid is not required in the solution. Nevertheless generating such a plot helps to understand the problem.

To generate a 3d plot of the solid, we draw the circular cross-sections of the solid of revolution. By hand, this can be done roughly by drawing ovals (circles look like ovals when observed at an angle) centered at the axis about which we revolve the 2d-region. We include a computer-generated plot below; the plot's precision is above what is expected on an exam.

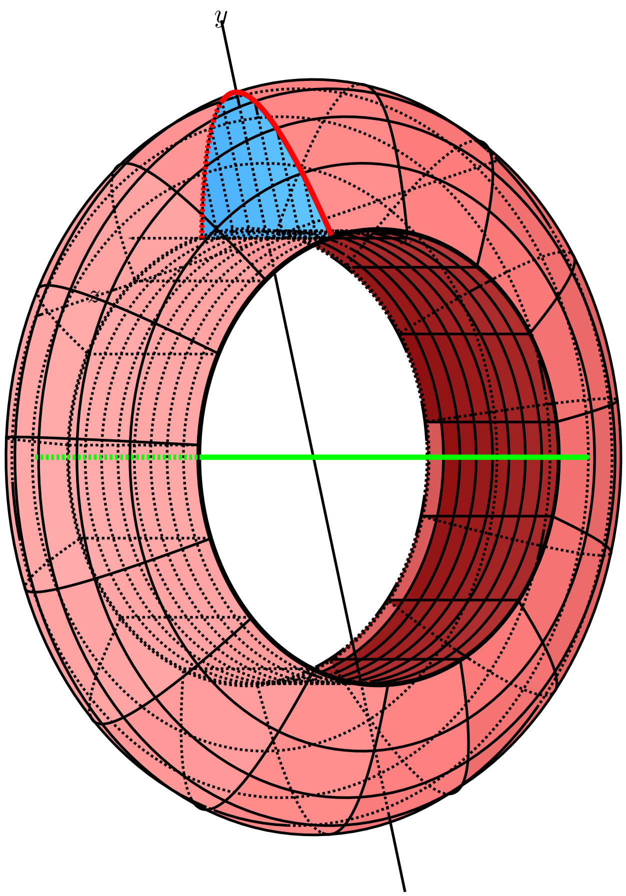


The volume of a solid (and in particular, of a solid of revolution) is computed by integrating the area $A(x) = \pi(\text{radius cross-section})^2 = \pi(-x^2+2)^2$ of the cross-section of the solid. Therefore the volume V equals

$$\begin{array}{rcll} V &=& \displaystyle\int_a^b A(x)\mathrm{d}x & \\ &=& \displaystyle\int_{-\sqrt{2}}^{\sqrt{2}} \pi(-x^2+2)^2\mathrm{d}x & \\ &=& \pi\left[\frac{1}{5}x^5 - \frac{4}{3}x^3 + 4x\right]_{-\sqrt{2}}^{\sqrt{2}} & \left|\ \text{step not required by problem}\right. \\ &=& \pi\dfrac{64}{15}\sqrt{2} & \left|\ \text{step not required by problem.}\right. \end{array}$$

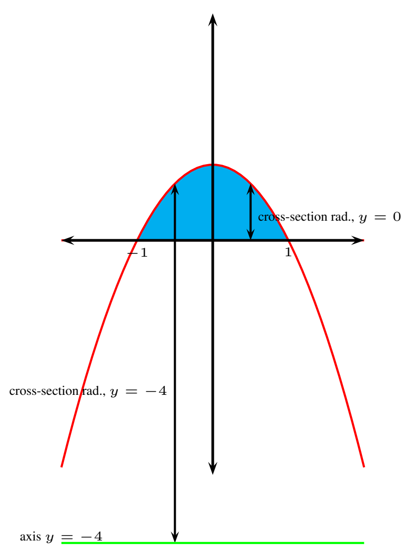
Rotation about $y = -3$**.** The cross-section of this solid of revolution is a washer with inner radius 3 and outer radius $-x^2 + 2 - (-3) = 5 - x^2$. Therefore the area of the cross-section is $\pi(5-x^2)^2 - \pi 3^2$ and the volume is computed via

$$\begin{array}{rcll} V &=& \displaystyle\int_a^b A(x)\mathrm{d}x & \\ &=& \displaystyle\int_{-\sqrt{2}}^{\sqrt{2}} \pi\left((5-x^2)^2 - 3^2\right)\mathrm{d}x & \\ &=& \pi\left[\frac{1}{5}x^5 - \frac{10}{3}x^3 + 16x\right]_{-\sqrt{2}}^{\sqrt{2}} & \left|\ \text{step not required by problem}\right. \\ &=& \pi\dfrac{304}{15}\sqrt{2} & \left|\ \text{step not required by problem.}\right. \end{array}$$

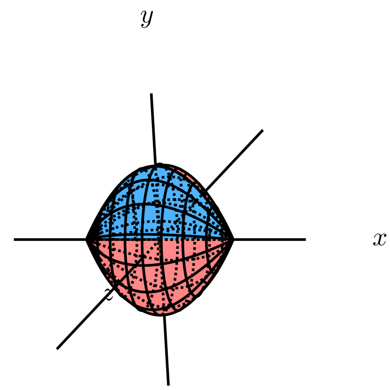


Solution. 82.5

First, we plot the 2d region. The two curves intersect when $-x^2 + 1 = 0$, i.e., when $x = \pm 1$.



Rotation about $y = 0$**.**

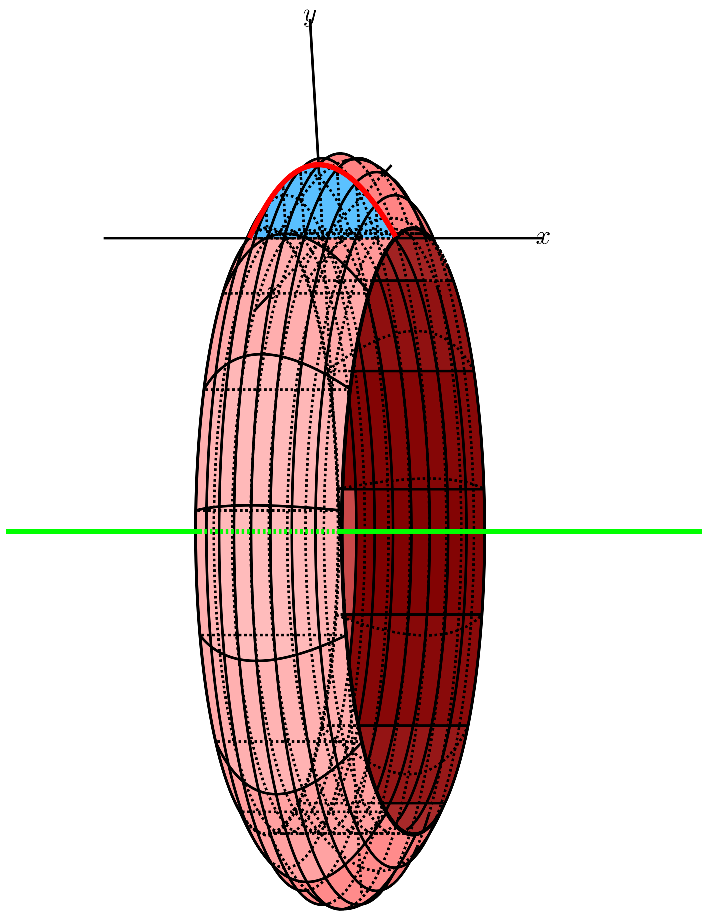


The volume of a solid (and in particular, of a solid of revolution) is computed by integrating the area $A(x) = \pi(\text{radius cross-section})^2 = \pi(-x^2+1)^2$ of the cross-section of the solid. Therefore the volume V equals

$$\begin{array}{rcll} V &=& \displaystyle\int_a^b A(x)\mathrm{d}x & \\ &=& \displaystyle\int_{-1}^1 \pi(-x^2+1)^2 \mathrm{d}x & \\ &=& \pi\left[\frac{1}{5}x^5 - \frac{2}{3}x^3 + x\right]_{-1}^1 & \left| \text{step not required by problem} \right. \\ &=& \pi\dfrac{16}{15} & \left| \text{step not required by problem.} \right. \end{array}$$

Rotation about $y = -4$. The cross-section of this solid of revolution is a washer with inner radius 4 and outer radius $-x^2+1-(-4) = 5-x^2$. Therefore the area of the cross-section is $\pi(5-x^2)^2 - \pi 4^2$ and the volume is computed via

$$\begin{array}{rcll} V &=& \displaystyle\int_a^b A(x)\mathrm{d}x & \\ &=& \displaystyle\int_{-1}^1 \pi\left((5-x^2)^2 - 4^2\right)\mathrm{d}x & \\ &=& \pi\left[\frac{1}{5}x^5 - \frac{10}{3}x^3 + 9x\right]_{-1}^1 & \left| \text{step not required by problem} \right. \\ &=& \dfrac{176}{15}\pi & \left| \text{step not required by problem.} \right. \end{array}$$



10.2.2 Problems Geared towards the Cylindrical Shells Method

Problem 83. 1. *Consider the region bounded by the curves $y = \sqrt{x}$, $x = 0$, $y = 2$. Use the method of cylindrical shells to find the volume of the solid obtained by rotating this region about the x-axis.*

answer: 8π

2. *Consider the region bounded by the curves $y = x^2$ and $y = 2 - x^2$. Use the method of cylindrical shells to find the volume of the solid obtained by rotating this region about the line $x = 1$.*

answer: $16\frac{\pi}{3}$