Precalculus
Homework Lecture 10

1. Evaluate the difference quotient and simplify your answer.

(a) $\dfrac{f(2+h)-f(2)}{h}$, where $f(x)=x^2-x-1$.

answer: $h+3$

(b) $\dfrac{f(a+h)-f(a)}{h}$, where $f(x)=x^2$.

answer: $h+2a$

(c) $\dfrac{f(a+h)-f(a)}{h}$, where $f(x)=x^3$.

answer: h^2+3a^2+3ah

(d) $\dfrac{f(a+h)-f(a)}{h}$, where $f(x)=x^4$.

answer: $6a^2h+4ah^2+h^3+4a^3$

(e) $\dfrac{f(x)-f(a)}{x-a}$, where $f(x)=\frac{1}{x}$.

answer: $-\frac{1}{ax}$.

(f) $\dfrac{f(x)-f(1)}{x-1}$, where $f(x)=\frac{x-1}{x+1}$.

answer: $\frac{1}{x+1}$.

2. Find the implied domain of the function.

(a) $f(x)=\dfrac{x+4}{x^2-4}$.

answer: $x\neq\pm 2$,
alternatively:
$x\in(-\infty,-2)\cup(-2,2)\cup(2,\infty)$

(b) $f(x)=\dfrac{2x^3-5}{x^2+5x+6}$.

answer: $x\neq -2,-3$,
alternatively:
$x\in(-\infty,-3)\cup(-3,-2)\cup(-2,\infty)$

(c) $f(t)=\sqrt[3]{3t-1}$.

answer: $x\in\mathbb{R}$ (the domain is all real numbers)

(d) $g(t)=\sqrt{5-t}-\sqrt{1+t}$.

answer: $x\in[-1,5]$.

(e) $h(x)=\dfrac{1}{\sqrt[6]{x^2-7x}}$.

answer: $x\in(-\infty,0)\cup(7,\infty)$.

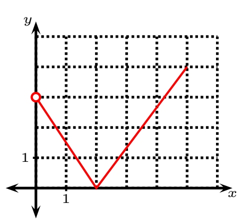
(f) $f(u)=\dfrac{u+1}{1+\frac{1}{u+1}}$.

answer: $u\neq -1,-2$ or
$u\in(-\infty,-2)\cup(-2,-1)\cup(-1,\infty)$..

(g) $F(x)=\sqrt{10-\sqrt{x}}$.

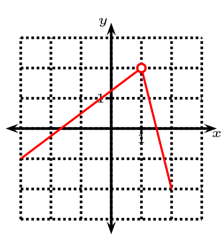
answer: $x\in[0,100]$

3. Write down a formula for a function whose graphs is given below. The graphs are up to scale. Please note that there is more than one way to write down a correct answer.



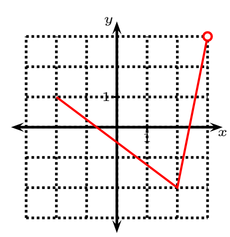
(a)

answer: $y=\begin{cases} -\frac{3}{2}x+3 & \text{if } 0<x\leq 2\\ \frac{4}{3}x-\frac{8}{3} & \text{if } 2<x\leq 6\end{cases}$



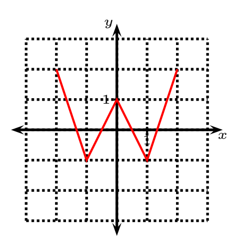
(b)

answer: $y=\begin{cases} \frac{3}{4}x+\frac{5}{4} & \text{if } -3\leq x<1\\ -4x+6 & \text{if } 1<x\leq 2\end{cases}$



(c)

answer: $y=\begin{cases} -\frac{3}{4}x-\frac{1}{2} & \text{if } -2\leq x<2\\ 5x-12 & \text{if } 2\leq x<3\end{cases}$



(d)

answer: $y=\begin{cases} -3x-4 & \text{if } -2\leq x<-1\\ 2x+1 & \text{if } -1\leq x<0\\ -2x+1 & \text{if } 0\leq x<1\\ 3x-4 & \text{if } 1\leq x\leq 2\end{cases}$

4. Decide whether the function f is even, odd, neither or both. Give a detailed explanation. The answer key has not been fully proofread, use with caution.

(a) $f(x) = x + 3x^3$

answer: odd

(b) $f(x) = x^2 + 3$

answer: even

(c) $f(x) = x^2 + x + 1$.

answer: neither

(d) $f(x) = 0$.

answer: both

(e) $f(x) = \frac{1}{x}$.

answer: odd

(f) $f(x) = \begin{cases} 5x+4 & \text{if } x > 0 \\ 5x-4 & \text{if } x < 0 \end{cases}$.

answer: odd

(g) $f(x) = \frac{1-x}{1+x} + \frac{1+x}{1-x}$.

answer: even

(h) $f(x) = \frac{1-x}{1+x} - \frac{1+x}{1-x}$.

answer: odd

(i) $f(x) = \frac{x-1}{x}$.

answer: neither

(j) $f(x) = x - \frac{1}{x}$.

answer: odd

(k) $f(x) = |x|$.

answer: even

(l) $f(x) = \sqrt{|x|}$.

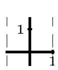
answer: even

Solution. 4.g.

To check whether a function f is even, odd or neither, we need to compare $f(x)$ to $f(-x)$. We have that

$$
\begin{aligned}
f(x) &= \frac{1-x}{1+x} + \frac{1+x}{1-x} \\
&= \frac{(1-x)(1-x)}{(1+x)(1-x)} + \frac{(1+x)(1+x)}{(1-x)(1+x)} \\
&= \frac{(1-x)^2 + (1+x)^2}{(1+x)(1-x)} \\
&= \frac{(1-\cancel{2x}+x^2) + (1+\cancel{2x}+x^2)}{1-x^2} \\
&= \frac{2+2x^2}{1-x^2} \\
&\text{Therefore} \\
f(-x) &= \frac{2+2(-x)^2}{1-(-x)^2} \\
&= \frac{2+2x^2}{1-x^2}.
\end{aligned}
$$

Thus we computed that $f(-x) = f(x)$, which shows that the function is even. Even functions have graphs that are symmetric across the y axis; a computer-generated plot of f confirms this symmetry.



Solution. 4.f.

We will show that this piecewise defined function is odd, although each of the individual pieces ($5x+4$ and $5x-4$), viewed over the entire real line, is neither even nor odd.

This problem can be solved both via algebra and graphically.

Solution via algebra. Recall that a function is even when $f(x) = f(-x)$ and odd when $f(-x) = -f(x)$. We have

$$
f(x) = \begin{cases} 5x+4 & \text{if } x > 0 \\ 5x-4 & \text{if } x < 0 \end{cases}
$$

and therefore

$$
\begin{aligned}
f(-x) &= \begin{cases} -5x+4 & \text{if } -x>0 \\ -5x-4 & \text{if } -x<0 \end{cases} && \text{Multiplying inequalities by } -1 \text{ reverses their direction} \\
&= \begin{cases} -5x+4 & \text{if } x<0 \\ -5x-4 & \text{if } x>0 \end{cases} && \text{swap the order of writing the cases} \\
&= \begin{cases} -5x-4 & \text{if } x>0 \\ -5x+4 & \text{if } x<0 \end{cases} \\
&= \begin{cases} -(5x+4) & \text{if } x>0 \\ -(5x-4) & \text{if } x<0 \end{cases} \\
&= -f(x).
\end{aligned}
$$

This shows that the function is odd.

Solution via plotting the function. This graphical solution is slightly informal, but shows a good understanding of the subject and is acceptable (and well perceived by graders) when taking exams.

We recall that a function is even if its graph is symmetric across the y axis and odd if its graph has a half-turn symmetry about the origin of the coordinate system. Plotting $f(x) = \begin{cases} 5x+4 & \text{if } x>0 \\ 5x-4 & \text{if } x<0 \end{cases}$ results in the following graph:

The graph is symmetric relative to rotation at $180°$Â around the origin so $f(x)$ is an odd function.