Calculus I
Lecture 2
Trigonometry Review

Todor Milev
https://github.com/tmilev/freecalc

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Outline

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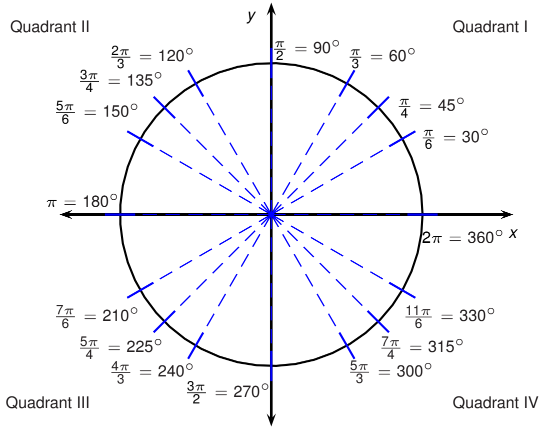
Degrees and radians

- Degrees is a unit for measuring angles, denoted by $^\circ$.
- The relationship between degrees and radians is:

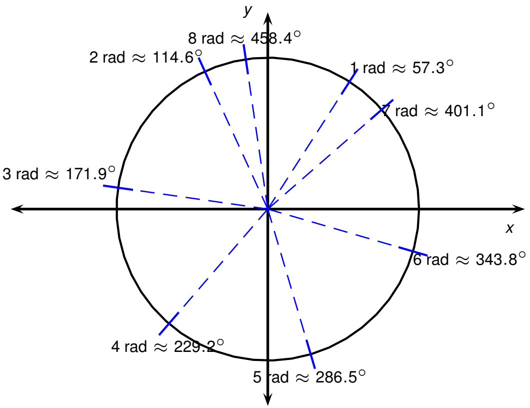
$$
\begin{aligned}
\pi \text{ rad} &= 180^\circ \\
1 \text{rad} &= \frac{180^\circ}{\pi} \approx 57.3^\circ \\
1^\circ &= \frac{\pi}{180} \text{ rad} \approx 0.017 \text{ rad.}
\end{aligned}
$$

- In other words, a half-turn is measured by πrad or 180°.
- Degrees are useful because the most frequently encountered fractions of a half turn are measured by a whole number of degrees.
- If a measurement unit is not specified, it is implied to be radians. For example, in $\sin 5$, the number 5 stands for 5 radians.



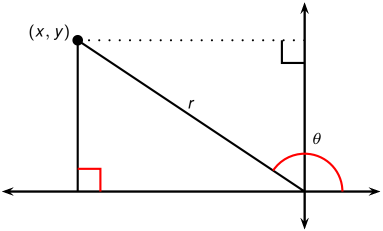
The most frequently encountered angles are given in the table below.

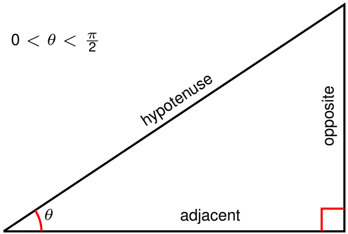
Deg.	0°	30°	45°	60°	90°	120°	135°	150°	180°	270°	360°
Rad.	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π



- Integer quantities of radians are not rational multiples of (the measure of) a half-turn and are not easy to compute with.
- For example to determine in which quadrant is an angle of k radians located one needs to know the numerical value of $\frac{k}{\pi}$, which requires knowledge of π with great numerical accuracy.

Trigonometric Functions and Right Angle Triangles

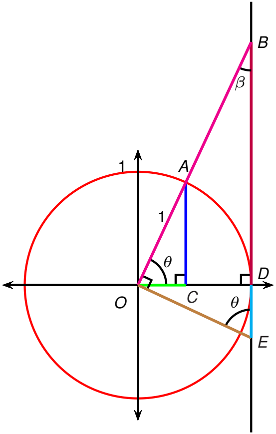




$\cos\theta = \frac{x}{r}$	$\sec\theta = \frac{r}{x}$	$\cos\theta = \frac{\text{adj}}{\text{hyp}}$	$\sec\theta = \frac{\text{hyp}}{\text{adj}}$
$\sin\theta = \frac{y}{r}$	$\csc\theta = \frac{r}{y}$	$\sin\theta = \frac{\text{opp}}{\text{hyp}}$	$\csc\theta = \frac{\text{hyp}}{\text{opp}}$
$\tan\theta = \frac{y}{x}$	$\cot\theta = \frac{x}{y}$	$\tan\theta = \frac{\text{opp}}{\text{adj}}$	$\cot\theta = \frac{\text{adj}}{\text{opp}}$
All angles		Acute angles	

- The trigonometric functions can be defined without requesting that the pt. (x, y) on the terminal arm of the angle lie on the unit circle.
- To do so we rescale by the distance r from the origin.
- The trig functions of acute θ (between 0 and $\frac{\pi}{2}$) can be interpreted as ratios of sides of right angle triangle with angle θ.

Geometric interpretation of all trigonometric functions

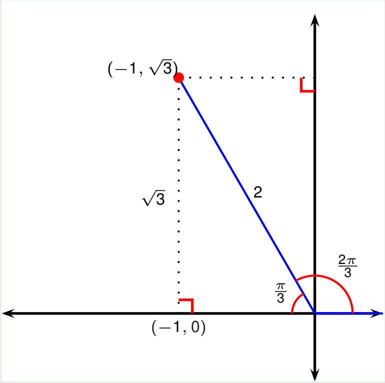


$$\beta = 180^\circ - 90^\circ - \theta$$
$$= 90^\circ - \theta$$
$$\angle OED = 180^\circ - 90^\circ - \beta$$
$$= 90^\circ - (90^\circ - \theta)$$
$$= \theta$$

Fix unit circle, center O, coordinates $(0,0)$. Let $\angle DOB = \theta$. Let OB intersect the circle at point A. Coordinates of A are $(\cos\theta, \sin\theta)$.

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{|AC|}{|OA|} = \frac{|AC|}{1} = |AC|$$
$$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{|OC|}{|OA|} = \frac{|OC|}{1} = |OC|$$
$$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{|BD|}{|OD|} = \frac{|BD|}{1} = |BD|$$
$$\cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{|DE|}{|OD|} = \frac{|DE|}{1} = |DE|$$
$$\sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{|OB|}{|OD|} = \frac{|OB|}{1} = |OB|$$
$$\csc\theta = \frac{\text{hyp}}{\text{opp}} = \frac{|OE|}{|DO|} = \frac{|OE|}{1} = |OE|$$

Example



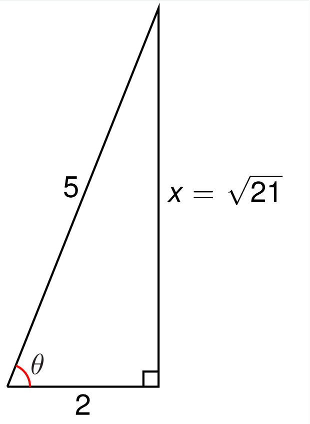
Find the exact values of the trigonometric functions of $\theta = \dfrac{2\pi}{3} = 120^\circ$.

$$\sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2} \quad \cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2} \quad \tan\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{-1} = -\sqrt{3}$$

$$\csc\left(\frac{2\pi}{3}\right) = \frac{2}{\sqrt{3}} \quad \sec\left(\frac{2\pi}{3}\right) = -\frac{2}{1} = -2 \quad \cot\left(\frac{2\pi}{3}\right) = -\frac{1}{\sqrt{3}}$$

Example

If $\cos\theta = \frac{2}{5}$ and $0 < \theta < \frac{\pi}{2}$, find the other five trigonometric functions of θ.

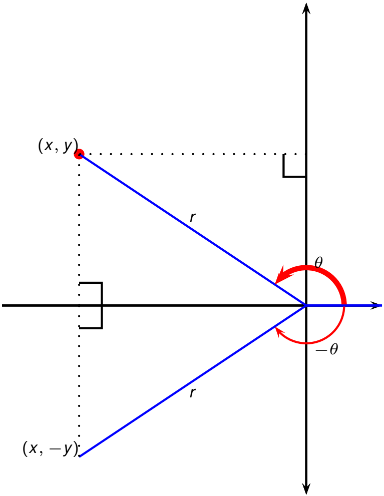


- Label the hypotenuse with length 5 and the adjacent side with length 2.
- Pythagorean theorem: $x^2 + 2^2 = 5^2$.
- Therefore $x^2 = 21$, so $x = \sqrt{21}$.

$$\sin\theta = \frac{\sqrt{21}}{5} \quad \tan\theta = \frac{\sqrt{21}}{2}$$

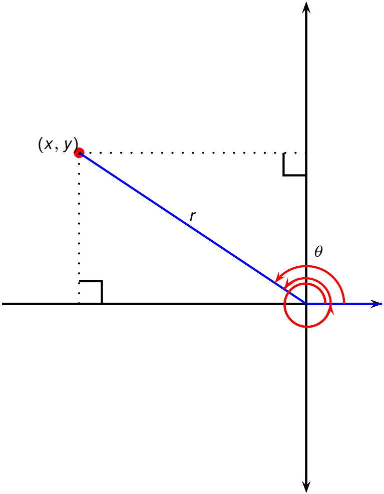
$$\csc\theta = \frac{5}{\sqrt{21}} \quad \sec\theta = \frac{5}{2}$$

$$\cot\theta = \frac{2}{\sqrt{21}}$$



$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{r}{y}$$
$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x}$$
$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

- Positive angles are obtained by rotating counterclockwise.
- Negative angles are obtained by rotating clockwise.
- If (x, y) is on the terminal arm of the angle θ, then $(x, -y)$ is on the terminal arm of $-\theta$.
- $\sin(-\theta) = \frac{-y}{r} = -\frac{y}{r} = -\sin\theta$.
- $\cos(-\theta) = \frac{x}{r} = \cos\theta$.
- sin is an odd function.
- cos is an even function.



$$\sin\theta = \frac{y}{r} \quad \csc\theta = \frac{r}{y}$$
$$\cos\theta = \frac{x}{r} \quad \sec\theta = \frac{r}{x}$$
$$\tan\theta = \frac{y}{x} \quad \cot\theta = \frac{x}{y}$$

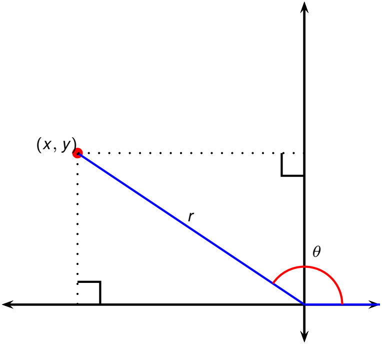
- 2π represents a full rotation.
- $\theta + 2\pi$ has the same terminal arm as θ.
- $\theta + 2\pi$ uses the same point (x, y) and the same length r.
- $\sin(\theta + 2\pi) = \sin\theta$.
- $\cos(\theta + 2\pi) = \cos\theta$.
- We say $\sin$ and $\cos$ are 2π-periodic.

Trigonometric Identities

Definition (Trigonometric Identity)

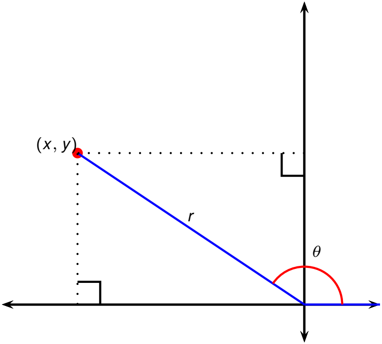
A trigonometric identity is an equality between the trigonometric functions in one or more variables that holds for all values of the involved variables in the domains of all of the expressions.

- By convention, when dealing with trigonometric identities we do not account for the domains of the involved expressions.
- For example, $\frac{\sin\theta}{\sin\theta} = 1$ is considered a valid trigonometric identity, although, when considered as a function, the left hand side is not defined for $\theta \neq 0$.



$$\sin\theta = \frac{y}{r} \quad \csc\theta = \frac{r}{y}$$
$$\cos\theta = \frac{x}{r} \quad \sec\theta = \frac{r}{x}$$
$$\tan\theta = \frac{y}{x} \quad \cot\theta = \frac{x}{y}$$

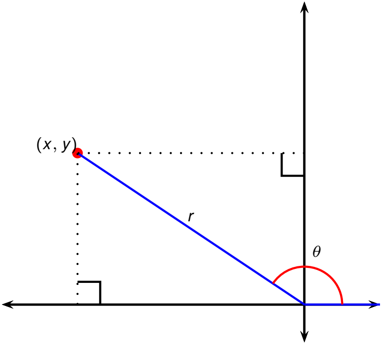
- $\csc\theta = \dfrac{1}{\sin\theta}$
- $\sec\theta = \dfrac{1}{\cos\theta}$
- $\cot\theta = \dfrac{1}{\tan\theta}$
- $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$
- $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$



$$\sin\theta = \frac{y}{r} \quad \csc\theta = \frac{r}{y}$$
$$\cos\theta = \frac{x}{r} \quad \sec\theta = \frac{r}{x}$$
$$\tan\theta = \frac{y}{x} \quad \cot\theta = \frac{x}{y}$$

$$\begin{aligned} & \sin^2\theta + \cos^2\theta \\ = \quad & \frac{y^2}{r^2} + \frac{x^2}{r^2} \\ = \quad & \frac{y^2 + x^2}{r^2} \\ = \quad & \frac{r^2}{r^2} \\ = \quad & 1 \end{aligned}$$

Therefore $\sin^2\theta + \cos^2\theta = 1$.



$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{r}{y}$$
$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x}$$
$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

Example ($\tan^2\theta + 1 = \sec^2\theta$)

Prove the identity
$\tan^2\theta + 1 = \sec^2\theta$.

$$\begin{aligned} \sin^2\theta + \cos^2\theta &= 1 \\ \frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} &= \frac{1}{\cos^2\theta} \\ \tan^2\theta + 1 &= \sec^2\theta \end{aligned}$$

The remaining identities are consequences of the addition formulas:

$$\begin{aligned}\sin(x+y) &= \sin x\cos y + \cos x\sin y\\ \cos(x+y) &= \cos x\cos y - \sin x\sin y\end{aligned}$$

Substitute $-y$ for y, and use the fact that $\sin(-y) = -\sin y$ and $\cos(-y) = \cos y$:

$$\begin{aligned}\sin(x-y) &= \sin x\cos y - \cos x\sin y\\ \cos(x-y) &= \cos x\cos y + \sin x\sin y\end{aligned}$$

The remaining identities are consequences of the addition formulas:

$$\begin{aligned}\sin(x+y) &= \sin x\cos y + \cos x\sin y\\ \cos(x+y) &= \cos x\cos y - \sin x\sin y\end{aligned}$$

To get the double angle formulas, substitute x for y:

$$\begin{aligned}\sin(2x) &= 2\sin x\cos x\\ \cos(2x) &= \cos^2 x - \sin^2 x\end{aligned}$$

Rewrite the second double angle formula in two ways, using $\cos^2 x = 1 - \sin^2 x$ and $\sin^2 x = 1 - \cos^2 x$:

$$\begin{aligned}\cos(2x) &= 2\cos^2 x - 1\\ \cos(2x) &= 1 - 2\sin^2 x\end{aligned}$$

To get the half-angle formulas, solve these equations for $\cos^2 x$ and $\sin^2 x$ respectively.

$$\cos^2 x = \frac{1+\cos(2x)}{2}, \qquad \sin^2 x = \frac{1-\cos(2x)}{2}$$

The remaining identities are consequences of the addition formulas:

$$\begin{array}{rcccc} \sin(x+y) & = & \sin x \cos y & + & \cos x \sin y \\ \cos(x+y) & = & \cos x \cos y & - & \sin x \sin y \end{array}$$

Divide the first equation by the second, and then cancel $\cos x \cos y$ from the top and bottom:

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

Do the same for the subtraction formulas:

$$\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

Here we explicitly permit the use of the Pythagorean identities and the double angle f-las:

$$
\begin{aligned}
\cos^2\theta + \sin^2\theta &= 1\\
\sin(2\theta) &= 2\sin\theta\cos\theta\\
\cos(2\theta) &= \cos^2\theta - \sin^2\theta
\end{aligned}
$$

Example

Prove the trigonometric identity.

$$(\sin\theta + \cos\theta)^2 = 1 + \sin(2\theta)$$

We need to transform both sides to the same expression. In this case, we choose to transform the left hand side to the right:

$$
\begin{aligned}
(\sin\theta + \cos\theta)^2 &= \sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta \quad \Big| \quad (A+B)^2 = A^2 + 2AB + B^2\\
&= 1 + \sin(2\theta)
\end{aligned}
$$

Strategy for proving trigonometric identities

An expression is rational trigonometric if it is written using $\sin\theta, \cos\theta$ and the four arithmetic operations.

Question

Is there a general method for proving all rational trigonometric identities in one variable?

- Given a number of variables and relations between them, there is an algorithm to check whether (rational) expressions in those variables are equal under the given relations.
- Thus, if we pick two variables s and c, and a single relation
$$s^2 + c^2 = 1$$
there is a standard method to verify whether two (rational) expressions in s and c are equal.
- The method is rather cumbersome for a human and is best suited for computers.

Strategy for proving trigonometric identities

An expression is rational trigonometric if it is written using $\sin\theta, \cos\theta$ and the four arithmetic operations.

Question

Is there a general method for proving all rational trigonometric identities in one variable?

- Yes.
- For expressions that depend only on $\sin\theta$ and $\cos\theta$, algebra tells us when two expressions in those are equal.
- Problems depending on $\cos\theta, \sin\theta$ alone will always be doable via easy ad-hoc tricks using
$$\sin^2\theta + \cos^2\theta = 1.$$
- The full method will not be needed in this course.
  - The full method: set $s = \sin\theta, c = \cos\theta$.
  - Check whether the two expressions in s, c are equal under the relation $s^2 + c^2 = 1$. (The method lies outside of present scope).

Strategy for proving trigonometric identities

An expression is rational trigonometric if it is written using $\sin\theta, \cos\theta$ and the four arithmetic operations.

Question

Is there a general method for proving all rational trigonometric identities in one variable?

- To prove a general trigonometric identity:
  - Use angle sum/double angle sum formulas to convert all formulas to trig expression depending only on $\sin\theta$, $\cos\theta$.
  - Use $\sin^2\theta + \cos^2\theta = 1$ to show the two formulas are equal (usage: ad-hoc).
  - You may need to use trig functions of angles smaller than θ, for example $\sin\left(\frac{\theta}{2}\right), \cos\left(\frac{\theta}{2}\right)$.
  - A fraction of θ such that all appearing angles are integer multiples of it will always work.

Example

Prove the identity $\tan\theta + \sec\theta = \dfrac{1+\tan\left(\frac{\theta}{2}\right)}{1-\tan\left(\frac{\theta}{2}\right)}$

All angles here are multiples of $\frac{\theta}{2}$, so set $\varphi = \frac{\theta}{2}$, $\theta = 2\varphi$.

$$
\begin{aligned}
\tan(2\varphi)+\sec(2\varphi) &= \frac{\sin(2\varphi)}{\cos(2\varphi)} + \frac{1}{\cos(2\varphi)}\\
&= \frac{\sin(2\varphi)+1}{\cos(2\varphi)}\\
&= \frac{2\sin\varphi\cos\varphi + \sin^2\varphi + \cos^2\varphi}{\cos^2\varphi - \sin^2\varphi}\\
&= \frac{(\cos\varphi+\sin\varphi)^{\cancel{2}}}{(\cos\varphi-\sin\varphi)\cancel{(\cos\varphi+\sin\varphi)}}\\
&= \frac{(\cos\varphi+\sin\varphi)\frac{1}{\cos\varphi}}{(\cos\varphi-\sin\varphi)\frac{1}{\cos\varphi}} = \frac{1+\frac{\sin\varphi}{\cos\varphi}}{1-\frac{\sin\varphi}{\cos\varphi}}\\
&= \frac{1+\tan\varphi}{1-\tan\varphi}
\end{aligned}
$$

$A^2+2AB+B^2 = (A+B)^2$

$A^2-B^2 = (A-B)(A+B)$

as desired.

Trigonometric equations

- Some problems will not ask you to prove a trigonometric identity, but rather to solve a trigonometric equation.
- Consider the problem of finding all values of x for which $\sin x = \sin(2x) = 2\sin x \cos x$.
- This is not a trigonometric identity - the two sides are different.
- However, there are values for x which the above equality holds.

Example

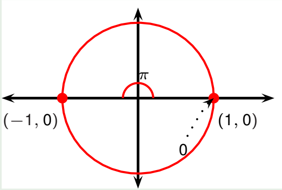
Find all values of θ in the interval $[0, 2\pi]$ such that $\sin\theta = \sin(2\theta)$.

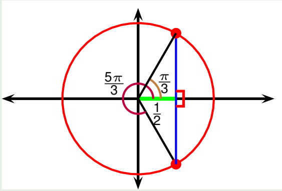
$$\begin{aligned}\sin\theta &= \sin(2\theta)\\ \sin\theta &= 2\sin\theta\cos\theta\\ 0 &= 2\sin\theta\cos\theta - \sin\theta\\ 0 &= \sin\theta(2\cos\theta - 1)\end{aligned}$$

$$\begin{aligned}\sin\theta &= 0\\ \theta &= 0 + 2k\pi\\ &\quad \text{or } \pi + 2k\pi\\ \theta &= 0 \text{ or } 2\pi \text{ or } \pi\end{aligned}$$

or

$$\begin{aligned}2\cos\theta - 1 &= 0\\ \cos\theta &= \tfrac{1}{2}\\ \theta &= \tfrac{\pi}{3} + 2k\pi \text{ or } \tfrac{5\pi}{3} + 2k\pi\\ \theta &= \tfrac{\pi}{3} \text{ or } \tfrac{5\pi}{3}\end{aligned}$$





Strategy for solving trigonometric equations

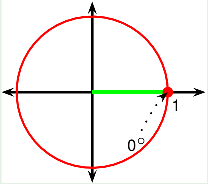
- Suppose we want to solve an algebraic trigonometric equation.
- More precisely, the equation should be an algebraic expressions of the trigonometric functions of a single variable.
- Here is a general strategy for solving such a problem:
  - Using trig identities, rewrite in terms of $\sin x$ and $\cos x$ only.
  - Suppose $x \in [2n\pi, (2n+1)\pi]$.
    - Set $\sin x = \sqrt{1-\cos^2 x}$ (allowed due to restrictions on x).
    - Set $\cos x = u$. Solve the resulting algebraic equation for u.
    - For the found solutions for u, solve $\cos x = u$.
    - Check whether your solutions satisfy $x \in [2n\pi, (2n+1)\pi]$.
  - Suppose $x \in [(2n-1)\pi, 2n\pi]$.
    - Set $\sin x = -\sqrt{1-\cos^2 x}$ (allowed due to restrictions on x).
    - Set $\cos x = u$. Solve the resulting algebraic equation for u.
    - For the found solutions for u, solve $\cos x = u$.
    - Check whether your solutions satisfy $x \in [(2n-1)\pi, 2n\pi]$.
- A similar strategy exists for $u = \sin x$ instead of $u = \cos x$.
- Problems requiring full algorithm may be too hard for Calc exams.

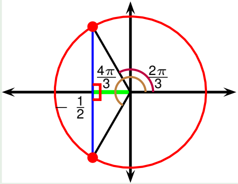
Example

Find all values of θ in the interval $\theta \in [0, 2\pi]$ for which

$$
\begin{aligned}
\cos(2\theta) &= \cos\theta \\
\cos^2\theta - \sin^2\theta - \cos\theta &= 0 && \Big| \text{ Express via } \cos\theta \\
\cos^2\theta - (1 - \cos^2\theta) - \cos\theta &= 0 \\
2\cos^2\theta - \cos\theta - 1 &= 0 && \Big| \text{ Set } \cos\theta = u \\
2u^2 - u - 1 &= 0 \\
(u-1)(2u+1) &= 0
\end{aligned}
$$

$$
\begin{aligned}
u - 1 &= 0 \\
\cos\theta &= 1 \\
\theta &= 0 + 2k\pi \\
\theta &= 0 \text{ or } 2\pi
\end{aligned}
\quad \text{or} \quad
\begin{aligned}
2u + 1 &= 0 \\
\cos\theta &= -\tfrac{1}{2} \\
\theta &= \tfrac{2\pi}{3} + 2k\pi \text{ or } \tfrac{4\pi}{3} + 2k\pi \\
\theta &= \tfrac{2\pi}{3} \text{ or } \tfrac{4\pi}{3}
\end{aligned}
$$





Definition (Complex numbers)

The set of complex numbers $\mathbb{C}$ is defined as the set

$$\{a + bi | a, b - \text{real numbers}\},$$

where the number i is a number for which

$$i^2 = -1 \quad .$$

The number i is called the imaginary unit. By definition, $\sqrt{-1} = i$.

- Complex addition/subtraction

$$(a + bi) \pm (c + di) = (a \pm c) + (b \pm d)i \quad .$$

- Complex multiplication

$$\begin{aligned}(a + bi)(c + di) &= ac + adi + bci + bdi^2 = ac + adi + bci - bd \\ &= (ac - bd) + i(ad + bc)\end{aligned}$$

You will not be tested on the material in the following slide.

Euler's Formula

Theorem (Euler's Formula)

$$e^{ix} = \cos x + i \sin x,$$

where $e \approx 2.71828$ is Euler's/Napier's constant .

Proof.

Recall $n! = 1 \cdot 2 \cdot 3 \cdot \dots \cdot (n-1) \cdot n$. Borrow from Calc II the f-las:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + \frac{(-1)^n x^{2n}}{(2n)!} + \dots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots$$

Euler's Formula

Theorem (Euler's Formula)

$$e^{ix} = \cos x + i \sin x,$$

where $e \approx 2.71828$ *is Euler's/Napier's constant .*

Proof.

Recall $n! = 1 \cdot 2 \cdot 3 \cdot \dots \cdot (n-1) \cdot n$. Borrow from Calc II the f-las:

$$\begin{array}{rclcccccc}
i\sin x & = & & ix & & -i\frac{x^3}{3!} & & +i\frac{x^5}{5!} & - \dots \\
\cos x & = & 1 & & -\frac{x^2}{2!} & & +\frac{x^4}{4!} & & + \dots \\
\hline
e^{ix} & = & 1 & +ix & -\frac{x^2}{2!} & -i\frac{x^3}{3!} & +\frac{x^4}{4!} & +i\frac{x^5}{5!} & - \dots
\end{array}$$

Rearrange. Plug-in $z = ix$. Use $i^2 = -1$. Multiply $\sin x$ by i. Add to get $e^{ix} = \cos x + i \sin x$. $\square$

You will not be tested on the material in the following slide.

Trigonometric Identities Revisited

- $e^{ix} = \cos x + i \sin x$ (Euler's Formula).
- $e^{ix} e^{iy} = e^{ix+iy} = e^{i(x+y)}$ (exponentiation rule: valid for $\mathbb{C}$).
- $e^0 = 1$ (exponentiation rule).
- $\sin(-x) = -\sin x$, $\cos(-x) = \cos x$ (easy to remember).

Example

$$\begin{aligned} \sin(x+y) &= \sin x \cos y + \sin y \cos x \\ \cos(x+y) &= \cos x \cos y - \sin x \sin y \quad . \end{aligned}$$

Proof.

$$\begin{aligned} e^{i(x+y)} &= \cos(x+y) + i \sin(x+y) \\ e^{ix} e^{iy} &= \cos(x+y) + i \sin(x+y) \\ (\cos x + i \sin x)(\cos y + i \sin y) &= \cos(x+y) + i \sin(x+y) \\ \cos x \cos y - \sin x \sin y + i(\sin x \cos y + \sin y \cos x) &= \cos(x+y) + i \sin(x+y) \end{aligned}$$

Compare coefficient in front of i and remaining terms to get the desired equalities. $\square$

Trigonometric Identities Revisited

- $e^{ix} = \cos x + i \sin x$ (Euler's Formula).
- $e^{ix} e^{iy} = e^{ix+iy} = e^{i(x+y)}$ (exponentiation rule: valid for $\mathbb{C}$).
- $e^0 = 1$ (exponentiation rule).
- $\sin(-x) = -\sin x$, $\cos(-x) = \cos x$ (easy to remember).

Example

$$\sin^2 x + \cos^2 x = 1$$

Proof.

$$\begin{aligned}
1 &= e^0 \\
&= e^{ix-ix} = e^{ix} e^{-ix} = (\cos x + i \sin x)(\cos(-x) + i \sin(-x)) \\
&= (\cos x + i \sin x)(\cos x - i \sin x) = \cos^2 x - i^2 \sin^2 x \\
&= \cos^2 x + \sin^2 x \quad .
\end{aligned}$$

$\square$

Trigonometric Identities Revisited

- $e^{ix} = \cos x + i \sin x$ (Euler's Formula).
- $e^{ix} e^{iy} = e^{ix+iy} = e^{i(x+y)}$ (exponentiation rule: valid for $\mathbb{C}$).
- $e^0 = 1$ (exponentiation rule).
- $\sin(-x) = -\sin x$, $\cos(-x) = \cos x$ (easy to remember).

Example

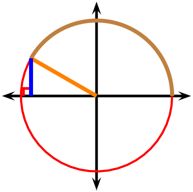
$$\begin{aligned} \sin(2x) &= 2 \sin x \cos x \\ \cos(2x) &= \cos^2 x - \sin^2 x \end{aligned} \quad .$$

Proof.

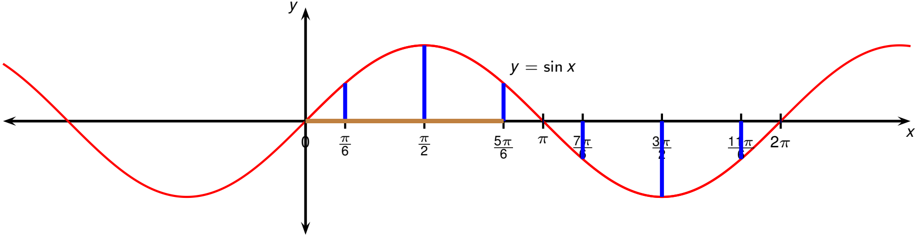
$$\begin{aligned} e^{i(2x)} &= \cos(2x) + i \sin(2x) \\ e^{ix} e^{ix} &= \cos(2x) + i \sin(2x) \\ (\cos x + i \sin x)^2 = (\cos x + i \sin x)(\cos x + i \sin x) &= \cos(2x) + i \sin(2x) \\ \cos^2 x - \sin^2 x + i(2 \sin x \cos x) &= \cos(2x) + i \sin(2x) \end{aligned}$$

Compare coefficient in front of i and remaining terms to get the desired equalities. □

Graph of $\sin x$

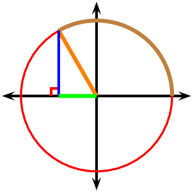


x	0	$\frac{\pi}{6}$	$\frac{\pi}{2}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{3\pi}{2}$	$\frac{11\pi}{6}$	2π
$\sin x$	0	$\frac{1}{2}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	-1	$-\frac{1}{2}$	0

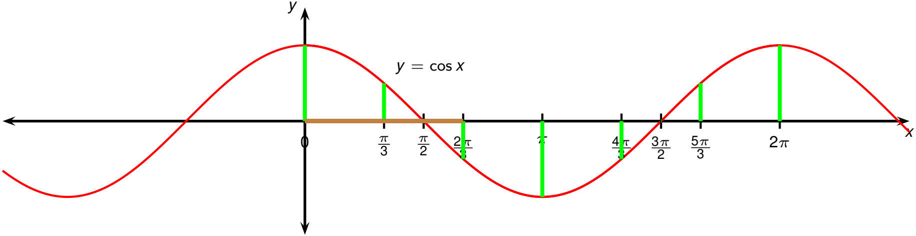


The graph of $\sin x$ is 2π-periodic so the rest of the graph can be inferred from the interval $[0, 2\pi]$.

Graph of $\cos x$

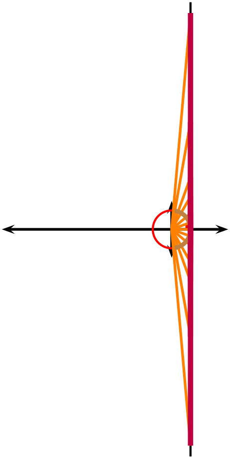


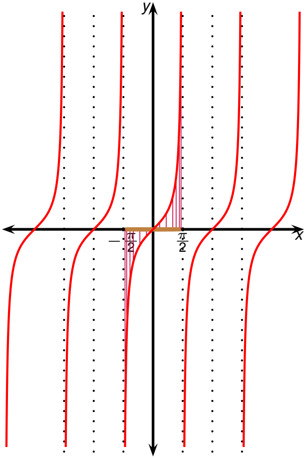
x	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	2π
$\cos x$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	0



The graph of $\cos x$ is 2π-periodic so the rest of the graph can be inferred from the interval $[0, 2\pi]$.

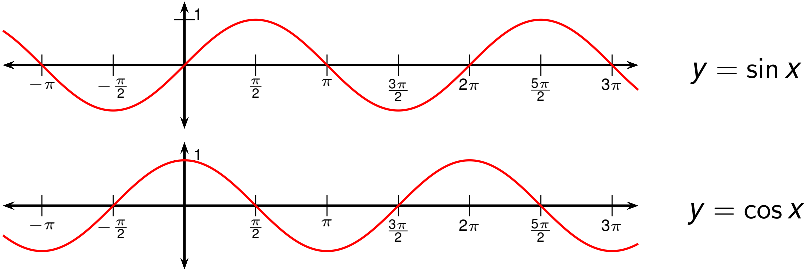
Graph of $\tan x$



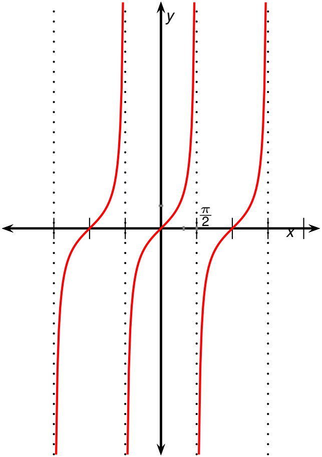


Near $\pm\frac{\pi}{2}$ the graph of $\tan x$ approaches $\pm\infty$. The graph of $\tan x$ is π-periodic so the rest of the graph can be inferred from the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

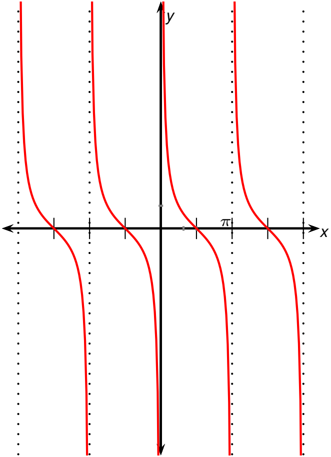
Graphs of the Trigonometric Functions



- $\sin x$ has zeroes at $n\pi$ for all integers n.
- $\cos x$ has zeroes at $\frac{\pi}{2} + n\pi$ for all integers n.
- $-1 \leq \sin x \leq 1$.
- $-1 \leq \cos x \leq 1$.
- If we translate the graph of $\cos x$ by $\frac{\pi}{2}$ units to the right we get the graph of $\sin x$. This is a consequence of $\cos\left(x - \frac{\pi}{2}\right) = \sin x$.

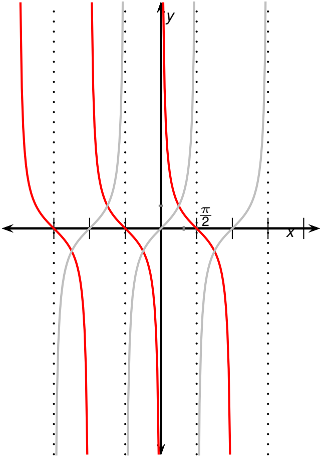


$y = \tan x$

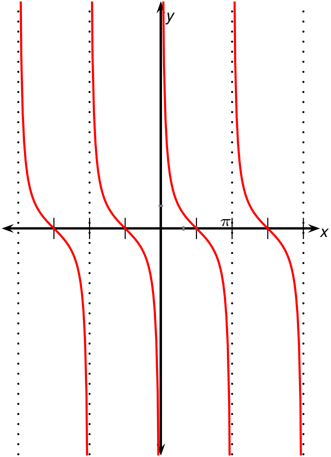


$y = \cot x$

If we move the graph of $\tan x$ by $\frac{\pi}{2}$ units to the left (or right) and reflect across the x axis, we get the graph of $\cot x$. This follows from $\tan\left(x \pm \frac{\pi}{2}\right) = -\cot x$.



$y = \tan x$



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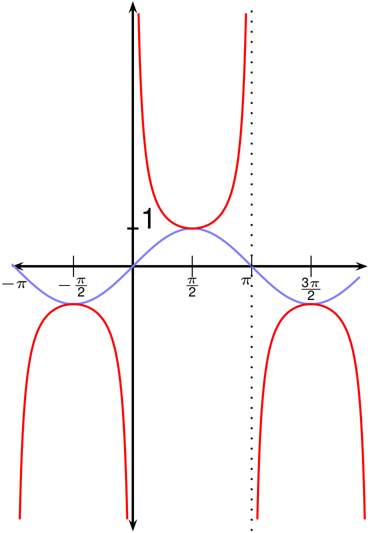
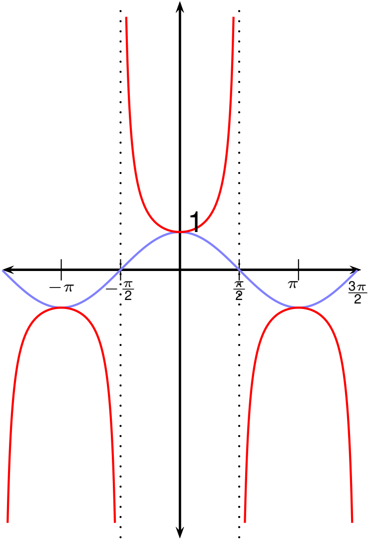

1
$-\pi$
$-\frac{\pi}{2}$
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$

$y = \csc x$


1
$-\pi$
$-\frac{\pi}{2}$
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$

$y = \sec x$