A brief discussion of different geometries of Julia sets

Timothy Minn Kang
Supervisor: Dr Davoud Cheraghi

September 30, 2020

1 Introduction

The mathematical field of complex dynamics is very diverse and began in the nineteenth century, motivated initially as a tool to help solve problems in functional analysis. Since then, with the exception of a hiatus in the twentieth century, this field has attracted a very significant amount of attention from mathematicians. This has been particularly evident since the 1960s. The interest in this field has been further stimulated by the ability to use computers to help solve problems and to produce simulations of high quality.
A concept of the greatest importance in complex dynamics is the Julia set, named after the French mathematician Gaston Julia. This set is of particular importance in complex dynamics because its elements are those of the complex plane where the behaviour of a function is chaotic. Informally this means that even a small change in value can result in a very different dynamical system. Another reason for the wide interest in the Julia set of a function is that it very often has a very fascinating geometry and appearance, when plotted.

In this report, we will begin by discussing the Julia set and a number of its properties after introducing several concepts needed to define the Julia set.
We will then consider several different functions and their Julia sets, which exhibit very different geometries. We will discuss these geometries and, in some instances, provide proofs or explanations as to why the Julia sets of certain functions have a particular geometry.
We will finally focus on 'cauliflower' Julia sets and those of certain quadratic polynomials, whose Julia sets will be sets known as Cantor sets.

2 Preliminaries and Background Material

Before we discuss the Julia set, there are several concepts which must be introduced.

Definition 2.1: We define $\hat{\mathbb{C}} := \mathbb{C} \bigcup \{\infty\}$.
Arithmetic with regard to ∞ is defined as follows:

- $\forall x \in \hat{\mathbb{C}}, x + \infty = \infty + x = \infty$
- $\forall x \in \hat{\mathbb{C}} \setminus \{0\}, x \times \infty = \infty$
- $\forall x \in \mathbb{C}, \dfrac{\infty}{x} = \infty$
- $\forall x \in \mathbb{C}, \dfrac{x}{\infty} = 0$

In this report, we will be considering holomorphic functions [1] which map from $\hat{\mathbb{C}}$[2] to itself.
Notation: Let $n \in \mathbb{Z}_{\geq 1}$ and $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. We will write f^n to mean the n^{th} iterate of f i.e. f 'applied' n times; $(f(p))^n$ to mean '$f(p)$ to the power of n', and $f^{(n)}(p)$ to denote the n^{th} derivative of f.

Definition 2.2: Let $p \in \hat{\mathbb{C}}$ and $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. We define the *orbit of p (under f)* to be the set $O = \{f^\theta(p) : \theta \in \mathbb{Z}_{\geq 1}\}$.

[1] For $\hat{\mathbb{C}}$, these functions are the rational maps.
[2] We consider $\hat{\mathbb{C}}$ because it is a *compact Riemann surface*, whereas $\mathbb{C}$ is not. A definition of a *Riemann surface* can be found in ...; it suffices to state that ...

(If this set is bounded, it is called a *bounded orbit.*)

Definition 2.3: $z \in \hat{\mathbb{C}}$ is a periodic point for $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ with period $n \in \mathbb{N} \setminus 0$ if $f^n(z) = z$ and no 'lower' iterate and n is the smallest such positive integer for which this holds. Any orbit of a periodic point is called a *periodic orbit.*
Moreover:

- if $|(f^n)'(z)| > 1$, z is called a *repelling* periodic point
- if $|(f^n)'(z)| < 1$, z is called an *attracting* periodic point
  - if $|(f^n)'(z)| = 0$, z is called a *superattracting* periodic point
  - if $0 < |(f^n)'(z)| < 1$, z is called a *geometrically attracting* periodic point
- if $|(f^n)'(z)| = 1$, z is called an *indifferent/ neutral* periodic point
- if $|(f^n)'(z)|$ is a root of unity, whilst neither f nor any of its iterates are the identity, z is called a *parabolic* periodic point

Definition 2.4: $z \in \hat{\mathbb{C}}$ is a periodic point for $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a fixed point of f if z is a periodic point with period 1.

Definition 2.5: Let $z_0 \to z_1 \to ... \to z_N = z_0$ be a periodic orbit with period $N \in \mathbb{Z}_{\geq 1}$. The multiplier of the orbit is defined as:

- $(f^N)'(z_0)$, if $p \in \mathbb{C}$
- $\dfrac{1}{\lim_{z\to\infty}(f^N)'(z)}$, if $p = \infty$

Definition 2.6: Suppose $z_0 \to z_1 \to z_2 \to ... \to z_K = z_0$ is an attracting periodic orbit of period K. The basin of attraction of the orbit $\{z_0, z_1...\}$ is the set $\mathfrak{O} = \{z \in \hat{\mathbb{C}} : (f^{Kj}(z_0))_{j\in\mathbb{Z}_{\geq 1}}$converges to a point of $\mathfrak{O}\}$.

Definition 2.7a [Local uniform convergence]: We say that a sequence $(f_n)_{n\in\mathbb{N}}$ of functions from $\hat{\mathbb{C}}$ to itself is *locally uniformly convergent with limit* $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ if $\forall p \in \hat{\mathbb{C}}$, there is a neighbourhood of p, U such that the restriction of $(f_n)_{n\in\mathbb{N}}$ to U converges uniformly in U.
Definition 2.7b [Normal family]: We say that a set of functions $\mathfrak{F}$ from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$ is a normal family if: any sequence of functions in $\mathfrak{F}$ $(f_n)_{n\in\mathbb{N}}$ contains a subsequence that converges locally uniformly on $\hat{\mathbb{C}}$.

N.B: The above definition does include the possibility of 'converging' to ∞, as $\infty \in \hat{\mathbb{C}}$ and $\hat{\mathbb{C}}$ is (topologically) compact.
Finally, we introduce a special set which will be crucial to the discussion of Julia sets in this report, particularly in Section 6:

Definition 2.8: Consider the interval $[0, 1]$.
Divide this interval into three subintervals of equal length $[0, \frac{1}{3}], [\frac{1}{3}, \frac{2}{3}], [\frac{2}{3}, 1]$. Remove the 'middle' subinterval to produce $C_1 := [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$.
Next, divide each interval of C_1 into three subintervals of equal length and remove the 'middle' subinterval from each one to produce C_2.
Proceed as such - that is, to produce $C_k, k \in \mathbb{N} \setminus \{0\}$, divide each interval of C_{k-1} into three subintervals of equal length and remove the 'middle' subinterval from each one.
We define the *ternary Cantor set*, $\mathfrak{C} := \cap_{k=1}^{\infty} C_k$.

Definition 2.9: A set $C \subset \hat{\mathbb{C}}$ is called a *Cantor set* is C is homeomorphic to $\mathfrak{C}$ or $\mathfrak{C} \times \mathfrak{C}$.

Defining the Julia Set of a Function

Definition 2.10: Let $p \in \hat{\mathbb{C}}$. Suppose that there exists a neighbourhood of p, U_p, such that the restriction of f and its iterates to U_p forms a normal family of functions. We define the Fatou set of a function f, $Fatou(f)$, to be the set of all such $p \in \hat{\mathbb{C}}$.

Definition 2.11 [Julia Set]: The Julia set of a function f, $J(f) = \hat{\mathbb{C}} \setminus Fatou(f)$.
Remarks:

1. The Fatou set of a holomorphic function f is an open set. To see this, let $p \in Fatou(f)$. Then by definition, we can find a neighbourhood of p, U_p, such that $\{f^n|_{U_p} : n \in \mathbb{Z}_{\geq 1}\}$ forms a normal family of functions. Let $p' \in U_p$. Then $\exists r > 0 \quad s.t. \quad B_r(p') \subset U$ and $\{f^n|_{B_r(p')} : n \in \mathbb{Z}_{\geq 1}\}$ forms a normal family of functions. Since the Fatou set of f is open, the Julia set of f, $J(f)$ must be closed.

2. There are many equivalent definitions of the Julia set; for example, another definition (found on page 269 of *'An Introduction to Chaotic Dynamical Systems'* by Robert Devaney [1] of $J(f)$ is that it is the closure of the set of repelling periodic points of f.

It will also be useful to introduce the following set:

Definition 2.12 [Filled Julia set]: Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be holomorphic. We define the filled Julia set of f, $K(f)$, to be the compact set whose boundary (in $\hat{\mathbb{C}}$) is $J(f)$, and whose interior is *the union of all components of $Fatou(f)$ with bounded orbit.*

3 Properties of the Julia set

Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be holomorphic. In this section we will be detailing properties that the Julia set, $J(f)$, has. The full proofs for most of them can be found in Section 4 of *'Dynamics in One Complex Variable'* [2] by John W Milnor but we will prove Proposition 3.12.

Proposition 3.1: $J(f)$ is fully invariant i.e. $z \in J(f) \Leftrightarrow f(z) \in J(f)$.

Proof: Found on page 44 of *'Dynamics in One Complex Variable'*[2]. ■

Proposition 3.2: $\forall n \in \mathbb{Z}_{\geq 1}, J(f^n) = J(f)$.

Proof: Found on page 44 of *'Dynamics in One Complex Variable'* [2]. ■

Proposition 3.3: All repelling and parabolic periodic points of f are elements of $J(f)$. Conversely, all attracting periodic points of f are elements of $Fatou(f)$.

Proof: Found on pages 45 and 46 of *'Dynamics in One Complex Variable'* [2]. ■

Proposition 3.4: If $deg(f) \in \mathbb{Z}_{\geq 2}$, then $J(f) \neq \emptyset$. [3]

Proof: Found on page 46 of *'Dynamics in One Complex Variable'* [2]. ■

Proposition 3.5: Suppose the degree of f is at least 2. Then, if the interior of $J(f)$ is non-empty, then $J(f) = \hat{\mathbb{C}}$.

Proof: Found on page 48 of *'Dynamics in One Complex Variable'* [2]. ■

Proposition 3.6: Suppose $\mathfrak{B}$ is the basin of attraction of an attracting period orbit of f. Then $J(f) = \partial\mathfrak{B}$ (the boundary of $\mathfrak{B}$). Moreover, every connected component of $Fatou(f)$ is either a connected component of $\mathfrak{B}$ or disjoint from $\mathfrak{B}$.

Proof: Found on page 48 of *'Dynamics in One Complex Variable'* [2]. ■

Proposition 3.7: Suppose $p \in J(f)$. Then the set $S = \{z \in \hat{\mathbb{C}} : \exists n \in \mathbb{Z}_{\geq 1}, f^n(z) = p\}$ is dense in $J(f)$.

Proof: Found on page 49 of *'Dynamics in One Complex Variable'* [2]. ■

Proposition 3.8: If f has degree $d \geq 2$, then $J(f)$ has no isolated points.

Proof: Found on page 49 of *'Dynamics in One Complex Variable'* [2]. ■

Proposition 3.9: If $deg(f) \geq 2$, then $J(f)$ is:

[3]Here, deg denotes the degree of f which is a rational map. That is, if $f = \frac{p}{q}$, where p, q are polynomials, then $deg(f) = max(deg(p), deg(q))$.

- either connected
- or has uncountably many disjoint connected components

Proof: Found on pages 49-51 of *'Dynamics in One Complex Variable'* [2]. ■

Proposition 3.10: Suppose $p \in J(f)$. Then the (forward) orbit of p, $\mathfrak{O} = \{p, f(p), f^2(p), ...\}$ is dense in $J(f)$.

Proof: Found on page 51 of *'Dynamics in One Complex Variable'* [2]. ■

We will provide a proof for the last property to be discussed about the Julia set; the proof of this property is posed as an exercise on pages 52 and 53 of *'Dynamics in One Complex Variable'* by Milnor [2].

Proposition 3.12: [Self similarilty] Suppose that $J(f) \neq \emptyset$ $p, q \in J(f)$. We say that (J, p) is *locally conformally isomorphic* to (J, q) if there is a neighbourhood N_p of p and a neighbourhood N_q of q for which we can find a conformal isomorphism $\phi : N_p \to N_q$. Moreover, $\phi(p) = q$ and $\phi(J \cap N_p) = \phi(J \cap N_q)$.
Let $z_0 \in J(f)$. Then $S = \{z \in J(f); (J, z)$ is locally conformally isomorphic $to(J, z_0)\}$ is dense in $J(f)$ unless this very unusual condition holds:

for any sequence $(z_j)_{j \in \mathbb{N} \cup \{0\}}$ such that $f(z_{j+1}) = z_j, j \in \mathbb{N} \cup \{0\}$, we can find $K > 0$ such that z_K is a critical point of f.

Proof: Suppose that this very unusual condition does not hold. Then there exists a sequence $(z_j)_{j \in \mathbb{N} \cup \{0\}}$ where $f(z_{j+1}) = z_j, j \in \mathbb{N} \cup \{0\}$ and $\forall j \in \mathbb{N}_{\geq 1}$ z_j is not a critical point of f.
From Proposition 3.7, we know that $\{z \in \hat{\mathbb{C}} : \exists n \in \mathbb{Z}_{\geq 1}, f^n(z) = p\}$ is dense in $J(f)$. Fix $K \in \mathbb{N}_{\geq 1}$. Then since none of $z_0, ..., z_K$ are critical points, we can find a neighbourhood of z_0, N_0, such that $f^K|_{N_0}$ (the restriction of f to N_0) is conformal and thus an isomorphism. Now, since f^K is holomorphic and thus continuous, we know that $f^K|_{N_0} : N_0 \to N_1$, where $N_1 = f^K(N_0)$ is a conformal isomorphism. Also, $N_1 \cup J(f) = f^K(N_0) \cup J(f) = f(N_0 \cup J(f))$. Since K was arbitrary, we have shown that $\forall j \in \mathbb{Z}_{\geq 1}$, (J, z_0) is locally conformally isomorphic to (J, z_j). This means that the set $\{z \in J(f); (J, z)$ is locally conformally isomorphic $to(J, z_0)\}$ is indeed dense in $J(f)$. ■

Informally, this proposition states that the geometry about a certain point of the Julia set of a function is generally (even if not always) observed everywhere in the Julia set. This will be seen in many of the examples discussed in Section 4.

4 Examples of functions and their Julia sets

We will now discuss the Julia sets of several different functions.

4.1 Computer algorithms for plotting Julia sets

Our first attempt at an algorithm to make a plot of the Julia set of a function made use of Proposition 3.7. This is sometimes used to plot the Julia sets of quadratic functions.
However, this algorithm would have several limitations in practice, because the number of 'preimages' of points would increase exponentially. Indeed, some points of the Julia set would be such that many iterations of the algorithm are needed to plot them.
Moreover, it is very difficult to implement for most polynomials of degree 3 or higher and for non-polynomial rational maps. In fact, it would be *impossible* to implement it for most polynomials of degree 5 or higher.

A more efficient and more accurate algorithm would plot the *filled* Julia set as opposed to the Julia set. This algorithm would use the property that any element of the filled Julia set of a 'function' has bounded orbit. We can make use of this property as we will be considering *mostly* polynomial functions, which are entire. To briefly explain the algorithm, the function is iterated on a subset of $\mathbb{C}$ a 'large' number of times. After doing so, the values of this subset of $\mathbb{C}$ which are considered by MATLAB to be 'infinity' are distinguished from those which are not and are *'bounded'*.
Both algorithms can be found on the following GitHub repository.

4.2 Plots and discussion of these examples

The plots shown (below) are of the *filled Julia set* of each function, shown mostly in yellow or green, and the Julia set of a function is itself the boundary between the regions in yellow/green and blue.

Example 4.2.1: If f is a rational map of degree 0 or 1, then either:

- $J(f) = \emptyset$, or
- $J(f)$ contains a single fixed point which is repelling or parabolic.

Proof: If f is constant, any iterate of f is just equal to f. This means that $\forall z \in \hat{\mathbb{C}}$ and any neighbourhood of z, U, the restriction of f and its iterates are constant and so any sequence of $\{f^n|_U(z) : n \in \mathbb{Z}_{\geq 1}\}$ is constant, so must converge uniformly and thus locally uniformly. The Fatou set is $\hat{\mathbb{C}}$ and therefore, the Julia set of f must be empty.

If f has degree 1, then f is a Möbius transformation. Let $f(z) = \dfrac{az+b}{cz+d}$, where $a, b, c, d \in \mathbb{C}$.

If f is the identity map, then $J(f) = \emptyset$.

Otherwise, f has two fixed points, z_1, z_2 and a long but simple calculation shows that their multipliers λ_1, λ_2 are such that $\lambda_2 = \dfrac{1}{\lambda_1}$. We deduce that $|\lambda_2| = \dfrac{1}{|\lambda_1|}$.

If $|\lambda_1| \neq |\lambda_2| \neq 1$, then one of $|\lambda_1|, |\lambda_2|$ is greater than 1. Thus one of z_1, z_2 is a repelling fixed point. Without loss of generality assume that z_1 is the repelling fixed point. Then Propositions 3.1 and 3.10 imply that $z_1 \in J(f)$ and that $\{z_1\}$ is dense in $J(f)$, so $J(f) = \{z_1\}$.

If $|\lambda_1| = |\lambda_2| = 1$, the fixed points z_1 and z_2 are either both parabolic or both non-parabolic; this is because $\lambda_1 = \dfrac{1}{\lambda_2}$. If λ_1, and hence λ_2, are *rational*, then they are parabolic. If they are both *irrational*, I learnt from Dr Davoud Cheraghi [3], my UROP supervisor, that $J(f) = \emptyset$. Otherwise, z_1, z_2 are parabolic. It follows from Proposition 3.3 that $\{z_1\}$ and $\{z_2\}$ are both dense in $J(f)$. This implies that in fact, $z_1 = z_2$, so $J(f) = \{z_1\}$. $\therefore If$J(f)$\neq \emptyset$, then $J(f)$ does consist of a single repelling or parabolic fixed point. ■

Example 4.2.2: The function $f(z) = z^n$, where $n \in \mathbb{N}_{\geq 2}$ has Julia set $\partial\mathbb{D} = \{z \in \hat{\mathbb{C}} : |z| = 1\}$. The Julia set for $n = 2$ has been plotted in Figure 1.

Proof: Suppose $z \in \mathbb{C}$ is such that $|z| < 1$. Then $\exists r > 0$ such that $U = B_r(z) \subset \mathbb{D}$. Consider $f|_U^m = z^{mn}, m \in \mathbb{Z}_{\geq 1}$. This sequence of functions converges uniformly (on U) to $g_1 : U \to \hat{\mathbb{C}}, g_1(\theta) = 0$. We thus deduce that $\mathbb{D} \in Fatou(f)$.

Suppose $z \in \mathbb{C}$ is such that $|z| > 1$. Then $\exists r > 0$such that$U = B_r(z) \cap \overline{\mathbb{D}} = \emptyset$. Consider $f|_U^m = z^{mn}, m \in \mathbb{Z}_{\geq 1}$. This sequence of functions converges uniformly (on U) to $g_2 : U \to \hat{\mathbb{C}}, g_2(\theta) = \infty$. We thus deduce that $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}} \in Fatou(f)$. If we consider $p \in \partial\mathbb{D}$ and fix $r \in \mathbb{R}_{>0}$, then we find that $f^m|_U$ converges pointwise to the following function, g:

$$g(z) = \begin{cases} 0 & \text{if } |z| < 0 \\ \text{z} & \text{if } |z| = 1 \\ \infty & \text{if } |z| > 1 \end{cases}$$

This function is discontinuous at $z \in \partial\mathbb{D}$, so there can be no local uniform convergence. Thus $J(f) = \mathbb{D}$. ■

The plot of the Julia set (and filled Julia set) of f is shown for the case $n = 2$.

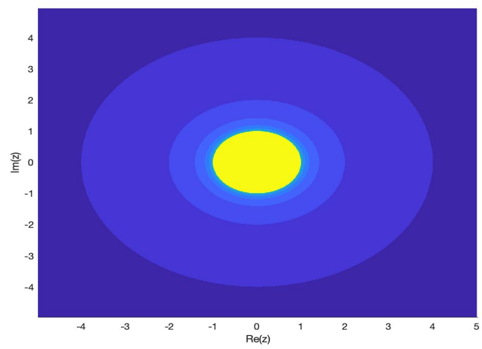


Figure 1: $K(f(z) = z^n), n = 2$

Example 4.2.3: Consider the following polynomials $P_n : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ which are defined through this recurrence relation: $P_{m+1}(z) = zP_m(z) - P_{m-1}(z), \forall m \in \mathbb{N}_{\geq 2}$ and $P_1 = z, P_2 = z^2 - 2$
The Julia sets of these polynomials are the same and they are the interval $[-2, 2]$. We have plotted the Julia set for P_2, shown in Figure 2.

Proof: *Step 1:* We first show that $\forall n \in \mathbb{N}_{\geq 2}, J(P_n) \subset [-2, 2]$.

Step 1a: We note that $\forall n \in \mathbb{N}_{\geq 2}$, 2 is a repelling fixed point of P_n.
We can prove that $\forall n \in \mathbb{N}_{\geq 2}$, 2 is a *fixed point* of P_2 by induction. First, it is clear that 2 is a fixed point for $P_2 = z^2 - 2$ and $P_3 = z^3 - 3z$ (which was found using the recurrence relation given previously). Next, we assume that 2 is a fixed point for $2, 3, ..., K$, where $K \in \mathbb{Z}_{\geq 3}$. Then,

$$P_{K+1}(2) = 2(P_K(2)) - P_{K-1}(2)$$
$$P_{K+1}(2) = 2(2) - 2 = 2$$

As 2 is a fixed point for P_{K+1}, 2 is a fixed point for $P_n, \forall n \in \mathbb{N}_{\geq 2}$.

We now show that $\forall n \in \mathbb{Z}_{\geq 2}, P'_n(2) \geq 2$ by induction.
For $P_2 = z^2 - 2$ and $P_3 = z^3 - 3z$, we find that $P'_2 = 2z$ and $P'_3 = 3z^2 - 3$. So $P'_2(2) = 4$ and $P'_3(2) = 9$.
We now let $K \in \mathbb{N}_{\geq 3}$ and suppose that for $2, 3, ...K, P'_n(2) \geq 2$. Then, differentiating the recurrence relation, we obtain:

$$P'_{K+1}(z) = P_K(z) + zP'_K(z) - P_{K-1}(z)$$

Letting $z = 2$, we find:

$$\begin{aligned} P'_{K+1}(2) = P_K(2) + 2P'_K(2) - P_{K-1}(2) \\ \Rightarrow P'_{K+1}(2) = 2 + P'_K(2) + (P'_K(2) - P_{K-1}(2)) \\ \Rightarrow P'_{K+1}(2) > 2 \end{aligned}$$

[4] We have thus proven that $\forall n \in \mathbb{N}_{\geq 2}, P'_n(2) > 2$. Thus 2 is a *repelling* fixed point for all these polynomials.

Step 1b: We show that the preimage of any point of $[-2, 2]$ is also in $[-2, 2]$. We make the following substitution: $\forall z \in \hat{\mathbb{C}}$, let $z = w + \frac{1}{w}$. We can then prove inductively that $\forall n \in \mathbb{Z}_{\geq 1}, P_n(z) = w^n + \frac{1}{w^n}$
Now suppose $z' \in [-2, 2]$. Let us find ... We have:

$$\begin{aligned} w^n + w^{-n} &= z' \\ \Rightarrow w^{2n} + 1 &= z'w^n \\ \Rightarrow w^{2n} - z'w^n + 1 &= 0 \\ \Rightarrow (w^n - \frac{z'}{2})^2 &= \frac{z'^2}{4} - 1 \\ \Rightarrow w^n = \frac{z' \pm \sqrt{z'^2 - 4}}{2} &= \frac{z' \pm (\sqrt{4 - z'^2})i}{2} \end{aligned}$$

The last equality follows from the fact that $|z'| < 2 \Rightarrow |z'|^2 < 4$, which means that $\sqrt{(}z'^2 - 4)$ cannot be real.
We thus have that:

$$\begin{aligned} |w^n|^2 = \frac{z'^2 + |z'^2 - 4|}{4} = \frac{z'^2 + (4 - z'^2)}{4} &= 1 \\ \Rightarrow |w^n| = |w|^n &= 1 \\ \Rightarrow |w| &= 1 \end{aligned}$$

This shows that $\frac{1}{w} = w*$ (where $w*$ is the complex conjugate of w). This means that $z = w + \frac{1}{w}$ must be real, and moreover $-2 \leq z \leq 2$. Because $2 \in [-2, 2]$ and any 'preimage' of 2 (with respect to f) must be in $[-2, 2]$, given Proposition 3.7, $\forall n \in \mathbb{Z}_{\geq 2}, J(P_n) \subset [-2, 2]$, which we can deduce if we consider any arbitrary point in [-2,2] and iterate it under f. It follows that $J(P_n) \subset [-2, 2], \forall n \in \mathbb{Z}_{\geq 2}$

Step 2: We now show that $\forall n \in \mathbb{Z}_{\geq 2}, J(P_n) = [-2, 2]$. We know from Proposition 3.6 that, since $\hat{\mathbb{C}} \setminus [-2, 2]$ is a connected component of $Fatou(f)$, this must be contained in $A(\infty)$. On the other hand, the orbit of any point in [-2,2] under f is a subset of [-2,2]. So $\partial A(\infty) = \hat{\mathbb{C}} \setminus [-2, 2] = J(f)$, again from Proposition 3.6.

■

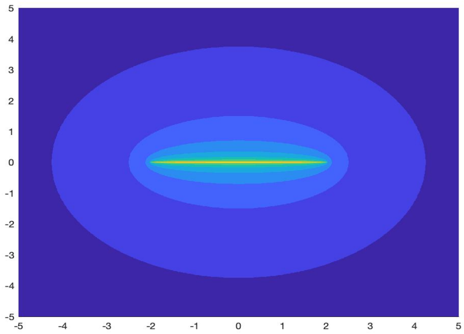


Figure 2: $J(p_n(z)), n = 2$

The next example is an application of Proposition 3.1.

Example 4.2.4: $f(z) = z^4 - 4z^2 + 2$ also has Julia set $[-2, 2]$.

Proof: Let $g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be defined as such: $g(z) = z^2 - 2$. Then note that $g(z) = [f(z)]^2$. By Proposition 3.1 we note that $J(f) = J(g)$ and since $J(g) = [-2, 2]$ (see Example 4.2.3), $J(f) = [-2, 2]$. ∎

The examples above illustrate that the Julia set of certain functions may be very 'rudimentary'.

However, the Julia sets of most functions are not nearly as 'rudimentary' as the ones above.

Example 4.2.5: $f(z) = z^2 - 6$ has a Cantor Julia set and $J(f) \subset [-3, -\sqrt{3}] \cup [\sqrt{3}, 3]$, shown in Figure 3.

Proof: To show that $J(f) \subset [-3, -\sqrt{(3)}] \cup [\sqrt{(3)}, 3]$, first note that 3 is a fixed point of $f(z)$, and moreover, since $f'(z) = 2z$, $f'(3) = 6$, so it is a repelling fixed point $\Rightarrow 3 \in J(f)$. Also, we note that

$$\forall z \in \subset [-3, -\sqrt{3}] \cup [\sqrt{3}, 3], f^{-1}(z) = \{\pm\sqrt{z+6}\} \subset [-3, -\sqrt{3}] \cup [\sqrt{3}, 3]$$

. By Proposition 3.7, and applying it to $p = 3$ we deduce that $J(f) \subset [-3, -\sqrt{3}] \cup [\sqrt{3}, 3]$. It is now sufficient to prove that $J(f)$ has no isolated points and is totally disconnected; we know that from Hocking and Young as cited by Milnor [**?**] that a Cantor set (in $\hat{\mathbb{C}}$) is a compact subset which has no isolated points and is totally disconnected, and we already know that $J(f)$ is both closed and bounded, so compact. The former condition is an immediate consequence of Proposition 3.8, as f has degree 2.

We must now prove that $J(f)$ is totally disconnected. But this is clear, since $J(f) \subset [-3, -\sqrt{3}] \cup [\sqrt{3}, 3]$ and, given Proposition 3.8, $J(f)$ must have uncountably many disjoint connected components, as it cannot be connected. It immediately follows that $J(f)$ must be totally disconnected. ∎

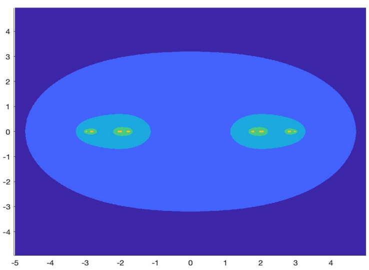


Figure 3: $J(f), f(z) = z^2 - 6$

Example 4.2.6: $z^2 + 2$ has a Cantor Julia set which is disjoint from $\mathbb{R}$, shown in Figure 4.

Proof Sketch: We use a similar proof to the one used for the previous example. To prove that $J(f) \cap \mathbb{R} = \emptyset$, we consider the fixed points $1 \pm \sqrt{7}i$. No preimage of these points (wrt f) can be real, and hence, $\{z \in \hat{\mathbb{C}} : \exists n \in \mathbb{Z}_{\geq 1}, f^n(z) = p\} \cap \mathbb{R} = \emptyset$.

[4]To deduce the third line of working, we have used the facts that 2 is a fixed point for these polynomials, and that $(P'_m(z)P'_{m-1}(z)) > 0, \forall m > 0$, which is easy to prove by induction.

We now prove that $J(f)$ both has no isolated points and is totally disconnected.
The former again follows from Proposition 3.8 and the fact that f is a quadratic polynomial.
To prove that $J(f)$ is totally disconnected, consider the following:
If $J(f)$ were not totally disconnected, it would have to be connected, by Proposition 3.9. But $J(f)$ cannot be disconnected, as any connected subset of $\hat{\mathbb{C}}$ containing both $1 \pm \sqrt{7}i$ (the fixed points of f) must intersect $\mathbb{R}$, which is impossible. ■

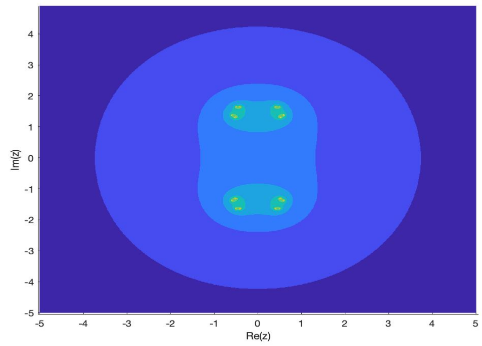


Figure 4: $J(f), f(z) = z^2 + 2$

Example 4.2.7: The Julia set of $f(z) = z^2 + i$ is shown in Figure 5; this set is called a *dendrite*.
Justification: We observe that 0 is a pre-periodic point of f i.e. 0 itself is not periodic, but $\exists N \in \mathbb{Z}_{\geq 1}$ such that $f^N(0)$ is periodic. Indeed,

$$\begin{aligned}
&f(0) = 0 + i = i \\
\Rightarrow\ &f^2(0) = f(f(0)) = f(i) = i^2 + i = -1 + i \\
\Rightarrow\ &f^3(0) = f(f^2(0)) = (-1+i)^2 + i = 1 - 2i - 1 + i = -i \\
\Rightarrow\ &f^4(0) = f(f^3(0)) = (-i)^2 + i = -1 + i \\
&\ldots
\end{aligned}$$

So, whilst 0 itself is not periodic, $f^2(0) = -1 + i$ is a periodic point of f with period 2.

We know from Chapter 3, Section 6 of 'An Introduction to Chaotic Dynamical Systems' that if 0 is a preperiodic point of a quadratic polynomial, then the Julia set of this polynomial is a dendrite. Since f is a quadratic polynomial, it is a dendrite.

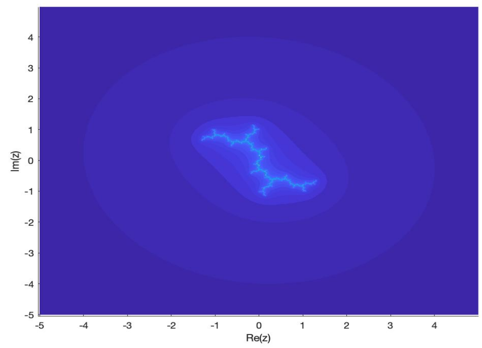


Figure 5: $K(f), f(z) = z^2 + i$

Example 4.2.8: The Julia set of $f(z) = z^2 - 1$ is depicted in Figure 6; this set is known as a 'basilica fractal'.
Comments: We notice that the general 'shape' of the Julia set of f (the boundary between the yellow and blue region in Figure) can be observed 'everywhere' except at certain points. These points are those for which the set described in Proposition 3.7 contains 0, the critical point of f. It follows from Proposition 3.12 that the general geometry of the Julia set (and filled Julia set)

may not be observed at these points.
Also, we see that the filled Julia set of f appears to have several disjoint components; these correspond to fixed points of not only f, but iterates of f.

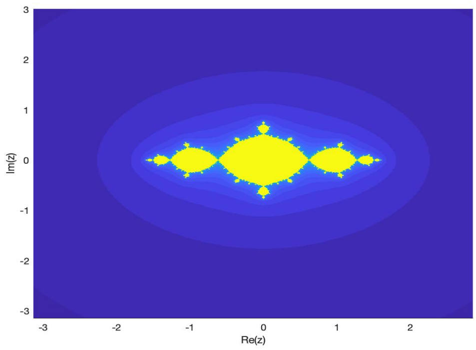


Figure 6: $K(f), f(z) = z^2 - 1$

Example 4.2.9: The Julia set of $f(z) = z^2 - (0.122 - 0.745i)$ is shown in Figure 7; this set is known as a 'Douady rabbit'.
Comment: The geometry of this Julia set is very similar to that of the Julia set of any quadratic polynomial such that the orbit of 0 is a superattracting periodic orbit of period 3 [2]. Although $f^3(0)$ is found not to be exactly equal to 0, we find that it is 'rather close' to 0 ($f^3(0) \in \mathbb{D}$, and thus $J(f)$ has a similar geometry.

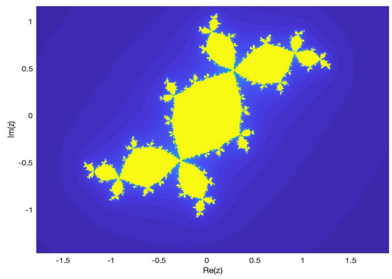


Figure 7: $K(f), f(z) = z^2 - (0.122 - 0.745i)$

Example 4.2.10: The function $f(z) = 1 - \dfrac{1}{z^2}$ has Julia set depicted in Figure 8.
Comment: We notice three dots which are very dark blue; these correspond to the values $0, -1, 1$; we note $f(0) = \infty$ and $f^2(-1) = f^2(1) = f(0) = \infty$. Also, $K(f)$ appears to have empty interior, which is indeed true, as the orbit of any element of $Fatou(f)$ under f must be ∞.

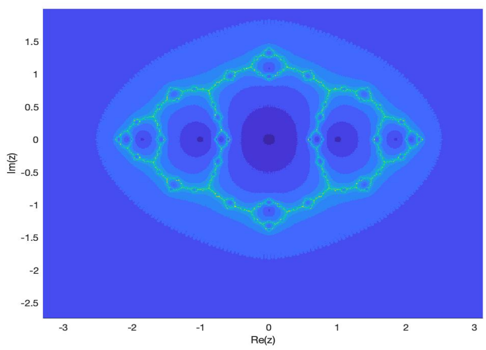


Figure 8: $f(z) = 1 - \frac{1}{z^2}$

Example 4.2.11: The Julia set, $J(f)$, of f when $f(z) = z^2 - \dfrac{1}{16z^2}$ is shown below in Figure

9.

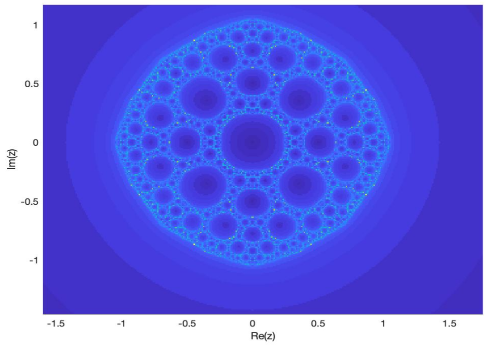


Figure 9: $J(f), f(z) = z^2 - \frac{1}{16z^2}$

Comment: $Fatou(f)$ is everywhere dense for this function; this Julia set is known as a *Sierpinski carpet*.
Furthermore, some iterate of f for any point of $Fatou(f)$ appears to be ∞; we note this as $K(f)$ again has empty interior (ie no elements of $Fatou(f)$ whose orbit under f is bounded).

Example 4.2.12: $f(z) = z^2 - \frac{1}{2}(1-i)$ has the filled Julia set shown in Figure 10.

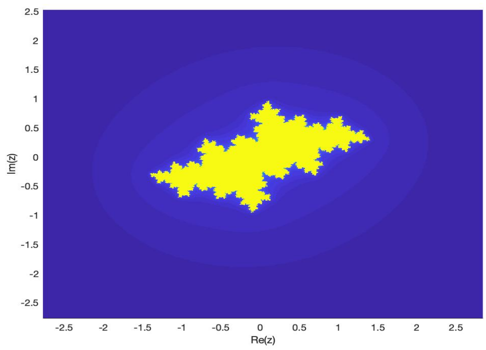


Figure 10: $K(f), f(z) = z^2 - (\frac{1-i}{2})$

Comment: It is interesting to see that, on an informal and heuristic note, that $K(f)$ has the shape in Figure 10 as it is being 'deformed' from the filled Julia set of the function z^2 (a disc) to either the Julia set of the function $z^2 - 1$ (a 'basilica fractal') or $z^2 + i$ (a dendrite).
However, $K(f)$ is a connected subset of $\hat{\mathbb{C}}$; the reason for this will be stated in Theorem 6.2.

Of course, it must be noted that there are many, many more different (and considerably more 'exciting') shapes that Julia sets of different functions (not mentioned in this report) can take. In the next section, we shall discuss a special kind of Julia set.

5 Cauliflower Julia Sets

The Julia sets of many functions resemble the shape of a cauliflower; one example of this can be found on page 121 of *'Dynamics in One Complex Variable'* by Milnor.

5.1 Parabolic Point Theory

Reminder: We say that p is a parabolic periodic point with period $n \in \mathbb{Z}_{\geq 1}$ for $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ if $(f^n)'(p)$ is a root of unity and no iterate of f is equal to the identity map. Suppose $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ has a parabolic fixed point p. Then since f is holomorphic, $f(z) = \lambda(z-p)[1 + \alpha(z-p)^m + ...]$, where λ is the multiplier of p and m is the smallest positive integer such that $\alpha \neq 0$. We call $m+1$ the *multiplicity* of the fixed point.
Definition 5.1a: We say that $w \in \mathbb{C}$ is an *attraction vector* for f if $m\alpha w^m = -1$.
Definition 5.1b: Let p be a parabolic fixed point of f. We say that $(f^k(z')_{k\in\mathbb{N}}$ (for $z' \in mat\hat{h}bbC$

converges to 0 from the direction w if $z_k := (f^k(z_0), k \in \mathbb{N}$ converges to p and $\lim_{k\to\infty}(k)^{\frac{1}{n}} z_k = w$. [5]

Definition 5.2: Let p be a parabolic fixed point for f and w be an attraction vector for p. We say that $A_w := \{z \in \hat{\mathbb{C}} : z$ converges to p from the direction w$\}$ is the parabolic basin of attraction of p associated with w.

The immediate basin of attraction A_w^0 is defined to be the unique connected component of A_w which is invariant under f; this component is unique, since any other connected component of A_w cannot be invariant under f, as all points of A_w must eventually belong to A_w^0.

Theorem 5.3: Suppose f is a non-identity rational map and p is a fixed point of f with multiplier 1 (and is thus parabolic). Each immediate basin of p contains at least one critical point of f.

Proof: This can be found in *'Dynamics in One Complex Variable'* by John W Milnor [**?**]. ■

5.2 Polynomials with 'Cauliflower' Julia Sets

We will focus on finding some polynomials whose Julia sets are 'Cauliflower' Julia sets.

Let us consider monic polynomials $p_n, n \in \mathbb{Z}_{\geq 2}$ with the following conditions:

- $\forall n \in \mathbb{Z}_{\geq 2}$, p_n has a single *real* critical (stationary) point
- $\forall n \in \mathbb{Z}_{\geq 2}$, $p_n(0) = 0$ (0 is a fixed point of p_n
- $\forall n \in \mathbb{Z}_{\geq 2}$, $(p_n)'(0) = 1$ (0 is a parabolic fixed point of p_n)

Remark: There are several other polynomials, and even other non-polynomial or non-rational functions, whose Julia set is a 'cauliflower' Julia set.

5.2.1 Determining formulae for p_n

Fix $n \in \mathbb{Z}_{\geq 2}$. We note that since we have only one critical point, $\beta \in \mathbb{R}$ say, that:

$$p_n'(z) = \alpha(z-\beta)^{(n-1)}$$

We set $\alpha = n$ to ensure that p_n is monic. We thus have:

$$p_n'(z) = n(z-\beta)^{(n-1)}$$

By substituting $z = 0$, we find that:

$$1 = n(-\beta)^{(n-1)}$$
$$\Rightarrow (\beta)^{(n-1)} = \frac{(-1)^{(n-1)}}{n}$$

If n is even, then $\beta = -\frac{1}{n^{\frac{1}{n-1}}}$

If n is odd, then $\beta = \pm\frac{1}{n^{\frac{1}{n-1}}}$ but let us set $\beta = -\frac{1}{n^{\frac{1}{n-1}}}$

Now, we also have that

$$p_n(z) = (z-\beta)^n + c$$

where $c \in \mathbb{C}$ is a constant.

Substituting $z = 0$, we find that:

$$0 = (-\beta)^n + c$$
$$\Rightarrow \text{c=-(-}\beta)^n = -\frac{1}{n^{\frac{n}{n-1}}}$$
$$\therefore \text{p}_n(z) = \left(z + \frac{1}{n^{\frac{1}{n-1}}}\right)^n - \frac{1}{n^{\frac{n}{n-1}}}$$

[5] We use the word 'direction' because any element of C can be associated with a vector in $\mathbb{R}^2$.

We will list the first few $p_n, n = 2, 3, 4, 5$.

$$p_2(z) = \left(z - \frac{1}{2}\right)^2 - \frac{1}{4}$$
$$p_3(z) = \left(z - \frac{1}{\sqrt{3}}\right)^3 - \frac{1}{3\sqrt{3}}$$
$$p_4(z) = \left(z - \frac{1}{4^{\frac{1}{3}}}\right)^4 - \frac{1}{4^{\frac{4}{3}}}$$
$$p_5(z) = \left(z - \frac{1}{5^{\frac{1}{4}}}\right)^5 - \frac{1}{5^{\frac{5}{4}}}$$

5.2.2 Properties of p_n

We will discuss two crucial properties which p_n has, which are essential for discussing the Julia sets and filled Julia sets of p_n.
Property 1: $\forall n \in \mathbb{Z}_{\geq 2}$, the parabolic fixed point 0 has only one immediate basin of attraction.

Proof: Fix $n \in \mathbb{Z}_{\geq 2}$. We note that $p_n(z)$ has a non-zero term in z^2, with coefficient a, which means that 0 has multiplicity $m + 1 = 2$.
We know that v is an attraction vector for 0 if and only if $mav^m = -1$. But $m = 1$, so the equation is $av = -1$. So $v = -\frac{1}{a}$ and so 0 can only have one attraction vector. ■

Property 2: $\forall n \in \mathbb{Z}_{\geq 2}$, p_n has n zeros. Moreover, these fixed points are:

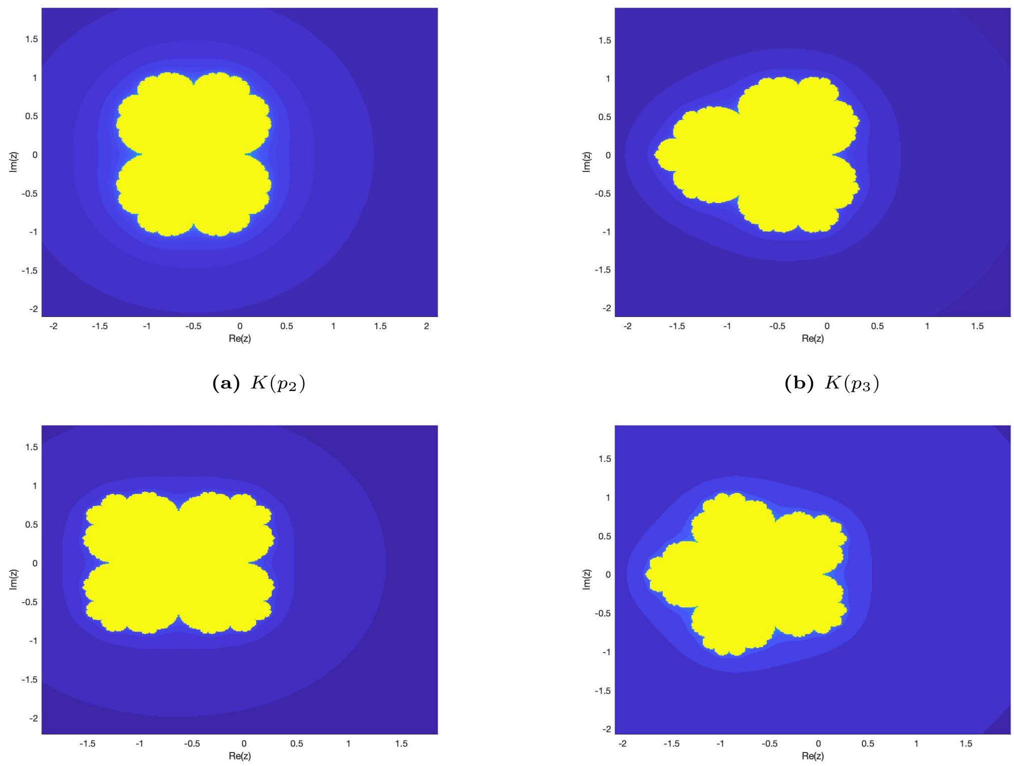
$$-\left(\frac{1}{n}\right)^{\frac{n}{n-1}} + (\frac{1}{n})^{\frac{1}{n-1}} e^j n, j = 0, 1, ..., (n-1)$$

and these zeros belong to $J(f)$.

Proof: We find these zeros by solving the equation for any $n \in \mathbb{Z}_{\geq 2}$. We know that $0 \in J(f)$ as it is parabolic (Proposition 3.3). It then follows from Proposition 3.1 that since $0 \in J(f)$, these fixed points must be, too. ■

5.2.3 Plotting and discussing $J(p_n)$ for $n = 2, 3, 4, 5$

We will plot and discuss the filled Julia sets for p_n, $n = 2, 3, 4, 5$.
The filled Julia set will be denoted by the region that is yellow.
Discussion: These polynomials all have *one* attracting basin, which is the interior of the filled Julia set. This is to be expected, as these polynomials p_n all have only a *single* critical point. By Theorem 5.3 any immediate basin contains *at least* one critical point of p_n). It thus follows that the filled Julia set only has one component.
We see that there are cusps. These correspond to the zeros of p_n. Moreover, for $n = 2, 3, 4, 5$, we see that there are 2,3,4,5 such cusps, respectively. These do all occur on the circle of radius $\left(\frac{1}{n}\right)^{\frac{1}{n-1}}$ centred at $-\left(\frac{1}{n}\right)^{\frac{1}{n-1}}$.



(a) $K(p_2)$ **(b)** $K(p_3)$

(c) $K(p_4)$ **(d)** $K(p_5)$

Figure 11: Parabolic stuff

6 Quadratic Maps with $|c| > 2$

In this section, we will consider the family of quadratic maps $\mathfrak{F} = \{f_c : c \in \mathbb{C}\}$ where:

$$\forall c \in \mathbb{C}, f_c : \hat{\mathbb{C}} \to \hat{\mathbb{C}}, f_c(z) = z^2 + c.$$

We will be considering the Julia set of f_c for $|c| > 2$. The reason is this: whilst $J(f_c)$ exhibits a range of different geometries for $|c| \geq 2$, for $|c| > 2$, $J(f_c)$ will always be a Cantor set. The aim of Section 6 is to prove this.

Proposition 6.1: Consider the quadratic map $f_c = z^2 + c$, where $c \in \mathbb{C}$. If $|c| > 2$, then the Julia set of f_c, $J(f_c)$, is a Cantor set.

To prove Proposition 6.1, we will need some more basic results.

Theorem 6.2: Let. $c \in \mathbb{C}$. (f_c) and $K(f_c)$ (the filled Julia set of f_c) connected if and only if the orbit of 0 under f is bounded.

Proof: This follows immediately from 'Theorem 9.5' (page 96) in *'Dynamics in One Complex Variable'* by John W. Milnor. ■

Lemma 6.3: Fix $c \in \mathbb{C}$. If $|c| > 2$, then the orbit of 0 under f_c is not bounded and the sequence $f_c^n(0)$ converges to ∞.

Proof: It suffices to prove that $\forall n \in \mathbb{Z}_{\geq 1}, |f_c^n(0)| \geq |c|(|c|-1)^{n-1}$. [6] We prove this by induction.

[Base case] For $n = 1$

$$f_c^n(0) = c$$
$$\Rightarrow |f_c^n(0)| = |c| = |c|(|c|-1)^{1-1}$$

[Inductive step] Suppose that for some $k \in \mathbb{Z}_{\geq 1}, |f_c^k(0)| \geq |c|(|c|-1)^{k-1}$. Then:

[6] Indeed, since $(|c|-1) > 1$, $\lim_{n\to\infty}(|c|(|c|-1)^{n-1}) = \infty$ $(|c| > 2)$.

$$\begin{aligned}|f_c^{k+1}(0)| &= |(f_c^k(0))^2 + c| \\ &\geq ||f_c^k(0)|^2 - |c|| \\ &\geq |c|^2(|c|-1)^{2(k-1)} - |c| \geq |c|^2(|c|-1)^{(k-1)} - |c|(|c|-1)^{(k-1)} \\ &= |c|[|c|-1]^k\end{aligned}$$

So if for $k \in \mathbb{Z}_{\geq 1}$, $|f_c^k(0)| \geq |c|(|c|-1)^{k-1}$, then $|f_c^{k+1}(0)| \geq |c|(|c|-1)^k$ $\therefore \forall n \in \mathbb{Z}_{\geq 1}, |f_c^n(0)| \geq |c|(|c|-1)^{n-1}$, by mathematical induction. We thus deduce that since $\lim_{n\to\infty}(|f_c^n(0)|) = \infty$ $(|c| > 2)$, $\lim_{n\to\infty}(f_c^n(0)) = \infty$. ■

Lemma 6.4: Consider $c \in \mathbb{C}$. If $|c| > 2$, then the Julia set of $f_c = z^2 + c$, $J(f_c)$, is not connected.
Sketch of Proof: Suppose $|c| > 2$. We know (from ...) that $J(f_c)$ is connected if and only if the orbit of 0 under f_c is bounded. But we know from ... that the orbit is not bounded and converges to ∞.

Corollary 6.5: If $|c| > 4$, then the Julia set of $f_c = z^2 + c$, $J(f_c)$, is totally disconnected.
Proof: We know from ... 3.[] that, because f_c is a quadratic polynomial (with degree 2), $J(f_c)$ is either connected or has uncountably many connected components. Since $J(f_c)$ is not connected, it must have uncountably many connected components and it thus follows that $J(f_c)$ must be totally disconnected.

Proof of Proposition 6.2: Suppose $|c| > 4$. Then it follows from Corollary 6.5 that $J(f_c)$, is totally disconnected. By ... we know also that $J(f_c)$ has no isolated points, as f_c is a quadratic polynomial (hence has degree 2). By ... it follows that $J(f_c)$ must be a Cantor set. □

Remarks:

- The Mandelbrot set, which is defined as $\mathfrak{M} = \{c \in \mathbb{C}$:orbit of 0 under $f_c$is bounded$\}$, gives the values of c for which $J(f_c)$ is not a Cantor set; it is often these values in which many are particularly interested.
- Although we have focused on quadratic polynomials of the form $z^2 + c$, this is sufficient to consider the behaviour of ALL quadratic polynomials, since relatively easy algebraic manipulation shows that every quadratic polynomial is conjugate [7] to a quadratic polynomial of the form $z^2 + c$. It follows that the Julia set of any quadratic polynomial is the same (up to expansion/dilation and rotation) as that of a polynomial of $\mathfrak{F}$.

Plots of examples

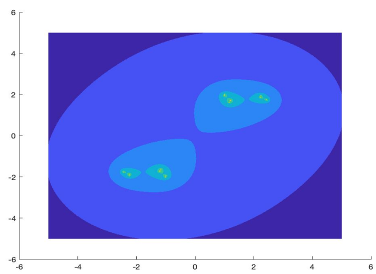


Figure 12: $J(z^2 + c), c = 4i$

[7] If two quadratic polynomials $f, g)$ are conjugate, then there exists a bijective map h such that $(f \circ h) = (h \circ g)$.

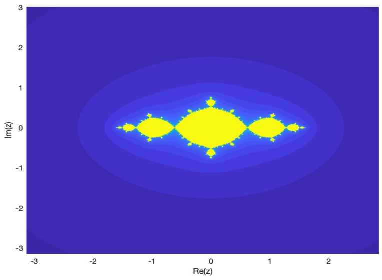


Figure 13: $J(z^2 + c), c = -1$

7 Conclusion

This project has been a brief insight into the concept of Julia sets in the mathematical discipline of complex dynamics. We have begun by moving from an intuitive idea of the Julia set as the subset of $\hat{\mathbb{C}}$ on which a function f exhibits 'chaotic' behaviour, to a formal definition involving concepts, namely the idea of normal families, applicable to other mathematical disciplines. It has then been very illuminating to discuss a wide range of properties of the Julia set and then to plot the (filled) Julia sets of a vast array of numerous examples of different functions.
Focusing particularly on polynomials with a 'cauliflower' Julia set with a given set of initial conditions, we have found that their (general) geometries may, to a considerable extent, be determined by discussing its parabolic fixed point (which it must have for the polynomials to exhibit this shape).
In discussing the family of quadratic polynomials $\mathfrak{F} = \{f_c(z) = z^2 + c | c \in \mathbb{C}\}$, we have found the considerably satisfying result that $\forall c \in \mathbb{C} : |c| > 2, J(f_c) and K(f_c)$ are connected.

This report is simply a very rudimentary overview of the various different kinds of Julia sets that can occur; indeed, there are several other varied, 'fascinating' geometries that the Julia sets and the filled Julia sets of functions not discussed in this report can possess.
The UROP I have been involved in has been nothing short of invaluable. As well as gaining knowledge in a field of mathematics which is very beautiful and exciting, it has helped me to develop a vast array of deeply valued skills, including research skills and communication skills.
I would finally like to sincerely thank my UROP supervisor, Dr Davoud Cheraghi, not only for his kindness in agreeing to supervise my UROP, but for his invaluable advice and knowledge with which he has provided me during the course of this UROP.

References

[1] Devaney, R. *An Introduction to Chaotic Dynamical Systems*. [ProQuest Ebook Central version] Second Edition. Westview Press, 2008.

[2] Milnor, John W. *Dynamics in One Complex Variable*. [ProQuest Ebook Central version] Third Edition. Princeton University Press, 2006.

[3] Dr Davoud Cheraghi. Senior Lecturer in Mathematics, Imperial College London. Personal communication, 4^{th} August 2020.