A business has created a mathematical model based on market data for its profit, P (in dollars), as function of the number of items sold, x. The model is given by the function

$$P(x) = -0.1x^2 + 150x - 1400.$$

How many units must be sold to maximize profit? What is the maximum profit?

Things you need to know about Quadratic Functions

General form: $f(x) = ax^2 + bx + c$

Vertex form: $f(x) = a(x-h)^2 + k$

Factored form: $f(x) = a(x - r_1)(x - r_2)$

Example 1: $f(x) = 2x^2 + 5x + 3$
Describe the following features of the quadratic function.

- The shape
- y-intercept
- Write it in vertex form
- Write it in factored form

Example 2: Find a formula for the parabola that goes through the points $(-5,0)$, $(3,0)$ and $(4,12)$

Example 3: A concert venue holds a maximum of 1,000 people with ticket prices at \$30, the average attendance is 650 people. It is predicted that for every dollar the ticket price is lowered approximately 25 more people will attend. Create a function to represent the revenue generated from ticket sales and use this to find the maximum possible revenue.

Example 4: Suppose a sunglass manufacturer determines the demand function for a certain line of sunglasses is given by $p = 50 - \frac{1}{4000}x$, where p is the price per pair, and x is the number of pairs sold. The fixed cost of producing this line of sunglasses is \$25,000 and each pair of sunglasses costs \$3 to produce. How many pairs of sunglasses should be produced and sold in order to maximize profits?

Warm-Ups:

Problem 1. A tank is being filled with a liquid. The amount of liquid, L in liters, after t minutes is given by $L(t) = 1.25t + 73$. Answer the following questions:

- Find $L^{-1}(x)$

  Solution.

$$\begin{aligned} y &= 1.25t + 73 \\ y - 73 &= 1.25t + 73 - 73 \\ y - 73 &= 1.25t \\ \frac{y-73}{1.25} &= \frac{1.25t}{1.25} \\ t &= \frac{y-73}{1.25} \end{aligned}$$

Thus $L^{-1}(y) = \frac{y-73}{1.25}$. ☐

- Find $L^{-1}(125)$

  Solution.

$$\begin{aligned} L^{-1}(125) &= \frac{125-73}{1.25} \\ &= \frac{52}{1.25} \\ &= 41.6. \end{aligned}$$

Thus $L^{-1}(125) = 41.6$. ☐

- Interpret what the inverse tells us.

  Solution. The inverse represents the amount of time in minutes it takes to fill the tank with a certain number of liters of liquid. ☐

Problem 2. Let $f(x) = \frac{1}{\sqrt{5x+1}}$ and $g(x) = -3x+2$.

- Find $(f \circ g)(x)$.

  Solution.

$$\begin{aligned} (f \circ g)(x) &= f(g(x)) \\ &= \frac{1}{\sqrt{5(-3x+2))+1}} \\ &= \frac{1}{\sqrt{-15x+10+1}} \\ &= \frac{1}{\sqrt{-15x+11}} \end{aligned}$$

☐

- Find $f^{-1}(x)$.

  Solution.

$$\begin{aligned} y &= \frac{1}{\sqrt{5x+1}} \\ y(\sqrt{5x+1}) &= 1 \\ \sqrt{5x+1} &= \frac{1}{y} \\ 5x+1 &= \frac{1}{y^2} \\ 5x &= \frac{1}{y^2} - 1 \\ x &= \frac{1}{5y^2} - \frac{1}{5}. \end{aligned}$$

Thus $f^{-1}(y) = \frac{1}{5y^2} - \frac{1}{5}$. ☐

Problem 3. Factor completely:

- $(x-11)^2 + 5(x-11) - 24$

Solution. Let $u = (x-11)^2$.

$$\begin{aligned}(x-11)^2 + 5(x-11) - 24 &= u^2 + 5u - 24 \\ &= (u+8)(u-3) \\ &= (x-11+8)(x-11-3) \\ &= (x-3)(x-14)\end{aligned}$$

□

- $3(8x+3)^2(7x-6) - (8x+3)(7x-6)^2$

  Solution. The greatest common factor between the two is $(8x+2)(7x-6)$.

$$\begin{aligned}3(8x+3)^2(7x-6) - (8x+3)(7x-6)^2 &= (8x+3)(7x-6)(3(8x+3)-(7x-6)) \\ &= (8x+3)(7x-6)(24x9-7x+6) \\ &= (8x+3)(7x-6)(17x+15)\end{aligned}$$

□

Concept Check:

Problem 4. Consider the following function:

$$f(x) = \sqrt{4-x} + 7$$

The domain of the function is $(-\infty, 4]$ and the range is $(-\infty, \infty)$.

1. Find $f^{-1}(x)$.

   Solution.

$$\begin{aligned}y &= \sqrt{4-x} + 7 \\ y - 7 &= \sqrt{4-x} \\ (y-7)^2 &= 4 - x \\ (y-7)^2 - 4 &= -x \\ -(y-7)^2 + 4 &= x\end{aligned}$$

   Thus $f^{-1}(y) = -(y-7)^2 + 4$. □

2. What is the domain of $f^{-1}(x)$?

   Solution. The rule tells us that we switch the domain and range from the original function. This implies that the domain of f^{-1} is the range of $f(x)$. Thus the domain for f^{-1} is $(-\infty, \infty)$. □

3. What is the range of $f^{-1}(x)$?

   Solution. The rule tells us that we switch the domain and range from the original function. This implies that the range of f^{-1} is the domain of $f(x)$. Thus the range of f^{-1} is $(-\infty, 4]$. □

Problem 5. • Find an equation for the parabola that passes through the points $(-1,0)$, $(3,0)$ and $(5,10)$.

Solution. Since we were given the x-intercepts of the equation, we will use intercept form, $y = a(x - r_1)(x - r_2)$. This gives us $y = a(x+1)(x-3)$ so we need to solve for a. We do this by using the point we haven't used yet.

$$\begin{aligned} 10 &= a(5+1)(5-3) \\ 10 &= a(6)(2) \\ \frac{10}{12} &= a \end{aligned}$$

Thus $a = \frac{5}{6}$ and our equation is $y = \frac{5}{6}(x+1)(x-3)$. □

- Find an equation for the parabola that has a vertex of $(4, 5)$ and passes through the point $(0, -6)$.

  Solution. Since we were given the vertex and a point, we will use vertex form, $y = a(x - h)^2 + k$, where (h, k) is the vertex. This give us $y = a(x-4)^2 + 5$ so we need to solve for a. We do this by using the point we haven't used yet.

$$\begin{aligned} -6 &= a(0-4)^2 + 5 \\ -6 &= a(-4)^2 + 5 \\ -11 &= a(-4)^2 \\ -11 &= 16a \\ \frac{-11}{16} &= a \end{aligned}$$

  Thus $a = \frac{-11}{16}$ and our equation is $y = \frac{-11}{16}(x-4)^2 + 5$ □

Problem 6. The braking distance $D(v)$ (in meters) of a certain car moving at a certain velocity v is given by $D(v) = \frac{v^2}{22}$. The car's velocity $B(t)$ (in meters per second) t seconds after starting is given by $9t$. How many meters will it take the car to stop if it has been driving for 30 seconds?

Problem 7. Scientists are studying the temperature on a distant planet. They find that the temperature T depends on the height h (in kilometers) above the planet's surface. The relationship is as follows:

$$T(h) = 48.5 - 2.5h.$$

Use this to answer the following questions:

1. In practical terms describe $T^{-1}(h)$.
2. Find $T^{-1}(h)$.
3. What is $T^{-1}(33)$?

Problem 8. A ball is thrown from an initial height of 2 meters with an initial upward velocity of 30 meters per second. The ball's height h (in meters) is given by

$$h(t) = 2 + 30t - 5t^2.$$

Use this to answer the following:

1. Find all values for which the ball is at 12 meters.
2. When does the ball touch the ground?

3. What is the maximum height the ball reaches and when does it reach that height?

Problem 9. Write the equation for the quadratic function that contains the following points $(-1,0)$, $(3,0)$ and $(5,4)$.

Problem 10. Write the equation for the quadratic function that contains has a vertex at $(3,4)$ and passes through the point $(8,-4)$.

Problem 11. The length of a rectangle is 8 yards less than 3 times it's width. The area of the rectangle is 35 square yards. Find the dimensions of the rectangle.

Problem 12. Answer the following questions about the following quadratic equation

$$x^2 - 6x + 11$$

1. Is the vertex a maximum or minimum?

2. Where does the vertex occur?

Problem 13. Find the zeros of the following quadratic equation

$$-3x^2 + 6x + 1$$

Problem 14. When an apple orchard owner plants 65 trees on an acre of ground, he gets an average yield of 1500 apples per tree. For each additional tree planted per acre, the annual yield per tree drops by 20 apples. Let x represent the number of additional trees above 65.

1. Determine an equation that represents the yield of apples the orchard owner will get per year.

2. How many trees should the owner plant in order to maximize the output of the orchard?

3. What is the maximum amount of apples will the orchard produce?

Problem 15. A charter flight charges a fee of $300 per person plus $2 per person for each unsold seat on the plane. The plane holds 200 passengers.

1. Determine an equation that represents the revenue based on number of passengers x

2. How many passengers should ride on the plane to maximize revenue?

3. What is the maximum revenue the company can make?