MATH 7251: High-Dimensional Probability **Spring 2022**

Lecture 3

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1 Economical Coverings of $\mathbb{R}^n$

Suppose K is an origin-symmetric convex body in $\mathbb{R}^n$. We want to construct $\{\boldsymbol{x}_i + K\}_{i=1}^{\infty}$ which is a covering of $\mathbb{R}^n$, that is, $\bigcup_{i=1}^{\infty} = \mathbb{R}^n$. We also want it to be efficient by using as few translations of K as possible. How do we define this when the covering is countably infinite? We use a measure of efficiency against the tiling of cubes:

$$\limsup_{r\to\infty} \frac{\{i : \boldsymbol{x}_i \in [-r,r]^n\} \cdot \mathrm{Vol}\,(K)}{(2r)^n}. \tag{1}$$

Note that the denominator is the n-dimensional volume of the cube $[-r,r]^n$. We are essentially checking how many centers of K lie inside a cube about the origin, normalized by the volumes of K and the cube. This quantity must be at least 1 because it is a covering (no empty spots remain).

Theorem 1.1: Rogers 1957

Let K be an origin-symmetric convex body in $\mathbb{R}^n$. Then, there exists a countable collection $\{\boldsymbol{x}_i\}_{i=1}^{\infty}$ such that $\{\boldsymbol{x}_i + K\}_{i=1}^{\infty}$ is a covering of $\mathbb{R}^n$ and

$$\limsup_{r\to\infty} \frac{\{i : \boldsymbol{x}_i \in [-r,r]^n\} \cdot \mathrm{Vol}\,(K)}{(2r)^n} = \mathcal{O}(n\log n). \tag{2}$$

Proof: The proof has two steps. First, we show that a random covering of sets slightly smaller than K is pretty good. Then, we deterministically cover the remaining points.

First, we need to choose the right number of points. Let $N = \lceil 4(2r)^n \cdot n\log n \cdot \mathrm{Vol}\,(K)^{-1}\rceil$. Let $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N$ be iid random vectors uniformly distributed across $[-r,r]^n$. We are interested in volumetric properties of the random set $\bigcup_{i=1}^{N}(\boldsymbol{x}_i + (1-\frac{1}{n\log n})K)$. Let $d > 0$ be the diameter of K.

Pick any fixed point $\boldsymbol{y} \in [-r+d, r-d]^n$. What is the probability that $\boldsymbol{y} \notin \bigcup_{i=1}^{N}(\boldsymbol{x}_i + (1-\frac{1}{n\log n})K)$? Since the $\boldsymbol{x}_i$ are independent and K is origin-symmetric,

$$\begin{aligned}
\Pr[\boldsymbol{y} \notin \bigcup_{i=1}^{N}(\boldsymbol{x}_i + (1-\frac{1}{n\log n})K)] &= \prod_{i=1}^{N} \Pr[\boldsymbol{y} \notin (\boldsymbol{x}_i + (1-\frac{1}{n\log n})K)] && (3)\\
&= \prod_{i=1}^{N}(1 - \Pr[\boldsymbol{x}_i \in (\boldsymbol{y} + (1-\frac{1}{n\log n})K)]) && (4)
\end{aligned}$$

$$= \prod_{i=1}^{N}(1 - \frac{\mathrm{Vol}\left(((1-\frac{1}{n\log n})K)\right)}{\mathrm{Vol}\left([-r,r]^n\right)}) \tag{5}$$

$$= \prod_{i=1}^{N}(1-(1-\frac{1}{n\log n})^n\frac{\mathrm{Vol}\,(K)}{(2r)^n}) \tag{6}$$

$$= (1-(1-\frac{1}{n\log n})^n\frac{\mathrm{Vol}\,(K)}{(2r)^n})^N \tag{7}$$

$$\leq \exp(-N(1-\frac{1}{n\log n})^n\frac{\mathrm{Vol}\,(K)}{(2r)^n}) \tag{8}$$

$$= \exp(-2n\log n), \tag{9}$$

where the last step follows from the definition of N. Note that this bound is independent of K.

Now, we will use a first-moment argument. For every $\boldsymbol{y} \in [-r,r]^n$, let $1_{\boldsymbol{y}}$ be the indicator of the event $\boldsymbol{y} \notin \bigcup_{i=1}^N(\boldsymbol{x}_i + (1-\frac{1}{n\log n})K)$. Then, we integrate over the cube and split it into the interior we considered before and the leftover shell.

$$\int_{[-r,r]^n}\int_\Omega 1_{\boldsymbol{y}}(\omega)d\omega d\boldsymbol{y} = \int_{\boldsymbol{y}\in[-r+d,r-d]^n}\int_\Omega 1_{\boldsymbol{y}}(\omega)d\omega d\boldsymbol{y} + \int_{\boldsymbol{y}\in[-r,r]^n\setminus[-r+d,r-d]^n}\int_\Omega 1_{\boldsymbol{y}}(\omega)d\omega d\boldsymbol{y} \tag{10}$$

$$\leq \exp(-2n\log n)\mathrm{Vol}\,(\text{interior}) + \mathrm{Vol}\,(\text{shell}) \tag{11}$$

$$= \exp(-2n\log n)\cdot(2r-2d)^n + [(2r)^n-(2r-2d)^n]. \tag{12}$$

Applying Fubini's Theorem, which allows us to exchange the order of integration,

$$\int_\Omega\int_{\boldsymbol{y}\in[-r,r]^n} 1_{\boldsymbol{y}}(\omega)d\omega d\boldsymbol{y} \leq \exp(-2n\log n)\cdot(2r-2d)^n + [(2r)^n-(2r-2d)^n]. \tag{13}$$

Since the measure integrates to 1 over Ω, there must exist a point $\omega_0 \in \Omega$ such that

$$\int_{\boldsymbol{y}\in[-r,r]^n} 1_{\boldsymbol{y}}(\omega_0)d\boldsymbol{y} \leq \exp(-2n\log n)(2r-2d)^n + [(2r)^n-(2r-2d)^n]. \tag{14}$$

As r becomes large, this value is at most $2\exp(-2n\log n)(2r)^n$. This can be interpreted as an upper bound on the volume of the set $[-r,r]^n \setminus \bigcup_{i=1}^N(\boldsymbol{x}_i(\omega_0) + (1-\frac{1}{n\log n})K)$. So, we have found a realization of covering centers $\{\boldsymbol{x}_i\}$ such that the volume of missing points is bounded above by the given expression (i.e., the covering is pretty good). This completes step one.

Now, we need to cover the missing points. We apply a deterministic argument. Consider a maximal packing of the missing points $[-r,r]^n\setminus\bigcup_{i=1}^N(\boldsymbol{x}_i(\omega_0)+(1-\frac{1}{n\log n})K)$ by parallel translations of $\frac{1}{n\log n}K$ – we are filling the gaps with very small versions of K. Using the volumetric argument from last time, we can estimate the size of this maximal packing by the ratio of the volumes:

$$\frac{2\exp(-2n\log n)(2r)^n}{(\frac{1}{n\log n})^n\mathrm{Vol}\,(K)} = 2\exp(-2n\log n + n\log(n\log n))(2r)^n\mathrm{Vol}\,(K)^{-1}. \tag{15}$$

When n is large, the expression in the exponent is negative. Then, it is at most

$$2(2r)^n \text{Vol}\,(K)^{-1} \leq N. \tag{16}$$

So, our maximal packing has at most N elements. Denote the centers of these elements by $\boldsymbol{z}_1, \boldsymbol{z}_2, \ldots, \boldsymbol{z}_M$ with $M \leq N$. Now,

$$\{\boldsymbol{x}_i(\omega_0)\}_{i=1}^N \cup \{\boldsymbol{z}_j\}_{j=1}^M \tag{17}$$

has at most $2N$ points total. We claim the covering generating by these centers covers the entire cube $[-r, r]^n$.

To see this, pick any point $\tilde{\boldsymbol{y}} \in [-r, r]^n$. If it belongs to the covering of the first set of points, there is nothing to prove. So, we assume $\tilde{\boldsymbol{y}} \notin \bigcup_{i=1}^N (\boldsymbol{x}_i(\omega_0) + K)$. Then, the interior of $\tilde{\boldsymbol{y}} + \frac{1}{n \log n} K$ does not intersect with $\bigcup_{i=1}^N (\boldsymbol{x}_i(\omega_0) + (1 - \frac{1}{n \log n})K)$. (We expanded slightly the neighborhood of $\tilde{\boldsymbol{y}}$ and shrank slightly the neighborhood of $(\boldsymbol{x}_i(\omega_0) + K)$). Because our packing is maximal, by definition we cannot insert the set $\tilde{\boldsymbol{y}} + \frac{1}{n \log n} K$ into the packing. So, there exists $j \leq M$ such that $\boldsymbol{z}_j + \frac{1}{n \log n} K$ intersects with $\tilde{\boldsymbol{y}} + \frac{1}{n \log n} K$. This implies that $\tilde{\boldsymbol{y}}$ is contained in $\boldsymbol{z}_j + \frac{2}{n \log n} K$, which is in turn contained in $\boldsymbol{z}_j + K$. Therefore, $\tilde{\boldsymbol{y}}$ must belong to the covering of the second set of points, proving the claim.

Finally, we do the computations: $N + M \leq 2N$ and

$$\frac{2N \text{Vol}\,(K)}{(2r)^n} = \frac{2 \cdot 4(2r)^n n \log n}{(2r)^n} = 8n \log n = \mathcal{O}(n \log n). \tag{18}$$

$\blacksquare$

The conclusion is that every sufficiently large cube in $\mathbb{R}^n$ can be covered by translations of K so that the covering density is of order $\mathcal{O}(n \log n)$. To pass the result to a countably infinite collection of centers in $\mathbb{R}^n$, we need to consider a sequence of cubes with side lengths tending to infinity sufficiently fast.

This result is close to optimal: Coxeter, Few, and Rogers showed that if covering of $\mathbb{R}^n$ by translations of the Euclidean ball is considered, then the covering density is $\Omega(n)$. Resolving this gap remains open.

A related question is as follows: What is the minimal number of points $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N$ such that the union $\bigcup_{i=1}^N (\boldsymbol{x}_i + \text{Int}(K))$ covers K? Hadwiger conjectured that for every K, $N \leq 2^n$. As an exercise, one can show that to cover the cube $[-1, 1]^n$ with parallel translations of its interior, one needs precisely 2^n copies.