Finding Perfect Matchings

1 Perfect Matchings in Non-Bipartite Graphs

We continue the proof from last time. We will show that $\det(T) \not\equiv 0$ implies G has a perfect matching. Recall that

$$\det(T) = \sum_{\sigma} (-1)^{\mathrm{sgn}(\sigma)} \prod_i t_{i,\sigma(i)}.$$

If $\det(T) \not\equiv 0$, then there exists at least one σ with

$$\prod_i t_{i,\sigma(i)} \neq 0.$$

What does σ "look like"? Well, for all i, the edge $(i, \sigma(i))$ exists in G. And, the union of the edges $(i, \sigma(i))$ forms a union of disjoint cycles because each i has one outgoing edge and one incoming edge because σ is a bijection. Finally, σ has no self-loops (assuming G had no self-loops).

Notice that each even-length cycle immediately gives a perfect matching for those nodes. We will show that if $\det(T) \not\equiv 0$ then there is always one σ in which all cycles are even. This gives a perfect matching on G.

For each σ with one or more odd cycle, define the leading odd cycle to be the one with the smallest index of any node. Say the leading odd cycle has indices $i_1, i_2, \ldots, i_{2k+1}$ for $k \geq 1$. Define a permutation σ' which reverses the edges in the leading odd cycles (*i.e.*, $\sigma'(i_j) = \sigma^{-1}(i_j)$ for j in the cycle) and leaves everything else the same. Then

$$\begin{aligned}
\prod_i t_{i,\sigma'(i)} &= \prod_{i \text{ in leading cycle}} t_{i,\sigma'(i)} \cdot \prod_{\text{other } i} t_{i,\sigma'(i)} \\
&= \prod_{i \text{ in leading cycle}} -t_{i,\sigma(i)} \cdot \prod_{\text{other } i} t_{i,\sigma(i)} \\
&= (-1)^{2k+1} \prod_i t_{i,\sigma(i)} \\
&= -\prod_i t_{i,\sigma(i)}.
\end{aligned}$$

So the term for σ' should cancel the term for σ in the determinant of T. For that, we can verify that $\mathrm{sgn}(\sigma) = \mathrm{sgn}(\sigma')$, so they do in fact cancel.

For the cancellation to be valid, we need that the mapping $\sigma \mapsto \sigma'$ is a bijection (so each σ' is only used to cancel one σ). Our choice to work on the leading odd cycle ensures that this is so.

Thus, the contributions of all σ with one or more odd cycles cancel out. So if $\det(T) \not\equiv 0$, there exists a σ with only even cycles. This results in a perfect matching.

2 Finding a Perfect Matching

2.1 Setup

Consider again the bipartite graph setting. So far, we only used that $\det(B) \equiv 0$ iff G has no perfect matching. Then, we applied Schwarz-Zippel. That is, if we draw all x_{ij} uniformly from $\{1, \ldots, 2n\}$, then

$\det(B) \neq 0$ with probability $\geq 1/2$. In this case, any set of size $2n$ would have worked. To find a perfect matching, we will draw from a more carefully chosen set.

For each i, j we draw c_{ij} i.i.d. uniformly from $\{1, \ldots, 2m\}$ and set $x_{ij} = 2^{c_{ij}}$. (That is, we are drawing from the first $2m$ powers of 2). Let's start by re-proving that this determines the existence of a perfect matching. Let $B_C :=$ the matrix B with values $2^{c_{ij}}$ substituted. Then

$$\begin{aligned}
\det(B_C) &= \sum_{\sigma} (-1)^{\operatorname{sgn}(\sigma)} \prod_i x_{i,\sigma(i)} \\
&= \sum_{\sigma \text{ that are perfect matchings}} (-1)^{\operatorname{sgn}(\sigma)} \prod_i 2^{c_{i,\sigma(i)}} \\
&= \sum_{\sigma \text{ that are perfect matchings}} (-1)^{\operatorname{sgn}(\sigma)} 2^{\operatorname{cost}(\sigma)}.
\end{aligned}$$

where $\operatorname{cost}(\sigma) = \sum_i c_{i,\sigma(i)}$. We interpret the c_{ij} as edge costs, so $\operatorname{cost}(\sigma)$ is the cost of a perfect matching.

2.2 Uniqueness

Note that $\det(B_C) \neq 0$ if there is a unique minimum-cost perfect matching (because then it would not be canceled out in the sum). We will show that this happens with probability at least 1/2.

We call an edge (i, j) a witness of non-uniqueness if the minimum cost of a perfect matching that must include (i, j) is the same as the minimum cost of a perfect matching that cannot include (i, j). If there are no witnesses, then the minimum cost matching is unique.

We will prove that each (i, j) is a witness with probability at most $1/2m$, so by a union bound we have our result. To do so, reveal the costs of all other edges first. Let γ^- be the cost of a minimum cost perfect matching not including (i, j) and $\hat{\gamma}$ the cost of a minimum cost perfect matching excluding both vertices $\{i, j\}$. These are independent of c_{ij}.

Conditioned on γ^- and $\hat{\gamma}$, (i, j) is a witness iff $\gamma^- = \hat{\gamma} + c_{ij}$ because the latter if the cost of the minimum cost matching forced to include (i, j). The probability that $c_{ij} = \gamma^- - \hat{\gamma}$ is at most $1/2m$ because there is a unique choice of c_{ij} that makes this true (which may or may not be in $\{1, \ldots, 2m\}$). Because this holds conditioned on any $\gamma^-, \hat{\gamma}$, it always holds. Thus (i, j) is a witness with probability $\leq 1/2m$ as desired.

2.3 Finding a Matching

To find a perfect matching, assume that c_{ij} are such that the minimum cost perfect matching is unique. We can assume this by repeating the draw of c_{ij} if necessary. Let γ^* be the cost of the unique minimum cost matching M^*.

If $(i, j) \in M^*$, then $G \setminus \{i, j\}$ has a unique minimum cost matching of cost $\gamma^* - c_{ij}$; otherwise, two different minimum cost matchings for $G \setminus \{i, j\}$ could be combined with (i, j) to give two different minimum cost matchings for G, a contradiction.

If $(i, j) \notin M^*$, then $G \setminus \{i, j\}$ has no minimum cost matching of cost at most $\gamma^* - c_{ij}$; otherwise, such a matching could be combined with (i, j) to give a perfect matching of cost at most γ^* for G, contradicting optimality or uniqueness of M^*.

In summary, $(i, j) \in M^*$ iff $G \setminus \{i, j\}$ has a minimum cost matching of cost $\gamma^* - c_{ij}$. To find M^*, it is enough to find the costs of the minimum cost matchings in G and $G \setminus \{i, j\}$ for all edges (i, j).

How do we compute the cost γ^* of the minimum cost perfect matching (assuming uniqueness)? Well, γ^* is the smallest γ such that $\det(B_C) \equiv 2^{\gamma} \pmod{2^{\gamma+1}}$. We could do this for all (i, j) and corresponding $G \setminus \{i, j\}$. However, this would be m determinant computations.

Let $B_C^{(i,j)}$ be the matrix B_C with row i and column j removed. We want to compute $\det(B_C^{(i,j)})$ for all (i,j). Then, γ_{ij}^* is the smallest γ such that $\det(B_C^{(i,j)}) \equiv 2^\gamma \pmod{2^{\gamma+1}}$. Recall (i,j) is included in M^* iff $\gamma^* = \gamma_{ij}^* + c_{ij}$.

Cramer's Rule for the matrix inverse expresses $B = A^{-1}$ as

$$b_{ij} = \frac{1}{\det(A)}(-1)^{i+j}\det(A^{(i,j)}).$$

By computing $\det(B_C)$ first, then inverting B_C, we can read off all of the $\det(B_C^{(i,j)})$. Thus, we can find a perfect matching in a bipartite graph with probability at least $1/2$ using only a single matrix inversion.