Random Graph Colorings

1 Card Shuffling with Pairwise Transpositions

Recall that we are trying to sample a complex object uniformly. We can simulate the uniform distribution using Markov chains, and we can analyze convergence speed using coupling.

We start with an arbitrary permutation of n cards. In each step, pick two positions i_1, i_2 i.i.d. uniformly (allow $i_1 = i_2$) and swap the cards at i_1 and i_2. Is this irreducible? Yes, by selection sort. Is it aperiodic? Yes, $i_1 = i_2$ gives self-loops. The stationary distribution is uniform because every node has degree n^2 and it is undirected.

How quickly does the distribution converge? Define a coupling. Let J_t be the number of positions for which $X_i(t) = Y_i(t)$, *i.e.*, card in position i in $X(t)$ equals the card in position i of $Y(t)$. An alternative view of the chain is that we pick an index i and a card c i.i.d. uniformly at random, then swaps the card at index i with card c.

Our coupling is to pick the same i and c in both chains. Does J ever increase with this coupling? No, at least one card is in the same position afterwards, and if both involved cards were in the same position nothing changes. J decreases when both chains have different cards in position i and have card c in different positions. The probability of this is at most the probability of choosing a disagreeing position times the probability of choosing a disagreeing card:

$$\begin{aligned}
\Pr[J_{t+1} \geq J_t + 1] &\geq (1 - \frac{J_t}{n}) \cdot (1 - \frac{J_t}{n}) \\
&= (1 - \frac{J_t}{n})^2 \\
\Pr[J_{t+1} = J_t] &\leq 1 - (1 - \frac{J_t}{n})^2.
\end{aligned}$$

This chain is a lower bound on the progress of the coupling. Initially, $J_0 \geq 0$, and we are done when $J_t = n$. Let T' be the first time t when $J_t = n$ for the chain $J_{t+1} = J_t + 1$ w.p. $(1 - J_t/n)^2$ and $J_{t+1} = J_t$ otherwise. We have

$$\mathbb{E}[T'] = \sum_{j=0}^{n-1} \mathbb{E}[T'_j].$$

T'_j is the time to go from j to $j+1$ and is distributed geometrically with parameter $(1-j/n)^2$. Thus

$$\begin{aligned}
\mathbb{E}[T'] &= \sum_{j=0}^{n-1} \frac{1}{(1-j/n)^2} \\
&= \sum_{j=0}^{n-1} \frac{n^2}{(n-j)^2} \\
&= \sum_{j=1}^{n} \frac{n^2}{j^2} \\
&= n^2 \sum_{j=1}^{n} \frac{1}{j^2} \\
&\leq n^2 \cdot \frac{\pi^2}{6}.
\end{aligned}$$

By Markov's Inequality, with probability at least 3/4 the chains are coupled after $\frac{2\pi^2}{3}n^2$ steps. Thus the chain mixes in $\mathcal{O}(n^2)$ steps. $\Omega(n \log n)$ is needed for this chain; using different techniques one can show that it mixes in $(n \log n)/2$ steps.

2 Random Graph Colorings

Given an undirected graph G, a valid coloring assigns a color to each node such that no two neighbors have the same color. It is $\mathcal{NP}$-hard to decide if G has a coloring using k colors for $k \geq 3$. Our goal is to make k large enough so that a coloring exists and is easy to find. We then sample nearly uniformly from all valid k-colorings.

Let Δ be the max degree of G. If $k \geq \Delta + 1$, then a coloring exists and is found by any greedy algorithm. We would like to randomly sample colorings in this regime.

This situation is often studied by physicists, in particular on regular structures like lattices. It is called Glauber Dynamics: In each round, pick a uniformly random node and uniformly random color. Try recoloring the node to that color. If illegal, do nothing.

If k is large enough the Markov graph will be connected, and it is aperiodic because it has self-loops. The stationary distribution is uniform because the Markov graph is nk-regular, beause we pick one vertex and one color. How quickly does this chain mix? The conjecture is that this mixes rapidly (*i.e.,* $poly(n,k)$ which is poly-log in the number of colorings) when $k \geq \Delta + 2$. Here, we will prove rapid mixing for $k \geq 2\Delta + 1$ via a coupling.

The first idea for a coupling is to pick the same node v in both chains, then correlate the color choices. We will keep track of the set S_t of nodes that have the same color in the two chains X, Y after t steps. Contrary to our earlier examples, we don't have monotone progress – sometimes S_t will shrink. We want to keep the probability of this small.

If we pick the same color for X, Y always, then it can happen frequently that the color is accepted in one chain and rejected in the other. This makes S_t smaller if v had the same color before. Basically, we want to pick the colors in such a way that we make S_t smaller in only few cases.

Formally, if $v \in S_t$ we choose the same color c for that X and Y. If $v \in S_t$, let C_X be the set of all colors of neighbors of v under X and similarly for C_Y. These are sets of illegal colors. Let $B_X := C_X \setminus C_Y$ and similarly for B_Y. These are sets of colors illegal in exactly one chain (note $B_X \cap B_Y = \emptyset$). Without loss of generality $|B_X| \geq |B_Y|$; let $B'_X \subseteq B_X$ be an arbitrary subset of B_X of size $|B'_X| = |B_Y|$. Let $\varphi : B'_X \to B_Y$ be an arbitrary bijection. When we choose color c in X, we choose $\varphi(c)$ if $c \in B'_X$, $\varphi^{-1}(c)$ if $c \in B_Y$, and c otherwise in Y. Importantly, the overall color mapping is a bijection so Y still chooses a uniformly random color as required.

Now, we analyze how likely S_t is to grow vs. shrink. Let m' be the number of edges between S_t and $\bar{S}_t$. If $v \notin S_t$, either S_t stays the same or becomes one smaller. When does S_t become smaller?

- $c \in B'_X$ ($\varphi(c) \in B_Y$): rejected in both
- $c \in B_Y$ ($\varphi(c) \in B'_X$): both accepted and change to different colors
- $c \in B_X \setminus B'_X$ ($\varphi(c) = c$): rejected in X and accepted in Y, ending in different colors
- $c \in C_X \cap C_Y$ ($\varphi(c) = c$): rejected in both
- $c \notin C_X \cup C_Y$ ($\varphi(c) = c$): accepted in both

Things get worse iff $c \in B_Y \cup (B_X \setminus B'_X)$. The size of this set is

$$|B_Y| + |B_X| - |B'_X| = |B_X|.$$

Therefore the probability that things get worse (conditioned on picking v) is at most $|B_X|/k$. We know $|B_X|$ is at most the number of neighbors of v that have different colors under X and Y. Call this quantity δ'_v. The overall probability of making S_t smaller is at most

$$\begin{aligned}\sum_{v \in S_t} \Pr[\text{choose v}] \cdot \frac{|B_X(v)|}{k} &\leq \frac{1}{nk} \sum_{v \in S_t} \delta'_v \\ &= \frac{m'}{nk}.\end{aligned}$$

If $v \notin S_t$, we calculate the probability of increasing S_t. Any color not in $C_X \cup C_Y$ guarantees progress. We can upperbound

$$|C_X \cup C_Y| \leq 2\delta_v - \delta'_v.$$

Thus the overall probability of making S_t larger is at least

$$\begin{aligned}\sum_{v \notin S_t} \Pr[\text{choose } v] \cdot \big(1 - \frac{|C_X \cup C_Y|}{k}\big) &\geq \frac{1}{nk} \sum_{v \notin S_t} (k - 2\Delta + \delta'_v) \\ &= \frac{1}{nk}(k - 2\Delta|\bar{S}_t|) + \frac{m'}{nk}.\end{aligned}$$

Finally,

$$\mathbb{E}[|S_{t+1}| - |S_t|] \geq \frac{1}{nk}(k - 2\Delta)|\bar{S_t}| + \frac{m'}{nk} - \frac{m'}{nk}$$

where the first two terms are from $v \notin S_t$ and the last is from $v \in S_t$. This is equal to

$$\frac{k - 2\Delta}{nk}|\bar{S}_t|.$$

Thus

$$\mathbb{E}[|\bar{S}_{t+1}| \mid S_t] \leq (1 - \frac{k - 2\Delta}{nk})|\bar{S}_t|.$$

Iterating expectations,

$$\begin{aligned}\mathbb{E}[|\bar{S}_t|] &\leq (1 - \frac{k - 2\Delta}{nk})|\bar{S}_0| \\ &\leq (1 - \frac{k - 2\Delta}{nk})^t n \\ &\leq \exp(-t(k - 2\Delta)/nk)n\end{aligned}$$

We want to show that this is at most $1/4$. Solving for t,

$$t \geq \frac{kn}{k - 2\Delta} \cdot \ln(4n)$$

is sufficient. Thus, our chain mixes rapidly (polylog in # of colorings). A more complicated analysis shows rapid mixing when $k > (1 + \varepsilon)\Delta$ when $\Delta = \mathcal{O}(\log n)$ and no cycle of length 10 or less.