# Matroid Intersection Minimax

## 1 Preliminaries

In this section, we'll recall some basic matroid properties and prove a useful lemma. Let $\mathcal{M} = (\mathcal{X}, \mathcal{I})$ be a matroid.

**Definition 1.** The **rank** of a set $S$ in $\mathcal{M}$, denoted $r_{\mathcal{M}}(S)$ is the maximum size of an independent set contained in $S$. That is,

$$r_{\mathcal{M}}(S) = \max_{\substack{T \subseteq S \\ T \in \mathcal{I}}} |T|$$

**Definition 2.** **Matroid deletion** is the removal of a subset of the ground set of a matroid. That is, for $T \subseteq \mathcal{X}$, we define $\mathcal{M} \setminus T = (\mathcal{X} \setminus T, \mathcal{I}')$ where $S \in \mathcal{I}'$ if and only if $S \in \mathcal{I}$ for $S \subseteq \mathcal{X} \setminus T$. In terms of the rank function,

$$r_{\mathcal{M} \setminus T}(S) = r_{\mathcal{M}}(S) \text{ with } S \subseteq \mathcal{X} \setminus T$$

**Definition 3.** **Matroid contraction** is the permanent inclusion of a certain set into the ground set; we then measure independence relative to this set. That is, for $T \subseteq \mathcal{X}$, we define $\mathcal{M}/T = (\mathcal{X}/T, \mathcal{I}')$ where $S \in \mathcal{I}'$if and only if $S \subseteq \mathcal{X} \setminus T$ and $T \cup S \in \mathcal{I}$. In terms of the rank function,

$$r_{\mathcal{M}/T}(S) = r_{\mathcal{M}}(S \cup T) - r_{\mathcal{M}}(T)$$

**Lemma 4.** *The rank function is submodular. That is, for any matroid $\mathcal{M} = (\mathcal{X}, \mathcal{I})$ and $S, T \subseteq \mathcal{X}$,*

$$r_{\mathcal{M}}(S) + r_{\mathcal{M}}(T) \geq r_{\mathcal{M}}(S \cup T) + r_{\mathcal{M}}(S \cap T)$$

*Proof.* Let $B$ be a base of $S \cap T$. We extend $B$ to a base $B_1$ of $S$ and a base $B_2$ of $T$ via the exchange property. Then, by the inclusion-exclusion principle, we have:

$$\begin{aligned} r_{\mathcal{M}}(S) + r_{\mathcal{M}}(T) - r_{\mathcal{M}}(S \cap T) &= |B_1| + |B_2| - |B| \\ &= |B_1 \cup B_2| \end{aligned}$$

*Claim* 5. $S \cup T \subseteq span(B_1 \cup B_2)$.

*Proof.* We have that $S \subseteq span(B_1)$ and $T \subseteq span(B_2)$ because they are bases. Thus $S \subseteq span(B_1 \cup B_2)$ and $T \subseteq span(B_1 \cup B_2)$, implying that $S \cup T \subseteq span(B_1 \cup B_2)$. □

Therefore,

$$\begin{aligned} |B_1 \cup B_2| &\geq r_{\mathcal{M}}(B_1 \cup B_2) \\ &= r_{\mathcal{M}}(span(B_1 \cup B_2)) \\ &\geq r_{\mathcal{M}}(S \cup T) \end{aligned}$$

Thus we obtain our desired inequality:

$$r_{\mathcal{M}}(S) + r_{\mathcal{M}}(T) - r_{\mathcal{M}}(S \cap T) \geq r_{\mathcal{M}}(S \cup T)$$

$\square$

# 2 Matroid Intersection Minimax

## 2.1 Matroid Intersection Minimax Theorem

**Theorem 6.** *Given matroids $\mathcal{M}_1 = (\mathcal{X}, \mathcal{I}_1)$ and $\mathcal{M}_2 = (\mathcal{X}, \mathcal{I}_2)$ with rank functions $r_1$ and $r_2$, let their intersection be $\mathcal{M}_1 \cap \mathcal{M}_2 = (\mathcal{X}, \mathcal{I}_1 \cap \mathcal{I}_2)$. Then, the size of the largest independent set in $\mathcal{M}_1 \cap \mathcal{M}_2$ is equal to the smallest difference in rank between two partitions of $\mathcal{X}$. That is,*

$$\max_{S \in \mathcal{I}_1 \cap \mathcal{I}_2} |S| = \min_{A \subseteq \mathcal{X}} r_1(A) - r_2(\bar{A})$$

*Note that this is a statement of duality. Furthermore, the set $A$ is constructed such that $\mathcal{M}_1$ is limiting on $A$ and $\mathcal{M}_2$ is limiting on $\bar{A}$ .*

*Proof.* We first show weak duality. Let $S \in \mathcal{I}_1 \cap \mathcal{I}_2$ and $A \subseteq \mathcal{X}$. Then we have that:

$$\begin{aligned} |S| &= |S \cap A| + |S \cap (\bar{A})| \\ &\leq r_1(A) + r_2(\bar{A}) \end{aligned}$$

That was pretty simple, but showing strong duality is not as easy. We will prove it via induction on $|\mathcal{X}|$ with the inductive hypothesis that the theorem is true for all matroids with ground set size less than $|\mathcal{X}|$. Let $\mathcal{M}_1 = (\mathcal{X} \cup \{e\}, \mathcal{I}_1)$ and $\mathcal{M}_2 = (\mathcal{X} \cup \{e\}, \mathcal{I}_2)$ with rank functions $r_1$ and $r_2$. By deleting $e$ from both matroids, the inductive hypothesis implies that:

$$\max_{\substack{S \in \mathcal{I}_1 \cap \mathcal{I}_2 \\ S \not\ni e}} |S| = \min_{A \subseteq \mathcal{X}} r_1(A) - r_2(\bar{A}) \tag{1}$$

Let $k$ be the value of Equation 1. When we add $e$ back into the ground set, the LHS either stays constant or increases by 1, while the RHS has to increase by 1. That is,

$$\forall B \subseteq \mathcal{X}, r_1(B) + r_2(\bar{B} \cup \{e\}) \geq k + 1 \tag{2}$$

Therefore, we can prove strong duality by showing that the LHS cannot possibly stay constant when $e$ is added into the ground set. Assume for the sake of contradiction that the value of the LHS stays constant when $e$ is added into the ground set. We now contract the matroids by $e$. Consider $\mathcal{M}_1' = \mathcal{M}_1/\{e\} = (\mathcal{X}/\{e\}, \mathcal{I}_1')$ and $\mathcal{M}_2' = \mathcal{M}_2/\{e\} = (\mathcal{X}/\{e\}, \mathcal{I}_2')$ with rank functions $r_1'$ and $r_2'$. Then, since the LHS cannot improve,

$$\max_{S \in \mathcal{I}_1' \cap \mathcal{I}_2'} |S \cup \{e\}| = \max_{S \in \mathcal{I}_1' \cap \mathcal{I}_2'} |S| + 1 \leq k$$

Therefore,

$$\max_{S \in \mathcal{I}_1' \cap \mathcal{I}_2'} |S| \leq k - 1$$

And,

$$r'_i(S) = r_i(S \cup \{e\}) - 1, i \in \{1, 2\}$$

So,

$$r'_1(S) + r'_2(\bar{S}) = r_1(S \cup \{e\}) + r_2(\bar{S} \cup \{e\}) - 2$$

We apply the inductive hypothesis to obtain:

$$\begin{aligned} \min_{S \subseteq \mathcal{X} \setminus \{e\}} r'_1(S) + r'_2(\bar{S}) &\leq k - 1 \\ \min_{S \subseteq \mathcal{X} \setminus \{e\}} r_1(S \cup \{e\}) + r_2(\bar{S} \cup \{e\}) &\leq k + 1 \end{aligned}$$

Fix $S$ attaining the minimum to obtain:

$$r_1(S \cup \{e\}) + r_2(\bar{S} \cup \{e\}) \leq k + 1 \tag{3}$$

Combine Equations 1 and 3 to obtain:

$$2k + 1 \geq r_1(A) + r_2(\bar{A}) + r_1(S \cup \{e\}) + r_2(\bar{S} \cup \{e\})$$

Invoking the submodularity of the rank function,

$$\begin{aligned} 2k + 1 &\geq r_1(A \cup S \cup \{e\}) + r_2(\bar{A} \cap \bar{S}) + r_1(A \cap S) + r_2(\bar{A} \cup \bar{S} \cup \{e\}) \\ &= r_1(A \cup S \cup \{e\}) + r_2(\overline{A \cup S}) + r_1(A \cap S) + r_2(\overline{A \cap S} \cup \{e\}) \end{aligned}$$

The terms are now in the form of Equation 2, reaching a contradiction as follows:

$$\begin{aligned} 2k + 1 &\geq k + 1 + k + 1 \\ 1 &\geq 2 \end{aligned}$$

□

## 2.2 Corollaries

**Corollary 7.** *The intersection of two matroid polytopes is integral for objective* **1**. *That is, for objective* **1**, *the intersection of two matroid polytopes behaves as the polytope of the matroid intersection.*

*Proof.* We have the definition of the matroid polytope intersection as the LP:

$$\begin{aligned} \max \quad & \sum_i x_i \\ \text{s.t.} \quad & \sum_{i \in T} x_i \leq r_1(T) \\ & \sum_{i \in T} x_i \leq r_2(T) \\ & \boldsymbol{x} \succeq 0 \end{aligned}$$

Let $S$ be the maximum cardinality independent set of $\mathcal{M}_1$ and $\mathcal{M}_2$. Then $\mathbf{1}$ is feasible for the above LP with value $|S|$. We take the dual LP:

$$\begin{aligned} \min \quad & \sum_{T \subseteq \mathcal{X}} r_1(T) y_T^1 + \sum_{T \subseteq \mathcal{X}} r_2(T) y_T^2 \\ \text{s.t.} \quad & \sum_{T \ni i} y_T^1 + \sum_{T \ni i} y_T^2 \geq 1 \\ & \boldsymbol{y} \succeq 0 \end{aligned}$$

This LP is a fractional version of the partition of $\mathcal{X}$ into $A$ and $\bar{A}$ that we saw earlier. Thus from the Matroid Intersection Minimax Theorem, we obtain that there exists an integral $A \subseteq \mathcal{X}$ such that $r_1(A)+r_2(\bar{A}) = |S|$. So, let $y_A^1 = 1, y_T^2 = 0$ for $T \neq A$ and $y_{\bar{A}}^2 = 1, y_T^2 = 0$ for $T \neq \bar{A}$. This $\boldsymbol{y}$ is feasible for the dual and the objective is:

$$\begin{aligned} \sum_{T \subseteq \mathcal{X}} r_1(T) y_T^1 + \sum_{T \subseteq \mathcal{X}} r_2(T) y_T^2 &= r_1(A) + r_2(\bar{A}) \\ &= |S| \end{aligned}$$

Thus by strong duality we have obtained an optimal integral solution. $\square$

**Corollary 8.** *Känig's Theorem: In a bipartite graph, the maximum cardinality of a matching is equal to the minimum cardinality of a vertex cover.*

*Proof.* An informal proof is that a bipartite graph is the intersection of two partition matroids, wherein we let $A$ be the subset of edges covered by the nodes on the left and $\bar{A}$ be the subset of edges covered by the nodes on the right. Then the cardinality of the maximum matching is $r_1(A) + r_2(\bar{A})$ so the cardinality of the minimum vertex cover must be equal. $\square$