Matroid Polytopes

1 Matroid Polytopes

As is often the case with tractable discrete problems, we can view the independent sets of a matroid as a polytope. This polytope is solvable, which paves the way for advanced applications of matroids. We define the **matroid polytope** $\mathcal{P}(\mathcal{M})$:

$$\sum_{j \in S} x_j \leq rank_{\mathcal{M}}(S) \ \forall S \subseteq \mathcal{X}$$
$$x_j \geq 0 \ \forall j \in \mathcal{X}$$

This polytope assigns a variable x_j to each element $j \in \mathcal{X}$, and sets each feasible $\boldsymbol{x}$ as a fractional subset of $\mathcal{X}$. Note that $0 \leq x_j \leq 1$ because the rank of a singleton is at most 1. Thus, the 0-1 indicator vector $\boldsymbol{x}^I$ for an independent set $I \in \mathcal{I}$ is in the polytope. However, it is not immediately clear that $\mathcal{P}(\mathcal{M})$ is the convex hull of $\mathcal{I}$.

Theorem 1. $\mathcal{P}(\mathcal{M}) = convHull(\{\boldsymbol{x}^I : I \in \mathcal{I}\})$.

Proof. It is clear that $\mathcal{P}(\mathcal{M}) \supseteq convHull(\{\boldsymbol{x}^I : I \in \mathcal{I}\})$. To show that $\mathcal{P}(\mathcal{M}) \subseteq convHull(\{\boldsymbol{x}^I : I \in \mathcal{I}\})$, we show that every vertex of $\mathcal{P}(\mathcal{M})$ is exactly $\boldsymbol{x}^I$ for $I \in \mathcal{I}$. Recall that it suffices to show that every linear function $\boldsymbol{w}^\intercal \boldsymbol{x}$ is maximized over $\mathcal{P}(\mathcal{M})$ at such an $\boldsymbol{x}^I$.

We can think of the greedy algorithm as an algorithm which computes an indicator vector $\boldsymbol{x}^* = \boldsymbol{x}^B \in \mathcal{P}(\mathcal{M})$. We will show that $\boldsymbol{x}^*$ maximizes $\boldsymbol{w}^\intercal \boldsymbol{x}$ over $\boldsymbol{x} \in \mathcal{P}(\mathcal{M})$.

Recall that $i \in \{1, \ldots, n\}$ is selected by the greedy algorithm if and only if $i \notin span(\{1, \ldots, i-1\})$. That is, i is selected if and only if $rank[1:i] - rank[1:i-1] = 1$. Thus $x_i^* = rank[1:i] - rank[1:i-1]$ for non-negative weights i and 0 for negative weights i. Let weights $1, \ldots, n$ be the non-negative ones. Then we have:

$$\sum_{i \in \mathcal{X}} w_i x_i^* = \sum_{i=1}^{n} w_i x_i^* = \sum_{i=1}^{n} w_i rank[1:i] - rank[1:i-1]$$

Consider an arbitrary $x \in \mathcal{P}(\mathcal{M})$. We have:

$$\sum_{i \in \mathcal{X}} w_i x_i \leq \sum_{i=1}^{n} w_i x_i$$

Now we can switch from a Riemann sum to a Lebesgue sum – that is, sum across the w's instead of the i's. The above is then equal to:

$$\sum_{i=1}^{n} (w_i - w_{i+1}) x[1:i] \leq \sum_{i=1}^{n} (w_i - w_{i+1}) rank[1:i]$$

We now switch back to a Riemann sum to obtain that the above is equal to:

$$\sum_{i=1}^{n} w_i (rank[1:i] - rank[1:i-1])$$

Thus we establish that $\boldsymbol{x}^*$ maximizes $\boldsymbol{w}^\intercal \boldsymbol{x}$ over $\boldsymbol{x} \in \mathcal{P}(\mathcal{M})$ and the theorem follows. $\square$

When given an independence oracle for $\mathcal{M}$, we can maximize linear functions over $\mathcal{P}(\mathcal{M})$ in time $\mathcal{O}(n \log n) + nT$ where T is the runtime of the oracle (this runtime is the same as the greedy algorithm, because they are equivalent problems). Thus by the equivalence of separation and optimization, we can implement a separation oracle for $\mathcal{P}(\mathcal{M})$ in time $poly(n, T)$, which is awesome due to the exponential constraints in $\mathcal{P}(\mathcal{M})$.

1.1 Matroid Base Polytope

The matroid polytope is the convex hull of independent sets, but what if we only want to study full-rank sets (e.g., spanning trees)? We define the **matroid base polytope** $\mathcal{P}_{base}(\mathcal{M})$:

$$\begin{aligned} \sum_{j \in S} x_j &\leq rank_{\mathcal{M}}(S) \ \forall S \subseteq \mathcal{X} \\ \sum_{j \in \mathcal{X}} x_j &= rank(\mathcal{M}) \\ x_j &\geq 0 \ \forall j \in \mathcal{X} \end{aligned}$$

Then the 0-1 indicator vector for every base of $\mathcal{M}$ is in the above polytope.

Theorem 2. *$\mathcal{P}_{base}(\mathcal{M}) = convHull(\{\boldsymbol{x}^B : B \text{ is a base of } \mathcal{M}\})$.*

Proof. It is clear that $\mathcal{P}_{base}(\mathcal{M}) \supseteq convHull(\{\boldsymbol{x}^B : B \text{ is a base of } \mathcal{M}\})$. To show that $\mathcal{P}_{base}(\mathcal{M}) \subseteq convHull(\{\boldsymbol{x}^B : B \text{ is a base of } \mathcal{M}\})$, first take $\boldsymbol{x} \in \mathcal{P}_{base}(\mathcal{M})$. Since $\boldsymbol{x} \in \mathcal{P}(\mathcal{M})$, $\boldsymbol{x}$ is a convex combination of independent sets $I_1, \ldots, I_k$ of $\mathcal{M}$.

Since $\|\boldsymbol{x}\|_1 = rank(\mathcal{M})$, and $\|\boldsymbol{x}^{I_\ell}\|_1 \leq rank(\mathcal{M})$ for all ℓ, it must be that $\|\boldsymbol{x}^{I_1}\|_1 = \|\boldsymbol{x}^{I_2}\|_1 = \cdots = \|\boldsymbol{x}^{I_k}\|_1 = rank(\mathcal{M})$. That is, each independent set has rank at most that of $\mathcal{M}$, but they average out to have the rank exactly of $\mathcal{M}$, so they must be equal. □

2 Matroid Intersection

Here, we will analyze an operation on matroids which does not produce a matroid, but does produce a solvable problem. Given $\mathcal{M}_1 = (\mathcal{X}, \mathcal{I}_1)$, $\mathcal{M}_2 = (\mathcal{X}, \mathcal{I}_2)$, then $\mathcal{M}_1 \cap \mathcal{M}_2 = (\mathcal{X}, \mathcal{I}_1 \cap \mathcal{I}_2)$. A few examples of matroid intersection include bipartite matching, which is an intersection of two partition matroids, and the minimum weight arborescence, which is an intersection of a graphic matroid and a partition matroid. Other examples include colorful spanning trees and orientations. Note that optimizing a linear function over $\mathcal{M}_1 \cap \mathcal{M}_2$ is equivalent to optimizing over $convHull(\{\boldsymbol{x}^I : I \in \mathcal{I}_1 \cap \mathcal{I}_2\})$; however it is not clear that this is a solvable polytope.

Theorem 3. *$\mathcal{P}(\mathcal{M}_1) \cap \mathcal{P}(\mathcal{M}_2) = convHull(\{\boldsymbol{x}^I : I \in \mathcal{I}_1 \cap \mathcal{I}_2\})$.*

Proof. As above, the $\supseteq$ direction is clear. But, for the $\subseteq$ direction, it is conceivable that $\mathcal{P}(\mathcal{M}_1) \cap \mathcal{P}(\mathcal{M}_2)$ has fractional vertices. As it turns out, this can't happen but is tough to prove and requires the exploitation of subtle structures of the matroid polytopes (link on course website). □

Thus for maximizing over matroid intersections, we have the following LP:

$$\begin{aligned} \max \quad & \sum_{i \in \mathcal{X}} w_i x_i \\ \text{s.t.} \quad & \sum_{j \in S} x_j \leq rank_{\mathcal{M}_1}(S) \ \forall S \subseteq \mathcal{X} \\ & \sum_{j \in S} x_j \leq rank_{\mathcal{M}_2}(S) \ \forall S \subseteq \mathcal{X} \\ & x_j \geq 0 \ \forall j \in \mathcal{X} \end{aligned}$$

Then given independence oracles for $\mathcal{M}_1$ and $\mathcal{M}_2$, we can optimize over the intersection in polynomial time (proof by equivalence of separation and optimization). But, 3-way matroid intersection is NP-hard by reduction from Hamiltonian Path in directed graphs. We can see that solving the Hamiltonian Path problem is equivalent to optimizing a modular function over the intersection of a graphic matroid and two partition matroids, but this cannot be done in polynomial time.