Geometric Duality

1 Convexity and Duality

There are two equivalent ways to represent a convex set, and their equivalence gives rise to a duality relationship.

Standard: A set is represented by the family of all the points in the set.

Dual: A set is represented by the set of halfspaces containing the set.

Theorem 1. *A closed convex set S is the intersection of all closed halfspaces $\mathcal{H}$ containing it.*

Proof. Clearly, $S \subseteq \bigcap_{H \in \mathcal{H}} H$. Now, consider $\boldsymbol{x} \notin S$. By the Separating Hyperplane Theorem, there exists a hyperplane separating $\boldsymbol{x}$ and S. Therefore, there is some $H \in \mathcal{H}$ with $\boldsymbol{x} \notin H$, so $\boldsymbol{x} \notin \bigcap_{H \in \mathcal{H}} H$. $\square$

Theorem 2. *A closed convex cone K is the intersection of all closed homogeneous halfspaces $\mathcal{H}$ containing it.*

Proof. For every non-homogeneous halfspace $\boldsymbol{a}^\intercal \boldsymbol{x} \leq b$ containing K, the smaller homogeneous halfspace $\boldsymbol{a}^\intercal \boldsymbol{x} \leq 0$ also contains K (because K is a cone). Therefore we can discard non-homogeneous halfspaces in the intersection. $\square$

Theorem 3. *A convex function is the pointwise supremum of all affine functions underestimating it everywhere.*

Proof. The epigraph of any convex function is convex, so it is the intersection of halfspaces $\mathcal{H}$. Each $H \in \mathcal{H}$ can be written $\boldsymbol{a}^\intercal \boldsymbol{x} - t \leq b$ for $\boldsymbol{a} \in \mathbb{R}^n, b \in \mathbb{R}$. Note that since $epi(f)$ goes on forever upwards, we need to be able to increase t to infinity and still satisfy the inequality – thus the subtraction.

This definition constraints each $(\boldsymbol{x}, t) \in epi(f)$ to $\boldsymbol{a}^\intercal \boldsymbol{x} - b \leq t$. Therefore, $f(\boldsymbol{x})$ is the lowest value of t such that $(\boldsymbol{x}, t) \in epi(f)$. So, $f(\boldsymbol{x})$ is the pointwise supremum of $\boldsymbol{a}^\intercal \boldsymbol{x} - b$ over all halfspaces $H(\boldsymbol{a}, b) \in \mathcal{H}$. $\square$

2 Duality of Convex Sets

2.1 Polar Duality

Definition 4. Let $S \subseteq \mathbb{R}^n$ be a closed convex set containing the origin. Then the **polar** of S is

$$S^\circ = \{\boldsymbol{y} : \boldsymbol{y}^\intercal \boldsymbol{x} \leq 1 \ \forall \boldsymbol{x} \in S\}$$

Note that every halfspace $\boldsymbol{a}^\intercal \boldsymbol{x} \leq b$ with $b \neq 0$ can be written as a normalized inequality $\boldsymbol{y}^\intercal \boldsymbol{x} \leq 1$ by dividing by b. Thus the polar of S is the normalized representation of halfspaces containing S. Some properties of the polar include:

1. $S^{\circ\circ} = S$

2. S° is a closed convex set containing the origin.

3. When $\mathbf{0}$ is in the interior of S, S° is bounded.

4. Polars scale as $\epsilon \to \frac{1}{\epsilon}$, so the bigger S is, the smaller the polar is.

5. When S is nonconvex, $S^{\circ} = (convHull(S))^{\circ}$ and $S^{\circ\circ} = convHull(S)$.

For polytopes, the vertices of the polytope are dual to the faces of the polar; thus the polar of the unit diamond is the unit cube. In this case, the polar is the convex hull of the rows of A. Also, if $S = S^{\circ}$, we say that S is **self-dual**. One example of this is that the 2-norm ball is self-dual.

2.2 Polar Duality for Cones

Since cones are more restrictive than general convex sets, polarity has a simplified form. Given a cone K,

$$K^{\circ} = \{\boldsymbol{y} : \boldsymbol{y}^{\mathsf{T}}\boldsymbol{x} \leq 0 \ \forall \boldsymbol{x} \in K\}$$

This follows from the original definition because cones are closed under scaling; that is, $\boldsymbol{y}^{\mathsf{T}}\boldsymbol{x} \leq 0 \iff \boldsymbol{y}^{\mathsf{T}}\boldsymbol{x} \leq 1$ because we can just re-scale the points. Because of this simplified form, we can say that K° is the set of all homogeneous halfspaces containing K. Intuitively, K° is the cone formed by the set of vectors that make an angle greater than 90 degrees with the borders of K.

This definition allows us to redefine Farkas' Lemma as:

$$\exists \boldsymbol{z} \in K^{\circ} \text{ s.t. } \boldsymbol{z}^{\mathsf{T}}\boldsymbol{w} > 0, \boldsymbol{w} \notin K$$

3 Duality of Convex Functions

Definition 5. Given a function $f : \mathbb{R}^n \to \mathbb{R}$, the **conjugate** of f is

$$f^*(\boldsymbol{y}) = \sup_{\boldsymbol{x}}(\boldsymbol{y}^{\mathsf{T}}\boldsymbol{x} - f(\boldsymbol{x}))$$

Intuitively, $f^*(\boldsymbol{y})$ is the minimal β such that the affine function $\boldsymbol{y}^{\mathsf{T}}\boldsymbol{x} - \beta$ underestimates $f(\boldsymbol{x})$ everywhere. In terms of convex sets, $f^*(\boldsymbol{y})$ is the distance we need to lower the hyperplane $\boldsymbol{y}^{\mathsf{T}}\boldsymbol{x} - t = 0$ to obtain a supporting hyperplane to $epi(f)$. Some properties include:

1. $f^{**} = f$ when f is convex.

2. f^* is a convex function.

3. Fenchel's Inequality: $\boldsymbol{x}\boldsymbol{y} \leq f(\boldsymbol{x}) + f^*(\boldsymbol{y})$ for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$.