Bipartite Matching

1 Bipartite Matching

1.1 Setup

Problem. Given a bipartite graph $G = (V, E)$ with $V = L \bigcup R$ and weights w_e on each edge $e \in E$, find a maximum weight matching between L and R.

Recall that a matching is a set of edges covering each node at most once. Let $n = |V|$ and $m = |E|$. Our focus here will be on using the polyhedral interpretation of this problem to gain insights about its structure.

1.2 Bipartite Matching LP Relaxation

We have the LP:

$$\begin{aligned} \max \quad & \sum_{e \in E} w_e x_e \\ \text{s.t.} \quad & \sum_{e \in \delta(v)} x_e \leq 1 \ \forall v \in V \\ & x_e \geq 0 \ \forall e \in E \end{aligned}$$

Note that this LP is a relaxation because we allow fractional matchings; as proved in last lecture, the vertices of the feasible polytope are the indicator vectors of matchings.

Theorem 1. *The feasible region of the matching LP is the convex hull of the indicator vectors of matchings. Thus, solving this LP is exactly equivalent to solving the combinatorial problem.*

Proof. It suffices to show that all vertices of the feasible polytope are integral. Consider $\boldsymbol{x} \in P$ non-integral, we will show that $\boldsymbol{x}$ is not a vertex. Let H be the subgraph formed by edges with $x_e \in (0, 1)$. H either contains a cycle, or else a maximal simple path. Informally, either way, we can add and subtract ϵ around the cycle and maintain feasibility, which shows that $\boldsymbol{x}$ is not a vertex. □

2 Total Unimodularity

A matrix $\mathbf{A}$ is called **totally unimodular** if every square submatrix has determinant 0, 1, or -1.

Theorem 2. *If $\mathbf{A} \in \mathbb{R}^{m \times n}$ is totally unimodular, and $\boldsymbol{b}$ is an integer vector, then $\{\boldsymbol{x} : \mathbf{A}\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{x} \geq 0\}$ has integer vertices.*

Proof. By Cramer's Rule. □

Claim 3. The constraint matrix of the bipartite matching LP is totally unimodular. This is another way we can prove integral vertices.

Proof. $\mathbf{A}_{ve} = 1$ if e is incident on v, and 0 otherwise. By induction on the size of the submatrix $\mathbf{A}'$, we can show that $\det(\mathbf{A}') = 0$. Thus $\mathbf{A}$ is totally unimodular. □

3 Duality of Bipartite Matching

The dual LP of bipartite matching is:

$$\begin{aligned} \min \ & \sum_{v \in V} y_v \\ \text{s.t. } & y_u + y_v \geq w_e \ \forall e = (u, v) \in E \\ & y_v \succeq 0 \ \forall v \in V \end{aligned}$$

The primal interpretation is that Player 1 is looking to build a set of projects. Each edge e is a project generating profit w_e. Each project e needs two resources, u and v. Each resource can be used by at most one project at a time. How can we choose a profit-maximizing set of projects? In the dual, Player 2 is looking to purchase resources and offers a price y_v for each resource. The prices should incentivize Player 1 to sell his resources, but Player 2 wants to pay as little as possible. Note that when edge weights are 1, binary solutions to the dual are vertex covers. Thus, the dual is a relaxation of the minimum cost vertex cover problem for bipartite graphs. Strong duality and the total unimodularity of the constraints leads to Konig's Theorem: the cardinality of the maximum matching is equal to the cardinality of the minimum vertex cover.