Convex Sets

1 Definitions

1.1 Convex Sets

Definition 1. A **convex combination** of a finite set of points $\boldsymbol{x_1}, \dots, \boldsymbol{x_k} \in \mathbb{R}^n$ is $c_1\boldsymbol{x_1} + \cdots + c_k\boldsymbol{x_k}$ with $\sum_{i=1}^k c_i = 1$ and $c_i \geq 0 \ \forall i$. A convex combination of an infinite set of points is defined as the expectation of some probability distribution defined over the set.

Definition 2. A set $S \subseteq \mathbb{R}^n$ is a **convex set** if the below equivalent conditions are met:

1. For every $\boldsymbol{x}, \boldsymbol{y} \in S$ and $\theta \in [0, 1]$, $\theta\boldsymbol{x} + (1-\theta)\boldsymbol{y} \in S$. That is, for every two points in S, the entire line segment between them lies in S.
2. Every convex combination of points in S lies in S.

While a convex set need not be topologically closed, interesting ones usually are.

Definition 3. A set $T \subseteq \mathbb{R}^n$ is the **convex hull** of a set $S \subseteq \mathbb{R}^n$ if the below equivalent conditions are met:

1. T is the smallest convex set containing S.
2. T is the intersection of all convex sets containing S.
3. T is the set of all convex combinations of points in S.

Note that a set is convex if and only if it is its own convex hull.

1.2 Affine Sets

Definition 4. An **affine combination** of a finite set of points $\boldsymbol{x_1}, \dots, \boldsymbol{x_k} \in \mathbb{R}^n$ is $c_1\boldsymbol{x_1} + \cdots + c_k\boldsymbol{x_k}$ with $\sum_{i=1}^k c_i = 1$.

Definition 5. Recall that a **linear subspace** is a vector space which is a subspace of $\mathbb{R}^n$ and therefore must include $\mathbf{0}$. A **shifted subspace** is obtained by centering a linear subspace at some $\boldsymbol{x_0}$, which therefore may or may not include $\mathbf{0}$.

Definition 6. A set $S \subseteq \mathbb{R}^n$ is an **affine set** if the below equivalent conditions are met:

1. For every $\boldsymbol{x}, \boldsymbol{y} \in S$ and $\theta \in \mathbb{R}$, $\theta\boldsymbol{x} + (1-\theta)\boldsymbol{y} \in S$. That is, for every two points in S, the entire line between them lies in S.
2. Every affine combination of points in S lies in S.
3. S is a shifted subspace.

4. S is the set of solutions to a set of $k \leq n$ linear equations (i.e., the intersection of k hyperplanes). That is, $S = \{\boldsymbol{x} : \mathbf{A}\boldsymbol{x} = \boldsymbol{b}\}$ for $\mathbf{A} \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$.

Definition 7. A set $T \subseteq \mathbb{R}^n$ is the **affine hull** of a set $S \subseteq \mathbb{R}^n$ if the below equivalent conditions are met:

1. T is the smallest affine set containing S.
2. T is the intersection of all affine sets containing S.
3. T is the set of all affine combinations of points in S.
4. T is the subspace with dimension equivalent to that of S.

Note that a set is affine if and only if it is its own convex hull.

Definition 8. The **affine dimension** of a set $S \subseteq \mathbb{R}^n$ is the dimension of its affine hull. If the affine dimension of S is also the dimension of S, S is said to have **full affine dimension**.

1.3 Conic Sets

Definition 9. A set $S \subseteq \mathbb{R}^n$ is a **conic set**, or **cone**, if the ray from the origin through any point in S lies in S. Note that cones must contain $\mathbf{0}$.

Definition 10. A **conic combination** of a finite set of points $\boldsymbol{x_1}, \ldots, \boldsymbol{x_k} \in \mathbb{R}^n$ is $c_1\boldsymbol{x_1} + \cdots + c_k\boldsymbol{x_k}$ with $c_i \geq 0 \; \forall i$.

Definition 11. A set $S \subseteq \mathbb{R}^n$ is a **convex cone** if every conic combination of points in S lies in S.

Definition 12. A cone $K \subseteq \mathbb{R}^n$ is a **pointed cone** if $\boldsymbol{x} \in K, \boldsymbol{x} \neq 0 \implies -\boldsymbol{x} \notin K$.

Definition 13. A cone $K \subseteq \mathbb{R}^n$ is a **proper cone** if it is convex, pointed, closed, and has full affine dimension.

Definition 14. A cone $K \subseteq \mathbb{R}^n$ is a **polyhedral cone** if it is the set of solutions to a finite set of homogenous linear inequalities $\mathbf{A}\boldsymbol{x} \leq \mathbf{0}$.

Definition 15. A set $T \subseteq \mathbb{R}^n$ is the **conic hull** of a set $S \subseteq \mathbb{R}^n$ if the below equivalent conditions are met:

1. T is the smallest convex cone containing S.
2. T is the intersection of all convex cones containing S.
3. T is the set of all conic combinations of points in S.

Note that a set is a convex cone if and only if it is its own conic hull.

2 Convexity-Preserving Set Operations

2.1 Intersections

The intersection of k convex sets is convex. For example, a polyhedron is a convex set generated by the intersection of k halfspaces. The PSD cone, comprised of all symmetric positive semidefinite matrices, is the intersection of k linear inequalities – that is, infinitely many halfspaces. In fact, we will prove that any closed convex set is the intersection of a (possibly infinite) set of halfspaces.

2.2 Affine Maps

If $f : \mathbb{R}^n \to \mathbb{R}^m$ is an affine function, then:

1. $f(S)$ is convex when $S \subseteq \mathbb{R}^n$ is a convex set.
2. $f^{-1}(T)$ is convex when $T \subseteq \mathbb{R}^m$ is a convex set.

This fact is often used to shortcuts proofs of convexity for sets. For example, an ellipsoid is the image of the (convex) unit ball after an affine function is applied. The polyhedron $\mathbf{A}\boldsymbol{x} \leq \boldsymbol{b}$ is the inverse image of the (convex) nonnegative orthant under $f(\boldsymbol{x}) = \boldsymbol{b} - \mathbf{A}\boldsymbol{x}$.

2.3 Perspective Functions

Define the perspective function $P : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be $P(\boldsymbol{x}, \boldsymbol{t}) = \frac{\boldsymbol{x}}{\boldsymbol{t}}$. Then:

1. $P(S)$ is convex when $S \subseteq \mathbb{R}^{n+1}$ is a convex set.
2. $P^{-1}(T)$ is convex when $T \subseteq \mathbb{R}^n$ is a convex set.

Note that P is a many-to-one function, so $P^{-1}(T)$ is the union of infinitely many possible input sets S such that $P(S) = T$. This second implication is surprising and quite powerful.