LP Duality

1 Introduction to Duality

1.1 Introduction

Let the **primal** LP be written in maximization standard form:

$$\begin{aligned} \max\ & \boldsymbol{c}^\intercal \boldsymbol{x} \\ \text{s.t. } & \mathbf{A}\boldsymbol{x} \le \boldsymbol{b} \\ & \boldsymbol{x} \ge 0 \end{aligned}$$

Then, the **dual** LP is written in minimization standard form:

$$\begin{aligned} \min\ & \boldsymbol{b}^\intercal \boldsymbol{y} \\ \text{s.t. } & \mathbf{A}^\intercal \boldsymbol{y} \ge \boldsymbol{c} \\ & \boldsymbol{y} \ge 0 \end{aligned}$$

Here, y_i is the dual variable corresponding to the primal constraint $\mathbf{A}_i \boldsymbol{x} \le b_i$, and $\mathbf{A}_j^\intercal \boldsymbol{y} \le c_j$ is the dual constraint corresponding to the primal variable x_j. Note that the primal LP has constraints on the rows of $\mathbf{A}$, while the dual LP has constraints on the columns of $\mathbf{A}$. In a nutshell, the dual variable measures the importance of the primal constraint, and vice versa. Furthermore, note that duality is an inversion: the dual of a dual is the primal.

1.2 Interpretations

1.2.1 Economic Interpretation

Recall the optimal production problem from the previous lecture. Let the primal LP be from the perspective of the production facility, which wants to maximize profit while staying within material constraints. Then, the dual LP is from the perspective of an outside buyer who wants to spend as little as possible while offering the facility an incentive to sell their raw materials rather than create products. The dual variable y_i represents the proposed price per unit of material i, corresponding to the primal constraint $\mathbf{A}_i \boldsymbol{x} \le b_i$ which represents the amount of material i available. The primal variable x_j represents the amount of product j that the production facility plans to make, corresponding to the dual constraint $\mathbf{A}_j^\intercal \boldsymbol{y} \le c_j$ which represents the possible profit of product j.

1.2.2 Physical Interpretation

Consider applying a force field $\boldsymbol{c}$ to a ball inside a bounded polytope $\mathbf{A}\boldsymbol{x} \le \boldsymbol{b}$. Eventually, the ball will come to rest against the "walls" – that is, the hyperplanes created by the LP constraints. At this point, the wall $\mathbf{A}_i \boldsymbol{x} \le b_i$ applies a force $-y_i \mathbf{A}_i$ to the ball. Since the ball is at rest, $\boldsymbol{c}^\intercal = \sum_i y_i \mathbf{A}_i = \boldsymbol{y}^\intercal \mathbf{A}$ – exactly the dual constraint. The objective function of the dual represents minimizing the work required to bring the ball back to the origin. We will see that, due to complementary slackness, only the walls adjacent to the ball push back at optimality.

2 Forms of Duality

2.1 Weak Duality

Theorem 1 (Weak Duality). *For every primal feasible $\boldsymbol{x}$ and dual feasible $\boldsymbol{y}$, $\boldsymbol{c}^\intercal \boldsymbol{x} \leq \boldsymbol{b}^\intercal \boldsymbol{y}$.*

Proof. When the primal and dual constraints are both satisfied:

$$\boldsymbol{c}^\intercal \boldsymbol{x} \leq \boldsymbol{y}^\intercal \mathbf{A} \boldsymbol{x} \leq \boldsymbol{y}^\intercal \boldsymbol{b} = \boldsymbol{b}^\intercal \boldsymbol{y}$$

□

Corollary 2. *If both LPs are feasible and bounded, the optimal primal solution is at most the optimal dual solution.*

Corollary 3. *The unboundedness of the primal implies the infeasiblity of the dual, and vice versa.*

Corollary 4. *If $\boldsymbol{x}^*$ is a feasible primal solution and $\boldsymbol{y}^*$ is a feasible dual solution, and $\boldsymbol{c}^\intercal \boldsymbol{x}^* = \boldsymbol{b}^\intercal \boldsymbol{y}^*$, then both are optimal.*

In the economic interpretation, **weak duality** shows that if selling the raw materials is more profitable than making any product, the money collected from the sale of the materials will be greater than from the production profit. In the physical interpretation, weak duality shows that the work required to bring the ball back to the origin is at least the potential energy difference between the origin and the primal optimum.

2.2 Strong Duality

Theorem 5 (Strong Duality). *If the primal LP is feasible and bounded, then so is the dual LP, and their optimums are equivalent.*

In the economic interpretation, **strong duality** shows that the outside buyer can offer prices which would make the facility indifferent between production and sale of raw materials. In the physical interpretation, strong duality shows that there is an assignment of forces to the walls which will bring the ball back to the origin with no wasted energy.

For an informal proof, consider the physical interpretation. When the ball is at rest, we expect only the walls touching the ball to help push the ball back. Thus $\mathbf{A}^\intercal \boldsymbol{y} = \boldsymbol{c}$ and $y_i(b_i - a_i \boldsymbol{x}) = 0$. Then:

$$\boldsymbol{y}^\intercal \boldsymbol{b} - \boldsymbol{c}^\intercal \boldsymbol{x} = \boldsymbol{y}^\intercal \boldsymbol{b} - \boldsymbol{y}^\intercal \mathbf{A} \boldsymbol{x} = \sum_i y_i(b_i - a_i \boldsymbol{x}) = 0$$

3 Complementary Slackness

Let $s_i = (\boldsymbol{b} - \mathbf{A}\boldsymbol{x})_i$ be the i^{th} primal slack variable and $t_j = (\mathbf{A}^\intercal \boldsymbol{y} - \boldsymbol{c})_j$ be the j^{th} dual slack variable. Then, **complementary slackness** says that a feasible $\boldsymbol{x}$ and $\boldsymbol{y}$ are optimal if and only if $x_j t_j = 0\ \forall j \in [n]$ and $y_i s_i = 0\ \forall i \in [m]$. That is, the dual variable is zero for the constraints which are not binding at the optimal primal solution, and vice versa. This is the meat behind the intuition that the dual variable measures the importance of the primal constraint (and vice versa), because only the "important" constraints will have nonzero dual variables associated with them.

In the economic interpretation, complementary slackness shows that the facility only produces products for which it is indifferent between sales and production, and only raw materials which are binding constraints on production are pricing at greater than 0. In the physical interpretation, complementary slackness shows that only the walls adjacent to the ball push back on it.

Proof. Given a primal and dual LP as in section 1, we can rewrite them using slack variables. We have the primal LP:

$$\begin{aligned} \max \quad & \boldsymbol{c}^\mathsf{T}\boldsymbol{x} \\ \text{s.t.} \quad & \mathbf{A}\boldsymbol{x} + \boldsymbol{s} = \boldsymbol{b} \\ & \boldsymbol{x} \geq 0 \\ & \boldsymbol{s} \geq 0 \end{aligned}$$

And the corresponding dual LP:

$$\begin{aligned} \min \quad & \boldsymbol{b}^\mathsf{T}\boldsymbol{y} \\ \text{s.t.} \quad & \mathbf{A}^\mathsf{T}\boldsymbol{y} - \boldsymbol{t} = \boldsymbol{c} \\ & \boldsymbol{y} \geq 0 \\ & \boldsymbol{t} \geq 0 \end{aligned}$$

Then:

$$\boldsymbol{y}^\mathsf{T}\boldsymbol{b} - \boldsymbol{c}^\mathsf{T}\boldsymbol{x} = \boldsymbol{y}^\mathsf{T}(\mathbf{A}\boldsymbol{x} + \boldsymbol{s}) - (\boldsymbol{y}^\mathsf{T}\mathbf{A} - \boldsymbol{t}^\mathsf{T})\boldsymbol{x} = \boldsymbol{y}^\mathsf{T}\boldsymbol{s} - \boldsymbol{t}^\mathsf{T}\boldsymbol{x}$$

By strong duality, this equation must equal zero, which is only possible if complementary slackness holds. $\square$