More LP Duality

1 Analysis of LPs

1.1 Recovery of the Primal

Complementary slackness allows us to recover the primal optimal from the dual optimal, and vice versa. To show this, let us first assume **non-degeneracy**: At every vertex of the primal [dual], there exists exactly n [m] tight constraints which are linearly independent. Under this condition, the primal LP has n variables, m generic constraints, and n non-negativity constraints (vice versa for the dual).

Let $\boldsymbol{y}$ be the dual optimal. Then, by non-degeneracy, exactly m of the $m+n$ total dual constraints are tight at $\boldsymbol{y}$. So, n dual constraints must be loose, implying that n primal constraints are tight. This creates a linear system of equations which can be solved to find the optimal primal $\boldsymbol{x}$.

1.2 Sensitivity Analysis

Oftentimes, we want to measure how the LP optimum changes with parameters $\boldsymbol{b}$ and $\boldsymbol{c}$. Let $OPT = OPT(\mathbf{A}, \boldsymbol{b}, \boldsymbol{c})$ be the optimal value of the LP. Then, when LP solutions $\boldsymbol{x}$ and $\boldsymbol{y}$ are unique, $\frac{\partial OPT}{\partial c_j} = x_j$ and $\frac{\partial OPT}{\partial b_i} = y_i$. In the economic interpretation, this shows that a small increase δ in the production parameter c_j increases profit by δx_j. Furthermore, a small increase δ in the material availability parameter b_i increases the price by δy_i. Thus x_j measures the impact of product j on profit, and y_i measures the impact of the amount of material i available on its price.

2 Examples of Duality

2.1 Shortest Path

Duality often provides insights into a problem that we would not have obtained otherwise, even in simple cases. For example, consider the shortest path problem: given $G = (V, E)$ where edge e has length $\ell_e \in \mathbb{R}_+$, find the minimum length path from $s \to t$. The primal LP solves this problem by sending one unit of flow from $s \to t$, from which we can read off the shortest path. The dual LP solves this problem by "stretching" the graph across the $s \to t$ axis as far as possible such that no edge breaks; then the shortest path is the first path to become taut.

2.2 Two-Player Zero-Sum Games

In a two-player zero-sum game such as rock-paper-scissors, we have a payoff matrix $\mathbf{A}$ which corresponds to the payoff for the row player based on what the column player does. For example, if the row player played rock and the column player played paper, the row player would have a payoff of -1. We assume that the players are using mixed strategies; that is, distributions over pure strategies. For example, one could play uniformly randomly between rock, paper, and scissors, which corresponds to a mixed strategy of $[\frac{1}{3}, \frac{1}{3}, \frac{1}{3}]$.

We assume the row player moves first with a mixed strategy $\boldsymbol{y}$. Then, the payoff is a function of the column strategy $\boldsymbol{y}^\intercal \mathbf{A}$. The best column strategy is that which maximizes $(\boldsymbol{y}^\intercal \mathbf{A})_j$. Thus, the row player solves an LP to determine the best $\boldsymbol{y}$ and corresponding payoff u.

$$\begin{aligned} \min \quad & \max_j (\boldsymbol{y}^\intercal \mathbf{A})_j \\ \text{s.t.} \quad & \sum_{i=1}^m y_i = 1 \\ & \boldsymbol{y} \geq 0 \end{aligned}$$

However, this is not an LP because a maximization is not a linear function. We can turn it into an LP by using u:

$$\begin{aligned} \min \quad & u \\ \text{s.t.} \quad & \boldsymbol{u} - \boldsymbol{y}^\intercal \mathbf{A} \geq 0 \\ & \sum_{i=1}^m y_i = 1 \\ & \boldsymbol{y} \geq 0 \end{aligned}$$

The corresponding column player LP is:

$$\begin{aligned} \min \quad & v \\ \text{s.t.} \quad & \boldsymbol{v} - \mathbf{A}\boldsymbol{x} \leq 0 \\ & \sum_{i=1}^n x_j = 1 \\ & \boldsymbol{x} \geq 0 \end{aligned}$$

These LPs are duals, which allows us to gain insight into the problem. In particular, by applying weak duality, we see that $u^* \geq v^*$, so two-player zero-sum games have a weak second mover advantage. Strong duality leads us to the **Minimax Theorem**: $u^* = v^*$ and there is no second mover advantage; each player can guarantee that their mixed-strategy payoff is exactly that of the other player. So, $y^*$and x^* are simultaneously optimal, called a **Nash equilibrium**.

3 Proof of Strong Duality

3.1 Background

Theorem 1 (Separating Hyperplane Theorem)**.** *If $A, B \in \mathbb{R}^n$ are disjoint convex sets, then there is a hyperplane separating them. If both are closed and at least one is compact, then there is a hyperplane strictly separating them.*

The proof of the Separating Hyperplane Theorem is a bit involved, so we will not show it here. One can see the difference between the two cases by considering the example where A is the set of everything below the x axis and B is the set of everything above the line $y = x^{-1}$. Since these sets meet at infinity, it is impossible to find a strictly separating hyperplane; however, if either set does not stretch to infinity, then a strictly separating hyperplane is simply the x axis.

Definition 2. A **convex cone** is a convex set which is closed under non-negative scaling and convex combinations.

Definition 3. The convex cone generated by $\boldsymbol{u_1}, \ldots, \boldsymbol{u_m} \in \mathbb{R}^n$ is the set of all their conic combinations (i.e., non-negative linear combinations of variables).

$$Cone(\boldsymbol{u_1}, \ldots, \boldsymbol{u_m}) = \left\{ \sum_{i=1}^{m} \alpha_i \boldsymbol{u_i} : \alpha_i \geq 0 \ \forall i \right\}$$

Lemma 4 (Farkas' Lemma). *Let $\mathcal{C}$ be the convex cone generated by $\boldsymbol{u_1}, \ldots, \boldsymbol{u_m} \in \mathbb{R}^n$ and let $\boldsymbol{w} \in \mathbb{R}^n$. Exactly one of the below is true:*

1. $\boldsymbol{w} \in \mathcal{C}$
2. *There exists a separating hyperplane between $\mathcal{C}$ and $\boldsymbol{w}$ which touches $\mathcal{C}$. That is, $\exists \boldsymbol{z} \in \mathbb{R}^n$ such that $\boldsymbol{z} \cdot \boldsymbol{u_i} \leq 0 \ \forall i$ and $\boldsymbol{z} \cdot \boldsymbol{w} > 0$.*

Corollary 5 (Theorem of the Alternative). *Let $\mathbf{U} = [\boldsymbol{u_1}, \ldots, \boldsymbol{u_m}]$ and let $\boldsymbol{w} \in \mathbb{R}^n$. Exactly one of the below is true:*

1. *$\mathbf{U}\boldsymbol{y} = \boldsymbol{w}, \boldsymbol{y} \geq 0$ has a solution.*
2. *$\mathbf{U}^\intercal \boldsymbol{z} \leq 0, \boldsymbol{z}^\intercal \boldsymbol{w} > 0$ has a solution.*

Farkas' Lemma and the Theorem of the Alternative say the same thing, but the former is the geometric version and the latter is algebraic. We'll use the Theorem of the Alternative for our proof because it conveniently casts the $\boldsymbol{u}_i$ vectors as a matrix.

3.2 Proof

To prove strong duality, we'll first show that either $OPT(dual) \leq v$ or $OPT(primal) > v$ for some $v \in \mathbb{R}$. Combined with weak duality, which says that $OPT(dual) \geq OPT(primal)$, this proves that $OPT(primal) = OPT(dual)$. Consider the primal LP:

$$\begin{aligned} \max \ & \boldsymbol{c}^\intercal \boldsymbol{x} \\ \text{s.t. } & \mathbf{A}\boldsymbol{x} \leq \boldsymbol{b} \end{aligned}$$

And the corresponding dual LP:

$$\begin{aligned} \min \ & \boldsymbol{b}^\intercal \boldsymbol{y} \\ \text{s.t.} & \mathbf{A}^\intercal \boldsymbol{y} = \boldsymbol{c} \\ & \boldsymbol{y} \geq 0 \end{aligned}$$

Given $v \in \mathbb{R}$, by the Theorem of the Alternative exactly one of the below is true:

1. The following system has a solution:

$$\begin{pmatrix} \mathbf{A}^\intercal & 0 \\ \boldsymbol{b}^\intercal & 1 \end{pmatrix} \boldsymbol{z} = \begin{pmatrix} \boldsymbol{c} \\ v \end{pmatrix}$$
$$\boldsymbol{z} \geq 0$$

   Let $\boldsymbol{y} \in \mathbb{R}^m_+, \delta \in \mathbb{R}_+$ such that $\boldsymbol{z} = \begin{pmatrix} \boldsymbol{y} \\ \delta \end{pmatrix}$. Then we have $\mathbf{A}^\intercal \boldsymbol{y} = \boldsymbol{c}$ and $\boldsymbol{b}^\intercal \boldsymbol{y} + \delta = v$, implying that $OPT(dual) \leq v$.

2. The following system has a solution:

$$\begin{pmatrix} \mathbf{A} & \boldsymbol{b} \\ 0 & 1 \end{pmatrix} \boldsymbol{z} \leq 0$$

$$\boldsymbol{z}^{\mathsf{T}} \begin{pmatrix} \boldsymbol{c} \\ v \end{pmatrix} > 0$$

Let $\boldsymbol{z}_1 \in \mathbb{R}^n, \boldsymbol{z}_2 \in \mathbb{R}, \boldsymbol{z}_2 \leq 0$ such that $\boldsymbol{z} = \begin{pmatrix} \boldsymbol{z}_1 \\ \boldsymbol{z}_2 \end{pmatrix}$. We observe two cases based on the value of $\boldsymbol{z}_2$:

(a) When $\boldsymbol{z}_2 \neq 0$ the primal feasible is:

$$\boldsymbol{x} = -\frac{\boldsymbol{z}_1}{z_2}$$

Then $OPT(primal) = \boldsymbol{c}^{\mathsf{T}} \boldsymbol{x} > v$.

(b) When $z_2 = 0$, the primal is either infeasible or unbounded, and the dual is infeasible (proof left as exercise).