Matroid Intersection Polytope Theorem

1 Proof Outline

Theorem 1. *Let $\mathcal{M}_1 = (\mathcal{X}, \mathcal{I}_1)$ and $\mathcal{M}_2 = (\mathcal{X}, \mathcal{I}_2)$ be matroids with rank functions $r_1(S)$ and $r_2(S)$. Then,*

$$\begin{aligned} \mathcal{P}(\mathcal{M}_1) \cap \mathcal{P}(\mathcal{M}_2) &= \mathcal{P}(\mathcal{M}_1 \cap \mathcal{M}_2) \\ &\triangleq convHull(\{\mathbf{1}_S : S \subseteq \mathcal{X}, S \in \mathcal{I}_1, S \in \mathcal{I}_2\}) \end{aligned}$$

Proof. The $\supseteq$ direction is easy, but the $\subseteq$ direction is more difficult. It suffices to show that $\mathcal{P}(\mathcal{M}_1 \cap \mathcal{M}_2)$ is vertex integral, that is, it has independent sets at its vertices. To do this, we can show that for an arbitrary objective, there is an integral optimal to the matroid intersection LP:

$$\begin{aligned} \max \ & \sum_i w_i x_i \\ \text{s.t.} \ & \sum_{i \in S} x_i \leq r_1(S) \ \forall S \subseteq \mathcal{X} \\ & \sum_{i \in S} x_i \leq r_2(S) \ \forall S \subseteq \mathcal{X} \\ & \boldsymbol{x} \succeq 0 \end{aligned}$$

Fix w and let x^* be the vertex optimal solution of the above LP. We can assume without loss of generality that x^* is the unique optimal solution.

Definition 2. A set $S \subseteq \mathcal{X}$ is **tight** for $\mathcal{M}_1$ if and only if $\sum_{i \in S} x_i^* = r_1(S)$, and likewise for $\mathcal{M}_2$. This is the same as having the corresponding constraint tight in the matroid intersection LP.

Definition 3. A family of sets is a **chain** if and only if, for any pair of them, one is a subset of the other. Note that if any set in a chain is removed from the chain, the resultant family of sets is still a chain.

We will show that the theorem follows from the below lemmas, then prove the lemmas.

Lemma 4. *The tight sets for $\mathcal{M}_1$ form a chain, and likewise for $\mathcal{M}_2$.*

Lemma 5. *Let $\mathbf{A} \in \mathbb{R}^{(k+\ell) \times n}$ be the incidence matrix of two chains $S_1 \subseteq S_2 \subseteq \cdots \subseteq S_k$ and $T_1 \subseteq T_2 \subseteq \cdots \subseteq T_\ell$. That is, $a_{ij} = 1$ if the element $j \in \mathcal{X}$ is included in set i. Then, $\mathbf{A}$ is totally unimodular, i.e., it and all its submatrices have determinant 1, 0, or -1.*

To show that the theorem follows, note that $\boldsymbol{x}^*$ is the solution to the system of equations:

$$\mathbf{A}\boldsymbol{x} = \begin{bmatrix} r_1(S_1) \\ r_1(S_2) \\ \vdots \\ r_1(S_k) \\ r_2(T_1) \\ r_2(T_2) \\ \vdots \\ r_2(T_\ell) \end{bmatrix}$$
$$x_i = 0 \text{ for some of the } i \in \mathcal{X}$$

This is a system of equations where the left hand side is totally unimodular and the right hand side is integer. Thus by Cramer's Rule, there must exist an integral solution $\boldsymbol{x}^*$. □

2 Proof of Lemma 1

We write the dual of the matroid intersection LP:

$$\begin{aligned} \min \quad & \sum_{S \subseteq \mathcal{X}} r_1(S) y_{1,S} + \sum_{S \subseteq \mathcal{X}} r_2(S) y_{2,S} \\ \text{s.t.} \quad & \sum_{S \ni i} y_{1,S} + \sum_{S \ni i} y_{2,S} \geq w_i \ \forall i \in \mathcal{X} \\ & \boldsymbol{y} \succeq 0 \end{aligned}$$

Let $\boldsymbol{y}^*$ be optimal for the above. Then, tight complementary slackness guarantees $y^*_{1,S} > 0 \iff S$ is tight for $\mathcal{M}_{1,\boldsymbol{x}^*}$, and similarly for 2.

We will now show that $\{S : y^*_{1,S} > 0\}$ is a chain. Assume for the sake of contradiction that it is not. Then, we have two sets S and T which . We decrease $y^*_{1,S}$ and $y^*_{1,T}$ by ϵ, then increase $y^*_{1,S\cap T}$ and $y^*_{1,S\cup T}$ by ϵ. By the dual constraint above, we have that for an element $i \in S \cap T$, the constraint doesn't change because $-2\epsilon + 2\epsilon = 0$, and for an element $i \in T \notin S$ we have $-\epsilon + \epsilon = 0$. Thus the constraint doesn't change. Now consider the objective value:

$$-\epsilon(r_1(S) + r_1(T)) + \epsilon(r_1(S \cup T) + r_1(S \cap T))$$

Applying the submodularity of the rank function, this is equal to:

$$\epsilon(r_1(S \cup T) + r_1(S \cap T) - r_1(S) - r_1(T))$$

Which must be less than or equal to zero. However, it is only equal to zero in the degenerate case, so we can "throw away" some redundant sets to remove degeneracy, thus achieving a decrease in the objective value and our desired contradiction.

3 Proof of Lemma 2

It is enough to show that $\det(\mathbf{A})$ is 1, 0, or -1 because any submatrix of $\mathbf{A}$ is still the incidence matrix of two chains. We can perform elementary row operations such that each row i contains only the elements such that S_i or T_i is the smallest set containing them. Then, in the S half of the matrix, no column has more than a single one, and similarly for the T half.

If there exists a column with no ones, $\det(\mathbf{A}) = 0$ and we are done. If not, we can get rid of the columns with a single one because the determinant will be the same as the incidence matrix of two smaller chains. Now, every element shows up once in the S half and once in the T half, and the halves are disjoint. This is exactly the incidence matrix of a bipartite graph where each S_i represents a left-hand node, each T_i represents a right-hand node, and the elements represent the edges incident to each node. We have previously shown this matrix to be totally unimodular, so $\mathbf{A}$ must be totally unimodular.