

Non-Linear Dynamic Inversion with Actuator Dynamics: an Incremental Control Perspective

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In this paper, we derive a Nonlinear Dynamic Inversion (NDI) control law for a non-linear system with first order linear actuators, and compare it to Incremental Nonlinear Dynamic Inversion (INDI), which has gained popularity in recent years. It is shown that for first order actuator dynamics, INDI approximates the corresponding NDI control law arbitrarily well under the condition of sufficiently fast actuators. If the actuator bandwidth is low compared to changes in the states, the derived NDI control law has the following advantages compared to INDI: 1) compensation of state derivative terms 2) well defined error dynamics and 3) exact tracking of a reference model, independent of error controller gains in nominal conditions. The comparison of the INDI control law with the well established control design method NDI adds to the understanding of incremental control. It is additionally shown how to quantify the deficiency of the INDI control law with respect to the exact NDI law for actuators with finite bandwidth. The results are confirmed through simulation results of the rolling motion of a fixed wing aircraft.

I. Introduction

One of the cornerstones of non-linear control theory is Nonlinear Dynamic Inversion (NDI), also called output feedback linearization, which has found applications in many different fields [1–5]. The concept is based on inverting the non-linearities of a system, such that the relation between a virtual control input and the output behaves as a linear system, in particular a cascade of integrators. This transformed system is straightforward to control with a linear control law. Very complex non-linear systems can be controlled perfectly through this method in theory, but the results deteriorate if the model is not accurate, if some of the system states cannot be measured accurately or actuator dynamics exist that are not considered.

Incremental Nonlinear Dynamic Inversion (INDI) is a control method that uses a local linearization of the model to derive a control law to control the defined output and its derivatives, by computation of an increment in the control input, neglecting any state-dependent terms [6]. Through the feedback of derivatives of the system output, such as angular acceleration in the case of inner loop control of an aircraft, unmodeled effects and disturbances are directly measured, and compensated for in the next control increment, which led to an increased popularity of the concept in flight control applications [7–13]. The controller responds under specific simplifications to the system to disturbances or unmodeled dynamics with the combined dynamics of the actuators and any filtering that is done on the output, and this has been observed in practical experiments as well [14].

It would seem that an INDI controller that also includes the model terms that are typically neglected, would provide better performance in both tracking and disturbance rejection. Wang et al. [15] suggest keeping a term with the state change " Δx " over one time step as part of the control law. The drawback of this approach is that when actuator dynamics cannot be neglected, the computed increment in input is not realized within one time sample. That makes the considered anticipation of the state change insufficient. In fact, this problem is hard to solve if one follows the 'traditional' derivation of the INDI control law, as will be further detailed in Section III. Li et al. [16] propose to add a large gain, such that if the actuator behaves as a first order system, it will achieve the desired value within one controller time step, effectively

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removing any actuator dynamics. This approach is not realistic, as controllers are typically run at a frequency much higher than the bandwidth of the actuators.

Zhou et al. [17] describe a method of including state dependent terms in the INDI control law in discrete time. The drawback of this approach is that, due to the discrete formulation, a control input is calculated that will solve for the virtual control within one time step. In most cases, the time constant of the actuators is larger than one time step of the controller, which would lead to large inputs to the actuators and possibly hidden oscillations between samples.

Several concepts have been proposed, that scale the control effectiveness matrix, for different reasons. Cordeiro et al. [18] noted that an input gain scaling, which was used to reduce the influence of noise, can reduce the closed loop bandwidth. In [19], it was shown that the input scaling also increases robustness with respect to time delays. Pfeifle and Fichter [20] proposed an additional gain that depends on the sampling time and the actuator time constant based on an alternative derivation of the incremental dynamic inversion control law. Raab et al. [21] related the meaning and value of an input gain scaling to the actuator parameters.

Raab et al. [21] also suggested that actuator dynamics can be included in the derivation of the controller, by taking an additional derivative of the system output. Essentially, this includes the actuator dynamics into the system dynamics, and allows one to also invert the actuator dynamics. The main benefits of the approach are the ability to incorporate actuator rate constraints in the control allocation and artificially choose faster or slower actuator dynamics, which is especially useful if different actuators are used to control coupled outputs. However, the state dependent effects were not taken into account in this paper.

In an extension, Bhardwaj et al. [22] based a reference model design on dynamic inversion including the actuator dynamics. From this reference model a feed-forward control signal was generated, that accounts for state-dependent model effects. This theoretically leads to perfect tracking of the reference model, if the reference model equals the plant and if the states of the reference model equal the plant states, i.e. if there are no model uncertainties and no disturbances. However, the benefit of this approach is deteriorated if the plant is not on the reference model, since if reference model states are different from the plant states, the feed-forward command will not be exact, and this will lead to unpredictable error dynamics. Another issue can arise if the effector positions influence the output dynamics, as this influence would depend on the control allocation, and hence some sort of feed-back from the control allocation to the reference model would be required.

In this paper, we derive an NDI control law for a non-linear system with first order linear actuators. We show that the resulting control law, through proper choice of the virtual control, has the same structure as an INDI control law, extended with a term that compensates for the state-based derivatives, based on the actuator dynamics. Through this different derivation, more insight is gained in the 'traditional' INDI derivation, and it is shown that when the actuators can be assumed to be fast in relation to the state dynamics the INDI law approximates the NDI control law.

II. General system with first order linear actuator dynamics

To introduce the concept, consider the general system with first order linear actuator dynamics given by:

$$\begin{aligned}\dot{x} &= f(x, u) \\ y &= h(x),\end{aligned}\tag{1}$$

where $x \in \mathbb{R}^n$ is the system state, $u \in \mathbb{R}^k$ is the actuator state, $y \in \mathbb{R}^m$ is the output, $f \in C^{(r+1)}(\mathbb{R}^n \times \mathbb{R}^k; \mathbb{R}^n)$ and $h \in C^{(r+1)}(\mathbb{R}^n; \mathbb{R}^m)$. The actuator dynamics are given by:

$$\dot{u} = \Omega (u_c - u)\tag{2}$$

where $\Omega \in \mathbb{R}^{(k \times k)}$ is a diagonal matrix with constant elements representing the bandwidth of the different actuators. Assume the system to have a relative degree of $r \in \mathbb{N}$ with respect to u for all elements of y , i.e. the r th derivative of y with respect to time is the first derivative of y which explicitly depends on u according to:

$$y^{(r)} = F(x, u)\tag{3}$$

with $F \in C^1(\mathbb{R}^n \times \mathbb{R}^k; \mathbb{R}^m)$. Traditionally, one would obtain derivatives of y , until finding the control input u in the equation, see for example [23]. In those cases, actuator dynamics are not considered, or assumed to be relatively fast such that they can be neglected. Instead, in this derivation the non-linear dynamic inversion is continued through the actuator dynamics by performing one more differentiation w.r.t. time:

$$y^{(r+1)} = F_x \dot{x} + F_u \dot{u}\tag{4}$$

where $F_x := \frac{\partial F(x,u)}{\partial x}$ and $F_u := \frac{\partial F(x,u)}{\partial u}$. Now the actuator relation in equation (2) is substituted into (4) to obtain a relation for $y^{(r+1)}$ that includes the actuator command.

$$y^{(r+1)} = F_x \dot{x} + F_u \Omega (u_c - u) \quad (5)$$

Choose now $y^{(r+1)} = v$, where v is a virtual control command. If it is assumed that $F_u \Omega$ has full row rank, the following choice of u_c will make $y^{(r+1)} = v$:

$$u_c = (F_u \Omega)^\dagger (v - F_x \dot{x}) + u \quad (6)$$

where $(F_u \Omega)^\dagger$ denotes a right inverse matrix that solves the linear equation system given in (5). The control signal u_c linearizes the response from the virtual control input v to the output y by a chain of integrators.

Often a linear controller is used to regulate the output y . In the paper, for the next step the virtual control input v is designed using a linear error controller such that the specified error dynamics are achieved. The order of the error dynamics in y will correspond to the sum of the relative degree and the order of the actuators, such that these could be specified by

$$e_y^{(r+1)} + \sum_{i=0}^r k_i e_y^{(i)} = 0 \quad (7)$$

where $e_y = y_{ref} - y$ is the error in y , $e_y^{(i)}$ is the i 'th derivative with respect to time, and $y^{(r+1)}$ is chosen as the virtual control v such that:

$$v = y_{ref}^{(r+1)} + \sum_{i=0}^r k_i e_y^{(i)} \quad (8)$$

Inserting v into (6) results in the control law:

$$u_c = (F_u \Omega)^\dagger \left(y_{ref}^{(r+1)} + \sum_{i=0}^r k_i e_y^{(i)} - F_x \dot{x} \right) + u \quad (9)$$

The resulting control law is a classical NDI control law with linear error controller, where the actuator dynamics were additionally included in the system dynamics and inverted. By inserting (9) into (5) the desired error dynamics are obtained as designed, because $F_u \Omega$ has full row rank such that $(F_u \Omega)(F_u \Omega)^\dagger = I_{m \times m}$:

$$\begin{aligned} y^{(r+1)} &= F_x \dot{x} + F_u \Omega (F_u \Omega)^\dagger \left(y_{ref}^{(r+1)} + \sum_{i=0}^r k_i e_y^{(i)} - F_x \dot{x} \right) \\ &= y_{ref}^{(r+1)} + \sum_{i=0}^r k_i e_y^{(i)} \end{aligned} \quad (10)$$

This leads to tracking of the corresponding derivative of a reference model $y_{ref}^{(r+1)}$, and the resulting error dynamics will correspond to (7).

This can be compared to the concept proposed by Bhardwaj et al. [22], where the state dependent term $F_x \dot{x}$ in the control law of Equation (9), is generated by a reference model using an additional feed forward term corresponding to $F_x \dot{x}_{ref}$. In that case, the feed forward from the reference model is only correct if the system is tracking the reference model. In addition, in case of disturbances which lead to a perturbation with differences between the desired and actual trajectory, the actual error dynamics will differ from the desired error dynamics, because they will be excited by the term $F_x (\dot{x} - \dot{x}_{ref})$ as can be seen in equation (10). However, in the proposed inversion based control law, where the state derivatives are used, the error dynamics will correspond to the desired error dynamics. A possible drawback of this, could be reduced stability margins when analyzing the actuator or sensor cuts of the linearized closed loop system due to the additional feedback, but this requires further investigation. It will become apparent in section III that the classical INDI approximates this NDI law under the condition that the actuators are fast with equal bandwidth.

A. Choice of Desired Error Dynamics

The error dynamics in Equation in (7) have a generic form, the gains of which could be chosen freely. This could lead to error dynamics that cannot be realistically obtained with real world actuators, due to physical constraints. It

therefore makes sense to design the error controller such as to include actuator dynamics with a bandwidth equal or comparable to the real actuators.

The order of the error dynamics in y corresponds to the sum of the relative degree of y and the order of the actuator dynamics. The desired dynamics can therefore be interpreted as cascaded dynamics composed of the slower desired system error dynamics and a faster desired inner loop dynamics with bandwidth Ω_y , corresponding to the actuator dynamics. It can hence make sense to express the desired error dynamics (7) in the Laplace domain as the product of dynamics due to the system physics and due to the actuators as follows:

$$E_y(s) \left(s^r + \sum_{i=0}^{r-1} K_i s^i \right) (sI + \Omega_y) = 0 \quad (11)$$

where $e_y = y_{ref} - y$ is the error in y with Laplace transform $E_y(s)$, $\left(s^r + \sum_{i=0}^{r-1} K_i s^i \right)$ describes the desired error dynamics in y and $(sI + \Omega_y)$ being the desired error dynamics occurring due to the first order actuators. The system error dynamics $\left(s^r + \sum_{i=0}^{r-1} K_i s^i \right)$ can be formulated in the time domain as

$$\mathcal{L}^{-1} \left(E_y(s) \left(s^r + \sum_{i=0}^{r-1} K_i s^i \right) \right) = e_y^{(r)} + \sum_{i=0}^{r-1} K_i e_y^{(i)} = 0 \quad (12)$$

where K_i are the error dynamics gain matrices and \mathcal{L}^{-1} is the inverse Laplace transform. Using (12) the combined error dynamics from equation (11) can be described in the time domain by:

$$\frac{d}{dt} \left(e_y^{(r)} + \sum_{i=0}^{r-1} K_i e_y^{(i)} \right) + \Omega_y \left(e_y^{(r)} + \sum_{i=0}^{r-1} K_i e_y^{(i)} \right) = e_y^{(r+1)} + \sum_{i=1}^r K_i e_y^{(i)} + \Omega_y \left(e_y^{(r)} + \sum_{i=0}^{r-1} K_i e_y^{(i)} \right) = 0 \quad (13)$$

Note that if a formulation as given in (7) is preferred, by reformulating (13) as

$$e_y^{(r+1)} + (K_r + \Omega_y) e_y^{(r)} + \sum_{i=1}^{r-1} \left((K_i + \Omega_y K_i) e_y^{(i)} \right) + \Omega_y K_0 e_y = 0 \quad (14)$$

the gain matrices k_i can be obtained directly. Inserting $e_y^{(r+1)} = y_{ref}^{(r+1)} - y^{(r+1)}$ into (13) and solving for $y^{(r+1)}$ provides the pseudo control input v that can be used in (6):

$$v = y_{ref}^{(r+1)} + \sum_{i=1}^r K_i e_y^{(i)} + \Omega_y \left(e_y^{(r)} + \sum_{i=0}^{r-1} K_i e_y^{(i)} \right) \quad (15)$$

with reference dynamics given in terms of the respective derivatives of y_{ref} , and error controller gains given by Ω_y and K_i . The final control law is then given by

$$u_c = (F_u \Omega)^{\dagger} \left(y_{ref}^{(r+1)} + \sum_{i=1}^r K_i e_y^{(i)} + \Omega_y \left(e_y^{(r)} + \sum_{i=0}^{r-1} K_i e_y^{(i)} \right) - F_x \dot{x} \right) + u \quad (16)$$

The resulting error dynamics will correspond to (11), which can be interpreted as cascaded dynamics composed of the slower desired system dynamics and a faster desired inner loop dynamics bandwidth Ω_y , due to the actuator dynamics. Since equation (16) essentially inverts for the actuator dynamics as well, the system could be made arbitrarily fast through the choice of Ω_y . However for practical applications, the actuators were designed to operate up to a certain bandwidth, and the choice Ω_y should be limited to the bandwidth of the actuators in the considered direction. In some cases the dynamics of a control effector is limited by the effector dynamics and not the physical actuator. Here an increase in effector bandwidth can be obtained by selecting Ω_y larger than the actuator bandwidth in the considered direction.

III. Comparison with Incremental Nonlinear Dynamic Inversion

This section compares the NDI control law, that was derived in section II, to an INDI controller, which has been derived for example by Bacon and Ostroff [6] and Sieberling et al. [24]. The INDI controller will be derived along the same lines, keeping the nomenclature the same as in the previous section, such that the controllers can be effectively compared. The classic INDI derivation contains some inaccuracies, which are pointed out in this section as well.

Again, consider the system in (1), and the r th derivative of the output as in Equation (3). Take the Taylor expansion of $F(x, u)$ with respect to x and u :

$$y^{(r)}(t) = F(x(t), u(t)) = F(x_0, u_0) + F_x(x(t) - x_0) + F_u(u(t) - u_0) + O(\Delta x, \Delta u) \quad (17)$$

where $F_x := \frac{\partial F(x, u)}{\partial x}$ and $F_u := \frac{\partial F(x, u)}{\partial u}$ and O denotes higher order terms. The term $F(x_0, u_0)$ can also be denoted by $y_0^{(r)}$. It should be noted that x and u are functions of time. The Taylor expansion in Equation (17) is performed with respect to x and u . For x_0 and u_0 it makes sense to choose a state and control input, corresponding to the system, a small time instance ago, i.e.:

$$\begin{aligned} x_0 &= x(t - \Delta t) \\ u_0 &= u(t - \Delta t) \end{aligned} \quad (18)$$

If $y^{(r)}$ is chosen as virtual control v and u is chosen as the control signal u_c , the following relation is obtained:

$$v = y_0^{(r)} + F_x(x - x_0) + F_u(u_c - u_0) + O(\Delta x, \Delta u) \quad (19)$$

It is assumed that the higher order terms can be neglected. The control law is obtained by solving (19) for u_c :

$$u_c = F_u^\dagger \left(v - y_0^{(r)} - F_x(x - x_0) \right) + u_0 \quad (20)$$

The term $F_x(x - x_0)$ is usually neglected, with several different arguments involving the bandwidth of the actuator and a sufficiently small sample time [24, 25], leading to the following control law:

$$u_c = F_u^\dagger \left(v - y_0^{(r)} \right) + u_0 \quad (21)$$

If the term $F_x(x - x_0)$, or some approximation of this is sought to be included, it is not clear how Δx in Equation (23) should be chosen. Wang et al. [15] proposed to use $\Delta x = \dot{x}_0 \Delta t$ with Δt chosen as the sample rate. However, because the actuator command will not be instantaneously reached, due to the actuator dynamics, the $F_x(x - x_0)$ term in the $y^{(r)}$ dynamics of Equation (17) will not be exactly canceled by the control signal. The resulting stability properties of the INDI control law are unclear, neither is it clear what was neglected from the Taylor expansion.

Now, we can specify desired error dynamics in y and derive the required virtual control. For example assume the desired error dynamics to be given by (12), then

$$v = y_{ref}^{(r)} + \sum_{i=0}^{r-1} K_i e_y^{(i)} \quad (22)$$

Combining Equations (21) and (22), the following control law is found:

$$u_c = F_u^\dagger \left(y_{ref}^{(r)} - y_0^{(r)} + \sum_{i=0}^{r-1} K_i e_y^{(i)} \right) + u_0 = F_u^\dagger \left(e_y^{(r)} + \sum_{i=0}^{r-1} K_i e_y^{(i)} \right) + u_0 \quad (23)$$

By comparing the above control law with the exact NDI law in (16), it is possible to identify which part was neglected. In the following, it is shown that under certain conditions relating to the actuators and the NDI control law parameters, the INDI law approximates the exact NDI law. Assume that:

- All actuators have the same bandwidth, i.e. $\Omega = \omega I_{k \times k}$, where ω is a positive scalar.
- Choose $\Omega_y = \omega I_{m \times m}$.

Then

$$(F_u \Omega)^\dagger \Omega_y = (F_u \omega I_{k \times k})^\dagger \omega I_{m \times m} = (F_u \omega)^\dagger \omega = (F_u)^\dagger \frac{1}{\omega} \omega = (F_u)^\dagger \quad (24)$$

Hence, the control law from equation (16) evaluates to:

$$u_c = \frac{1}{\omega} (F_u)^\dagger \left(y_{ref}^{(r+1)} + \sum_{i=1}^r K_i e_y^{(i)} - F_x \dot{x} \right) + (F_u)^\dagger \left(e_y^{(r)} + \sum_{i=0}^{r-1} K_i e_y^{(i)} \right) + u \quad (25)$$

It is seen here that in the limit where the actuator bandwidth ω approaches infinity the NDI law turns exactly into the INDI control law:

$$\begin{aligned} u_c &= \lim_{\omega \rightarrow \infty} \left(\frac{1}{\omega} (F_u)^\dagger \left(y_{ref}^{(r+1)} + \sum_{i=1}^r K_i e_y^{(i)} - F_x \dot{x} \right) + (F_u)^\dagger \left(e_y^{(r)} + \sum_{i=0}^{r-1} K_i e_y^{(i)} \right) + u \right) \\ &= (F_u)^\dagger \left(e_y^{(r)} + \sum_{i=0}^{r-1} K_i e_y^{(i)} \right) + u \end{aligned} \quad (26)$$

The NDI law of Section II perfectly inverts the system, so Equation (26) shows exactly what the error is that one would make applying the INDI control law with the error controller design of Equation (22) if the actuators cannot be assumed to be fast. These missing terms will lead to errors in reference model tracking, and lead to error dynamics that are different from the designed error dynamics.

On the other hand, this comparison shows that the derivation that was performed in Section II is a useful alternative to arrive at the incremental non-linear dynamic inversion law as it does not require any ad-hoc arguments. As such, the derivation provides new theoretical support for the INDI control method, while also providing a means to compensate for model-dependent terms ($F_x \dot{x}$) in the control law. In INDI, these terms are not compensated for, and depending on the system, this may lead to significant errors in tracking and unpredictable error dynamics.

As shown in the Appendix VIII, the INDI law does not hold in the limit, i.e. it is only valid for $\omega < \infty$, hence the above equation essentially states that for a fixed time, the INDI law approximates the true NDI law arbitrarily well by the choice of sufficiently high actuator bandwidth.

It can further be shown, that the input scaling as suggested by Cordeiro et al. [18] and [19] for reducing the influence of noise and increasing the robustness with respect to time delays, can be interpreted as a modification of the innermost bandwidth of y : $\Omega_y = \Lambda \omega$, where the input scaling gain matrix $\Lambda \in \mathbb{R}^{m \times m}$ is diagonal. If this relation is inserted into (16), and it is assumed that the actuators have equal bandwidth, i.e. $\Omega = \omega I_{k \times k}$, and taking the limit of the bandwidth going to infinity, we obtain the conventional INDI control law with the scaling gain matrix that was proposed by Cordeiro et al. [18]:

$$\begin{aligned} u_c &= \lim_{\omega \rightarrow \infty} \left(\frac{1}{\omega} (F_u)^\dagger \left(y_{ref}^{(r+1)} + \sum_{i=1}^r K_i e_y^{(i)} - F_x \dot{x} \right) + \frac{1}{\omega} (F_u)^\dagger \Lambda \omega \left(e_y^{(r)} + \sum_{i=0}^{r-1} K_i e_y^{(i)} \right) + u \right) \\ &= (F_u)^\dagger \Lambda \left(e_y^{(r)} + \sum_{i=0}^{r-1} K_i e_y^{(i)} \right) + u \end{aligned} \quad (27)$$

Cordeiro et al. [18] observed that choosing the diagonal elements of Λ smaller than 1 reduces the closed-loop bandwidth. With this formulation of the control law, the scaling gain can be directly identified as a modification of the desired innermost bandwidth. If the scaling factor is 1, as in conventional INDI, this corresponds to a desired innermost bandwidth equal to the actuator bandwidth.

IV. Comparison with Incremental Nonlinear Dynamic Inversion including Actuators

Smeyr et al. [14] showed that when actuator dynamics are present, $y^{(r)}$ follows v with the dynamics of the actuator, in the case that the $F_x \Delta x$ term and the higher order terms can be neglected in (19), and in the case that all actuators have the same bandwidth i.e. $\Omega = \omega I_{k \times k}$, such that:

$$y^{(r+1)} = \omega(v - y^{(r)}). \quad (28)$$

Then for the virtual control the following relation can be obtained:

$$v = \frac{1}{\omega} y^{(r+1)} + y^{(r)} \quad (29)$$

Choosing the same error dynamics like in the NDI control law, given by (14) with $\Omega_y = \omega I_{m \times m}$, solving for $y^{(r+1)}$ according to (15), and inserting into the INDI control law given by (21), results in the control law

$$u_c = F_u^\dagger \frac{1}{\omega} \left(y_{ref}^{(r+1)} + \sum_{i=1}^r K_i e_y^{(i)} + \omega \left(e_y^{(r)} + \sum_{i=0}^{r-1} K_i e_y^{(i)} \right) \right) + u_0 \quad (30)$$

This control law equals the NDI control law in (16), if $\Omega = \omega I_{k \times k}$, $\Omega_y = \omega I_{m \times m}$ and $F_x \dot{x}$ is neglected. This approach requires all actuators to have the same bandwidth, otherwise Equation (28) does not hold.

In [21] an approach can be found to resolve this problem. With the error dynamics proposed in (14), the control law proposed in [21] can be formulated using the notation of this paper as

$$u_c = (F_u \Omega)^\dagger \left(y_{ref}^{(r+1)} + \sum_{i=1}^r K_i e_y^{(i)} + \Omega_y \left(e_y^{(r)} + \sum_{i=0}^{r-1} K_i e_y^{(i)} \right) \right) + u \quad (31)$$

This is again equal to the corresponding NDI law in (16) with $F_x \dot{x}$ neglected. Compared to equation (30), it does not have the assumption of equal actuators and it offers the option to specify a desired Ω_y .

V. Simple Example

In the following, the mechanics of deriving the control law is demonstrated on a simple SISO linear system. The roll rate of a fixed wing aircraft is to be controlled. Consider the roll dynamics given by:

$$\dot{p} = L_p p + L_u u \quad (32)$$

where p is the roll rate, L_p is the roll damping, u is the aileron deflection, and L_u is the aileron roll efficiency. L_p is usually large, and hence contributes significantly to the dynamics. This example demonstrates how it is taken into account by the proposed control concept. The ailerons are driven by an actuator with the following first order dynamics:

$$\dot{u} = \omega (u_c - u) \quad (33)$$

where u_c is the actuator command and ω is the actuator bandwidth. The output to be controlled is $y = p$. The relative degree r of the system is 1 as given by (32). The control relation as given by (5) is:

$$\ddot{p} = L_p \dot{p} + L_u \omega (u_c - u) \quad (34)$$

Solving for u_c as explained in (6), gives the following control law.

$$u_c = \frac{1}{L_u \omega} (\ddot{p}_d - L_p \dot{p}) + u \quad (35)$$

A reference model is chosen based on the desired dynamics:

$$\begin{aligned} \dot{p}_{ref} &= -L_{p,d} (\delta - p_{ref}) \\ \dot{\delta} &= \omega_d (p_c - \delta) \end{aligned} \quad (36)$$

where $L_{p,d}$ is the desired roll damping and δ is a generalized roll acceleration due to the aileron deflection.

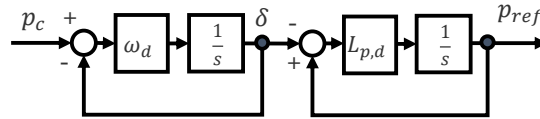


Fig. 1 Block diagram of the reference model

The dynamics of δ is given by the desired build up in roll acceleration due to the aileron dynamics ω_d . p_c is the pilot roll rate command, since in steady state, $p_{ref} = p_c$. Note that \ddot{p}_{ref} can easily be calculated using the reference dynamics in (36):

$$\ddot{p}_{ref} = -L_{p,d} (\omega_d (p_c - \delta) - \dot{p}_{ref}) \quad (37)$$

To choose \ddot{p}_d , the desired error dynamics have to be formulated. Since the actuators are of first order, and the system dynamics are first order, a product of 2 first order systems are chosen as follows:

$$e_p (s + \omega_d) (s - L_{p,d}) = 0 \quad (38)$$

where $e_p = p_{ref} - p$ and which in the time domain is:

$$\ddot{e}_p + \omega_d \dot{e}_p - L_{p,d} (\dot{e}_p + \omega_d e_p) = \ddot{e}_p + k_1 \dot{e}_p + k_0 e_p = 0 \quad (39)$$

where

$$\begin{aligned} k_1 &= (\omega_d - L_{p,d}) \\ k_0 &= -L_{p,d} \omega_d \end{aligned} \quad (40)$$

Solving for $\ddot{p} = \ddot{p}_d$:

$$\ddot{p}_d = \ddot{p}_{ref} + k_1 \dot{e}_p + k_0 e_p \quad (41)$$

will lead to the final control law by substituting (41) into (35):

$$u_c = \frac{1}{L_u \omega} (\ddot{p}_{ref} + k_1 \dot{e}_p + k_0 e_p - L_p \dot{p}) + u \quad (42)$$

which can be ordered in terms of the contributions:

$$u_c = \frac{1}{L_u \omega} \left(\underbrace{\ddot{p}_{ref}}_{\text{Feed-forward}} + \underbrace{k_1 \dot{e}_p + k_0 e_p}_{\text{Error control}} - \underbrace{L_p \dot{p}}_{\text{Model part}} \right) + u \quad (43)$$

This can be compared to the standard INDI control law for the same example

$$u_c = \frac{1}{L_u} (\dot{e}_p - L_{p,d} e_p) + u \quad (44)$$

with reference dynamics:

$$\dot{p}_{ref} = -L_{p,d} (p_c - p_{ref}) \quad (45)$$

A. Simulation results

For the example described above, simulations were conducted to validate the approach. In the simulations, it was assumed that state information was available without noise, and there were no uncertainties in the parameters. The parameters were $L_u = 0.25$, $L_p = -6.6 = 1/2 L_{p,d}$ and $\omega = 20 \text{ rad/s}$. In the following we will compare:

- *NDI* as given in Section II, for this example given by (42), with reference dynamics given by (36) and (37).
- *INDI* as given in Section III, for this example given by Equation (44) with reference dynamics given by (45).
- *INDI with actuators* as given in Section IV, for this example given by (42) without $L_p \dot{p}$, with reference dynamics given by (36) and (37).

First of all Figure 2 compares the responses in roll rate p to a step input of $5^\circ/\text{s}$ in p_c for the INDI and NDI control law. The NDI law makes the plant correctly follow the reference model with reference inputs, while this is not the case for INDI with its corresponding reference dynamics.

Figure 3 reveals the error dynamics of the classical INDI control law (44) and the proposed NDI law (43). The closed loop system response for both control laws was simulated with a roll rate command $p_c = 0^\circ/\text{s}$ for an initial value perturbation of $p = 5^\circ/\text{s}$, i.e. the initial value of the plant was $5^\circ/\text{s}$, while the initial value of the reference model was $p_{ref} = 0^\circ/\text{s}$. The simulation shows that the resulting error dynamics of the NDI controller correspond exactly to the desired error dynamics given by (39). The resulting error dynamics of the INDI controller do not correspond to the specified error dynamics $\dot{e}_p - L_{p,d} e_p = 0$.

Figure 4 compares the NDI control law with the INDI control law where actuator dynamics are taken into account in the error controller design. Both laws use the same reference dynamics. This extended INDI controller still does not realize the perfect tracking of the reference signal of the NDI law, the reason being that in the INDI controller the L_p term is neglected. In [22], this issue was fixed for the extended INDI controller by adding an additional reference

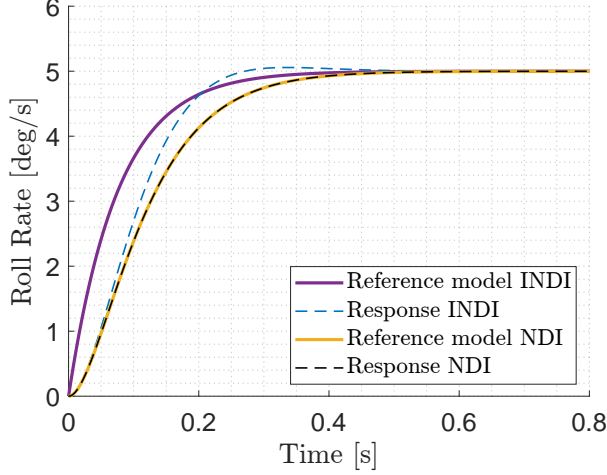


Fig. 2 Step response of the INDI and NDI control law.

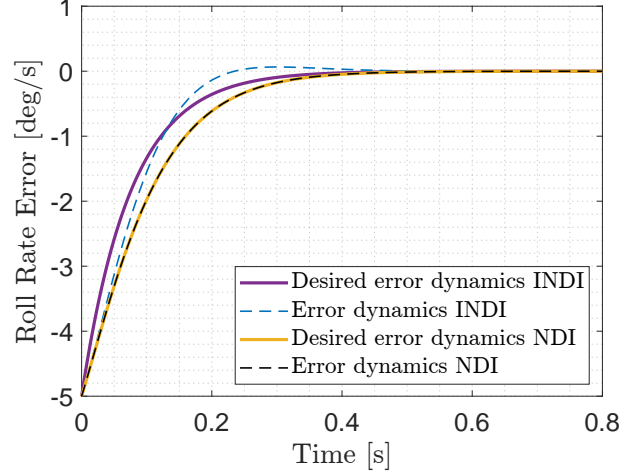


Fig. 3 Comparison of desired and actual error dynamics considering a $5^\circ/s$ initial value perturbation.

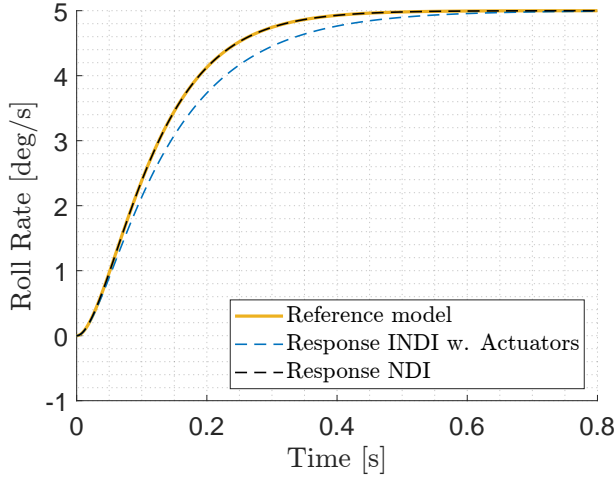


Fig. 4 Step response of the INDI with Actuators in the design and NDI control law.

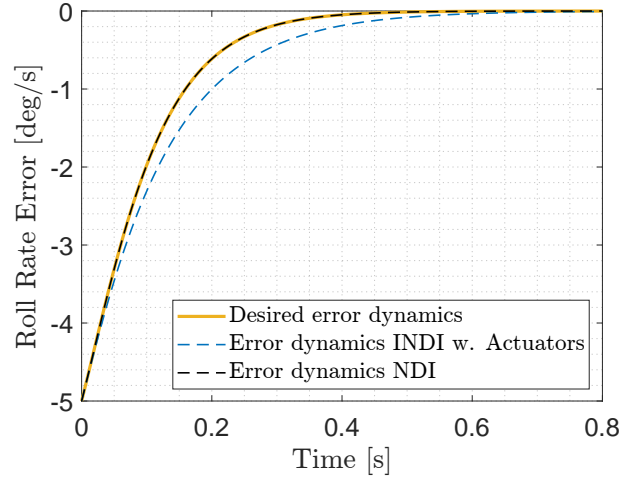


Fig. 5 Comparison of desired and actual error dynamics considering a $5^\circ/s$ initial value perturbation.

model based feed-forward term. The resulting control law in [22] can be reformulated to resemble the NDI control law given by (43), with the only difference that the term $L_p \dot{p}$ is substituted with $L_p \dot{p}_{ref}$ from a physical reference model as the one given in Equation (36). Under the condition that $\dot{p} = \dot{p}_{ref}$, i.e. that no error exist, and the plant is exactly equal to the reference model, this leads to perfect tracking of reference inputs, similar to the NDI law. However, the condition that $F_x \dot{x}$ equals $F_x \dot{x}_{ref}$, might not hold in the following cases: 1) dynamics and couplings which are not modeled in the reference dynamics but are present in the plant, 2) a control allocation mismatch for the reference model and the INDI controller due to disturbances, model uncertainties, and utilization of over-actuated channels for secondary control objectives, leading to different responses, and 3) disturbances.

In Figure 5 the discrepancy is shown for the case that the reference model state is zero, but the plant state is not, i.e. $F_x \dot{x}$ does not correspond to $F_x \dot{x}_{ref}$. Here, the closed-loop system responses of the NDI control law and the INDI with actuators were simulated for a roll rate command $p_c = 0^\circ/s$ with an initial value of the plant of $5^\circ/s$, while the initial value of the reference model was $p_{ref} = 0^\circ/s$, such that $\dot{p} \neq \dot{p}_{ref} = 0$. It is revealed that for the NDI law (43), the dynamics with which the error $p_{ref} - p$ declines, corresponds exactly to the desired error dynamics specified by (39). The INDI control law with actuators was designed with the same desired error dynamics like the NDI law and it is seen that the resulting error dynamics does not correspond to these desired dynamics.

Finally, Figures 6 and 7 investigate the influence of the actuator bandwidth on the error dynamics of the classical

INDI and the NDI control law. The same simulations as before with $p_c = 0^\circ/s$ and an initial condition $p = 5^\circ/s$ was performed, but with varying actuator bandwidth. The simulations show that:

- 1) For the INDI law (44) the dynamics of $p_{ref} - p$ do not exactly correspond to the error dynamics that were specified in the design, i.e. $\dot{e} - L_{p,d}e_p = 0$, but for increasing actuator bandwidth these dynamics are approached by the resulting error dynamics.
- 2) The desired NDI error dynamics given by (39) approach the INDI error dynamics given by $\dot{e} - L_{p,d}e_p = 0$, for increasing actuator bandwidth.

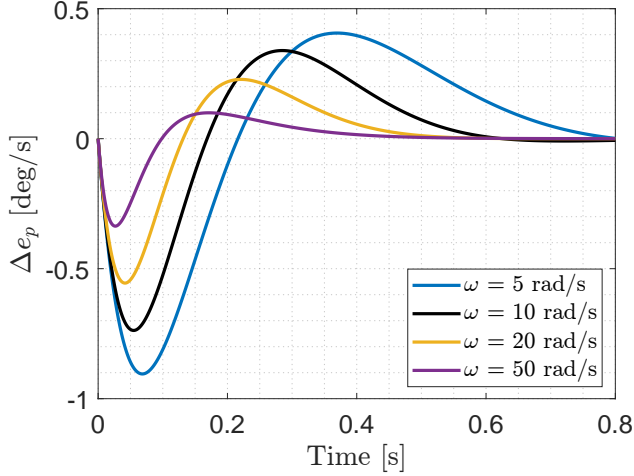


Fig. 6 Difference between actual INDI error dynamics and design INDI error dynamics.

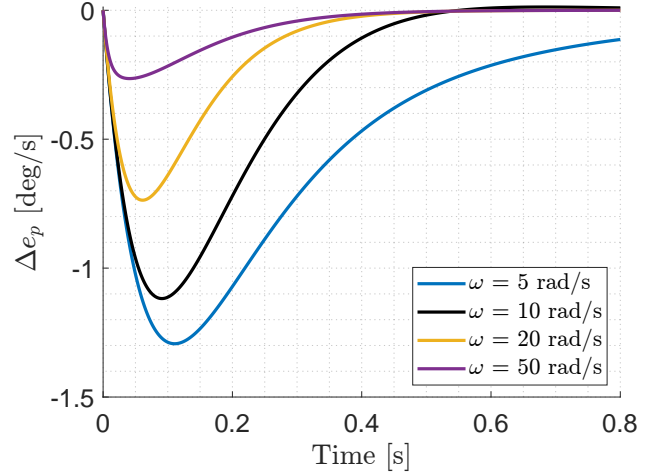


Fig. 7 Difference between NDI error dynamics and design INDI error dynamics.

Consider for Statement 1) Figure 6, which reveals the *difference* between the error e_p resulting from simulation of the closed-loop system with the INDI control law, and the initial value response of the desired error dynamics $\dot{e} - L_{p,d}e_p = 0$. It shows that for increasing bandwidth of the actuator this difference decreases, meaning that the resulting error dynamics approach the specified desired dynamics of the INDI law. For Statement 2) consider Figure 7, which depicts the *difference* between the desired INDI error dynamics and the desired NDI error dynamics, for increasing actuator bandwidth. Hence, the conclusion can be drawn that that with increasing bandwidth the NDI error dynamics approach the INDI error dynamics.

VI. Discussion

One of the potential issues that could be raised with this control design, is the reliance on additional state information. The question is if the respective signals can be measured or otherwise obtained. This is something that is dependent on the system under consideration. In terms of the system output, Equation (26) shows that the requirement is the same as for an INDI controller, the output up until $y^{(r)}$ should be available. Additionally, the NDI control law requires the state derivative information \dot{x} . Depending on the system, this may overlap with the derivative of the output. In many cases, the requirements on available signals may therefore be the same, or slightly increased compared to a regular INDI controller.

In some cases, filtering of these noisy derivative signals may be required. Filtering leads to delay, which will lead to deterioration of the controller performance. For an INDI controller this problem can be circumvented with filter synchronization of output and input filters [14], but this cannot be applied to the feedback of \dot{x} in the proposed controller, as there is no signal to synchronize with. Instead a complementary filter could be used, where the high frequencies are coming from a model, and the low frequencies from the filtered measurement [26, 27]. The proposed NDI control law additionally requires information about the actuator bandwidth, which might also be uncertain. In addition, the actuators might not be first order. In that case, an effective bandwidth of the actuator can be used for the control law design.

VII. Conclusion

The proposed derivation of the NDI control law considering first order linear actuator dynamics, which one could call Actuator-NDI (ANDI), theoretically provides a perfect inversion of a system with such actuators, also in case of different bandwidth. It allows to assign an innermost desired bandwidth for the tracking variable different than the actuator bandwidth. The ANDI control law leads to an INDI control law if infinitely fast actuators are considered. This provides a better theoretical foundation for INDI control, from which it can be readily seen what the impact is of the terms that are neglected in the INDI derivation. Moreover, the derived NDI control law compensates for state-dependent terms, which are not taken into account with the classical INDI formulation. Compared to the reference model based feed-forward control from literature, the benefit of the proposed control law is twofold. First, the inversion is based on the actual vehicle states, and therefore provides exact tracking performance without the reference model having to exactly model and match all plant states. Secondly, it provides predictable and consistent error dynamics.

VIII. Acknowledgements

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Appendix: NDI and INDI limit

Here it is shown how the error dynamics of NDI and INDI behaves in the limit of the actuator bandwidth going to infinity. Consider first the NDI law from equation (26) together with the system output dynamics as given in equation (5), with the same assumptions as in Section III, i.e.

- All actuators have the same bandwidth, i.e. $\Omega = \omega I_{k \times k}$, where ω is a non-zero scalar
- Choose $\Omega_y = \omega I_{m \times m}$

Then the following limit for the actuator bandwidth going to infinity can be formulated:

$$\lim_{\omega \rightarrow \infty} \left(y^{(r+1)} \right) = \lim_{\omega \rightarrow \infty} (F_x \dot{x} + F_u I_{k \times k} \omega (u_c - u)) \quad (46)$$

By inserting (26) into the above equation, the limit of the error dynamics can be deduced:

$$\begin{aligned} \lim_{\omega \rightarrow \infty} \left(y^{(r+1)} \right) &= \lim_{\omega \rightarrow \infty} (F_x \dot{x} + F_u I_{k \times k} \omega (u_c - u)) \\ \lim_{\omega \rightarrow \infty} \left(y^{(r+1)} \right) &= \lim_{\omega \rightarrow \infty} \left(F_x \dot{x} + F_u I_{k \times k} \omega \left(\frac{1}{\omega} (F_u)^+ \left(y_{ref}^{(r+1)} + \sum_{i=1}^r K_i e_y^{(i)} - F_x \dot{x} \right) + (F_u)^+ \left(e_y^{(r)} + \sum_{i=0}^{r-1} K_i e_y^{(i)} \right) + u - u \right) \right) \\ \lim_{\omega \rightarrow \infty} \left(y^{(r+1)} \right) &= \lim_{\omega \rightarrow \infty} \left(F_x \dot{x} + \frac{1}{\omega} \omega \left(y_{ref}^{(r+1)} + \sum_{i=1}^r K_i e_y^{(i)} - F_x \dot{x} \right) + \omega \left(e_y^{(r)} + \sum_{i=0}^{r-1} K_i e_y^{(i)} \right) \right) \\ \lim_{\omega \rightarrow \infty} \left(y^{(r+1)} \right) &= \lim_{\omega \rightarrow \infty} \left(\left(y_{ref}^{(r+1)} + \sum_{i=1}^r K_i e_y^{(i)} \right) + \omega \left(e_y^{(r)} + \sum_{i=0}^{r-1} K_i e_y^{(i)} \right) \right) \\ \lim_{\omega \rightarrow \infty} \left(e_y^{(r+1)} + \sum_{i=1}^r K_i e_y^{(i)} + \omega \left(e_y^{(r)} + \sum_{i=0}^{r-1} K_i e_y^{(i)} \right) \right) &= 0 \end{aligned} \quad (47)$$

which is the limit of the desired error dynamics given in equation (13).

If the INDI law from (23) is inserted into (46), the following can be derived:

$$\begin{aligned} \lim_{\omega \rightarrow \infty} \left(y^{(r+1)} \right) &= \lim_{\omega \rightarrow \infty} \left(F_x \dot{x} + F_u I_{k \times k} \omega \left((F_u)^\dagger \left(e_y^{(r)} + \sum_{i=0}^{r-1} K_i e_y^{(i)} \right) + u - u \right) \right) \\ \lim_{\omega \rightarrow \infty} \left(y^{(r+1)} \right) &= \lim_{\omega \rightarrow \infty} \left(F_x \dot{x} + F_u I_{k \times k} \omega \left((F_u)^\dagger \left(e_y^{(r)} + \sum_{i=0}^{r-1} K_i e_y^{(i)} \right) \right) \right) \\ \lim_{\omega \rightarrow \infty} \left(-y^{(r+1)} + F_x \dot{x} + \omega \left(e_y^{(r)} + \sum_{i=0}^{r-1} K_i e_y^{(i)} \right) \right) &= 0 \end{aligned} \quad (48)$$

which does not correspond to the desired error dynamics. Hence, even in the limit, the INDI does not produce the correct error dynamics. The INDI only approximates the true NDI law arbitrarily well as given in Equation (26).

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