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$$\mathcal{L}(d|\{\vec{b}_G\}) = \int d\theta_i \, \exp\left[-\frac{1}{2}\underbrace{\left(P^{(d)} - P^{(t)}_{const} - \sum_i \theta_i P^{(t)}_{linear,i}\right) \cdot C^{-1} \left(P^{(d)} - P^{(t)}_{const} - \sum_j \theta_j P^{(t)}_{linear,j}\right)^T}\right]$$

I'm using the definition:

$A \star B = A \cdot C^{-1} \cdot B^T$

$$= P^{(d)} \star P^{(d)} - P^{(d)} \star P^{(t)}_{const} - P^{(d)} \star \sum_i \theta_i P^{(t)}_{linear,i} - P^{(t)}_{const} \star P^{(d)} + P^{(t)}_{const} \star P^{(t)}_{const} + P^{(t)}_{const} \star \sum_i \theta_i P^{(t)}_{lin,i}$$

$$- \sum_i \theta_i P^{(t)}_{lin,i} \star P^{(d)} + \sum_i \theta_i P^{(t)}_{lin,i} \star P^{(t)}_{const} + \sum_{i,j} \theta_i \theta_j P^{(t)}_{lin,i} \star P^{(t)}_{lin,j}$$

We can rearrange these terms as:

$$\sum_{i,j} \theta_i \theta_j P^{(t)}_{lin,i} \star P^{(t)}_{lin,j} + \sum_i \theta_i \left[P^{(t)}_{const} \star P^{(t)}_{lin,i} - P^{(d)} \star P^{(t)}_{lin,i} - P^{(t)}_{lin,i} \star P^{(d)} + P^{(t)}_{lin,i} \star P^{(t)}_{const} \right]$$

$$+ P^{(d)} \star P^{(d)} - P^{(d)} \star P^{(t)}_{const} - P^{(t)}_{const} \star P^{(d)} + P^{(t)}_{const} \star P^{(t)}_{const}$$

I will define the following matrices:

$$\mathcal{L}(d|\vec{b}_G) = \int d\theta_i \, \exp\left[-\frac{1}{2} \sum \theta_i A_{ij} \theta_j + \sum \theta_i B_i + C\right]$$ such that:

$$\mathcal{L}(d|D_G) = \int d\vec{\theta}\, \exp\left[-\frac{1}{2}\sum_{ij}\theta_i A_{ij}\theta_j + \sum_i \ldots + C\right]$$ such that:

$$A_{ij} = P^{(t)}_{lim,i} \star P^{(t)}_{lim,j}\,, \quad B_i = -\frac{1}{2}\left(P^{(t)}_{const} \star P^{(t)}_{lim,i} - P^{(d)} \star P^{(t)}_{lim,i} - P^{(t)}_{lim,i} \star P^{(d)} + P^{(t)}_{lim,i} \star P^{(t)}_{const}\right)$$

$$C = -\frac{1}{2}\left(P^{(d)} \star P^{(d)} - P^{(d)} \star P^{(t)}_{const} - P^{(t)}_{const} \star P^{(d)} + P^{(t)}_{const} \star P^{(t)}_{const}\right)$$

Our likelihood is then:

$$\mathcal{L}(d|\theta_i, \theta_G) = \exp\left[-\frac{1}{2}\theta_i A_{ij}\theta_j + \theta_i B_i + C\right]$$

$$\mathcal{L}(d|\theta_G) = \int d\vec{\theta}\; e^{-\frac{1}{2}\vec{\theta}\cdot A\cdot\vec{\theta}^{t} + B\cdot\vec{\theta}^{t} + C}$$

$$= e^{C}\int d\vec{\theta}\; e^{-\frac{1}{2}\vec{\theta}\cdot A\cdot\vec{\theta}^{t} + B\cdot\vec{\theta}^{T}} = e^{C}\left(\frac{(2\pi)^n}{\det A}\right)^{1/2} e^{\frac{1}{2}B\cdot A^{-1}\cdot B} = (2\pi)^{n/2}\exp\left[\frac{1}{2}B\cdot A^{-1}\cdot B + C - \frac{1}{2}\log\det A\right]$$

The marginalised likelihood is:

$$\mathcal{L}(d \mid \Theta_G) = (2\pi)^{n/2} \exp\left[\frac{1}{2}\mathbb{B}\cdot\mathbb{A}^{-1}\cdot\mathbb{B} + \mathbb{C} - \frac{1}{2}\log\det\mathbb{A}\right]$$