



Progressive lattice sieving

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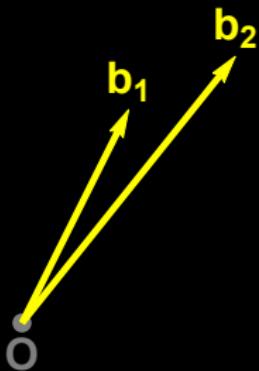
Lattices

What is a lattice?



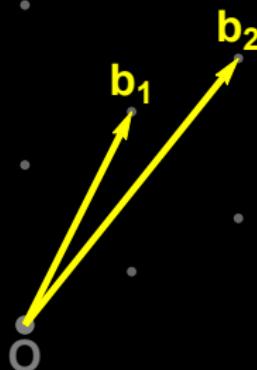
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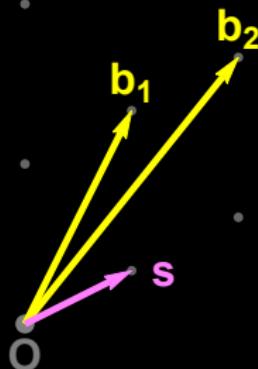
Lattices

What is a lattice?



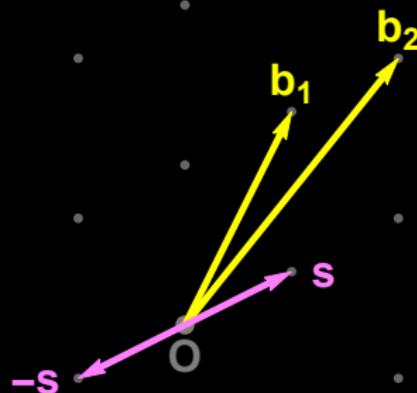
Lattices

Shortest Vector Problem (SVP)



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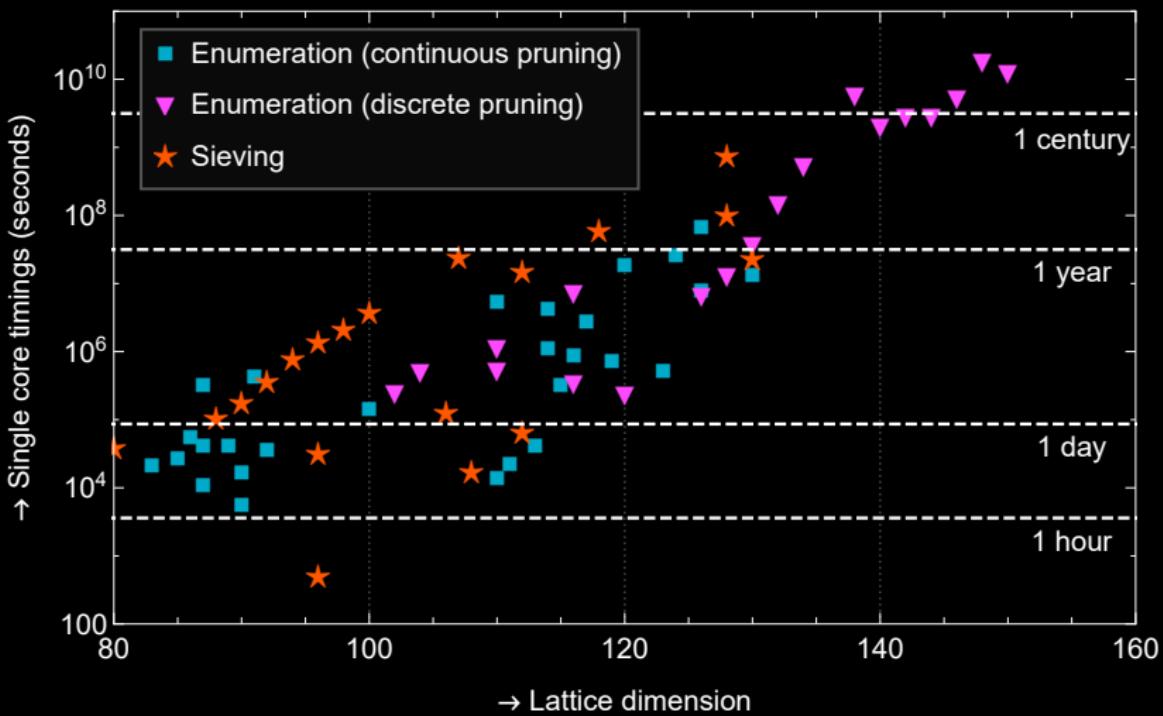
SVP hardness

Theory

	Algorithm	$\log_2(\text{Time})$	$\log_2(\text{Space})$
Proven SVP	Enumeration [Poh81, Kan83, ..., MW15, AN17]	$O(n \log n)$	$O(\log n)$
	AKS-sieve [AKS01, NV08, MV10, HPS11]	$3.398n$	$1.985n$
	ListSieve [MV10, MDB14]	$3.199n$	$1.327n$
	Birthday sieves [PS09, HPS11]	$2.465n$	$1.233n$
	Enumeration/DGS hybrid [CCL17]	$2.048n$	$0.500n$
	Voronoi cell algorithm [AEVZ02, MV10b]	$2.000n$	$1.000n$
	Quantum sieve [LMP13, LMP15]	$1.799n$	$1.286n$
	Quantum enum/DGS [CCL17]	$1.256n$	0.500n
Heuristic SVP	Discrete Gaussian sampling [ADRS15, ADS15, AS18]	1.000n	$1.000n$
	The Nguyen–Vidick sieve [NV08]	$0.415n$	$0.208n$
	The GaussSieve [MV10, ..., IKMT14, BNvdP16, YKYC17]	$0.415n$	$0.208n$
	Triple sieve [BLS16, HK17]	$0.396n$	$0.189n$
	Two-level sieve [WLTB11]	$0.384n$	$0.256n$
	Three-level sieve [ZPH13]	$0.3778n$	$0.283n$
	Overlattice sieve [BGJ14]	$0.3774n$	$0.293n$
	Triple sieve with NNS [HK17, HKL18]	$0.359n$	0.189n
Hyperplane LSH [Cha02, Laa15, ..., LM18, Duc18]		$0.337n$	$0.337n$
Graph-based NNS [EPY99, DCL11, MPLK14, Laa18]		$0.327n$	$0.282n$
Hypercube LSH [TT07, Laa17]		$0.322n$	$0.322n$
Quantum sieve [LMP13, LMP15]		$0.312n$	$0.208n$
May–Ozerov NNS [MO15, BGJ15]		$0.311n$	$0.311n$
Spherical LSH [AINR14, LdW15]		$0.298n$	$0.298n$
Cross-polytope LSH [TT07, AILRS15, BL16, KW17]		$0.298n$	$0.298n$
Spherical LSF [BDGL16, MLB17, ALRW17, Chr17]		$0.292n$	$0.292n$
Quantum NNS sieve [LMP15, Laa16]		$0.265n$	$0.265n$

SVP hardness

Practice [SVP17]



SVP hardness

NIST submissions

Title	Si	En	Submitters
CRYSTALS-Dilithium	•		Lyubashevsky, Ducas, Kiltz, Lepoint, Schwabe, Seiler, Stehlé
CRYSTALS-Kyber	•		Schwabe, Avanzi, Bos, Ducas, Kiltz, Lepoint, Lyubashevsky, Schanck, ...
Ding Key Exchange	•		Ding, Takagi, Gao, Wang
(R.)EMBLEM	•		Seo, Park, Lee, Kim, Lee
FALCON	•		Prest, Fouque, Hoffstein, Kirchner, Lyubashevsky, Pornin, Ricosset, ...
FrodoKEM	•		Naehrig, Alkim, Bos, Ducas, Easterbrook, LaMacchia, Longa, Mironov, ...
Giophantus	•		Akiyama, Goto, Okumura, Takagi, Nuida, Hanaoka, Shimizu, Ikematsu
HILA5	•		Saarinen
KCL	•		Zhao, Jin, Gong, Sui
KINDI	•		El Bansarkhani
LAC	•		Lu, Liu, Jia, Xue, He, Zhang
LIMA	•		Smart, Albrecht, Lindell, Orsini, Osheter, Paterson, Peer
Lizard	•		Cheon, Park, Lee, Kim, Song, Hong, Kim, Kim, Hong, Yun, Kim, Park, ...
LOTUS	•		Phong, Hayashi, Aono, Moriai
NewHope	•		Pöppelmann, Alkim, Avanzi, Bos, Ducas, De La Piedra, Schwabe, Stebila
NTRUEncrypt	◦	◦	Zhang, Chen, Hoffstein, Whyte
NTRU-HRSS-KEM	•		Schanck, Hüsing, Rijneveld, Schwabe
NTRU Prime	•		Bernstein, Chuengsatiansup, Lange, Van Vredendaal
pqNTRUSign	◦	◦	Zhang, Chen, Hoffstein, Whyte
qTESLA	•		Bindel, Akleylek, Alkim, Barreto, Buchmann, Eaton, Gutoski, Krämer, ...
Round2	•		Garcia-Morchon, Zhang, Bhattacharya, Rietman, Tolhuizen, Torre-Arce
SABER	•		D'Anvers, Karmakar, Roy, Vercauteren
Three Bears	•		Hamburg
Titanium	•		Steinfeld, Sakzad, Zhao
Totals:	21	3	Total: 24 proposals estimate SVP hardness with sieving/enumeration

*Not included in this overview: Compact LWE, DRS, Mersenne, Odd Manhattan, Ramstake, ...

SVP hardness

Overview

Problem: How hard is SVP in high dimensions?

- Two main approaches: *enumeration* and *sieving*
 - ▶ Enumeration: memory-efficient, asymptotically slow
 - ▶ Sieving: memory-intensive, asymptotically fast
- Theoretically (large n): sieving > enumeration
- Practically (small n): enumeration > sieving
- NIST submissions: (mostly) sieving

SVP hardness

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Problem: Can sieving still be improved?

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- Theoretically: Probably not... [BDGL16, ALRW17, HKL18]

SVP hardness

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Problem: Can sieving still be improved?

- Theoretically: Probably not... [BDGL16, ALRW17, HKL18]
- Practically: Yes! **(this work)**, [Duc18]

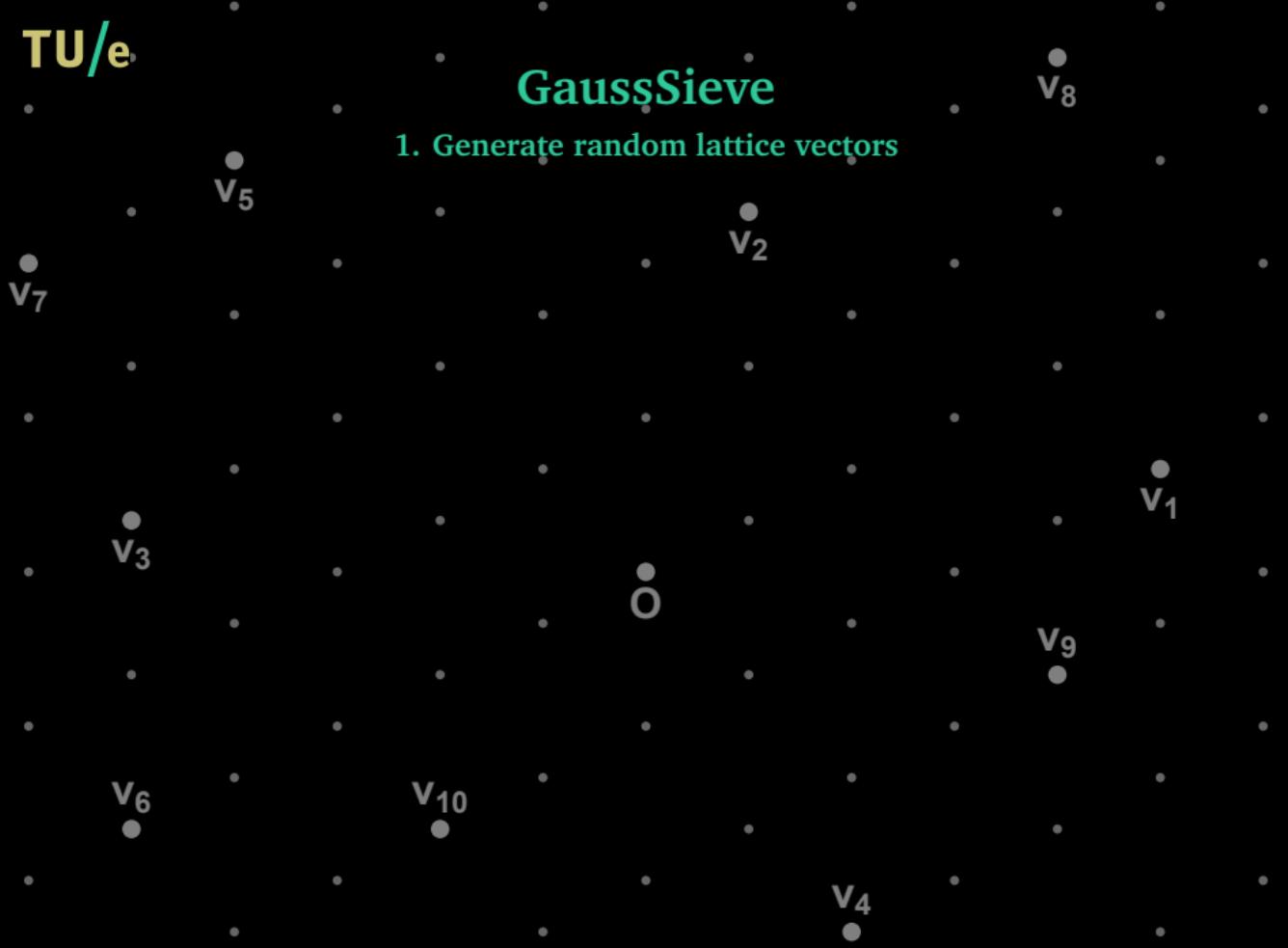
GaussSieve

1. Generate random lattice vectors

O

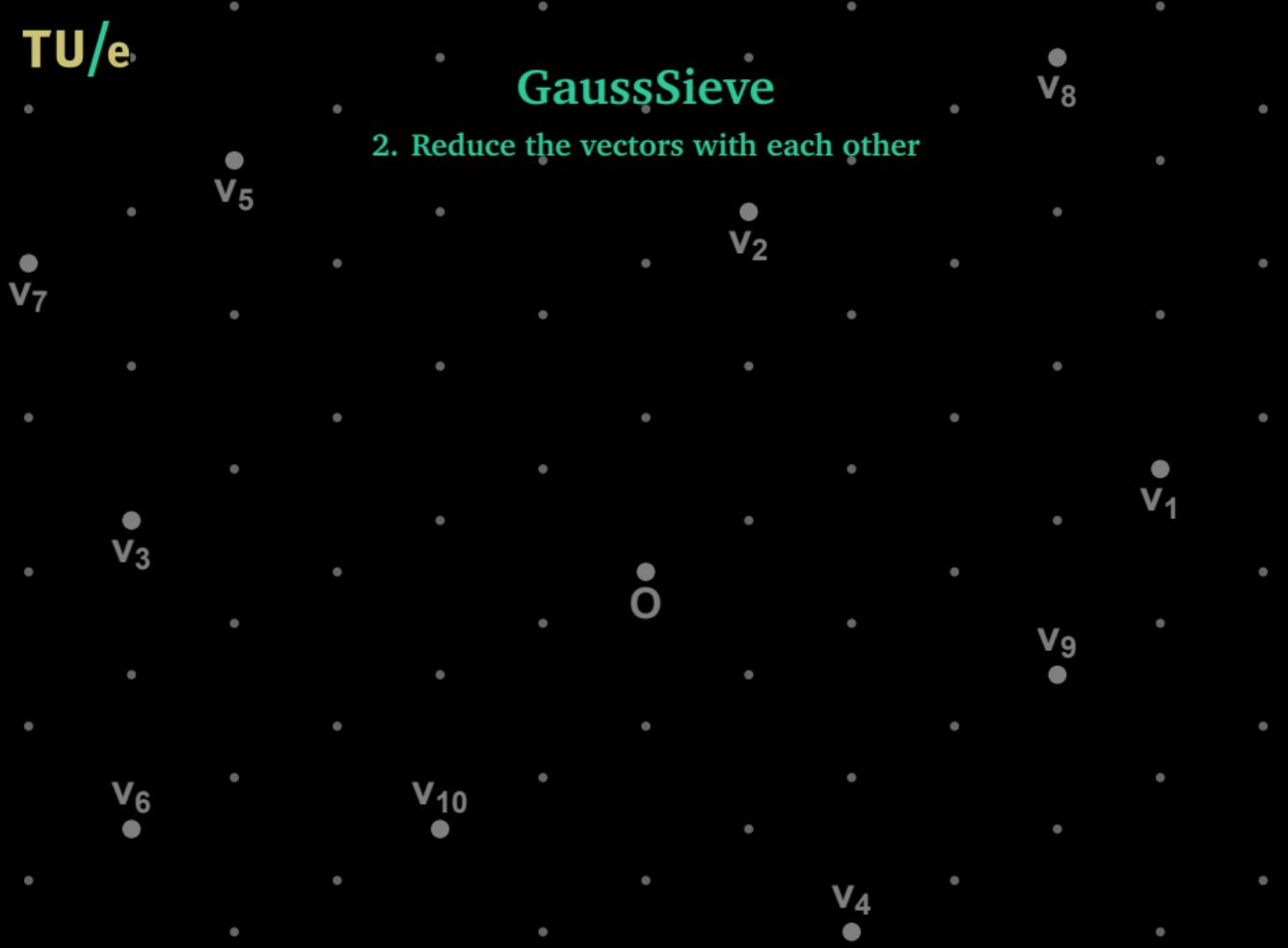
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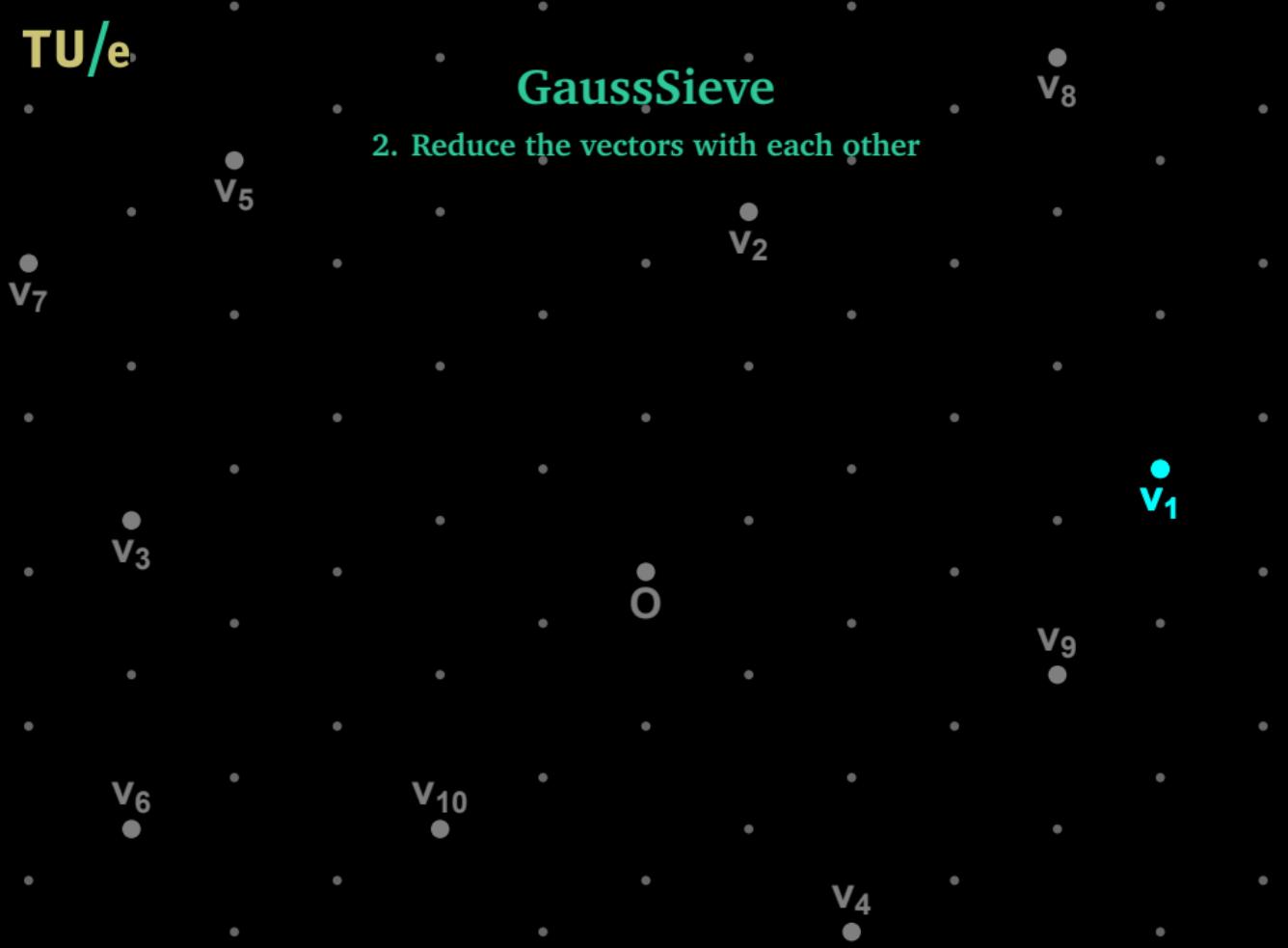
GaussSieve

2. Reduce the vectors with each other



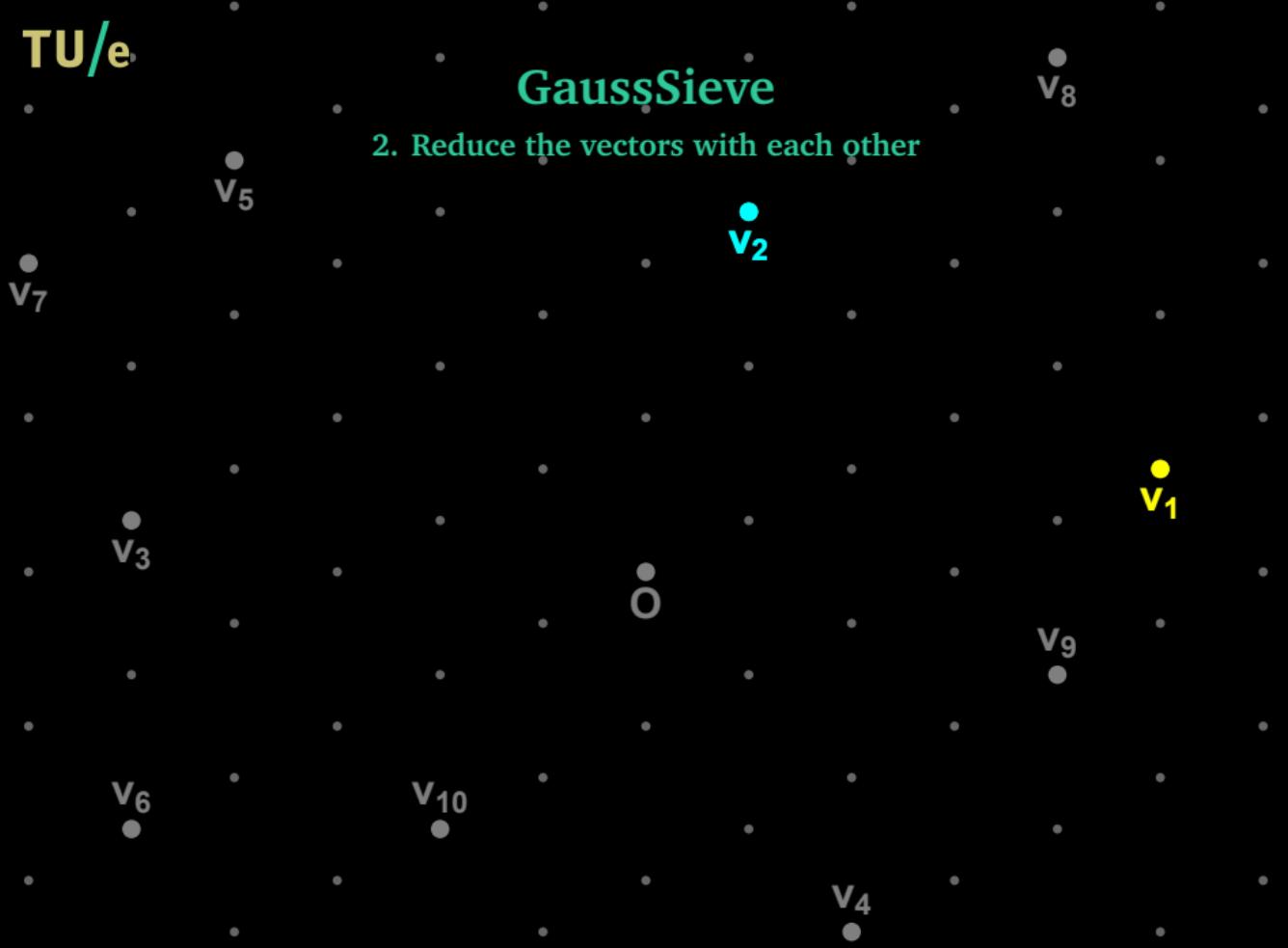
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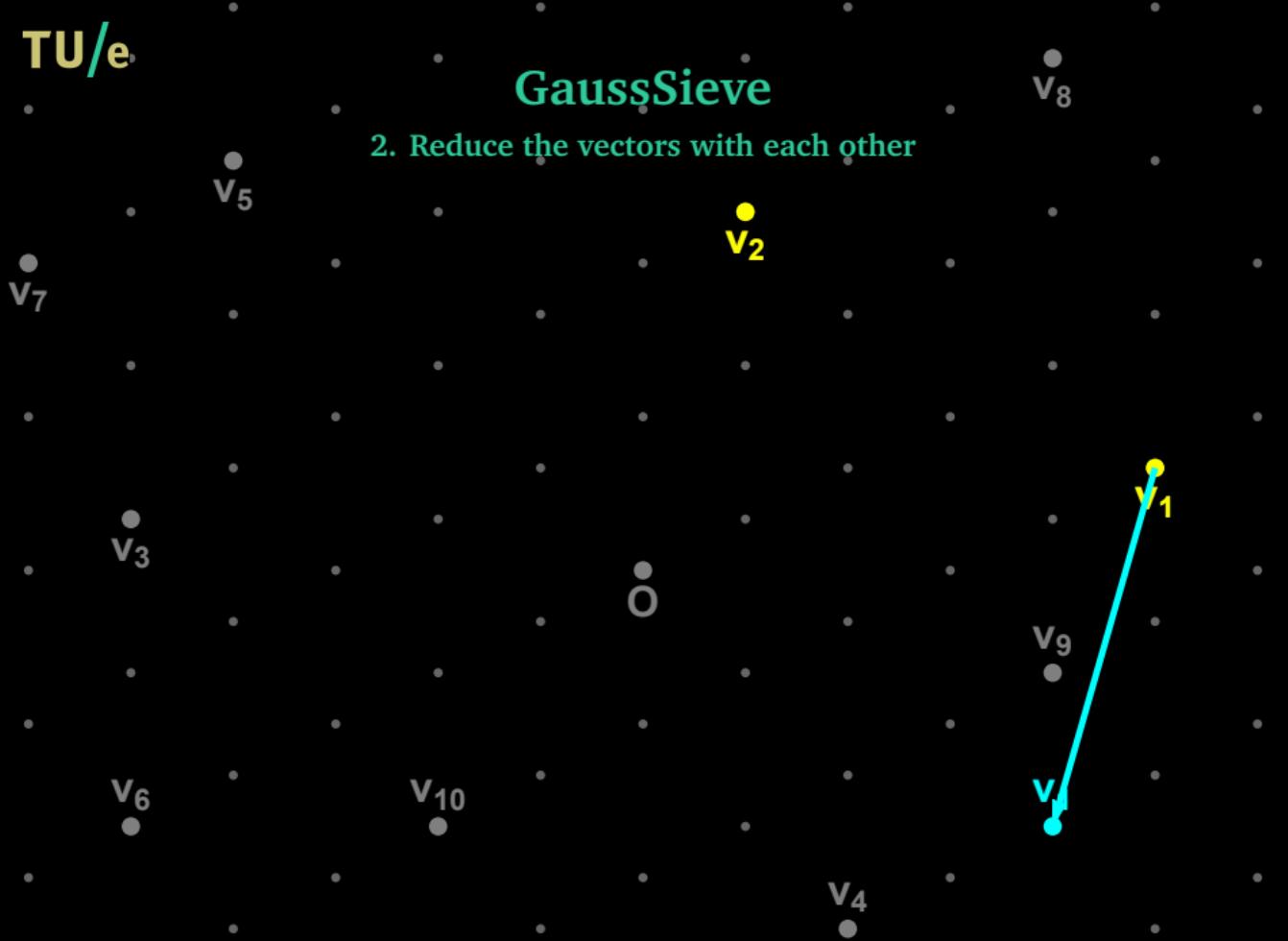
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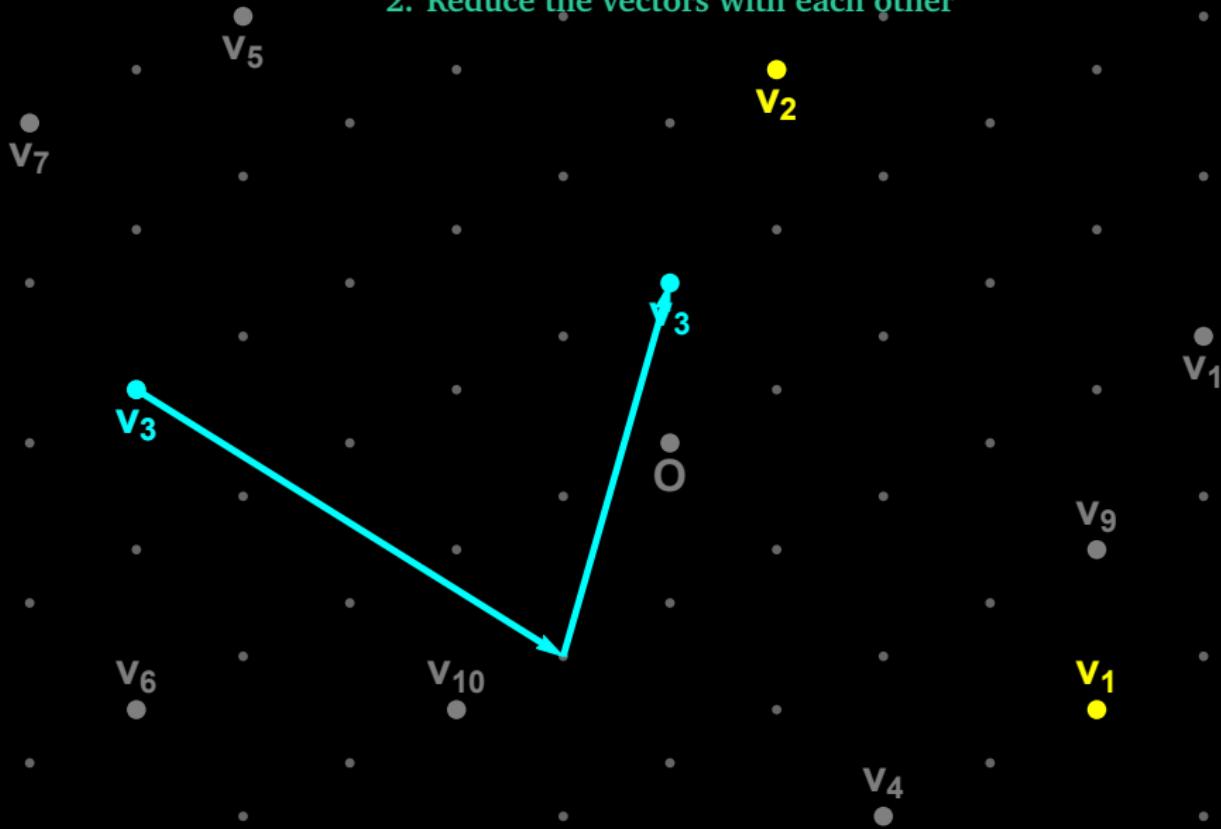
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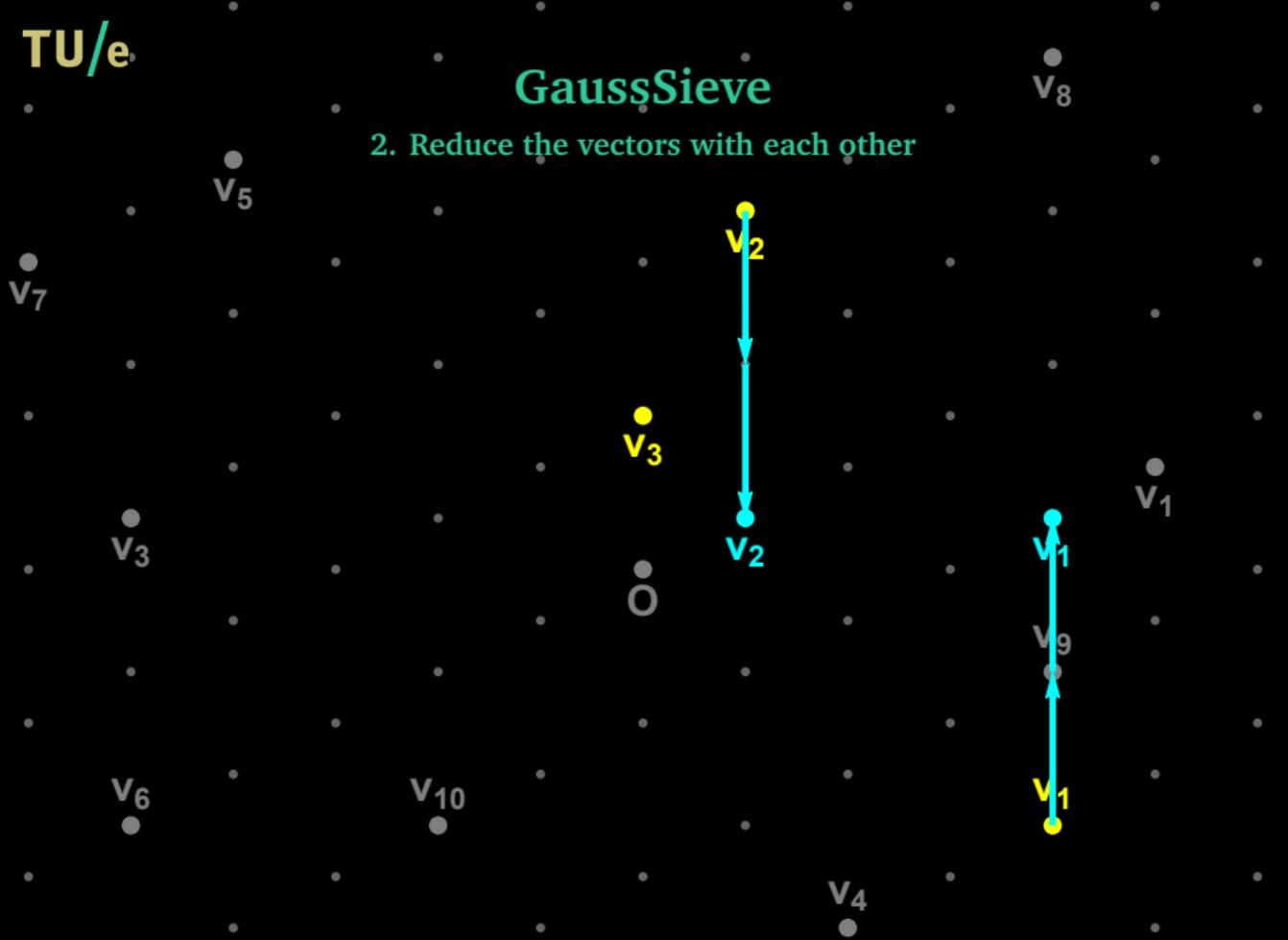
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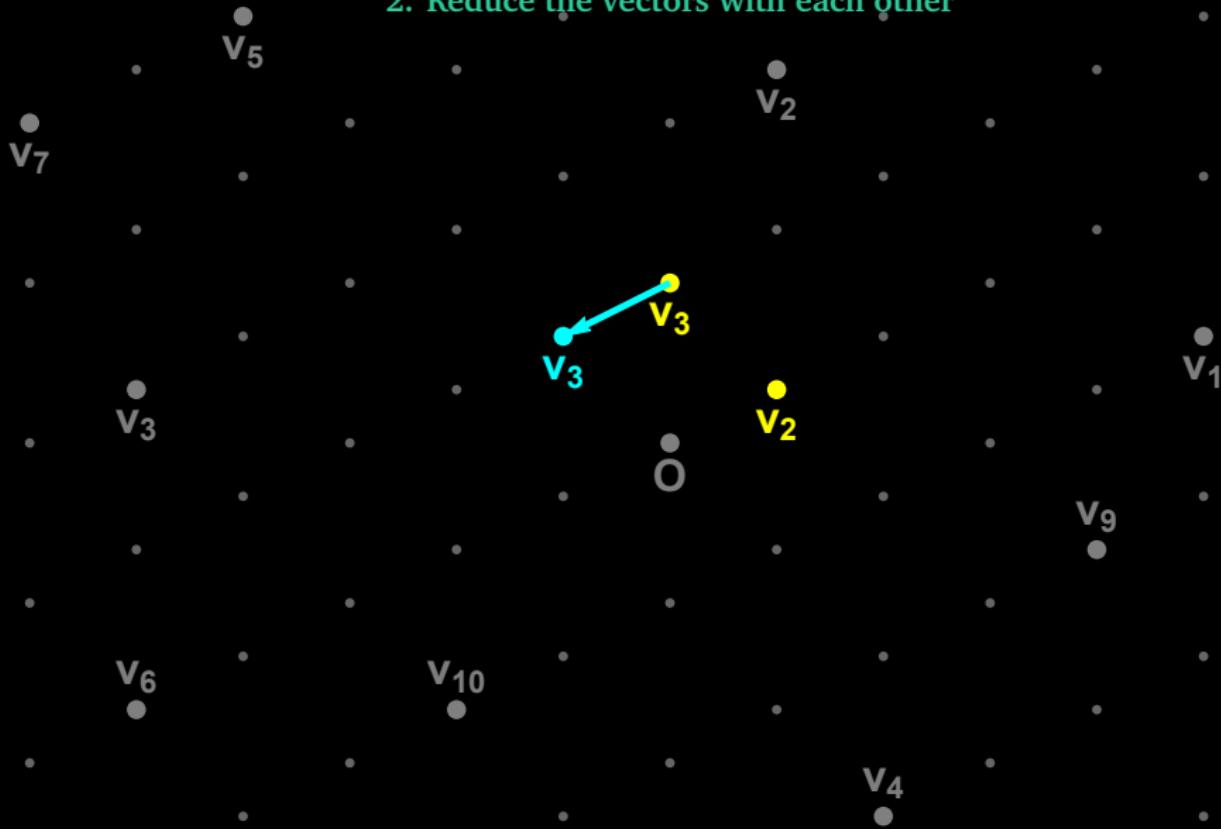
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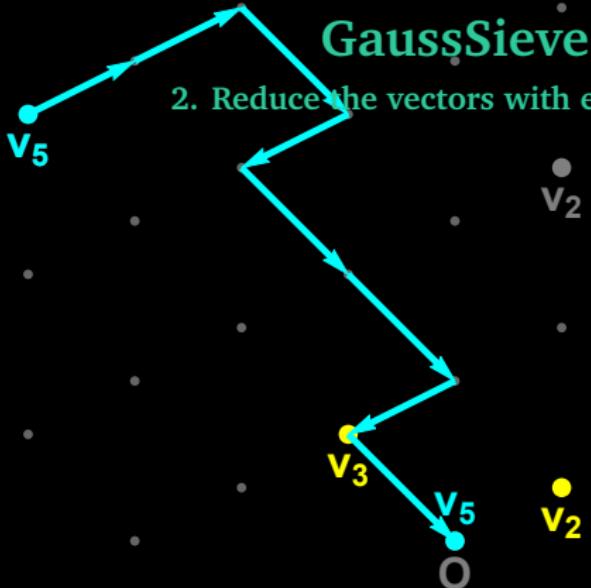
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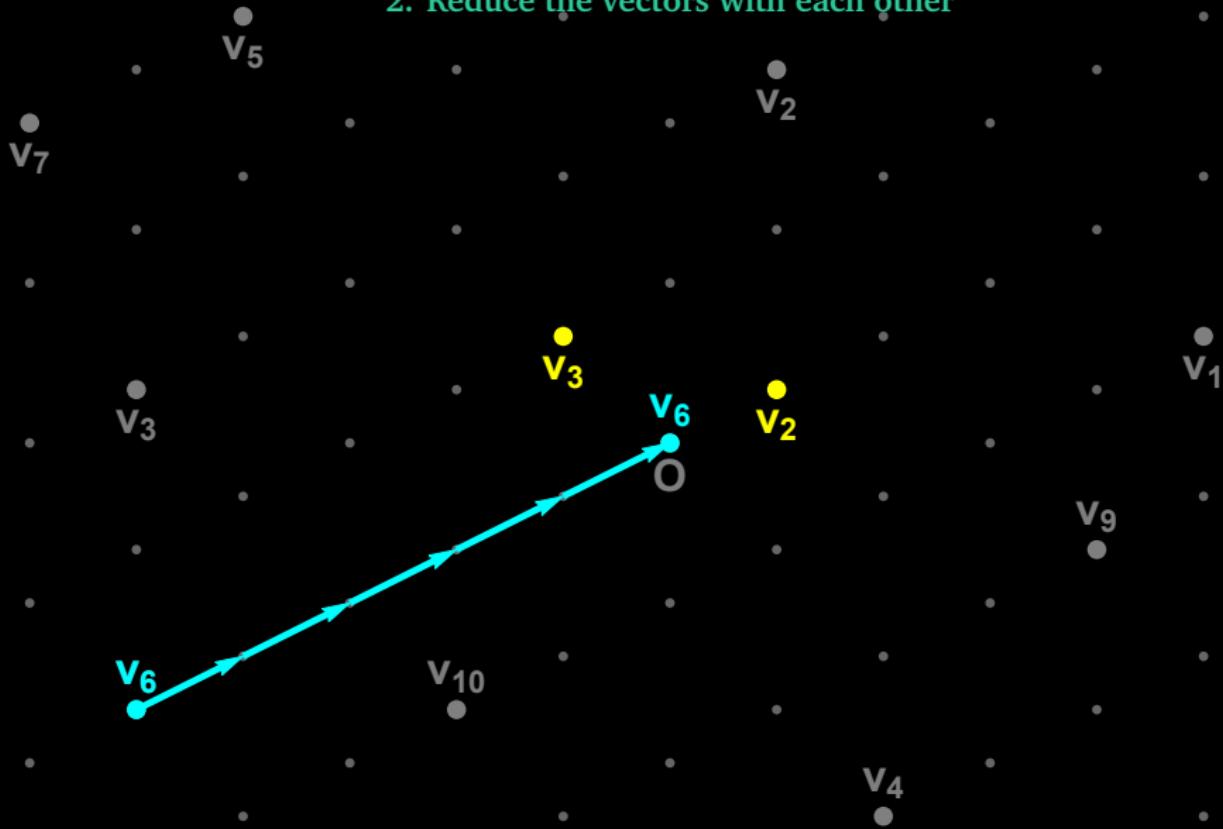
GaussSieve

2. Reduce the vectors with each other

 v_6 v_{10} v_4 v_3 v_7 v_2 v_1 v_9 v_8 v_4

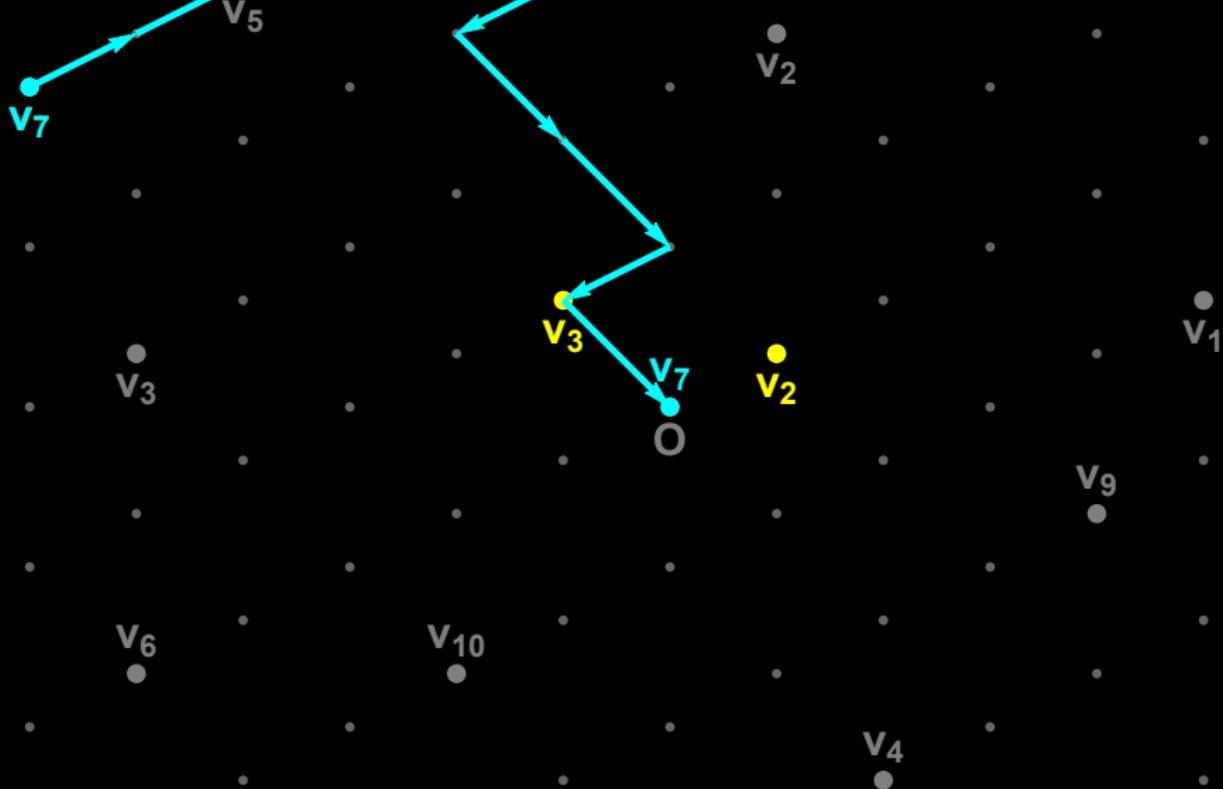
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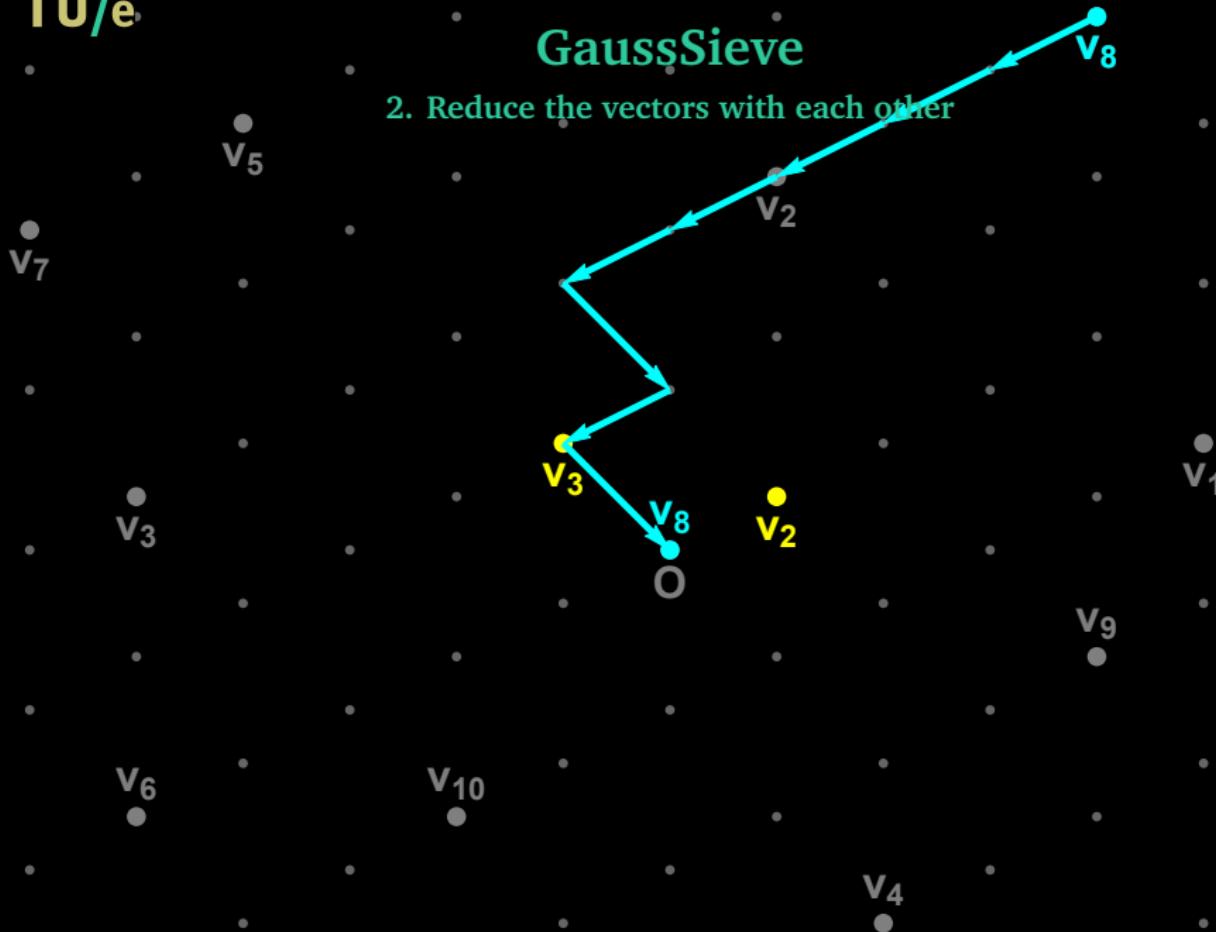
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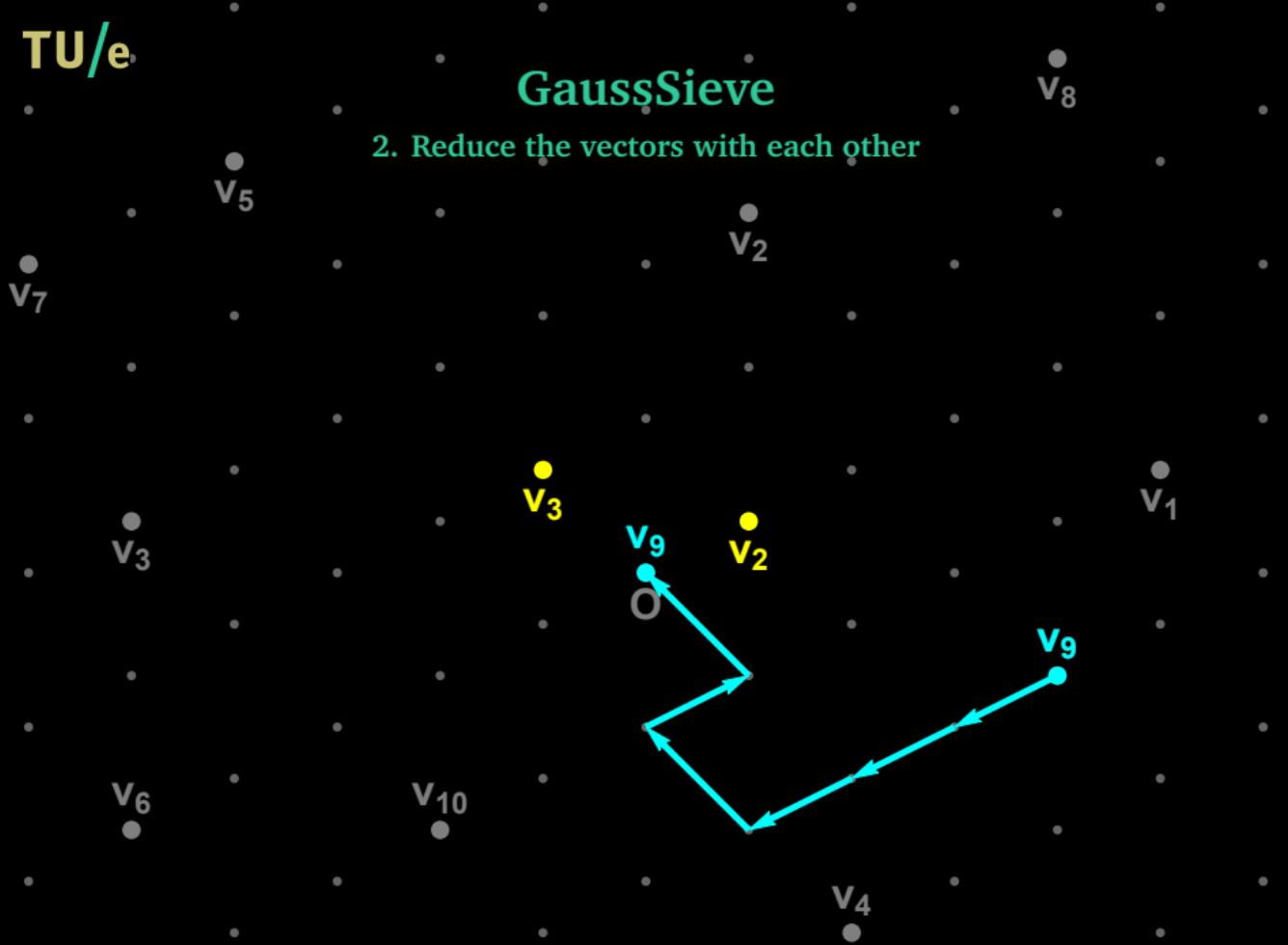
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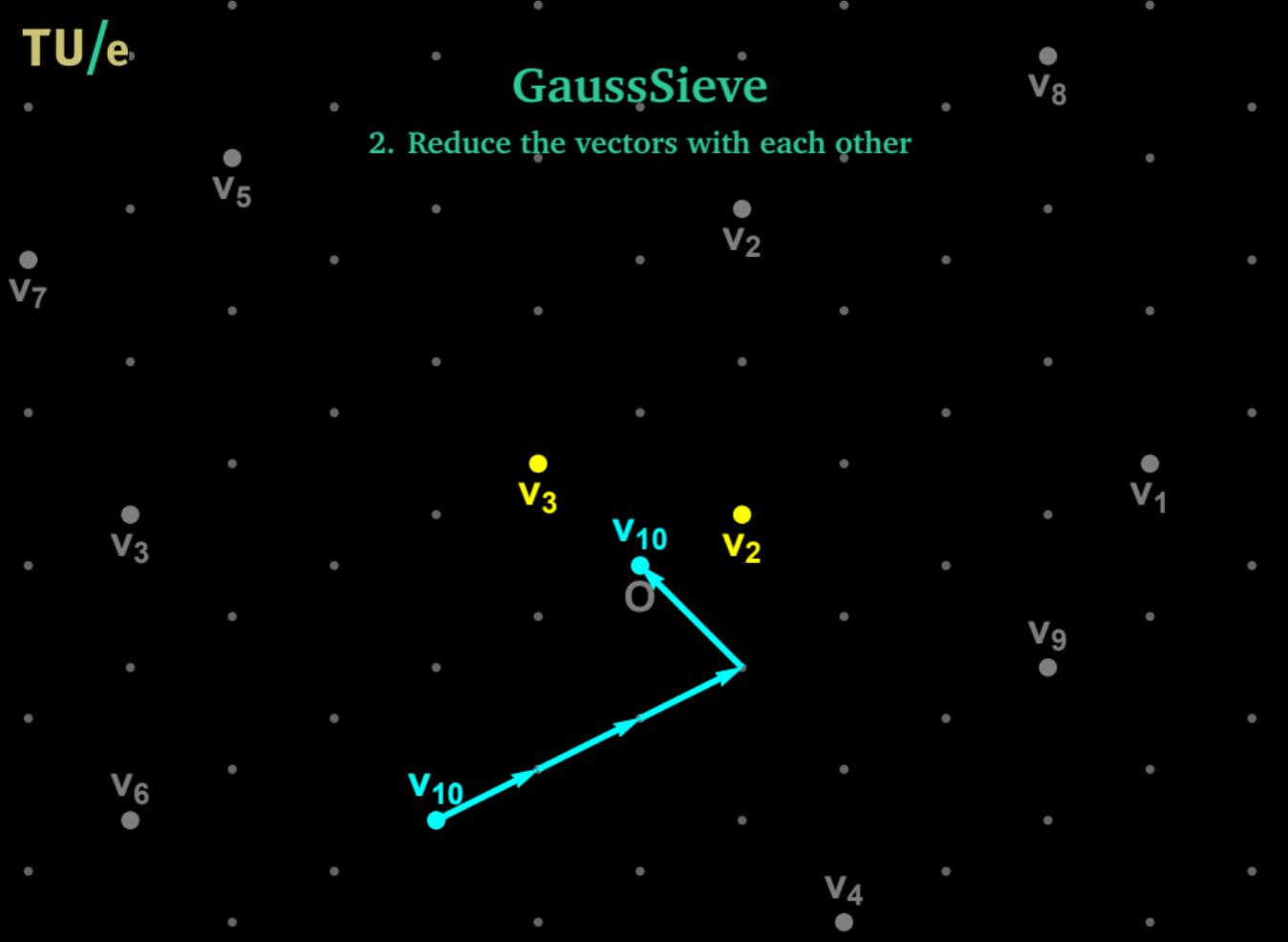
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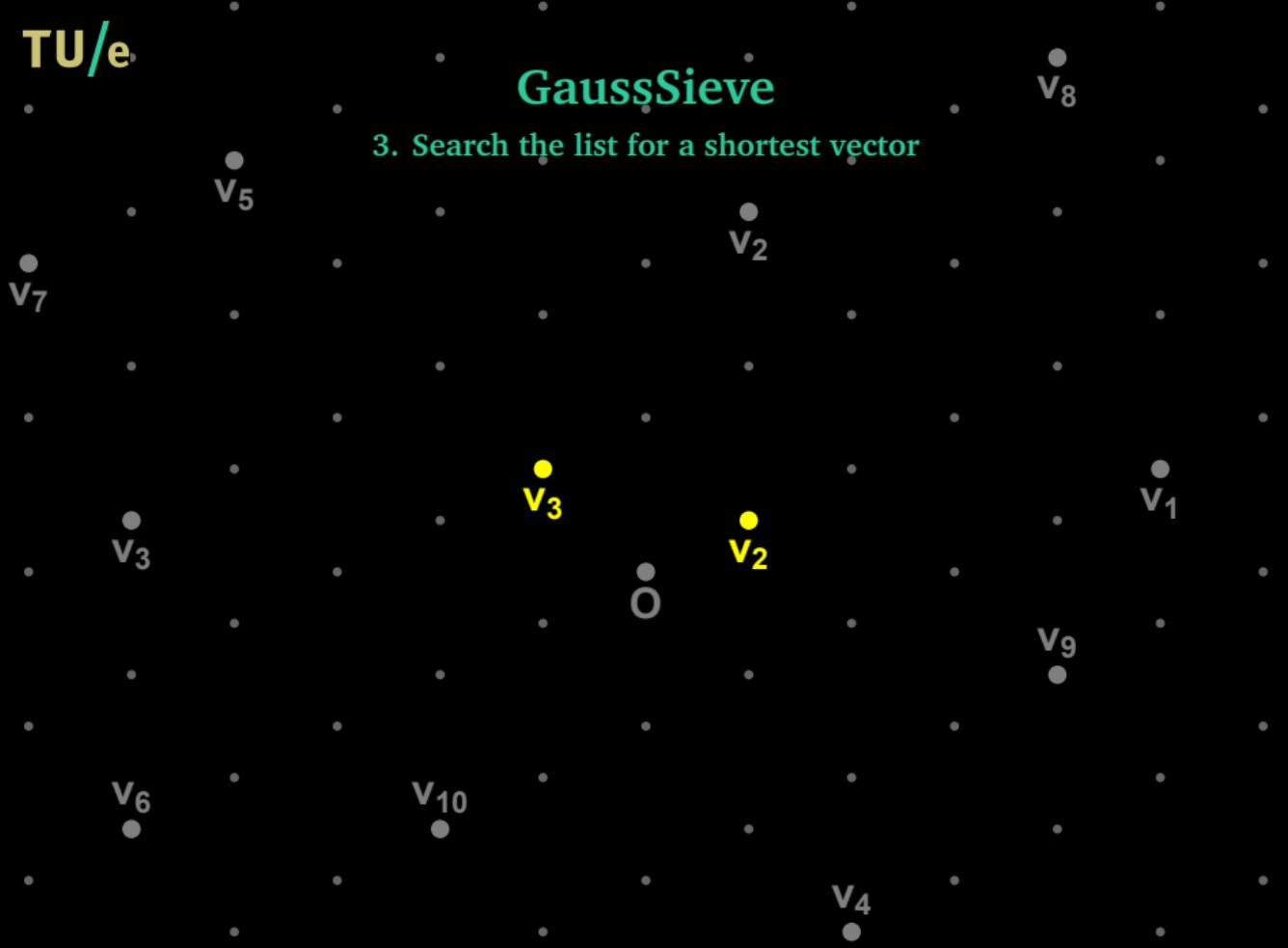
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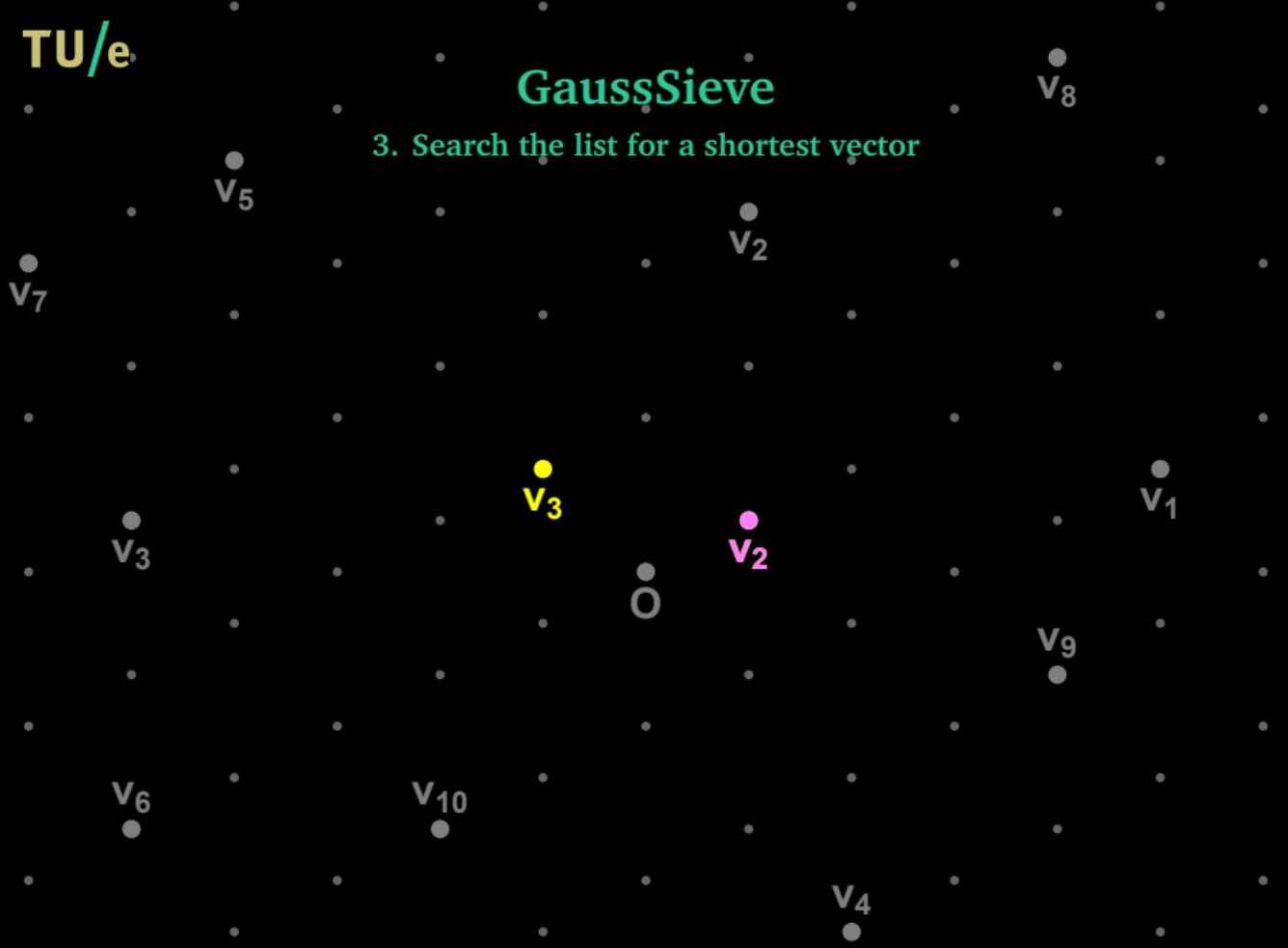
GaussSieve

3. Search the list for a shortest vector



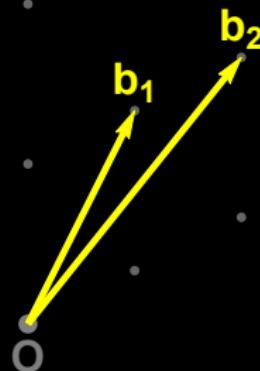
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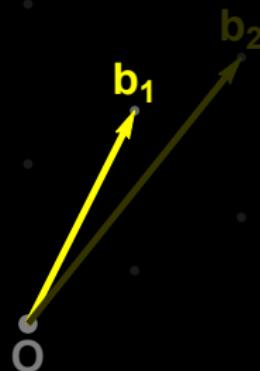
ProGaussSieve

1. Generate random vectors on sublattice



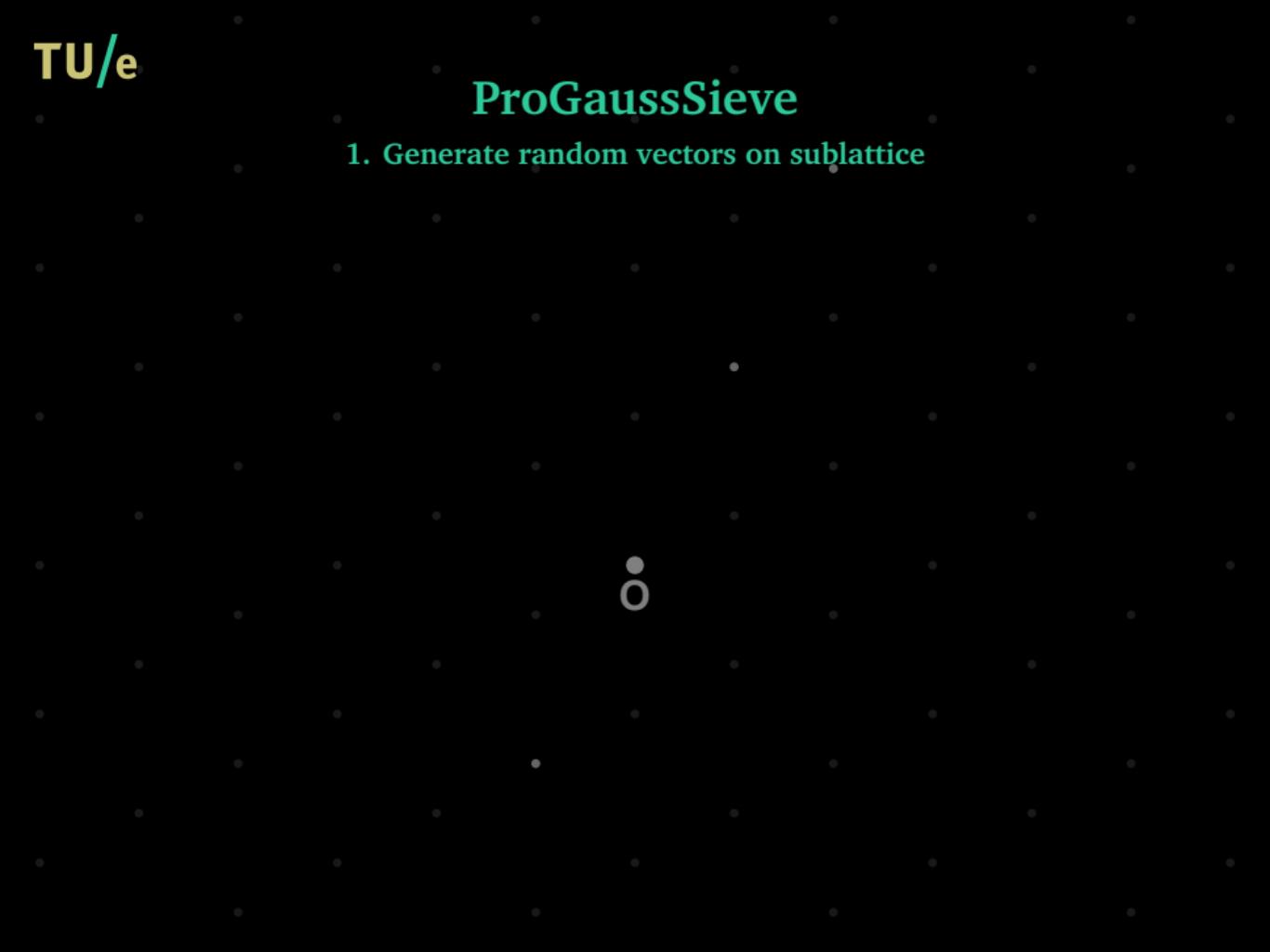
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O

ProGaussSieve

1. Generate random vectors on sublattice

O

v₁

v₂

ProGaussSieve

2. Reduce the vectors with each other

v_1

O

v_2

ProGaussSieve

2. Reduce the vectors with each other

v_1

O

v_2

ProGaussSieve

2. Reduce the vectors with each other

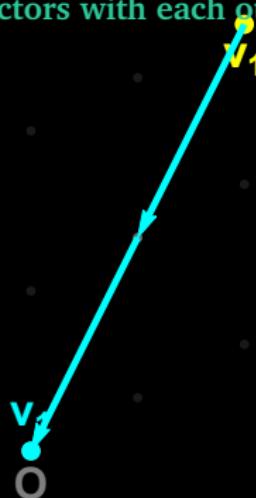
v₁

O

v₂

ProGaussSieve

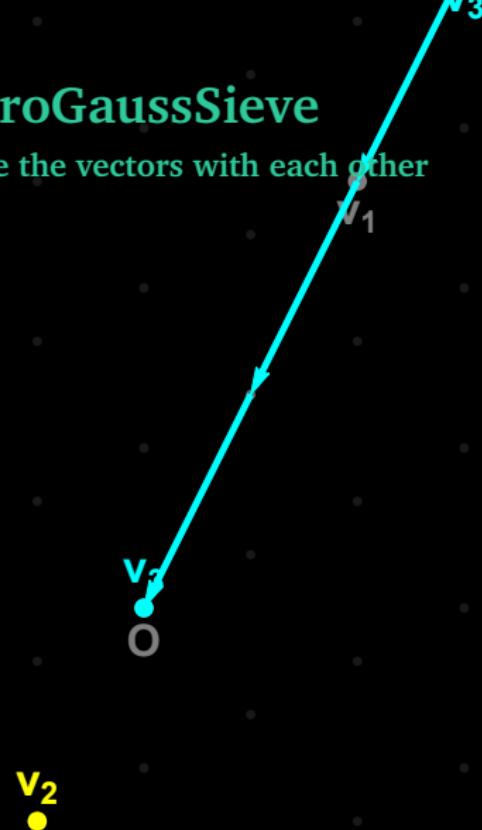
2. Reduce the vectors with each other



v₂

ProGaussSieve

2. Reduce the vectors with each other



v_2

ProGaussSieve

2. Reduce the vectors with each other

v_1

O

v_2

ProGaussSieve

2. Reduce the vectors with each other

O

v_2

ProGaussSieve

3. Generate random vectors on full lattice

O

v_0

ProGaussSieve

3. Generate random vectors on full lattice



ProGaussSieve

4. Reduce the vectors with each other



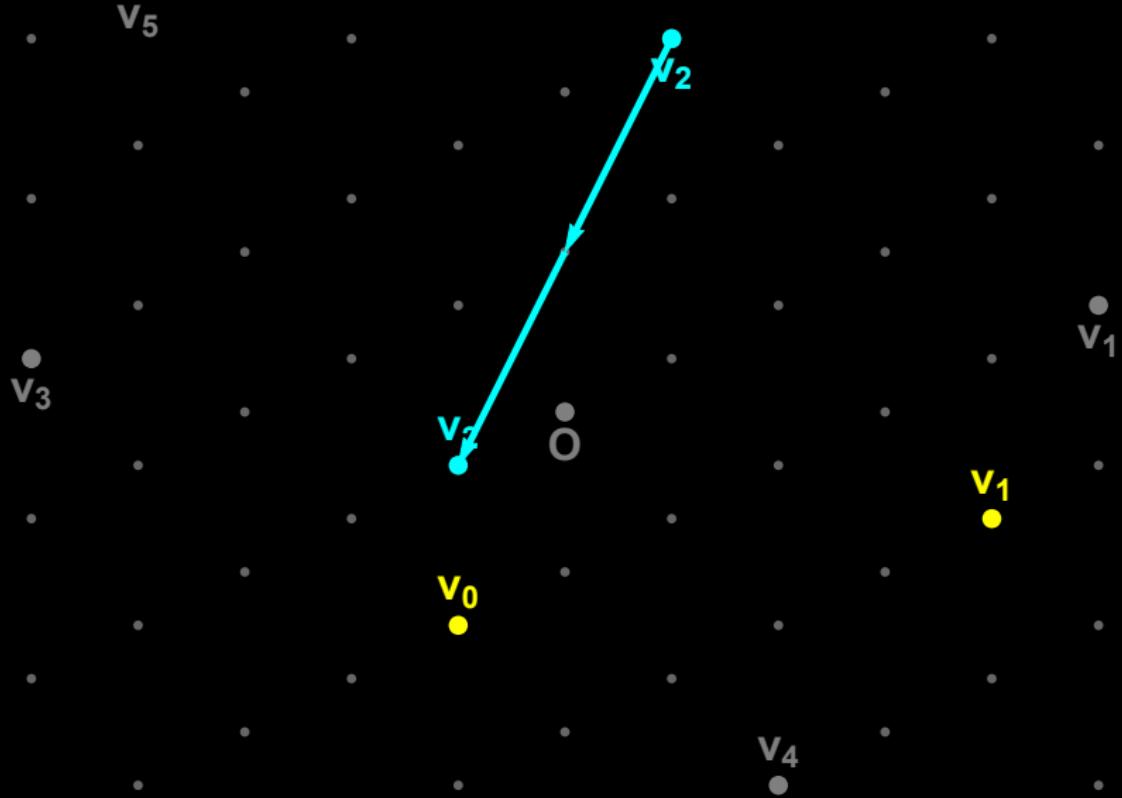
ProGaussSieve

4. Reduce the vectors with each other



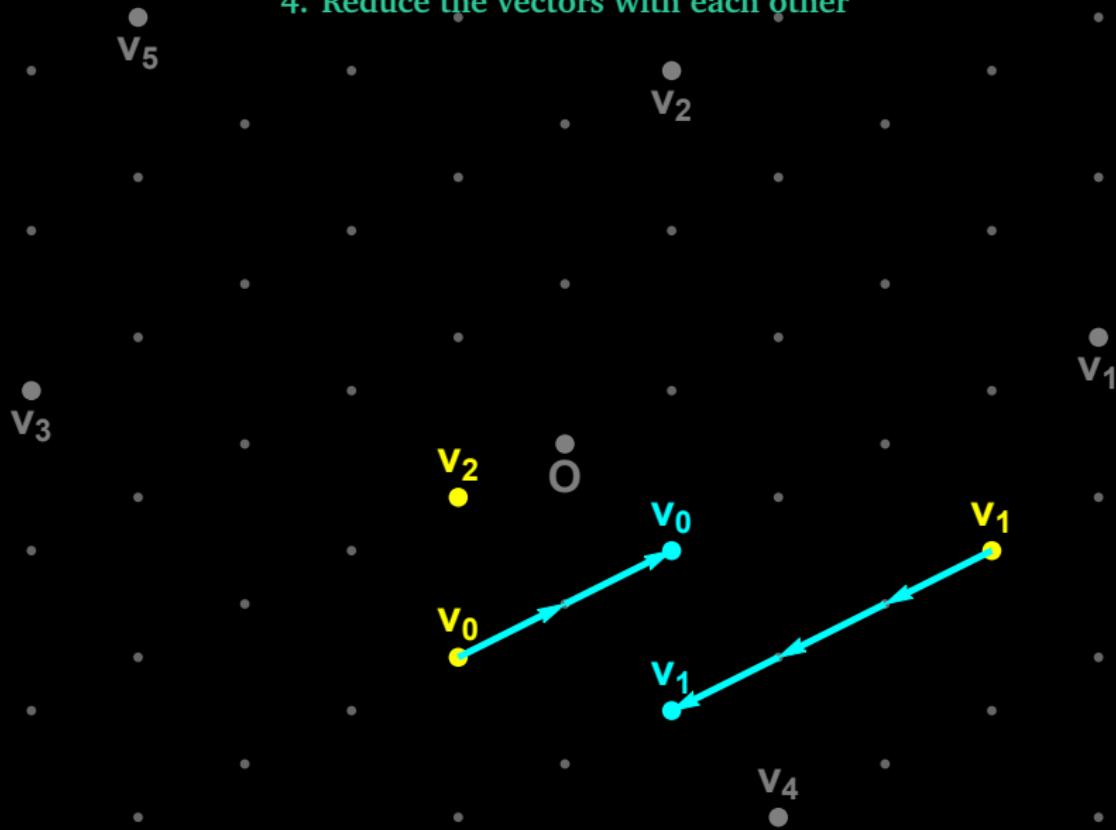
ProGaussSieve

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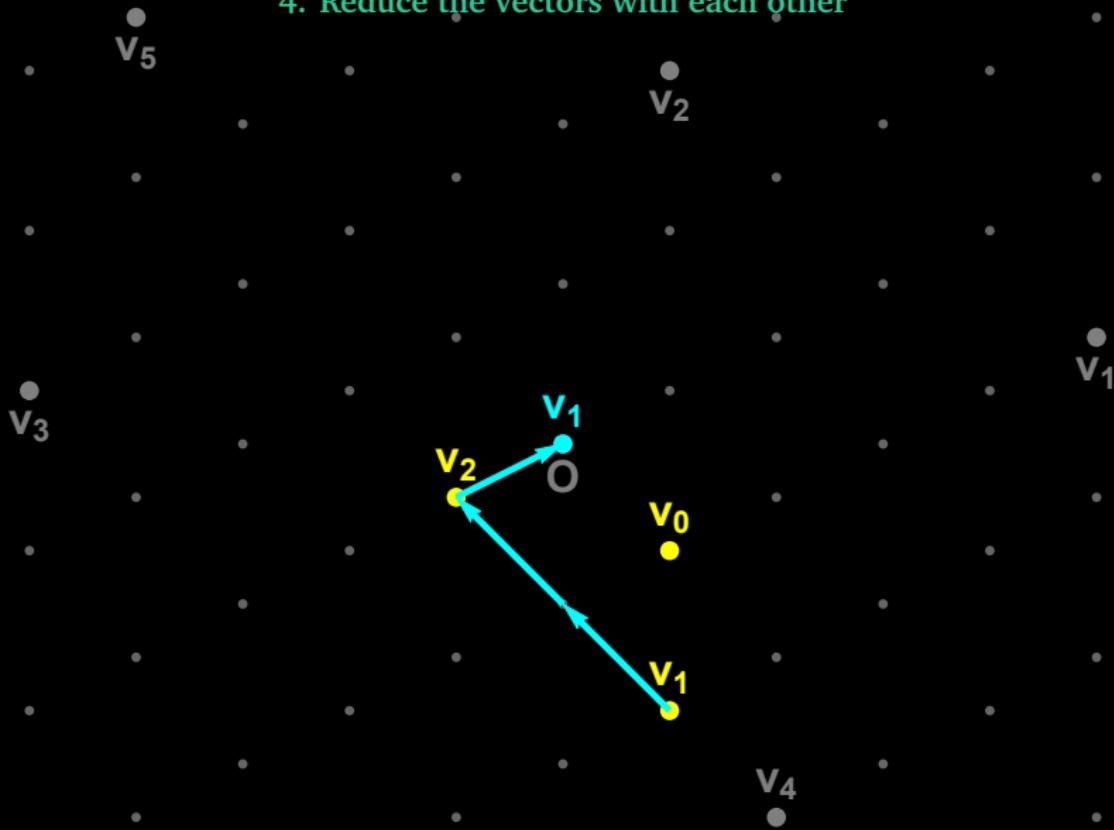
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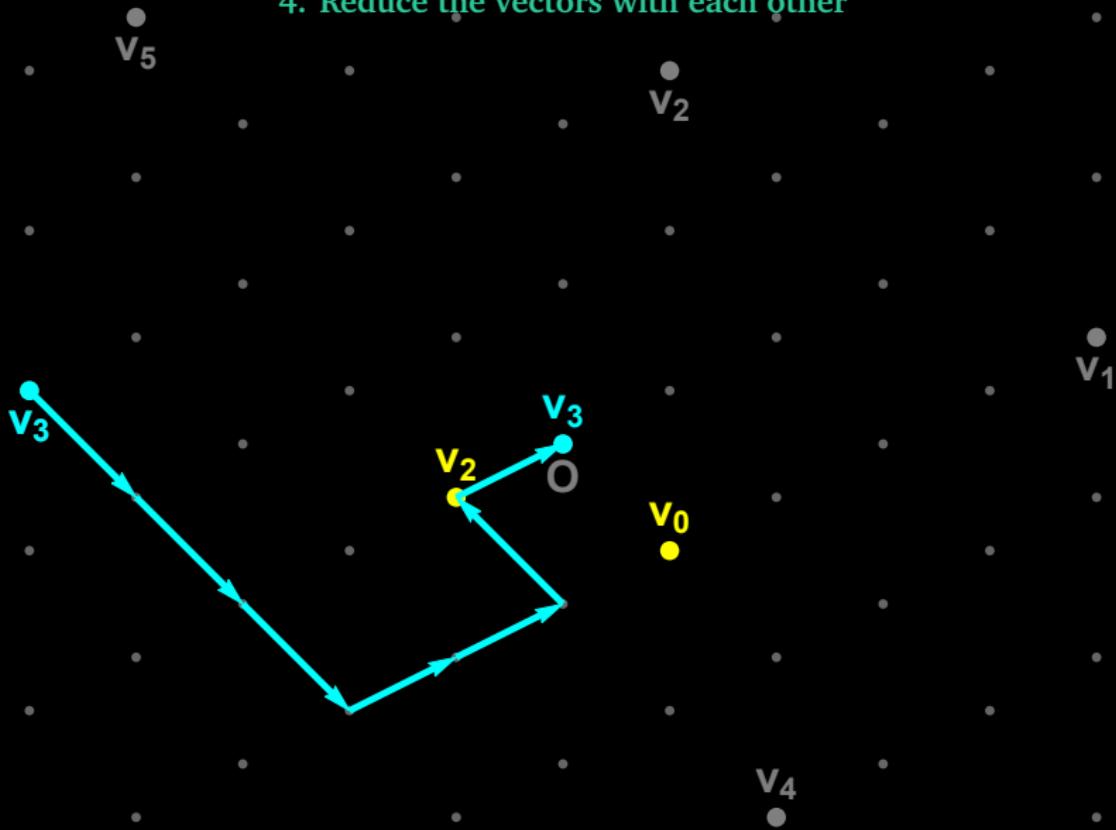
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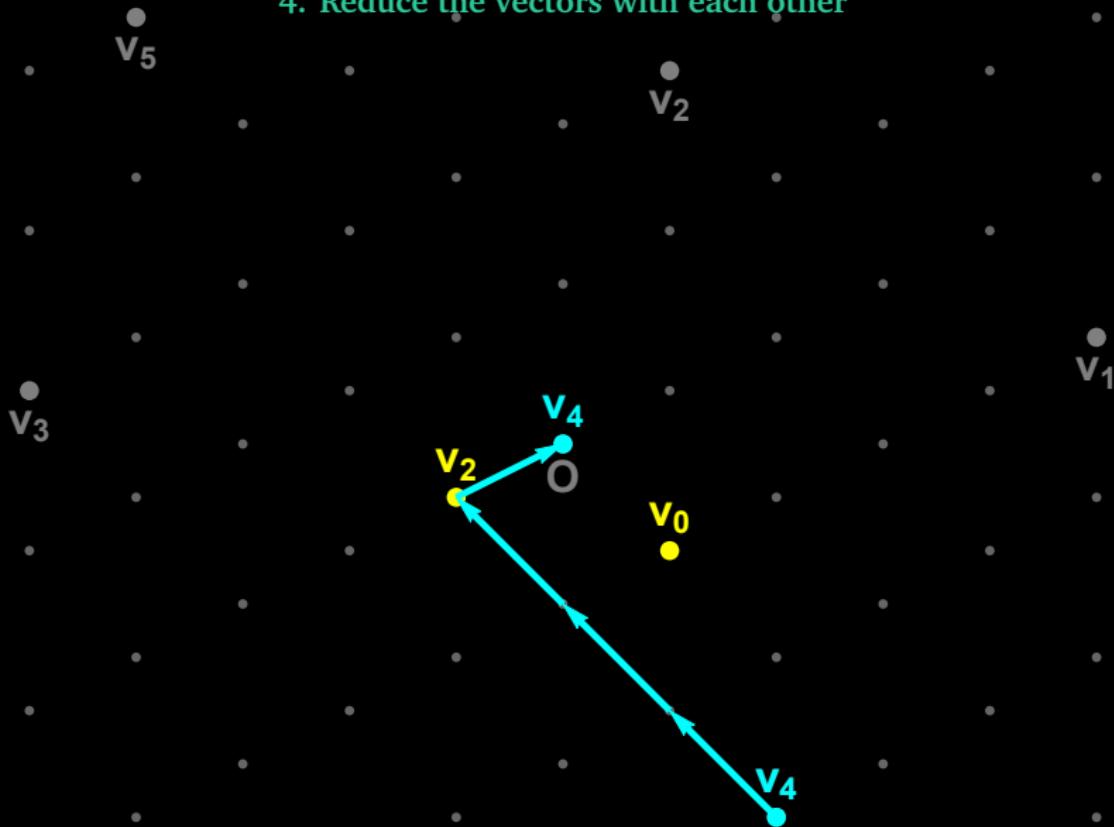
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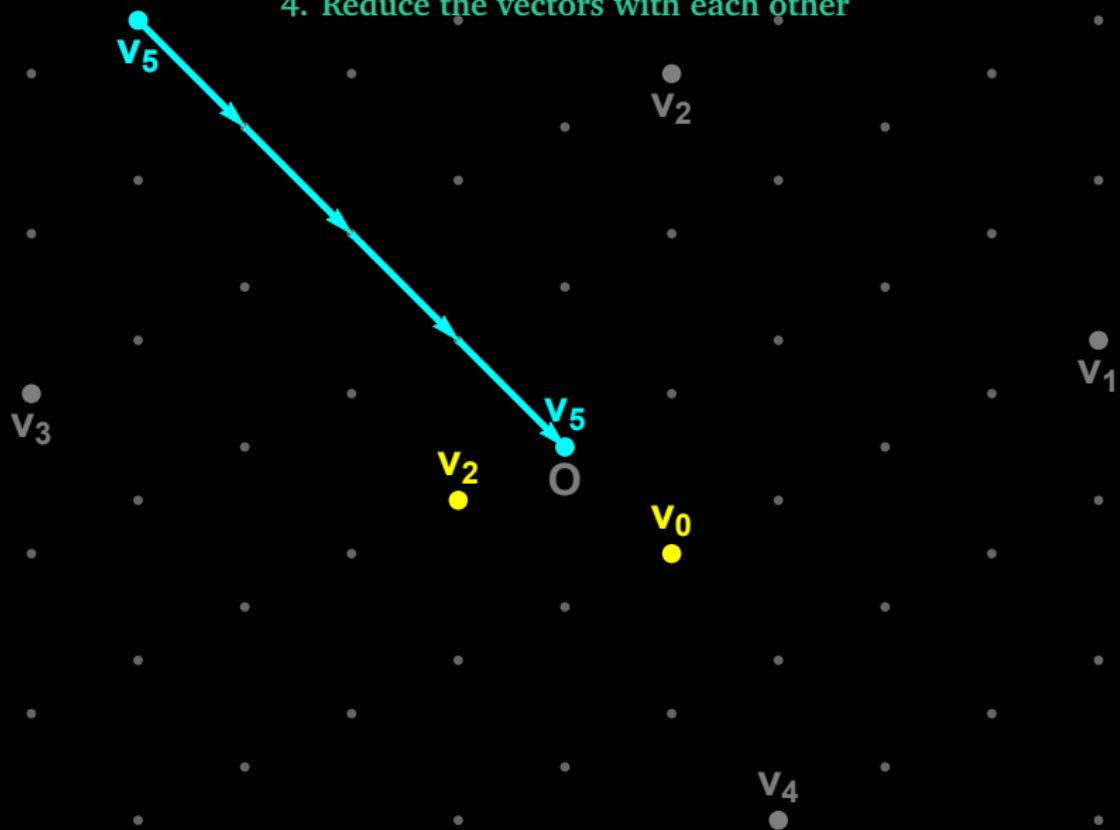
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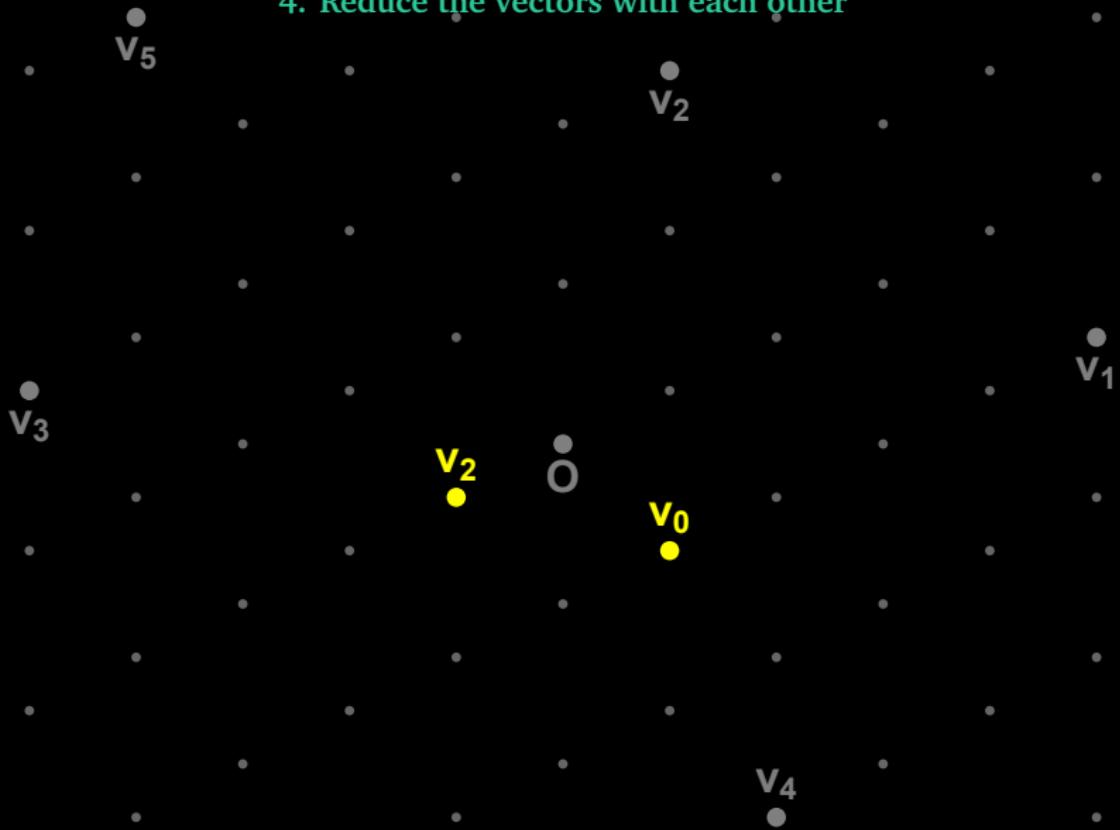
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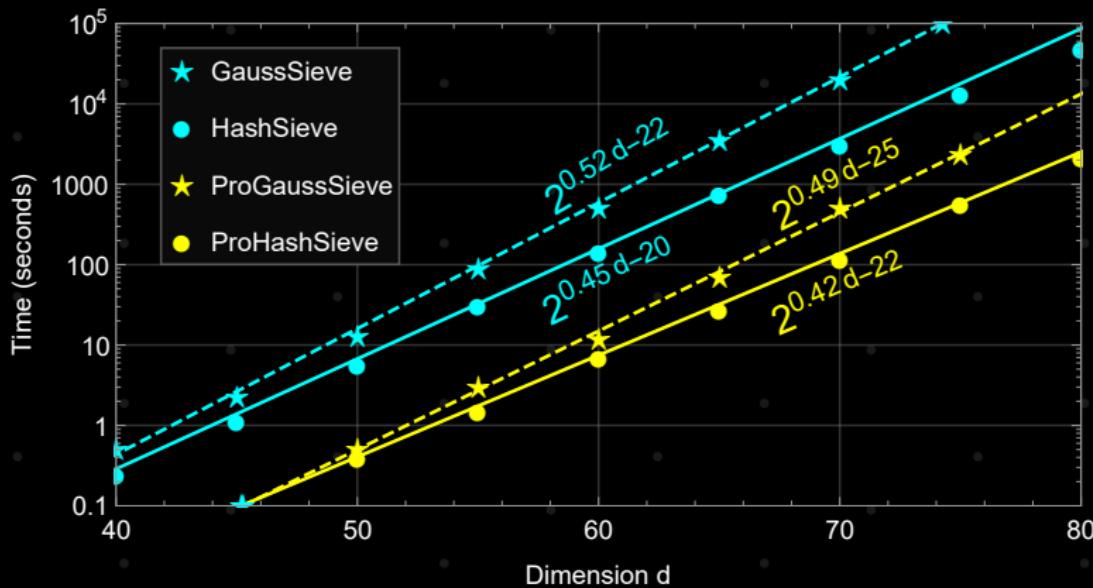
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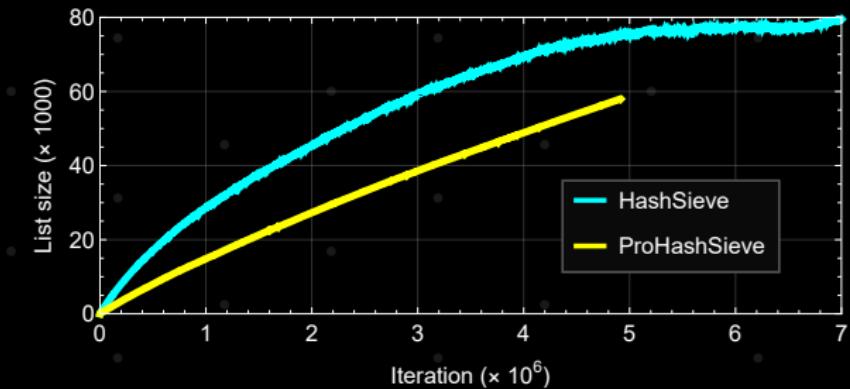
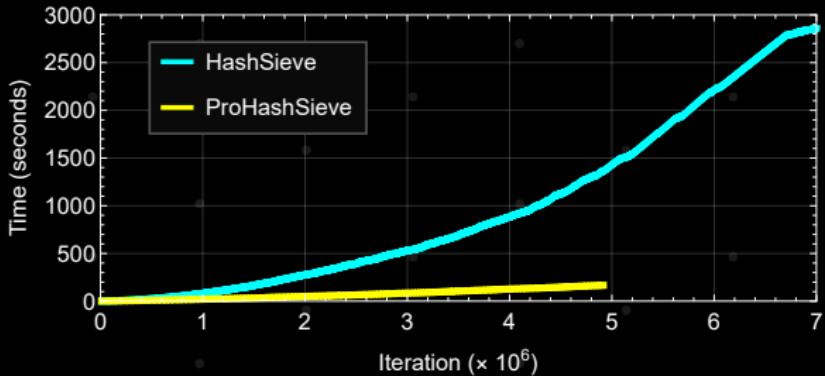
Progressive sieving

Time complexities

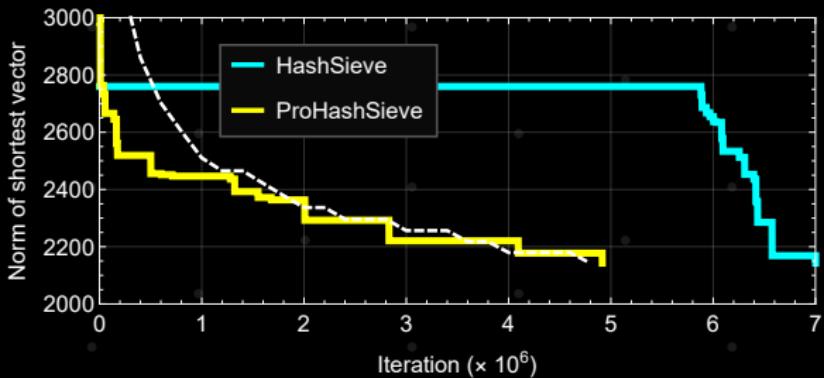
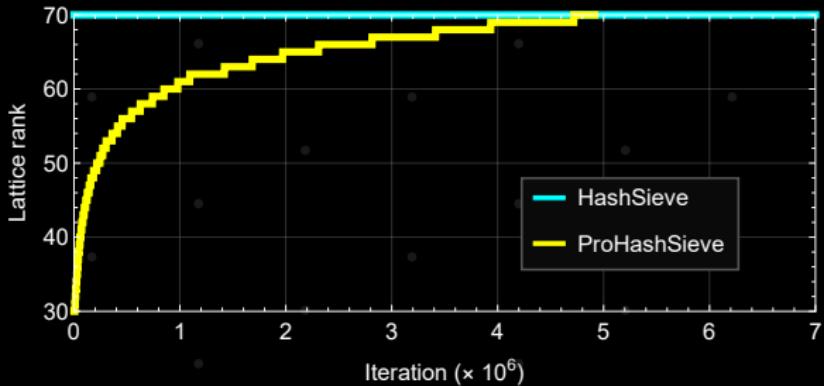


Progressive sieving

Execution profiles ($n = 70$)



Progressive sieving

Execution profiles ($n = 70$)

Progressive sieving

Effects of basis reduction ($n = 70$)

Exact SVP	← GaussSieve →			← HashSieve →		
	LLL	BKZ-10	BKZ-30	LLL	BKZ-10	BKZ-30
Standard sieving	19100	18100	16500	3300	3050	2900
Progressive sieving	595	440	390	165	125	115
Speedup factor	32×	41×	42×	20×	24×	25×

Approximate SVP $(\gamma = 1.1)$	← GaussSieve →			← HashSieve →		
	LLL	BKZ-10	BKZ-30	LLL	BKZ-10	BKZ-30
Standard sieving	18500	17200	15600	3180	2960	2700
Progressive sieving	120	40	3	65	20	2
Speedup factor	150×	400×	5000×	50×	150×	1000×

Conclusion

Progressive lattice sieving

- Uses recursive approach (rank reduction)
- Finds approximate solutions faster
- Benefits more from reduced bases
- Better predictability
- Faster, using slightly less memory
- No theoretical/asymptotic improvements...
 - ▶ Best classical time: $(3/2)^{n/2+o(n)} \approx 2^{0.292n+o(n)}$
 - ▶ Best quantum time: $(13/9)^{n/2+o(n)} \approx 2^{0.265n+o(n)}$

Questions?

