Analytic function:

Def^n A function f defined in a region D is said to be analytic at a point 'a' if f is differentiable at every point of some neighbourhood of 'a'.

Cauchy Riemann equations:

Statement:

Suppose $f(z) = u(z) + iv(z)$ is analytic in an openset O. Write $z = x + iy$. Then u_x, u_y, v_x and v_y exists at every point of O. Moreover, they satisfy the pair of equations

$$u_x = v_y \text{ and } u_y = -v_x.$$

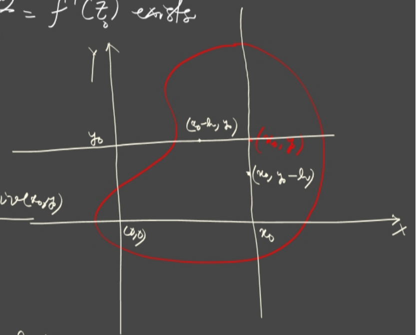
These equations are called pair of Cauchy-Riemann equations.

Proof: We shall prefer to write:

$$u(z) = u(x,y)$$
$$v(z) = v(x,y)$$

Take a point $z_0 \in O$. Since f is analytic at every point of O. So by definition of analyticity, the function is differentiable at $z = z_0$.

Therefore $\lim\limits_{z \to z_0} \dfrac{f(z) - f(z_0)}{z - z_0} = f'(z_0)$ exists



Suppose $z \to z_0$ along the line $y = y_0$

$$\lim_{(x,y_0) \to (x_0,y_0)} \frac{u(x,y_0) + iv(x,y_0) - u(x_0,y_0) - iv(x_0,y_0)}{(x - x_0) + i(y_0 - y_0)} = f'(z_0)$$

$$\Rightarrow \lim_{x \to x_0} \frac{u(x,y_0) - u(x_0,y_0)}{x - x_0} + i \lim_{x \to x_0} \frac{v(x,y_0) - v(x_0,y_0)}{x - x_0} = f'(z_0)$$

$$\Rightarrow u_x + iv_x \Big|_{(x_0,y_0)} = f'(z_0) \quad \text{---} \ (1)$$

$\therefore$ u_x and v_x exists at (x_0, y_0).

Similarly, along the line $x = x_0$

$$\lim_{(x_0,y) \to (x_0,y_0)} u(x_0,y) + iv(x_0,y)$$