Symmetric Rank-K Update
CS 378 Programming for Correctness and Performance

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1 Operation

Consider the operation

$$C := A^T A + \widehat{C}$$

where C is a $n \times n$ symmetric matrix, stored in the lower triangle, and A is a $n \times n$ matrix. This is a special case of matrix-matrix multiplication, which takes one input matrix A and its transpose A^T, multiplying them together and updating the value of matrix C with product. The result is a rank-k update of matrix C. We will refer to this operation as SYRK where SYRK stands for symmetric rank-k update. The SYmmetric indicates that we are using and updating a symmetric matrix C.

2 Precondition and Postcondition

In the precondition

$$C = \widehat{C}$$

$\widehat{C}$ denotes the original contents of C. This allows us to express the state upon completion, the postcondition, as

$$C = A^T A + \widehat{C}.$$

It is explicitly stated that C is a symmetrix matrix stored in the lower triangle.

3 Partitioned Matrix Expressions and Loop Invariants

There are two PMEs for this operation.

3.1 PME 1

To derive the first PME, partition

$$A \rightarrow \left(\begin{array}{c} A_T \\ \hline A_B \end{array}\right).$$

yielding,

$$A^T \rightarrow \left(A_T^T \,\middle|\, A_B^T \right)$$

,

Substituting these into the postcondition yields

$$C \ = \left(A_T^T \,\middle|\, A_B^T \right) \left(\frac{A_T}{A_B} \right) + \widehat{C}$$

or equivalently,

$$C = A_T^T A_T + A_B^T A_B + \widehat{C}$$

which we refer to as the first PME for this operations.

From this, we can choose two loop invariants:

Invariant 1:

$$C = A_T^T A_T + \widehat{C} \tag{1}$$

(The top part of A has been fully computed and the bottom part has been left alone).

Invariant 2:

$$C = A_B^T A_B + \widehat{C} \tag{2}$$

(The bottom part of A has been fully computed and the top part has been left alone).

3.2 PME 2

To derive the second PME, partition

$$A \rightarrow \left(A_L \,\middle|\, A_R \right)$$

, yielding

$$A^T \rightarrow \left(\frac{A_L^T}{A_R^T} \right)$$

, and partitioning

$$C \rightarrow \left(\begin{array}{c|c} C_{TL} & * \\ \hline C_{BL} & C_{BR} \end{array} \right)$$

, where we are ignoring the upper right quadrant as C is symmetric and stored in the lower triangle. Substituting these into the postcondition yields

$$\left(\begin{array}{c|c} C_{TL} & * \\ \hline C_{BL} & C_{BR} \end{array} \right) = \left(\begin{array}{c|c} A_L^T A_L + C_{TL} & * \\ \hline A_R^T A_L + C_{BL} & A_R^T A_R + C_{BR} \end{array} \right)$$

which we refer to as the second PME.

From this, we can choose two more loop invariants:

Invariant 3:

$$\left(\begin{array}{c|c} C_{TL} & * \\ \hline C_{BL} & C_{BR} \end{array}\right) = \left(\begin{array}{c|c} A_L^T A_L + \widehat{C}_{TL} & * \\ \hline \widehat{C}_{BL} & \widehat{C}_{BR} \end{array}\right). \tag{3}$$

(The top part has been partially finished and the bottom part has been left untouched).

Invariant 4:

$$\left(\begin{array}{c|c} C_{TL} & C_{TR} \\ \hline C_{BL} & C_{BR} \end{array}\right) = \left(\begin{array}{c|c} A_L^T A_L + \widehat{C}_{TL} & * \\ \hline A_R^T A_L + \widehat{C}_{BL} & \widehat{C}_{BR} \end{array}\right). \tag{4}$$

(The top part has been completely finished and the bottom part has been partially finished).

4 Deriving the Algorithms

4.1 Loop Invariant (1)

The unblocked algorithm for loop invariant (1) is showing in Figure 1. While the blocked algorithm for loop invariant (1) is showing in Figure 2.

4.2 Loop Invariant (3)

The unblocked algorithm for loop invariant (3) is showing in Figure 3. While the blocked algorithm for loop invariant (3) is showing in Figure 4.

Step	Algorithm: $[C] := \text{SYRK\_UNB\_VAR1}(A, C)$
1a	$\{C = \widehat{C} \}$
4	$A \rightarrow \left(\begin{array}{c} A_T \\ \hline A_B \end{array}\right)$, $A^T \rightarrow \left(\begin{array}{c\|c} A_T^T & A_B^T \end{array}\right)$ where A_T has 0 rows, A_T^T has 0 columns
2	$\{C = A_T^T A_T + \widehat{C} \}$
3	while $m(A_T) < m(A)$ do
2,3	$\{ \quad C = A_T^T A_T + \widehat{C} \wedge m(A_T) < m(A) \}$
5a	$\left(\begin{array}{c} A_T \\ \hline A_B \end{array}\right) \rightarrow \left(\begin{array}{c} A_0 \\ \hline a_1^T \\ A_2 \end{array}\right)$ where a_1 has 1 row
6	$\{ \quad C = A_0^T A_0 + \widehat{C} \}$
8	$C := a_1 a_1^T + \widehat{C}$
7	$\{ \quad C = A_0^T A_0 + a_1 a_1^T + \widehat{C} \}$
5b	$\left(\begin{array}{c} A_T \\ \hline A_B \end{array}\right) \leftarrow \left(\begin{array}{c} A_0 \\ a_1^T \\ \hline A_2 \end{array}\right)$
2	$\{ \quad C = A_T^T A_T + \widehat{C} \}$
	endwhile
2,3	$\{C = A_T^T A_T + \widehat{C} \wedge \neg(m(A_T) < m(A)) \}$
1b	$\{[C] = \text{syrk}(A, \widehat{C}) \}$

Figure 1: Unblocked Algorithm for Loop Invariant (1)

Step	Algorithm: $[C] := \text{SYRK\_BLK\_VAR1}(A, C)$
1a	$\{C = \widehat{C}$ $\}$
4	$A \rightarrow \left(\begin{array}{c} A_T \\ \hline A_B \end{array}\right)$, $A^T \rightarrow \left(\begin{array}{c\|c} A_T^T & A_B^T \end{array}\right)$ where A_T has 0 rows, A_T^T has 0 columns
2	$\{C = A_T^T A_T + \widehat{C}$ $\}$
3	while $m(A_T) < m(A)$ do
2,3	$\{$ $C = A_T^T A_T + \widehat{C} \wedge m(A_T) < m(A)$ $\}$
5a	**Determine block size** b $\left(\begin{array}{c} A_T \\ \hline A_B \end{array}\right) \rightarrow \left(\begin{array}{c} A_0 \\ \hline A_1 \\ A_2 \end{array}\right)$ where A_1 has b rows
6	$\{$ $C = A_0^T A_0 + \widehat{C}$ $\}$
8	$C := A_1^T A_1 + \widehat{C}$
7	$\{$ $C = A_0^T A_0 + A_1^T A_1 + \widehat{C}$ $\}$
5b	$\left(\begin{array}{c} A_T \\ \hline A_B \end{array}\right) \leftarrow \left(\begin{array}{c} A_0 \\ A_1 \\ \hline A_2 \end{array}\right)$
2	$\{$ $C = A_T^T A_T + \widehat{C}$ $\}$
	endwhile
2,3	$\{C = A_T^T A_T + \widehat{C} \wedge \neg(m(A_T) < m(A))$ $\}$
1b	$\{[C] = \text{syrk}(A, \widehat{C})$ $\}$

Figure 2: Blocked Algorithm for Loop Invariant (1)

Step	Algorithm: $[C] := \textsc{syrk\_unb\_var}3(A, C)$
1a	$\{ C = \widehat{C} \}$
4	$A \rightarrow \left(A_L \,\middle|\, A_R \right)$, $A^T \rightarrow \left(\begin{array}{c} A_L^T \\ \hline A_R^T \end{array} \right)$, $C \rightarrow \left(\begin{array}{c\|c} C_{TL} & * \\ \hline C_{BL} & C_{BR} \end{array} \right)$ where A is $n \times n$, A^T is $n \times n$, C is $n \times n$
2	$\left\{ \left(\begin{array}{c\|c} C_{TL} & C_{TR} \\ \hline C_{BL} & C_{BR} \end{array} \right) = \left(\begin{array}{c\|c} A_L^T A_L + \widehat{C}_{TL} & * \\ \hline \widehat{C}_{BL} & \widehat{C}_{BR} \end{array} \right) \right\}$
3	while $n(A_L) < n(A)$ do
2,3	$\left\{ \left(\begin{array}{c\|c} C_{TL} & C_{TR} \\ \hline C_{BL} & C_{BR} \end{array} \right) = \left(\begin{array}{c\|c} A_L^T A_L + \widehat{C}_{TL} & * \\ \hline \widehat{C}_{BL} & \widehat{C}_{BR} \end{array} \right) \wedge n(A_L) < n(A) \right\}$
5a	$A \rightarrow \left(A_0 \,\middle|\, \alpha_1 \;\; A_2 \right)$, $A^T \rightarrow \left(\begin{array}{c} A_0^T \\ \hline \alpha_1^T \\ A_2^T \end{array} \right)$, $\left(\begin{array}{c\|c} C_{TL} & * \\ \hline C_{BL} & C_{BR} \end{array} \right) \rightarrow \left(\begin{array}{c\|cc} C_{00} & * & * \\ \hline c_{10}^T & \gamma_{11} & c_{12}^T \\ C_{20} & c_{21} & C_{22} \end{array} \right)$ where a_1 has 1 column, γ_{11} is 1×1
6	$\left\{ \left(\begin{array}{ccc} C_{00} & * & * \\ c_{10}^T & \gamma_{11} & c_{12}^T \\ C_{20} & c_{21} & C_{22} \end{array} \right) = \left(\begin{array}{ccc} A_0^T A_0 + C_{00} & * & * \\ c_{10}^T & \gamma_{11} & c_{12}^T \\ C_{20} & c_{21} & C_{22} \end{array} \right) \right\}$
8	$c_{10}^T := \alpha_1^T A_0 + \widehat{c_{10}^T}$ $\gamma_{11} := \alpha_1^T \alpha_1 + \widehat{\gamma_{11}}$ $c_{01} := c_{10}^{T\,T}$
7	$\left\{ \left(\begin{array}{ccc} C_{00} & c_{01} & * \\ c_{10}^T & \gamma_{11} & * \\ C_{20} & c_{21} & C_{22} \end{array} \right) = \left(\begin{array}{ccc} A_0^T A_0 + C_{00} & A_0^T \alpha_1 + c_{01} & * \\ \alpha_1^T A_0 + c_{10}^T & \alpha_1^T \alpha_1 + \gamma_{11} & * \\ C_{20} & c_{21} & C_{22} \end{array} \right) \right\}$
5b	$A \leftarrow \left(A_0 \;\; \alpha_1 \,\middle|\, A_2 \right)$, $A^T \leftarrow \left(\begin{array}{c} A_0^T \\ \alpha_1^T \\ \hline A_2^T \end{array} \right)$, $\left(\begin{array}{c\|c} C_{TL} & * \\ \hline C_{BL} & C_{BR} \end{array} \right) \leftarrow \left(\begin{array}{cc\|c} C_{00} & c_{01} & * \\ c_{10}^T & \gamma_{11} & * \\ \hline C_{20} & c_{21} & C_{22} \end{array} \right)$
2	$\left\{ \left(\begin{array}{c\|c} C_{TL} & C_{TR} \\ \hline C_{BL} & C_{BR} \end{array} \right) = \left(\begin{array}{c\|c} A_L^T A_L + \widehat{C}_{TL} & * \\ \hline \widehat{C}_{BL} & \widehat{C}_{BR} \end{array} \right) \right\}$
	endwhile
2,3	$\left\{ \left(\begin{array}{c\|c} C_{TL} & C_{TR} \\ \hline C_{BL} & C_{BR} \end{array} \right) = \left(\begin{array}{c\|c} A_L^T A_L + \widehat{C}_{TL} & * \\ \hline \widehat{C}_{BL} & \widehat{C}_{BR} \end{array} \right) \wedge \neg (n(A_L) < n(A)) \right\}$
1b	$\{ C = A^T A + \widehat{C} \}$

Figure 3: Unblocked Algorithm for Loop Invariant (3)

Step	Algorithm: $[C] := \textsc{syrk\_blk\_var3}(A, C)$
1a	$\{ C = \widehat{C} \}$
4	$A \rightarrow \left(A_L \,\middle|\, A_R \right)$, $A^T \rightarrow \left(\begin{array}{c} A_L^T \\ \hline A_R^T \end{array} \right)$, $C \rightarrow \left(\begin{array}{c\|c} C_{TL} & * \\ \hline C_{BL} & C_{BR} \end{array} \right)$ where A is $n \times n$, A^T is $n \times n$, C is $n \times n$
2	$\left\{ \left(\begin{array}{c\|c} C_{TL} & C_{TR} \\ \hline C_{BL} & C_{BR} \end{array} \right) = \left(\begin{array}{c\|c} A_L^T A_L + \widehat{C}_{TL} & * \\ \hline \widehat{C}_{BL} & \widehat{C}_{BR} \end{array} \right) \right\}$
3	**while** $n(A_L) < n(A)$ **do**
2,3	$\left\{ \left(\begin{array}{c\|c} C_{TL} & C_{TR} \\ \hline C_{BL} & C_{BR} \end{array} \right) = \left(\begin{array}{c\|c} A_L^T A_L + \widehat{C}_{TL} & * \\ \hline \widehat{C}_{BL} & \widehat{C}_{BR} \end{array} \right) \wedge n(A_L) < n(A) \right\}$
5a	**Determine block size b** $\left(A_L \,\middle|\, A_R \right) \rightarrow \left(A_0 \,\middle|\, A_1 \;\; A_2 \right)$, $\left(\begin{array}{c\|c} C_{TL} & * \\ \hline C_{BL} & C_{BR} \end{array} \right) \rightarrow \left(\begin{array}{c\|cc} C_{00} & * & * \\ \hline C_{10} & C_{11} & C_{12} \\ C_{20} & C_{21} & C_{22} \end{array} \right)$ where A_1 has b columns, C_{11} is $b \times b$
6	$\left\{ \left(\begin{array}{ccc} C_{00} & * & * \\ C_{10} & C_{11} & C_{12} \\ C_{20} & C_{21} & C_{22} \end{array} \right) = \left(\begin{array}{ccc} A_0^T A_0 + C_{00} & * & * \\ C_{10} & C_{11} & C_{12} \\ C_{20} & C_{21} & C_{22} \end{array} \right) \right\}$
8	$C_{10} := A_1^T A_0 + \widehat{C}_{10}$ $C_{11} := A_1^T A_1 + \widehat{C}_{11}$ $C_{01} := C_{10}^T$
7	$\left\{ \left(\begin{array}{ccc} C_{00} & C_{01} & * \\ C_{10} & C_{11} & * \\ C_{20} & C_{21} & C_{22} \end{array} \right) = \left(\begin{array}{ccc} A_0^T A_0 + C_{00} & A_0^T A_1 + C_{01} & * \\ A_1^T A_0 + C_{10} & A_1^T A_1 + C_{11} & * \\ C_{20} & C_{21} & C_{22} \end{array} \right) \right\}$
5b	$A \rightarrow \left(A_L \,\middle|\, A_R \right) \leftarrow \left(A_0 \;\; A_1 \,\middle|\, A_2 \right)$, $\left(\begin{array}{c\|c} C_{TL} & * \\ \hline C_{BL} & C_{BR} \end{array} \right) \leftarrow \left(\begin{array}{cc\|c} C_{00} & C_{01} & * \\ C_{10} & C_{11} & * \\ \hline C_{20} & C_{21} & C_{22} \end{array} \right)$
2	$\left\{ \left(\begin{array}{c\|c} C_{TL} & C_{TR} \\ \hline C_{BL} & C_{BR} \end{array} \right) = \left(\begin{array}{c\|c} A_L^T A_L + \widehat{C}_{TL} & * \\ \hline \widehat{C}_{BL} & \widehat{C}_{BR} \end{array} \right) \right\}$
	endwhile
2,3	$\left\{ \left(\begin{array}{c\|c} C_{TL} & C_{TR} \\ \hline C_{BL} & C_{BR} \end{array} \right) = \left(\begin{array}{c\|c} A_L^T A_L + \widehat{C}_{TL} & * \\ \hline \widehat{C}_{BL} & \widehat{C}_{BR} \end{array} \right) \wedge \neg (n(A_L) < n(A)) \right\}$
1b	$\{ C = A^T A + \widehat{C} \}$

Figure 4: Blocked Algorithm for Loop Invariant (3)