FLORIDA STATE UNIVERSITY

Assignment 1: Relations and Their Properties

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1 EXERCISE

For the relation: $R = \{(1,3),(1,4),(2,3),(2,4),(3,1),(3,4)\}$ on the set $A = \{1,2,3,4\}$, explain/show whether or not the relation is the following:
(a) reflexive,
(b) symmetric
(c) antisymmetric
(d) transitive

1.1 SOLUTION

(a) reflexive:
Assume that the relation above is reflexive, then $\forall a \in A \implies (a,a) \in R$ however $1 \in A$ and $(1,1) \notin R$ ■
(b) symmetric:
If R was symmetric then, $\forall (a,b) \in R \implies (b,a) \in R$, however $(1,4) \in R \land (4,1) \notin R$ ■
(c) antisymmetric:
If R was antisymmetric, then $\forall x \forall y[((x,y) \in R \land (y,x)) \in R \implies (x = y)]$, however $(1,3) \in R \land (3,1) \in R \land (1 \neq 3)$ ■
(d) transitive:
If R was transitive, then $\forall x \forall y \forall z[(x,y) \in R \land (y,z) \in R \implies (x,z) \in R]$, however $(1,3) \in R \land (3,1) \in R \land (1,1) \notin R$ ■

2 Exercise

Let the sets be relations on the real numbers: $R_1 = \{(a,b) \in \mathbb{R}^2 | a \geq b\}$, the "greater than or equal to" relation, and let $R_2 = \{(a,b) \in \mathbb{R}^2 | a \neq b\}$, the "unequal to" relation. Find:
(a) $R_1 \cap R_2$
(b) $R_1 - R_2$
(c) $R_1 \oplus R_2$

2.1 Solution

(a) $R_1 \cap R_2$**:**
Let $R = R_1 \cap R_2$, then by definition

$$R = \{(a,b) \in \mathbb{R}^2 | (a \geq b) \wedge (a \neq b)\} \implies R = \{(a,b) \in \mathbb{R}^2 | (a > b)\} \blacksquare$$

(b) $R_1 - R_2$**:**
Let $R = R_1 - R_2$, then by definition:

$$R = R_1 \cap \overline{R_2} = \{(a,b) \in \mathbb{R}^2 | (a \geq b) \wedge \neg(a \neq b)\} = \{(a,b) \in \mathbb{R}^2 | (a \geq b) \wedge (a = b)\} = \{(a,b) \in \mathbb{R}^2 | (a = b)\} \blacksquare$$

(c) $R_1 \oplus R_2$**:**
Let $R = R_1 \oplus R_2$, then:

$$R = (R_1 - R_2) \cup (R_2 - R_1)$$

where,

$$(R_2 - R_1) = R_2 \cap \overline{R_1} = \{(a,b) \in \mathbb{R}^2 | (a \neq b) \wedge \neg(a \geq b)\} = \{(a,b) \in \mathbb{R}^2 | (a \neq b) \wedge (a < b)\} = \{(a,b) \in \mathbb{R}^2 | (a < b)\}$$

and

$$R_1 - R_2 = \{(a,b) \in \mathbb{R}^2 | (a = b)\}$$

from the previous example. Therefore:

$$R = \{(a,b) \in \mathbb{R}^2 | (a = b) \vee (a < b)\} = \{(a,b) \in \mathbb{R}^2 | (a \leq b)\} \blacksquare$$

3 Exercise

(a) How many **binary** relations are there on the set $\{a,b,c\}$?
(b) If $R = \{(1,1),(1,2),(2,4),(3,1),(3,0)\}$, $S = \{(1,2),(2,0),(3,1),(0,0),(4,3)\}$ find $R \circ S$

3.1 Solution

(a) How many binary relations are there on the set $\{a,b,c\}$**?**
By definition a binary operation is a subset of the cartesian product of A, therefore we have 2^n where $n = |AXA| = |A|x|A| = 3x3 = 9$ therefore $2^9 \blacksquare$
(b) If $R = \{(1,1),(1,2),(2,4),(3,1),(3,0)\}$**,** $S = \{(1,2),(2,0),(3,1),(0,0),(4,3)\}$ **find** $R \circ S$
By definition,

$$R \circ S = \{(a,c) | (a,b) \in R \wedge (b,c) \in S (\exists b)\}$$

or

$$R \circ S = \{(1,2),(1,0),(2,3),(3,2),(3,0)\}$$

4 Exercise

R is the relation represented by the matrix $M_R = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, find the matrix for:

(a) R^{-1}
(b) $\overline{R}$
(c) $R \circ R$

4.1 Solution

(a) $R^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

(b) $\overline{R} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

(c) $R \circ R = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

5 Exercise

(a) The relation R is on $\{1,2,3\}$. Represent the relation

$$R = \{(1,1),(2,1),(2,2),(2,3),(3,2)\}$$

with a matrix.
(b) By looking at the matrix, is the relation R reflexive? Why or why not?
(c) Draw the directed graph that represents the relation R.

5.1 Solution

(a) Represent the above relation as a matrix.

$$M_R = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

(b) Does this matrix represent a reflexive relation?
No, $\exists i | M_{i,i} \neq 1$ aka $M_{3,3} = 0$.

(c) Draw the directed graph that represents the relation R.