FLORIDA STATE UNIVERSITY

# Assignment 2: Closure of Relations

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## 1 EXERCISE

If possible, give an example of a nonempty relation R on the set $A = \{0, 1\}$ that satisfies the following. Give the Matrix representation.
(a) $R$ is reflexive and antisymmetric. Explain why.
(b) $R$ is irreflexive, but $R^2$ is not irreflexive. Explain why.
(c) R is asymmetric and reflexive. Explain why.

### 1.1 SOLUTION

**(a) reflexive and antisymmetric:**
Let $R = \{(0, 0), (1, 1)\}$. $R$ is reflexive from the fact that the identiity relation for both elements in A is in R. The relation is antisymmetric due to the fact that every element in A is only related to itself.

$$M_R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

**(b) irreflexive and $R^2$ not irreflexive:**
Let $R = \{(0, 1), (1, 0)\}$. Then $R$ is irreflexive because it doesn't conatin the identity relation for 0 or 1. Also $R^2 = \{(0, 0), (1, 1)\}$

$$M_R = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$M_{R^2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

**(c) asymmetric and reflexive:**
Assume $R$ is reflexive and asymmetric. From the fact that $R$ is asymmetric then for all $x, y \in A$ $x \sim y \implies y \nsim x$ however if we set $x$ and $y$ to any $a \in A$ we see that we have a contradiction from the fact that $R$ is suppose to be reflexive. This contradiction implies that R cannot be both reflexive and asymmetric.

# 2 Exercise

Find the transitive closure of the given relation, $R$, on $\{a, b, c, d, e\}$, using either algorithm presented in Section 9.4 of the Rosen textbook and videos (i.e. using a matrix).

$$R = \{(e, d), (d, b), (c, a), (b, d), (a, c)\}$$

## 2.1 Solution

First we need to convert $R$ into $M_R$:

$$M_R = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Then we run through Warshall's Algorithm as follows:

```java
/**
Warshall's Algorithm in Java
Sorry, I couldn't figure out how to write formatted text so I wrote code
    instead :)
*/
public static boolean[] findTransitiveClosure(boolean[] matrix) {
   boolean[] W = matrix;
   int N = matrix.length;
   for (int k = 1; k <= N; k++) {
      for (int i = 1; i <= N; i++) {
         for (int j = 1; j <= N; j++) {
            W[i][j] = W[i][j] || (W[i][k] && W[k][i]);
         }
      }
   }
   return W;
}
```

$$M_{t(R)} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

# 3 Exercise

Let $R$ be the relation $\{(a,b)|a \geq b\}$ on the set of integers.
Find the symmetric closure of $R$, write your answer in set notation.
Explain how you determined this.

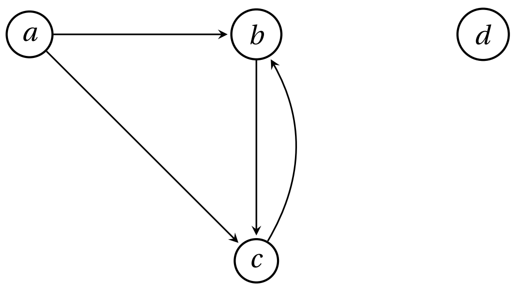
## 3.1 Solution

The symmetric closure is give by the formula $S = R \cup R^{-1}$ therefore:

$$S = R \cup R^{-1} = \{(a,b)|a \geq b\} \cup \{(b,a)|a \geq b\} = \mathbb{Z}$$

# 4 Exercise

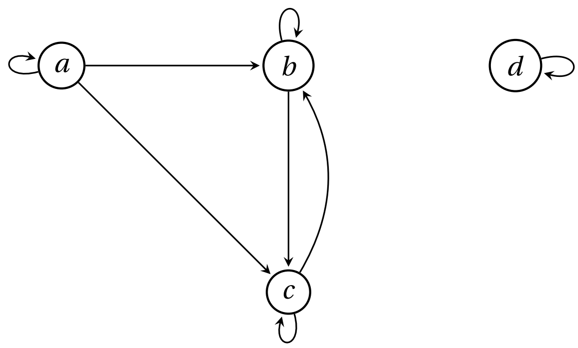
Given the directed graph below,



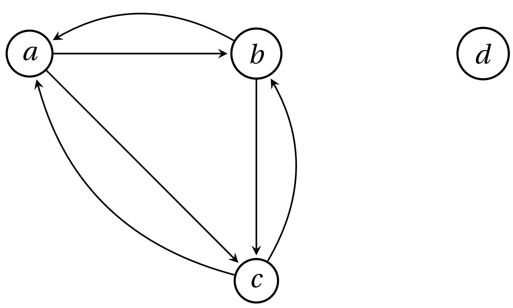

draw the directed graph of
(a) The reflexive closure of the relation.
(b) The symmetric closure of the relation.
(c) The transitive closure of the relation.

## 4.1 Solution

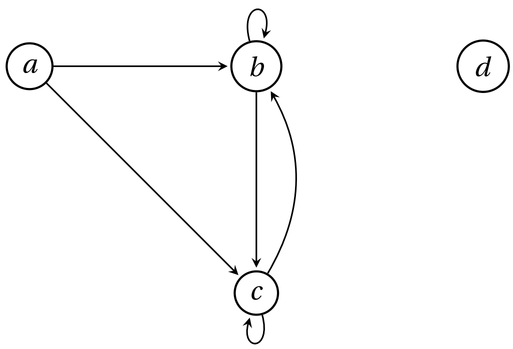
**(a) reflexive closure:**




**(b) symmetric closure:**




**(c) transitive closure:**




## 5 Exercise

Let R be represented by the matrix

$$M_R = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

(a) Give the matrix that represents the reflexive closure of R, r(R):
(b) Give the matrix that represents the symmetric closure of R, s(R):

### 5.1 Solution

**(a) reflexive closure:**

$$M_{r(R)} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

**(b) symmetric closure:**

$$M_{s(R)} = M_R \cup M_{R^{-1}} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \cup \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

## 6 Exercise

Prove directly that if the relation $R$ is symmetric, then its transitive closure, $t(R) = R^*$, is also symmetric.

### 6.1 Solution

Let $R$ be a symmetic relation on the set $S$, let $t(R)$ be the transitive closure of $R$, and let $(a,b) \in t(R)$. By the definition of a transitive closure $\exists n \in \mathbb{N}$ such that $(a,b) \in R^n$. Therefor there must have been some chain in R that produced this, thus there was some $x_0, x_1, ..., x_n$ such that:

$$x_0 = a$$

$$x_n = b$$

For $k = 0, ..., n-1 | (x_k, x_{k+1}) \in R$
Now we can essentially reverse each relation in this chain from the fact that R is symmetric, or:
For $k = 0, ..., n$ let $y_k = x_{n-k}$
Then:

$$y_0 = x_n = b$$

$$y_n = x_0 = a$$

For $k = 0, ..., n-1 | (x_{n-k-1}, x_{n-k}) \in R$ thus $(y_{k+1}, y_k) \in R$
Since R is symmetric, for $k = 0, ..., n-1 | (y_k, y_{k+1}) \in R$ thus $(b,a) \in R^n$ and $(b,a) \in t(R)$ therefore the transitive closure is also symmetric.