Combinatorics

Timothy Mou

June 11, 2023

Contents

These are some notes about miscellaneous math and combinatorics topics for competitive programming.

§1 Binomial Coefficients

Binomial coefficients can be computed in $O(1)$ with $O(N)$ precomputation (need to use trick to compute inverse factorials quickly). If k is small, we can also compute $\binom{n}{k}$ in $O(k)$.

- $$\sum_{i=0}^{N} \binom{N}{i} = 2^N.$$
- $$\sum_{i=0}^{N} \binom{N}{i} i = N \cdot 2^{N-1}$$

  This can be seen combinatorially; this is the number of subsets where we pick a "leader" for each subset.
- $$\sum_{i=0}^{N} \binom{N}{i}(i+1) = \sum_{i=0}^{N} \binom{N}{i} i + \sum_{i=0}^{N} \binom{N}{i} = N \cdot 2^{N-1} + 2 \cdot 2^{N-1} = (N+2)2^{N-1}.$$

  Follows from the previous identity.

§2 Stirling Numbers

Stirling numbers of the second kind, denoted $S(n, k)$ are the number of ways to partition $[n]$ into k indistinguishable nonempty blocks. They obey the recurrence

$$S(n, k) = k \cdot S(n - 1, k) + S(n - 1, k - 1),$$

with the base cases $S(n, n) = 1$, $S(n, 0) = S(0, n) = 0$ for $n > 0$. They can also be calculated explicitly in $O(k \log n)$ with the formula

$$S(n, k) = \frac{1}{k!} \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} i^n.$$