Local thermodynamical equilibrium, β frame and transport coefficients

F. Becattini and E. Grossi
Universita' di Firenze and INFN
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L. Bucciantini, L. Tinti, F. Matera

Outline

- Local equilibrium in quantum relativistic theory
- The β frame
- Revising the method of generating "Kubo"formulae

Our approach features

- Full quantum-relativistic treatment

$$T^{\mu\nu}(x) \equiv \mathrm{tr}\left(\hat{\rho}\,\hat{T}^{\mu\nu}(x)\right)_{\mathrm{ren.}}$$

- Covariance maintained throughout

- Kinetic-theory free

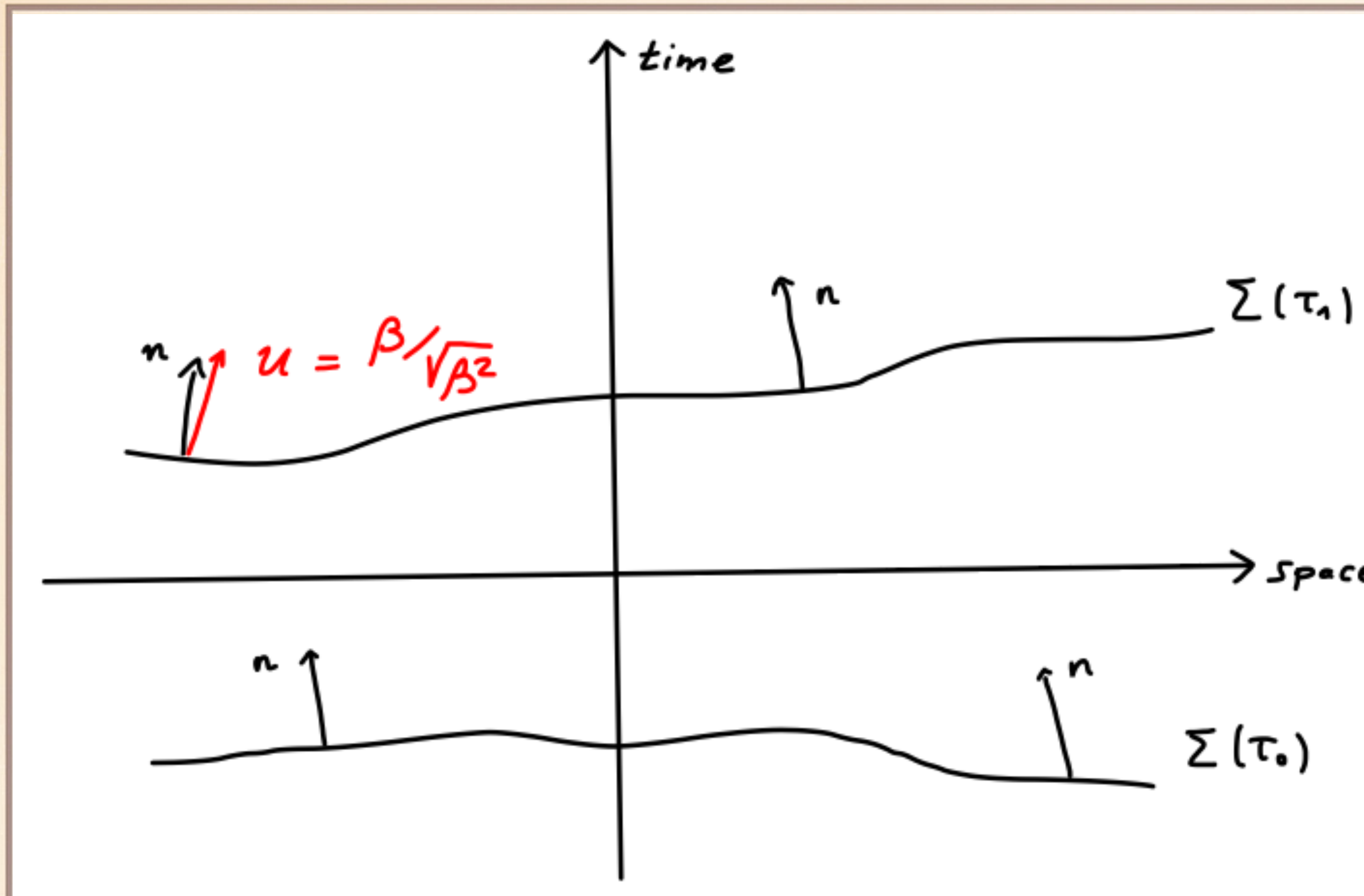
Local thermodynamical equilibrium

Maximization of entropy with fixed *densities* of conserved currents

$$S = -\mathrm{tr}(\widehat{\rho}_{LE} \log \widehat{\rho}_{LE}) \qquad \text{max. w.r.t. } \widehat{\rho}_{LE}$$

$$n_\mu \mathrm{tr}(\widehat{\rho}_{\mathrm{LE}}\, \widehat{T}^{\mu\nu}(x))_{\mathrm{ren}} = n_\mu \langle \widehat{T}^{\mu\nu}(x)\rangle_{\mathrm{LE}} \equiv n_\mu T^{\mu\nu}_{\mathrm{LE}}(x) = n_\mu T^{\mu\nu}(x)$$
$$n_\mu \mathrm{tr}(\widehat{\rho}_{\mathrm{LE}} \widehat{j}^\mu(x))_{\mathrm{ren}} = n_\mu \langle \widehat{j}^\mu(x)\rangle_{\mathrm{LE}} \equiv n_\mu j^\mu_{\mathrm{LE}}(x) = n_\mu j^\mu(x)$$

As entropy is a non-conserved global quantity in LTE, it requires the specification of one frame, namely a 3d spacelike hypersurface with its normal *n* at any time (a spacetime foliation)



The vector field *n(x)* must be vorticity-free:

$$\epsilon_{\mu\nu\rho\sigma} n^\nu (\partial^\rho n^\sigma - \partial^\sigma n^\rho) = 0$$

Maximize:

$$-\mathrm{tr}(\widehat{\rho} \log \widehat{\rho}) + \int_{\Sigma(\tau)} \mathrm{d}\Sigma \, n_\mu \left[\left(\langle \widehat{T}^{\mu\nu}(x) \rangle - T^{\mu\nu}(x) \right) \beta_\nu(x) - \left(\langle \widehat{j}^\mu(x) \rangle - j^\mu(x) \right) \xi(x) \right]$$

The 5 Lagrange multiplier functions β(x) and ξ(x) become the primordial thermodynamic fields

$$\beta^\mu(x) = \frac{1}{T} u^\mu(x)$$

$$\xi(x) = \frac{\mu(x)}{T(x)}$$

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The solution

$$\widehat{\rho}_{\rm LE} = \frac{1}{Z_{\rm LE}} \exp\left[- \int_{\Sigma(\tau)} {\rm d}\Sigma \; n_\mu \left(\widehat{T}^{\mu\nu}(x)\beta_\nu(x) - \xi(x)\widehat{j}^\mu(x) \right) \right]$$

$$n_\mu T^{\mu\nu}_{\rm LE}[\beta,\xi,n] = n_\mu T^{\mu\nu} \qquad n_\mu j^\mu_{\rm LE}[\beta,\xi,n] = n_\mu j^\mu,$$

This operator, as expected, depends on the hypersurface Σ
It is independent if

$$\partial_\mu \beta_\nu + \partial_\nu \beta_\mu = 0 \qquad \partial_\mu \xi = 0.$$

β Killing vector field

Global equilibrium

$$\beta_\mu = b_\mu + \omega_{\mu\nu} x^\nu \qquad \xi = \text{const.}$$

Reproduces all known forms of global equilibrium

$$\begin{cases} b = (1/T, \underline{0}) \\ \omega = 0 \end{cases} \rightarrow e^{-\hat{H}/T}$$

$$\begin{cases} b = \text{cost.} \\ \omega = 0 \end{cases} \rightarrow e^{-b\cdot\hat{P}}$$

$$\begin{cases} b = (1/T, \underline{0}) \\ \omega = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \underline{\omega} \end{array}\right) \end{cases} \rightarrow e^{-\hat{H}/T + \underline{\omega}\cdot\hat{J}/T}$$

If $\beta^\mu = b^\mu$ const.

$$\langle \hat{T}^{\mu\nu} \rangle_{eq} = (\rho + p)\frac{\beta^\mu \beta^\nu}{\beta^2} - p g^{\mu\nu}$$

LOCAL equilibrium depends on **Σ**

A choice has to be made. The simplest is to define LTE on the hypersurface orthogonal to β itself

$$\boxed{n = \text{vers}\,\beta}$$

The β frame

$$\beta_\mu T^{\mu\nu}_{\rm LE}[\beta,\xi] = \beta_\mu T^{\mu\nu} \qquad \beta_\mu j^\mu_{\rm LE}[\beta,\xi] = \beta_\mu j^\mu,$$

Equations defining β and ξ

This is possible if β is vorticity-free. If it is not, the procedure is more complicated but it still can be done

Features of the β frame

- Simplest form of entropy density $S = -\mathrm{tr}(\hat{\rho}_{LE} \log \hat{\rho}_{LE})$

$$\log Z = \int_\Sigma \mathrm{d}\Sigma_\mu \, \Phi^\mu \quad \Rightarrow \quad s^\mu = \phi^\mu + T^{\mu\nu}_{\mathrm{LE}}\beta_\nu - \xi j^\mu_{\mathrm{LE}} + s^\mu_T(n)$$

$$s^\mu n_\mu = n_\mu \phi^\mu + n_\mu T^{\mu\nu}\beta_\nu - \xi n_\mu j^\mu .$$

$$\sqrt{\beta^2} s = \beta \cdot \phi + \beta_\mu \beta_\nu T^{\mu\nu} - \xi \beta_\mu j^\mu ,$$

- *vers* β is the four-velocity of an ideal relativistic thermometer, that is an object able to reach equilibrium with respect to energy and momentum echange with the fluid in x

Relativistic hydro in the β frame

Equation of state

$$p(\beta^2, \xi)$$

Ideal s.e.t.

$$T^{\mu\nu}_{\rm id} = -2\frac{\partial p}{\partial \beta^2}\beta^\mu \beta^\nu - p g^{\mu\nu},$$

$$-2\frac{\partial p}{\partial \beta^2} = \frac{\rho + p}{\beta^2} = \frac{h}{\beta^2}.$$

All gradients in u, T can be recasted as gradients of β

$$\sqrt{\beta^2}\Delta_{\mu\nu} D\beta^\nu + \frac{1}{2}\nabla_\mu \beta^2 = \beta_\lambda \Delta_{\mu\nu}(\partial^\lambda \beta^\nu + \partial^\nu \beta^\lambda)$$

$$\sqrt{\beta^2}\Delta_{\mu\nu} D\beta^\nu + \frac{1}{2}\nabla_\mu \beta^2 = \beta^2 A^\mu + \frac{1}{2}\nabla_\mu \frac{1}{T^2} = \frac{1}{T^2}\left(A_\mu - \frac{1}{T}\nabla_\mu T\right).$$

$$\begin{aligned}\nabla_\mu u^\nu &= \nabla_\mu \frac{\beta^\nu}{\sqrt{\beta^2}} = \beta^\nu \left(-\frac{1}{2}\right)(\beta^2)^{-3/2}\nabla_\mu \beta^2 + \frac{1}{\sqrt{\beta^2}}\nabla_\mu \beta^\nu \\ &= \frac{1}{\sqrt{\beta^2}}\left(-\frac{\beta^\nu \beta^\rho}{\beta^2}\nabla_\mu \beta_\rho + \nabla_\mu \beta^\nu\right) = \frac{1}{\sqrt{\beta^2}}\Delta^{\rho\nu}\nabla_\mu \beta_\rho,\end{aligned}$$

$$\nabla_\mu u^\mu = \frac{1}{\sqrt{\beta^2}}\nabla_\mu \beta^\mu.$$

The transport coefficients

$$\partial_\mu T^{\mu\nu} = \partial_\mu \mathrm{tr}(\widehat{\rho}\,\widehat{T}^{\mu\nu})_{\mathrm{ren}} = \mathrm{tr}(\widehat{\rho}\,\partial_\mu \widehat{T}^{\mu\nu})_{\mathrm{ren}} = 0,$$

where **ρ** is the fixed density operator in the Heisenberg picture

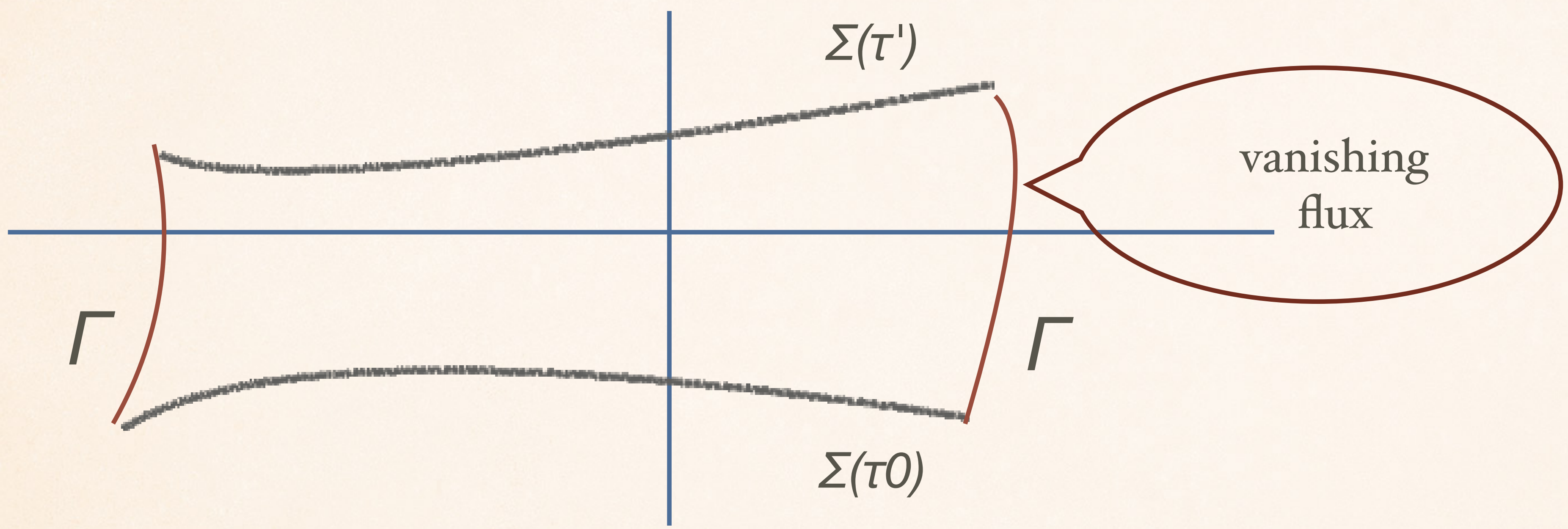
The hydrodynamical problem is to determine the evolution of the mean values starting from LTE condition at some time *τ0*

$$\hat{\rho} \equiv \hat{\rho}_{LE}(\tau_0) = \frac{1}{Z_{LE}(\tau_0)} \quad \exp\left[-\int_{\tau_0} d\Sigma\, n_\mu \left(\hat{T}^{\mu\nu}\beta_\nu - \xi\, \hat{j}^\mu \right)\right]$$

Gauss theorem

$$-\int_{\Sigma(\tau_0)} \mathrm{d}\Sigma \, n_\mu \left(\widehat{T}^{\mu\nu}\beta_\nu - \widehat{j}^\mu \xi\right) = -\int_{\Sigma(\tau')} \mathrm{d}\Sigma \, n_\mu \left(\widehat{T}^{\mu\nu}\beta_\nu - \widehat{j}^\mu \xi\right) + \int_\Omega \mathrm{d}\Omega \, \left(\widehat{T}^{\mu\nu} d_\mu \beta_\nu - \widehat{j}^\mu d_\mu \xi\right),$$

covariant derivative



$$\widehat{\rho} = \frac{1}{Z} \exp\left[-\int_{\Sigma(\tau')} \mathrm{d}\Sigma \, n_\mu \left(\widehat{T}^{\mu\nu}\beta_\nu - \widehat{j}^\mu \xi\right) + \int_\Omega \mathrm{d}\Omega \, \left(\widehat{T}^{\mu\nu} d_\mu \beta_\nu - \widehat{j}^\mu d_\mu \xi\right)\right]$$

If the system stays close to LTE at any time, the second term is a correction and an expansion can be made from present LTE

Expansion and Approximations

$$\widehat{\rho} = \frac{1}{Z} \exp\left[- \int_{\Sigma(\tau')} \mathrm{d}\Sigma \, n_\mu \left(\widehat{T}^{\mu\nu} \beta_\nu - \widehat{j}^\mu \xi \right) + \int_\Omega \mathrm{d}\Omega \, \left(\widehat{T}^{\mu\nu} d_\mu \beta_\nu - \widehat{j}^\mu d_\mu \xi \right) \right]$$

$\hat{A}$ $\hat{B}$

For small B, using the linear response theory:

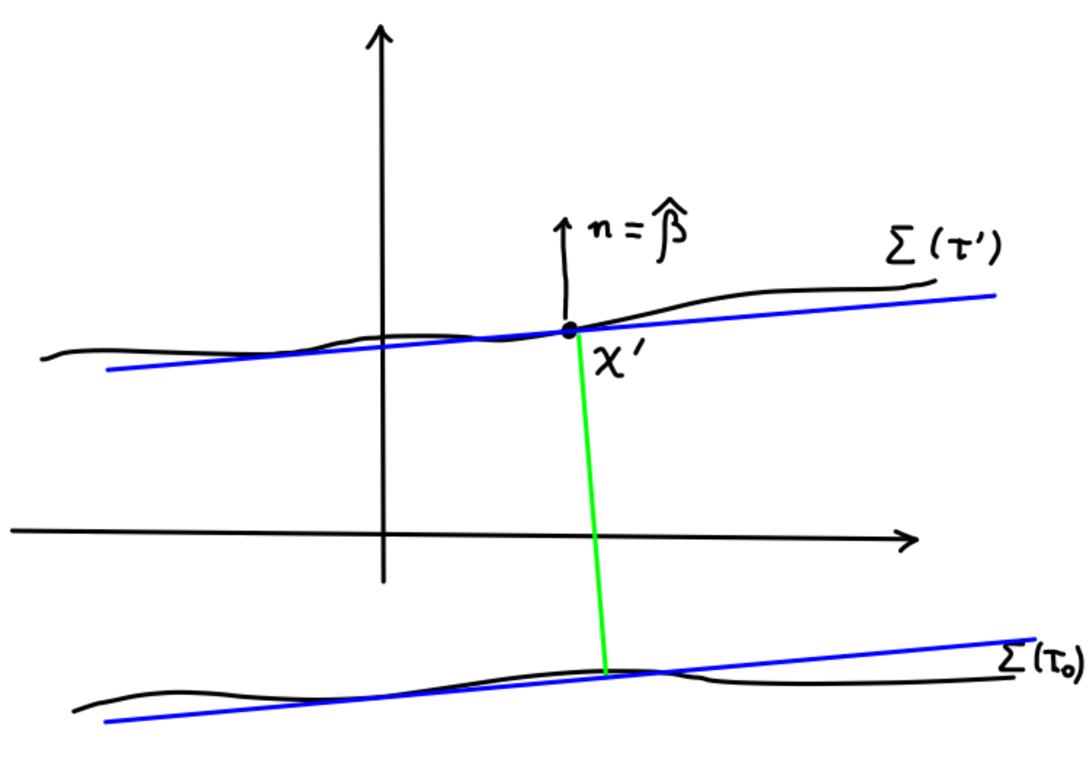
$$\langle \widehat{O}(x') \rangle \simeq \langle \widehat{O}(x') \rangle_{\mathrm{LE}} - \langle \widehat{O}(x') \rangle_{\mathrm{LE}} \langle \widehat{B} \rangle_{\mathrm{LE}} + \int_0^1 \mathrm{d}z \, \langle \widehat{O}(x') \mathrm{e}^{z\widehat{A}} \widehat{B} \mathrm{e}^{-z\widehat{A}} \rangle_{\mathrm{LE}}$$

If we want to calculate the mean value of an operator O in $x'=(\tau',\sigma')$

$$\widehat{A} = -\int_{\Sigma(\tau')} \mathrm{d}\Sigma \; n_\mu \left(\widehat{T}^{\mu\nu}\beta_\nu - \widehat{j}^\mu \xi\right) \simeq -\beta_\nu(\tau',\sigma') \int_{\Sigma(\tau')} \mathrm{d}\Sigma \; n_\mu \widehat{T}^{\mu\nu} + \xi(\tau',\sigma') \int_{\Sigma(\tau')} \mathrm{d}\Sigma \; n_\mu \widehat{j}^\mu = -\beta_\nu(x')\widehat{P}^\nu + \xi(x')\widehat{Q}$$

$$\widehat{\rho}_{\mathrm{LE}} \simeq \frac{1}{Z_{\mathrm{LE}}} \exp[-\widehat{A}] \simeq \frac{1}{Z} \exp[-\beta(x')\cdot\widehat{P} + \xi(x')\widehat{Q}] \equiv \widehat{\rho}_{\mathrm{eq(x')}}$$

Geometrical construction in the spacetime: approximating the integration boundaries with parallel hyperplanes



$$\int_\Omega \mathrm{d}\Omega \; \left(\widehat{T}^{\mu\nu} d_\mu \beta_\nu - \widehat{j}^\mu d_\mu \xi\right) \to \int_{T\Omega} \mathrm{d}^4 x \; \left(\widehat{T}^{\mu\nu} \partial_\mu \beta_\nu - \widehat{j}^\mu \partial_\mu \xi\right)$$

The Kubo formulae

After standard manipulations

$$\langle \widehat{O}(x') \rangle - \langle \widehat{O}(x') \rangle_{\rm LE} \simeq iT \int_{\tau_0}^{\tau'} {\rm d}^4x \int_{\tau_0}^{\tau} {\rm d}\theta \, \Big(\langle [\widehat{O}(x'), \widehat{T}^{\mu\nu}(\theta, \mathbf{x})] \rangle_{\beta(x')} \partial_\mu \beta_\nu(x) - \langle [\widehat{O}(x'), \widehat{j}^{\mu}(\theta, \mathbf{x})] \rangle_{\beta(x')} \partial_\mu \xi(x) \Big)$$

Define

$$\delta\beta \equiv \beta - \beta_{\rm eq} = \beta - \beta(x') \qquad \delta\xi \equiv \xi - \xi_{\rm eq} = \xi - \xi(x')$$

In the hydro limit, the perturbation varies on a scale much larger than the operators correlation length, thus only the smallest Fourier component can be retained

$$\delta\beta_\nu(x) \simeq A_\nu \frac{1}{2i} ({\rm e}^{-iK\cdot(x-x')} - {\rm e}^{iK\cdot(x-x')})$$

NOTE that it is crucial to have a perturbation which is periodical or vanishes at the boundary Γ to have no contribution from the timelike boundary of Ω

$$\partial_\mu \beta_\nu \simeq K_\mu A_\nu \frac{1}{2}(\mathrm{e}^{-iK\cdot(x-x')} + \mathrm{e}^{iK\cdot(x-x')}) = \partial_\mu \beta_\nu(x') \mathrm{Re}\ \mathrm{e}^{-iK\cdot(x-x')} = \mathrm{Re}\ \partial_\mu \beta_\nu(x') \mathrm{e}^{-iK\cdot(x-x')}$$

$$\begin{aligned}\langle \widehat{O}(x')\rangle - \langle \widehat{O}(x')\rangle_{\mathrm{LE}} \simeq & \partial_\mu \beta_\nu(x') \lim_{K\to 0} \mathrm{Im}\, T \int_{\tau_0}^{\tau'} \mathrm{d}^4 x \int_{\tau_0}^{\tau} \mathrm{d}\theta\, \langle [\widehat{T}^{\mu\nu}(\theta,\mathbf{x}), \widehat{O}(x')]\rangle_{\beta(x')} \mathrm{e}^{-iK\cdot(x-x')} \\ & -\partial_\mu \xi(x') \lim_{K\to 0} \mathrm{Im}\, T \int_{\tau_0}^{\tau'} \mathrm{d}^4 x \int_{\tau_0}^{\tau} \mathrm{d}\theta \langle [\widehat{j}^\mu(\theta,\mathbf{x}), \widehat{O}(x')]\rangle_{\beta(x')} \mathrm{e}^{-iK\cdot(x-x')}\end{aligned}$$

The final Kubo formulae in a covariant form

$$\delta O(x') \equiv \langle \hat{O}(x')\rangle - \langle \hat{O}(x')\rangle_{LE} \cong (\hat{O}, \hat{T}^{\mu\nu})_{\beta(x')} \partial_\mu \beta_\nu(x') - (\hat{O}, \hat{j}^\mu)_{\beta(x')} \partial_\mu \xi(x')$$

$$(\widehat{X}, \widehat{Y}) \equiv n^\alpha \frac{\partial}{\partial K^\alpha}\Big|_{n\cdot k=0} \lim_{k_T\to 0} \mathrm{Im}\, iT \int_{-\infty}^{\tau'} \mathrm{d}^4 x\, \langle [\widehat{X}(x'), \widehat{Y}(x)]\rangle_{\beta(x')} \mathrm{e}^{-iK\cdot(x-x')}$$

First-order expansion of the currents

Operator decomposition onto $\beta(x')$

$$\widehat{T}^{\mu\nu}(x) = \widehat{\varepsilon}(x)\frac{\beta^\mu(x')\beta^\nu(x')}{\beta^2(x')} + \widehat{q}^\mu(x)\frac{\beta^\nu(x')}{\sqrt{\beta^2(x')}} + \widehat{q}^\nu(x)\frac{\beta^\mu(x')}{\sqrt{\beta^2(x')}} + \widehat{\Pi}^{\mu\nu}(x) - \widehat{\pi}(x)\Delta^{\mu\nu}(x')$$

$$\widehat{j}^\mu(x) = \widehat{n}(x)\frac{\beta^\mu(x')}{\sqrt{\beta^2(x')}} + \widehat{\nu}^\mu(x)$$

$$\begin{aligned}\delta T^{\mu\nu} = & \left[(\widehat{\varepsilon},\widehat{\varepsilon})\frac{D\beta^2}{2\sqrt{\beta^2}} - (\widehat{\varepsilon},\widehat{\pi})\nabla\cdot\beta - (\widehat{\varepsilon},\widehat{n})D\xi\right]\frac{\beta^\mu\beta^\nu}{\beta^2} + \left[(\widehat{q}^\mu,\widehat{q}^\rho)\left(\frac{\nabla_\rho\beta^2}{2\sqrt{\beta^2}} + D\beta_\rho\right) - (\widehat{q}^\mu,\widehat{\nu}^\rho)\nabla_\rho\xi\right]\frac{\beta^\nu}{\sqrt{\beta^2}} \\ & + \left[(\widehat{q}^\nu,\widehat{q}^\rho)\left(\frac{\nabla_\rho\beta^2}{2\sqrt{\beta^2}} + D\beta_\rho\right) - (\widehat{q}^\nu,\widehat{\nu}^\rho)\nabla_\rho\xi\right]\frac{\beta^\mu}{\sqrt{\beta^2}} + (\widehat{\Pi}^{\mu\nu},\widehat{\Pi}^{\rho\sigma})\partial_\rho\beta_\sigma \\ & - \left[(\widehat{\pi},\widehat{\varepsilon})\frac{D\beta^2}{2\sqrt{\beta^2}} - (\widehat{\pi},\widehat{\pi})\nabla\cdot\beta - (\widehat{\pi},\widehat{n})D\xi\right]\Delta^{\mu\nu}\end{aligned}$$

$$\delta j^\mu = \left[(\widehat{n},\widehat{\varepsilon})\frac{D\beta^2}{2\sqrt{\beta^2}} - (\widehat{n},\widehat{\pi})\nabla\cdot\beta - (\widehat{n},\widehat{n})D\xi\right]\frac{\beta^\mu}{\sqrt{\beta^2}} + (\widehat{\nu}^\mu,\widehat{q}^\rho)\left(\frac{\nabla_\rho\beta^2}{2\sqrt{\beta^2}} + D\beta_\rho\right) - (\widehat{\nu}^\mu,\widehat{\nu}^\rho)\nabla_\rho\xi$$

Using Lorentz invariance

$$\begin{aligned}\delta T^{\mu\nu} = &\left[(\widehat{T}^{00},\widehat{T}^{00})_0 \frac{D\beta^2}{2\sqrt{\beta^2}} - (\widehat{T}^{00},\widehat{\Theta})_0 \nabla\cdot\beta - (\widehat{T}^{00},\widehat{j}^0)_0 D\xi\right]\frac{\beta^\mu\beta^\nu}{\beta^2}\\
&+\left[(\widehat{T}^{0i},\widehat{T}^{0i})_0\left(\frac{\nabla_\rho\beta^2}{2\sqrt{\beta^2}} + \Delta^{\mu\rho}D\beta_\rho\right) - (\widehat{T}^{0i},\widehat{j}^i)_0\nabla^\mu\xi\right]\frac{\beta^\nu}{\sqrt{\beta^2}} + (\mu\leftrightarrow\nu)\\
&+2(\widehat{T}^{ij},\widehat{T}^{ij})_{0\ i\neq j}\left[\frac{1}{2}(\Delta^{\mu\rho}\Delta^{\nu\sigma}+\Delta^{\nu\rho}\Delta^{\mu\sigma}) - \frac{1}{3}\Delta^{\mu\nu}\Delta^{\rho\sigma}\right]\partial_\rho\beta_\sigma\\
&-\left[(\widehat{\Theta},\widehat{T}^{00})_0\frac{D\beta^2}{2\sqrt{\beta^2}} - (\widehat{\Theta},\widehat{\Theta})_0\nabla\cdot\beta - (\widehat{\Theta},\widehat{j}^i)_0 D\xi\right]\Delta^{\mu\nu}\end{aligned}$$

$$\widehat{\Theta} = (1/3)\textstyle\sum_i \widehat{T}^{ii}$$

$$\delta j^\mu = \left[(\widehat{j}^0,\widehat{T}^{00})_0\frac{D\beta^2}{2\sqrt{\beta^2}} - (\widehat{j}^0,\widehat{\Theta})_0\nabla\cdot\beta - (\widehat{j}^0,\widehat{j}^0)_0 D\xi\right]\frac{\beta^\mu}{\sqrt{\beta^2}} + (\widehat{j}^i,\widehat{T}^{0i})_0\left(\frac{\nabla^\mu\beta^2}{2\sqrt{\beta^2}} + \Delta^{\mu\rho}D\beta_\rho\right) - (\widehat{j}^i,\widehat{j}^i)\nabla^\mu\xi$$

$$(\hat{X},\hat{Y})_0 = i\,T\,\frac{d}{d\omega}\Big|_{\omega=0}\lim_{\underline{k}\to 0}\int_{-\infty}^{0} d^4x\,\langle[\hat{X}(0),\hat{Y}(x)]\rangle_T\, e^{-i\omega t}\, e^{i\underline{k}\cdot\underline{x}}$$

Transport coefficients can be inferred from the above formulae. But first...

Enforcing the β frame

If the β field is not vorticous (like our small wavelength perturbation) then n= *vers* β and

$$\beta_\mu \delta T^{\mu\nu} = 0 \qquad \beta_\mu \delta j^\mu = 0$$

$$n_\mu \mathrm{tr}(\widehat{\rho}_{\mathrm{LE}} \widehat{T}^{\mu\nu}(x))_{\mathrm{ren}} = n_\mu \langle \widehat{T}^{\mu\nu}(x) \rangle_{\mathrm{LE}} \equiv n_\mu T^{\mu\nu}_{\mathrm{LE}}(x) = n_\mu T^{\mu\nu}(x)$$
$$n_\mu \mathrm{tr}(\widehat{\rho}_{\mathrm{LE}} \widehat{j}^{\mu}(x))_{\mathrm{ren}} = n_\mu \langle \widehat{j}^{\mu}(x) \rangle_{\mathrm{LE}} \equiv n_\mu j^{\mu}_{\mathrm{LE}}(x) = n_\mu j^{\mu}(x)$$

These are 5 equations. Very similar to Landau matching conditions, though not exactly the same (see later on)

If the β field is vorticous, then

$$(\beta_\mu - \varpi_{\mu\lambda} x^\lambda) \delta T^{\mu\nu} = 0 \qquad (\beta_\mu - \varpi_{\mu\lambda} x^\lambda) \delta j^\mu = 0$$

$$\varpi_{\nu\lambda}(x) = -\frac{1}{2}(\partial_\nu \beta_\lambda - \partial_\lambda \beta_\nu) - \frac{1}{6}\left(x^\rho \partial_\rho \partial_\nu \beta_\lambda - x^\rho \partial_\rho \partial_\lambda \beta_\nu\right) + \ldots$$

Contracting the previous relation with β_ν

$$\begin{cases} (\widehat{\varepsilon},\widehat{\varepsilon})\dfrac{D\beta^2}{2\sqrt{\beta^2}} - (\widehat{\varepsilon},\widehat{\pi})\nabla\cdot\beta - (\widehat{\varepsilon},\widehat{n})D\xi = 0 \\ (\widehat{n},\widehat{\varepsilon})\dfrac{D\beta^2}{2\sqrt{\beta^2}} - (\widehat{n},\widehat{\pi})\nabla\cdot\beta - (\widehat{n},\widehat{n})D\xi = 0 \end{cases}$$

Solving

$$\begin{cases} \dfrac{D\beta^2}{2\sqrt{\beta^2}} &= \dfrac{(\widehat{\varepsilon},\widehat{\pi})(\widehat{n},\widehat{n}) - (\widehat{n},\widehat{\pi})(\widehat{\varepsilon},\widehat{n})}{(\widehat{\varepsilon},\widehat{\varepsilon})(\widehat{n},\widehat{n}) - (\widehat{\varepsilon},\widehat{n})^2}\nabla\cdot\beta \\ D\xi &= \dfrac{(\widehat{\varepsilon},\widehat{n})(\widehat{\varepsilon},\widehat{\pi}) - (\widehat{\varepsilon},\widehat{\varepsilon})(\widehat{n},\widehat{\pi})}{(\widehat{\varepsilon},\widehat{\varepsilon})(\widehat{n},\widehat{n}) - (\widehat{\varepsilon},\widehat{n})^2}\nabla\cdot\beta \end{cases}$$

Replacing into the transverse trace part

$$\begin{aligned} &\left[-(\widehat{\pi},\widehat{\varepsilon})\frac{D\beta^2}{2\sqrt{\beta^2}} + (\widehat{\pi},\widehat{\pi})\nabla\cdot\beta + (\widehat{\pi},\widehat{n})D\xi\right]\Delta^{\mu\nu} \\ &= \left[(\widehat{\pi},\widehat{\pi}) - \frac{(\widehat{n},\widehat{n})(\widehat{\varepsilon},\widehat{\pi})^2 - (\widehat{n},\widehat{\pi})(\widehat{\varepsilon},\widehat{n})(\widehat{\varepsilon},\widehat{\pi}) - (\widehat{\varepsilon},\widehat{n})(\widehat{\varepsilon},\widehat{\pi})(\widehat{\pi},\widehat{n}) + (\widehat{\varepsilon},\widehat{\varepsilon})(\widehat{\pi},\widehat{n})^2}{(\widehat{\varepsilon},\widehat{\varepsilon})(\widehat{n},\widehat{n}) - (\widehat{\varepsilon},\widehat{n})^2}\right]\Delta^{\mu\nu}\nabla\cdot\beta \end{aligned}$$

consistent with previous results,
under reexamination

Bulk viscosity coefficient

$$\zeta = \frac{1}{T}\frac{\begin{vmatrix}(\widehat{\varepsilon},\widehat{\varepsilon}) & (\widehat{\varepsilon},\widehat{\pi}) & (\widehat{\varepsilon},\widehat{n})\\ (\widehat{\pi},\widehat{\varepsilon}) & (\widehat{\pi},\widehat{\pi}) & (\widehat{\pi},\widehat{n})\\ (\widehat{n},\widehat{\varepsilon}) & (\widehat{n},\widehat{\pi}) & (\widehat{n},\widehat{n})\end{vmatrix}}{\begin{vmatrix}(\widehat{\varepsilon},\widehat{\varepsilon}) & (\widehat{\varepsilon},\widehat{n})\\ (\widehat{n},\widehat{\varepsilon}) & (\widehat{n},\widehat{n})\end{vmatrix}} = \frac{1}{T}\frac{\begin{vmatrix}(\widehat{T}^{00},\widehat{T}^{00})_0 & (\widehat{T}^{00},\widehat{\Theta})_0 & (\widehat{T}^{00},\widehat{j}^0)_0\\ (\widehat{\Theta},\widehat{T}^{00})_0 & (\widehat{\Theta},\widehat{\Theta})_0 & (\widehat{\Theta},\widehat{j}^0)_0\\ (\widehat{j}^0,\widehat{T}^{00})_0 & (\widehat{j}^0,\widehat{\Theta})_0 & (\widehat{j}^0,\widehat{j}^0)_0\end{vmatrix}}{\begin{vmatrix}(\widehat{T}^{00},\widehat{T}^{00})_0 & (\widehat{T}^{00},\widehat{j}^0)_0\\ (\widehat{j}^0,\widehat{T}^{00})_0 & (\widehat{j}^0,\widehat{j}^0)_0\end{vmatrix}}$$

For an uncharged fluid

$$\zeta = \frac{1}{T}\frac{\begin{vmatrix}(\widehat{\varepsilon},\widehat{\varepsilon}) & (\widehat{\varepsilon},\widehat{\pi}))\\ (\widehat{\pi},\widehat{\varepsilon}) & (\widehat{\pi},\widehat{\pi})\end{vmatrix}}{(\widehat{\varepsilon},\widehat{\varepsilon})} = \frac{1}{T}\frac{\begin{vmatrix}(\widehat{T}^{00},\widehat{T}^{00})_0 & (\widehat{T}^{00},\widehat{\Theta})_0\\ (\widehat{\Theta},\widehat{T}^{00})_0 & (\widehat{\Theta},\widehat{\Theta})_0\end{vmatrix}}{(\widehat{T}^{00},\widehat{T}^{00})_0}$$

β frame vs Landau frame

If $\langle \hat{T}^{\mu\nu}_{\mathrm{LE}} \rangle = \langle \hat{T}^{\mu\nu}_{\mathrm{id}} \rangle$ they coincide

Yet, in general

$$T^{\mu\nu}_{\mathrm{LE}}(x) = \mathrm{tr}(\widehat{\rho}_{\mathrm{LE}}\widehat{T}^{\mu\nu}(x))_{\mathrm{ren}} = \frac{1}{Z_{\mathrm{LE}}}\mathrm{tr}\left(\exp\left[-\int \mathrm{d}\Sigma_\mu \left(\widehat{T}^{\mu\nu}\beta_\nu - \xi\widehat{j}^\mu\right)\right]\widehat{T}^{\mu\nu}(x)\right)_{\mathrm{ren}}$$

Taylor expand about *x*

$$\begin{aligned}&\exp\left[-\int \mathrm{d}\Sigma_\mu \left(\widehat{T}^{\mu\nu}\beta_\nu - \xi\widehat{j}^\mu\right)\right]\\ \simeq &\exp\left[-\beta_\nu(x)\widehat{P}^\nu + \xi(x)\widehat{Q} - \nabla_\lambda\beta_\nu(x)\int_{T\Sigma}\mathrm{d}\Sigma_\mu(y)\,\widehat{T}^{\mu\nu}(y)(y^\lambda - x^\lambda) + \nabla_\lambda\xi(x)\int_{T\Sigma}\mathrm{d}\Sigma_\mu(y)\,\widehat{j}^\mu(y)(y^\lambda - x^\lambda) + \dots\right]\end{aligned}$$

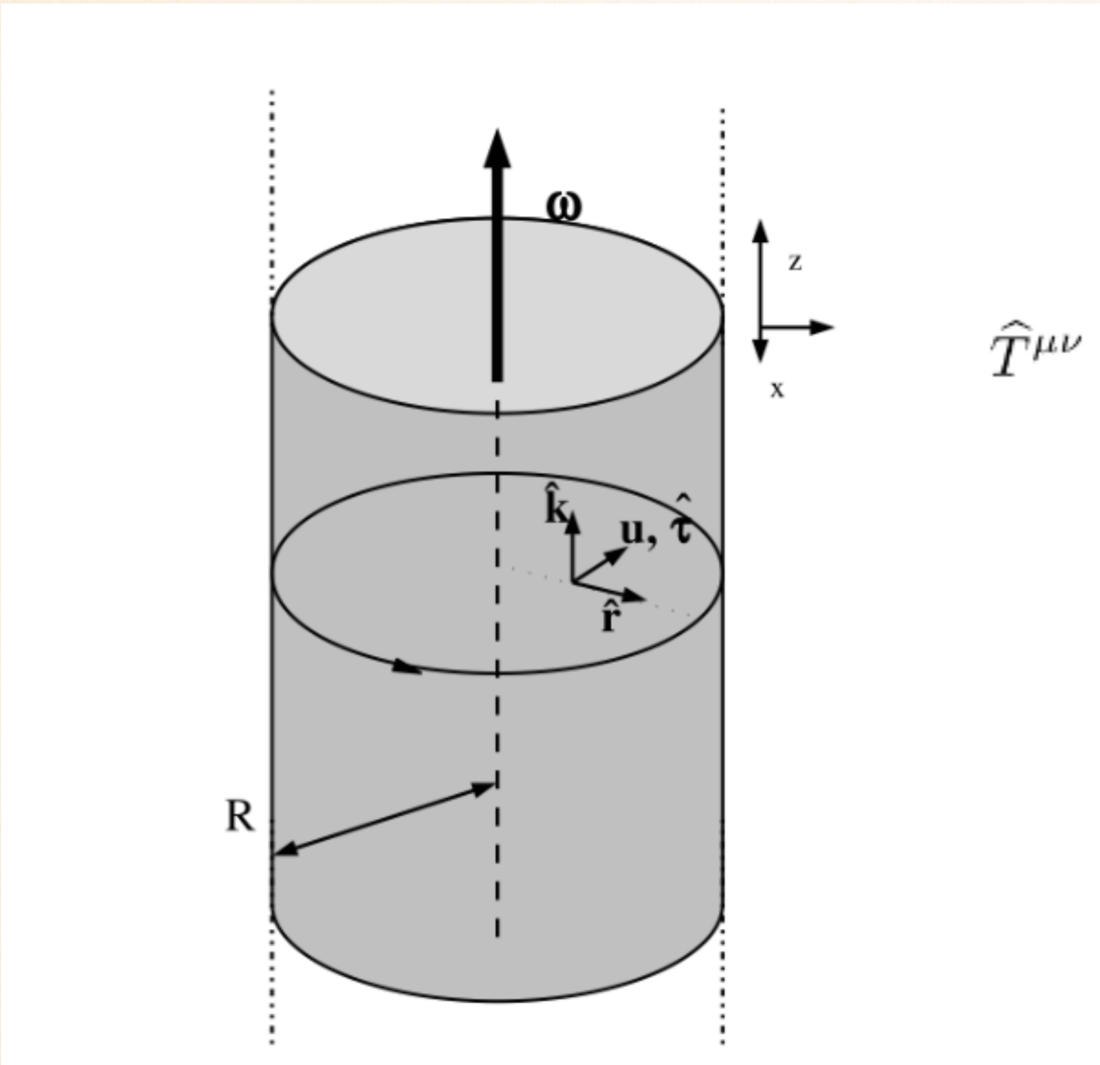
Then

$$T^{\mu\nu}_{\rm LE}(x) \simeq \frac{1}{Z_{\rm eq}(\beta(x),\xi(x))}\,{\rm tr}\left(\exp\left[-\beta_\nu(x)\widehat{P}^\nu + \xi(x)\widehat{Q}\right]\widehat{T}^{\mu\nu}(x)\right)_{\rm ren} + \mathcal{O}(\partial\beta,\partial\xi).$$

$$T^{\mu\nu}_{\rm LE}(x) \simeq T^{\mu\nu}_{\rm id}(x) + \mathcal{O}(\nabla\beta,\nabla\xi) = (\rho+p)_{\rm eq}\frac{1}{\beta^2}\beta^\mu(x)\beta^\nu(x) - p_{\rm eq}g^{\mu\nu} + \mathcal{O}(\partial\beta,\partial\xi),$$

Is there some leading correction? We think that the smallest is 2nd order in the gradient: no correction to transport coefficients

β # Landau. An equilibrium calculation: the free scalar field



$$\widehat{\rho} = \frac{1}{Z} \exp[-\widehat{H}/T + \omega \cdot \widehat{J}_z/T]\mathsf{P}_V,$$

$$\widehat{T}^{\mu\nu} = \partial^{(\mu}\widehat{\psi}\,\partial^{\nu)}\widehat{\psi} - g^{\mu\nu}\widehat{\mathcal{L}} \qquad \widehat{\mathcal{L}} = \frac{1}{2}\left(\partial_\mu\widehat{\psi}\,\partial^\mu\widehat{\psi} - m^2\widehat{\psi}^2\right)$$

Keeping in mind

$$\beta_\mu = b_\mu + \omega_{\mu\nu}x^\nu \qquad \omega_{\mu\nu} = -\partial_\mu\beta_\nu + \partial_\nu\beta_\mu$$

in this case

$$\omega_{\mu\nu} = \begin{pmatrix} 0 & 0 \\ 0 & \omega/T \end{pmatrix}$$

It can be shown by explicit calculation

$$T = \begin{pmatrix} u \cdot T \cdot u & u \cdot T \cdot \hat{\tau} & 0 & 0 \\ u \cdot T \cdot \hat{\tau} & \hat{\tau} \cdot T \cdot \hat{\tau} & 0 & 0 \\ 0 & 0 & \hat{r} \cdot T \cdot \hat{r} & 0 \\ 0 & 0 & 0 & k \cdot T \cdot k \end{pmatrix}$$

$$\hat{r} \cdot T \cdot \hat{r} \neq \hat{k} \cdot T \cdot \hat{k}$$

$$\hat{r} \cdot T \cdot \hat{r} - k \cdot T \cdot k = \sum_{M=-\infty}^{+\infty} \sum_{p_T} \int \mathrm{d}p_L \, \frac{2}{(2\pi)^2 \, \varepsilon \, R^2 \, J_M'^2(p_T R)} \frac{1}{\mathrm{e}^{\beta(\varepsilon - M\omega)} - 1} \left[p_T^2 J_M'(p_T r)^2 - p_L^2 J_M(p_T r)^2 \right],$$

$$\Rightarrow \quad \langle \hat{T}^{\mu\nu} \rangle_{eq} \neq (\rho + p) u^\mu u^\nu - p g^{\mu\nu}$$

$$\langle \hat{T}^{\mu\nu} \rangle_{eq} = -\frac{\partial p}{\partial \beta^2} \beta^\mu \beta^\nu - p(\beta^2, \xi) g^{\mu\nu} + O((\omega/T)^2)$$