Multi-meson systems in lattice QCD

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University of Washington

eXtremeQCD 2008

- Multi-hadron systems in QCD
- $n<13$ pions and kaons
  - Two and three body interactions
  - Pion and kaon condensates
  - In-medium screening of the $Q\bar{Q}$ potential

Many body* QCD

* Many body: more than two hadrons

Many body* QCD

- Nuclear physics from QCD

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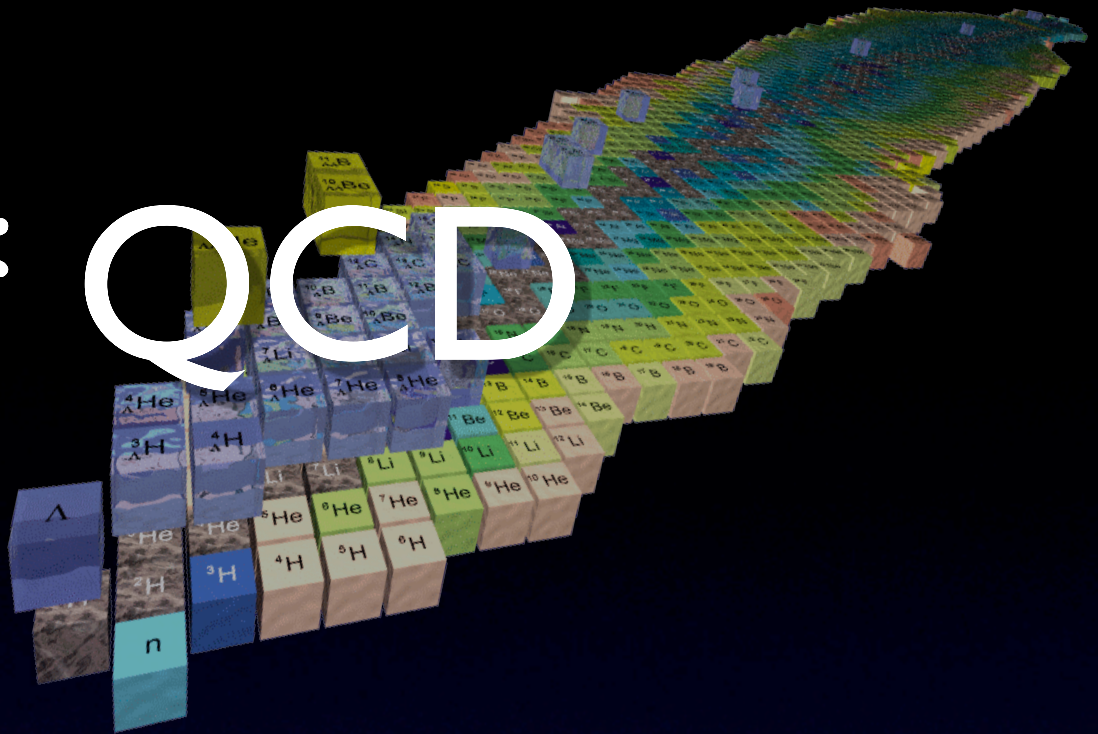

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  - Spectra

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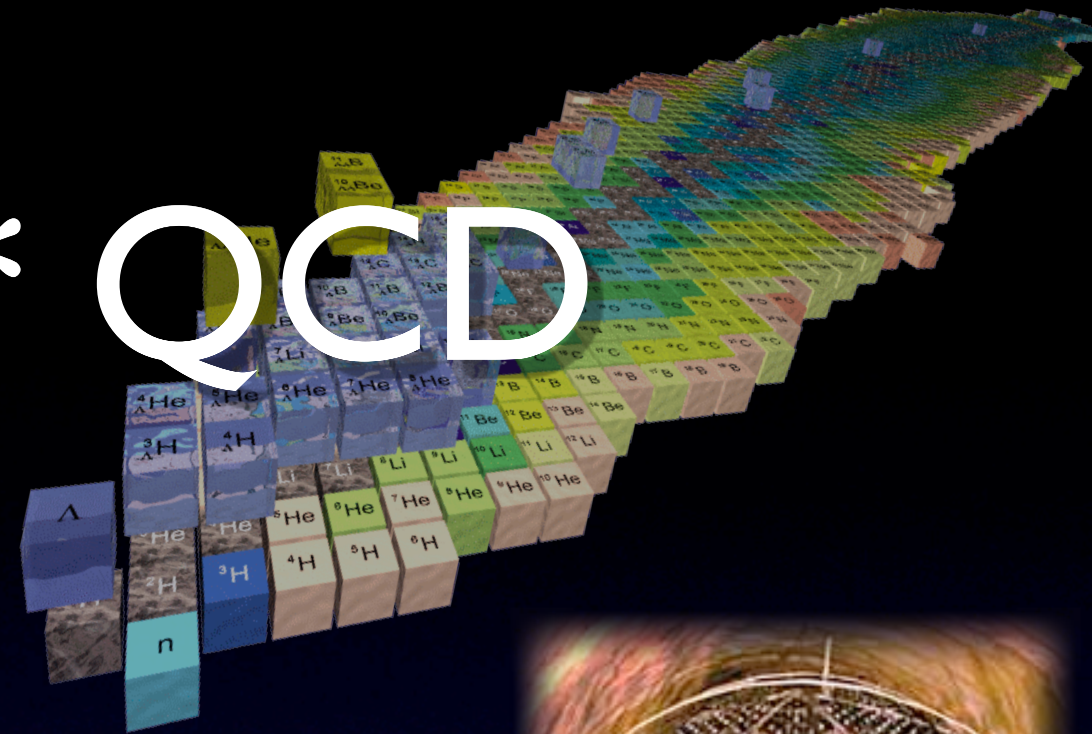

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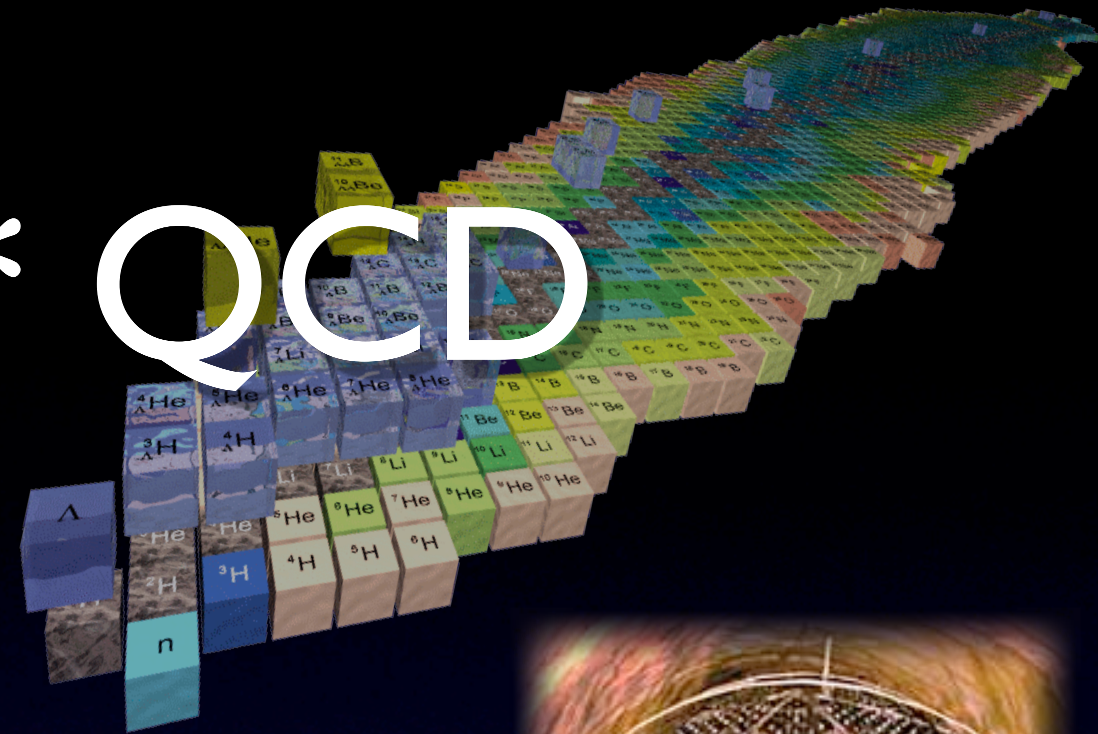

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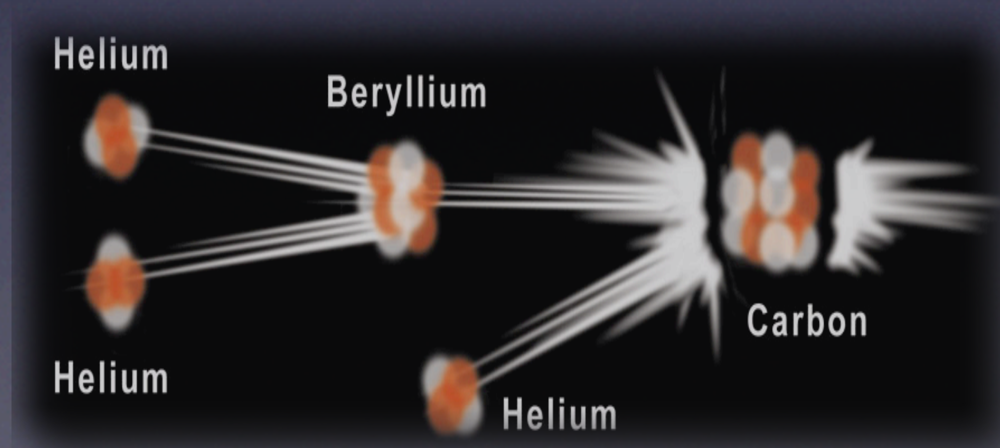

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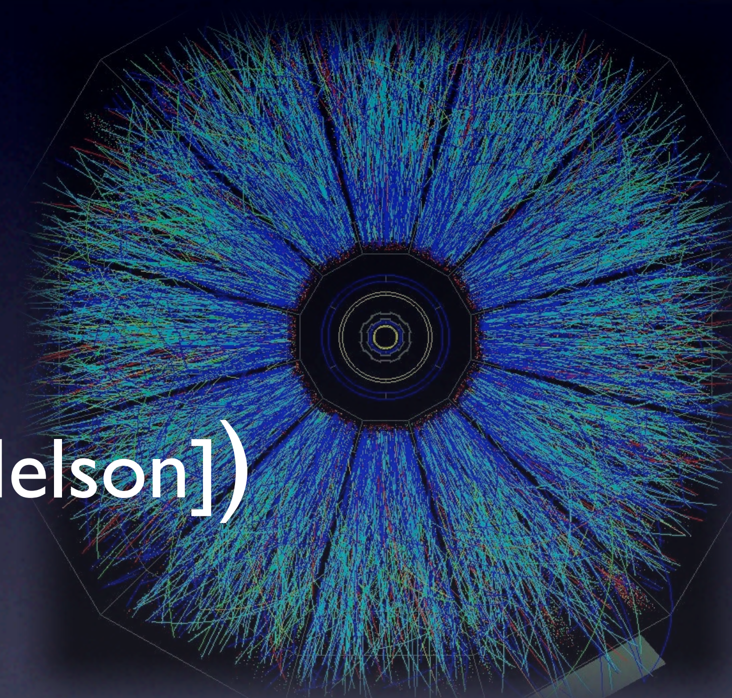

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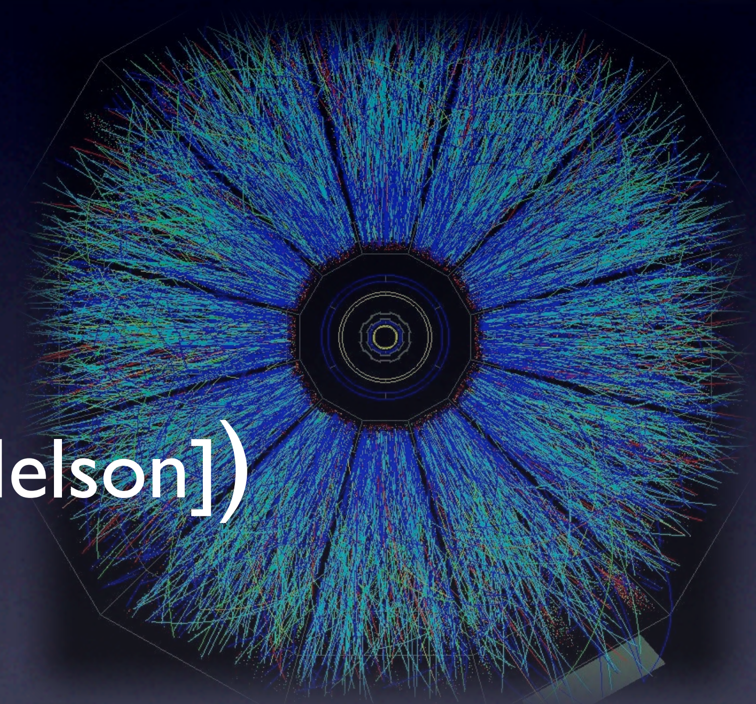
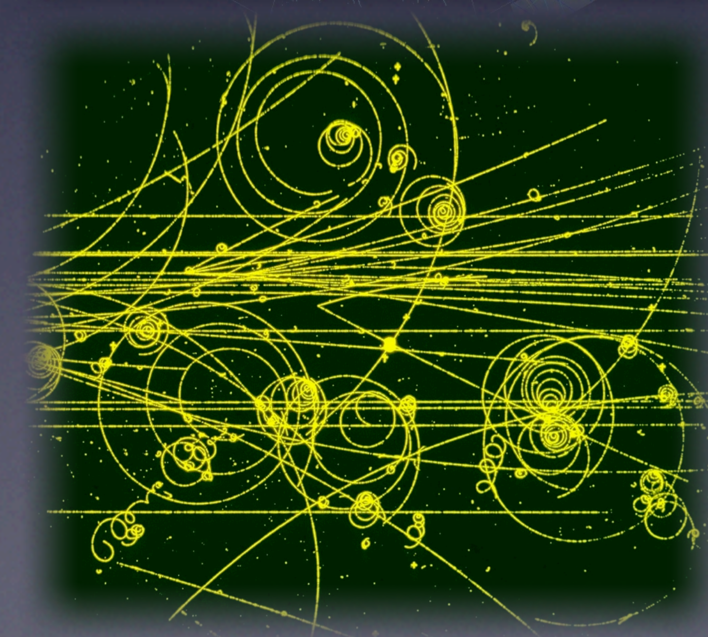
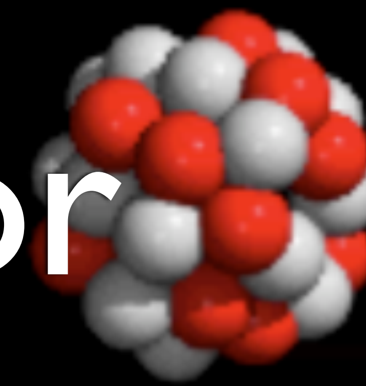
- Many body decays:





Many body correlator



- How can we compute the mass of ^{208}Pb in QCD?

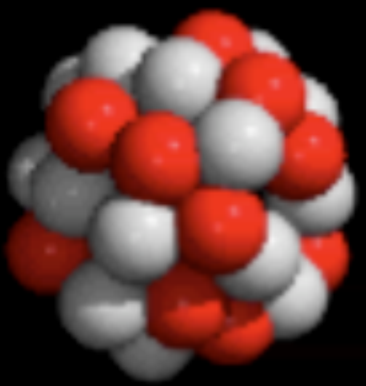
$$\langle 0 | T\, q_1(t) \ldots q_{624}(t)\, \overline{q}_1(0) \ldots \overline{q}_{624}(0) | 0 \rangle$$

- Large time behaviour gives GS energy (up to EW effects)

$$\stackrel{t \to \infty}{\longrightarrow} \# \exp(-M_{Pb} t)$$

- But...

Many difficulties



- Interpolators and contractions
- Statistical noise
- Signals for very massive states
- Small energy splittings
- Numerical precision

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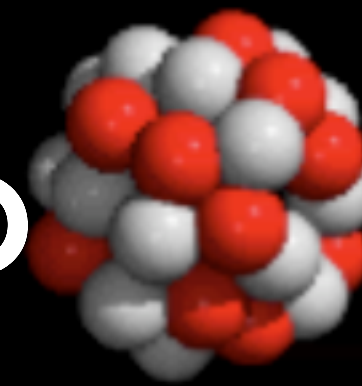
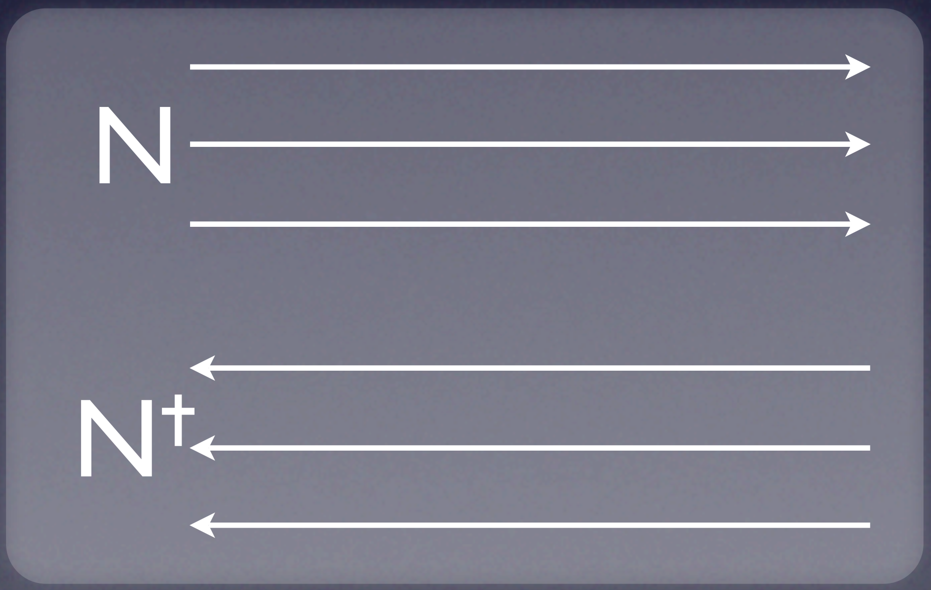
Signal-to-noise ratio



- QCD functional integrals done by importance sampling: propagators
- Variance in correlator determined by

$$\sigma^2\langle C\rangle = \langle C\, C^\dagger\rangle - |\langle C\rangle|^2$$



- For nucleon:

$$s/n \sim \exp[-(M_N - 3/2 m_\pi)t]$$

- For nucleus A:

$$s/n \sim \exp[-A(M_N - 3/2 m_\pi)t]$$

[Lepage '89]

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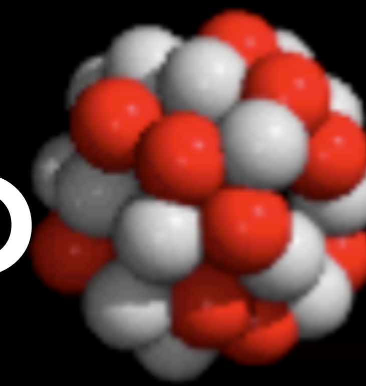

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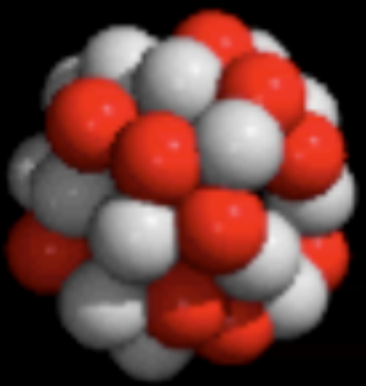
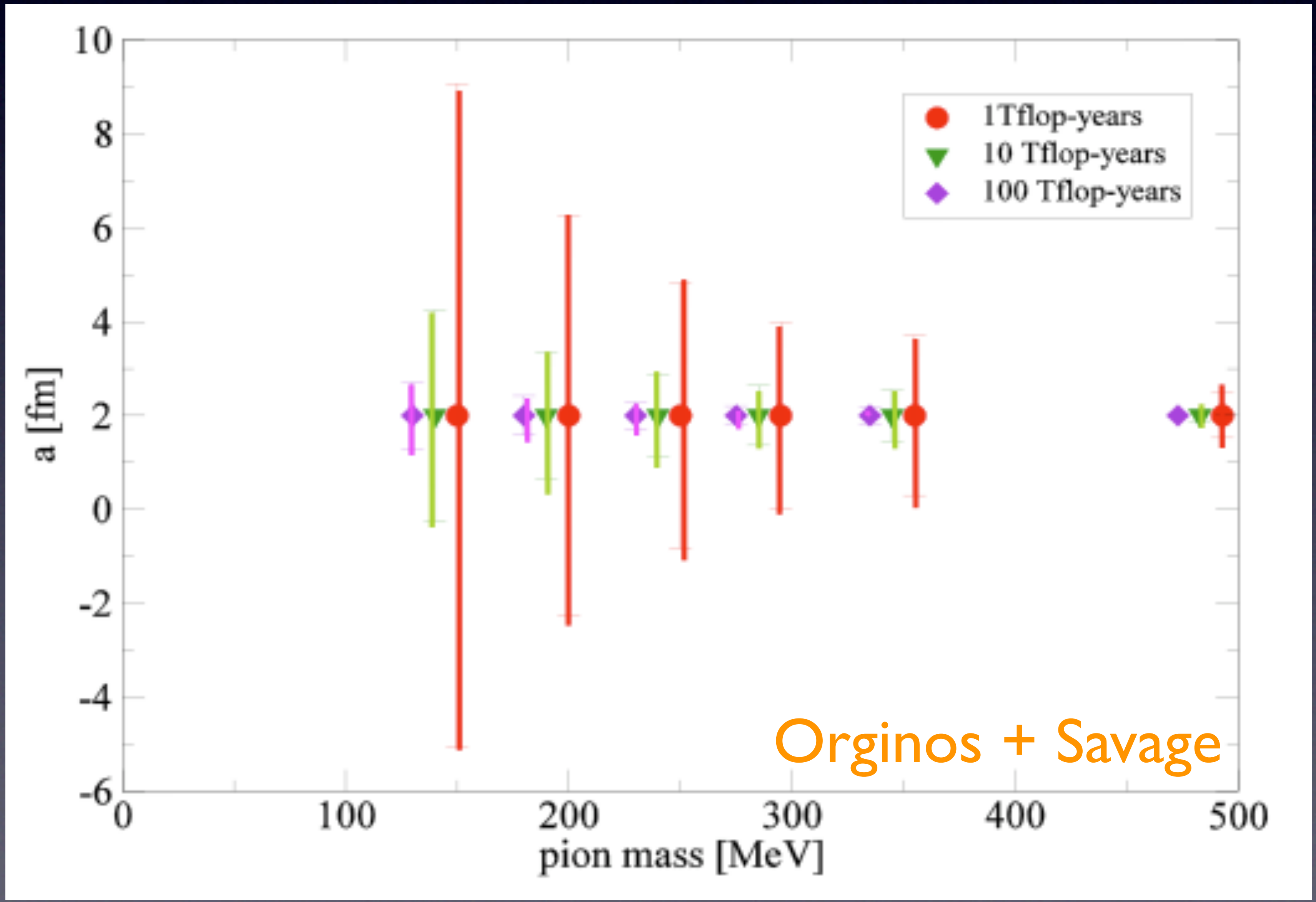


- Exponentially bad problem
  - ex: NN scattering length



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Phys Rev D78:014507, 2008
arXiv:0807.1856
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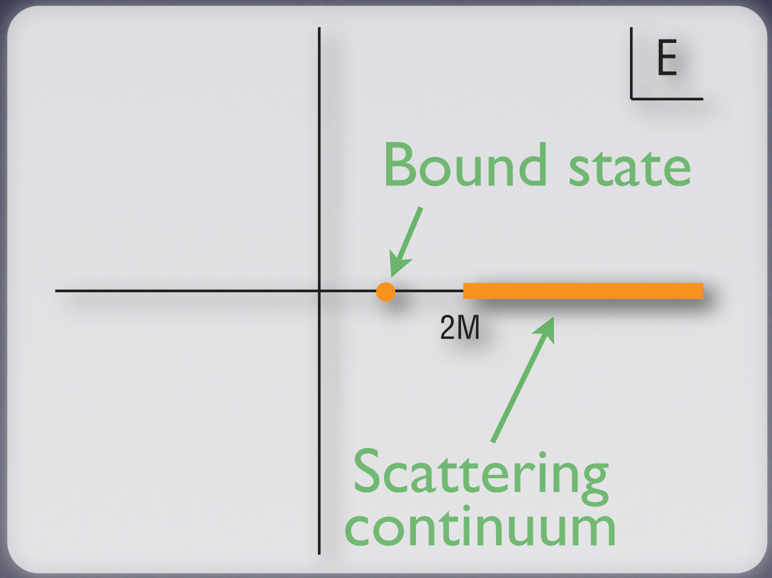
Multi-meson systems:

Poor man's nuclear physics

Hadron scattering

- Maiani-Testa: *extracting multi-hadron S-matrix elements from Euclidean lattice calculations of corresponding Green functions is impossible*
- Lüscher: volume dependence of two-particle energy levels ⇒ scattering phase-shift up to inelastic threshold



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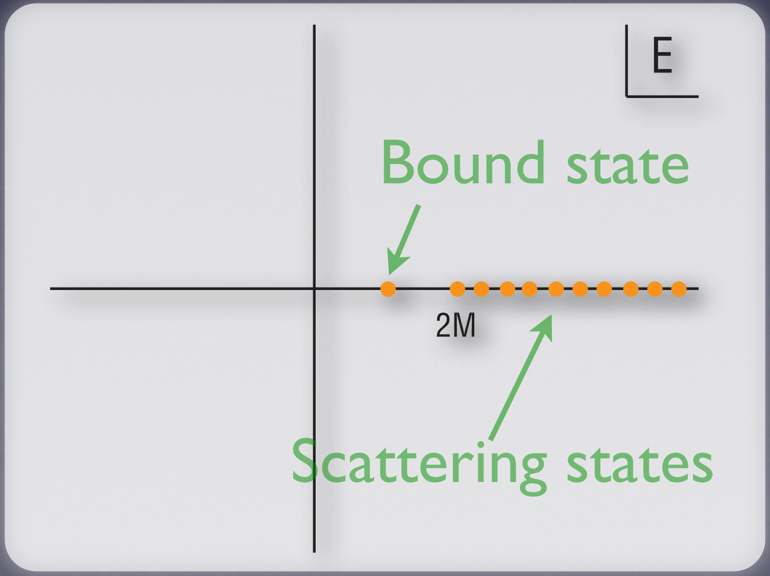
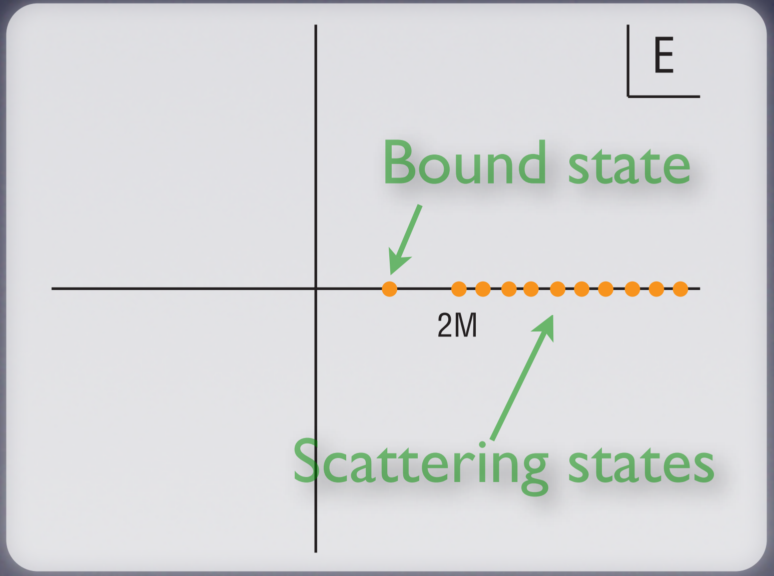


Hadron scattering

- Maiani-Testa: *extracting multi-hadron S-matrix elements from Euclidean lattice calculations of corresponding Green functions is impossible*
- Lüscher: volume dependence of two-particle energy levels ⇒ scattering phase-shift up to inelastic threshold
- Exact relation provided $r \ll L$
- Used for $\pi\pi$, KK, NN, ΛN



Multi-boson energies

[Beane, WD & Savage; WD & Savage]

- Large volume expansion of GS energy of *n* meson system to $1/L^7$
    - 2 & 3 body interactions (*N* body: $L^{-3(N-1)}$)
    - Relativistic up to particle production
    - *n*=2: reproduces expansion of Lüscher
- Can include higher PW, higher body, excited states, fermions

[Bogoliubov '47][Huang, Yang '57]

Multi-boson energies

[WD+Savage arXiv:0801.0763]

$$\Delta E_n = \frac{4\pi\bar{a}}{M\,L^3}\,{}^nC_2\Bigg\{1-\left(\frac{\bar{a}}{\pi\,L}\right)\mathcal{I}+\left(\frac{\bar{a}}{\pi\,L}\right)^2\left[\mathcal{I}^2+(2n-5)\mathcal{J}\right]$$

$$-\left(\frac{\bar{a}}{\pi\,L}\right)^3\left[\mathcal{I}^3+(2n-7)\mathcal{I}\mathcal{J}+\left(5n^2-41n+63\right)\mathcal{K}\right]$$

$$+\left(\frac{\bar{a}}{\pi\,L}\right)^4\Big[\mathcal{I}^4-6\mathcal{I}^2\mathcal{J}+(4+n-n^2)\mathcal{J}^2+4(27-15n+n^2)\mathcal{I}\,\mathcal{K}$$

$$+(14n^3-227n^2+919n-1043)\mathcal{L}+16(n-2)\left(\mathcal{T}_0+n\mathcal{T}_1\right)\Big]\Bigg\}$$

$$+{}^nC_3\frac{1}{L^6}\,\hat{\eta}_3^L+{}^nC_3\,\frac{6\pi\bar{a}^3}{M^3L^7}\,(n+3)\,\mathcal{I}\;+\;\mathcal{O}\left(L^{-8}\right)$$

$$\bar{a}=a+\frac{2\pi}{L^3}a^3\,r$$

$$\hat{\eta}_3^L=\bar{\eta}_3^L\left[1-\frac{6\bar{a}}{\pi\,L}\mathcal{I}\right]+\frac{72\pi\bar{a}^4r}{M\,L}\mathcal{I}$$

$\mathcal{I},\mathcal{J},\ldots$

geometric constants

Many mesons in LQCD

- Consider n π^+ correlator (m_u=m_d)

$$C_n(t) = \left\langle 0 \left| \left[\sum_{\mathbf{x}} \overline{d}\gamma_5 u(\mathbf{x},t)\overline{u}\gamma_5 d(0,0) \right]^n \right| 0 \right\rangle$$
$$\rightarrow A e^{-E_n t}$$

- $n!^2$ Wick contractions: $(13!)^2 \sim 10^{19}$

$$C_3(t) = \mathrm{tr}\left[\Pi\right]^3 \;-\; 3\,\mathrm{tr}\left[\Pi\right]\mathrm{tr}\left[\Pi^2\right] + \;2\,\mathrm{tr}\left[\Pi^3\right]$$

$$\Pi = \sum_{\mathbf{x}} \gamma_5 S(\mathbf{x},t;0)\gamma_5 S^\dagger(\mathbf{x},t;0)$$

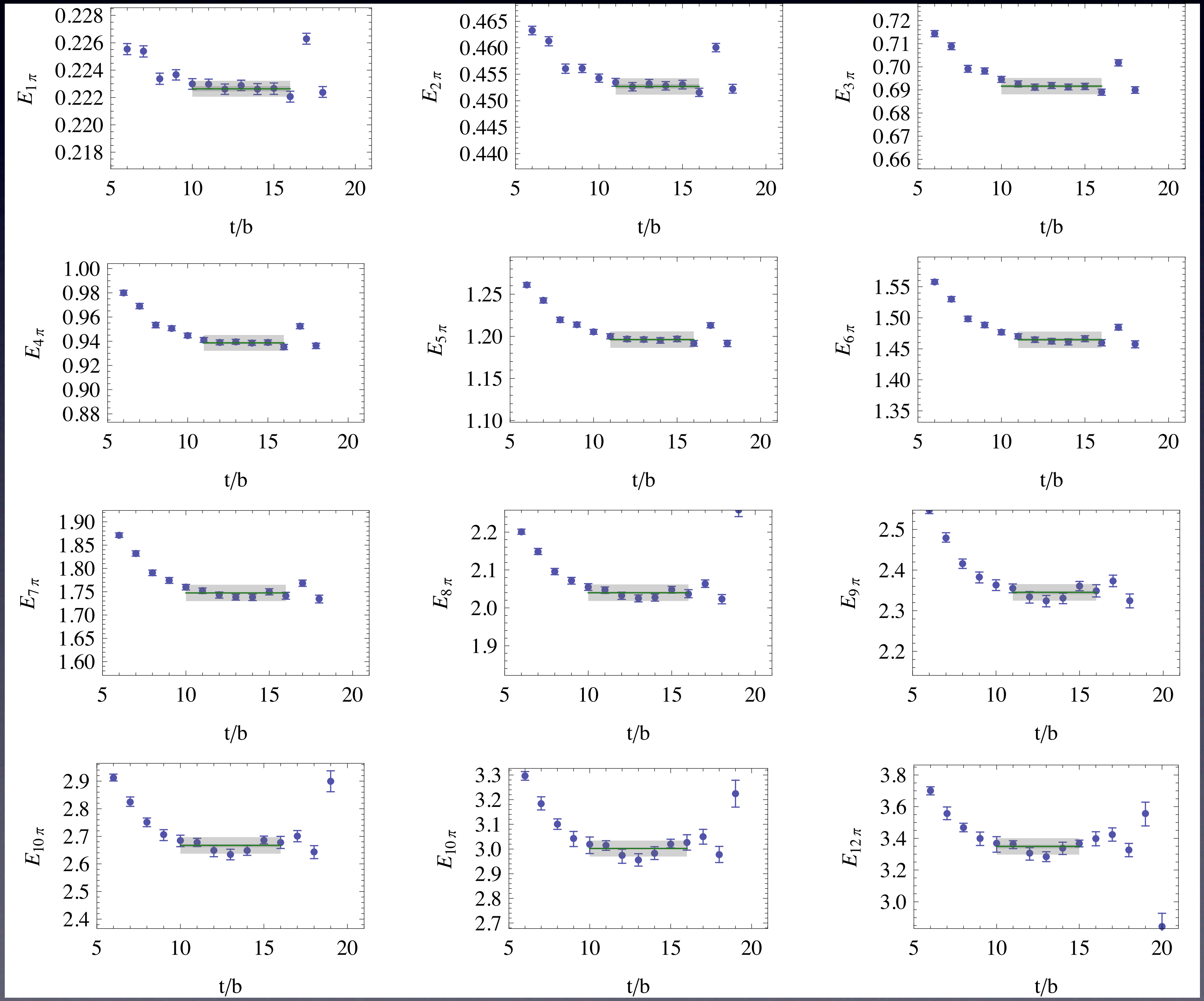
- π^+ contractions: only a single quark propagator

Lattice details

- Calculations use MILC gauge configurations
  - L=2.5 fm, a=0.12 fm, *rooted* staggered
  - also L=3.5 fm and a=0.09 fm
- NPLQCD: domain-wall quark propagators
  - $m_\pi \sim$ 291, 318, 352, 358, 491, 591 MeV
  - 24 propagators / lattice in best case
- I_z=n=1,...,12 pion and (S=n) kaon systems

n-meson energies

- Effective energy plots: $\log[C_n(t)/C_n(t+1)]$



m_π = 352 MeV

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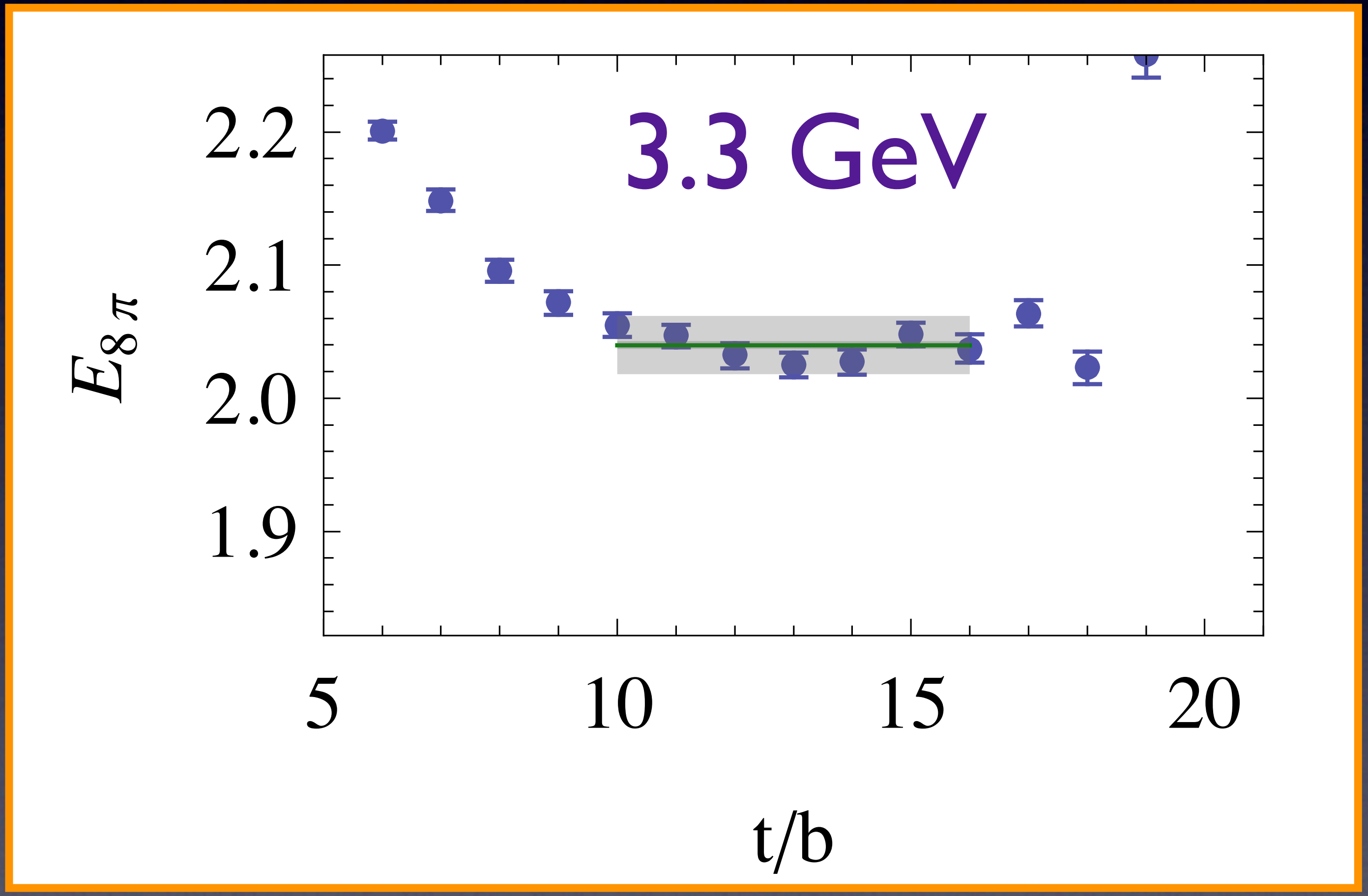

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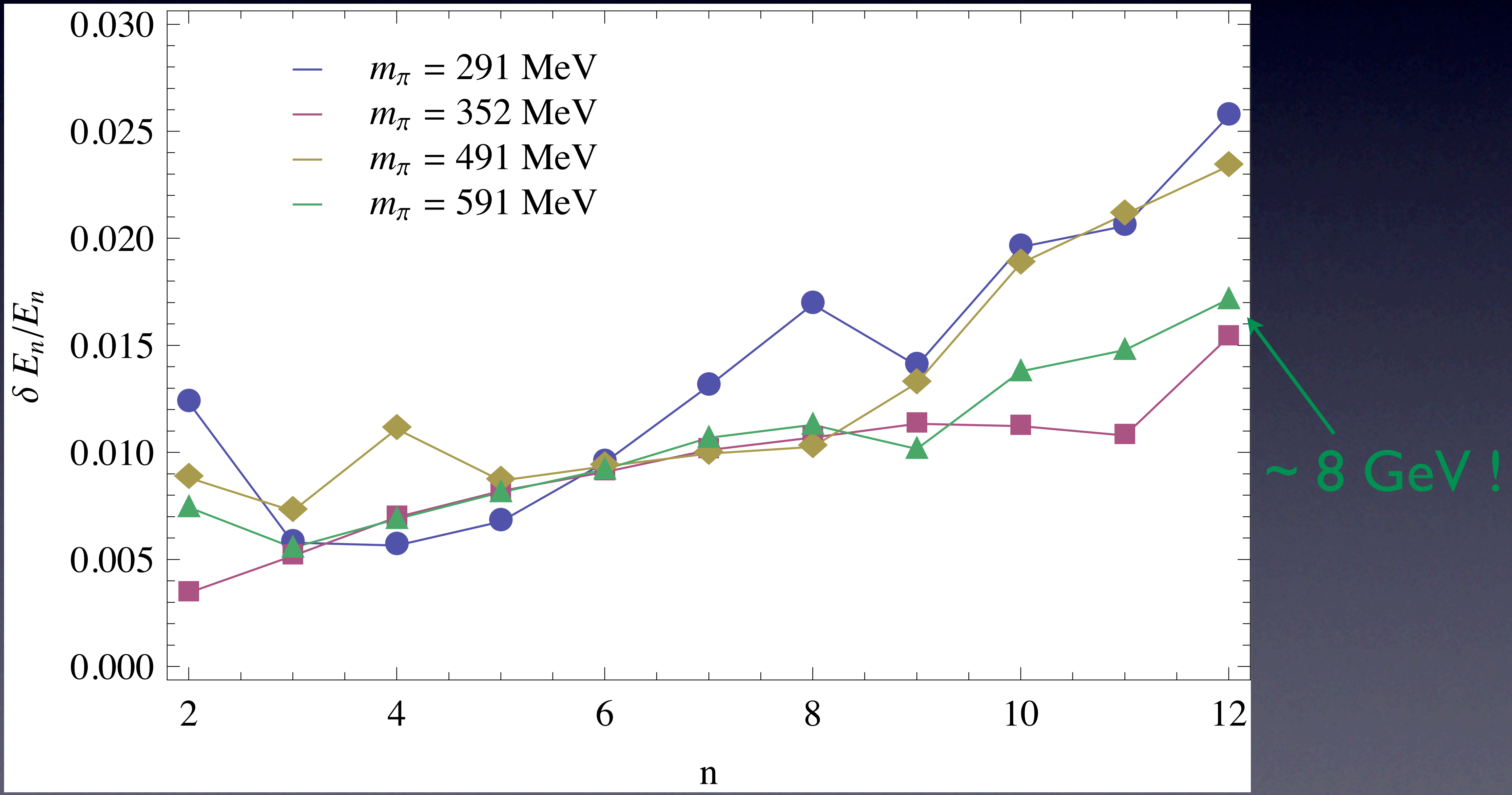
n-meson energies

- Clean signals for n=1,...,12



Multi-boson energies

- Multiple extractions of $\overline{a}$ and $\hat{\eta}_3^L$

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& - \left(\frac{\overline{a}}{\pi\,L}\right)^3 \left[\mathcal{I}^3 + (2n-7)\mathcal{I}\mathcal{J} + \left(5n^2 - 41n + 63\right)\mathcal{K}\right] \\
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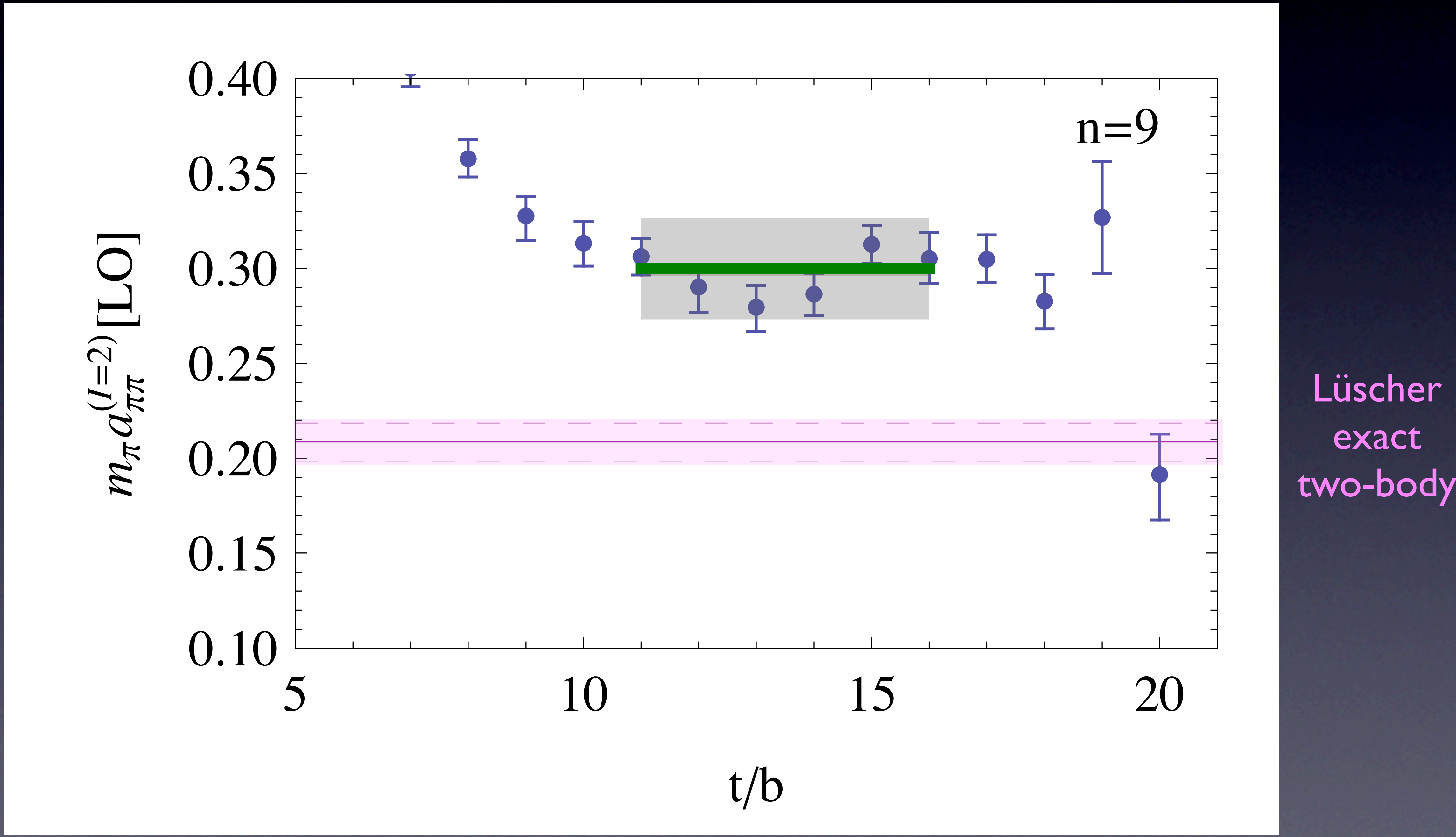
Pion scattering

- Extractions of $m_\pi a$ from four orders in L



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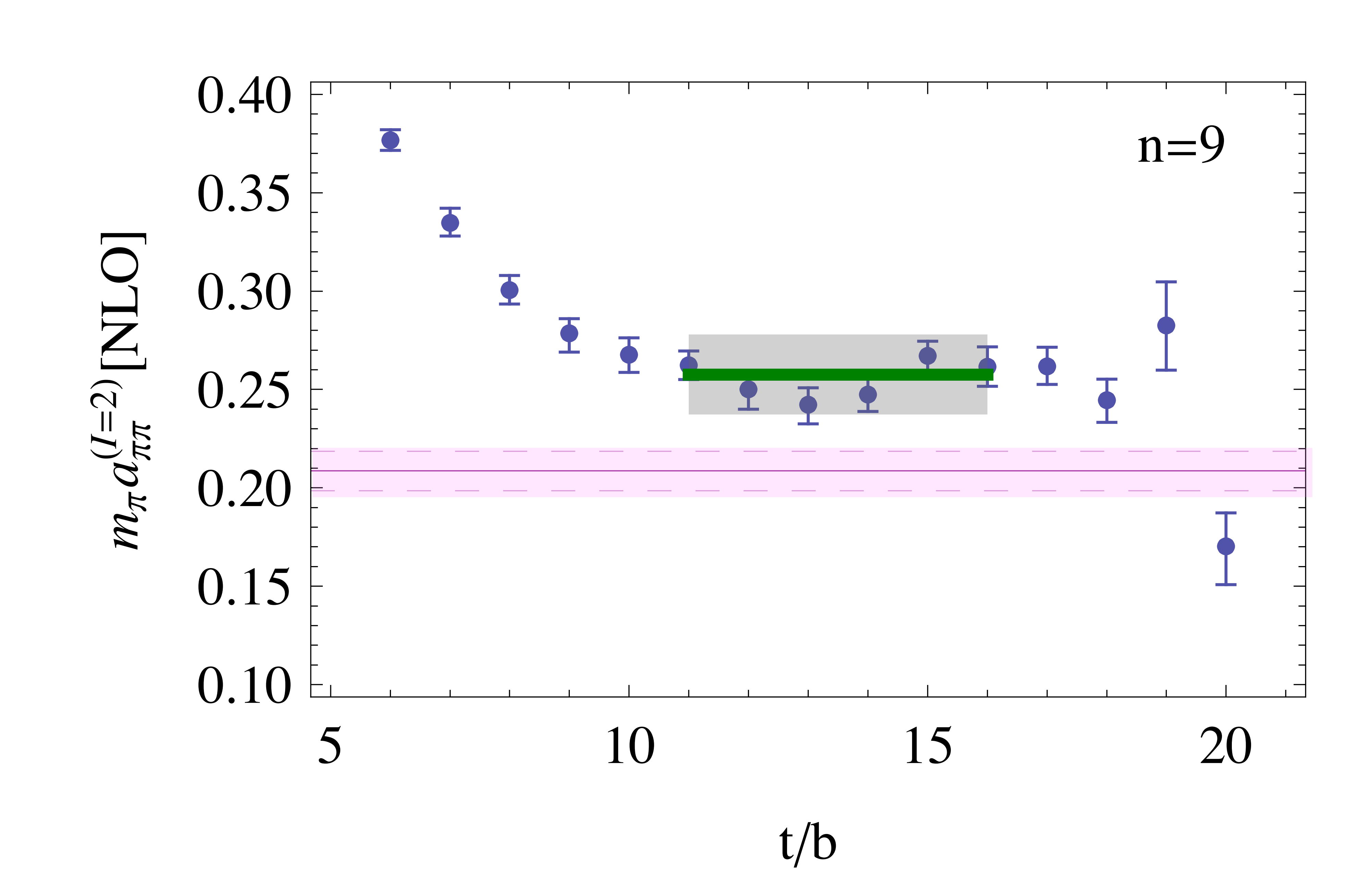

Lüscher
exact
two-body

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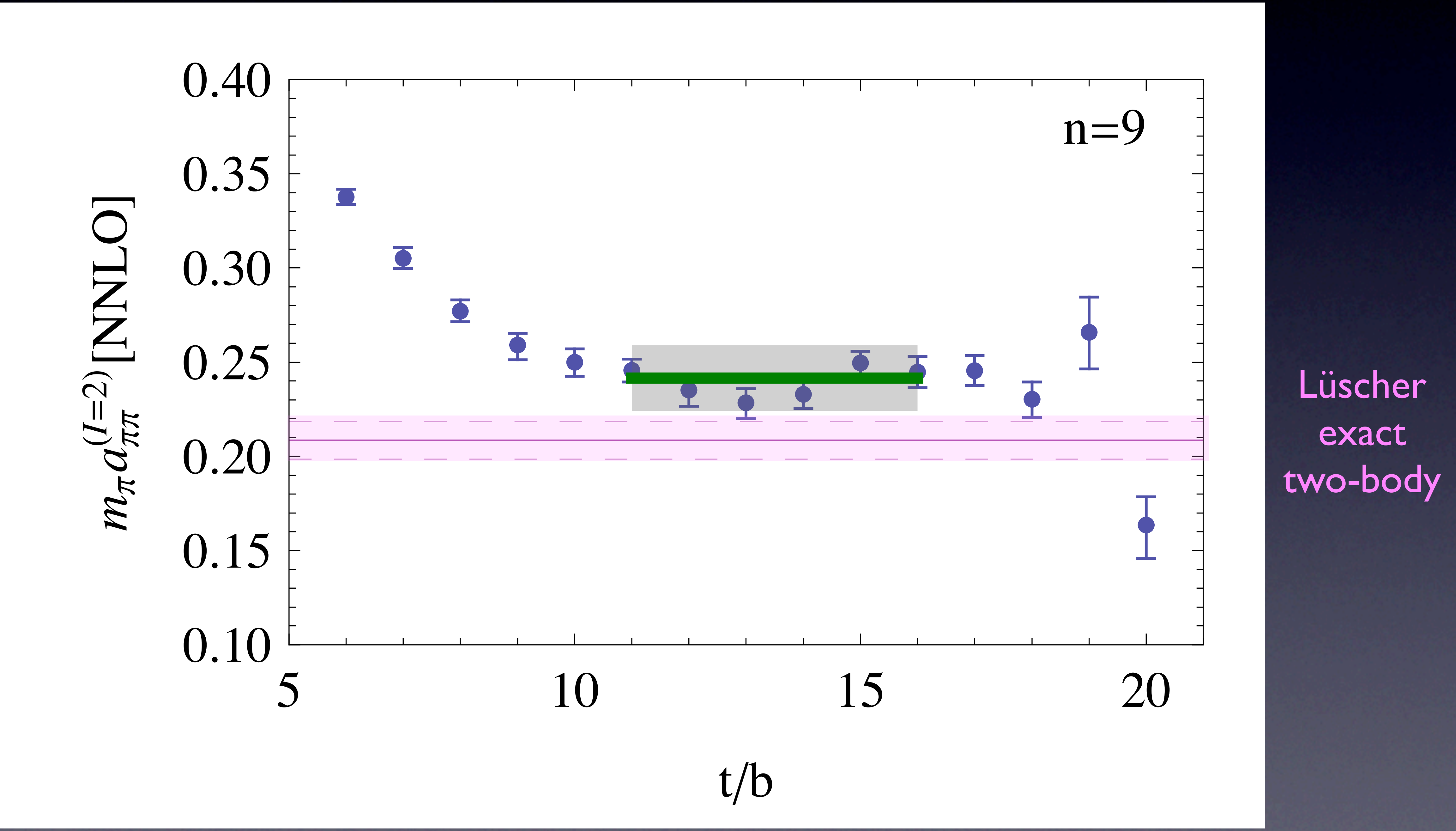
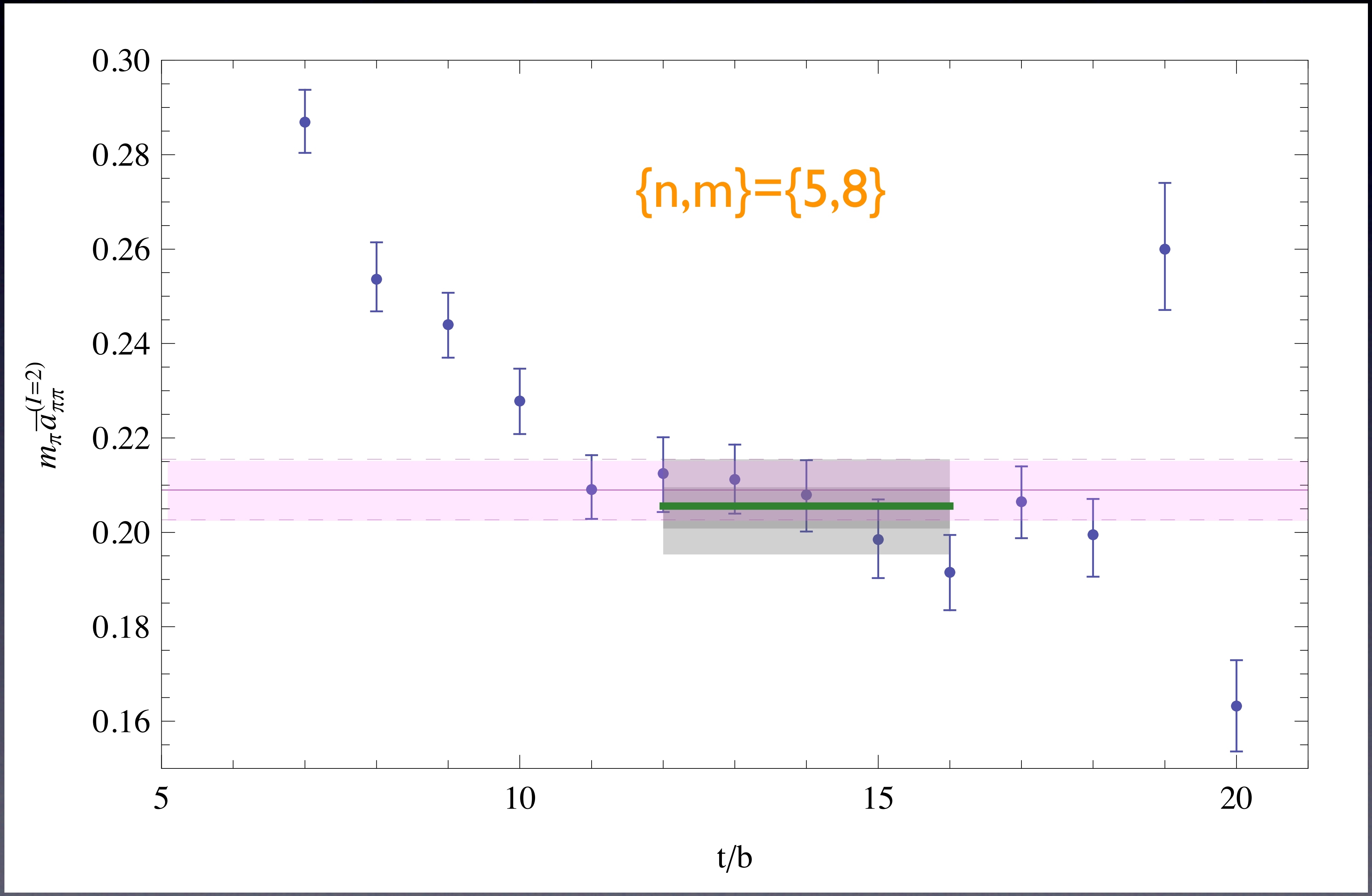


N^3LO

- Two energies to cancel 3-body: 45 combinations



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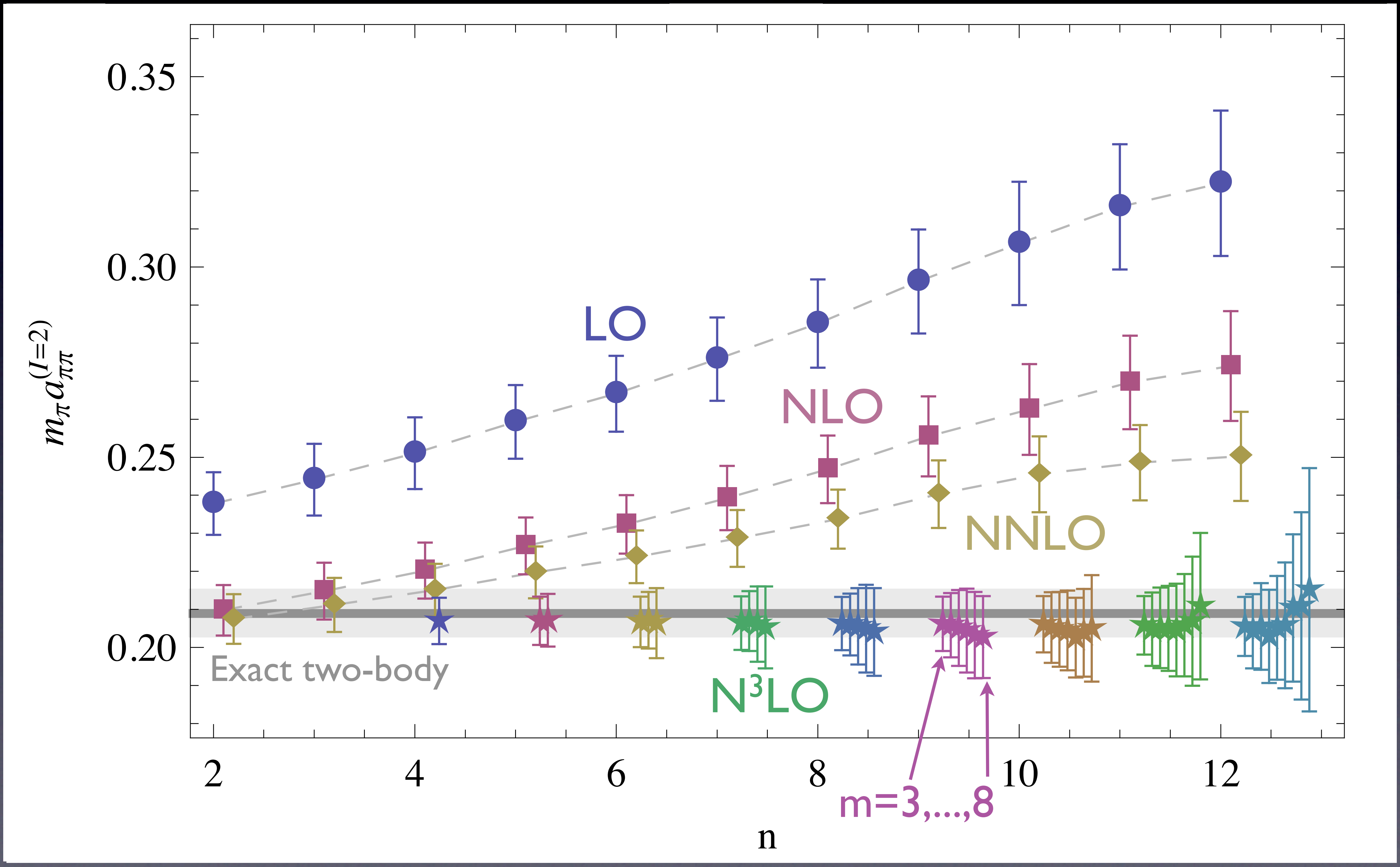

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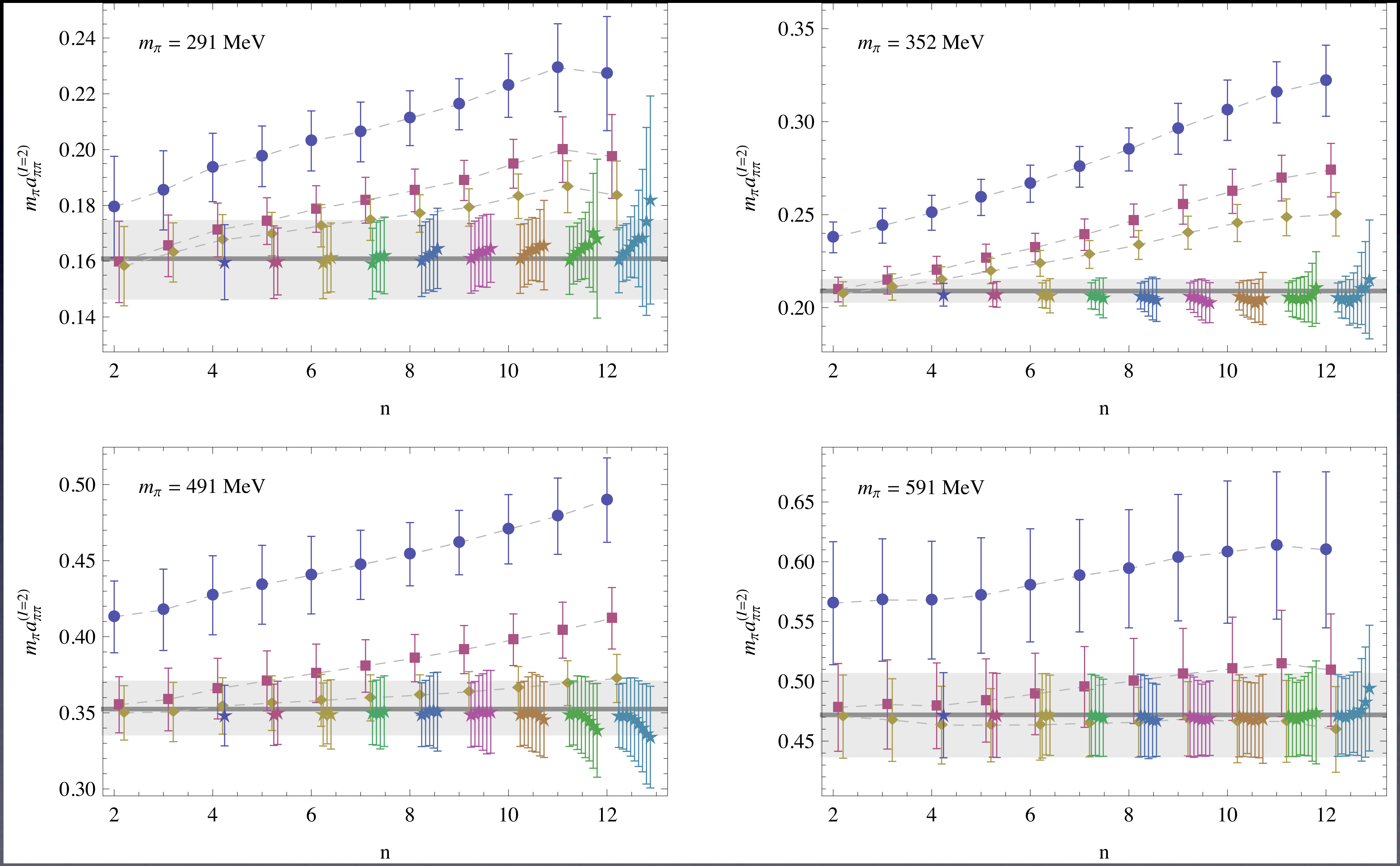

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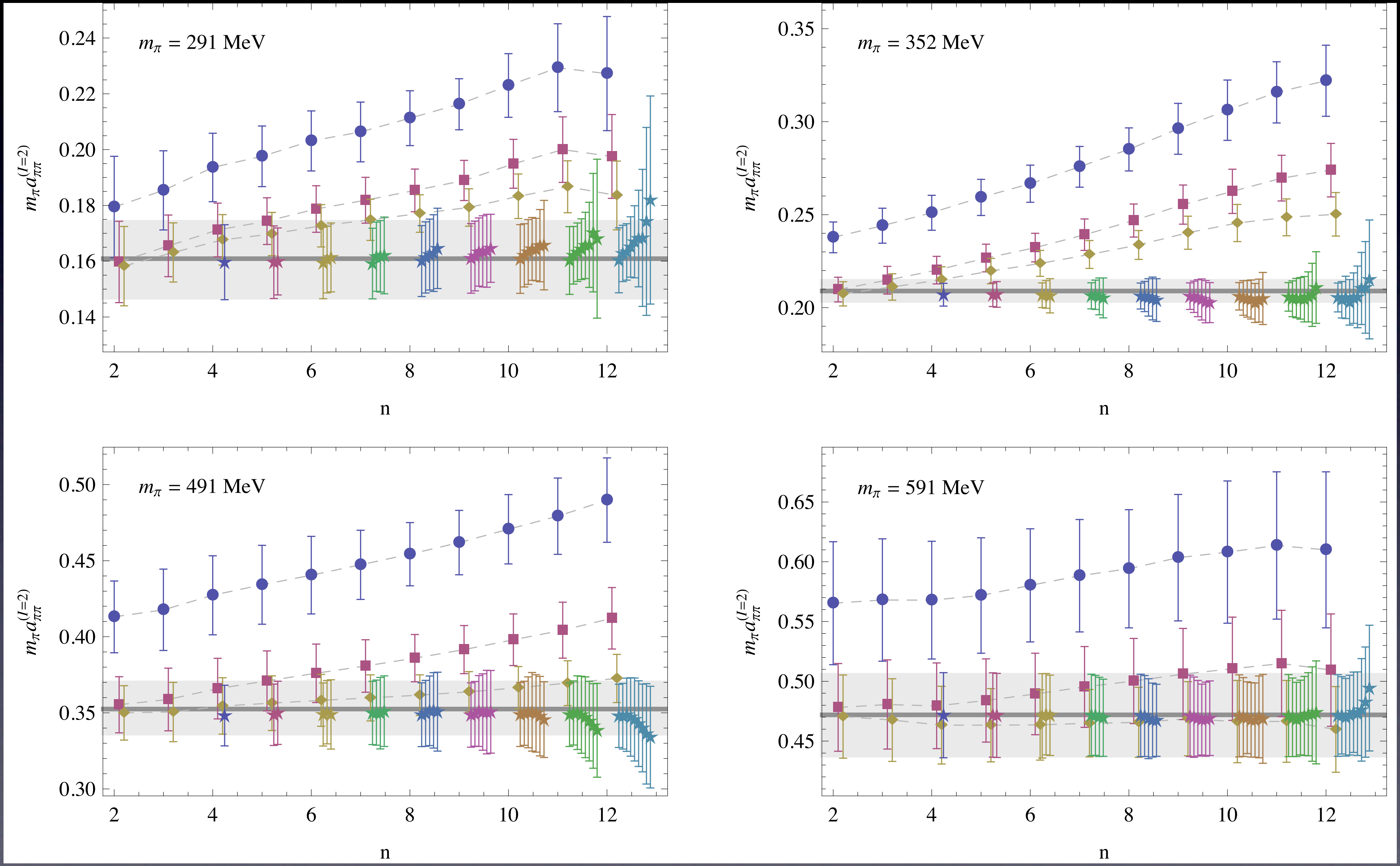


Expansion shows no sign of breakdown?

Scattering lengths

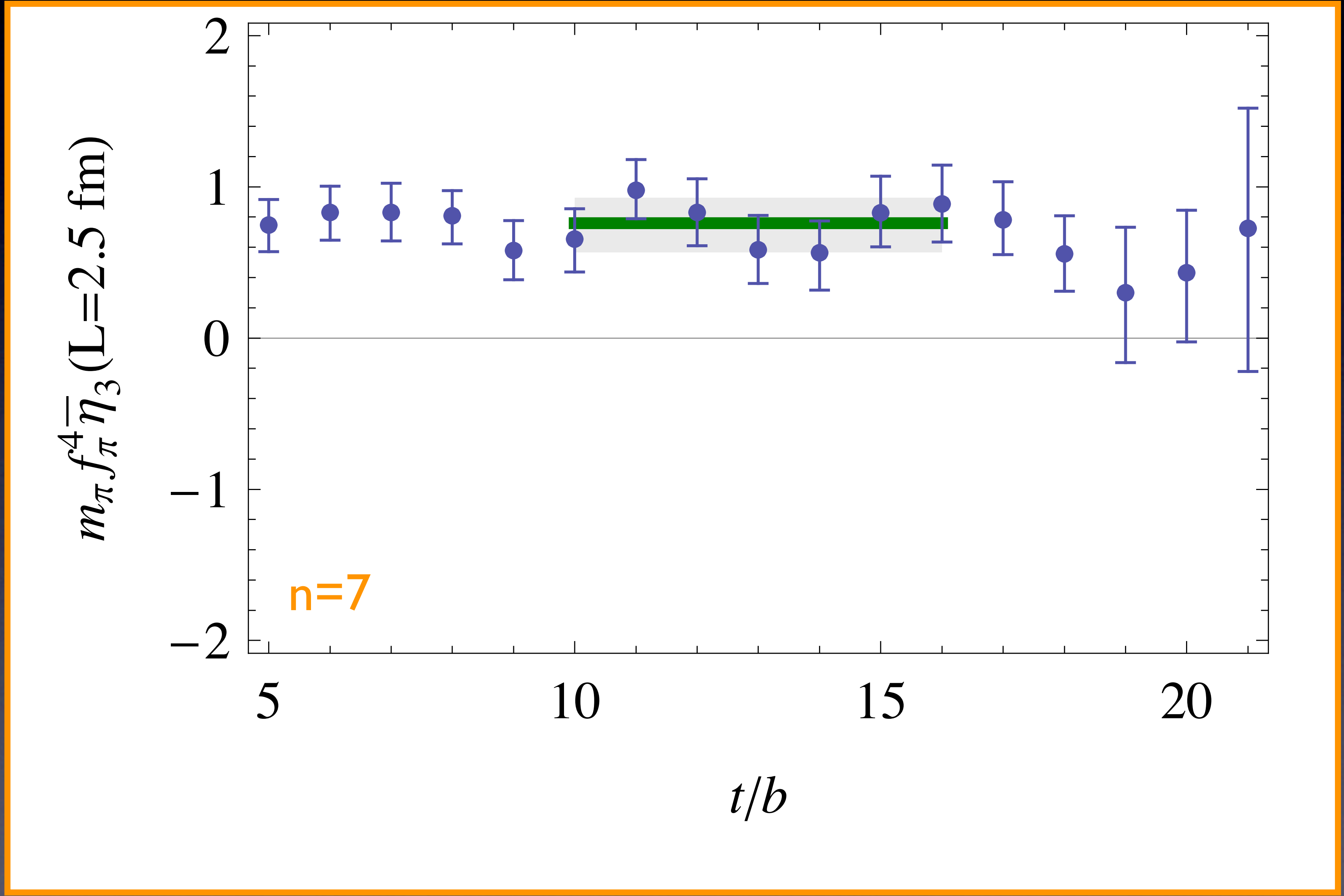
- Scattering lengths equally well extracted for two mesons or ten mesons
- Well described by analytic prediction
- Shows presence of contribution that scales as $\binom{n}{3}$
  - varies by two-orders of magnitude

Three meson interactions

- At $1/L^6$, point-like three-boson interaction must occur [Braaten, Nieto '95]
  - RGI 3BI: $\overline{\eta}_3^{(L)}$ physically meaningful
  - Depends logarithmically on L

- Naive dimensional-analysis $m_\pi f_\pi^4 \, \overline{\eta}_3^{(L)} \sim 1$
- Combinations of energy shifts isolates the RGI interaction

$\pi^+\pi^+\pi^+$ interaction



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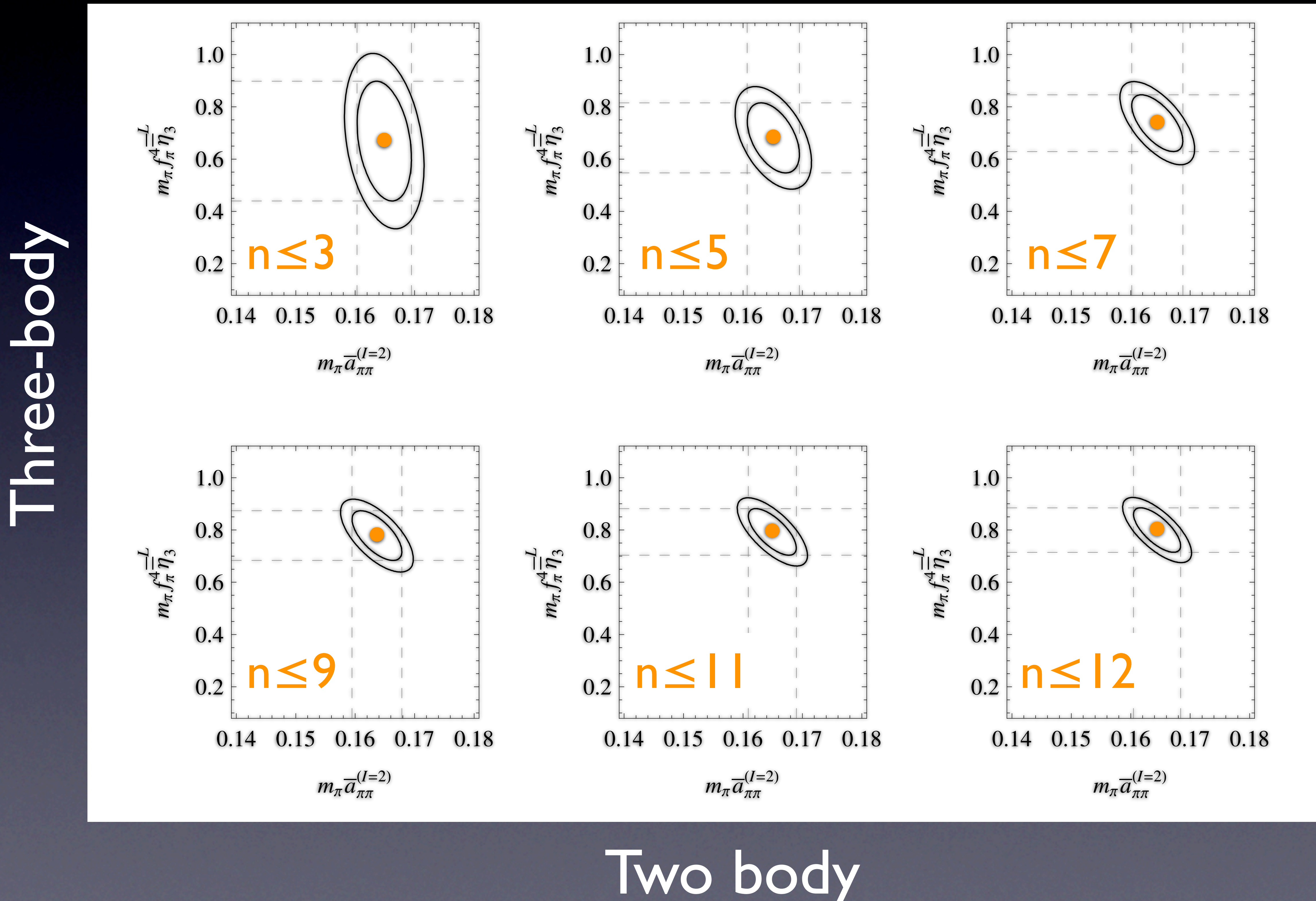
n correlations

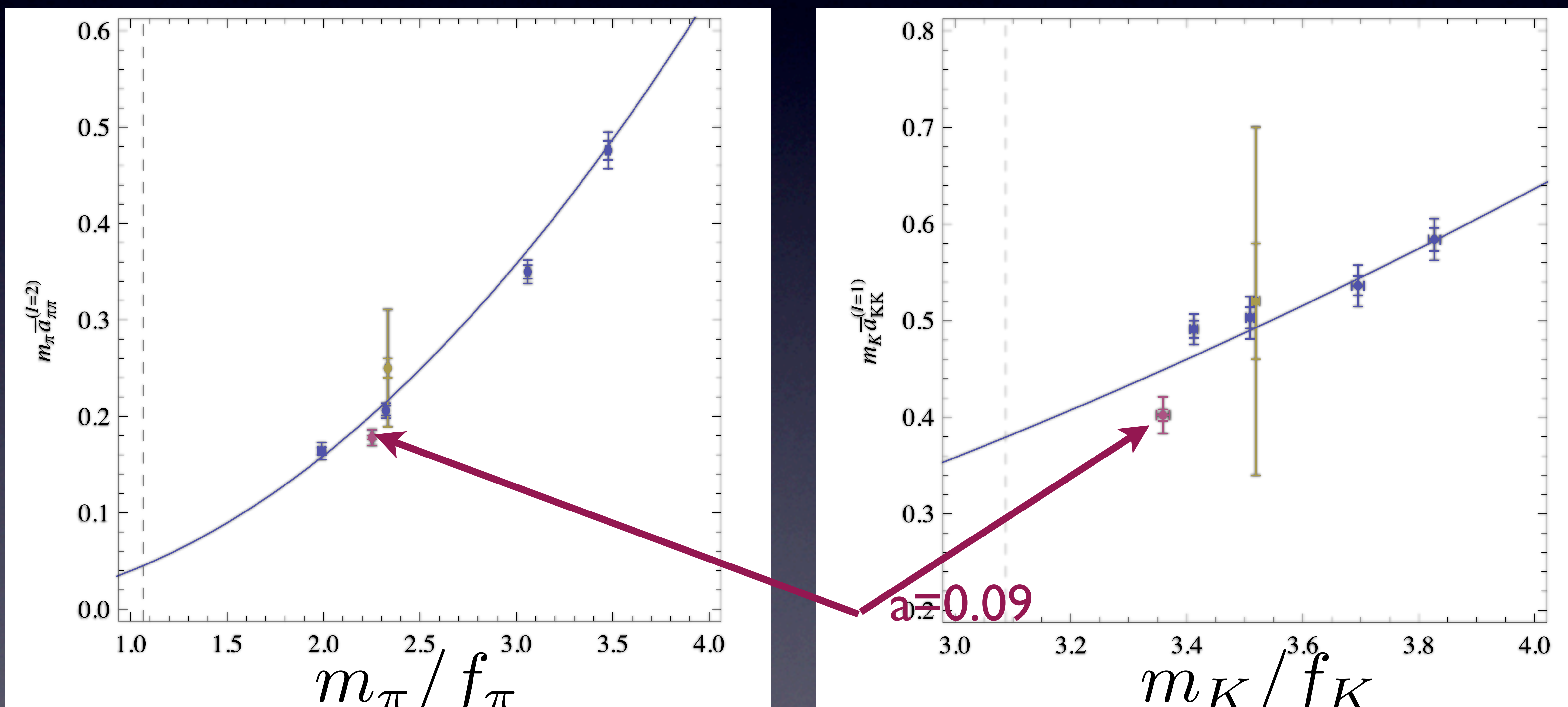
- Fit effective energies to extract two parameters: a, η_3 from 1/L expansion
- Use 12 eff. energies in *n*-*t*-correlated analysis
  - Large correlation matrix: correlated χ^2
  - Reduces uncertainties as *n* pion correlators "explore more of the lattice"

n correlations

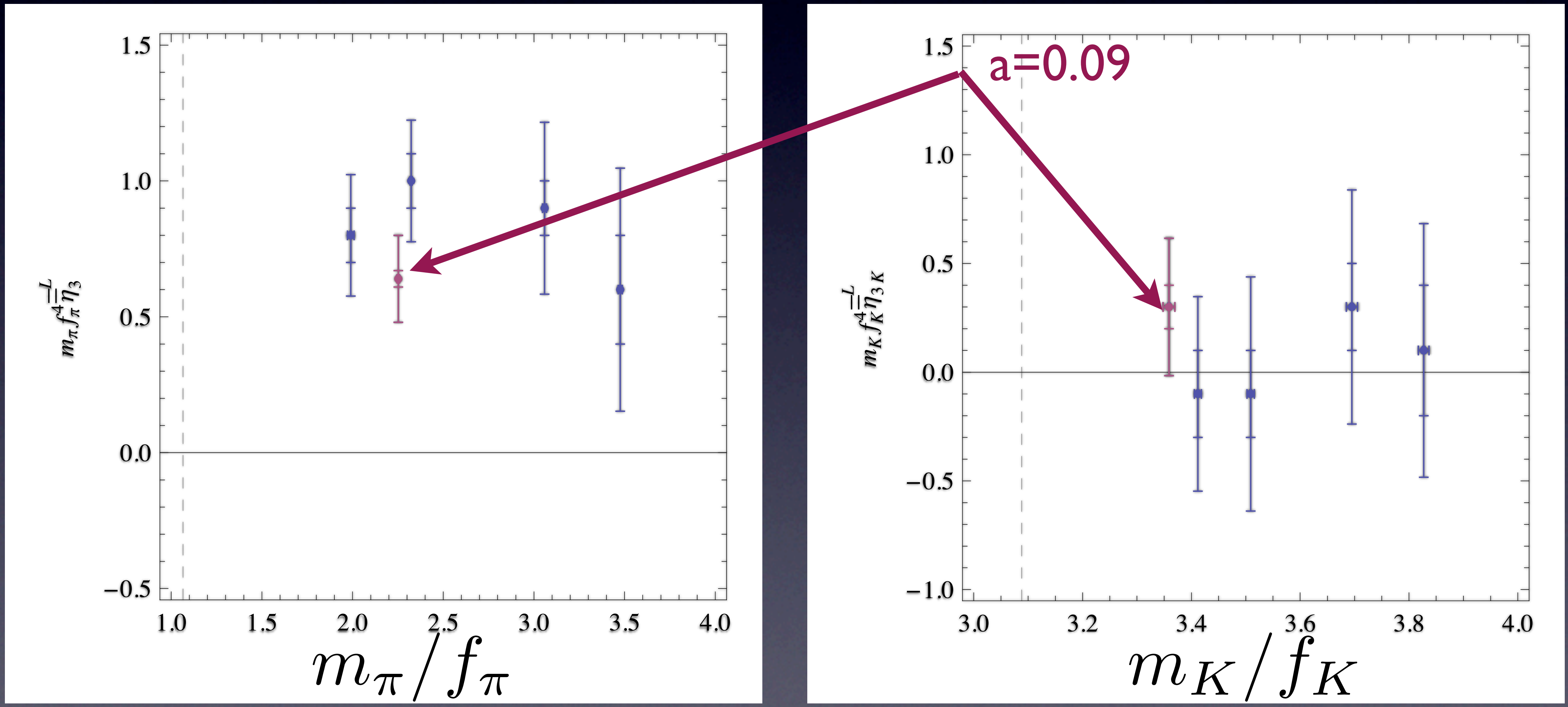


$2\pi^+$ and $2K^-$ interaction

curves: Weinberg



$3\pi^+$ and $3K^-$ interaction



Naïve dimension analysis: 1

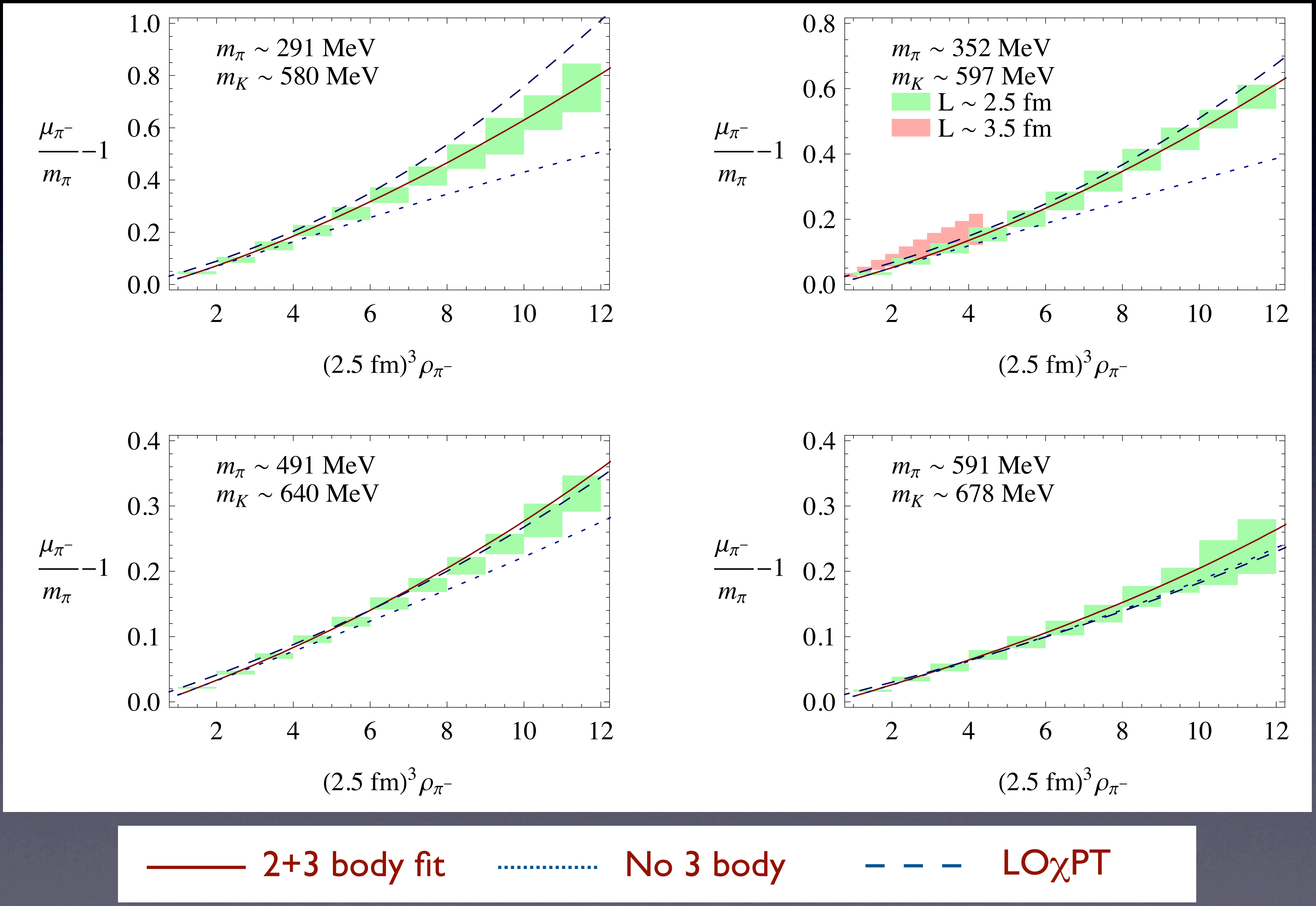
Equation of State

- 1/L expansion: analytic form of EOS
- Chemical potential

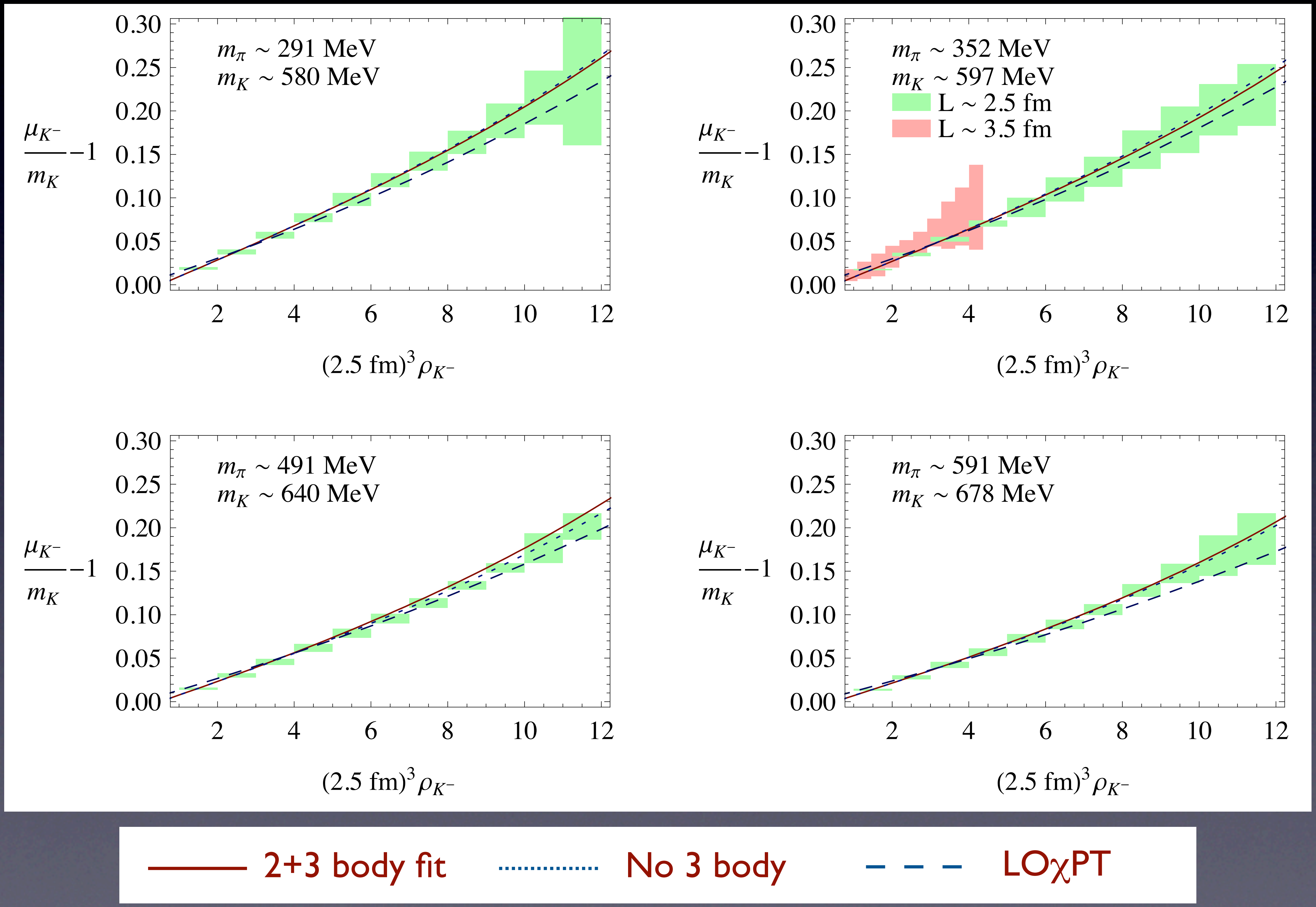
$$\mu = \left. \frac{d\ E}{d\ n} \right|_{V\,const}$$

- $\mu(\rho)$ numerically using finite difference
- Compare with LOχPT [Son & Stephanov]

Isospin Chemical Potential



Kaon Chemical Potential

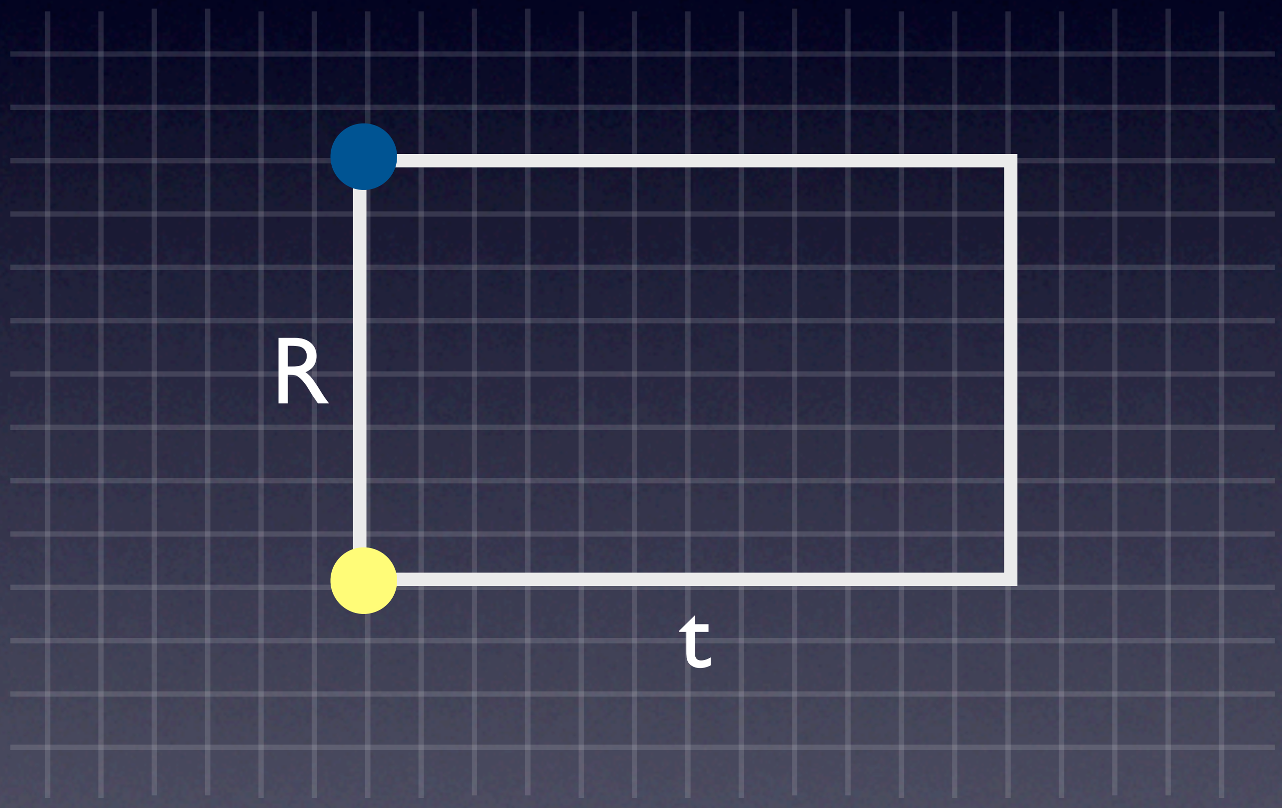


Chemical potentials

- π^- and K^- chemical potentials
  - Important contribution from $\pi\ \pi\ \pi$
  - Results consistent with LOχPT
    - supports Kaplan/Nelson analysis for n-stars
- Other μ_I results [de Forcrand et al, Kogut&Sinclair]

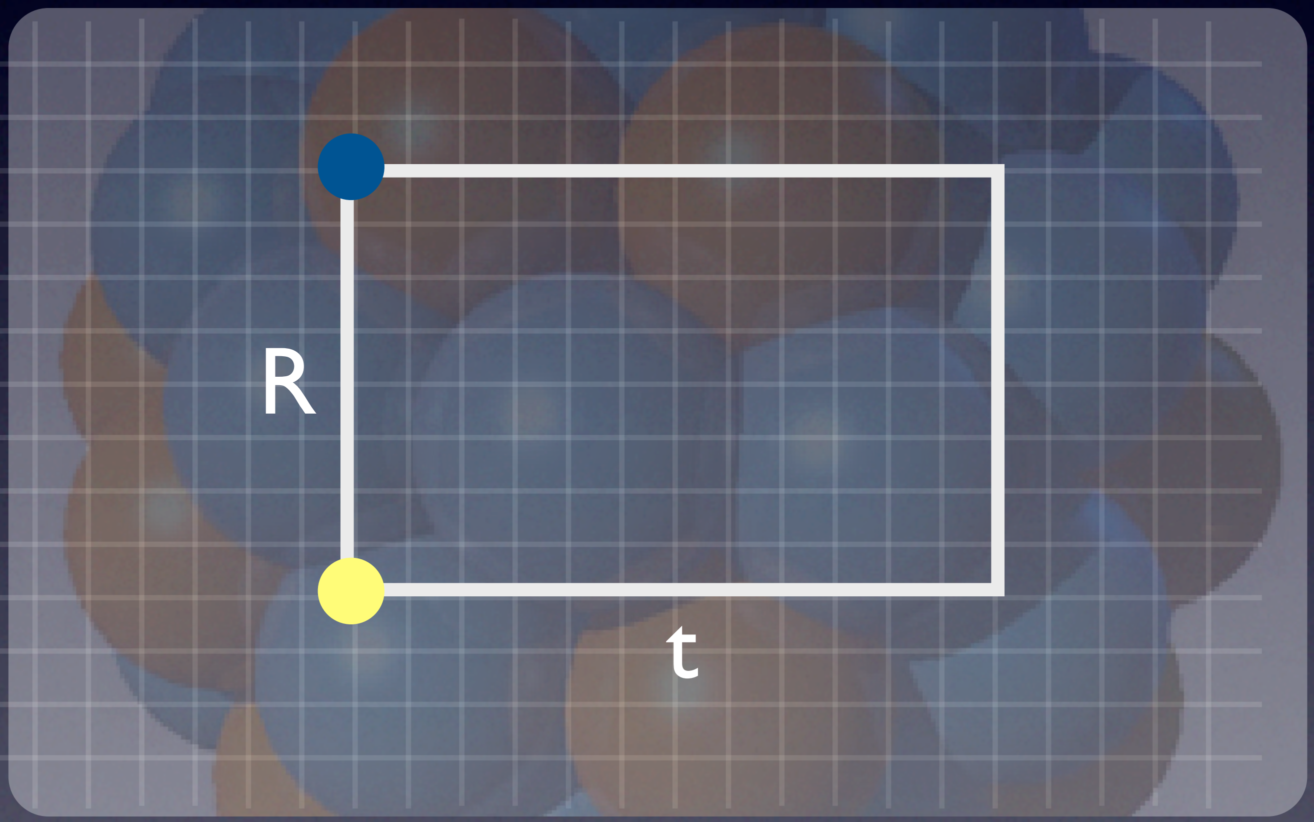
In medium $Q\bar{Q}$ potential

- Static quark potential



In medium $Q\bar{Q}$ potential

- Static quark potential



- Modified by condensate? Hadronic screening?

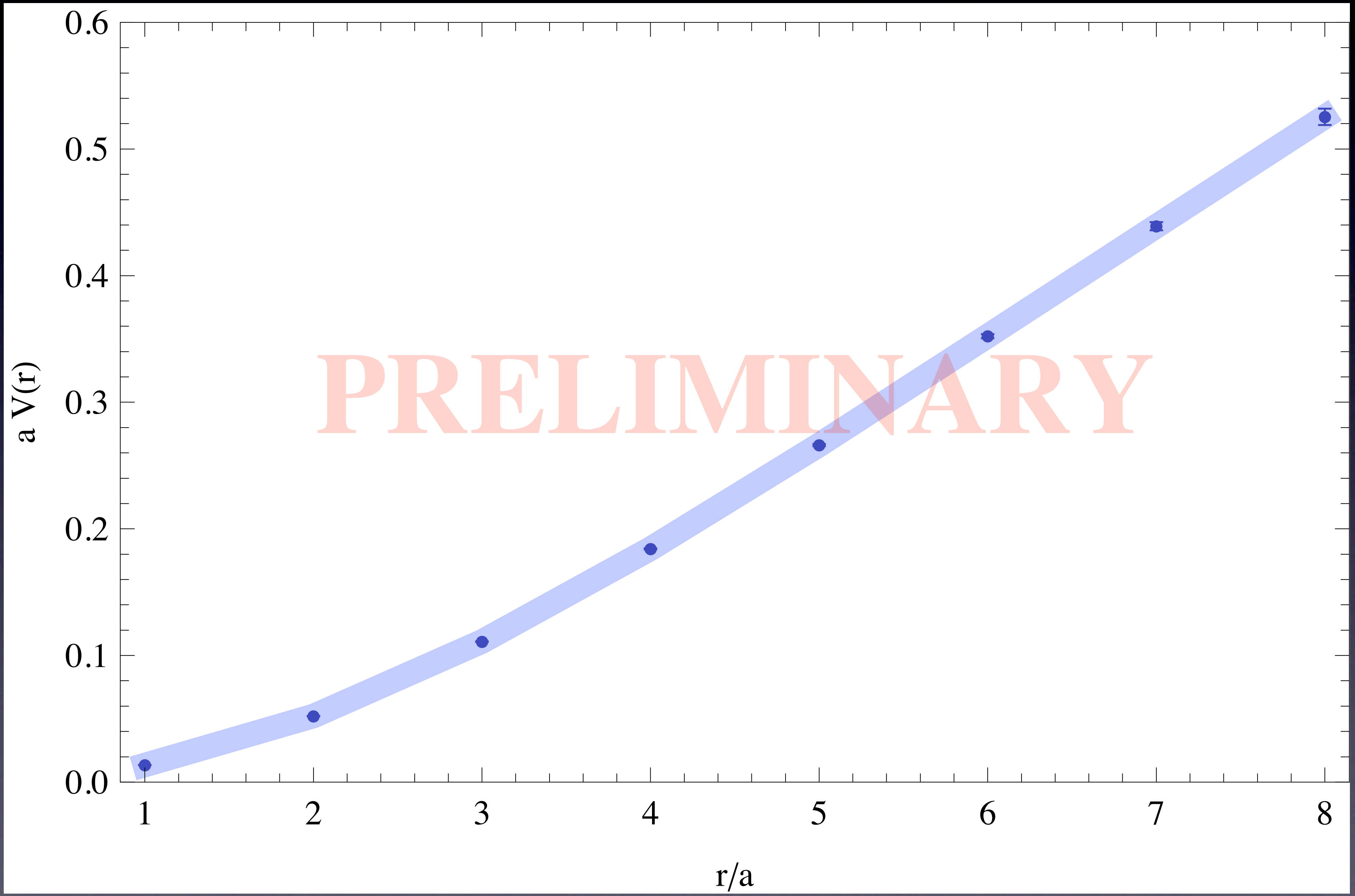
In medium $Q\bar{Q}$ potential

$$C_V(r,t) = \langle 0|W[R,t_0,t]|0\rangle \longrightarrow Z\ \exp\left[-V(r)(t-t_0)\right]$$

$$C_V(r,t;n) = \left\langle 0 \middle| W[R,t_0,t] \left[\sum_{\vec{x}} \overline{d}\gamma_5 u(\vec{x},t)\overline{u}\gamma_5 d(\vec{0},0)\right]^n \middle| 0 \right\rangle$$

$$R(r,t;n) = \frac{C_V(r,t;n)}{C_V(r,t)C_n(t)} \longrightarrow \#\ \exp\left[-\delta V(r,n)(t-t_0)\right]$$

Vacuum Static Potential



- Gauge field smeared for optimal signal at large r

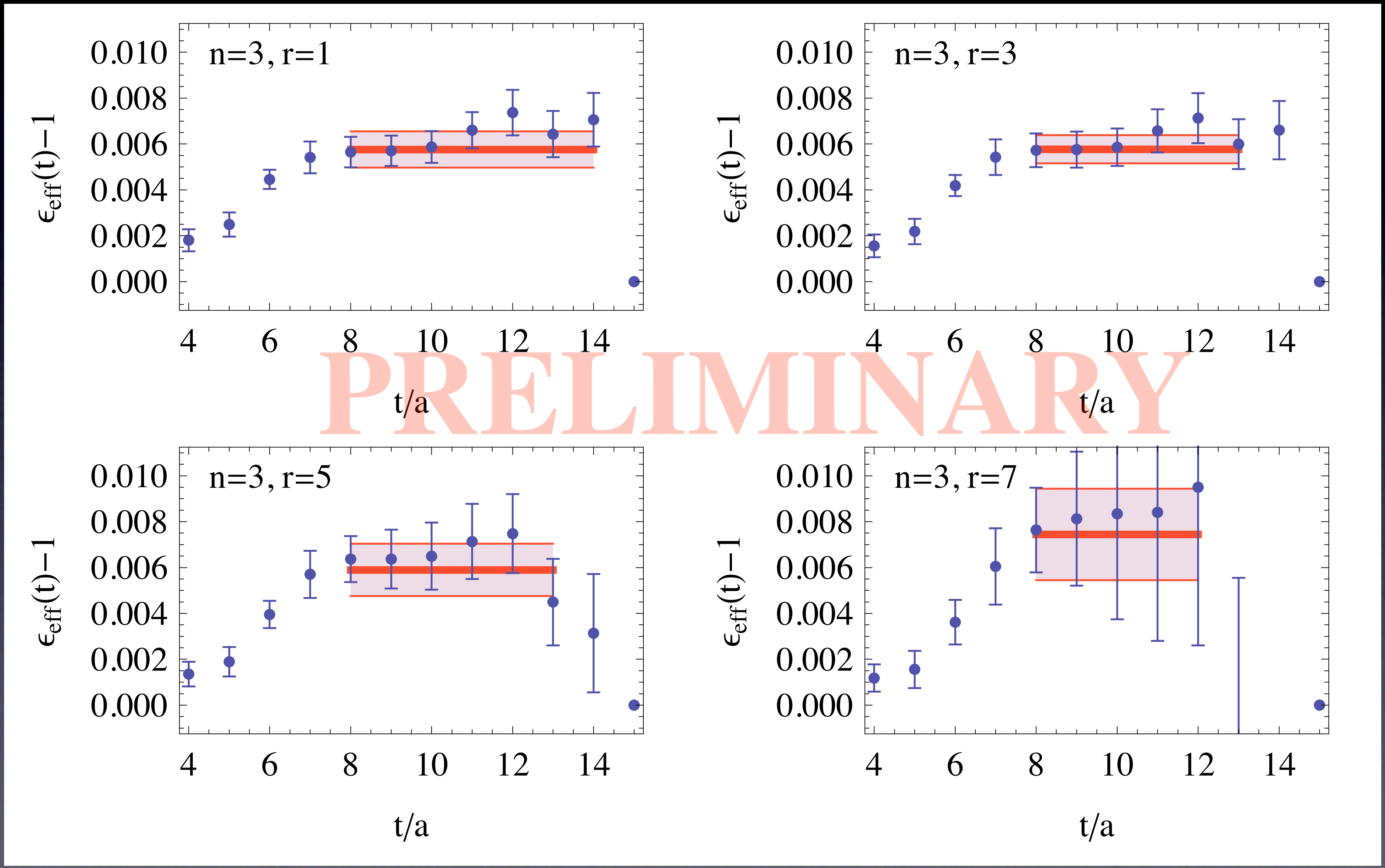
Dielectric "Constant"

- Effects of medium shown by

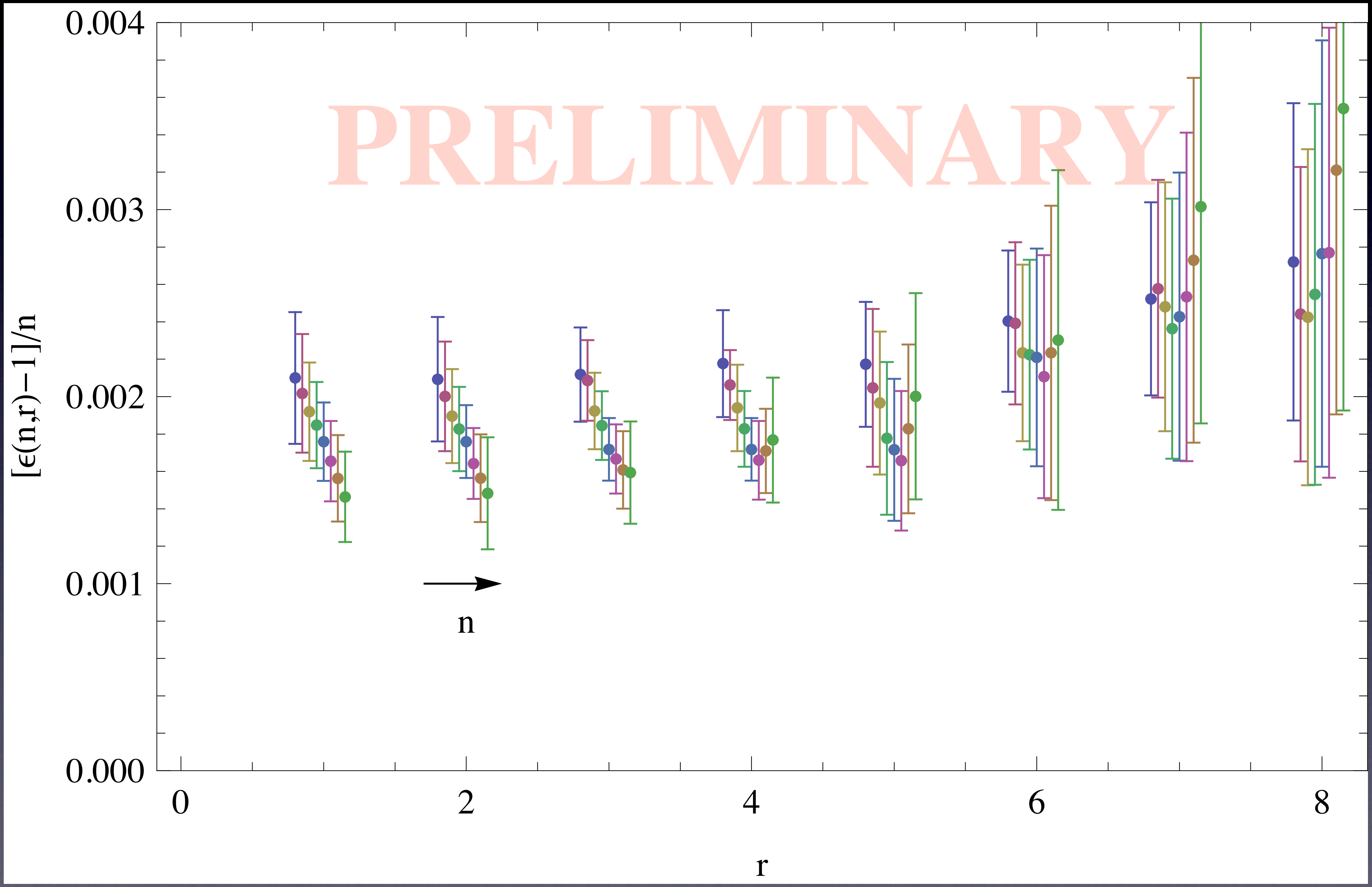
$$\epsilon(r,n) \equiv \left[1 + \frac{\delta V(r,n)}{V(r)}\right]^{-1}$$

- Would be constant for a dielectric medium

Dielectric "Constant"

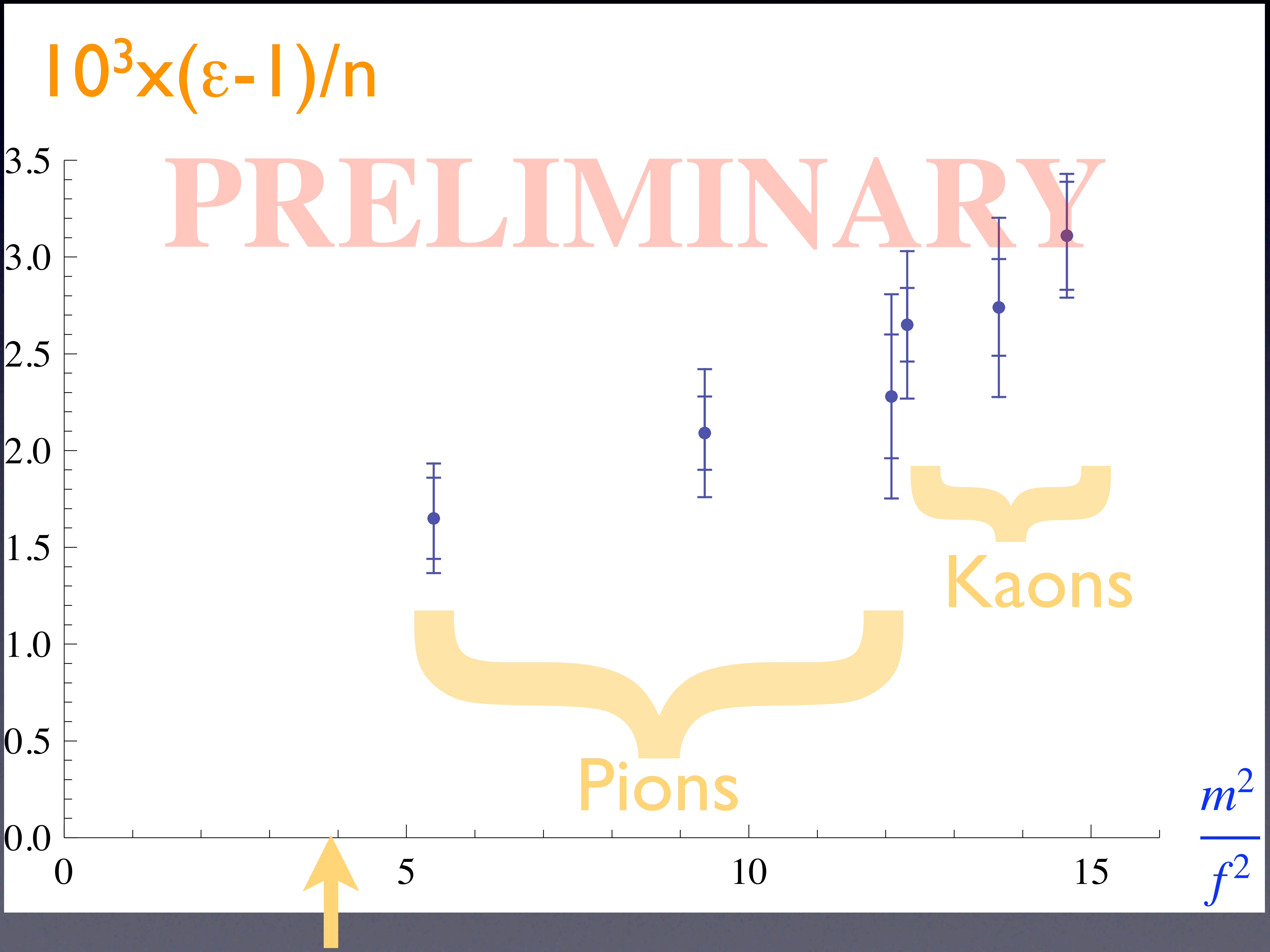


Dielectric "Constant"



$\varepsilon(n,r) \sim 1 + A\, n$??

Dielectric "Constant"



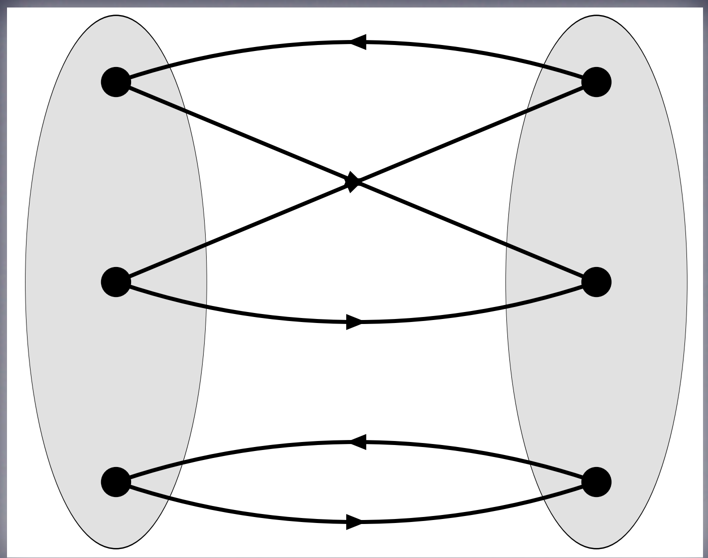
Dielectric "Constant"

- Work in progress
- Well described as a dielectric medium
  - Small vs large R
  - *Effects of gauge field smearing TBD*
- $J/\psi-\pi$ scattering
- Relevance to experiment?

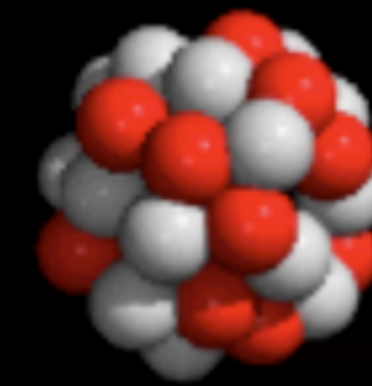
Summary

- Multi-hadron systems
  - Two and three meson interactions
  - Equation of state of meson gases
  - Hadronic screening of static quark potential
- More to come...

Contractions and Interpolators



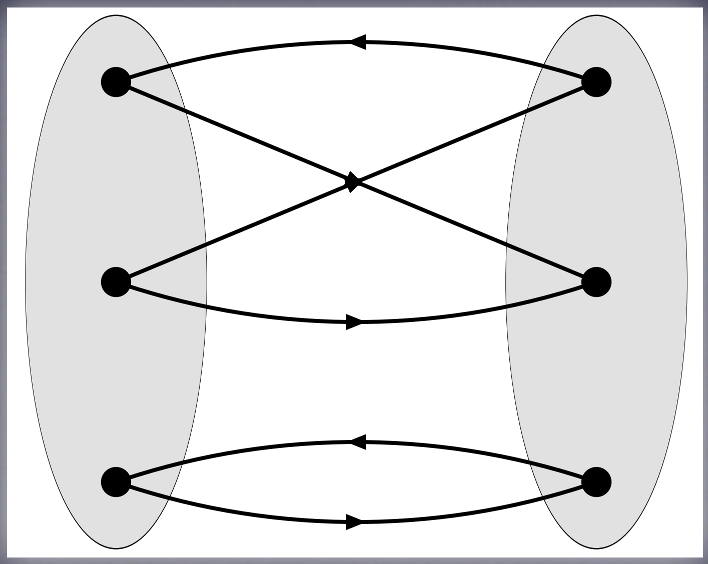
Contractions and Interpolators



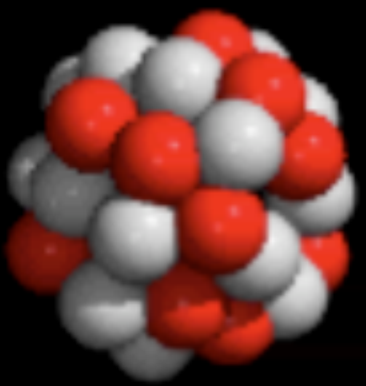
- Number of contractions for $^{A}Xy_{Z}$:

  $$(A+Z)!(2A-Z)!$$

  - Pb: 10^{1289}
  - evaluation with fewer operations
- Need many propagators
- What interpolating operators?



Contractions and Interpolators



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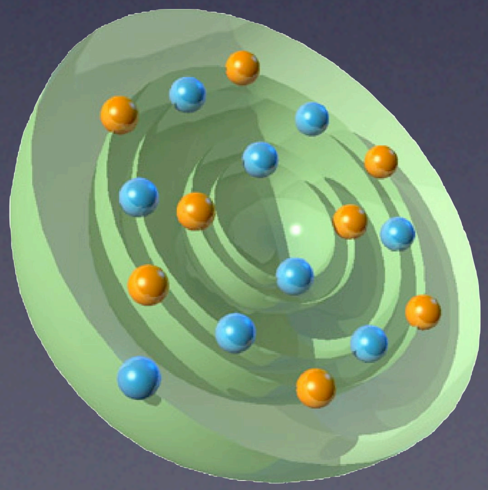
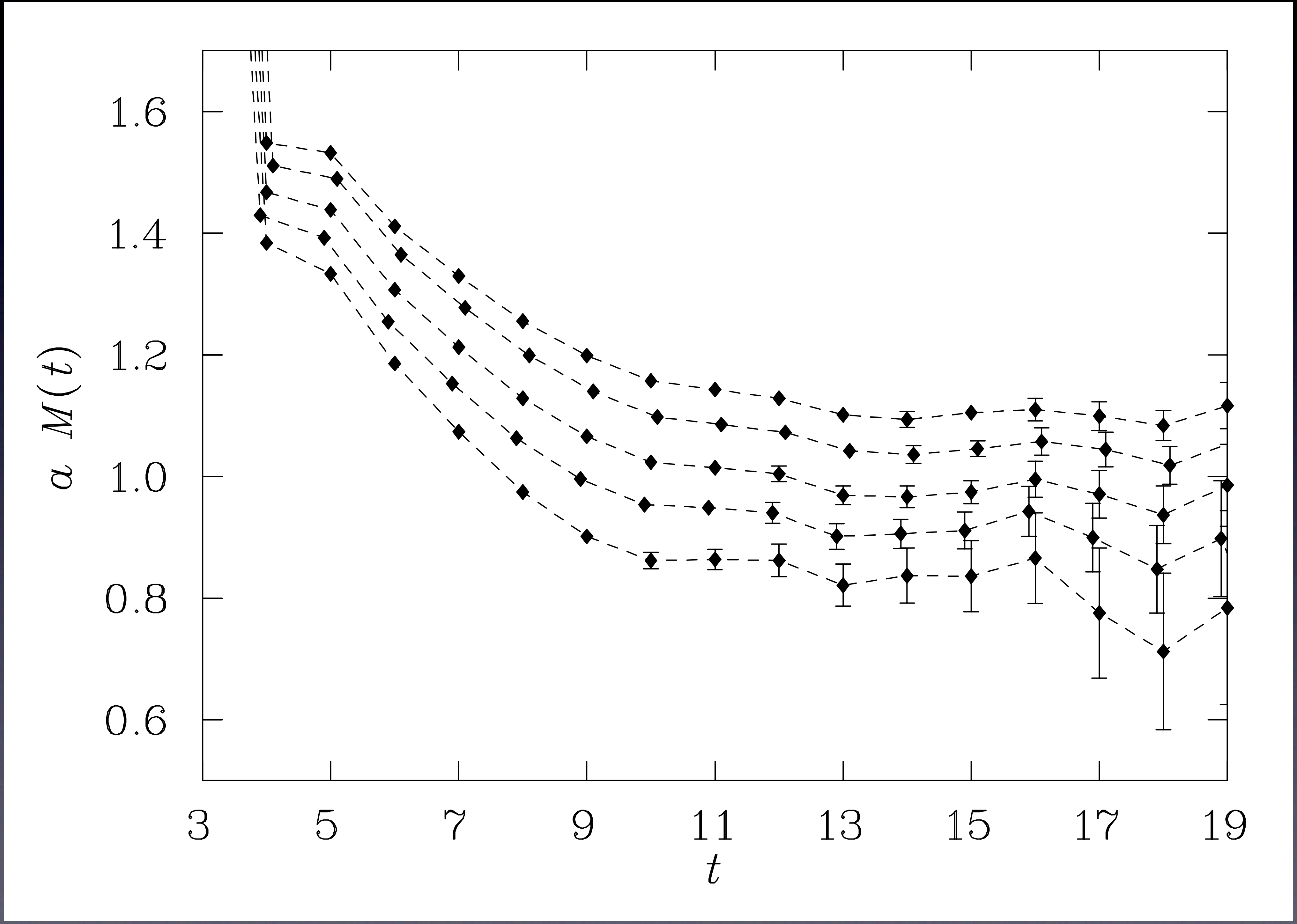
$$(A+Z)!(2A-Z)!$$

  - Pb: 10^{1289}
  - evaluation with fewer operations
- Need many propagators
- What interpolating operators?
  - The shell model+!

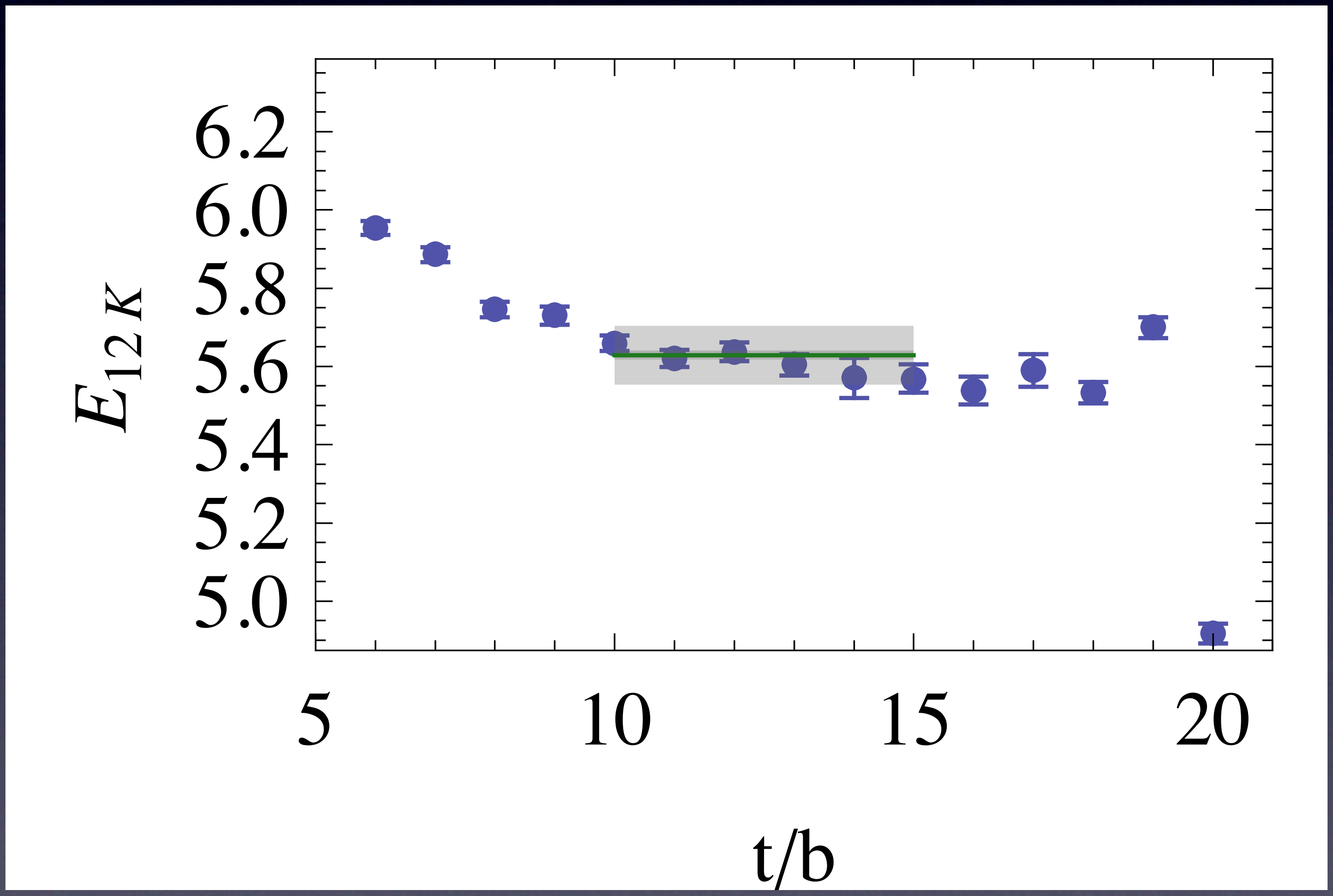


An example



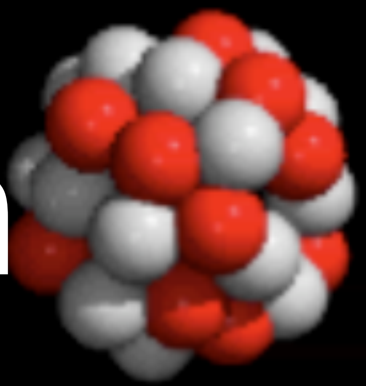
Massive states

- Nuclear masses >> typical lattice scales

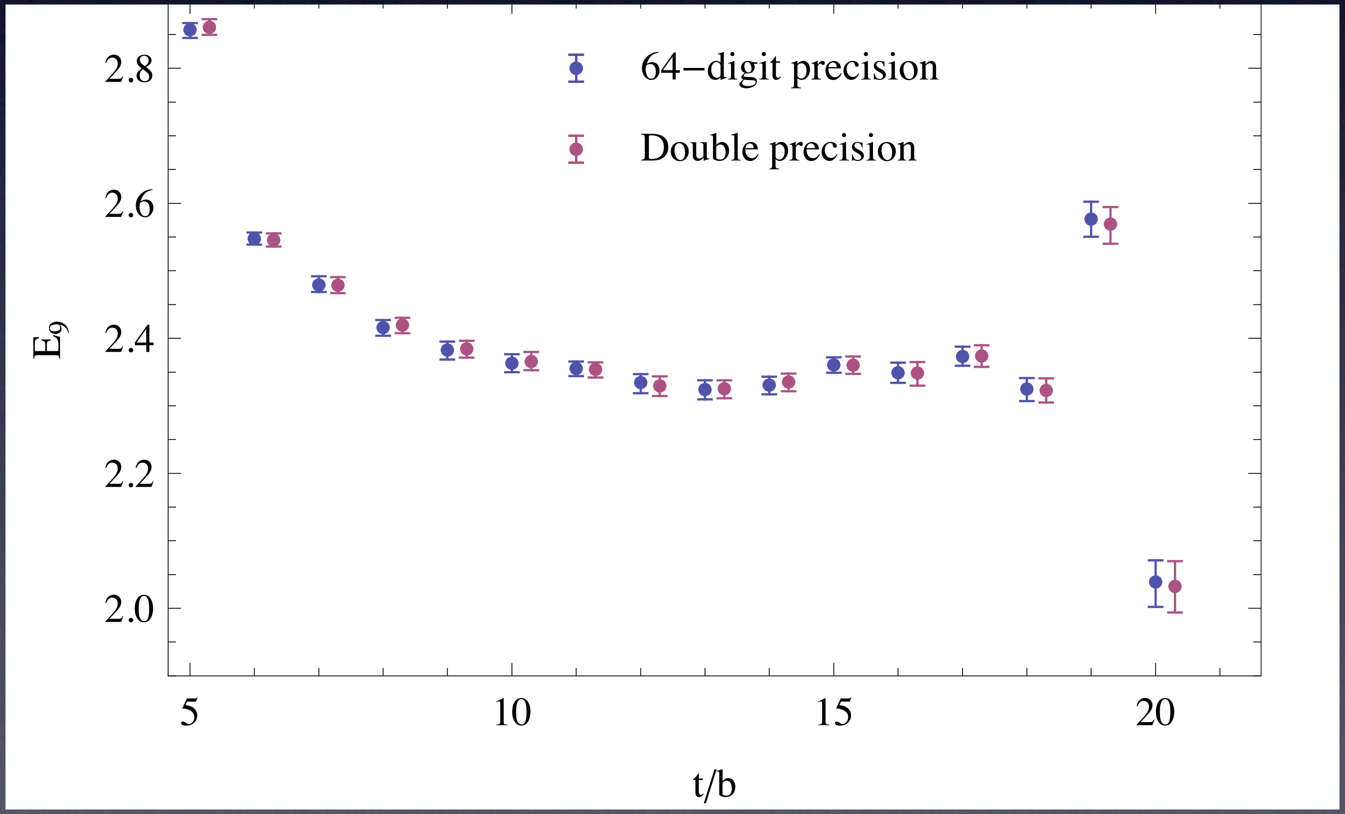


- Twelve kaon energy: 5.6/a ~ 8.8 GeV

Numerical precision

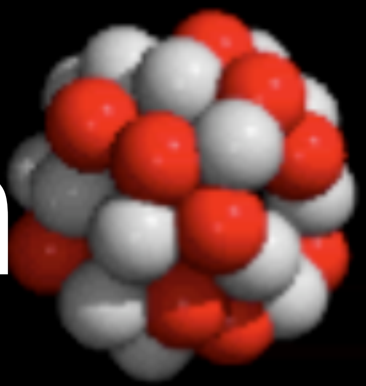


- Double precision is not enough: not gauge invariant
- Need very high precision contraction code



- Propagator precision??

Numerical precision



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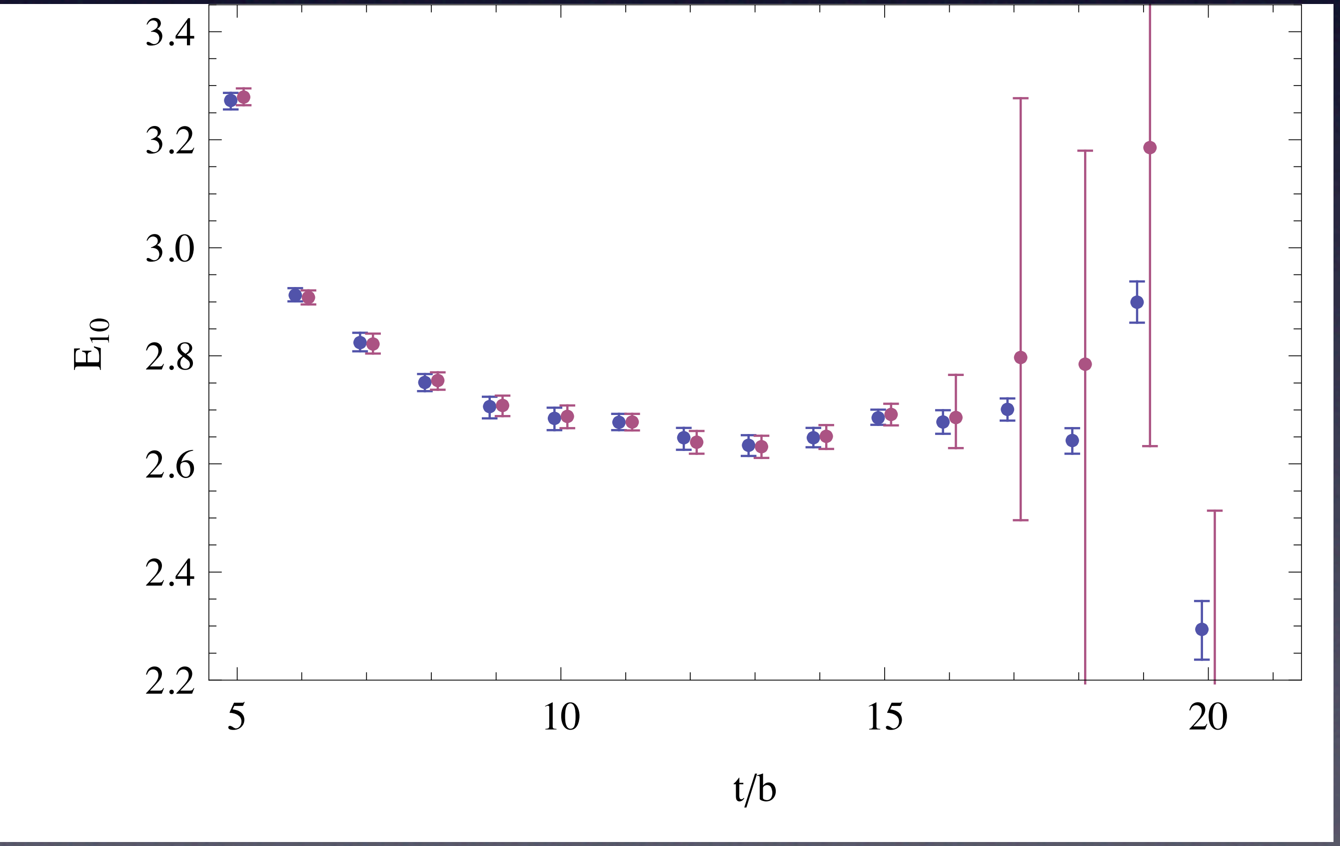

- Propagator precision??

Numerical precision

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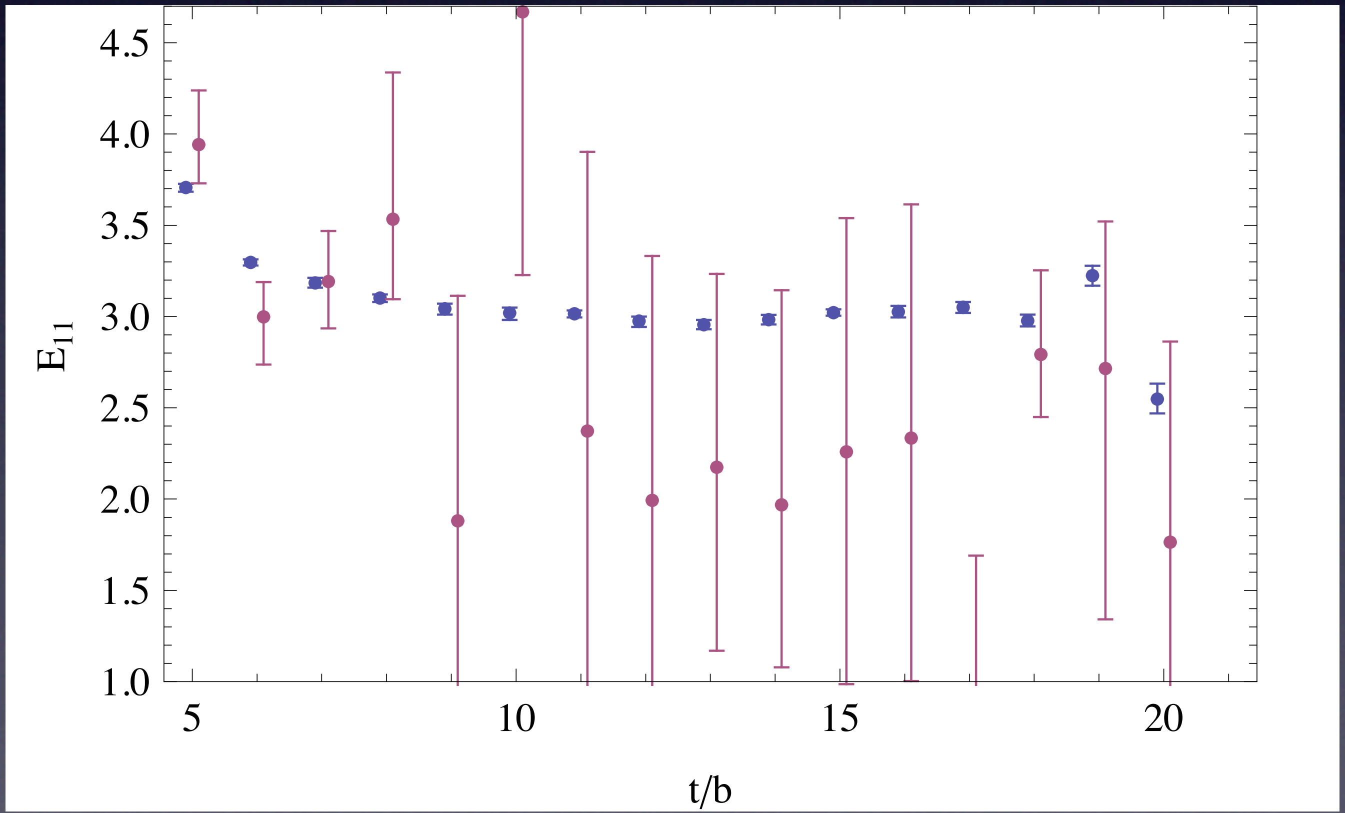


- Propagator precision??

n=13 & matrix identites

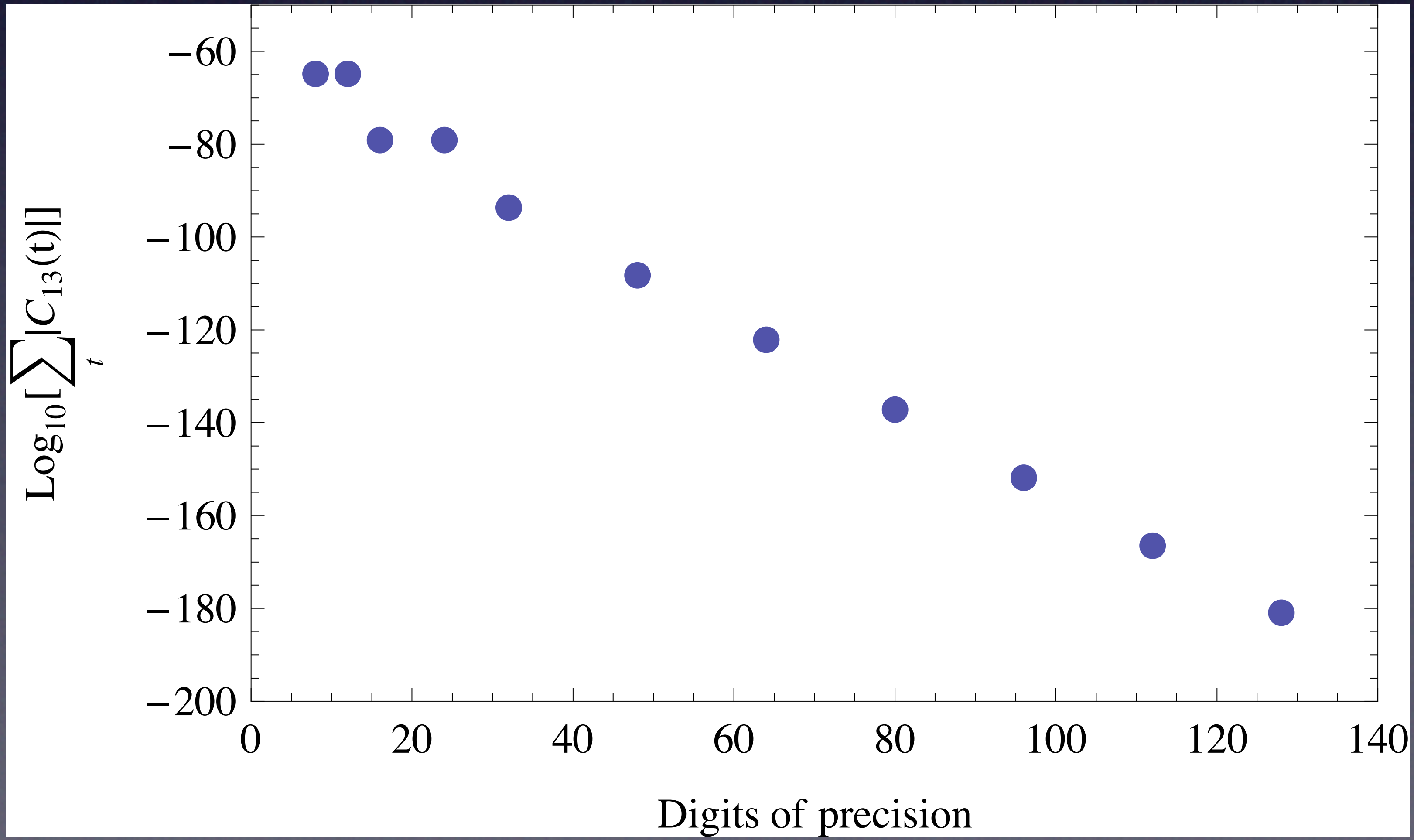
- Pauli principle: *$C_{13}(t)$* correlator vanishes
- Matrix Identity: vanishes ∀ 12x12 matrices

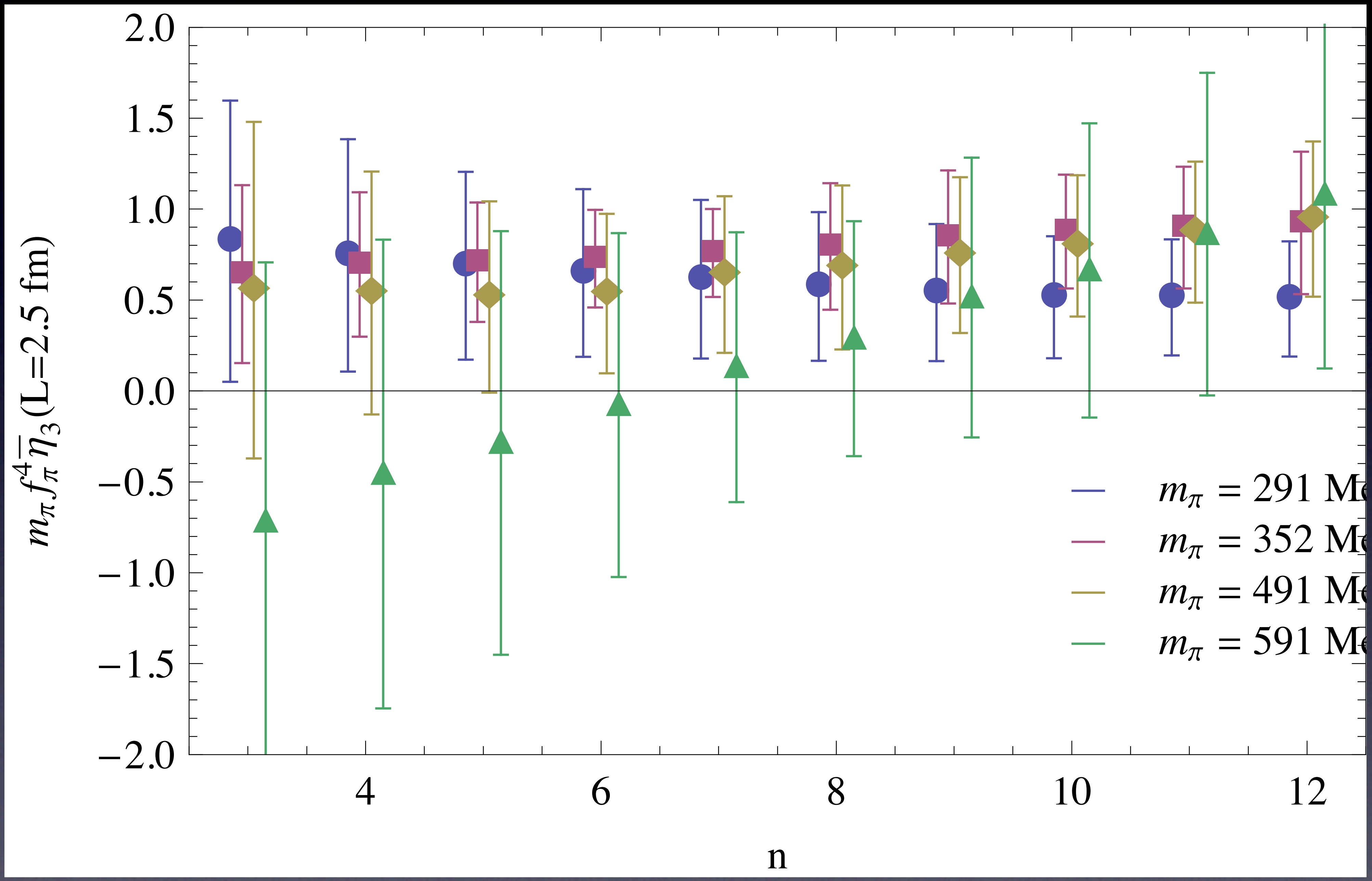
$$
\begin{aligned}
C_{13}(t) = {} & T_1^{13} - 78T_2T_1^{11} + 572T_3T_1^{10} + 2145T_2^2T_1^9 - 4290T_4T_1^9 - 25740T_2T_3T_1^8 + 30888T_5T_1^8 \\
& -25740T_2^3T_1^7 + 68640T_3^2T_1^7 + 154440T_2T_4T_1^7 - 205920T_6T_1^7 + 360360T_2^2T_3T_1^6 \\
& -720720T_3T_4T_1^6 - 864864T_2T_5T_1^6 + 1235520T_7T_1^6 + 135135T_2^4T_1^5 - 1441440T_2T_3^2T_1^5 \\
& +1621620T_4^2T_1^5 - 1621620T_2^2T_4T_1^5 + 3459456T_3T_5T_1^5 + 4324320T_2T_6T_1^5 - 6486480T_8T_1^5 \\
& +1601600T_3^3T_1^4 - 1801800T_2^3T_3T_1^4 + 10810800T_2T_3T_4T_1^4 + 6486480T_2^2T_5T_1^4 - 12972960T_4T_5T_1^4 \\
& -14414400T_3T_6T_1^4 - 18532800T_2T_7T_1^4 + 28828800T_9T_1^4 - 270270T_2^5T_1^3 + 7207200T_2^2T_3^2T_1^3 \\
& -16216200T_2T_4^2T_1^3 + 20756736T_5^2T_1^3 + 5405400T_2^3T_4T_1^3 - 14414400T_3^2T_4T_1^3 - 34594560T_2T_3T_5T_1^3 \\
& -21621600T_2^2T_6T_1^3 + 43243200T_4T_6T_1^3 + 49420800T_3T_7T_1^3 + 64864800T_2T_8T_1^3 - 103783680T_{10}T_1^3 \\
& -9609600T_2T_3^3T_1^2 + 32432400T_3T_4^2T_1^2 + 2702700T_2^4T_3T_1^2 - 32432400T_2^2T_3T_4T_1^2 \\
& -12972960T_2^3T_5T_1^2 + 34594560T_3^2T_5T_1^2 + 77837760T_2T_4T_5T_1^2 + 86486400T_2T_3T_6T_1^2 \\
& -103783680T_5T_6T_1^2 + 55598400T_2^2T_7T_1^2 - 111196800T_4T_7T_1^2 - 129729600T_3T_8T_1^2 \\
& -172972800T_2T_9T_1^2 + 283046400T_{11}T_1^2 + 135135T_2^6T_1 + 3203200T_3^4T_1 - 16216200T_4^3T_1 \\
& -7207200T_2^3T_3^2T_1 + 24324300T_2^2T_4^2T_1 - 62270208T_2T_5^2T_1 + 86486400T_6^2T_1
\end{aligned}
$$

$$\boxed{T_i^j = \mathrm{tr}\left[X^i\right]^j}$$

n=13 & matrix identites

- Pauli principle: $C_{13}(t)$ correlator vanishes
- Matrix Identity: vanishes ∀ 12x12 matrices
- Numerically





$m_\pi f_\pi^4 \overline{\eta}_3$(L=2.5 fm)
n
m_π = 291 Me
m_π = 352 Me
m_π = 491 Me
m_π = 591 Me

Analytic details

- Hamiltonian formulation = pionless EFT

$$
\begin{aligned}
H \;=\; & \sum_{\mathbf{k}} h^{\dagger}_{\mathbf{k}}\, h_{\mathbf{k}} \left(\frac{|\mathbf{k}|^2}{2M} - \frac{|\mathbf{k}|^4}{8M^3} \right) \\
& + \frac{1}{(2!)^2} \sum_{\mathbf{Q},\mathbf{k},\mathbf{p}} h^{\dagger}_{\frac{\mathbf{Q}}{2}+\mathbf{k}} h^{\dagger}_{\frac{\mathbf{Q}}{2}-\mathbf{k}}\, h_{\frac{\mathbf{Q}}{2}+\mathbf{p}}\, h_{\frac{\mathbf{Q}}{2}-\mathbf{p}} \left(\frac{4\pi a}{M} + \frac{\pi a}{M}\left(ar - \frac{1}{2M^2}\right) \left(|\mathbf{k}|^2 + |\mathbf{p}|^2 \right) \right) \\
& + \frac{\eta_3(\mu)}{(3!)^2} \sum_{\mathbf{Q},\mathbf{k},\mathbf{p},\mathbf{r},\mathbf{s}} h^{\dagger}_{\frac{\mathbf{Q}}{3}+\mathbf{k}} h^{\dagger}_{\frac{\mathbf{Q}}{3}+\mathbf{p}}\, h^{\dagger}_{\frac{\mathbf{Q}}{3}-\mathbf{k}-\mathbf{p}}\, h_{\frac{\mathbf{Q}}{3}+\mathbf{r}} h_{\frac{\mathbf{Q}}{3}+\mathbf{s}}\, h_{\frac{\mathbf{Q}}{3}-\mathbf{r}-\mathbf{s}} \,,
\end{aligned}
$$

- Calculate to 5th order
- Three loop diagrams: 9d integer sums