STOCHASTIC QUANTIZATION AT FINITE CHEMICAL POTENTIAL

Gert Aarts

with Nucu Stamatescu

arXiv:0807.1597 [hep-lat]

Swansea University

Swansea University
Prifysgol Abertawe

first collaboration meeting below sea level

INTRODUCTION

QCD AT NONZERO BARYON DENSITY

QCD at finite μ: complex fermion determinant

$$\det M(\mu) = [\det M(-\mu)]^*$$

partition function: $$Z = \int DU \, e^{-S_B(U)} \det M$$

importance sampling not possible

- reweighting
- Taylor expansion
- analytical continuation
- density of states
- canonical ensemble
- ...

here: $\boxed{\text{stochastic quantization}}$

STOCHASTIC QUANTIZATION

LANGEVIN DYNAMICS

- alternative nonperturbative numerical approach
- weight $=$ equilibrium distribution of stochastic process

think: $\boxed{\textit{Brownian motion}}$

particle in a fluid: friction (γ) and kicks (η)
Langevin equation:

$$\frac{d}{dt}\vec{v}(t) = -\gamma\vec{v}(t) + \vec{\eta}(t) \qquad \langle \eta_i(t)\eta_j(t')\rangle = 2kT\gamma\delta_{ij}\delta(t-t')$$

equilibrium solution/noise average:

$$\lim_{t\to\infty} \frac{1}{2}\langle v_i(t)v_j(t)\rangle = \frac{1}{2}\delta_{ij}kT$$

STOCHASTIC QUANTIZATION

LANGEVIN DYNAMICS

apply to field theory (Parisi and Wu '81)

$$\frac{\partial \phi(x,\theta)}{\partial \theta} = -\frac{\delta S[\phi]}{\delta \phi(x,\theta)} + \eta(x,\theta)$$

Gaussian noise

$$\langle \eta(x,\theta) \rangle = 0 \qquad \langle \eta(x,\theta)\eta(x',\theta') \rangle = 2\delta(x-x')\delta(\theta-\theta')$$

corresponding Fokker-Planck equation

$$\frac{\partial P[\phi,\theta]}{\partial \theta} = \int d^d x \, \frac{\delta}{\delta \phi(x,\theta)} \left(\frac{\delta}{\delta \phi(x,\theta)} + \frac{\delta S[\phi]}{\delta \phi(x,\theta)} \right) P[\phi,\theta]$$

stationary solution: $\qquad P[\phi] \sim e^{-S}$

STOCHASTIC QUANTIZATION

LANGEVIN DYNAMICS

- real action: formal proofs of convergence
  (but can also use importance sampling)
- complex action: no formal proofs available
  (but other methods in serious trouble)

force $\delta S/\delta\phi$ complex: $\boxed{\text{complex Langevin dynamics}}$

example: real scalar field $\phi \to \mathrm{Re}\,\phi + i\mathrm{Im}\,\phi$

$$\frac{\partial \mathrm{Re}\,\phi}{\partial\theta} = -\mathrm{Re}\,\frac{\delta S}{\delta\phi} + \eta \qquad \frac{\partial \mathrm{Im}\,\phi}{\partial\theta} = -\mathrm{Im}\,\frac{\delta S}{\delta\phi}$$

observables: analytic extension

$$\langle O(\phi)\rangle \to \langle O(\mathrm{Re}\,\phi + i\mathrm{Im}\,\phi)\rangle$$

(PRE)HISTORY

- Parisi and Wu '81
- Damgaard and Hüffel, Physics Reports '87

application to finite μ:

effective three-dimensional spin models

- Karsch and Wyld '85
- Ilgenfritz '86
- Bilic, Gausterer, Sanielevici '88

Finite chemical potential

WHAT WE DID

three models of the form

$$Z = \int DU e^{-S_B} \det M \qquad\qquad \det M(\mu) = [\det M(-\mu)]^*$$

- QCD in hopping expansion
- SU(3) one link model
- U(1) one link model

observables:

- (conjugate) Polyakov loops
- density
- phase of determinant

THREE MODELS

I: QCD IN HOPPING EXPANSION

fermion matrix:

$$M = 1 - \kappa \sum_{i=1}^{3} \text{space} - \kappa \left(e^{\mu} \Gamma_{+4} U_{x,4} T_4 + e^{-\mu} \Gamma_{-4} U_{x,4}^{-1} T_{-4} \right)$$

hopping expansion:

$$\begin{aligned} \det M &\approx \det \left[1 - \kappa \left(e^{\mu} \Gamma_{+4} U_{x,4} T_4 + e^{-\mu} \Gamma_{-4} U_{x,4}^{-1} T_{-4} \right) \right] \\ &= \prod_{\mathbf{x}} \det \left(1 + h e^{\mu/T} \mathcal{P}_{\mathbf{x}} \right)^2 \det \left(1 + h e^{-\mu/T} \mathcal{P}_{\mathbf{x}}^{-1} \right)^2 \end{aligned}$$

with $h = (2\kappa)^{N_\tau}$ and the (conjugate) Polyakov loops $\mathcal{P}_{\mathbf{x}}^{(-1)}$

full gauge dynamics included

THREE MODELS

II: SU(3) ONE LINK MODEL

$$Z = \int dU e^{-S_B} \det M \qquad \text{link } U \in \text{SU(3)}$$

$$S_B = -\frac{\beta}{6}\left(\mathrm{Tr}\, U + \mathrm{Tr}\, U^{-1}\right)$$

determinant:

$$\begin{aligned}\det M &= \det\left[1 + \kappa\left(e^{\mu}\sigma_+ U + e^{-\mu}\sigma_- U^{-1}\right)\right] \\ &= \det\left(1 + \kappa e^{\mu} U\right) \det\left(1 + \kappa e^{-\mu} U^{-1}\right)\end{aligned}$$

with $\sigma_\pm = (\mathbb{1} \pm \sigma_3)/2$

- det in colour space remaining
- exact evaluation by integrating over the Haar measure

THREE MODELS

III: U(1) ONE LINK MODEL

U(1) model: link $U = e^{ix}$ with $-\pi < x \leq \pi$

$$S_B = -\frac{\beta}{2}\left(U + U^{-1}\right) = -\beta \cos x$$

determinant:

$$\det M = 1 + \frac{1}{2}\kappa\left[e^{\mu}U + e^{-\mu}U^{-1}\right] = 1 + \kappa\cos(x - i\mu)$$

partition function:

$$Z = \int_{-\pi}^{\pi} \frac{dx}{2\pi}\, e^{\beta\cos x}\left[1 + \kappa\cos(x - i\mu)\right]$$

- all observables can be computed analytically

Complex Langevin dynamics

Langevin update:

$$U(\theta+\epsilon) = R(\theta)\,U(\theta) \qquad\qquad R = \exp\left[i\lambda_a\left(\epsilon K_a + \sqrt{\epsilon}\eta_a\right)\right]$$

- drift term

$$K_a = -D_a S_{\rm eff} \qquad\qquad S_{\rm eff} = S_B + S_F \qquad\qquad S_F = -\ln\det M$$

- noise

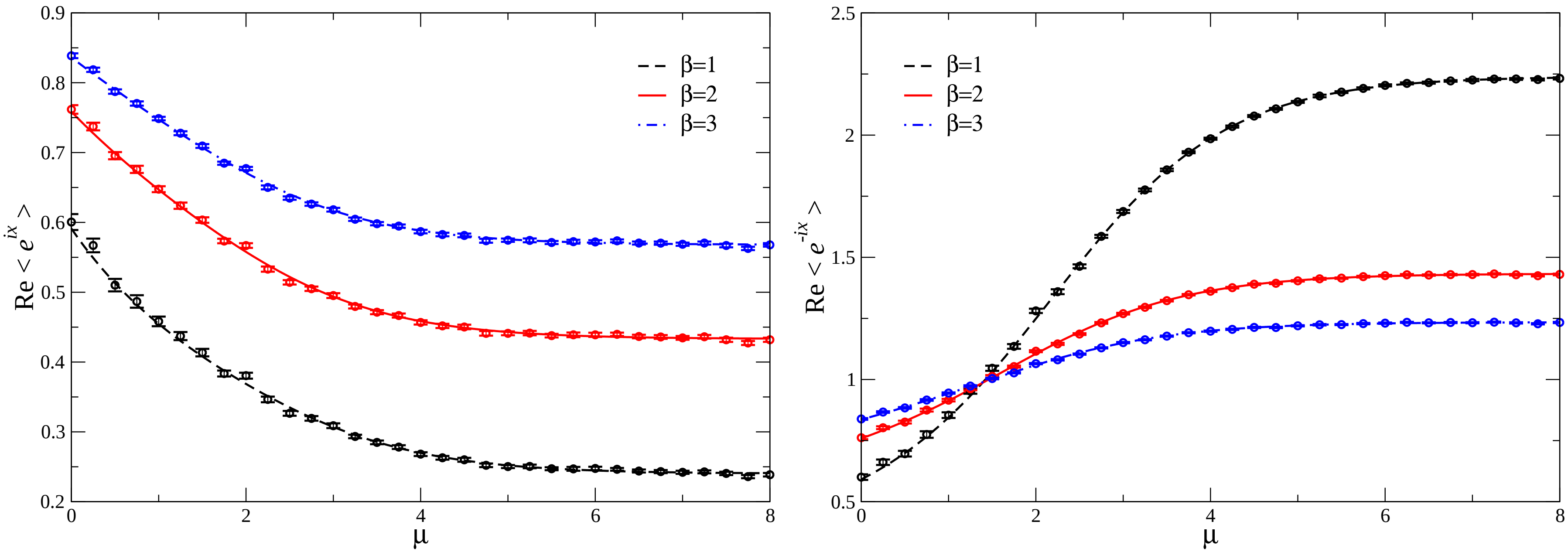
$$\langle\eta_a\rangle = 0 \qquad\qquad \langle\eta_a\eta_b\rangle = 2\delta_{ab}$$

real action: $\Rightarrow K^\dagger = K \Leftrightarrow U \in$ SU(3)

complex action: $\Rightarrow K^\dagger \neq K \Leftrightarrow U \in$ SL$(3,\mathbb{C})$

(conjugate) Polyakov loops

U(1) one link model

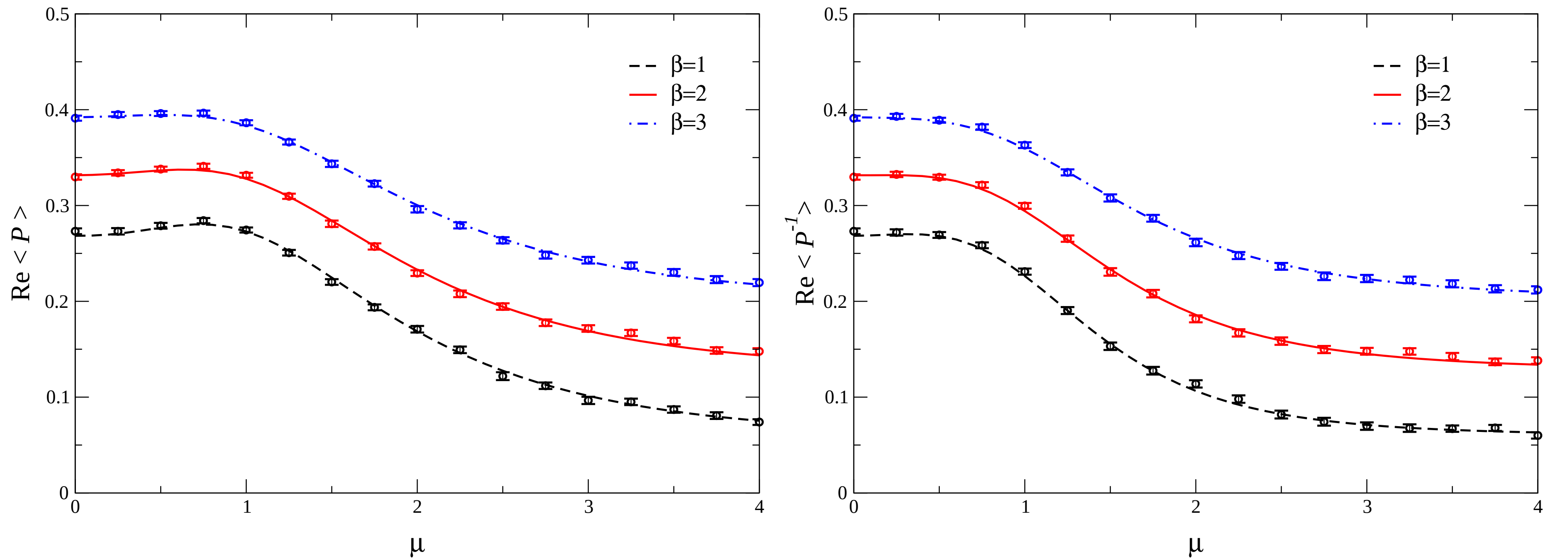


- data points: complex Langevin
  stepsize $\epsilon = 5 \times 10^{-5}$, 5×10^{7} time steps
- lines: exact results

excellent agreement for all μ

(CONJUGATE) POLYAKOV LOOPS

SU(3) ONE LINK MODEL

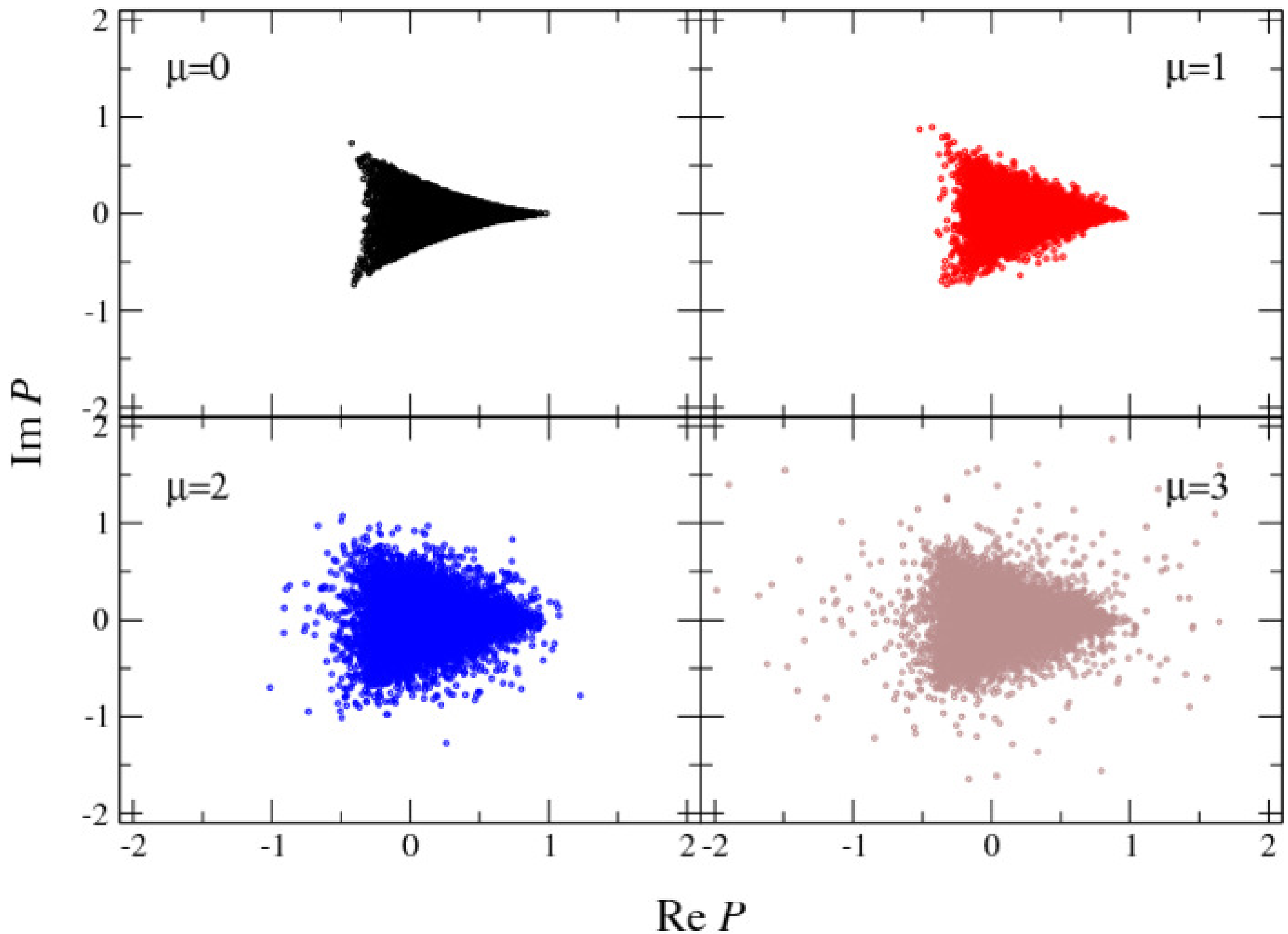


- data points: complex Langevin
  stepsize $\epsilon = 5 \times 10^{-5}$, 5×10^{7} time steps
- lines: exact results

excellent agreement for all μ

(CONJUGATE) POLYAKOV LOOPS

SU(3) ONE LINK MODEL

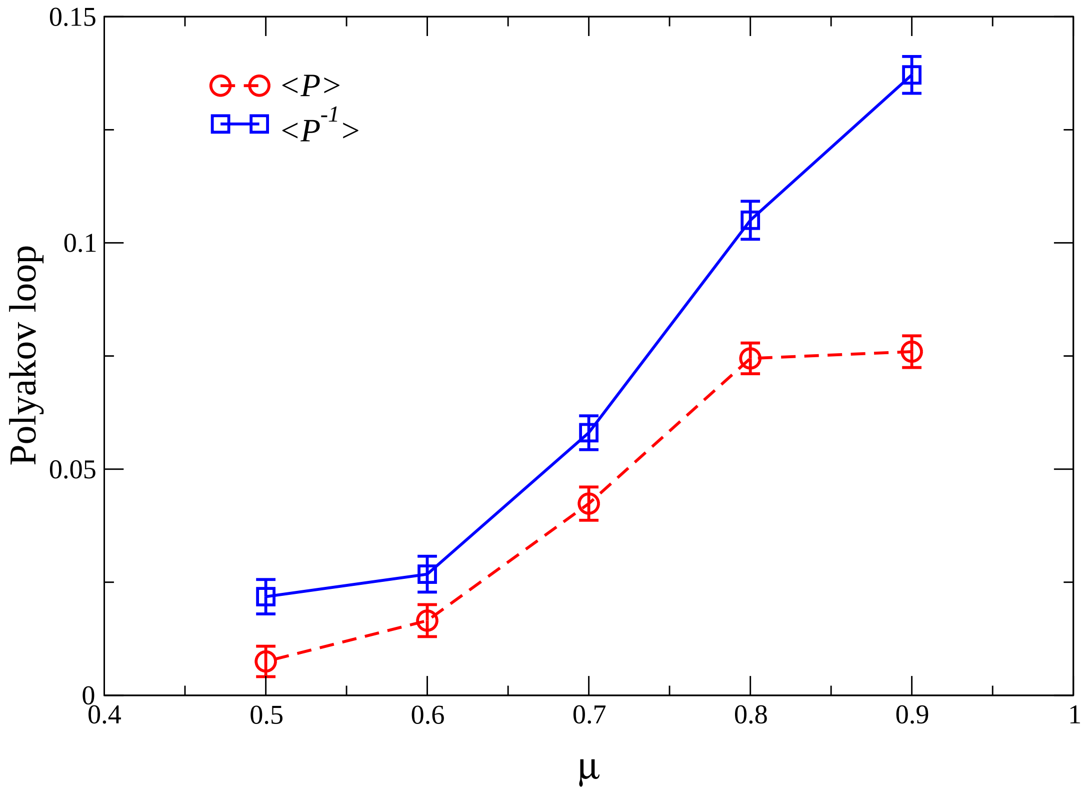


scatter plot of P during Langevin evolution

(CONJUGATE) POLYAKOV LOOPS

QCD IN HOPPING EXPANSION

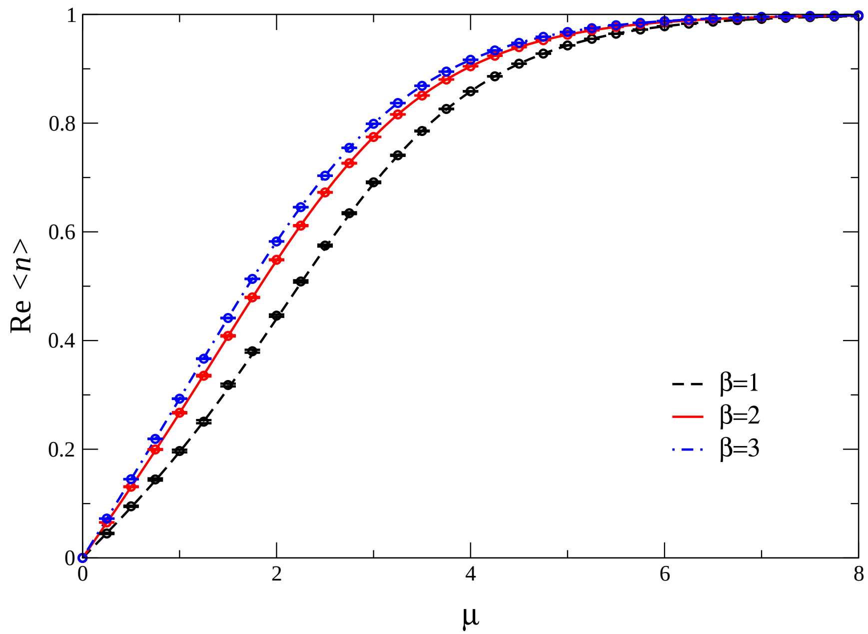
first results on 4^4 lattice at $\beta = 5.6$, $\kappa = 0.12$, $N_f = 3$



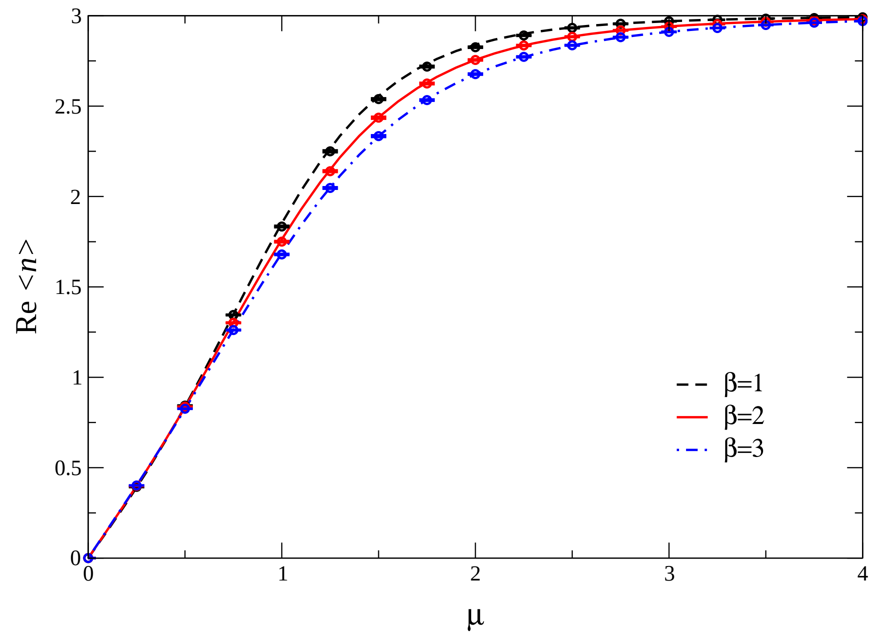
low-density “confining” phase $\Rightarrow$ high-density “deconfining” phase

DENSITY

U(1) ONE LINK MODEL



SU(3) ONE LINK MODEL

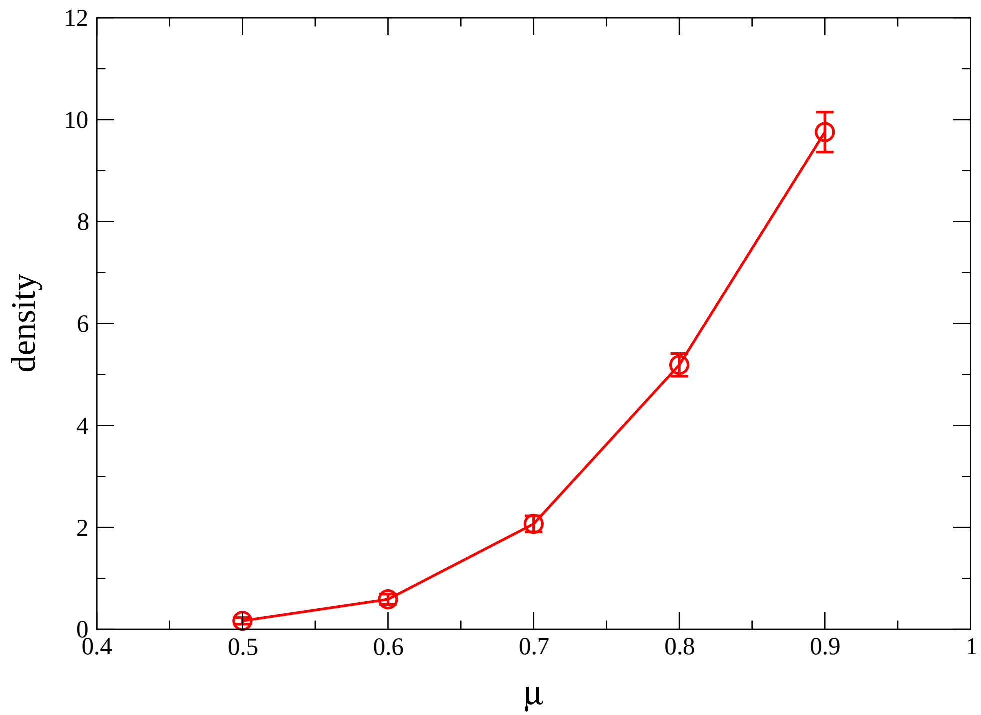


- linear increase at small μ
- saturation at large μ

excellent agreement for all μ

DENSITY

QCD IN HOPPING EXPANSION

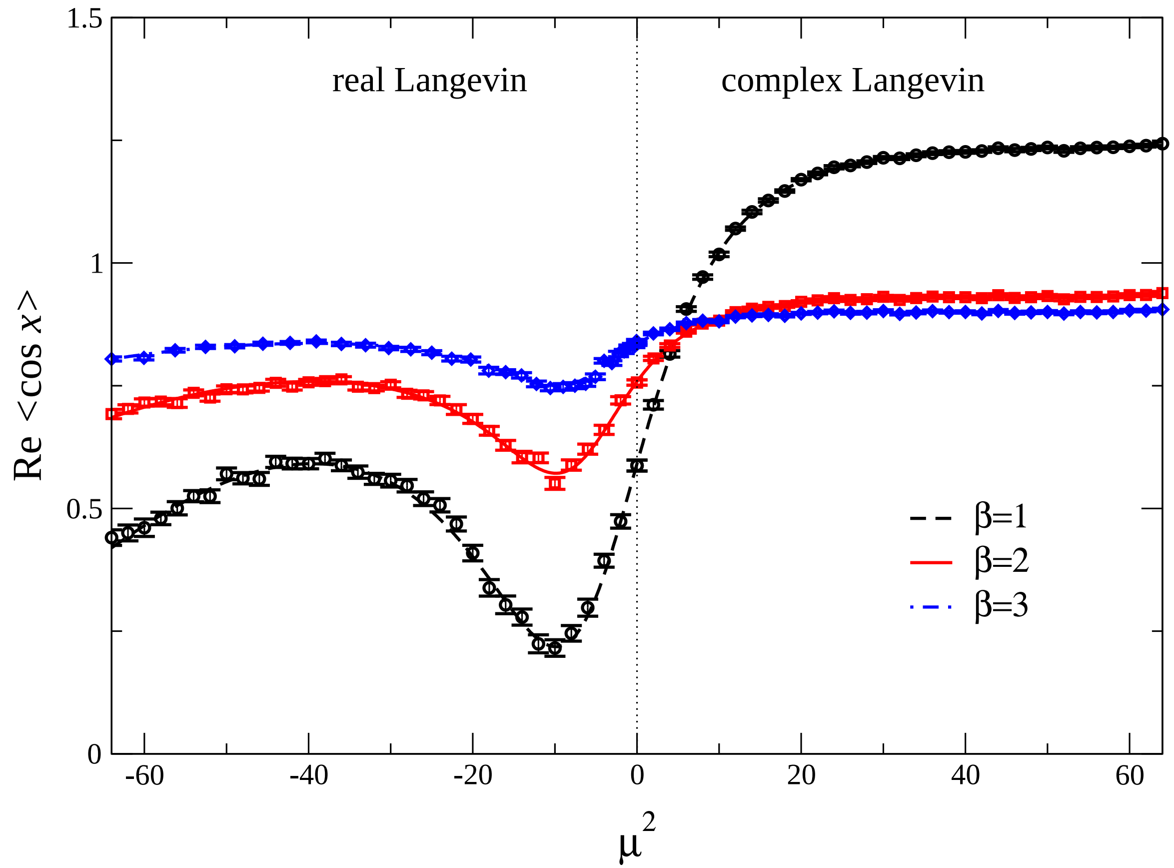


first results on 4^4 lattice at $\beta = 5.6$, $\kappa = 0.12$, $N_f = 3$

low-density phase $\Rightarrow$ high-density phase

Real vs. complex Langevin

U(1) one link model



plaquette as a function of μ^2

$\mu^2 < 0$: imaginary chemical potential $\Leftrightarrow$ real action

NUMERICAL STABILITY/RUNAWAYS

PROBLEM IN THE 80'S

- one link models: no problem

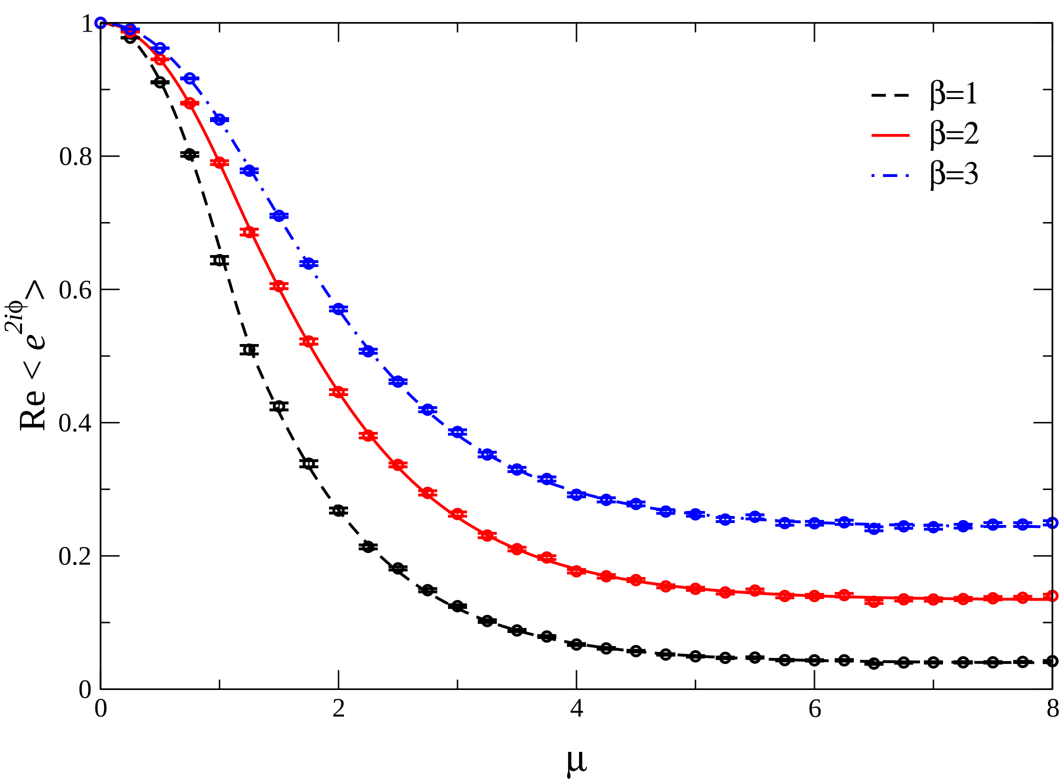
- field theory: runaways practically eliminated
  careful with numerical precision and roundoff errors
  dynamical step size

SIGN PROBLEM

U(1) ONE LINK MODEL

$$\det M(\mu) = [\det M(-\mu)]^* = |\det M(\mu)| e^{i\phi}$$

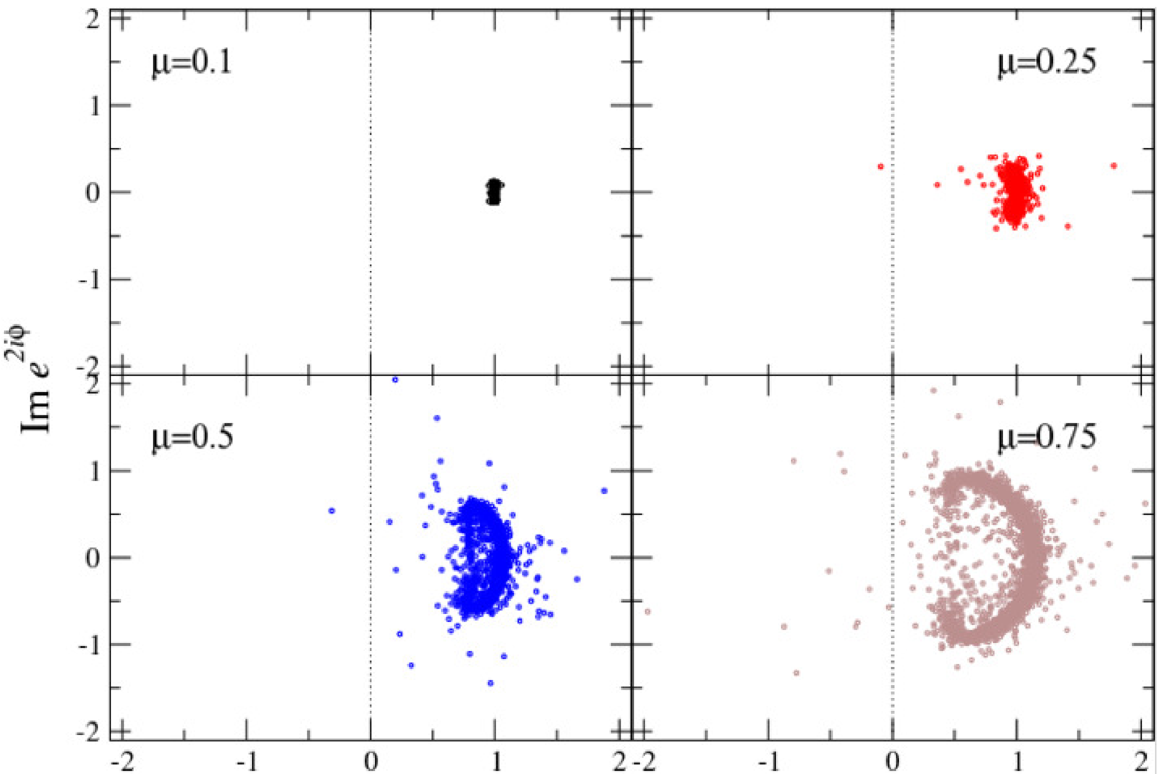
average phase factor: $\langle e^{2i\phi} \rangle = \left\langle \frac{\det M(\mu)}{\det M(-\mu)} \right\rangle$



SIGN PROBLEM

U(1) ONE LINK MODEL

$$\det M(\mu) = [\det M(-\mu)]^* = |\det M(\mu)|e^{i\phi}$$



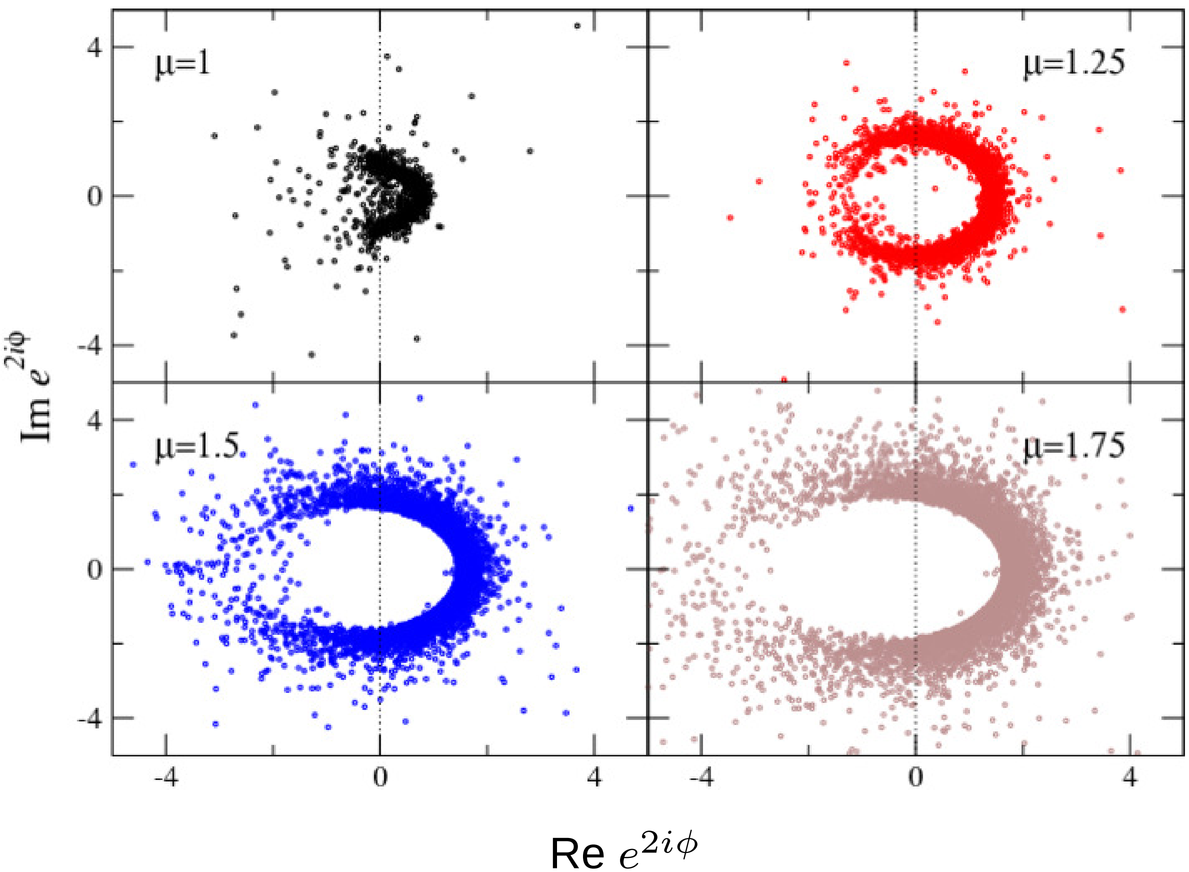
Re $e^{2i\phi}$

scatter plot of $e^{2i\phi}$ during Langevin evolution

SIGN PROBLEM

U(1) ONE LINK MODEL

$$\det M(\mu) = [\det M(-\mu)]^* = |\det M(\mu)| e^{i\phi}$$

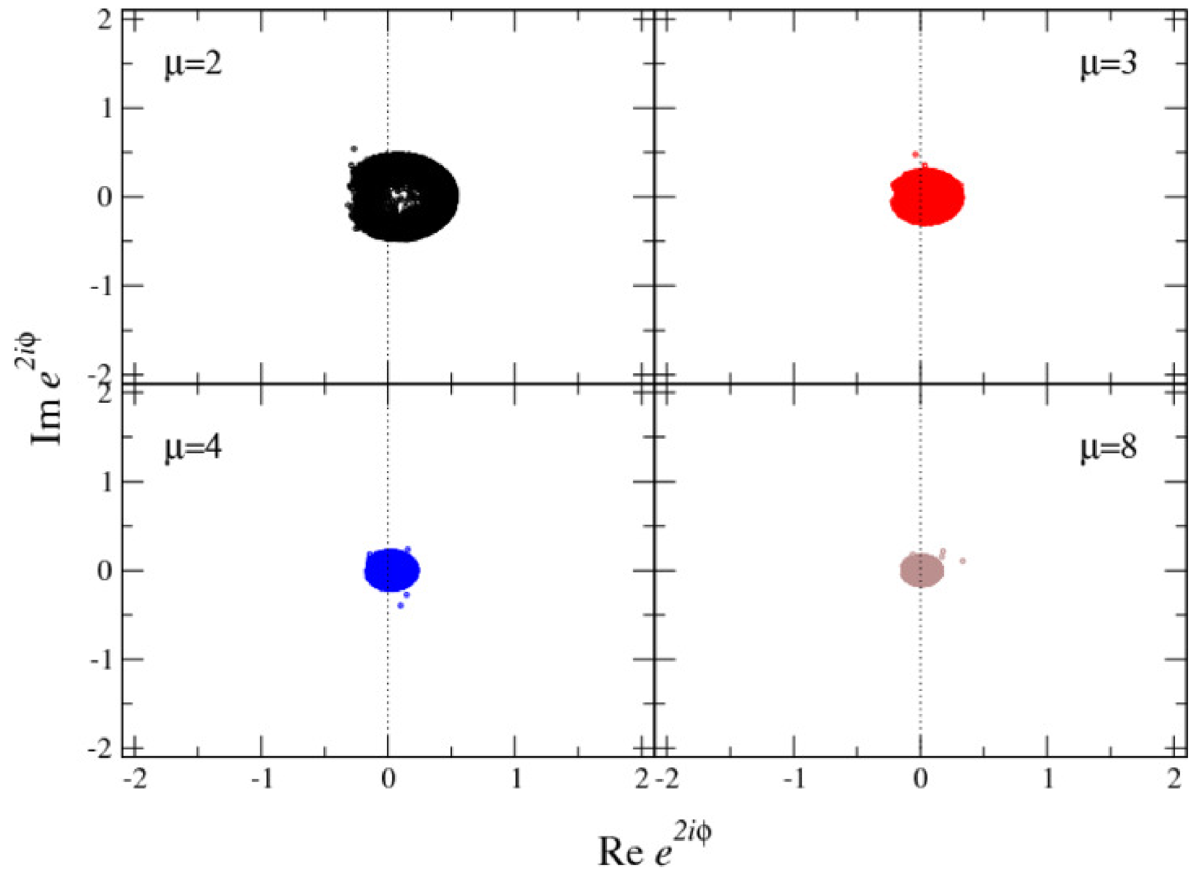


scatter plot of $e^{2i\phi}$ during Langevin evolution

Sign problem

U(1) ONE LINK MODEL

$$\det M(\mu) = [\det M(-\mu)]^* = |\det M(\mu)| e^{i\phi}$$

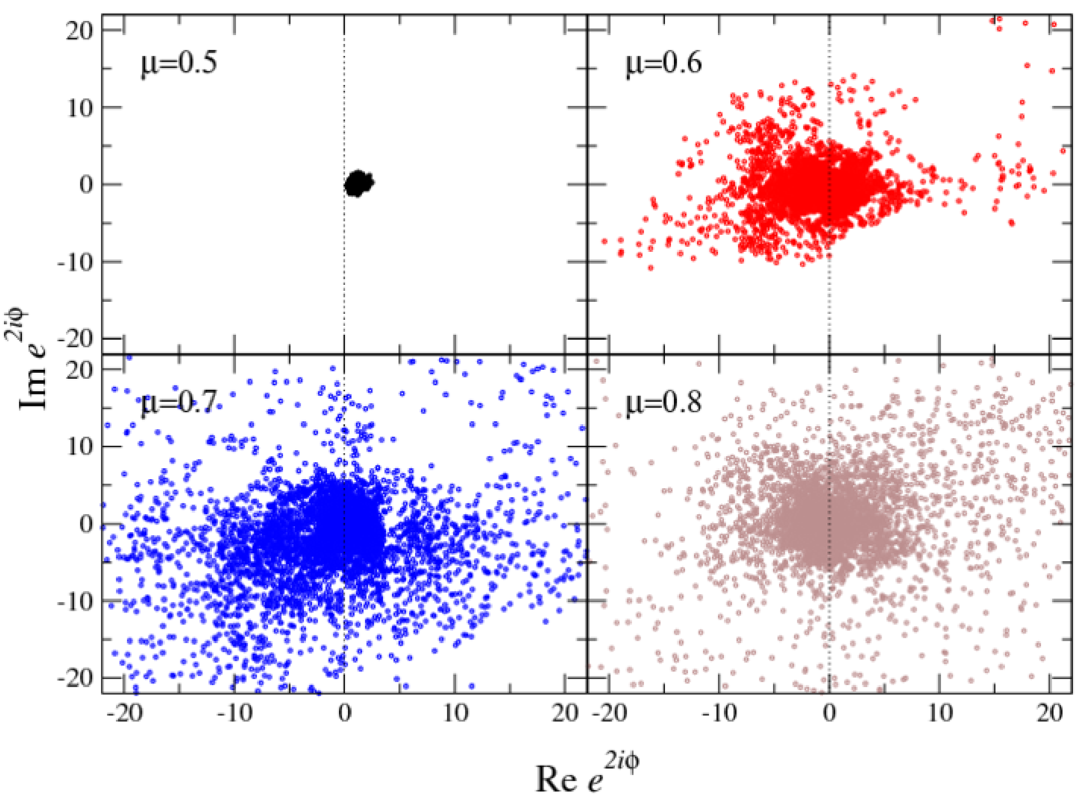


scatter plot of $e^{2i\phi}$ during Langevin evolution

SIGN PROBLEM

QCD IN HOPPING EXPANSION

average phase factor: $\langle e^{2i\phi} \rangle = \left\langle \frac{\det M(\mu)}{\det M(-\mu)} \right\rangle$

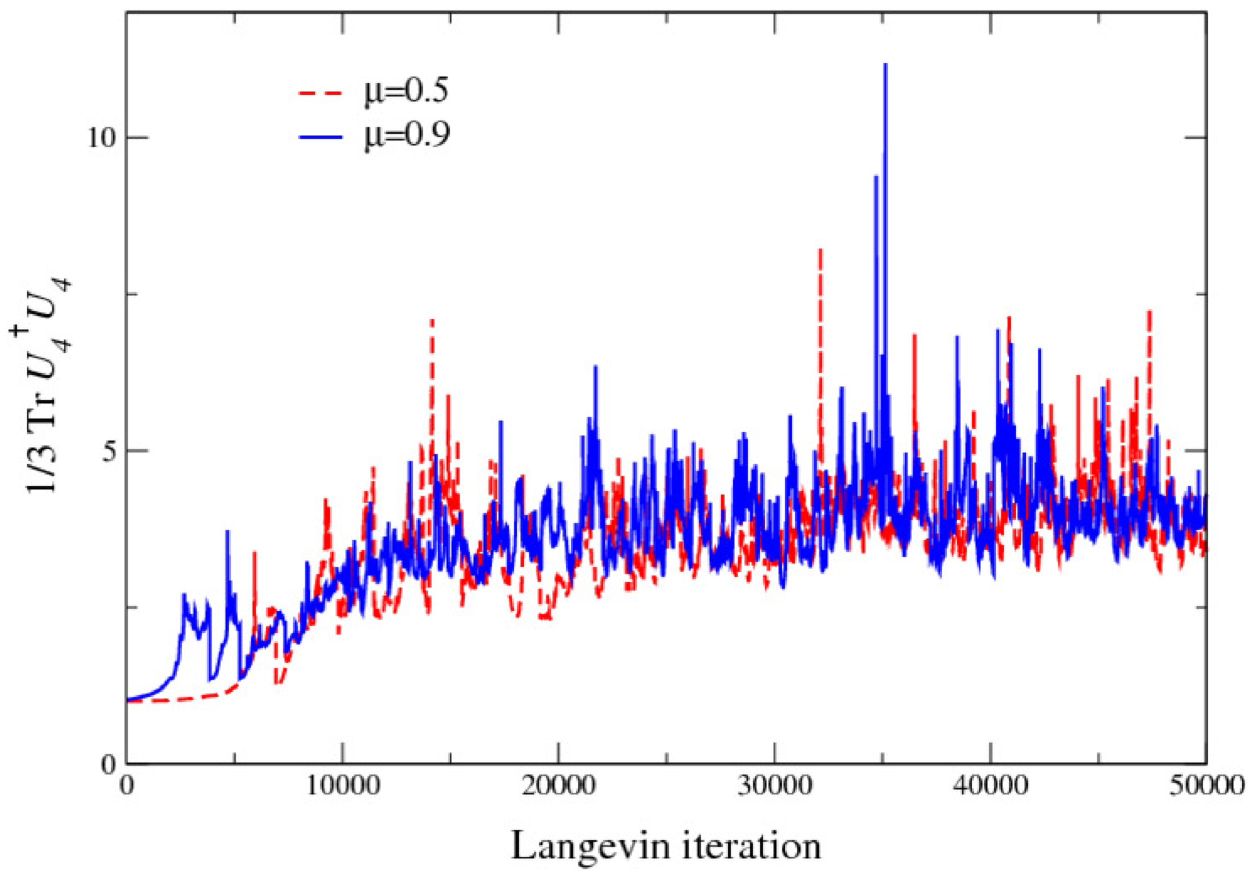


scatter plot of $e^{2i\phi}$ during Langevin evolution

SU(3) $\to$ SL(3,$\mathbb{C}$)

QCD IN HOPPING EXPANSION

$$\frac{1}{3}\mathrm{Tr}\, U^{\dagger}U \geq 1 \qquad\qquad = 1 \text{ if } U \in \mathrm{SU(3)}$$



Why does it (apparently) work?

- one link models: excellent
- precise agreement with exact results
- sign problem not a problem
- well defined distributions
- field theory encouraging

$\boxed{\text{why?}}$

- classical flow
- Fokker-Planck equation

in U(1) model

CLASSICAL FLOW

U(1) ONE LINK MODEL

link $U = e^{ix}$ complexification $x \to z = x + iy$

Langevin dynamics: $\dot{x} = K_x + \eta$ $\dot{y} = K_y$

classical forces: $K_x = -\mathrm{Re}\frac{\partial S}{\partial x}\Big|_{x\to z}$ $K_y = -\mathrm{Im}\frac{\partial S}{\partial x}\Big|_{x\to z}$

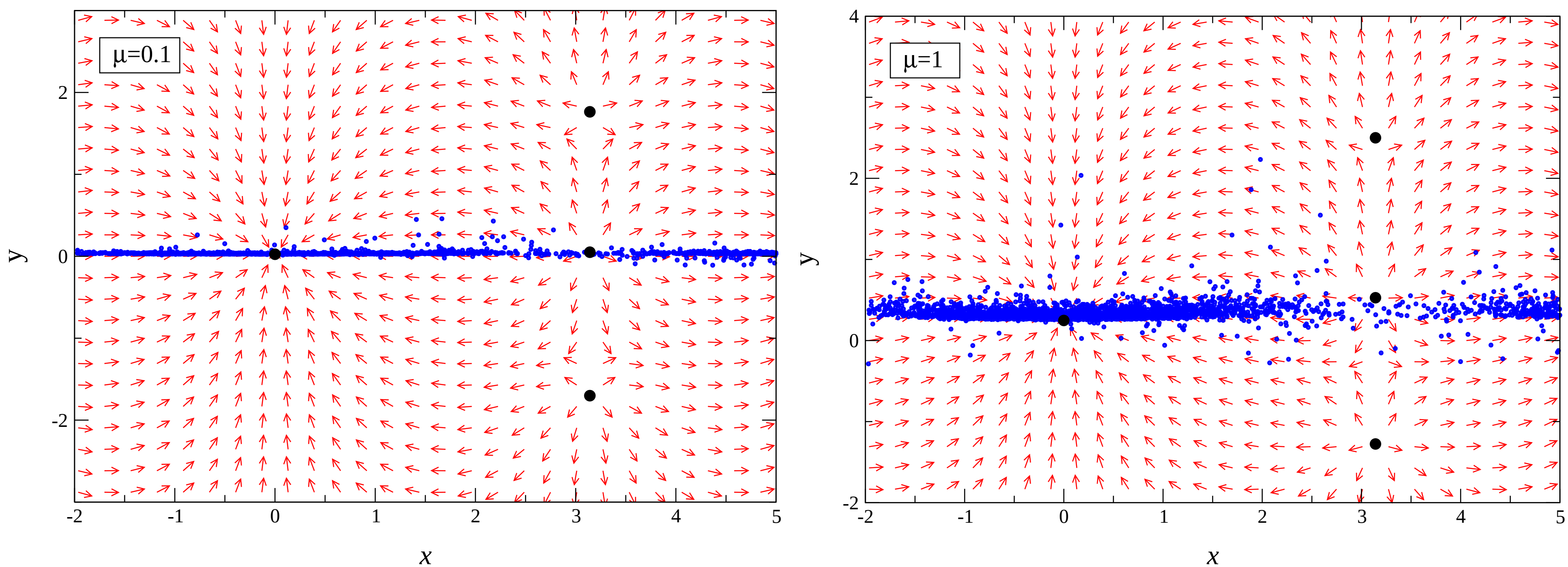
- classical fixed points: $K_x = K_y = 0$
- one stable fixed point at $x = 0$, $y = y_s(\mu)$
- unstable fixed points at $x = \pi$, $y = y_u(\mu)$

structure is independent of μ!

Classical flow

U(1) one link model

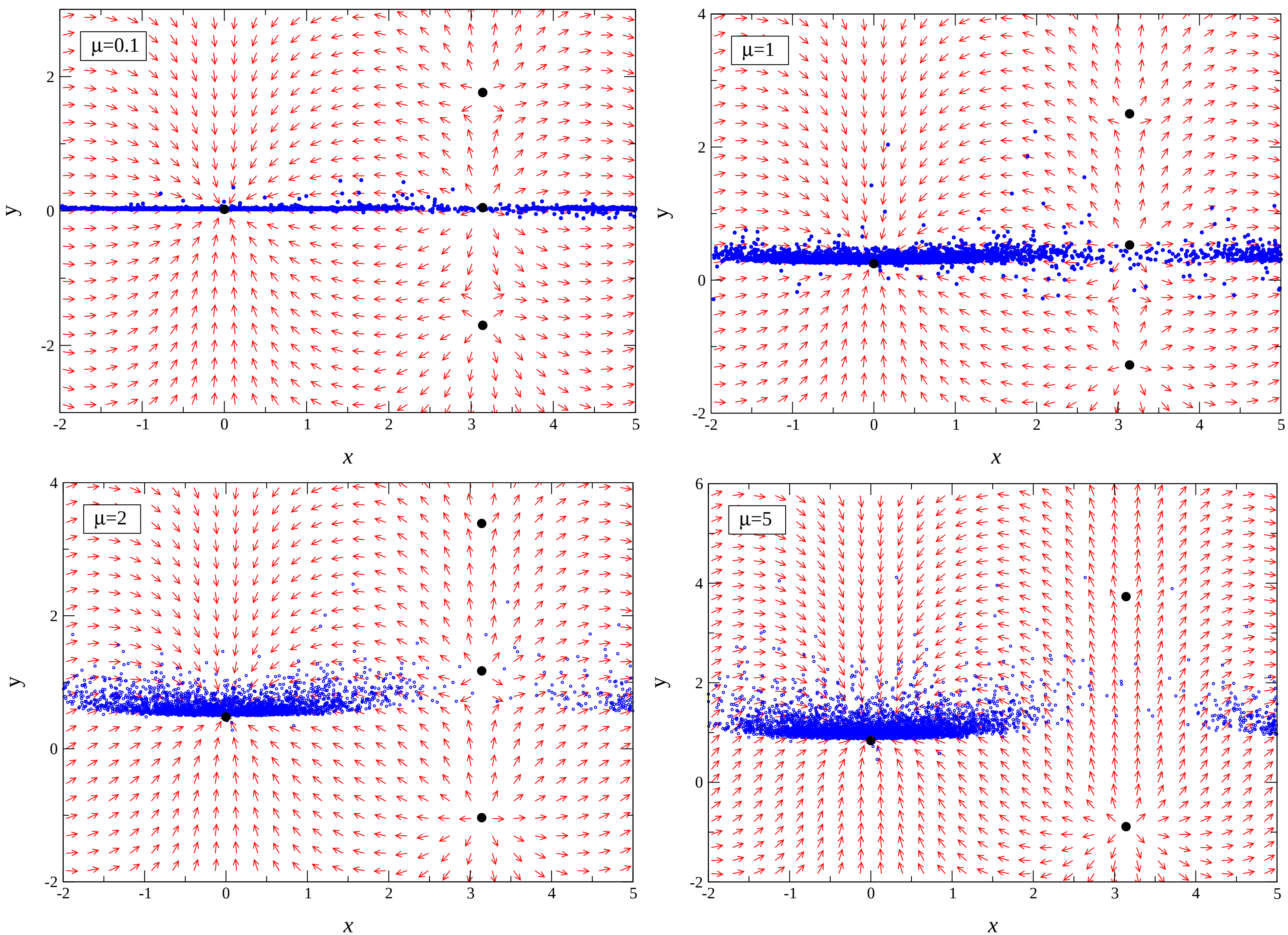
flow diagrams and Langevin evolution



- black dots: classical fixed points
- $\mu = 0$: dynamics only in x direction
- $\mu > 0$: spread in y direction

CLASSICAL FLOW

U(1) ONE LINK MODEL



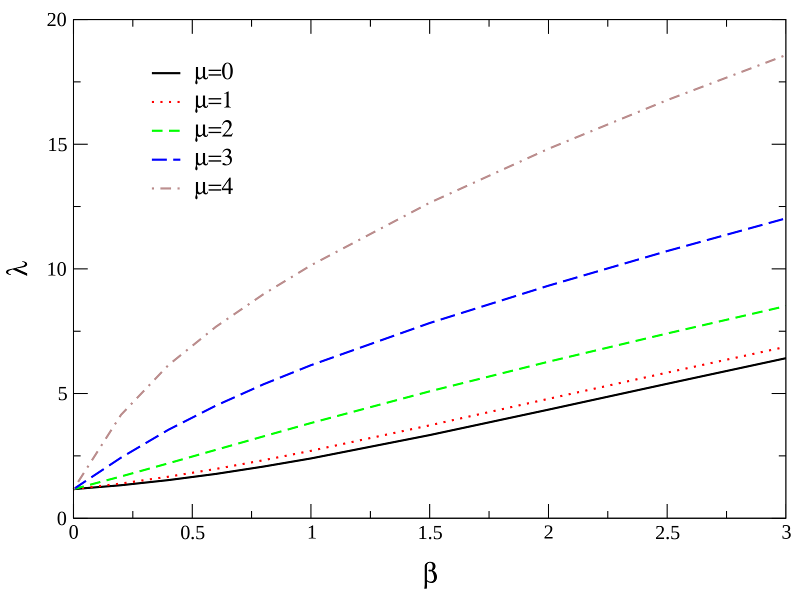
COMPLEX FOKKER-PLANCK EQUATION

U(1) ONE LINK MODEL

complex Fokker-Planck equation:

$$\frac{\partial P(x,\theta)}{\partial \theta} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} + \frac{\partial S}{\partial x}\right) P(x,\theta)$$

all eigenvalues are real $\Leftrightarrow \det M(\mu) = [\det M(-\mu)]^*$



smallest nonzero eigenvalue

all eigenvalues ≥ 0 (!)

open question: real Fokker-Planck equation for $\rho(x, y, \theta)$

SUMMARY

finite chemical potential: complex action
stochastic quantization and complex Langevin dynamics

- one link models: excellent
- field theory: encouraging

detailed study of

- (sign problem and) phase of the determinant

why? partly understood in simple models

- classical flow qualitatively unchanged
- complex FP equation: eigenvalues ≥ 0

to do: more field theory