# Phase of the Fermion Determinant at Nonzero Chemical Potential

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# Contents

# I. Extreme Domains in QCD

The Chiral Domain

The Microscopic Domain

The Microscopic Domain of the Dirac Spectrum

Applications

# Chiral Domain of QCD

$$\begin{aligned} m_\pi &\ll F_\pi, \qquad \mu_k \ll F_\pi, \qquad T < T_c, \\ V &\gg \Lambda_{QCD}^{-4}. \end{aligned}$$

- QCD is an expansion in $m_\pi/F_\pi$ and $\mu_k/F_\pi$ in this domain.
- The chiral Lagrangian does not depend on the baryon chemical potential but depends on the isospin and strangeness chemical potential.
- The temperature dependence of the QCD partition function is given by the thermodynamics of a weakly interacting pion gas.
- Also known as the $p$ -domain

# Microscopic Domain of QCD

$$m_\pi \ll L^{-1}, \qquad \mu_k \ll L^{-1}$$
$$L = V^{1/4} \gg \Lambda_{QCD}^{-1}, \qquad T \le T_c$$

- The QCD partition function factorizes in a zero momentum mode part and a nonzero momentum mode part.
- The zero momentum mode part is the large $N$ limit of a chiral random matrix theory of $N \times N$ matrices with the global symmetries of QCD..
- The temperature dependence is through the chiral condensate and the pion decay constant.
- The partition function depends on the isospin and strangeness chemical potential but not on the baryon chemical potential.
- Also known as the $\epsilon$ -domain.

# Extreme Microscopic Domain of QCD

$$\begin{aligned} m_\pi \ll L^{-1}, &\qquad \mu_k \ll L^{-1} \\ L = V^{1/4} \gg \Lambda_{QCD}^{-1}, &\qquad T \leq T_c \\ m_\pi F_\pi L^2 \gg 1, &\qquad \mu F_\pi L^2 \gg 1 \end{aligned}$$

- ✓ This domain matches to the mean field limit of the chiral domain.
- ✓ It is the “strong coupling” limit of chiral random matrix theory which is dual to the weak coupling limit of the corresponding nonlinear $\sigma$-model.

# The Microscopic Domain of the Dirac Spectrum

$$m_\pi, \ \mu_k \quad \text{arbitrary}$$

$$z \ll \frac{F_\pi^2}{2\Sigma L^2} \qquad \mu_z \ll L^{-1}$$

$$L = V^{1/4} \gg \Lambda_{QCD}^{-1}, \qquad T \leq T_c$$

- The Compton wavelength of the Goldstone particles corresponding to $z$ is much larger than the size of the box.
- The partition function that describes the QCD Dirac spectrum depends on $\mu_z$ .
- This partition function is equivalent to the large $N$ ‘limit of a chiral random matrix theory.
- $m_\pi$ and $\mu_k$ can be in the microscopic domain, in the chiral domain or outside the chiral domain.

# Large $N_c$ Limit of the Microscopic Domain

Density of Dirac eigenvalues according to Banks-Casher

$$\rho(0) = \frac{V}{\pi\Sigma} \Longrightarrow \Delta\lambda \equiv \frac{1}{\rho(0)}$$

Number of eigenvalues in the microscopic domain

$$\frac{F_\pi^2}{2\Sigma L^2}\frac{1}{\Delta\Lambda} = \pi F_\pi^2 L^2 \sim N_c$$

Estimate for $N_c = 3$ : $L = 3\,\mathrm{fm}$ , $\quad F_\pi \approx 0.5\,\mathrm{fm}^{-1}$

$$\Longrightarrow \pi F_\pi^2 L^2 \approx 7.$$

# Uses of the Microscopic Domain

- Distribution of the small Dirac eigenvalues is a measure for for chiral symmetry breaking. For example, this is used to determine the critical number of flavors for the conformal phase. Fodor-Holland-Kuti-Nogradi-Schroeder-2008
- Eigenvalue distributions show if quarks are massless. One possible measure is the validity flavor-topology duality. Fukaya-et-al-2006
- The chiral condensate and the pion decay constant can be determined from the distribution of the lowest Dirac eigenvalue.
  Wettig-et al-1999, Damgaard-3t al-2005 Osborn-Wettig-2005, Akemann et al-2006
- Knowing the properties of the small Dirac eigenvalues can be exploited to obtain better estimators the fermion determinant. Luescher-Palombi-2008
- Chiral Random Matrix Theory has contributed significantly to our understanding of QCD at nonzero chemical potential: quenching, phase diagram, phase of the fermion determinant, alternative to the Banks-Casher relation.
  Jackson-JV-1995, Stephanov-1996, Halasz-et al-1998, Osborn-Splittorff-JV-2005, Splittorff-JV-2006/2008

# II. Phase of the Fermion Determinant

Phase and Dirac Eigenvalues

Sign Problem in QCD at $\mu \neq 0$

Phase Factor and Partition Functions

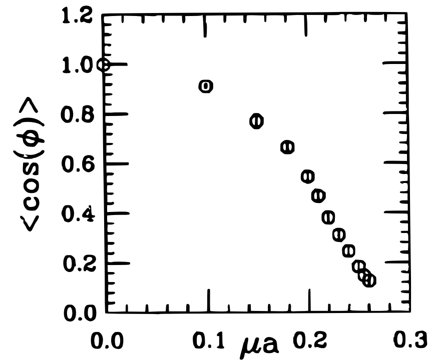
Lattice QCD in 1d

# Phase Factor and Dirac Eigenvalues

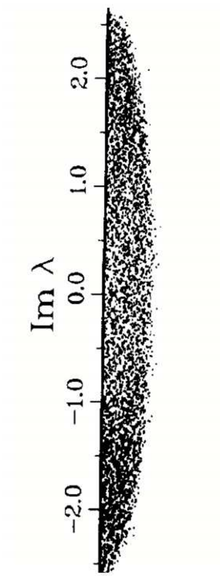
$$\det(D+m+\mu\gamma_0) = e^{i\theta}|\det(D+m+\mu\gamma_0)|$$

$\prod_k(\lambda_k + m)$

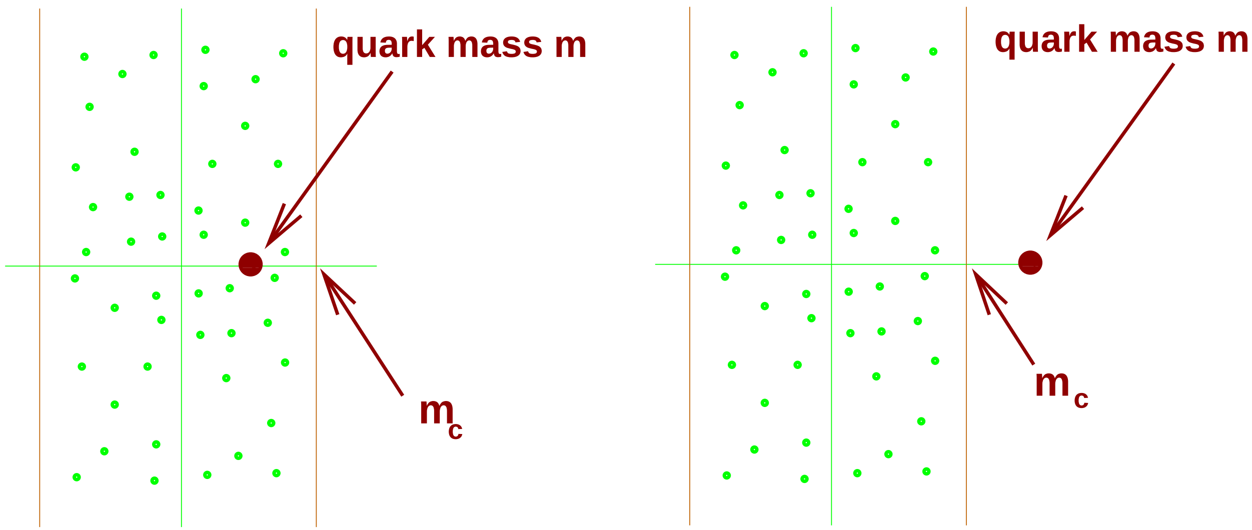
phase factor




Toussaint-1990




Barbour et al. 1986




Scatter plot of Dirac eigenvalues

$$m < m_c \qquad \text{then} \qquad \langle e^{i\theta}\rangle \sim 0$$

# Sign Problem in QCD at $\mu \neq 0$

Order parameters: $\langle \theta^2 \rangle$ , $\langle e^{i\theta} \rangle$ , $\langle e^{2i\theta} \rangle$ , $\langle e^{-2i\theta} \rangle$ , $\cdots$ ,
the statistical distribution of $\theta$ .

Averages can be taken with respect to different partition functions:

Quenched partition function, $\langle \cdots \rangle_{\mathrm{q}}$
Phase quenched partition function, $\langle \cdots \rangle_{\mathrm{pq}}$ (most relevant)
Full QCD partition function, $\langle \cdots \rangle_{N_f}$

Seriousness of the sign problem: *No problem* if $\langle \theta^2 \rangle^{1/2} < \frac{\pi}{2}$
*Mild* if $\langle \theta^2 \rangle^{1/2} > \frac{\pi}{2}$ but $\langle \theta^2 \rangle \sim V^0$ ,
*Serious* if $\langle \theta^2 \rangle \sim V$ ,
$\langle e^{i\theta} \rangle \sim e^{-VF}$ .

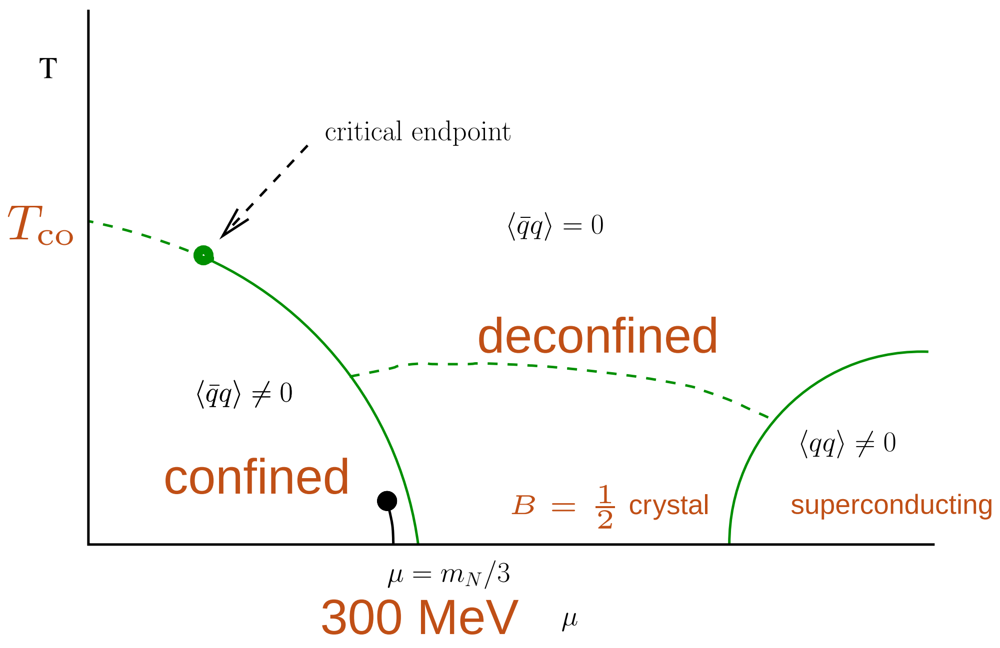
We will see next that the latter possibility arises naturally.

# Phase Factor and Partition functions

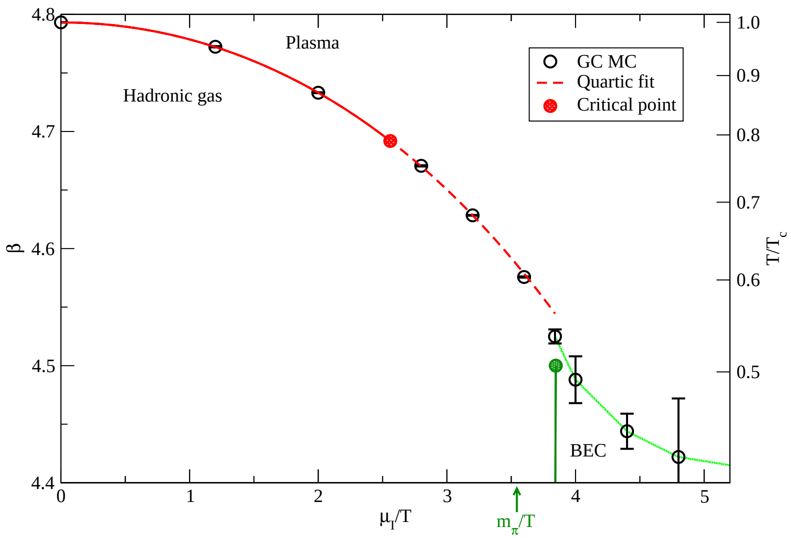
$$\langle e^{2i\theta}\rangle_{\rm pq} \quad = \quad \frac{\langle(\det(D+m+\mu\gamma_0))^2\rangle}{\langle|\det(D+m+\mu\gamma_0)|^2\rangle} \equiv \frac{Z^{\rm QCD}_{N_f=2}}{Z^{|\rm QCD|}_{N_f=2}} = \frac{Z^{\rm QCD}_{N_f=2}(\mu)}{Z^{\rm QCD}_{N_f=2}(\mu_I=\mu)}$$
$$\sim \quad e^{-V(F_{\rm QCD}-F_{|\rm QCD|})}.$$

- Phase quenched QCD is QCD at nonzero isospin chemical potential:
  $|\det(D+m+\mu\gamma_0)|^2 = \det(D+m+\mu\gamma_0)\det(D+m-\mu\gamma_0)$.
- Sign problem remains for $N_c \to \infty$ :
  $F_{\rm QCD}(\mu) = F_{|\rm QCD|}(\mu)[1+O(\frac{1}{N_c})]$. (Cohen-2004),
  but $F_{\rm QCD} \sim O(N_c)$ .

# Phase Diagram of QCD and |QCD|




Schematic QCD phase diagram.




Phase diagram of phase quenched QCD (de Forcrand-Stephanov-Wenger-2007). Agrees with earlier work by Kogut and Sinclair.

$Z_{|\rm QCD|}$ has a phase transition at $\mu = m_\pi/2$ so that the free energies of the two theories are completely different.

An nonzero temperature the free energies are different for any nonzero value of the chemical potential.

# Remarks

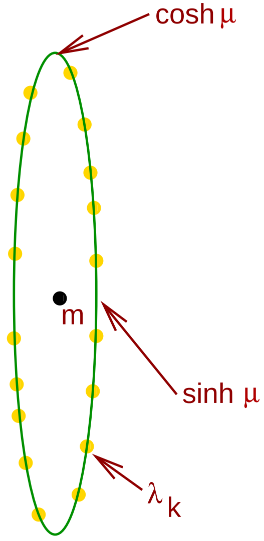
- Eigenvalues are distributed more or less homogeneously inside a strip.
- The strip has a hard edge.
- Convergence of the average phase factor. What is the asymptotic $p$ dependence of the ratio
$$\frac{\langle \prod_{k=-p}^{p} (\lambda_k^{\rm QCD} + m) \rangle}{\langle \prod_{k=-p}^{p} (\lambda_k^{|\rm QCD|} + m) \rangle} \quad ?$$
- If the chemical potential is in the microscopic domain (i.e. $\mu^2 F_\pi^2 V = \text{fixed}$ for $V \to \infty$), this ratio is determined by eigenvalues in the microscopic domain (i.e., $\lambda_k \ll 1/F_\pi \sqrt{V}$).
- Random matrix theory suggest that for finite $\mu$ the convergence might be as slow as $O(\sqrt{N/p})$.
- The phase factor is essential for physical observables.

# $U(1)$ Gauge Theory in one Dimension

Dirac operator:

$$D = \begin{pmatrix} mI & e^{\mu} & \cdots & e^{-\mu}U^{\dagger} \\ -e^{-\mu} & mI & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & mI & e^{\mu} \\ -e^{\mu}U/2 & \cdots & -e^{-\mu} & mI \end{pmatrix}$$

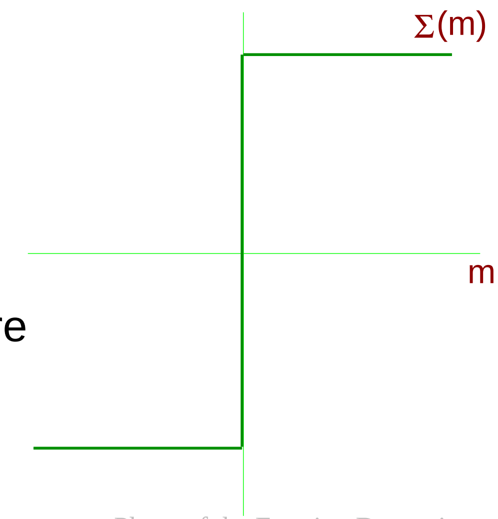
$U(1)$
matrix




Dirac spectrum of 1d QCD

$$\Sigma(m) = \frac{\left\langle \sum_k \frac{1}{\lambda_k + m} \prod_k (\lambda_k + m) \right\rangle}{\langle \prod_k (\lambda_k + m) \rangle}$$

determinant with
a complex phase




The chiral condensate has a discontinuity in region where there are no eigenvalues

Ravagli-JV-2007

# Phase Factor in Extreme QCD

- √ Lattice QCD   Allton-et al-2005, Ejiri-2006/2008, Fodor-Schmidt-2007
- √ Average phase factor for one-plaquette QCD.   Aarts-2008
- √ Average phase factor in 1d QCD.   Ravagli-JV-2007
- √ Large $N_c$ -limit of average phase factor.
- √ Hard thermal loop expansion of the average phase factor.
  Fraga-Villavicencio-2008
- √ Average phase factor in chiral perturbation theory.
  Splittorff-JV-2007
- √ Average phase factor in the microscopic domain of QCD.
  Splittorff-JV-2006
- √ Average phase factor in chiral random matrix theory.   Ravagli-JV-2007, Han-Stephanov-2008

# III. Phase Factor in Chiral Perturbation Theory

One Loop Result

Comparison with Lattice Results

Probability Distribution of the Phase

# One Loop Chiral Perturbation Theory

The chiral Lagrangian depends on the the isospin chemical potential but not on the the quark number chemical potential.

To one loop order we find:

$$\langle \det{}^2(D+m+\mu\gamma_0)\rangle \quad \sim \quad e^{-VF^{(0)}_{N_f=2}} \prod_k \prod_p \frac{1}{\sqrt{m_k^2+\vec{p}^2+p_0^2}},$$

$$\langle |\det(D+m+\mu\gamma_0)|^2\rangle \quad \sim \quad e^{-VF^{(0)}_{pq}} \prod_k \prod_p \frac{1}{\sqrt{m_k^2+\vec{p}^2+(p_0-2i\mu)^2}}.$$

Only charged Goldstone bosons contribute to the ratio of the two partition functions. This results in

$$\langle e^{2i\theta}\rangle_{\rm pq} \quad = \quad \frac{(m_\pi-2\mu)(m_\pi+2\mu)}{m_\pi^2} e^{h(m_\pi^2L^2,\mu^2L^2)},$$

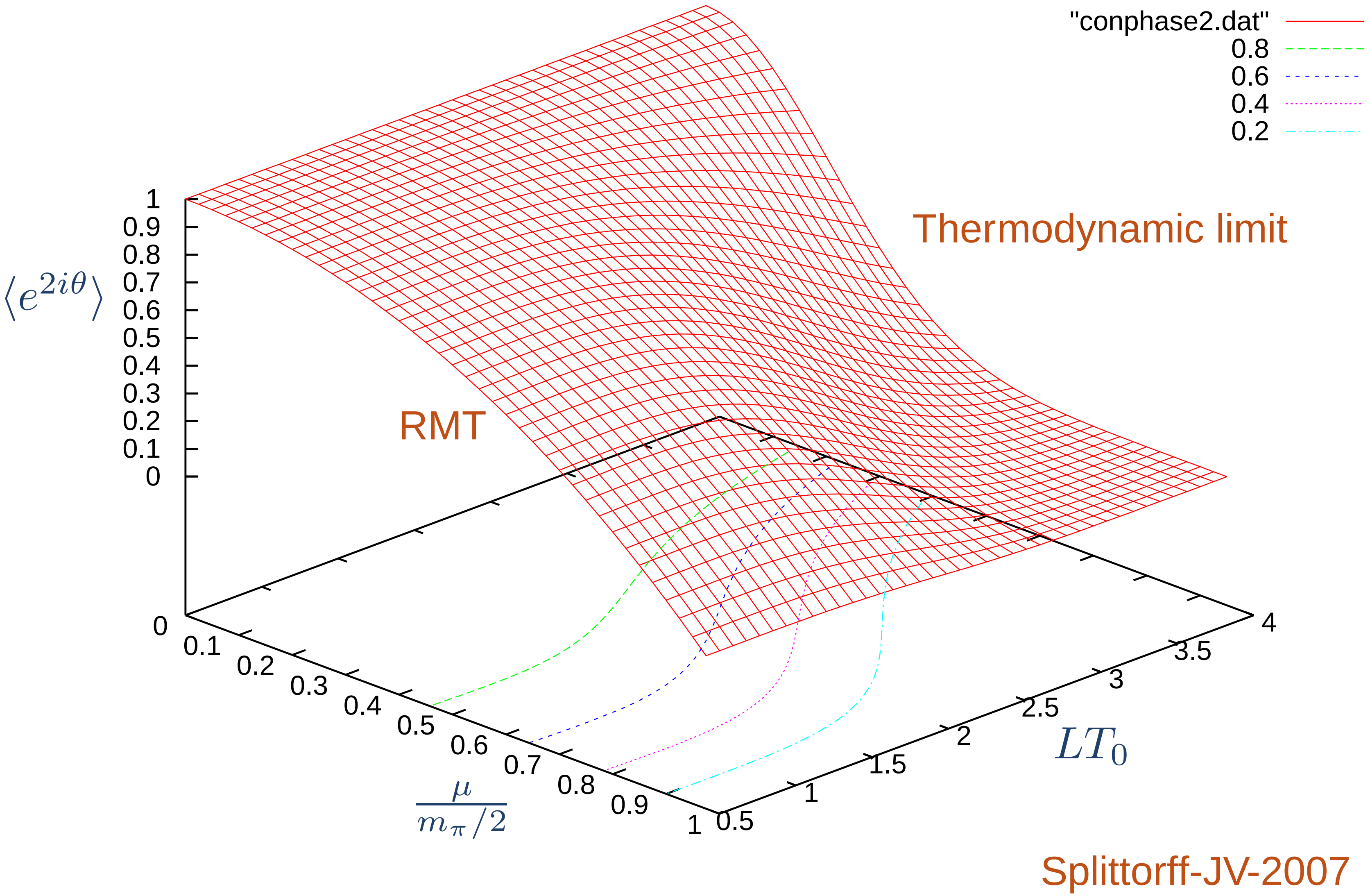
with $h$ a finite function.

Splittorff-JV-2007

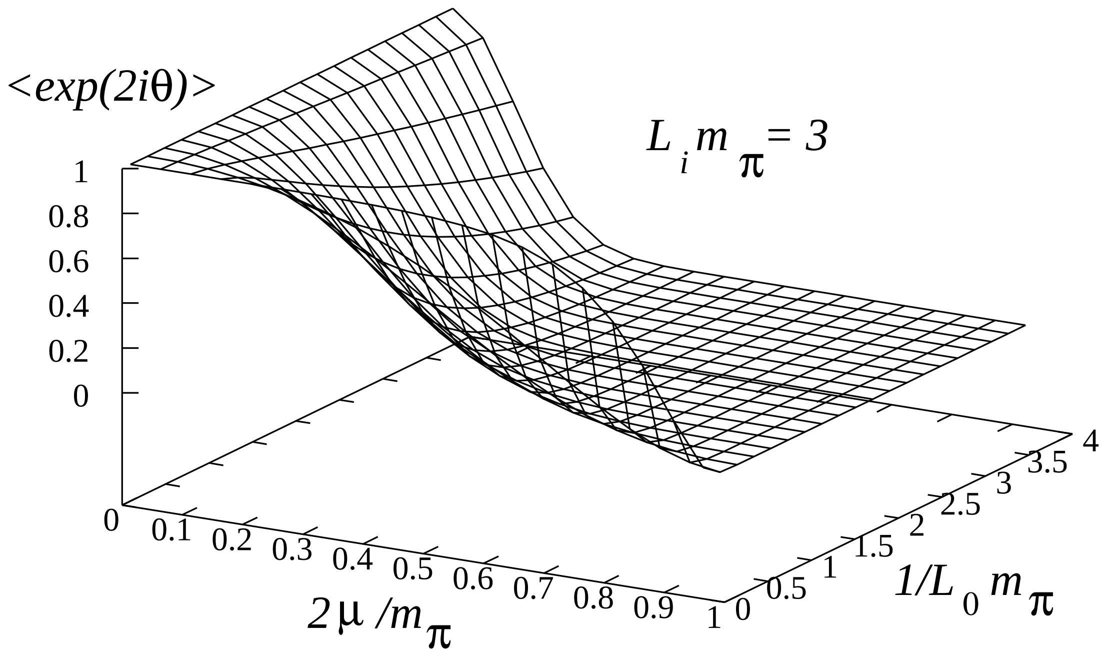
zero momentum contribution

can be derived from random matrix theory

# One-Loop Result



# Temperature Dependence of $\langle \exp(i\theta) \rangle$

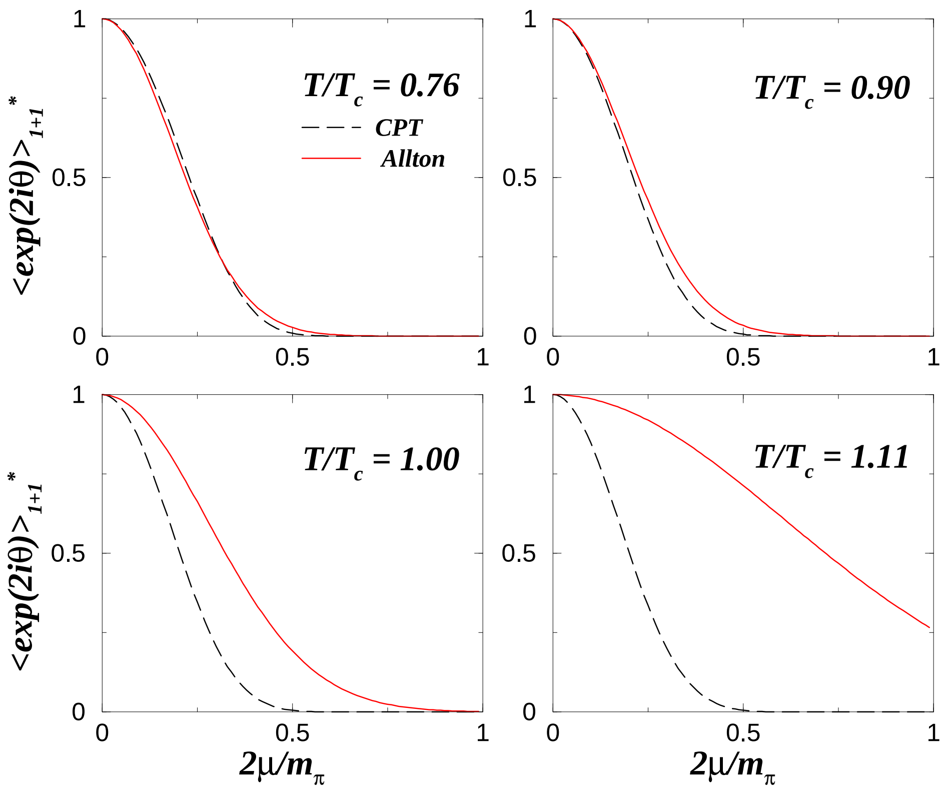


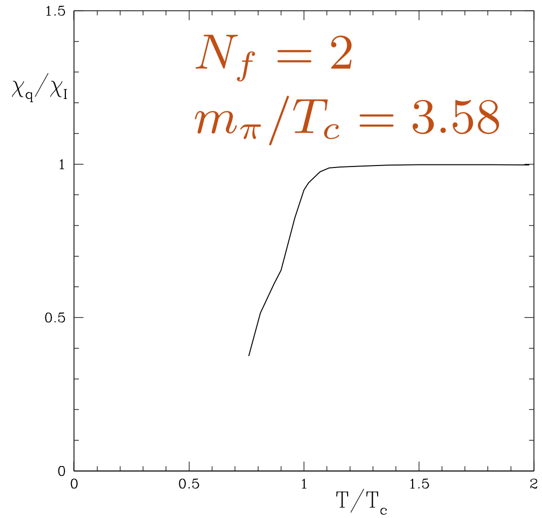

Splittorff-JV-2007

Average phase factor for $N_f = 2$ as a function of the chemical potential and the temperature ( $1/L_0$ ).

In the chiral domain, simulations are possible for small chemical potentials or low temperatures.

# Comparison with Lattice Simulations




Ratio of quark and isospin susceptibility $(\chi_q/\chi_I)$ to second order in $\mu$ (data: Allton et al. 2005)

Average phase factor in lattice QCD using the lowest order Taylor expansion (Allton-et-al.-2005) compared to one loop chiral perturbation theory in a box equal to the size of the lattice.

$$\langle e^{2i\theta}\rangle_{1+1^*} = \frac{Z_{\rm QCD}(\mu)}{Z_{|\rm QCD|}(\mu)}$$
$$\sim e^{V\mu^2(\chi_q-\chi_I)}.$$

# Probability Distribution of the Phase

The density of the phase angle is defined by

$$\rho(\phi) = \langle \delta(\phi - \overbrace{\mathrm{Im} \log \det(D + m + \mu\gamma_0)}^{\theta}) \rangle_{N_f}$$

Notice that $\phi \in \langle -\infty, \infty \rangle$ .

- According to the Central Limit Theorem we expect that $\rho(\phi)$ is a Gaussian. Ejiri-2007.
- If the average is over dynamical quarks, the phase density is complex, Splittorff-JV-2007

$$\langle \delta(\phi - \theta) e^{iN_f \theta} |\det{}^{N_f}(D + m + \mu\gamma_0)| \rangle$$
$$= e^{iN_f \phi} \langle \delta(\phi - \theta) |\det{}^{N_f}(D + m + \mu\gamma_0)| \rangle \ .$$

- Observables are determined by correlations with the phase of the fermion determinant. Knowing the Gaussian distribution is clearly not sufficient.

# Derivation of the Phase Density

$$\begin{aligned}\rho_{N_f}(\phi) &= \langle \delta(\phi - \operatorname{Im}\log\det(D+m+\mu\gamma_0))\rangle_{N_f} \\ &= \langle \sum_n e^{in(\phi - \operatorname{Im}\log\det(D+m+\mu\gamma_0))}\rangle_{N_f}\end{aligned}$$

The phase density therefore follows from the moments of the phase factor.

$$\langle e^{2in\theta}\rangle_{N_f} = \frac{1}{Z_{N_f}}\left\langle \frac{\det^{n+N_f}(D+m+\mu\gamma_0)}{\det^{n}(D^\dagger+m+\mu\gamma_0)}\right\rangle$$

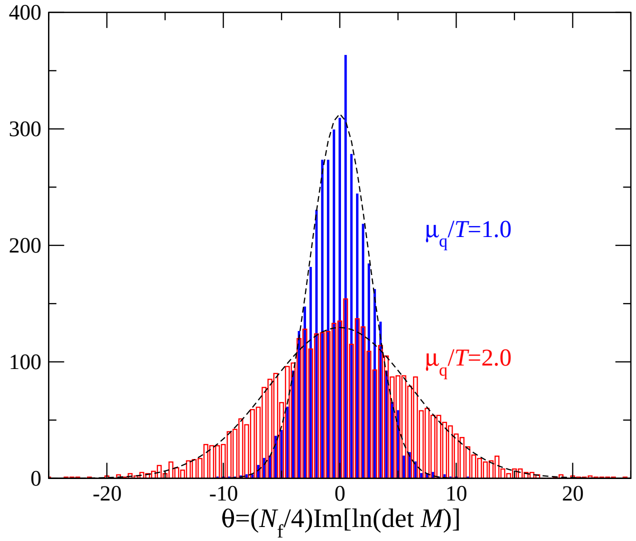
We have $2n(n+N_f)$ charged Goldstone particles. They are fermions. All uncharged Goldstone particles are bosons. We thus find

$$\langle e^{2in\theta}\rangle_{N_f} = e^{n(n+N_f)\underbrace{[G_0(\mu=0)-G_0(\mu)]}_{-\Delta G})}$$

# Phase Density

By Poisson resummation we obtain

$$\rho(\phi) = \sum_n e^{in\phi} e^{-n(n+N_f)\Delta G} = \frac{e^{\frac{1}{4}N_f^2\Delta G}}{\sqrt{\pi \Delta G}} e^{iN_f\phi - \frac{\phi^2}{\Delta G}}.$$




Phase density in lattice QCD.

Ejiri-2007

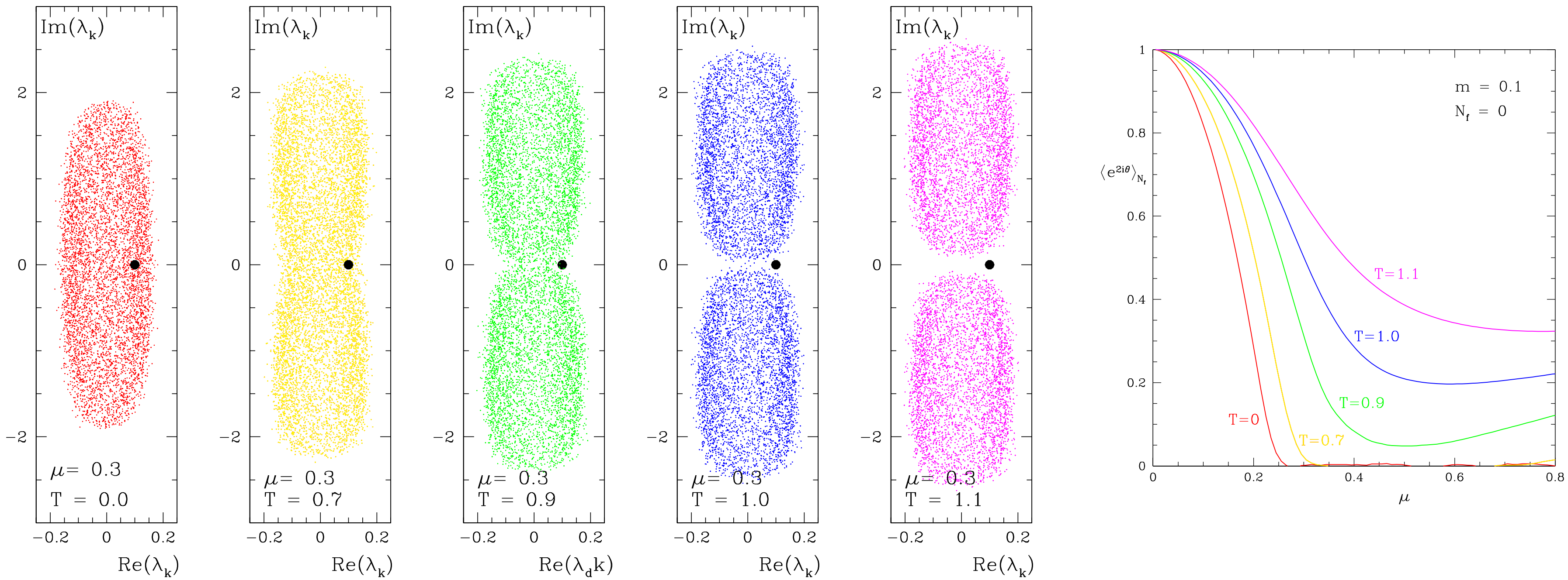
- Gaussian distribution modified by a phase.
- $\Delta G \sim VT^2\mu^2$ .
- Agrees (up to the overall phase) with lattice results by Ejiri obtained by Tailor expansion of the phase angle.

# IV. Phase Diagram of Average Phase Factor

$T$ -dependence of Phase Factor

Phase Diagram

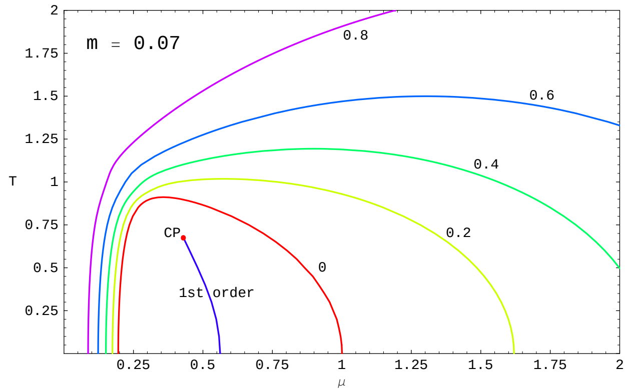
# Temperature Dependence of $\langle e^{2i\theta} \rangle$



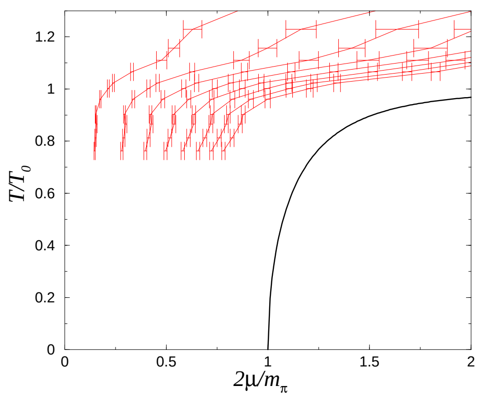

Scatter plot of Dirac eigenvalues obtained from a schematic chiral random matrix model. This random matrix model has the spectral flow of QCD and is equivalent to the zero momentum limit of a chiral Lagrangian.

The average phase factor becomes nonzero when the quark mass is outside the spectral support. The quark mass is indicated by the black dot. Ravagli-JV-2007

# Ingredients for Phase Diagram of the Average Phase Factor




Analytical random matrix result for phase diagram of average phase factor. Curves show contours of equal average phase factor. Han-Stephanov-2008




Lattice results showing contour lines with equal variance of the phase of the fermion determinant.

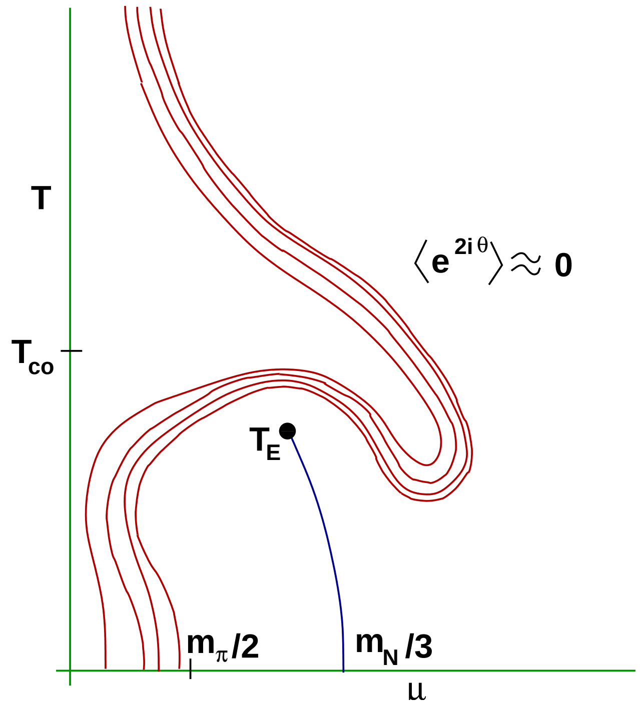
Allton-et al-2005, Splittorff-2006

- √ Lattice simulations are feasible around $T_{\rm co}$ and small chemical potential.
  Fodor-Katz-2002, Allton-et al-2002, d'Elia-Lombardo-2002,
  De Forcrand-Philipsen-2002,Gavai-Gupta-2003
- √ Weak coupling result of QCD valid for high temperatures

$$F_{\rm QCD}(\mu,T) - F_{|\rm QCD|}(\mu,T) \sim \alpha_s^2\mu^2T^2.$$

Ipp-Rebhan-2003, Vuorinen-2003

# Phase Diagram of the Average Phase Factor




Schematic “Phase diagram” of average phase factor at finite volume. Contour lines are curves with equal average phase factor.

There is a target of opportunity starting from $T_{co}$ to the region just above the critical end point.

Interesting physics requires lattice methods that can deal with the sign problem.

# V. Quenched Average Phase Factor and Analyticity in $\mu$

Quenched RMT result

Phase Factor at Imaginary Chemical Potential

# Quenched Average Phase Factor

- The *quenched* average phase factor is given by

$$\langle e^{2i\theta} \rangle_{\rm q} = \left\langle \frac{\prod_k (\lambda_k + m)}{\prod_k (\lambda_k^* + m)} \right\rangle_{\rm q}.$$

- This expression contains integrable poles.
- Is the quenched average phase factor analytic in $\mu$ ?
- We can answer this question in the microscopic domain of QCD where the QCD partition function is given by chiral random matrix theory.
- Using a version of the random matrix model proposed by Osborn (2004) the model is analytically solvable in terms of complex orthogonal polynomials.

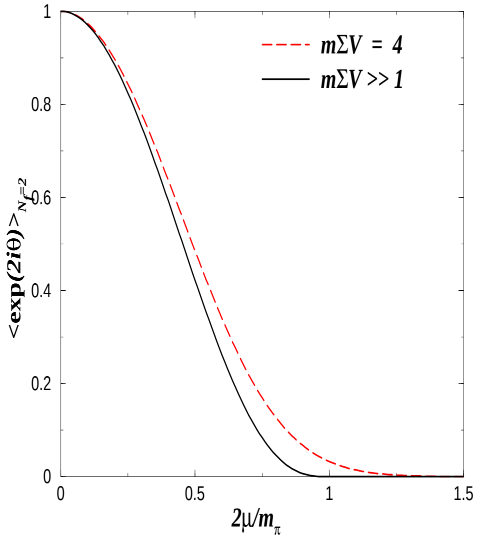
# Quenched RMT Result

$$\langle e^{2i\theta}\rangle_{N_f=0} = 1 - 4\hat{\mu}^2 I_0(\hat{m})K_0(\hat{m})$$

$$\hat{m} = mV\Sigma$$
$$\hat{\mu} = \mu - F_\pi\sqrt{V}$$

$$-e^{-2\hat{\mu}^2}\frac{1}{4\hat{\mu}^2}e^{-\frac{\hat{m}^2}{8\hat{\mu}^2}}\int_{\hat{m}}^{\infty} dxx\exp[-\frac{x^2}{4\hat{\mu}^2}]K_0\left(\frac{x\hat{m}}{4\hat{\mu}^2}\right)(I_0(x)\hat{m}I_1(\hat{m}) - xI_1(x)I_0(\hat{m}))\,,$$

Splittorff-JV-2007




Splittorff-JV-2006

- Reduces to mean field result for $N_f$ flavors,

$$\left(1 - \frac{4\mu^2}{m_\pi^2}\right)^{N_f+1}, \quad \mu < m_\pi/2,$$

  for $\hat{\mu} \to \infty,\ \hat{m} \to \infty$: and is exponentially suppressed for $\mu > m_\pi/2$.
- This expression has an essential singularity at $\mu = 0$ .
- What about analytical continuation to imaginary chemical potential?

# Average Phase Factor at Imaginary Chemical Potential

Analytical continuation of phase factor (Splittorff-Svetitsky-2007)

( $\det^*(D+m+mu\gamma_0) = \det(D+m-\mu\gamma_0)$ )

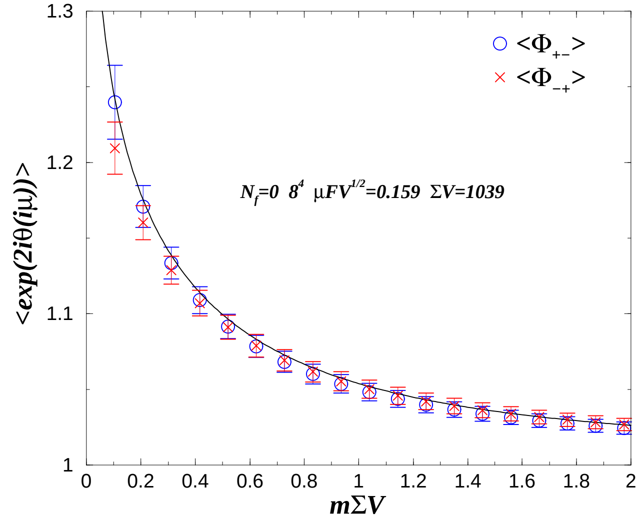
$$\left\langle \frac{\det(D+m+i\mu\gamma_0)}{\det(D+m-i\mu\gamma_0)} \right\rangle$$

Has been evaluated analytically in the microscopic domain of QCD. In the quenched case we find

$$1 - 4\hat{\mu}^2 I_0(\hat{m}) K_0(\hat{m}).$$

Damgaard-Splittorff-2006

Splittorff-JV-2006




“Phase” of the fermion determinant for imaginary chemical potential.

Splittorff-Svetitsky-2007

# Discussion of Quenched Phase Factor

$$\langle e^{2i\theta}\rangle_{N_f=0} = 1 - 4\hat{\mu}^2 I_0(\hat{m})K_0(\hat{m})$$

Splittorff-JV-2007

$$-e^{-2\hat{\mu}^2}\frac{1}{4\hat{\mu}^2}e^{-\frac{\hat{m}^2}{8\hat{\mu}^2}}\int_{\hat{m}}^{\infty} dx x \exp[-\frac{x^2}{4\hat{\mu}^2}]K_0\left(\frac{x\hat{m}}{4\hat{\mu}^2}\right)(I_0(x)\hat{m}I_1(\hat{m}) - xI_1(x)I_0(\hat{m}))\,,$$

- √ The first two terms can be obtained by analytical continuation from imaginary chemical potential.
- √ The second term has an essential singularity at $\mu = 0$ and cannot be obtained by analytical continuation.
- √ The second term nullifies the first term for $\mu > m_\pi/2$.
- √ The quenched average phase factor is also nonanalytic for QCD in 1d.
- √ The question is why the average phase factor is nonanalytic, and whether this should be a warning sign for other observables.

# VI . Conclusions

- √ The sign problem is severe when the quark mass is inside the support of the Dirac eigenvalues (i.e. for $\mu > m_\pi/2)$ .

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# VI . Conclusions

- The sign problem is severe when the quark mass is inside the support of the Dirac eigenvalues (i.e. for $\mu > m_\pi/2)$ .
- For $T < F_\pi$ , the sign problem becomes manageable in the microscopic domain of QCD ( $\mu^2 F_\pi^2 V \sim O(1)$ ).
- In the domain of validity of chiral perturbation theory the distribution of the phase of the quark determinant is a Gaussian modified by a complex phase.

# VI . Conclusions

- The sign problem is severe when the quark mass is inside the support of the Dirac eigenvalues (i.e. for $\mu > m_\pi/2)$ .
- For $T < F_\pi$ , the sign problem becomes manageable in the microscopic domain of QCD ( $\mu^2 F_\pi^2 V \sim O(1)$ ).
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- √ In the microscopic domain of QCD the quenched average phase factor is nonanalytic in $\mu$ . We expect tjat this nonanalyticity does not occur in observables that are derivatives of the free energy.

# Spectral Density for $N_f = 1$

The spectral density can be decomposed as

$$\hat{\rho}_{N_f=1}(\hat{x},\hat{y},\hat{m};\hat{\mu}) = \hat{\rho}_Q(\hat{x},\hat{y};\hat{\mu}) + \hat{\rho}_R(\hat{x},\hat{y},\hat{m};\hat{\mu}),$$

with ($\hat{z} = \hat{x} + i\hat{y}$)

$$\begin{aligned}\hat{\rho}_R(\hat{x},\hat{y},\hat{m};\hat{\mu}) \quad &= \quad \frac{|\hat{z}|^2}{2\pi\hat{\mu}^2} e^{-(\hat{z}^2+\hat{z}^{*\,2})/(8\hat{\mu}^2)} \\ &\times K_0(\frac{|\hat{z}|^2}{4\hat{\mu}^2})\frac{I_0(\hat{z})}{I_0(\hat{m})}\int_0^1 dt\, t e^{-2\hat{\mu}^2 t^2} I_0(\hat{z}^* t) I_0(\hat{m}t).\end{aligned}$$

Quenched spectral density

$$\hat{\rho}_Q(\hat{x},\hat{y};\hat{\mu}) = \hat{\rho}_U(\hat{x},\hat{y},\hat{x}+i\hat{y};\hat{\mu}).$$

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