Relativistic conformal hydrodynamics and holography

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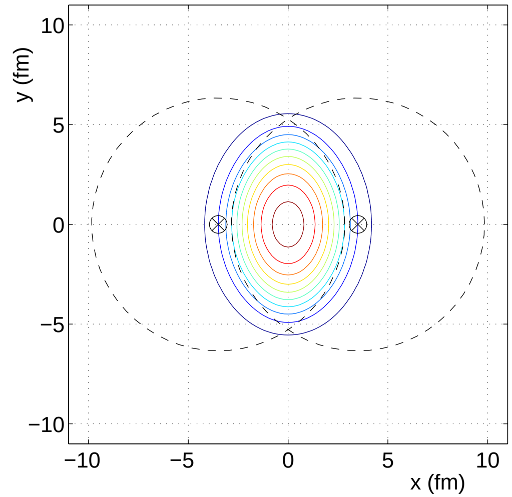
Motivation

- Relativistic Heavy Ion Collisions
- Traditional path: kinetic description $\Rightarrow$ hydrodynamics
- Discovery of sQGP: hydrodynamics but no kinetic description
    - i.e QFT $\Rightarrow$ hydrodynamics.
- Strong coupling regime of some SUSY gauge theories can be studied using AdS/CFT (holographic) correspondence.
    - i.e., instead of QFT $\Rightarrow$ kinetic description (Boltzmann) $\Rightarrow$ hydrodynamics,
QFT $\Rightarrow$ holographic description $\Rightarrow$ hydrodynamics
- This talk:
    - Introduction
    - Hydrodynamics as an effective theory
    - Finding kinetic coeff. by matching to AdS/CFT.

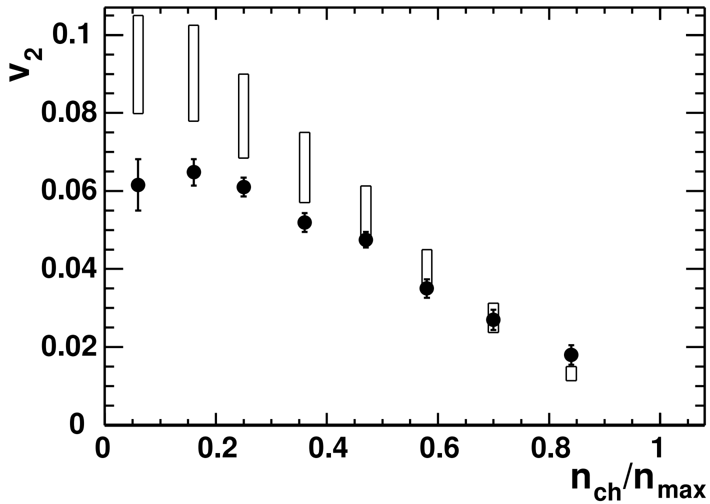
Hydrodynamic modeling of R.H.I.C. and v2

Approach: take an equation of state, initial conditions, and solve hydrodynamic equations to get particle yields, spectra, etc.

from Kolb/Heinz review



- v2 – a measure of elliptic flow is a key observable.
- Pressure gradient is large in-plane. This translates into momentum anisotropy. To do this the plasma must do work, i.e., pressure$\times \Delta V$

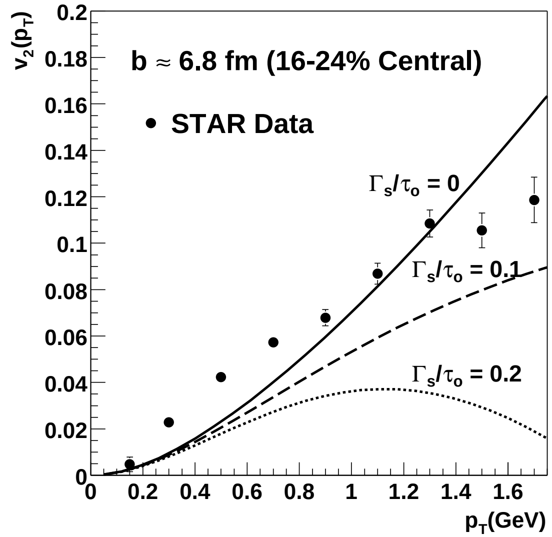


- v2 is large $\rightarrow$ 1st conclusion, there is pressure, and it builds very early.
  I.e., plasma thermalizes early ($< 1\mathrm{fm}/c$).
- BIG theory question: HOW does it thermalize? and why so fast/early?
- Need to understand initial conditions
- Mechanism of thermalization?
  Plasma instabilities?

Small viscosity and sQGP (liquid)

Another surprise: where is the viscosity?

Teaney



- Ideal hydro already agrees with data.
- Adding even a small viscous correction makes the agreement worse (Teaney, Romatschke, . . .)
- If the plasma was weakly interacting the viscosity $\frac{\eta}{T^3} \sim (\text{coupling})^{-2}$ would be large.
- Conclusion: the plasma must be strongly coupled – it is a liquid.

- Can there be an ideal liquid, can $\eta = 0$? What if coupling $\to \infty$?
- Policastro, Kovtun, Son, Starinets found that in an $\mathcal{N} = 4$ super-Yang-Mills theory at ∞ coupling $\eta = s/(4\pi)$. And so is in a class of theories with infinite coupling. Special to AdS/CFT, or a universal lower bound?
- If $\frac{\eta}{s} = \frac{1}{4\pi}$ is the lowest bound – data suggests RHIC produced an almost perfect fluid.
- Need viscous (3D) hydro simulation to confirm.
- Second-order corrections?

Scales and hydrodynamics

- Hydrodynamics is an effective macroscopic theory, describing transport of energy, momentum and other conserved quantities.
- The domain of validity is large distance and time scales (small k and ω).
- If the underlying kinetic description exists, there is a mean free path, $\ell_{\rm mfp}$. The scale where hydrodynamics applies is greater than $\ell_{\rm mfp}$.
- In a strongly coupled system (e.g., sQGP at RHIC) kinetic description may not exist. Then the domain of validity is set by a typical microscopic scale, e.g., T^{-1}.
- Hydrodynamics can be described as an expansion in gradients.
- To lowest order – ideal hydrodynamics.
- The expansion parameter – $k\ell_{\rm micro}$.

Hydrodynamic degrees of freedom and equations

- Densities of conserved quantities. In any field theory at least energy and momentum densities T^{00} and T^{0i}.
- Convenient covariant variables:
  - ε – T^{00} in the local rest frame (where $T^{0i} = 0$); and
  - u^μ – local 4-velocity (the velocity of the local rest frame).

Then, by Lorentz covariance:

$$T^{\mu\nu} \equiv \varepsilon\, u^\mu u^\nu + T_\perp^{\mu\nu}$$

where $T_\perp^{\mu\nu}$ – has only spatial components in local rest frame (i.e., $u_\mu T_\perp^{\mu\nu} = 0$).

- The components of $T_\perp^{\mu\nu}$ are *not* independent variables, but (local, instantaneous) functions of ε and u^μ.

$$T_\perp^{\mu\nu} = P(\varepsilon)\Delta^{\mu\nu} + \text{terms with gradients}$$

where the symmetric, transverse ($\perp$) tensor with no derivatives is

$$\Delta^{\mu\nu} \equiv g^{\mu\nu} + u^\mu u^\nu \,,$$

- 4 variables and 4 equations: $\nabla_\mu T^{\mu\nu} = 0$.

First order order hydrodynamics

- Without gradient terms – ideal hydrodynamics.
- To first order in gradients:

$$T_\perp^{\mu\nu} = P(\varepsilon)\Delta^{\mu\nu} \underbrace{- \eta(\varepsilon)\sigma^{\mu\nu} - \zeta(\varepsilon)\Delta^{\mu\nu}(\nabla\cdot u) + \text{higher derivs.}}_{\text{viscous stress } \Pi_{\mu\nu}}$$

where viscous strain (traceless, or shear part of it):

$$\sigma^{\mu\nu} = 2^{\langle}\nabla^\mu u^{\nu\rangle}$$

$$\langle A^{\mu\nu}\rangle \stackrel{\text{def}}{=} \frac{1}{2}\Delta^{\mu\alpha}\Delta^{\nu\beta}(A_{\alpha\beta} + A_{\beta\alpha}) - \frac{1}{d-1}\Delta^{\mu\nu}\Delta^{\alpha\beta}A_{\alpha\beta}$$

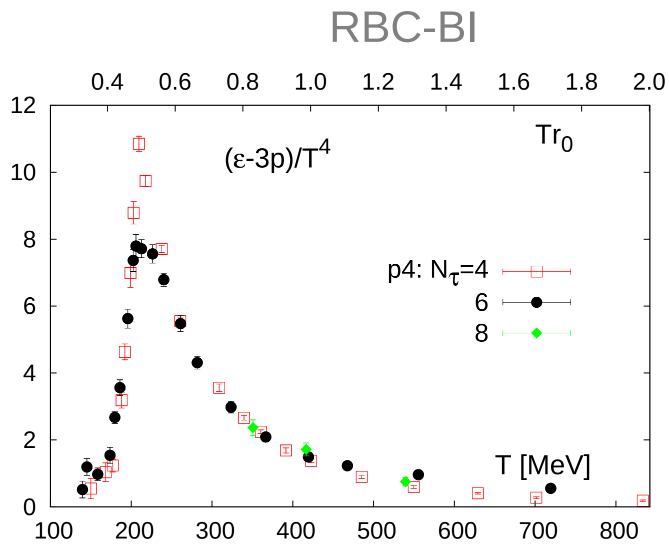
($\Delta^{\mu\nu}$ projects on $\perp u^\mu$).

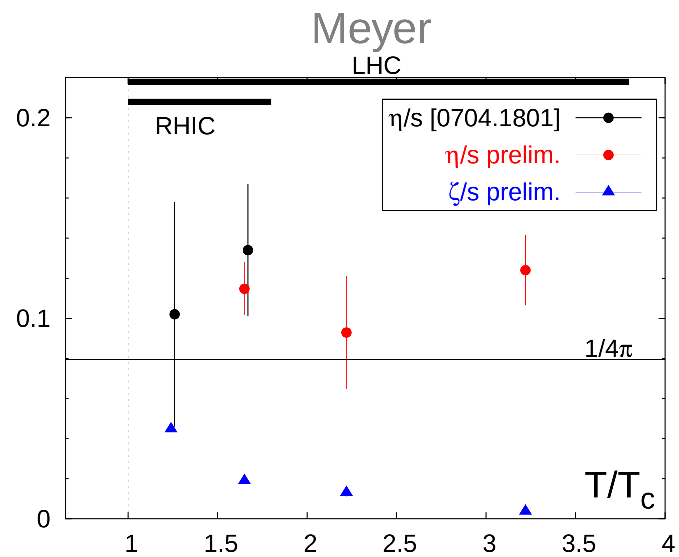
- η and ζ – shear and bulk viscosities.
- T^{ij} – rate of momentum transfer (flow), i.e., force/area

$\zeta(\nabla\cdot u)$ – contribution to isotropic pressure due to gradients;
$\eta\sigma^{\mu\nu}$ – drag force due to the gradients of velocity $\perp$ to the velocity – shear stress.

Conformal theories

- Why could this be relevant to QCD?
  - QCD at $T > 2T_c$ is almost conformal (but still strongly coupled).





  - AdS/CFT

Scale invariance and Weyl symmetry

- Consider a field theory with no scale, self-similar under dilation $x \to \lambda x$ (accompanied by appropriate rescaling of fields). $\lambda = \mathrm{const}$ here.

- Examples: ferromagnet at a critical point, $N = 4$ SUSY YM.

- Instead of coordinate rescaling one can formally do $g_{\mu\nu} \to \lambda^{-2} g_{\mu\nu}$.

- One can then promote $g_{\mu\nu} \to \lambda^{-2} g_{\mu\nu}$ to *local* symmetry, i.e., generalize the theory to curved space in such a way that the action (as a functional of background metric) is invariant under *local* Weyl transformations (in addition to GR transforms):

$$g_{\mu\nu} \longrightarrow e^{-2\omega(x)} g_{\mu\nu}.$$

- For example, since $T^{\mu\nu} \equiv \delta S / \delta g_{\mu\nu}$

$$T^{\mu}_{\mu} = g_{\mu\nu} T^{\mu\nu} = -(1/2) \delta S / \delta\omega = 0$$

Conformal hydrodynamics (to 1st order)

- Using just tracelessness $T^\mu_\mu = 0$ constrains these coefficients $(\Delta^\mu_\mu = d-1)$:
  - $P = \dfrac{\varepsilon}{d-1}$; - $\zeta = 0.$
- To use Weyl invariance we need transformation properties of hydro variables:
  - By dimensions: $T \to e^\omega T$ and $\varepsilon = \# \cdot T^d$. (We shall use T below.)
  - $g_{\mu\nu} u^\mu u^\nu = -1$ means $u^\mu \to e^\omega u^\mu$.
- Since $T^{\mu\nu}\sqrt{-g} = \delta S / \delta g_{\mu\nu}$,

$$T^{\mu\nu} \to e^{(d+2)\omega}\, T^{\mu\nu};$$

  - More nontrivially, $\sigma^{\mu\nu} \equiv 2^{\langle} \nabla^\mu u^{\nu\rangle}$ transforms *homogeneously*

$$\sigma^{\mu\nu} \to e^{3\omega} \sigma^{\mu\nu},$$

hence $\eta = \# \cdot T^{d-1}$.

Second-order hydrodynamics

- Need to find all possible contributions to $T_\perp^{\mu\nu}$ with 2 derivatives, transforming *homogeneously* under Weyl transform.
  - Also: use 0-th order equations:

$$D\ln T = -\frac{1}{d-1}(\nabla_\perp \cdot u), \quad Du^\mu = -\nabla_\perp^\mu \ln T,$$

to convert temporal derivatives ($D \equiv u^\mu \nabla_\nu$) into spatial $(\nabla_\perp^\mu \equiv \Delta^{\mu\alpha}\nabla_\alpha)$.

- $\exists$ five such terms:

$$\mathcal{O}_1^{\mu\nu} = R^{\langle\mu\nu\rangle} - (d-2)\left(\nabla^{\langle\mu}\nabla^{\nu\rangle}\ln T - \nabla^{\langle\mu}\ln T\,\nabla^{\nu\rangle}\ln T\right),$$
$$\mathcal{O}_2^{\mu\nu} = R^{\langle\mu\nu\rangle} - (d-2)u_\alpha R^{\alpha\langle\mu\nu\rangle\beta}u_\beta\,,$$
$$\mathcal{O}_3^{\mu\nu} = \sigma^{\langle\mu}{}_\lambda\sigma^{\nu\rangle\lambda}\,, \qquad \mathcal{O}_4^{\mu\nu} = \sigma^{\langle\mu}{}_\lambda\Omega^{\nu\rangle\lambda}\,, \qquad \mathcal{O}_5^{\mu\nu} = \Omega^{\langle\mu}{}_\lambda\Omega^{\nu\rangle\lambda}\,.$$

where $\Omega^{\mu\nu} = \Delta^{\mu\alpha}\Delta^{\nu\beta}\nabla_{[\alpha}u_{\beta]}$ – vorticity.

  - Only $\mathcal{O}_1^{\mu\nu}$ and $\mathcal{O}_2^{\mu\nu}$ contribute in *linearized* hydrodynamics.
  - $\mathcal{O}_2^{\mu\nu} = 0$ in flat space.

Second-order kinetic coefficients

- Convenient to use this combination $\mathcal{O}_1^{\mu\nu} - \mathcal{O}_2^{\mu\nu} - (1/2)\mathcal{O}_3^{\mu\nu} - 2\mathcal{O}_5^{\mu\nu}$ equal to

$$
{}^{\langle} D\sigma^{\mu\nu\rangle} + \frac{1}{d-1}\sigma^{\mu\nu}(\nabla\cdot u)
$$

- Stress tensor to 2-nd order:

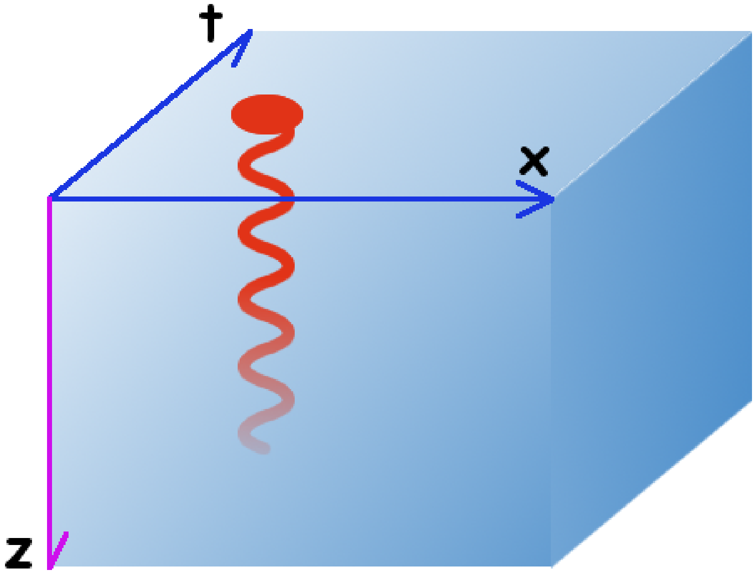
$$
\begin{aligned}
T_{\perp}^{\mu\nu} &= P\Delta^{\mu\nu} - \eta\sigma^{\mu\nu} \\
&\quad + \eta\tau_{\Pi}\left[{}^{\langle} D\sigma^{\mu\nu\rangle} + \frac{1}{d-1}\sigma^{\mu\nu}(\nabla\cdot u)\right] + \kappa\left[R^{\langle\mu\nu\rangle} - (d-2)u_{\alpha}R^{\alpha\langle\mu\nu\rangle\beta}u_{\beta}\right] \\
&\quad + \lambda_1\sigma^{\langle\mu}{}_{\lambda}\sigma^{\nu\rangle\lambda} + \lambda_2\sigma^{\langle\mu}{}_{\lambda}\Omega^{\nu\rangle\lambda} + \lambda_3\Omega^{\langle\mu}{}_{\lambda}\Omega^{\nu\rangle\lambda}\,.
\end{aligned}
$$

- The five new coefficients are τ_{Π}, κ, $\lambda_{1,2,3}$.

- Nonlinear term $\sigma^{\mu\nu}\nabla\cdot u$ has until recently been often omitted. We see this term is necessary for conformal invariance.

AdS/CFT

The 4d $N = 4$ SUSY YM theory in strong coupling limit can be represented by a semiclassical gravitational theory in 5d.



$$S = \int d^5x \sqrt{-g}(R - 2\Lambda)$$

- Recipe for calculating a correlator of, e.g., $T^{\mu\nu}$:

Vary boundary value at $z = 0$ of $g^{\mu\nu}$, then

$$\langle T^{\mu\nu}(x) \rangle = \frac{\delta S}{\delta g_{\mu\nu}(x, 0)}.$$

Kinetic coefficients from AdS/CFT

- Example: match the following correlator in hydrodynamics:

$$\langle T^{xy}T^{xy}\rangle(\omega,k)_{\rm ret} = P - i\eta\omega + \eta\tau_\Pi\omega^2 - \frac{\kappa}{2}[(d-3)\omega^2 + k^2]\,.$$

to gravity calculation and find

- $$P = \frac{\pi^2}{8}N_c^2T^4, \quad \eta = \frac{\pi}{8}N_c^2T^3, \quad \underbrace{\tau_\Pi = \frac{2-\ln 2}{2\pi T}\,, \quad \kappa = \frac{\eta}{\pi T}\,.}_{\text{new}}$$

  - Nontrivial cross-checks in sound and shear channels.

- Using solution to nonlinear equations found by Heller and Janik (asymptotics at large τ of Bjorken boost-invariant flow):

  - $$\lambda_1 = \frac{\eta}{2\pi T}$$

- Bhattacharyya, Hubeny, Minwalla, Rangamani: $\lambda_2 = \dfrac{2\eta\ln 2}{\pi T}; \qquad \lambda_3 = 0.$

- In kinetic (weakly coupled) theory:

$$\tau_\Pi \sim \frac{\eta}{Ts} \gg \frac{1}{T}.$$

$$\kappa = 0(?)$$

Müller-Israel-Stewart

- Truncate the gradient expansion at second order.
- Use $\Pi^{\mu\nu} = -\eta\sigma^{\mu\nu}$ in second-order terms.
- Resulting equations are hyperbolic (causal) even outside of domain of validity (large gradients) – good for simulations.
- Transverse momentum modes (shear) obey diffusion equation similar to:

$$\partial_t \rho = -\boldsymbol{\nabla} \boldsymbol{j}$$

with

$$\boldsymbol{j} = -D\boldsymbol{\nabla}\rho$$

Which means $\partial_t \rho = D\boldsymbol{\nabla}^2\rho$ - parabolic. Disturbance propagates with infinite speed? Problem even for nonrelativistic case?

- Now use instead:

$$\boldsymbol{j} = -D\boldsymbol{\nabla}\rho - \tau\partial_t \boldsymbol{j}$$

- This system is hyperbolic, with characteristic velocity:

$$v_{\text{disc}} = \sqrt{D/\tau}$$

- The problem is only in the regime ($k\ell \gtrsim 1$) where hydrodynamics is inapplicable. There are no actual modes which propagate with v_{disc}.

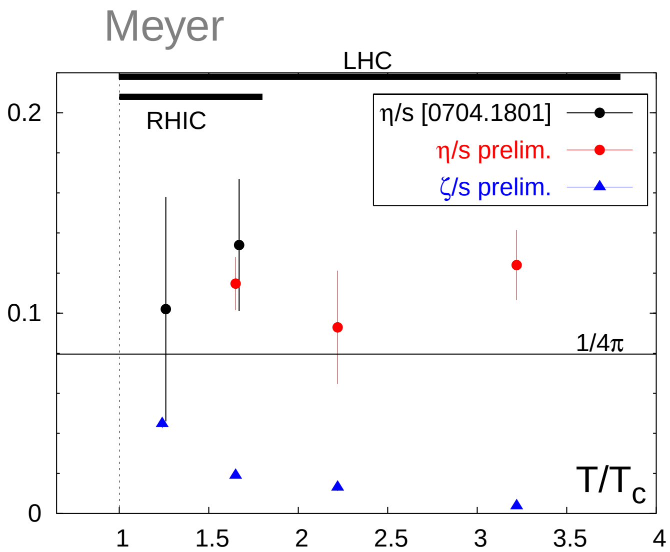
Summary

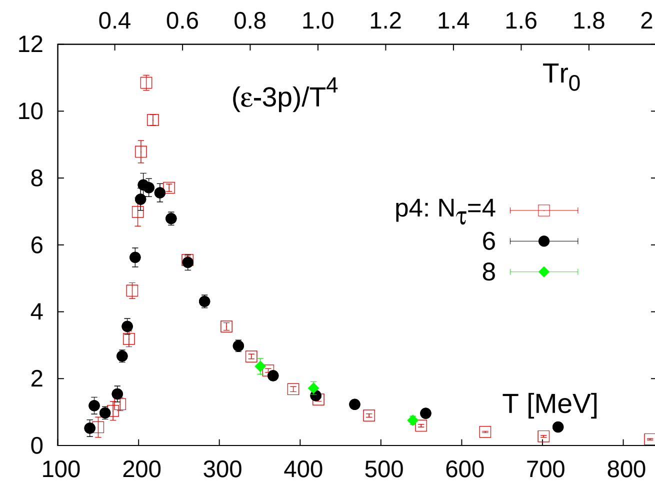
- Hydrodynamics is an expansion in gradients of hydrodynamic variables.
- In conformal theories (e.g., QCD above $2T_c$) the form of the equations (stress tensor) are restricted.
- To first order: only one viscosity coefficient η.
- To second order: only 5 (in curved space) coefficients.
- For $N = 4$ SUSY YM at strong coupling (and large N_c) the coefficients have been determined using AdS/CFT.

Appendix

Viscosity on the lattice

- Difficult problem: need to get large *real*-time behavior of a correlation function, from Euclidean (*imaginary*) time measurements.
- Numerical noise must be very low.
- Must assume that extrapolation to large times (low frequencies) is smooth.





- At $T \sim 1 - 3.5\, T_c$ η/s is close to $1/(4\pi)$
- The bulk viscosity vanishes quickly above $T \sim 2T_c$. The latter is in agreement with trace anomaly calculation by RBC-BI $\rightarrow$

Entropy and the second law