From fermions to loop and dimer models

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University of Bern

XQCD08 – Raleigh NC, 21-23 July 2008

The phase diagram of $N_f = 8$ QCD at finite isospin density

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with

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Lattice 2007 – Regensburg, 31 July 2007

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Definition of the model

- Euclidean lagrangian density in 2D [Gross, Neveu '74]

$$\mathcal{L} = \sum_{\alpha=1}^{N} \bar{\psi}^{\alpha}(x) \not{\partial}\ \psi^{\alpha}(x) - \frac{g^2}{2} \left(\sum_{\alpha=1}^{N} \bar{\psi}^{\alpha}(x) \psi^{\alpha}(x) \right)^2 ,$$

  where $\psi^{\alpha}(x)$ are 2-component Dirac spinors and α flavour index.

- Introduce a scalar field $\sigma(x)$ conjugate to $\sum_{\alpha=1}^{N} \bar{\psi}^{\alpha}(x) \psi^{\alpha}(x)$:

$$\mathcal{L} = \sum_{\alpha=1}^{N} \bar{\psi}^{\alpha}(x) \not{\partial}\ \psi^{\alpha}(x) + \frac{1}{2g^2} \sigma(x)^2 + \sigma(x) \sum_{\alpha=1}^{N} \bar{\psi}^{\alpha}(x) \psi^{\alpha}(x).$$

Properties

- The Gross-Neveu model
  - is renormalisable and asymptotically free,

$$\beta(g) = -\frac{N-1}{2\pi} g^3 + O(g^5),$$

  - has a $O(2N) \times \Gamma$-symmetry where Γ is the discrete chiral symmetry

$$\Gamma : \qquad \psi \to \gamma_5 \psi, \quad \bar{\psi} \to -\bar{\psi}\gamma_5, \quad \sigma \to -\sigma,$$

  - exhibits spontaneous breaking of the discrete chiral symmetry

    $\Rightarrow$ fermions acquire non-vanishing mass $\sigma_0 = \langle \sigma \rangle$ (dimensional transmutation).

Note: there is no Goldstone boson due to Γ being a discrete symmetry.

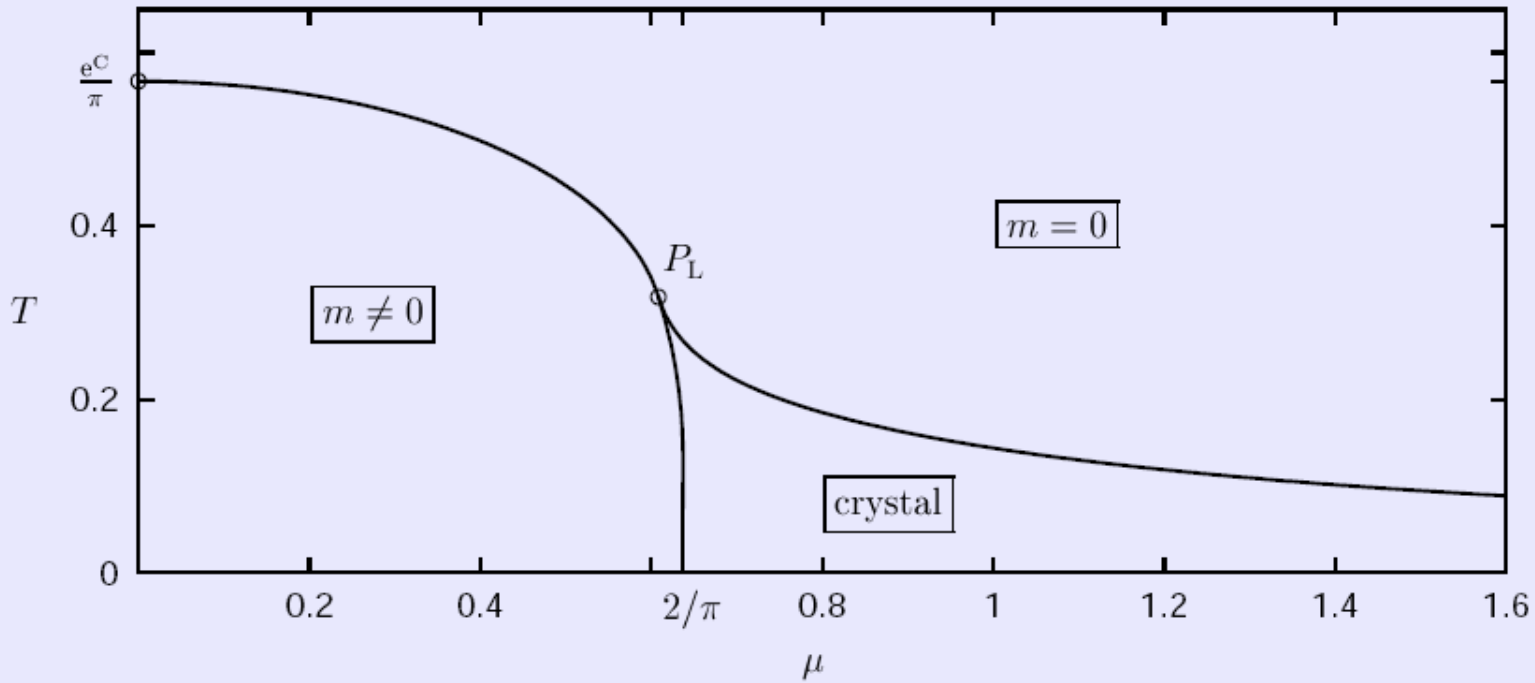
Spectrum of the GN model

- To leading order in $1/N$ the spectrum consists of [Dashen, Hasslacher, Neveu '75; Feinberg, Zee '97]
  - single fermions,
  - n-fermion bound states,
  - baryons (kink-antikink state).

- The GN model possesses a rich μ-T phase structure [Dashen, Ma, Rajaraman '75; Wolff '85; Karsch, Kogut, Wyld '87].

The revised phase diagram

- The structure of cold baryonic matter has only recently been clarified [Thies, Schön, Brzoska, Schnetz, Urlichs '00-'06; Dunne, Baym '08].
- In addition to the massive and massless Fermi gas, there is a new baryonic crystal phase at low temperature:

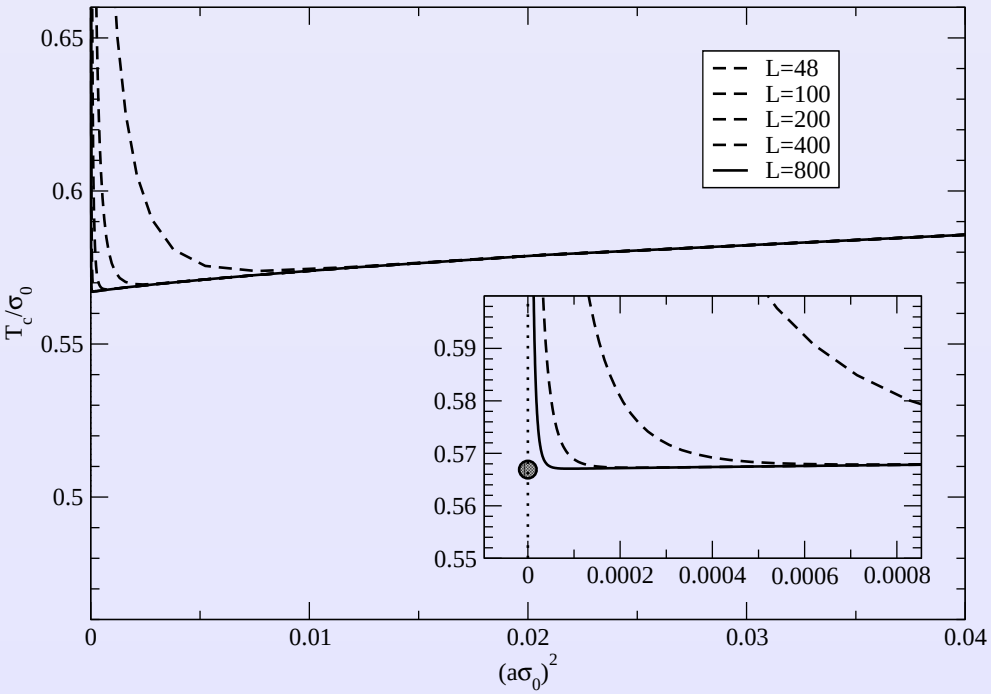


- $\mu_c = \frac{2}{\pi}$ consistent with m_B, no first order transition at $\mu \neq 0$.

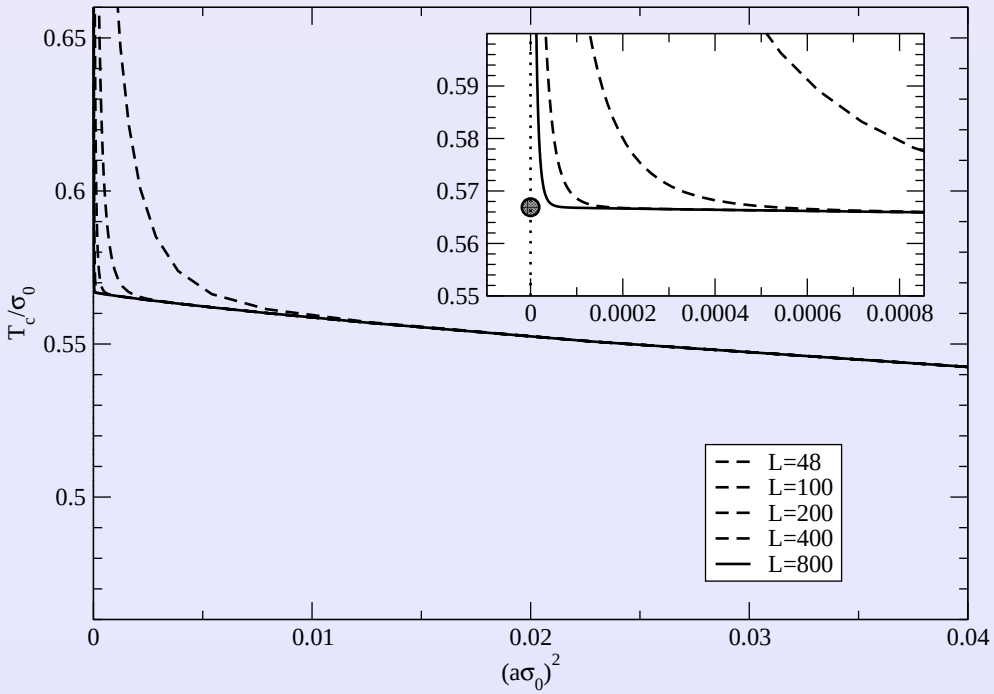
GN model on the lattice

- One can use staggered or overlap fermions, both of which preserve the discrete chiral symmetry [de Forcrand, Wenger '06].
- Scaling of T_c/σ_0 vs $(a\sigma_0)^2$ $\Rightarrow$ universality at work:

staggered operator



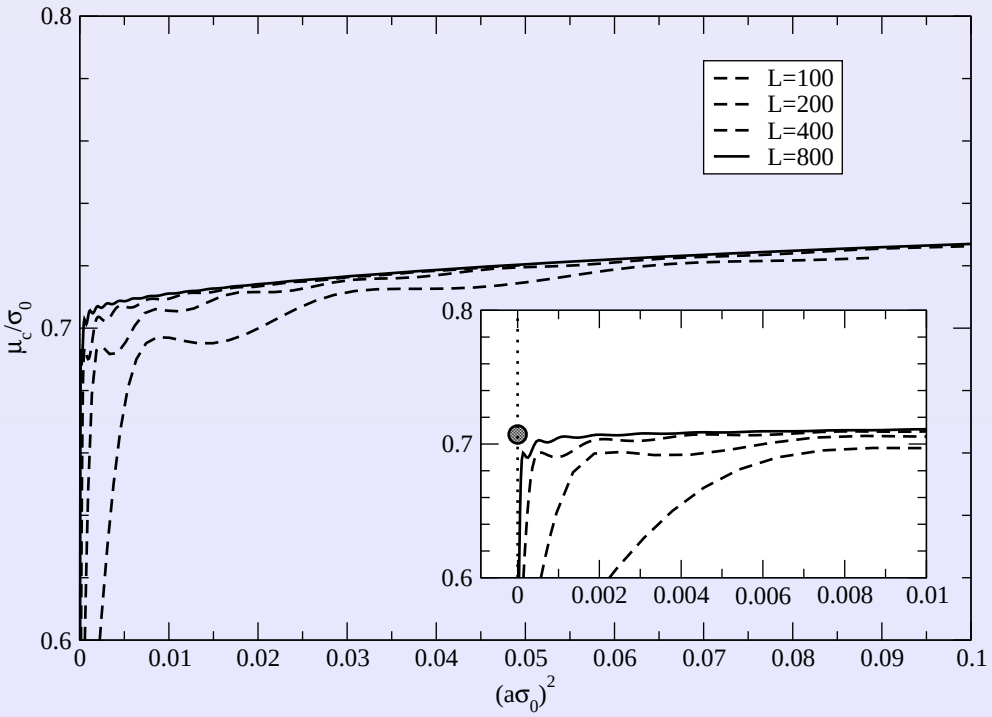
overlap operator



Homogeneous mean field results

- Scaling of μ_c/σ_0 vs $(a\sigma_0)^2$:

staggered operator



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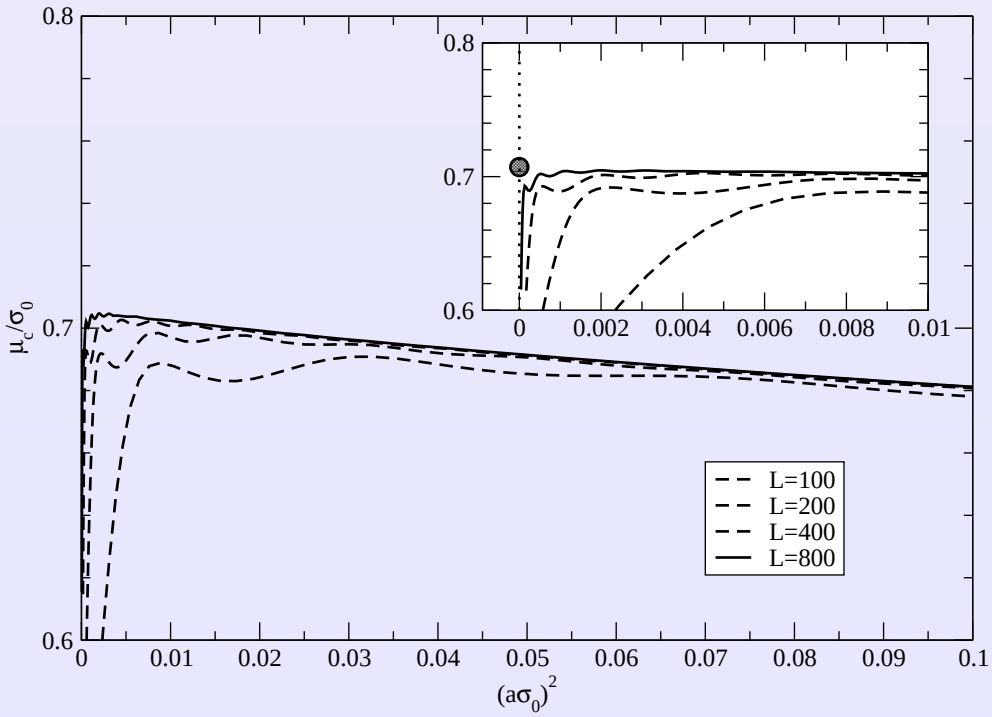
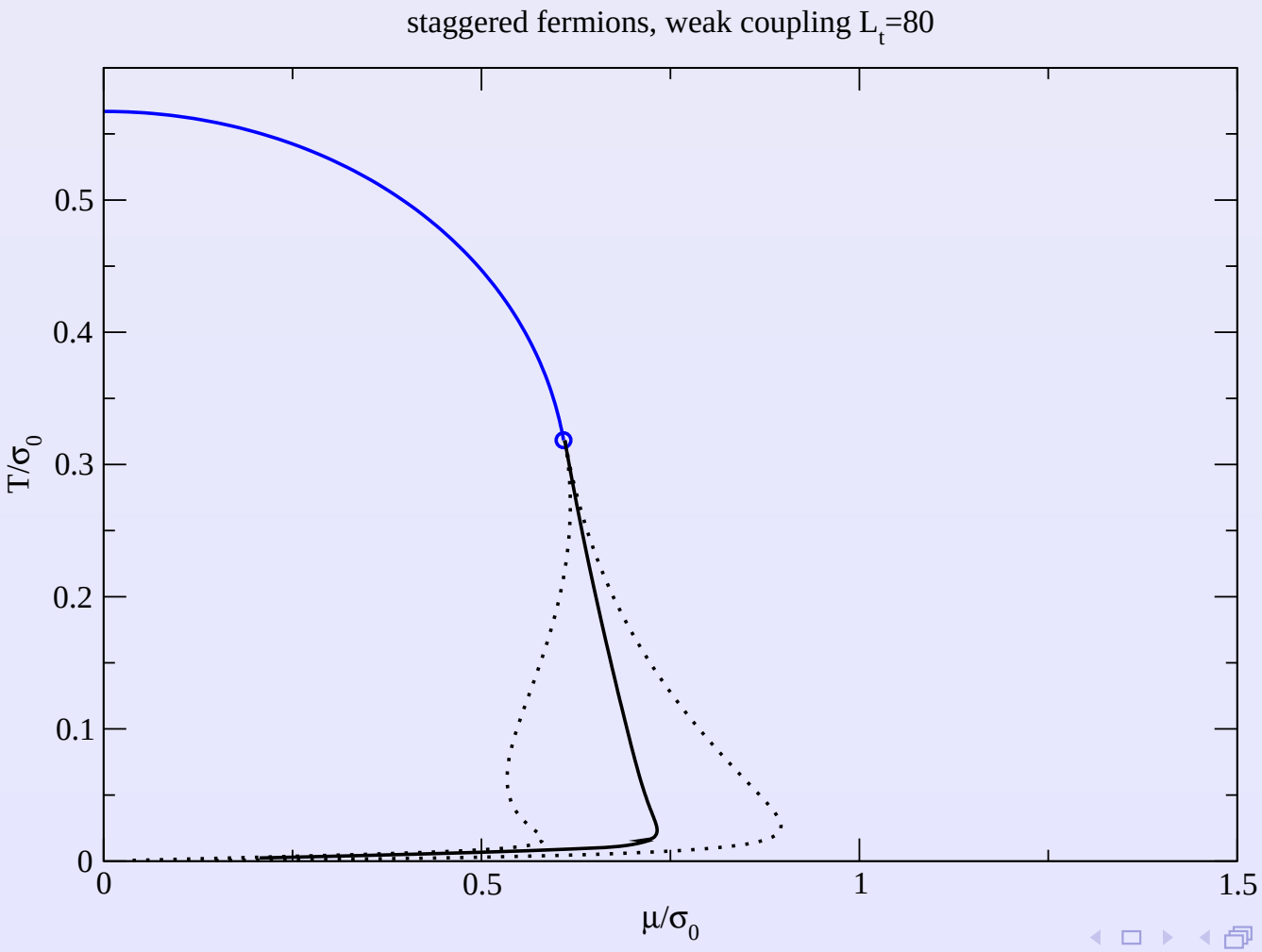


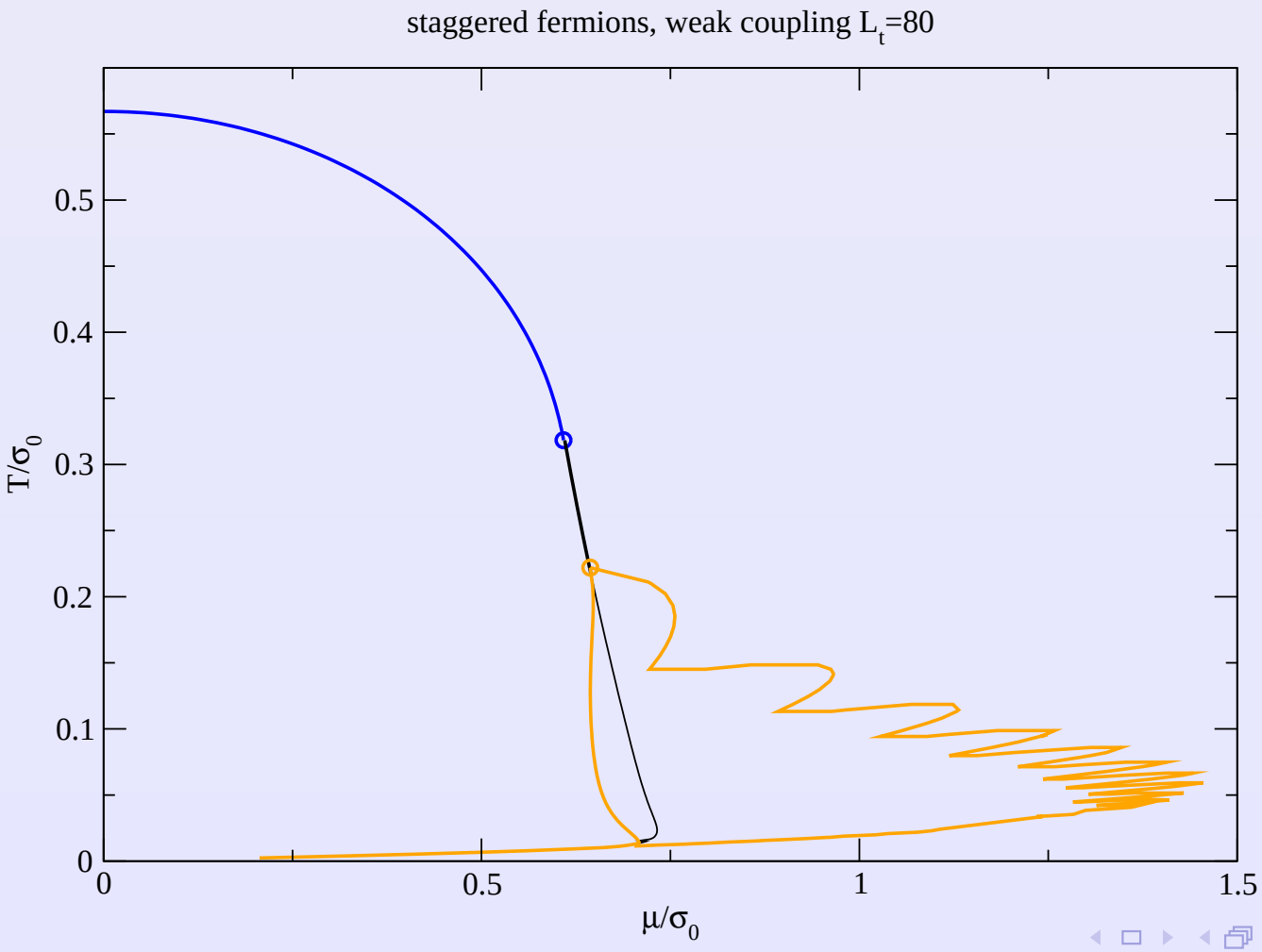
Crystal phase results

- Phase diagram:



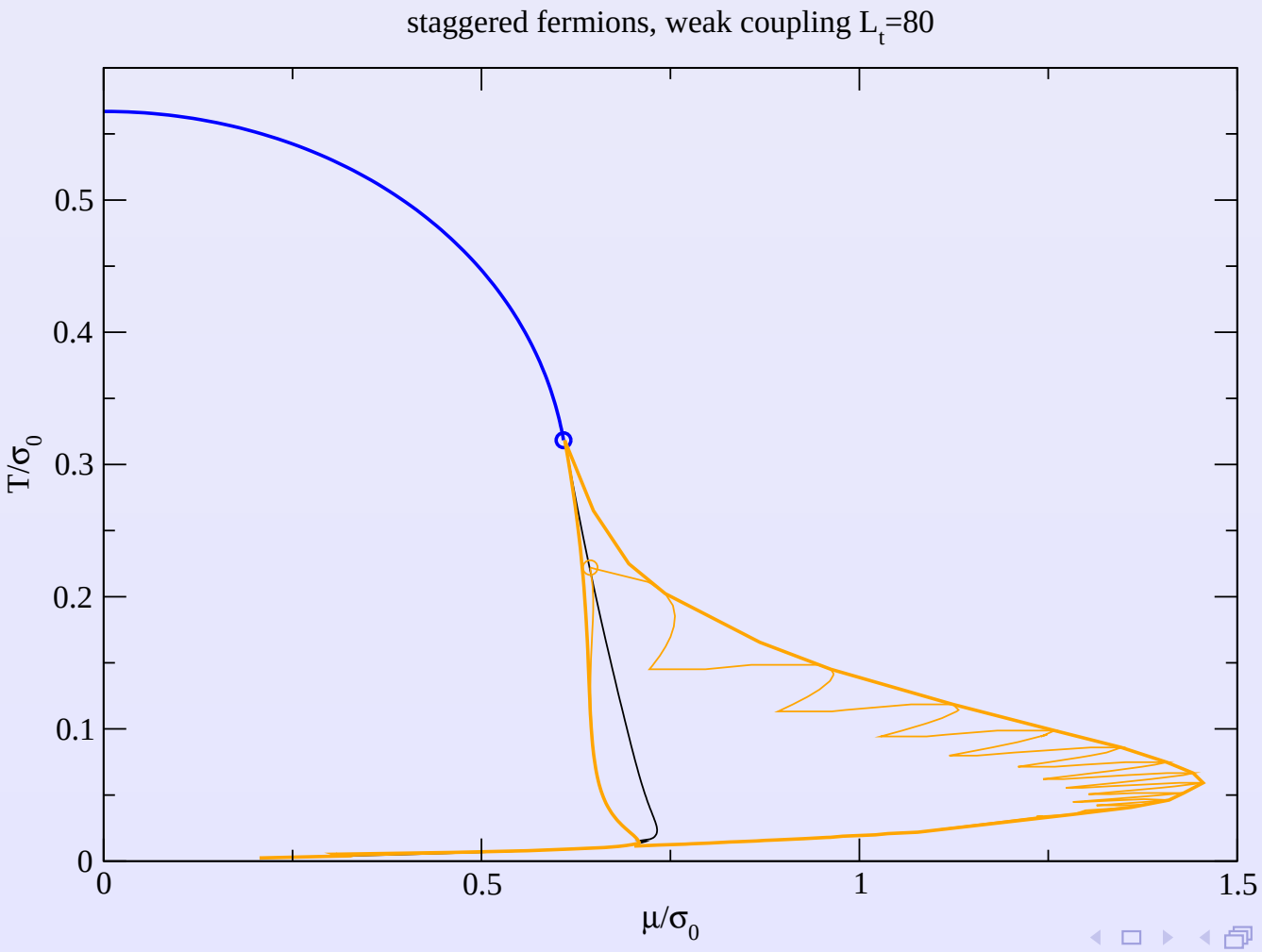
Crystal phase results

- Phase diagram with crystal phase, unit cell $l_x = 80$:



Crystal phase results

- Phase diagram with crystal phase, thermodynamic limit:



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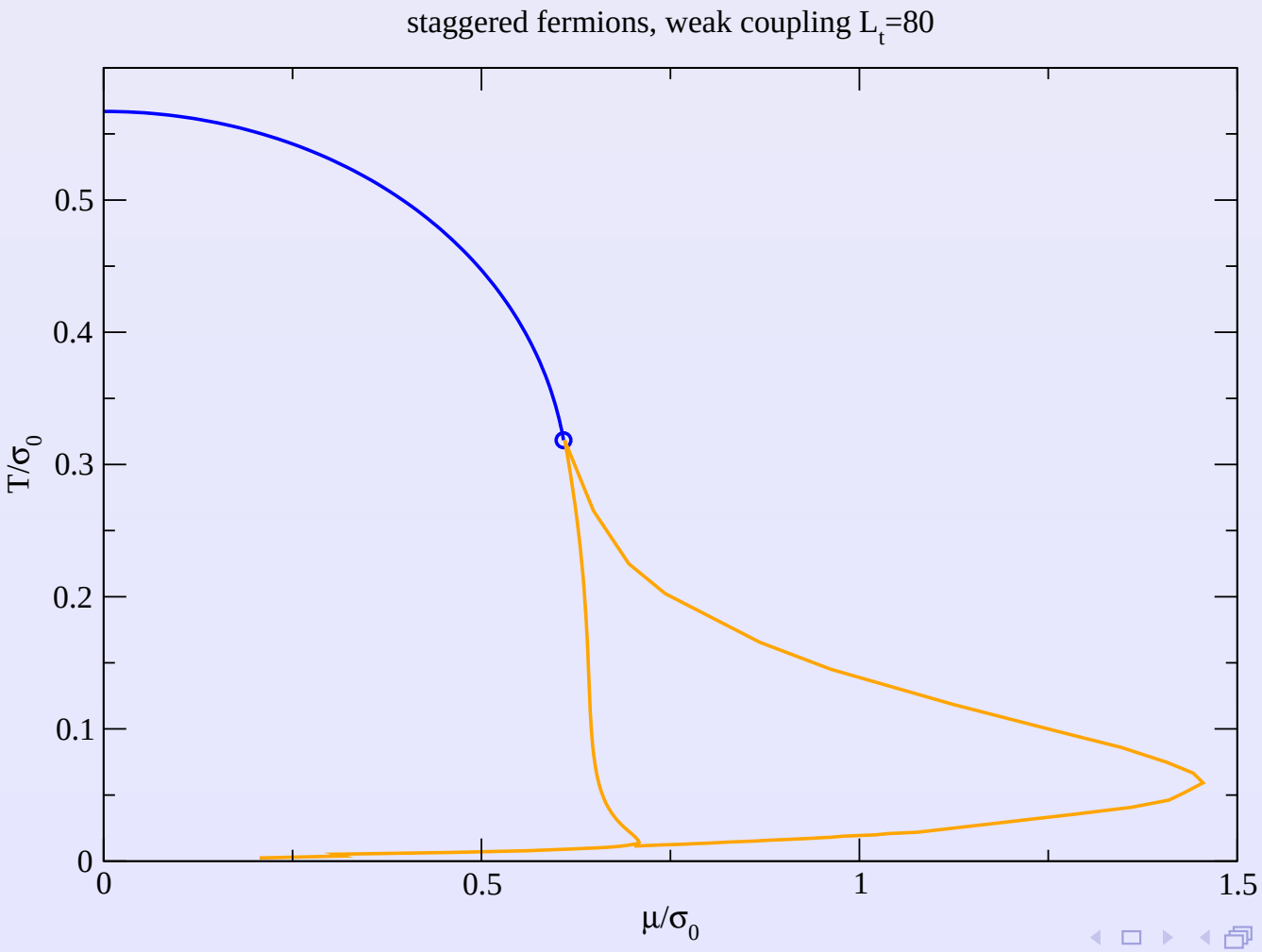


GN model with Majorana fermions

- Most natural formulation in terms of Majorana fermions.
- For the Wilson lattice discretisation:

$$\mathcal{L} = \frac{1}{2}\xi^T \mathcal{C}(\gamma_\mu \tilde{\partial}_\mu - \frac{1}{2}\partial^* \partial + m)\xi - \frac{g^2}{4}\left(\xi^T \mathcal{C}\xi\right)^2 .$$

- The discrete chiral symmetry $\xi \to \gamma_5 \xi$ is broken explicitly:

  $\Rightarrow$ restored in the continuum by fine tuning $m \to m_c$.

- For even N each pair of Majorana fermions may be considered as one Dirac fermion ($\mathcal{C} = -\mathcal{C}^T$):

$$\psi = \frac{1}{\sqrt{2}}(\xi_1 + i\xi_2), \quad \bar{\psi} = \frac{1}{\sqrt{2}}(\xi_1^T - i\xi_2^T)\mathcal{C}.$$

GN model with Majorana fermions

- At $g = 0$, integrating the fermions yields the Pfaffian

$$Z_0 = \mathrm{Pf}\left[\mathcal{C}(\gamma_\mu \tilde{\partial}_\mu + m - \frac{1}{2}\partial^* \partial)\right].$$

- At $g \neq 0$ we introduce a scalar field $\sigma \propto \xi^T \mathcal{C} \xi$ as before:

$$S = \frac{1}{2}\sum_x \xi^T(x)\mathcal{C}(2 + m + \sigma(x))\xi(x) - \sum_{x,\mu} \xi^T(x)\mathcal{C}\frac{1-\gamma_\mu}{2}\xi(x+\hat{\mu}).$$

- Using the nilpotency of even Grassmann elements we can expand the Boltzmann factor

$$\int \mathcal{D}\xi \prod_x \left(1 + \varphi(x)\xi^T(x)\mathcal{C}\xi(x)\right) \prod_{x,\mu} \left(1 + \xi^T(x)\mathcal{C}\frac{1-\gamma_\mu}{2}\xi(x+\hat{\mu})\right)$$

where $\varphi(x) = 2 + m + \sigma(x)$.

GN model as a closed loop model

- At each site, the fields $\xi^T \mathcal{C}$ and ξ must be exactly paired to give a contribution to the path integral.
- Only closed, non-intersecting paths survive the integration

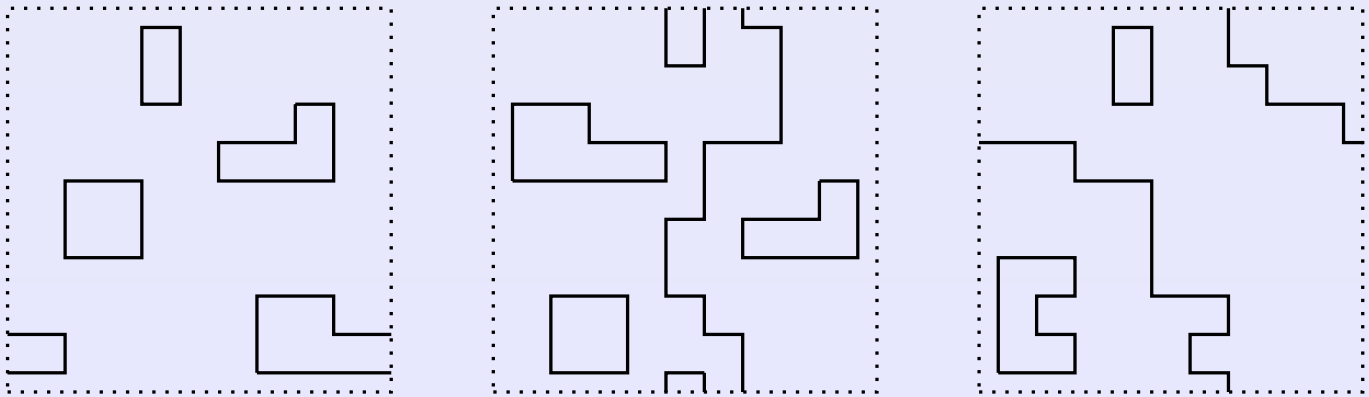
  $\Rightarrow$ closed loop representation in terms of monomers and dimers.
- The weight ω of each loop l can be calculated analytically [Stamatescu '82; Wolff '07]:

$$\omega(l) = \text{sign}[l] \left(\frac{1}{\sqrt{2}}\right)^c, \quad c = \text{number of corners}$$

- Each empty site carries the monomer weight $\varphi(x)$.

Boundary conditions

- The factor sign[/] is $+1$ unless the loop winds around the boundary.
- In that case, sign[/] depends on the fermionic boundary conditions being periodic or anti-periodic:



$\mathcal{L}_{00}$ $\mathcal{L}_{01}$ $\mathcal{L}_{11}$

$\Rightarrow$ classify all configurations into equivalence classes $\mathcal{L}_{00}, \mathcal{L}_{10}, \mathcal{L}_{01}, \mathcal{L}_{11}$.

Boundary conditions

- Partition function summing over all non-oriented, self-avoiding loops

$$Z_{\mathcal{L}} = \sum_{\{l\}\in\mathcal{L}} \omega[l] \prod_{x\notin l} \varphi(x), \quad \mathcal{L} \in \mathcal{L}_{00} \cup \mathcal{L}_{10} \cup \mathcal{L}_{01} \cup \mathcal{L}_{11}$$

  represents a fermionic system with fluctuating boundary conditions [Wolff '07].

- For fixed fermionic boundary conditions we have, e.g.,

$$Z_{\xi}^{-+} = Z_{\mathcal{L}_{00}} + Z_{\mathcal{L}_{10}} - Z_{\mathcal{L}_{01}} + Z_{\mathcal{L}_{11}}.$$

Simulation algorithms

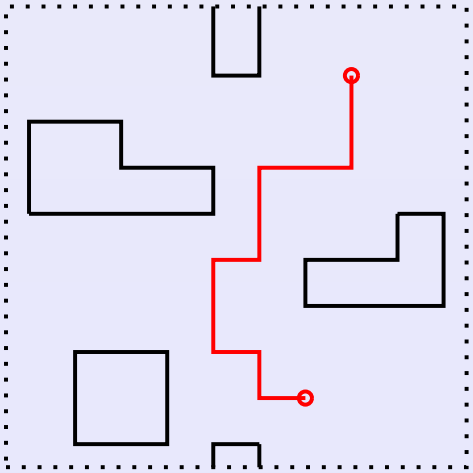
- *Problem*: local loop updates involving plaquette moves
  - can not change between $\mathcal{L}_{00}, \mathcal{L}_{10}, \mathcal{L}_{01}, \mathcal{L}_{11}$,
  - are highly inefficient.
- *Solution 1*: interpret closed loops as Peierls contours of an Ising system,

  $\Rightarrow$ efficient cluster algorithms can be constructed [Wolff '07].
- *Solution 2*: enlarge the configuration space by one open string [Prokof'ev, Svistunov '01]:

  $\Rightarrow$ worm algorithm.
- Both solutions almost completely eliminate critical slowing down at $g \to 0$.

Worm algorithm

- The open string corresponds to the insertion of a Majorana fermion pair $\{\xi^T(x)\mathcal{C}, \xi(y)\}$ at position x and y:



$\Rightarrow$ open string samples directly the correlation function.

- Head and tail of the worm can move around.
- Local moves are determined by Metropolis or heat bath.

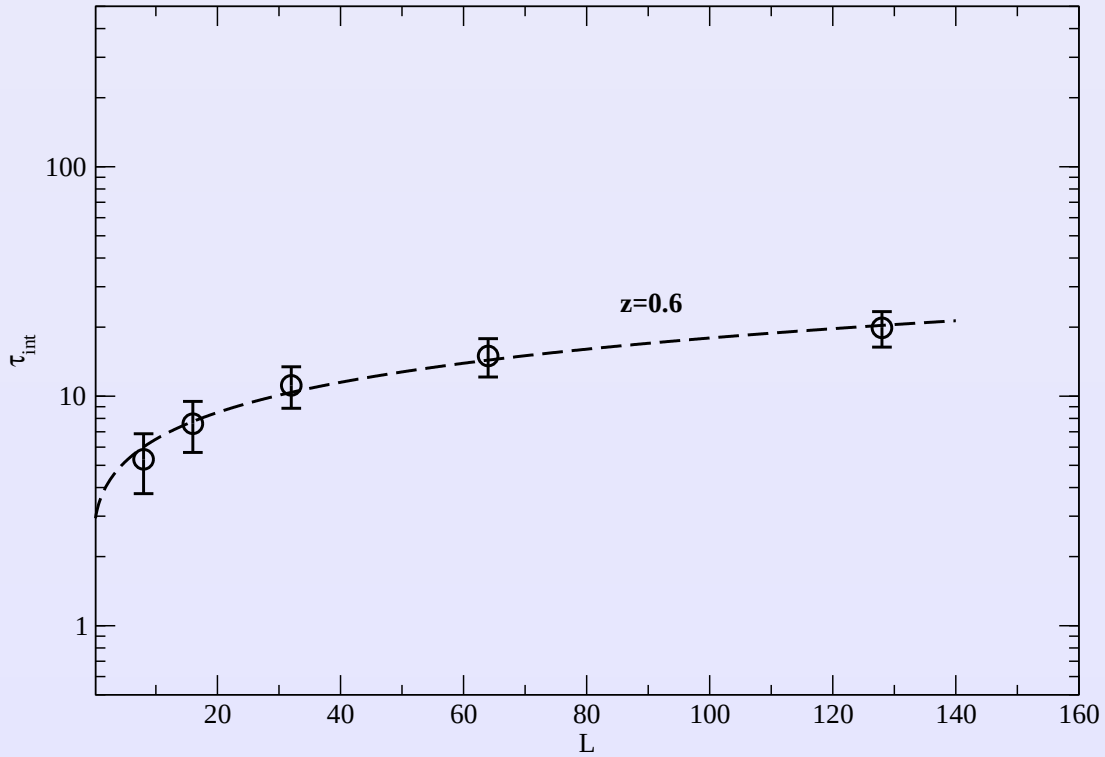
Worm algorithm

- Worm can be made to break up exisiting loops and reconnect them.
- Worm can arbitrarily wind around the lattice

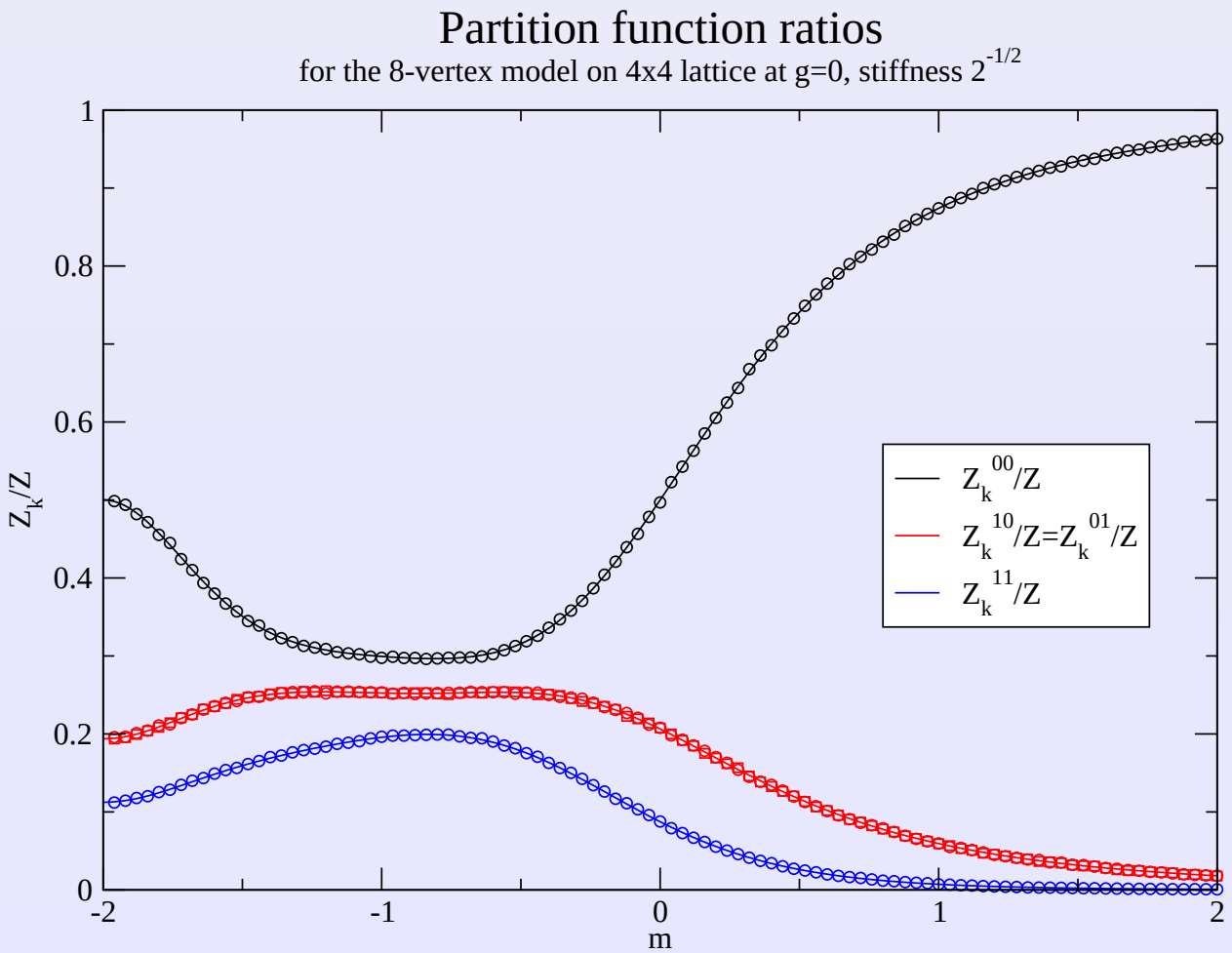
  $\Rightarrow$ all sectors $\mathcal{L}_{00}, \mathcal{L}_{10}, \mathcal{L}_{01}, \mathcal{L}_{11}$ are visited.
- Each time the open loop closes we make contact with the original partition function.
- Similar ideas have been around for a long time [Thun et al. '8?; Evertz, Lana, Marcu '93; Prokof'ev, Svistunov '01].

Autocorrelation times for worm agorithm

- Measure $\langle \xi^T \mathcal{C} \xi \rangle_{Z_\xi}$ for which $\langle K(x) \rangle_{loop}$ (monomer density) is an improved estimator.
- Elimination of critical slowing down:



Consistency check



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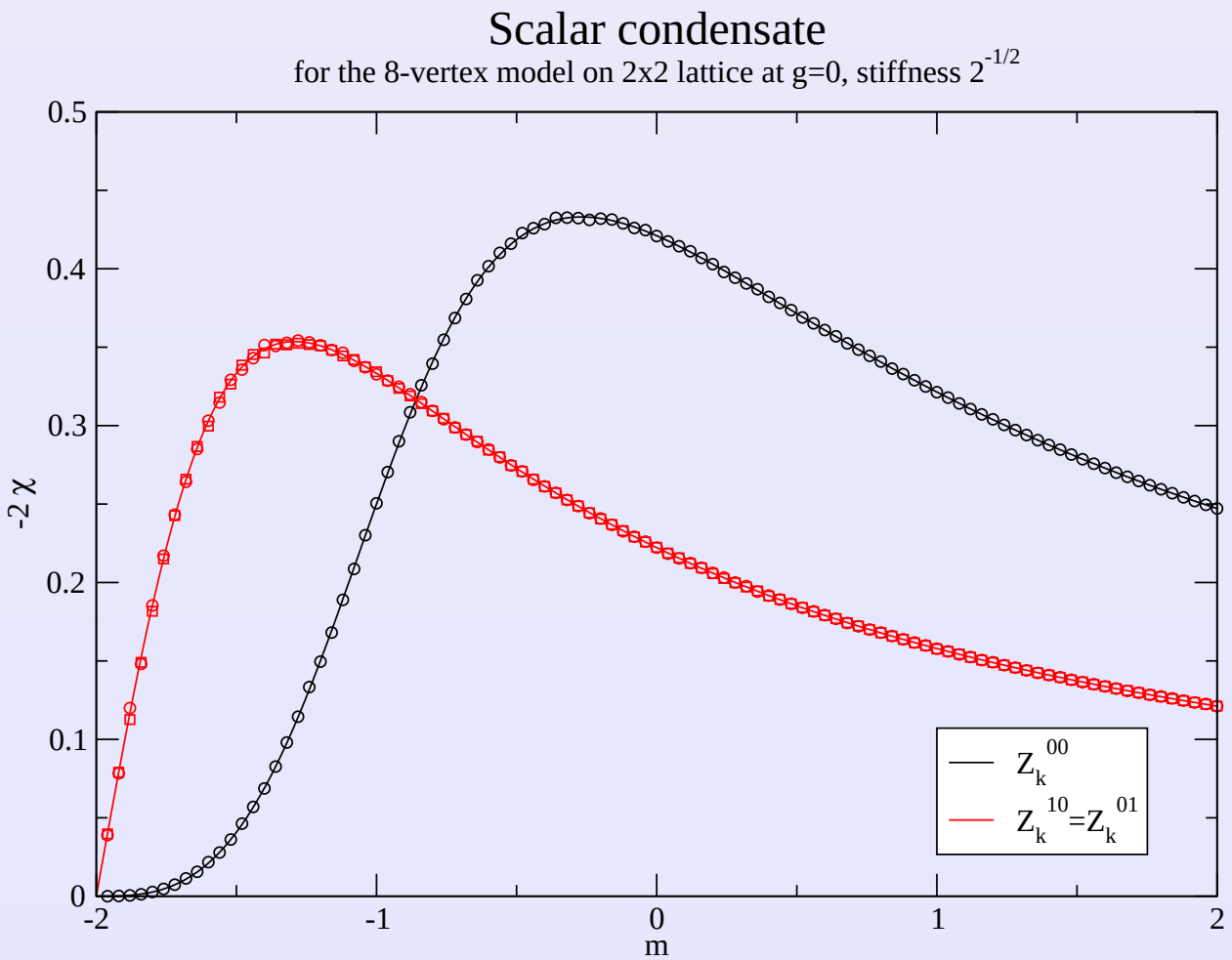
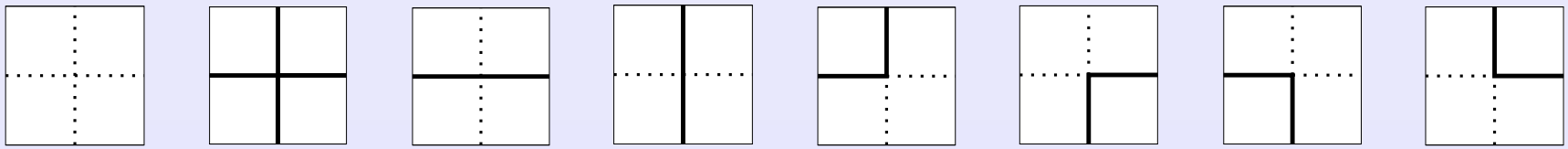


Equivalence to the 8-vertex model

- The loop formulation of the GN model is equivalent to a special case of the 8-vertex model [Lieb '67; Sutherland '70; Fan, Wu '70]:

$$Z_{8-vertex} = \sum_{l \in \mathcal{L}} \prod_{x} \omega(x).$$

- Generically we have eight vertices with weights:



ω_1 ω_2 ω_3 ω_4 ω_5 ω_6 ω_7 ω_8

- For the GN model we have [Scharnhorst '96; Gattringer '98; Wolff '07]:

$$\omega_1 = \varphi(x), \qquad \omega_3 = \omega_4 = 1,$$
$$\omega_2 = 0, \qquad \omega_5 = \omega_6 = \omega_7 = \omega_8 = \frac{1}{\sqrt{2}}.$$

8-vertex model

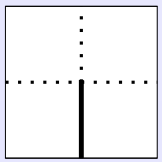
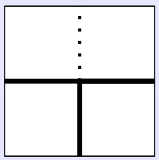
- The 8-vertex model can be solved analytically in two particular cases [Fan, Wu '70; Baxter '71; Samuel '80]:
  - in the 'zero field' limit where

$$\begin{aligned} \omega_1 &= \omega_2, & \omega_3 &= \omega_4, \\ \omega_5 &= \omega_6, & \omega_7 &= \omega_8, \end{aligned}$$

  - under the free fermion condition where

$$\omega_1\omega_2 + \omega_3\omega_4 = \omega_5\omega_6 + \omega_7\omega_8.$$

- For the simulation we introduce open tiles:

Examples of 8-vertex models

- Ising model on the dual lattice,
- Ising model in high-temperature expansion,
- Ising model with next-to-nearest and four-spin coupling,

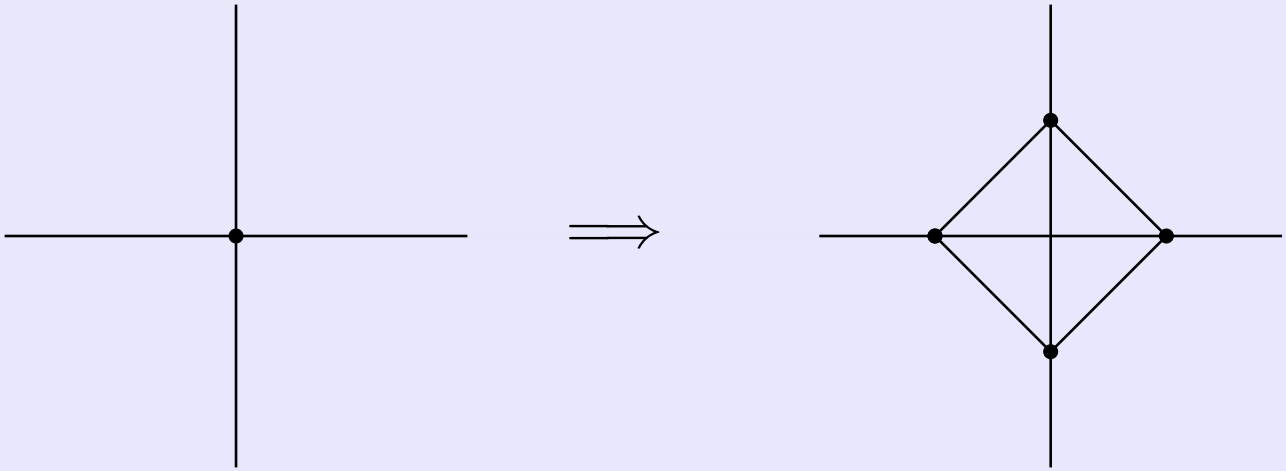
  [Wu '71; Kadanoff, Wegner '71; Jüngling '75]
- GN model with Majorana Wilson fermions,
- QED_2 at $\beta = 0$ with Wilson fermions,

  [Salmhofer '9?]
- GN model with Dirac Wilson fermions (Thirring).

  [Scharnhorst '96; Gattringer '98; Wolff '07]

Equivalence of the 8-vertex model with a closely packed dimer model

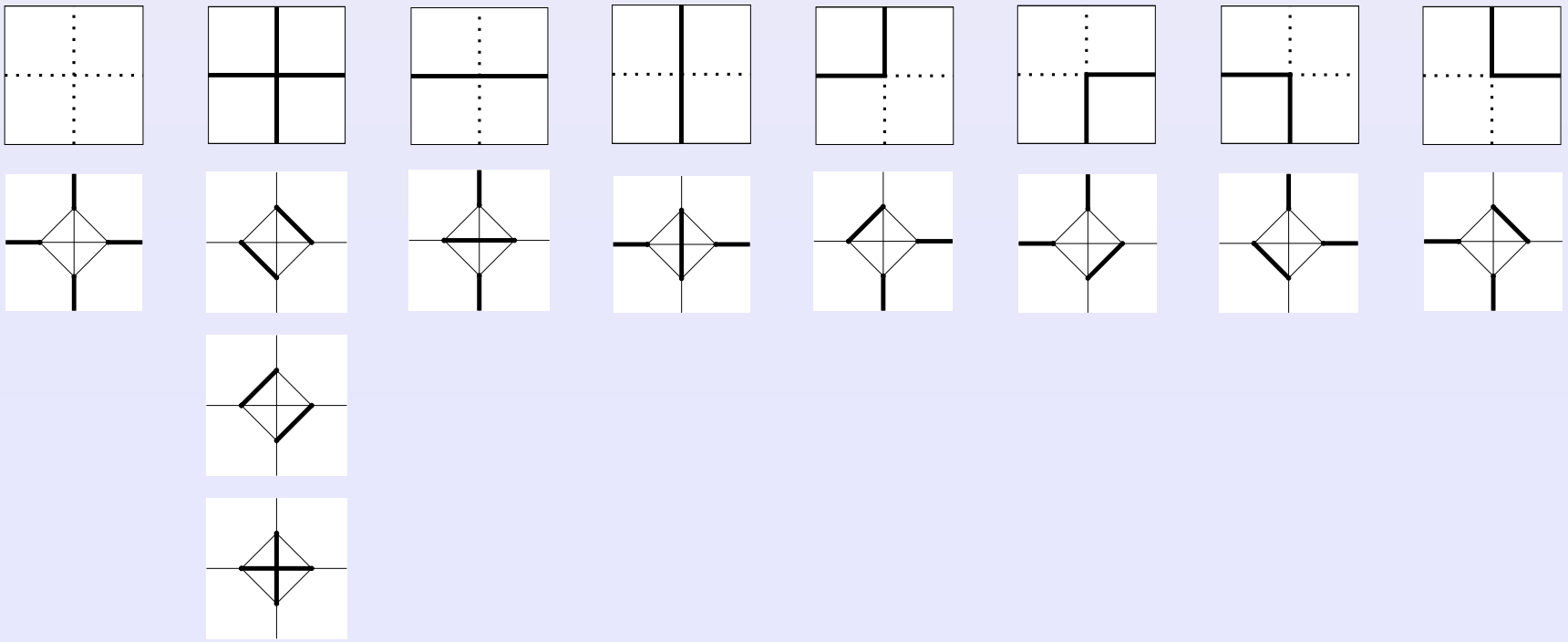
- The 8-vertex model can be reformulated in terms of a closely packed dimer model [Kasteleyn '61 & '63; Fisher '61 & '66; Muse '73].
- Several possibilities exist, one on the 'terminal city' can be made consistent with $\omega_2 = 0$.
- Decorate each vertex with a 'terminal city':



- Occupy the decorated lattice with hard core dimers carrying fugacities z_i.

Equivalence of the 8-vertex model with a closely packed dimer model

- Correspondence between the 8-vertex and closely packed dimer configurations:



$\Rightarrow$ links the model to its exact solution in terms of Pfaffians.

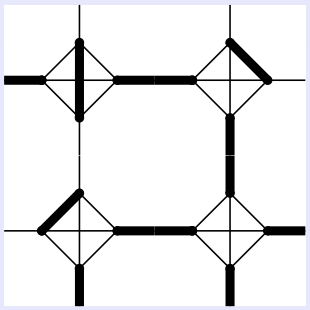
Counting CP dimer coverings

- Terminal lattice is not planar:
  - there is no Kasteleyn orientation,
  - third dimer configuration for vertex 2 contributes with a minus sign.
- Assures the correct counting for the Ising model on the terminal lattice.
- Assures correct counting for the GN model.
- First glimpse of sign problem in higher dimensions.
- Formally this is all fine, practically (i.e. for MC) it's better to have positive weights.

  $\Rightarrow$ introduce dimer interaction to suppress vertex 2.

Simulation algorithms for the CP dimer model

- Several algorithms exist to simulate closely packed (CP) dimer models [Evertz, Lana, Marcu '93; Syluasien; Sandvik, Moessner; Adams, Chandrasekharan '03]:
  - loop cluster algorithms,
  - directed loop algorithm.
- Idea:
  Instead of an open string, introduce two monomers.
- Monomers move around the lattice and flip dimers:



1. Choose lattice site randomly and put a monomer pair.

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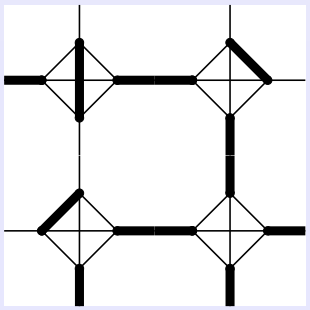

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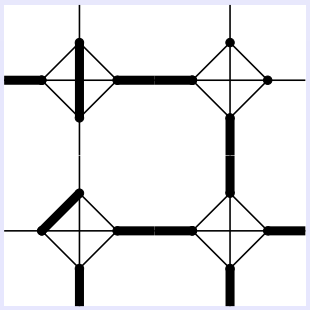

2. Choose direction randomly and move monomer.

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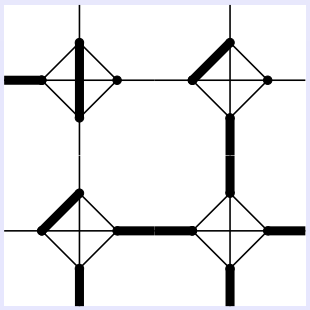

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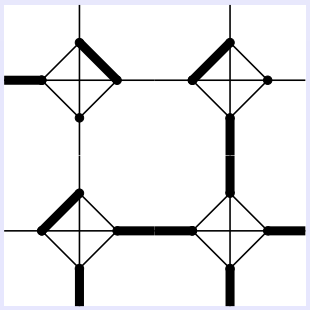

3. Newly met dimer provides direction for next monomer move.

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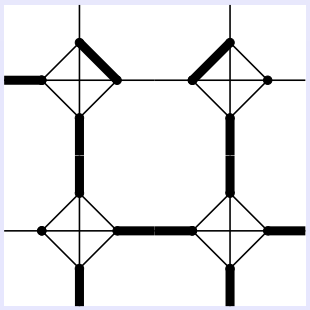

4. Repeat until the two monomers meet again.

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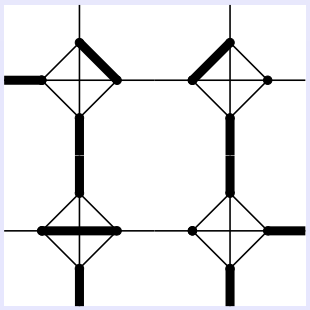

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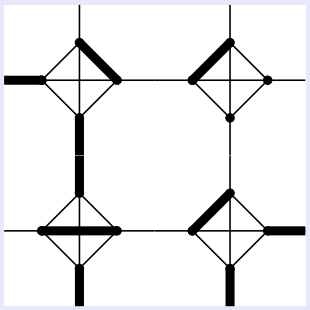

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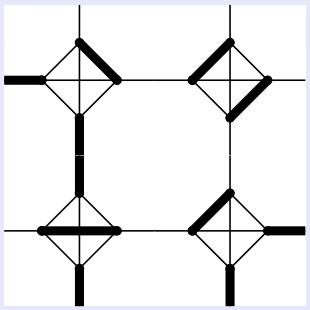
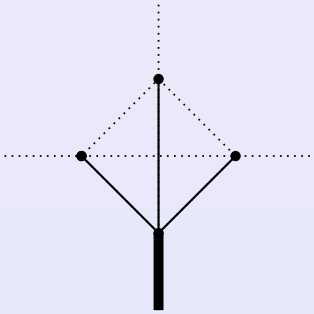


4. Repeat until the two monomers meet again.

Simulation algorithms for the CP dimer model

- Directed loop algorithm:



1. enter the loop through link i,
2. choose link $j \neq i$ with probability $p_{ij} = a_{ij}/z_i$,
3. flip dimer from direction i to j.

- Detailed balance leads to a set of equations:

$$\sum_j p_{ij} = z_i, \quad \forall i\,,$$

$$a_{ij} = a_{ji}, \quad \forall i,j\,.$$

$\Rightarrow$ directed loop equations

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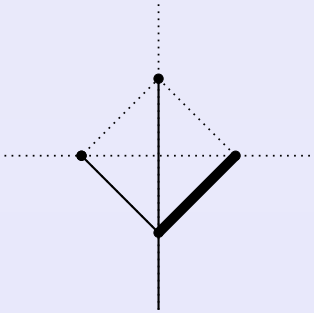

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Summary and outlook

- The Gross-Neveu model can be reformulated as a loop model or a model of closely packed hard core dimers.
- For both formulations very efficient algorithms are available which essentially eliminate critical slowing down:
- So far, the method is restricted to two dimensions, but it is worthwile to think about higher dimensions.
  - $\Rightarrow$ improved estimators might be able to handle the sign problem.

Large-N limit

- In the large-N limit with $\lambda = g^2 N$ fixed, the model can be solved analytically:
  - Integrate out the fermions to obtain $Z = \int_{[d\sigma]} \exp\{-S_{\text{eff}}\}$,

$$S_{\text{eff}} = N \left\{ \int_{[dx]} \frac{\sigma(x)^2}{2\lambda} - \text{Tr} \log\left[\not{\partial} + \sigma\right] \right\}.$$

  - The minimum of the effective potential is given by

$$\partial_{\sigma(x)} S_{\text{eff}}/N = \frac{\sigma(x)}{\lambda} - \partial_{\sigma(x)} \text{Tr} \log\left[\not{\partial} + \sigma\right] = 0, \ \forall x.$$

Gap equation and the spectrum

- For constant σ this reduces to a single equation

$$\frac{\sigma}{\lambda} = \partial_\sigma \mathrm{Tr} \log \left[\partial \!\!\!/ + \sigma\right],$$

or in momentum space $\frac{1}{\lambda} = \int_{[dk]} \frac{2}{k^2+\sigma^2}$.

$\Rightarrow$ *gap equation (self consistency equation)*

- Equivalent equations via Hartree-Fock, Schwinger-Dyson, Bethe-Salpeter approaches.
- To leading order in $1/N$ the spectrum consists of [Dashen, Hasslacher, Neveu '75; Feinberg, Zee '97]
  - single fermions,
  - n-fermion bound states,
  - baryons (kink-antikink state).

The phase structure

- The GN model possesses a rich μ-T phase structure:

  [Dashen, Ma, Rajaraman '75; Wolff '85; Karsch, Kogut, Wyld '87]

  - Mermin-Wagner-Coleman theorems forbid spontaneous breaking of
    - continuous symmetry at $T = 0$,
    - discrete symmetry at $T \neq 0$.
  - fluctuations are expected to destroy any long range order

    $\Rightarrow$ free massless boson propagator is logarithmic in 2D,
  - however, fluctuations are suppressed at large N:

$$\langle \bar{\psi}(x)\psi(x)\bar{\psi}(y)\psi(y)\rangle \sim 1 + \frac{1}{N}\ln|x-y| + O(1/N^2)$$

    becomes constant as $N \to \infty$.

$\Rightarrow$ take large-N limit before thermodynamic limit!