One finds, using (7):

$$\Delta \mathbf{p} = [-\mu_x(0)(\omega\tau)^{-1}(\sin \omega\tau) + \mu_y(0)(\omega\tau)^{-1}(1 - \cos \omega\tau), 0, \mu_z(0)]\, \tau\partial_z B_z, \quad (9)$$

showing that the x component of the momentum is a factor of order $1/\omega\tau$ smaller than the z component. Typically, $\tau \simeq 10^{-3}$ sec, so, for $\omega \simeq 10^{11}$ rad/sec, this factor is about 10^{-8}. Hence, the classically expected trace of the magnetic moments on a screen behind the magnet will be a very narrow vertical line.

[a] We regret to note that Richard Mattuck died on 5 May 1982, to our sorrow and loss.

[1] See, for example, E. Merzbacher, *Quantum Mechanics* (Wiley, New York, 1970), p. 252.

[2] R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics* (Addison-Wesley, New York, 1964), Vol II, p. 35-3.

[3] Unpublished communication.

Hyperfine splitting in the ground state of hydrogen

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The hyperfine structure of atomic hydrogen is derived in a simple and self-contained way that makes the theory accessible to advanced undergraduates in a first course on quantum mechanics.

I. INTRODUCTION

The electron and the proton in atomic hydrogen constitute tiny magnetic dipoles, whose interaction energy varies according to the relative orientation of their dipole moments. If the spins are parallel (or, more precisely, if they are in the triplet state), the energy is somewhat higher than it is when the spins are antiparallel (the singlet state). The difference is not large, amounting to a mere 6×10^{-6} eV, as compared with a binding energy of 13.6 eV and typical fine structure splitting on the order of 10^{-4} eV.[1] Nevertheless, this "hyperfine" splitting is of substantial interest—indeed, before the discovery of the 3 °K cosmic background radiation,[2] the 21-cm line resulting from hyperfine transitions in atomic hydrogen was widely regarded as the most pervasive and distinctive radiation in the universe.[3]

Hyperfine structure is seldom treated in elementary quantum mechanics texts, and this is unfortunate because the calculation is really quite simple—easier, certainly, than the fine structure, and perhaps more illuminating. The reason for avoiding it probably has to do with a rather subtle point in classical electrodynamics (a matter, in fact, of some interest in its own right), to wit, the calculation of the energy of interaction between two magnetic dipoles. My purpose here is to show that this problem is by no means inaccessible to advanced undergraduates and, in fact, serves as a nice application both of electrodynamics and of quantum theory.[4]

Because students generally find electric dipoles easier to think about than magnetic dipoles, I begin (in Sec. II) with a calculation of the electric field of an electric dipole. Then, in Sec. III, I apply essentially the same techniques to obtain the magnetic field of a magnetic dipole. In Sec. IV I work out the formula for the interaction energy of two magnetic dipoles. Up to this point the calculation lies entirely within the realm of classical electrodynamics; the quantum mechanics enters only in the final step, where the classical interaction energy is interpreted as the hyperfine structure Hamiltonian, and the energies of the singlet and triplet spin states are evaluated in first-order perturbation theory. The result is compared with experimental measurements of fantastic precision, and some comments are made concerning hyperfine splitting in positronium, muonium, and muonic hydrogen. Section V offers some physical insight into the question of why the singlet configuration has the lower energy.

II. FIELD OF AN ELECTRIC DIPOLE

In mks units, the potential of an ideal electric dipole is given by[5]

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{r}}{r^2}, \quad (1)$$

where $\mathbf{p}$ is the dipole moment and $\mathbf{r}$ is the vector from the dipole to the point of observation (r is its magnitude and $\hat{r} = \mathbf{r}/r$). Taking the gradient of V, we obtain the dipole field

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}[3(\mathbf{p}\cdot\hat{r})\hat{r} - \mathbf{p}]. \quad (2)$$

But this familiar result[6] cannot be correct, for it is incompatible with the following general theorem,[7] which applies to all static charge configurations.

Theorem 1. The average electric field over a spherical volume of radius R, due to an arbitrary distribution of stationary charges within the sphere, is

$$\mathbf{E}_{\rm av} = -\frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}}{R^3}, \quad (3)$$

where $\mathbf{p}$ is the total dipole moment with respect to the center of the sphere.

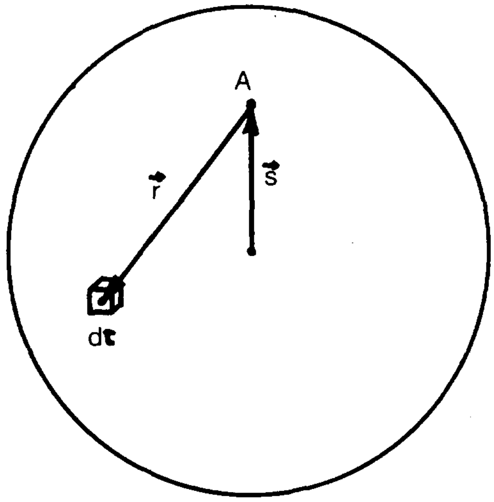

Fig. 1. Average field, over a sphere, due to a point charge at A.

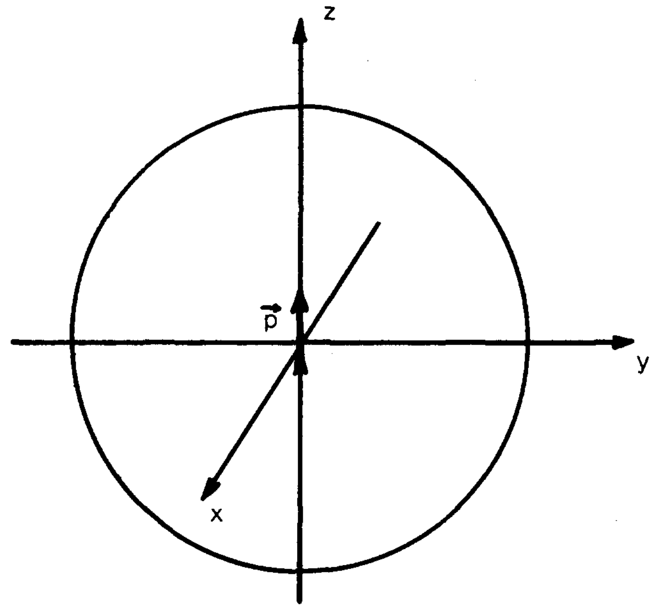

Fig. 2. Ideal dipole at center of sphere.

Proof: The average field due to a *single* charge q located at point A within the sphere (Fig. 1) is given by

$$\mathbf{E}_{\mathrm{av}} = \frac{1}{\tau}\frac{1}{4\pi\epsilon_0}\int \frac{q}{r^2}\hat{r}\,d\tau,$$

where $\tau = \frac{4}{3}\pi R^3$ is the volume of the sphere. But this is exactly the same as the field that would be produced at A by a *uniformly* charged sphere carrying a charge density $\rho = -q/\tau$. The latter is easily obtained by application of Gauss's law[8]:

$$\mathbf{E} = \frac{1}{3\epsilon_0}\rho\mathbf{s},$$

where $\mathbf{s}$ is the vector from the center of the sphere to A. It follows that the average field over a sphere due to a *single* point charge is

$$\mathbf{E}_{\mathrm{av}} = -\frac{1}{4\pi\epsilon_0}\frac{q\mathbf{s}}{R^3}.$$

For an *arbitrary* distribution of charges within the sphere, $q\mathbf{s}$ is replaced by

$$\sum q_i\mathbf{s}_i = \mathbf{p}$$

(the total dipole moment of the sphere), and the theorem is proved.

Let us apply this theorem to the simplest possible case: an ideal dipole at the origin, pointing in the z direction (Fig. 2). If we take the dipole field (2) as it stands, we have

$$\mathbf{E}_{\mathrm{av}} = \frac{1}{\tau}\frac{1}{4\pi\epsilon_0}\mathbf{p}\int \frac{1}{r^3}(3\cos^2\theta - 1)r^2\sin\theta\,dr\,d\theta\,d\phi. \quad (4)$$

But the θ integral gives zero, while the r integral is infinite, so the result is indeterminate. Evidently Eq. (2) is incorrect—or at best ambiguous. What has gone wrong? The source of the problem is the point $r = 0$, where the potential of the dipole is singular. Our formula for the field is unobjectionable everywhere *else*, but at that one point we must be more careful.

An ideal dipole is, after all, the point limit of a real (extended) dipole, so let us approach it from that perspective. The usual model—equal and opposite charges $\pm q$ separated by a displacement $\mathbf{s}$, with $\mathbf{p} = q\mathbf{s}$—is cumbersome, for our present purposes. More tractable is a uniformly polarized sphere, of radius a, polarization $\mathbf{P}$, and dipole moment

$$\mathbf{p} = \tfrac{4}{3}\pi a^3\mathbf{P}. \quad (5)$$

It is well known[9] that the field *outside* such a sphere is given precisely by Eq. (2):

$$\mathbf{E}_{\mathrm{out}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}[3(\mathbf{p}\cdot\hat{r})\hat{r} - \mathbf{p}] \quad \text{for } r > a, \quad (6)$$

while (surprisingly) the field *inside* the sphere is *uniform* (Fig. 3):

$$\mathbf{E}_{\mathrm{in}}(\mathbf{r}) = -\frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}}{a^3} \quad \text{for } r < a. \quad (7)$$

In the ideal dipole limit ($a \to 0$) the interior region shrinks to zero, and one might suppose that this contribution disappears altogether. However, $\mathbf{E}_{\mathrm{in}}$ itself *blows up*, in the same limit, and in just such a way that its *integral* over the sphere.

$$\int \mathbf{E}_{\mathrm{in}}\,d\tau = \left(-\frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}}{a^3}\right)\left(\tfrac{4}{3}\pi a^3\right) = -\frac{\mathbf{p}}{3\epsilon_0}, \quad (8)$$

remains constant, no matter how small the sphere be-

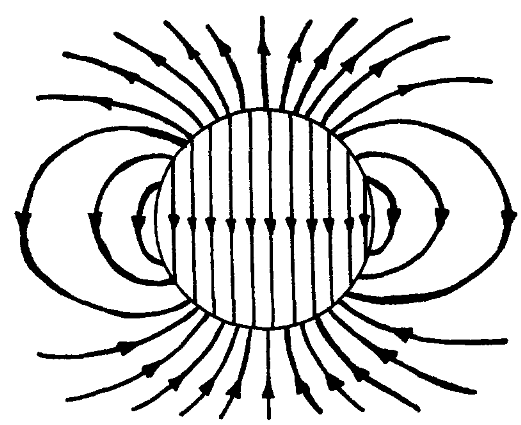

Fig. 3. Field of a uniformly polarized sphere.

comes. We recognize here the defining conditions for a Dirac delta function; evidently, as $a \to 0$ the field inside the sphere goes to

$$\mathbf{E}_{\text{in}}(\mathbf{r}) = -\frac{1}{3\epsilon_0}\mathbf{p}\delta^3(\mathbf{r}). \tag{9}$$

We may write the field of an ideal dipole as follows:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}[3(\mathbf{p}\cdot\hat{r})\hat{r} - \mathbf{p}] - \frac{1}{3\epsilon_0}\mathbf{p}\delta^3(\mathbf{r}), \tag{10}$$

on the understanding that the first term applies only to the region *outside* an infinitesimal sphere about the point $r = 0$. With the radial integral thus truncated, Eq. (4) now yields zero unambiguously—but there is an extra contribution to $\mathbf{E}_{\text{av}}$, coming from the delta function:

$$\mathbf{E}_{\text{av}} = \frac{1}{\tau}\int\left(-\frac{1}{3\epsilon_0}\mathbf{p}\delta^3(\mathbf{r})\right)d\tau = -\frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}}{R^3}, \tag{11}$$

which is exactly what Theorem 1 requires. Although the delta function only affects the field at the point $r = 0$, it is crucial in establishing the consistency of the theory.[10]

III. FIELD OF A MAGNETIC DIPOLE

The vector potential of an ideal magnetic dipole is given by[11]

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{r}}{r^2}, \tag{12}$$

where $\mathbf{m}$ is the dipole moment. Taking the curl of this potential we obtain the dipole field:

$$\mathbf{B}(\mathbf{r}) = (\mu_0/4\pi)(1/r^3)[3(\mathbf{m}\cdot\hat{r})\hat{r} - \mathbf{m}]. \tag{13}$$

This familiar result[12] is identical in form to the *electric* field of an *electric* dipole [Eq. (2)], and once again it cannot be correct, for it is incompatible with the following general theorem.[13]

Theorem 2. The average magnetic field over a spherical volume of radius R, due to an arbitrary configuration of steady currents within the sphere, is

$$\mathbf{B}_{\text{av}} = (\mu_0/4\pi)2\mathbf{m}/R^3, \tag{14}$$

where $\mathbf{m}$ is the total dipole moment of the sphere.

Proof: By definition

$$\mathbf{B}_{\text{av}} = \frac{1}{\tau}\int \mathbf{B}\,d\tau, \tag{15}$$

where $\tau = \frac{4}{3}\pi R^3$, as before. Writing $\mathbf{B}$ as the curl of $\mathbf{A}$, and invoking the vector identity[14]

$$\int_{\text{volume}}(\nabla\times\mathbf{A})d\tau = -\int_{\text{surface}}\mathbf{A}\times d\mathbf{a},$$

we have

$$\mathbf{B}_{\text{av}} = -\frac{1}{\tau}\int \mathbf{A}\times d\mathbf{a},$$

where $d\mathbf{a}$ is an infinitesimal element of area at the surface of the sphere, pointing in the radial direction. Now, the vector potential is itself an integral over the current distribution[15]:

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int\frac{\mathbf{J}}{r}d\tau, \tag{16}$$

and hence

$$\mathbf{B}_{\text{av}} = -\frac{1}{\tau}\frac{\mu_0}{4\pi}\int\int\frac{1}{r}(\mathbf{J}\times d\mathbf{a})d\tau. \tag{17}$$

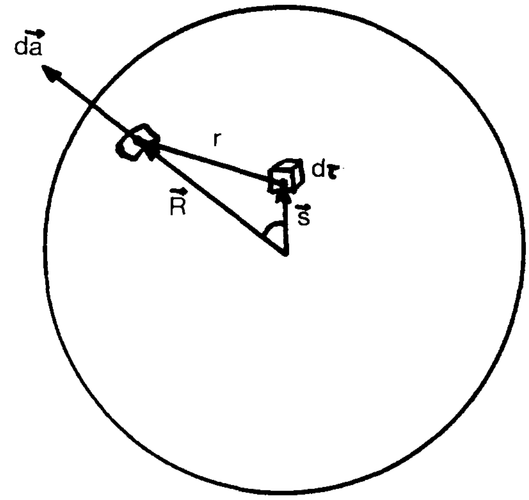

Fig. 4. Geometry for Eq. (17).

I propose to do the surface integral first, setting the polar axis along the vector ($\mathbf{s}$) from the center to $d\tau$ (Fig. 4), so that

$$r = (R^2 + s^2 - 2Rs\cos\theta)^{1/2},$$
$$d\mathbf{a} = R^2\sin\theta\,d\theta\,d\phi\,\hat{R},$$

and therefore

$$\int\frac{1}{r}d\mathbf{a} = \int(R^2 + s^2 - 2Rs\cos\theta)^{-1/2} \times R^2\sin\theta\,d\theta\,d\phi\,(\cos\theta\,\hat{s}) = \tfrac{4}{3}\pi\mathbf{s}.$$

Finally, the volume integral yields

$$\mathbf{B}_{\text{av}} = -\frac{\mu_0}{4\pi R^3}\int(\mathbf{J}\times\mathbf{s})d\tau = \frac{\mu_0}{4\pi}\frac{2\mathbf{m}}{R^3},$$

where

$$\mathbf{m} = \tfrac{1}{2}\int(\mathbf{s}\times\mathbf{J})d\tau \tag{18}$$

is the total dipole moment of the sphere.[16] Q.E.D.

Suppose we wish to check Theorem 2 for the simplest possible case: an ideal magnetic dipole $\mathbf{m}$ at the origin. If we attempt to calculate the average magnetic field, using Eq. (13), we obtain the same indeterminate integral as before [Eq. (4)]. Once again, the source of the problem is the point $r = 0$; there is an extra delta-function contribution to the field, which Eq. (13) ignores.

In order to obtain this extra term, we treat the ideal dipole as the point limit of a uniformly magnetized sphere, of radius a, magnetization $\mathbf{M}$, and dipole moment

$$m = \tfrac{4}{3}\pi a^3\mathbf{M}. \tag{19}$$

The field *outside* such a sphere is given precisely by Eq. (13):

$$\mathbf{B}_{\text{out}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{1}{r^3}[3(\mathbf{m}\cdot\hat{r})\hat{r} - \mathbf{m}] \quad \text{for } r > a, \tag{20}$$

while the field *inside* the sphere is *uniform*[17]:

$$\mathbf{B}_{\text{in}}(\mathbf{r}) = (\mu_0/2\pi)\mathbf{m}/a^3 \quad \text{for } r < a. \tag{21}$$

In the ideal dipole limit ($a \to 0$) the interior region shrinks to zero, but the field goes to infinity; their product remains constant:

$$\int\mathbf{B}_{\text{in}}\,d\tau = \left(\frac{\mu_0}{2\pi}\frac{\mathbf{m}}{a^3}\right)\left(\tfrac{4}{3}\pi a^3\right) = \tfrac{2}{3}\mu_0\mathbf{m}. \tag{22}$$

As $a \to 0$, therefore, the field inside the sphere goes to a delta function:

$$\mathbf{B}_{\text{in}}(\mathbf{r}) = \tfrac{2}{3}\mu_0 \mathbf{m}\delta^3(\mathbf{r}) . \tag{23}$$

The magnetic field of an ideal dipole can thus be written

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{1}{r^3}[3(\mathbf{m}\cdot\hat{r})\hat{r} - \mathbf{m}] + \tfrac{2}{3}\mu_0 \mathbf{m}\delta^3(\mathbf{r}) , \tag{24}$$

with the understanding that the first term applies only to the region outside an infinitesimal sphere at the origin. The *average* field (over a sphere of radius R) comes exclusively from the delta-function term:

$$\mathbf{B}_{\text{av}} = \frac{1}{\tau}\int [\tfrac{2}{3}\mu_0 \mathbf{m}\delta^3(\mathbf{r})]\,d\tau = \frac{\mu_0}{2\pi}\frac{\mathbf{m}}{R^3} ,$$

which is exactly what Theorem 2 requires. Once again, although it only affects the one point $r = 0$, the delta-function contribution is essential for the consistency of the theory.[18]

IV. HYPERFINE STRUCTURE IN THE GROUND STATE OF HYDROGEN

The energy of a magnetic dipole $\mathbf{m}$, in the presence of a magnetic field $\mathbf{B}$, is given by the familiar formula[19]

$$H = -\mathbf{m}\cdot\mathbf{B} . \tag{25}$$

In particular, the energy of one magnetic dipole ($\mathbf{m}_1$) in the field of another magnetic dipole ($\mathbf{m}_2$) is

$$H = -\frac{\mu_0}{4\pi}\frac{1}{r^3}[3(\mathbf{m}_1\cdot\hat{r})(\mathbf{m}_2\cdot\hat{r}) - \mathbf{m}_1\cdot\mathbf{m}_2] - \tfrac{2}{3}\mu_0 \mathbf{m}_1\cdot\mathbf{m}_2\delta^3(\mathbf{r}) , \tag{26}$$

where $\mathbf{r}$ is their separation. The formula is symmetric in its treatment of $\mathbf{m}_1$ and $\mathbf{m}_2$, as of course it *should* be—it represents the energy of interaction of the two dipoles. In most applications $\mathbf{m}_1$ and $\mathbf{m}_2$ are physically separated, and the delta-function term can be ignored; however, it is precisely this part which accounts for hyperfine splitting in the ground state of hydrogen.

In first-order perturbation theory, the change in energy of a quantum state is given by the expectation value of the perturbing Hamiltonian.[20] The ground-state wave function for atomic hydrogen is[21]

$$\psi_0 = (\pi a^3)^{-1/2} e^{-r/a}|s\rangle , \tag{27}$$

where $a = 0.52917706$ Å is the Bohr radius[22] and $|s\rangle$ denotes the spin of the electron. Treating the dipole–dipole interaction [Eq. (26)] as a perturbation, the energy of the ground state is shifted by an amount

$$E' = \int \psi_0^* H \psi_0 \, d\tau . \tag{28}$$

Because ψ_0 (and indeed *any* $l = 0$ state) is spherically symmetrical,[23] the θ integral gives zero, just as it did in Eq. (4). Accordingly

$$E' = -\tfrac{2}{3}\mu_0\langle \mathbf{m}_1\cdot\mathbf{m}_2\rangle|\psi_0(0)|^2 = -\tfrac{2}{3}(\mu_0/\pi a^3)\langle \mathbf{m}_1\cdot\mathbf{m}_2\rangle . \tag{29}$$

Here $\mathbf{m}_1$ is the magnetic dipole moment of the proton and $\mathbf{m}_2$ is that of the electron; they are proportional to the respective spins:

$$\mathbf{m}_1 = \gamma_p \mathbf{S}_p, \quad \mathbf{m}_2 = -\gamma_e \mathbf{S}_e , \tag{30}$$

where the γ's are the two gyromagnetic ratios[24] (the minus records the negative charge of the electron). Thus

$$E' = \tfrac{2}{3}(\mu_0/\pi a^3)\gamma_e\gamma_p\langle \mathbf{S}_e\cdot\mathbf{S}_p\rangle . \tag{31}$$

In the presence of such "spin–spin coupling," the z components of $\mathbf{S}_e$ and $\mathbf{S}_p$ are no longer separately conserved; the "good" quantum numbers for the system are rather the eigenvalues of the *total* angular momentum

$$\mathbf{J} = \mathbf{S}_e + \mathbf{S}_p . \tag{32}$$

Now

$$J^2 = (\mathbf{S}_e + \mathbf{S}_p)^2 = S_e^2 + S_p^2 + 2\mathbf{S}_e\cdot\mathbf{S}_p ,$$

so that

$$\mathbf{S}_e\cdot\mathbf{S}_p = \tfrac{1}{2}(J^2 - S_e^2 - S_p^2) .$$

The electron and proton carry spin-$\frac{1}{2}$, so the eigenvalues of S_e^2 and S_p^2 are $\frac{3}{4}\hbar^2$. The two spins combine to form a spin-1 triplet ($J^2 = 2\hbar^2$) and a spin-0 singlet ($J^2 = 0$).[25] Thus

$$\langle \mathbf{S}_e\cdot\mathbf{S}_p\rangle = \begin{cases} \frac{1}{4}\hbar^2 & \text{(triplet)} \\ -\frac{3}{4}\hbar^2 & \text{(singlet)} \end{cases} \tag{33}$$

and hence

$$E' = \frac{2}{3}\frac{\mu_0\hbar^2}{\pi a^3}\gamma_e\gamma_p \begin{cases} \frac{1}{4} & \text{(triplet)} \\ -\frac{3}{4} & \text{(singlet)} \end{cases} . \tag{34}$$

Evidently the singlet state, in which the spins are antiparallel, carries a somewhat lower energy than the triplet combination (Fig. 5). The energy gap is

$$\Delta E_{\text{hyd}} = \tfrac{2}{3}(\mu_0\hbar^2/\pi a^3)\gamma_e\gamma_p . \tag{35}$$

Now, the gyromagnetic ratios are given by

$$\gamma = (e/2m)\, g , \tag{36}$$

where e is the proton charge, m is the mass of the particle, and g is its "g factor" (2.0023 for the electron, 5.5857 for the proton).[26] So, finally,

$$\Delta E_{\text{hyd}} = (\mu_0\hbar^2 e^2/6\pi a^3)\, g_e\, g_p/m_e m_p = 5.884\times 10^{-6}\ \text{eV} . \tag{37}$$

The frequency of the photon emitted in a transition from the triplet to the singlet state is then

$$\nu = \Delta E/h = 1422.8\ \text{MHz} , \tag{38}$$

and its wavelength is

$$\lambda = c/\nu = 21.07\ \text{cm} . \tag{39}$$

The experimental value is[27]

$$\nu = 1420.405\,751\,766\,7\ \text{MHz} ; \tag{40}$$

the 0.2% discrepancy is attributable to quantum electrodynamical corrections.[28]

It is instructive to express the hyperfine splitting [Eq. (37)] in terms of the binding energy ($R = 13.6058$ eV) of the ground state:

$$\Delta E_{\text{hyd}} = \tfrac{8}{3}(R^2/m_p c^2)\, g_e\, g_p . \tag{41}$$

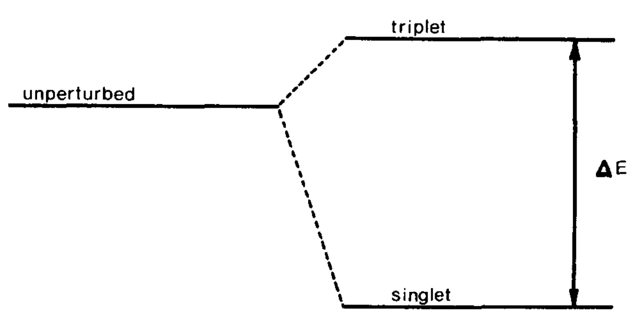

Fig. 5. Hyperfine splitting in the ground state of hydrogen.

By constrast, the *fine* structure goes like $(R^2/m_e c^2)$, and is therefore typically greater by a factor on the order of $(m_p/m_e) = 1836$. In the case of positronium, where the proton is replaced by a positron, the fine and hyperfine splittings are roughly equal in size. If we apply Eq. (37) to positronium (using the reduced mass, of course, in calculating the "Bohr radius"), we obtain

$$\Delta E_{\text{pos}} = \tfrac{1}{8}(1 + m_e/m_p)^3 (g_e/g_p)(m_p/m_e) \times \Delta E_{\text{hyd}} = 4.849 \times 10^{-4}\ \text{eV}, \tag{42}$$

as compared with an experimental value of 8.411×10^{-4} eV.[29] The large discrepancy is due primarily to pair annihilation, which splits the levels by an additional amount, $\frac{3}{4}\Delta E_{\text{pos}}$,[30] and does not occur, of course, in hydrogen. Muonium (in which a *muon* substitutes for the proton) offers a cleaner application of Eq. (37). The g factor of the muon is 2.0023,[22] (identical, up to corrections of very high order, with that of the electron), so

$$\Delta E_{\text{muon}} = \left(\frac{1 + m_e/m_p}{1 + m_e/m_\mu}\right)^3 \frac{g_\mu}{g_p} \frac{m_p}{m_\mu} \Delta E_{\text{hyd}} = 1.8493 \times 10^{-5}\text{eV}, \tag{43}$$

which compares very well with the experimental value[31]

$$1.845\,888\,5 \times 10^{-5}\text{eV}.$$

The 0.2% discrepancy is, again, a quantum electrodynamical correction.[32] Incidentally, the hyperfine splitting in muonic hydrogen (muon substituting for electron) would be "gigantic":

$$\Delta E_{\mu\text{hyd}} = \left(\frac{1 + m_p/m_e}{1 + m_p/m_\mu}\right)^3 \frac{g_\mu}{g_e} \frac{m_e}{m_\mu} \Delta E_{\text{hyd}} = 0.182\,96\ \text{eV}, \tag{44}$$

which corresponds to a wavelength of 67 800 Å, in the infrared region. However, as far as I know this quantity has not yet been measured directly in the laboratory.[33]

V. WHY IS THE SINGLET LEVEL LOWER?

In the singlet state, the proton and electron spins are antiparallel, which is to say that their magnetic moments are parallel. Why should this be the configuration of lowest energy? On a formal level, it is a consequence of the sign of the delta-function term in the interaction energy [Equation (26)]; *electric* dipoles, by contrast, would line up *antiparallel* [compare Eqs. (10) and (24)]. But we would like to understand this on a more intuitive basis.

Imagine two compass needles, held a distance r apart, at angles θ_1 and θ_2 to the line joining them (Fig. 6). If r is substantially greater than the length of each needle, they interact essentially as ideal magnetic dipoles, and the energy of the system is given by the first term in Eq. (26):

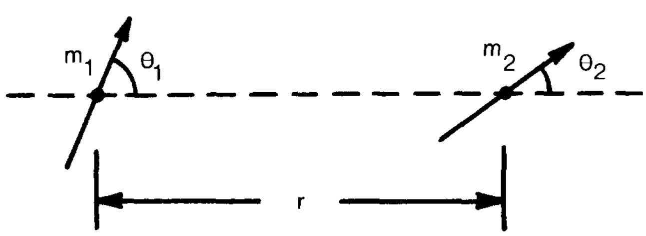

Fig. 6. Interaction of two magnetic dipoles.

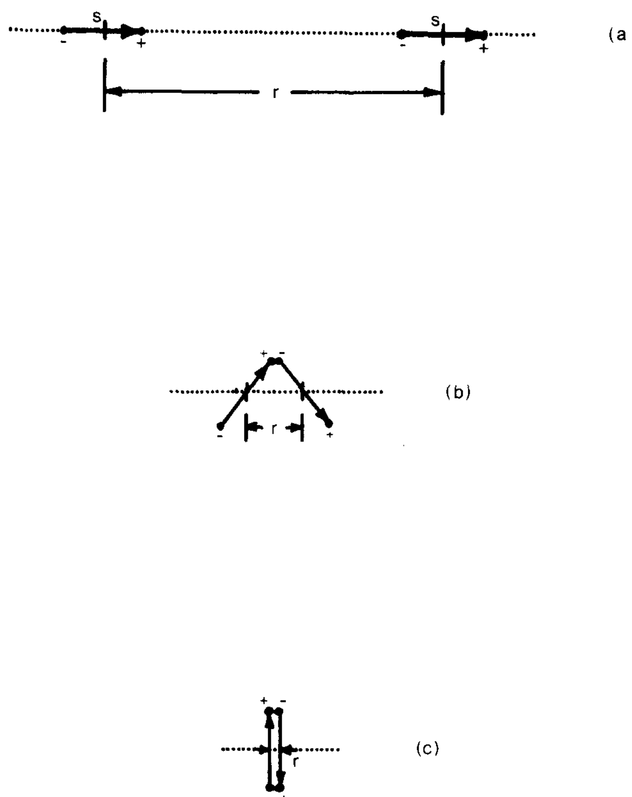

Fig. 7. Stable configuration for electric dipoles: (a) $r > s$; (b) $r < s$; and (c) $r \to 0$.

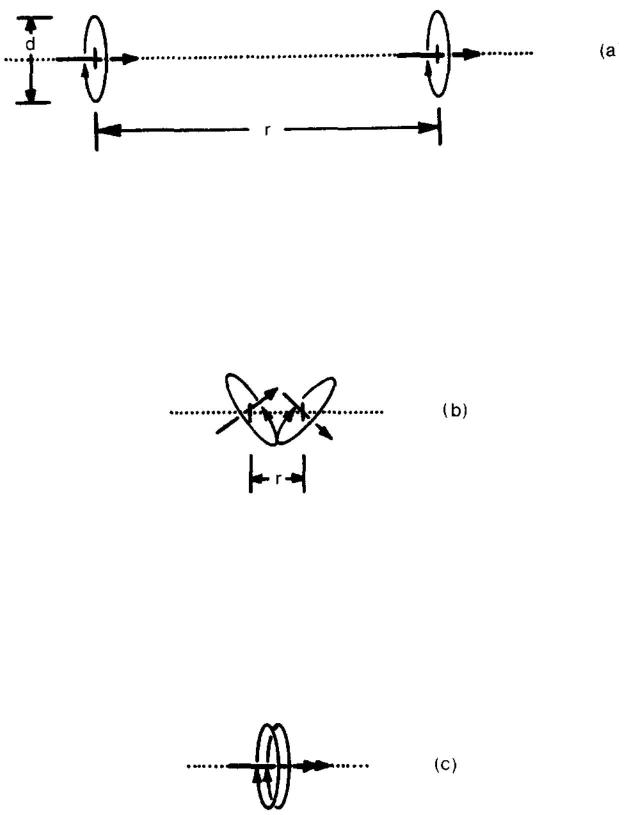

Fig. 8. Stable configuration for magnetic dipoles: (a) $r > d$; (b) $r < d$; and (c) $r \to 0$.

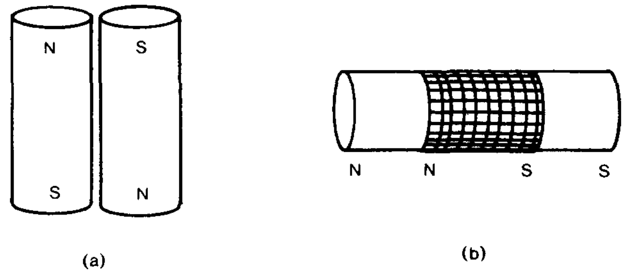

Fig. 9. Stable configuration for bar magnet: (a) solid; (b) penetrable.

$$W = \frac{\mu_0}{4\pi} \frac{1}{r^3} m_1 m_2 [\cos(\theta_1 - \theta_2) - 3\cos\theta_1 \cos\theta_2] . \quad (45)$$

It is easy to show that the minimum occurs at $\theta_1 = \theta_2 = 0$ (or, equivalently, at $\theta_1 = \theta_2 = \pi$); if they are free to rotate, then, the compass needles will tend to line up parallel to one another, along the common axis. And the same goes for electric dipoles.

But what happens as we bring them closer together? Consider first the case of electric dipoles—plus and minus charges separated by a distance s. As long as r is greater than s, they line up along the axis [Fig. 7(a)], but when the positive end of one meets the negative end of the other, these ends stick together and move off the line of centers [Fig. 7(b)], until finally, as $r \to 0$, the two dipoles are oriented *anti*parallel to one another, and *perpendicular* to the line joining them [Fig. 7(c)].

If we now repeat the process with *magnetic* dipoles—represented by circular current loops of diameter d—no such reversal occurs. Since parallel currents attract, there will occur a time when the circles tilt over to touch one another [Fig. 8(b)], but as $r \to 0$ the loops come together with their currents in the same direction [Fig. 8(c)]. The stable configuration for superimposed *magnetic* dipoles, then, is one in which they lie *parallel* to each other, and to the line joining them.

Of course, if you conducted this last experiment using long thin bar magnets, instead of current loops, to represent the magnetic dipoles, you would arrive at the opposite conclusion: they line up *anti*parallel, like electric dipoles, with the north pole of one against the south pole of the other [Fig. 9(a)]. But this is an artifact of the solidity of iron, having nothing to do with the magnetic coupling. If the two bars could interpenetrate, they would prefer to line up parallel to one another, as before [Fig. 9(b)].

On purely classical grounds then, superimposed magnetic dipoles tend to align themselves parallel to one another, and for this reason we should expect the singlet configuration in the ground state of hydrogen to carry the lower energy. It is well to remember that the hyperfine splitting is due to a *contact* interaction, and one's intuitions, based on the coupling of distant dipoles, may be deceiving. We have seen this in the contrasting behavior of electric and magnetic dipoles at very short range. Indeed, if the magnetic moments of electrons and protons were actually due to north and south magnetic monopoles, analogous to electric dipoles, then the triplet configuration would carry the lower energy.

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[8]Reference 5, p. 61.

[9]E. M. Purcell, *Electricity and Magnetism* (McGraw-Hill, New York, 1965), p. 327.

[10]If the ideal dipole is represented by infinitesimally displaced plus and minus charges, the delta-function term pertains to the region between them, where (unlike all other points on the axis) the field is opposite to **p**.

[11]Reference 5, p. 320.

[12]Reference 6, p. 177.

[13]Reference 5, p. 389–391. A somewhat different proof is given here.

[14]This follows from the divergence theorem, $\int \nabla \cdot \mathbf{V}\, d\tau = \int \mathbf{V} \cdot d\mathbf{a}$, if you let $\mathbf{V} = \mathbf{A} \times \mathbf{c}$, where $\mathbf{c}$ is an arbitrary constant vector.

[15]Reference 6, p. 175.

[16]Reference 5, p. 321. The entire proof of Theorem 2 can be cast into the (perhaps more familiar) language of line currents by the standard transcription $\int (\;)\mathbf{J}\, d\tau \to I \int (\;) d\mathbf{l}$.

[17]Reference 6, p. 208.

[18]If the ideal dipole is represented by an infinitesimal current loop, the delta-function term pertains to the region inside the loop, where (unlike all other points in the plane) the field is parallel to **m**.

[19]Reference 4, p. 186.

[20]Reference 1, p. 264.

[21]Reference 1, p. 229.

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