Chiral fermions and chemical potential

Some thoughts

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Introduction of chemical potential on the lattice for chiral fermions

- Overlap Dirac operator at nonzero chemical potential and random matrix theory; Jacques Bloch, Tilo Wettig; hep-lat/0604020
- Domain-wall and overlap fermions at nonzero quark chemical potential; Jacques Bloch, Tilo Wettig; arXiv:0709.4630 [hep-lat]
- Energy density for chiral lattice fermions with chemical potential; Christof Gattringer , Ludovit Liptak; arXiv:0704.0092 [hep-lat]
- Thermodynamics of the ideal overlap quarks on the lattice; Debasish Banerjee, R.V. Gavai, Sayantan Sharma; arXiv:0803.3925 [hep-lat]

Basic idea in hep-lat/0604020

- Follow Hasenfratz and Karsch (Phys. Lett. B125:308, 1983)
  Replace U_4 by $e^{\mu}U_4$ and $U_4^\dagger$ by $e^{-\mu}U_4^\dagger$ everywhere in the Wilson-Dirac kernel.
- Extend the definition of the sign function.

Massless overlap Dirac operator

Massless overlap Dirac operator in even dimensions is

$$D_o = \frac{1}{2}\left(1 + \gamma_{d+1}\epsilon[H_w(U,m)]\right)$$

- H_W is the hermitian Wilson Dirac operator.
- U is the background gauge field.
- m is the negative Wilson mass taken to be in the range $[0,2]$.
- ϵ function on a Hermitian matrix is defined as follows: If

  $$H_w = V\Lambda V^\dagger$$

  where V is the unitary matrix that diagonalizes H_w and $\Lambda_{ij} = \lambda_i\delta_{ij}$ with real λ_i being the eigenvalues of H_w; then

  $$\epsilon(H_w) = V\frac{\Lambda}{|\Lambda|}V^\dagger$$

  where $|\Lambda|_{ij} = |\lambda_i|\delta_{ij}$.

Wilson Dirac operator with a chemical potential

In the presence of a chemical potential, μ, H_w is not hermitian. It takes the form

$$H_w = \begin{pmatrix} B & C_R \\ C_L & -B \end{pmatrix}$$

$$\begin{aligned}[C_L]_{x\alpha i,y\beta j} = & \ \tfrac{1}{2}\textstyle\sum_{k=1}^{d-1} \sigma_k^{\alpha\beta}[\delta_{y,x+\hat{k}}(U_k(x))_{ij} - \delta_{x,y+\hat{k}}(U_k^\dagger(y))_{ij}] \\ & +\tfrac{1}{2}\sigma_d^{\alpha\beta}[\delta_{y,x+\hat{d}}e^{\mu}(U_d(x))_{ij} - \delta_{x,y+\hat{d}}e^{-\mu}(U_d^\dagger(y))_{ij}]\end{aligned}$$

$$\begin{aligned}[C_R]_{x\alpha i,y\beta j} = & -\tfrac{1}{2}\textstyle\sum_{k=1}^{d-1} [\sigma_k^\dagger]^{\alpha\beta}[\delta_{y,x+\hat{k}}(U_k(x))_{ij} - \delta_{x,y+\hat{k}}(U_k^\dagger(y))_{ij}] \\ & -\tfrac{1}{2}[\sigma_d^\dagger]^{\alpha\beta}[\delta_{y,x+\hat{d}}e^{\mu}(U_d(x))_{ij} - \delta_{x,y+\hat{d}}e^{-\mu}(U_d^\dagger(y))_{ij}]\end{aligned}$$

$$\begin{aligned}[B]_{x\alpha i,y\beta j} = & \ \tfrac{1}{2}\delta_{\alpha\beta}\textstyle\sum_{k=1}^{d-1} [2\delta_{xy}\delta_{ij} - \delta_{y,x+\hat{k}}(U_k(x))_{ij} - \delta_{x,y+\hat{k}}(U_k^\dagger(y))_{ij}] \\ & +\tfrac{1}{2}\delta_{\alpha\beta}[2\delta_{xy}\delta_{ij} - \delta_{y,x+\hat{d}}e^{\mu}(U_d(x))_{ij} - \delta_{x,y+\hat{d}}e^{-\mu}(U_d^\dagger(y))_{ij}] - m\delta_{x\alpha i,y\beta j}\end{aligned}$$

- $C_L^\dagger(\mu) = C_R(-\mu)$
- $B^\dagger(\mu) = B(-\mu)$

Bloch and Wettig, hep-lat/0604020

The definition of the ϵ function for a hermitian matrix is extended to a general complex matrix as follows: If

$$H_w = V\Lambda V^{-1}$$

where V is a complex matrix that diagonalizes H_w and $\Lambda_{ij} = \lambda_i \delta_{ij}$ with complex λ_i being the eigenvalues of H_w; then

$$\epsilon(H_w) = V\epsilon(\Lambda)V^{-1} = V\left[\lim_{L_s\to\infty} \frac{e^{L_s\Lambda}-1}{e^{L_s\Lambda}+1}\right]V^{-1}$$

where $\left[e^{L_s\Lambda}\right]_{ij} = e^{L_s\lambda_j}\delta_{ij}$.

If

$$\lambda_j = R_j + iI_j$$

then

$$\lim_{L_s\to\infty} \frac{e^{L_s\lambda_j}-1}{e^{L_s\lambda_j}+1} V^{-1} = \frac{R_j}{|R_j|} = \frac{\mathrm{Re}\ \lambda_j}{|\mathrm{Re}\ \lambda_j|}$$

A domain-wall justification

The domain wall action for massless fermions can be wrriten as (H. Neuberger, hep-lat/9710089)

$$S = -\sum_{s=1}^{2L_s} \bar{\Phi}_s (D\Phi)_s$$

$$\begin{pmatrix} \bar{\Phi}_1 & \bar{\Phi}_2 & \cdots & \cdots & \bar{\Phi}_{2L_s-1} & \bar{\Phi}_{2L_s} \end{pmatrix} = \begin{pmatrix} \bar{\chi}_1^R & \bar{\chi}_1^L & \cdots & \cdots & \bar{\chi}_{L_s-1}^R & \bar{\chi}_{L_s}^L \end{pmatrix}$$

$$D = \begin{pmatrix}
C_R & B+1 & 0 & 0 & 0 & 0 & \dots & \dots & 0 & 0 \\
B+1 & -C_L & -1 & 0 & 0 & 0 & \dots & \dots & 0 & 0 \\
0 & -1 & C_R & B+1 & 0 & 0 & \dots & \dots & 0 & 0 \\
0 & 0 & B+1 & -C_L & -1 & 0 & \dots & \dots & 0 & 0 \\
0 & 0 & 0 & -1 & C_R & B+1 & \dots & \dots & 0 & 0 \\
0 & 0 & 0 & 0 & B+1 & -C_L & \vdots & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \ddots & \ddots & \dots \\
0 & 0 & 0 & 0 & 0 & 0 & \dots & \dots & B+1 & -C_L
\end{pmatrix}$$

The physical fermion is

$$\bar{\psi} = \begin{pmatrix} \bar{\chi}_1^R & \bar{\chi}_{L_s}^L \end{pmatrix}$$

Pseudofermions

The contribution from all the unphysical fermions are subtracted by the pseudofermion action

$$D^{\rm pf} = \begin{pmatrix} C_R & B+1 & 0 & 0 & 0 & 0 & \ldots & \ldots & 0 & 1 \\ B+1 & -C_L & -1 & 0 & 0 & 0 & \ldots & \ldots & 0 & 0 \\ 0 & -1 & C_R & B+1 & 0 & 0 & \ldots & \ldots & 0 & 0 \\ 0 & 0 & B+1 & -C_L & -1 & 0 & \ldots & \ldots & 0 & 0 \\ 0 & 0 & 0 & -1 & C_R & B+1 & \ldots & \ldots & 0 & 0 \\ 0 & 0 & 0 & 0 & B+1 & -C_L & \vdots & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ddots & \ddots & \ldots \\ 1 & 0 & 0 & 0 & 0 & 0 & \ldots & \ldots & B+1 & -C_L \end{pmatrix}$$

Fermion determinant

Neuberger, hep-lat/9710089

$$\det D = (\det B)^k \det \left[\frac{1 - T^{-k}}{2} - \frac{1 + T^{-k}}{2} \gamma_5 \right]$$

$$\det D^{\mathrm{pf}} = (\det B)^k \det \left[- \left(1 + T^{-k}\right) \gamma_5 \right]$$

$$T = \begin{pmatrix} \frac{1}{B+1} & \frac{1}{B+1} C_L \\ C_R \frac{1}{B+1} & C_R \frac{1}{B+1} C_L + B + 1 \end{pmatrix}$$

$$\frac{\det D}{\det D^{\mathrm{pf}}} = \det \frac{1}{2} \left[1 + \gamma_5 \tanh \frac{T^{-L_s} - 1}{T^{-L_s} + 1} \right]$$

Using $\tanh$ to define ϵ

T is a general complex matrix in the presence of a chemical potential and we are interested in

$$\frac{T^{-L_s} - 1}{T^{-L_s} + 1}$$

Let

$$T = VEV^{-1}$$

where V is the general complex matrix that diagonalizes T and $E_{ij} = e_i \delta_{ij}$ is the diagonal matrix made up of the complex eigenvalues, e_i, of T.

Then

$$\frac{T^{-L_s} - 1}{T^{-L_s} + 1} = V \frac{E^{-L_s} - 1}{E^{-L_s} + 1} V^{-1}$$

$$\lim_{L_s \to \infty} \frac{e_i^{-L_s} - 1}{e_i^{-L_s} + 1} = \begin{cases} 1 & \text{if } |e_i| < 1 \\ -1 & \text{if } |e_i| > 1 \end{cases}$$

Overlap Dirac operator with chemical potential

$$\lim_{L_s \to \infty} \frac{T^{-L_s} - 1}{T^{-L_s} + 1} = \epsilon \left[- \ln T \right]$$

$$\lim_{L_s \to \infty} \frac{\det D}{\det D^{\mathrm{pf}}} = \det D_o$$

$$D_o = \frac{1}{2} \left[1 + \gamma_5 \epsilon \left(- \ln T \right) \right]$$

$-\ln T \to H_w$ as the lattice spacing in the $(d+1)$ direction goes to zero.

The definition of the overlap Dirac operator with a chemical potential in hep-lat/0604020 ***seems*** justified.

Isospin chemical potential

$$H_w(\mu) = \begin{pmatrix} B(\mu) & C_R(\mu) \\ C_L(\mu) & -B(\mu) \end{pmatrix}$$

- $C_L^\dagger(\mu) = C_R(-\mu)$
- $B^\dagger(\mu) = B(-\mu)$

$$H_w^\dagger(\mu) = H_w(-\mu)$$

$$\epsilon\left(H_w(-\mu)\right) = \left[\epsilon\left(H_w(\mu)\right)\right]^\dagger$$

$$\det D_o(\mu) = \left[\det D_o(-\mu)\right]^*$$

A reminder of the derivation of the overlap Dirac operator

Overlap fermions provides a solution to the problem of putting chiral fermions on the lattice.

Assume there is no chemical potential.

Form two many body operators:

$$\mathcal{H}_- = a^\dagger \gamma_5 a$$
$$\mathcal{H}_+ = a^\dagger H_w a$$

Then

$$\det C_L = \langle b - | b+ \rangle$$
$$\det C_R = \langle t - | t+ \rangle$$

where $|b\pm >$ are the normalized lowest energy states of $\mathcal{H}_\pm$ and $|t\pm >$ are the normalized highest energy states of $\mathcal{H}_\pm$.

Phases of these states have to be fixed such that

$$\det C_L = \det C_R^\dagger$$

The computation

Let

$$V = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

be the unitary matrix that diagonalizes H_w with $\begin{pmatrix} \alpha \\ \gamma \end{pmatrix}$ and $\begin{pmatrix} \beta \\ \delta \end{pmatrix}$ spanning the positive and negative eigenvalues of H_w respectively.

$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ span the positive and negative eigenvalues of γ_5 respectively.

Therefore, $\det C_L = \delta$ and $\det C_R = \alpha$ up to a phase.

Since V is unitary, one can show that

$$\det V = \frac{\det \alpha}{\det \delta^\dagger}$$

Since $\det V \det V^\dagger = 1$, it follows that

$$\det \alpha \det \alpha^\dagger = \det \delta \det \delta^\dagger$$

and therefore $\det C_L \det C_L^\dagger = \det C_R \det C_R^\dagger$ are the same and independent of the phase choice.

Derivation of the overlap Dirac operator

$$\epsilon(H_w)V = \begin{pmatrix} \alpha & -\beta \\ \gamma & -\delta \end{pmatrix}$$

$$\gamma_5\epsilon(H_w)V = \begin{pmatrix} \alpha & -\beta \\ -\gamma & \delta \end{pmatrix}$$

$$D_o V = \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}$$

$$\det D_o \frac{\det \alpha}{\det \delta^\dagger} = \det \alpha \det \delta$$

$$\det D_o = \det \delta \det \delta^\dagger$$

Addition of the chemical potential

$$V = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

$$\epsilon(H_w)V = \begin{pmatrix} \alpha & -\beta \\ \gamma & -\delta \end{pmatrix}$$

$$\gamma_5\epsilon(H_w)V = \begin{pmatrix} \alpha & -\beta \\ -\gamma & \delta \end{pmatrix}$$

$$D_oV = \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}$$

$$\det D_o \det V = \det \alpha \det \delta$$

$$\det D_o = \det V^{-1} \det \alpha \det \delta$$

Remarks

- H_w is not hermitian. $\mathcal{H}_+$ is not a hermitian many body operator.
- $a^\dagger$ should really be replaced by a^{-1}. It carries the same meaning. a^{-1} is the creation operator and is the inverse of a, the annihilation operator.
- If (a, a^{-1}) obey canonical anticommutation relations, and if $b = V^{-1}a$, then (b, b^{-1}) also obey canonical anticommutation relations.
- $\det C_L \det C_R$ is not real and positive.
- There should be no ambiguity in the definition of $\det C_L \det C_R$.
- Under $V \rightarrow DV$ where D is an arbitary complex diagonal matrix, $\det V^{-1} \det \alpha \det \delta$ and therefore $\det D_o$ is invariant.
- The propagator

$$G_o = D_o^{-1} - 1 = \begin{pmatrix} 0 & \beta\delta^{-1} \\ \gamma\alpha^{-1} & 0 \end{pmatrix}$$

is clearly chiral and is invariant under $V \rightarrow DV$.

Eigenvalues of $S = \gamma_5 \epsilon$

$$\epsilon^2 = 1$$

Let

$$S\psi = s\psi$$

Then

$$\epsilon\psi = s\gamma_5\psi \quad \Rightarrow \quad \psi = s\epsilon\gamma_5\psi \quad \Rightarrow \quad \frac{1}{s}\left[\gamma_5\psi\right] = S\left[\gamma_5\psi\right]$$

- There is a pairing of eigenvalues of the form, $(s, 1/s)$.
- $s = \pm 1$ are not paired.
- $s = -1$ corresponds to a zero mode of D_o.
- If ϵ is hermitian, S is unitary and all eigenvalues lie on the unit circle.
- In the presence of μ. eigenvalues inside the unit circle have partners outside the unit circle.

$\det D_O(\mu)$

Assume we are in the zero topological sector.

When $\mu = 0$, let $s_j = e^{i\phi_j}$ with $0 \leq \phi_j < \pi$ be half the eigenvalues of S. Then,

$$\det D_o(0) = \prod_j \cos^2 \frac{\phi_j}{2}$$

When $\mu \neq 0$, let us assume that $|s_j| < 1$ be half the eigenvalues of S. Then,

$$\det D_o(0) = \prod_j \frac{1}{4} \left[2 + s_j + s_j^{-1}\right]$$

Whether $\mu = 0$ or $\mu \neq 0$, s_j close to -1 cause a suppression and this is just the role of almost zero modes.

What if all the s_j with $|s_j| < 1$ get close to zero? (A poosible scenario as μ is increased)
Then, the determinant gets very large (opposite of suppression).
In addition, if the phase of s_j gets uniformly distributed on the unit circle, then the determinant will remain large.

Phase of $\det D_O(\mu)$

- Phase of the fermion determinant results in the sign problem.
- What happens in large N_c QCD with finite number of fermions flavors?
- Can we work in the quenched approximation even with a chemical potential in the large N_c limit?
- The fermion determinant should still be one power of N_c less than the vacuum polarization.
- But the fermion determinant will have a factor of N_c and this implies that the phase of the determinant can be anywhere on the unit circle. If so, phase averaging is a problem.
- What is the phase distribution of the eigenvalues s_j with $|s_j| > 1$?