Lattice calculation of the QCD EOS with asqtad fermions

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Outline

- The QCD equation of state (EOS) at zero chemical potential ($\mu = 0$)
  - Properties of QGP from the experiment. The significance of the equation of state.
  - Nonzero temperature QCD on the lattice.
  - Integral method for the calculation of the EOS on the lattice.
  - The EOS results
- The QCD EOS at nonzero chemical potential ($\mu \neq 0$)
  - The Taylor expansion method
  - EOS results at $\mu/T = \mathrm{const}$
  - The isentropic EOS

- **THE QCD EQUATION OF STATE AT ZERO CHEMICAL POTENTIAL**

QCD Matter at Extreme Conditions

- QCD – the theory of the strong interactions, as a consequence of the nonperturbative structure of the vacuum has the properties of quark confinement and dynamical chiral symmetry breaking.
- At high temperatures and/or densities the vacuum structure changes and the hadron constituents – quarks and gluons – are expected to be deconfined and the chiral symmetry restored. The new phase of nuclear matter is called "quark-gluon plasma" (QGP).
- Where to find QGP:
  - Early Universe
  - Early stages of supernova explosions
  - Neutron stars interior
  - Physics experiments - heavy-ion collisions (RHIC, CERN, etc.)

QGP at Experimental Conditions

- QGP's nonperturbative character at $T \approx T_c$:
  - Dimensional arguments estimate $\varepsilon_c \approx 1\,\mathrm{GeV/fm}^3$ and $T_c \approx 170\,\mathrm{MeV}$. (Density at total overlap of several light hadrons within typical hadron volume of 1-3 fm^3.)
  - $T_c/\Lambda_{QCD} \approx 0.5$, which means that at experimentally accessible temperatures $T/T_c = 1-3$ the system is still in a QCD non-perturbative regime

    $$g \equiv \sqrt{4\pi\alpha_s} = O(1).$$

    QGP $\rightarrow$ sQGP. Evidence for strong interactions.
  - The most adequate tool to study sQGP is a nonperturbative one – Lattice QCD. Perturbation theory is only a rough guide.

The significance of the EOS of QGP

- In heavy-ion collisions after thermalization the system evolves hydrodynamically and its behavior will depend on the EOS ($\varepsilon(T)$ and $p(T)$).

- The hydrodynamical models that include a QGP phase and a resonance gas for the hadronic phase connected by a first order phase transition all assume an ideal gas EOS for the QGP phase. They reproduce the low p_T proton elliptic flow.

- However still there is no consistent picture that describes the heavy-ion collisions at RHIC. A more realistic EOS from lattice calculations as an input to the hydrodynamic models is an obvious direction for comparison with data.

Nonzero Temperature Lattice QCD

The quantum statistical Gibbs ensemble partition function $Z(T)$ at temperature T and the Euclidean path integral formulation of QFT are related by

$$Z(T) = \mathrm{Tr}\, \mathrm{e}^{-\mathrm{H}/\mathrm{T}} = \int \prod_{\mathrm{x}} \mathrm{d}\phi(\mathrm{x})\, \mathrm{e}^{-\mathrm{S}_\mathrm{E}(\phi,\mathrm{T})},$$

where $S_E(\phi, T)$ is the classical action at imaginary time

$$t = -i/T,$$

for a field configuration $\phi(x)$ on a space-time lattice of dimensions $N^3 \times N_t$. The lattice temporal extent and temperature are related through

$$T = 1/(a_t N_t).$$

On the lattice:

$$\mathbf{S_E}(\mathbf{U}, \mathbf{\Psi}, \overline{\mathbf{\Psi}}) = \mathbf{S_G}(\mathbf{U}) + \underbrace{\mathbf{S_F}(\mathbf{U}, \mathbf{\Psi}, \overline{\mathbf{\Psi}})}_{\overline{\mathbf{\Psi}}\mathbf{M}\mathbf{\Psi}}.$$

The expectation value of an observable $\mathbf{O}(\mathbf{U}, \mathbf{\Psi}, \overline{\mathbf{\Psi}})$ is given by

$$\langle \mathbf{O} \rangle = \frac{\mathbf{1}}{\mathbf{Z}} \int [\mathbf{dU}][\mathbf{d\Psi}][\mathbf{d}\overline{\mathbf{\Psi}}]\mathbf{O}(\mathbf{U}, \mathbf{\Psi}, \overline{\mathbf{\Psi}})\mathbf{e}^{-\mathbf{S_E}(\mathbf{U},\mathbf{\Psi},\overline{\mathbf{\Psi}})} = \frac{\mathbf{1}}{\mathbf{Z}} \int [\mathbf{dU}]\mathbf{O}(\mathbf{U})\det(\mathbf{M})\, \mathbf{e}^{-\mathbf{S_G}(\mathbf{U})}.$$

Lattice actions

- Gauge action: 1-loop improved Symanzik action. Discretization errors – $O(\alpha_s^2 a^2, a^4)$.

$$\mathbf{S_G} = \beta \sum_{\mathbf{x},\mu<\nu} (\mathbf{1} - \mathbf{P}_{\mu\nu}) + \beta_{\mathrm{rt}} \sum_{\mathbf{x},\mu<\nu} (\mathbf{1} - \mathbf{R}_{\mu\nu}) + \beta_{\mathrm{ch}} \sum_{\mathbf{x},\mu<\nu<\sigma} (\mathbf{1} - \mathbf{C}_{\mu\nu\sigma}),$$

- Fermion action: Asqtad staggered quark action – tree level improved, taste violations supressed. Discretization erros – $O(\alpha_s^2 a, a^4)$.

$$\begin{aligned} S_F &= \overline{\Psi} M \Psi \\ M &= 2m_f + \underbrace{\sum_i c_i (V_i - V_i^\dagger)}_{\text{fat link}} + \underbrace{w(L - L^\dagger)}_{\text{Lepage term}} + \underbrace{v(N - N^\dagger)}_{\text{Naik term}} \end{aligned}$$

- Simulation algorithm: Hybrid Molecular Dynamics R algorithm.

The Symanzik Improved Gauge action

- Symanzik improved gauge action: 1x1 plaquette loops, 2x1 rectangles and the 3D chairs. The last two loops are introduced in the action with appropriate parameters in order to cancel the a^2 artifacts at tree level.

$$\mathbf{S_G} = \beta \sum_{\mathbf{x},\mu<\nu} (\mathbf{1} - \mathbf{P}_{\mu\nu}) + \beta_{\rm rt} \sum_{\mathbf{x},\mu<\nu} (\mathbf{1} - \mathbf{R}_{\mu\nu}) + \beta_{\rm ch} \sum_{\mathbf{x},\mu<\nu<\sigma} (\mathbf{1} - \mathbf{C}_{\mu\nu\sigma}),$$

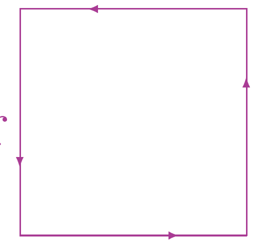
where $\beta = \frac{10}{g^2}$, $\beta_{\rm rt} = -\frac{\beta}{20u_0^2}(1 + 0.4805\alpha_s)$, $\beta_{\rm ch} = -\frac{\beta}{u_0^2}0.03325$ with $\alpha_s = -\frac{4\ln(u_0)}{3.0684}$

and $u_0 = \langle P \rangle^{1/4}$.

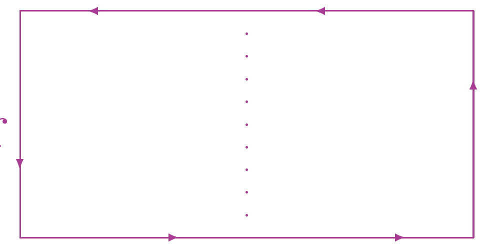
- The action coefficients are calculated using lattice tadpole improved perturbation theory.
- The tadpole improvement refers to the technique for summing to all orders the large perturbative contributions of the specific for lattice QCD tadpole diagrams. At tree level this simply means $U_\mu \to U_\mu/u_0$ in the action.
- Thus the discretization errors are reduced to $O(\alpha_s^2 a^2)$.

Lattice Loop Terms in the Gauge Action

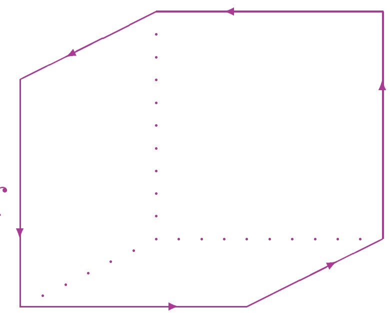
$$P_{\mu\nu} = \frac{1}{3}\mathrm{Re}\,\mathrm{Tr}$$



$$R_{\mu\nu} = \frac{1}{3}\mathrm{Re}\,\mathrm{Tr}$$

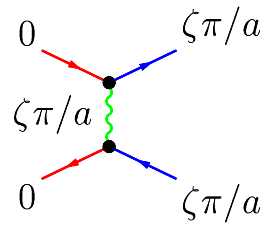


$$C_{\mu\nu\sigma} = \frac{1}{3}\mathrm{Re}\,\mathrm{Tr}$$



The Asqtad Quark Action

- There is a taste symmetry violation in the standard staggered action, which can be understood as quark changing their taste as they interact with high momentum ($\sim \pi/a$) gluons.



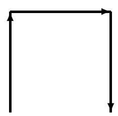
- Asqtad quark action reduces the taste symmetry breaking by means of terms which suppress the taste changing interactions. Lattice artifacts of order a^2 are removed at tree level and the leading errors are $O(\alpha_s a^2)$.

$$\begin{aligned} S_f^A &= \overline{\Psi} M \Psi \\ M &= 2m_f + \underbrace{\sum_i c_i (V_i - V_i^\dagger)}_{\text{fat link}} + \underbrace{w(L - L^\dagger)}_{\text{Lepage term}} + \underbrace{v(N - N^\dagger)}_{\text{Naik term}} \end{aligned}$$

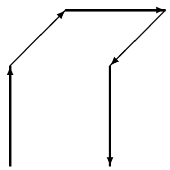
- To simulate n_f flavors - $(\det M)^{n_f/4}$.

Lattice Gauge Paths in the Asqtad Action

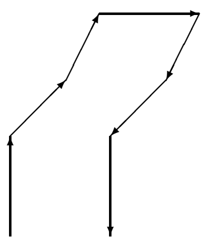
- Link – V_1, term in the fat link



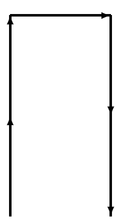
- Staple – V_3, term in the fat link



- 5-Staple – V_5, term in the fat link



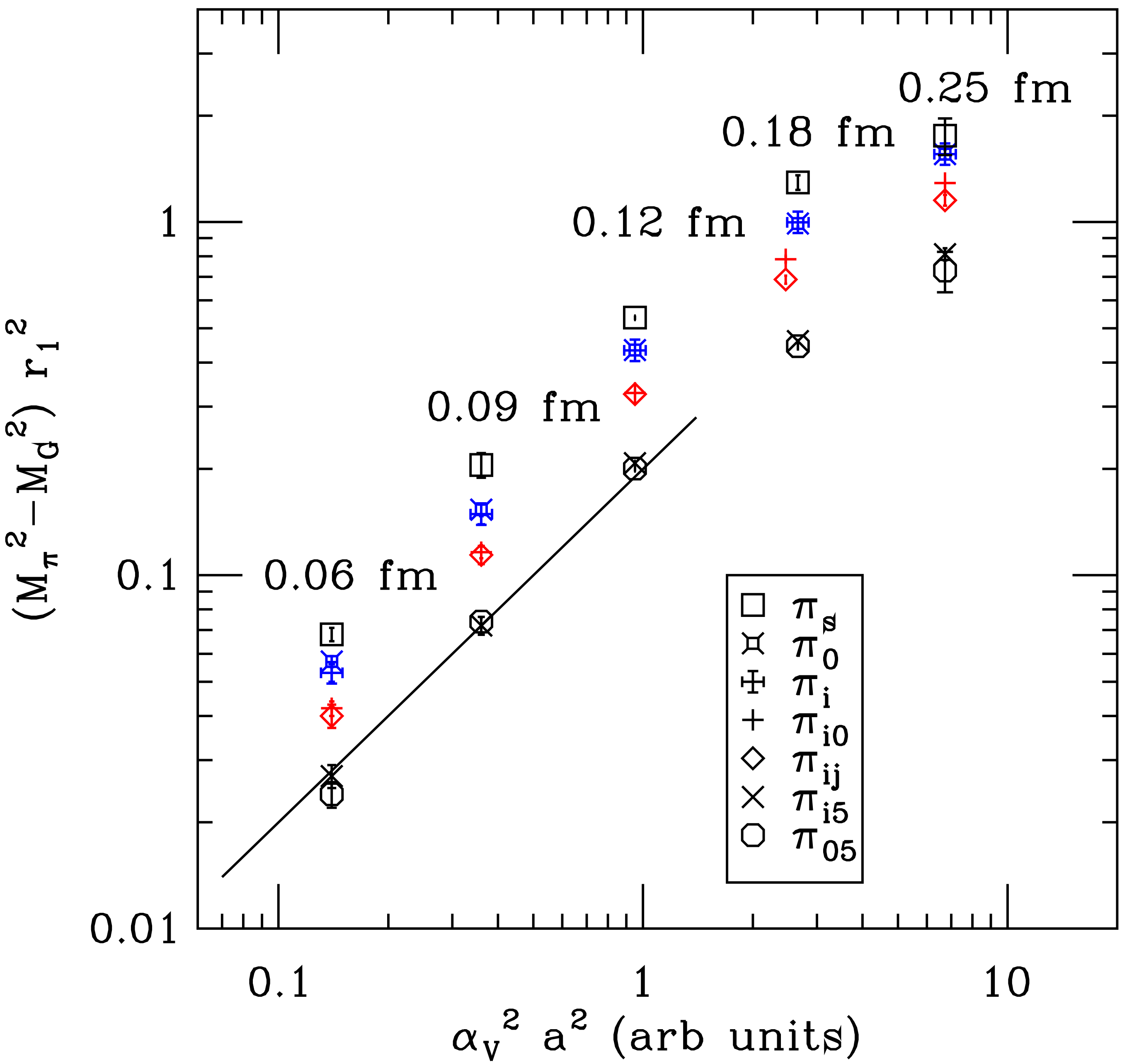
- 7-Staple – V_7, term in the fat link



- Lepage term – L, corrects for small $O(a^2)$ error introduced by the fat link

- Naik term – N, improves dispersion relation

Taste splitting of the pion multiplet



The EOS on the Lattice using the Integral Method

Start from the thermodynamic identities:

$$\varepsilon V = -\left.\frac{\partial \ln Z}{\partial(1/T)}\right|_V, \qquad \frac{p}{T} = \left.\frac{\partial \ln Z}{\partial V}\right|_T \approx \frac{\ln Z}{V}, \qquad I = \varepsilon - 3p = -\frac{T}{V}\frac{d\ln Z}{d\ln a},$$

where $V = N_s^3 a^3$, $T = \frac{1}{N_t a}$. The partition function is

$$Z = \int dU \exp\left\{-S_g + \sum_f (n_f/4)\mathrm{Tr}\ln[M(am_f, U, u_0)]\right\}.$$

with $M(am_f, U, u_0)$ the fermion matrix corresponding to the Asqtad quark action with 2 degenerate light quark flavors and 1 heavy quark flavor.

Thus:

$$\begin{aligned} Ia^4 \;=\; & -6\frac{d\beta_{\mathrm{pl}}}{d\ln a}\Delta\langle P\rangle - 12\frac{d\beta_{\mathrm{rt}}}{d\ln a}\Delta\langle R\rangle - 16\frac{d\beta_{\mathrm{ch}}}{d\ln a}\Delta\langle C\rangle \\ & - \sum_f \frac{n_f}{4}\left[\frac{d(m_f a)}{d\ln a}\Delta\left\langle\bar\psi\psi\right\rangle_f + \frac{du_0}{d\ln a}\Delta\left\langle\bar\psi\,\frac{dM}{du_0}\,\psi\right\rangle_f\right]. \end{aligned}$$

The EOS on the Lattice using the Integral Method

$$pa^4 = \int_{\ln a_0}^{\ln a} (-Ia'^4) d \ln a'$$

where $\ln a_0$ is determined by where (the zero-temperature corrected) $Ia^4 = 0$ at coarse lattice spacings.

The energy density is given by:

$$\varepsilon a^4 = (I + 3p)a^4$$

Observables to calculate: all gauge loops plus the fermion quantities in the zero- and nonzero-temperature phases

$$\left\langle \bar{\psi}\psi \right\rangle_f = \left\langle 2aM^{-1} \right\rangle_f$$

$$\left\langle \bar{\psi} \frac{dM}{du_0} \psi \right\rangle_f = \left\langle \frac{dM}{du_0} M^{-1} \right\rangle_f .$$

Choosing the Action Parameters

- Action parameters to choose: β, m_s, m_{ud} and u_0. Changing the parameters changes the lattice scale a and the physics on the lattice.
- Simulations at different parameters and scales represent the same physics if:
  - $m_{\eta_{ss}}/m_\phi = \mathrm{const}$ - fixes the heavy quark mass
  - $m_\pi/m_\rho = \mathrm{const}$ - fixes the light quark mass
- We want a guark-gluon system for which we change the temperature ($T = 1/(aN_t)$) without changing the physics. We have to choose the parameters of the action in a way that lets us stay on a chosen constant physics trajectory at zero temperature. We approximate two such trajectories:
  - $m_{ud} \approx 0.2 m_s$, ($m_\pi/m_\rho \approx 0.4$)
  - $m_{ud} \approx 0.1 m_s$, ($m_\pi/m_\rho \approx 0.3$)

  Both trajectories have m_s tuned to the physical strange quark mass within 20 %.

Parameterizing the Constant Physics Trajectories

- Construction of a constant physics trajectory:
  - At anchor points in β, tune m_π/m_ρ and m_η/m_ϕ.
  - Between anchor points the trajectory is interpolated, using a one-loop RG inspired formula.
- The $m_{ud} = 0.2 m_s$ trajectory – 3 anchor points $\beta = 6.467$, 6.76, and 7.092:

$$am_s = \begin{cases} 0.082 \ \exp\left((\beta - 6.4674) \frac{\ln(0.050/0.0820)}{(6.76-6.4674)} \right), & \beta \in [6.467, 6.76] \\ 0.05 \ \exp\left((\beta - 6.76) \frac{\ln(0.031/0.05)}{(7.092-6.76)} \right), & \beta \in [6.76, 7.092] \end{cases}$$

$$am_{ud} = \begin{cases} 0.01675 \ \exp\left((\beta - 6.4674) \frac{\ln(0.010/0.01675)}{(6.76-6.4674)} \right), & \beta \in [6.467, 6.76] \\ 0.01 \ \exp\left((\beta - 6.76) \frac{\ln(0.00673/0.01)}{(7.092-6.76)} \right), & \beta \in [6.76, 7.092]. \end{cases}$$

Parameterizing the Constant Physics Trajectories

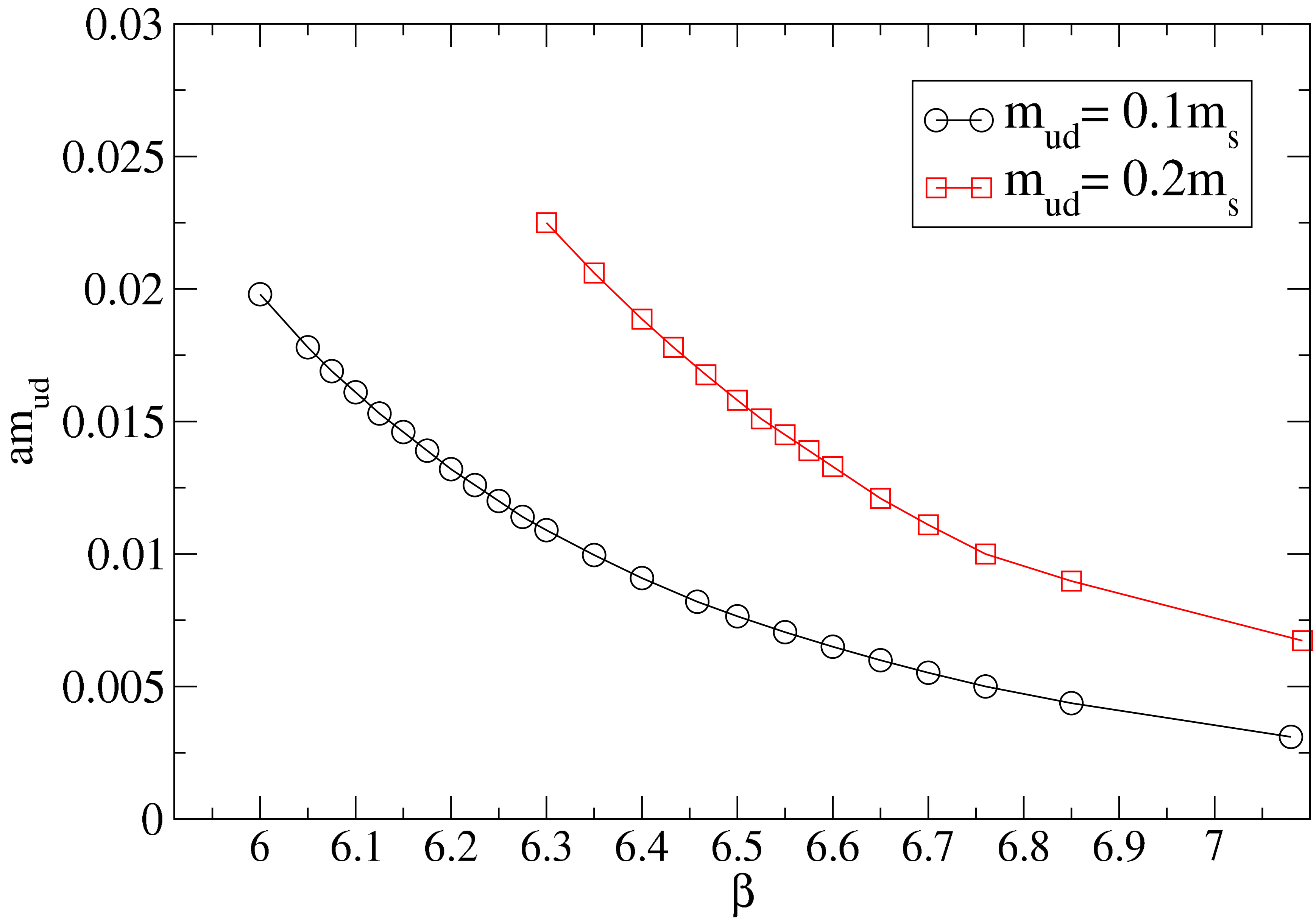
- The $m_{ud} = 0.1 m_s$ trajectory – 2 anchor points $\beta \in [6.458, 6.76]$:

$$am_s = 0.05 \exp\left((\beta - 6.76)\frac{\ln(0.082/0.05)}{(6.458 - 6.76)}\right)$$

$$am_{ud} = 0.005 \exp\left((\beta - 6.76)\frac{\ln(0.0082/0.005)}{(6.458 - 6.76)}\right).$$

- For both trajectories, for values of β out of the above intervals, the formulas are used as extrapolations appropriately.

Constant physics trajectories



Determination of the Lattice Spacing

- The lattice spacing a can be calculated from $1S - 2S$ Υ splittings

$$a = (a\Delta E)_{\rm lat}/\Delta E_{\rm exp}$$

- Measurements from about 30 zero temperature ensembles are fitted to

$$\frac{a}{r_1} = \frac{c_0 f(g^2) + c_2 g^2 f^3(g^2) + c_4 g^4 f^3(g^2)}{1 + d_2 g^2 f^2(g^2)},$$

where $r_1 = 0.318(7)(4)$ fm. The definition of

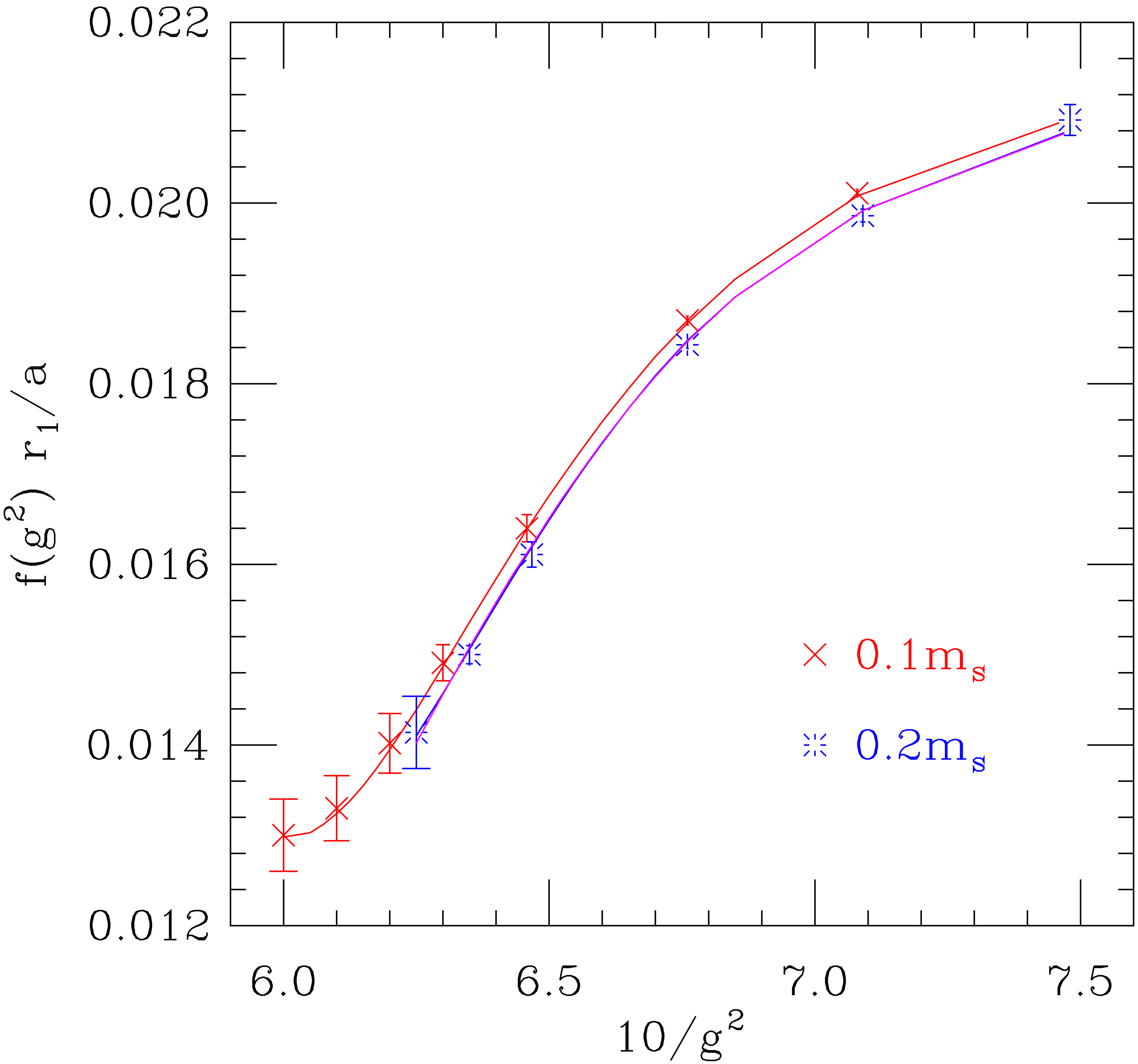
$$f(g^2) = (b_0 g^2)^{-b_1/(2b_0^2)} e^{-1/(2b_0 g^2)}$$

involves the universal beta-function coefficients for massless three-flavor QCD, b_0 and b_1. The coefficients c_0, c_2 and c_4 are

$$\begin{aligned}
c_0 &= c_{00} + (c_{01u} am_{ud} + c_{01s} am_s)/f(g^2) + c_{02}(2am_{ud} + am_s)^2/f^2(g^2) \\
c_2 &= c_{20} + c_{21}(2am_{ud} + am_s)/f(g^2) \\
c_4 &= c_{40} \\
d_2 &= d_{20},
\end{aligned}$$

The fit has $\chi^2/DOF \approx 1.3$ and a CL 0.13.

Constant physics trajectories



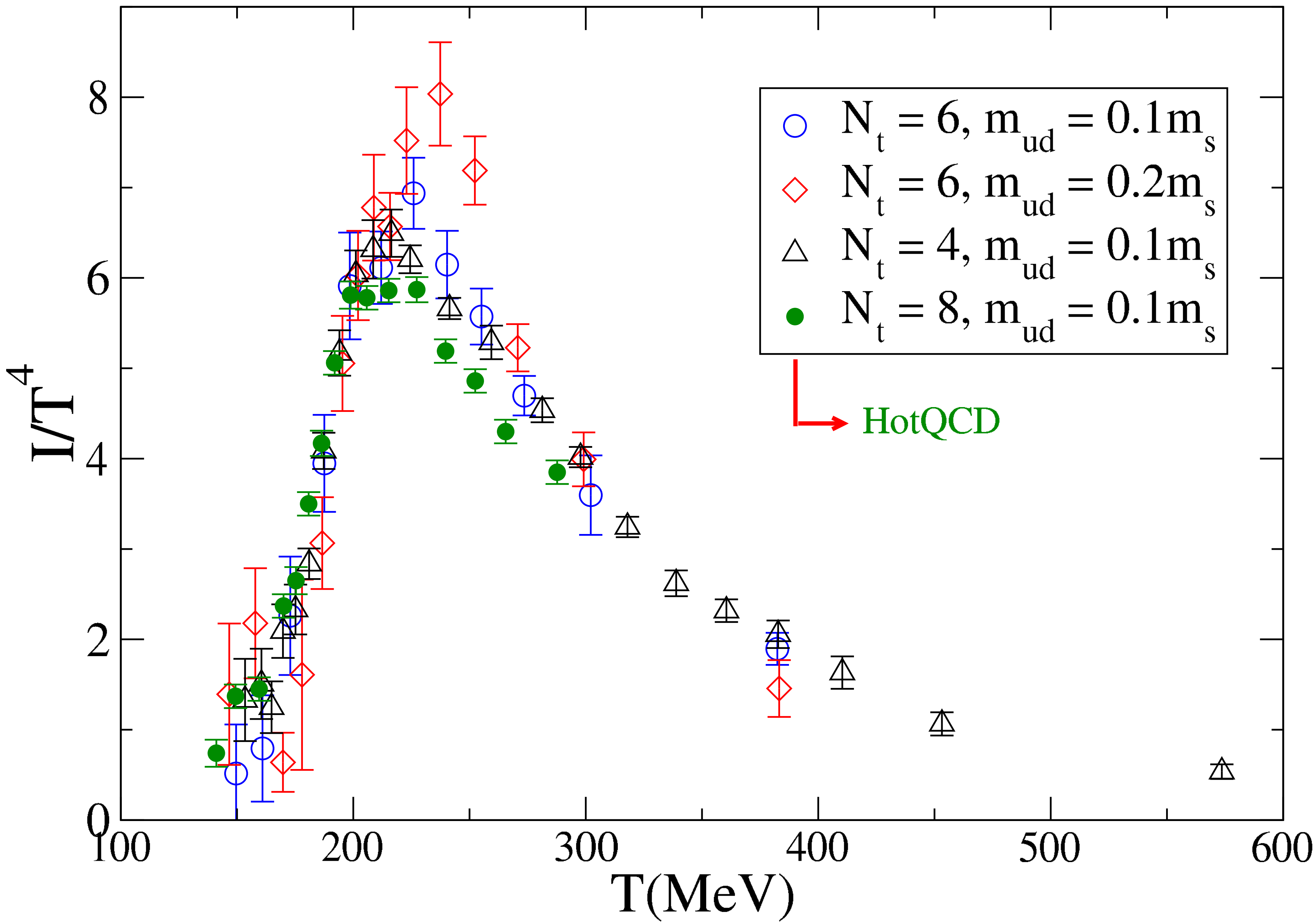
Simulations Overview

- We simulate 2+1 flavor QCD with $m_{ud} = 0.1m_s$ and $0.2m_s$ along trajectories of constant physics using improved gauge and quark actions. Our system is at thermal equilibrium and zero chemical potential.

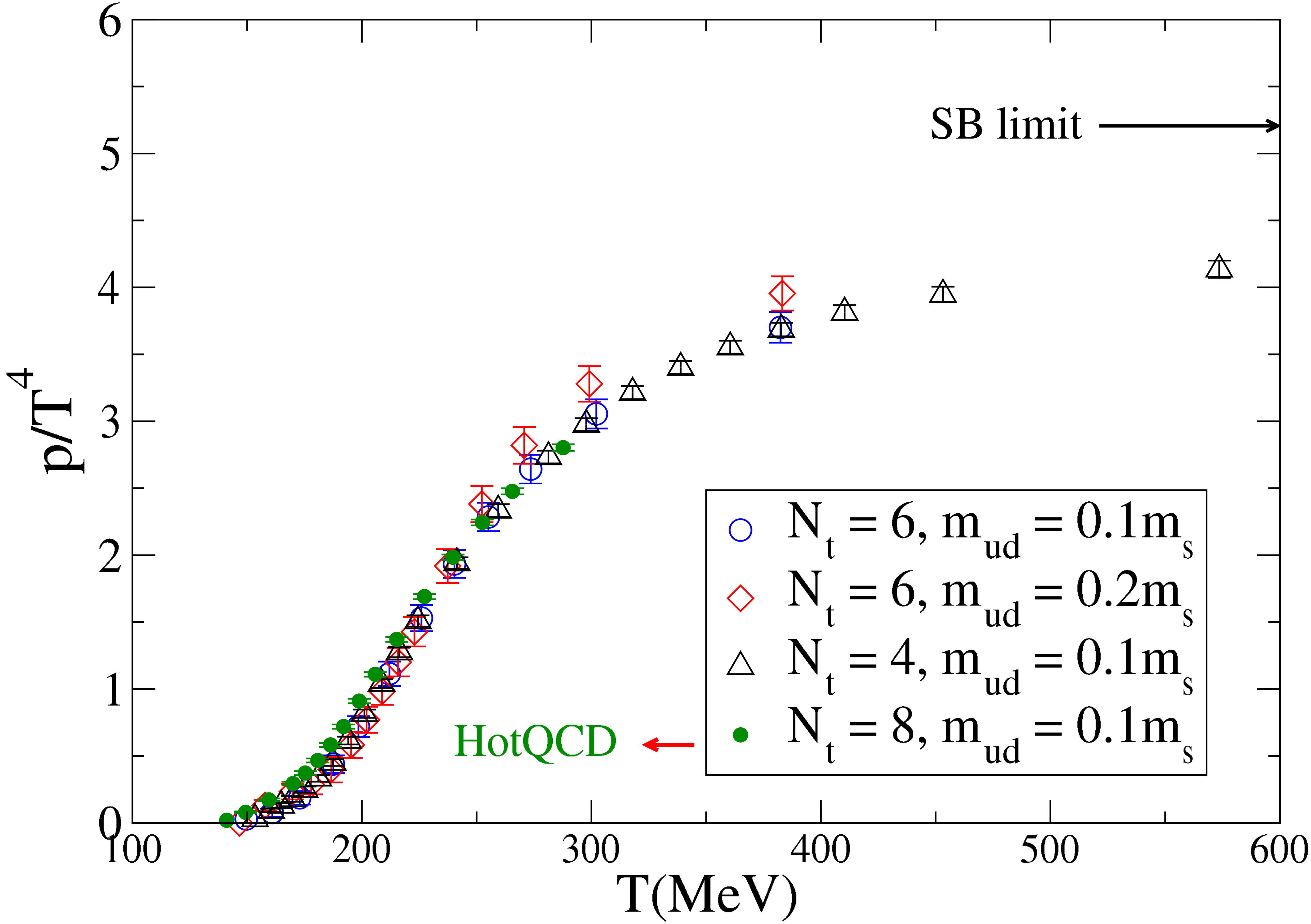
- Simulation algorithm – the inexact dynamical R-algorithm at $N_t = 4$ and 6. Step-size of the equations of motion integration is the min of $2/(3m_{ud})$ and 0.02, in some cases even smaller. Estimated step-size errors are up to the size of the statistical errors. For $N_t = 8$ the exact RHMC algorithm is used.

- Temperature $1/(aN_t)$ is changed by varying a (0.09 – 0.39 fm) along the trajectory and keeping $N_t = \mathrm{const}$. The cases of $N_t = 4, 6$ and 8 (HotQCD) are interesting to compare since at smaller N_t the taste splitting in the improved staggered action is worse - we want to know how this affects the EOS.

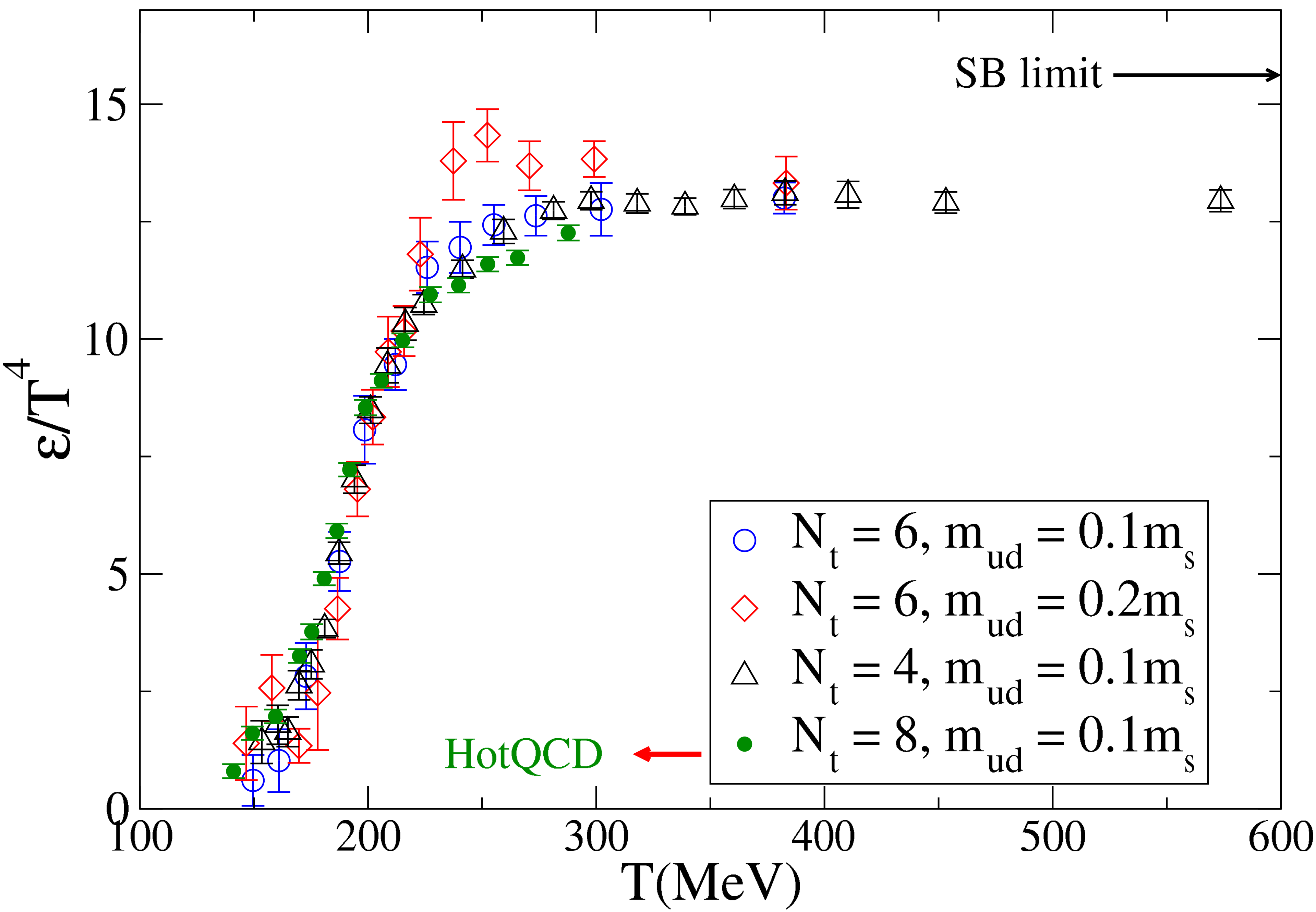
EOS results – Interaction measure



EOS results – Pressure



EOS results – Energy density



▶ THE QCD EQUATION OF STATE AT NONZERO CHEMICAL POTENTIAL

The EOS with 2+1 flavors at non-zero chemical potential

- We use the Taylor expansion method (C.R. Allton *et. al*, Phys.Rev. D66(2002) 074507).
- **Pressure**:

$$\frac{p}{T^4} = \frac{\ln Z}{VT^3} = \sum_{n,m=0}^{\infty} c_{nm}(T) \left(\frac{\bar{\mu}_l}{T}\right)^n \left(\frac{\bar{\mu}_h}{T}\right)^m .$$

Due to the CP symmetry the series nonzero terms are even in $n+m$. The nonzero coefficients are

$$c_{nm}(T) = \frac{1}{n!}\frac{1}{m!}\frac{N_\tau^3}{N_\sigma^3}\frac{\partial^{n+m}\ln \mathcal{Z}}{\partial(\mu_l N_\tau)^n \partial(\mu_h N_\tau)^m}|_{\mu_{l,h}=0} \quad ,$$

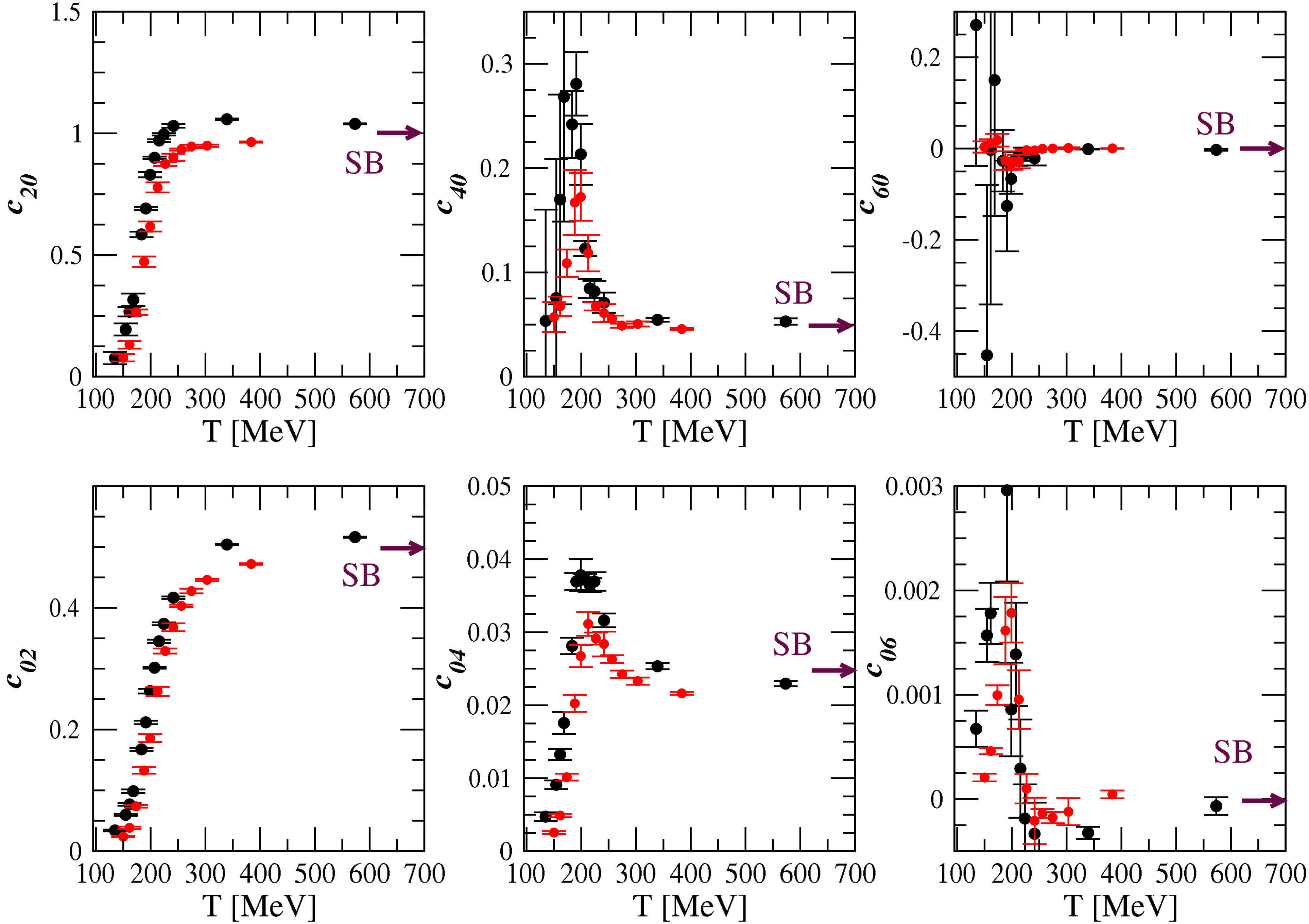
- **Interaction measure**:

$$\frac{I}{T^4} = -\frac{N_t^3}{N_s^3}\frac{d\ln Z}{d\ln a} = \sum_{n,m}^{\infty} b_{nm}(T) \left(\frac{\bar{\mu}_l}{T}\right)^n \left(\frac{\bar{\mu}_h}{T}\right)^m ,$$

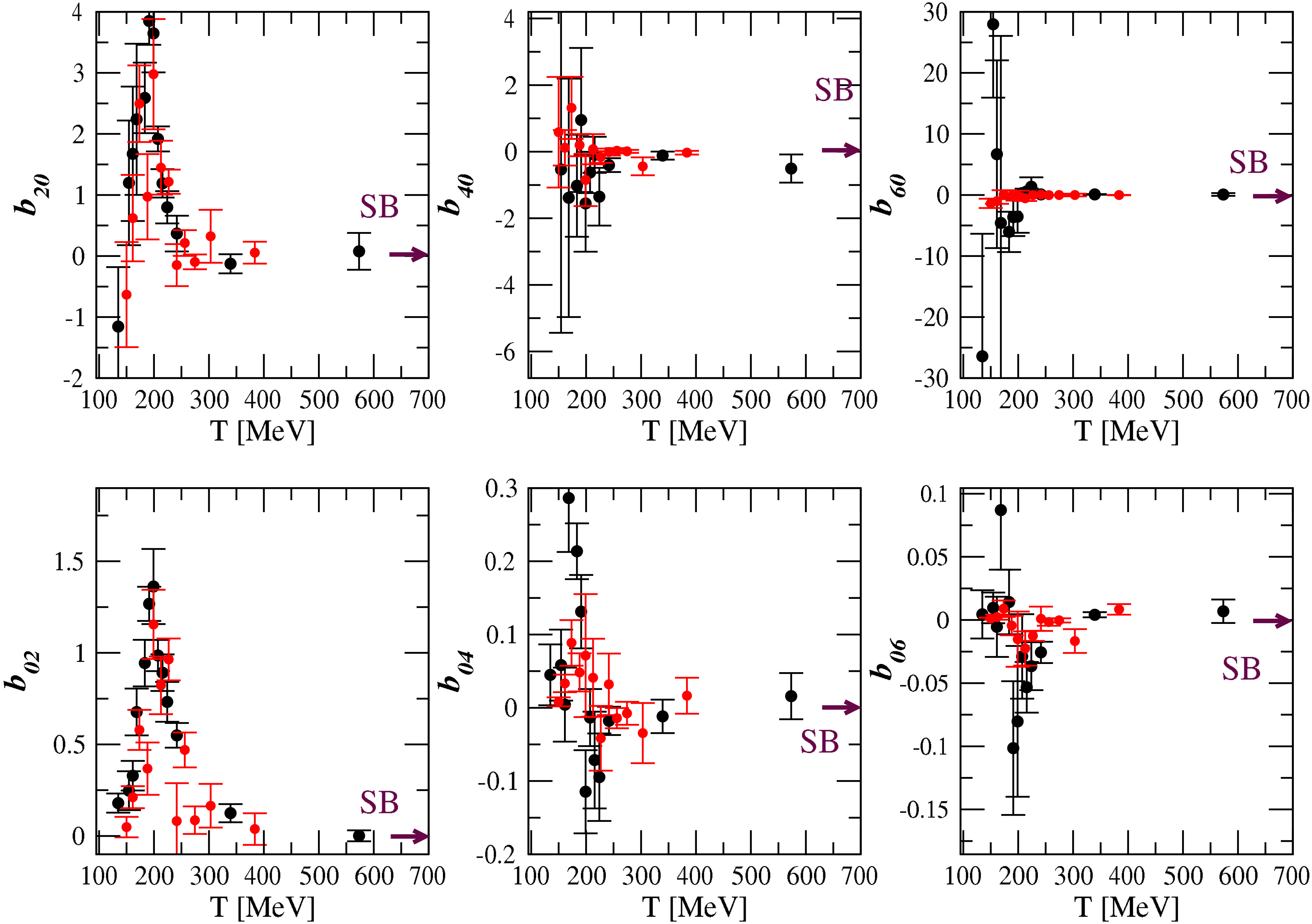
where again only even terms are nonzero and

$$b_{nm}(T) = -\frac{1}{n!m!}\frac{N_t^3}{N_s^3}\frac{\partial^{n+m}}{\partial(\mu_l N_t)^n \partial(\mu_h N_t)^m}\bigg|_{\mu_{l,h}=0} \left(\frac{d\ln Z}{d\ln a}\right) .$$

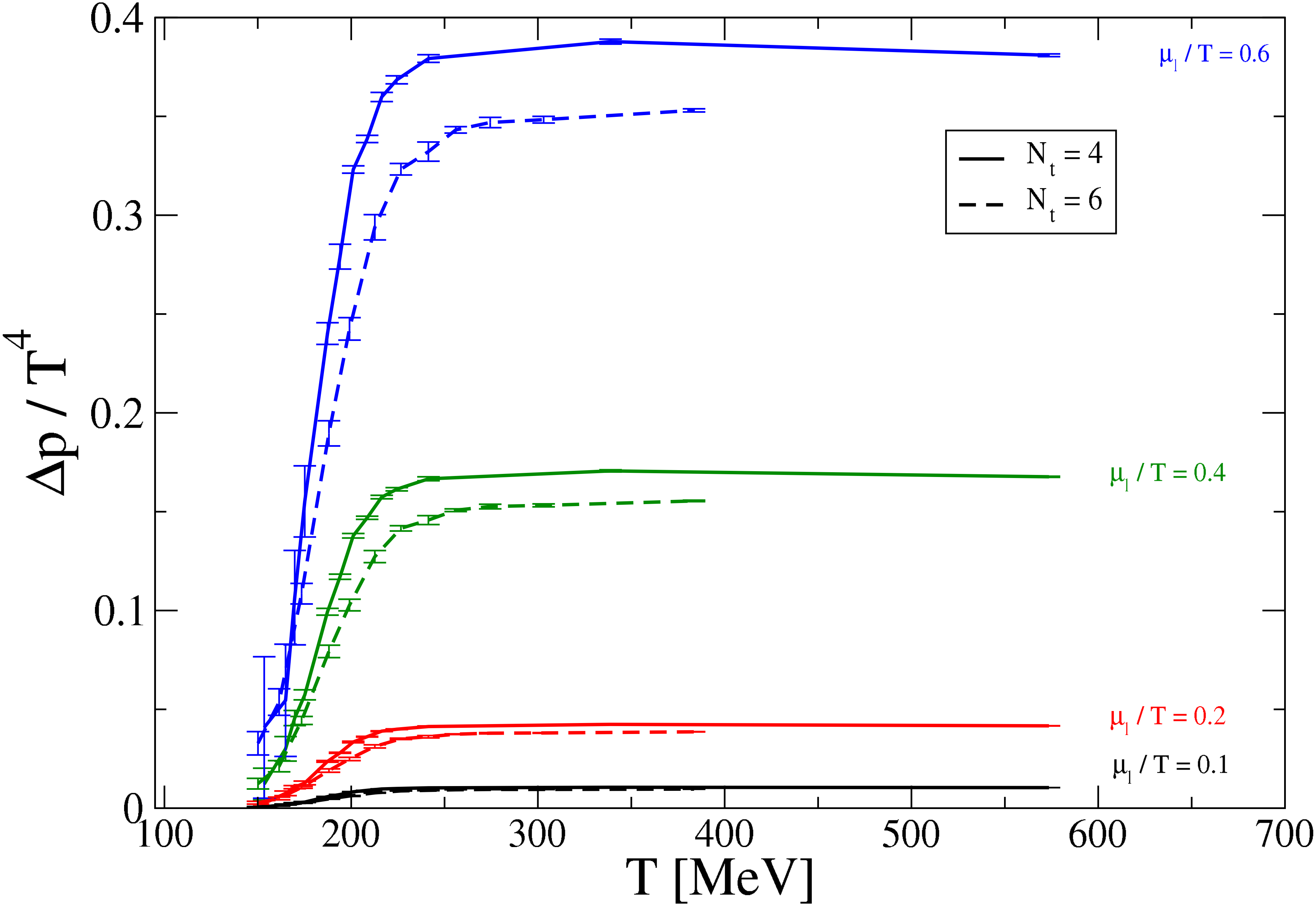
Some of the pressure expansion coefficients: $m_{ud} = 0.1 m_s$, $N_t = 4$(black) and 6(red)

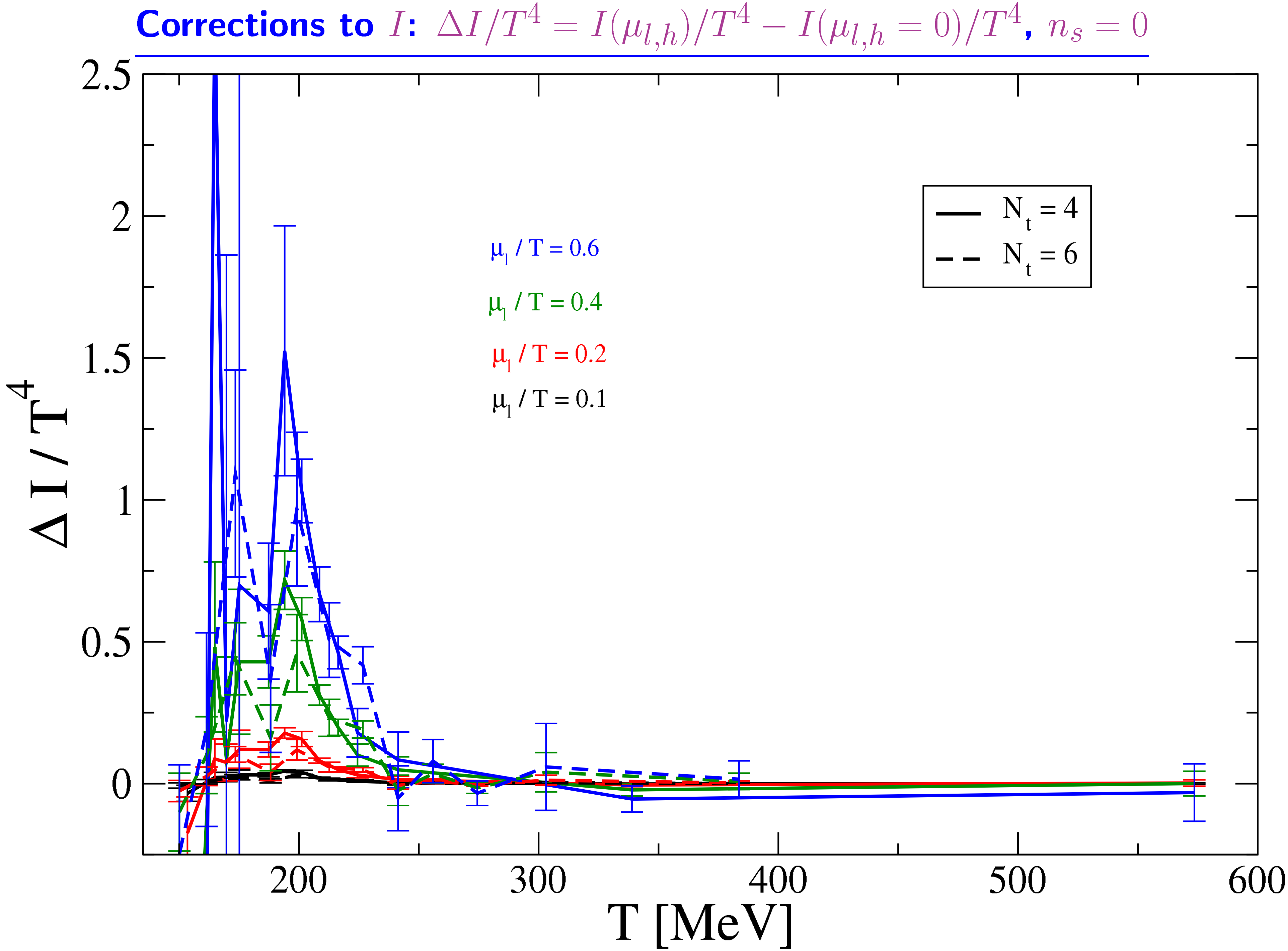


Some of the I expansion coefficients: $m_{ud} = 0.1 m_s$, $N_t = 4$(black) and 6(red)



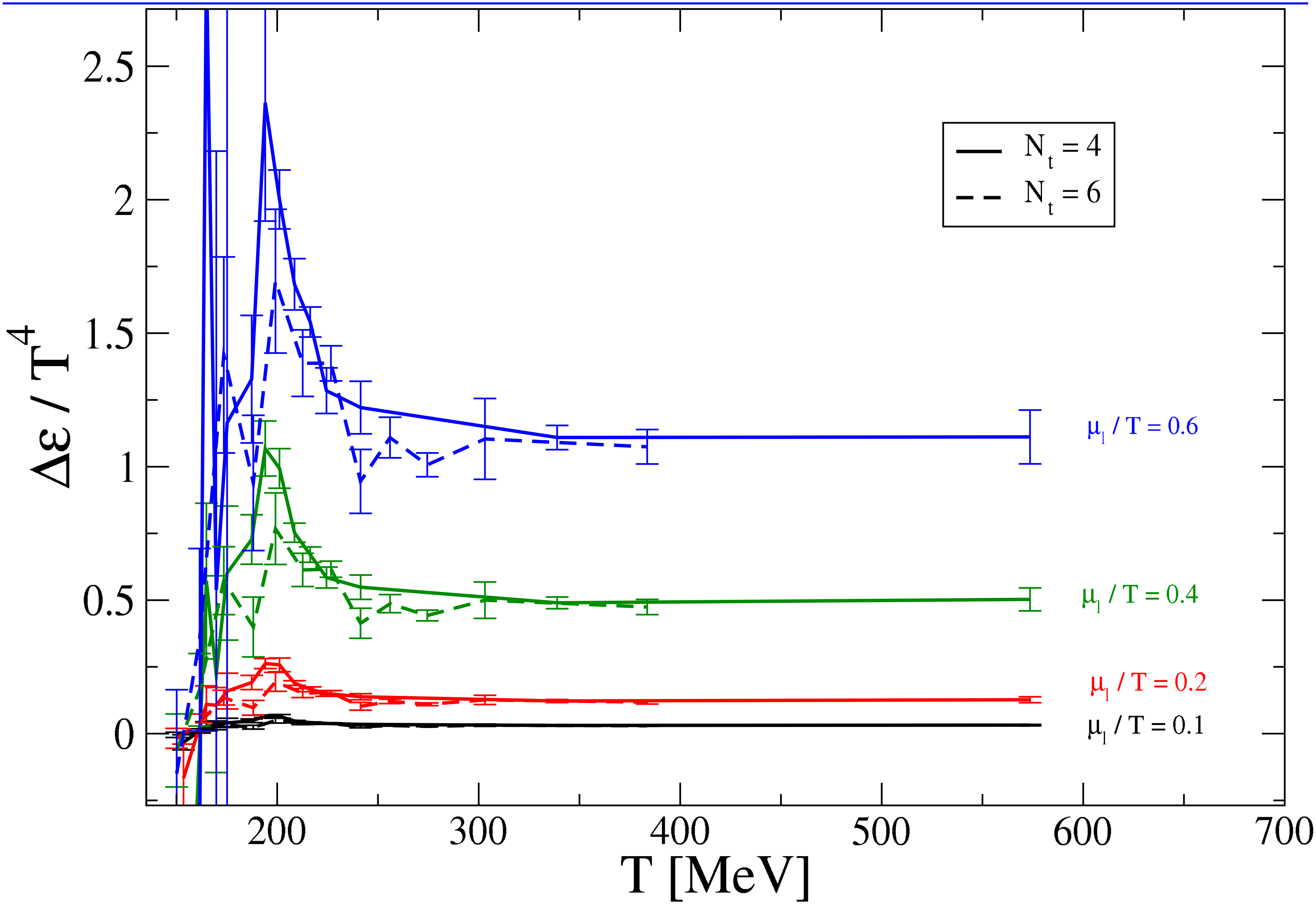
Corrections to pressure: $\Delta p/T^4 = p(\mu_{l,h})/T^4 - p(\mu_{l,h}=0)/T^4$, $n_s = 0$





Corrections to I: $\Delta I/T^4 = I(\mu_{l,h})/T^4 - I(\mu_{l,h}=0)/T^4$, $n_s = 0$
$\Delta I / T^4$
T [MeV]
$N_t = 4$
$N_t = 6$
$\mu_l / T = 0.6$
$\mu_l / T = 0.4$
$\mu_l / T = 0.2$
$\mu_l / T = 0.1$
2.5
2
1.5
1
0.5
0
200
300
400
500
600

Corrections to energy density: $\Delta\varepsilon = /T^4 = \varepsilon(\mu_{l,h})/T^4 - \varepsilon(\mu_{l,h} = 0)/T^4$, $n_s = 0$

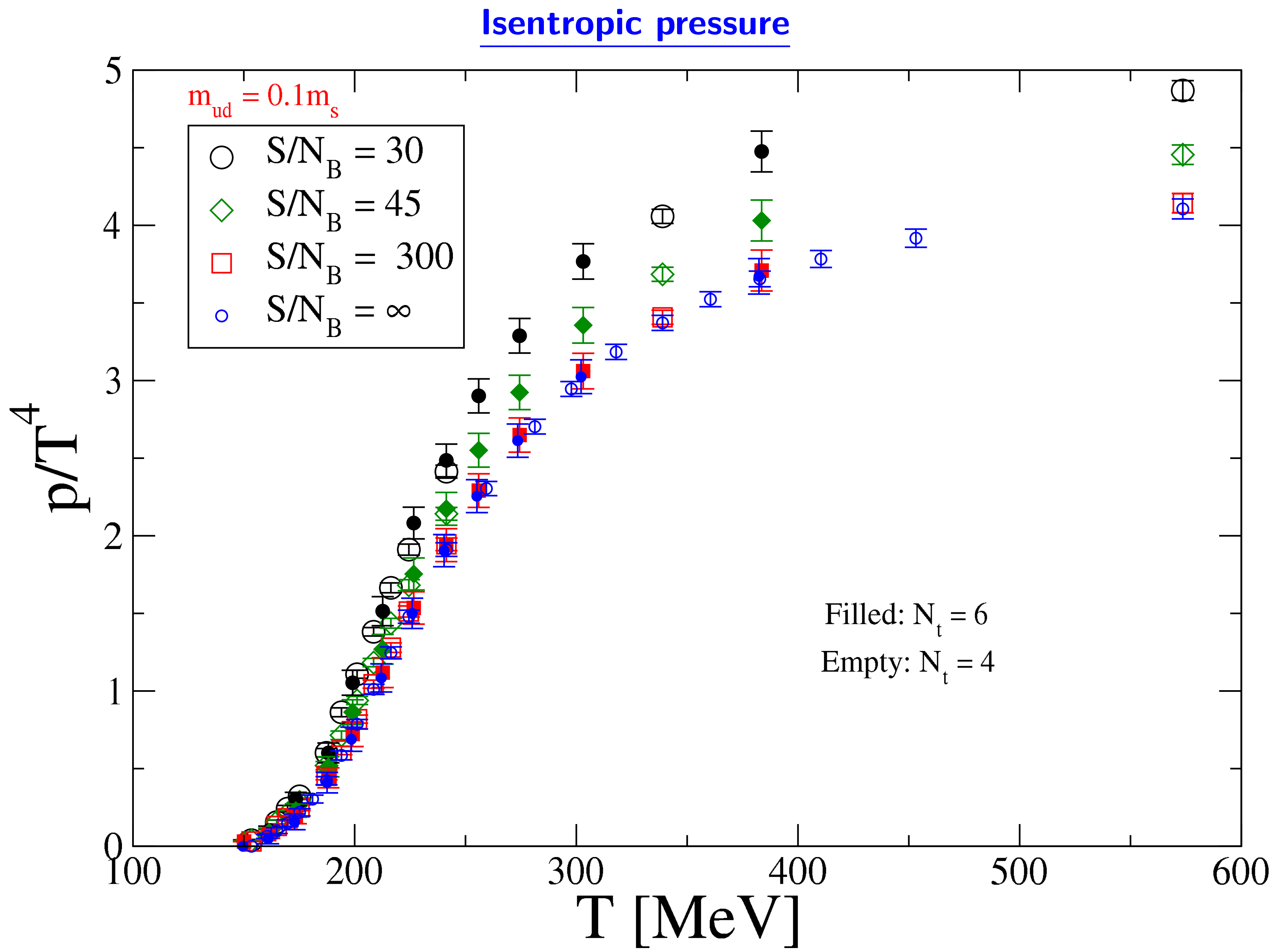


The isentropic trajectories for different s/n_B

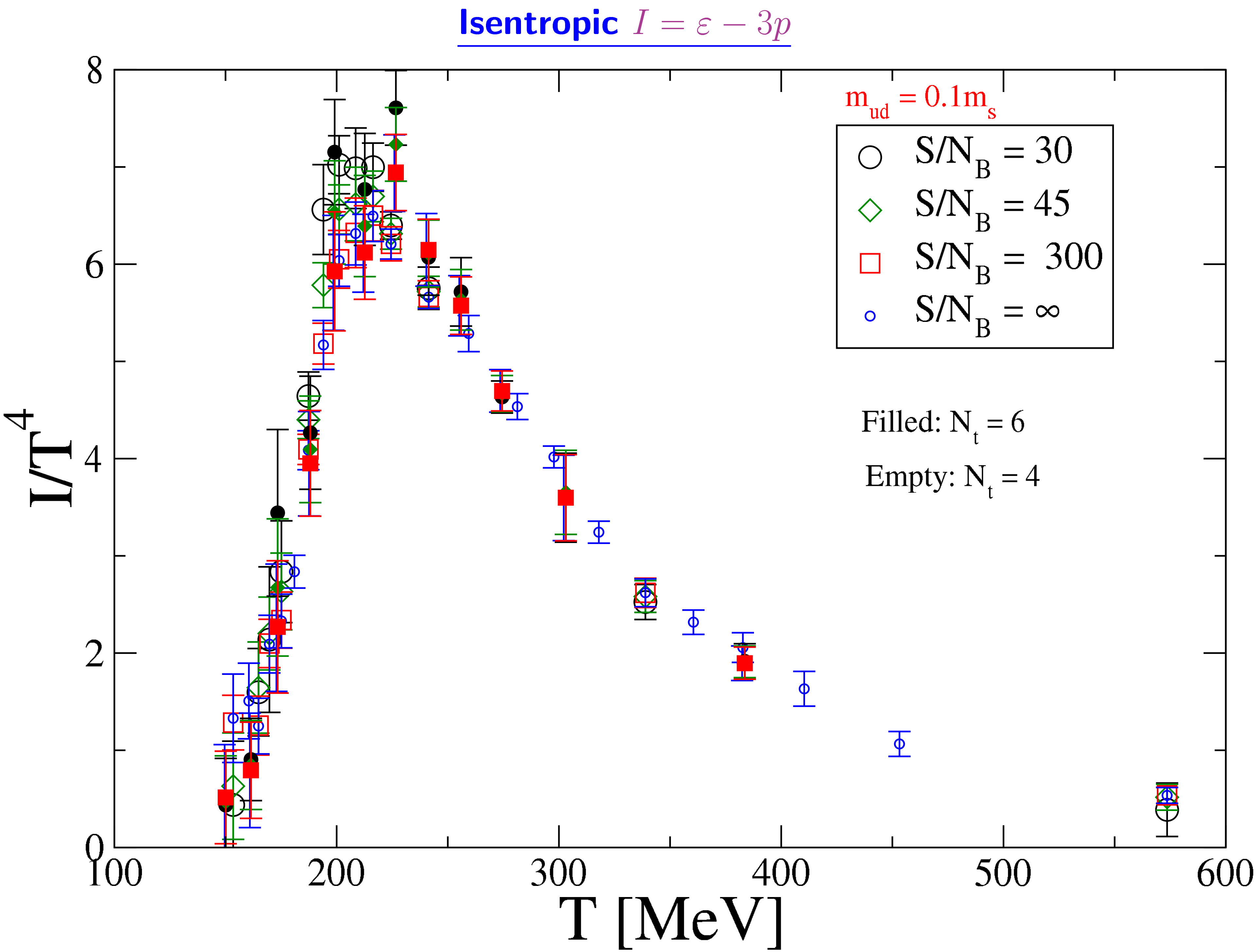
- The AGS, SPS and RHIC produce matter expands isentropically, *i.e.*, the entropy and n_B are constant. This implies that s/n_B remains constant. For the experiments mentioned, s/n_B is approximately 30, 45 and 300.
- The isentropic trajectories in the (μ_l, μ_h, T) space, obtained by numerically solving the system

$$\frac{s}{n_B}(\mu_l, \mu_h) = C$$
$$\frac{n_s}{T^3}(\mu_l, \mu_h) = 0,$$

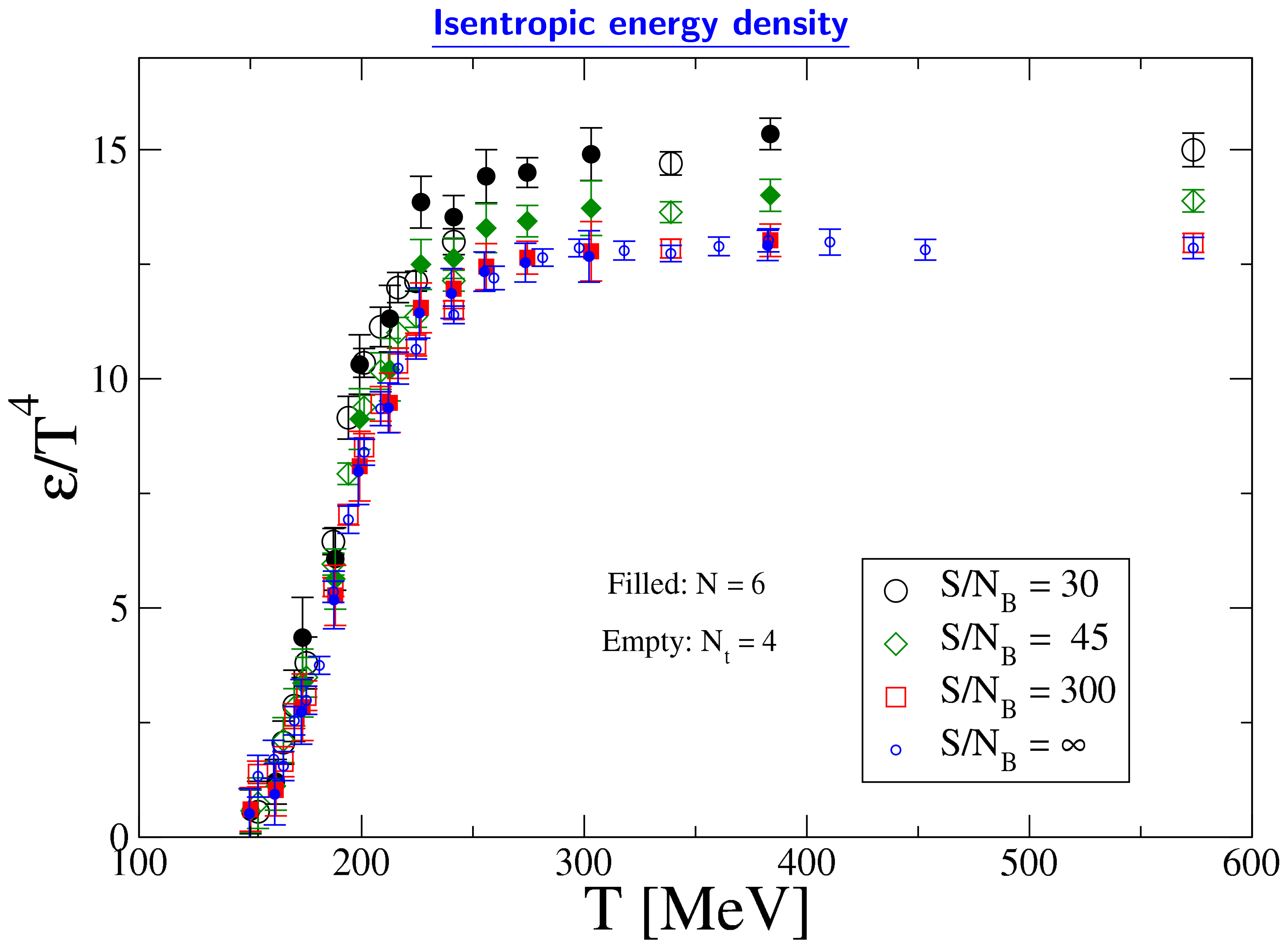
with $C = 30$, 45, 300.

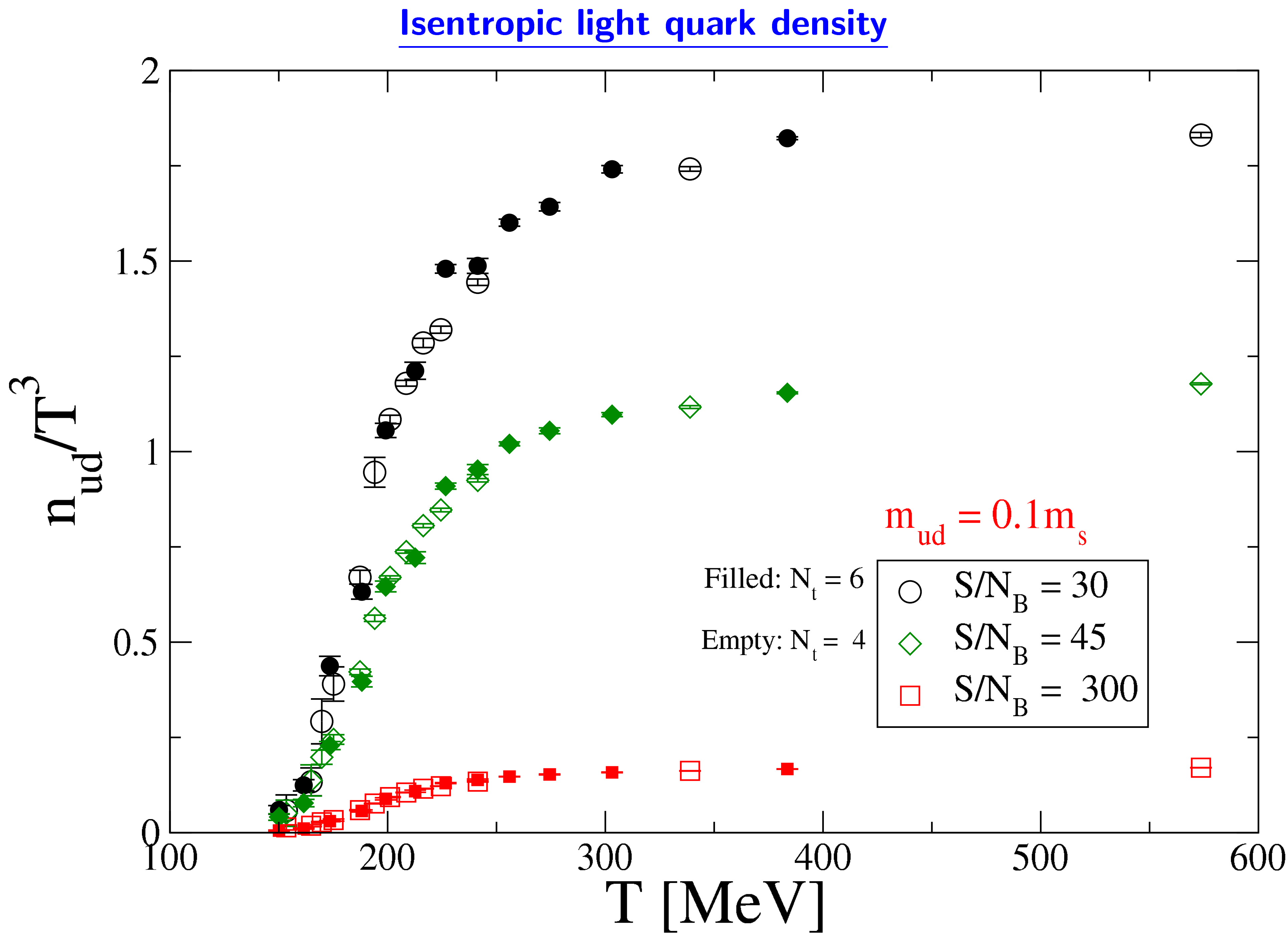


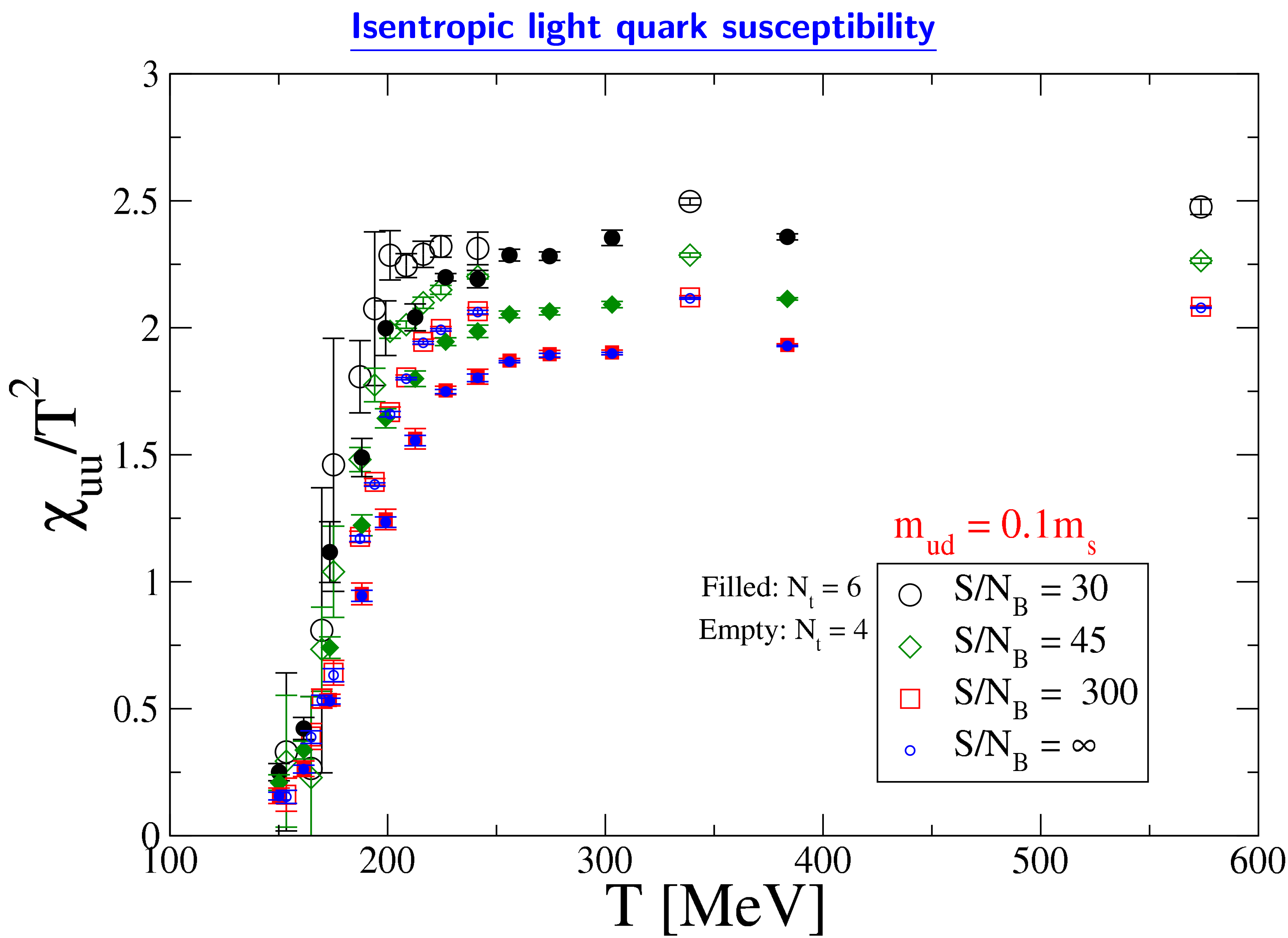
Isentropic pressure
$m_{ud} = 0.1 m_s$
$S/N_B = 30$
$S/N_B = 45$
$S/N_B = 300$
$S/N_B = \infty$
Filled: $N_t = 6$
Empty: $N_t = 4$
p/T^4
T [MeV]
0
1
2
3
4
5
100
200
300
400
500
600

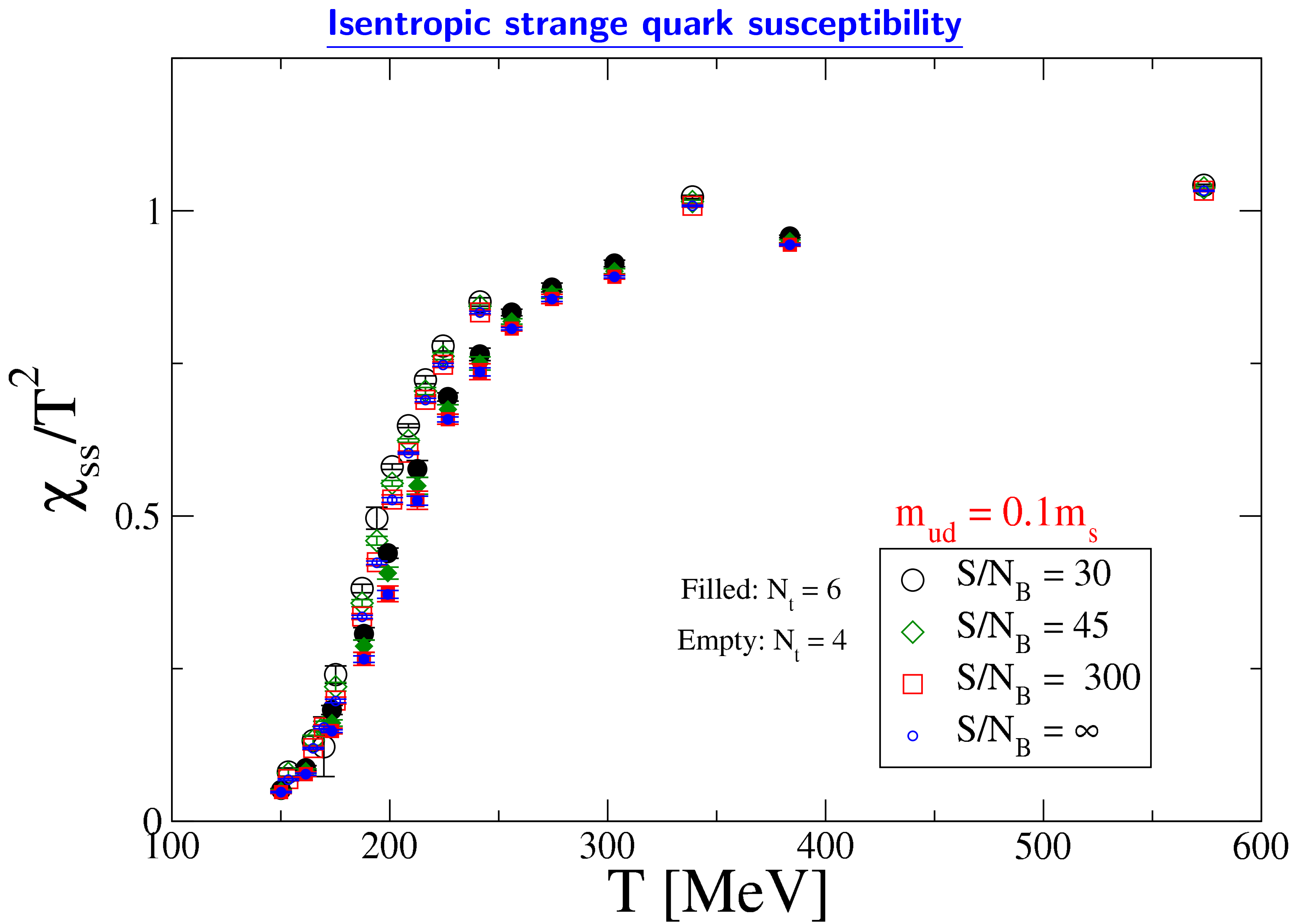


Isentropic $I = \varepsilon - 3p$
$m_{ud} = 0.1 m_s$
$S/N_B = 30$
$S/N_B = 45$
$S/N_B = 300$
$S/N_B = \infty$
Filled: $N_t = 6$
Empty: $N_t = 4$
I/T^4
T [MeV]
0
2
4
6
8
100
200
300
400
500
600


Isentropic energy density
ε/T^4
T [MeV]
Filled: N = 6
Empty: N_t = 4
S/N_B = 30
S/N_B = 45
S/N_B = 300
S/N_B = ∞
0
5
10
15
100
200
300
400
500
600

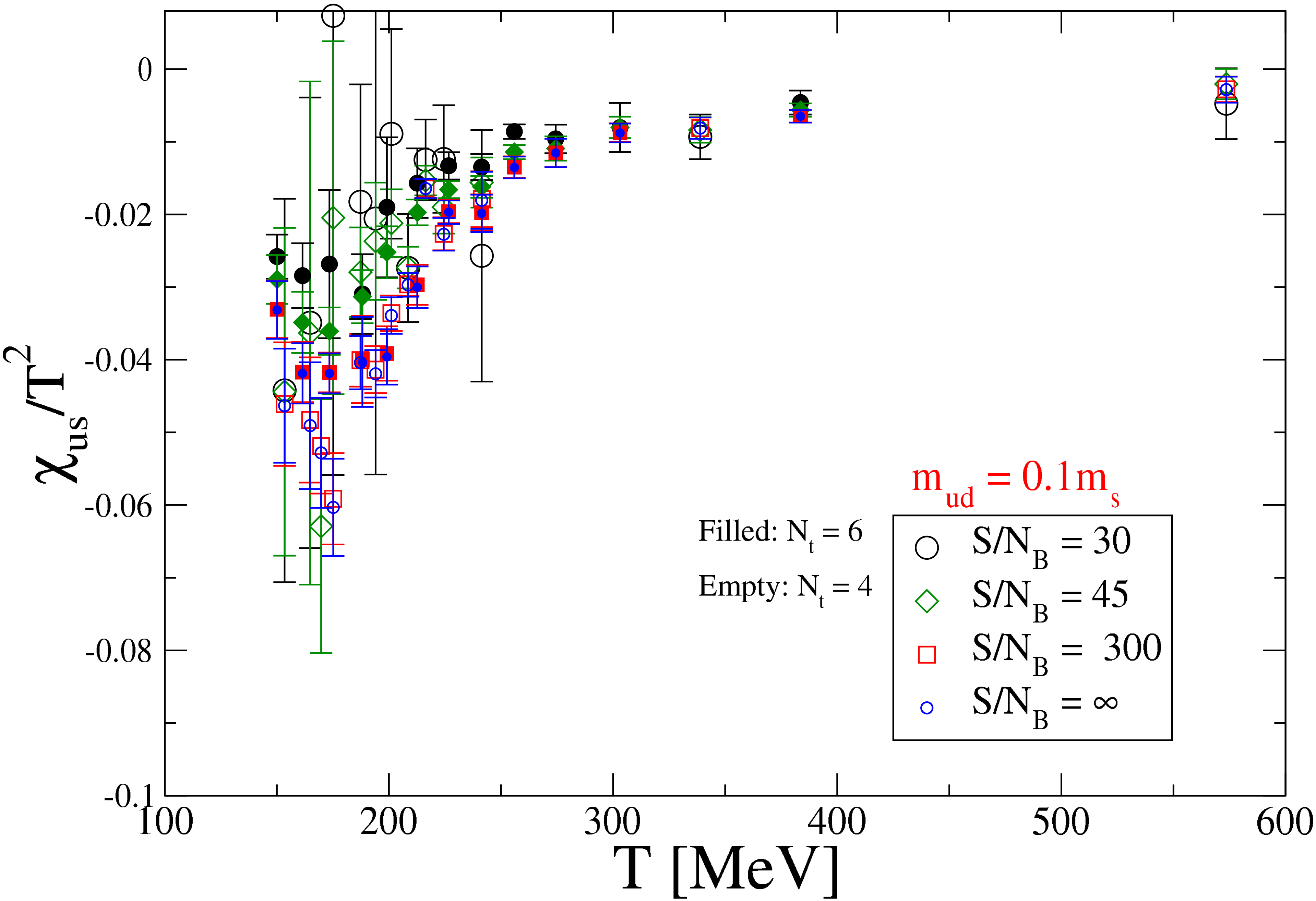

Isentropic light quark density
n_{ud}/T^3
T [MeV]
$m_{ud} = 0.1m_s$
Filled: $N_t = 6$
Empty: $N_t = 4$
$S/N_B = 30$
$S/N_B = 45$
$S/N_B = 300$
0
0.5
1
1.5
2
100
200
300
400
500
600


Isentropic light quark susceptibility
χ_{uu}/T^2
T [MeV]
$m_{ud} = 0.1 m_s$
Filled: $N_t = 6$
Empty: $N_t = 4$
$S/N_B = 30$
$S/N_B = 45$
$S/N_B = 300$
$S/N_B = \infty$



Isentropic strange quark susceptibility
χ_{ss}/T^2
T [MeV]
$m_{ud} = 0.1 m_s$
Filled: $N_t = 6$
Empty: $N_t = 4$
$S/N_B = 30$
$S/N_B = 45$
$S/N_B = 300$
$S/N_B = \infty$
0
0.5
1
100
200
300
400
500
600

Isentropic off-diagonal quark susceptibility



Conclusions

- We have calculated the EOS for 2+1 dynamical flavors of improved staggered quarks ($m_{ud}/m_s = 0.1$ and 0.2) along trajectories of constant physics, at $N_t = 4$, 6, and 8.
- Our results show that the different N_t results are quite similar except in the crossover region where the interaction measure is a bit higher on the finer $N_t = 6$ lattice.
- We also do not see significant differences between the EOS results from the two physics trajectories.
- We find deviations from the 3 flavor Stefan–Boltzmann limit in the temperature region that we have studied.
- Non-zero chemical potential EOS study was done for the $m_{ud}/m_s = 0.1$ trajectory and the isentropic EOS was determined.

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