QCD at Finite Density

(Nuclear/Quark Matter)

QCD at Finite Density

Partition function

$$Z = \mathrm{Tr}\left[e^{-\beta(H-\mu N)}\right] \qquad \beta = 1/T \qquad N = \int d^3x\, \psi^\dagger \psi$$

Path integral representation (euclidean)

$$Z = \int DA_\mu \det(iD\!\!\!/ + i\mu\gamma_4)e^{-S} = \int DA_\mu e^{i\phi} |\det(iD\!\!\!/ + i\mu\gamma_4)| e^{-S}$$

Sign problem: importance sampling does not work

Also: No general theorems (a la Vafa-Witten)

Phase structure much richer

(breaking of translational, rotational, parity, isospin, ... symmetry)

Evading the Sign Problem I

QCD like theories with extra "C" symmetry

$$\det(D)\det(CDC^{-1}) = \det(D)\det(D)^* = |\det(D)|^2$$

QCD with $SU(2)_F$ symmetry at non-zero μ_{I_3}

QCD with $N_c = 2$ colors

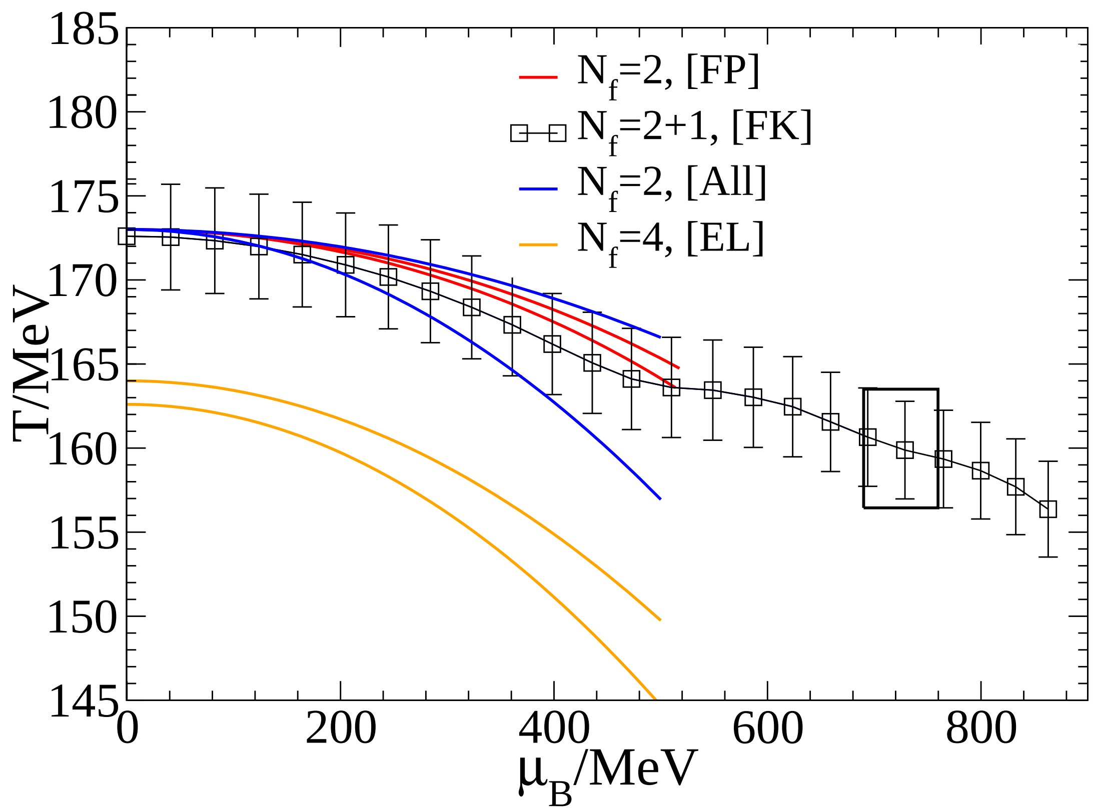
QCD with fermions in the adjoint representation

These theories have some common features

Charged Goldstone Bosons, Bose condensation

No Higgs phase (color superconductivity)

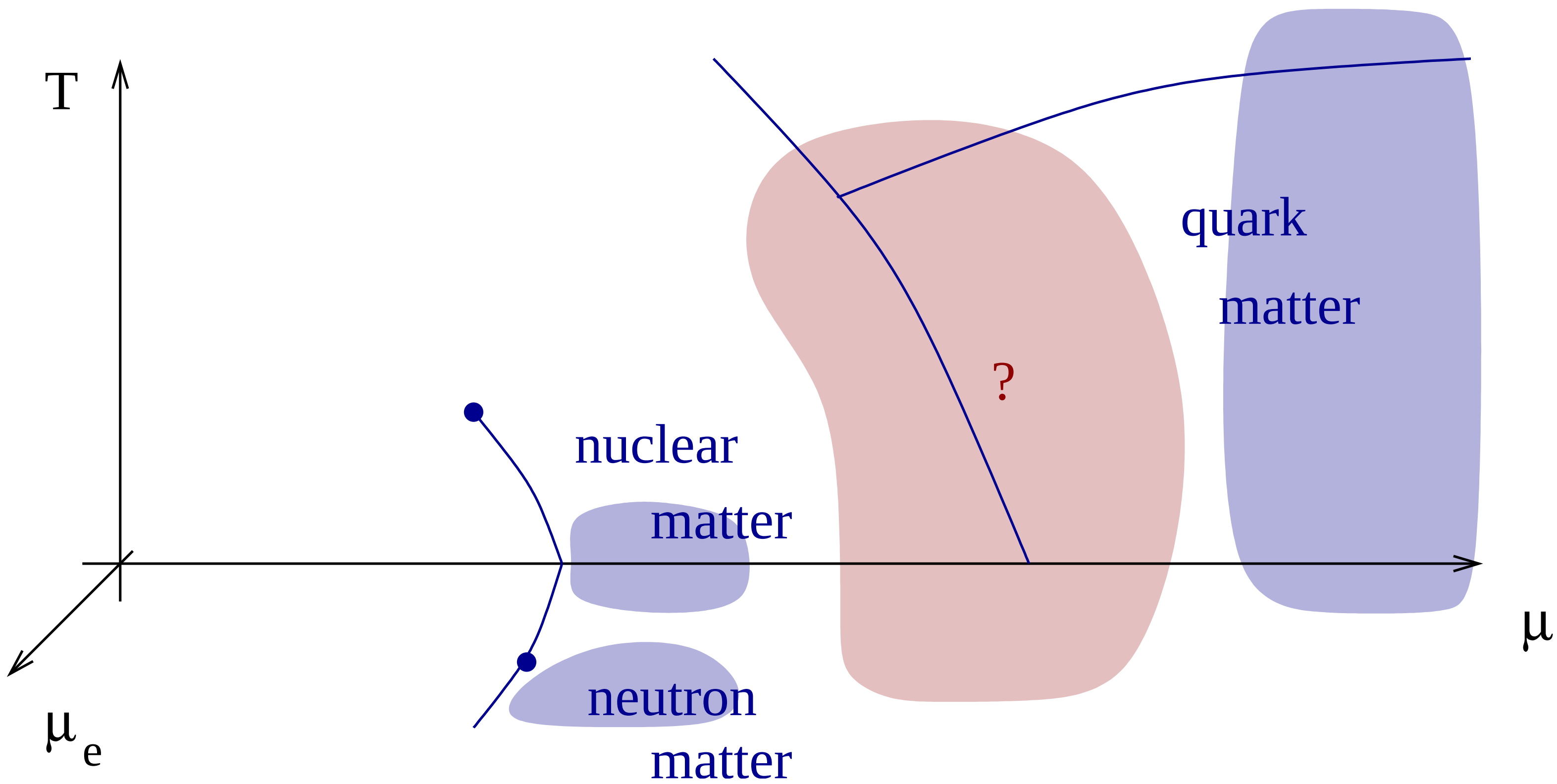
Evading the Sign Problem II



[FK] Improved re-weighting, [FP] imaginary chemical potential

[All] Taylor expansions

Schematic Phase Diagram



Dense Baryonic Matter

Low Density

- Equation of state of nuclear/neutron matter
- Neutron/proton superfluidity, pairing gaps

Moderate Density

- Pion/kaon condensation, hyperon matter
- Pairing, equation of state at high density

High Density

- Quark matter
- Color superconductivity, Color-flavor-locking

Dense Baryonic Matter

Low Density (constrained by NN interaction, phenomenology)

Equation of state of nuclear/neutron matter

Neutron/proton superfluidity, pairing gaps

Moderate Density (very poorly known)

Pion/kaon condensation, hyperon matter

Pairing, equation of state at high density

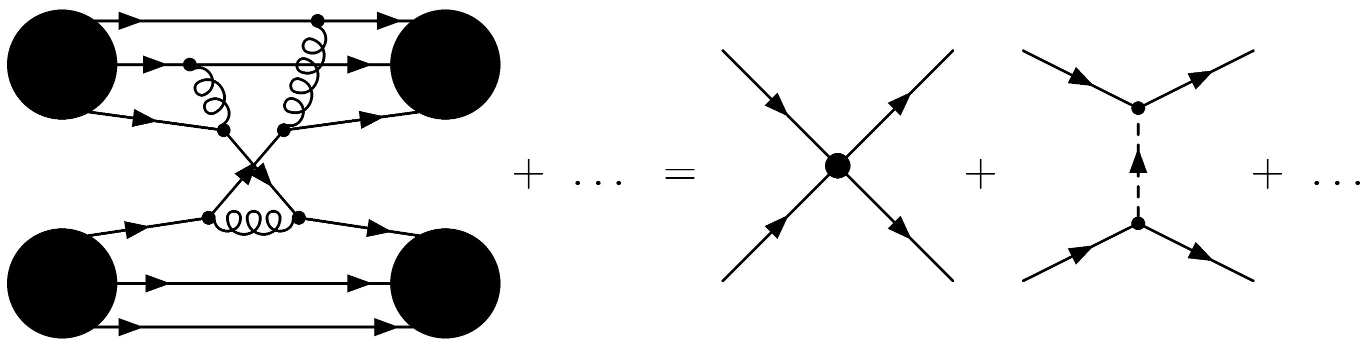
High Density (weak coupling methods apply)

Quark matter

Color superconductivity, Color-flavor-locking

Low Density: Nuclear Effective Field Theory

	Nucleons are point particles
Low Energy Nucleons:	Interactions are local
	Long range part: pions



	Systematically improvable
Advantages:	Symmetries manifest (Chiral, gauge, ...)
	Connection to lattice QCD

Effective Field Theory

Effective field theory for point-like, non-relativistic neutrons

$$\mathcal{L}_{\mathrm{eff}} = \psi^\dagger \left(i\partial_0 + \frac{\nabla^2}{2M} \right) \psi - \frac{C_0}{2} (\psi^\dagger \psi)^2 + \frac{C_2}{16} \left[(\psi\psi)^\dagger (\psi \overleftrightarrow{\nabla}^2 \psi) + h.c. \right] + \ldots$$

Simplifications: neutrons only, no pions (very low energy)

Effective range expansion

$$p \cot \delta_0 = -\frac{1}{a} + \frac{1}{2} \sum_n r_n p^{2n}$$

Coupling constants

$$C_0 = \frac{4\pi a}{M}, \qquad C_2 = \frac{4\pi a^2}{M} \frac{r}{2}, \quad \ldots \qquad a = -18\,\mathrm{fm},\ r = 2.8\,\mathrm{fm}$$

Neutron Matter: Universal Limit

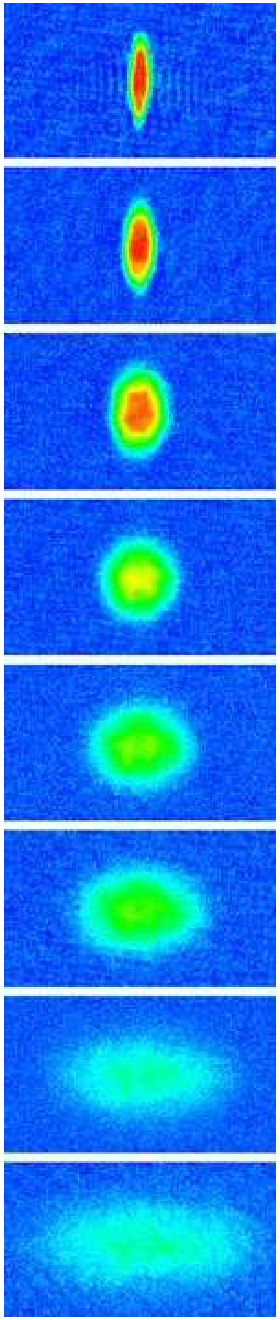
Consider limiting case ("Bertsch" problem)

$$(k_F a) \to \infty \qquad\qquad (k_F r) \to 0$$

Why is this limit interesting? (Close to real world!)

- Scale (and conformal) invariance
- Universal equation of state $E/A = \xi (E/A)_0$
- Cross section saturates QM unitarity bound
- Perfect fluid?
- Connection to cold atoms

Cold Fermi Gases

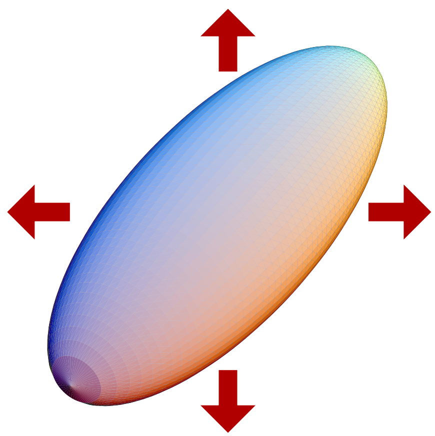


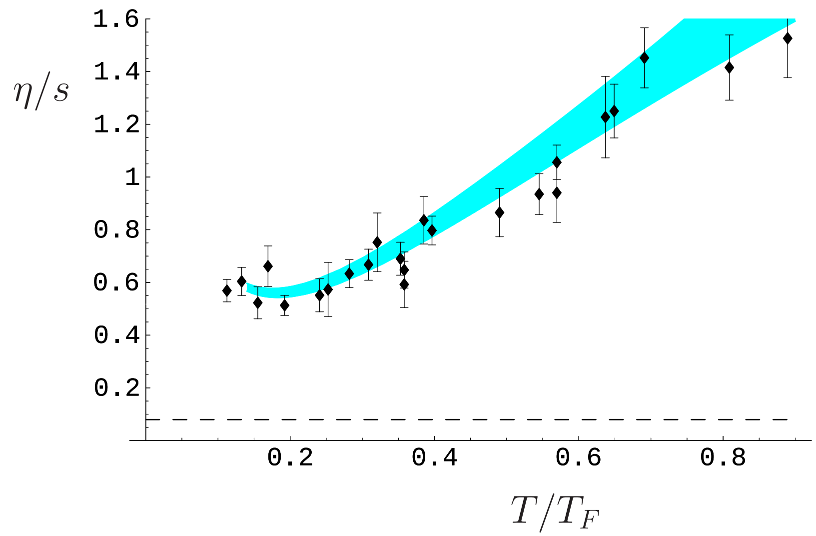
Universal equation of state

$$(E/A) = 0.42(E/A)_0$$

Experiment, Quantum MC, ϵ expansion

Transport: η/s from damping of collective modes





Epsilon Expansion

Bound state wave function $\psi \sim 1/r^{d-2}$. Nussinov & Nussinov

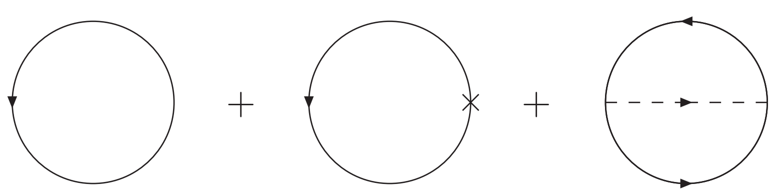
$d \geq 4$: Non-interacting bosons $\xi(d=4)=0$

$d \leq 4$: Effective lagrangian for atoms $\Psi = (\psi_\uparrow, \psi_\downarrow^\dagger)$ and dimers ϕ

$$\mathcal{L} = \Psi^\dagger \left(i\partial_0 + \frac{\sigma_3 \nabla^2}{2m} \right) \Psi + \mu \Psi^\dagger \sigma_3 \Psi + \Psi^\dagger \sigma_+ \Psi \phi + h.c.$$

Nishida & Son (2006)

Perturbative expansion: $\phi = \phi_0 + g\varphi$ $(g^2 \sim \epsilon)$



$O(1)$ $O(1)$ $O(\epsilon)$

$$\xi = \frac{1}{2}\epsilon^{3/2} + \frac{1}{16}\epsilon^{5/2}\ln\epsilon - 0.0246\,\epsilon^{5/2} + \ldots$$

$\xi = 0.475$ $\Delta = 0.62 E_F$

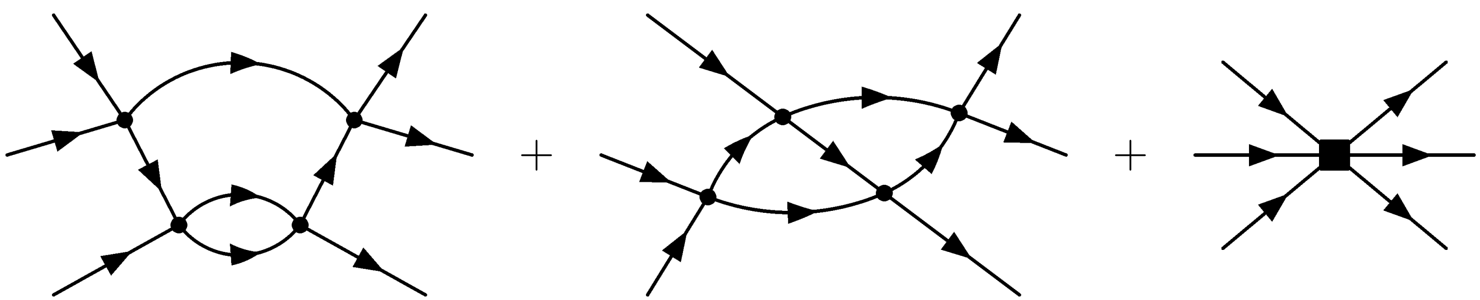
Nuclear Matter

isospin symmetric matter: first order onset transition

$$\rho_0 \simeq 0.14\,\mathrm{fm}^{-3} \qquad (k_F \simeq 250\,\mathrm{MeV}) \qquad B/A = 15\,\mathrm{MeV}$$

can be reproduced using accurate V_{NN} (V_{3N} crucial, $V_{4N} \approx 0$)

EFT methods: explain need for V_{3N} if $N_f > 1$ (and $V_{4N} \ll V_{3N}$)



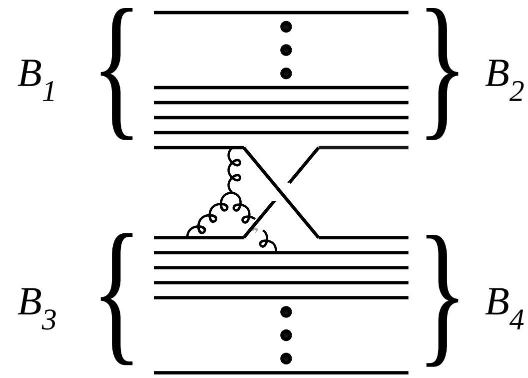
systematic calculations difficult since $k_F a \gg 1$, $k_F r \sim 1$

Nuclear Matter at large N_c

Nucleon nucleon interaction is $O(N_c)$

$$m_N = O(N_c) \qquad r_N = O(1)$$

$$V_{NN} = O(N_c)$$



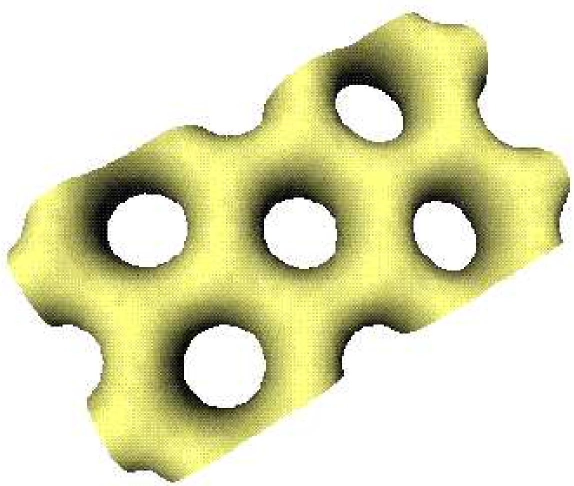
Get $SU(2N_f)$ (Wigner symmetry) relations

$$C_0(\psi^\dagger\psi)^2 \gg C_T(\psi^\dagger\vec{\sigma}\psi)^2$$

Dense matter: $k_F = O(1)$ $(E_F \sim 1/N_c)$

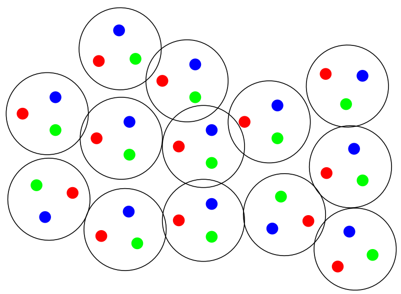
crystallization

Note: $E \sim N_c$ (no phase transition?)



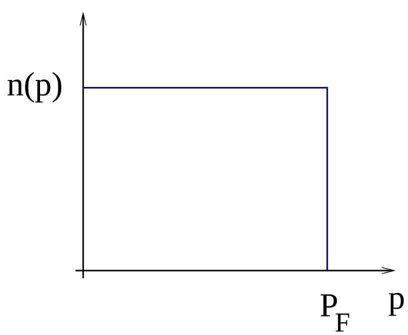
Very Dense Matter

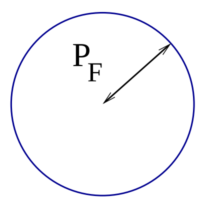
Consider baryon density $n_B \gg 1\,\mathrm{fm}^{-3}$



quarks expected to move freely

Ground state: cold quark matter (quark fermi liquid)





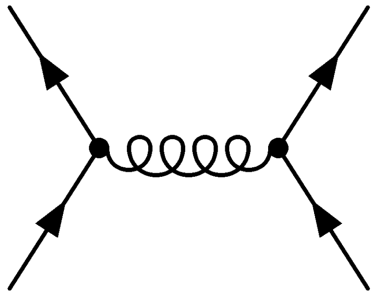
only quarks with $p \sim p_F$ scatter

$p_F \gg \Lambda_{QCD} \rightarrow$ coupling is weak

No chiral symmetry breaking, confinement, or dynamically generated masses

High Density: Pairing in Quark Matter

QQ scattering in perturbative QCD

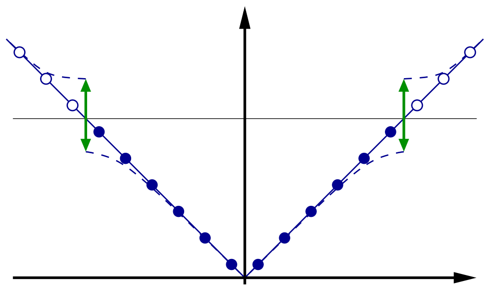


$$(\vec{T})_{ac}(\vec{T})_{bd} = -\frac{1}{3}(\delta_{ab}\delta_{cd}-\delta_{ad}\delta_{bc})+\frac{1}{6}(\delta_{ab}\delta_{cd}+\delta_{ad}\delta_{bc})$$

$$[3] \quad \times \quad [3] \quad = \quad [\bar{3}] \quad + \quad [6]$$

Fermi surface: pairing instability in weak coupling

$$\Phi_{ij}^{ab,\alpha\beta} = \langle \psi_i^{a,\alpha} C \psi_j^{b,\beta} \rangle$$



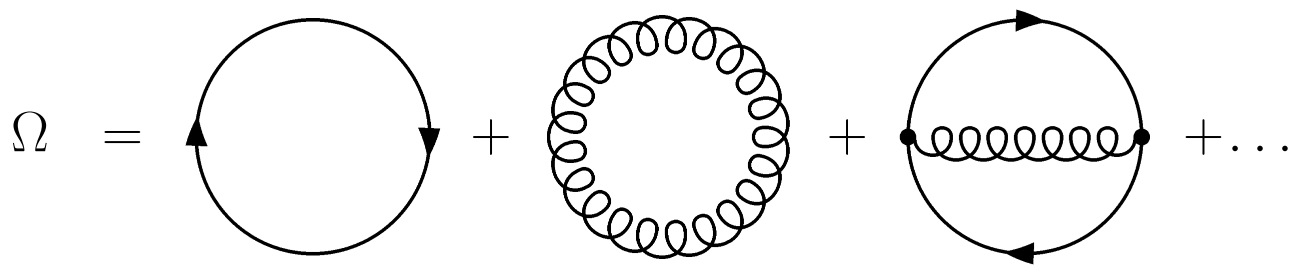
Phase structure in perturbation theory

Minimize $\Omega(\Phi_{ij}^{ab,\alpha\beta})$

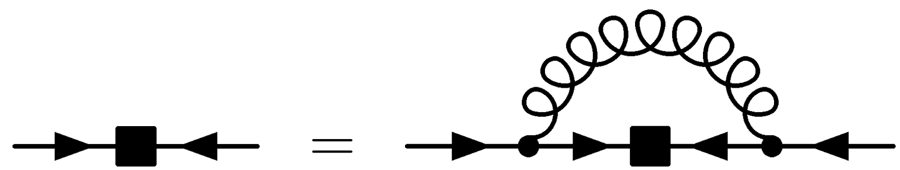
In practice: consider $\Phi_{ij}^{ab,\alpha\beta}$ with residual symmetries

Superconductivity

Thermodynamic potential



Variational principle $\delta\Omega/\delta\Phi$ gives gap equation



$$\Delta(p_0) = \frac{g^2}{18\pi^2}\int dq_0 \log\left(\frac{\Lambda_{BCS}}{|p_0 - q_0|}\right)\frac{\Delta(q_0)}{\sqrt{q_0^2 + \Delta(q_0)^2}}$$

$$\Lambda_{BCS} = c_i 256\pi^4 \mu g^{-5}$$

(c_i depends on phase)

$$\Delta_i = 2\Lambda_{BCS}\exp\left(-\frac{3\pi^2}{\sqrt{2}g}\right)$$

$N_f = 2$: 2SC Phase

$N_f = 2$, color-anti-symmetric: spin-0 BCS condensate

$$\langle \psi_i^b C\gamma_5 \psi_j^c \rangle = \Phi^a \epsilon^{abc} \epsilon_{ij}$$

Order parameter $\phi^a \sim \delta^{a3}$ breaks $SU(3)_c \to SU(2)$

- $SU(2)_L \times SU(2)_R$ unbroken
- 4 gapped, 2 (almost) gapless fermions
- light $U(1)_A$ Goldstone boson
- $SU(2)$ confined ($\Lambda_{conf} \ll \Delta$)

$N_f = 3$: CFL Phase

Consider $N_f = 3$ $(m_i = 0)$

$$\langle q_i^a q_j^b \rangle = \phi\, \epsilon^{abI} \epsilon_{ijI}$$

$$\langle ud \rangle = \langle us \rangle = \langle ds \rangle$$

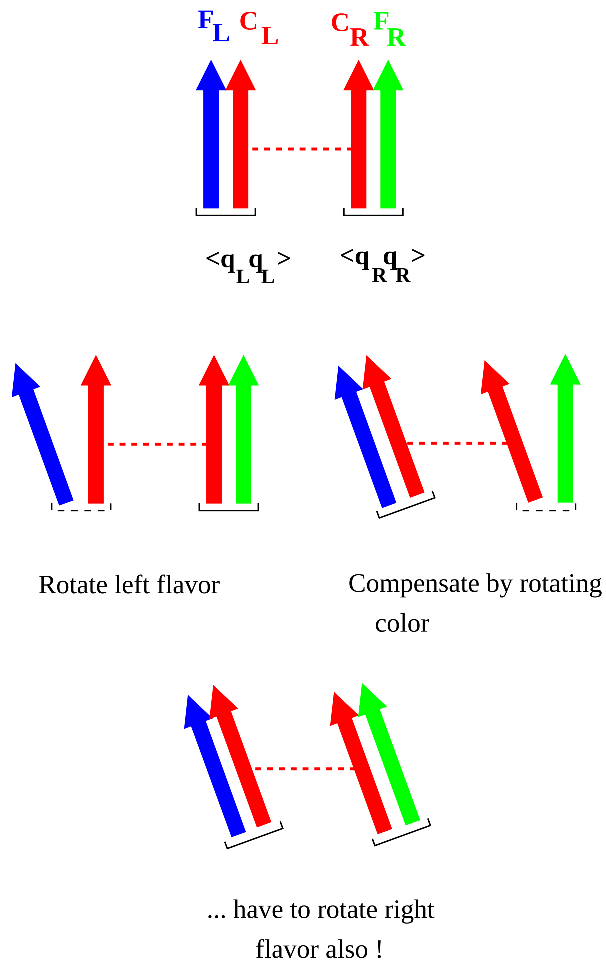
$$\langle rb \rangle = \langle rg \rangle = \langle bg \rangle$$

Symmetry breaking pattern:

$$SU(3)_L \times SU(3)_R \times [SU(3)]_C \times U(1) \to SU(3)_{C+F}$$

All quarks and gluons acquire a gap

$[8] + [1]$ fermions, Q integer



$$\langle \psi_L \psi_L \rangle = -\langle \psi_R \psi_R \rangle$$

CFL Phase: What does it look like?

CFL phase is fully gapped

transparent insulator

CFL is a superfluid

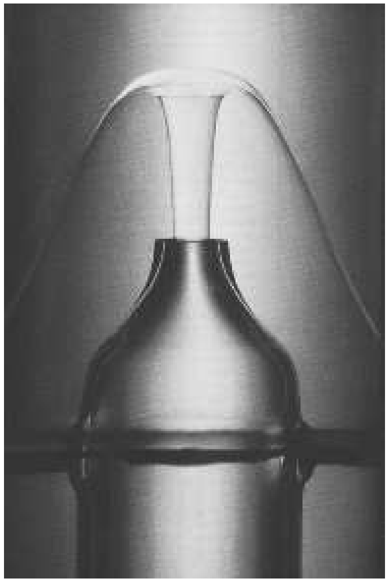
rotational vortices

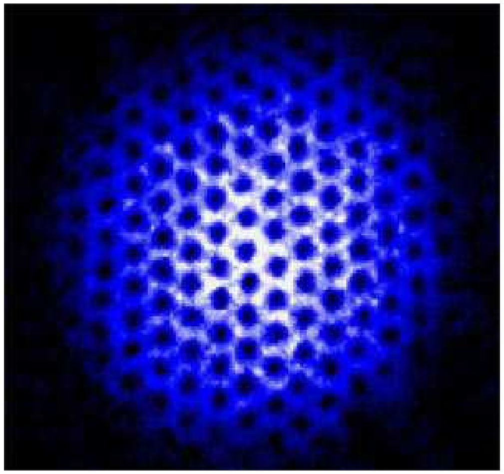
CFL is not an electric superconductor

magnetic flux only partially expelled

CFL is “confined”

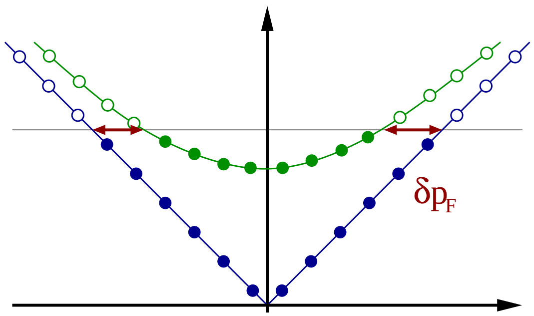
excitations: mesons and baryons





Towards the real world: Non-zero strange quark mass

Have $m_s > m_u, m_d$: Unequal Fermi surfaces



$$\delta p_F \simeq \frac{m_s^2}{2p_F}$$

Also: If $p_F^s < p_F^{u,d}$ have unequal densities

Charge neutrality not automatic

Strategy

Consider $N_f = 3$ at $\mu \gg \Lambda_{QCD}$ (CFL phase)

Study response to $m_s \neq 0$

Constrained by chiral symmetry

Effective theory: (CFL) baryon chiral perturbation theory

$$\begin{aligned}
\mathcal{L} \;=\;& \frac{f_\pi^2}{4}\left\{\mathrm{Tr}\left(\nabla_0\Sigma\nabla_0\Sigma^\dagger\right) - v_\pi^2\,\mathrm{Tr}\left(\nabla_i\Sigma\nabla_i\Sigma^\dagger\right)\right\} \\
&+ A\left\{\left[\mathrm{Tr}\left(M\Sigma\right)\right]^2 - \mathrm{Tr}\left(M\Sigma M\Sigma\right) + h.c.\right\} \\
&+ \mathrm{Tr}\left(N^\dagger i v^\mu D_\mu N\right) - D\,\mathrm{Tr}\left(N^\dagger v^\mu \gamma_5 \left\{\mathcal{A}_\mu, N\right\}\right) \\
&- F\,\mathrm{Tr}\left(N^\dagger v^\mu \gamma_5 \left[\mathcal{A}_\mu, N\right]\right) + \frac{\Delta}{2}\left\{\mathrm{Tr}\left(NN\right) - \left[\mathrm{Tr}\left(N\right)\right]^2\right\}
\end{aligned}$$

$$\nabla_0\Sigma = \partial_0\Sigma + i\hat{\mu}_L\Sigma - i\Sigma\hat{\mu}_R$$

$$D_\mu N = \partial_\mu N + i[\mathcal{V}_\mu, N]$$

$$f_\pi^2 = \frac{21 - 8\log 2}{18}\,\frac{\mu^2}{2\pi^2} \quad v_\pi^2 = \frac{1}{3} \quad A = \frac{3\Delta^2}{4\pi^2} \quad D = F = \frac{1}{2}$$

Phase Structure and Spectrum

Phase structure determined by effective potential

$$V(\Sigma) = \frac{f_\pi^2}{2}\mathrm{Tr}\left(\hat{\mu}_L \Sigma \hat{\mu}_R \Sigma^\dagger\right) - A\mathrm{Tr}(M\Sigma^\dagger) - B_1\left[\mathrm{Tr}(M\Sigma^\dagger)\right]^2 + \ldots$$

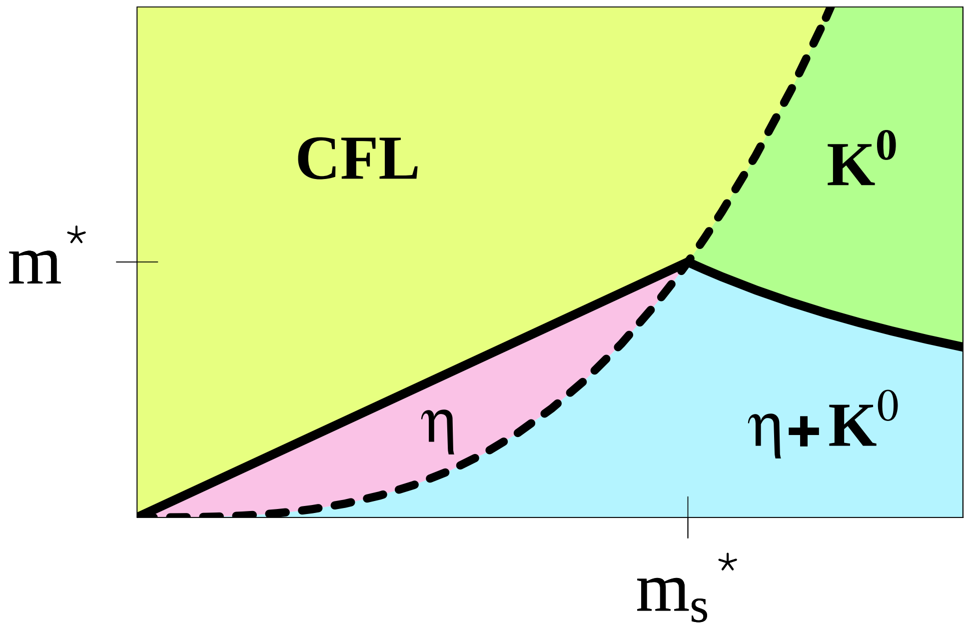
$$V(\Sigma_0) \equiv min$$

Fermion spectrum determined by

$$\mathcal{L} = \mathrm{Tr}\left(N^\dagger i v^\mu D_\mu N\right) + \mathrm{Tr}\left(N^\dagger \gamma_5 \rho_A N\right) + \frac{\Delta}{2}\left\{\mathrm{Tr}\left(NN\right) - \left[\mathrm{Tr}\left(N\right)\right]^2\right\},$$

$$\rho_{V,A} = \frac{1}{2}\left\{\xi\hat{\mu}_L\xi^\dagger \pm \xi^\dagger\hat{\mu}_R\xi\right\} \qquad\qquad \xi = \sqrt{\Sigma_0}$$

Phase Structure of CFL Phase



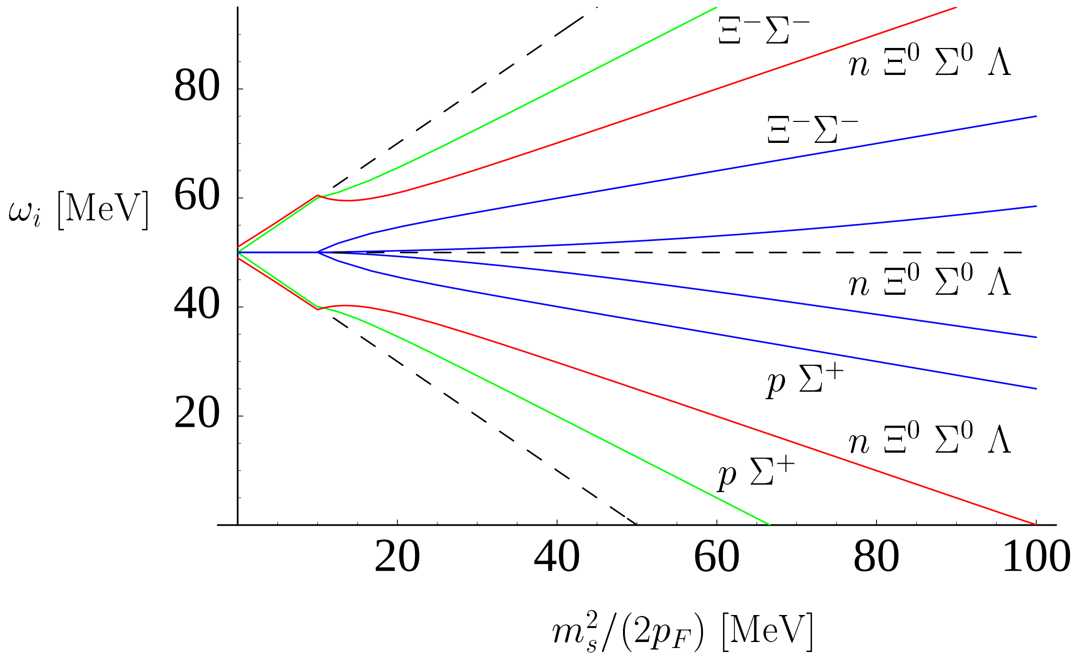
$$m_s^{crit} \sim 3.03\, m_d^{1/3} \Delta^{2/3}$$

$$m^* \sim 0.017\, \alpha_s^{4/3} \Delta$$

QCD realization of s-wave meson condensation

Driven by strangeness over-saturation of CFL state

Fermion Spectrum

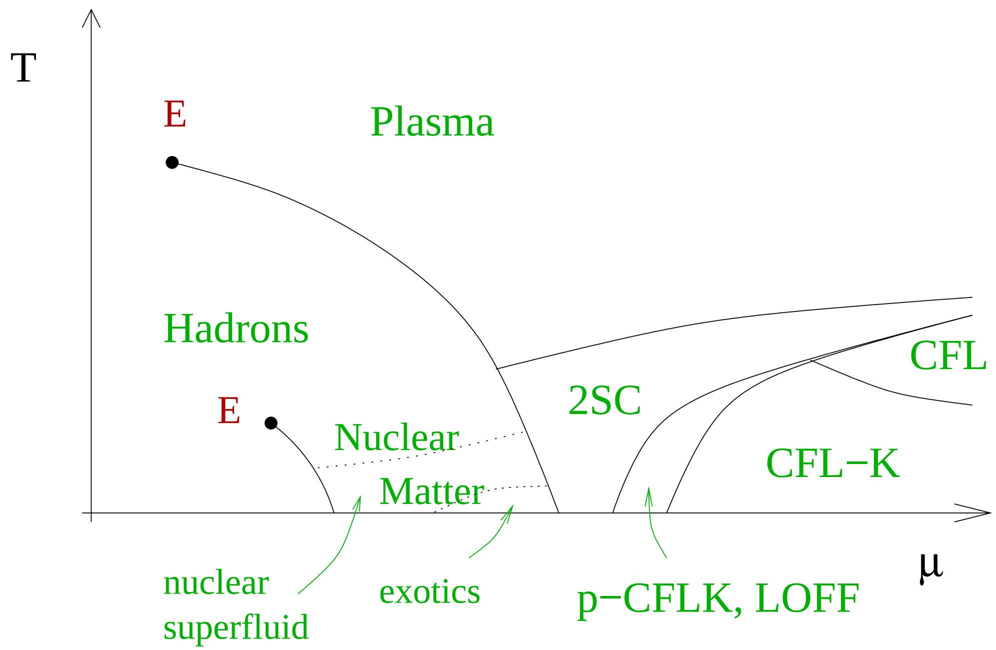


$$m_s^{crit} \sim (8\mu\Delta/3)^{1/2}$$

gapless fermion modes (gCFLK)

(chromomagnetic) instabilities ?

Phase Diagram: $m_s \neq 0$

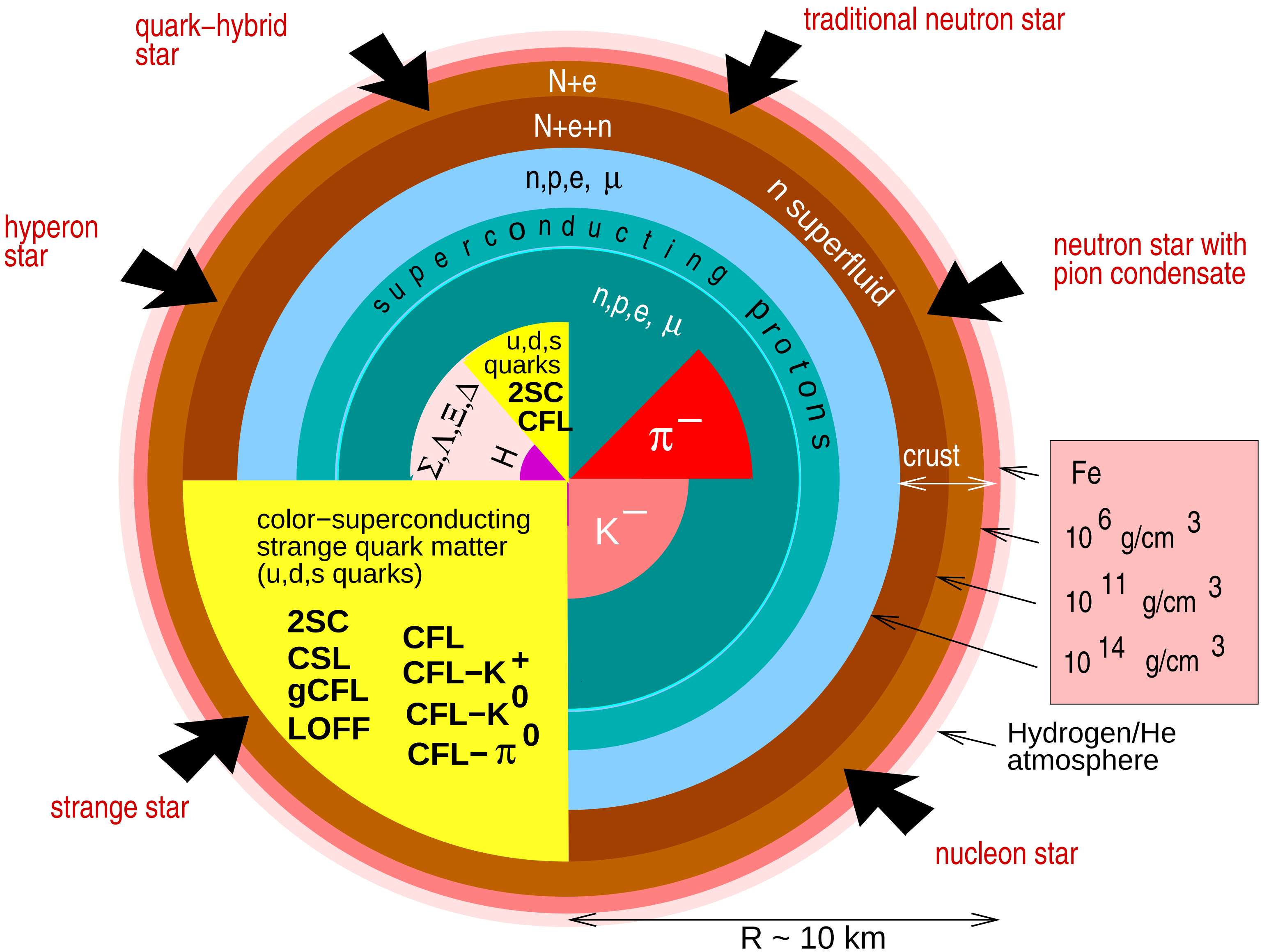


Phase structure at moderate μ (and $m_s, \mu_e \neq 0$) complicated and poorly understood.Systematic calculations

$$m_s^2 \ll \mu^2, m_s \Delta \ll \mu^2, g \ll 1$$

Use neutron stars to rule out certain phases

Composition of Neutron Stars



F. Weber (2005)

Observational Constraints

Mass-radius relationship, maximum mass

Equation of state

Cooling behavior

Phase structure, low energy degrees of freedom

Rotation

Equation of state, Viscosity

Spin-down, glitches

Superfluidity

Resources

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