Instanton constituents in sigma models and Yang-Mills theory

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Part I: The $O(3)$ sigma model

a scalar field in 2D ...

$$S = \int d^2x \, \frac{1}{2}(\partial_\mu \phi^a)^2 \qquad a = 1, 2, 3: \text{ global } O(3) \text{ symmetry}$$

...with a constraint

$$\phi^a \phi^a = 1 \qquad \text{(circumvent Derrick's theorem)}$$

nontrivial properties:

- asymptotic freedom
- dynamical mass gap
- topology and instantons

condensed matter physics and toy model for gauge theories

Topology

finite action:

$$r \to \infty : \quad \phi^a \to \text{const.}$$

as a mapping:

$$\phi : \mathbb{R}^2 \cup \{\infty\} \simeq S^2_x \longrightarrow S^2_c$$

winding number/degree: all such ϕ's are characterized by an integer Q = how often S^2_c is wrapped by S^2_x through ϕ

here:

$$Q = \frac{1}{8\pi} \int d^2x \, \epsilon_{\mu\nu} \epsilon_{abc} \phi^a \partial_\mu \phi^b \partial_\nu \phi^c \in \mathbb{Z}$$

topological quantum number = invariant under small deformations of ϕ

Classical solutions

Bogomolnyi trick…

$$(\partial_\mu \phi^a \pm \epsilon_{\mu\nu}\epsilon_{abc}\phi^b\partial_\nu\phi^c)^2 = (\partial_\mu\phi^a)^2 \pm 2\epsilon_{\mu\nu}\epsilon_{abc}\phi^a\partial_\mu\phi^b\partial_\nu\phi^c + (\partial_\mu\phi^a)^2$$

…and bound (integrated):

$$S \geq 4\pi|Q|$$

where the equality holds iff

$$\partial_\mu\phi^a = \mp\,\epsilon_{\mu\nu}\epsilon_{abc}\phi^b\partial_\nu\phi^c \qquad \text{‘selfduality equations’}$$

first order (instead of second order in eqns. of motion)

classical solutions:

instantons = localised in both directions

Complex structure

introduce complex coordinates both in space and color space:

$$\begin{array}{lll} x_{1,2} & \longrightarrow & z = x_1 + ix_2 \\ \phi^a & \longrightarrow & u = \dfrac{\phi^1 + i\phi^2}{1 - \phi^3} \end{array} \qquad \begin{array}{llll} \mathrm{N}: & \phi^a = (0,0,1) & u = \infty \\ \mathrm{S}: & \phi^a = (0,0,-1) & u = 0 \end{array}$$

$\Rightarrow$ self-duality equations become Cauchy-Riemann conditions on u

$\Rightarrow$ any meromorphic function $u(z)$ is a solution

topological charge: $Q =$ number of zeroes or poles

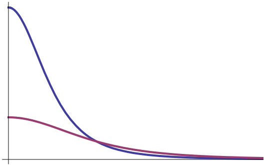
topological charge density:

$$q(x) = \frac{1}{\pi} \frac{1}{(1 + |u|^2)^2} \left| \frac{\partial u}{\partial z} \right|^2$$

Charge 1 instantons

- simplest functions:

$$\left.\begin{aligned} u(z) &= \tfrac{\lambda}{z-z_0} \\ u(z) &= \tfrac{z-z_0}{\lambda} \end{aligned}\right\} q(x) = \frac{1}{\pi}\frac{\lambda^2}{(|z-z_0|^2+\lambda^2)^2}$$



Belavin-Polyakov monopole

are $Q = 1$ instantons: location z_0, size λ

1 pole and 1 zero to cover S^2_c, one of them at infinity

- both, pole and zero, at finite z:

$$u(z) = \frac{z - z_I}{z - z_{II}}$$

constituents at $z = \{z_I, z_{II}\}$? $\rightsquigarrow$ 'instanton quarks'?!

NO! same profile $q(x)$ as above $\Rightarrow$ one lump

conjecture: 2 complex moduli per $Q \rightsquigarrow$ locations of 2 constituents?!

Finite temperature

= one compact direction, say: Im $z = x_2 \sim x_2 + \beta, \qquad \beta = 1/k_B T$

- instantons:

  use that higher charge solutions = products

$$u(z) = \prod_{k=1}^{Q} \frac{\lambda}{z - z_{0,k}} \qquad Q \text{ poles}$$

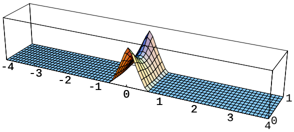
  and infinitely many copies: $z_{0,k} \equiv z_0 + k \cdot i\beta, \quad k \in \mathbb{Z}$

- a regularized $u(z)$ is:

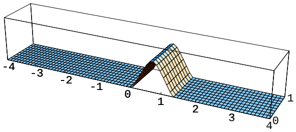
$$u(z) = \frac{\lambda}{\exp((z - z_0)\frac{2\pi}{\beta}) - 1}$$

has residues λ at $z = z_0 + k \cdot i\beta$
and charge 1 over $S^1 \cdot R^1$

small λ



large λ



Boundary conditions

$q(x)$ and action density invariant under global $SO(3)$ rotations

an $SO(2)$ subgroup: $\phi \to \begin{pmatrix} \text{rotation} & \\ \text{with } \omega & \\ & 1 \end{pmatrix} \phi, \quad u \to e^{2\pi i\omega} u$

- let the fields ϕ and u be periodic up to that $SO(2)$ subgroup: FB '07

$$u(z + i\beta) = e^{2\pi i\omega} u(z) \qquad \omega \in [0, 1]$$

$q(x)$ strictly periodic

- novel solution:

$$u(z) = \frac{e^{\omega(z-z_0)\frac{2\pi}{\beta}} \cdot \lambda}{\exp((z-z_0)\frac{2\pi}{\beta}) - 1} \qquad \text{has residues } e^{2\pi i\omega k}\lambda \text{ at } z = z_0 + k \cdot i\beta$$

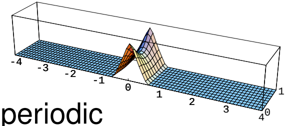
$\Rightarrow$ 'different orientation' of the instanton copies

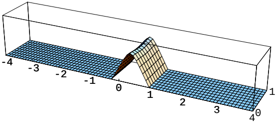
$\Rightarrow$ nontrivial overlaps $\Rightarrow$ instanton constituents

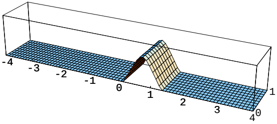
Topological profiles

$\ln q(x)$: $(z_0 = 0$, cut off below $e^{-5})$

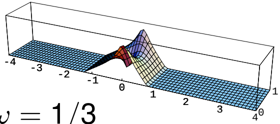
$\lambda = \beta$ $\lambda = 10\beta$ $\lambda = 100\beta$

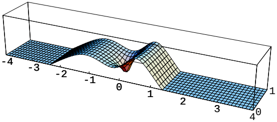


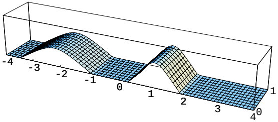




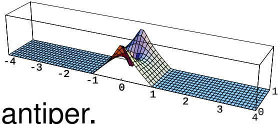
periodic

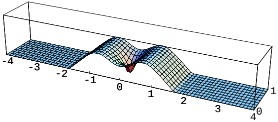


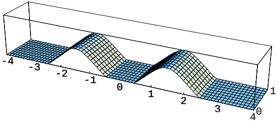




$\omega = 1/3$







antiper.

$\Rightarrow$ for large size λ: 2 lumps with action ω and $\bar{\omega} = 1 - \omega$

'Dissociation'

- rewrite:

$$u(z) \;=\; \frac{1}{\exp(-\omega(z-z_1)\frac{2\pi}{\beta}) - \exp(\bar{\omega}(z-z_2)\frac{2\pi}{\beta})}$$

  locations: $z_1 = z_0 - \beta\,\frac{\ln\lambda}{2\pi\omega}, \quad z_2 = z_0 + \beta\,\frac{\ln\lambda}{2\pi\bar{\omega}}$

  instanton size $\rightarrow$ constituent distance: $z_2 - z_1 \sim \ln\lambda$

  constituent size: fixed by β and ω

- really locations of topological lumps?

  YES: corrections of the second term at $z = z_1$ are exp. small

- individual constituent:

$$u(z) = \exp(\omega z\,\frac{2\pi}{\beta}) \qquad \bar{\omega} \text{ analogous}$$

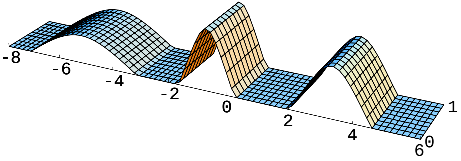
  top. charge:

$$q(x) = \frac{\pi\omega^2}{\beta^2\cosh^2(\omega\,\mathrm{Re}\,z\,\frac{2\pi}{\beta})} \qquad \text{(static)} \qquad Q = \omega$$

- possible values for Q: $0, 1, \ldots$ $\omega, 1+\omega, \ldots$ $1-\omega, 2-\omega, \ldots$
  asympt. $\phi_{-\infty} \to \phi_{+\infty}$: N→ N, S→ S N→ S S→ N
  constituents alternate
- why instanton quarks not visible for zero temperature, i.e. on $\mathbb{R}^2$?

  $\beta \to \infty$: constituents large and overlap!

  no other scale competing with their distance
- generalisations: *CP*(*N*) models FB et al. in progress



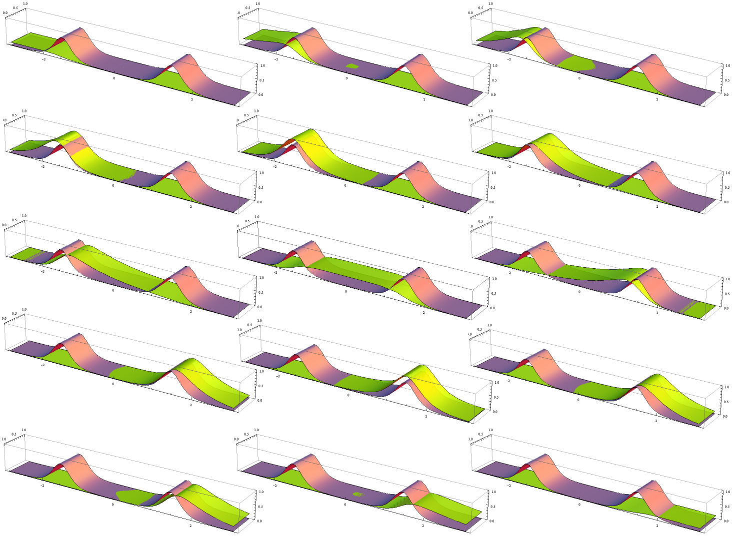
N constituents as expected

- realisation in condensed matter:
  cylinder of ..?.. with quasi-periodic bc.s

Fermionic zero modes

gauge field description $\Rightarrow$ couple fermions $\Rightarrow$ zero modes

phase-bc.s $\psi(x_0 + i\beta) = e^{2\pi i \zeta}\psi(x_0)$, evolution with ζ: FB et al. in progress



pure Yang-Mills theory in (Euclidean) 4D:

$$S = \int \frac{1}{2}\,\mathrm{tr}\,F_{\mu\nu}^2 \geq |Q| = |\int \frac{1}{2}\,\mathrm{tr}\,F_{\mu\nu}\tilde{F}_{\mu\nu}|$$

dual field strength $\tilde{F}^a_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^a_{\rho\sigma} \quad (\vec{E}^a \rightleftharpoons \vec{B}^a)$

integer Q: instanton number/topological charge

topology:

$$A_\mu \stackrel{r\to\infty}{\to} i\Omega^{-1}\partial_\mu\Omega \qquad \ldots\text{pure gauge}$$

$$Q = \deg(\Omega : S^3_{r\to\infty} \to SU(N)) \qquad \ldots\text{winding number}$$

Instantons

(anti)selfdual: $F^a_{\rho\sigma} = \pm \tilde{F}^a_{\mu\nu}$ first order, nonlinear

charge 1: axially symmetric ansatz and solution BPST

$$A^a_\mu = \eta^a_{\mu\nu} \frac{2x_\nu}{x^2+\rho^2} \qquad \mathrm{tr} F^2 = \frac{\rho^4}{(x^2+\rho^2)^4} \qquad \eta^a_{\mu\nu} \in \{-1, 0, 1\}$$

size ρ
localized in space and time
algebraic decay, similar to $O(3)$ instantons on $\mathbb{R}^2$

instanton liquid model from semiclassical path integral Shuryak

- chiral symmetry breaking
- axial anomaly
- topological susceptibility
- confinement?

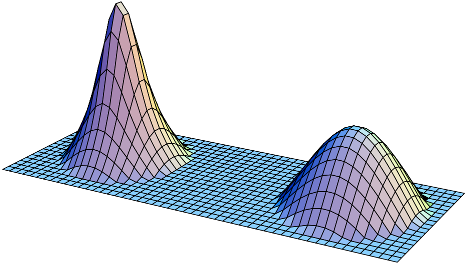
Finite temperature: Calorons

- use higher charge solutions of same color orientation — CFTW

⇒ first calorons — Harrington-Shepard '78

- most general calorons: need ADHM formalism and Nahm transform

⇒ calorons of nontrivial holonomy — Kraan,van Baal; Lee, Lu '98



space-space plot of action density for $SU(2)$, intermediate holonomy

⇒ 2 lumps, almost static

N_c for gauge group $SU(N_c)$, like quarks in baryons

⇒ magnetic monopoles of opposite magnetic charge

in fact dyons with same electric as magn. charge (selfdual)

Role of the holonomy

relative gauge orientation of instanton copies in the ADHM constr.

$\Rightarrow A_\mu$ periodic up to a gauge transformation $e^{2\pi i\omega\sigma_3/2}$ (cf. $O(3)$)

gauge theory: compensated by time-dependent transf. $e^{2\pi i\omega\sigma_3 x_0/2}$

$\Rightarrow$ introduces an asymptotic gauge field A_0

$\Rightarrow$ asymptotic Polyakov loop = holonomy

$$\mathcal{P}(\vec{x}) \equiv \mathcal{P}\exp\left(i\int_0^\beta dx_0\, A_0\right) \to e^{2\pi i\omega\sigma_3/2} \equiv \mathcal{P}_\infty$$

'environment'

acts like a Higgs field, in the group: vev ω, direction σ_3

- monopoles have masses ω/β and $\bar{\omega}/\beta$, $\qquad \bar{\omega} = 1 - \omega$
- $A_\mu^{a=3}$: power law decay (massless 'photon'),
  $A_\mu^{a=1,2}$: exponential decay (massive 'W-bosons')

- Polyakov loop in the bulk: $\mathcal{P}(\vec{x}) = \pm \mathbb{1}_2$ at the monopoles
  Higgs field vanishes = 'false vacuum'
  necessary for top. reasons — Ford et al.; Reinhardt; Jahn et al.

- index theorem valid — Nye, Singer
  localisation depending on bc.s:

$$\psi(x_0 + i\beta) = e^{2\pi i z}\psi(x_0) \qquad (A_\mu \text{ still periodic})$$

  $z \in \{-\frac{\omega}{2}, \frac{\omega}{2}\}$ incl. periodic: localised at monopole — Garcia Perez et al.
  $z \in$ rest incl. antiperiodic: localised at antimonopole

  a zero in their profiles at the 'other' monopole, topological — FB

- calorons can be studied on the lattice by cooling — Ilgenfritz et al. '02, FB et al.

- physical relevance of $\mathcal{P}_\infty$:
  conjecture: holonomy $\mathrm{tr}\,\mathcal{P}_\infty \rightleftharpoons$ deconfinement order param. $\langle \mathrm{tr} P \rangle_x$

Calorons and the dynamics of YM theories

- eff. potential at 1-loop: triv. holonomy favored! Gross, Pisarski, Jaffe; Weiss

overruled by caloron gas contribution: Diakonov et al.

$\Rightarrow$ minima at $\mathcal{P} = \pm\mathbb{1}_2$ become unstable for low enough temperature

$\Rightarrow$ onset of confinement

- gas of calorons and anticalorons put on the lattice: Gerhold et al.

$\Rightarrow$ linearly rising interquark potential just for nontrivial holonomy!

- confinement from a gas of purely selfdual dyons Diakonov, Petrov
  unphysical (top. charge builds up)

- dyons of all magnetic and electric charges $(q_i, e_i) \in (\pm 1, \pm 1)$
  interaction from excess in action $(S - S_{\text{naive}})$: FB in progress

$$V = \sum_{i \neq j} \frac{q_i q_j - e_i e_j}{|\vec{x}_i - \vec{x}_j|}$$

electric interaction is anti-Coulomb!

Summary

sigma models in 2D and YM in 4D admit instantons

instanton constituents for $S^1 \times R^1$ and $S^1 \times R^3$ = finite T

with fractional charges, say ω and $1 - \omega$ in the lowest models

when in compact direction periodic up to a subgroup, say $e^{2\pi i\omega..}$

= subgroup of global and local symmetry

Yang-Mills theory:

can be made periodic $\rightarrow$ holonomy $\mathcal{P}_\infty$

$\Rightarrow$ caloron constituents as building blocks of semiclass. models at finite temperature: (de)confinement?!

sigma models:

quasi-periodic bc.s stay

$\Rightarrow$ spin chains? skyrmion lattices? Quantum Hall effect?