Scale invariant fluid dynamics for the unitary Fermi gas

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Hydroynamics

Hydrodynamics (undergraduate version): Newton's law for continuous, deformable media.

Fluids: Gases, liquids, plasmas, ...

Hydrodynamics (postmodern): Effective theory of non-equilibrium long-wavelength, low-frequency dynamics of any many-body system.

$\tau \sim \tau_{micro}$ $\qquad$ $\tau \sim \lambda$

$\tau \gg \tau_{micro}$: Dynamics of conserved charges.

Water: $(\rho, \epsilon, \vec{\pi})$

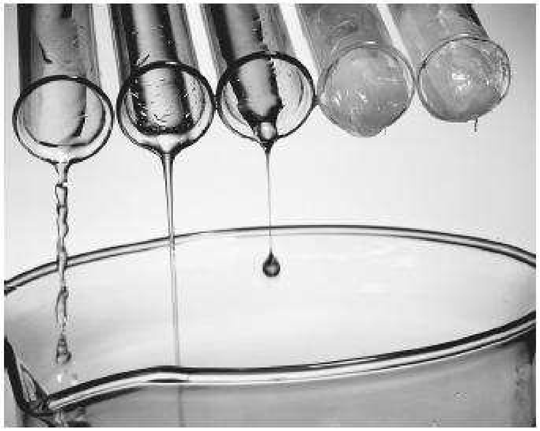
Simple non-relativistic fluid

Simple fluid: Conservation laws for mass, energy, momentum

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}(\rho \vec{v}) = 0$$

$$\frac{\partial \epsilon}{\partial t} + \vec{\nabla} \vec{j}^{\epsilon} = 0$$

$$\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j}\Pi_{ij} = 0$$



Constitutive relations: Energy momentum tensor

$$\Pi_{ij} = P\delta_{ij} + \rho v_i v_j + \eta \left(\partial_i v_j + \partial_j v_i - \frac{2}{3}\delta_{ij}\partial_k v_k \right) + O(\partial^2)$$

reactive dissipative 2nd order

Expansion $\Pi^0_{ij} \gg \delta\Pi^1_{ij} \gg \delta\Pi^2_{ij}$

Regime of applicability

Expansion parameter $Re^{-1} = \frac{\eta(\partial v)}{\rho v^2} = \frac{\eta}{\rho L v} \ll 1$

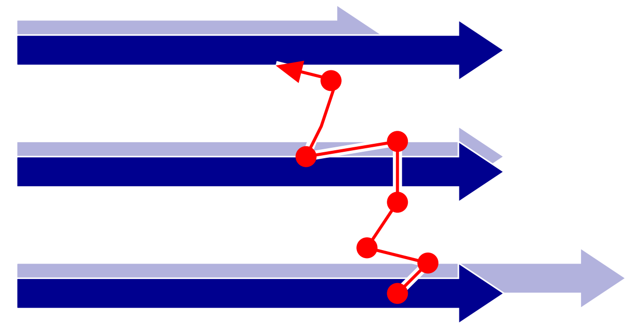
$$\frac{1}{Re} = \underbrace{\frac{\eta}{\hbar n}}_{\text{fluid property}} \times \underbrace{\frac{\hbar}{mvL}}_{\text{flow property}}$$

Consider $mvL \sim \hbar$: Hydrodynamics requires $\eta/(\hbar n) < 1$

Shear viscosity in kinetic theory

Kinetic theory: conserved quantities carried by quasi-particles

$$\frac{\partial f_p}{\partial t} + \vec{v}\cdot\vec{\nabla}_x f_p + \vec{F}\cdot\vec{\nabla}_p f_p = C[f_p]$$



$$\boxed{\eta \sim \frac{1}{3} n\,\bar{p}\,l_{mfp}}$$

Weakly interacting gas: $l_{mfp} \sim 1/(n\sigma)$ $\quad\Rightarrow\quad$ $\eta \sim \frac{1}{3}\frac{\bar{p}}{\sigma}$

$$\boxed{\eta(\sigma \to 0) \to \infty}$$

Strongly interacting gas:

$$\boxed{\frac{\eta}{n} \sim \bar{p} l_{mfp} \geq \hbar}$$

but: kinetic theory not reliable!

Holographic duals: Transport properties

Thermal (conformal) field theory $\equiv AdS_5$ black hole

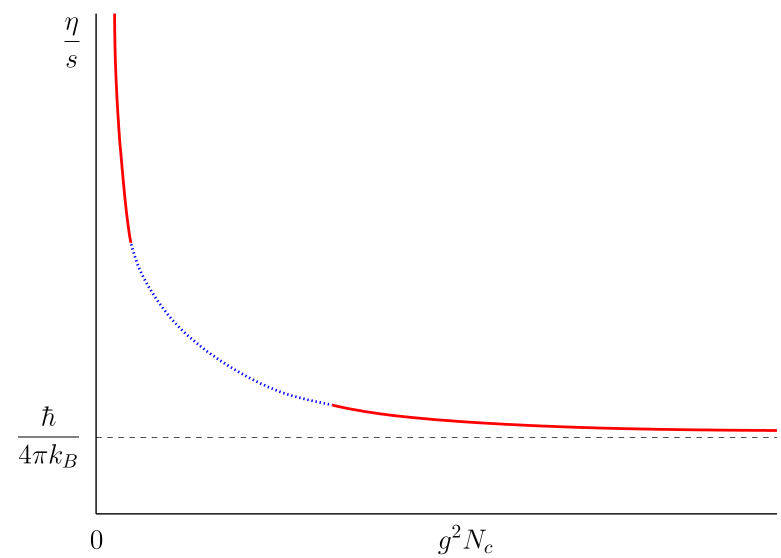
CFT entropy $\Leftrightarrow$ Hawking-Bekenstein entropy $\sim$ area of event horizon

shear viscosity $\Leftrightarrow$ Graviton absorption cross section $\sim$ area of event horizon

Strong coupling limit

$$\boxed{\frac{\eta}{s} = \frac{\hbar}{4\pi k_B}}$$

Son and Starinets (2001)



Strong coupling limit universal? Provides lower bound for all theories?

Answer appears to be no; e.g. theories with higher derivative gravity duals.

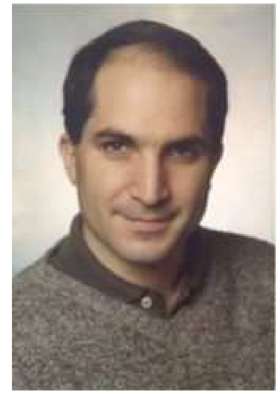
Effective theories for fluids (Unitary Fermi Gas, $T > T_F$)

$$\mathcal{L} = \psi^\dagger \left(i\partial_0 + \frac{\nabla^2}{2M} \right)\psi - \frac{C_0}{2}(\psi^\dagger\psi)^2$$

$$\frac{\partial f_p}{\partial t} + \vec{v}\cdot\vec{\nabla}_x f_p = C[f_p] \qquad \omega < T$$

$$\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j}\Pi_{ij} = 0 \quad \omega < T\frac{n}{\eta}$$

Effective theories (Strong coupling)



$$\mathcal{L} = \bar{\lambda}(i\sigma \cdot D)\lambda - \frac{1}{4} G^a_{\mu\nu} G^a_{\mu\nu} + \ldots \;\Leftrightarrow\; S = \frac{1}{2\kappa_5^2} \int d^5x \sqrt{-g}\mathcal{R} + \ldots$$

$$SO(d+2,2) \rightarrow Schr(d) \qquad\qquad AdS_{d+3} \rightarrow \mathcal{X}_{d+3}$$

$$\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j}\Pi_{ij} = 0 \;\; (\omega < T)$$

Outline

I. Conformal second order hydrodynamics

II. Fluctuations

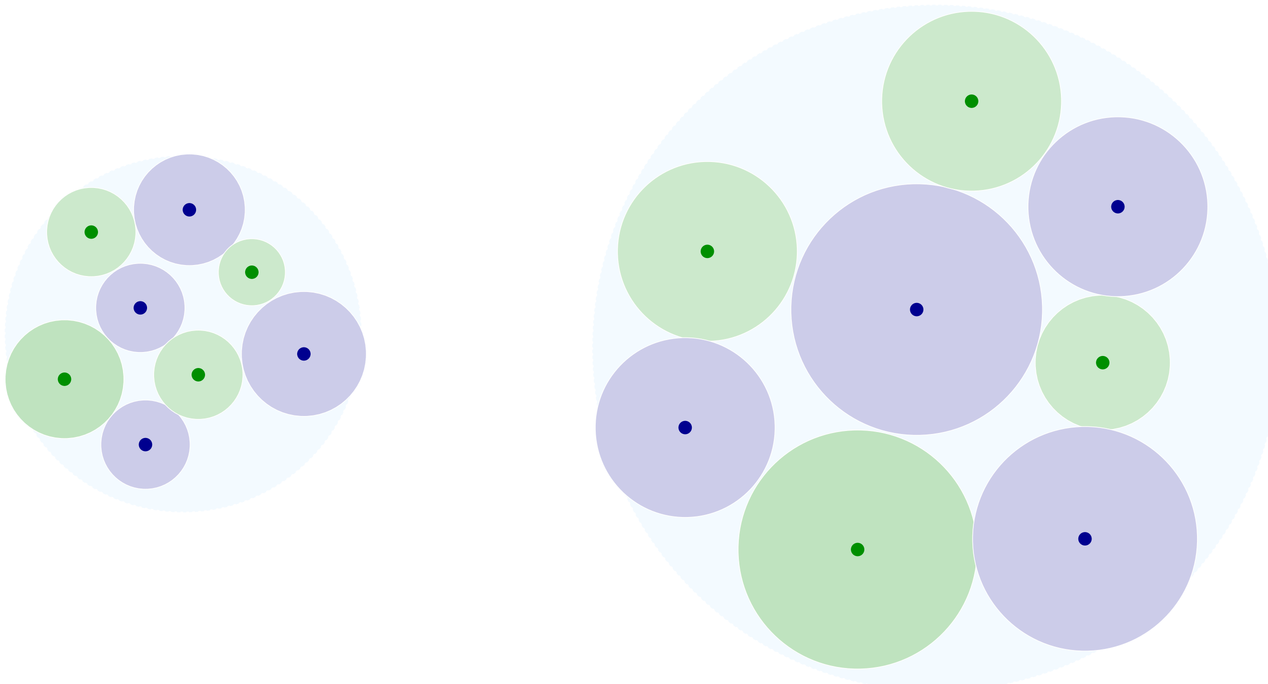
III. Kinetic theory

IV. Experiment

V. Outlook: QGP vs Cold Atoms

I. Scale invariant fluid dynamics

Many body system: Effective cross section $\sigma_{tr} \sim n^{-2/3}$ (or $\sigma_{tr} \sim \lambda^2$)



Systems remains hydrodynamic despite expansion

Scale and conformal symmetry

Gallilean boosts	$\vec{x}' = \vec{x} + \vec{v}t$	$t' = t$
scale trafo	$\vec{x}' = e^s \vec{x}$	$t' = e^{2s} t$
conformal trafo	$\vec{x}' = \vec{x}/(1+ct)$	$1/t' = 1/t + c$

Ideal fluid dynamics

$$\Pi^0_{ij} = P\delta_{ij} + \rho v_i v_j, \qquad \boxed{P = \frac{2}{3}\mathcal{E}}$$

First order viscous hydrodynamics

$$\delta^{(1)}\Pi_{ij} = -\eta\sigma_{ij}, \quad \sigma_{ij} = \left(\nabla_i v_j + \nabla_j v_i - \frac{2}{3}\delta_{ij}(\nabla\cdot v)\right), \qquad \boxed{\zeta = 0}$$

Second order conformal hydrodynamics

Relaxation of shear stress is a second order hydro term. Complete list

$$
\begin{aligned}
\delta^{(2)}\Pi^{ij} \;&=\; \eta\tau_\pi \left[{}^{\langle} D\sigma^{ij\rangle} + \frac{2}{3}\,\sigma^{ij}(\nabla\cdot v) \right] \\
&\quad + \lambda_1 \sigma^{\langle i}{}_k \sigma^{j\rangle k} + \lambda_2 \sigma^{\langle i}{}_k \Omega^{j\rangle k} + \lambda_3 \Omega^{\langle i}{}_k \Omega^{j\rangle k} \\
&\quad + \gamma_1 \nabla^{\langle i} T \nabla^{j\rangle} T + \gamma_2 \nabla^{\langle i} P \nabla^{j\rangle} P + \gamma_3 \nabla^{\langle i} \nabla^{j\rangle} T + \ldots
\end{aligned}
$$

$$
D = \partial_0 + v\cdot\nabla \qquad A^{\langle ij\rangle} = \frac{1}{2}\left(A_{ij} + A_{ji} - \frac{2}{3} g_{ij} A^k{}_k\right) \qquad \Omega^{ij} = \left(\nabla_i v_j - \nabla_j v_i\right)
$$

New transport coefficients $\tau_\pi, \lambda_i, \gamma_i$

Can be written as a relaxation equation for $\pi^{ij} \equiv \delta\Pi^{ij}$

$$
\pi^{ij} = -\eta\sigma^{ij} - \tau_\pi \left[{}^{\langle} D\pi^{ij\rangle} + \frac{5}{3}(\nabla\cdot v)\pi^{ij} \right] + \ldots
$$

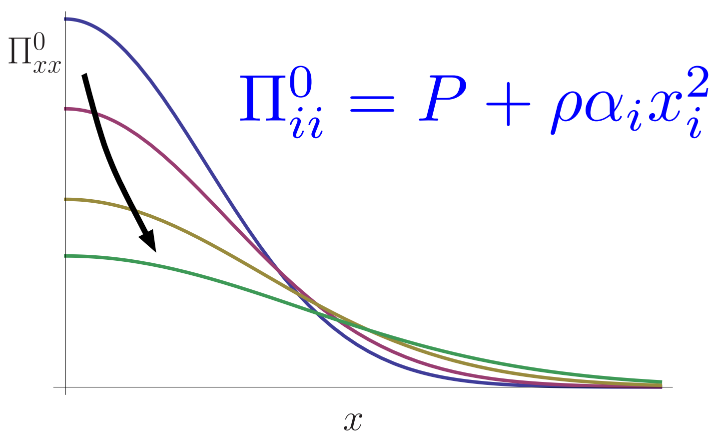
Chao, Schaefer (2011)

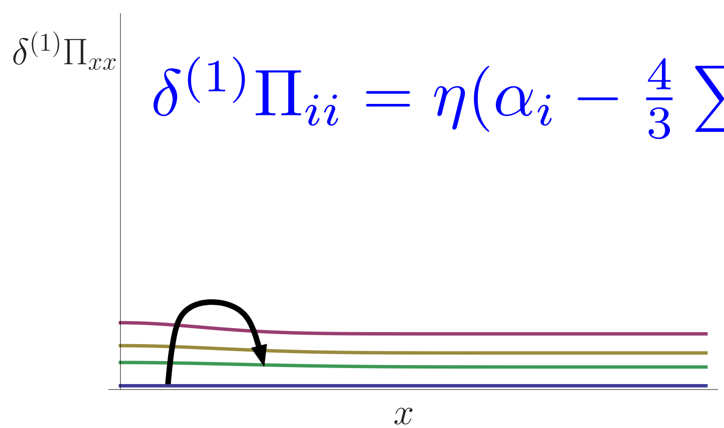
Why second order fluid dynamics?

Scaling ("Hubble") expansion

$$\rho(x_i,t) = \rho_0(b_i(t)x_i), \qquad v_i(x_j,t) = \alpha_i(t)x_i, \qquad \alpha_i(t) = \dot{b}_i(t)/b_i(t)$$

Compare ideal and dissipative stresses





Ideal stresses propagate with speed $\sim c_s$, dissipative stresses propagate with infinite speed. Hydro always breaks down in the dilute corona.

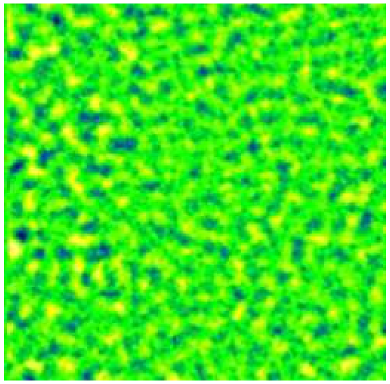
Solved by relaxation time $\tau_\pi \sim \frac{\eta}{P}$.

II. Fluctuations

If hydrodynamics is an effective (field?) theory

then where are the loop corrections?

Thermal fluctuations



Hydrodynamic variables fluctuate

$$\langle \delta v_i(x,t)\delta v_j(x',t)\rangle = \frac{T}{\rho}\delta_{ij}\delta(x-x')$$

Linearized hydrodynamics propagates fluctuations as shear or sound

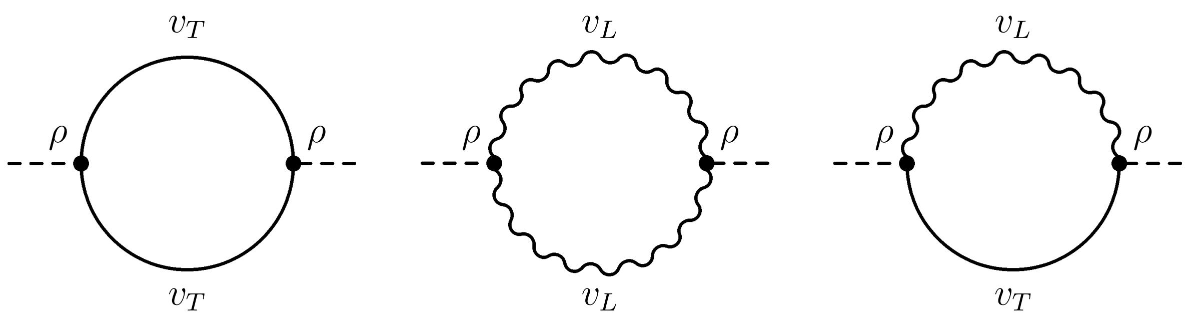
$$\langle \delta v_i^T \delta v_j^T \rangle_{\omega,k} = \frac{2T}{\rho}(\delta_{ij} - \hat{k}_i\hat{k}_j)\frac{\nu k^2}{\omega^2 + (\nu k^2)^2} \qquad \textit{shear}$$

$$\langle \delta v_i^L \delta v_j^L \rangle_{\omega,k} = \frac{2T}{\rho}\hat{k}_i\hat{k}_j \frac{\omega k^2 \Gamma}{(\omega^2 - c_s^2 k^2)^2 + (\omega k^2 \Gamma)^2} \qquad \textit{sound}$$

$v = v_T + v_L$: $\quad \nabla \cdot v_T = 0,\ \nabla \times v_L = 0$ $\qquad \nu = \eta/\rho, \quad \Gamma = \frac{4}{3}\nu + \ldots$

Hydro Loops: “Breakdown” of second order hydro

Response function $G_R^{xyxy} = \langle \theta(t)[\Pi^{xy}, \Pi^{xy}] \rangle_{\omega,k}$ $\qquad \Pi_{xy} = \rho v_x v_y$



$$G_R^{xyxy} = P + \delta P + i\omega[\eta + \delta\eta] + \omega^2 \left[\eta\tau_\pi + \delta(\eta\tau_\pi)\right]$$

$$\delta\eta \sim T \left(\frac{\rho}{\eta}\right)^2 \left(\frac{P}{\rho}\right)^{1/2} \qquad\qquad \delta(\eta\tau_\pi) \sim \frac{1}{\sqrt{\omega}} \left(\frac{\rho}{\eta}\right)^{3/2}$$

Hydro Loops: "Breakdown" of second order hydro

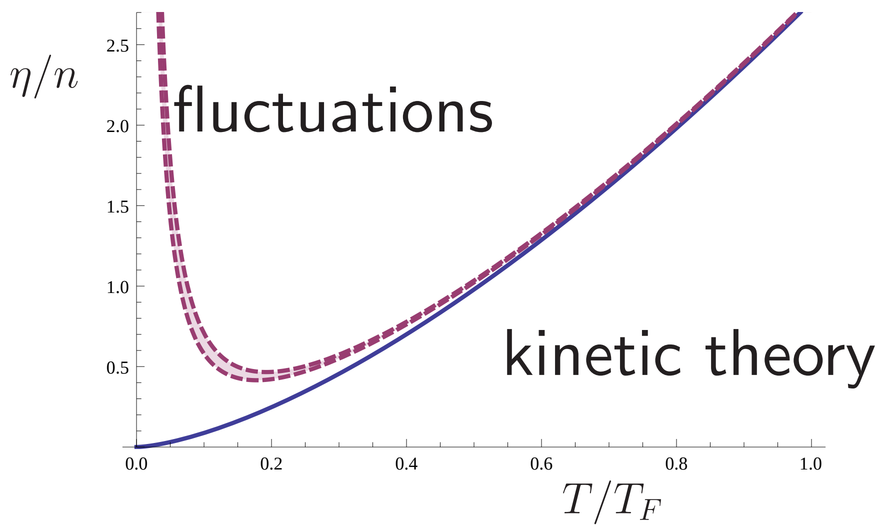
$$\delta\eta \sim T\left(\frac{\rho}{\eta}\right)^2\left(\frac{P}{\rho}\right)^{1/2} \qquad\qquad \delta(\eta\tau_\pi) \sim \frac{1}{\sqrt{\omega}}\left(\frac{\rho}{\eta}\right)^{3/2}$$

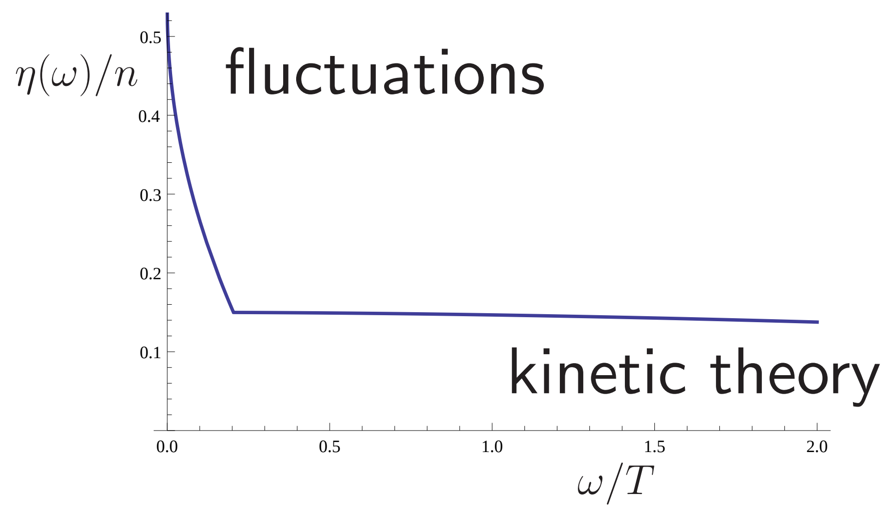
Small shear viscosity enhances fluctuation corrections.

Small η leads to large $\delta\eta$: There must be a bound on η/n.

Relaxation time diverges: 2nd order hydro without fluctuations inconsistent.

Fluctuation induced bound on η/n





$$(\eta/n)_{min} \simeq 0.3$$

spectral function

non-analytic $\sqrt{\omega}$ term

see also Kovtun, Moore, Romatschke (2011)

III. Linear response and kinetic theory

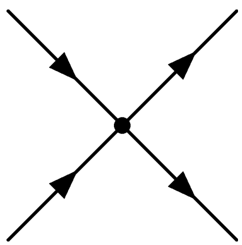
Consider background metric $g_{ij}(t, \mathbf{x}) = \delta_{ij} + h_{ij}(t, \mathbf{x})$. Linear response

$$\delta\Pi^{ij} = -\frac{1}{2} G_R^{ijkl} h_{kl}$$

Kubo relation: $\eta(\omega) = \dfrac{1}{\omega} \mathrm{Im} G_R^{xyxy}(\omega, 0)$

Kinetic theory: Boltzmann equation

$$\left(\frac{\partial}{\partial t} + \frac{p^i}{m} \frac{\partial}{\partial x^i} - \left(g^{il} \dot{g}_{lj} p^j + \Gamma^i_{jk} \frac{p^j p^k}{m} \right) \frac{\partial}{\partial p^i} \right) f(t, \mathbf{x}, \mathbf{p}) = C[f]$$

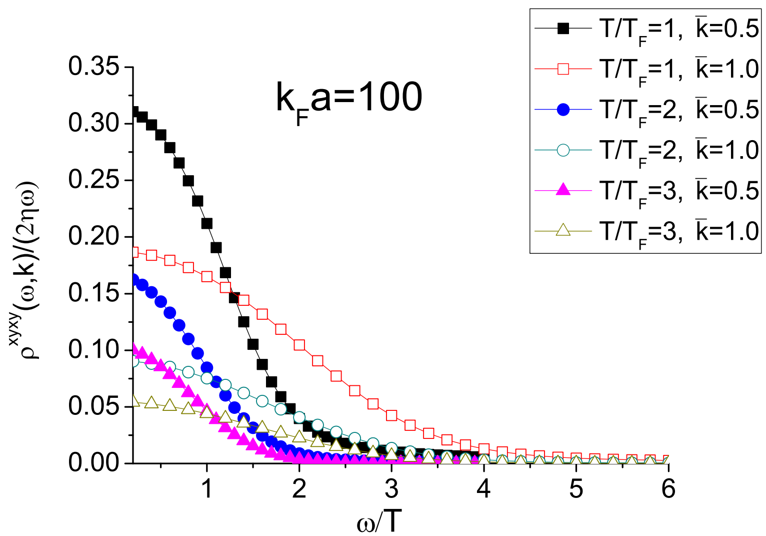
$C[f] =$ 

Kinetic theory

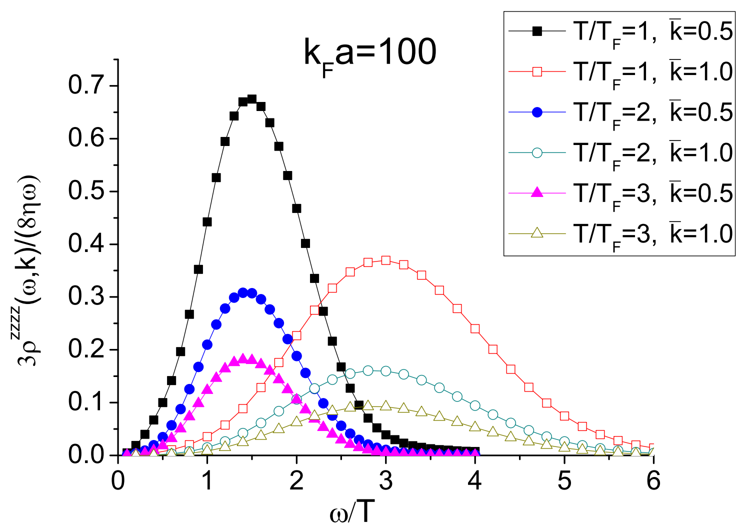
linearize $f = f_0 + \delta f$, solve for δf, $\hookrightarrow \delta\Pi_{ij}$, $\hookrightarrow G_R$, $\hookrightarrow \eta(\omega)$

$$\eta(\omega) = \frac{\eta}{1+\omega^2\tau_\pi^2} \qquad \eta = \frac{15}{32\sqrt{\pi}}(mT)^{3/2} \qquad \tau_\pi = \frac{\eta}{nT}$$

shear channel



sound channel



Braby, Chao, T.S. (2010)

Second order hydrodynamics from kinetic theory

Boltzmann equation (BGK approximation)

$$
\begin{aligned}
\delta^{(2)}\Pi^{ij} \;=\; & \frac{\eta^2}{P}\left[\langle D\sigma^{ij}\rangle + \frac{2}{3}\,\sigma^{ij}(\nabla\cdot v)\right] \\
& + \frac{\eta^2}{P}\left[\sigma^{\langle i}{}_k\sigma^{j\rangle k} + \sigma^{\langle i}{}_k\Omega^{j\rangle k}\right] + O(\kappa\eta\nabla^i\nabla^j T)
\end{aligned}
$$

relaxation time $\tau_\pi = \dfrac{\eta}{P} \simeq \dfrac{\eta}{nT}$

Shear & bulk viscosity: Sum rules

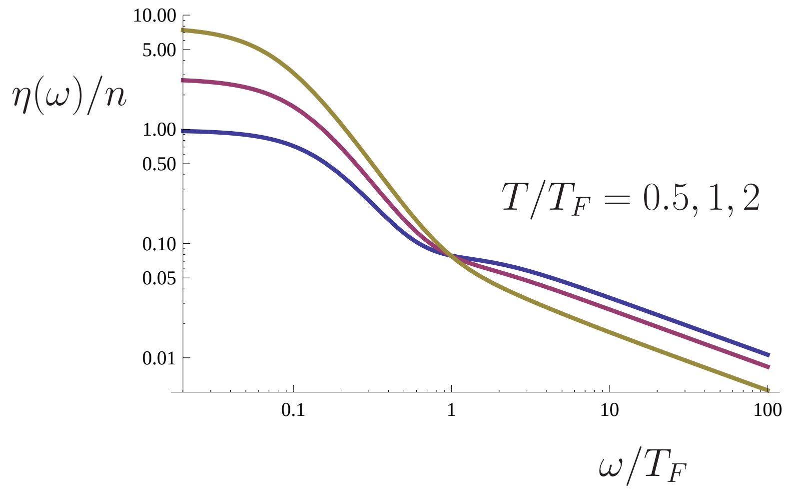
Randeria & Taylor proved the sum rules (corrected by Enss & Zwerger)

$$\frac{1}{\pi}\int dw\,\left[\eta(\omega)-\frac{C}{15\pi\sqrt{m\omega}}\right]=\frac{\mathcal{E}}{3}-\frac{C}{10\pi ma}$$

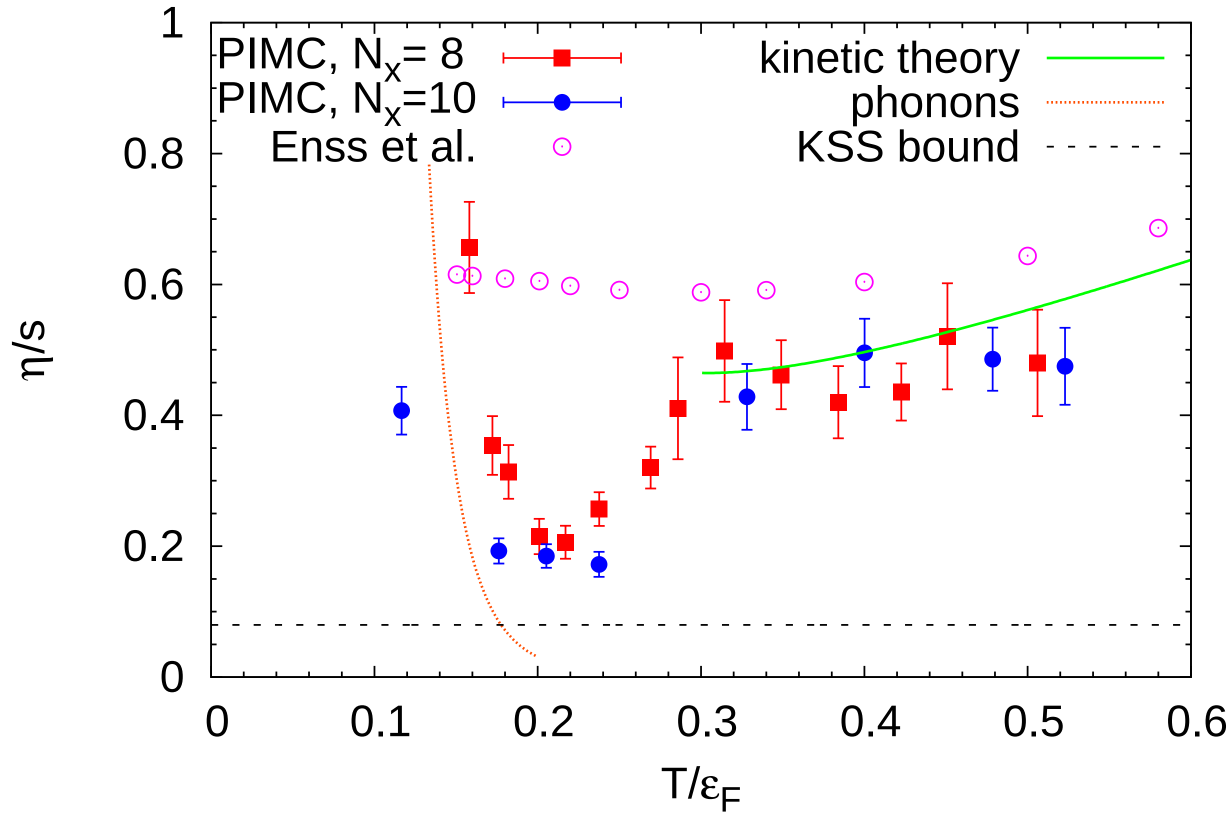
$$\frac{1}{\pi}\int dw\,\zeta(\omega)=\frac{1}{72\pi ma^2}\left(\frac{\partial C}{\partial a^{-1}}\right)$$

where C is Tan's contact, $n_k \sim C/k^4$.

Model spectral function: Kinetic theory for $\omega < T$, OPE for $\omega > T$.

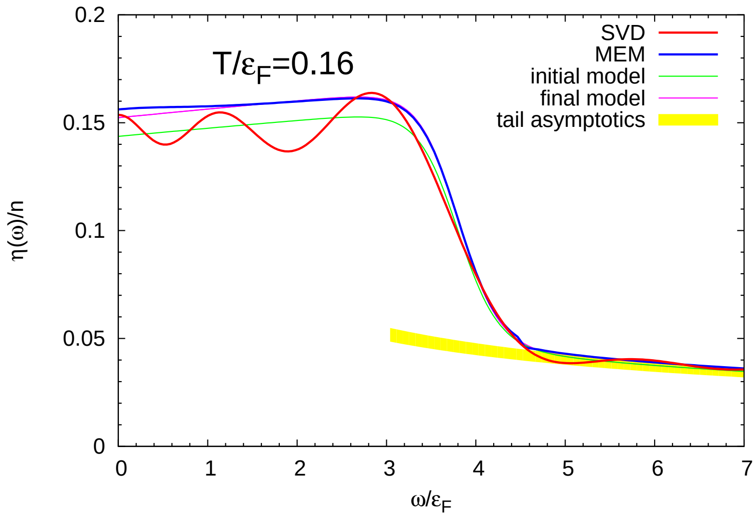


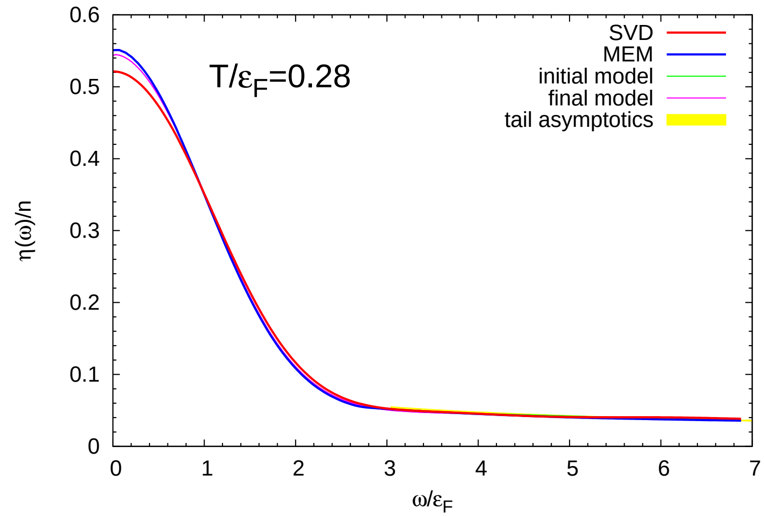
Lattice data: η/s



Wlazlowski, Magierski & Drut, arXiv:1204.0270

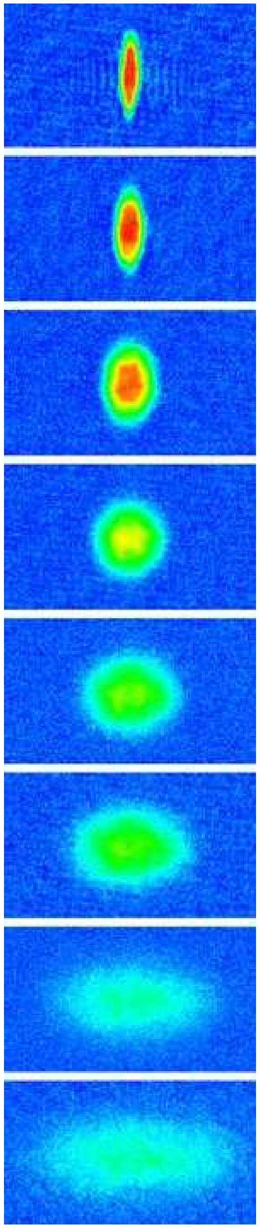
Lattice data: Spectral functions



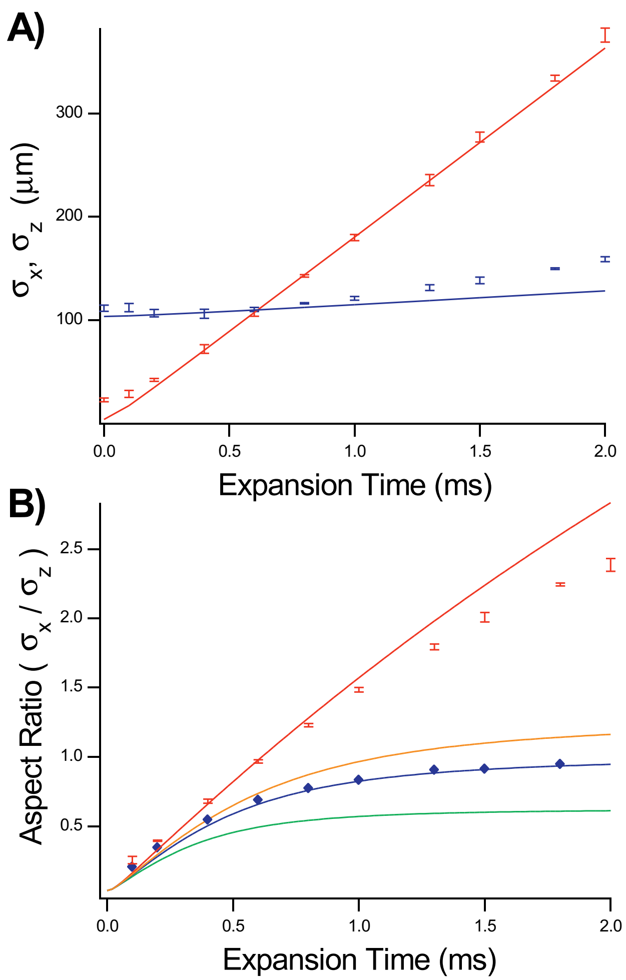


Wlazlowski, Magierski & Drut, arXiv:1204.0270

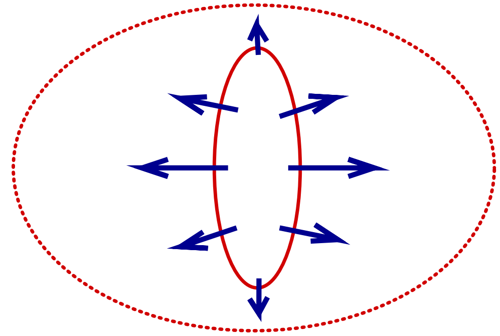
IV. Experiments: Flow and Collective Modes



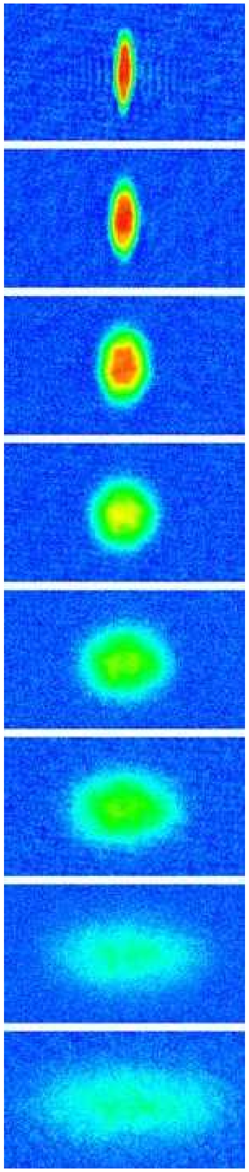
O'Hara et al. (2002)



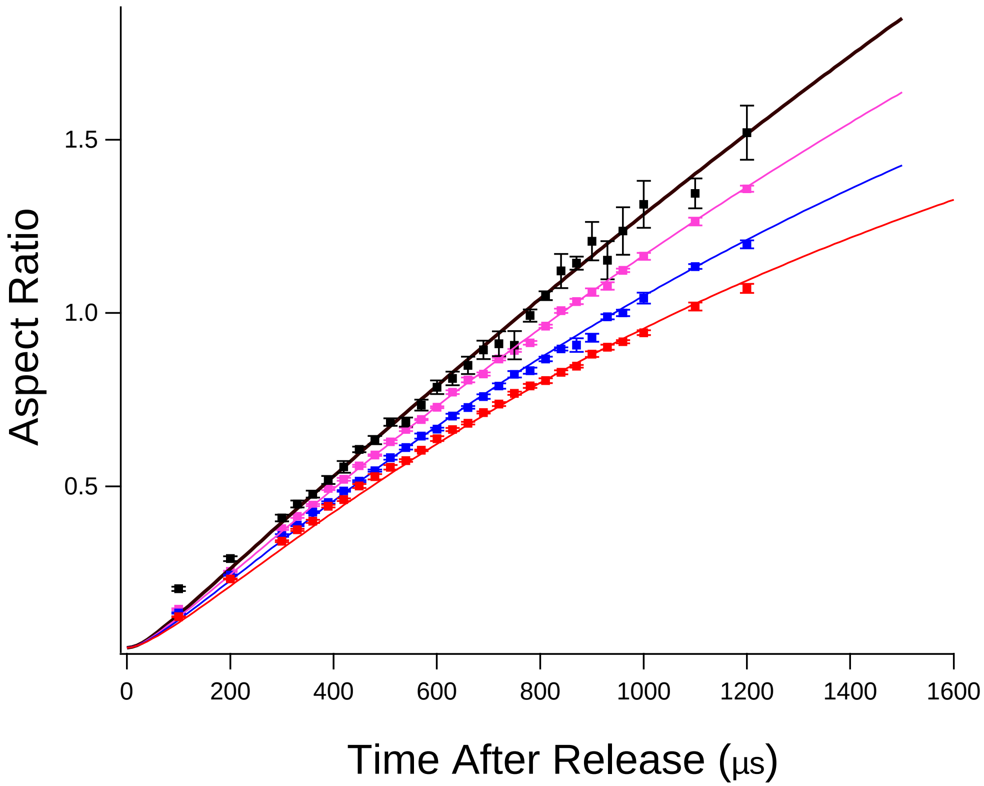
Hydrodynamic expansion converts coordinate space anisotropy to momentum space anisotropy



Elliptic flow: High T limit



Quantum viscosity $\eta = \eta_0 \dfrac{(mT)^{3/2}}{\hbar^2}$



$$\eta = \eta_0 (mT)^{3/2}$$

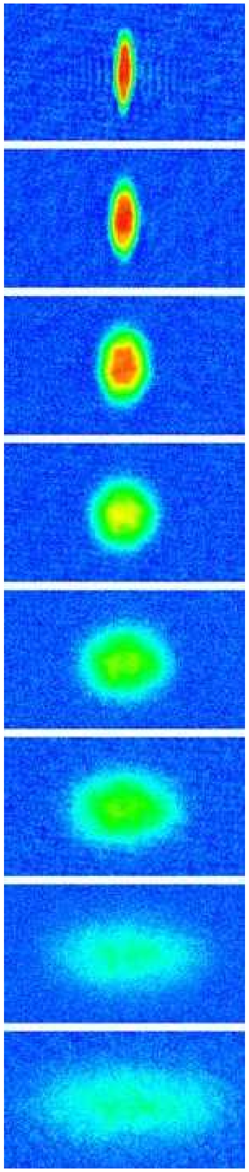
$$\tau_\pi = \eta / P$$

Cao et al., Science (2010)

fit: $\eta_0 = 0.33 \pm 0.04$

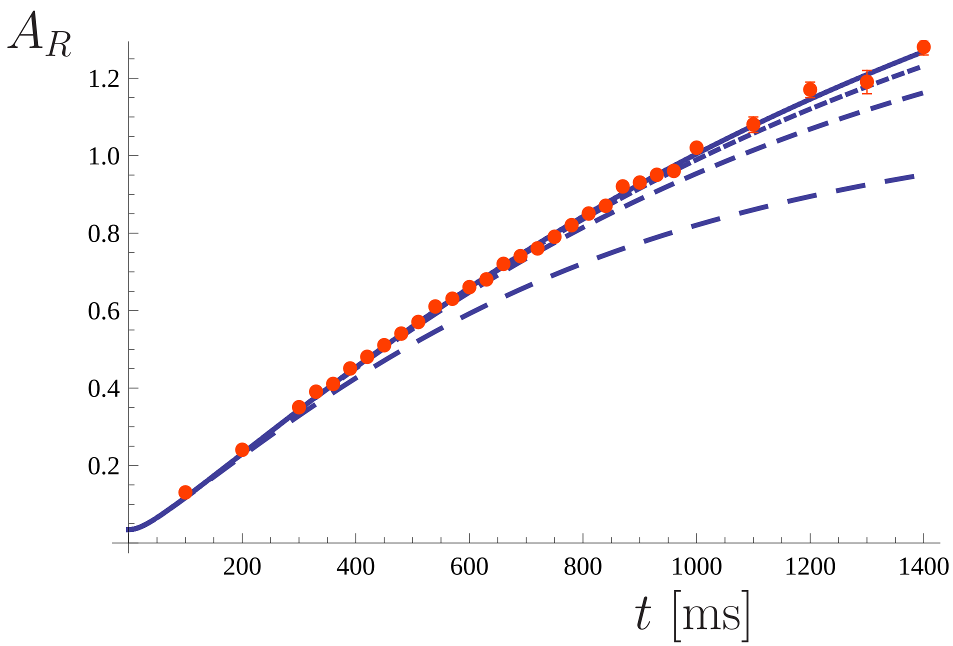
theory: $\eta_0 = \frac{15}{32\sqrt{\pi}} = 0.26$

Elliptic flow: Freezeout?



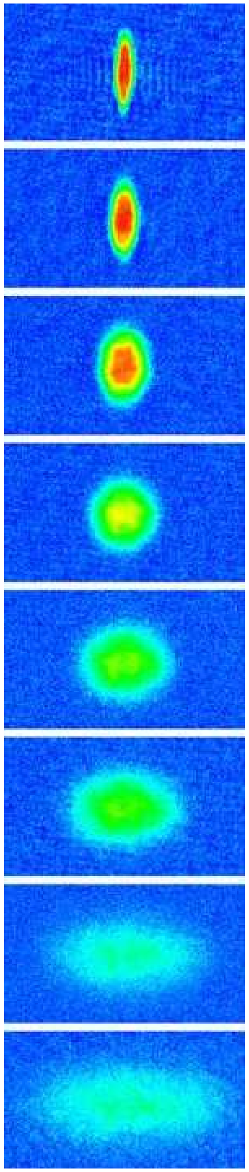
switch from hydro to (weakly collisional) kinetics

at scale factor $b_\perp^{fr} = 1, 5, 10, 20$

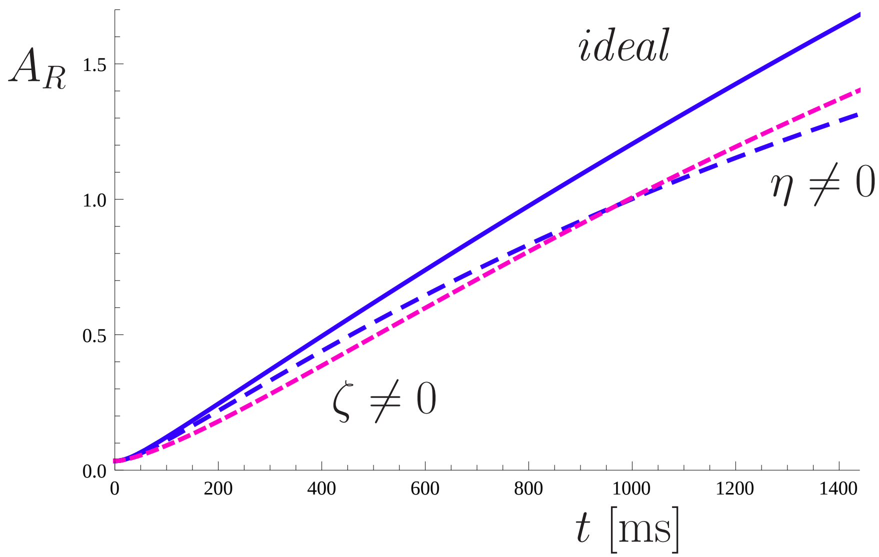


no freezeout seen in the data

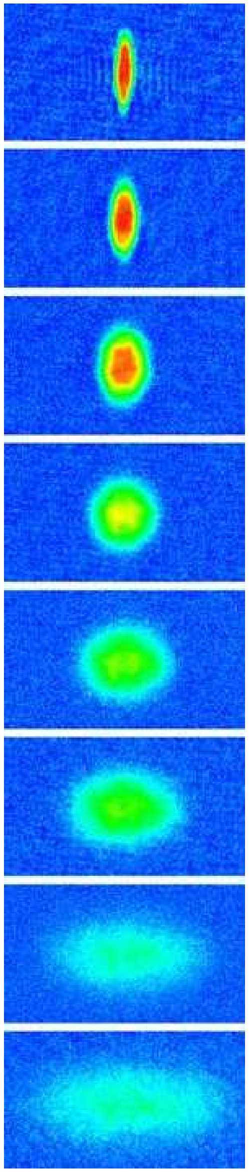
Elliptic flow: Shear vs bulk viscosity



Dissipative hydro with both η, ζ

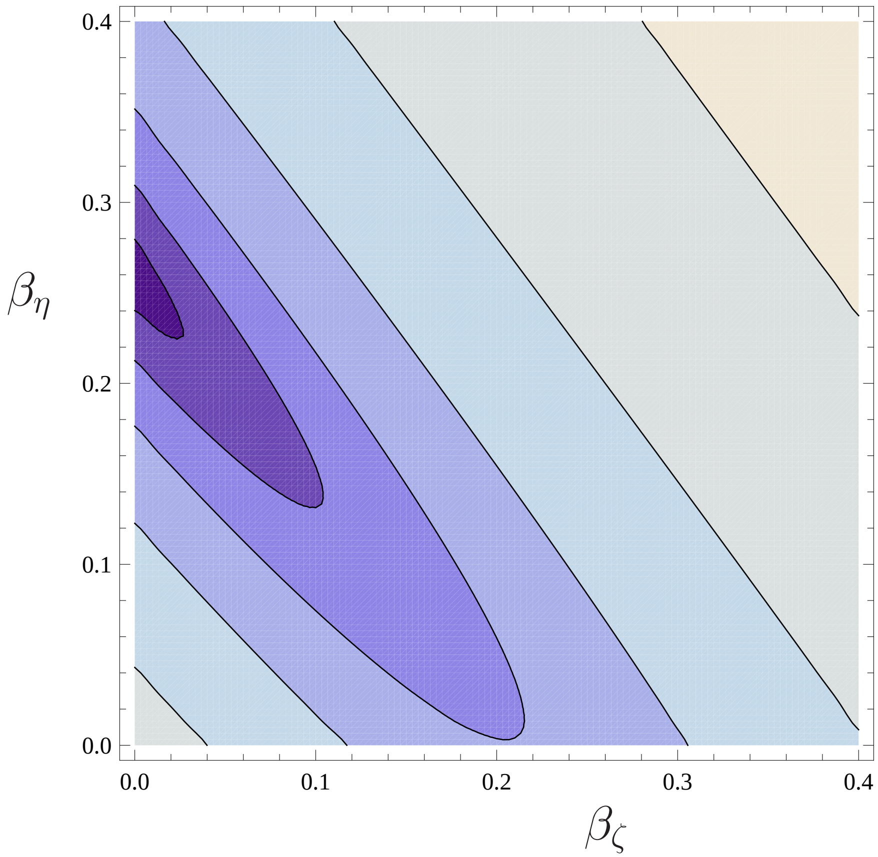


Elliptic flow: Shear vs bulk viscosity



Dissipative hydro with both η, ζ

$$\beta_{\eta,\zeta} = \frac{[\eta,\zeta]}{n}\,\frac{E_F}{E}\,\frac{1}{(3\lambda N)^{1/3}}$$



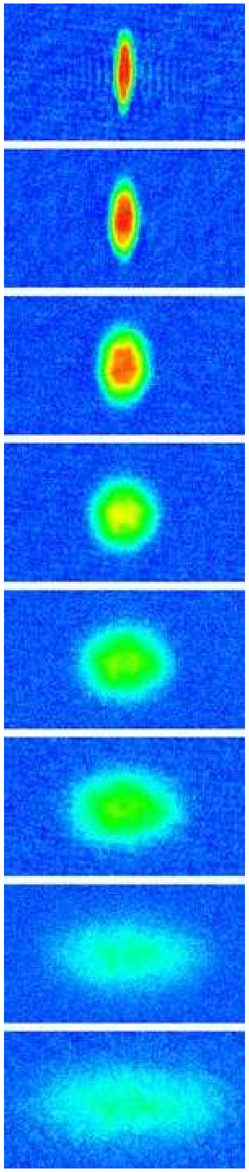
$\eta \gg \zeta$

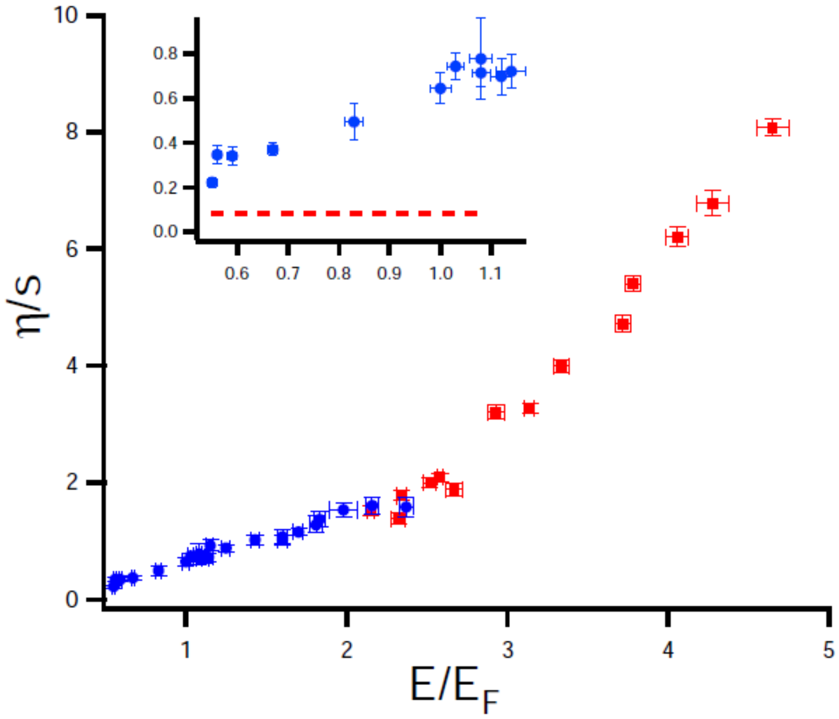
Dusling, Schaefer (2010)

Viscosity to entropy density ratio

consider both collective modes (low T)

and elliptic flow (high T)





Cao et al., Science (2010)

$$\eta/s \leq 0.4$$

Outlook

Experimental determination of transport properties: Collective modes and elliptic flow give $\langle \eta/s \rangle \lesssim 0.4$.

Local analysis requires second order hydro or hydro+kinetic. (I am working on this.)

Shear viscous relaxation time can be measured by comparing collective modes and elliptic flow.

Can we observe breaking of scale invariance and the return of bulk viscosity away from unitarity? Can we measure η and ζ_3 in the superfluid phase?