

# Fluid Dynamics and Viscosity in Strongly Correlated Fluids

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**Key Words** Non-equilibrium Physics, Kinetic Theory, Holographic Duality

**Abstract** We review the modern view of fluid dynamics as an effective low energy, long wavelength theory of many body systems at finite temperature. We introduce the notion of a nearly perfect fluid, defined by a ratio  $\eta/s$  of shear viscosity to entropy density of order  $\hbar/k_B$  or less. Nearly perfect fluids exhibit hydrodynamic behavior at all distances down to the microscopic length scale of the fluid. We summarize arguments that suggest that there is fundamental limit to fluidity, and review the current experimental situation with regard to measurements of  $\eta/s$  in strongly coupled quantum fluids.

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*παντα ρει (everything flows)*

Heraclitus

*The mountains flowed before the Lord.*

Prophet Deborah, Judges, 5:5

## 1 Fluid Dynamics

### 1.1 Fluid Dynamics as an effective theory

Fluid dynamics is often described as a consequence of applying Newton's laws to a continuous deformable medium. However, the ideas underlying fluid dynamics are much more general. Fluid dynamics describes classical and quantum liquids, gases, and plasmas. It accounts for the low energy properties of magnetic materials, liquid crystals, crystalline solids, supersolids, and many other systems. Indeed, fluid dynamics is now understood as an effective theory for the large distance, long time properties of any material (1,2). The only requirement for the applicability of fluid dynamics is that the system relaxes to approximate local thermodynamic equilibrium on the time scale of the observation. This is the idea captured by the two quotations above: In principle everything behaves as a fluid, but in some systems observing fluid dynamic behavior may require divine patience (3).

Fluid dynamics is based on the observation that there are two basic time scales associated with the behavior of a many body system. The first is a microscopic time scale  $\tau_{fluid}$  that characterizes the rate at which a generic disturbance relaxes. In a typical molecular liquid this rate is governed by the collision rate between molecules. The second time scale  $\tau_{coll}$  is associated with the relaxation of conserved charges. Since conserved charges cannot relax locally, but have to

decay by diffusion or collective motion, this time grows with the length scale  $\lambda$  of the disturbance,  $\tau_{coll} \sim \lambda$ . Fluid dynamics is based on the separation of scales  $\tau_{fluid} \ll \tau_{coll}$ , and  $\omega_{fluid} = \tau_{fluid}^{-1}$  can be viewed as the breakdown scale of fluid dynamics as an effective theory.

In a simple non-relativistic fluid the conserved charges are the mass density  $\rho$ , the momentum density  $\vec{\pi}$ , and the energy density  $\mathcal{E}$ . The momentum density can be used to define the fluid velocity,  $\vec{u} = \vec{\pi}/\rho$ . By Gallilean invariance the energy density can then be written as the the sum of the internal energy density and kinetic energy density,  $\mathcal{E} = \mathcal{E}_0 + \frac{1}{2}\rho u^2$ . The conservation laws are

$$\frac{\partial \rho}{\partial t} = -\vec{\nabla} \cdot \vec{\pi}, \quad (1)$$

$$\frac{\partial \pi_i}{\partial t} = -\nabla_j \Pi_{ij}, \quad (2)$$

$$\frac{\partial \mathcal{E}}{\partial t} = -\vec{\nabla} \cdot \vec{j}^\epsilon. \quad (3)$$

In order for these equations close we have to specify constitutive relations for the stress tensor  $\Pi_{ij}$  and the energy current  $\vec{j}^\epsilon$ . Since fluid dynamics is an effective long wavelength theory we expect that the currents can be systematically expanded in gradients of the hydrodynamic variables  $\rho$ ,  $\vec{u}$  and  $\mathcal{E}_0$ . In the case of the stress tensor the leading, no-derivative, terms are completely fixed rotational symmetry and Gallilean invariance. We have

$$\Pi_{ij} = \rho u_i u_j + P \delta_{ij} + \delta \Pi_{ij}, \quad (4)$$

where  $P = P(\rho, \mathcal{E}_0)$  is the equation of state and  $\delta \Pi_{ij}$  contains gradient terms. The approximation  $\delta \Pi_{ij} = 0$  is called ideal fluid dynamics. Ideal fluid dynamics is time reversal invariant and the entropy is conserved. If gradient terms are included then time reversal invariance is broken and the entropy increases. We will refer to  $\delta \Pi_{ij}$  as the dissipative stresses. At first order in the gradient expansion  $\delta \Pi_{ij}$

can be written as  $\delta\Pi_{ij} = -\eta\sigma_{ij} - \zeta\delta_{ij}\langle\sigma\rangle$  with

$$\sigma_{ij} = \nabla_i u_j + \nabla_j u_i - \frac{2}{3}\delta_{ij}\langle\sigma\rangle, \quad \langle\sigma\rangle = \vec{\nabla} \cdot \vec{u}. \quad (5)$$

This expression contains two transport coefficients, the shear viscosity  $\eta$  and the bulk viscosity  $\zeta$ . The energy current is given by  $\vec{j}^\epsilon = \vec{u}w + \delta\vec{j}^\epsilon$ , where  $w = P + \mathcal{E}$  is the enthalpy. At leading order in the gradient expansion  $\delta j_i^\epsilon = u_j \delta\Pi_{ij} - \kappa \nabla_i T$ , where  $\kappa$  is the thermal conductivity. The second law of thermodynamics implies that  $\eta, \zeta$  and  $\kappa$  must be positive.

We can now establish the expansion parameter that controls the fluid dynamic description. We first note that the ideal stress tensor contains two terms, which are related to the pressure  $P$  and the inertial stress  $\rho u_i u_j$ . The relative importance of these two terms is governed by the Mach number  $Ma = v/c_s$ , where  $c_s^2 = (\partial P)/(\partial \rho)_{\bar{s}}$  is the speed of sound and  $\bar{s} = s/n$  is the entropy per particle. Flows with  $Ma \sim 1$  are called compressible, and flows with  $Ma \ll 1$  are incompressible. We are most interested in expanding systems, which are certainly compressible.

The validity of hydrodynamics requires that dissipative terms are small relative to ideal terms. We will focus on the role of shear viscosity, because it is the dominant source of dissipation in the systems considered here, and because both  $\zeta$  and  $\kappa$  can become zero in physically realizable limits. In particular,  $\zeta$  vanishes in a scale invariant fluid like the unitary gas, and  $\kappa$  vanishes in a relativistic fluid with zero baryon chemical potential like the pure gluon plasma. In the case  $Ma \sim 1$  the expansion parameter is

$$Re^{-1} = \frac{\eta \nabla u}{\rho u^2} = \frac{\eta}{\rho u L}, \quad (6)$$

where  $Re$  is called the Reynolds number and  $L$  is a characteristic length scale

of the flow. Before continuing we briefly comment on incompressible flows. The expansion parameter in this case is  $Ma^2/Re$ . Flows with  $Ma \ll 1$  and  $Re^{-1} \ll 1$  are nearly ideal, turbulent flows. The regime  $Ma^2/Re \ll 1$  but  $Re^{-1} \gtrsim 1$  is that of very viscous flow. Today interest in very viscous flow is often related to classical fluids in confined geometries. A typical example is the problem of bacterial swimming (4).

We note that  $Re^{-1}$  can be written as

$$Re^{-1} = \frac{\eta}{\hbar n} \times \frac{\hbar}{muL}, \quad (7)$$

where both factors are dimensionless. The first factor is solely a property of the fluid, and the second factor characterizes the flow. For a typical classical flow the second factor is much smaller than one, and the validity of fluid dynamics places no constraints on  $\eta/(\hbar n)$ . For the types of experiments that we will explore in Sects. 2 and 3 the second factor is of order one, and the applicability of fluid dynamics requires  $\eta \lesssim \hbar n$ . We note that in relativistic flows the inertial term is  $\Pi_{ij} = sT u_i u_j$ , and the analogous requirement is  $\eta \lesssim \hbar s/k_B$ . We will refer to fluids that satisfy this condition as nearly perfect fluids (5–8). We will show that nearly perfect fluids exhibit hydrodynamic behavior on remarkably short length and time scales, comparable to microscopic scales like the inverse temperature or the inverse Fermi wave vector. In the following, we will use units in which  $\hbar = k_B = 1$ .

The long wavelength expansion can be extended beyond the first order in gradients of the hydrodynamic variables. The classical higher order equations are known as Burnett and super-Burnett equations (9,10). Explicit forms of second order terms based on kinetic theory were derived by Grad in the non-relativistic case (11), and by Israel, Stewart and others for relativistic fluids (12). Histori-

cally, these theories have not been used very frequently. One reason is that the effects are not very large. In the case of the Navier-Stokes equation dissipative terms can exponentiate and alter the motion qualitatively, even if at any given time gradient corrections are small. A simple example is a collective oscillation of a fluid, see Sect. 2.2. Without viscosity the mode cannot decay, but if dissipation is present the motion is exponentially damped. Typically, second order terms do not exponentiate, and the gain in accuracy from including higher order terms is frequently offset by uncertainties in higher order transport coefficients or the need for additional boundary conditions.

The second reason is that the classical equations at second order are unstable to short wavelength perturbations. In relativistic fluid dynamics problems with acausality and instability already appear at the Navier-Stokes level. These difficulties are not fundamental: Fluid dynamics is an effective theory, and unstable or acausal modes occur outside the domain of validity of the theory. It is nevertheless desirable to construct schemes that have second or higher order accuracy and satisfy causality and stability requirements. A possible solution is to promote the dissipative currents to hydrodynamic variables and postulate a set of relaxation equations for these quantities. Consider the dissipative stress tensor and define  $\pi_{ij} = \delta\Pi_{ij}$ . The relaxation equation for  $\pi_{ij}$  is

$$\tau_R \dot{\pi}_{ij} = -\pi_{ij} - \eta \sigma_{ij} + \dots, \quad (8)$$

where  $\dots$  contains second order terms like  $(\nabla \cdot u)\sigma_{ij}$  and  $\sigma_{ik}\sigma_{kj}$ . To second order accuracy this equation is equivalent to  $\delta\Pi_{ij} = -\eta\sigma_{ij} + \tau_R\eta\dot{\sigma}_{ij} + \dots$ , which is part of the standard Burnett theory. Physically, equ. (8) describes the relaxation of the dissipative stresses to the Navier-Stokes form. The resulting equations are stable and causal, and the sensitivity to higher order gradients can be checked

by varying second order coefficients like  $\tau_R$  (13).

Equ. (8) was first proposed by Maxwell as a model for very viscous fluids. Cattaneo observed that relaxation equations can be used to restore causality and studied a relaxation model in the context of Fourier's law  $\delta\vec{j}^\epsilon = -\kappa\vec{\nabla}T$  (14,15). Relaxation equations were derived from kinetic theory by Müller (16), Israel and Stewart (17), and others. To achieve the expected scaling of second order terms with  $Re^{-2}$  it is important to include a full set of second order terms that respect the symmetries of the theory. This problem was addressed for relativistic scale invariant fluids by Baier et al. (18), and in the non-relativistic case in (19).

It is well known that the low energy expansion in effective field theories is not a simple power series in the expansion parameter  $\omega/\Lambda$ . Quantum fluctuations lead to non-analytic terms. In the case of fluid dynamics  $\Lambda = \omega_{fluid}$  and non-analyticities arise due to thermal fluctuations of the hydrodynamic variables. As a consequence the dissipative currents  $\delta\Pi_{ij}$  and  $\delta\vec{j}^\epsilon$  contain not only gradient terms but also stochastic contributions. The magnitude of the stochastic terms is determined by fluctuation-dissipation theorems. We have

$$\langle \Pi_{ij}(t, \vec{x}) \Pi_{kl}(t', \vec{x}') \rangle = 2\eta T \left( \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} - \frac{2}{3} \delta_{ij} \delta_{kl} \right) \delta(t - t') \delta(\vec{x} - \vec{x}'), \quad (9)$$

$$\langle j_i^\epsilon(t, \vec{x}) j_j^\epsilon(t', \vec{x}') \rangle = 2\kappa T^2 \delta_{ij} \delta(t - t') \delta(\vec{x} - \vec{x}'), \quad (10)$$

where  $\langle . \rangle$  denotes a thermal average and we have neglected bulk viscosity. A calculation of the response function in stochastic fluid dynamics shows that the hydrodynamic expansion contains non-analyticities that are small compared to the Navier-Stokes term, but bigger than second order terms (20,21). This implies that, strictly speaking, the second order theory is only consistent if stochastic terms are included. There are some studies of fluctuating fluid dynamics (22), but in particle and nuclear physics this problem has only recently attracted interest



(23).

## 1.2 Microscopic models of fluids: Kinetic Theory

Within fluid dynamics the equation of state and the transport coefficients are parameters that have to be extracted from experiment. If a more microscopic description of the fluid is available then we can compute these parameters in terms of more fundamental quantities. The simplest microscopic description of a fluid is kinetic theory. Kinetic theory is itself an effective theory that describes the long distance behavior of an underlying classical or quantum many-body system. It is applicable whenever there is a range of energies and momenta in which the excitations of the fluid are long-lived quasi-particles. Kinetic theory can be used to relate properties of these quasi-particles, their masses, lifetimes, and scattering cross sections, to the equation of state and the transport coefficients. Kinetic theory can also be used to extend the description of collective effects such as sound or macroscopic flow into the regime where fluid dynamics breaks down.

The basic object in kinetic theory is the quasi-particle distribution function  $f_p(\vec{x}, t)$ . Hydrodynamic variables can be written as integrals of  $f_p$  over  $d\Gamma = d^3p/(2\pi)^3$ . For example, the off-diagonal component of the stress tensor is given by

$$\Pi_{ij}(\vec{x}, t) = \int d\Gamma_p p_i v_j f_p(\vec{x}, t), \quad (i \neq j), \quad (11)$$

where  $\vec{v} = \vec{\nabla}_p E_p$  is quasi-particle velocity. Similar expressions exist for other conserved currents. The equation of motion for  $f_p(\vec{x}, t)$  is the Boltzmann equation

$$\left( \frac{\partial}{\partial t} + \vec{v} \cdot \vec{\nabla}_x + \vec{F} \cdot \vec{\nabla}_p \right) f_p(\vec{x}, t) = C[f_p], \quad (12)$$

where  $\vec{F} = -\vec{\nabla}_x E_p$  is the force and  $C[f_p]$  is the collision term. Taking moments of the Boltzmann equation we can derive the conservation laws (1-3). In order to

extract the constitutive relations we have to assume that the distribution function is close the equilibrium distribution  $f_p(\vec{x}) = f_p^0(\vec{x}) + \delta f_p(\vec{x})$ , and that spatial variation of  $f_p(x)$  is small. The equilibrium distribution can be expressed in terms of the conserved charges or, more conveniently, in terms of the corresponding intensive quantities  $\mu, T$  and  $\vec{u}$ . We find

$$f_p^0(\vec{x}) = \frac{1}{\exp [\beta (E_p - \vec{u} \cdot \vec{p} - \mu)] \pm 1}, \quad (13)$$

where  $\beta = 1/T$ , the  $\pm$  sign corresponds to fermions and bosons, respectively, and  $\beta, \mu, \vec{u}$  are functions of  $\vec{x}$  and  $t$ .

In order to identify the expansion parameter we have to understand the scales involved in the collision term. If  $\delta f_p \ll f_p^0$  we can use  $C[f_p^0] = 0$  to linearize the collision term. The linearized collision term is a hermitean, negative semi-definite, operator that can be expanded in terms of its eigenvalues and eigenvectors. We will call the inverse eigenvalues collision times. At long times we can approximate the collision term by the longest collision time  $\tau_0$  and write

$$C[f_p^0 + \delta f_p] \simeq -\frac{\delta f_p}{\tau_0}, \quad (14)$$

where we have used the fact that at late times  $\delta f_p$  will be dominated by its projection on the lowest eigenvector. Equ. (14) is called the BGK or relaxation time approximation (24). We can define a mean free path by  $l_{mfp} = \tau_0 \bar{v}$  where  $\bar{v} = \langle v^2 \rangle^{1/2}$ . The expansion parameter for the gradient expansion is given by the Knudsen number

$$Kn = \frac{l_{mfp}}{L} \quad (15)$$

where  $L \sim \nabla^{-1}$  as in equ. (6). The systematic determination of the constitutive equation via an expansion in  $Kn$  is called the Chapman-Enskog expansion (25).

We find, for example,

$$\eta = \frac{1}{3} n l_{mfp} \bar{p}, \quad (16)$$

and  $\tau_R = \tau_0 = \eta/P$  (19,25). In order to estimate the Reynolds number we can use  $Ma = u/c_s \sim 1$ . In kinetic theory we find  $c_s^2 = \frac{5}{9} \langle v^2 \rangle$  and  $Kn \sim Re^{-1}$ . The Knudsen expansion is equivalent to the Reynolds number expansion in fluid dynamics.

Fluid dynamics corresponds to the long time behavior of kinetic theory. It is also interesting to examine the short time behavior. Consider the response of the fluid to an external shear strain  $h_{xy}$  with frequency  $\omega$  and wave number  $k$ . The solution of the Boltzmann equation is of the form

$$\delta f_p(\omega, k) = \frac{1}{2T} \frac{-i\omega p_x v_y}{-i\omega + i\vec{v} \cdot \vec{k} + \tau_0^{-1}} f_p^0 h_{xy}. \quad (17)$$

This result can be used to compute the spectral function of correlators of conserved currents. For  $k = 0$  the term  $(-i\omega + \tau_0^{-1})$  in the denominator of equ. (17) leads to a Lorentzian shape of the spectral function, which is a signature of the presence of quasi-particles. The spectral function also provides information about the breakdown of kinetic theory for large  $\omega$  and  $k$ . There is no intrinsic scale in the Boltzmann equation other than the collision time  $\tau_0$  which sets the scale for the hydrodynamic expansion. The high energy scale is set by matching the Boltzmann equation to the equation of motion for a non-equilibrium Green function in quantum field theory (26). Instead of carrying this out explicitly, we can compare the kinetic spectral functions in equ. (17) to the spectral functions in quantum field theory, see Sect. 2.1. The result is that the breakdown scale is  $\omega_{micro} \sim T$ . This should be compared to the hydrodynamic scale  $\omega_{fluid} \sim \tau_0^{-1} \sim \eta/P$ . For a typical fluid these scales are well separated, but for a nearly perfect fluid the two scales are comparable. At least parametrically, in a nearly perfect fluid there is

no room for kinetic theory, that means there is no regime in which kinetic theory is more accurate than fluid dynamics.

The collision term is determined by the quasi-particle cross section  $\sigma$ , and a rough estimate of the mean free path is given by  $l_{mfp} = 1/(n\sigma)$ . Using equ. (16) we find  $\eta \sim \bar{p}/\sigma$ . This result has a number of interesting consequences:

- The viscosity of a dilute gas is independent of its density. The physical reason for this behavior is that viscosity is determined by the rate of momentum diffusion. The number of particles is proportional to  $n$ , but the mean free path scales as  $1/n$ . As a result, the diffusion rate is constant. Maxwell was so surprised by this result that he tested it by measuring the damping rate of a torsion pendulum in a sealed container as a function of the air pressure (27,28). He confirmed that  $\eta$  is not a function of  $P$  at fixed  $T$ . Of course, if the air is very dilute then  $l_{mfp} > L$  and the hydrodynamic description breaks down. In this limit, called the Knudsen regime, damping is proportional to pressure.
- The result  $\eta \sim 1/\sigma$  also implies that viscosity of a weakly coupled gas is very large. This is counter-intuitive because we think of viscosity as friction between fluid layers. Consider a fluid sheared between two parallel plates in the  $xz$  plane. The force per unit area is

$$\frac{F}{A} = \eta \nabla_y u_x . \quad (18)$$

We naively expect this force to grow with the strength of the interaction. Our intuition is shaped by very viscous fluids, for which viscosity is indeed determined by force chains and solid friction. This is not entirely inappropriate, because the word viscosity is derived from the Latin word for

mistletoe, viscum album.

### 1.3 Matching and Kubo relations

In the case of kinetic theory we can derive the equations of fluid dynamics from the underlying microscopic theory. In more complicated cases, for example if the short distance description is a strongly coupled field theory, this may not be possible. In this case we can rely on the fact that fluid dynamics is a general long distance effective theory, and compute the transport coefficients based on the idea of matching. Matching expresses the requirement that in the regime of validity of the effective theory, correlation functions must agree with correlators in the microscopic theory. Consider the retarded correlation function of the stress tensor

$$G_R^{xyxy}(\omega, \mathbf{k}) = -i \int dt \int d^3x e^{i\omega t - i\vec{k} \cdot \vec{x}} \Theta(t) \langle [\Pi^{xy}(t, \vec{x}), \Pi^{xy}(0, 0)] \rangle. \quad (19)$$

In linear response theory this function controls the stress induced by an external strain. In fluid dynamics  $\Pi_{xy} \simeq \rho u_x u_y$  and we can compute the correlation function from linearized hydrodynamics and fluctuation relations. We find

$$G_R^{xyxy}(\omega, k) = P - i\eta\omega + \tau_R\eta\omega^2 - \frac{\kappa_R}{2}k^2 + O(\omega^3, \omega k^2), \quad (20)$$

where  $\tau_R$  is the relaxation time defined in equ. (8) and  $\kappa_R$  is another second order transport coefficient (21). Equ. (20) implies the Kubo relation

$$\eta = - \lim_{\omega \rightarrow 0} \lim_{k \rightarrow 0} \frac{d}{d\omega} \text{Im} G_R^{xyxy}(\omega, \vec{k}). \quad (21)$$

This equation can be applied to field theory, based on equ. (19) and the microscopic definition of the stress tensor. This method is used to compute transport coefficients on the lattice, both in relativistic and non-relativistic field theories

(29–34). The difficulty with using Kubo’s formula is that imaginary time Monte Carlo simulations do not provide direct access to correlation functions for real frequencies. Measuring the shear viscosity requires analytic continuation of imaginary time data, which leads to uncertainties that are difficult to quantify. We note that some transport coefficients, like the parameter  $\kappa_R$  in equ. (20) can be measured directly from imaginary time data.

Equation (20) confirms that the expansion parameter of the hydrodynamic expansion is  $\omega/\omega_{fluid}$  with  $\omega_{fluid} \simeq \eta/P \simeq \tau_R^{-1}$ . Note that fluctuations introduce non-analytic terms at order  $\omega^{3/2}$  (20, 21). This is a breakdown of the gradient expansion, but not a breakdown of hydrodynamics. For example, at second order in the low energy expansion the  $\omega^{3/2}$  term is completely determined by  $\eta$  and  $P$ , and the relaxation time  $\tau_R$  can be extracted by matching  $G_R(\omega)$  to the low energy expansion in fluid dynamics.

#### 1.4 Microscopic models of fluids: Holography

Kinetic theory provides explicit theoretical realizations of weakly coupled fluids. Holographic dualities and the AdS/CFT correspondence have led to controlled realizations of strongly coupled fluids. The basic idea originated from the study of black holes. It had been known for some time that black holes have entropy, and that the process of a perturbed black hole settling down to a stationary configuration bears some resemblance to dissipative relaxation in fluids. Indeed, it was shown that one can assign a shear viscosity and electric conductivity to the “stretched horizon”, an imaginary surface that hovers just above the event horizon (35).

These ideas were made precise in the context of the AdS/CFT correspondence

(36), see the reviews (37–40). In the simplest case one considers a Schwarzschild black hole embedded in five dimensional Anti-de-Sitter ( $\text{AdS}_5$ ) space. The full spacetime has additional compact dimensions, which are required by string theory but play no role in our discussion. Black holes in  $\text{AdS}_5$  do not evaporate and the black hole is in thermal equilibrium. This means that the rate of Hawking radiation balances the amount of energy falling back into the black hole. Based on its causal structure we can view  $\text{AdS}_5$  as having a “boundary” which is four-dimensional Minkowski space. Matter on the boundary is in thermal equilibrium with the black hole spacetime.

The AdS/CFT correspondence asserts that the boundary is described by an ordinary quantum field theory, and that the correlation functions of this field theory have a dual description in terms of boundary correlation functions of a gravitational theory in the bulk. The correspondence is simplest if the boundary theory is strongly coupled and contains a large number  $N$  of degrees of freedom. In this case the bulk theory is just classical gravity. The partition function of the boundary QFT is

$$Z_{QFT}[J_i] = \exp(-S[\phi_i|_{\partial M} = J_i]) , \quad (22)$$

where  $J_i$  is a set of sources in the field theory,  $S$  is the gravitational action,  $\phi_i$  is a dual set of fields in the gravitational theory, and  $\partial M$  is the boundary of  $\text{AdS}_5$ . The fields  $\phi_i$  satisfy classical equations of motions subject to boundary conditions on  $\partial M$ .

The original construction involves a black hole in  $\text{AdS}_5$  and is dual to a relativistic fluid governed by a generalization of QCD known as  $\mathcal{N} = 4$  super Yang-Mills theory. This theory is considered in the limit of a large number of colors  $N_c$ . The gravitational theory is Einstein gravity with additional matter fields that are not

relevant here. The AdS<sub>5</sub> black hole metric is

$$ds^2 = \frac{(\pi T R)^2}{u} \left( -f(u) dt^2 + d\vec{x}^2 \right) + \frac{R^2}{4u^2 f(u)} du^2, \quad (23)$$

where  $\vec{x}, t$  are Minkowski space coordinates, and  $u$  is a “radial” coordinate where  $u = 1$  is the location of the black hole horizon and  $u = 0$  is the boundary.  $T$  is the temperature,  $R$  is the AdS radius, and  $f(u) = 1 - u^2$ . In the boundary theory the metric couples to the stress tensor  $\Pi_{\mu\nu}$ . Correlation functions of the stress tensor can be found by linearizing the bulk action around the AdS<sub>5</sub> solution,  $g_{AB} = g_{AB}^0 + \delta g_{AB}$ , where  $A, B = 1, \dots, 5$ . Small oscillations of the off-diagonal strain  $\delta g_x^y$  are particularly simple. We consider harmonic dependence on the Minkowski coordinates  $\delta g_x^y = \phi_p(u) e^{ikx - i\omega t}$ . Fluctuations are governed by the wave equation

$$\phi_p''(u) - \frac{1+u^2}{uf(u)} \phi_p'(u) + \frac{\omega^2 - k^2 f}{(2\pi T)^2 u f(u)^2} \phi_p(u) = 0. \quad (24)$$

This differential equation has two linearly independent solutions. The retarded correlation function corresponds to picking a solution which is purely infalling at the horizon (37). For small (or very large)  $\omega, k$  this solution can be found analytically (41, 42).  $G_R(\omega, k)$  is computed by inserting the solution into the Einstein-Hilbert action, and then computing the variation with respect to the boundary value of  $\delta g_x^y$ . The result is of the form given in equ. (20) with (18, 43)

$$P = \frac{sT}{4}, \quad \eta = \frac{s}{4\pi}, \quad \tau_R = \frac{2 - \log(2)}{2\pi T}. \quad (25)$$

Note that in the case of a relativistic fluid  $\eta$  is naturally expressed in units of the entropy density  $s$ , not the density  $n$ . This is because a relativistic fluid need not have a conserved particle number. As a rough comparison we can use the fact that for a weakly interacting relativistic gas  $s/n = 3.6$ . We observe that the



AdS/CFT correspondence describes a very good fluid. In particular,  $\eta/s < 1$  and  $\tau_R \sim T^{-1}$ . This is a remarkable result because the AdS/CFT correspondence has provided the first reliable theoretical description of a nearly perfect fluid.

There are many aspects of the strongly coupled fluid that can be studied using AdS/CFT:

- The spectral function  $\eta(\omega) = -\frac{1}{\omega} \text{Im} G_R(\omega)$  does not show evidence for quasi-particles (42, 44). Instead of a Lorentzian of width  $1/\tau_R$  one finds a smooth function that interpolates between the hydrodynamic limit  $\eta(0) = \eta$  and the high frequency limit  $\eta(\omega) \sim \omega^3$ . Because of non-renormalization theorems, the  $\omega \rightarrow \infty$  limit is given by the correlation function in free field theory.
- The relaxation time can be written as  $\tau_R = c\eta/P$  with  $c = (2 - \log(2))/2 \simeq 0.65$ . This can be compared to the Israel-Stewart result  $\tau_R = 1.5\eta/P$ . We observe that the relaxation time is very short, but in units of  $\eta/P$  it is only a factor 2.3 smaller than kinetic theory would predict. The AdS/CFT correspondence has also been used to compute other second order transport coefficients (18).
- The validity of the hydrodynamic expansion is controlled by the location of the poles of  $G_R(\omega)$  in the complex  $\omega$  plane. The hydrodynamic pole of the shear correlator is located at  $\omega \simeq iD_\eta k^2$ , where  $D_\eta = \eta/(sT)$  is the momentum diffusion constant. Non-hydrodynamic poles correspond to so-called quasi-normal modes of the linearized Einstein equations. These quasi-normal modes come in complex conjugate pairs and are located at a minimum distance of order  $T$  from the real axis (45). This confirms that the expansion parameter in a nearly perfect fluid is  $\omega/T$ .

- Using the AdS/CFT correspondence one can study the approach to equilibrium in great detail. For initial conditions that lead to Bjorken flows the approach to hydrodynamics is very rapid. After the quasi-normal modes are damped, on time scales on the order of  $(\tau T) \lesssim 1$ , the Navier-Stokes description is very accurate, even though non-equilibrium contributions to the pressure can be large (46, 47). This is sometimes referred to as rapid “hydrodynamization”.
- Heller et al. studied the large order behavior of the hydrodynamic expansion for a Bjorken-like flow. They find that the gradient expansion is an asymptotic series, and that the radius of convergence is zero (48). The coefficients of high order terms, and the leading singularity in the Borel plane, are governed by the lowest quasi-normal mode. We note that this phenomenon is unrelated to the non-analytic terms in the expansion mentioned above. The calculation is performed in the large  $N_c$  limit of the field theory, so that non-analytic terms in the gradient expansion are suppressed (49). Heller et al. speculate that the large order behavior is analogous to the factorial divergence of large orders of perturbation theory in quantum field theory.

Finally, we note that one can directly derive the equation of fluid dynamics by promoting the parameters that label the near horizon metric to hydrodynamic variables (50). Solving the resulting Einstein equations order-by-order in gradients provides an alternative derivation of the second order transport coefficients discussed above. This method provides a general connection between solutions of the Einstein equation and the Navier-Stokes equation, referred to as the fluid-gravity correspondence (51).

## 1.5 Viscosity bounds

The AdS/CFT correspondence provides an explicit, albeit somewhat theoretical, example of a nearly perfect fluid. This leads to two questions: Can nearly perfect fluids be realized in the laboratory, and is there a fundamental limit to fluidity? We will address the first question in Sects. 2 and 3 below. There are several arguments that the answer to the second question is affirmative. We summarize these arguments here:

*Uncertainty relation (5):* Kinetic theory predicts that  $\eta = \frac{1}{3}nl_{mpf}\bar{p}$ , and that low viscosity corresponds to a short mean free path. However, the uncertainty relation suggests that the product of mean free path and mean momentum cannot become arbitrarily small. Using  $l_{mpf}\bar{p} \gtrsim 1$  implies  $\eta/n \gtrsim 1/3$ . This argument was originally presented in the context of relativistic fluids. In these systems the inverse Reynolds number is given by  $\eta/(s\tau T)$ . Using the entropy per particle of a weakly interacting relativistic Bose gas,  $s/n = 3.6$ , we get  $\eta/s \gtrsim 0.09$ .

There are several issues with this argument. First, it is based on applying kinetic theory in a regime where there are no well-defined quasi-particles and the theory is not applicable. Second, there is no obvious reason why the entropy per particle cannot be much larger than the free-gas value (52). Finally, a bound on transport coefficients related to the uncertainty relation was first proposed Mott in connection with electric conductivity (53). A minimal conductivity implies that the metal-insulator transition must be continuous. However, this prediction is known to be false. Continuous metal-insulator transitions have been observed (54), and the physical mechanism of these transitions can be understood in terms of Anderson localization.

*Holographic dualities (6):* It was observed that the value  $\eta/s = 1/(4\pi)$  is

obtained in the strong coupling limit of a large class of holographic theories. These theories are characterized by the fact that the dual gravitational description involves the Einstein-Hilbert action (37,55,56). Kovtun, Son, and Starinets (KSS) conjectured that the strong coupling result is an absolute lower bound for the ratio  $\eta/s$  in all fluids,

$$\frac{\eta}{s} \geq \frac{1}{4\pi} \quad (26)$$

This idea is a significant step forward compared to the argument based on the uncertainty relation. The value  $1/(4\pi)$  is the result of a reliable calculation. Holographic dualities explain why the relevant quantity is  $\eta/s$ , and they account for the difference between momentum and charge diffusion. The diffusion constant goes to zero in the strong coupling limit (57–59), while the ratio  $\eta/s$  remains finite.

It was discovered, however, that there are holographic theories that provide counterexamples to the KSS conjecture (60,61). Finite coupling corrections increase the ratio  $\eta/s$ , but there are cases in which calculable finite  $N_c$  corrections lower  $\eta/s$ . In terms of the dual description these theories correspond to gravitational theories that contain a certain higher derivative correction to the Einstein-Hilbert action known as the Gauss-Bonnet term (62). While this result rules out the KSS conjecture, there are compelling arguments for a weaker version of the viscosity bound. Given that the violation of the KSS bound can be related to the Gauss-Bonnet term one has to study constraints on the Gauss-Bonnet coefficient  $\lambda_{GB}$ . It was found that large values of  $\lambda_{GB}$  lead to violations of causality. For the class of theories that are known to violate the KSS bound causality implies the slightly weaker bound  $\eta/s \geq \frac{16}{25} \frac{1}{4\pi}$  (63). It seems likely that this is not the final word from holographic dualities. Generalizations of Gauss-Bonnet gravity,

so-called Lovelock theories, have been studied (64), and lower values of  $\eta/s$  may be possible.

*Fluctuations* (20, 21): Shear viscosity is related to momentum diffusion, and  $\eta/s = 0$  would imply that mean free path for momentum transport is zero. However, in fluid dynamics momentum can also be carried by collective modes such as sound and shear waves. Indeed, if the viscosity is small this process becomes more efficient because the damping rate of sound and shear modes is small. This suggests that the physical viscosity of the fluid cannot be zero.

This argument can be made more precise using the low energy expansion of hydrodynamic correlation functions. Fluctuations not only contribute to non-analytic terms in  $G_R(\omega)$ , but they also correct the polynomial terms that determine the transport coefficients. The retarded shear stress correlator in a relativistic fluid is of the form  $G_R(\omega) = P + \delta P + i\omega(\eta + \delta\eta) + \dots$  where  $\delta P$  is a correction to the pressure and

$$\eta + \delta\eta = \eta + \frac{17}{120\pi^2} \frac{\Lambda_K D_\eta s^2 T^3}{\eta^2}. \quad (27)$$

is the physical viscosity. Here,  $\Lambda_K$  is the breakdown momentum of the hydrodynamic description and  $D_\eta = \eta/(sT)$  is the momentum diffusion constant. The gradient expansion requires  $\Lambda_K D_\eta \lesssim 1$ . We observe that  $\delta\eta \sim 1/\eta^2$ , so the physical viscosity cannot become arbitrarily small. The bound for  $\eta/s$  depends on the equation of state. For a quark gluon plasma  $\eta/s \gtrsim 0.1$  (20), and in a non-relativistic Fermi gas  $\eta/s \gtrsim 0.2$  (21).

The bound is interesting, because it sheds some light on what is special about shear viscosity. The stress tensor is quadratic in the fluid velocity and has a leading order, non-linear coupling to shear waves. Other currents do not have non-linear mode couplings at leading order. The bound is not universal, but it is

complementary to the holographic bounds in the sense that it only operates at finite  $N$ , whereas the holographic bounds are rigorous at infinite  $N$ .

It is difficult to summarize the situation regarding the proposed viscosity bounds. There is strong evidence that viscosity is different from other transport coefficients. We can find systems for which bulk viscosity, conductivity, or diffusion constants vanish, but there are physical effects, the universality of the graviton coupling in holographic theories, and the universality of the stress tensor in stochastic fluid dynamics, that make it difficult to find scenarios in which the shear viscosity vanishes. The precise value of the bound is not known, but we will see that empirically the value  $\eta/s = 1/(4\pi)$  found in simple holographic theories is a good approximation for the viscosity of the best quantum fluids that can be studied in the laboratory.

## 2 Non-relativistic Fluids

### 2.1 The unitary Fermi gas

In the following two sections we describe theoretical and experimental results regarding the transport properties of the two best fluids that have been studied in the laboratory (7). These two fluids are ultracold atomic Fermi gases magnetically tuned to a Feshbach resonance, and the quark gluon plasma produced in relativistic heavy ion collisions at the relativistic heavy ion collider (RHIC) in Brookhaven, New York, and the large hadron collider (LHC) at CERN in Geneva, Switzerland.

Ultracold Fermi gases are composed of atoms with half-integer total spin. Experiments focus on alkali atoms such as  $^6\text{Li}$ . These atoms can be confined in all-optical or magneto-optical traps. We will concentrate on systems in which

two hyperfine states are macroscopically occupied. Because the density and temperature are very low details of the atomic interaction and the atomic structure are not resolved, and the atoms can be described as the two components of a point-like non-relativistic spin 1/2 fermion. The fermions are governed by the effective Lagrangian

$$\mathcal{L} = \psi^\dagger \left( i\partial_0 + \frac{\vec{\nabla}^2}{2m} \right) \psi - \frac{C_0}{2} (\psi^\dagger \psi)^2. \quad (28)$$

The coupling constants  $C_0$  is related to the  $s$ -wave scattering length  $a$ . At low temperature and density neither higher partial waves nor range corrections are important. The two-body  $s$ -wave scattering matrix is

$$\mathcal{M} = \frac{4\pi}{m} \frac{1}{1/a + iq}, \quad (29)$$

where  $q$  is the relative momentum. The precise relation between  $C_0$  and  $a$  depends on the regularization scheme. In dimensional regularization  $C_0 = 4\pi a/m$ . In the limit of weak coupling this result follows from the Born approximation.

Of particular interest is the “unitarity” limit  $a \rightarrow \infty$ . In this limit the system has no dimensionful parameters and the theory is scale invariant (65). The scattering amplitude behaves as  $1/(iq)$ , which saturates the  $s$ -wave unitarity bound. The two-body wave function scales as  $1/r$  and the many body system is strongly correlated even if the density is low. Experimentally, the unitarity limit can be studied using magnetically tuned Feshbach resonances (66,67).

We note that even at unitarity the dilute Fermi gas has well-defined quasi-particles if the temperature is large. The average scattering amplitude scales as  $\sigma \sim \langle q^{-2} \rangle \sim \lambda_{dB}^2$ , where  $\lambda_{dB} \sim (mT)^{-1/2}$  is the thermal wave length. In the high temperature limit the average cross section is small, and the collisional width of a fermion quasi-particle is  $\Gamma \sim zT$  (68), where  $z = (n\lambda^3)/2 \ll 1$  is the fugacity. In

this regime the shear viscosity can be computed using kinetic theory. The result is (69, 70)

$$\eta = \frac{15}{32\sqrt{\pi}}(mT)^{3/2}. \quad (30)$$

As expected, the viscosity is independent of density and grows with temperature. The ratio  $\eta/n$  scales as  $1/z$  and is parametrically large. We also find  $\eta/s \sim 1/(z \log(1/z))$ .

In the regime  $z \gtrsim 1$  the unitary gas is strongly coupled. At  $z \sim 12$  the system undergoes a phase transition to a superfluid (71). In the superfluid phase the  $U(1)$  symmetry of the effective Lagrangian equ. (28) is spontaneously broken, and at low temperature there is a well defined bosonic quasi-particle related to the  $U(1)$  Goldstone mode. Momentum diffusion due to Goldstone modes can be studied using kinetic theory, and we find  $\eta \sim T^{-5}$  (72). Combined with equ. (30) this result indicates that the viscosity has a minimum in the vicinity of the critical temperature. In this regime there are no reliable calculations of transport properties, but T-matrix calculations suggest that  $\eta/n$  reaches a value of about 0.5 (73). We note that at  $T_c$  the entropy per particle is very close to 1. Lower values of the shear viscosity,  $\eta/s \simeq 0.2$ , have been found in quantum Monte Carlo calculations (34).

In kinetic theory the viscosity spectral function has a Lorentzian line shape with width  $\tau_R^{-1} = P/\eta$  (74). In the strongly coupled regime the shape of the spectral function is not known, but one can determine the asymptotic behavior for  $\omega \rightarrow \infty$  as well as the frequency sum rule. The sum rule is given by (73, 75)

$$\frac{2}{\pi} \int d\omega \left[ \eta(\omega) - \frac{\mathcal{C}}{15\pi\sqrt{m\omega}} \right] = \frac{2}{3} \mathcal{E}, \quad (31)$$

where  $\mathcal{C}$  is a short distance coefficient known as the contact density that measures the strength of short range correlations (76), and the subtraction term inside the



integral corresponds to the high frequency tail of the spectral function (77). In the high temperature limit  $\mathcal{C} = 4\pi n^2 \lambda_{dB}^2$ , and one can check that the high frequency tail smoothly matches kinetic theory for  $\omega \sim T$ . We can now identify the relevant scales that limit the fluid dynamic and kinetic descriptions,  $\omega_{fluid} \sim zT$  and  $\omega_{micro} \sim T$ . For  $z \ll 1$  we find the expected hierarchy of scales, but in the strongly correlated regime both scales are comparable.

## 2.2 Flow and viscosity

Fluid dynamics can be observed in experiments that involve releasing the gas from a deformed trap. In typical experiments the trap corresponds to a harmonic confinement potential  $V = \frac{1}{2}m(\omega_{\perp}^2 x_{\perp}^2 + \omega_z^2 z^2)$  with an aspect ratio  $\omega_{\perp}/\omega_z \sim (20 - 30)$ . In hydrostatic equilibrium pressure gradients along the transverse direction are much bigger than pressure gradients along the longitudinal direction. Hydrodynamic evolution after the gas is released converts this difference into different expansion velocities, and during the late stages of the evolution the cloud is elongated along the transverse direction. The observation of this effect led to the discovery of nearly perfect fluidity in ultracold gases (78). Shear viscosity counteracts the differential acceleration and leads to a less deformed final state. The shear viscosity can be measured by studying the time evolution of the cloud radii (79,80).

An alternative approach is based on recapturing the gas after release from the trap. This excites a transverse breathing mode. Hydrodynamic behavior can be verified by measuring the frequency of the collective mode. In an ideal fluid  $\omega = \sqrt{10/3}\omega_{\perp}$ , whereas in a weakly collisional gas  $\omega = 2\omega_{\perp}$  (81,82). The transition from ballistic behavior in the weak coupling limit to hydrodynamic behavior in

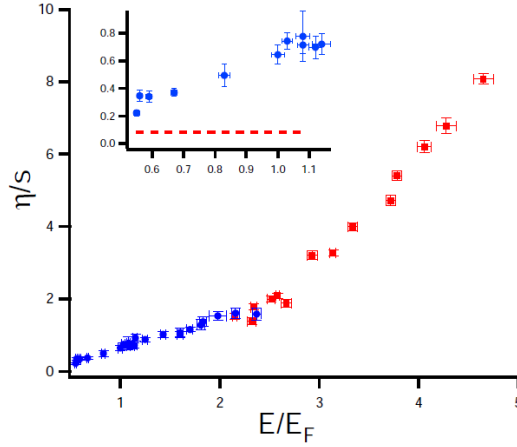


Figure 1: Measurements of  $\eta/s$  in the dilute Fermi gas at unitarity using collective modes (blue circles) and elliptic flow (red squares), from (80).

the unitary gas was observed experimentally in (83, 84). In the hydrodynamic regime damping of collective modes is governed by dissipative terms. The rate of energy dissipation is

$$\dot{E} = - \int d^3x \left\{ \frac{1}{2} \eta(x) (\sigma_{ij})^2 + \zeta(x) \langle \sigma \rangle^2 + \frac{\kappa(x)}{T} (\vec{\nabla} T)^2 \right\}. \quad (32)$$

At unitarity the system is scale invariant and the bulk viscosity is predicted to vanish (65, 85). This prediction was experimentally verified in (86). Thermal conductivity does not contribute to damping because the gas is isothermal. As a consequence the damping rate is a measure of shear viscosity.

Both the expansion and the collective mode experiments involve approximate scaling flows. The motion is analogous to the Hubble flows in cosmology, and to the Bjorken expansion of a quark gluon plasma (QGP). Consider the Euler equation for the acceleration of an ideal fluid. We have  $\dot{\vec{u}} \simeq -\vec{\nabla} P/\rho = -\vec{\nabla} \mu/m$ , where we have used the Gibbs-Duhem relation  $dP = nd\mu$ . Since the external potential is harmonic, the chemical potential is harmonic, too. As a consequence

the velocity field is linear, and the cloud expands in a self-similar fashion. Since the fluid velocity is linear the shear stress  $\sigma_{ij}$  is spatially constant and the rate of dissipation is sensitive to the spatial integral of  $\eta(x)$

$$\langle \eta \rangle = \int d^3x \eta(x). \quad (33)$$

Using measurements of the trap integrated entropy we can extract the ratio  $\langle \eta \rangle / \langle s \rangle$ . This analysis was originally performed in (87,88). A more recent analysis that combines collective mode data at low  $T$  with expansion data at high  $T$  is shown in Fig. 1 (80). The high temperature data matches expectations from kinetic theory. The viscosity drops with  $T$  and the ratio of trap averages reaches  $\langle \eta \rangle / \langle s \rangle \lesssim 0.4$ .

It is clearly desirable to unfold these measurements and determine local values of  $\eta/s$ . The main difficulty is a reliable treatment of the low density corona. In this regime  $\eta$  is independent of density and the integral in equ. (33) is ill defined. This signals the breakdown of fluid dynamics in the dilute region. The problem also appears if one applies the Navier-Stokes equation to an expanding gas cloud. In the dilute regime  $\eta$  is not a function of density and the viscous stresses  $\eta\sigma_{ij}$  are independent of position. This implies that while ideal stresses propagate with the speed of sound, viscous stresses propagate with infinite speed. As discussed in Sect. 1.1 this problem can be solved by including a finite relaxation time (79,89). In the low density regime the viscous relaxation time  $\tau_R \simeq \eta/(nT)$  is large. Since the dissipative stresses are zero initially, taking a finite relaxation time into account suppresses the contribution of the corona. A schematic version of this idea was used in Cao et al. (80), but a more systematic treatment is needed.

### 3 Relativistic Fluids

#### 3.1 The quark gluon plasma

The QGP is a hot and dense systems of quarks and gluons governed by the QCD Lagrangian

$$\mathcal{L} = -\frac{1}{4}G_{\mu\nu}^a G_{\mu\nu}^a + \sum_f \bar{q}_f (i\gamma^\mu D_\mu - m_f) q_f, \quad (34)$$

where  $G_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc}A_\mu^b A_\nu^c$  is the QCD field strength tensor,  $g$  is the coupling constant and  $f^{abc}$  are the  $SU(3)$  structure constants. The covariant derivative acting on the quark fields is  $iD_\mu q = (i\partial_\mu + gA_\mu^a \frac{\lambda^a}{2})q$  and  $m_f$  is the quark mass. At the temperature scale probed in RHIC or LHC experiments the three light flavors, up, down, and strange, are thermally populated, while the heavy flavors, mainly charm and bottom, are produced in hard collisions and can serve as probes of the medium.

Asymptotic freedom implies that at very high temperature the QGP can be described in terms of quark and gluon quasi-particles. A typical gluon has a thermal momentum of order  $T$ . Soft gluons with momenta much less than  $T$  are modified by the interaction with hard particles. As a consequence, electric gluons acquire a Debye screening mass  $m_D \sim gT$ . In perturbation theory there is no static screening of magnetic fields, but magnetic gluons are dynamically screened for momenta greater than  $(m_D^2 \omega)^{1/3}$ , where  $\omega$  is the frequency. The static magnetic sector of QCD is non-perturbative even if the temperature is very large. Confinement in three-dimensional pure gauge theory generates a mass scale of order  $g^2 T$ . This mass scale determines the magnetic screening scale in the QGP,  $m_M \sim g^2 T$ .

Perturbation theory in the quark gluon plasma is based on the separation of

scales  $m_M \ll m_D \ll T$ . Strict perturbation theory in  $g$  only works for very small values of the coupling constant,  $g \lesssim 1$  (90). On the other hand, quasi-particle models that rely on the separation of scales describe the thermodynamics of the plasma quite well, even for temperatures close to the phase transition to a hadronic gas (91).

The dispersion relation for the bosonic modes in the plasma evolves smoothly from quasi-gluons with masses  $m \sim m_D$  at momenta  $q \gtrsim gT$  to collective oscillations, plasmons, at low  $q$ . The energy of the plasmon in the limit  $q \rightarrow 0$  is  $\omega_P = m_D/\sqrt{3}$ , and the plasmon width is  $\Gamma \sim g^2 T$  (92). The calculation of the collisional width of quasi-particles with momenta of order  $T$  is a complicated, non-perturbative problem, but the width remains parametrically small,  $\Gamma \sim g^2 \log(1/g) T$  (93).

Momentum diffusion is controlled by binary scattering between quarks and gluons. The cross section is proportional to  $g^4$ , and the IR divergence due to the exchange of massless gluons is regulated by dynamic screening. As a consequence the shear viscosity scales as  $\eta \sim T^3/(g^4 \log(1/g))$ . A detailed calculation in  $N_f = 3$  QCD gives (94–96)

$$\eta = \frac{kT^3}{g^4 \log(\mu^*/m_D)}, \quad (35)$$

where  $k = 106.67$ . The scale inside the logarithm is sensitive to bremsstrahlung processes like  $gg \rightarrow ggg$ . Arnold et al. find  $\mu^* = 2.96T$  (95,96). The time scale for momentum diffusion is  $\eta/(sT) \sim 1/(g^4 \log(1/g)T)$ . This scale is parametrically large, but the precise value is very sensitive to the coupling constant. In  $N_f = 3$  QCD we get  $\eta/s \simeq 9.2/(g^4 \log(1/g))$ . Using  $g \simeq 2$ , which corresponds to  $\alpha_s \simeq 0.3$ , and  $\log(1/g) \gtrsim 1$  we conclude that  $\eta/s \lesssim 0.6$ .

At  $T \simeq 150$  MeV the quark gluon plasma undergoes a crossover transition to

a hadronic resonance gas (97,98). The resonance gas is strongly coupled, but as the temperature is reduced further the system evolves to a weakly coupled gas of mostly pions, kaons, and nucleons. The viscosity of a pion gas is parametrically large,  $\eta/s \sim (f_\pi/T)^4$ , where  $f_\pi \simeq 93$  MeV is the pion decay constant (99). Similar to the arguments in the case of cold Fermi gases we therefore expect that  $\eta/s$  has a minimum in the vicinity of  $T_c$ . In this regime the only reliable theoretical approach is lattice gauge theory. As in the case of non-relativistic fermions the calculations are difficult because one has to extract the viscosity spectral function from imaginary time data. In the case of pure gauge theory Meyer finds  $\eta/s = 0.102(56)$  at  $T = 1.24T_c$  and  $\eta/s = 34(33)$  at  $T = 1.65T_c$  (30).

Useful constraints on the spectral function are provided by sum rules. Romatschke and Son showed that (100)

$$\frac{2}{\pi} \int d\omega [\eta(\omega) - \eta_{T=0}(\omega)] = \frac{2}{5} \mathcal{E}, \quad (36)$$

where  $\eta_{T=0}(\omega)$  is the spectral function at zero temperature. The high frequency behavior can be studied in perturbation theory. We find  $\eta(\omega) \sim \omega^3$  at both zero and non-zero temperature. Finite temperature effect were studied in (101, 102). We note that in non-relativistic theories the tail of the spectral function is determined by short range correlations, whereas in a relativistic theory the high frequency behavior is determined by the  $gg$  and  $q\bar{q}$  continuum. In kinetic theory the shape of the spectral function at small frequency is a Lorentzian with a width proportional to  $1/\eta$ . The lattice calculation in (30) does not find a quasi-particle peak, but the resolution is insufficient to draw a final conclusions. A spectral function broadly consistent with the existence of quasi-particles was observed in a study of the electric conductivity of the quark gluon plasma (103).

### 3.2 Flow, higher moments of flow, and viscosity

Experimental information on transport properties of the quark gluon plasma comes from the observation of hydrodynamic flow in heavy ion collisions at collider energies (104,105). There are several observations that support the assumption that heavy ion collisions create a locally thermalized system:

- The overall abundances of produced particles is described by a simple thermal model that depends on only two parameters, the temperature  $T$  and baryon chemical potential  $\mu$  at freezeout (106,107).
- For transverse momenta  $p_\perp \lesssim 2$  GeV the spectra  $dN/d^3p$  of produced particles follow a modified Boltzmann distribution characterized by the freezeout temperature and a collective radial expansion velocity (104,108). Radial flow manifests itself in the fact that the spectra of heavy hadrons, which acquire a larger momentum boost from the collective expansion, have a larger apparent temperature than the spectra of light hadrons.
- In non-central collisions the azimuthal distribution of produced particles shows a strong anisotropy called elliptic flow (104,109). Elliptic flow represents the collective response of the quark gluon plasma to pressure gradients in the initial state, which in turn are related to the geometry of the overlap region of the colliding nuclei.

Analyzing the azimuthal distribution is the main tool for constraining the shear viscosity of the plasma. We define harmonics of the particle distribution

$$p_0 \frac{dN}{d^3p} \Big|_{p_z=0} = v_0(p_\perp) \left( 1 + 2v_1(p_\perp) \cos(\phi) + 2v_2(p_\perp) \cos(2\phi) + \dots \right), \quad (37)$$

where  $p_z$  is the longitudinal (beam) direction,  $p_\perp$  is the transverse momentum, and  $\phi$  is the angle relative to the impact parameter direction. The coefficient  $v_2$  is

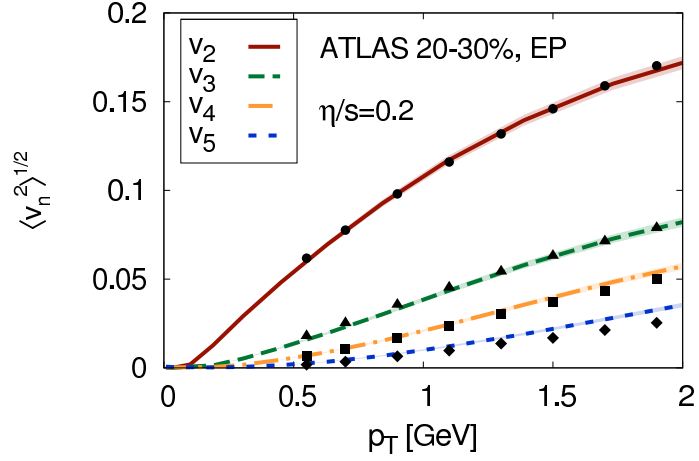


Figure 2: Fourier coefficients  $v_2, \dots, v_5$  of the azimuthal charged particle distribution as a function of the transverse momentum  $p_\perp$  measured in  $Pb + Pb$  collisions at the LHC (114). The lines show a hydrodynamic analysis using  $\eta/s = 0.2$  (115). called elliptic flow, and the higher moments are known as triangular, quadrupolar, etc. flow. Substantial elliptic flow, reaching about  $v_2(p_\perp = 2 \text{ GeV}) \simeq 20\%$  in semi-central collisions, was discovered in the early RHIC data (110,111) and confirmed at the LHC (112). More recently, it was realized that fluctuations in the initial energy density generates substantial higher harmonics, including odd Fourier moments like  $v_3$  (113).

Viscosity tends to equalize the radial flow velocity and suppress elliptic flow and higher flow harmonics. An estimate of the relevant scales can be obtained from simple scaling solutions of fluid dynamics. The simplest solution of this type was proposed by Bjorken, who considered a purely longitudinal expansion (116). Bjorken assumed that the initial entropy density is independent of rapidity, and that the subsequent evolution is invariant under boosts along the  $z$  axis. The Bjorken solution provides a natural starting point for more detailed numerical and analytical studies (104,117). Bjorken flow is characterized by a flow profile



of the form  $u_\mu = \gamma(1, 0, 0, u_z) = (t/\tau, 0, 0, z/\tau)$ , where  $\gamma = (1 - u_z^2)^{1/2}$  is the boost factor and  $\tau = (t^2 - z^2)^{1/2}$  is the proper time. This velocity field solves the relativistic Navier-Stokes equation. Energy conservation then determines the evolution of the entropy density. We find

$$-\frac{\tau}{s} \frac{ds}{d\tau} = 1 - \frac{4}{3} \frac{\eta}{sT\tau}, \quad (38)$$

where we have neglected bulk viscosity. In ideal hydrodynamics  $s \sim T^3$  and  $T \sim 1/\tau^{1/3}$ . The validity of the gradient expansion requires that the viscous correction is small (5)

$$\frac{\eta}{s} \ll \frac{3}{4}(T\tau). \quad (39)$$

It is usually assumed that in the QGP  $\eta/s$  is approximately constant. For the Bjorken solution  $T\tau \sim \tau^{2/3}$  grows with time, and equ. (39) is most restrictive during the early stages of the evolution. Using an equilibration time  $\tau_0 = 1$  fm and an initial temperature  $T_0 = 300$  MeV gives  $\eta/s \lesssim 0.6$ . We conclude that fluid dynamics can only be applied to heavy ion collisions if the QGP behaves as a nearly perfect fluid.

At late time the expansion becomes three dimensional and  $T\tau$  is independent of time. The fluid is composed of hadronic resonances which have cross sections that reflect hadronic sizes and are approximately independent of energy. In that case  $\eta \sim T/\sigma$ . Using  $s \sim T^3$  and  $T \sim 1/\tau$  we find that the dissipative correction  $\eta/(sT\tau)$  grows with proper time as  $\tau^2$ . This shows that fluid dynamics also breaks down at late times. At RHIC and LHC energies the duration of the fluid dynamic phase is 5-10 fm/c, depending on collision energy and geometry. We note that in contrast to the situation in heavy ion collisions there is no freeze-out in the cold atomic gas experiments. At unitarity the mean cross section increases as the temperature drops, and the fluid parameter  $\eta/(nT\tau)$  is approximately constant

during the evolution.

In heavy ion collisions we can only observe the final distribution of hadrons. In principle one could imagine reconstructing azimuthal harmonics of the stress tensor from the measured particle distribution, but this would require very complete coverage and particle identification, and it has not been attempted. In any case, hadrons continue to interact after the fluid freezes out, and some rearrangement of momentum takes place. This means that we need a prescription for converting hydrodynamic variables to kinetic distribution functions. What is usually done is that on the freezeout surface the conserved densities in fluid dynamics are matched to kinetic theory (118).

In ideal fluid dynamics the distribution functions are Bose-Einstein or Fermi-Dirac distributions characterized by the local temperature and fluid velocity. Viscosity modifies the stress tensor, and via the matching to kinetic theory this modification changes the distribution functions  $f_p$ . The value of  $\eta/s$  only constrains the  $p_i v_j$  moment of the distribution function. The full distribution function can only be reconstructed if the collision term is specified. Using the BGK collision term a very simple formula for the leading correction  $\delta f_p$  is obtained

$$\delta f_p = \frac{1}{2T^3} \frac{\eta}{s} f_0 (1 \pm f_0) p_\alpha p_\beta \sigma^{\alpha\beta}, \quad (40)$$

where the  $\pm$  sign refers to Bose/Fermi distributions. This result is a reasonable approximation to more microscopic theories (95). The shift in the distribution function leads to a modification of the single particle spectrum. In the case of the Bjorken expansion and at large  $p_\perp$  we find

$$\frac{\delta(dN)}{dN_0} = \frac{1}{3\tau_f T_f} \frac{\eta}{s} \left( \frac{p_\perp}{T_f} \right)^2, \quad (41)$$

where  $dN_0$  is the number of particles produced in ideal fluid dynamics,  $\delta(dN)$  is the dissipative correction, and  $\tau_f$  is the freezeout time. In a system with strong longitudinal expansion viscous corrections tend to equalize the momentum flow by pushing particles to higher  $p_\perp$ . Since the single particle distribution enters into the denominator of  $v_2$  this effect tends to suppress  $v_2$  at large  $p_\perp$ . The effect from the numerator, dissipative corrections due to the  $\cos(2\phi)$  component of the radial flow, act in the same direction (119). What is important is that corrections to the spectrum are controlled by the same parameter  $\eta/(s\tau T)$  that governs the derivative expansion in fluid dynamics. This is a reflection of the fact that in the regime where kinetic theory can be matched to fluid dynamics we have  $Kn \sim Re^{-1}$ .

We obtain a number of simple predictions that are confirmed by experiment (120): Dissipative corrections grow with  $p_\perp$ , they are larger in small systems that freeze out earlier, and they are larger for higher harmonics which are more sensitive to gradients of the radial flow profile. Quantitative predictions that provide not just bounds on  $\eta/s$  but reliable measurements of transport properties of the plasma require a number of ingredients (121):

- An initial state model that incorporates the nuclear geometry and fluctuations in the initial energy deposition. The simplest possibility is a Monte-Carlo implementation of the Glauber model (122), but some calculations also include saturation effects, quantum fluctuations of the initial color field, and pre-equilibrium evolution of the initial field (115). Alternatively, one may try to describe the pre-equilibrium stage using kinetic theory (123,124) or the AdS/CFT correspondence (125). At the end of the initial stage the stress tensor is matched to fluid dynamics.

- Second order dissipative fluid dynamics in 2+1 (boost invariant) or 3+1 dimensions. Calculations must include checks to ensure insensitivity to poorly constrained second order transport coefficients and a realistic equation of state (EOS). A realistic EOS has to match lattice QCD results at high temperature, and a hadronic resonance gas below  $T_c$  (126). The resonance gas EOS must allow for chemical non-equilibrium effects below the chemical freezeout temperature  $T_{chem} \simeq T_c$ .
- Kinetic freezeout and a kinetic afterburner. At the kinetic freezeout temperature the fluid is converted to hadronic distribution functions. Ideally, these distribution functions are evolved further using a hadronic cascade (127, 128), but at a minimum one has to include feed-down from hadronic resonance decays.

First estimates of  $\eta/s$  from the RHIC data were obtained in (129–131). A more recent analysis of LHC data is shown in Fig. 2 (115). The authors find  $\eta/s \simeq 0.2$  at the LHC, and  $\eta/s \simeq 0.12$  from a similar analysis of RHIC data. Similar results were obtained by other authors. Song et al. report an average value of  $\eta/s \simeq (0.2 - 0.24)$  at the LHC and  $\eta/s \simeq 0.16$  at RHIC (132). Luzum and Ollitrault have tried to constrain the allowed range of  $\eta/s$ , obtaining  $0.07 \leq \eta/s \leq 0.43$  at RHIC (133). Given the complexity of the analysis, uncertainties are difficult to quantify. A survey of the main sources of error in the determination of  $\eta/s$  can be found in (134). It is interesting that the extracted values of  $\eta/s$  are lower at RHIC than they are at the LHC, as one would expect based on asymptotic freedom. We should emphasize, however, that given the uncertainties it is too early to conclude this with high confidence.

## 4 Frontiers

In absolute units the shear viscosity of the ultracold Fermi gas and the quark gluon plasma differ by more than 25 orders of magnitude (7). The approximate universality of  $\eta/s$  in strongly coupled fluids, and the near agreement with the value predicted by the AdS/CFT correspondence in the strong coupling limit of a large class of field theories is quite remarkable. Much work remains to be done in order to determine to what extent this observation can be made precise, and what it implies about the structure of strongly correlated quantum systems. In this outlook we can only give a very brief summary of some of these issues.

*Transport coefficients:* There is an ongoing effort to map out the full density and temperature dependence of  $\eta/s$  in both the ultracold gases and the quark gluon plasma, and to determine other transport coefficients, like the bulk viscosity and diffusion coefficients. There are a number of experimental puzzles that remain to be addressed. In the case of heavy ion collisions, nearly ideal flow is even more pervasive than one would expect. Strong flow is also observed in photons, electrons from heavy quark decays, and hadrons emitted in high-multiplicity  $p + Pb$  collisions at LHC energies, see (135) for a recent summary and original references. In the case of cold atomic gases we now have very accurate data for the dependence of  $\langle\eta\rangle$  on the total energy of the cloud (136). These data have not been unfolded. It was observed that the scaling of  $\langle\eta\rangle$  with the total energy is remarkably simple,  $\langle\eta\rangle/\langle n\rangle \sim aE + bE^3$  for all energies above the critical point, but the origin of this scaling behavior is not understood.

*Quasi-particles:* We would like to understand whether nearly perfect fluidity,  $\eta/s \sim 1/(4\pi)$ , necessarily implies the absence of quasi-particles, as is the case in the AdS/CFT correspondence. The most direct way to study this issue is to

determine the spectral function. Since the only local probe of the stress tensor is the graviton, this will likely require numerical studies. We are also interested in pushing weak coupling descriptions into the regime where the quasi-particle picture breaks down, for example by using the renormalization group.

*Viscosity bound:* It is not known whether there is a fundamental lower limit for  $\eta/s$ . Part of the issue may well be that we need to define more carefully what we mean by a fluid, and that we need to understand how these defining characteristics are reflected in microscopic theories. We would also like to know what kinds of theories have holographic duals, and what aspects of the field theory lead to the emergence of certain universal features, like the strong coupling limit  $\eta/s = 1/(4\pi)$ .

*Other strongly correlated fluids:* In addition to the two fluids discussed in this review there are a number of other systems that are potentially of interest. One interesting class is two dimensional fluids, for example the electron gas in graphene (137), or the so-called strange metal phase of the high  $T_c$  superconductors (138).

*Equilibration at strong and weak coupling:* Empirical evidence suggests that equilibration in heavy ion collisions takes place on a very short time scale,  $\tau_{eq} \sim 1$  fm. Rapid equilibration is natural in holographic theories (46), but it is difficult to make contact with asymptotic freedom and the well-established theory and phenomenology of parton distribution functions. Understanding equilibration in weak coupling is a complicated problem that involves many competing scales, and even establishing the parametric dependence of the equilibration time on  $\alpha_s$  is difficult, see (139) for a recent overview.

*Anomalous hydrodynamics:* There are a number of novel hydrodynamic effects that have been discovered in recent years. An example is the chiral magnetic effect. Topological charge fluctuations in the initial state of a heavy ion collision, combined with the magnetic field generated by the highly charged ions, can manifest themselves in electric charge fluctuations in the final state (140). This effect is now understood as part of a broader class of anomalous hydrodynamic effects (141). Anomalous transport coefficients were originally discovered in the context of holographic dualities in (142), and interpreted using general arguments based on fluid dynamics in (143).

Acknowledgments: This work was supported in parts by the US Department of Energy grant DE-FG02-03ER41260. We would like to acknowledge useful discussions with Peter Arnold, Dam Son, Derek Teaney, and John Thomas.

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