Extracting transport properties of the QGP from lattice simulations

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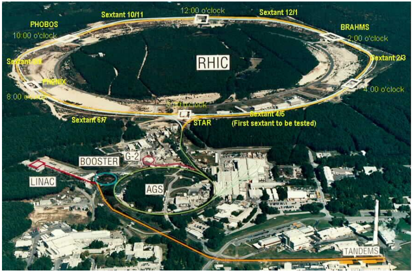
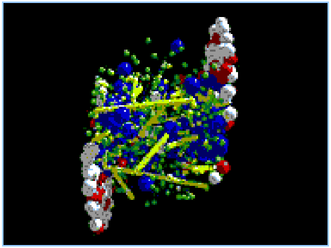
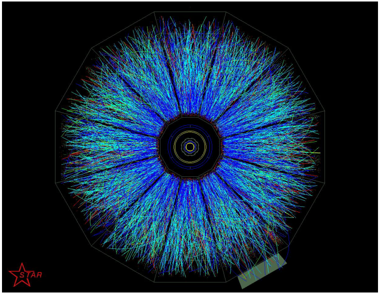
arXiv:0805.4567; arXiv:0806.3914

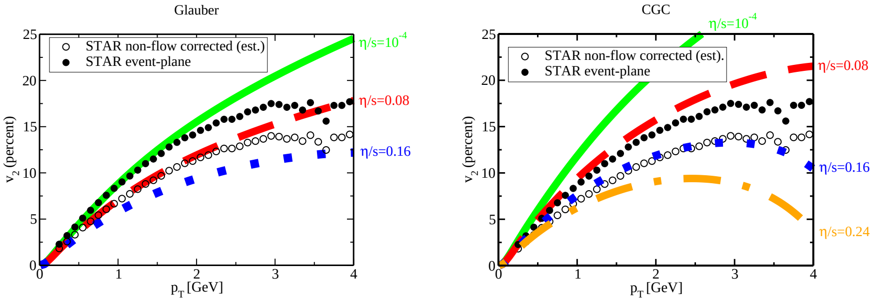
Heavy Ion collisions at RHIC







Hydrodynamics, ideal and viscous



Teaney '03; P. & U. Romatschke, '07; Luzum, Romatschke, '08;
Heinz, Song '07; Dusling, Teaney '07

anisotropic flow incompatible with $\eta/s \gtrsim 0.2$

Hydrodynamics

Macroscopic form of the energy-momentum tensor:

$$T^{\mu\nu} = -Pg^{\mu\nu} + (\epsilon + P)u^\mu u^\nu + \Delta T^{\mu\nu}$$

$$\Delta T^{\mu\nu} = \eta(\Delta^\mu u^\nu + \Delta^\nu u^\mu) + (\tfrac{2}{3}\eta - \zeta)H^{\mu\nu}\partial_\rho u^\rho$$

- u^μ is the velocity of energy-transport
- $\eta, \zeta =$ shear and bulk viscosities respectively
- $H^{\mu\nu} = u^\mu u^\nu - g^{\mu\nu}, \quad \Delta_\mu = \partial_\mu - u_\mu u^\beta \partial_\beta$

Kubo formula for the shear viscosity

- couple $T_{0i}(x)$ to a small external velocity field $v_i(x)$
- switch off the external field $\Rightarrow$ the system relaxes to equilibrium
- the Kubo formula is a matching condition between the linearized hydro. description and the linear response formalism of the QFT:

$$\boxed{\eta = -\lim_{\omega\to 0} \frac{1}{\omega}\,\mathrm{Im}\, D^{\mathrm{R}}_{12,12}(\omega,\mathbf{0}),\ D^{\mathrm{R}} \text{ the retarded correlator of } T_{12}}$$

Dispersion relation and analytic continuation: (Karsch, Wyld, '86)

- $D^{\mathrm{R}}_{12,12}(k) = -\int d\omega \frac{\rho(\omega)}{\omega - k_0 - i\epsilon}, \qquad \mathrm{Im}\, D^{\mathrm{R}}_{12,12}(k) = -\pi\rho(k)$
- $\mathcal{D}_{12,12}(\omega_n,\mathbf{k}) = \int d\omega \frac{\rho(\omega,\mathbf{k})}{\omega + i\omega_n}$ (the Euclidean correlator)

$$\boxed{\mathcal{D}_{12,12}(\tau,\mathbf{k}) = \int_0^\infty d\omega \frac{\cosh\omega(L_0/2-\tau)}{\sinh \omega L_0/2}\rho(\omega,\mathbf{k}), \qquad \eta = \pi \lim_{\omega\to 0} \rho(\omega,\mathbf{0})/\omega}$$

The QCD energy-momentum tensor

Separating the traceless part $\theta_{\mu\nu}$ from the trace part S for gluons, denoted 'g', and quarks, denoted 'f',

$$
\begin{aligned}
T_{\mu\nu} &\equiv \theta^{\mathrm{g}}_{\mu\nu} + \theta^{\mathrm{f}}_{\mu\nu} + \tfrac{1}{4}\delta_{\mu\nu}(\theta^{\mathrm{g}} + \theta^{\mathrm{f}}), \\
\theta^{\mathrm{g}}_{\mu\nu} &= \tfrac{1}{4}\delta_{\mu\nu} F^a_{\rho\sigma} F^a_{\rho\sigma} - F^a_{\mu\alpha} F^a_{\nu\alpha}, \\
\theta^{\mathrm{f}}_{\mu\nu} &= \tfrac{1}{4}\textstyle\sum_f \bar{\psi}_f \overleftrightarrow{D}_\mu \gamma_\nu \psi_f + \bar{\psi}_f \overleftrightarrow{D}_\nu \gamma_\mu \psi_f - \tfrac{1}{2}\delta_{\mu\nu} \bar{\psi}_f \overleftrightarrow{D}_\rho \gamma_\rho \psi_f, \\
\theta^{\mathrm{g}} &= \beta(g)/(2g)\; F^a_{\rho\sigma} F^a_{\rho\sigma}, \qquad \theta^{\mathrm{f}} = [1 + \gamma_m(g)] \textstyle\sum_f \bar{\psi}_f m \psi_f
\end{aligned}
$$

- $\overleftrightarrow{D}_\mu = \overrightarrow{D}_\mu - \overleftarrow{D}_\mu$
- $\beta(g)$ is the beta-function
- $\gamma_m(g)$ is the anomalous dimension of the mass operator
- all expressions are written in Euclidean space.

The energy-momentum tensor in Wilson lattice QCD

Wilson action:

$$S_{\rm g} = \frac{1}{g_0^2}\sum_{x,\mu,\nu} \mathrm{Tr}\,\{1 - P_{\mu\nu}(x)\},$$

$$S_{\rm f} = \sum_x \bar\psi(x)\psi(x) - \kappa\textstyle\sum_\mu \left[\bar\psi(x)U_\mu(x)(1-\gamma_\mu)\psi(x{+}a\hat\mu) + \bar\psi(x{+}a\hat\mu)U_\mu^\dagger(x)(1+\gamma_\mu)\psi(x)\right]$$

$\Rightarrow$ [see e.g. “Lattice sum rules”, HM, ’06]

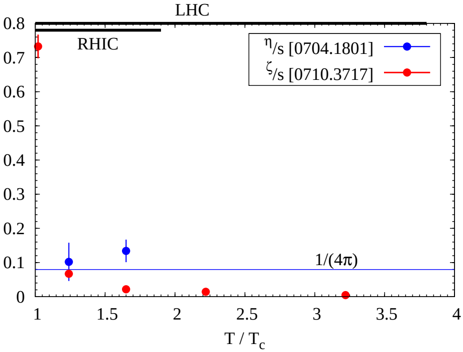
$$a^3\sum_{\mathbf{x}}\theta_{00}^{\rm g}(x_\odot) = \frac{2Z_{\rm g}(g_0)}{ag_0^2}\sum_{\mathbf{x}}\mathrm{Re\,Tr}\left[\textstyle\sum_k P_{0k}(x) - \sum_{k<l}\frac{1}{2}[P_{kl}(x)+P_{kl}(x+a\hat0)]\right]$$

$$a^3\sum_{\mathbf{x}}\theta_{00}^{\rm f}(x_\odot) = \frac{3}{4}Z_{\rm f}(g_0)\sum_{\mathbf{x},\mu}\bar\psi(x)U_0(x)(1-\gamma_0)\psi(x+a\hat0) + \bar\psi(x+a\hat0)(1+\gamma_0)U_0(x)^{-1}\psi(x)$$
$$-\frac{1}{3}(\bar\psi(x)U_k(x)(1-\gamma_k)\psi(x+a\hat k) + \bar\psi(x+a\hat k)U_k(x)^{-1}(1+\gamma_k)\psi(x))$$

$$a^3\sum_{\mathbf{x}}\theta^{\rm g}(x_\odot) = -\frac{2}{a}\frac{dg_0^{-2}}{d\log a}\mathrm{Re\,Tr}\left[\textstyle\sum_k P_{0k}(x) + \sum_{k<l}\frac{1}{2}[P_{kl}(x)+P_{kl}(x+a\hat0)]\right]$$

$$a^3\sum_{\mathbf{x}}\theta^{\rm f}(x_\odot) = \frac{d\kappa}{d\log a}\sum_{\mathbf{x},\mu}\bar\psi(x)U_\mu(x)(1-\gamma_\mu)\psi(x+a\hat\mu) + \bar\psi(x+a\hat\mu)U_\mu(x)^{-1}(1+\gamma_\mu)\psi(x)$$

- $Z_{\rm g}(g_0) = \frac{1}{2}\frac{\partial\log(\beta_\sigma/\beta_\tau)(a_\sigma,a_\tau)}{\partial\log a_\tau}$, $Z_{\rm f}(g_0) = \frac{\partial(\kappa_\sigma-\kappa_\tau)(a_\sigma,a_\tau)}{\partial\log a_\tau}$



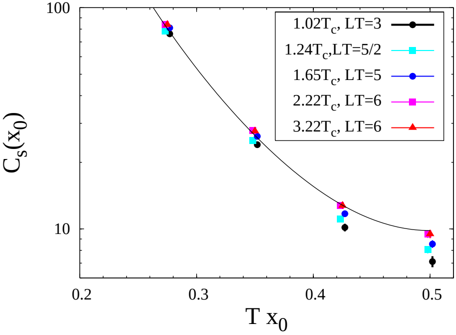
Perturbative and AdS/CFT calculations:

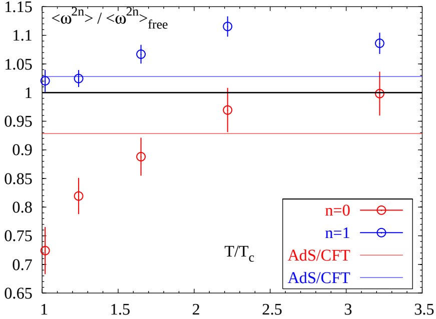
$$\eta/s,\ \zeta/s = \begin{cases} \dfrac{0.484}{\pi^2\alpha_s^2\log(0.608/\alpha_s)}, & \dfrac{1.25\alpha_s^2}{\pi^2\log(4.06/\alpha_s)} \qquad N_f = 0\ PT \\ 1/(4\pi), & 0 \qquad \mathcal{N} = 4\ SYM. \end{cases}$$

Arnold, Moore, Yaffe '03; Arnold, Dogan, Moore '06;

Policastro, Son, Starinets '01; Kovtun, Son, Starinets '04

Insensitivity of $\langle T_{12} T_{12} \rangle$ to interactions

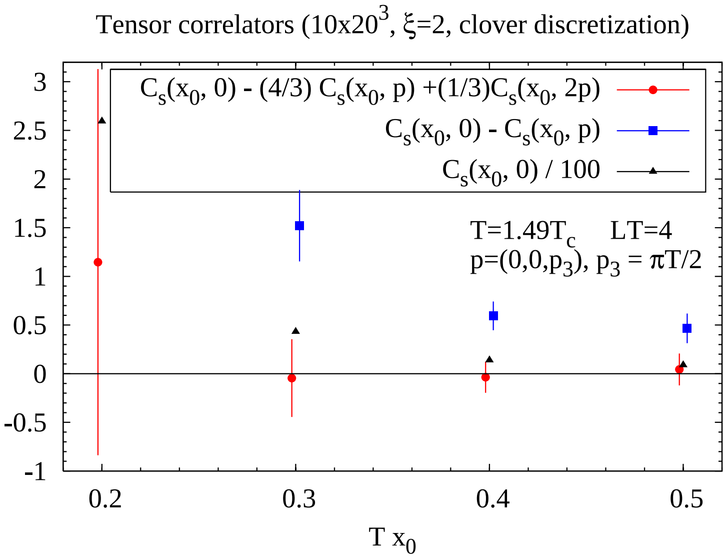




$$\langle \omega^{2n} \rangle \equiv \int_0^\infty d\omega \omega^{2n} \frac{\rho(\omega)}{\sinh \omega/2T} = T^5 \left. \frac{d^{2n}C}{dx_0^{2n}} \right|_{x_0=L_0/2}$$

- due to the large UV contribution, the Euclidean correlator is not very sensitive to interactions...

Attempt to subtract the UV contribution of $\langle T_{12} T_{12} \rangle$



Universal properties of spectral functions

Ward Identities: if $\mathbf{q} = (0, 0, q)$, $\partial_\mu T_{\mu\nu} = 0 \Rightarrow$

$$\rho_{00,00}(\omega, \mathbf{q}) = \frac{q^4}{\omega^4}\,\rho_{33,33}(\omega, \mathbf{q}) \quad \Rightarrow \quad \rho_{00,00}(\omega, \mathbf{q}) \overset{\omega\to\infty}{\sim} q^4$$

$$\rho_{01,01}(\omega, \mathbf{q}) = \frac{q^2}{\omega^2}\,\rho_{13,13}(\omega, \mathbf{q}) \quad \Rightarrow \quad \rho_{01,01}(\omega, \mathbf{q}) \overset{\omega\to\infty}{\sim} q^2\omega^2.$$

Hydrodynamics prediction at small $(\omega, \mathbf{q})$:

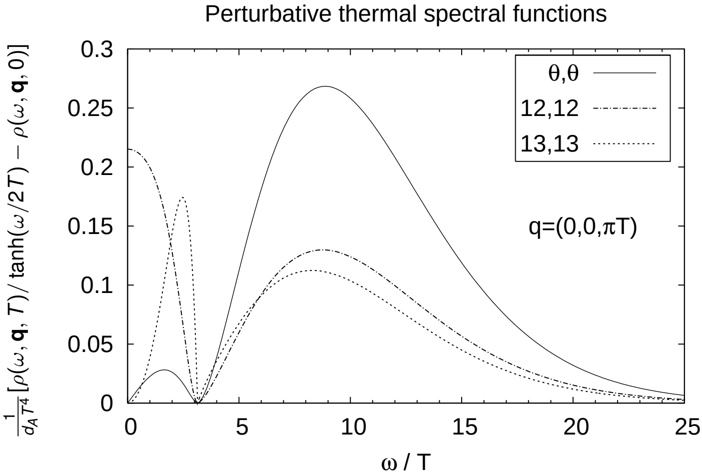
shear channel: $$\frac{\rho_{01,01}(\omega, \mathbf{q})}{\omega} \overset{\omega,q\to 0}{\sim} \frac{\eta}{\pi}\,\frac{q^2}{\omega^2 + (\eta q^2/(\epsilon + P))^2},$$

sound channel: $$\frac{\rho_{00,00}(\omega, \mathbf{q})}{\omega} \overset{\omega,q\to 0}{\sim} \frac{\Gamma_s}{\pi}\,\frac{(\epsilon + P)\,q^4}{(\omega^2 - v_s^2 q^2)^2 + (\Gamma_s \omega q^2)^2},$$

$$\Gamma_s = \frac{\frac{4}{3}\eta + \zeta}{\epsilon + P} = \frac{\frac{4}{3}\eta + \zeta}{Ts}$$

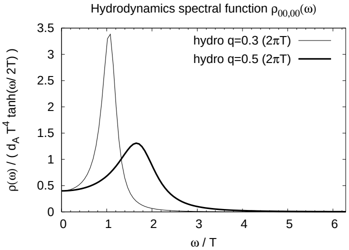
for a derivation see D. Teaney, 2006

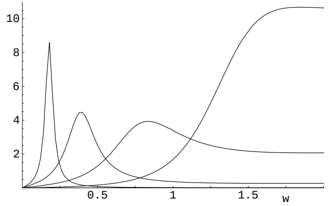
Spectral functions at finite q in free theory HM, 2008



$$\rho_{\theta,\theta}(\omega, q, T=0) = \left(\frac{11\alpha_s N_c}{6\pi}\right)^2 \frac{d_A\theta(\omega-q)}{4(4\pi)^2}(\omega^2 - q^2)^2$$

Sound channel spectral function $\rho_{00,00}(\omega, \mathbf{q})$





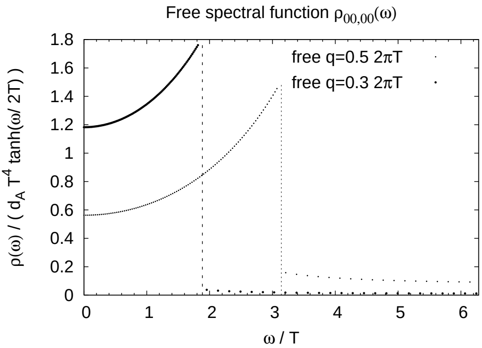
strongly coupled $\mathcal{N} = 4$ SYM

$\frac{2\rho}{\pi d_A T^4}$ for $q/(2\pi T) = 0.3, 0.6, 1.0, 1.5$

(Kovtun, Starinets 2006)

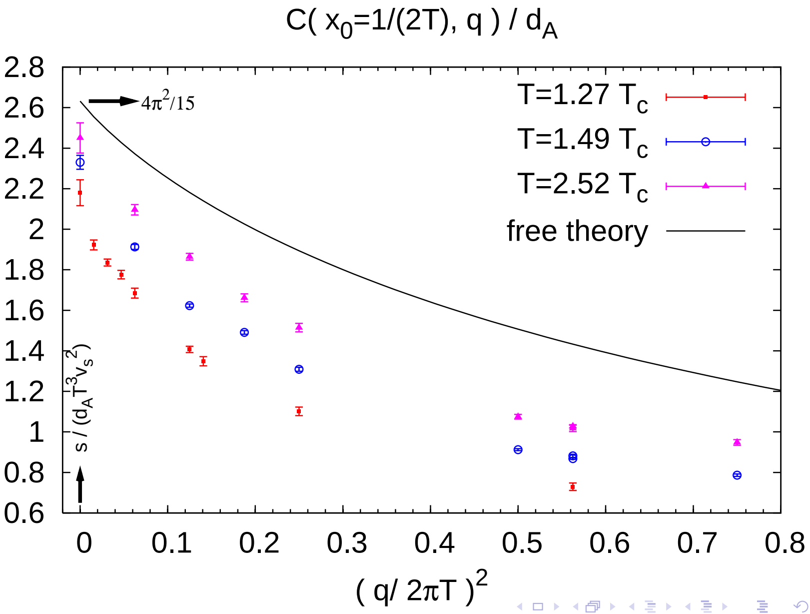
$\leftarrow$ Hydro. spectral function:

$v_s^2 = \frac{1}{3}$, $s = \frac{3}{4} s_{SB}$, $T\Gamma_s = \frac{1}{3\pi}$



SU(N_c) gauge theory (HM, 2008)

Energy density two-point function



Matrix elements of $L^{-3/2}\int d^3\mathbf{x}\, e^{i\mathbf{q}\cdot\mathbf{x}}\widehat{T}_{00}(\mathbf{x})$

- following [Pivovarov '99],

$$\int d^4x e^{iqx}\,\langle T_{\mu\nu}(x)T_{\rho\sigma}(0)\rangle = \eta_{\mu\nu,\rho\sigma}(q)T(q\cdot q) + f_{\mu\nu,\alpha\beta}(q)T_s(q\cdot q)$$

- $\Rightarrow \quad C_{00,00}(x_0,\mathbf{q}) = \mathbf{q}^2\int\frac{dq_0}{2\pi}e^{-iq_0x_0}[\frac{4}{3}T(q_0^2+\mathbf{q}^2) + T_S(q_0^2+\mathbf{q}^2)]$
- $\frac{\partial}{\partial x_0}\left(\frac{\partial}{\partial q} - \frac{4}{q}\right)C_{00,00}(x_0,q) = qx_0 C_{00,00}(x_0,q)$
- on the other hand, if $v_n(\mathbf{q}) \equiv \langle n,\mathbf{q}|L^{-3/2}\int d^3\mathbf{x}\, e^{i\mathbf{q}\cdot\mathbf{x}}\widehat{T}_{00}(\mathbf{x})|\Omega\rangle$,

$$\text{Källen-Lehmann} \;\Rightarrow\; C_{00,00}(x_0,\mathbf{q}) = \sum_n |v_n|^2(\mathbf{q})e^{-E_n(\mathbf{q})x_0}$$

- it follows $E_n^2(\mathbf{q}) = E_n^2(0) + \mathbf{q}^2$, $\boxed{|v_n|^2(\mathbf{q}) = \frac{E_n^2(0)a_n\mathbf{q}^4}{E_n(\mathbf{q})}.}$

$$\eta_{\mu\nu} = q_\mu q_\nu - q^2\delta_{\mu\nu},\;\; \eta_{\mu\nu,\rho\sigma} = \eta_{\mu\rho}\eta_{\nu,\rho} + \eta_{\mu\sigma}\eta_{\nu\rho} - \frac{2}{3}\eta_{\mu\nu}\eta_{\rho\sigma},\;\; f_{\mu\nu,\rho\sigma} = \eta_{\mu\nu}\eta_{\rho\sigma}.$$

Lesson from this exercise

All the $T = 0$ contributions vanish as $\sim \mathbf{q}^4$ when $\mathbf{q} \to 0 \quad \Rightarrow$

$$\boxed{\frac{d}{dq^2} C_{00,00}(x_0, \mathbf{q}) \neq 0 \text{ is a purely thermal effect.}}$$

But can it be attributed to the *sound peak alone* for sufficiently small **q** ?

- if it is the case, $\Gamma_s T$ can be extracted by a one-parameter fit, the others being known from thermodynamics.

Simple extraction of the sound attenuation length

Sound peak: $$\frac{\rho_{00,00}(\omega,\mathbf{q})}{\omega} = \frac{\epsilon+P}{\pi}\,\frac{\Gamma_s q^4}{(\omega^2-v_s^2q^2)^2+(\Gamma_s q^2\omega)^2}$$

Using this Ansatz, $$C(x_0=\frac{1}{2T},q)\approx\frac{(c_v/T^3)}{1+\left(\frac{\Gamma_s q}{2v_s}\right)^2}$$

So given a fit $C(\frac{1}{2T},q) = \mathrm{intercept} - \mathrm{slope}\frac{q^2}{T^2}$,

$$T^2\Gamma_s^2 = \frac{4v_s^4}{s/T^3}\cdot\ \mathrm{slope}.$$

In this way, I find at $1.27T_c$

$$T\Gamma_s \approx 0.24(3)$$

with no significant dependence on the fit range, as long as $q_{\rm max} \le \frac{3}{8}2\pi T$.

More sophisticated Ansatz

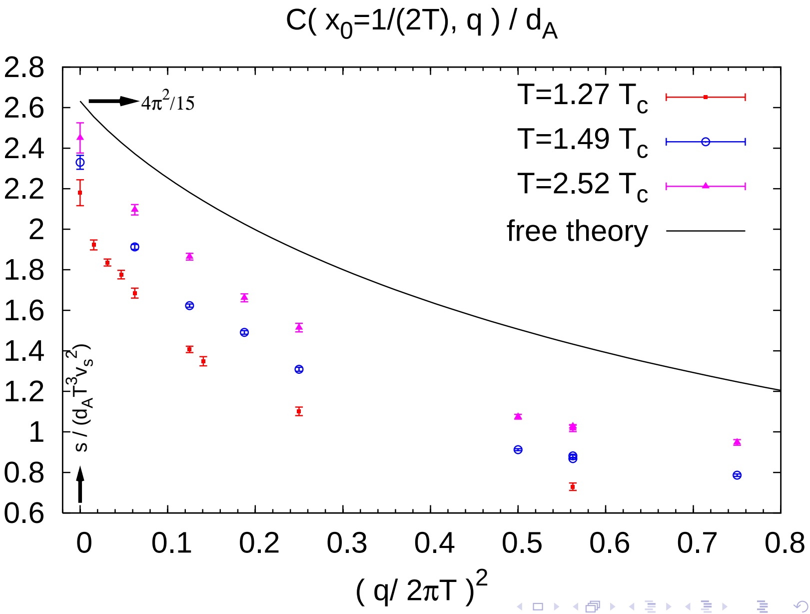
$$\widehat{\rho}(\omega, q, T) =$$

$$\frac{\frac{2}{\pi}\tanh(\frac{\omega}{2T})\,\Gamma_s\,(\epsilon+P)q^4}{(\omega^2-v_s^2q^2)^2+(\Gamma_s\omega q^2)^2}\cdot(1-\tanh^2\omega/2T)$$

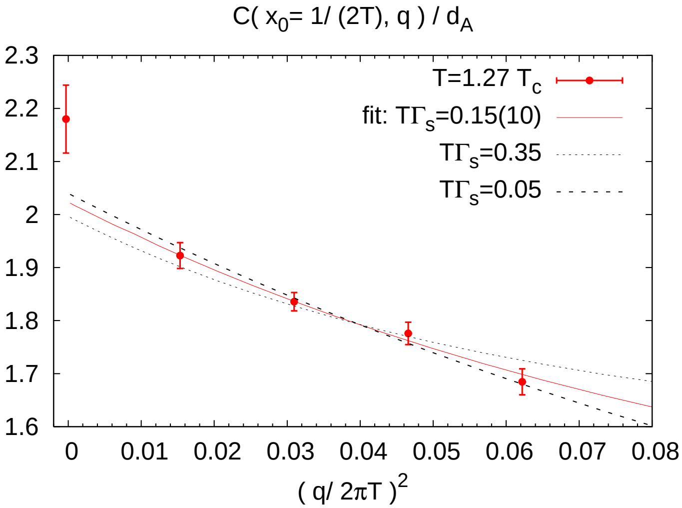
$$+\frac{2d_Aq^4}{15(4\pi)^2}\tanh\frac{\omega}{2T}\quad\cdot\tanh^2\omega/2T$$

- thermodynamics known $\Rightarrow$ only one free parameter, Γ_s (!)
- the $\tanh^2$ are arbitrary, but for small enough q, it will not matter
- caveat: the behavior for $\omega \approx \Lambda_{\overline{\rm MS}} \approx T$ is not known
- we *know* that the sound peak dominates the Euclidean correlator $C(x_0 \approx \frac{1}{2T}, q)$ for sufficiently small q
- it is *assumed* that $\frac{d}{dq^2}C(\frac{1}{2T}, q = 0)$ is also dominated by the sound peak rather than the $\omega \approx T$ region.
- it is at least true at weak coupling and in strongly coupled SYM.

Energy density two-point function



Fitting the small momenta ($T = 1.27T_c, 12 \times 48^3, \xi = 2$)



- used $v_s^2 = 0.263(18)$ [Boyd. et al '96, CPPACS '01], $\frac{s}{T^3} = 4.54(14)$

Lattice sum rules & the speed of sound v_s

Lattice sum rule: [HM, '07]

$$\langle a^4 \sum_x \theta(x)\theta(0)\rangle_{T-0} = T^5 \frac{\partial}{\partial T}\frac{\epsilon - 3P}{T^4} + u(g_0)(\epsilon - 3P).$$

$$u(g_0) \equiv \frac{d^2\beta}{d(\log a)^2}\left(\frac{d\beta}{d\log a}\right)^{-1} \sim 2b_1 g_0^4 + \mathrm{O}(g_0^6)$$

$$v_s^2 = \frac{\partial P}{\partial \epsilon} \quad \Leftrightarrow \quad \frac{s}{v_s^2} = \frac{\partial \epsilon}{\partial T} = c_v$$

Using the lattice sum rule, one easily obtains

$$\frac{1}{v_s^2} - 3 = (4 - u(g_0))\frac{\epsilon - 3P}{\epsilon + P} + \frac{\langle a^4 \sum_x \theta(x)\theta(0)\rangle_{T-0}}{\epsilon + P}.$$

Conclusions

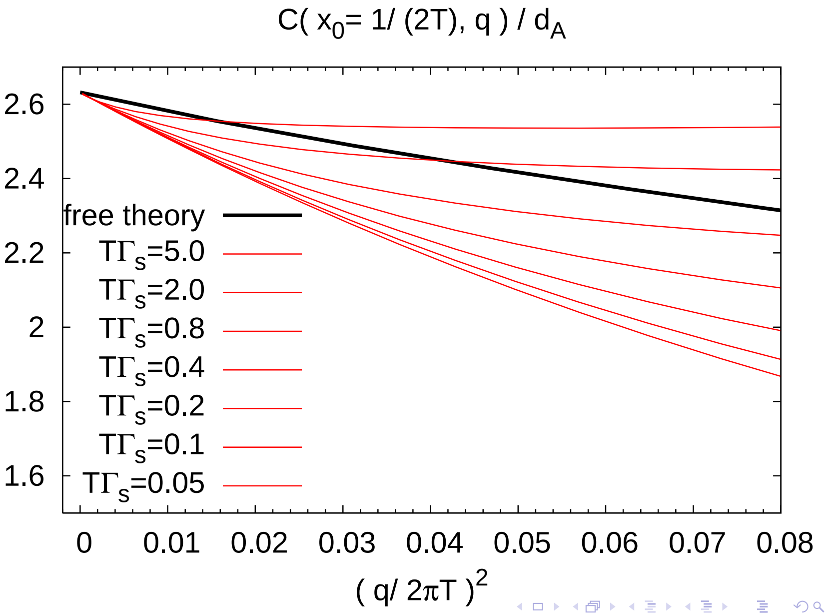
1. subpercent-level precision gluonic correlators have been achieved thanks to multi-level algorithm, variance-reduced discretizations of $T_{\mu\nu}$, treelevel improvement and anisotropic lattices

2. new method based on the energy density correlator at $0 < \mathbf{q} \ll 2\pi T$ (sound channel) $\Rightarrow$
 the functional form of ρ is to a large extent known *a priori*

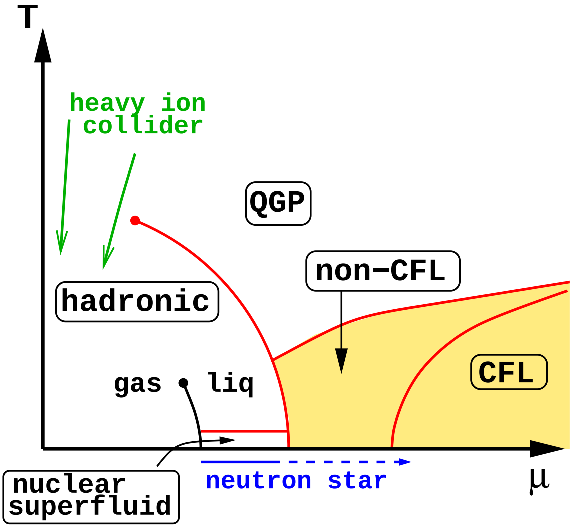
3. preliminary result: $\Gamma_s T = \frac{\frac{4}{3}\eta+\zeta}{s} = 0.15(10)$ at $1.3T_c$

4. shear channel: the contribution of
 $\rho_{01,01}(\omega, \mathbf{q})/\omega \approx \frac{1}{\pi}\frac{\eta q^2}{\omega^2+(\frac{\eta q^2}{\epsilon+P})^2}$ to $C_{01,01}(x_0, q)$
 is independent of η (unfortunately).

Sensitivity to Γ_s

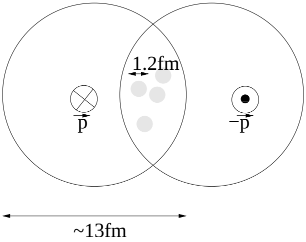


Phase diagram of QCD

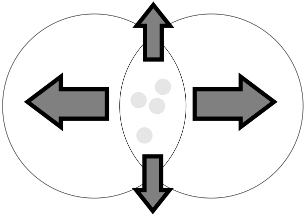


Alford, Schmitt, Rajagopal, Schäfer 0709.4635

Heavy ion collisions and elliptic flow

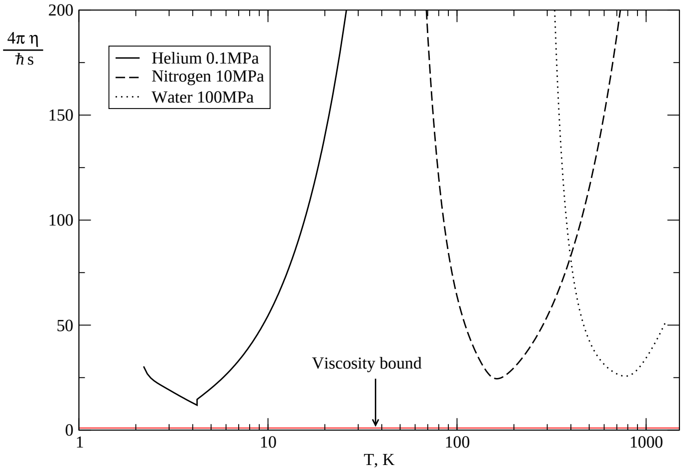


- transv. section of the collision of two gold nuclei
- impact vector $\vec{b}$ determined experimentally by ϕ distribution of particles



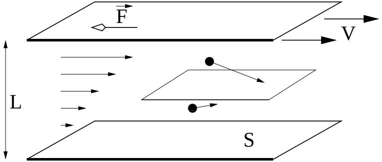
- pressure gradient is greater in the $(\vec{b}, \hat{z})$ plane
- $\Rightarrow$ excess of particles produced in that plane

Shear viscosity of ordinary substances



from Kovtun, Son, Starinets PRL 94:111601,2005

Shear viscosity of a dilute gas

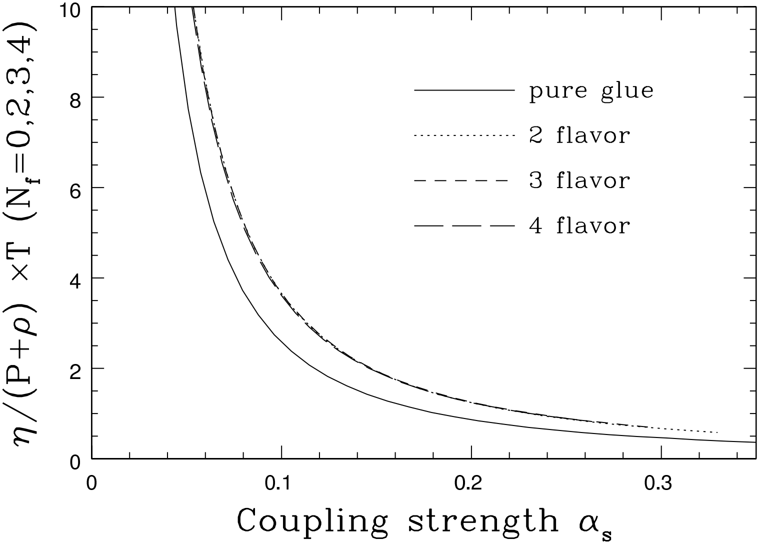


- $F/S = \eta V/L$
- the molecules flying through a horizontal slice have a longitudinal momentum characteristic of their surface of last scattering
- those moving $\downarrow$ have greater longit. momentum than those moving $\uparrow$
- $\Rightarrow$ transfer of longit. momentum from top to bottom

$$\boxed{\eta = \frac{1}{3}\frac{\bar{p}}{\sigma}}, \qquad \sigma = \text{cross-section}$$

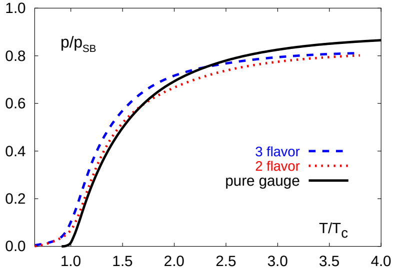
- η is independent of the density for a dilute gas
- bulk viscosity: $\zeta \propto$ density (requires 3-body collisions).

Pure gauge theory vs. full QCD in LO PT



G. Moore, SEWM 04

Thermodynamics

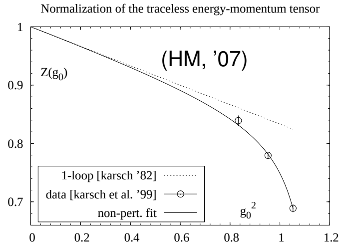


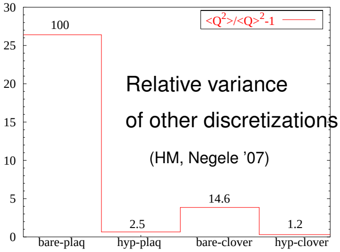
Karsch, *Hard Probes* '06

$$T_{00} = \theta_{00} + \tfrac{1}{4}\theta,$$

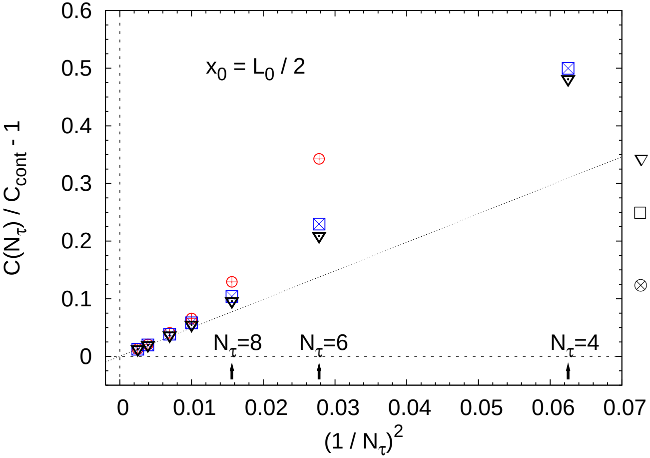
$$\begin{cases} \epsilon - 3P = \langle\, \theta \,\rangle_T - \langle\, \theta \,\rangle_0, \\ \epsilon + P = \tfrac{4}{3}\langle\, \theta_{00} \,\rangle_T \end{cases}$$

Renorm. factor, plaq. discretization





Cutoff effects in lattice perturbation theory



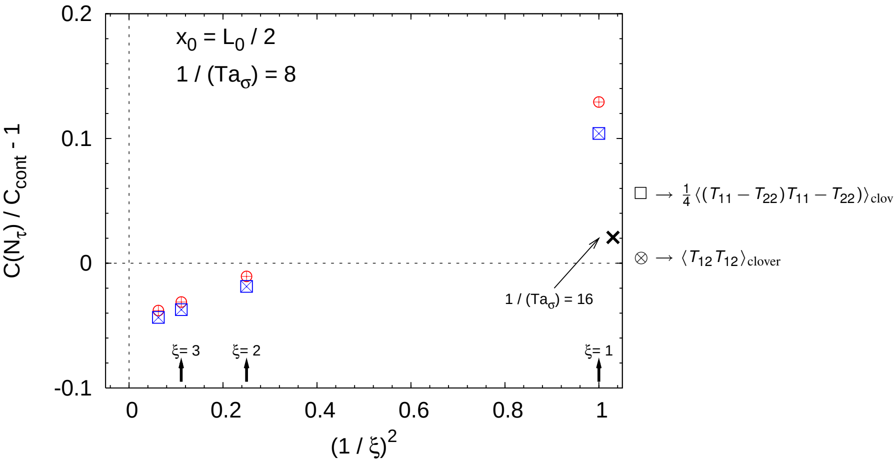
$\nabla \rightarrow \frac{1}{4}\langle (T_{11}-T_{22})T_{11}-T_{22})\rangle_{\rm plaq}$

$\Box \rightarrow \frac{1}{4}\langle (T_{11}-T_{22})T_{11}-T_{22})\rangle_{\rm clov}$

$\otimes \rightarrow \langle T_{12}T_{12}\rangle_{\rm clover}$

- clover & plaquette discretizations have comparable cutoff effects
- $N_\tau \geq 8$ is mandatory
- use these results to remove treelevel cutoff effects on non-pert. correlators.

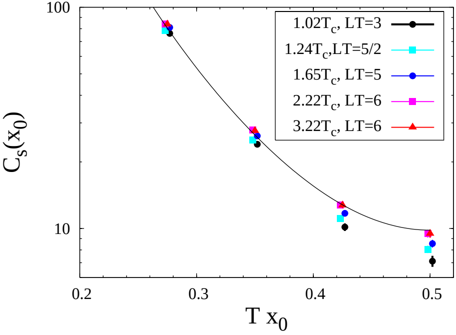
Perturbative cutoff effects: anisotropic lattice

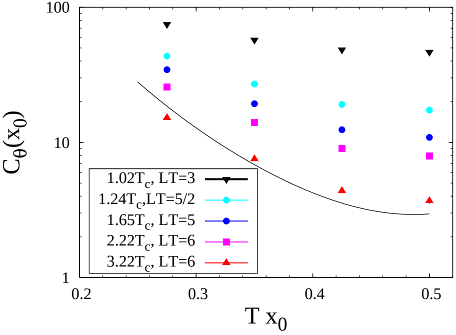


- for fixed L_0/a_σ, $\xi = 2$ or 3 is optimal
- ($N_\tau = 16$, $\xi = 2$) is as good as ($N_\tau = 16$, $\xi = 1$) and saves a factor 8 in the spatial volume.

The finite T Euclidean correlators ($N_\tau = 8,\ \xi = 1$)

$\langle T_{12} T_{12} \rangle_c$ $\langle \theta\ \theta \rangle_c$

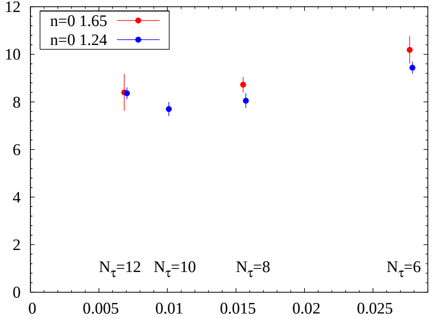




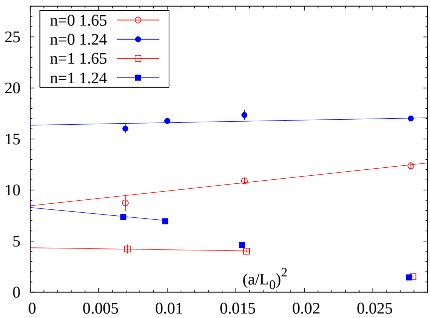
- near-conformal behaviour in one channel, large deviations in the other
- small errors thanks to multi-level algorithm (HM '03)

Taking the continuum limit

$\langle T_{12} T_{12} \rangle_c$



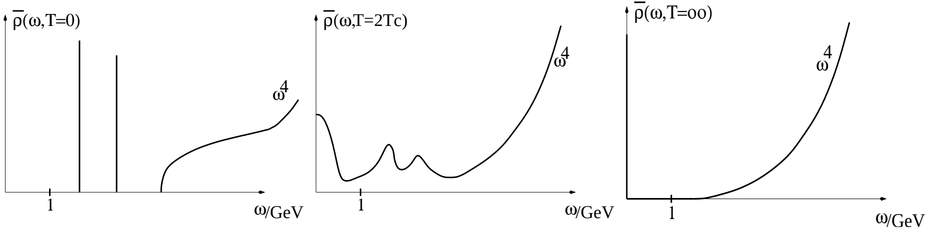
$\langle \theta\ \theta \rangle_c$



$$\langle \omega^{2n} \rangle \equiv \int_0^\infty d\omega \omega^{2n} \frac{\rho(\omega)}{\sinh \omega/2T} = T^5 \left. \frac{d^{2n}C}{dx_0^{2n}} \right|_{x_0=L_0/2}$$

- ... tree-level improvement works well.

Expected form of the spectral functions



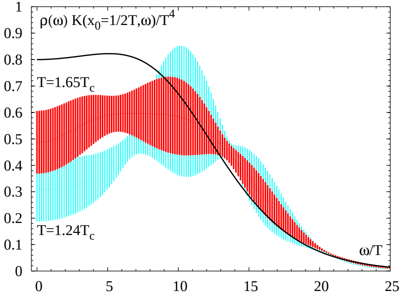
Free theory: $\boxed{\rho_{\mu\nu,\rho\sigma}(\omega) \leftrightarrow \langle T_{\mu\nu} T_{\rho\sigma} \rangle}$

- $\rho_{12,12}(\omega, T=0) = \frac{d_A}{10(4\pi)^2}\omega^4$
- $\rho_{\theta,\theta}(\omega, T=0) = \left(\frac{11\alpha_s N_c}{6\pi}\right)^2 \frac{d_A}{4(4\pi)^2}\omega^4$
- $\rho_{12,12}(\omega, T) = \frac{d_A}{10(4\pi)^2} \frac{\omega^4}{\tanh \frac{\omega}{4T}} + \left(\frac{2\pi}{15}\right)^2 d_A T^4\, \omega\delta(\omega)$
- $\rho_{\theta,\theta}(\omega, T) = \frac{d_A}{4(4\pi)^2} \left(\frac{11\alpha_s N_c}{6\pi}\right)^2 \frac{\omega^4}{\tanh \frac{\omega}{4T}}.$

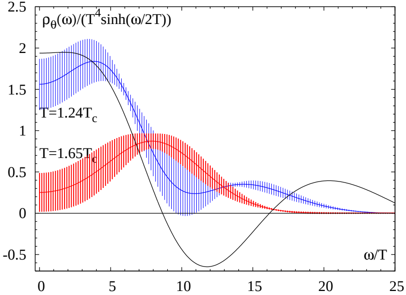
The reconstructed spectral functions (2007)

$$\widehat{\rho}(\omega) = m(\omega)[1 + \textstyle\sum_{\ell=1}^{N} c_\ell u_\ell(\omega)], \quad m(\omega \gg T) \text{ as predicted by PT}$$

$\langle T_{12} T_{12} \rangle_c \ \ (N_\tau = 8,\ \xi = 1)$

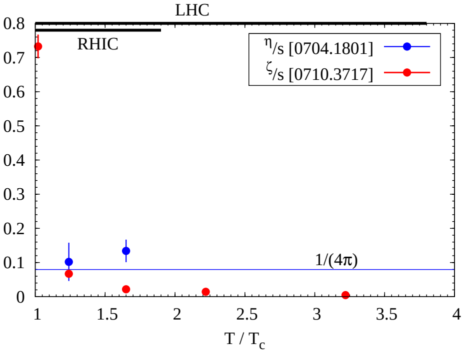


$\langle \theta\ \theta \rangle_c \ \ (N_\tau = 12,\ \xi = 1)$



- $\eta / T^3 = \frac{\pi}{2} \times$ *intercept*
- black curve = normalized $\mathcal{N} = 4$ SYM spectral function.
- $\zeta / T^3 = (\frac{\pi}{18} \times$ *intercept*$)$ increasing for $T \to T_c$
- black curve $= \hat{\delta}(0, \omega)$

2007 Results



Perturbative and AdS/CFT calculations:

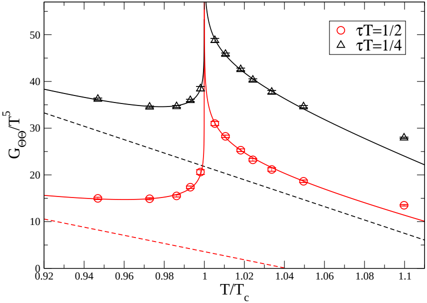
$$\eta/s,\ \zeta/s = \begin{cases} \dfrac{0.484}{\pi^2\alpha_s^2 \log(0.608/\alpha_s)}, & \dfrac{1.25\alpha_s^2}{\pi^2 \log(4.06/\alpha_s)} \qquad N_f = 0\ PT \\ 1/(4\pi), & 0 \qquad \mathcal{N} = 4\ SYM. \end{cases}$$

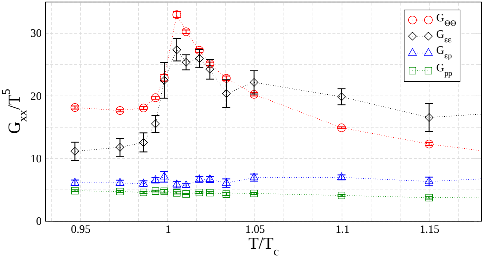
Arnold, Moore, Yaffe '03; Arnold, Dogan, Moore '06;

Policastro, Son, Starinets '01; Kovtun, Son, Starinets '04

The $\langle\theta\theta\rangle$ correlator at $T = T_c(1 \pm \epsilon)$ in SU(2)

Hübner, Karsch, Pica; see C. Pica's talk





- beautiful confirmation of the 3d Ising class critical behavior
- what happens to ζ at the phase transition is still an open question.

Backup slides

The “inverse problem”: solving for $\rho(\omega)$

$$\boxed{\text{Given } C_i = \int_0^\infty d\omega \rho(\omega) \frac{\cosh \omega L_0 \tau_i/2}{\sinh \omega L_0/2} \quad (i = 1, \ldots N), \text{ reconstruct } \rho(\omega).}$$

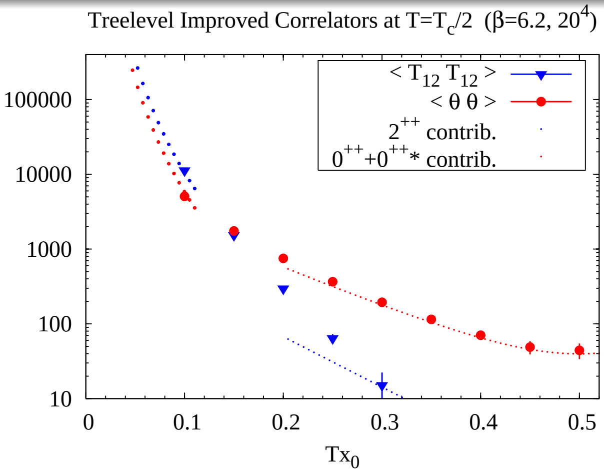
- our estimator for ρ: $\widehat{\rho}(\omega) = m(\omega)[1 + \widehat{a}(\omega)]$, $\widehat{a}(\omega) = \sum_{\ell=1}^{N} c_\ell u_\ell(\omega)$
- $m(\omega) > 0$; $m(\omega \gg T)$ has the correct perturbative behavior
- basis u_ℓ determined by singular-value decomposition of $M(x_0, \omega) \stackrel{\text{def}}{=} K(x_0, \omega) m(\omega)$: $\quad M^{\mathrm{t}} = U \mathrm{w} V^{\mathrm{t}}, \quad u_\ell(\omega) =$columns of U
- we are only able to reconstruct a ’fudged’ version of the genuine $a(\omega)$:

$$\boxed{\widehat{a}(\omega) = \int \widehat{\delta}(\omega, \omega') a(\omega') d\omega' \qquad \widehat{\delta}(\omega, \omega') = \sum_{\ell=1}^{N} u_\ell(\omega) u_\ell(\omega').}$$

- $\widehat{\delta}(\omega, \omega')$ is called the resolution function (how complete is the basis?)
- we would like the resolution function to resemble a delta-function.

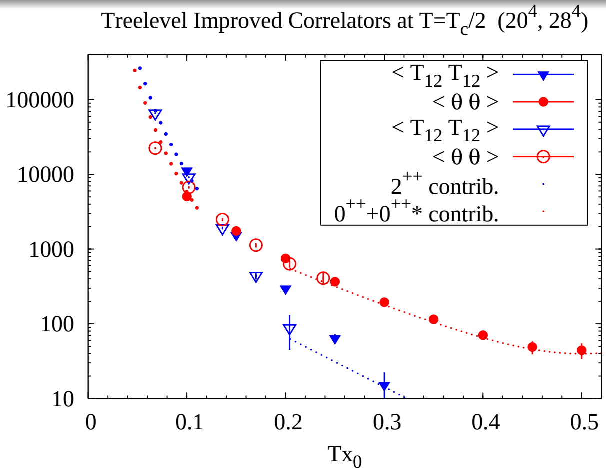
Backus & Gilbert, (geophysics, 1968)

Correlators in the confined phase



- in the scalar channel, the two stable glueballs almost saturate the correlator beyond 0.5fm
- calculation made possible by the multi-level algorithm (HM '03)

Correlators in the confined phase



- in the scalar channel, the two stable glueballs almost saturate the correlator beyond 0.5fm
- calculation made possible by the multi-level algorithm (HM '03)

Spectral functions in free theory: analytic expressions

For a polynomial P, define

$$\mathcal{I}([P],\omega,q,T) = \theta(\omega-q)\int_0^1 dz\,\frac{P(z)\,\sinh\frac{\omega}{2T}}{\cosh\frac{\omega}{2T}-\cosh\frac{qz}{2T}} + \theta(q-\omega)\int_1^\infty dz\,\frac{P(z)\,\sinh\frac{\omega}{2T}}{\cosh\frac{qz}{2T}-\cosh\frac{\omega}{2T}}\,.$$

Then, for instance,

$$\rho_{\theta,\theta}(\omega,q,T) = \left(\frac{11\alpha_s N_c}{6\pi}\right)^2 \frac{d_A}{4\,(4\pi)^2}\,(\omega^2-q^2)^2\;\mathcal{I}([1],\omega,q,T)\,,$$

$$\mathcal{I}([1],\omega,q,T) = -\frac{\omega}{q}\theta(q-\omega) + \frac{2T}{q}\log\frac{\sinh(\omega+q)/4T}{\sinh|\omega-q|/4T}$$

- in all other channels, spectral function is a linear combination of polylogarithms

HM, 2008

Lattice perturbation theory

The "clover" discretization of $T_{\mu\nu}$:

$$\theta_{00}(x) = \frac{1}{g_0^2} \mathrm{Re\,Tr} \left\{ \widehat{Z}_{\mathrm{d}\tau}(g_0,\xi_0) \sum_k \widehat{F}_{0k}(x)^2 - \widehat{Z}_{\mathrm{d}\sigma}(g_0,\xi_0) \sum_{k<l} \widehat{F}_{kl}(x)^2 \right\}.$$

Perturbation theory on the anisotropic lattice:

$$a_\sigma^3 \sum_{\mathbf{y}} e^{i\mathbf{q}\cdot\mathbf{y}} \langle\, \widehat{F}^a_{\mu\nu}(0)\widehat{F}^a_{\rho\sigma}(0)\; \widehat{F}^b_{\alpha\beta}(y)\widehat{F}^b_{\gamma\delta}(y)\,\rangle_0$$

$$= \frac{d_A}{a_\sigma^5} \int_{-\pi}^{\pi} \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \Big[\tilde\phi_{\mu\nu\alpha\beta}(\mathbf{p},\tau)\tilde\phi_{\rho\sigma\gamma\delta}(\mathbf{p}+\mathbf{q},\tau) + \tilde\phi_{\mu\nu\gamma\delta}(\mathbf{p},\tau)\tilde\phi_{\rho\sigma\alpha\beta}(\mathbf{p}+\mathbf{q},\tau)\Big],$$

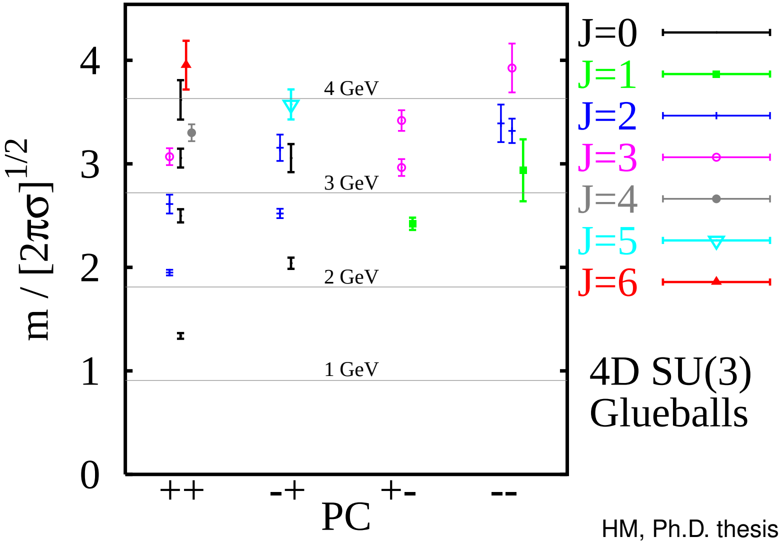
where

$$\tilde\phi_{\mu\nu\alpha\beta}(\mathbf{p},\tau) \equiv \xi_0 \int_{-\pi}^{\pi} \frac{dp_0}{2\pi} \frac{e^{ip_0\tau}}{\xi_0^2\hat{p}_0^2 + \hat{\mathbf{p}}^2} \times$$

$$\Big[\delta_{\nu\beta}\xi_0^{\delta_\mu+\delta_\alpha} \cos^2(p_\nu/2)\ \sin p_\mu\ \sin p_\alpha + \delta_{\mu\alpha}\xi_0^{\delta_\nu+\delta_\beta} \cos^2(p_\mu/2)\ \sin p_\nu\ \sin p_\beta$$

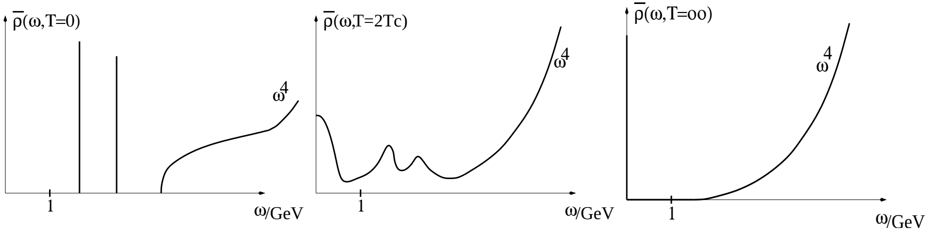
$$-\delta_{\mu\beta}\xi_0^{\delta_\nu+\delta_\alpha}\ \cos^2(p_\mu/2)\ \sin p_\nu\ \sin p_\alpha - \delta_{\nu\alpha}\xi_0^{\delta_\mu+\delta_\beta} \cos^2(p_\nu/2)\ \sin p_\mu\ \sin p_\beta\Big].$$

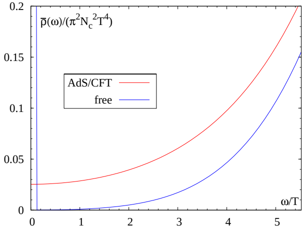
Spectrum of the pure gauge theory



HM, Ph.D. thesis 04

Expected form of the spectral function for η

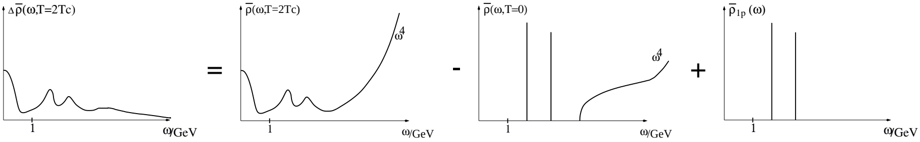




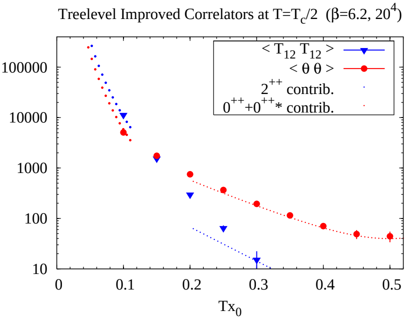
- at $T \gg T_c$, perturbation theory predicts a peak at $\omega = 0$ of width $O(\alpha_s NT)$
- “strong coupling $\leftrightarrow$ smooth spectral function”
  “weak coupling $\leftrightarrow$ spiky spectral function”
- in strongly coupled $\mathcal{N} = 4$ SYM, $\rho(\omega)$ is very smooth
- $\rho(\omega)$ for ζ: no peak at $\omega = 0 \Rightarrow$ more favorable.

Strategy I, pictorially

$$\boxed{\Delta\rho = \rho - \rho_{T=0} + \rho_{1\text{p}}}$$



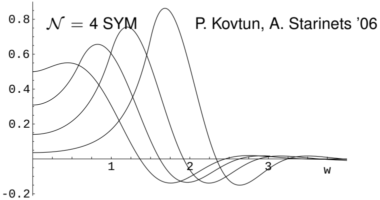
1. compute $\rho_{T=0}(\omega)$
2. compute the one-particle contributions to $\rho(\omega)$, $\rho_{1\text{p}}$
3. compute $\Delta C(x_0, T) \equiv C(x_0, T) - \int d\omega K(\omega x_0, \omega/T)[\rho_0(\omega) - \rho_{1\text{p}}(\omega)]$
4. $\Delta C = \int d\omega K(\omega x_0, \omega L_0)\Delta\rho(\omega)$
5. $\eta = \pi\Delta\rho(\omega)/\omega|_{\omega\to 0}$



Subtracting $\rho(\omega, \mathbf{p} \neq \mathbf{0})$

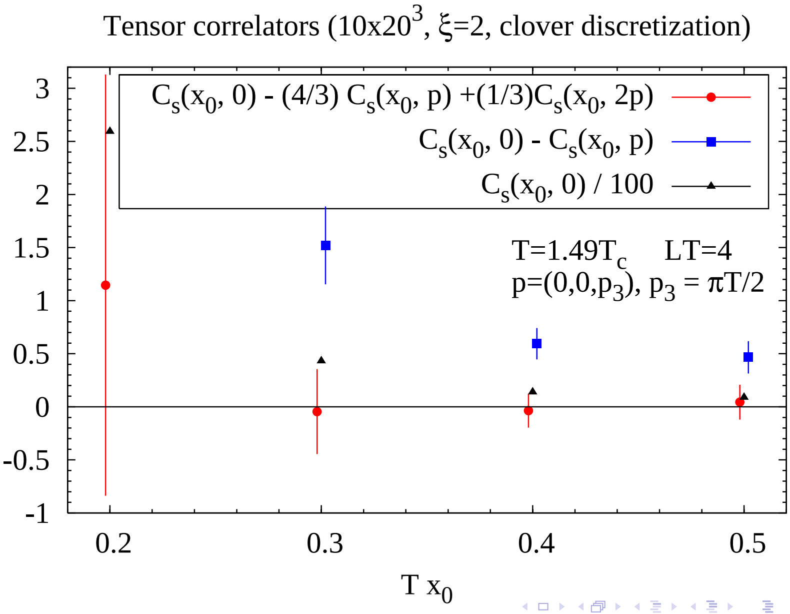
- $\mathbf{q} = (0, 0, q)$: $\rho_{12,12}(\omega, q, T = 0) = \frac{d_A \theta(\omega - q)}{10(4\pi)^2}(\omega^2 - q^2)^2$
- exploit the linearity of the problem
  to cancel the ω^4 contribution, e.g. via
- $\boxed{C(x_0, \mathbf{q} = \mathbf{0}) - \frac{4}{3} C(x_0, \mathbf{q}) + \frac{1}{3} C(x_0, 2\mathbf{q})}$

- this inverse problem is better conditioned
- if $\mathbf{q} = (0, 0, q)$ large enough, estimate $\rho(\omega, \mathbf{p})/\omega|_{\omega=0}$ in PT (or neglect it).
- if $\mathbf{q} = (q, 0, 0)$, hydro predicts $\rho(\omega, \mathbf{q})/\omega|_{\omega=0} = 0$



Finite-temperature part of the tensor spectral function $(\rho(\omega) - \rho_0(\omega))/\omega$, plotted in units of $\pi^2 N_c^2 T^4$ as a function of $\omega/2\pi T$. Different curves correspond to values of the momentum $\mathbf{p}/2\pi T$ equal to 0, 0.6, 1.0, and 1.5.

Differences of $\mathbf{p} \neq \mathbf{0}$ correlators

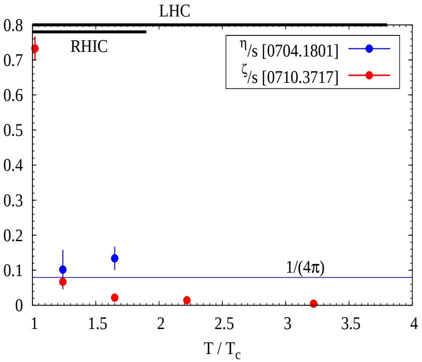


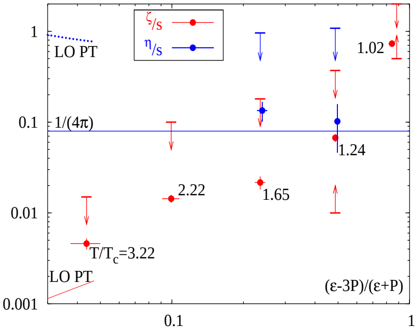
Non-zero momentum correlators (anisotropic lattices)

$\langle T_{12}\ T_{12}\rangle,\ \ 20^3 \times 10,\ \xi = 2$

$\langle \theta\ \theta\rangle,\ \ 16^3 \times 12,\ \xi = 2$

Summary





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Arnold, Moore, Yaffe '03; Arnold, Dogan, Moore '06;

Policastro, Son, Starinets '01; Kovtun, Son, Starinets '04

Universal properties of spectral functions

Ward Identities: $\mathbf{q} = (0, 0, q)$

$$\rho_{00,00}(\omega, \mathbf{q}) = \frac{q^4}{\omega^4} \rho_{33,33}(\omega, \mathbf{q})$$

Hydrodynamics prediction at small $(\omega, \mathbf{q})$:

$$\frac{\rho_{01,01}(\omega, \mathbf{q})}{\omega} \underset{\omega,q\to 0}{\sim} \frac{\eta}{\pi} \frac{q^2}{\omega^2 + (\eta q^2/(\epsilon + P))^2},$$

$$\frac{\rho_{00,00}(\omega, \mathbf{q})}{\omega} \underset{\omega,q\to 0}{\sim} \frac{\Gamma_s(\epsilon + P)}{\pi} \frac{q^4}{(\omega^2 - v_s^2 q^2)^2 + (\Gamma_s \omega q^2)^2},$$

$$\Gamma_s = \frac{\frac{4}{3}\eta + \zeta}{\epsilon + P} = \frac{\frac{4}{3}\eta + \zeta}{Ts}$$

Teaney, 2006