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Comments on fusion matrix in $N = 1$ super Liouville field theory

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Abstract

We study several aspects of the $N = 1$ super Liouville theory. We show that certain elements of the fusion matrix in the Neveu–Schwarz sector are related to the structure constants according to the same rules which we observe in rational conformal field theory. We collect some evidences that these relations should hold also in the Ramond sector. Using them the Cardy–Lewellen equation for defects is studied, and defects are constructed.

1. Introduction

During the last decades we got deep understanding of the properties of rational conformal field theories having a finite number of primaries. Many important relations were obtained between basic notions of RCFT. In particular we would like to mention the Verlinde formula [1], relating matrix of modular transformation and fusion coefficients, Moore–Seiberg relations between elements of fusion matrix, braiding matrix and matrix of modular transformations [2–4]. We have formulas for boundary states [5], and defects [6,7] in rational conformal field theories. Situation in non-rational conformal field theories is much more complicated. The infinite and

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even uncountable number of primary fields is the main reason that progress in this direction is very slow. One of the well studied non-rational theories is Liouville field theory. Liouville field theory has attracted a lot of attention since Polyakov's suggestion to study strings in non-critical dimension. Three-point correlation function (DOZZ formula) [8,9] and fusing matrix [10,11] were found exactly. Other important examples of the non-rational CFT are $N = 1$ superconformal Liouville theory, conformal and superconformal Toda theories and more general para-Toda theories. It is interesting to mention that all of them play a role in the recently established AGT correspondence, giving the structure constants and conformal blocks in terms of the one-loop and instanton Nekrasov's partition functions. Originally this correspondence was proposed in [12] for the Liouville field theory, and afterwards generalized for the Toda field theory [13]. Later the AGT correspondence was developed also for super- and para extensions. Namely for $N = 1$ Super Liouville field theory it was suggested in [14] and for para-Toda (Liouville) in [15]. Further study of the AGT correspondence for the super and para extensions can be found in [16–21].

Many data have been collected also in $N = 1$ superconformal Liouville theory. In particular three-point functions [22,23] and the NS sector fusion matrices [24,25] have been found exactly. Some attempts to find the fusion matrix also in the Ramond sector can be found in [26,27].

In this paper we study some of the Moore–Seiberg relations for the fusion matrix of the $N = 1$ Super Liouville field theory. Recall some basic facts on the fusion matrix. It is defined as a matrix of transformation of conformal blocks [28] in s and t channels [4]:

$$\mathcal{F}^s_p\begin{bmatrix} k & j \\ i & l \end{bmatrix} = \sum_q F_{p,q}\begin{bmatrix} k & j \\ i & l \end{bmatrix}\mathcal{F}^t_q\begin{bmatrix} l & j \\ i & k \end{bmatrix}. \tag{1}$$

Here we write all formulas in the absence of the multiplicities *i.e.* for the fusion numbers $N^i_{jk} = 0, 1$. Fusion matrix plays an important role in conformal field theories, *e.g.* it enters in the conformal bootstrap [4,29], and Cardy–Lewellen [30] equations.

Our task here is to study the following relations, proved in rational CFT, in $N = 1$ super Liouville field theory:

$$F_{0,i}\begin{bmatrix} j & k \\ j & k^* \end{bmatrix} F_{i,0}\begin{bmatrix} k^* & k \\ j & j \end{bmatrix} = \frac{F_j F_k}{F_i}, \tag{2}$$

where

$$F_i \equiv F_{0,0}\begin{bmatrix} i & i^* \\ i & i \end{bmatrix} = \frac{S_{00}}{S_{0i}} \tag{3}$$

and

$$C^p_{ij} = \frac{\eta_i\eta_j}{\eta_0\eta_p} F_{0,p}\begin{bmatrix} j & i \\ j & i^* \end{bmatrix}, \qquad \eta_i = \sqrt{C_{ii^*}/F_i}\,, \tag{4}$$

which using (2) can be written also as

$$C^p_{ij} = \frac{\xi_i\xi_j}{\xi_0\xi_p}\frac{1}{F_{p,0}\begin{bmatrix} j^* & j \\ i & i \end{bmatrix}}, \qquad \xi_i = \eta_i F_i = \sqrt{C_{ii^*}F_i}\,. \tag{5}$$

Let us explain notations. First of all 0 denotes vacuum field and i^* is the field conjugate to i in a sense that $N^0_{ii^*} = 1$. Then S_{ij} is a matrix of the modular transformations, C^p_{ij} are structure constants, C_{ii^*} are two-point functions.

The relation (2) is a consequence of the pentagon identity for fusion matrix [2–4]. The expression (3) results from the two different ways of calculation of the quantum dimension [3]. The equations (4) and (5) result from the bootstrap equation combined with the pentagon identity [5,29,31,32].

These relations were examined in the Liouville field theory. The eq. (2) in the Liouville field theory was tested in [33]. The expressions (4) and (5) were examined in the Liouville field theory in [32,34]. In [32], (4) and (5) in the Liouville field theory were checked using the relation of the fusion matrix with boundary three-point function. In [34], eq. (4) was checked using the following star-triangle integral identity for the double Sine-functions $S_b(x)$:

$$\int \frac{dx}{i} \prod_{i=1}^{3} S_b(x+a_i)S_b(-x+b_i) = \prod_{i,j=1} S_b(a_i+b_j)\,, \tag{6}$$

where

$$\sum_i (a_i+b_i) = Q\,. \tag{7}$$

Recently it was found in [35] the supersymmetric generalization of this formula (eq. (63) in text).

Our first aim here is to calculate the elements of the fusion matrix in the NS sector constructed in [24,25] with one of the intermediate entries set to the vacuum. For this purpose we find convenient to define general expressions for the fusion matrix and structure constants, composed from the supersymmetric double Gamma and double Sine-functions, which reduce to the known elements of the NS sector fusion matrix and structure constants for the certain choices of the types of the supersymmetric double functions. Using the supersymmetric version of the star-triangle identity (63) we found constraints which should be satisfied by the types of the supersymmetric double functions to ensure that the elements of the fusion matrix with one of the entries set to the vacuum give rise to the corresponding structure constant according to the pattern of the equations (4) and (5). We checked that the elements of the fusion matrix in the NS sector indeed satisfy these constraints, and thus established equations (4) and (5) for the NS sector of the $N=1$ Super Liouville field theory.

Next we turn to the fusion matrix in the Ramond sector. Since the general expression for fusion matrix in the Ramond sector is absent, we check the equations (4) and (5) for the elements of the fusion matrix with a degenerate entry, computed in [36,37]. Setting the intermediate state to the vacuum we find that at least these particular elements of the fusion matrix in the Ramond sector again satisfy (5). This drastically simplifies the Cardy–Lewellen equations. It enables us easily to construct topological defects in the $N=1$ super Liouville field theory.

The paper is organized as follows. In section 2 we review basic facts on $N=1$ super Liouville theory. In section 3 we define general expressions for the fusion matrix and structure constants. We compute the elements of the fusion matrix with one of the intermediate states set to the vacuum and find conditions under which they give rise to the structure constants. In section 4 we specialize the formulae obtained in section 3 to the fusion matrices of the NS sector found in [24,25]. We write down the equations (2)–(5) for the elements of the fusion matrix in the NS sector. In section 5 we analyze the Ramond sector for a degenerate entry. In section 6 we apply formulae obtained in section 5 to solve the Cardy–Lewellen equations for topological defects. In appendix some useful formulas are collected.

2. $N = 1$ Super Liouville field theory

Let us review basic facts on the $N = 1$ Super Liouville field theory [38].

$N = 1$ super Liouville field theory is defined on a two-dimensional surface with metric g_{ab} by the local Lagrangian density

$$\mathcal{L} = \frac{1}{2\pi} g_{ab}\partial_a\varphi\partial_b\varphi + \frac{1}{2\pi}(\psi\bar{\partial}\psi + \bar{\psi}\partial\bar{\psi}) + 2i\mu b^2\bar{\psi}\psi e^{b\varphi} + 2\pi\mu^2 b^2 e^{2b\varphi}\,. \tag{8}$$

Here φ is a bosonic field and ψ is its fermionic superpartner, b is a dimensionless Liouville coupling constant and μ is a two-dimensional cosmological constant.

The energy–momentum tensor and the superconformal current are

$$T = -\frac{1}{2}(\partial\varphi\partial\varphi - Q\partial^2\varphi + \psi\partial\psi)\,, \tag{9}$$

$$G = i(\psi\partial\varphi - Q\partial\psi)\,. \tag{10}$$

The superconformal algebra is

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n}\,, \tag{11}$$

$$[L_m, G_k] = \frac{m-2k}{2}G_{m+k}\,, \tag{12}$$

$$\{G_k, G_l\} = 2L_{l+k} + \frac{c}{3}\left(k^2 - \frac{1}{4}\right)\delta_{k+l}\,, \tag{13}$$

with the central charge

$$c_L = \frac{3}{2} + 3Q^2 \tag{14}$$

where

$$Q = b + \frac{1}{b}\,. \tag{15}$$

Here k and l take integer values for the Ramond sector and half-integer values for the Neveu–Schwarz sector.

NS–NS primary fields $N_\alpha(z,\bar{z})$ in this theory, $N_\alpha(z,\bar{z}) = e^{\alpha\varphi(z,\bar{z})}$, have conformal dimensions

$$\Delta_\alpha^{NS} = \frac{1}{2}\alpha(Q-\alpha)\,. \tag{16}$$

Introduce also the field

$$\tilde{N}_\alpha(z,\bar{z}) = G_{-1/2}\bar{G}_{-1/2}N_\alpha(z,\bar{z})\,. \tag{17}$$

The R–R is defined as

$$R_\alpha(z,\bar{z}) = \sigma(z,\bar{z})e^{\alpha\varphi(z,\bar{z})}\,, \tag{18}$$

where σ is the spin field.[1]

[1] Sometimes the Ramond field is defined as $R_\alpha^\pm(z,\bar{z}) = \sigma^\pm(z,\bar{z})e^{\alpha\varphi(z,\bar{z})}$, but in this paper the second field R^- is not important.

The dimension of the R–R operator is

$$\Delta^R_\alpha = \frac{1}{16} + \frac{1}{2}\alpha(Q-\alpha)\,. \tag{19}$$

The NS–NS and R–R operators with the same conformal dimensions are proportional to each other, namely we have

$$N_\alpha = \mathcal{G}_{NS}(\alpha) N_{Q-\alpha}\,, \tag{20}$$

$$R_\alpha = \mathcal{G}_{R}(\alpha) R_{Q-\alpha}\,, \tag{21}$$

where $\mathcal{G}_{NS}(\alpha)$ and $\mathcal{G}_R(\alpha)$ are so-called reflection functions. They also give two-point functions. The elegant way to write the reflection functions is to introduce NS and R generalization of the ZZ function [39] in the bosonic Liouville theory:

$$W_{NS}(\alpha) = \frac{2(\pi\mu\gamma(bQ/2))^{-\frac{Q-2\alpha}{2b}}\pi(\alpha-Q/2)}{\Gamma(1+b(\alpha-Q/2))\Gamma(1+\frac{1}{b}(\alpha-Q/2))}\,, \tag{22}$$

$$W_{R}(\alpha) = \frac{2\pi(\pi\mu\gamma(bQ/2))^{-\frac{Q-2\alpha}{2b}}}{\Gamma(1/2+b(\alpha-Q/2))\Gamma(1/2+\frac{1}{b}(\alpha-Q/2))}\,. \tag{23}$$

The reflection functions can be written

$$\mathcal{G}_{NS}(\alpha) = \frac{W^{NS}(Q-\alpha)}{W^{NS}(\alpha)}\,, \tag{24}$$

$$\mathcal{G}_{R}(\alpha) = \frac{W^{R}(Q-\alpha)}{W^{R}(\alpha)}\,. \tag{25}$$

The functions (22) and (23) satisfy also the relations

$$W_{NS}(\alpha)W_{NS}(Q-\alpha) = -4\sin\pi b(\alpha-Q/2)\sin\pi\frac{1}{b}(\alpha-Q/2)\,, \tag{26}$$

$$W_{R}(\alpha)W_{R}(Q-\alpha) = 4\cos\pi b(\alpha-Q/2)\cos\pi\frac{1}{b}(\alpha-Q/2)\,. \tag{27}$$

The physical delta function normalizable states have $\alpha = \frac{Q}{2} + iP$.

For the super conformal theory, characters are defined for the NS sector, for the R sector and the $\widetilde{\mathrm{NS}}$ sector. The corresponding characters for generic P which have no null-states are

$$\chi^{NS}_P(\tau) = \sqrt{\frac{\theta_3(q)}{\eta(q)}}\frac{q^{P^2/2}}{\eta(\tau)}\,, \tag{28}$$

$$\chi^{\widetilde{NS}}_P(\tau) = \sqrt{\frac{\theta_4(q)}{\eta(q)}}\frac{q^{P^2/2}}{\eta(\tau)}\,, \tag{29}$$

$$\chi^{R}_P(\tau) = \sqrt{\frac{\theta_2(q)}{2\eta(q)}}\frac{q^{P^2/2}}{\eta(\tau)}\,, \tag{30}$$

where $q = \exp(2\pi i\tau)$ and

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)\,. \tag{31}$$

Modular transformation of characters (28)–(30) is well-known:

$$\chi_P^{NS}(\tau) = \int \chi_{P'}^{NS}(-1/\tau) e^{-2i\pi PP'} dP' . \tag{32}$$

$$\chi_P^{\widetilde{NS}}(\tau) = \int \chi_{P'}^{R}(-1/\tau) e^{-2i\pi PP'} dP' . \tag{33}$$

$$\chi_P^{R}(\tau) = \int \chi_{P'}^{\widetilde{NS}}(-1/\tau) e^{-2i\pi PP'} dP' . \tag{34}$$

The degenerate states are given by the momenta:

$$\alpha_{m,n} = \frac{1}{2b}(1-m) + \frac{b}{2}(1-n) \tag{35}$$

with even $m-n$ in the NS sector and odd $m-n$ in the R sector. They have null-states at the level $\frac{mn}{2}$. Hence the degenerate characters read:

$$\chi_{m,n}^{NS} = \chi_{\frac{1}{2}(nb+mb^{-1})}^{NS} - \chi_{\frac{1}{2}(nb-mb^{-1})}^{NS} , \tag{36}$$

$$\chi_{m,n}^{\widetilde{NS}} = \chi_{\frac{1}{2}(nb+mb^{-1})}^{\widetilde{NS}} - (-)^{rs} \chi_{\frac{1}{2}(nb-mb^{-1})}^{\widetilde{NS}} , \tag{37}$$

$$\chi_{m,n}^{R} = \chi_{\frac{1}{2}(nb+mb^{-1})}^{R} - \chi_{\frac{1}{2}(nb-mb^{-1})}^{R} . \tag{38}$$

Modular transformations of (36)–(38) are

$$\chi_{m,n}^{NS}(\tau) = \int \chi_P^{NS}(-1/\tau) 2\sinh(\pi m P/b)\sinh(\pi nbP) dP , \tag{39}$$

$$\chi_{m,n}^{\widetilde{NS}}(\tau) = \int \chi_P^{R}(-1/\tau) 2\sinh(\pi m P/b)\sinh(\pi nbP) dP , \quad m,n \quad \text{even} , \tag{40}$$

$$\chi_{m,n}^{\widetilde{NS}}(\tau) = \int \chi_P^{R}(-1/\tau) 2\cosh(\pi m P/b)\cosh(\pi nbP) dP , \quad m,n \quad \text{odd} . \tag{41}$$

Note that the vacuum component of the matrix of modular transformation specified by $(m,n) = (1,1)$ in formulae (39)–(41) coincide with the right hand side of (26) and (27).

The structure constants in $N = 1$ super Liouville field theory are computed in [22,23]:

$$\langle N_{\alpha_1}(z_1,\bar z_1) N_{\alpha_2}(z_2,\bar z_2) N_{\alpha_3}(z_3,\bar z_3)\rangle = \frac{C_{NS}(\alpha_1,\alpha_2,\alpha_3)}{|z_{12}|^{2(\Delta^N_{\alpha_1}+\Delta^N_{\alpha_2}-\Delta^N_{\alpha_3})}|z_{23}|^{2(\Delta^N_{\alpha_2}+\Delta^N_{\alpha_3}-\Delta^N_{\alpha_1})}|z_{13}|^{2(\Delta^N_{\alpha_1}+\Delta^N_{\alpha_3}-\Delta^N_{\alpha_2})}} , \tag{42}$$

$$\langle \tilde N_{\alpha_1}(z_1,\bar z_1) N_{\alpha_2}(z_2,\bar z_2) N_{\alpha_3}(z_3,\bar z_3)\rangle = \frac{\tilde C_{NS}(\alpha_1,\alpha_2,\alpha_3)}{|z_{12}|^{2(\Delta^N_{\alpha_1}+\Delta^N_{\alpha_2}-\Delta^N_{\alpha_3}+1/2)}|z_{23}|^{2(\Delta^N_{\alpha_2}+\Delta^N_{\alpha_3}-\Delta^N_{\alpha_1}-1/2)}|z_{13}|^{2(\Delta^N_{\alpha_1}+\Delta^N_{\alpha_3}-\Delta^N_{\alpha_2}+1/2)}} , \tag{43}$$

$$\langle R_{\alpha_1}(z_1,\bar z_1) R_{\alpha_2}(z_2,\bar z_2) N_{\alpha_3}(z_3,\bar z_3)\rangle = \frac{C_R(\alpha_1,\alpha_2|\alpha_3) + \tilde C_R(\alpha_1,\alpha_2|\alpha_3)}{|z_{12}|^{2(\Delta^R_{\alpha_1}+\Delta^R_{\alpha_2}-\Delta^N_{\alpha_3})}|z_{23}|^{2(\Delta^R_{\alpha_2}+\Delta^N_{\alpha_3}-\Delta^R_{\alpha_1})}|z_{13}|^{2(\Delta^R_{\alpha_1}+\Delta^N_{\alpha_3}-\Delta^R_{\alpha_2})}} , \tag{44}$$

where $z_{ij} = z_i - z_j$, and

$$C_{NS}(\alpha_1,\alpha_2,\alpha_3)=\lambda^{(Q-\sum_{i=1}^3\alpha_i)/b}\times \tag{45}$$
$$\frac{\Upsilon'_{NS}(0)\Upsilon_{NS}(2\alpha_1)\Upsilon_{NS}(2\alpha_2)\Upsilon_{NS}(2\alpha_3)}{\Upsilon_{NS}(\alpha_1+\alpha_2+\alpha_3-Q)\Upsilon_{NS}(\alpha_1+\alpha_2-\alpha_3)\Upsilon_{NS}(\alpha_2+\alpha_3-\alpha_1)\Upsilon_{NS}(\alpha_3+\alpha_1-\alpha_2)},$$

$$\tilde{C}_{NS}(\alpha_1,\alpha_2,\alpha_3)=\lambda^{(Q-\sum_{i=1}^3\alpha_i)/b}\times \tag{46}$$
$$\frac{\Upsilon'_{NS}(0)\Upsilon_{NS}(2\alpha_1)\Upsilon_{NS}(2\alpha_2)\Upsilon_{NS}(2\alpha_3)}{\Upsilon_R(\alpha_1+\alpha_2+\alpha_3-Q)\Upsilon_R(\alpha_1+\alpha_2-\alpha_3)\Upsilon_R(\alpha_2+\alpha_3-\alpha_1)\Upsilon_R(\alpha_3+\alpha_1-\alpha_2)},$$

$$C_R(\alpha_1,\alpha_2|\alpha_3)=\lambda^{(Q-\sum_{i=1}^3\alpha_i)/b}\times \tag{47}$$
$$\frac{\Upsilon'_{NS}(0)\Upsilon_R(2\alpha_1)\Upsilon_R(2\alpha_2)\Upsilon_{NS}(2\alpha_3)}{\Upsilon_R(\alpha_1+\alpha_2+\alpha_3-Q)\Upsilon_R(\alpha_1+\alpha_2-\alpha_3)\Upsilon_{NS}(\alpha_2+\alpha_3-\alpha_1)\Upsilon_{NS}(\alpha_3+\alpha_1-\alpha_2)},$$

$$\tilde{C}_R(\alpha_1,\alpha_2|\alpha_3)=\lambda^{(Q-\sum_{i=1}^3\alpha_i)/b}\times \tag{48}$$
$$\frac{\Upsilon'_{NS}(0)\Upsilon_R(2\alpha_1)\Upsilon_R(2\alpha_2)\Upsilon_{NS}(2\alpha_3)}{\Upsilon_{NS}(\alpha_1+\alpha_2+\alpha_3-Q)\Upsilon_{NS}(\alpha_1+\alpha_2-\alpha_3)\Upsilon_R(\alpha_2+\alpha_3-\alpha_1)\Upsilon_R(\alpha_3+\alpha_1-\alpha_2)},$$

and

$$\lambda=\pi\mu\gamma\left(\frac{bQ}{2}\right)b^{1-b^2}. \tag{49}$$

Fusion matrix in the NS sector is computed in [24,25]. Let us denote

$$F_{\alpha_s,\alpha_t}\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{bmatrix}_1^1 \equiv F_{N_{\alpha_s},N_{\alpha_t}}\begin{bmatrix}N_{\alpha_3} & N_{\alpha_2}\\ N_{\alpha_4} & N_{\alpha_1}\end{bmatrix},$$
$$F_{\alpha_s,\alpha_t}\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{bmatrix}_1^2 \equiv F_{N_{\alpha_s},\tilde{N}_{\alpha_t}}\begin{bmatrix}N_{\alpha_3} & N_{\alpha_2}\\ N_{\alpha_4} & N_{\alpha_1}\end{bmatrix}, \tag{50}$$

$$F_{\alpha_s,\alpha_t}\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{bmatrix}_2^1 \equiv F_{\tilde{N}_{\alpha_s},N_{\alpha_t}}\begin{bmatrix}N_{\alpha_3} & N_{\alpha_2}\\ N_{\alpha_4} & N_{\alpha_1}\end{bmatrix},$$
$$F_{\alpha_s,\alpha_t}\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{bmatrix}_2^2 \equiv F_{\tilde{N}_{\alpha_s},\tilde{N}_{\alpha_t}}\begin{bmatrix}N_{\alpha_3} & N_{\alpha_2}\\ N_{\alpha_4} & N_{\alpha_1}\end{bmatrix}. \tag{51}$$

To write the fusion matrix we use the following convention. The functions $\Upsilon_i, \Gamma_i, S_i$ will be understood $\Upsilon_{NS}, \Gamma_{NS}, S_{NS}$ for $i=1 \bmod 2$, and $\Upsilon_R, \Gamma_R, S_R$ for $i=0 \bmod 2$. Now we can write the fusion matrix:

$$F_{\alpha_s,\alpha_t}\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{bmatrix}_j^i = \tag{52}$$
$$\frac{\Gamma_i(2Q-\alpha_t-\alpha_2-\alpha_3)\Gamma_i(Q-\alpha_t+\alpha_3-\alpha_2)\Gamma_i(Q+\alpha_t-\alpha_2-\alpha_3)\Gamma_i(\alpha_3+\alpha_t-\alpha_2)}{\Gamma_j(2Q-\alpha_1-\alpha_s-\alpha_2)\Gamma_j(Q-\alpha_s-\alpha_2+\alpha_1)\Gamma_j(Q-\alpha_1-\alpha_2+\alpha_s)\Gamma_j(\alpha_s+\alpha_1-\alpha_2)}$$
$$\times\frac{\Gamma_i(Q-\alpha_t-\alpha_1+\alpha_4)\Gamma_i(\alpha_1+\alpha_4-\alpha_t)\Gamma_i(\alpha_t+\alpha_4-\alpha_1)\Gamma_i(\alpha_t+\alpha_1+\alpha_4-Q)}{\Gamma_j(Q-\alpha_s-\alpha_3+\alpha_4)\Gamma_j(\alpha_3+\alpha_4-\alpha_s)\Gamma_j(\alpha_s+\alpha_4-\alpha_3)\Gamma_j(\alpha_s+\alpha_3+\alpha_4-Q)}$$
$$\times\frac{\Gamma_{\rm NS}(2Q-2\alpha_s)\Gamma_{\rm NS}(2\alpha_s)}{\Gamma_{\rm NS}(Q-2\alpha_t)\Gamma_{\rm NS}(2\alpha_t-Q)}\frac{1}{i}\int_{-i\infty}^{i\infty}d\tau\, J_{\alpha_s,\alpha_t}\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{bmatrix}_j^i,$$

$$J_{\alpha_s,\alpha_t}\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{bmatrix}_1^1 = \tag{53}$$
$$\frac{S_{NS}(Q+\tau-\alpha_1)S_{\mathrm{NS}}(\tau+\alpha_4+\alpha_2-\alpha_3)S_{NS}(\tau+\alpha_1)S_{NS}(\tau+\alpha_4+\alpha_2+\alpha_3-Q)}{S_{NS}(Q+\tau+\alpha_4-\alpha_t)S_{NS}(\tau+\alpha_4+\alpha_t)S_{NS}(Q+\tau+\alpha_2-\alpha_s)S_{NS}(\tau+\alpha_2+\alpha_s)}$$
$$+\frac{S_R(Q+\tau-\alpha_1)S_R(\tau+\alpha_4+\alpha_2-\alpha_3)S_R(\tau+\alpha_1)S_R(\tau+\alpha_4+\alpha_2+\alpha_3-Q)}{S_R(Q+\tau+\alpha_4-\alpha_t)S_R(\tau+\alpha_4+\alpha_t)S_R(Q+\tau+\alpha_2-\alpha_s)S_R(\tau+\alpha_2+\alpha_s)},$$

$$J_{\alpha_s,\alpha_t}\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{bmatrix}_2^1 = \tag{54}$$
$$\frac{S_{NS}(Q+\tau-\alpha_1)S_{\mathrm{NS}}(\tau+\alpha_4+\alpha_2-\alpha_3)S_{NS}(\tau+\alpha_1)S_{NS}(\tau+\alpha_4+\alpha_2+\alpha_3-Q)}{S_{NS}(Q+\tau+\alpha_4-\alpha_t)S_{NS}(\tau+\alpha_4+\alpha_t)S_R(Q+\tau+\alpha_2-\alpha_s)S_R(\tau+\alpha_2+\alpha_s)}$$
$$-\frac{S_R(Q+\tau-\alpha_1)S_R(\tau+\alpha_4+\alpha_2-\alpha_3)S_R(\tau+\alpha_1)S_R(\tau+\alpha_4+\alpha_2+\alpha_3-Q)}{S_R(Q+\tau+\alpha_4-\alpha_t)S_R(\tau+\alpha_4+\alpha_t)S_{NS}(Q+\tau+\alpha_2-\alpha_s)S_{NS}(\tau+\alpha_2+\alpha_s)},$$

$$J_{\alpha_s,\alpha_t}\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{bmatrix}_1^2 = \tag{55}$$
$$\frac{S_{NS}(Q+\tau-\alpha_1)S_{\mathrm{NS}}(\tau+\alpha_4+\alpha_2-\alpha_3)S_{NS}(\tau+\alpha_1)S_{NS}(\tau+\alpha_4+\alpha_2+\alpha_3-Q)}{S_R(Q+\tau+\alpha_4-\alpha_t)S_R(\tau+\alpha_4+\alpha_t)S_{NS}(Q+\tau+\alpha_2-\alpha_s)S_{NS}(\tau+\alpha_2+\alpha_s)}$$
$$-\frac{S_R(Q+\tau-\alpha_1)S_R(\tau+\alpha_4+\alpha_2-\alpha_3)S_R(\tau+\alpha_1)S_R(\tau+\alpha_4+\alpha_2+\alpha_3-Q)}{S_{NS}(Q+\tau+\alpha_4-\alpha_t)S_{NS}(\tau+\alpha_4+\alpha_t)S_R(Q+\tau+\alpha_2-\alpha_s)S_R(\tau+\alpha_2+\alpha_s)},$$

$$J_{\alpha_s,\alpha_t}\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{bmatrix}_2^2 = \tag{56}$$
$$\frac{S_{NS}(Q+\tau-\alpha_1)S_{\mathrm{NS}}(\tau+\alpha_4+\alpha_2-\alpha_3)S_{NS}(\tau+\alpha_1)S_{NS}(\tau+\alpha_4+\alpha_2+\alpha_3-Q)}{S_R(Q+\tau+\alpha_4-\alpha_t)S_R(\tau+\alpha_4+\alpha_t)S_R(Q+\tau+\alpha_2-\alpha_s)S_R(\tau+\alpha_2+\alpha_s)}$$
$$+\frac{S_R(Q+\tau-\alpha_1)S_R(\tau+\alpha_4+\alpha_2-\alpha_3)S_R(\tau+\alpha_1)S_R(\tau+\alpha_4+\alpha_2+\alpha_3-Q)}{S_{NS}(Q+\tau+\alpha_4-\alpha_t)S_{NS}(\tau+\alpha_4+\alpha_t)S_{NS}(Q+\tau+\alpha_2-\alpha_s)S_{NS}(\tau+\alpha_2+\alpha_s)}.$$

3. Values of the fusion matrix for the intermediate vacuum states

In this section we define some general expressions for the fusion matrix and structure constants composed from the supersymmetric double functions, which reduce to the expressions (52) for the fusion matrix and (45)–(48) for the structure constants in the NS sector for certain choices of the types of the supersymmetric double functions. We find constraints on the types of the supersymmetric double functions which guarantee that these general expressions in the vacuum limit of the fusion matrix intermediate states obey to the equations (4)–(5).

3.1. $\alpha_s \to 0$

Motivated by the form of structure constants (45)–(48) and fusing matrix (52) we define the following general expressions for the fusion matrix:

$$F^{\mathcal{I}}_{\alpha_s,\alpha_t}\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{bmatrix} = \frac{M^{\mathcal{I}}}{i}\int_{-i\infty}^{i\infty} d\tau\, J^{\mathcal{I}}_{\alpha_s,\alpha_t}\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{bmatrix} \tag{57}$$

with

$$M^{\mathcal{I}} = \frac{\Gamma_A(2Q-\alpha_t-\alpha_2-\alpha_3)\Gamma_B(Q-\alpha_t+\alpha_3-\alpha_2)\Gamma_C(Q+\alpha_t-\alpha_2-\alpha_3)\Gamma_D(\alpha_3+\alpha_t-\alpha_2)}{\Gamma_E(2Q-\alpha_1-\alpha_s-\alpha_2)\Gamma_{NS}(Q-\alpha_s-\alpha_2+\alpha_1)\Gamma_E(Q-\alpha_1-\alpha_2+\alpha_s)\Gamma_{NS}(\alpha_s+\alpha_1-\alpha_2)} \times \frac{\Gamma_B(Q-\alpha_t-\alpha_1+\alpha_4)\Gamma_C(\alpha_1+\alpha_4-\alpha_t)\Gamma_D(\alpha_t+\alpha_4-\alpha_1)\Gamma_A(\alpha_t+\alpha_1+\alpha_4-Q)}{\Gamma_{NS}(Q-\alpha_s-\alpha_3+\alpha_4)\Gamma_F(\alpha_3+\alpha_4-\alpha_s)\Gamma_{NS}(\alpha_s+\alpha_4-\alpha_3)\Gamma_F(\alpha_s+\alpha_3+\alpha_4-Q)} \times \frac{\Gamma_{\mathrm{NS}}(2Q-2\alpha_s)\Gamma_{\mathrm{NS}}(2\alpha_s)}{\Gamma_{\mathrm{L}}(Q-2\alpha_t)\Gamma_{\mathrm{L}}(2\alpha_t-Q)}, \tag{58}$$

$$J^{\mathcal{I}}_{\alpha_s,\alpha_t}\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{bmatrix} = \frac{S_{\nu_1}(Q+\tau-\alpha_1)S_{\mathrm{K}}(\tau+\alpha_4+\alpha_2-\alpha_3)S_{\nu_2}(\tau+\alpha_1)S_{\nu_3}(\tau+\alpha_4+\alpha_2+\alpha_3-Q)}{S_{\mu_1+1}(Q+\tau+\alpha_4-\alpha_t)S_{\mu_2+1}(\tau+\alpha_4+\alpha_t)S_{\mu_3+1}(Q+\tau+\alpha_2-\alpha_s)S_{\mathrm{K}}(\tau+\alpha_2+\alpha_s)} + \eta\frac{S_{\nu_1+1}(Q+\tau-\alpha_1)S_{\mathrm{K}+1}(\tau+\alpha_4+\alpha_2-\alpha_3)S_{\nu_2+1}(\tau+\alpha_1)S_{\nu_3+1}(\tau+\alpha_4+\alpha_2+\alpha_3-Q)}{S_{\mu_1}(Q+\tau+\alpha_4-\alpha_t)S_{\mu_2}(\tau+\alpha_4+\alpha_t)S_{\mu_3}(Q+\tau+\alpha_2-\alpha_s)S_{\mathrm{K}+1}(\tau+\alpha_2+\alpha_s)}, \tag{59}$$

where $\eta = (-1)^{(1+\sum_i(\nu_i+\mu_i))/2}$. $\mathcal{I}$ denotes fusion matrices of different structures, and capital Latin letters here take values NS and R. The expressions similar to (59) were considered also in [27] in construction of the Racah–Wigner coefficients.

Define also the following general expression for structure constants:

$$C_{\mathcal{I}}(\alpha_1,\alpha_2,\alpha_3) = \lambda^{(Q-\sum_{i=1}^3\alpha_i)/b} \times \frac{\Upsilon'_{NS}(0)\Upsilon_L(2\alpha_1)\Upsilon_E(2\alpha_2)\Upsilon_F(2\alpha_3)}{\Upsilon_A(\alpha_1+\alpha_2+\alpha_3-Q)\Upsilon_B(\alpha_1+\alpha_2-\alpha_3)\Upsilon_C(\alpha_2+\alpha_3-\alpha_1)\Upsilon_D(\alpha_3+\alpha_1-\alpha_2)}. \tag{60}$$

Now consider the limit:

$$\alpha_s = \epsilon \to 0, \qquad \alpha_3 = \alpha_4, \qquad \alpha_1 = \alpha_2\,. \tag{61}$$

In this limit using formulae from appendix and the definition (60) we get for the factor in front of integral:

$$M^{\mathcal{I}} \to C_{\mathcal{I}}(\alpha_t,\alpha_1,\alpha_3)\frac{W_{NS}(Q)W_F(\alpha_3)W_L(\alpha_t)}{2\pi W_E(Q-\alpha_1)} \times \frac{S_B(Q-\alpha_t+\alpha_3-\alpha_1)S_D(\alpha_3+\alpha_t-\alpha_1)S_{\mathrm{E}}(2\alpha_1)}{S_{\mathrm{F}}(2\alpha_3)S_{NS}(\epsilon)}. \tag{62}$$

Let us now evaluate the integral part of (57) in the limit (61). For this purpose we will use the formula [35]

$$\sum_{\nu=0,1}(-1)^{\nu(1+\sum_i(\nu_i+\mu_i))/2}\int\frac{dx}{i}\prod_{i=1}^{3}S_{\nu+\nu_i}(x+a_i)S_{1+\nu+\mu_i}(-x+b_i) = 2\prod_{i,j=1}S_{\nu_i+\mu_j}(a_i+b_j)\,, \tag{63}$$

$$\sum_i(\nu_i+\mu_i) = 1 \bmod 2\,, \tag{64}$$

and

$$\sum_i (a_i + b_i) = Q\,. \tag{65}$$

First note that in the limit (61) the arguments of S_K's in numerator and denominator coincide and they get canceled.

For the rest of S's in this limit we get for a_i in the argument of $S_{\nu_i}(\tau + a_i)$ and b_i in the argument of $S_{\mu_i+1}(-\tau + b_i)$:

$$\begin{aligned} &a_1 = Q - \alpha_1\,, && b_1 = \alpha_t - \alpha_3\,,\\ &a_2 = \alpha_1\,, && b_2 = Q - \alpha_3 - \alpha_t\,,\\ &a_3 = 2\alpha_3 + \alpha_1 - Q\,, && b_3 = -\alpha_1\,. \end{aligned} \tag{66}$$

From (66) we obtain

$$\begin{aligned} a_1 + b_1 &= Q - \alpha_1 + \alpha_t - \alpha_3\,,\\ a_1 + b_2 &= 2Q - \alpha_1 - \alpha_3 - \alpha_t\,,\\ a_1 + b_3 &= Q - 2\alpha_1\,, \end{aligned} \tag{67}$$

$$\begin{aligned} a_2 + b_1 &= \alpha_1 + \alpha_t - \alpha_3\,,\\ a_2 + b_2 &= Q + \alpha_1 - \alpha_3 - \alpha_t\,,\\ a_2 + b_3 &= \epsilon\,, \end{aligned} \tag{68}$$

$$\begin{aligned} a_3 + b_1 &= \alpha_3 + \alpha_t + \alpha_1 - Q\,,\\ a_3 + b_2 &= \alpha_1 + \alpha_3 - \alpha_t\,,\\ a_3 + b_3 &= 2\alpha_3 - Q\,. \end{aligned} \tag{69}$$

Note that

$$\begin{aligned} a_1 + b_1 &= Q - (a_3 + b_2)\,,\\ a_1 + b_2 &= Q - (a_3 + b_1)\,, \end{aligned} \tag{70}$$

and

$$\sum_i (a_i + b_i) = Q\,. \tag{71}$$

Let us impose also

$$\begin{aligned} &\nu_1 + \mu_1 = \nu_3 + \mu_2 \quad \mathrm{mod}\ 2\,,\\ &\nu_1 + \mu_2 = \nu_3 + \mu_1 \quad \mathrm{mod}\ 2\,,\\ &\nu_2 + \mu_3 = 1 \quad \mathrm{mod}\ 2\,. \end{aligned} \tag{72}$$

Assuming also that (64) is satisfied we get from (63) using formulas (67)–(72)

$$\frac{1}{i}\int_{-i\infty}^{i\infty} d\tau\, J^{\mathcal{I}}_{\alpha_s,\alpha_t}\begin{bmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{bmatrix} \to \frac{2S_{\nu_2+\mu_1}(\alpha_1+\alpha_t-\alpha_3)S_{\nu_3+\mu_3}(2\alpha_3-Q)S_{\mathrm{NS}}(\epsilon)}{S_{\nu_1+\mu_3}(2\alpha_1)S_{\nu_2+\mu_2}(\alpha_3+\alpha_t-\alpha_1)}\,. \tag{73}$$

Requiring additionally that

$$\begin{aligned} &\nu_2+\mu_1=B\,, \\ &\nu_2+\mu_2=D\,, \\ &\nu_1+\mu_3=E\,, \\ &\nu_3+\mu_3=F\,, \end{aligned} \tag{74}$$

where these equalities as before are understood in a sense, that odd sums identified with the NS sector, and even sums identified with the Ramond sectors, we get

$$F^{\mathcal{I}}_{0,\alpha_t}\begin{bmatrix}\alpha_3 & \alpha_1\\ \alpha_3 & \alpha_1\end{bmatrix} = C_{\mathcal{I}}(\alpha_t,\alpha_1,\alpha_3)\frac{W_{NS}(Q)W_L(\alpha_t)}{\pi W_E(Q-\alpha_1)W_F(Q-\alpha_3)}\,. \tag{75}$$

3.2. $\alpha_t \to 0$ *limit*

Consider the same fusing matrix, but parametrized in the form

$$F^{\mathcal{I}}_{\alpha_s,\alpha_t}\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{bmatrix} = \frac{\mathcal{R}^{\mathcal{I}}}{i}\int_{-i\infty}^{i\infty} d\tau\, J^{\mathcal{I}}_{\alpha_s,\alpha_t}\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{bmatrix} \tag{76}$$

with

$$\begin{aligned} &\mathcal{R}^{\mathcal{I}} = \\ &\frac{\Gamma_E(2Q-\alpha_t-\alpha_2-\alpha_3)\Gamma_{NS}(Q-\alpha_t+\alpha_3-\alpha_2)\Gamma_E(Q+\alpha_t-\alpha_2-\alpha_3)\Gamma_{NS}(\alpha_3+\alpha_t-\alpha_2)}{\Gamma_A(2Q-\alpha_1-\alpha_s-\alpha_2)\Gamma_B(Q-\alpha_s-\alpha_2+\alpha_1)\Gamma_C(Q-\alpha_1-\alpha_2+\alpha_s)\Gamma_D(\alpha_s+\alpha_1-\alpha_2)} \\ &\times\frac{\Gamma_{NS}(Q-\alpha_t-\alpha_1+\alpha_4)\Gamma_F(\alpha_1+\alpha_4-\alpha_t)\Gamma_{NS}(\alpha_t+\alpha_4-\alpha_1)\Gamma_F(\alpha_t+\alpha_1+\alpha_4-Q)}{\Gamma_B(Q-\alpha_s-\alpha_3+\alpha_4)\Gamma_C(\alpha_3+\alpha_4-\alpha_s)\Gamma_D(\alpha_s+\alpha_4-\alpha_3)\Gamma_A(\alpha_s+\alpha_3+\alpha_4-Q)} \\ &\times\frac{\Gamma_{\mathrm{L}}(2Q-2\alpha_s)\Gamma_{\mathrm{L}}(2\alpha_s)}{\Gamma_{\mathrm{NS}}(Q-2\alpha_t)\Gamma_{\mathrm{NS}}(2\alpha_t-Q)}\,, \end{aligned} \tag{77}$$

$$\begin{aligned} &J^{\mathcal{I}}_{\alpha_s,\alpha_t}\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{bmatrix} = \\ &\frac{S_{\nu_1}(Q+\tau-\alpha_1)S_{\mathrm{K}}(\tau+\alpha_4+\alpha_2-\alpha_3)S_{\nu_2}(\tau+\alpha_1)S_{\nu_3}(\tau+\alpha_4+\alpha_2+\alpha_3-Q)}{S_{\mu_1+1}(Q+\tau+\alpha_4-\alpha_t)S_K(\tau+\alpha_4+\alpha_t)S_{\mu_2+1}(Q+\tau+\alpha_2-\alpha_s)S_{\mu_3+1}(\tau+\alpha_2+\alpha_s)} \\ &+\eta\frac{S_{\nu_1+1}(Q+\tau-\alpha_1)S_{\mathrm{K}+1}(\tau+\alpha_4+\alpha_2-\alpha_3)S_{\nu_2+1}(\tau+\alpha_1)S_{\nu_3+1}(\tau+\alpha_4+\alpha_2+\alpha_3-Q)}{S_{\mu_1}(Q+\tau+\alpha_4-\alpha_t)S_{K+1}(\tau+\alpha_4+\alpha_t)S_{\mu_2}(Q+\tau+\alpha_2-\alpha_s)S_{\mu_3}(\tau+\alpha_2+\alpha_s)}\,, \end{aligned} \tag{78}$$

where $\eta=(-1)^{(1+\sum_i(\nu_i+\mu_i))/2}$.

We change here notations for the capital Latin letters denoting different spin structures. This is done to keep parametrization for the capital Latin letters in the formula for structure constants (60). Alternatively we could keep the same parametrization in formula for fusing matrix and change the notations in formula for structure constants.

Consider the limit

$$\alpha_t=\epsilon\to 0, \qquad \alpha_3=\alpha_2, \qquad \alpha_4=\alpha_1\,. \tag{79}$$

In this limit using formulas in appendix and (60) we have for the factor in front of integral

$$\begin{aligned} &\mathcal{R}^{\mathcal{I}} \to \frac{2}{\pi\epsilon^2 C_{\mathcal{I}}(\alpha_s,\alpha_2,\alpha_1)}\frac{W_{NS}(0)W_E(Q-\alpha_2)W_L(Q-\alpha_s)}{W_F(\alpha_1)}\times \\ &\frac{S_{\mathrm{F}}(2\alpha_1)}{S_B(Q-\alpha_s-\alpha_2+\alpha_1)S_D(\alpha_s+\alpha_1-\alpha_2)S_{\mathrm{E}}(2\alpha_2)S_{\mathrm{NS}}(\epsilon)}\,. \end{aligned} \tag{80}$$

Consider now the limit of the integrand (78).

In the limit (79) the arguments of S_K's in numerator and denominator coincide and they get canceled.

For the rest of S's in this limit we get for a_i in the argument of $S_{\nu_i}(\tau + a_i)$ and b_i in the argument of $S_{\mu_i+1}(-\tau + b_i)$:

$$\begin{aligned} a_1 &= Q - \alpha_1\,, & b_1 &= -\alpha_1\,, \\ a_2 &= \alpha_1\,, & b_2 &= \alpha_s - \alpha_2\,, \\ a_3 &= 2\alpha_2 + \alpha_1 - Q\,, & b_3 &= Q - \alpha_2 - \alpha_s\,. \end{aligned} \tag{81}$$

From (81) we easily obtain:

$$\begin{aligned} a_1 + b_1 &= Q - 2\alpha_1\,, \\ a_1 + b_2 &= Q - \alpha_1 + \alpha_s - \alpha_2\,, \\ a_1 + b_3 &= 2Q - \alpha_1 - \alpha_s - \alpha_2\,, \end{aligned} \tag{82}$$

$$\begin{aligned} a_2 + b_1 &= \epsilon\,, \\ a_2 + b_2 &= \alpha_1 + \alpha_s - \alpha_2\,, \\ a_2 + b_3 &= Q - \alpha_2 - \alpha_s + \alpha_1\,, \end{aligned} \tag{83}$$

$$\begin{aligned} a_3 + b_1 &= 2\alpha_2 - Q\,, \\ a_3 + b_2 &= \alpha_2 + \alpha_1 + \alpha_s - Q\,, \\ a_3 + b_3 &= \alpha_2 + \alpha_1 - \alpha_s\,. \end{aligned} \tag{84}$$

Note that

$$\begin{aligned} a_1 + b_3 &= Q - (a_3 + b_2)\,, \\ a_1 + b_2 &= Q - (a_3 + b_3)\,, \end{aligned} \tag{85}$$

and

$$\sum_i (a_i + b_i) = Q\,. \tag{86}$$

Assume that

$$\begin{aligned} &\nu_1 + \mu_3 = \nu_3 + \mu_2 \quad \mod 2\,, \\ &\nu_1 + \mu_2 = \nu_3 + \mu_3 \quad \mod 2\,, \\ &\nu_2 + \mu_1 = 1 \quad \mod 2\,. \end{aligned} \tag{87}$$

Under these conditions we get from the theorem (63), using formulas (82)–(87)

$$\frac{1}{i}\int_{-i\infty}^{i\infty} d\tau\, J^{\mathcal{I}}_{\alpha_s,\alpha_t}\begin{bmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{bmatrix} = \frac{2S_{\nu_2+\mu_2}(\alpha_1+\alpha_s-\alpha_2)S_{\nu_3+\mu_1}(2\alpha_2-Q)S_{\mathrm{NS}}(\epsilon)}{S_{\nu_1+\mu_1}(2\alpha_1)S_{\nu_2+\mu_3}(\alpha_2+\alpha_s-\alpha_1)}\,. \tag{88}$$

Requiring additionally that

$$\begin{aligned}
&\nu_2 + \mu_3 = B\,, \\
&\nu_2 + \mu_2 = D\,, \\
&\nu_3 + \mu_1 = E\,, \\
&\nu_1 + \mu_1 = F\,,
\end{aligned} \tag{89}$$

where these equalities as before are understood in a sense, that odd sums identified with the NS sector, and even sums identified with the Ramond sectors, we get

$$\tilde{F}^{\mathcal{I}}_{\alpha_s,\epsilon}\begin{bmatrix} \alpha_2 & \alpha_2 \\ \alpha_1 & \alpha_1 \end{bmatrix} = \lim_{\epsilon\to 0}\epsilon^2 F^{\mathcal{I}}_{\alpha_s,\epsilon}\begin{bmatrix} \alpha_2 & \alpha_2 \\ \alpha_1 & \alpha_1 \end{bmatrix} = \frac{4}{\pi C_{\mathcal{I}}(\alpha_s,\alpha_2,\alpha_1)}\frac{W_{NS}(0)W_L(Q-\alpha_s)}{W_F(\alpha_1)W_E(\alpha_2)}\,. \tag{90}$$

4. NS sector fusion matrix

Here we specialize results of the previous section to the fusion matrix (52) and structure constants (45)–(48) of the NS sector. We show that the corresponding types of the supersymmetric double functions satisfy the constraints leading to the equations (4), (5). We explicitly write down the equations (4), (5) for the $N = 1$ super Liouville theory. We find also the analogue of the equation (2) in the $N = 1$ super Liouville theory.

Recall that structure constants in the NS sector are given by eq. (45) and (46) and fusion matrix by (52).

Remember that $NS = 1,\ \mathrm{mod}\,2$ and $R = 0,\ \mathrm{mod}\,2$. Putting $A = B = C = D = L = E = F = NS$, $\nu_1 = \nu_2 = \nu_3 = 1$, $\mu_1 = \mu_2 = \mu_3 = 0$, and using (75), we obtain for the $(i = 1, j = 1)$ component of the NS sector fusing matrices in the limit (61)

$$F_{0,\alpha_t}\begin{bmatrix} \alpha_3 & \alpha_1 \\ \alpha_3 & \alpha_1 \end{bmatrix}^1_1 = C_{NS}(\alpha_t,\alpha_1,\alpha_3)\frac{W_{NS}(Q)W_{NS}(\alpha_t)}{\pi W_{NS}(Q-\alpha_1)W_{NS}(Q-\alpha_3)}\,. \tag{91}$$

Putting $A = B = C = D = R$, $L = E = F = NS$, $\nu_1 = \nu_2 = \nu_3 = 1$, $\mu_1 = \mu_2 = 1$, $\mu_3 = 0$, and using (75), we obtain for the $(i = 2, j = 1)$ component of the NS sector fusing matrices in the limit (61)

$$F_{0,\alpha_t}\begin{bmatrix} \alpha_3 & \alpha_1 \\ \alpha_3 & \alpha_1 \end{bmatrix}^2_1 = \tilde{C}_{NS}(\alpha_t,\alpha_1,\alpha_3)\frac{W_{NS}(Q)W_{NS}(\alpha_t)}{\pi W_{NS}(Q-\alpha_1)W_{NS}(Q-\alpha_3)}\,. \tag{92}$$

It is obvious to see that both choices of the ν_i and μ_i satisfy the conditions (72), (64), (74).

Putting $A = B = C = D = L = E = F = NS$, $\nu_1 = \nu_2 = \nu_3 = 1$, $\mu_1 = \mu_2 = \mu_3 = 0$, and using (90), we obtain for the $(i = 1, j = 1)$ component of the NS fusing matrices in the limit (79)

$$\begin{aligned}
\tilde{F}_{\alpha_s,0}\begin{bmatrix} \alpha_2 & \alpha_2 \\ \alpha_1 & \alpha_1 \end{bmatrix}^1_1 &= \lim_{\epsilon\to 0}\epsilon^2 F_{\alpha_s,\epsilon}\begin{bmatrix} \alpha_2 & \alpha_2 \\ \alpha_1 & \alpha_1 \end{bmatrix}^1_1 \\
&= \frac{4}{\pi C_{NS}(\alpha_s,\alpha_2,\alpha_1)}\frac{W_{NS}(0)W_{NS}(Q-\alpha_s)}{W_{NS}(\alpha_1)W_{NS}(\alpha_2)}\,.
\end{aligned} \tag{93}$$

Putting $A = B = C = D = R$, $L = E = F = NS$, $\nu_1 = \nu_2 = \nu_3 = 1$, $\mu_1 = 0$, $\mu_2 = \mu_3 = 1$, and using (90), we obtain for the $(i = 1, j = 2)$ component of the NS fusing matrix in the limit (79)

$$\tilde{F}_{\alpha_s,0}\begin{bmatrix}\alpha_2 & \alpha_2\\ \alpha_1 & \alpha_1\end{bmatrix}_2^1 = \lim_{\epsilon\to 0}\epsilon^2 F_{\alpha_s,\epsilon}\begin{bmatrix}\alpha_2 & \alpha_2\\ \alpha_1 & \alpha_1\end{bmatrix}_2^1$$
$$= \frac{4}{\pi\tilde{C}_{NS}(\alpha_s,\alpha_2,\alpha_1)}\frac{W_{NS}(0)W_{NS}(Q-\alpha_s)}{W_{NS}(\alpha_1)W_{NS}(\alpha_2)}\,. \tag{94}$$

It is again obvious to see that both sets of the values of ν_i and μ_i satisfy the conditions (64), (87) and (89).

Note also the relations:

$$F_{0,\alpha_s}\begin{bmatrix}\alpha_1 & \alpha_2\\ \alpha_1 & \alpha_2\end{bmatrix}_1^1 \tilde{F}_{\alpha_s,0}\begin{bmatrix}\alpha_2 & \alpha_2\\ \alpha_1 & \alpha_1\end{bmatrix}_1^1 = \frac{S(0)S(\alpha_s)}{\pi^2 S(\alpha_1)S(\alpha_2)}\,, \tag{95}$$

$$F_{0,\alpha_s}\begin{bmatrix}\alpha_1 & \alpha_2\\ \alpha_1 & \alpha_2\end{bmatrix}_1^2 \tilde{F}_{\alpha_s,0}\begin{bmatrix}\alpha_2 & \alpha_2\\ \alpha_1 & \alpha_1\end{bmatrix}_2^1 = \frac{S(0)S(\alpha_s)}{\pi^2 S(\alpha_1)S(\alpha_2)}\,, \tag{96}$$

where $S(\alpha) = \sin\pi b(\alpha - Q/2)\sin\pi\frac{1}{b}(\alpha - Q/2)$.

Remembering the relation (39) and that the vacuum field is given by the pair $(1,1)$ we see that the function $S(\alpha)$ coincides with the vacuum component of the matrix of modular transformations. We see that the relations (91)–(96) indeed have the structure of the equations (2), (4) and (5).

5. Fusion matrix in the Ramond sector

The fusion matrix in the Ramond sector unfortunately is not known in general. Although for some attempts see [26]. But for the degenerate primaries (35) fusion matrix can be computed via direct solutions of the corresponding differential equation for conformal blocks. In particular the necessary elements of the fusion matrix when one of the entries is the simplest degenerate field $R_{-b/2}$ are computed in [36,37]. The degenerate field $R_{-b/2}$ possesses the OPE:

$$N_\alpha R_{-b/2} = C_{N_\alpha R_{-b/2}}^{R_{\alpha-b/2}} R_{\alpha-b/2} + C_{N_\alpha R_{-b/2}}^{R_{\alpha+b/2}} R_{\alpha+b/2}\,, \tag{97}$$

$$R_\alpha R_{-b/2} = C_{R_\alpha R_{-b/2}}^{N_{\alpha-b/2}} N_{\alpha-b/2} + C_{R_\alpha R_{-b/2}}^{N_{\alpha+b/2}} N_{\alpha+b/2}\,. \tag{98}$$

The corresponding structure constant can be computed in the Coulomb gas formalism using the screening integrals:

$$C_{N_\alpha R_{-b/2}}^{R_{\alpha-b/2}} = 1\,, \tag{99}$$

$$C_{N_\alpha R_{-b/2}}^{R_{\alpha+b/2}} = \pi\mu b^2\gamma(bQ/2)\gamma(1-b\alpha)\gamma(b\alpha - bQ/2) = \frac{\mathcal{G}_{NS}(\alpha)}{\mathcal{G}_R(\alpha+b/2)}\,, \tag{100}$$

$$C_{R_\alpha R_{-b/2}}^{N_{\alpha-b/2}} = 1\,, \tag{101}$$

$$C_{R_\alpha R_{-b/2}}^{N_{\alpha+b/2}} = 2i\pi\mu b^2\gamma(bQ/2)\gamma(1/2-b\alpha)\gamma(b\alpha - b^2/2) = 2i\frac{\mathcal{G}_R(\alpha)}{\mathcal{G}_{NS}(\alpha+b/2)}\,. \tag{102}$$

The fusion matrices can be computed having explicit expression of the conformal blocks with degenerate entries:

$$F_{R_{\alpha-b/2},0}\begin{bmatrix}R_{-b/2} & R_{-b/2}\\ N_\alpha & N_\alpha\end{bmatrix} = \frac{\Gamma(\alpha b - b^2/2 + 1/2)\Gamma(-b^2)}{\Gamma(\alpha b - b^2)\Gamma(1/2 - b^2/2)}\,, \tag{103}$$

$$F_{R_{\alpha+b/2},0}\begin{bmatrix}R_{-b/2} & R_{-b/2}\\ N_\alpha & N_\alpha\end{bmatrix} = \frac{\Gamma(-\alpha b + b^2/2 + 3/2)\Gamma(-b^2)}{\Gamma(1-\alpha b)\Gamma(1/2 - b^2/2)}\,, \tag{104}$$

$$F_{N_{\alpha-b/2},0}\left[\begin{array}{cc} R_{-b/2} & R_{-b/2} \\ R_\alpha & R_\alpha \end{array}\right] = \frac{\Gamma(\alpha b - b^2/2)\Gamma(-b^2)}{\Gamma(\alpha b - b^2 - 1/2)\Gamma(1/2 - b^2/2)}, \tag{105}$$

$$F_{N_{\alpha+b/2},0}\left[\begin{array}{cc} R_{-b/2} & R_{-b/2} \\ R_\alpha & R_\alpha \end{array}\right] = \frac{\Gamma(-\alpha b + b^2/2 + 1)\Gamma(-b^2)}{2i\Gamma(1/2 - \alpha b)\Gamma(1/2 - b^2/2)}. \tag{106}$$

It is an easy exercise to check that the values of the structure constants (99)–(102) and fusion matrices (103)–(106) satisfy the relations:

$$\begin{aligned} C^{R_{\alpha-b/2}}_{N_\alpha R_{-b/2}} F_{R_{\alpha-b/2},0}\left[\begin{array}{cc} R_{-b/2} & R_{-b/2} \\ N_\alpha & N_\alpha \end{array}\right] &= \frac{\Gamma(\alpha b - b^2/2 + 1/2)\Gamma(-b^2)}{\Gamma(\alpha b - b^2)\Gamma(1/2 - b^2/2)} \\ &= \frac{W_{NS}(0) W_R(\alpha - b/2)}{W_{NS}(\alpha) W_R(-b/2)}, \end{aligned} \tag{107}$$

$$\begin{aligned} C^{R_{\alpha+b/2}}_{N_\alpha R_{-b/2}} F_{R_{\alpha+b/2},0}\left[\begin{array}{cc} R_{-b/2} & R_{-b/2} \\ N_\alpha & N_\alpha \end{array}\right] &= \frac{\pi \mu b^2 \gamma(bQ/2)\Gamma(-b^2)\Gamma(\alpha b - b^2/2 - 1/2)}{\Gamma(1/2 - b^2/2)\Gamma(\alpha b)} \\ &= \frac{W_{NS}(0) W_R(\alpha + b/2)}{W_{NS}(\alpha) W_R(-b/2)}, \end{aligned} \tag{108}$$

$$\begin{aligned} C^{N_{\alpha-b/2}}_{R_\alpha R_{-b/2}} F_{N_{\alpha-b/2},0}\left[\begin{array}{cc} R_{-b/2} & R_{-b/2} \\ R_\alpha & R_\alpha \end{array}\right] &= \frac{\Gamma(\alpha b - b^2/2)\Gamma(-b^2)}{\Gamma(\alpha b - b^2 - 1/2)\Gamma(1/2 - b^2/2)} \\ &= \frac{W_{NS}(0) W_{NS}(\alpha - b/2)}{W_R(\alpha) W_R(-b/2)}, \end{aligned} \tag{109}$$

$$\begin{aligned} C^{N_{\alpha+b/2}}_{R_\alpha R_{-b/2}} F_{N_{\alpha+b/2},0}\left[\begin{array}{cc} R_{-b/2} & R_{-b/2} \\ R_\alpha & R_\alpha \end{array}\right] &= \frac{\pi \mu b^2 \gamma(bQ/2)\Gamma(\alpha b - b^2/2)\Gamma(-b^2)}{\Gamma(\alpha b + 1/2)\Gamma(1/2 - b^2/2)} \\ &= \frac{W_{NS}(0) W_{NS}(\alpha + b/2)}{W_R(\alpha) W_R(-b/2)}. \end{aligned} \tag{110}$$

One expects that similar relations should hold also for general expressions of the corresponding elements of fusion matrix in the RR sector. For example the fusion matrix with four RR entries should satisfy the relations

$$\begin{aligned} &\lim_{\epsilon\to 0}\epsilon^2 F_{N_{\alpha_s},N_\epsilon}\left[\begin{array}{cc} R_{\alpha_2} & R_{\alpha_2} \\ R_{\alpha_1} & R_{\alpha_1} \end{array}\right] \\ &\quad = \frac{4}{\pi(C_R(\alpha_s|\alpha_2,\alpha_1) + \tilde{C}_R(\alpha_s|\alpha_1,\alpha_2))} \frac{W_{NS}(0) W_{NS}(Q-\alpha_s)}{W_R(\alpha_1) W_R(\alpha_2)}, \end{aligned} \tag{111}$$

$$F_{0,N_{\alpha_t}}\left[\begin{array}{cc} R_{\alpha_3} & R_{\alpha_1} \\ R_{\alpha_3} & R_{\alpha_1} \end{array}\right] = (C_R(\alpha_t|\alpha_1,\alpha_3) + \tilde{C}_R(\alpha_t|\alpha_1,\alpha_3)) \frac{W_{NS}(Q) W_{NS}(\alpha_t)}{\pi W_R(Q-\alpha_1) W_R(Q-\alpha_3)}. \tag{112}$$

One can hope that constraints like (111) and (112) may help to obtain the general expressions for the corresponding elements of the fusion matrix.

6. Defects in Super-Liouville theory

Two-point functions with a defect X insertion can be written as

$$\langle \Phi_i(z_1,\bar{z}_1) X \Phi_i(z_2,\bar{z}_2)\rangle = \frac{D^i}{(z_1-z_2)^{2\Delta_i}(\bar{z}_1-\bar{z}_2)^{2\Delta_i}}, \tag{113}$$

where

$$D^i = \mathcal{D}^i C_{ii} \tag{114}$$

and C_{ii} is a two-point function. They satisfy the Cardy–Lewellen equation for defects [7,32,40,41]

$$\sum_k D^0 D^k \left(C^k_{ij} F_{k0} \begin{bmatrix} j & j \\ i & i \end{bmatrix} \right)^2 = D^i D^j \,. \tag{115}$$

Denote

$$D_{NS}(\alpha) = \langle N_\alpha X N_\alpha \rangle \,, \tag{116}$$

$$D_R(\alpha) = \langle R_\alpha X R_\alpha \rangle \,. \tag{117}$$

Let us take $j = R_{-b/2}$. Using (97), (98) and (107)–(110) one can obtain:

$$\Psi_{NS}(\alpha)\Psi_R(-b/2) = \Psi_R(\alpha - b/2) + \Psi_R(\alpha + b/2) \,, \tag{118}$$

$$\Psi_R(\alpha)\Psi_R(-b/2) = \Psi_{NS}(\alpha - b/2) + \Psi_{NS}(\alpha + b/2) \,, \tag{119}$$

where

$$\frac{D_{NS}(\alpha)}{D_{NS}(0)} = \Psi_{NS}(\alpha) \left(\frac{W_{NS}(0)}{W_{NS}(\alpha)} \right)^2 \,, \tag{120}$$

$$\frac{D_R(\alpha)}{D_{NS}(0)} = \Psi_R(\alpha) \left(\frac{W_{NS}(0)}{W_R(\alpha)} \right)^2 \,. \tag{121}$$

The solution of the equations (118) and (119) is

$$\Psi_{NS}(\alpha; m, n) = \frac{\sin(\pi m b^{-1}(\alpha - Q/2)) \sin(\pi n b(\alpha - Q/2))}{\sin(\pi \frac{mb^{-1}Q}{2}) \sin(\frac{\pi n b Q}{2})} \,, \tag{122}$$

$$\Psi_R(\alpha; m, n) = \frac{\sin(\pi m(\frac{1}{2} + b^{-1}(\alpha - Q/2))) \sin(\pi n(\frac{1}{2} + b(\alpha - Q/2)))}{\sin(\frac{\pi m b^{-1} Q}{2}) \sin(\frac{\pi n b Q}{2})} \,, \tag{123}$$

where $m - n$ is even.

Substituting (122) and (123) in (120) and (121) we obtain

$$D_{NS}(\alpha; m, n) = \frac{\sin(\pi m b^{-1}(\alpha - Q/2)) \sin(\pi n b(\alpha - Q/2))}{W_{NS}(\alpha)^2} \,, \tag{124}$$

$$D_R(\alpha; m, n) = \frac{\sin(\pi m(\frac{1}{2} + b^{-1}(\alpha - Q/2))) \sin(\pi n(\frac{1}{2} + b(\alpha - Q/2)))}{W_R(\alpha)^2} \,. \tag{125}$$

Dividing by two-point functions (24) and (25) we obtain

$$\mathcal{D}_{NS}(\alpha; m, n) = \frac{\sin(\pi m b^{-1}(\alpha - Q/2)) \sin(\pi n b(\alpha - Q/2))}{\sin(\pi b^{-1}(\alpha - Q/2)) \sin(\pi b(\alpha - Q/2))} \,, \tag{126}$$

$$\mathcal{D}_R(\alpha; m, n) = \frac{\sin(\pi m(\frac{1}{2} + b^{-1}(\alpha - Q/2))) \sin(\pi n(\frac{1}{2} + b(\alpha - Q/2)))}{\cos(\pi b^{-1}(\alpha - Q/2)) \cos(\pi b(\alpha - Q/2))} \,. \tag{127}$$

To obtain the continuous family of defects we use the strategy developed in [42,43]. Namely consider $D_R(-b/2)$ as a parameter characterizing a defect. More precisely we define

$$A = \frac{D_R(-b/2)}{D_{NS}(0)} \left(\frac{W_R(-b/2)}{W_{NS}(0)} \right)^2 . \tag{128}$$

Denoting also

$$D_{NS}(\alpha) = \frac{\tilde{\Psi}_{NS}(\alpha)}{W_{NS}(\alpha)^2} , \tag{129}$$

$$D_R(\alpha) = \frac{\tilde{\Psi}_R(\alpha)}{W_R(\alpha)^2} , \tag{130}$$

we obtain

$$A\tilde{\Psi}_{NS}(\alpha) = \tilde{\Psi}_R(\alpha - b/2) + \tilde{\Psi}_R(\alpha + b/2) , \tag{131}$$

$$A\tilde{\Psi}_R(\alpha) = \tilde{\Psi}_{NS}(\alpha - b/2) + \tilde{\Psi}_{NS}(\alpha + b/2) . \tag{132}$$

The solution of (131) and (132) is given by

$$\tilde{\Psi}_{NS}(\alpha; u) = \cosh(\pi(2\alpha - Q)u) , \tag{133}$$

$$\tilde{\Psi}_R(\alpha; u) = \cosh(\pi(2\alpha - Q)u) , \tag{134}$$

with a parameter u related to A by

$$2\cosh 2\pi bu = A . \tag{135}$$

Substituting (133) and (134) in (129) and (130) we obtain

$$D_{NS}(\alpha; u) = \frac{\cosh(\pi(2\alpha - Q)u)}{W_{NS}(\alpha)^2} , \tag{136}$$

$$D_R(\alpha; u) = \frac{\cosh(\pi(2\alpha - Q)u)}{W_R(\alpha)^2} . \tag{137}$$

Dividing by two-point functions (24) and (25) we obtain

$$\mathcal{D}_{NS}(\alpha; u) = \frac{\cosh(\pi(2\alpha - Q)u)}{\sin(\pi b^{-1}(\alpha - Q/2)) \sin(\pi b(\alpha - Q/2))} , \tag{138}$$

$$\mathcal{D}_R(\alpha; u) = \frac{\cosh(\pi(2\alpha - Q)u)}{\cos(\pi b^{-1}(\alpha - Q/2)) \cos(\pi b(\alpha - Q/2))} . \tag{139}$$

7. Discussion

As we mentioned at the end of section 5 one of the immediate problems is construction of the fusion matrix in the Ramond sector. One can try to write an Ansatz using expressions in section 3 and also the Racah–Wigner coefficients in [27] and to match it with the numerous criteria which fusion matrix should satisfy. It includes *e.g.* the pentagon identity, reducing to the delta function when one of the external entries set to the vacuum, matching with the known values at the degenerate entries, and also, as we extensively discussed in this paper, the equations (2), (4), (5) at the intermediate vacuum state.

These methods can be very useful to construct fusion matrix in the parafermionic Liouville field theory [44]. Parafermionic Liouville field theory is the simplest generalization of the supersymmetric Liouville theory. Whereas the supersymmetric Liouville theory is the Liouville field theory coupled to the Ising model, the parafermionic Liouville field theory is the Liouville field theory coupled to the parafermions. The structure constants of the parafermionic Liouville field theory at the level N are written in [44], mimicking the formulas (45)–(48), using the following generalization of the Υ_{NS} and Υ_R functions

$$\Upsilon_k^{(N)}(x)=\prod_{j=1}^{N-k}\Upsilon_b\left(\frac{x+kb^{-1}+(j-1)Q}{N}\right) \times \prod_{j=N-k+1}^{N}\Upsilon_b\left(\frac{x+(k-N)b^{-1}+(j-1)Q}{N}\right). \tag{140}$$

It is easy to check that these functions can be written as

$$\Upsilon_k^{(N)}(x)=\frac{1}{\Gamma_k^{(N)}(x)\Gamma_{N-k}^{(N)}(Q-x)}, \tag{141}$$

where

$$\Gamma_k^{(N)}(x)=\prod_{j=1}^{N-k}\Gamma_b\left(\frac{x+kb^{-1}+(j-1)Q}{N}\right) \times \prod_{j=N-k+1}^{N}\Gamma_b\left(\frac{x+(k-N)b^{-1}+(j-1)Q}{N}\right). \tag{142}$$

The functions $\Gamma_k^{(N)}(x)$ have the property

$$\frac{\Gamma_k^{(N)}(x+Q)}{\Gamma_k^{(N)}(x)}=W_k(x)=\frac{2\pi b^{\frac{(b-b^{-1})x}{N}}b^{\frac{2k}{N}-1}}{\Gamma\left(\frac{k}{N}+\frac{bx}{N}\right)\Gamma\left(1-\frac{k}{N}+\frac{b^{-1}x}{N}\right)}, \tag{143}$$

which is very similar to (149) and (150). Recall that these properties played crucial role in calculations in section 3. Therefore one can try to write an Ansatz for fusion matrix in the parafermionic Liouville field theory, mimicking (52), using the corresponding para version of the double Gamma and double Sine functions and matching it with the abovementioned criteria.

We should mention that there is also another route to construct directly the braiding/fusion matrix using chiral vertex operators and examining their braiding properties. This method was applied to the Liouville field theory in [11], and then generalized to the NS sector of $N=1$ super Liouville field theory in [25]. This method has an advantage to be constructive, but problem is that it is not an easy task to construct the chiral vertex operators. The attempt [26] to extend this program to the Ramond sector of the $N=1$ super Liouville field theory has encountered numerous problems. To implement it in the para Liouville field theory could be even more difficult. But nevertheless it may happen that we will get correct answer if somehow find a way to combine both methods.

It is well known that in the AGT correspondence Wilson lines in the $N=2$ $SU(N)$ ($SU(2)$) superconformal gauge theory on S^4 correspond to topological defects in Toda (Liouville) conformal field theory [45]. This was established noting that the Wilson line expectation values found by Pestun [46] coincide with the correlation function with defects in Toda (Liouville) conformal field theory. On the other hand, as we mentioned in introduction, it is found that $N=2$ $SU(N)$ ($SU(2)$) superconformal gauge theories on $S^4/\mathbb{Z}_p$ correspond to parafermionic Toda (Liouville) field theories. In particular $N = 2$ $SU(2)$ superconformal gauge theories on $S^4/\mathbb{Z}_2$ correspond to supersymmetric Liouville field theory. Thus having the topological defects in super Liouville theory one can test the AGT correspondence of $SU(2)$ superconformal gauge theory on $S^4/\mathbb{Z}_2$ with supersymmetric Liouville field theory in the presence of the Wilson lines. For this purpose one should generalize the Pestun [46] calculation of the Wilson line expectation value to $S^4/\mathbb{Z}_2$ and compare with the defect eigenvalues derived in section 6.

The Lagrangian of the $N = 1$ super Liouville field theory with the topological defect was introduced in [47]. In [41] the light and heavy semiclassical limits were used to match two-point correlation function with the Lagrangian approach for the bosonic Liouville theory in the presence of the defects. It is an interesting task to match, using various semiclassical techniques, the results of section 6 with the Lagrangian of [47].

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Appendix A. Useful formulae

The function $\Gamma_b(x)$

The function $\Gamma_b(x)$ is a close relative of the double Gamma function studied in [48,49]. It can be defined by means of the integral representation

$$\log \Gamma_b(x) = \int_0^\infty \frac{dt}{t} \left(\frac{e^{-xt} - e^{-Qt/2}}{(1-e^{-bt})(1-e^{-t/b})} - \frac{(Q-2x)^2}{8e^t} - \frac{Q-2x}{t} \right) . \tag{144}$$

Important properties of $\Gamma_b(x)$ are

1. Functional equation: $\Gamma_b(x+b) = \sqrt{2\pi} b^{bx-\frac{1}{2}} \Gamma^{-1}(bx) \Gamma_b(x)$.
2. Analyticity: $\Gamma_b(x)$ is meromorphic, poles: $x = -nb - mb^{-1}, n, m \in \mathbb{Z}^{\geq 0}$.
3. Self-duality: $\Gamma_b(x) = \Gamma_{1/b}(x)$.

The function $\Upsilon_b(x)$ may be defined in terms of $\Gamma_b(x)$ as follows

$$\Upsilon_b(x) = \frac{1}{\Gamma_b(x)\Gamma_b(Q-x)} . \tag{145}$$

It has the following property:

$$\Upsilon_b'(0) = \Upsilon_b(b) = \frac{2\pi}{\Gamma_b^2(Q)} . \tag{146}$$

In the super Liouville theory are important the functions

$$\Gamma_1(x) \equiv \Gamma_{\rm NS}(x) = \Gamma_b\left(\frac{x}{2}\right) \Gamma_b\left(\frac{x+Q}{2}\right) , \tag{147}$$

$$\Gamma_0(x) \equiv \Gamma_{\rm R}(x) = \Gamma_b\left(\frac{x+b}{2}\right) \Gamma_b\left(\frac{x+b^{-1}}{2}\right) . \tag{148}$$

They have the properties:

$$\frac{\Gamma_{\mathrm{NS}}(2\alpha)}{\Gamma_{\mathrm{NS}}(2\alpha-Q)} = W_{NS}(\alpha)\lambda^{\frac{Q-2\alpha}{2b}}, \tag{149}$$

$$\frac{\Gamma_{\mathrm{R}}(2\alpha)}{\Gamma_{\mathrm{R}}(2\alpha-Q)} = W_{\mathrm{R}}(\alpha)\lambda^{\frac{Q-2\alpha}{2b}}, \tag{150}$$

where $W_{NS}(\alpha)$, $W_{\mathrm{R}}(\alpha)$ are defined in (22) and (23), and $\lambda = \pi\mu\gamma\left(\frac{bQ}{2}\right)b^{1-b^2}$.

$\Gamma_{\mathrm{NS}}(x)$ has a pole in zero:

$$\Gamma_{\mathrm{NS}}(x) \sim \frac{\Gamma_{\mathrm{NS}}(Q)}{\pi x}. \tag{151}$$

The structure constants in the super Liouville theory are defined in terms of the functions:

$$\Upsilon_1(x) \equiv \Upsilon_{\mathrm{NS}}(x) = \Upsilon_b\left(\frac{x}{2}\right)\Upsilon_b\left(\frac{x+Q}{2}\right) = \frac{1}{\Gamma_{\mathrm{NS}}(x)\Gamma_{\mathrm{NS}}(Q-x)}, \tag{152}$$

$$\Upsilon_0(x) \equiv \Upsilon_{\mathrm{R}}(x) = \Upsilon_b\left(\frac{x+b}{2}\right)\Upsilon_b\left(\frac{x+b^{-1}}{2}\right) = \frac{1}{\Gamma_{\mathrm{R}}(x)\Gamma_{\mathrm{R}}(Q-x)}. \tag{153}$$

They have the properties:

$$\frac{\Upsilon_{\mathrm{NS}}(2x)}{\Upsilon_{\mathrm{NS}}(2x-Q)} = \mathcal{G}_{NS}(x)\lambda^{-\frac{Q-2x}{b}}, \tag{154}$$

$$\frac{\Upsilon_{\mathrm{R}}(2x)}{\Upsilon_{\mathrm{R}}(2x-Q)} = \mathcal{G}_{R}(x)\lambda^{-\frac{Q-2x}{b}}, \tag{155}$$

where $\mathcal{G}_{NS}(x)$ and $\mathcal{G}_{R}(x)$ are defined in (24) and (25).

The zeroes of Υ_{NS}, Υ_{R} are

$$\Upsilon_{\mathrm{NS}}(x) = 0 \quad \text{at} \quad x = -mb - nb^{-1}, \quad x = Q + mb + nb^{-1} \quad (m+n \quad \text{even}), \tag{156}$$

$$\Upsilon_{\mathrm{R}}(x) = 0 \quad \text{at} \quad x = -mb - nb^{-1}, \quad x = Q + mb + nb^{-1} \quad (m+n \quad \text{odd}). \tag{157}$$

We need also the values of the derivative $\Upsilon'_{\mathrm{NS}}(0)$ in zero:

$$\Upsilon'_{\mathrm{NS}}(0) = \frac{\pi}{\Gamma^2_{\mathrm{NS}}(Q)}. \tag{158}$$

To write fusion matrix we need also the functions:

$$S_1(x) \equiv S_{\mathrm{NS}}(x) = \frac{\Gamma_{\mathrm{NS}}(x)}{\Gamma_{\mathrm{NS}}(Q-x)}, \tag{159}$$

$$S_0(x) \equiv S_{\mathrm{R}}(x) = \frac{\Gamma_{\mathrm{R}}(x)}{\Gamma_{\mathrm{R}}(Q-x)}. \tag{160}$$

They have the properties:

$$\frac{S_{\mathrm{NS}}(2x)}{S_{\mathrm{NS}}(2x-Q)} = W_{NS}(x)W_{NS}(Q-x), \tag{161}$$

$$\frac{S_{\mathrm{R}}(2x)}{S_{\mathrm{R}}(2x-Q)} = W_R(x)W_R(Q-x). \tag{162}$$

And finally we need the following properties which can be easily obtained from the definitions and properties above:

$$\Gamma_{\rm A}(2Q-2\alpha)\Gamma_{\rm A}(Q-2\alpha)=\frac{W_A(Q-\alpha)\lambda^{-\frac{Q-2\alpha}{2b}}}{\Upsilon_{\rm A}(2\alpha)S_{\rm A}(2\alpha)}\,, \tag{163}$$

$$\Gamma_{\rm A}(2\alpha-Q)\Gamma_{\rm A}(Q-2\alpha)=\frac{\lambda^{-\frac{Q-2\alpha}{2b}}}{\Upsilon_{\rm A}(2\alpha)W_A(\alpha)}\,, \tag{164}$$

$$\Gamma_{\rm A}(2\alpha)\Gamma_{\rm A}(2\alpha-Q)=\frac{S_{\rm A}(2\alpha)\lambda^{-\frac{Q-2\alpha}{2b}}}{\Upsilon_{\rm A}(2\alpha)W_A(\alpha)}\,, \tag{165}$$

$$\Gamma_{\rm A}(2Q-2\alpha)\Gamma_{\rm A}(2\alpha)=\frac{W_A(Q-\alpha)\lambda^{-\frac{Q-2\alpha}{2b}}}{\Upsilon_{\rm A}(2\alpha)}\,, \tag{166}$$

where A takes values NS or R.

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