Folded Sheet of Paper

Sunaina Pai

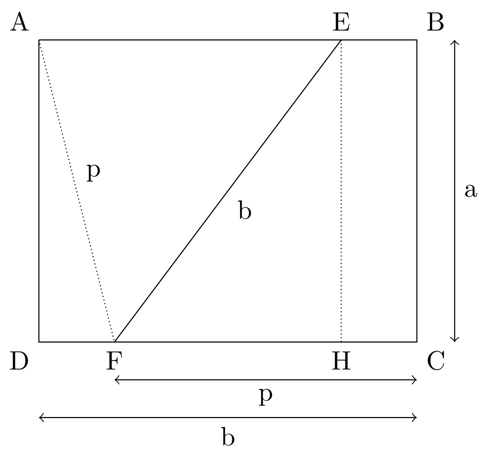
Problem

A rectangular sheet of paper is folded so that two diagonally opposite corners come together. If the crease formed is the same length as the longer side of the sheet, what is the ratio of the longer side of the sheet to the shorter side?

Source: http://www.qbyte.org/puzzles/puzzle01.html#p1

Solution

Let us represent the sheet of paper as a rectangle $ABCD$ where $AB > BC$. Let us represent the crease with the line segment EF such that the crease EF meets AB and DC at E and F, respectively. Let H be a point on DC such that $EH \perp DC$.



Let us define some of the line segments with single letter variables:

$$AB = DC = b,$$
$$AD = BC = a,$$
$$AF = FC = p.$$

Note that when the sheet is folded to meet at corners A and C, FC coincides with AF. This justifies $AF = FC$.

Since the length of the crease is same as longer side, we get

$$EF = b.$$

We compute DF as

$$DF = b - p. \tag{1}$$

By applying Pythagorean theorem to the right-angled $\triangle ADF$, we get

$$\begin{aligned} AF^2 = AD^2 + DF^2 &\implies p^2 = a^2 + (b-p)^2 \\ &\iff p^2 = a^2 + b^2 + p^2 - 2bp \\ &\iff p = \frac{a^2 + b^2}{2b}. \end{aligned}$$

If we turn the rectangle upside down, the problem remains the same, i.e. EB appears in the modified problem where DF is in the original problem. Therefore by symmetry and (1),

$$DF = EB = HC = b - p.$$

Now we compute FH as

$$\begin{aligned} FH &= DC - DF - HC \\ &= b - 2(b-p) \\ &= b - 2b + 2p \\ &= 2p - b \\ &= \frac{2\left(a^2 + b^2\right)}{2b} - b \\ &= \frac{a^2 + b^2}{b} - b \\ &= \frac{a^2 + b^2 - b^2}{b} \\ &= \frac{a^2}{b}. \end{aligned}$$

By applying Pythagorean theorem to the right-angled $\triangle EFH$ we get

$$\begin{aligned} EH^2 + FH^2 = EF^2 &\implies a^2 + \left(\frac{a^2}{b}\right)^2 = b^2 \\ &\iff b^4 - a^2b^2 - a^4 = 0. \end{aligned} \tag{2}$$

Let $\left(\frac{b}{a}\right)^2 = x$. Then $b^2 = a^2x$. Substituting this in (2) we get

$$\begin{aligned} b^4 - a^2b^2 - a^4 = 0 &\iff a^4x^2 - a^4x - a^4 = 0 \\ &\iff a^4(x^2 - x - 1) = 0. \end{aligned}$$

Since $a \neq 0$, we have $x^2 - x - 1 = 0$. Therefore

$$x = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}.$$

Since $x > 0$, we ignore the negative result for x and we get

$$\begin{aligned} x = \frac{1+\sqrt{5}}{2} &\iff \left(\frac{b}{a}\right)^2 = \frac{1+\sqrt{5}}{2} \\ &\iff \frac{b}{a} = \sqrt{\frac{1+\sqrt{5}}{2}}. \end{aligned}$$