Triangular Area

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Problem

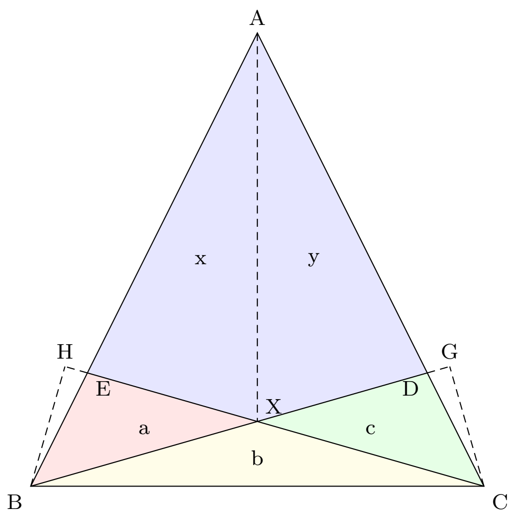
In $\triangle ABC$, produce a line from B to AC, meeting at D, and from C to AB, meeting at E. Let BD and CE meet at X. Let $\triangle BXE$ have area a, $\triangle BXC$ have area b, and $\triangle CXD$ have area c. Find the area of quadrilateral $AEXD$ in terms of a, b, and c.

Source: http://www.qbyte.org/puzzles/puzzle01.html#p2

Solution

Draw a line segment AX. Let the area of $\triangle AXE$ be x and the area of $\triangle AXD$ be y. We need to find $x + y$ in terms of a, b, and c.

Extend the line segments CE and BD such that BH meets CE and CG meets BD at right angles.



We know that the triangles with collinear bases have a common height and also their areas are in the ratio of their respective bases that are collinear.

From $\triangle BXE$ and $\triangle BXC$ we have

$$\frac{EX}{CX} = \frac{a}{b}.$$

From $\triangle BXC$ and $\triangle CXD$ we have

$$\frac{BX}{DX} = \frac{b}{c}.$$

From $\triangle AXB$ and $\triangle AXD$ we have

$$\begin{aligned} \frac{BX}{DX} = \frac{a+x}{y} \implies & \frac{b}{c} = \frac{a+x}{y} \\ \iff & by = cx + ac. \end{aligned} \tag{1}$$

From $\triangle AXE$ and $\triangle AXC$ we have

$$\begin{aligned} \frac{EX}{CX} = \frac{x}{y+c} \implies & \frac{a}{b} = \frac{x}{y+c} \\ \iff & bx = ay + ac. \end{aligned} \tag{2}$$

Substituting (2) in (1) we get

$$\begin{aligned} by = c\left(\frac{ay+ac}{b}\right) + ac \iff & b^2y - acy = ac^2 + abc \\ \iff & y = ac\left(\frac{b+c}{b^2-ac}\right). \end{aligned} \tag{3}$$

Substituting (1) in (2) we get

$$\begin{aligned} bx = a\left(\frac{cx+ac}{b}\right) + ac \iff & b^2x - acx = a^2c + abc \\ \iff & x = ac\left(\frac{a+b}{b^2-ac}\right). \end{aligned} \tag{4}$$

Adding (3) and (4) we get the area of quadrilateral $AEXD$ as

$$x + y = ac\left(\frac{b+c}{b^2-ac}\right) + ac\left(\frac{a+b}{b^2-ac}\right) = ac\left(\frac{a+2b+c}{b^2-ac}\right).$$