MP307 Practical 2017/2018: Queuing Theory - Markov Chains 2

A number of Maple procedures relevant to queueing theory are available in the Maple worksheet `Practical2.mw` that can be downloaded from the MP307 Blackboard page.
These may be read directly into your Maple session by opening this worksheet within a Maple session.

The procedures contained in this worksheet are:

- `Uniform()`. This generates a pseudo-random number with uniform probability on [0,1].
- `Nearneigh(pup,pdown,r)`. This generates the transition matrix `P` for a nearest neighbour model with maximum size `r` and probability `pup` of one step up transition and `pdown` for one step down transition. The output is the transition matrix.
- `Equilibrium(P)`. This computes the equilibrium probabilities for a given transition matrix `P`. If the system is not ergodic then an error message appears. The output is a globally defined vector `pi`.
- `Queue(P,n0,nit)`. This simulates a queue with any transition matrix `P` with initial queue size `n0` for `nit` iterations. The output is a list of simulated queue data.
- `Qplot(qdata,pi)`. This generates up to 50 animations of the queue for any input data list `qdata`. The second argument is optional and consists of the equilibrium probabilities `pi`(if they exist and have been calculated via `Equilibrium(P)` above). The output is a Maple animation where up to 50 plots are shown of the queue evolving. The normalised frequency of events is also plotted which can be compared to a plot of the equilibrium probabilities `pi` if provided. It is recommended that you chose the plot display/window in the Maple options menu.

Some help to understand how the above functions works and how to solve the questions below can be found in `Practical2.mw`.

Notice

Solutions to the questions marked with **(*)** has to be shown (and explained) to the instructor at the practicals in order to get 6% that count towards the overall mark.

1. Simulate the Markov process with transition matrix:

$$\begin{pmatrix} 1/2 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 1/4 & 0 & 1/2 & 0 & 1/4 \\ 1/4 & 0 & 1/4 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 1/2 \end{pmatrix}$$

   This is the same example as in Practical 1. Observe the long-time behaviour for initial queue size 0 and 1. Observe what happens if the first row is changed to

$$\begin{pmatrix} 1/4 & 1/4 & 1/2 & 0 & 0 & 0 \end{pmatrix}$$

2. (*) In the file `Practical2.mw` a queue is observed 1000 time intervals (same as Question 4 of Practical 1). Compare the actual frequencies of events with a simulated one. Use the probabilities found in Practical 1.

   Next, consider the queue observed over 10000 time intervals in Question 5 of Practical 1, where the size of the queue after each time step is given in the file `qdata.mw`.

   (a) Construct a simple model for this queue as a Markov chain with only nearest neighbour interactions.

   - Estimate the transition probabilities.
   - What is the expected behaviour of the queue as time continues?
   - Is the system ergodic ?

   (b) Suppose that the time step used is known to be 10 sec.

   - What is the average waiting time for customer service/arrival?
   - What is average number of servicings/arrivals per minute?

   (c) Compare the actual frequencies of events with simulated ones.

3. (*) **Telephone Exchange Queue**. A telephone exchange consists of N operators on N lines. Assume that the calling and servicing patterns are Poisson with parameters $\alpha_k = \alpha,\ \beta_k = k\beta$ respectively for queue size of $0 \le k \le N$ and for some $\alpha, \beta > 0$. The equilibrium distribution for the queue is:

$$\pi_k = \frac{\rho^k}{k!}\left(1 + \rho + \frac{\rho^2}{2!} + ... + \frac{\rho^N}{N!}\right)^{-1},\ \rho = \frac{\alpha}{\beta}$$

Of particular interest, in this model, is the frequency with which the exchange becomes saturated over a long period of time i.e. π_N. This should be a small number for an efficient exchange since it provides a measurement of the number of calls lost to the system.

(a) A telephone exchange has 5 operators who can handle 40 calls an hour each on average. If calls arrive at the exchange on average every 40 secs, show that there is approximately a 5.5% chance that the exchange will become saturated after a long time.

(b) Simulate this system over 3 hours with 10 sec time intervals. For what proportion of your simulated events do you find all operators are busy?

Maple Notes

You may also find it useful to perform some do loops. The simplest sort of example is :

```
> for i from 1 to 10 do
> maple statement(s);
> end do;
```

where `i` is a counter which runs from 1 to 10 in this example.
Note the use of the format `for ... do ... end do;`.
You may also need the Maple conditional if statement which is illustrated by following example:

```
> if x<=2 then
> maple statement(s);
> elif x < 4 then
> maple statement(s);
> else
> maple statement(s);
> end if;
```

Note the use of the format `if ... elif ... else ... end if;`.