MP307 Practical 2017/2018: Queueing Theory - Markov Chains I

Download the Maple file `Practical1.mw` from the MP307 Blackboard page and use it to answer the following questions.

Notice

Solutions to the questions marked with **(*)** has to be shown (and explained) to the instructor at the practicals in order to get 6% that count towards the overall mark.

1. Consider the two state telephone system discussed in class with transition matrix $P(i,j)$ for a given time step:
$$\begin{pmatrix} 1/2 & 1/2 \\ 2/3 & 1/3 \end{pmatrix}$$
Find Prob($i \to j$ in 5 steps). Find the equilibrium probabilities π_0 and π_1.

2. **(*)** A finite queue of maximum size 3 is observed with the following transition matrix $P(i,j)$ for a given time step:
$$\begin{pmatrix} 1/3 & 0 & 2/5 & 4/15 \\ 1/4 & 0 & 3/10 & 9/20 \\ 0 & 2/3 & 1/5 & 2/15 \\ 1/5 & 0 & 2/5 & 2/5 \end{pmatrix}$$
Find Prob($i \to j$ in 10 steps). Find the equilibrium probabilities $\pi_0, \pi_1, \pi_2, \pi_3$.

3. **(*)** Consider the random walk on 6 sites with the following transition matrix.
$$\begin{pmatrix} 1/2 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 1/4 & 0 & 1/2 & 0 & 1/4 \\ 1/4 & 0 & 1/4 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 1/2 \end{pmatrix}$$
Is the system ergodic? Compare your result to that for the modified random walk with transition matrix below and explain the observed difference in behaviour.
$$\begin{pmatrix} 1/4 & 1/4 & 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 1/4 & 0 & 1/2 & 0 & 1/4 \\ 1/4 & 0 & 1/4 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 1/2 \end{pmatrix}$$

4. **(*)** A queue is observed over 1000 time intervals where the size of the queue after each time step is given. Construct a simple model for this queue as a Markov chain with only nearest neighbour interactions. What is the expected behaviour of the queue as time continues? Is the system ergodic?

5. **(*)** A queue is observed over 10000 one-second time intervals with data as given in the Maple worksheet `qdata.mw` that can be downloaded from the MP307 Blackboard page. Construct a Poisson nearest neighbour model with a single arrival and servicing pattern and hence answer the following questions:

   (a) What is the average time taken for 1 customer to arrive?

   (b) What is the average number of customer servicings per second?

   (c) What is your estimate for the equilibrium probability $P(n \geq 4)$, where n is the queue size in this model?

   (d) Suppose that two equivalent servers are introduced. What would the equilibrium probability $P(n \geq 4)$ then be?