TECHNISCHE HOCHSCHULE DEGGENDORF

Lattice Boltzmann Method

Huseyn Gurbanov
Tejas Shinde

1. INTRODUCTION: -

Lattice Boltzmann Method is a dynamic method that simulates the macroscopic behavior of fluids by using a simple mesoscopic model. It inherited the main principles of Lattice Gas Automaton (LGA) and made improvements. From lattice gas automaton, it is possible to derive the macroscopic Navier-Stokes equations.

Specialty of Lattice Boltzmann Method & Difference from the traditional macroscopic
numerical calculation method:

1. It is based on and starts from non-equilibrium statistical mechanics and Discrete model
2. It connected dynamic lattice model, whose time, space, and velocity phase space are fully
discrete, with Boltzmann equation.
3. The implementation of this method can describe the law of fluid motion without Solving Navier-Stokes equations
4. Most of the calculations are local and more suitable for parallel
5. However, one of the disadvantages of LBM is requiring a lot of memory to store the distribution function, which is also the main chokepoint of LBM.

From the Geometry side:

• LBM is well suited for mass-conservative fluid simulation of complex boundaries
• LBM can well realize mass-conserving mobile boundary problems and it is very attractive
for soft material simulation

This is graph which represent how we approached to the problem from the side of perspective and solving Lid Driven Cavity problem.

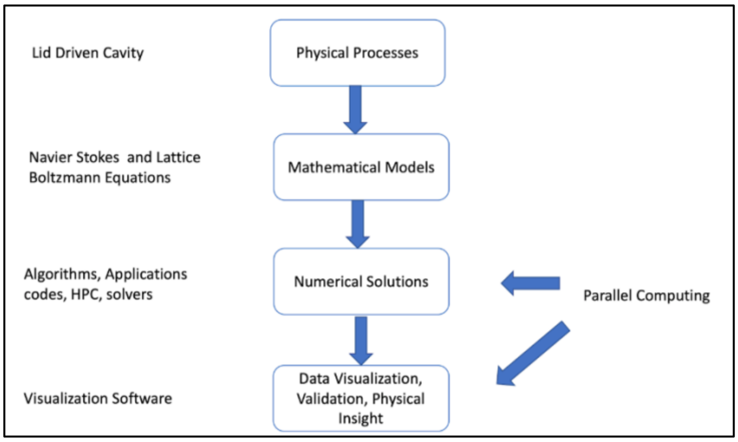


Figure 1: Flow Chart to tackle the Fluid Problem

The lid-driven cavity is a well-known benchmark problem for slow moving incompressible fluid flow, and it is used as test benchmark problem to test CFD codes for validating computational methods. While the boundary conditions are relatively simple, the flow features created are quite interesting and complex.

2. PROBLEM STATEMENT:

LID DRIVEN CAVITY: -

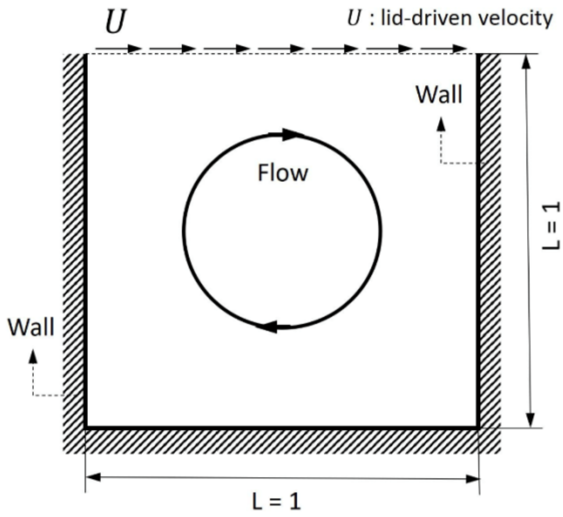


Figure 2: Geometry of LID-DRIVEN CAVITY

The lid driven cavity is a classical fluid dynamics problem, in this problem we are dealing with a square cavity consisting of three rigid walls with no-slip conditions and a lid moving with tangential velocity.
The three walls are rigid that is the velocities of the wall are zero and only the upper wall boundary moves in the x direction with constant velocity.

3. MATHEMATICAL MODEL: -

Lattice Boltzmann method: -

The Boltzmann equation talks about the evolution of the probability distribution function.

3.1 The Boltzmann transport equation: -

The equation is derived using conservation principles is given as: -

$$\frac{\partial f}{\partial t}+\frac{\partial f}{\partial x_i}\left(\frac{dx_i}{dt}\right)+\frac{\partial f}{\partial \xi_i}\left(\frac{d\xi_i}{dt}\right)=\Omega(f)$$

$\frac{dx_i}{dt} = \xi_i$ - microscopic particle velocity

$\frac{F_i}{\rho} = \frac{d\xi_i}{dt}$ - Body force per unit mass

F_i - Body force per unit volume

LBM is mesoscopic technique that makes use of microscopic and macroscopic properties. So, this $\frac{dx_i}{dt} = \xi_i$ is the microscopic particle velocity.

<u>3.2 Moments</u>: - Integral of Distribution function over entire velocity space weighted with some function of the microscopic velocity.
The probability distribution function is connected to macroscopic variables density rho and velocity u through moments.

The moments are given as: -

Density - $\rho(\vec{x},t) = \int f(\vec{x},\vec{\xi},t)\,\mathrm{d}\xi$

Momentum Density –

$$\rho(\vec{x},t)\vec{\mathrm{u}}(\vec{\mathrm{x}},\mathrm{t}) = \int f(\vec{x},\vec{\xi},t)\,\overrightarrow{*\,\xi}\;\mathrm{d}\xi$$

3.3 $\boldsymbol{\Omega(f)}$ – *Collision operator* –

Obtained from the BGK operator (Bhatnagar, Gross and Krook) the simplest one that can be used for Navier stokes simulation.

$$\Omega(f) = -\frac{(f - f^{eq})}{\tau}$$

It relaxes the populations towards an equilibrium f eq i at a rate determined by the relaxation time

τ – relaxation time representing amount of time it consumed to return to equilibrium state
f^{eq} - Equilibrium Distribution Function

3.4 Equilibrium Distribution Function

The equilibrium function is given by: -

$$\boldsymbol{f_\alpha^{eq}(\vec{x},t) = \omega_\alpha\rho\left(1+\frac{\overrightarrow{u.C_\alpha}}{c_s^2}+\frac{(\vec{u}\cdot C_\alpha)^2}{2c_s^4}-\frac{\overline{u.u}}{2c_s^2}\right)}$$

ρ – macroscopic density $(\vec{x},t)$
$\vec{u}$ - macroscopic velocity $(\vec{x},t)$
c_s - Speed of sound
$c_s^2 = \frac{1}{3}\left(\frac{\Delta x}{\Delta t}\right)^2$ - Δx = 1 and Δt = 1 therefore $c_s^2 = \frac{1}{3}$

ω_α - weights according to the Discrete velocity set , for example – D2Q9 , α = 0 then ω_α = 4/9 ,
α = 1-4 then ω_α = 1/9,
α = 5-8 then ω_α = 1/36
$\sum \omega_\alpha$ = 1

The equilibrium is such that its moments are the same as those of

Density - $\rho(\vec{x},t) = \sum_{\alpha} f_{\alpha}(\vec{x},t)$

Momentum Density –
$\rho(\vec{x},t)\vec{\mathrm{u}}(\vec{\mathrm{x}},\mathrm{t}) = \sum_{\alpha} f_{\alpha}(\vec{x},t)\, c_{\alpha}$

The equilibrium f_{α}^{eq} depends on the local quantities of density ρ and fluid velocity u only. These are calculated from the local values of f_{α} by (with the fluid velocity found as

$$\vec{\mathrm{u}}(\vec{\mathrm{x}},\mathrm{t}) = \frac{\rho(\vec{x},t)\vec{\mathrm{u}}(\vec{\mathrm{x}},\mathrm{t})}{\rho(\vec{x},t)}$$

The link between the Lattice Boltzmann Equation and the Navier Stokes Equation can be determined using the Chapman Enskog analysis. Through this, we can show that the LBE results in macroscopic behavior according to the Navier Stokes Eq, with the kinematic shear viscosity given by the relaxation time as

$$v = c_s^2 \Delta t \cdot \left(\frac{\tau}{\Delta t} - \frac{1}{2}\right)$$

and the kinematic bulk viscosity given as

$$V_b = \frac{2v}{3}$$

3.5 Lattice Arrangements: -

the lattice arrangements in 2D and 3D and corresponding magnitudes and weights.

3.5.1 D2Q9: -

This model is very common, especially for solving fluid flow problems. It has nine velocity vectors, with the central particle speed equal to zero. The representation of discrete velocity sets is shown below: -

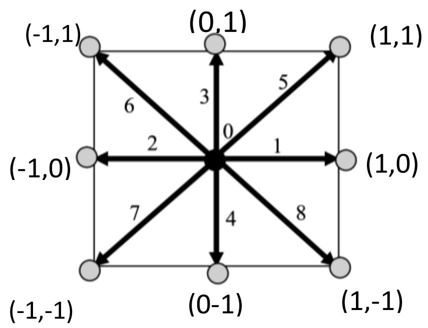


Figure 3: Discrete Velocity Set D2Q9 and corresponding velocities

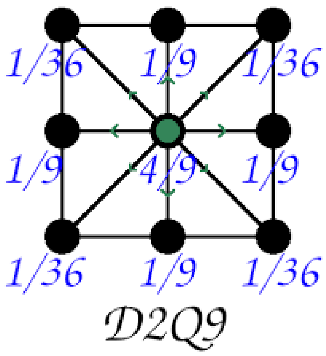


Figure 4: Weights of the velocities

3.5.2 D3Q19 -

This model has 19 velocity vectors, with a central vector of speed zero.

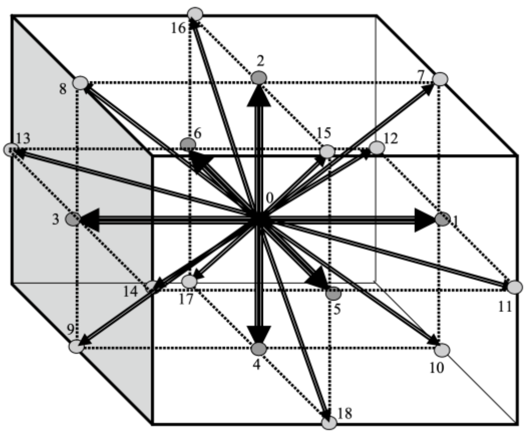


Figure 5: Discrete Velocity Set D3Q19

The weighting factors are as follows:
for f0 it is 12/36,
for f1 to f6 it is 2/36, and
for f7 to f18 it is 1/36

3.6 Discretized Boltzmann Equation: -

The Boltzmann Force Free equation is given as follows: -

$$\frac{\partial f}{\partial t} + \xi_i \frac{\partial f}{\partial x_i} = \Omega(f) = \; -\frac{(f - f^{eq})}{\tau} * \Delta t$$

The above equation is the workhorse of the lattice Boltzmann method and replaces the Navier–Stokes equation in CFD simulations. It is possible to derive the Navier– Stokes equation from the Boltzmann equation.

1. The equation is a linear partial differential equation.

2. The equation resembles an advection equation with a source term.
3. The left-hand side of the equation represents the advection (streaming).
4. The right-hand-side term represents the collision process, the source term.
The discretized Boltzmann Equation can be written as: -

$$f_\alpha(\vec{x} + C_\alpha \Delta t, t + \Delta t) - f_\alpha(\vec{x}, t) = -\frac{\left(f_\alpha(\vec{x}, t) - f_\alpha^{eq}(\vec{x}, t)\right)}{\tau / \Delta t}$$

We can decompose this equation into two distinct parts that are performed in succession:

3.6.1. Collision (or relaxation),

The First part is the collision step, and the equation is as given as follows:

$$\boldsymbol{f_\alpha^*(\vec{x}, t) = f_\alpha(\vec{x}, t) - \frac{\Delta t}{\tau}(f_\alpha(\vec{x}, t) - f_\alpha^{eq}(\vec{x}, t))}$$

Where $f_\alpha^*(\vec{x}, t)$ represents the distribution function after collisions and $f_\alpha^{eq}(\vec{x}, t)$ is found from through. And it is convenient and efficient to implement collision in the form

$$f_\alpha^*(\vec{x}, t) = f_\alpha(\vec{x}, t)\left(1 - \frac{\Delta t}{\tau}\right) - f_\alpha^{eq}(\vec{x}, t) * \left(\frac{\Delta t}{\tau}\right)$$

3.6.2. Streaming (or propagation)

The second part is streaming. Post collision function is transferred into post streaming function

$$\boldsymbol{f_\alpha^*(\vec{x} + \vec{c}_\alpha \Delta t, t + \Delta t) = f_\alpha^*(\vec{x}, t)}$$

For example: - Consider this initial distribution function or pre collision distributions

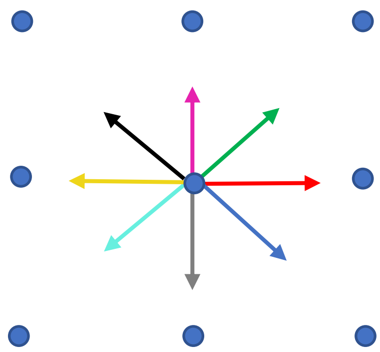


Figure 7: Initial distribution function

now we undergo collision so these distributions will now rearrange the probabilities, and we will have one distribution getting longer one shorter and this may be the representation after the collision step,

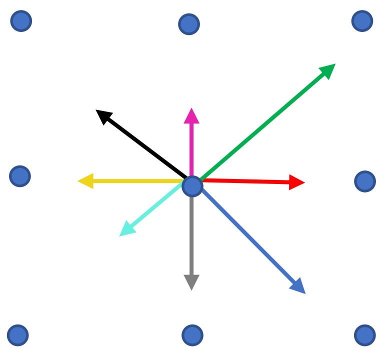
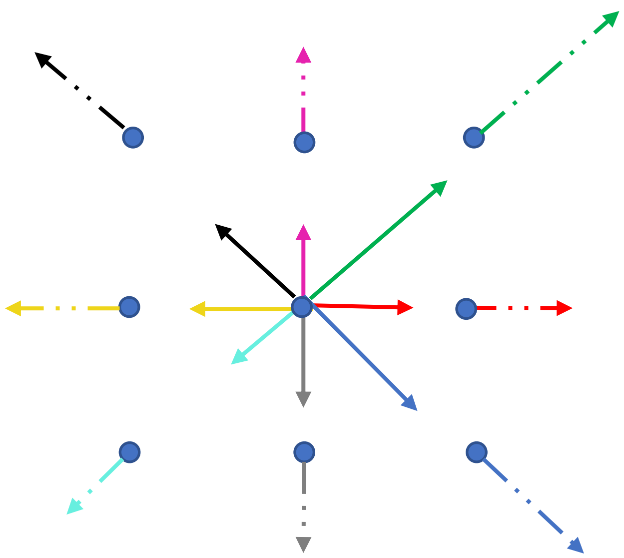


Figure 8: Post collision Distribution Functions

now we have the streaming step, streaming means distributions will be pushed out, so here in the equation you can see the function at position X have to be moved into the neighboring position $\vec{x} + \vec{c}_\alpha \Delta t$, so after streaming steps the distributions are moved to the neighboring point as shown



4. Boundary Condition

In the LBE, the boundary conditions apply at boundary nodes which are sites with at least one link to a solid and a fluid node. Rather than specifying the macroscopic variables of interest, such as rho and u, LB boundary conditions apply to the mesoscopic populations f, giving more degrees of freedom than the set of macroscopic variables.

4.1 Bounce-Back Boundary Condition

In order to obtain a no-slip boundary condition for a specified boundary, we use the "Bounce Back Method. The simplest scheme is to place a wall halfway between a wall grid point and a fluid grid point and then "bounce-back" particles that stream into the wall. For instance, f_4, f_7, and f_8 stream into the wall, and are bounced back by setting $f_5=f_7$, $f_2=f_4$, and $f_6=f_8$.

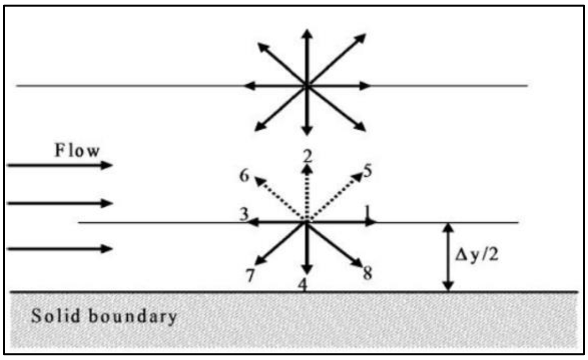
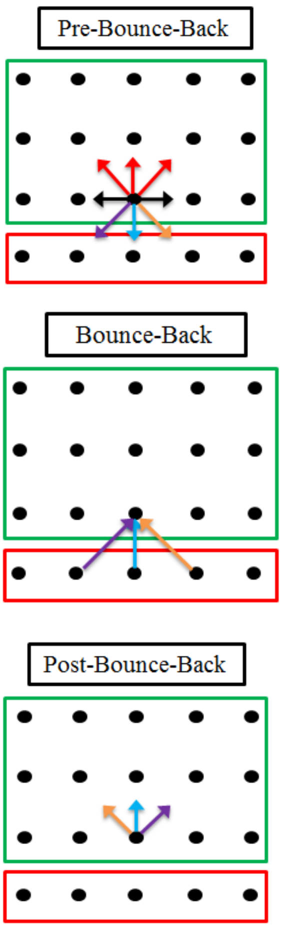
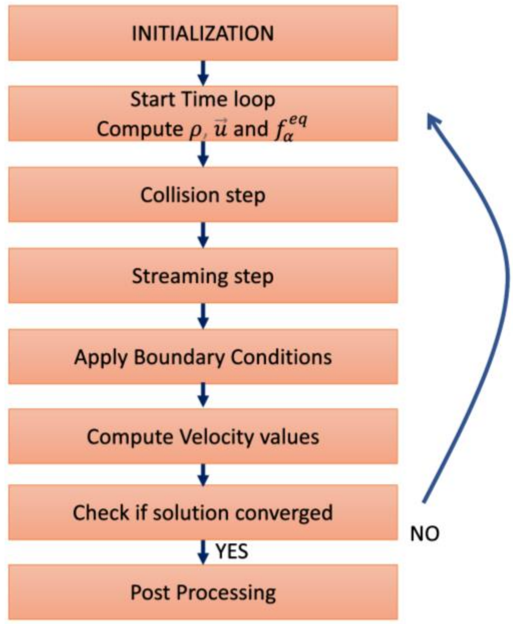


Illustration of the Half way bounce back B.C.



5. IMPLEMENTATION –

The Time step algorithm



The core LBM algorithm consists of a cyclic sequence of sub steps, with each cycle corresponding to one time step.

These sub steps are also visualized in fig above-

1.Compute the macroscopic moments $\rho(\vec{x},t)$ and $\vec{u}(\vec{x},t)$ from $f_\alpha(\vec{x},t)$ via

2. Obtain the equilibrium distribution $f_\alpha^{eq}(\vec{x},t)$ from

3. Perform collision (relaxation) as shown in

4. Perform streaming (propagation) via

5. Apply boundary conditions

6. Increase the time step, setting t to $t+\Delta t$, and go back to step 1 until the last time step or convergence has been reached.

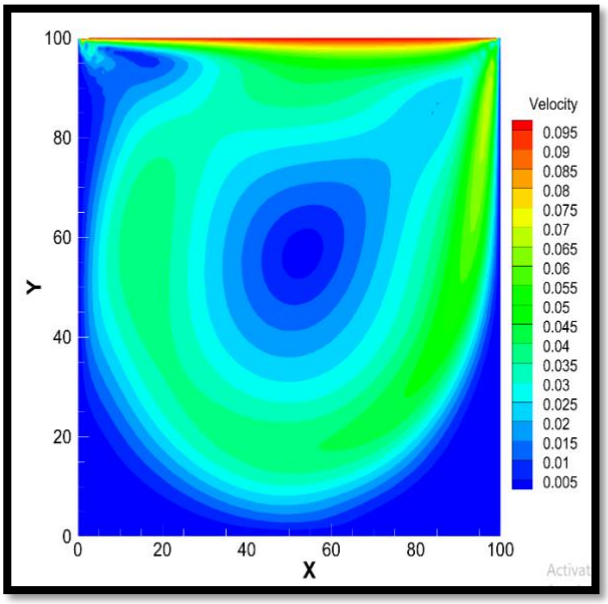
X, Y, U, V, Velocity components of Data.dat output file for visualization respectively.
When we are loading this file to "Tecplot" or any .dat file visualization software we can see below results in terms of U, V and velocity value changing.

1	1	2.41267e-06	-1.6637e-06	2.93068e-06
2	1	9.85327e-06	-5.01565e-06	1.10564e-05
3	1	2.12781e-05	-6.30179e-06	2.21917e-05
4	1	3.50113e-05	-7.36385e-06	3.57773e-05
5	1	4.91404e-05	-6.97094e-06	4.96323e-05
6	1	6.21798e-05	-5.87511e-06	6.24567e-05
7	1	7.27144e-05	-4.40173e-06	7.28475e-05
8	1	7.99431e-05	-3.14321e-06	8.00049e-05
9	1	8.4179e-05	-1.28921e-06	8.41889e-05
10	1	8.46854e-05	5.9549e-07	8.46875e-05
11	1	8.1447e-05	2.61217e-06	8.14889e-05
12	1	7.46571e-05	4.58588e-06	7.47978e-05
13	1	6.42451e-05	6.66701e-06	6.45902e-05
14	1	5.01253e-05	8.69903e-06	5.08745e-05
15	1	3.18125e-05	1.05561e-05	3.35181e-05
16	1	9.8378e-06	1.23149e-05	1.57619e-05
17	1	-1.65293e-05	1.4129e-05	2.1745e-05
18	1	-4.70678e-05	1.63267e-05	4.98191e-05
19	1	-8.19588e-05	1.88805e-05	8.41054e-05
20	1	-0.000121531	2.11243e-05	0.000123353
21	1	-0.000165811	2.37978e-05	0.00016751
22	1	-0.000215631	2.63914e-05	0.00021724
23	1	-0.00027071	2.90987e-05	0.000272269
24	1	-0.000331322	3.20607e-05	0.00033287
25	1	-0.000397763	3.52283e-05	0.00039932
26	1	-0.000470319	3.83714e-05	0.000471881
27	1	-0.000548853	4.16802e-05	0.000550433
28	1	-0.000633338	4.5035e-05	0.000634937
29	1	-0.000724377	4.8178e-05	0.000725977
30	1	-0.000821529	5.11706e-05	0.000823121
31	1	-0.000924578	5.40343e-05	0.000926155
32	1	-0.00103335	5.70913e-05	0.00103493
33	1	-0.00114768	5.96695e-05	0.00114923

33	1	-0.00114768	5.96695e-05	0.00114923
34	1	-0.00126648	6.19283e-05	0.001268
35	1	-0.00138946	6.37913e-05	0.00139093
36	1	-0.00151549	6.53226e-05	0.0015169
37	1	-0.00164321	6.65256e-05	0.00164456
38	1	-0.00177232	6.69518e-05	0.00177359
39	1	-0.00190085	6.64204e-05	0.00190201
40	1	-0.00202755	6.54624e-05	0.00202861
41	1	-0.00215135	6.3562e-05	0.00215229
42	1	-0.00226998	6.08268e-05	0.00227079
43	1	-0.00238257	5.74347e-05	0.00238326
44	1	-0.00248644	5.2861e-05	0.002487
45	1	-0.00258059	4.76305e-05	0.00258103
46	1	-0.00266303	4.11632e-05	0.00266335
47	1	-0.00273197	3.41468e-05	0.00273218
48	1	-0.00278596	2.63693e-05	0.00278609
49	1	-0.00282368	1.75486e-05	0.00282373
50	1	-0.00284386	8.29233e-06	0.00284387
51	1	-0.00284523	-1.58049e-06	0.00284523
52	1	-0.00282676	-1.17261e-05	0.00282679
53	1	-0.00278794	-2.24422e-05	0.00278803
54	1	-0.00272863	-3.29154e-05	0.00272883
55	1	-0.00264919	-4.34742e-05	0.00264955
56	1	-0.00254945	-5.35724e-05	0.00255001
57	1	-0.00243056	-6.3256e-05	0.00243138
58	1	-0.00229326	-7.239e-05	0.0022944
59	1	-0.0021392	-8.08304e-05	0.00214073
60	1	-0.00197006	-8.82089e-05	0.00197203
61	1	-0.00178803	-9.43477e-05	0.00179052
62	1	-0.00159582	-9.91855e-05	0.0015989
63	1	-0.00139546	-0.000102446	0.00139922
64	1	-0.00118967	-0.000104397	0.00119424
65	1	-0.000981388	-0.000104808	0.000986969

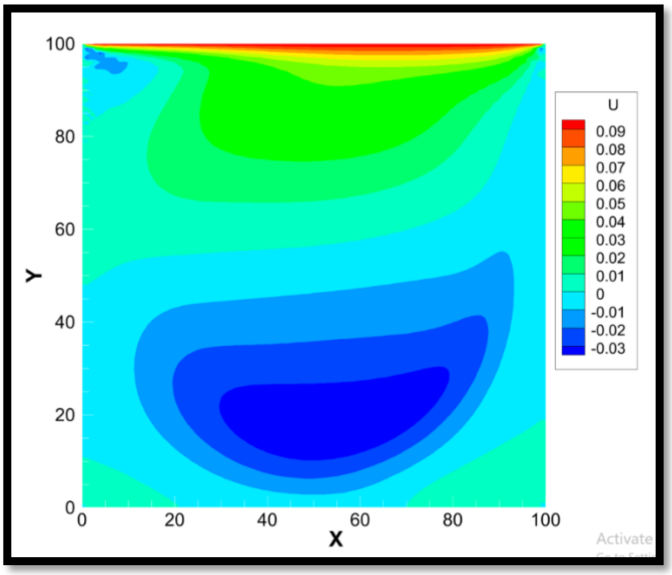
Import there Data file into Tecplot or Paraview and visualize the velocity by plotting contours.

6.Results: -

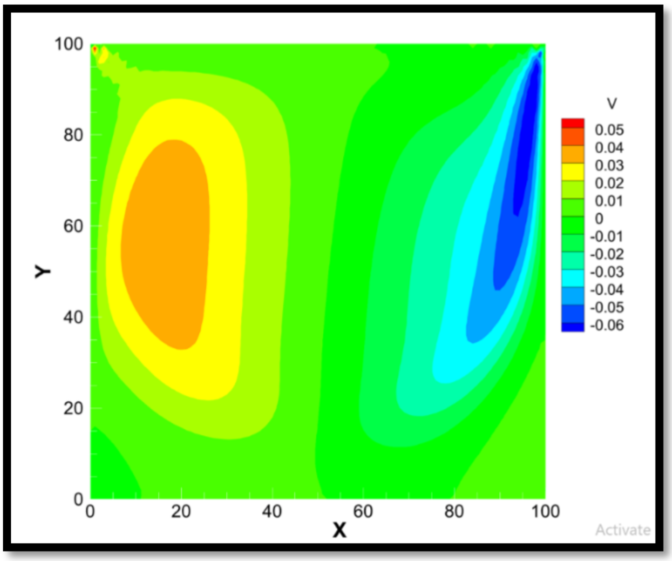


We can see that for lower Reynolds numbers, the flow separates near the bottom left and right corners and two vortices are formed. As the Reynolds number increases, there is more inertia in the flow, causing it to separate earlier along the wall and create larger corner vortices. Increasing the Reynolds number further, a third vortex forms in the top left corner. For the highest Reynolds number (10,000), two vortices are present in the bottom corners in addition to the one in the top left corner.

First, we check the magnitude of the velocity in the cavity, plotted with the rainbow color scale, and the direction of the flow, indicated with the vector plot. We see that the velocity approaches U = 1 at the top of the cavity, where the fluid flow is being driven by the moving wall. The fluid is pushed into the wall on the right, where it flows downward before moving back up the left side of the cavity. This motion creates a large vortex in the center of the cavity. As the Reynolds number increases to 10,000, we see that the velocities are higher in the cavity and the vortex extends more obviously into the bottom of the cavity.



This contour plot shows how the fluid particles moves in +ve X and -ve X direction. The Fluid particles in the region near to the moving lid the velocity is high as shown by the red color. And in the bottom with dark blue color the fluid moves from right to the left.



Similarly, this contour plot shows the velocity in y direction. On the right side of the fluid domain (blue colour), the fluid is travelling downwards due to the inertia force and on the left side the fluid is moving upwards to fill the cavity formed by the fluid in that region moving in +ve X direction.

7.References:

1.Lattice Boltzmann Method
Fundamentals and Engineering Applications with Computer Codes- A.A Mohamad

2.The Lattice Boltzmann Method - Principles and Practice
Timm Krüger, Halim Kusumaatmaja, Alexandr Kuzmin, Orest Shardt, Goncalo Silva, Erlend Magnus Viggen