PIP1B Supervision 1: Exchange and Competitive Equilibrium

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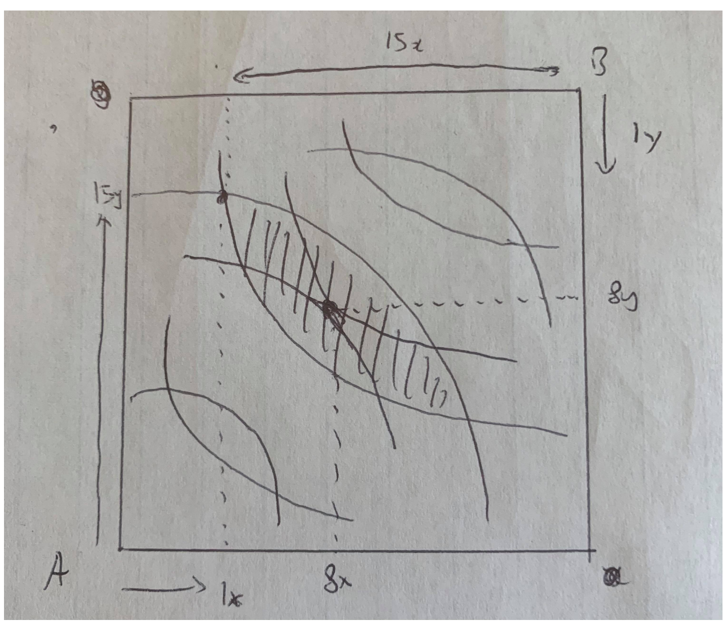
1 Pareto Efficiency

Suppose agents A and B with identical Cobb-Douglas preferences of $u(x,y) = \sqrt{xy}$ with endowments $(10,1)$ and $(1,10)$. A Pareto efficient allocation is where they have the same MRS. When we have allocation $(10,10)$ and $(1,1)$, we get $MRS_A = -\frac{MU_x}{MU_y} = -\frac{0.5\sqrt{y}/\sqrt{x}}{0.5\sqrt{x}/\sqrt{y}} = -\frac{10}{10} = -1$ and $MRS_B = -1$. This is Pareto efficient. When we have $(6,5)$ and $(5,6)$, we get $MRS_A = -\frac{6}{5}$ and $MRS_B = -\frac{5}{6}$. This is Pareto inefficient, and yet agent B has a higher utility of $\sqrt{30}$ instead of $\sqrt{11}$. Hence a Pareto-efficient allocation does not always result in a given agent's utility being higher than in a Pareto-inefficient allocation.

2 Alan and Betty

(a) We can see that the set of Pareto-improving trades available given Alan and Betty start from their endowments is given by the area above their indifference curves at the endowment which overlap i.e. the shaded disc. If they trade 7 units of x for 7 units of y, we get an efficient outcome where the indifference curves are tangential. We can see this by looking at the fact that both will have the same marginal rate of substitution, since $MRS_A(8,8) = -\frac{MU_x}{MU_y} = -\frac{y}{x} = -\frac{8}{8} = -1$ and $MRS_B(8,8) = -\frac{8}{8} = -1$.



(b) A contract curve describes the set of points which represent Pareto efficient allocations - that is, $MRS_A = MRS_B$. Thus we have the equation of the contract curve in (3).

$$\frac{y_A}{x_A} = \frac{y_B}{x_B} \tag{1}$$

$$y_A(16 - x_A) = (16 - y_A)x_A \tag{2}$$

$$x_A = y_A \tag{3}$$

(c) The net demand function is the gross demand function minus the endowment. The gross demand function for Alan is given by $\max x_A^{\frac{1}{4}} y_A^{\frac{1}{4}}$ subject to $x_A p_x + y_A p_y = p_x + 15p_y$.

$$\mathcal{L}(x_A, y_A) = x_A^{\frac{1}{4}} y_A^{\frac{1}{4}} + \lambda(x_A p_x + y_A p_y - p_x + 15p_y) \tag{4}$$
$$\frac{d\mathcal{L}}{dx_A} = \frac{1}{4} x_A^{-\frac{3}{4}} y_A^{\frac{1}{4}} + \lambda p_x = 0 \tag{5}$$
$$\frac{d\mathcal{L}}{dy_A} = \frac{1}{4} x_A^{\frac{1}{4}} y_A^{-\frac{3}{4}} + \lambda p_y = 0 \tag{6}$$
$$\frac{d\mathcal{L}}{d\lambda} = x_A p_x + y_A p_y - p_x + 15p_y = 0 \tag{7}$$

From (5) and (6) we can see that $\frac{y_A}{x_A} = \frac{p_x}{p_y}$. We can substitute this into (6) to produce $x_A p_x + \left(\frac{x_A p_x}{p_y}\right) p_y = p_x + 15p_y$.

$$x_A = \frac{p_x + 15p_y}{2p_x} \tag{8}$$
$$y_A = \frac{p_x + 15p_y}{2p_y} \tag{9}$$
$$e_A^x = \frac{p_x + 15p_y}{2p_x} - 1 \tag{10}$$
$$e_A^y = \frac{p_x + 15p_y}{2p_y} - 15 \tag{11}$$

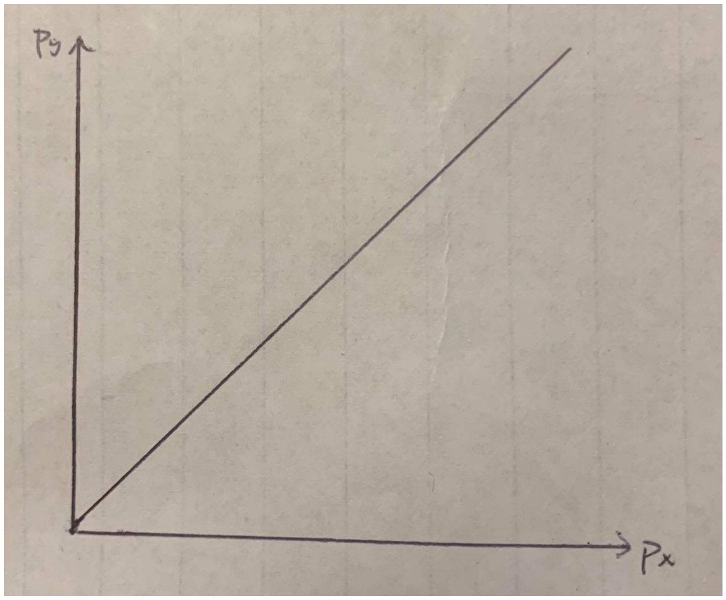
In a similar fashion, we can find the net demand for Betty too, by looking for the gross demand given by $\max x_B^{\frac{1}{4}} y_B^{\frac{1}{4}}$ such that $x_B p_x + y_B p_y = 15p_x + p_y$.

$$\mathcal{L}(x_B, y_B) = x_B^{\frac{1}{4}} y_B^{\frac{1}{4}} + \lambda(x_B p_x + y_B p_y - 15p_x + p_y) \tag{12}$$
$$\frac{d\mathcal{L}}{dx_B} = \frac{1}{4} x_B^{-\frac{3}{4}} y_B^{\frac{1}{4}} + \lambda p_x = 0 \tag{13}$$
$$\frac{d\mathcal{L}}{dy_B} = \frac{1}{4} x_B^{\frac{1}{4}} y_B^{-\frac{3}{4}} + \lambda p_y = 0 \tag{14}$$
$$\frac{d\mathcal{L}}{d\lambda} = x_B p_x + y_B p_y - 15p_x + p_y = 0 \tag{15}$$
$$x_B = \frac{15p_x + p_y}{2p_x} \tag{16}$$
$$y_B = \frac{15p_x + p_y}{2p_y} \tag{17}$$
$$e_B^x = \frac{15p_x + p_y}{2p_x} - 15 \tag{18}$$
$$e_B^y = \frac{15p_x + p_y}{2p_y} - 1 \tag{19}$$

At the competitive equilibrium, the excess demands for each good sum to equal 0.

$$e_A^x = -e_B^x \tag{20}$$
$$\frac{p_x + 15p_y}{2p_x} - 1 = 15 - \frac{15p_x + p_y}{2p_x} \tag{21}$$
$$p_x = p_y \tag{22}$$

By normalising prices to 1, we can see that the gross demands will be $(8, 8)$ for Alan and Betty. We also have the equilibrium price line $p_x = p_y$. We can see that the equilibrium is efficient, since we know from before that at $(8, 8)$, $MRS_A = -1 = MRS_B$.



(d) Walras' Law says that the sum of the value of excess demands across all markets must equal 0 - that is, $p_x(e_A^x + e_B^x) + p_y(e_A^y + e_B^y) = 0$. That is clear from below, since $p_x = p_y$.

$$p_x(e_A^x + e_B^x) = p_x\left(\frac{16p_x + 16p_y}{2p_x} - 16\right) = \frac{16p_x + 16p_y}{2} - 16p_x = 0 \tag{23}$$

$$p_y(e_A^y + e_B^y) = p_y\left(\frac{16p_x + 16p_y}{2p_y} - 16\right) = \frac{16p_x + 16p_y}{2} - 16p_y = 0 \tag{24}$$

(e) For Alan's new utility function, we can find the new competitive equilibrium prices and quantity. They face the following problems.

$$\max x_A^{\frac{3}{4}} y_A^{\frac{1}{4}} \text{ subject to } p_x x_A + p_y y_A = p_x + 15p_y \tag{25}$$

$$\max x_B^{\frac{1}{4}} y_B^{\frac{1}{4}} \text{ subject to } p_x x_B + p_y y_B = 15p_x + p_y \tag{26}$$

We can solve for the first order conditions for Alan given by $\mathcal{L} = x_A^{\frac{3}{4}} y_A^{\frac{1}{4}} - \lambda(p_x x_A + p_y y_A - p_x 15p_y)$.

$$\frac{3}{4} x_A^{-\frac{1}{4}} y_A^{\frac{1}{4}} - \lambda p_x = 0 \tag{27}$$

$$\frac{1}{4} x_A^{\frac{3}{4}} y_A^{-\frac{3}{4}} - \lambda p_y = 0 \tag{28}$$

From (27) divided by (28), we know that $3\frac{y_A}{x_A} = \frac{p_x}{p_y}$. Substituting this back into the budget constraint, we get that $x_A = \frac{3(p_x+15p_y)}{4p_x}$ and $y_A = \frac{p_x+15p_y}{4p_y}$. Since we know Betty has the same utility function as before, we can produce both Alan and Betty's respective excess demands, which we know for each good sums to 0.

$$e_A^x = -e_B^x \tag{29}$$

$$\frac{3(p_x + 15p_y)}{4p_x} - 1 = -\frac{15p_x + p_y}{2p_x} + 15 \tag{30}$$

$$e_A^y = -e_B^y \tag{31}$$

$$\frac{p_x + 15p_y}{4p_y} - 15 = -\frac{15p_x + p_y}{2p_y} + 1 \tag{32}$$

$$p_x = \frac{47p_y}{31} \tag{33}$$

If we normalise $p_y = 1$, we can write $p_x = \frac{47}{31}$. The equilibrium quantities for Alan and Betty will be $(\frac{384}{47}, \frac{128}{31})$ and $(\frac{368}{47}, \frac{368}{31})$ respectively. That is, Alan has more of x and less of y than previously, and the converse holds for Betty.

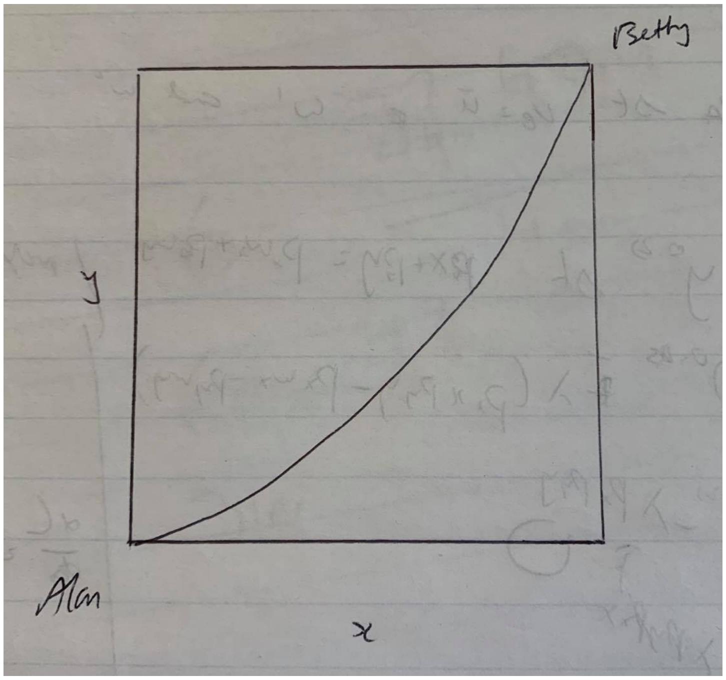
The contract curve is the set of all points where $MRS_A = MRS_B$. We know the latter from before.

$$MRS_A = MRS_B \tag{34}$$

$$\frac{\frac{3}{4}x_A^{-\frac{1}{4}}y_A^{\frac{1}{4}}}{\frac{1}{4}x_A^{\frac{3}{4}}y_A^{-\frac{3}{4}}} = \frac{y_B}{x_B} \tag{35}$$

$$\frac{3y_A}{x_A} = \frac{y_B}{x_B} \tag{36}$$

As we know that $x_A + x_B = 16$ and $y_A + y_B = 16$, we can find the contract curve $y_A = \frac{8x_A}{24 - x_A}$.



3 Walras' Law

Suppose n consumers and k goods. The excess demand function is given by $z = \sum_i^n (x_i - w_i)$. We can see that the value of excess demands is $pz = \sum_i^n p(x_i - w_i)$. Since we know that at the competitive equilibrium consumers are maximising their utility given their budget constraint, they will be at $px_i = pw_i$. As such, $pz = 0$, fulfilling Walras' Law.

4 Competitive Equilibrium

A competitive equilibrium must be Pareto-efficient - if not, agents for whom at least one could benefit would make an exchange, which is possible since this is a competitive market.

Suppose a two good, two agent economy. In competitive equilibrium, we know that consumers will be consuming such that $MRS_A = \frac{p_1}{p_2}$ and $MRS_B = \frac{p_1}{p_2}$. Consequently, $MRS_A = MRS_B$. This is the requirement for Pareto-efficiency. As such, the statement holds. Indeed, this could be generalised to an n consumers and k goods situation too.