Geometric ergodicity of Markov processes: a copula-based approach

THESIS

submitted in partial fulfillment of the
requirements for the degree of

MASTER OF SCIENCE
in
MATHEMATICS AND BUSINESS STUDIES

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Leiden, The Netherlands, April 15, 2022

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April 15, 2022

Abstract

This thesis considers a copula-based approach to establish a condition for geometric ergodicity of Markov processes. Markov chains are popular in financial applications and statistical inference and can be generated by a unique copula and invariant distribution functions. One significant assumption in many applications is geometric ergodicity of the Markov process. The assumption has been verified for many popular copula models for univariate chains. We derive a new condition that guarantees geometric ergodicity also for multivairate chains. Using these conditions, we demonstrate some Archimedean copulas, such as Clayton and Gumbel, are geometrically ergodic.

Contents

Chapter 1

Introduction

Markov chains are stochastic processes in state space which move from one state to another with Markov property. This property shows the probability distribution of next state is determined only by the current state. There are considerable theory and applications of Markov chains in non-linear time series analysis. One line of research is the copula-based methods to analyze different dependence structures of multivariate time series, such as financial applications [3], because copulas can be used for contemporaneous dependence and temporal dependence. As Sklar's theorem shows that any multivariate distribution can be decomposed into a unique copula and its marginal distribution functions, discrete-time Markov chains can be expressed by a copula function C and marginal distributions F. Here the copula captures dependence structures of the Markov process.

There are a number of published papers contributing to copula-based Markov models. The work of Chen and Fang [3] provides a general estimation method of copula-based semiparametric stationary Markov models in univariate cases. Nagler et al. [4] propose a class of stationary vine copula models for multivariate Markov chains, and give an estimation method based on the stepwise maximum likelihood estimator. In these copula-based time series models, copulas are used to generate the distributions of these Markov processes.

In this context, one of the important conditions in most copula-based models is that these Markov processes are required to be stationary and geometrically ergodic. The stationary assumption offers invariance and reduces the complexity of copula model. The significant condition for geometric ergodicity of Markov chains is the one established well in Meyn and Tweedie [5], which offers the "Foster-Lyapunov" drift criteria. On the basis of this work, Jarner and Roberts [6] give drift conditions through hitting times of Markov chains to polynomial rates. The work of Beare [2] connects the property of geometric ergodicity with associated Archimedean copulas in univariate case. Under the regular variation conditions, the Archimedean generator is required to vary regularly at 0 and 1 and used to demonstrate the drift conditions in Meyn and Tweedie, which means the Markov chain is geometric ergodic.

Our work also is based on the drift conditions of Meyn and Tweedie [5]. The aim of our thesis is to obtain a general condition for geometric ergodicity via the copula. Inspired by the work of Beare [2] on geometric ergodicity of bivariate Archimedean copulas, we derive a drift condition which works for multivariate copulas. We propose this idea first in the univariate case and then extend it to the multivariate case.

In our work, the condition is expressed by the tail density and conditional tail density based on the copula approach. The idea for our conditions is established on the work of Joe et al. [7], which suggests the extreme value copulas can be determined by their tail dependence functions and builds the relation between regular variation and tail dependence functions. Moreover, **Proposition** 2.5 and **Proposition** 2.6 in the paper of Li and Wu [8] show the expressions of tail dependence functions for Archimedean copulas. Then via these nice expressions, we verify the geometric ergodicity condition for Archimedean copulas in the multivariate case, including Gumbel and Clayton copulas.

The thesis is organized in the following structure. First we introduce some basic knowledge of Markov chains from Meyn and Tweedie [5] in Chapter 2. Chapter 3 provides an overview of copulas, including the concepts of copulas and archimedean copulas, and estimation of copulas. And besides, we briefly consider the copula-based Markov chains. In Chapter 4, we derive a general condition on the copula requiring that the integral of ”conditional tail density” is less than 1. Chapter 5 gives necessary definitions and properties of conditional tail densities. We introduce a simple expression of Archimedean copulas conditional tail densities, which is derived from the tail copula function of Li and Wu [8]. Based on the formulas in Chapter 5, we demonstrate that the integral of Archimedean copulas is less than one in Chapter 6.

Chapter 2

Markov chains

In this chapter, I will briefly introduce the Markov chain on general (non-countable) state spaces and its geometric ergodicity. The main references are Lawler [9], and Meyn and Tweedie [5].

2.1 Markov chains

2.1.1 Basic concepts

Assuming a triple $\{\Omega, \mathscr{F}, P\}$ is probability space, $(X, \mathscr{B}(X))$ is a measurable space, where X is a general state space, and $\mathscr{B}(X)$ is a σ-algebra generated by a countable subset of X. A stochastic process $(U_t)_{t \in Z_+} : \{\Omega, \mathscr{F}\} \to (X, \mathscr{B}(X))$ is a measurable function for all $t \in Z_+$.

Definition 1. *(Markov transition function [5]) A probability $P(x, A)$ is a Markov transition probability of measurable space $(X, \mathscr{B}(X))$ if*
(i) for all $x \in X$, $P(x, \cdot)$ is a probability measure,
(ii) for all $A \in \mathscr{B}(X)$, $P(\cdot, A)$ is a measurable function and $P(\cdot, A) \geq 0$.

The Markov transition $P(x, A) = P(U_t \in A | U_{t-1} = x)$ shows the Markov property satisfies by stochastic process $(U_t)_{t \in Z_+}$. This property indicates, the state of future system U_t is mainly affected by the present state of the system U_{t-1}. Given U_{t-1}, other past information, such as the original state U_0, does not have an influence.

Definition 2. *(Time-homogeneous Markov chain [5]) A stochastic process $(U_t)_{t \in Z_+}$ is called time-homogeneous Markov chain with Markov transition function $P(x, A)$ and initial measure μ, if for any $n \in \mathbb{N}$, measurable sets $A_i \in X_i, i = 1, \ldots, n$, and a fixed* *starting point* $x \in A_0$,

$$P(U_1 \in A_1, \dots, U_n \in A_n) = \int_{y_0 \in A_0} \cdots \int_{y_{n-1} \in A_{n-1}} \mu(dy_0) P(y_0, dy_1) \dots P(y_{n-1}, A_n). \tag{2.1}$$

Time-homogeneous Markov chains applied in most time series models show the memoryless property, which means, given the present state, memory of the past has no influence on the future. The equation (2.1) indicates that the probabilities $P(x, A)$ only depend on the values of x and the sets of A and don't relate to any time points. This property is called homogeneity.

Note 1. *We generally define a sequence* $\boldsymbol{U} = \{U_0, U_1, U_2, \dots\}$, *and all possible values of* $(U_t)_{t \in Z}$ *is called the state space.*

Definition 3. *(n-step transition probability [5])* $P^n(x, A)$ *is called n-step transition probability if for* $n \geq 1$, *any* $x \in X$ *and* $A \in \sigma(X)$

$$P^n(x, A) = \int_X P(x, dy) P^{n-1}(y, A).$$

2.1.2 State types

Definition 4. *(Occupation time and return time [5]) For any set* $A \in \sigma(U_t)$, *the occupation time* η_A *is the number of arrivals of set* A*:*

$$\eta_A = \sum_{n=1}^{\infty} 1\{U_t \in A\}.$$

The return time τ_A *is the time of first arrival in* A*:*

$$\tau_A = min\{t \geq 1 : U_t \in A\}.$$

Definition 5. *(Return time probabilities [5])* $L(x, A)$ *is called return time probabilities to a set* A *from the state* x, *defined as*

$$L(x, A) = P_x(\tau_A < \infty) = P_x(\boldsymbol{U} \text{ ever enters } A).$$

Definition 6. *(*ϕ*-irreducibility [5]) A measure* ϕ *is called irreducible, if for any state* x, *we have*

$$\phi(A) > 0 \to L(x, A) > 0.$$

From a more intuitive perspective, there are two states communicating and each state has a positive probability of reaching the other state. This property is reflexive, symmetric, and transitive. Based on this relation, the state space can be divided into disjoint sets called communication classes. The chain is called irreducible if there only exists one communication class. Thus the fundamental idea of irreducibility is that no matter the starting point, a Markov chain will reach all parts of the space.

Note 2. *If the Markov chain is reducible, it is possible to discuss each communication class separately.*

Definition 7. *(Aperiodicity [5]) Assuming an irreducible Markov chain, let for any state $x \in X$,*

$$d(x) = g.c.d.\{n \geq 1 : P^n(x,x) > 0\}. \tag{2.2}$$

where g.c.d. refers to the greatest common divisor of non-zero integers. An irreducible chain on a countable space X is called aperiodic, if $d(x) \equiv 1$ for all $x \in X$.

The $d(x)$ in the equation (2.2) refers to the greatest common divisor of the number of all possible steps that can return to the original point. If c is the greatest common divisor of non-zero integers $a_1, \ldots, a_n$ for some $n \geq 2$, there exists some non-zero integers $b_1, \ldots, b_n$ such that $a_i = cb_i$ for $\forall\, i \in [1, n]$.

It is obvious that for an irreducible Markov chain, an aperiodic state means that the chain is aperiodic. For instance, assuming in a Markov chain with ϕ-irreducibility, if the number of steps for state 0 to return is 3 and 4, the greatest common divisor is 1 from the equation (2.2). Thus the state 0 is aperiodic and other states in this Markov chain are aperiodic.

2.1.3 Small sets and petite sets

Definition 8. *(Small sets [10]) A set $B \in \sigma(U_t)$ is called small if there is $t > 0$, $\delta > 0$, and a non-trivial measure ν such that for $\forall x \in B$, the t-step transition probability satisfies*

$$P^t(x, A) \geq \delta\nu(A).$$

Definition 9. *(Petite sets [5]) Let $\mu(n)$ be a distribution over $\mathbb{N}$. A set $B \in \sigma(U_t)$ is petite if there is a measure ν, such that for $\forall x \in B$,*

$$\sum_{n=0}^{\infty} P^n(x, A)\mu(n) \geq \nu(B). \tag{2.3}$$

If the lower bound is attained in equation (2.3), for all $x \in B$, the elements of Markov chain (U_t) are independent, because the Markov transitions do not depend on the starting point x. With a small set B, the properties of a Markov chain can be studied from any starting point $x \in B$, since chains regenerate and forget their past with some probability.

The proof of the following properties can be found in Meyn.

Property 1. *If the Markov chain is irreducible and aperiodic, then all petite sets are small.*

Lemma 1. *If $A \in \sigma(U_t)$ is ν-small, then A is petite.*

Lemma 2. *The Union of two petite sets is petite.*

2.1.4 Drift

Definition 10. *(Drift for Markov chains [5]) The drift operator $\triangle$ is for a non-negative measure function V, is defined as*

$$\triangle V(x) = \int P(x, dy)V(y) - V(x).$$

The drift operator $\triangle V(x) = E[V(U_t)|U_{t-1} = x] - V(x)$ is used to enlarge or reduce the distance from the expected next state to the current state by function V, when given the current state of system. It can be used to establish a condition to measure the ergodicity of Markov chains.

2.2 Concepts of ergodicity

Definition 11. *[5] Let $(U_t)_{t \in Z}$ be a ϕ-irreducible and aperiodic Markov chain with transition probabilities P. The Markov chain is called ergodic if there exists some small set A such that for every $x \in A$,*

$$\lim_{t \to \infty} P^t(x, A) = P^{\infty}(A) > 0.$$

The Markov chain is called geometrically ergodic, if there exists some small set A, $\beta < 1$ and a finite constant M, such that for every $x \in A$,

$$|P^t(x, A) - P^{\infty}(A)| \leq M\beta^t.$$

We expect Markov chains to converge to a stationary distribution in order to simplify problems. The ergodicity means the process convergences to a stationary distribution arbitrarily slow. In contrast, the geometric ergodicity indicates the process convergences to a stationary distribution exponentially fast. It is evident that geometric ergodicity implies ergodicity. Here we introduce a geometric ergodic theorem to study the ergodic property of Markov chains.

Theorem 1. *(Geometric Ergodic [5]) Let the Markov chain $(U_t)_{t\in Z}$ be ϕ-irreducible and aperiodic. It is geometrically ergodic, if and only if for some petite set A and some constant $b < \infty$, $\beta > 0$, there exists a drift function $V \geq 1$ satisfying*

$$\triangle V(x) = \int P(x, dy)V(y) - V(x) \leq -\beta V(x) + b1_A(X). \tag{2.4}$$

Theorem 1, is referred to as minorisation condition. It is not only an equivalence of some well-known geometrically ergodic theorem [5] , but also develops a equivalence with drift conditions. To use this theorem to show the geometrically ergodic property, one should start with setting a drift function $V \geq 1$. Then given some $b < \infty$, $\beta > 0$, we find a set A which is supposed to be a petite set.

Chapter 3

Copulas

In this section, I'll give an overview of copulas and the relationship with Markov chains. A copula is a multivariate distribution function used to describe the non-linear relationship between variables with different margins.

3.1 Bivariate copulas

Definition 12. *[11] Considering two random variables u and v, a bivariate copula $C(u,v)$ is a function that satisfies:*
(i) $C(u,v) = [0,1] \times [0,1]$.
(ii) For any $(u,v) \in [0,1]^2$, $C(u,0) = C(0,v) = 0$.
(iii) For any $0 < u_1 < u_2 < 1$ *and* $0 < v_1 < v_2 < 1$,

$$C(u_2,v_2) - C(u_2,v_1) - C(u_1,v_2) + C(u_1,v_1) \geq 0.$$

(iv) For any $u, v \in [0,1]$,

$$C(u,1) = u, C(1,v) = v.$$

In general, a copula function also is expressed by the Sklar Formula, as shown below.

Theorem 2. *(Sklar's Theorem [12]) Let two variables X and Y with marginal distribution $F_X(x) = P(X \leq x)$, $F_Y(y) = P(Y \leq y)$ and joint distribution function $F_{X,Y}(x,y) = P(X \leq x, Y \leq y)$. There exists a copula C, such that*

$$F_{X,Y}(x,y) = C(F_X(x), F_Y(y))$$

When F_X and F_Y are continuous, then $U = F_X(X) \sim U(0,1)$ and $V = F_Y(Y) \sim U(0,1)$. Hence, the copula C is uniquely given by

$$\begin{aligned} P(U \le u, V \le v) &= P(F_X(x) \le u, F_Y(y) \le v) \\ &= P(X \le F_X^{-1}(u), Y \le F_Y^{-1}(v)) \\ &= F_{X,Y}(F_X^{-1}(u), F_Y^{-1}(v)) = C(u,v). \end{aligned}$$

Note 3. *Sklar Theorem explains that Copulas are models which capture the dependence between random variables. Thus, one can express the joint distribution in terms of the marginal distributions of each random variable and their copula function.*

This formula indicates copulas are joint distribution functions of uniform random variables U, V. From this point of view, the distribution function of two random variables can be broken into three parts: two marginal distributions and a copula. Since the copulas connect the joint distribution of random variables to their marginal distributions, we can construct a distribution by specifying marginal distributions and dependence structure.

Here are some important properties of the a bivariate copula $C(u,v)$:

Property 2. *$C(u,v)$ is monotonically non-decreasing with respect to each variable.*

Property 3. *(Fréchet-Hoeffding bounds) For any $u, v \in [0,1]$, there exist the following bounds:*

$$max(u + v - 1, 0) \le C(u,v) \le min(u,v).$$

Definition 13. *(Reflected copula[13]) Given the copula C of two variables (U, V), the reflected copula $\hat{C}$ is defined as the copula of $(1-U, 1-V)$; that is,*

$$\begin{aligned} \hat{C}(u,v) &= P(1-U \le u, 1-V \le v) \\ &= P(U \ge 1-u, V \ge 1-v). \end{aligned}$$

Property 4. *(Inclusion-exclusion principle) Given $(U, V) \sim C$ and its reflected copula $\hat{C}$, there exists*

$$\hat{C}(u,v) = u + v - 1 + C(1-u, 1-v).$$

Proof.

$$\begin{aligned} P(U > u, V > v) &= 1 - P(U \le u \quad or \quad V \le v) \\ &= 1 - P(U \le u) - P(V \le v) + P(U \le u, V \le v) \\ &= 1 - u - v + C(u,v). \end{aligned}$$

Then by the definition of reflected copula, we have

$$\begin{aligned}\hat{C}(u,v) &= P(U \geq 1-u, V \geq 1-v) \\ &= 1-(1-u)-(1-v)+C(u,v) \\ &= u+v-1+C(u,v).\end{aligned}$$

□

Definition 14. *(Tail dependence coefficient [12]) The upper tail dependence coefficient of a bivariate copula $C(u,v)$ is defined as*

$$\lambda^U = \lim_{t\to 1} P(V > t | U > t) = \lim_{t\to 1} \frac{1-2t+C(t,t)}{1-t},$$

and the lower tail dependence coefficient is defined as

$$\lambda^L = \lim_{t\to 0} P(V \leq t | U \leq t) = \lim_{t\to 0} \frac{C(t,t)}{t}.$$

Given random variables, tail dependence of a copula can be used to measure the external dependence in joint distribution independent of marginal distributions. If $\lambda^U > 0$, the copula C has the upper tail dependence;that is, for t close to 1, there is a high probability that V is greater that t given that U is greater that t. If a copula is reflection symmetric, that is $C(u,v) = \hat{C}(u,v)$ [14], the upper and lower tail dependence coefficients are equivalent $\lambda^U = \lambda^L$.

3.2 Multivariate copulas

Let F be the joint distribution of a random vector $(X_1, \ldots, X_t)$ and $F_1, \ldots, F_t$ be its continuous marginal distributions, that is $F_i(X_i) = U_i, U_i \sim U(0,1)$. Using the Sklar's Formula, there exists a unique copula function C

$$C(u_1, \ldots, u_t) = F(F_1^{-1}(u_1), \ldots, F_t^{-1}(u_t)).$$

The function $c(u_1, \ldots, u_t) = \frac{\partial C(u_1,\ldots,u_t)}{\partial u_1 \ldots \partial u_t}$ is the probability density function of a copula. The probability density function $f(F_1^{-1}(u_1), \ldots, F_t^{-1}(u_t))$ has the form

$$f(F_1^{-1}(u_1), \ldots, F_t^{-1}(u_t)) = c(u_1, \ldots, u_t) \prod_{i=1}^{t} f_i(F_i^{-1}(u_i)), \tag{3.1}$$

where f_i is the marginal density function.

3.3 Families of copulas

Copulas are widely used in two main families, one is the family of elliptical copulas and the other is the family of Archimedean copulas.

3.3.1 Elliptical copulas[1]

The family of elliptic Copula functions is mainly used to characterize the dependence with symmetric relations.

Definition 15. *(Elliptical distribution) The random vector $X = (X_1, \dots, X_t)$ has an elliptical distribution if its density satisfies*

$$f(x) = |\Sigma|^{-\frac{1}{2}} g[(y-\mu)^T \Sigma^{-1} (y-\mu)], \quad \textit{for any } y \in R^t,$$

where Σ is a symmetric positive semi-definite matrix which determines the scale and the correlation of the random variables, and μ is a location parameter vector. The function $g : [0, \infty) \to [0, \infty)$, called characteristic generator, is used to generate various elliptical distributions.

The copula generated by elliptical distribution is a elliptical copula. The fist popular one is the Gaussian copula which has the copula distribution

$$C(u_1, \dots, u_t; \rho) = \Phi_\rho(\Phi^{-1}(u_1), \dots, \Phi^{-1}(u_t)),$$

and density

$$\begin{aligned} c(u_1, \dots, u_t; \rho) &= \frac{\partial^t C(u_1, \dots, u_t; \rho)}{\partial u_1 \dots \partial u_t} \\ &= |\rho|^{-\frac{1}{2}} exp(-\frac{1}{2}(\Phi^{-1}(u_1), \dots, \Phi^{-1}(u_t))^T (\rho^{-1} - I)(\Phi^{-1}(u_1), \dots, \Phi^{-1}(u_t))), \end{aligned}$$

where Φ_ρ is a cdf of multivariate normal distribution, ρ is its correlation coefficient matrix, and I is the identity matrix.

3.3.2 Archimedean copulas[2]

Definition 16. *A copula is called Archimedean copula with generator function $\phi^{-1} : [0, 1] \to [0, \infty]$, if*

$$C(u_1, \dots, u_t) = \phi(\phi^{-1}(u_1) + \phi^{-1}(u_2) + \dots + \phi^{-1}(u_t)),$$

where ϕ^{-1} is continuous and monotonically decreasing convex function with $\phi^{-1}(1) = 0$ and satisfies $\sum_{i=1}^{d} \phi^{-1}(u_i) \leq \phi^{-1}(0)$.

Definition shows that an Archimedean copula is constructed by choosing a suitable continuous and strictly decreasing convex function on $[0,1]$. Common Archimedean generators are the following functions:

$$
\begin{aligned}
&Gumbel: \phi^{-1}(u) = (-\ln u)^{\alpha},\\
&Clayton: \phi^{-1}(u) = \frac{u^{-\alpha}-1}{\alpha},\\
&Frank: \phi^{-1}(u) = -\ln\frac{e^{-\alpha u}-1}{e^{-\alpha}-1}.
\end{aligned}
$$

Lemma 3. *Let C be an Archimedean copula on variables $U_1,\ldots,U_t$ with generator ϕ,*

$$C(u_1,\ldots,u_t) = \phi(\phi^{-1}(u_1)+\phi^{-1}(u_2)+\cdots+\phi^{-1}(u_t)).$$

Then the copula $\tilde{C}$ on variables $U_1,\ldots,U_{t-m}$ is also Archimedean:

$$\tilde{C}(u_1,\ldots,u_{t-m}) = \phi(\phi^{-1}(u_1)+\cdots+\phi^{-1}(u_{t-m})).$$

Proof. According to the definition of Archimedean copula, we have

$$
\begin{aligned}
\tilde{C}(u_1,\ldots,u_{t-m}) &= P(U_1 \le u_1,\ldots,U_{t-m} \le u_{t-m})\\
&= P(U_1 \le u_1,\ldots,U_{t-m} \le u_{t-m}, U_{t-m+1} \le 1,\ldots,U_t \le 1)\\
&= C(u_1,\ldots,u_{t-m},1,\ldots,1)\\
&= \phi(\phi^{-1}(u_1)+\cdots+\phi^{-1}(u_{t-m})+\phi^{-1}(1)+\cdots+\phi^{-1}(1))\\
&= \phi(\phi^{-1}(u_1)+\cdots+\phi^{-1}(u_{t-m}))
\end{aligned}
$$

where $\phi^{-1}(1) = 0$ based on the definition of Archimedean generators. □

3.4 Estimation of copulas

The major estimation methods of Copulas can be divided into parametric methods and semi-parametric methods.

The parameter estimation method specifies both marginal distribution functions and the type of copula functions, then parameters of these distributions and copulas are what to be estimated. Based on the maximum likelihood estimation, the main parameter estimation of Copula method are the maximum likelihood(ML) [13] and the inference function for margins (IFM) [15].

3.4.1 Maximum likelihood estimation

Suppose the marginal distributions of random variables $(X_1, \ldots, X_d)$ are $F_i(x_i|\theta_i)$ with density $f_i(x_i|\theta_i)$, where θ_i is the corresponding unknown parameters of random variable X_i. Let the copula be $C(u_1, u_2, \ldots, u_d; \alpha)$ with corresponding density function $c(u_1, u_2, \ldots, u_d; \alpha)$, where α represents the copula parameters. By equation (3.1), the log-likelihood is

$$\begin{aligned}
\ln L(\theta_1, \ldots, \theta_d, \alpha) &= \ln \prod_{i=1}^{n} c[u_{i,1}, \ldots, u_{i,d}; \alpha] \prod_{j=1}^{d} f_j(x_{i,j}|\theta_j) \\
&= \Sigma_{i=1}^{n} c[u_{i,1}, \ldots, u_{i,d}; \alpha] + \Sigma_{j=1}^{d} \Sigma_{i=1}^{n} \ln f_j(x_{i,j}|\theta_j). \qquad (3.2)
\end{aligned}$$

The joint maximum likelihood estimator is obtained by solving for the maximum point of the log-likelihood function, that is

$$(\hat{\theta}_1, \ldots, \hat{\theta}_d, \hat{\alpha}) = \arg\max \ln L(\theta_1, \ldots, \theta_d, \alpha).$$

3.4.2 Inference function for margins (IFM) estimation

Compared to the joint maximum likelihood method, two-step estimation is easier for computations [16]. This method is consists of two steps: from equation (3.2), it is obvious that the marginal parameters θ_j and the copula parameter α can be estimation independently. Thus, the marginal parameters θ_j can be estimated by ML method at first

$$\hat{\theta}_j = \arg\max \Sigma_{i=1}^{n} \ln f_j(x_{i,j}|\theta_j).$$

In the second step the copula parameter α is estimated by substituting $\hat{\theta}_j$ for θ_j in equation (3.2) and then maximizing the likelihood function. Eventually the estimator of IFM is

$$\hat{\alpha} = \arg\max \Sigma_{i=1}^{n} c[F(x_i, \hat{\theta}_1), \ldots, F(x_i, \hat{\theta}_d); \alpha].$$

3.5 Copula-based Markov chains

The copula-based approach is a general method to analyze the time dependence of time series, used in many recent papers. Let $(U_t)_{t \in Z}$ be a time-homogeneous Markov chain with marginal distribution F. This stochastic process is determined by the joint distribution of (U_1, U_2) denoted $H(u_1, u_2)$. Considering the Sklar Formula, we can express this joint distribution $H(u_1, u_2) = P(U_1 \leq u_1, U_2 \leq u_2)$ in terms of the marginal distributions F and the copula function C

$$H(u_1, u_2) = C(F(u_1), F(u_2)).$$

It is flexible to generate temporal dependence models based on copulas, because the choices of the copula function and the marginal distribution in the Markov chains are independent of each other. Therefore, copula-based Markov chains have been successfully applied in various fields, including financial time series [17] [3], model diagnostic procedures and statistical inference [18] [19].

Further, recent results and theorems of copula-based Markov process require and discuss ergodicity or geometric ergodicity of $(U_t)_{t\in Z}$ as well. The work of Chen and Fang [3] indicates that first-order nonlinear stationary Markov chains generated by several specific copulas including the Gaussian copula and the Clayton copula are geometrically ergodic. Our goal is to find a general approach to show some multivariate copula-based Markov chains are geometrically ergodic.

Chapter 4

Geometric ergodicity and copulas

In this chapter, our goal is to transform the definition of geometric ergodicity into a general condition on the copula. Specifically, we derive a condition on the "conditional tail density" of the copula in **Sec** 4.1 Univariate case, **Sec** 4.2 Two-variables case, **Sec** 4.3 Multivariate case.

4.1 Univariate case

We first build intuition in the univariate case. Let $(U_t)_{t\in Z}$ be a stationary Markov chain with $U_t \sim U[0,1]$. We assume that the joint (copula) density c of (U_t, U_{t-1}) is positive on $(0,1)^2$, which means irreducibility and aperiodicity of $\{U_t\}$ hold [20].

Then we want to prove that there exists a suitable measure function V, a constant ρ, a small set A satisfying the drift conditions (**Theorem** 1). We generalize an argument of Beare that was specific for Archimedean copulas. This idea is to take a set $A = [\underline{\xi}, \bar{\xi}]$ and find $\underline{\xi}$, $\bar{\xi}$ and a function V satisfying **Theorem** 1, and then prove this set A is a small set. To construct the drift function V, we separately consider the regions near 0 and 1.

4.1.1 Case 1: $u_2 \to 0$

For our drift function, we take $V(u) = u^{-\alpha}1(u \leq u_2) + 1(u > u_2)$, for $\alpha > 0$. Then following equation (2.4), we want to have

$$\int P(x, dy)V(y) \leq \rho V(x) + b1_A(x),$$

with $\rho = 1 - \beta \in (0,1)$. If we take $x \notin A$, the condition would be $\int P(x,dy)V(y) \leq \rho V(x)$.

It holds

$$
\begin{aligned}
\lim_{u_2\to 0}\frac{E(V(U_1)|U_2=u_2)}{V(u_2)} &= \lim_{u_2\to 0}\frac{\int_0^1 V(u_1)c(u_1,u_2)du_1}{V(u_2)}\\
&= \lim_{u_2\to 0}\frac{\int_0^{u_2} u_1^{-\alpha}c(u_1,u_2)du_1}{V(u_2)} + \lim_{u_2\to 0}\frac{\int_{u_2}^1 c(u_1,u_2)du_1}{V(u_2)}\\
&= \lim_{u_2\to 0}\frac{\int_0^{u_2} u_1^{-\alpha}c(u_1,u_2)du_1}{V(u_2)} + 0,
\end{aligned}
$$

where $\lim_{u_2\to 0} V(u_2) = \infty$, and

$$
\int_{u_2}^1 c(u_1,u_2)du_1 \le \int_0^1 c(u_1,u_2)du_1 = 1.
$$

Thus, we could get

$$
\lim_{u_2\to 0}\frac{\int_{u_2}^1 c(u_1,u_2)du_1}{V(u_2)} = 0.
$$

Given u_2, since U_2 is uniform on $[0,1]$, i.e.$\frac{\partial}{\partial u_2}P(U_2 \le u_2) = 1$, the conditional density

$$
P(u_1|u_2) = \frac{c(u_1,u_2)}{f_{U_2}(u_2)} = c(u_1,u_2).
$$

Then we change the variable $u_1 = u_2 w$, which gives,

$$
\begin{aligned}
\lim_{u_2\to 0}\frac{\int_0^{u_2} u_1^{-\alpha}c(u_1,u_2)du_1}{V(u_2)} &= \lim_{u_2\to 0}\frac{\int_0^1 u_2^{-\alpha}w^{-\alpha}c(u_2w,u_2)d(u_2w)}{u_2^{-\alpha}}\\
&= \int_0^1 w^{-\alpha}\lim_{u_2\to 0} c(u_2w,u_2)u_2dw\\
&= \int_0^1 w^{-\alpha}\lambda_L(w,1)dw = \rho_0. \qquad (4.1)
\end{aligned}
$$

In summary, we obtain

$$
\lim_{u_2\to 0}\frac{E(V(U_1)|U_2=u_2)}{V(u_2)} = \rho_0 \in [0,1), \qquad \text{if} \quad \rho_0 < 1.
$$

Then for arbitrary $\rho \in (\rho_0, 1)$, there exists $\underline{\xi} \in (0,1)$ such that the drift function $V(u) = u^{-\alpha}1(u \le \underline{\xi}) + 1(u > \underline{\xi})$ satisfies $E(V(U_1)|U_2=u_2) \le \rho V(u_2)$, for all $u_2 \in (0, \underline{\xi})$.

4.1.2 Case 2: $u_2 \to 1$

Similarly, we take $V(u)$ with $1-u$ in place of u, for $\alpha > 0$, $V(u) = (1-u)^{-\alpha}1(u \geq u_2) + 1(u < u_2)$. Then

$$
\begin{aligned}
\lim_{u_2\to 1} \frac{E(V(U_1)|U_2 = u_2)}{V(u_2)} &= \lim_{u_2\to 1} \frac{\int_0^1 V(u_1)c(u_1,u_2)du_1}{V(u_2)} \\
&= \lim_{u_2\to 1} \frac{\int_0^{u_2} c(u_1,u_2)du_1}{V(u_2)} + \lim_{u_2\to 1} \frac{\int_{u_2}^1 (1-u_1)^{-\alpha}c(u_1,u_2)du_1}{V(u_2)} \\
&= 0 + \lim_{u_2\to 1} \frac{\int_{u_2}^1 (1-u_1)^{-\alpha}c(u_1,u_2)du_1}{V(u_2)} \\
&= \lim_{u_0\to 0} \frac{\int_{1-u_0}^1 (1-u_1)^{-\alpha}c(u_1,1-u_0)du_1}{V(1-u_0)},
\end{aligned}
$$

where $\lim_{u_2\to 1} \frac{1}{V(u_2)} = \lim_{u_2\to 1}(1-u_2)^{\alpha} = 0$.
Then using the changes of variable $u_1 = 1 - u_0 w$, we get

$$
\begin{aligned}
&\lim_{u_0\to 0} \frac{\int_{1-u_0}^1 (1-u_1)^{-\alpha}c(u_1,1-u_0)du_1}{V(1-u_0)} \\
&= \lim_{u_0\to 0} \frac{\int_{1-u_0}^1 u_0^{-\alpha}w^{-\alpha}c(1-u_0w,1-u_0)d((1-u_0)w)}{(1-(1-u_0))^{-\alpha}} \\
&= \lim_{u_0\to 0} \frac{\int_1^0 u_0^{-\alpha}w^{-\alpha}c(1-u_0w,1-u_0)(-u_0)dw}{u_0^{-\alpha}} \\
&= \int_0^1 w^{-\alpha} \lim_{u_0\to 0} c(1-u_0w,1-u_0)u_0dw \\
&= \int_0^1 w^{-\alpha}\lambda_U(w,1)dw = \rho_1. \qquad (4.2)
\end{aligned}
$$

Hence, we get

$$
\lim_{u_2\to 1} \frac{E(V(U_1)|U_2 = u_2)}{V(u_2)} = \rho_1 \in (0,1), \qquad \text{if} \quad \rho_1 < 1.
$$

Consequently, for any $\rho \in (\rho_1, 1)$, there is $\bar{\xi} \in (0,1)$, for $\forall u_2 \in (\bar{\xi}, 1)$, such that the drift function $V(u) = (1-u)^{\alpha}1(u \geq \bar{\xi}) + 1(u < \bar{\xi})$ satisfies $E(V(U_1)|U_2 = u_2) \leq \rho V(u_2)$.

4.1.3 Summary

Eventually, there exists $\rho \in (max\{\rho_0, \rho_1\}, 1)$ and $\underline{\xi}, \bar{\xi} \in (0,1)$ using the combined drift function

$$V(u) = u^{-\alpha}1(u \leq \underline{\xi}) + 1(u \in (\underline{\xi}, \bar{\xi})) + (1-u)^{-\alpha}1(u \geq \bar{\xi}),$$

such that for all $u_2 \in (0, \underline{\xi}) \bigcup (\bar{\xi}, 1)$, it holds $E(V(U_1)|U_2 = u_2) \leq \rho V(u_2)$.

In addition, for all $u_2 \in (\underline{\xi}, \bar{\xi})$, we could find if $\sup_{u_2 \in [\underline{\xi}, \bar{\xi}]} \sup_{u_1 \in (0,1)} c(u_1, u_2) < \infty$,

$$b = \sup_{u_2 \in [\underline{\xi}, \bar{\xi}]} \int V(u_1) c(u_1, u_2) du_1 < \infty,$$

satisfying $E(V(U_1)|U_2 = u_2) \leq b$.

Finally combining together, we obtain that

$$E(V(U_1)|U_2 = u_2) \leq \rho V(u_2) + b1_A(u_2) \qquad \forall u_2 \in (0,1),$$

where $A = (\underline{\xi}, \bar{\xi})$.

4.2 Two-variables case

We now extend this argument to a bivariable Markov chain $\mathbf{U}_t = (U_{t,1}, U_{t,2}) \sim U[0,1]^2$. Here we also want to find a small set $A = ([\underline{\xi_1}, \bar{\xi_1}] \times (0,1)) \cup ((0,1) \times [\underline{\xi_2}, \bar{\xi_2}])$ and a function $V \geq 1$, such that the drift conditions is satisfied. The set A is shown in Fig 4.1

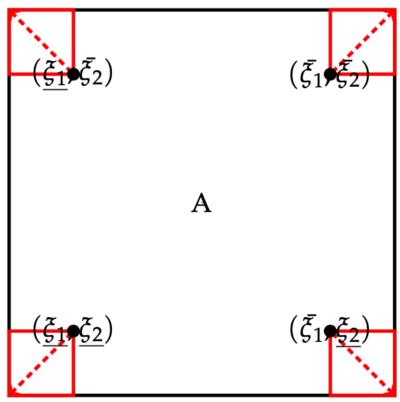


Figure 4.1: *Two-variables*

4.2.1 Case 1: $\mathbf{u}_2 \to (0,0)$

If $\mathbf{u}_2 \to (0,0)$, the drift function is set to

$$V(\mathbf{u}_t) = u_{t,1}^{-\alpha} u_{t,2}^{-\alpha} \mathbb{1}(\mathbf{u}_t \leq \mathbf{u}_2) + \mathbb{1}((u_{t,1} > u_{2,1}) \cup (u_{t,2} > u_{2,2})). \tag{4.3}$$

Here we start with the joint distribution of two variables in the Markov chain defined as $\tilde{c}(\mathbf{u}_i) = \tilde{c}(u_{i,1}, u_{i,2})$, then following the idea in the univariate case:

$$\begin{aligned}
&\lim_{\mathbf{u}_2 \to (0,0)} \frac{E(V(\mathbf{U}_1)|\mathbf{U}_2 = \mathbf{u}_2)}{V(\mathbf{u}_2)} \\
&= \lim_{\mathbf{u}_2 \to (0,0)} \frac{\int_0^1 \int_0^1 V(\mathbf{u}_1) \frac{c(\mathbf{u}_1, \mathbf{u}_2)}{\tilde{c}(\mathbf{u}_2)} d\mathbf{u}_1}{V(\mathbf{u}_2)} \\
&= \lim_{\mathbf{u}_2 \to (0,0)} \frac{\int_0^{u_{2,1}} \int_0^{u_{2,2}} u_{1,1}^{-\alpha} u_{1,2}^{-\alpha} c(\mathbf{u}_1, \mathbf{u}_2) d\mathbf{u}_1}{V(\mathbf{u}_2)\tilde{c}(\mathbf{u}_2)} + \lim_{\mathbf{u}_2 \to (0,0)} \frac{\int_{u_{2,1}}^1 \int_{u_{2,2}}^1 c(\mathbf{u}_1, \mathbf{u}_2) d\mathbf{u}_1}{V(\mathbf{u}_2)\tilde{c}(\mathbf{u}_2)} \\
&= \lim_{\mathbf{u}_2 \to (0,0)} \frac{\int_0^{u_{2,1}} \int_0^{u_{2,2}} u_{1,1}^{-\alpha} u_{1,2}^{-\alpha} c(\mathbf{u}_1, \mathbf{u}_2) d\mathbf{u}_1}{V(\mathbf{u}_2)\tilde{c}(\mathbf{u}_2)} + 0,
\end{aligned}$$

where $\lim_{\mathbf{u}_2 \to (0,0)} V(\mathbf{u}_2) = \infty$.

Setting $\mathbf{w} = (w_1, w_2, w_3, w_4)$, $\mathbf{w_1} = (w_1, w_2)$ and $\mathbf{w_2} = (w_3, w_4)$, then we change the variable with $\mathbf{u}_1 = (w_1 u, w_2 u) = \mathbf{w_1} u$ and $\mathbf{u}_2 = (w_3 u, w_4 u) = \mathbf{w_2} u$,

$$\begin{aligned}
&\lim_{\mathbf{u}_2 \to (0,0)} \frac{\int_0^{u_{2,1}} \int_0^{u_{2,2}} (u_{1,1})^{-\alpha} (u_{1,2})^{-\alpha} c(\mathbf{u}_1, \mathbf{u}_2) d\mathbf{u}_1}{V(\mathbf{u}_2)\tilde{c}(\mathbf{u_2})} \\
&= \lim_{u \to 0} \frac{\int_0^{w_3} \int_0^{w_4} (w_1 u)^{-\alpha} (w_2 u)^{-\alpha} c(\mathbf{w} \cdot u) u^2 dw_1 dw_2}{(w_3 u)^{-\alpha} (w_4 u)^{-\alpha} \tilde{c}(w_3 u, w_4 u)} \\
&= \int_0^{w_3} \int_0^{w_4} (w_1/w_3)^{-\alpha} (w_2/w_4)^{-\alpha} \lim_{u \to 0} u^2 \frac{c(\mathbf{w} \cdot u)}{\tilde{c}(\mathbf{w_2} u)} dw_1 dw_2 \\
&= \int_0^{w_3} \int_0^{w_4} (w_1/w_3)^{-\alpha} (w_2/w_4)^{-\alpha} \lambda_0(\mathbf{w_1}|\mathbf{w_2}) dw_1 dw_2,
\end{aligned}$$

where $\lambda_{(0,0)}(\mathbf{w_1}|\mathbf{w_2}) = \lim_{u \to 0} u^2 \frac{c(\mathbf{w} \cdot u)}{\tilde{c}(\mathbf{w_2} u)}$ is a lower conditional tail density function (see **Chapter** 4).

If $\lambda_{(0,0)}(\mathbf{w_1}|\mathbf{w_2})$ could make the above equation belong to (0,1), there exists $\underline{\xi_1}, \underline{\xi_2} \in (0,1)$, and $\rho \in (0,1)$, such that for all $\mathbf{u}_2 \in (0, \underline{\xi_1}) \times (0, \underline{\xi_2})$, we have $\underline{E(V(\mathbf{U}_1)}|\mathbf{U}_2 = \mathbf{u}_2) \leq \rho V(\mathbf{u}_2)$.

4.2.2 Case 2: $\mathbf{u}_2 \to (0,1)$ or $(1,0)$ or $(1,1)$

For results in other cases, such as case $\mathbf{u}_2 \to (0,1),(1,0)$ or $(1,1)$, we suggest that their results are similar to the case $\mathbf{u}_2 \to (0,0)$. By the definition of copula density, if $c_{0,0}$ is the copula density of $(U_{t,1}, U_{t,2})$, then $c_{1,1}$ is the copula density of $(1-U_{t,1}, 1-U_{t,2})$. Thus, the proof of $\lim_{\mathbf{u_2}\to(1,1)} \int\int \frac{\mathbf{u_1}^{-\alpha} c(\mathbf{u_1},\mathbf{u_2})}{\mathbf{u_2}^{-\alpha}\tilde{c}_{(1,1)}(\mathbf{u_2})} d\mathbf{u_1} < 1$ in the case $\mathbf{u}_2 \to (1,1)$ will be similar to show

$$\lim_{\mathbf{u_2}\to(0,0)} \int\int \frac{\mathbf{u_1}^{-\alpha} c(\mathbf{u_1},\mathbf{1}-\mathbf{u_2})}{\mathbf{u_2}^{-\alpha}\tilde{c}_{(0,0)}(\mathbf{1}-\mathbf{u_2})} d\mathbf{u_1} < 1.$$

In the end, if we want to find out a small set S satisfying the drift conditions, we could proof it in any regions near 0 or 1.

Corollary 1. *Considering two-variables case, we can get 4 corners around the regions of 0 or 1 at all. In the case ($\mathbf{u_2} \to (0,0)$), this conditional tail density is called lower conditional tail density*

$$\lambda_{(0,0)}(\mathbf{w_1}|\mathbf{w_2}) = \lim_{u\to 0} u^2 \frac{c(\mathbf{w}u)}{\tilde{c}(\mathbf{w_2}u)} dw_1 dw_2.$$

In the case($\mathbf{u_2} \to (1,1)$), this one is called upper conditional tail density

$$\lambda_{(1,1)}(\mathbf{w_1}|\mathbf{w_2}) = \lim_{u\to 0} u^2 \frac{c(\mathbf{1}-\mathbf{w}u)}{\tilde{c}(1-\mathbf{w_2})u)} dw_1 dw_2.$$

In addition, the conditional tail density in corners $(0,1)$ and $(1,0)$ are respectively denoted as

$$\lambda_{(0,1)}(\mathbf{w_1}|\mathbf{w_2}) = \lim_{u\to 0} u^2 \frac{c((w_1, 1-w_2, w_3, 1-w_4)u)}{\tilde{c}((w_3, 1-w_4)u)} dw_1 dw_2$$

and

$$\lambda_{(1,0)}(\mathbf{w_1}|\mathbf{w_2}) = \lim_{u\to 0} u^2 \frac{c((1-w_1, w_2, 1-w_3, w_4)u)}{\tilde{c}((1-w_3, w_4)u)} dw_1 dw_2.$$

4.2.3 Summary

Consequently, for all $\mathbf{u_2} \in B = [0, \underline{\xi_1}] \times [0, \underline{\xi_2}] \cup [0, \underline{\xi_1}] \times [\bar{\xi_2}, 1] \cup [\bar{\xi_1}, 1] \times [0, \underline{\xi_2}] \cup [\bar{\xi_1}, 1] \times [\bar{\xi_2}, 1]$, the combined drift function satisfies $E(V(\mathbf{U}_1)|\mathbf{U}_2 = \mathbf{u}_2) \le \rho V(\mathbf{u}_2)$. For all $\mathbf{u_2} \in A = [0,1]^d \setminus B = ([\underline{\xi_1}, \bar{\xi_1}] \times (0,1)) \cup ((0,1) \times [\underline{\xi_2}, \bar{\xi_2}])$, there exists

$$b = \sup_{\mathbf{u_2}\in A} \int\int V(\mathbf{u_1}) c(\mathbf{u}_1, \mathbf{u}_2) d\mathbf{u}_1 < \infty, \quad \text{if} \sup_{\mathbf{u_2}\in A} \sup_{\mathbf{u}_1} \frac{c(\mathbf{u}_1, \mathbf{u}_2)}{c(\mathbf{u}_2)} < \infty$$

such that $E(V(\mathbf{U}_1)|\mathbf{U}_2 = \mathbf{u}_2) \leq b$.

Therefore, for all $\mathbf{u_2} \in [0,1]^2$, it holds $E(V(\mathbf{U}_1)|\mathbf{U}_2 = \mathbf{u}_2) \leq \rho V(\mathbf{u}_2) + b1_A(\mathbf{u_2})$. By **Lemma** 4, the set in two-variables case is a small set, then this bivariable Markov chain is geometrically ergodic.

4.3 Multivariate case

The above proof is useful for multivariate case. Now if there are d-dimension variables, considering $\mathbf{U}_t = (U_{t,1}, U_{t,2}, \ldots, U_{t,d}) \sim U[0,1]^d$, the object in this case is to find a small set $A = \bigcup_{i=1}^d [\underline{\xi}_i, \bar{\xi}_i] \times [0,1]^{d-1}$.

Analogous to the case of two variables, if setting $c_{0,\ldots,0}$ as the copula density of $(U_{t,1}, \ldots, U_{t,d})$ and $c_{0,\ldots,0,1}$ as the copula density of $(U_{t,1}, \ldots, U_{d-1}, 1-U_d)$, the proof of $c_{0,\ldots,0,1}$ will be similar to the proof of $c_{0,\ldots,0}$. Hence, we only focus on the $(0,\ldots,0)$ corner and $c_{0,\ldots,0}$ to get the geometric ergodicity of multivariate case in our proof.

When $\mathbf{u}_2 \to \mathbf{0}$, we set the drift function

$$V(\mathbf{u}) = \prod_{i=1}^{d} u_i^{-\alpha} \mathbb{1}(\mathbf{u} \leq \mathbf{u}_2) + \mathbb{1}(\mathbf{u} \in [0,1]^d \setminus \{\mathbf{u} : \mathbf{u} \leq \mathbf{u}_2\})$$

Then we follow the same idea in two-variables case,

$$\begin{aligned}
\lim_{\mathbf{u}_2 \to \mathbf{0}} \frac{E(V(\mathbf{U}_1)|\mathbf{U}_2 = \mathbf{u}_2)}{V(\mathbf{u}_2)} &= \lim_{\mathbf{u}_2 \to \mathbf{0}} \frac{\int_0^1 \cdots \int_0^1 V(\mathbf{u}_1) \frac{c(\mathbf{u}_1, \mathbf{u}_2)}{\tilde{c}(\mathbf{u}_2)} d\mathbf{u}_1}{V(\mathbf{u}_2)} \\
&= \lim_{u_2 \to \mathbf{0}} \frac{\int_0^{u_{2,1}} \cdots \int_0^{u_{2,d}} \prod_{i=1}^d u_{1,i}^{-\alpha} c(\mathbf{u}_1, \mathbf{u}_2) d\mathbf{u}_1}{V(\mathbf{u}_2)\tilde{c}(\mathbf{u}_2)} + 0,
\end{aligned}$$

where $\lim_{u_2 \to 0} V(\mathbf{u}_2) = \infty$.

Defining $\mathbf{w} = (\mathbf{w_1}, \mathbf{w_2}) = ((w_1, \ldots, w_d), (w_{d+1}, \ldots, w_{2d}))$, we change the variable with

$$(\mathbf{u}_1, \mathbf{u}_2) = \mathbf{w} \cdot \mathbf{u} = (w_1 u, \ldots, w_d u, w_{d+1} u, \ldots, w_{2d} u),$$

$$\begin{aligned}
&\lim_{\mathbf{u}_2 \to \mathbf{0}} \frac{\int_0^{u_{2,1}} \cdots \int_0^{u_{2,d}} \prod_{i=1}^d u_{1,i}^{-\alpha} c(\mathbf{u}_1, \mathbf{u}_2) d\mathbf{u}_1}{V(\mathbf{u}_2)\tilde{c}(\mathbf{u}_2)} \\
&= \lim_{u \to 0} \frac{\int_0^{w_{d+1}} \cdots \int_0^{w_{2d}} \prod_{i=1}^d (w_i u)^{-\alpha} c(\mathbf{w} \cdot u) u^d dw_1, \ldots dw_d}{\prod_{i=d+1}^{2d} (w_i u)^{-\alpha} \tilde{c}(\mathbf{w_2} u))} \\
&= \int_0^1 \cdots \int_0^1 \prod_{i=1}^d w_i^{-\alpha} \prod_{i=d+1}^{2d} w_i^{\alpha} \lim_{u \to 0} u^d \frac{c(\mathbf{w} \cdot u)}{\tilde{c}(\mathbf{w_2} u))} dw_1, \ldots dw_d, \qquad (4.4)
\end{aligned}$$

where $\lambda_{(0,\dots,0)}(\mathbf{w_1}|\mathbf{w_2}) = \lim_{u\to 0} u^d \frac{c(\mathbf{w}u)}{\bar{c}(\mathbf{w_2}u)}$.

If for the equation (4.4), there exists $\underline{\xi_1},\dots,\underline{\xi_d} \in (0,1)$ and $\xi \in (0,1)$, such that $E(V(\mathbf{U}_1)|\mathbf{U}_2 = \mathbf{u}_2) \le \rho V(\mathbf{u}_2)$ for all $\mathbf{u}_2 \in \bigotimes_{i=1}^{d}(0,\underline{\xi_i})$.

In the last step, we must show the set A is a small set. This idea is referred from the Beare [2].

Lemma 4. *Assuming the joint density c is strictly positive and continuous, c is bounded away from 0 on A^2. Then A is small.*

Proof. Consider a copula of an arbitrary d-variables process $\mathbf{U}_i = (U_{i1},\dots,U_{id})$. Let $k = \inf_{(\mathbf{u},\mathbf{v})\in A^2} \frac{c(\mathbf{u},\mathbf{v})}{c(\mathbf{v})} > 0$. Define a nontrivial measure ν which satisfies $\nu B = k\int_B 1_A(\mathbf{v})d\mathbf{v}$. For any $\mathbf{u_1} \in A$ and any B

$$
\begin{aligned}
&P(\mathbf{u_1}, B) = P(\mathbf{U_1} \in B|\mathbf{U_2} = \mathbf{u_2}) \\
&= \int_B \frac{c(\mathbf{u},\mathbf{v})}{c(\mathbf{v})} d\mathbf{v} \ge \int_B \frac{c(\mathbf{u},\mathbf{v})}{c(\mathbf{v})} 1_A(\mathbf{v}) d\mathbf{v} \ge k\int_B 1_A(\mathbf{v})d\mathbf{v} = \nu B,
\end{aligned}
$$

Since this equation holds, we could say A is ν- small. According to **Proposition 5.5.3 of Meyn**, A is ν-petite. □

The Markov chain (U_t) is geometrically ergodic. We therefore have the following result.

Theorem 3. *Let a Markov chain $(U_t)_{t\in Z}$ be ϕ-irreducible and aperiodic with $U_t = (U_{t,1},\dots,U_{t,d})$. If there exists $\alpha > 0$ and the conditional tail density $\lambda_{\mathbf{v}}(\mathbf{w_1}|\mathbf{w_2})$ of $(U_t)_{t\in Z}$ (see in **Definition** 18) is such that*

$$
\sup_{\mathbf{v}\in[0,1]^d} \sup_{\mathbf{w_2}\ge 0} \int_0^1 \cdots \int_0^1 \prod_{i=1}^{d} w_i^{-\alpha} \prod_{i=d+1}^{2d} w_i^{\alpha} \lambda_{\mathbf{v}}(\mathbf{w_1}|\mathbf{w_2}) dw_1, \dots dw_d < 1,
$$

then the Markov process $(U_t)_{t\in Z}$ is geometrically ergodic.

Chapter 5

Conditional tail densities

5.1 Non-conditional and conditional tail densities

In **Definition** 14, we introduce non-conditional tail densities for bivariate copulas. Here before we define the notions of conditional tail density, it will be more clear to define the notion of tail density in multivariate case. In multivariate case, we take d-dimensional variables $U = (u_1, \ldots, u_d)$ as an instance.

Definition 17. *[8] Let C be a copula for a random vector $U = (u_1, \ldots, u_d)$ with a continuous density c. The upper and lower tail tail density functions are respectively:*

$$\lambda^U = \lim_{u \to 1} u^{d-1} c(1 - uw_i, 1 \leq i \leq d),$$
$$\lambda^L = \lim_{u \to 0} u^{d-1} c(uw_i, 1 \leq i \leq d).$$

From the proof in two-variables case ($\mathbf{u_2} \to (0,0), (0,1), (1,0)$ or $(1,1)$), we get the conditional tail density functions $\lambda_{(0,0)}(\mathbf{w_1}|\mathbf{w_2}) = \lim_{u\to 0} u^2 \frac{c(\mathbf{w}\cdot u)}{\tilde{c}(w_3 u, w_4 u)}$, and so on(see in **Corollary** 1). The conditional tail density of multivariate copula is similarly defined here, as shown below.

Definition 18. *Let C be a $2d$-dimensional copula with a continuous density $c(U_i, U_j)$ and a continuous density $\tilde{c}(U_i)$ for any $i, j > 0$. Setting $\mathbf{w} = (w_1, \ldots, w_{2d})$, $\mathbf{w_1} = (w_1, \ldots, w_d)$ and $\mathbf{w_2} = (w_{d+1}, \ldots, w_{2d})$, the conditional tail densities of corner $(0, \ldots, 0)$ is*

$$\lambda_{(0,\ldots,0)}(\mathbf{w_1}|\mathbf{w_2}) = \lim_{u \to 0} u^d \frac{c(\mathbf{w}u)}{\tilde{c}(\mathbf{w_2}u)}.$$

Since $c_{0,\ldots,0}$ and $c_{0,1,0,\ldots,0}$ are the copula density of $(U_{t,1}, U_{t,1}, U_{t,2}, \ldots, U_{t,d})$ and $(U_{t,1}, 1 - U_{t,1}, U_{t,2}, \ldots, U_{t,d})$ separately, the expression of conditional tail density of other corners in the set $A = \bigotimes_{i=1}^{d}(0,1)$ can be derived from $\lambda_{(0,\ldots,0)}(\mathbf{w_1}|\mathbf{w_2})$.

5.2 Basic properties

Property 5. *Let C be a* $2d$*-dimensional copula with conditional tail density* $\lambda_{(0,\ldots,0)}(\mathbf{w_1}|\mathbf{w_2})$ *or* $\lambda_{(1,\ldots,1)}(\mathbf{w_1}|\mathbf{w_2})$ *and so on, and continuous density function c.*
1. The conditional tail density function is a homogeneous function of order $-d$*; that is for any* $t \geq 0$, $\lambda_{(0,\ldots,0)}(t\mathbf{w_1}|t\mathbf{w_2}) = t^{-d}\lambda_{(0,\ldots,0)}(\mathbf{w_1}|\mathbf{w_2})$ *.*
2. If the conditional tail density is non-zero and differentiable, then it is directionally decreasing.
3. The conditional tail density is either identically zero or positive everywhere.

Proof. We only prove the results for $\lambda_{(0,\ldots,0)}(\mathbf{w_1}|\mathbf{w_2})$:
1. For any $t \geq 0$, we have

$$\begin{aligned}\lambda_{(0,\ldots,0)}(t\mathbf{w_1}|t\mathbf{w_2}) &= \lim_{u\to\mathbf{0}} u^d \frac{c(t\mathbf{w}u)}{\tilde{c}(t\mathbf{w_2}u)} \\ &= \lim_{u\to 0} \frac{(tu)^d c(\mathbf{w}(tu))}{t^d \tilde{c}(\mathbf{w_2}(tu))} = t^{-d}\lambda_{(0,\ldots,0)}(\mathbf{w_1}|\mathbf{w_2}).\end{aligned}$$

2. According to the homogeneity property of $\lambda_{(0,\ldots,0)}(\mathbf{w_1}|\mathbf{w_2})$, Euler's Theorem shows that

$$(-d)\lambda_{(0,\ldots,0)}(\mathbf{w_1}|\mathbf{w_2}) = \sum w_j \frac{\partial \lambda_{(0,\ldots,0)}(\mathbf{w}|\mathbf{w_2})}{\partial w_j}. \tag{5.1}$$

For $\lambda_{(0,\ldots,0)}(\mathbf{w_1}|\mathbf{w_2}) > 0$, equation (5.1) implies that the derivative of $\lambda_{(0,\ldots,0)}(\mathbf{w_1}|\mathbf{w_2})$ is non-positive along any direction originated from 0, that is $\lambda_{(0,\ldots,0)}(\mathbf{w_1}|\mathbf{w_2})$ is strictly decreasing through w_j originated from 0.
3. Joe and Li [8] showed that the tail dependence function and the upper tail density are either identically zero or positive everywhere. It is also true for conditional tail density $\lambda_{(0,\ldots,0)}(\mathbf{w_1}|\mathbf{w_2})$ in our paper.
If there exist a positive $\mathbf{v}$ such that $\lambda_{(0,\ldots,0)}(\mathbf{v}|\mathbf{w_2}) = 0$, then for any $t > 0$, $\lambda_{(0,\ldots,0)}(t\mathbf{v}|t\mathbf{w_2}) = 0$ by the homogeneous property. Since $\lim_{t\to 0}\lambda_{(0,\ldots,0)}(t\mathbf{v}|t\mathbf{w_2}) = 0$, then for any $\mathbf{w_1} > 0$, we have $\lambda_{(0,\ldots,0)}(\mathbf{w_1}|\mathbf{w_2}) = 0$. That means $\lambda_{(0,\ldots,0)}(\mathbf{w_1}|\mathbf{w_2})$ is either zero completely or positive everywhere. □

Lemma 5. *Let C be a copula of 2d-variables with* $\mathbf{w} = (w_1, \ldots, w_{2d})$, $\mathbf{w_1} = (w_1, \ldots, w_d)$ *and* $\mathbf{w_2} = (w_{d+1}, \ldots, w_{2d})$*. If there exists the lower tail density of C* $\lambda^L(\mathbf{w}u)$ *and* $\tilde{\lambda}^L(\mathbf{w_2}u)$ *a.e. non-zero, the lower conditional tail density of C is*

$$\begin{aligned}\lambda_{0,\ldots,0}(\mathbf{w_1}|\mathbf{w_2}) &= \lim_{u\to 0} u^d \frac{c(\mathbf{w}\cdot u)}{\tilde{c}(\mathbf{u_2})} d\mathbf{w} = \lim_{u\to 0} \frac{u^{2d-1}c(\mathbf{w}\cdot u)}{u^{d-1}\tilde{c}(\mathbf{u_2})} \\ &= \frac{\lim_{u\to 0} u^{2d-1}c(\mathbf{w}\cdot u)}{\lim_{u\to 0} u^{d-1}\tilde{c}(\mathbf{u_2})} = \frac{\lambda^L(\mathbf{w}u)}{\tilde{\lambda}^L(\mathbf{w_2}u)}.\end{aligned}$$

5.3 Conditional tail densities of Archimedean copulas

In the previous part, we gave the definition and properties of conditional tail densities. In this section we consider specific conditional tail densities for Archimedean copulas, which are derived from Lemma 5 and the expressions of tail densities of Archimedean copulas (see Proposition 2.5 and 3.3 in [7]). They have made assumptions about regular variation of the tails of copula in order to derive multivariate tail distributions.

Definition 19. *[7] A positive function $f(x)$ is said to be regularly varying with index $\beta > 0$ at ∞, if for any $t > 0$*

$$\lim_{x\to\infty} \frac{f(tx)}{f(x)} = t^{-\beta}.$$

If $\beta = 0$, $f(x)$ is said to be slowly varying, that is

$$\lim_{x\to\infty} \frac{f(tx)}{f(x)} = 1.$$

The function $f(x)$ is regularly varying with index $\beta > 0$ at 0, if and only if

$$\lim_{x\to 0} \frac{xf'(x)}{f(x)} = \beta. \tag{5.2}$$

A positive function f can also be said to be regularly varying at any point x_0 with index β, if $f(x_0 - x^{-1})$ is regularly varying at ∞.

Proposition 1. *Let $C(u) = \phi(\sum \phi^{-1}(u_i))$ be an Archimedean copula, where the inverse of generator ϕ is regularly varying at ∞ with tail index β. Setting $\mathbf{w} = (w_1, \ldots, w_{2d})$, $\mathbf{w_1} = (w_1, \ldots, w_d)$ and $\mathbf{w_2} = (w_{d+1}, \ldots, w_{2d})$, the lower tail dependence function $b^L(\mathbf{w})$ and lower tail density $\lambda^L(\mathbf{w}) = \lim_{u\to 0} u^{2d-1} c(\mathbf{w}u)$ of C are*

$$b^L(\mathbf{w}) = (\sum_{i=1}^{2d} w_i^{-1/\beta})^{\beta},$$

$$\lambda^L(\mathbf{w}) = \prod_{i=2}^{2d}(1 + \frac{i-1}{\beta})(\prod_{i=1}^{2d} w_i)^{-1-1/\beta}(\sum_{i=1}^{2d} w_i^{-1/\beta})^{-\beta-2d}.$$

The lower conditional tail density $\lambda_0(\mathbf{w})$ of C is

$$\lambda_{(0,\ldots,0)}(\mathbf{w_1}|\mathbf{w_2}) = \prod_{i=1+d}^{2d}(1 + \frac{i-1}{\beta})(\prod_{i=1}^{d} w_i)^{-1-1/\beta} \frac{(\sum_{i=1+d}^{2d} w_i^{-1/\beta})^{\beta+d}}{(\sum_{i=1}^{2d} w_i^{-1/\beta})^{\beta+2d}} \tag{5.3}$$

Proof. The tail dependence functions and the tail densities of Archimedean copulas were derived from Li and Wu(see Propositions 2.5 and 2.6)[8].
For $\lim_{u\to 0} u^{d-1}\tilde{c}(\mathbf{u_2}) = \lim_{u\to 0} u^{d-1}\tilde{c}(u_{2,1},\ldots,u_{2,d}) \neq 0$, by the algebraic limit theorem and **Lemma** 5 we have

$$\begin{aligned}
\lambda_{(0,\ldots,0)}(\mathbf{w_1}|\mathbf{w_2}) &= \lim_{u\to 0} u^d \frac{c(\mathbf{w}\cdot u)}{\tilde{c}(\mathbf{u_2})} d\mathbf{w} \\
&= \frac{\lambda^L((w_1,\ldots,w_{2d})u)}{\tilde{\lambda}^L((w_{d+1},\ldots,w_{2d})u)} \\
&= \frac{\prod_{i=2}^{2d}(1+\frac{i-1}{\beta})(\prod_{i=1}^{2d} w_i)^{-1-1/\beta}(\sum_{i=1}^{2d} w_i^{-1/\beta})^{-\beta-2d}}{\prod_{i=2}^{d}(1+\frac{i-1}{\beta})(\prod_{i=1+d}^{2d} w_i)^{-1-1/\beta}(\sum_{i=d+1}^{2d} w_i^{-1/\beta})^{-\beta-d}} \\
&= \prod_{i=1+d}^{2d}(1+\frac{i-1}{\beta})(\prod_{i=1}^{d} w_i)^{-1-1/\beta}\frac{(\sum_{i=1}^{2d} w_i^{-1/\beta})^{-\beta-2d}}{(\sum_{i=d+1}^{2d} w_i^{-1/\beta})^{-\beta-d}},
\end{aligned}$$

where using **Lemma** 3,

$$\begin{aligned}
\tilde{\lambda}^L(\mathbf{w_2}u) &= \frac{\partial b^L((w_{d+1},\ldots,w_{2d})u)}{\partial w_{d+1}\ldots\partial w_{2d}} \\
&= \prod_{i=2}^{d}(1+\frac{i-1}{\beta})(\prod_{i=1+d}^{2d} w_i)^{-1-1/\beta}(\sum_{i=d+1}^{2d} w_i^{-1/\beta})^{-\beta-d}.
\end{aligned}$$

□

Proposition 2. *Let $C=\phi(\sum_{i=1}^{2d}\phi^{-1}(u_i))$ be an Archimedean copula, where generator ϕ^{-1} is regularly varying at 1 with tail index $\beta>1$. Setting $\mathbf{w}=(w_1,\ldots,w_{2d})$, $\mathbf{w_1}=(w_1,\ldots,w_d)$ and $\mathbf{w_2}=(w_{d+1},\ldots,w_{2d})$, the upper exponent function $a^U(\mathbf{w})$ and upper tail density $\lambda^U(\mathbf{w})=\lim_{u\to 0} u^{2d-1}c(\mathbf{1}-\mathbf{w}u)$ of C are*

$$\begin{aligned}
a^U(\mathbf{w}) &= (\sum_{i=1}^{2d} w_i^{\beta})^{1/\beta}, \\
\lambda^U(\mathbf{w}) &= (-1)^{(2d-1)}\frac{\partial a^U}{\partial w_1\ldots\partial w_{2d}} = \prod_{i=2}^{2d}((i-1)\beta-1)(\prod_{i=1}^{2d} w_i)^{\beta-1}(\sum_{i=1}^{2d} w_i^{\beta})^{-2d+1/\beta}.
\end{aligned}$$

The upper conditional tail density $\lambda_{(1,\ldots,1)}(\mathbf{w_1}|\mathbf{w_2})=\lim_{u\to 0} u^d\frac{c(\mathbf{1}-\mathbf{w}u)}{\tilde{c}(\mathbf{1}-\mathbf{w_2}u)}$ of C is

$$\lambda_{(1,\ldots,1)}(\mathbf{w_1}|\mathbf{w_2}) = \prod_{i=1+d}^{2d}((i-1)\beta-1)(\prod_{i=1}^{d} w_i)^{\beta-1}\frac{(\sum_{i=1}^{2d} w_i^{\beta})^{-2d+1/\beta}}{(\sum_{i=d+1}^{2d} w_i^{\beta})^{-d+1/\beta}} \qquad (5.4)$$

Lemma 6. *(Theorem 8.33 [13]) Consider an Archimedean copula with Laplace transform $\phi(u)$. If $\phi'(0) = -\infty$, the Archimedean copula has upper tail dependence; otherwise, if $\phi'(0)$ is finite, then λ^U is zero, which means this copula does not have upper tail dependence.*

Lemma 7. *(Theorem 3.1 [21]) Assume an Archimedean copula C with generator $\phi^{-1}(u)$. There exists a limit denoted by $\theta = -\lim_{u\to 0} \frac{u(\phi^{-1})'(u)}{\phi^{-1}(u)}$. By equation (5.2), if $\theta = 0$, the generator ϕ^{-1} is slowly varying at 0:*

$$\lim_{u\to 0} \frac{\phi^{-1}(tu)}{\phi^{-1}(u)} = 1, t \in (0, \infty).$$

Then the copula dose not have lower tail dependence. While $\theta > 0$, there exists lower tail dependence.

Example 1. *Consider variables $U_i = (u_{i,1}, u_{i,2})$, which has an Archimedean copula with Clayton Laplace transform $\phi(u) = (1+u)^{-1/\beta}$. It is easily shown that $\phi(s)$ is regularly varying at ∞ with its index $1/\beta$, that is*

$$\lim_{u\to\infty} \frac{\phi(tu)}{\phi(u)} = \lim_{u\to\infty} (\frac{1+tu}{1+u})^{-1/\beta} = t^{-1/\beta}.$$

*According to the **Proposition 1**, the lower conditional tail density function $\lambda_{(0,0)}(\mathbf{w_1}|\mathbf{w_2})$ is*

$$\lambda_{(0,0)}(\mathbf{w_1}|\mathbf{w_2}) = (6\beta^2 + 5\beta + 1)(w_1 w_2)^{-1-\beta} \frac{(w_3^{-\beta} + w_4^{-\beta})^{1/\beta+2}}{(w_1^{-\beta} + w_2^{-\beta} + w_3^{-\beta} + w_4^{-\beta})^{1/\beta+4}},$$

for $\beta > 0$ with $\mathbf{w_1} = (w_1, w_2)$ and $\mathbf{w_2} = (w_3, w_4)$.

*Since the derivative of Laplace transform is $\phi'(u) = -\frac{(1+u)^{-1/\beta-1}}{\beta}$, then $\phi^{-1}(0) = -\frac{1}{\beta}$. According to the **Lemma** 6, when $\phi'(0)$ of Clayton copula is finite, Clayton copula does not have upper tail density and upper conditional tail density functions.*

Example 2. *Consider an Archimedean copula with Gumbel Laplace transform $\phi(u) =$ $\exp\{-u^{1/\beta}\}$. The generator $\phi^{-1}(u) = (-\log u)^{\beta}$ is regularly varying at 1 with its index β, that is*

$$\lim_{u\to\infty} \frac{\phi^{-1}(1-(tu)^{-1})}{\phi^{-1}(1-u^{-1})} = \lim_{u\to\infty} \frac{(-\ln(1-(tu)^{-1}))^{\beta}}{(-\ln(1-(u)^{-1}))^{\beta}} = t^{-\beta}.$$

*According to the **Proposition 2**, the upper conditional tail density function $\lambda_{(1,1)}(\mathbf{w_1}|\mathbf{w_2})$ is, for $\beta > 1$*

$$\lambda_{(1,1)}(\mathbf{w_1}|\mathbf{w_2}) = (6\beta^2 - 5\beta + 1)(w_1 w_2)^{\beta-1} \frac{(w_1^{\beta} + w_2^{\beta} + w_3^{\beta} + w_4^{\beta})^{1/\beta-4}}{(w_3^{\beta} + w_4^{\beta})^{1/\beta-2}},$$

with $\mathbf{w_1} = (w_1, w_2)$ *and* $\mathbf{w_2} = (w_3, w_4)$.

Next we calculate the limit

$$\begin{aligned}
\theta &= -\lim_{u \to 0} \frac{u(\phi^{-1})'(u)}{\phi^{-1}(u)} \\
&= -\lim_{u \to 0} \frac{u(\beta(-\ln u)^{\beta-1} 1/u)}{(-\ln u^{\beta})} \\
&= \lim_{u \to 0} \frac{\beta}{\ln u} = 0.
\end{aligned}$$

Using the ***Lemma*** *7, it is obviously to show that there is asymptotic independence in the neighbourhood of 0, that is the lower tail density and the lower conditional tail density of Gumbel copula are zero.*

Chapter 6

Integral of conditional tail densities

6.1 Numerical experiments in the two-variables case

By programming in R, we use the package 'cubature' to calculate multivariate integration. In our experiments, we use 'adaptIntegrate' command. The settings in this command are as follows: function is the integral of Clayton copula and Gumbel copula, the lowerLimit is $(0,0)$, and the upperLimit is the value of $(0,0) < (w_3, w_4) < (1,1)$, which is randomly given. Here we also try to fix different maximum tolerance, actually it dose not have an important impact on the integral. Thus, we randomly fix the maximum tolerance from $1e-5$ to $1e-3$.

6.1.1 Integral of Clayton copulas

Consider the Clayton copula in the two-variables case. According to the **Example** 1, the lower conditional tail density function $\lambda_{(0,0)}(\mathbf{w_1}|\mathbf{w_2})$ with $\mathbf{w_1} = (w_1, w_2)$ and $\mathbf{w_2} = (w_3, w_4)$ is

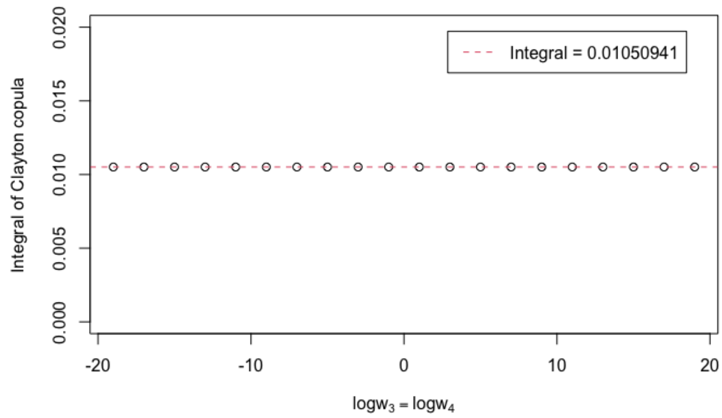
$$\lambda_{(0,0)}(\mathbf{w_1}|\mathbf{w_2}) = (6\beta^2 + 5\beta + 1)(w_1 w_2)^{-1-\beta} \frac{(w_3^{-\beta} + w_4^{-\beta})^{1/\beta+2}}{(w_1^{-\beta} + w_2^{-\beta} + w_3^{-\beta} + w_4^{-\beta})^{1/\beta+4}},$$

for its index $1/\beta > 0$. Thus, the integral of Clayton copula is

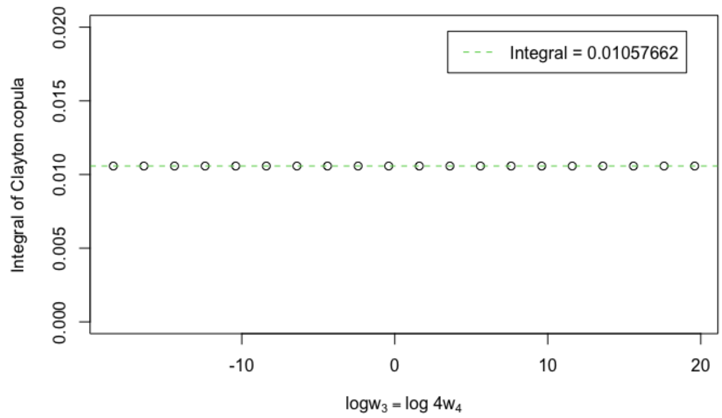
$$\begin{aligned}
&\int_0^{w_3}\int_0^{w_4} (\frac{w_1 w_2}{w_3 w_4})^{-\alpha} \lambda_{(0,0)}(\mathbf{w_1}|\mathbf{w_2}) dw_1 dw_2 \\
&= \int_0^{w_3}\int_0^{w_4} (\frac{w_1 w_2}{w_3 w_4})^{-\alpha-1-\beta} (6\beta^2 + 5\beta + 1) \frac{(w_3^{-\beta} + w_4^{-\beta})^{1/\beta+2}}{(w_1^{-\beta} + w_2^{-\beta} + w_3^{-\beta} + w_4^{-\beta})^{1/\beta+4}} dw_1 dw_2.
\end{aligned}$$

The numerical results of integral of Clayton copula conditional tail densities are presented below.

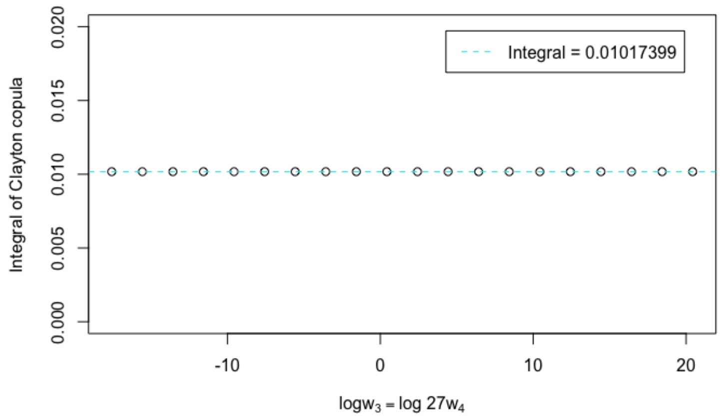
(i) Given some β, w_3, $w_4 > 0$ and $\alpha \to 0$, we discover the integral only depends on the ratio of w_3, w_4. For instance, taking $\beta = 1$, $\alpha = 10^{-5}$, the x-axis in (a) to (c) represents $\log(w_3)$ and y-axis represents the integral of Clayton copula. Here we separately take $\frac{w_3}{w_4} = 1$ in fig.(a), $\frac{w_3}{w_4} = 4$ in fig.(b) and $\frac{w_3}{w_4} = 27$ in fig.(c). These figures shows the integral is a constant for certain ratios.



(a)

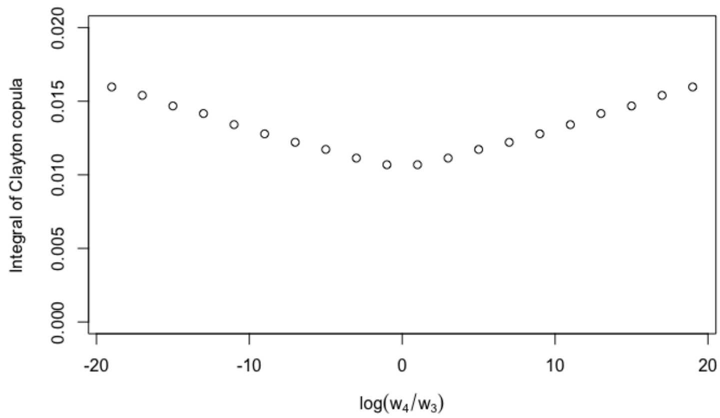


(b)



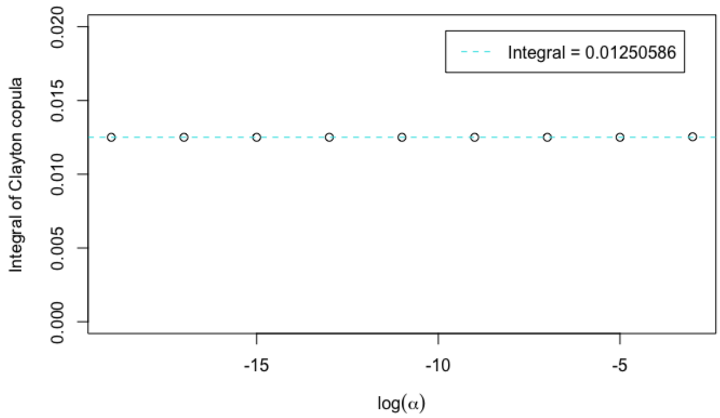
(c)

(*ii*) Further we investigate the effect of the ratio of w_3/w_4 on the integral. For $\beta = 1$, $\alpha = 10^{-5}$, we take $w_3 = 1$ constantly and $w_4 \in (10^{-19}, 10^{19})$. In fig. 6.1(d), x-axis represents $\log(w_4/w_3)$ and y-axis represents the integral of Clayton copulas. The points in the below figure indicate that the integral gets the minimum when $w_3/w_4 = 1$.



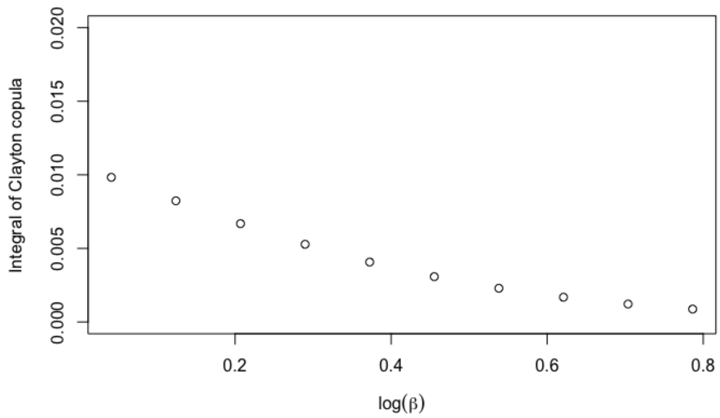
(d)

(*iii*) Here we ignore different directions of convergence, only set $w_3/w_4 = 1$ and $\beta = 0.5$. In fig. 6.1(e), x-axis represents $\log(\alpha)$ and y-axis represents the integral of Clayton copulas. By taking $\alpha \to 0$, we discover that the value of α has no obvious impact on the integral.



(e)

(*iv*) We tend to find out the influence of β on the integral. Thus, setting $w_3/w_4 = 1$ and $\alpha = 0.01$. The x-axis and y-axis represent $\log(\beta)$ and the integral of Clayton copulas, respectively in fig. 6.1(f). The figure shows integral of Clayton copula is decreasing as β increases for all real β.



(f)

6.1.2 Integral of Gumbel copulas

Consider the Gumbel copulas in two-variables case. According to the **Example** 2, the upper conditional tail density function $\lambda_{(1,1)}(\mathbf{w_1}|\mathbf{w_2})$ with $\mathbf{w_1} = (w_1, w_2)$

and $\mathbf{w_2} = (w_3, w_4)$ is

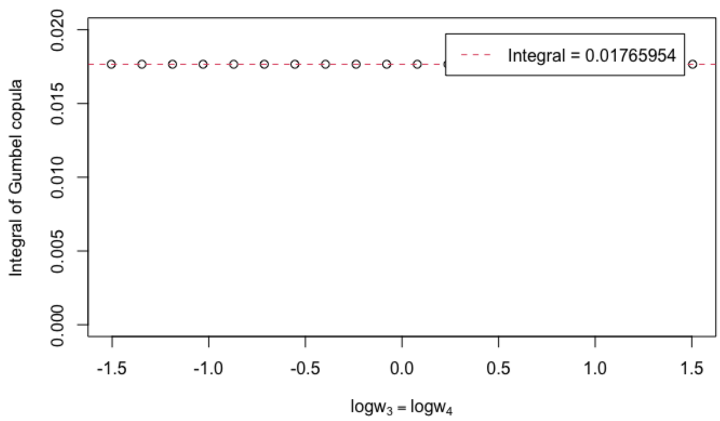
$$\lambda_{(1,1)}((\mathbf{w_1}|\mathbf{w_2}) = (6\beta^2 - 5\beta + 1)(w_1 w_2)^{\beta-1} \frac{(w_1^\beta + w_2^\beta + w_3^\beta + w_4^\beta)^{1/\beta - 4}}{(w_3^\beta + w_4^\beta)^{1/\beta-2}}, \quad (6.1)$$

for $\beta > 1$. We have the integral of Gumbel copula is

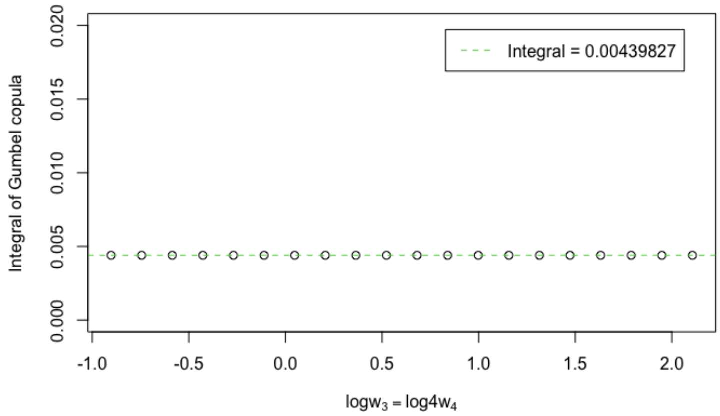
$$\int_0^{w_3} \int_0^{w_4} (\frac{w_1 w_2}{w_3 w_4})^{-\alpha} \lambda_{(1,1)}((\mathbf{w_1}|\mathbf{w_2}) dw_1 dw_2$$
$$= \int_0^{w_3} \int_0^{w_4} (\frac{w_1 w_2}{w_3 w_4})^{-\alpha-1-\beta} (6\beta^2 - 5\beta + 1)(w_1 w_2)^{\beta-1} \frac{(w_1^\beta + w_2^\beta + w_3^\beta + w_4^\beta)^{1/\beta-4}}{(w_3^\beta + w_4^\beta)^{1/\beta-2}} dw_1 dw_2.$$

By programming in R, the simulation results of integral of Gumbel copula conditional tail densities have the similar results with the Clayton copulas.

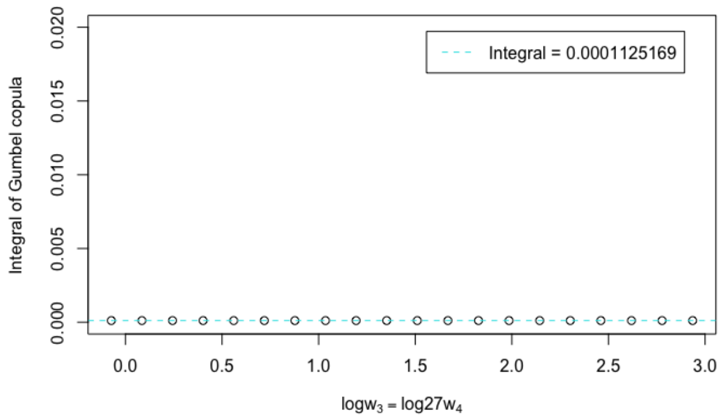
(i) The ration of w_3, w_4 depends on the integral of upper conditional tail density for β, w_3, $w_4 > 0$ and $\alpha \to 0$. For example, we take $\beta = 2$, $\alpha = 10^{-5}$, and some w_3, w_4 satisfying separately $\frac{w_3}{w_4} = 1$ in fig.(g), $\frac{w_3}{w_4} = 4$ in fig.(h), $\frac{w_3}{w_4} = 27$ in fig.(i). In fig 6.1(g) to fig 6.1(i), the x-axis represents the $\log(w_3)$ and y-axis represents the integral. These figures show that the integral keep constant if the ratio of w_3, w_4 is given; that is is the integral would converge to a constant from a given direction.



(g)

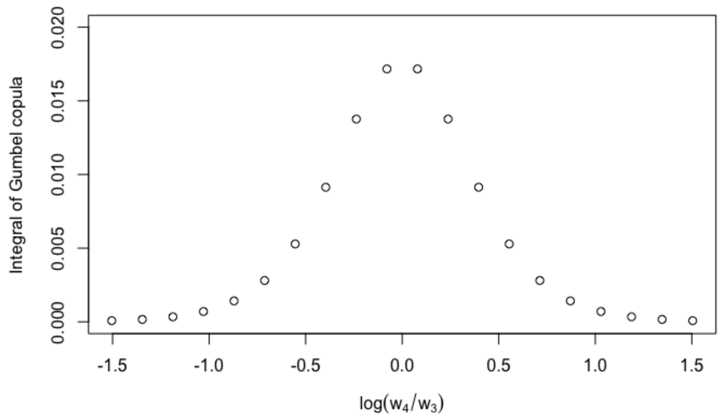

Integral = 0.00439827
Integral of Gumbel copula
0.000
0.005
0.010
0.015
0.020
-1.0
-0.5
0.0
0.5
1.0
1.5
2.0
logw3 = log4w4

(h)


Integral = 0.0001125169
Integral of Gumbel copula
0.000
0.005
0.010
0.015
0.020
0.0
0.5
1.0
1.5
2.0
2.5
3.0
logw3 = log27w4

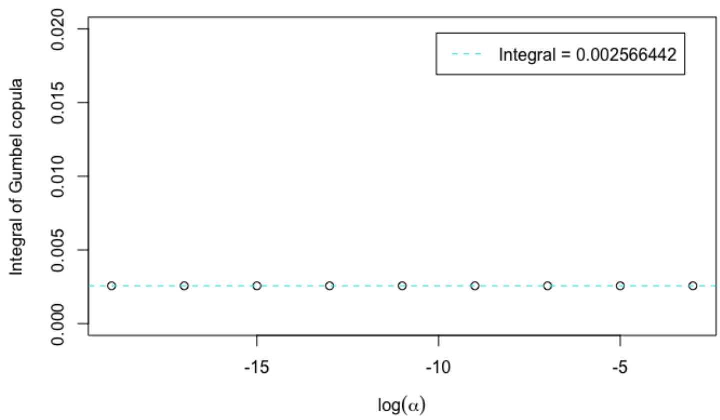
(i)

(*ii*) Now, we illustrate how the ratio of w_3/w_4 impact on the integral of Gumbel copulas. We take $\beta = 2$, $\alpha = 10^{-5}$, $w_3 \equiv 1$ and $w_4 \in (10^{-19}, 10^{19})$ here for convenience. In fig. 6.1(j), x-axis represents $\log(w_4/w_3)$ and y-axis represents the integral. This figure shows that the maximum integral of Gumbel copulas is obtained when $w_3/w_4 = 1$.



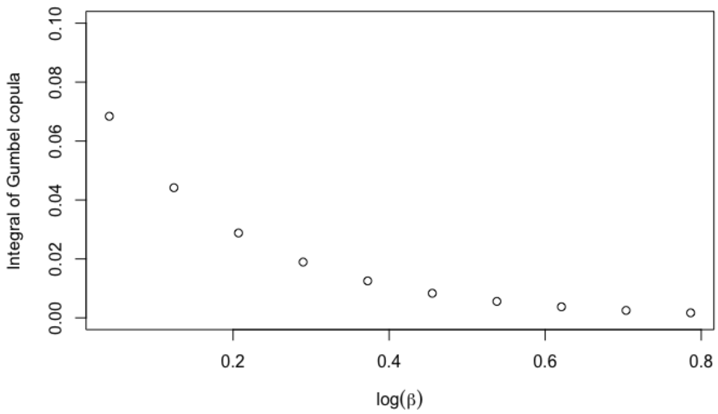
(j)

(*iii*) In this part, we take $w_3/w_4 = 1$, $\beta = 5$ and some α, which satisfies $\alpha \to 0$. We discover that the y-axis(the integral of Gumbel copulas) keeps stable when α is increasing in fig. 6.1(k), where the the horizontal axis as $\log(\alpha)$. Hence α has no impact on the integral of Gumbel copulas.



(k)

(iv) Here setting $w_3 = w_4 = 1$, $\alpha = 0.01$. The fig. 6.1(l), shows the integral of Gumbel copula is monotonically decreasing as β increases, where x-axis and y-axis are separately acted as the $\log(\beta)$ and the integral of Gumbel copulas.



(l)

6.1.3 Integral of t copulas

Consider a t copula in two-dimensional symmetric case with mean 0, dispersion matrix Σ and degree of freedom v. Based on the **Proposition 3.1** [8], the upper tail density function of a d-dimensional copula with $\mathbf{w} = (w_1, \ldots, w_d)$ is

$$\lambda^U(\mathbf{w}) = |\Sigma|^{-\frac{1}{2}} v^{(1-d)} \frac{\Gamma(\frac{v+d}{2})}{\Gamma(\frac{v+1}{2})\pi^{(d-1)/2}} \frac{[(\mathbf{w}^{-1/v})^T \Sigma^{-1} \mathbf{w}^{-1/v}]^{-\frac{v+d}{v}}}{\prod_{i=1}^d w_i^{\frac{v+1}{v}}} \tag{6.2}$$

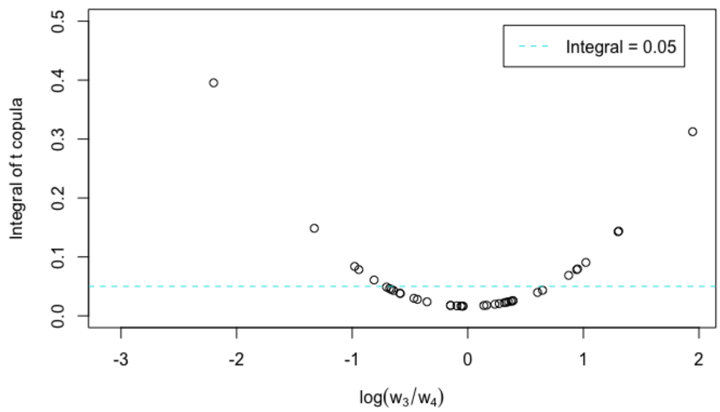
Thus the upper conditional tail density function $\lambda_{(1,1)}((\mathbf{w_1}|\mathbf{w_2})$ with $\mathbf{w_1} = (w_1, w_2)$, $\mathbf{w_2} = (w_3, w_4)$ and $\mathbf{w} = (w_1, w_2, w_3, w_4)$is

$$\begin{aligned}
\lambda_{(1,1)}(\mathbf{w_1}|\mathbf{w_2}) &= \frac{\lambda^U((w_1, w_2, w_3, w_4)u)}{\tilde{\lambda}^U((w_3, w_4)u)} \\
&= (\frac{|\Sigma_1|}{|\Sigma_2|})^{-\frac{1}{2}} \frac{v_1(v_1/2+1)\Gamma(v_1/2)\Gamma(\frac{v_2+1}{2})}{v_1^2\Gamma(\frac{v_1+1}{2})\Gamma(v_2/2)\pi} \\
&\times \frac{\prod_{i=3}^4 w_i^{\frac{v_2+1}{v_2}} [(\mathbf{w}^{-1/v_1})^T \Sigma_1^{-1} \mathbf{w}^{-1/v_1}]^{-\frac{v_1+4}{v_1}}}{\prod_{i=1}^4 w_i^{\frac{v_1+1}{v_1}} [(\mathbf{w_2}^{-1/v_2})^T \Sigma_2^{-1} \mathbf{w_2}^{-1/v_2}]^{-\frac{v_2+2}{v_2}}}
\end{aligned}$$

where the degree of freedom v_1, v_2, dispersion matrix Σ_1, Σ_2 respectively in (w_1, w_2, w_3, w_4) case and (w_3, w_4) case. Thus the integral of t copula is less than $\int_0^{w_3} \int_0^{w_4} (\frac{w_1 w_2}{w_3 w_4})^{-\alpha} \lambda_{(1,1)}((\mathbf{w_1}|\mathbf{w_2}) dw_1 dw_2$.

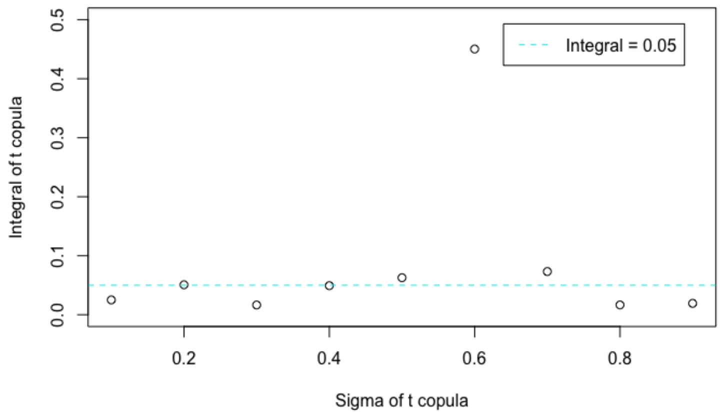
By generating random numbers with the t copula by R, we have the simulation results given below.

(i) Here setting the correlation of any variables 0.5 and the degree of freedom 3 and 1, we generate random numbers under the t copulas. In fig. 6.1(m), x-axis and y-axis separately represents the ratio of $log(w_3/w_4)$ and the integral. This figure shows that the integral of t copulas is smaller than 1. In addition, the integral of t copula get the minimum as the ratio $log(w_3/w_4) = 0$.



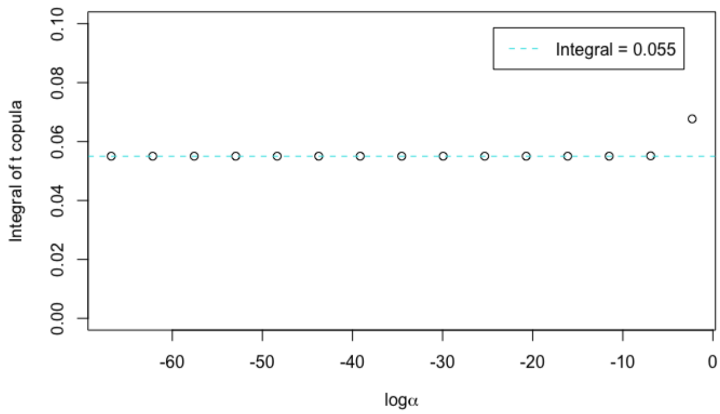
(m)

(ii) Now we consider how the matrix Σ impact on the integral of t copulas. To simplify the issues, we only consider the symmetric matrix Σ and set the values from 0.1 to 0.9. The fig. 6.1(n) shows that the y-axis integral is smaller than 1 and the x-axis is the value of Σ.



(n)

(*iii*) Given some $\alpha \to 0$, we know that the integral of t copulas (y-axis) is a constant when α (x-axis is $log(\alpha)$) increases from fig. 6.1(o). Therefore, α would not affect integral results of t copulas.



(o)

6.1.4 Conclusion

According to the numerical approximations, we found that the integral of Clayton copulas or Gumbel copulas seems to be no more that 1. Thus, we think any Markov chain generated by Clayton copulas and Gumbel copulas is geometrically ergodic when $w_{d+1}, \ldots, w_{2d} < 1$. In addition, the integrals of Clayton copulas and of Gumbel copulas are only depended on the directions of convergence, that is the ratio of $\frac{w_{2j}}{w_j}$ for $j \in [1, d]$. We tend to give a general proof the integral of Clayton copulas or Gumbel copulas is less than 1 in $w_{d+1}, \ldots, w_{2d} > 1$.

6.2 General proof in d-variables case

We will obtain the integral of Archimedean copulas, such as Clayton copulas and Gumbel copulas, based on Dominated Convergence Theorem.

Consider variables $U_i = (u_{i,1}, u_{i,2}, \ldots, u_{i,d})$, which has an Archimedean copula. Based on the results from experiments, we will only consider $w_{d+1}, \ldots, w_{2d} > 1$ when calculating below.

Lemma 8. *(Dominated convergence theorem) Let f_n be a sequence of real measurable functions. If satisfying*
(i) $\forall x$, $f_n(x) \to f(x)$;
(ii) there exists a non-negative function $F(x)$ such that $\forall x, n \Rightarrow |f_n(x)| \leq F(x)$, where $\int F(x)dx < \infty$;
then $\lim_{n\to\infty} \int f_n(x)dx = \int f(x)dx$.

Theorem 4. *Suppose that variables $U_i = (u_{i,1}, u_{i,2}, \ldots, u_{i,d})$ have an Archimedean copula. And suppose there is $\alpha > 0$ such that $\int$, the conditional tail density $\lambda(\mathbf{w_1}|\mathbf{w_2})$ of this copula satisfies its integral is bounded by 1, i.e.,*

$$\lim_{\alpha\to 0} \int_0^{w_{d+1}} \cdots \int_0^{w_{2d}} \left(\frac{\prod_{i=1}^{d} w_i}{\prod_{i=d+1}^{2d} w_i}\right)^{-\alpha} \lambda(\mathbf{w_1}|\mathbf{w_2}) dw_1 \ldots dw_d < 1,$$

for all $\beta \geq 0$ and some $\alpha > 0$.

Proof. To simplify calculations, we use the integral

$$\begin{aligned}
&\int_0^{w_{d+1}} \cdots \int_0^{w_{2d}} \left(\frac{\prod_{i=1}^{d} w_i}{\prod_{i=d+1}^{2d} w_i}\right)^{-\alpha} \lambda(\mathbf{w_1}|\mathbf{w_2}) dw_1 \ldots dw_d \\
&= \left(\prod_{i=d+1}^{2d} w_i\right)^{\alpha} \int_0^{w_{d+1}} \cdots \int_0^{w_{2d}} \left(\prod_{i=1}^{d} w_i\right)^{-\alpha} \lambda(\mathbf{w_1}|\mathbf{w_2}) dw_1 \ldots dw_d,
\end{aligned}$$

In our setting, the sequence of measurable functions $f_n(w_1, \ldots, w_d)$ is

$$f_n(w_1, \ldots, w_d) = \left(\prod_{i=1}^{d} w_i\right)^{-\frac{1}{n}} \lambda(\mathbf{w_1}|\mathbf{w_2}), \tag{6.3}$$

and $f(w_1, \dots, w_d) = \lambda(\mathbf{w_1}|\mathbf{w_2})$.

(i)It is obvious that

$$\begin{aligned}\lim_{n\to\infty} f_n(w_1, \dots, w_d) &= \lim_{n\to\infty} (\prod_{i=1}^{d} w_i)^{-\frac{1}{n}} \lambda(\mathbf{w_1}|\mathbf{w_2}) \\ &= (\prod_{i=1}^{d} w_i)^{\lim_{n\to\infty}(-\frac{1}{n})} \lambda(\mathbf{w_1}|\mathbf{w_2}) = f(w_1, \dots, w_d).\end{aligned}$$

(ii)Assume that we have found a dominated function $F(w_1, \dots, w_d)$ which is integrable.

Note that $\lambda(\mathbf{w_1}|\mathbf{w_2})$ is the density of a sub-distribution function, i.e., satisfying that the distribution function may integrate less than 1. Consequently, the integral of Archimedean copula conditional tail densities satisfies

$$\begin{aligned}&\lim_{\alpha\to 0} \int_0^{w_{d+1}} \cdots \int_0^{w_{2d}} (\frac{\prod_{i=1}^{d} w_i}{\prod_{i=d+1}^{2d} w_i})^{-\alpha} \lambda(\mathbf{w_1}|\mathbf{w_2}) dw_1 \dots dw_d \\ &= \lim_{n\to\infty} (\prod_{i=d+1}^{2d} w_i)^{1/n} \int_0^{w_{d+1}} \cdots \int_0^{w_{2d}} f_n(w_1, \dots, w_d) dw_1 \dots dw_d \\ &= 1 \cdot \int_0^{w_{d+1}} \cdots \int_0^{w_{2d}} \lambda(\mathbf{w_1}|\mathbf{w_2}) \\ &< 1.\end{aligned}$$

for any $w_{d+1}, \dots, w_{2d} > 1$. $\square$

Corollary 2. *If variables $U_i = (u_{i,1}, u_{i,2}, \dots, u_{i,d})$ is modeled by Clayton Laplace transform $\phi(u) = (1+u)^{-1/\beta}$, it has a lower conditional tail density $\lambda_{\mathbf{0}}(\mathbf{w_1}|\mathbf{w_2})$ which satisfies*

$$\lim_{\alpha\to 0} \int_0^{w_{d+1}} \cdots \int_0^{w_{2d}} (\frac{\prod_{i=1}^{d} w_i}{\prod_{i=d+1}^{2d} w_i})^{-\alpha} \lambda_{\mathbf{0}}(\mathbf{w_1}|\mathbf{w_2}) dw_1 \dots dw_d < 1$$

for all $\beta > 0$ and some $\alpha > 0$, given $w_{d+1}, \dots, w_{2d} > 1$.

Proof. Indeed, we only need to find a dominated function $F(w_{1,d})$ which is integral for Clayton copulas.

According to the **Proposition** 1, the lower conditional tail density function $\lambda_{\mathbf{0}}(\mathbf{w_1}|\mathbf{w_2})$ is for $w_{d+1}, \ldots, w_{2d} > 1$,

$$\begin{aligned}
\lambda_{\mathbf{0}}(\mathbf{w_1}|\mathbf{w_2}) &= \prod_{i=1+d}^{2d}(1+(i-1)\beta)(\prod_{i=1}^{d} w_i)^{-1-\beta}\frac{(\sum_{i=1+d}^{2d} w_i^{-\beta})^{1/\beta+d}}{(\sum_{i=1}^{2d} w_i^{\beta})^{1/\beta+2d}} \\
&\leq \nu_\beta(\prod_{i=1}^{d} w_i)^{-1-\beta}\frac{d^{1/\beta+d}}{(\sum_{i=1}^{d} w_i^{\beta})^{1/\beta+2d}} \\
&= \nu_\beta(\prod_{i=1}^{d} w_i)^{-1-\beta}\frac{d^{1/\beta+d}}{(d(\sum_{i=1}^{d}\frac{1}{d}w_i^{\beta}))^{1/\beta+2d}},
\end{aligned}$$

where $\nu_\beta = \prod_{i=1+d}^{2d}(1+(i-1)\beta)$. Using the inequality of arithmetic and geometric means, i.e., $\frac{x_1+\cdots+x_d}{d} \geq \sqrt[d]{x_1\ldots x_d}$ for $x_1\ldots,x_d \geq 0$, we have

$$\begin{aligned}
\lambda_{\mathbf{0}}(\mathbf{w_1}|\mathbf{w_2}) &\leq \nu_\beta(\prod_{i=1}^{d} w_i)^{-1-\beta}\frac{d^{1/\beta+d}}{d^{1/\beta+2d}(\prod_{i=1}^{d} w_i^{-\beta})^{\frac{1}{d}(1/\beta+2d)}} \\
&= d^{-d}\nu_\beta(\prod_{i=1}^{d} w_i)^{-1-\beta}(\prod_{i=1}^{d} w_i)^{2\beta+\frac{1}{d}} \\
&< \nu_\beta(\prod_{i=1}^{d} w_i)^{\beta+\frac{1}{d}-1}.
\end{aligned}$$

So the dominating function $F(w_1,\ldots,w_d) = \nu_\beta(\prod_{i=1}^{d} w_i)^{\beta+\frac{1}{d}-1}$ is integrable if $\beta > 0$, since

$$\begin{aligned}
&\int_0^{w_{d+1}}\cdots\int_0^{w_{2d}} F(w_1,\ldots,w_d)dw_1\ldots dw_d \\
&= \nu_\beta\int_0^{w_{d+1}}(w_1)^{\beta+\frac{1}{d}-1}dw_1\cdots\int_0^{w_{2d}}(w_d)^{\beta+\frac{1}{d}-1}dw_d \\
&= \nu_\beta\frac{(\prod_{i=1+d}^{2d} w_i)^{\beta+\frac{1}{d}}}{\beta^d} < \infty.
\end{aligned}$$

□

Corollary 3. *If variables $U_i = (u_{i,1}, u_{i,2}, \ldots, u_{i,d})$ is modeled by Gumbel Laplace transform $\phi(u) = \exp\{-u^{1/\beta}\}$, it has a upper conditional tail density $\lambda_{\mathbf{1}}(\mathbf{w_1}|\mathbf{w_2})$ which satisfies*

$$\lim_{\alpha\to 0}\int_0^{w_{d+1}}\cdots\int_0^{w_{2d}}(\frac{\prod_{i=1}^{d} w_i}{\prod_{i=d+1}^{2d} w_i})^{-\alpha}\lambda_{\mathbf{1}}(\mathbf{w_1}|\mathbf{w_2})dw_1\ldots dw_d < 1$$

for $\forall\epsilon > 0$, $\beta > 1$ and some $\alpha > 0$, given $w_{d+1},\ldots,w_{2d} > 1$.

Proof. The proof is quite similar to that of Clayton copula, which only requires to find a dominated function.

Then according to the **Proposition** 2, the upper conditional tail density function of Gumbel copulas $\lambda_{\mathbf{1}}(\mathbf{w_1}|\mathbf{w_2})$ is

$$\begin{aligned}
\lambda_{\mathbf{1}}(\mathbf{w_1}|\mathbf{w_2}) &= \prod_{i=1+d}^{2d}((i-1)\beta-1)(\prod_{i=1}^{d} w_i)^{\beta-1}\frac{(\sum_{i=d+1}^{2d} w_i^{\beta})^{d-1/\beta}}{(\sum_{i=1}^{2d} w_i^{\beta})^{2d-1/\beta}} \\
&\leq \nu_\beta(\prod_{i=1}^{d} w_i)^{\beta-1}\frac{(\sum_{i=d+1}^{2d} w_i^{\beta})^{d-1/\beta}}{(\sum_{i=d+1}^{2d} w_i^{\beta})^{2d-1/\beta}} \\
&= \nu_\beta(\prod_{i=1}^{d} w_i)^{\beta-1}\frac{1}{(\sum_{i=d+1}^{2d} w_i^{\beta})^{d}} \\
&\leq \nu_\beta(\prod_{i=1}^{d} w_i)^{\beta-1} d^{-d} \\
&< \nu_\beta(\prod_{i=1}^{d} w_i)^{\beta-1},
\end{aligned}$$

for $\beta > 1$ and $w_{d+1}, \ldots, w_{2d} > 1$.

It is therefore natural to obtain the dominated function of $f_n(w_1, \ldots, w_d)$ is $F(w_1, \ldots, w_d) = \nu_\beta(\prod_{i=1}^{d} w_i)^{\beta-1}$, which satisfies

$$\int_0^{w_{d+1}} \cdots \int_0^{w_{2d}} F(w_1, \ldots, w_d) dw_1 dw_d < \infty$$

with $\beta > 1$. □

Chapter 7

Conclusion

In this thesis, we have proposed a copula-based condition for geometric ergodicity of Markov chains. First according to the **Theorem** 3 from Meyn and Tweedie, we establish a geometric ergodicity criterion of Markov chains in the univariate case with tail densities based on copula approach in Chapter 4. We show that if the tail density is either the density of a sub-distribution function or zero, there exists a suitable drift function V and a petite set, such that the Markov chain is geometric ergodic. Then we extend the condition of geometric ergodicity to multivariate case with the conditional tail density. The new condition requires the supremum of integral of conditional tail densities functio is smaller than 1. In order to verify the multivariate condition, we derive the expressions of conditional tail densities for Archimedean copulas, including Gumbel and Clayton copulas in Chapter 5. By the numerical experiments of Chapter 6, we get any Markov chain generated by Gumbel copulas, Clayton copulas and student t copulas in 2-dimensional case is geometrically ergodic and these results are only depended on the directions of convergence. Based on these findings, we conclude that the established condition with the conditional tail density shows that the Markov processes generated by Archimedean copulas are geometrically ergodic.

Acknowledgements

These words are for those who have helped me greatly during my journey of Master degree.

First, thanks a lot to my supervisor Thomas Nagler. I would never accomplish my thesis without his teaching and supporting. He is so kind to support my work during the hard coronavirus outbreak.

Great appreciate for my parents' support. Without the moral support and financial assistance of my parents, I would not have been able to engage in my studies and enjoy the grateful life in Leiden.

Thanks for my roommates Hansha Leng and Long Long. I can fully enjoy the Dutch landscape and visit prestigious museums in their company.

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