# Near-Optimal Dimension Reduction for Facility Location

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# Dimension Reduction

**Theorem (Johnson-Lindenstrauss lemma [Johnson-Lindenstrauss 84])**

*For all* $n > 0$, $\varepsilon \in (0,1)$, *there exists a random linear map* $\pi\colon \mathbb{R}^d \to \mathbb{R}^m$, *for* $m = O(\varepsilon^{-2} \log n)$, *such that for every* $X \subset \mathbb{R}^d$ *with* $|X| = n$, *with high probability*

$$\forall x, y \in X, \|\pi(x) - \pi(y)\| \in (1 \pm \varepsilon) \|x - y\| .$$

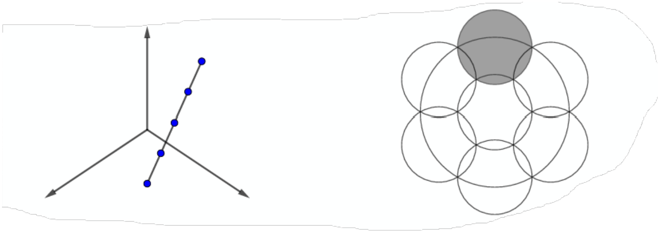
- JL mapping: $\pi\colon x \mapsto \frac{1}{\sqrt{m}} Gx$, where $G \in \mathbb{R}^{m \times d}$ with $g_{i,j} \sim i.i.d. N(0,1)$.
- $m = O(\varepsilon^{-2} \log n)$ is tight [Larsen, Nelson, FOCS 2017].
- Curse of dimensionality: target dimension $m = \Theta(\log n)$ is always too high to afford.
  - TSP does not admit a PTAS in dimension $m = \Theta(\log n)$ [Trevisan, SIAM J. Comput. 00].
  - Many problems have $2^{2^d}$ dependence in the (low-dimensional) PTAS.

# Doubling Dimension

[Gupta, Krauthgamer, Lee, FOCS 03]

Solution: Use *intrinsic dimension*.

- $\mathrm{ddim}(X) :=$ the minimum $t \geq 0$, such that $\forall r > 0$, each ball in $X$ of radius $r$ can be covered by at most $2^t$ balls of radius $r/2$.



- Fundamental question: refine JL lemma such that $m$ only depends on $\mathrm{ddim}(X)$.
- Preserve objective value for specific computational problems instead of pairwise distances. (i.e. $\mathrm{opt}(\pi(X)) \approx \mathrm{opt}(X)$)

# Related Work on JL

- Preserve objective value for specific computational problems instead of pairwise distances. (i.e. $\mathrm{opt}(\pi(X)) \approx \mathrm{opt}(X)$)

| Problems | Approximation | Target Dimension | References |
|---|---|---|---|
| Nearest Neighbor | $1+\varepsilon$ | $O(\varepsilon^{-2}\mathrm{ddim})$ | IN07 |
| $k$-Center Clustering | $1+\varepsilon$ | $O(\varepsilon^{-2}(\log k + \mathrm{ddim}))$ | JKS24 |
| $k$-Median / $k$-Means | $1+\varepsilon$ | $O(\varepsilon^{-2}\log k)$ | MMR19 |
| Max-Cut | $1+\varepsilon$ | $O(1/\varepsilon^2)$ | CJK23 |
| MST | $1+\varepsilon$ | $O(\varepsilon^{-2}\mathrm{ddim}\log\log n)$ | NSIZ21 |
| UFL | $O(1)$ | $O(\varepsilon^{-2}\mathrm{ddim})$ | NSIZ21 |
| UFL | $1+\varepsilon$ | $\tilde{O}(\varepsilon^{-2}\mathrm{ddim})$ | This work |

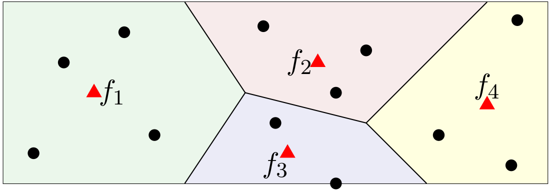
# Uniform Facility Location (UFL)

Input: Point set $X \subset \mathbb{R}^d$ with bounded doubling dimension $\mathrm{ddim}$, *opening cost* $\mathfrak{f} > 0$.
Goal: Find a set of *facilities* $F \subset \mathbb{R}^d$, so as to minimize the objective

$$\mathrm{cost}(X, F) := \underbrace{\mathfrak{f} \cdot |F|}_{\text{opening cost}} + \underbrace{\sum_{x \in X} \mathrm{dist}(x, F)}_{\text{connection cost}},$$

where $\mathrm{dist}(x, F) := \min_{y \in F} \mathrm{dist}(x, y)$ and $\mathrm{dist}(x, y) := \|x - y\|_2$.

- W.l.o.g., $\mathfrak{f} \equiv 1$ (by scaling the point set).
- Denote the *optimal value* by $\mathrm{ufl}(X)$.




**Problem (Dimension reduction for UFL)**

*Given $\varepsilon, \delta > 0$, find target dimension $m = f(\mathrm{ddim}, \varepsilon, \delta)$, such that $\Pr[\mathrm{ufl}(\pi(X)) \in (1 \pm \varepsilon)\,\mathrm{ufl}(X)] \geq 1 - \delta$, where $\pi\colon \mathbb{R}^d \to \mathbb{R}^m$ is the JL mapping.*

# Results

**Theorem (Dimension reduction for UFL)**

$\Pr[\mathrm{ufl}(\pi(X)) \in (1 \pm \varepsilon)\,\mathrm{ufl}(X)] \geq 1 - \delta$, *for target dimension* $m := O(\varepsilon^{-2}\mathrm{ddim} \cdot \log(\delta^{-1}\varepsilon^{-1}\mathrm{ddim}))$.

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- Previous results: $O(1)$-approximation with $m = O(\varepsilon^{-2}\mathrm{ddim})$ [Narayanan, Silwal, Indyk, Zamir, ICML 21] or $(1+\varepsilon)$-approximation with $m = O(\varepsilon^{-2}\log n)$ [Makarychev, Makarychev, Razenshteyn, STOC 19].

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**Theorem (PTAS for UFL)**

*There is an algorithm that computes a* $(1+\varepsilon)$*-approximate solution for UFL, running in time* $(2^{m'}d + 2^{2^{m'}}) \cdot \tilde{O}(n)$, *for* $m' = O\left(\mathrm{ddim} \cdot \log(\mathrm{ddim}/\varepsilon)\right)$.

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- Facilities are allowed to be picked from the ($d$-dimensional) ambient space.
- Previous result: Running in time $2^{2^{O(\mathrm{ddim}^2)}} d \cdot \tilde{O}(n)$, with facilities restricted to the same doubling metric [Cohen-Addad, Feldmann, Saulpic, JACM 21].

# Proof Overview

- Upper bound $\mathrm{ufl}(\pi(X))$: $\mathrm{ufl}(\pi(X)) \leq \mathrm{cost}(\pi(X), \pi(F^*)) \lesssim \mathrm{cost}(X, F^*) = \mathrm{ufl}(X)$.

**Proposition** ([Johnson, Lindenstrauss 84] [Makarychev, Makarychev, Razenshteyn, STOC 19])

$\forall x, y \in \mathbb{R}^d$ *and* $t > 0$,

$$\mathbb{E}\left[\max\left\{0, \frac{\|\pi(x) - \pi(y)\|}{\|x - y\|} - (1 + t)\right\}\right] \leq \frac{1}{mt} e^{-t^2 m/2}.$$

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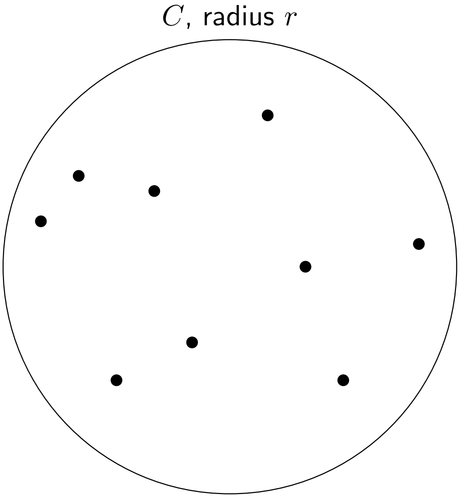
- Upper bound $\mathrm{ufl}(\pi(X))$: $\mathrm{ufl}(\pi(X)) \le \mathrm{cost}(\pi(X), \pi(F^*)) \lesssim \mathrm{cost}(X, F^*) = \mathrm{ufl}(X)$.
- Lower bound $\mathrm{ufl}(\pi(X))$
  (1) Partition $X$ into "light" clusters $\mathbf{\Lambda} = \{C_1, C_2, \ldots, C_{|\mathbf{\Lambda}|}\}$, where $\mathrm{ufl}(C_i) = \Theta(\mathrm{ddim}/\varepsilon)^{O(\mathrm{ddim})}$.
  (2) Sub-additivity: $\mathrm{ufl}(X) \le \sum_{C \in \mathbf{\Lambda}} \mathrm{ufl}(C)$.
  (3) On each cluster $C \in \mathbf{\Lambda}$, $\mathrm{ufl}(C) \lesssim \mathrm{ufl}(\pi(C))$ in expectation.
  (4) $\sum_{C \in \mathbf{\Lambda}} \mathrm{ufl}(\pi(C)) \lesssim \mathrm{ufl}(\pi(X))$.

# Step 1: Hierarchical Decomposition




Recursively partition a large cluster into small sub-clusters of half radius.

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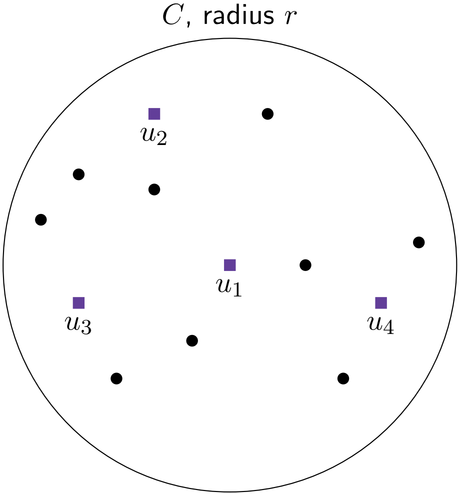


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- Construct $(r/2)$-net $N$ on $C$.

**Definition ($\rho$-net)**

- $\rho$-packing: $\forall x, y \in N$, $\mathrm{dist}(x, y) \geq \rho$.
- $\rho$-covering: $\forall x \in C$, $\exists y \in N$, s.t. $\mathrm{dist}(x, y) \leq \rho$.
- $N$ is a $\rho$-net if it is both $\rho$-packing and $\rho$-covering for $C$.

**Proposition (Packing property)**

*If $S$ is $\rho$-packing then*
$|S| \leq (2\,\mathrm{diam}(S)/\rho)^{\mathrm{ddim}(S)}$.

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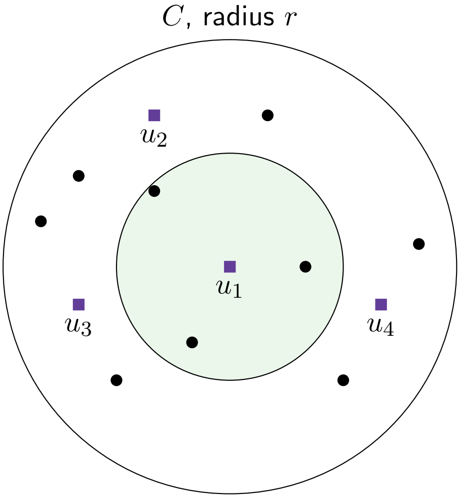


Recursively partition a large cluster into small sub-clusters of half radius.

- Construct a level $i$ child cluster $C_u \leftarrow C \cap B(u, r/2) \setminus \bigcup_{v \in N:\sigma(v)<\sigma(u)} B(v, r/2)$.

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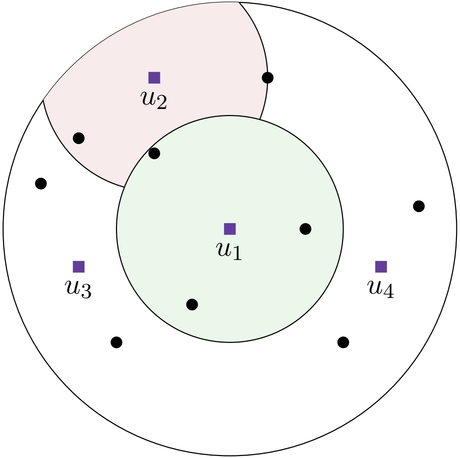
$C$, radius $r$

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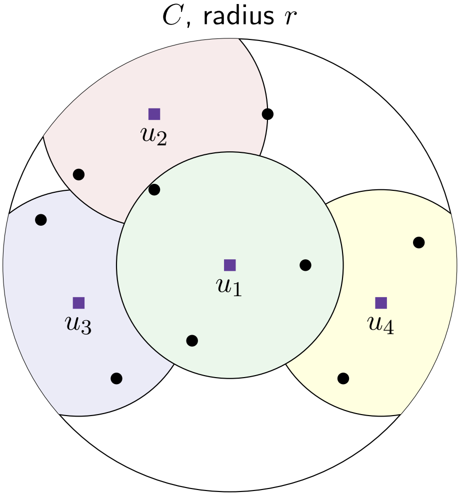


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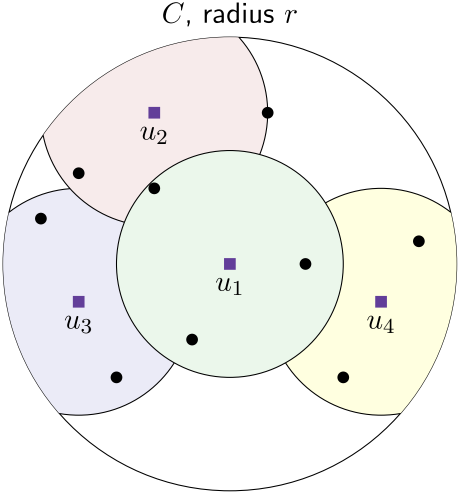
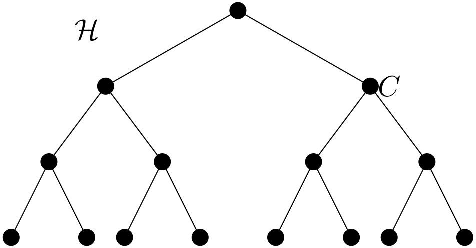
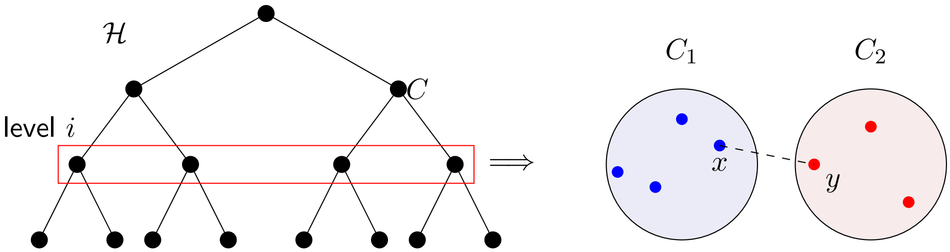



- Node $\leftrightarrow$ cluster.
  - Root: $X$;
  - leaves: singletons;
  - level $i$: diameter $\Theta(2^i)$.
- Each node (cluster) has $2^{O(\mathrm{ddim})}$ child nodes (clusters).

# Cut




- Cut [Talwar, STOC 04] [Cohen-Addad, Feldmann, Saulpic, JACM 21]: a pair of points $(x, y)$ is cut by cluster $C$ if $x \in C$ and $y \notin C$.
- Badly cut: $(x, y)$ is *badly cut* if $(x, y)$ is cut by some cluster $C$ with $\mathrm{diam}(C) \geq \frac{\mathrm{ddim}}{\varepsilon^2} \cdot \|x - y\|$.
- There is a (random) hierarchical decomposition, such that $\forall x, y \in X$, $\Pr[(x, y) \text{ is badly cut}] \leq O(\varepsilon^2)$.

# Step 2: Eliminating Badly-Cut Pairs

Fix facility set $F \subseteq X$.
Goal: $\forall x \in X$, if $(x, F(x))$ is cut by cluster $C$, then $x$ and $F(x)$ are separated ($\|x - F(x)\| \gtrsim \mathrm{diam}(C)$).

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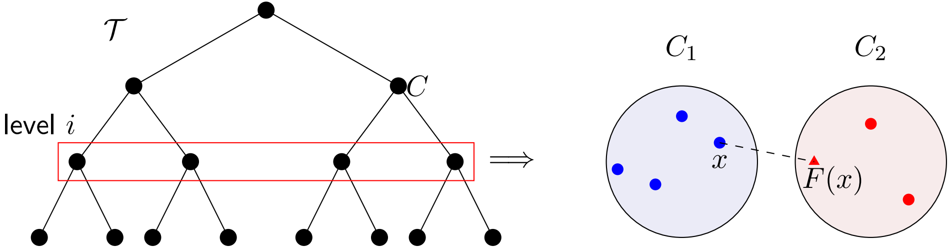
- For each level $i$:
  - For each $x \in X$, if $x \in C, F(x) \notin C$ and $\mathrm{diam}(C) \geq \frac{\mathrm{ddim}}{\varepsilon^2} \cdot \|x - F(x)\|$, then "move" $x$ into the same level $i$ cluster as $F(x)$.



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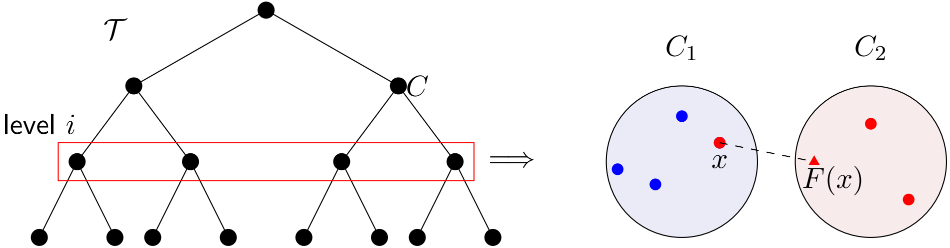

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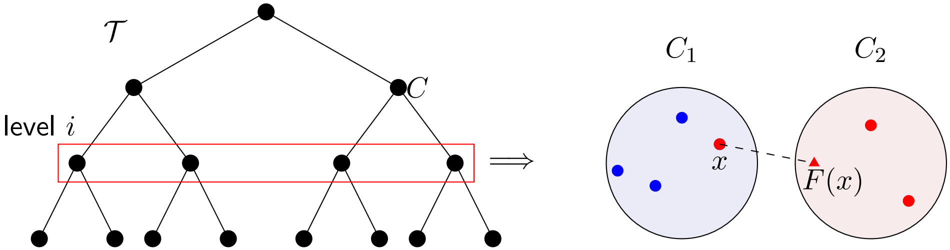
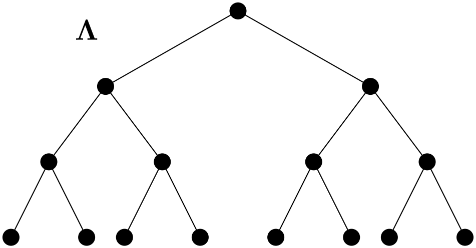



- On the new decomposition $\mathcal{T}$, $(x, F(x))$ is not badly-cut.
  - Separation: If $(x, F(x))$ is cut by cluster $C$, then $\|x - F(x)\| \geq \frac{\varepsilon^2}{\mathrm{ddim}} \mathrm{diam}(C)$.

# Step 3: Partition

Given threshold $\kappa = \Theta(\mathrm{ddim}/\varepsilon)^{O(\mathrm{ddim})}$, find the lowest level "heavy cluster" $(\mathrm{ufl}(C) \geq \kappa)$ in a bottom-up manner.



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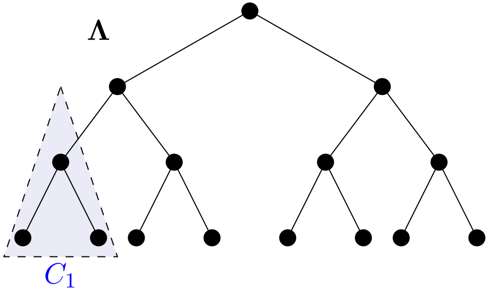

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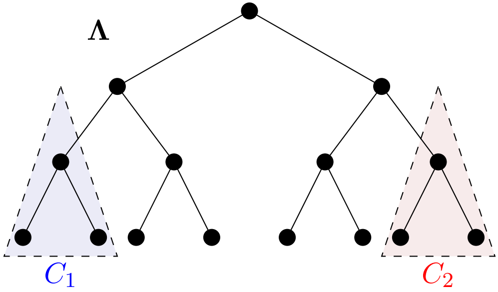

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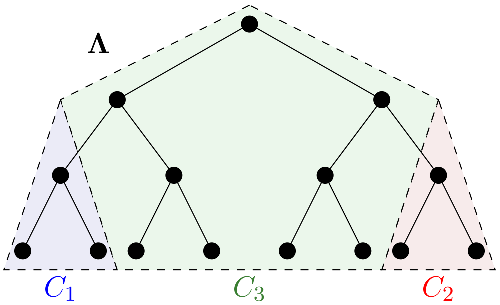

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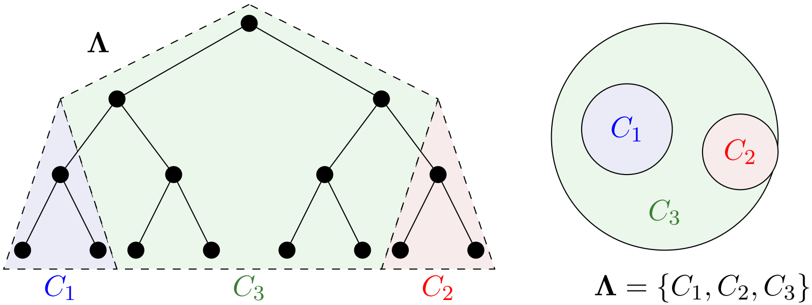

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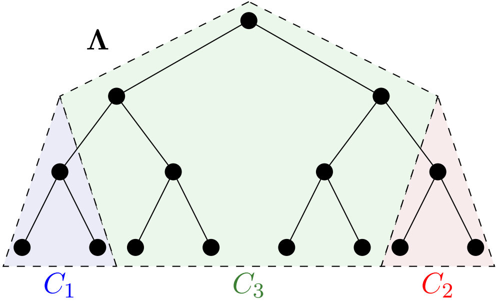
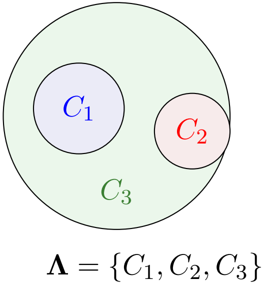



- Bounded optimal value: $\forall C \in \mathbf{\Lambda}$, $\kappa \leq \mathrm{ufl}(C) \leq 2^{O(\mathrm{ddim})}\kappa$.

# JL on Each Cluster

- Bounded optimal value: $\forall C \in \mathbf{\Lambda}$, $\kappa \le \mathrm{ufl}(C) \le 2^{O(\mathrm{ddim})}\kappa$.
- Equivalent to $\tau$-median on $C$, where $\tau = 2^{O(\mathrm{ddim})}\kappa = \Theta(\mathrm{ddim}/\varepsilon)^{O(\mathrm{ddim})}$.

**Lemma (Refined from [Makarychev, Makarychev, Razenshteyn, STOC 19])**

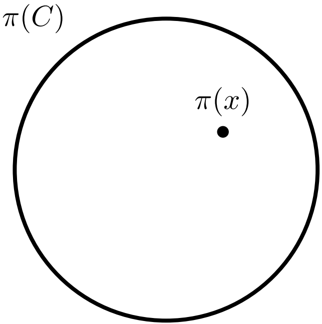
*Assume* $\mathrm{ufl}(C) \le \tau$. *Then for* $m = \Omega(\varepsilon^{-2}\log(1/\varepsilon))$,

$$\Pr\left[\mathrm{ufl}(\pi(C)) \le \frac{1}{1+\varepsilon}\,\mathrm{ufl}(C)\right] \le \tau^3 \cdot e^{-\Omega(\varepsilon^2 m)}.$$

- Choose target dimension $m = O(\varepsilon^{-2}\log\tau) = O(\varepsilon^{-2}\mathrm{ddim}\cdot\log(\mathrm{ddim}/\varepsilon))$.
- On each cluster $C \in \mathbf{\Lambda}$, $\mathrm{ufl}(C) \lesssim \mathrm{ufl}(\pi(C))$ in expectation.

# Lower Bound $\mathrm{ufl}(\pi(X))$

- Goal: $\sum_{C\in\mathbf{\Lambda}} \mathrm{ufl}(\pi(C)) \lesssim \mathrm{ufl}(\pi(X))$.
- Idea: Open a "local" facility set $F'_{\pi(C)}$ for each $\pi(C)$ and show $\mathrm{dist}(\pi(x), F'_{\pi(C)}) \lesssim \mathrm{dist}(\pi(x), F^*_{\pi}), \forall x \in C$.

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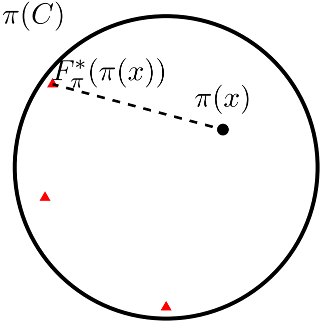


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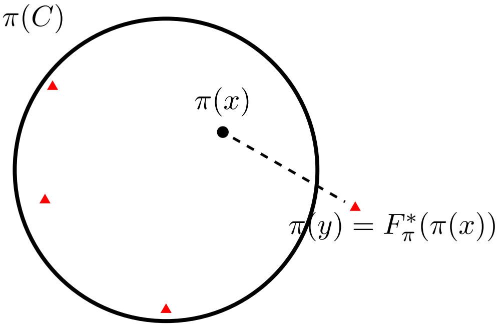


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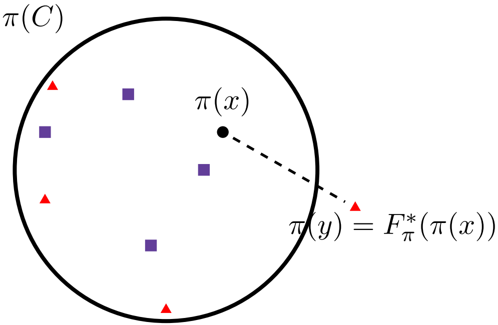


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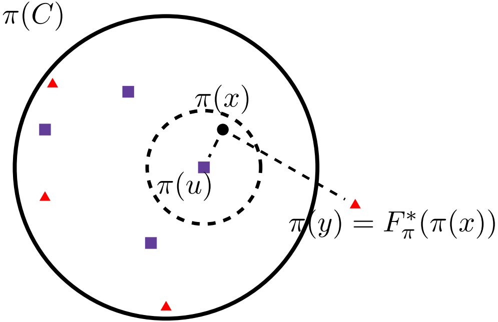


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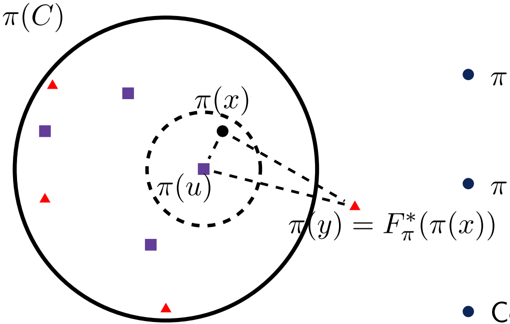
- Goal: $\sum_{C \in \mathbf{\Lambda}} \mathrm{ufl}(\pi(C)) \lesssim \mathrm{ufl}(\pi(X))$.
- Idea: Open a "local" facility set $F'_{\pi(C)}$ for each $\pi(C)$ and show $\mathrm{dist}(\pi(x), F'_{\pi(C)}) \lesssim \mathrm{dist}(\pi(x), F^*_\pi), \forall x \in C$.

$$F'_{\pi(C)} := (F^*_\pi \cap \pi(C)) \cup \pi(\overbrace{N_C}^{\sigma \cdot \mathrm{diam}(C)\text{-net on } C}).$$




- $\pi(x)$ and $\pi(u)$ are close.
  - $\|x - u\| \le \sigma \cdot \mathrm{diam}(C) \implies \|\pi(x) - \pi(u)\| \le O(\sigma) \cdot \mathrm{diam}(C)$ w.h.p.
- $\pi(x)$ and $\pi(y)$ are separated.
  - $(x, y)$ is not badly cut $\implies \|x - y\| \ge \frac{\varepsilon^2}{\mathrm{ddim}} \cdot \mathrm{diam}(C)$.
  - $\|\pi(x) - \pi(y)\| \ge \frac{\Omega(\varepsilon^2)}{\mathrm{ddim}} \cdot \mathrm{diam}(C)$ w.h.p.
- Conclusion: $\|\pi(x) - \pi(u)\| \le \|\pi(x) - \pi(y)\|$.

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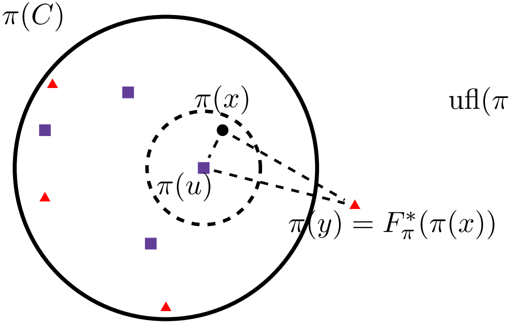


$$\mathrm{ufl}(\pi(C)) \le \sum_{x\in C} \mathrm{dist}(\pi(x), F'_{\pi(C)}) + \left|F'_{\pi(C)}\right|$$

$$\le \underbrace{\sum_{x\in C} \mathrm{dist}(\pi(x), F^*_\pi)}_{\text{connection cost on } \pi(C)} + \underbrace{|F^*_\pi \cap \pi(C)|}_{\text{opening cost on } \pi(C)} + |N_C|.$$

$$\implies \sum_{C\in\mathbf{\Lambda}} \mathrm{ufl}(\pi(C)) \le \mathrm{ufl}(\pi(X)) + \text{err}.$$

# PTAS for UFL

- Construct the partition $\mathbf{\Lambda}$.
  - Hierarchical decomposition $\mathcal{H}$.
  - Eliminate badly-cut pairs $(x, F(x)) \Longrightarrow \mathcal{T}$.
  - Construct $\mathbf{\Lambda}$ via $\mathcal{T}$.
- Construct near-optimal clustering $\mathcal{C}$.
  - For each $C \in \mathbf{\Lambda}$, construct the near optimal clustering $\mathcal{C}_C$ for $\pi(C)$.
  - $\mathcal{C} := \bigcup_{C \in \mathbf{\Lambda}} \mathcal{C}_C$.
- Construct facility set $F$.
  - For each $X_i \in \mathcal{C}$, compute the (approximate) 1-median center $f_i$ for $X_i$.
  - $F := \{f_1, f_2, \ldots, f_s\}$.

# Thank you!