Solving the Rebalancing Premium Puzzle and the Low-Risk Anomaly

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Abstract

Volatility is usually considered as a synonym for risk. However, recently developed investment strategies based on the concept of volatility harvesting claim to use volatility as a source of additional return. Proponents of these strategies refer to the additional return they offer over buy-and-hold investments as rebalancing premium or rebalancing bonus. In this paper our analysis of performance of rebalanced portfolios is consistent with multi-period capital growth theory. We identify that over short horizons there is a risk premium associated with rebalancing and zero expected additional growth over buy-and-hold portfolios. We point out that at longer time horizons a bonus from rebalancing does appear under some conditions. We clearly identify these conditions when comparing expected growth rates of optimally rebalanced portfolio with the growth rate of the best asset in this portfolio. Finally, we provide insights on how and when it is possible to add value from rebalancing in active portfolio management. As fire can be either dangerous, if uncontrolled, or useful to run a mechanical engine if controlled, in the very same way it should be possible to put volatility to work in a controlled manner in order to produce growth.

1 Introduction

Over the last decade convincing evidence has led to a growing consensus that part of the alpha of active managers could be explained by the existence of systematic factor risk premia Ang (2015). A good review on factor investing is given in Cazalet and Roncalli (2014); Harvey (2015); Ang (2015).

In a quest to apply alternative risk premia in asset allocation and portfolio construction framework an observation that rebalancing volatile assets in a portfolio may produce an extra return with respect to a passive buy-and-hold portfolio awake the interest of a number of authors [P. Bouchey et al. (2012); Hallerbach (2014); Pal and Wong (2013)]. Indeed the idea that rebalancing a portfolio to some predefined fixed weight may increase the long term growth of an invested portfolio is not new. Historically it can be traced back to optimal-growth portfolio strategies in 1960s by Claude Shannon, the father of Information Theory. Although Shannon never published on the subject, he gave a historic talk at the MIT in the mid 1960s on the topic of maximizing the growth rate of wealth. By using a simple Wiener example he detailed a method on how to grow portfolio wealth by rebalancing weights to some predefined allocation. This followed the line of thought of the works by Kelly [Kelly (1956)] and Breiman [Breiman (1961)] relating the information asymmetry to optimal bet sizing in order to minimize the time necessary for the wealth to achieve a specific goal.

In this quickly developing area of quantitative asset management there is a significant confusion in terminology and often a lack of agreement about when rebalancing premium exists and when it does not. No wonder that the general investment community is left perplexed and is somewhat distrustful of the innovations in this area. In our views, the confusion stems principally from the fact that the rebalancing premium is by construct a multi-period effect while traditional tools and metrics to tackle premia (e.g. beta and alpha) are just single period averages.

Indeed it is clear that properties related to volatility harvesting and rebalancing premium are intimately related to the concept of optimal growth portfolios [L.Maclean, E.Thorp, and W. Ziemba (2010)]. One of the first formalisms used to address volatility harvesting is Stochastic Portfolio Theory [Fernholz (2002); G. Oderda (2013)]. In this context it can be shown that if the Market remains diverse (i.e. none of the stocks dominates the Market in terms of capitalization), then eventually a rebalancing effect can be observed.

In this paper we aim to debunk some common misconceptions and long standing issues concerning rebalancing. In doing this, to foster intuition we focus initially on a two asset case and drop Fernholz diversity condition, i.e. in our context a cap-weighted market could be dominated by the best performing asset.

We focus on capital growth dynamics of two strategies in stationary markets: a contrarian strategy based on a constantly rebalanced portfolio, and a trend following strategy corresponding to a buy and hold portfolio. In this we will abstract from more challenging issues of real non stationary markets and the effects of transaction costs. We believe that additional layers of complexity should be added only once the right level of intuition is established. Our objective is to extract some simple rules on regimes allowing the relative dominance of one strategy with respect to the other.

We approach the problem from two horizon limits. In section 2 we show that on short time horizon, there is no expected rebalancing bonus but a risk premium related to rebalancing emerges. In this case frequent local gains are compensated by a negative skew of the resulting distribution of relative in-sample growth rates. When we extend the time horizon and the cumulative growth becomes large, in some cases the rebalancing bonus emerges. In section 3 we probe the applicability of the short time limit in realistic scenarios. In section 4 we identify regimes when a rebalanced portfolio is expected to grow faster than the best asset in this portfolio, in doing this we lay down the path to the low-risk anomaly explaination. Armed with our results we arrive at a number of interesting theoretical and practical conclusions in terms of optimal allocation in section 5, where we summarize when rebalancing makes sense and provide a diagram of the best investment choices for the case of two risky and one risk free assets and perfect information. In order to not dilute concepts with technicalities we omit obvious algebra and relegate nontrivial derivations to appendix.

2 Rebalancing as a risk premium

Authors analyzing rebalancing return and its components usually consider the limit of small variations in asset prices over the considered time horizon. This approximation is implicit in the derivation of the so-called dispersion discount in Hallerbach (2014). Yet, the approximation is not often explicitly pointed out while it bears some remarquable consequences as we will outline in the sequel. In this paper we refer to this regime as the short term or short sample limit.

Another common misunderstanding in the literature regarding rebalancing return stems from the confusion between model parameters and in-sample realization of these parameters. This distinction becomes especially important in the short sample limit. We will show here that in the short term approximation rebalancing premium does not come as a free lunch but genuinely behaves as a risk premium, i.e. frequent limited gains are counterbalanced by rare but potentially severe losses drawn from a left-skewed distribution of returns.

To get an easy insight of our thesis, we start from a very simple two assets set-up. We then build two reference portfolios: a buy-and-hold portfolio where allocation's weights fluctuate according to assets returns; a rebalanced portfolio, the weights of which are reset to the initial ones at the end of each rebalancing period. We start with a buy-and-hold portfolio initially identical to the rebalanced portfolio with weights ω_{1o} and ω_{2o}. Both portfolios have the same return over the first period $t = 1$.

Instead of starting with a predefined breakdown of volatility premium and dispersion discount, we simply find a second order expansion of realized growth rates of rebalanced and buy-and-hold portfolios. In our simple two assets case considering T rebalancing periods (derivation for N assets is given in Appendix A) we analyze the realized growth rate g_p per rebalancing period[1]. The realized growth rate for the rebalanced portfolio g_p^{rb} is given by:

$$\begin{aligned} g_p^{rb} &= \tfrac{1}{T}\sum_{t=1}^{T} g_{p_t}^{rb} = \tfrac{1}{T}\sum_{t=1}^{T} log\left(1+\omega_{1o}r_{1t}+\omega_{2o}r_{2t}\right) \\ &\overset{\text{2nd Order}}{\approx} \bar{r}_p - \tfrac{1}{2}\left(\tfrac{T-1}{T}\widehat{\sigma_p}^2 + \bar{r}_p^2\right) \end{aligned} \tag{1}$$

where the sample mean return of the rebalanced portfolio $\bar{r}_p$ is given by:

$$\bar{r}_p = \frac{1}{T}\sum_{t=1}^{T}\left(\omega_{1o}r_{1t}+\omega_{2o}r_{2t}\right)$$

and sample rebalanced portfolio variance $\widehat{\sigma_p}^2$ is defined as:

$$\widehat{\sigma_p}^2 = \frac{1}{T-1}\sum_{t=1}^{T}\left(\omega_{1o}r_{1t}+\omega_{2o}r_{2t}-\bar{r}_p\right)^2$$

where r_{1t} and r_{2t} are simple returns of the two assets over period t.

[1]Note that the growth rate per period is not the same as geometric average return $r_p^{geom} = \exp(g_p) - 1$, although these two quantities are often confused

Similarly for the realized growth rate g_p^{bh} of the buy-and-hold portfolio we get:

$$\begin{aligned} g_p^{bh} &= \tfrac{1}{T} log\left(1+\omega_{1o}\left(\prod_{t=1}^{T}(1+r_{1t})-1\right)+\omega_{2o}\left(\prod_{t=1}^{T}(1+r_{2t})-1\right)\right) \\ &= \tfrac{1}{T} log\left(\omega_{1o}\prod_{t=1}^{T}(1+r_{1t})+\omega_{2o}\prod_{t=1}^{T}(1+r_{2t})\right) \\ &\overset{\text{2nd Order}}{\approx} \overline{r}_p-\tfrac{T}{2}\overline{r}_p^2-\tfrac{T-1}{2T}\left(\omega_{1o}\widehat{\sigma_1}^2+\omega_{2o}\widehat{\sigma_2}^2\right)+\tfrac{T-1}{2}\left(\omega_{1o}\overline{r}_1^2+\omega_{2o}\overline{r}_2^2\right) \end{aligned} \tag{2}$$

where $\overline{r}_i$ and $\widehat{\sigma_i}$ are the realized mean return and sample standard deviation of asset i.

While to get a usable expression for the rebalanced portfolio we only need $\sigma_i << 1$ for the second order expansion in equation 1, we must assume $T\overline{r}_i << 1$ in order to drop cross terms of the higher order in the first step of equation 2. In appendix A whe show in detail why this approximation is needed and point out that it is identical to $\sqrt{T}\sigma_i << 1$. For the difference of the realized growth rates we get:

$$g_p^{rb}-g_p^{bh}=\frac{1}{2}\frac{T-1}{T}\left[\omega_{1o}\left(\widehat{\sigma_1}^2-T\overline{r}_1^2\right)+\omega_{2o}\left(\widehat{\sigma_2}^2-T\overline{r}_2^2\right)-\left(\widehat{\sigma_p}^2-T\overline{r}_p^2\right)\right] \tag{3}$$

This result is quite similar to the one obtained by Hallerbach in [Hallerbach (2014)]. Importantly, we identify the limits of applicability $\sqrt{T}\sigma_i << 1$ of both equation 3 and the result in Hallerbach (2014). The expression in equation 3 can be grouped into terms identified as volatility premium and dispersion discount as in Hallerbach (2014). However, we find that such grouping makes it difficult (if not impossible) to compare premium and discount terms[2], we will thus proceed through an alternate way.

While the difference of the realised growth rates in equation 3 can be either positive or negative, it is important to make assesment of this value on average over multiple observation. Therefore, we define rebalancing bonus as the expected value of the difference $g_p^{rb}-g_p^{bh}$

Up to this point we made no assumption about the distributions of asset returns r_i in this section. If we assume geometric Brownian motion (normally distributed returns with zero mean and no serial dependence $N(\sigma_i, \mu_i = 0)$, we obtain that there is no rebalancing bonus in the short term. By taking the expected value of equation 3 and noting that the expected value of the square of the sample realized mean $\overline{r}_i$ over T observations is:

$$\mathbb{E}\left(\overline{r_i}^2\right)=\frac{\mathbb{E}\left(\widehat{\sigma}_i^2\right)}{T}=\frac{\sigma_i^2}{T}>0$$

we get the result that on average in the limit $\sqrt{T}\sigma_i << 1$ the rebalancing bonus is zero[3]:

$$\boxed{\mathbb{E}\left(g_p^{rb}-g_p^{bh}\right)=0} \tag{4}$$

[2]Note that our expression contains a correction for volatility premium, consistent with the fact that for T=1 the difference in the growth rates should be zero by definition

[3]It is obvious from above that if returns have a serial dependence characteristic to a mean-reverting process, a positive rebalancing bonus will be observed

Once again we emphasize the importance of distinguishing between the parameters of return distributions σ_i, μ_i and the sample observations $\widehat{\sigma_i}$, $\overline{r_i}$. Over multiple observation periods $\widehat{\sigma_i}$ will have an expected value σ_i and independent observations will be distributed according to χ^2 distribution with $(T-1)$ degrees of freedom. At the same time, $\overline{r_i}^2$ will have an expected value of σ_i^2/T and will be distributed as χ^2 with one degree of freedom. The resulting combined distribution describing realized observations of growth rate differences will then be characterized by a strong negative skew. If we attempt to profit from rebalancing in the short term, we will observe frequent small positive returns over buy and hold strategy, which will be eventually be offset by rare but large negative returns. This is well evidenced by a numerical experiment for two assets and T=10 repeated 100000 times. The histogram of the realized difference of growth rates $g_{rb} - g_{bh}$ and approximation of this difference by equation 3 is shown in figure 1.

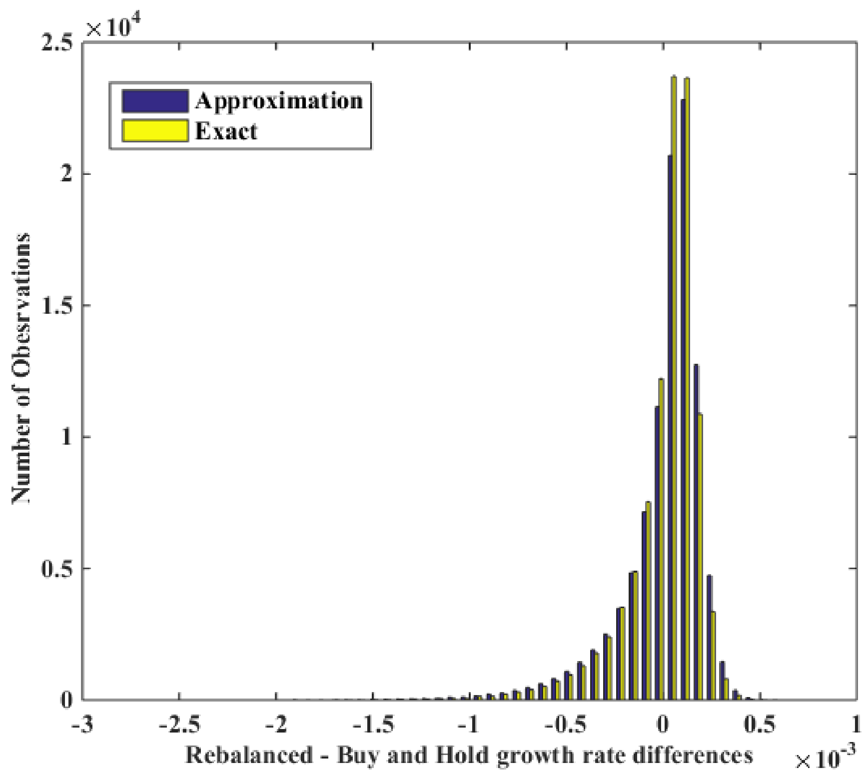

Figure 1: Distribution of the realized growth rate difference $g_p^{rb} - g_p^{bh}$ for a portfolio of two uncorrelated assets with normally distributed returns with zero means and 2% per period volatilities. 100000 simulations over $T = 10$ rebalancing periods are considered.

What if we consider assets with returns following geometric Brownian motion but with nonzero expected returns? It appears that for proponents of rebalancing things only get worse. In this case:

$$\mathbb{E}\left(\overline{r_i}^2\right) = \mu_i^2 + \frac{\sigma_i^2}{T}$$

Therefore, on average we have:

$$\boxed{\mathbb{E}\left(g_p^{rb} - g_p^{bh}\right) = -T\omega_{1o}\omega_{2o}\left(\mu_1 - \mu_2\right)^2} \tag{5}$$

When expected returns are different there is a small (second order in difference of μ) rebalancing discount even when we do not take into account transaction costs. From this result it would appear that, if we intend to keep our portfolio allocations for a short time, it is better not to rebalance the portfolio.

This result seemingly contradicts conclusions from a number of previous studies claiming that regular rebalancing leads to a better performance. In the next section we will evaluate what happens when the approximation $\sqrt{T}\sigma_i << 1$ breaks down.

3 Probing the limits: when simulations provide more insight

In order to check the case $\sqrt{T}\sigma_i \simeq 1$ we can proceed with the expansion of buy and hold growth rate to the fourth order. After some tedious algebra we indeed obtain a positive expected value for the rebalancing premium but choose not show it here to save space. Instead, we rely on a simple Monte Carlo simulation to probe when the effective rebalancing premium comes into force and the approximation breaks. We analyze this by plotting the expected difference in the numerical experiment described below when the number of rebalancing periods increases. The parameters used for this exercise are estimated from the total return series of Starbucks and Apple stocks over a horizon from January 1993 to May 2015. Obviously the following example is not general but it provides some useful insights.

We select i.i.d. normally distributed asset returns with the daily expected return of 11.4 and 12.1 basis points for assets 1 and 2, while daily volatility is 2.6% and 3% respectively. The correlation between asset returns is 0.25. Initially the portfolio holds 50% of each asset. We compare the difference between daily expected growth rates of a portfolio rebalanced daily and buy-and-hold portfolio for horizons between 2 weeks and 1000 years. The results of simulations are summarized in figure 2. Each point is an average over 200000 independent simulations. Observed rebalancing bonus per day for Apple and Starbucks over this horizon is also shown in the figure.

Despite the simplicity of the numerical experiment described above, we can make some important conclusions. First, the short horizon approximation used in section 2 has very limited applicability. In our example, the result in equation 3 works for horizons up to five months and then it breaks down as the difference between approximated and exact calculations becomes statistically significant. Many authors consider implicitly the second order expansion of buy-and-hold portfolio's multi-period growth rate and apply the results of this expansion to multi-year periods. In the light of our example, conclusions of these authors should be re-examined. As we show, second order results apply only for very short horizons when realistic asset parameters are considered. Second, we demonstrate that, while it is futile to look for advantages of rebalanced over buy-and-hold portfolios over a short horizon in case of i.i.d normally distributed returns, the rebalancing bonus may indeed appear when longer horizons $\sqrt{T}\sigma_i \simeq 1$ are considered. Another important observation concerns the limiting value of the rebalancing bonus at long horizons.

The expected growth rate of a buy-and-hold portfolio approaches the expected growth rate of the best asset in the portfolio in the long term. The asset with the higher expected growth rate in a portfolio thus plays an important role in defining the maximum possible value of rebalancing premium.

Before moving on to consider when the expected growth of the rebalanced portfolio is higher than the expected growth of the best asset, we should point out some other observations from the numerical experiment defined above. Can we compare rebalancing bonus at different rebalancing frequencies? We already noticed that rebalancing bonus appears when $\sqrt{T}\sigma \simeq 1$. If instead of rebalancing daily, we do this once a month (roughly once every 20 days), the number of rebalancing

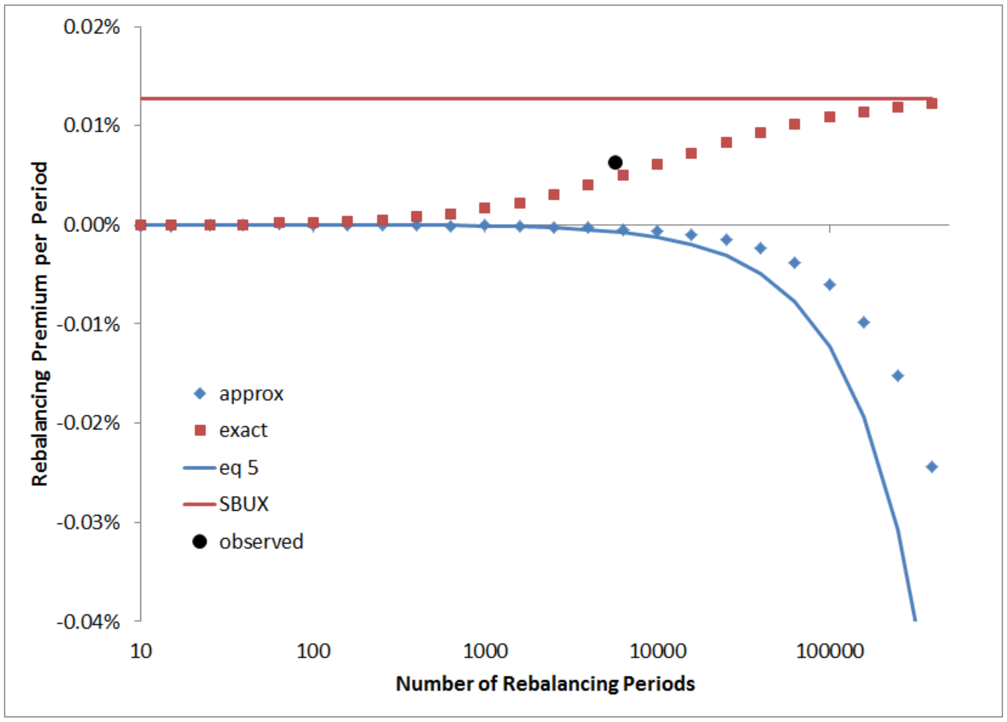

Figure 2: Rebalancing bonus per day versus horizon for two assets with normally distributed daily returns $N(0.00114, 0.026)$ and $N(0.00121, 0.03)$ and correlation 0.25. Each point represents an average over 200000 independent simulations. Blue diamonds represent approximate results using equation 3, while red squares show results of exect calculation. Blue line is the expectation according to equation 5 and red line shows expected difference between the growth rate of rebalanced portfolio and that of the best asset (SBUX). Black dot shows the actual rebalancing bonus observed for AAPL and SBUX over a horizon from January 1993 to May 2015

periods over T days will decrease to $T/20$ while the volatility of monthly returns will increase to $\sigma\sqrt{20}$. Overall, we see that if $\sqrt{T}\sigma = 1$, then $\sqrt{\frac{T}{20}}\sigma\sqrt{20} = 1$. It appears that in a frictionless market the rebalancing bonus should not depend on rebalancing frequency. We verify this observation in numerical simulations summarized in figure 3.

When we include realistic two-way transaction costs of 30 bps, the situation obviously changes. Monthly rebalancing becomes profitable for horizons longer than one year, while daily rebalancing bonus is only sufficient to offset transaction costs after four years or more. Before embracing rebalancing as a source of excess returns one needs to be aware that even if it is positive, it may take a while before one observes the benefit in practice.

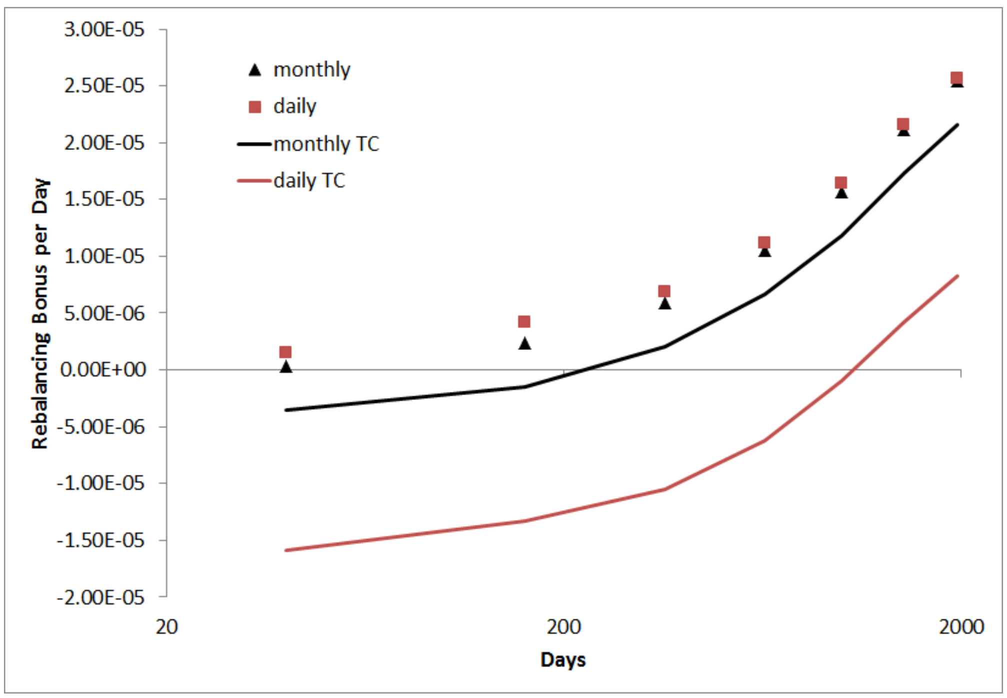

Figure 3: Rebalancing bonus for two assets with normally distributed returns $N(0.00114, 0.026)$ and $N(0.00121, 0.03)$ and correlation 0.25. Daily (red squares) and monthly (black triangles) rebalancing frequencies are compared in a frictionless market. Lines show greater impact of transaction costs on rebalancing bonus.

4 Beating the Best Asset: a path to the low-risk anomaly solution

As explained in previous sections, analytical Taylor seiries expansion of the growth rate of buy-and-hold portfolios is inadequate when horizon T becomes large (i.e. approaches the limit $\sqrt{T}\sigma \simeq 1$). Can we obtain additional insight into rebalancing problem without resorting to numerics? It turns out that one can say a lot more when comparing rebalanced portfolio growth rates with the growth rates of individual assets. In fact, when we move into the long horizon limit, this becomes the most relevant question to ask. In the long term, the growth of the buy-and-hold portfolio will be determined by its fastest growing asset and the dependence on the initial allocation of weights in the buy-and-hold portfolio will eventually disappear.

As before, the growth rate of the rebalanced portfolio is given by equation 1. We assume that asset 1 has a higher expected growth rate and that one period returns of both assets are small $|r_{it}| << 1$

$$g_1 = \frac{1}{T}\sum_{t=1}^{T} log\left(1 + r_{1t}\right) \overset{\text{2nd Order}}{\approx} \frac{1}{T}\sum_{t=1}^{T}\left(r_{1t} - \frac{1}{2}r_{1t}^2\right) \tag{6}$$

Before proceeding further to the expected value of the difference between growth rates of the rebalanced portfolio and the fastest growing asset we make an assumption of stationary i.i.d. normally distributed asset i returns with mean μ_i and standard deviation σ_i over one period:

$$r_{it} = \mu_i + \sigma_i \varepsilon_{it} \tag{7}$$

where ε_{it} is a standart normal i.i.d. variable $N(0,1)$ with zero mean and unit variance. Assuming correlation between the two assets to be ρ we have the following expressions for the expected values:

$$\begin{aligned}
\mathbb{E}(\sum_{t=1}^{T} \varepsilon_{it}) &= 0 \\
\mathbb{E}(\sum_{t=1}^{T} \varepsilon_{it}^2) &= 1 \\
\mathbb{E}(\sum_{t=1}^{T} \varepsilon_{it}\varepsilon_{jt}) &= \rho \\
\mathbb{E}(\sum_{t=1}^{T}\sum_{t\neq k} \varepsilon_{it}\varepsilon_{jk}) &= 0
\end{aligned} \tag{8}$$

After some simple algebra we obtain expected growth rates:

$$\begin{aligned}
\mathbb{E}(g_1) &= \mu_1 - \tfrac{1}{2}\left(\mu_1^2 + \sigma_1^2\right) \\
\mathbb{E}(g_{rb}) &= \omega_{1o}\mu_1 + \omega_{2o}\mu_2 - \tfrac{1}{2}\omega_{1o}^2\left(\mu_1^2 + \sigma_1^2\right) - \tfrac{1}{2}\omega_{2o}^2\left(\mu_2^2 + \sigma_2^2\right) - \omega_{1o}\omega_{2o}\left(\mu_1\mu_2 + \sigma_1\sigma_2\rho\right)
\end{aligned} \tag{9}$$

Assuming full investment $\omega_{1o}+\omega_{2o}=1$ and comparing expected value of the rebalanced portfolio growth rate with that of asset 1, we determine the weight limits when the rebalanced portfolio is expected to outperform the best asset in the long run:

$$0<\omega_{2o}<2\frac{\mu_2-\mu_1+\mu_1^2+\sigma_1^2-\mu_1\mu_2-\sigma_1\sigma_2\rho}{\mu_1^2+\sigma_1^2+\mu_2^2+\sigma_2^2-2\mu_1\mu_2-2\sigma_1\sigma_2\rho} \tag{10}$$

The optimal weights, for which the expected growth rate of the rebalanced portfolio is maximized are given by:

$$\begin{aligned}\omega_{1o} &= \frac{\mu_1-\mu_2+\mu_2^2+\sigma_2^2-\mu_1\mu_2-\sigma_1\sigma_2\rho}{\mu_1^2+\sigma_1^2+\mu_2^2+\sigma_2^2-2\mu_1\mu_2-2\sigma_1\sigma_2\rho}\\ \omega_{2o} &= \frac{\mu_2-\mu_1+\mu_1^2+\sigma_1^2-\mu_1\mu_2-\sigma_1\sigma_2\rho}{\mu_1^2+\sigma_1^2+\mu_2^2+\sigma_2^2-2\mu_1\mu_2-2\sigma_1\sigma_2\rho}\end{aligned} \tag{11}$$

Before moving further, let us consider two special cases for the parameters in equations 11.

In the case where $\mu_1=\mu_2$ the optimal growth portfolio turns out to be the rebalanced minimum variance portfolio between two assets, shading some light on the so-called low-risk anomaly in R. Haugen and N. Baker (1991) [see also R. Haugen and N. Baker (2012)]. In this respect the anomaly disappears as the dominance of a Minimum Variance allocation framework emerges as a natural phenomenon implicit in the compounding process. In absence of any valuable information on future expected returns, the rebalanced Minimum Variance Portfolio turns out to be the most sensible choice if one targets the long term growth of capital. In a separate paper we go a step further and obtain full efficient frontier of Markovitz portfolio theory based on information theoretic considerations and the result in equation 11 [V. Dubikovskyy and G. Susinno (2015)].

Another special case, which is even more relavant for the discussion about rebalancing, concerns a combination of one risky and one risk-free asset. If we set $\sigma_2=0$ and denote $\mu_2=r_f$ the risk-free return, we obtain the optimal weight for the risky asset to be held with the risk-free asset in an optimal rebalanced portfolio. To simplify notation, we drop the subscript for the risky asset in what follows: $\mu_1=\mu$, $\sigma_1=\sigma$, and $\omega_{1o}=\omega_o$

$$\omega_o=\frac{(\mu-r_f)(1-r_f)}{\sigma^2+(\mu-r_f)^2} \tag{12}$$

The optimal expected growth of this rebalanced portfolio is then given by:

$$\begin{aligned}\mathbb{E}(g_{rb}) &= r_f+\omega_o\left(\mu-r_f\right)-\tfrac{1}{2}\omega_o^2\sigma^2-\tfrac{1}{2}\left(r_f+\omega_o(\mu-r_f)\right)^2\\ &= r_f-\tfrac{1}{2}r_f^2+\tfrac{1}{2}\tfrac{(\mu-r_f)^2(1-r_f)^2}{\sigma^2+(\mu-r_f)^2}=g_f+\tfrac{1}{2}\tfrac{SR^2(1-r_f)^2}{\tau+SR^2}\end{aligned} \tag{13}$$

where growth rate of the risk free asset is $g_f=r_f-\frac{1}{2}r_f^2$, Sharpe ratio of the risky asset is $SR=\sqrt{\tau}\frac{\mu-r_f}{\sigma}$ and τ is the number of rebalancing periods per year[4]. An important property of

[4]Note that $\sqrt{\tau}$ scaling factor is needed to annualize return and volatility as Sharpe ratio is usually defined for

the rebalanced portfolio is immediately obvious from equation 13. If we have to choose one risky asset from many, we should select the asset with the highest Sharpe ratio as it yields the highest growth rate in the optimally rebalanced portfolio, which includes a risk-free asset. Incredibly, from a solution to a simple two-asset rebalancing problem we directly obtain one of the most important results in modern portfolio theory. Indeed, this is a multi-period case of the Tobin's mutual fund theorem [J.Tobin (1958)]. Note, that this result does not require an *ad hoc* assumption that investors are averse to asset price volatility. In our problem setting investors are only concerned with maximizing the expected growth rate of their wealth. For such an investor risk is no longer simply synonymous with volatility. More generally and more intuitively the risk can be associated with the probability of negative returns or negative growth rates and, thus, it is related to the accuracy of forecasting return expectations. For a more detailed discussion on this see V. Dubikovskyy and G. Susinno (2015).

Another important characteristic of equation 13 is that optimal expected growth is positive even for assets with negative Sharpe ratio (when expected return of the risky asset is smaller than the risk-free return). In this case, according to equation 12 it is optimal to short the risky asset and invest the proceeds at the risk free rate. In addition, if optimal weight in equation 12 is greater than one, the optimal strategy is to borrow money at the risk-free rate and to leverage investment into the risky asset. It is easy to prove that if short-selling and leverege are allowed and if borrowing at the risk-free rate and trading for free is possible, it is more profitable to invest in a constant-weight portfolio of a risk-free and risky asset than in the risky asset alone. In this framework, knowledge of future expected returns and variance fully determines the choice of leverage. There is no need to assume a certain arbitrary trade-off between risk and variance to find an optimal leverage.

If neither leverage nor shorting is allowed, we can determine the limits when rebalancing is more profitable than holding the risky asset. From equation 11, in a long-only non-leveraged case, when $\omega_o \leq 0$ we hold risk-free asset instead of rebalancing and when $\omega_o \geq 1$ it is better to fully invest in the risky asset[5].

$$\begin{cases} r_f < \mu < \sigma^2/2 & \iff \mathbb{E}(g_1) < g_f < \mathbb{E}(g_{rb}) \\ \sigma^2/2 < \mu - r_f < \sigma^2 & \iff g_f < \mathbb{E}(g_1) < \mathbb{E}(g_{rb}) \\ \sigma^2 < \mu - r_f & \iff g_f < \mathbb{E}(g_{rb}) < \mathbb{E}(g_1) \end{cases} \tag{14}$$

Equation 14 provides an important insight into a problem of allocating capital between a risky and a risk-free asset. Let's take $r_f = 0$ as is proper in the current investment environment.

If the expected return of the risky asset is less than zero, it does not make sense to invest. This should not be a surprise to anyone. However, positive arithmetic mean μ does not guarantee growth of the investment in the risky asset. In fact, if $0 < \mu < \sigma^2/2$ the buy-and-hold strategy will eventually lose all initial investment. While this result is familiar to many investors, very few

annualized quantities

[5]Here we dropped usually much smaller $(\mu - r_f)^2$ terms

actually know that it is possible to obtain positive growth for a regularly rebalanced portfolio of cash and a risky asset even if by itself the risky asset has a negative expected growth rate. It is even less obvious that an optimal rebalanced portfolio will in a long run outperform any buy and hold combination of a riky asset and cash if $\mu < \sigma^2$. Finally, if the expected return is large enough $\mu > \sigma^2$, the investor should fully invest in the risky asset to maximize the long term wealth.

5 When rebalancing pays off

We should point out that the methodology adopted in section 4 is readily extendable to any number of assets. Here we consider the case of two risky assets and cash given perfect knowledge about future expected returns, volatilities, and correlations. Short-selling and leverage are not allowed. This case is practically useful and is well suited for an intuitive graphical representation [P. Laureti, M. Medo, and Y.-C. Zhang (2009)] . We clearly separate the parameter space into regions where rebalancing adds value from the regions where it is best to hold one asset with the highest growth rate. Schematically, the regions are shown in the diagram in figure 4. A point on this diagram, showing excess expected returns for the two assets, will determine what investment choice will produce highest growth in the long term. As is well known in option pricing, asset variance turns out to be a useful measuring stick in the return space.

Below we focus on explaining optimal investment choices corresponding to each region. Light grey areas in the parameter space is where rebalancing adds value. In white areas it is better to buy-and-hold a single asset.

Region A covers the most obvious case - when expected excess returns of both risky assets are negative, we should just hold the risk-free asset. Regions B, C, and D have a common characteristic in that they require rebalancing between the risk-free asset and risky assets. As shown in equation 13, the risky asset with maximum Sharpe ratio maximizes expected growth when combined with the risk free asset. Region B corresponds to the set of parameters, for which asset 2 has the highest Sharpe ratio, in region C - it is asset 1 that dominates. In region D the risky holding with the highest Sharpe ratio is a rebalanced portfolio of two risky assets. Clearly, the risky assets have to be held in proportions that produce the maximum Sharpe ratio in order to maximize the expected growth rate in equation 13. Region D is the only region, where all three assets are held and rebalanced. In region G we should hold an optimal combination of risky assets according to equation 11. In regions E and F the best expected growth is produced by fully investing into the corresponding risky asset. The lines separating region B from E and region C from F are defined by 100% optimal weight of the risky asset in equation 12. Similarly, the lines separating region G from E and F correspond to 100% weight of one of the assets in equation 11. Within region G we identify the lines reflecting some well-known portfolio construction choices. The dashed line corresponds to minimum variance portfolio of two risky assets, while the dotted line - to their equal weight combination.

Having identified regions for optimal investment in the parameter space, we turn to a simple example to illustrate how much difference the correct selection can make. We consider total returns of two stocks: Apple Inc. as asset 1 (AAPL), and Starbucks Corporation as asset 2 (SBUX) over the period from January 1993 to May 2015. Both companies were big success stories over the period considered. If you were to invest 100$ in AAPL in January 1993, your investment would grow to 6800$, while initial 100$ investment in SBUX would be worth 9800$ today. Holding a monthly rebalanced equal weight combination of the two assets demonstrates the power of rebal-

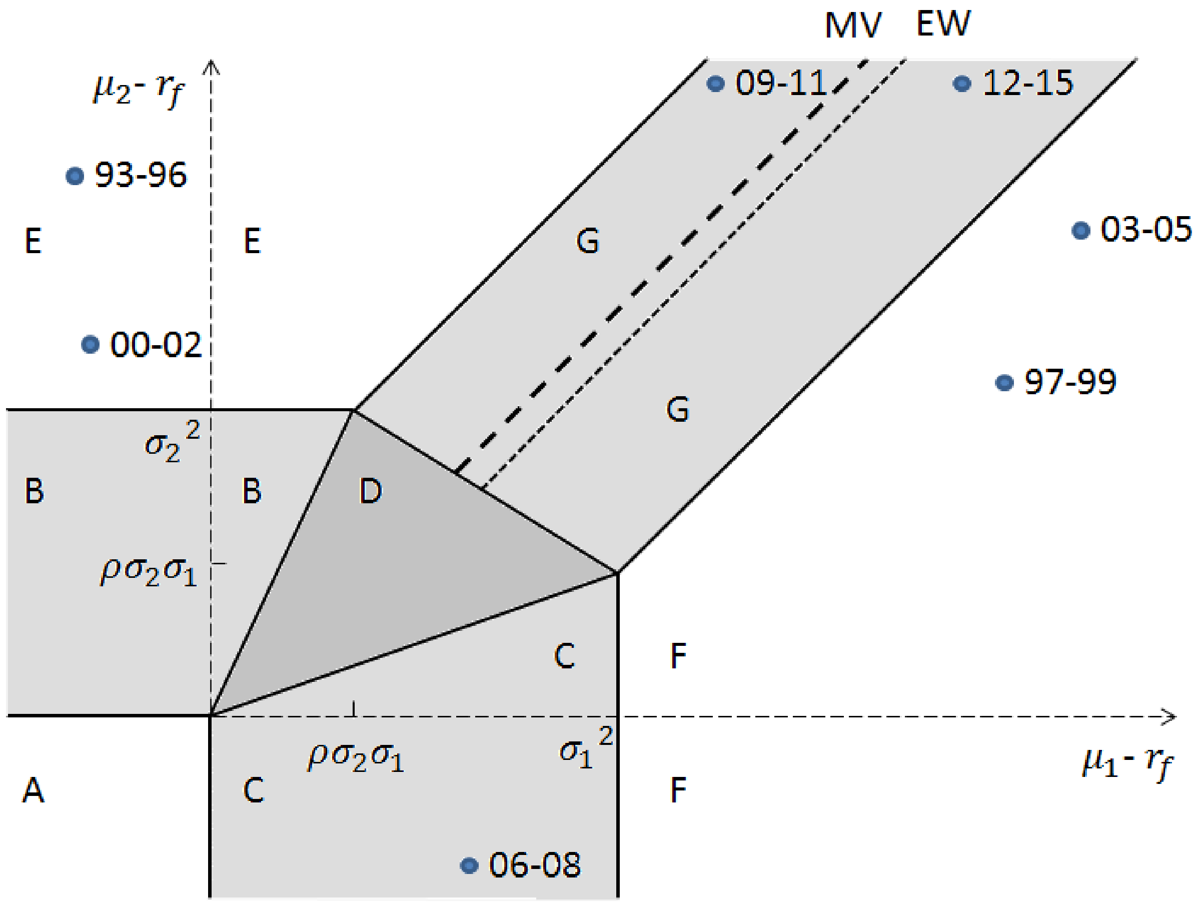

Figure 4: Optimal investment choices given perfect information for 2 risky and a risk-free asset. Grey regions - rebalancing yields higher expected growth (region B - between risk-free and asset 2, region C - between risk-free and asset 1, region G - between risky assets, region D - risk-free and the maximum Sharpe ratio combination of risky assets). White regions - single asset holding is optimal (region A - risk-free asset, region E - asset 2, region F - asset 1). Dots with numbers show realized returns of a pair (AAPL, SBUX) over corresponding periods

ancing in the long term. 100$ invested into this rebalanced portfolio will be worth 17000$ today. The difference appears to be impressive. However, let us consider another case, where we make investment decisions roughly every three years. The points identified in figure 4 by two numbers show where the realized returns of AAPL and SBUX fall over the period specified. What if every three years we were able to correctly predict the region in the parameter space where the expected returns of the pair of stocks would fall? Now we are not claiming the perfect knowledge but just an ability to correctly identify the region for each 3-year period. Also we do not expect to hold the opimal weight but choose equal weight combinations in relevant regions. If we were to invest according to such region selection, we would do as follows: from 1993 to the end of 1996 and from 2000 to 2002 we would hold SBUX, from 1997 to 1999 and from 2003 to 2005 we would hold AAPL, from 2006 to 2008 we would rebalance between 50% in cash and 50% in AAPL, and only starting from 2009 we would have a monthly rebalanced equal-weight combination of the two assets. Increadibly, correct identification of the regions in the parameter space would help us turn our initial investment of 100$ in 1993 into 370000$ today or about 54 times more than the investment into AAPL would bring. The cumulative wealth curves are shown for each case cosdidered above in figure 5. 30 bps two-way transaction costs were included in calculation.

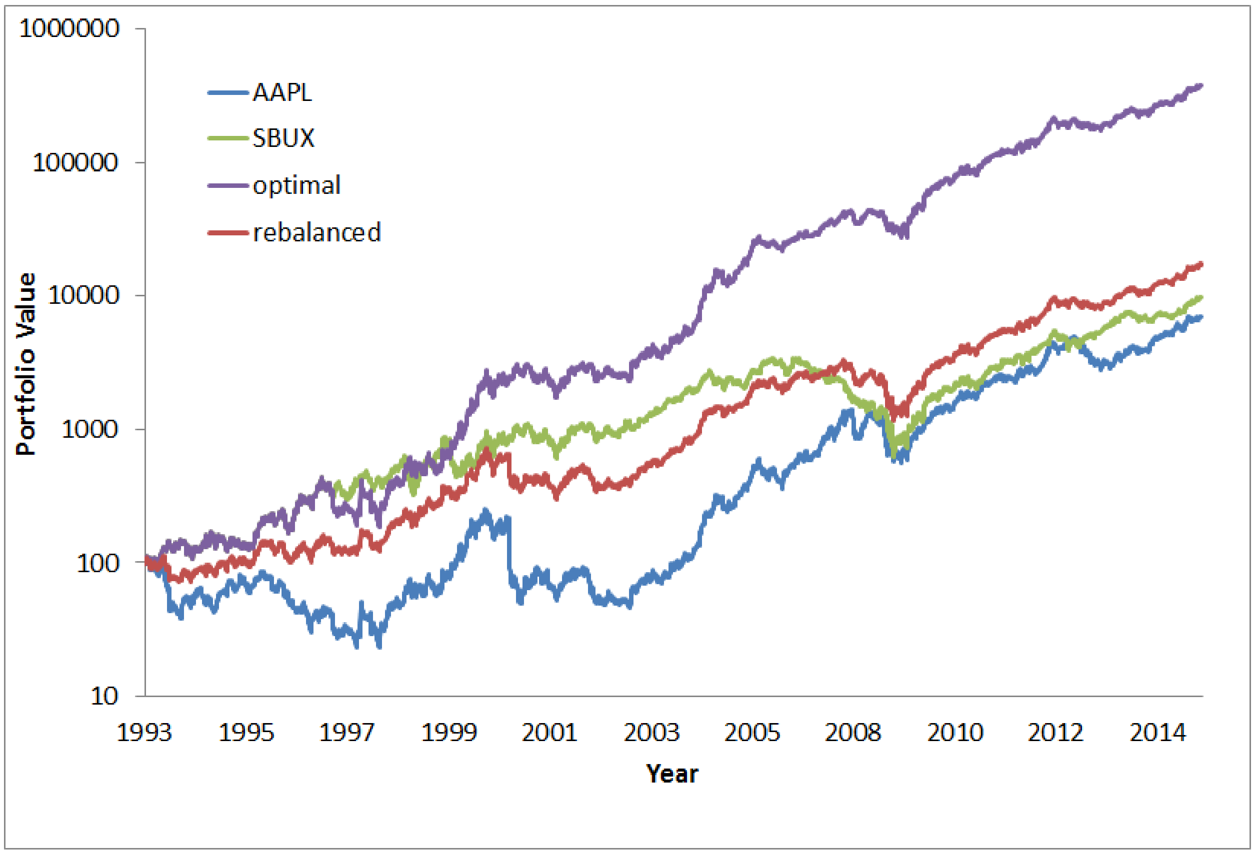

Figure 5: Cumulative wealth growth curves for investments into cash, AAPL, and SBUX from January 1993 to May 2105. Blue (AAPL) and green (SBUX) lines show 100% buy-and-hold investments into corresponding stocks. Red line shows growth of capital of an equal weighted monthly rebalanced portfolio of AAPL and SBUX. Violet line shows potential performance if regions in figure 4 were to be correctly identified every three years

Realistically, even the correct prediction of a region for a set of assets may not be possible. However, schematic representation in figure 4 may also guide us in portfolio construction if we take into account uncertainty in expected return forecasts. If instead of given μ_1 and μ_2 we have a distribution of likely outcomes, we can construct a portfolio that maximizes growth taking an integral over such a distribution. This can be easily accomplished even in a multiple stock setting by a Monte Carlo integration technique. An application of this approach to portfolio construction is a topic of a separate article.

It is well known that expected returns are difficult to forecast accurately. However, we can better forecast variances and to a certain degree correlations. One relevant question to ask then is which assets one should select from the point of view of their volatilities and correlations in order to ensure that rebalancing adds value. What we strive to achieve is to make our regions where rebalancing is optimal (grey areas) bigger in the parameter space. Selecting very volatile assets is one way to do this. Hence the name volatility harvesting. If you have volatile assets you are more likely to end up in a region where periodic rebalancing is beneficial. One way to expand regions D and G is to choose assets with low correlations. Thus, portfolios well diversified into assets with low pairwise correlations are more likely to benefit from rebalancing.

Our simple graphical representation of optimal investment choices has already yielded a number of insights. Another observation concerns certain risk based portfolio construction techniques. It is often claimed that risk based investing takes no views on expected returns. However, in order to be optimal from the point of view of expected growth rate maximization, each risk based methodology implies a certain view on expected returns [Qian (2005); S. Maillard, T. Roncalli, and J. Teiletche (2010)]. As pointed out above, rebalanced minimum variance portfolio (MV) construction implies that the expected returns are assumed to be equal. If they are, then in region G minimum variance portfolio is the optimal growth portfolio. Therefore, in figure 4 MV line corresponds to $\mu_1 = \mu_2$.

Similarly, equally weighted portfolio (EW) fully invested in risky assets implies that the expected growth rates of the two assets are the same:

$$\mu_1 - \frac{\sigma_1^2}{2} = \mu_2 - \frac{\sigma_2^2}{2} \tag{15}$$

EW portfolio is in the middle of the region G in figure 4. This means that EW portfolio is ideal if we expect assets to have equal growth rates. Such a view expressed by equation 15 is also least sensitive to misspecification. Among all portfolios where rebalancing is expected to dominate, departures from equal growth rate expectations are least likely to bring us into a regime where buy and hold is the best strategy.

Equal risk contribution (ERC) portfolio is identical to maximum diversification portfolio in the two-asset case. This portfolio lies between MV and EW lines in figure 4 and it implies the following expectation on future mean returns:

$$\mu_1 - \mu_2 = \frac{(1+\rho)\sigma_1\sigma_2(\sigma_1 - \sigma_2)}{\sigma_1 + \sigma_2} \tag{16}$$

It is clear that in the two asset case for the ERC portfolio to be expected growth optimal it is necessary to assume that expected returns of assets are proportional to their volatilities[6].

[6] Note that this assumption is necessary but not sufficient for optimality

6 Conclusions

This work started as an attempt to clarify a widespread confusion about rebalancing premium. In order to do so we looked at a simplified case of only two assets. In the process we discovered that the insights we obtain in this simple setting allow to tackle some of the long standing controversies over return from rebalancing. We showed how the confusion stems principally from the fact that the rebalancing premium is by construct a multi-period effect while conventional tools and metrics to describe rebalancing premia are single period averages. By using an adapted formalism and by carefully considering the difference between the one period model set-up and the satistics of in sample realizations (this is what would be observed in a real allocation exercise) the premium appears as a genuine premium with frequent small gains couterbalanced by rare but large losses. Our results allowed us to describe market regimes for which rebalancing works and regimes for which it does not and enabled us to clarify the implicit return assumptions of fashionable allocation schemes such as the Equal Risk Contribution or Minimum Variance portfolios.

It is worthwhile to sketch out future research work within the framework we propose. The concept should be extended to the multi-asset case, take into account transaction costs and realistic asset properties such as mean reversion. In particular, it will be of great interest to analyze optimal rebalanced portfolios with the use of results from information theory by changing the focus from risk aversion to more general expected return uncertainty.

A Rebalancing Premiea: the Multi-Assets Case

Average growth rate g_p^{rb} over T periods, we will have:

$$\begin{aligned} g_p^{rb} &= \tfrac{1}{T}\sum_{t=1}^{T} g_{p_t}^{rb} = \tfrac{1}{T}\sum_{t=1}^{T} log\left(1+\sum_{i=1}^{N}\omega_{io}r_{it}\right) \\ &\overset{\text{2nd Order}}{\approx} \tfrac{1}{T}\sum_{t=1}^{T}\sum_{i=1}^{N}\omega_{io}r_{it} - \tfrac{1}{2T}\sum_{t=1}^{T}\left(\sum_{i=1}^{N}\omega_{io}r_{it}\right)^2 \\ &= \overline{r}_p - \tfrac{1}{2}\left(\widehat{\sigma_p}^2 + \overline{r}_p^2\right) \end{aligned} \tag{17}$$

where the sample mean return of the rebalanced portfolio $\overline{r}_p$ is given by:

$$\overline{r}_p = \frac{1}{T}\sum_{t=1}^{T}\sum_{i=1}^{N}\omega_{io}r_{it}$$

and the following relation is used for the sample rebalanced portfolio variance $\widehat{\sigma_p}^2$:

$$\widehat{\sigma_p}^2 = \frac{1}{T-1}\sum_{t=1}^{T}\left(\sum_{i=1}^{N}\omega_{io}r_{it} - \overline{r}_p\right)^2 = \frac{T}{T-1}\left(\frac{1}{T}\sum_{t=1}^{T}r_{pt}^2 - \overline{r}_p^2\right)$$

Similarly for the growth rate g_p^{bh} of the buy-and-hold portfolio we get:

$$\begin{aligned} g_p^{bh} &= \tfrac{1}{T}log\left(1+\sum_{i=1}^{N}\omega_{io}\left(\prod_{t=1}^{T}(1+r_{it})-1\right)\right) \\ &\overset{\text{2nd Order}}{\approx} \tfrac{1}{T}log\left(1+\sum_{i=1}^{N}\omega_{io}\left(\sum_{t=1}^{T}r_{it} + \sum_{t>k}r_{it}r_{ik}\right)\right) \\ &\overset{\text{2nd Order}}{\approx} \tfrac{1}{T}\sum_{i=1}^{N}\sum_{t=1}^{T}\omega_{io}r_{it} - \tfrac{T}{2}\left(\sum_{i=1}^{N}\omega_{io}\overline{r}_i\right)^2 + \tfrac{1}{2T}\sum_{i=1}^{N}\omega_{io}\sum_{t\neq k}r_{it}r_{ik} \\ &= \overline{r}_p - \tfrac{T}{2}\overline{r}_p^2 + \tfrac{T}{2}\sum_{i=1}^{N}\omega_{io}\overline{r}_i^2 - \tfrac{1}{2T}\sum_{i=1}^{N}\omega_{io}\sum_{t=1}^{T}r_{it}^2 \\ &= \overline{r}_p - \tfrac{T}{2}\overline{r}_p^2 - \tfrac{T-1}{2T}\sum_{i=1}^{N}\omega_{io}\widehat{\sigma}_i^2 + \tfrac{T-1}{2}\sum_{i=1}^{N}\omega_{io}\overline{r}_i^2 \end{aligned} \tag{18}$$

On the second line of the expression 18 we make a strong assumption $T\overline{r}_i << 1$ to take only the terms of the first and second order in r_{it}. The need for this assumption arrises due to larger number of terms associated with higher orders in r_{it}. Thus, we have T terms of the first order, $T(T-1)/2$ terms of the second order and $T(T-1)(T-2)/6$ terms of the third order. In order to drop third order terms we need to assume $r_{it}(T-2)/3 << 1$ or $\overline{r}_i(T-2)/3 << 1$. For $T >> 1$ and a stationary process with the assumption of small mean compared to volatility we have a simpler requirement $\sqrt{T}\sigma_i << 1$ for all assets. The necessity of this approximation may have been overlooked by other authors studying the expansion of rebalancing premium to the second order.

Note that in order to arrive to the final expression in 18 we used the following steps to simplify the double sum over cross-products of returns:

$$\sum_{t>k} r_{it} r_{ik} = \frac{1}{2} \sum_{t \neq k} r_{it} r_{ik} + \frac{1}{2} \sum_{t=1}^{T} r_{it}^2 - \frac{1}{2} \sum_{t=1}^{T} r_{it}^2 = \frac{1}{2} \left(\sum_{t=1}^{T} r_{it} \right)^2 - \frac{1}{2} \sum_{t=1}^{T} r_{it}^2 = \frac{T^2}{2} \bar{r}_i^2 - \frac{T-1}{2} \widehat{\sigma}_i{}^2 - \frac{T}{2} \bar{r}_i^2 \tag{19}$$

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