Some Light Analysis

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1 Foundations of Calculus

We assume f, g, and h are all real-valued functions of some real variable x.

Definition 1.1 (Limit)**.**

$$\lim_{x\to a} f(x) = L \iff \forall\varepsilon > 0,\ \exists\delta > 0 : \forall x \neq a\ |x-a| < \delta \implies |f(x) - L| < \varepsilon$$

Remark. We extend our standing assumptions such that

$$\lim_{x\to a} f(x) = L_1 \text{ and } \lim_{x\to a} g(x) = L_2$$

Theorem 1.1 (Linearity of the limit)**.**

$$\lim_{x\to a}[f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$$

$$\lim_{x\to a} cf(x) = c \lim_{x\to a} f(x)$$

Proof. To show that the limit preserves addition, we must show that for all $\varepsilon > 0$, there exists some $\delta > 0$ such that if $|x - a| < \delta$ then $|(f(x) + g(x)) - (L_1 + L_2)| < \varepsilon$. By definition of the limit, we can choose a $\delta_1 > 0$ and a $\delta_2 > 0$ such that $|x-a| < \delta_1 \implies |f(x)-L| < \frac{\varepsilon}{2}$ and $|x-a| < \delta_2 \implies |g(x)-L| < \frac{\varepsilon}{2}$. Choose $\delta = \min\{\delta_1, \delta_2\}$, and assume $|x - a| < \delta$. Thus we have

$$|f(x) - L_1| < \frac{\varepsilon}{2} \text{ and } |g(x) - L_2| < \frac{\varepsilon}{2}.$$

Adding inequalities, we obtain that

$$|f(x) - : wL_1| + |g(x) - L_2| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

and, by the Triangle Inequality and transitivity of inequality, we obtain

$$|f(x) + g(x) - (L_1 + L_2)| < \varepsilon.$$

The proof for the second part is much simpler. We must show that for all $\varepsilon > 0$, there exists some $\delta > 0$ such that if $|x - a| < \delta$ then $|cf(x) - cL_1| < \varepsilon$.

Given $\varepsilon > 0$. If $c = 0$, then clearly $|0f(x) - 0L_1| = 0 < \varepsilon$, regardless of δ. Otherwise, let $\delta_1 > 0$ be that value such that

$$|x - a| < \delta_1 \implies |f(x) - L_1| < \frac{\varepsilon}{|c|}.$$

Choose $\delta = \delta_1$. Assume $|x - a| < \delta$. Thus

$$|f(x) - L_1| < \frac{\varepsilon}{|c|} \implies |c||f(x) - L_1| < \varepsilon \implies |cf(x) - cL_1| < \varepsilon.$$

□

Theorem 1.2 (Limit preserves multiplication).

$$\lim_{x\to a} f(x)g(x) = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right) = L_1 L_2$$

Proof. Given $\varepsilon > 0$, we must show there exists some $\delta > 0$ such that $|x - a| < \delta \implies |f(x) - L| < \varepsilon$.

Let $\delta_1, \delta_2, \delta_3$ such that:

$$\begin{aligned}
|x - a| < \delta_1 &\implies |f(x) - L_1| < \frac{\varepsilon}{2|L_2|} \\
|x - a| < \delta_2 &\implies |g(x) - L_2| < \frac{\varepsilon}{2(1 + |L_1|)} \\
|x - a| < \delta_3 &\implies |f(x) - L_1| < 1.
\end{aligned}$$

Choose $\delta = \min\{\delta_1, \delta_2, \delta_3\}$, and assume $|x - a| < \delta$. Note that

$$\begin{aligned}
|f(x)| &= |f(x) - L_1 + L_2| && \text{Adding and subtracting 0} \quad (1) \\
&\leq |f(x) - L_1| + |L_1| && \text{Triangle Inequality} \quad (2) \\
&< 1 + |L_1| && |x - a| < \delta_3 \quad (3)
\end{aligned}$$

Using this fact, we show that

$$\begin{aligned}
|f(x)g(x) - L_1L_2| &= |f(x)g(x) + f(x)L_2 - f(x)L_2 - L_1L_2| \\
&= |f(x)[g(x) - L_2] - L_2[f(x) - L_1]| \\
&\leq |f(x)[g(x) - L_2]| + |L_2[f(x) - L]| \\
&= |f(x)||g(x) - L_2| + |L_2||f(x) - L| \\
&< (1 + |L_1|)\frac{\varepsilon}{2(1 + |L_1)} + |L_2|\frac{\varepsilon}{2|L_2|} \\
&= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}$$

□

Remark. Many inequalities in these proofs skip some steps. For example, to rigorously derive the third-to-last line of the previous proof, take the right side of the statements involving δ_1 and δ_2, multiplying one by $|L_2|$ and the other by $1 + |L_1|$ (by right δ_3 inequality), and sum them.

Theorem 1.3 (Equivalency of $h \to 0$ and $x \to a$)**.**

$$\lim_{x \to a} f(x) = L \iff \lim_{x \to 0} f(a + x) = L$$

Proof. First, we will show $\lim_{x \to a} f(x) = L$ implies $\lim_{x \to 0} f(a + x) = L$.

Given $\varepsilon > 0$. By making the substitution, $x \to x + a$, and noting $(a+x) - a = x$, we can find $\delta_1 > 0$ such that $|x| < \delta_1 \implies |f(a + x) - L| < \varepsilon$. Choosing $\delta = \delta_1$, and assuming $|x| < \delta$, we have $|f(a + x) - L| < \varepsilon$.

The other direction is omitted for similarity.

$\square$

2 Derivatives

Definition 2.1 (Derivative)**.** The derivative of $f(x)$, denoted $f'(x)$ or $\frac{df}{dx}$, is defined as follows.

$$\frac{df}{dx} = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

The derivative can also be defined by:

$$\frac{df}{dx} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a},$$

which can be derived straightforwardly in Theorem 1.3.

Theorem 2.1 (Linearity of the derivative)**.**

$$\frac{d}{dx} c f(x) = c \frac{df}{dx} \text{ and } \frac{d}{dx} \left[f(x) + g(x) \right] = \frac{df}{dx} + \frac{dg}{dx}$$

Proof. First, we will show the derivative preserves scalar multiplication.

$$\begin{aligned}
\lim_{h \to 0} \frac{cf(x + h) - cf(x)}{h} &= \lim_{h \to 0} \frac{c[f(x + h) - f(x)]}{h} && \text{Factoring out } c \\
&= c \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} && \text{Linearity of limit} \\
&= cf'(x) && \text{Def. of derivative}
\end{aligned}$$

Showing that the derivative preserves addition completes the proof.

$$\begin{aligned}
\frac{d}{dx}\left[f(x)+g(x)\right] &= \lim_{h\to 0}\frac{[f(x+h)+g(x+h)]-[f(x)+g(x)]}{h} && \text{Definition of sum}\\
&= \lim_{h\to 0}\frac{f(x+h)-f(x)+g(x+h)-g(x)}{h} && \text{Rearranging}\\
&= \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}+\frac{g(x+h)-g(x)}{h}\right] && \text{Splitting frac.}\\
&= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}+\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} && 2.1\\
&= \frac{df}{dx}+\frac{dg}{dx} && \text{Def. of derivative}
\end{aligned}$$

□

Theorem 2.2 (Product rule for derivatives)**.**

$$\frac{d}{dx}f(x)g(x) = f'(x)g(x)+g'(x)f(x)$$

Proof.

$$\begin{aligned}
\frac{d}{dx}f(x)g(x) &= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}\\
&= \lim_{h\to 0}\frac{f(x+h)g(x+h)+f(x)g(x+h)-f(x)g(x+h)-f(x)g(x)}{h}\\
&= \lim_{h\to 0}\frac{g(x+h)[f(x+h)-f(x)]+f(x)[g(x+h)-g(x)]}{h}\\
&= \lim_{h\to 0}\left[g(x+h)\frac{f(x+h)-f(x)}{h}+f(x)\frac{g(x+h)-g(x)}{h}\right]\\
&= \lim_{h\to 0}g(x+h)\frac{f(x+h)-f(x)}{h}+\lim_{h\to 0}f(x)\frac{g(x+h)-g(x)}{h}\\
&= g(x)\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}+f(x)\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\\
&= g(x)f'(x)+f'(x)g(x)
\end{aligned}$$

□

Theorem 2.3 (Chain rule)**.**

$$\frac{d}{dx}(f\circ g)(x) = f'(g(x))g'(x)$$

Remark. The proof of the chain rule is notoriously frustrating. This one is almost verbatim from Wikipedia, though several others exist.

Proof.

$$\frac{d}{dx}(f \circ g)(x) = \lim_{x \to a} \frac{f(g(x)) - f(g(a))}{x - a} \tag{4}$$

The ultimate goal of our proof is to multiply and divide by $g(x) - g(a)$, and consider the limit of the product. Unfortunately, because $g(x) - g(a)$ might be equal to zero, we have slightly more work to do. We define an auxiliary function $Q(y)$ such that:

$$Q(y) = \begin{cases} \frac{f(y)-f(g(a))}{y-g(a)}, & \text{if } y \neq g(a) \\ f'(g(a)), & \text{if } y = g(a) \end{cases}$$

The equality

$$\frac{f(g(x)) - f(g(a))}{x - a} = Q(g(x))\frac{g(x) - g(a)}{x - a},$$

now holds, because if $g(x) = g(a)$, the whole right hand expression is zero, otherwise the factors cancel. Then

$$\begin{aligned}
\frac{d}{dx}(f \circ g)(x) &= \lim_{x \to a} \frac{f(g(x)) - f(g(a))}{x - a} \\
&= \lim_{x \to a} Q(g(x)) \left(\frac{g(x) - g(a)}{x - a} \right) \\
&= \left(\lim_{x \to a} Q(g(x)) \right) \left(\lim_{x \to a} \frac{g(x) - g(a)}{x - a} \right) \\
&= \left(\lim_{x \to a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} \right) g'(x) \\
&= f'(g(x))g'(x),
\end{aligned}$$

where the second-to-last inequality comes from the fact that the value of x at a is irrelevant for the limit. ∎

3 Integrals and Antiderivatives

Definition 3.1 (Antiderivative)**.** We define $\int f(x)dx$ to be the function such that $\frac{d}{dx} \int f(x)dx$.

4 Fundamental Theorem of Calculus

4.1 Partial Inverses

To develop a deep understanding of the Fundamental Theorem, we will begin by reviewing some important definitions from the theory of functions.

Definition 4.1 (Onto)**.** We say that some function $f : X \to Y$ is *onto* if and only if for every element y of Y, there exists some element x of X such that $f(x) = y$.

Definition 4.2 (One-to-one)**.** We say that some function $f : X \to Y$ is *one-to-one* if and only if, for some elements x_1 and x_2 of X, we have that if $f(x_1) = f(x_2)$ then $x_1 = x_2$.

Definition 4.3 (Bijective)**.**

4.2 Derivative/Integral is a Partial Inverse

4.3 The Computation Theorem