NATIONAL SUN YAT-SEN UNIVERSITY, DEPARTMENT OF MATHEMATICS

Reliability Analysis Assignment 5 (personal)

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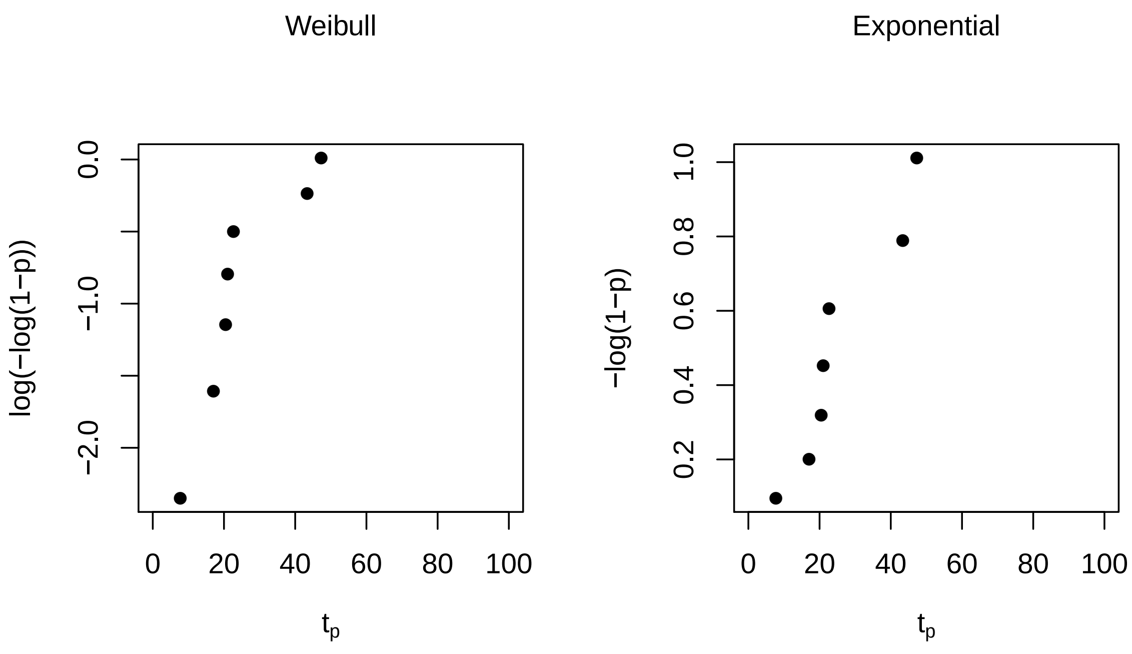
7.6

(a)



(b) The exponential probability plot fits better than the Weibull one.

(c) Let $X_{(1)}, ..., X_{(r)}$ be the exact-timed data and $X_{(r+1)}, ..., X_{(n)}$ be the right censored data. Assume both set of data have exponential distributions with θ, then

$$L(\theta) = \left(\prod_{i=1}^{r} \frac{1}{\theta} e^{-\frac{x_{(i)}}{\theta}}\right)\left(\prod_{i=r+1}^{n} \frac{1}{\theta} e^{-\frac{x_{(i)}}{\theta}}\right) = \left(\frac{1}{\theta}\right)^r e^{-\frac{\sum_{i=1}^n x_{(i)}}{\theta}} = \left(\frac{1}{\theta}\right)^r e^{-\frac{\sum_{i=1}^n x_i}{\theta}}$$

$$\Rightarrow \; logL(\theta) = log\left(\frac{1}{\theta}\right)^r - \frac{\sum_{i=1}^n x_i}{\theta}$$

$$\text{Let } \; \frac{\partial logL(\theta)}{\partial\theta} = -\frac{r}{\theta} + \frac{\sum_{i=1}^n x_i}{\theta^2} = 0 \;\Rightarrow\; \widehat{\theta} = \frac{\sum_{i=1}^n x_i}{r}$$

$$\because \; \left.\frac{\partial^2 logL(\theta)}{\partial\theta^2}\right|_{\theta=\widehat{\theta}} = \frac{\sum_{i=1}^n x_i - 2\sum_{i=1}^n x_i}{\theta^3} < 0$$

$$\therefore \; \widehat{\theta}_{mle} = \frac{\sum_{i=1}^n x_i}{r} \approx 82.8$$

$$\Rightarrow \; Var\left(\widehat{\theta}_{mle}\right) = Var\left(\frac{\sum_{i=1}^n x_i}{r}\right) = Var\left(\frac{2 \cdot \frac{\sum_{i=1}^n x_i}{\theta} \cdot \frac{\theta}{2}}{r}\right)$$

$$\because \; 2 \cdot \frac{\sum_{i=1}^n x_i}{\theta} \sim \chi^2_{2r}$$

$$\Rightarrow \; Var\left(\widehat{\theta}_{mle}\right) = \frac{\theta^2}{4r^2} \cdot 4r = \frac{\theta^2}{r}$$

$$\Rightarrow \; se_{\widehat{\theta}_{mle}} = \frac{\theta}{\sqrt{r}}$$

$$\Rightarrow \; \widehat{se}_{\widehat{\theta}_{mle}} = \frac{\widehat{\theta}}{\sqrt{r}} = \frac{\sum_{i=1}^n x_i}{r\sqrt{r}} \approx 31.3$$

(d)

$$\frac{\widehat{\theta} - \theta}{\widehat{se}_{\widehat{\theta}}} \sim N(0,1) \;\Rightarrow\; P\left(\left|\frac{\widehat{\theta} - \theta}{\widehat{se}_{\widehat{\theta}}}\right| < z_{1-\frac{\alpha}{2}}\right) = 1 - \alpha$$

$$\Rightarrow \; \widehat{\theta} - z_{1-\frac{\alpha}{2}} \widehat{se}_{\widehat{\theta}} < \theta < \widehat{\theta} + z_{1-\frac{\alpha}{2}} \widehat{se}_{\widehat{\theta}}$$

$\therefore$ The 95% C.I. of θ is $[21.452, 144.148]$

(e)

$$F(t_p) = 1 - e^{\frac{t_p}{\theta}} \;\Rightarrow\; t_p = -\theta log(1-p) \;\;\therefore\;\; \widehat{t}_{0.1_{mle}} \approx 8.7$$

(f)

$$t_{0.1} = -\theta log(1 - 0.1) \approx 0.105\theta$$

By (d),

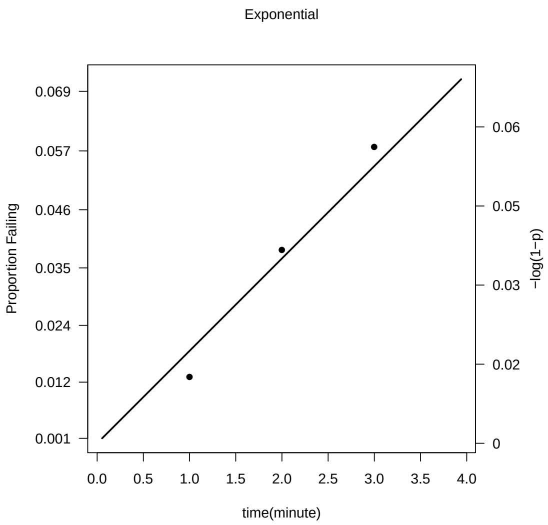
$$0.105\left(\widehat{\theta} - z_{1-\frac{\alpha}{2}} \widehat{se}_{\widehat{\theta}}\right) < 0.105\theta < 0.105\left(\widehat{\theta} + z_{1-\frac{\alpha}{2}} \widehat{se}_{\widehat{\theta}}\right)$$

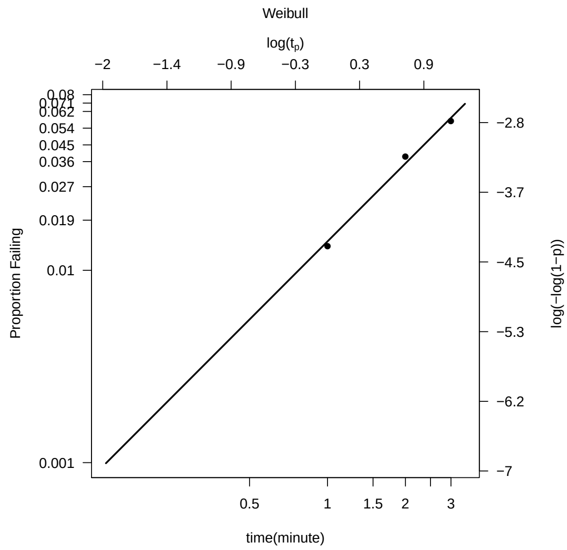
$$0.105 \cdot 21.452 < t_{0.1} < 0.105 \cdot 144.148$$

Therefore, $[2.252, 15.136]$ is the 95% C.I. of $t_{0.1}$.

8.2

(a) and (b)





(c) Both of the log-likelihood of the two distributions are -54.8, thus there are no difference between them.

8.3

By the equation (8.3), we perform a likelihood-ratio test.

$$H_0 : (\beta, \eta) = (\beta_0, \eta_0) \;\; ; \;\; H_1 : (\beta, \eta) \neq (\beta_0, \eta_0)$$

If

$$-2log\left[\frac{L(\beta_0, \eta_0)}{L(\widehat{\beta}_0, \widehat{\eta}_0)}\right] > \chi^2_{1-\alpha,2}$$

than, H_0 would be rejected. Under the condition of the null hypothesis, the likelihood ratio would be

$$-2log\left\{\frac{\prod_{i=1}^{3}\left[\left(1-e^{-\left(\frac{t_i}{\widehat{\eta}'}\right)}\right)-\left(1-e^{-\left(\frac{t_{i-1}}{\widehat{\eta}'}\right)}\right)\right]^{l_i}\prod_{j=1}^{3}\left[e^{-\left(\frac{t_j}{\widehat{\eta}'}\right)}\right]^{l_j}}{\prod_{i=1}^{3}\left[\left(1-e^{-\left(\frac{t_i}{\widehat{\eta}}\right)^{\widehat{\beta}}}\right)-\left(1-e^{-\left(\frac{t_{i-1}}{\widehat{\eta}}\right)^{\widehat{\beta}}}\right)\right]^{l_i}\prod_{j=1}^{3}\left[e^{-\left(\frac{t_j}{\widehat{\eta}}\right)^{\widehat{\beta}}}\right]^{l_j}}\right\} = 0.7232654$$

, where $\widehat{\beta}$ and $\widehat{\eta}$ are the mle of parameters of Weibull distribution and $\widehat{\eta}$ is the mle of parameters of exponential distribution. Let $\alpha = 0.05$, $\chi^2_{1-\alpha,2} = \chi^2_{0.95,2} = 5.9915$. Hence, H_0 is not rejected.

8.4

(a) Based on $Z_{\widehat{F}(2)} \sim N(0,1)$, the 95% C.I. is

$$\left[\widehat{F}(2) \pm z_{1-\frac{0.05}{2}} \widehat{se}_{\widehat{F}(2)}\right].$$

$$\widehat{F}(2) = 1 - e^{-e^{\frac{log2-3.162}{0.743}}} = 0.035, \text{ where } \widehat{\mu}_{mle} = 3.162 \text{ and } \widehat{\sigma}_{mle} = 0.743$$

By equation (8.10),

$$\widehat{se}_{\widehat{F}(2)} = \sqrt{\left(\frac{\partial F(2)}{\partial \mu}\right)^2 \widehat{Var}\left(\widehat{\mu}\right) + 2\left(\frac{\partial F(2)}{\partial \mu}\right)\left(\frac{\partial F(2)}{\partial \sigma}\right)\widehat{Cov}\left(\widehat{\mu},\widehat{\sigma}\right) + \left(\frac{\partial F(2)}{\partial \sigma}\right)^2 \widehat{Var}\left(\widehat{\sigma}\right)},$$

$\widehat{se}_{\widehat{F}(2)} = 0.011$. Thus, $[0.014, 0.056]$ is the 95% C.I.

(b) Based on $Z_{logit} \sim N(0,1)$,

$$\left[\frac{\widehat{F}(2)}{\widehat{F}(2) + \left(1 - \widehat{F}(2)\right) w}, \frac{\widehat{F}(2)}{\widehat{F}(2) + \frac{1-\widehat{F}(2)}{w}}\right],$$

where

$$w = e^{z_{1-\frac{0.05}{2}} \frac{\widehat{se}_{\widehat{F}(2)}}{\widehat{F}(2)\left(1-\widehat{F}(2)\right)}} = 1.847 \ .$$

Thus, $[0.019, 0.063]$ is the 95% C.I. of $F(2)$.

(c) The confidence interval based on $Z_{\widehat{F}(2)} \sim N(0,1)$ is thinner than the other on under the same level.

9.3

(a) and (b)

$$\widehat{\mu}^* = 10.3901 \text{ and } \widehat{\sigma}^* = 0.3460$$

(c)

$$\widehat{\mu} = 10.2299 \text{ and } \widehat{\sigma} = 0.3164$$

$$Z_{log\left(\widehat{t}^*_{0.1}\right)} = \frac{log\left(\widehat{t}^*_{0.1}\right) - log\left(\widehat{t}_{0.1}\right)}{\widehat{se}_{log\left(\widehat{t}^*_{0.1}\right)}}$$

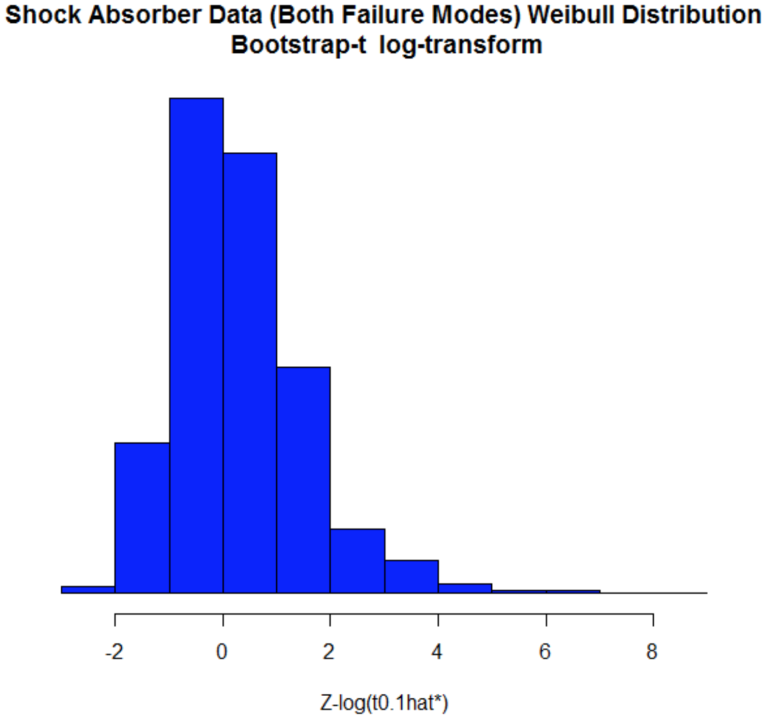
$$\widehat{t}_{0.1} = e^{\widehat{\mu}+\widehat{\sigma}\Phi^{-1}_{sev}(0.1)} \text{ and } \widehat{t}^*_{0.1} = e^{\widehat{\mu}^*+\widehat{\sigma}^*\Phi^{-1}_{sev}(0.1)}$$

$$log\left(\widehat{t}_{0.1}\right) = \widehat{\mu} + \widehat{\sigma}\Phi^{-1}_{sev}(0.1) = 9.518 \text{ and } log\left(\widehat{t}^*_{0.1}\right) = \widehat{\mu}^* + \widehat{\sigma}^*\Phi^{-1}_{sev}(0.1) = 9.611$$

$$\widehat{se}_{log\left(\widehat{t}^*_{0.1}\right)} = \frac{\widehat{se}_{\widehat{t}^*_{0.1}}}{\widehat{t}^*_{0.1}}$$

, where $\widehat{se}_{\widehat{t}^*_{0.1}}$ can be calculated by equation (8.10).

(d) and (e)



The 95% C.I. of $t_{0.1}$ is $[8472.97, 17252.16]$ which differs from the C.I. in table 9.2 for the condition in this problem based on resampling from the original data.