Deep Learning and Inverse Problems
Summer 2024
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Problem Set 3

Issued:	Tuesday, April	30, 2024,	1:00 pm,
Due:	Tuesday, May	7, 2024,	1:00 pm.

Problem 1 (Tikhonov-Regularized Least-Squares)**.** Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be an invertible matrix. Let $\mathbb{R}^n \ni \mathbf{y} = \mathbf{A}\mathbf{x}^* + \mathbf{e}$ be a noisy measurement of a signal $\mathbf{x}^* \in \mathbb{R}^n$. Here, $\mathbf{e} \sim \mathcal{N}(\mathbf{0}, \sigma^2 \mathbf{I})$ is a Gaussian noise term. In this problem, we discuss the regularized least-squares estimator

$$\hat{\mathbf{x}}_\lambda(\mathbf{y}) = \arg\min_{\mathbf{x}} ||\mathbf{A}\mathbf{x} - \mathbf{y}||_2^2 + \lambda\, ||\mathbf{x}||_2^2, \tag{1}$$

of the true signal $\mathbf{x}^*$. Here, the factor $\lambda \geq 0$ is the regularization weight.
Hint: To solve this problem, it might be useful to review Section 3.3 from the lecture notes.

(a) Consider the singular-value decompositon $\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T$ of the matrix $\mathbf{A}$. Here $\mathbf{U}, \mathbf{V} \in \mathbb{R}^{n \times n}$ are orthonormal and $\mathbf{\Sigma}$ is a diagonal matrix with $\mathbf{\Sigma} = \operatorname{diag}(\sigma_1, ..., \sigma_n)$. Show that for all $\lambda \geq 0$ and for fixed $\mathbf{y}$, the vector

$$\hat{\mathbf{x}}_\lambda(\mathbf{y}) = \mathbf{V}\operatorname{diag}\left(\frac{\sigma_1}{\sigma_1^2 + \lambda}, ..., \frac{\sigma_n}{\sigma_n^2 + \lambda}\right)\mathbf{U}^T\mathbf{y}$$

is a solution of the regularized least-squares problem (1).

(b) Is the solution from (a) the unique minimizer? If yes, why? If no, state another minimizer.

(**Hint:** Strict convexity.)

(c) Show that the expected mean-squared error $\mathbb{E}_{\mathbf{e}}\left[||\hat{\mathbf{x}}_\lambda(\mathbf{y}) - \mathbf{x}^*||_2^2\right]$ (expectation is over the random noise $\mathbf{e}$) of the estimator $\hat{\mathbf{x}}_\lambda(\mathbf{y})$ satisfies

$$\mathbb{E}_{\mathbf{e}}\left[||\hat{\mathbf{x}}_\lambda(\mathbf{y}) - \mathbf{x}^*||_2^2\right] = \sum_{i=1}^{n}\left(1 - \frac{\sigma_i^2}{\sigma_i^2 + \lambda}\right)^2 (\mathbf{v}_i^T\mathbf{x}^*)^2 + \sigma^2 \sum_{i=1}^{n}\left(\frac{\sigma_i}{\sigma_i^2 + \lambda}\right)^2,$$

where $\mathbf{v}_i \in \mathbb{R}^n$ denotes the i-th column of the matrix $\mathbf{V}$.

Hint: First use the result from (a) to show that

$$\hat{\mathbf{x}}_\lambda(\mathbf{y}) = \mathbf{V}\operatorname{diag}\left(\frac{\sigma_1^2}{\sigma_1^2 + \lambda}, ..., \frac{\sigma_n^2}{\sigma_n^2 + \lambda}\right)\mathbf{V}^T\mathbf{x}^* + \mathbf{V}\operatorname{diag}\left(\frac{\sigma_1}{\sigma_1^2 + \lambda}, ..., \frac{\sigma_n}{\sigma_n^2 + \lambda}\right)\mathbf{U}^T\mathbf{e}.$$

(d) Assume you know a good regularization weight $\bar{\lambda}$ for the noise $\mathbf{e} \sim \mathcal{N}(\mathbf{0}, \bar{\sigma}^2)$. If we change the noise model to $\mathbf{e} \sim \mathcal{N}(\mathbf{0}, \tilde{\sigma}^2)$ with $\tilde{\sigma}^2 > \bar{\sigma}^2$, how would you adapt the regularization weight λ? Would you make it larger or smaller than $\bar{\lambda}$? Justify your answer!

Problem 2 (Regularizing Deconvolution)**.** Here, we numerically solve the deblurring problem discussed in the lecture notes. For simplicity, we consider the 1D case. Let $\mathbf{x}^* \in \mathbb{R}^n$ be a 1D signal. Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be the matrix that implements convolution with the Gaussian kernel from Figure 3.1 of the notes. We want to use Tikhonov reulgarization to recover the signal $\mathbf{x}^*$ from a noisy measurement $\mathbf{y} = \mathbf{A}\mathbf{x}^* + \mathbf{e}$, where $\mathbf{e}$ is Gaussian noise. In this problem, your task is to write code to reproduce the bias-variance tradeoff (Figure 3.1 of the lecture notes). To that end, implement the matrix $\mathbf{A}$ to carry out the Gaussian convolution and make use of the results derived in problem 1.
Hint: Feel free to use generative AI models such as Copilot or ChatGPT to help you with the coding.