UNRAVELING DIRICHLET'S DIVISOR PROBLEM

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Abstract. The divisor function $\tau(n)$ counts the number of divisors for an integer n, including 1 and itself. It is part of the Dirichlet's Divisor Problem, which involves predicting the α in which $\sum_{n\leq x}\tau(n) - x\log(x) + (2\gamma-1)x + \Delta(x)$, where $\Delta(x) \leq C(\epsilon)X^{\alpha+\epsilon}$ holds. Note that it is known that $\frac{1}{4} \leq \alpha \leq \frac{1}{3}$. In addition, it is conjectured that $\alpha = \frac{1}{4}$ is admissible. We numerically estimate through the use of a log-log plot that this bound holds for $\alpha = 0.285$ and a constant C of $e^{3.301}$ for up to $x = 10^7$.

Contents

1. Introduction

The divisor function and its generalizations are important in mathematics and occur in various problems pertaining to number theory. More specifically, they are a tremendous aid in studying the distribution of prime numbers and asymptotic behaviour of the Riemann zeta function. This survey article discusses the problems that involve the divisor function, such as the Dirichlet's Divisor Problem.

The divisor function $\tau(n)$ is a function that counts the number of divisors for an integer n, including 1 and itself. This function can be expressed as

$$\tau(n) = \sum_{d|n} 1,$$

where $d \mid n$ means that d is a factor of n. The notation $d(n)$ is also sometime used for the divisor function. Table 1 below lists some values of $\tau(n)$.

$\boldsymbol{n}$	1	5	25	6	$2\cdot 3\cdot 5\cdot 7$	$2^3\cdot 5^2$
$\boldsymbol{\tau(n)}$	1	2	3	4	16	12

Table 1. Some values of the divisor function $\tau(n)$

There are several generalization of the divisor function. Note that one number can be a product of different variations of factors. For instance,

$$\begin{aligned} 10 &= 1\cdot 1\cdot 10 = 1\cdot 10\cdot 1 = 10\cdot 1\cdot 1 \\ &= 1\cdot 2\cdot 5 = 2\cdot 1\cdot 5 = 2\cdot 5\cdot 1 \\ &= 1\cdot 5\cdot 2 = 5\cdot 1\cdot 2 = 5\cdot 2\cdot 1. \end{aligned}$$

This highlights that there are nine ways to write 10 as a product of three positive factors and that the order of the factors matter. In general, the function $\tau_k(n)$ counts the number of ways to write n as a product of k positive factors as such:

$$\tau_k(n) = \sum_{n=n_1 n_2 \cdots n_k} 1,$$

where the sum runs over all k-tupple $n_1, \cdots, n_k$ of natural numbers that multiply to n. For the special case $k = 2$, we write $\tau(n)$ for $\tau_2(n)$.

2. Dirichlet's Divisor Problem

The Dirichlet's Divisor Problem consists of estimating the summatory function using

$$D(x) = \sum_{n \leq x} \tau(n)$$

in which $\tau(n)$ represents the divisor function. Peter Gustav Lejeune Dirichlet established in 1849 that this summatory function can be estimated with the following expression:

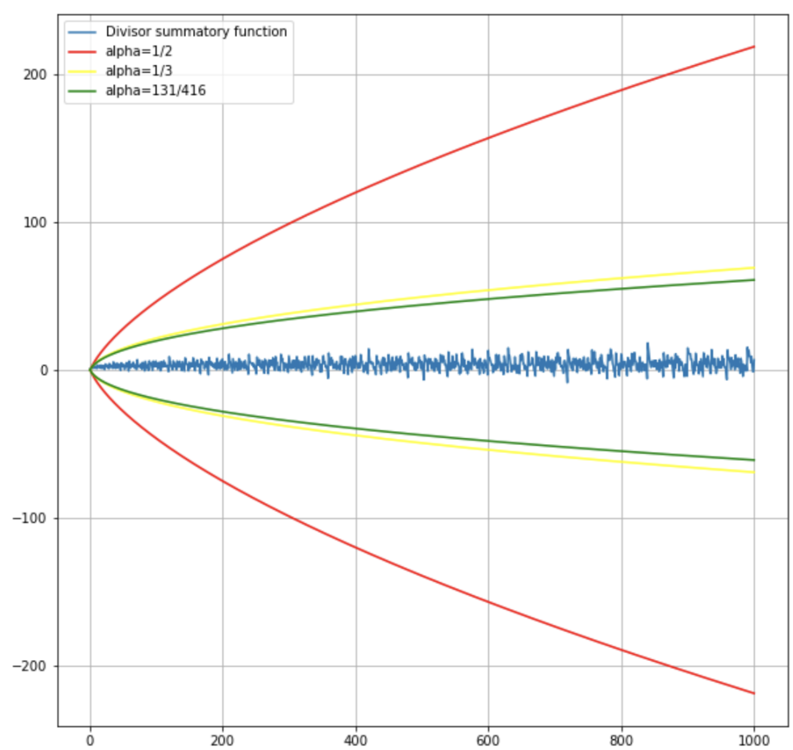
$$x \log(x) + (2\gamma - 1)x + \Delta(x)$$

where γ is the Euler constant and $\Delta(x)$ is the error term. Mathematicians are interested in finding the smallest α for which

$$|\Delta(x)| \leq C x^{\alpha+\epsilon}$$

where it holds for any $\epsilon > 0$ and for some constant C depending on ϵ. The following graph and table highlights significant discoveries concerning this exponent α:

Graph of the error term in the Dirichlet's Divisor Problem



Name	α	**Date**
Dirichlet	$\frac{1}{2}$	1849
Voronoi	$\frac{1}{3}$	1904
Huxley	$\frac{131}{416}$	2003

TABLE 2. Table of the error term in the Dirichlet's Divisor Problem

3. NUMERICALLY DETERMINING THE EXPONENT IN THE DIRICHLET'S DIVISOR PROBLEM

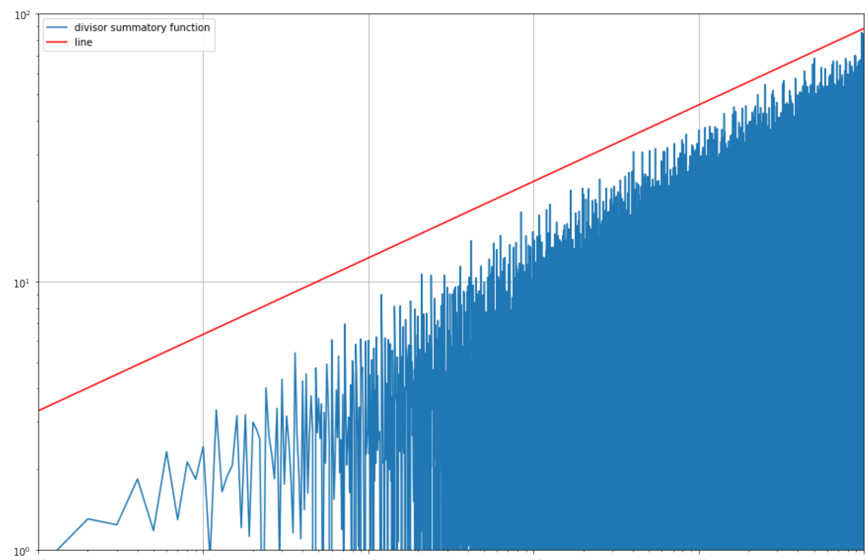
We can try to numerically determine the exponent in the Dirichlet's Divisor Problem by creating a log-log plot. A log-log plot is a two-dimensional graph of the logarithmic transformation for both the x and y-axis of the numeric data. It is useful in determining a power relationship of the form $y = cx^{\alpha}$, since the curve simply becomes relatively straight after taking log's of both sides, as shown below.

$$y = cx^{\alpha}$$
$$log(y = cx^{\alpha})$$
$$log(y) = log(c) + \alpha log(x)$$
$$\text{Let Y = log(y) and X = log(x).}$$
$$Y = \alpha X + log(c).$$

The y-intercept provides the constant while the slope gives the exponent. For example, the curve $y = 5x^{1/4}$ will transform into a straight line with a y-intercept of log(5) and a slope of 1/4.

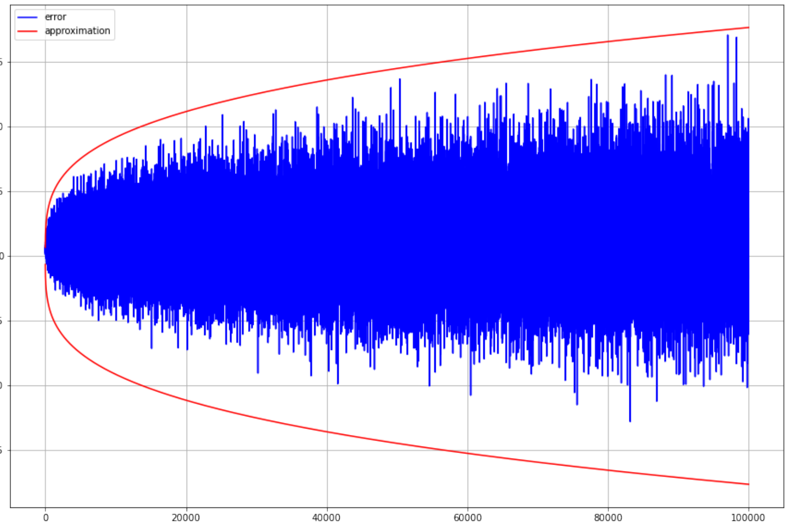
We can create a log-log plot using the divisor summatory function to approximate the bounds, as shown in the graph below.

Graph of the Dirichlet's Divisor Problem as a Log-Log Plot of the Error Term



This graph is created using $\alpha = 0.285$ and the y-intercept as 3.301. As a result, the error term in the Dirichlet's Divisor Problem can be bounded by $y = \pm e^{3.301} x^{0.285}$. Note that this alpha within known estimations of the error term in the Dirichlet's Divisor Problem, so $0.25 \leq 0.285 \leq 0.333$. Using this approximation, we can see that the divisor summatory function is bounded up to 10^5.

Graph of the error term in the Dirichlet's Divisor Problem



Although it is very time-consuming to graph beyond 10^5, we can make an educated guess on whether it is bounded using Python. Note that the error value is 94.1924 at 10^7 while the bound has a value of 2682.91 at 10^7. Since the error value is significantly less than the bound value, this curve is possibly a good bound up to 10^7.

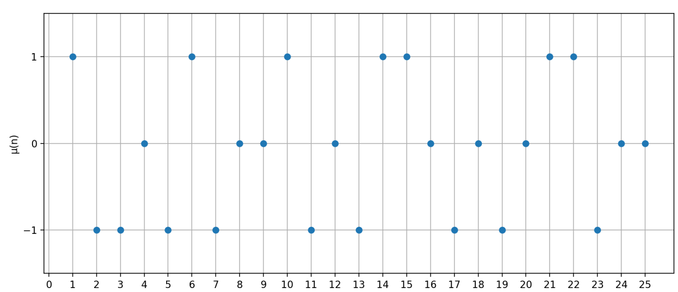
4. Connection to the Möbius Function

In 1832, August Ferdinand Möbius introduced the Möbius function $\mu(n)$, a multiplicative function as followed:

$$\mu(n) = \begin{cases} (-1)^{\upsilon(n)} & \textit{if } n \textit{ is square-free,} \\ 0 & \textit{otherwise,} \end{cases}$$

where $\upsilon(n)$ denotes the number of distinct prime divisors of n. With this definition, we can show, for example, that $\mu(2021) = (-1)^2 = 1$, because $2021 = 43 * 47$ and therefore, a product of two distinct prime factors.

Graph of Möbius Function $\mu(n)$ for $n \leq 25$



The Mertens function, named after Franz Mertens, is the partial sum of the Möbius function, defined as

$$M(x) = \sum_{k \leq x} \mu(k).$$

Thomas Joannes Stieltjes conjectured in 1885 that

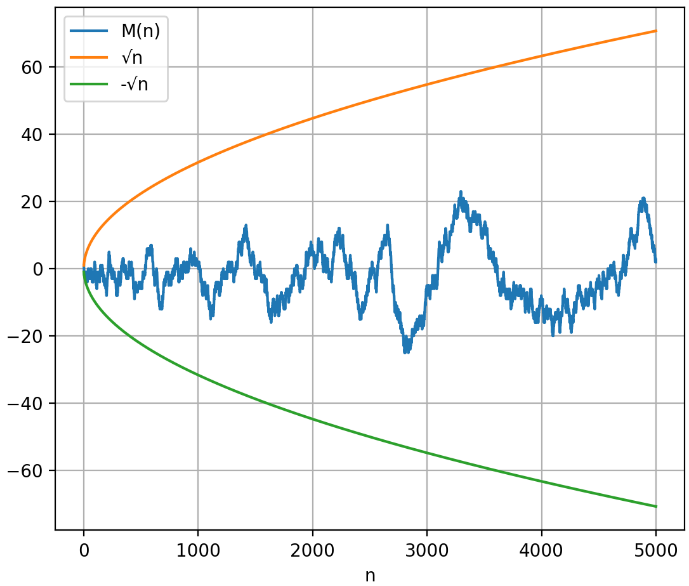
$$M(x) = O(x^{\frac{1}{2}})$$

In other words, he believed that the Mertens function was bounded by $\pm C\sqrt{n}$, where C is some constant. Known as the Mertens conjecture, it was later disproven by Andrew Odlyzko and Herman te Riele and revised to be

$$M(x) = O(x^{\frac{1}{2}+\epsilon}),$$

for any $\epsilon > 0$. Refering back to the graph of the error term in the Dirichlet's Divisor Problem, we can see that both graphs fluctuate between positive and negative values.

Graph of Former Version of the Merten Conjecture (Disproven) $\mu(n)$ for $n \leq 25$



5. Acknowledgements

I would like to thank David Nguyen for providing invaluable guidance throughout this research.

6. Appendix

Link to Jupyter Notebook: https://bit.ly/3W1KC6G

References

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