2.1

a

$Y = X^3$ and $f_X(x) = 42x^5(1-x), \quad 0 < x < 1$

Y is increasing on the interval $(0, 1)$, so we will not need to split the pdf into a piecewise

$f_Y(y) = f_X(g^{-1}(y))|\frac{d}{dy}g^{-1}(y)|$

$g(x) = x^3$
$Y = g(X) = X^3$
$g^{-1}(y) = \sqrt[3]{y}$
$\frac{d}{dy}g^{-1}(y) = y^{-2/3}/3$

$f_Y(y) = f_X(\sqrt[3]{y})|y^{-2/3}/3|$
$f_Y(y) = 42(\sqrt[3]{y})^5(1 - (\sqrt[3]{y}))|y^{-2/3}/3|$
$f_Y(y) = 42y^{5/3}(1 - \sqrt[3]{y})|y^{-2/3}/3|$
$f_Y(y) = (42y^{5/3} - 42y^2)|y^{-2/3}/3|$

$y^{-2/3}$ is always positive

$f_Y(y) = (42y^{5/3} - 42y^2)(y^{-2/3}/3)$
$f_Y(y) = 14(y - y^{4/3})$
$f_Y(y) = 14y(1 - y^{1/3})$
$\square$

$\int_0^1 f_Y(y)dy = \int_0^1 14y(1 - y^{1/3})dy$

$= 14\int_0^1 y - y^{4/3}$

$= 14(y^2/2 - 3y^{7/3}/7)\Big|_0^1$

$= 7y^2 - 6y^{7/3}\Big|_0^1$

$= 7 - 6 - 0 + 0 = 1$
$\square$

b

$Y = 4X + 3$ and $f_X(x) = 7e^{-7x}, \quad 0 < x < \infty$

Y is increasing on the interval $(0, \infty)$, so we will not need to split the pdf into a piecewise

$f_Y(y) = f_X(g^{-1}(y))|\frac{d}{dy}g^{-1}(y)|$

$g(x) = 4x + 3$
$g^{-1}(y) = y/4 - 3/4$

$\frac{d}{dy}g^{-1}(y) = 1/4$

$f_Y(y) = f_X(y/4 - 3/4)|1/4|$
$f_Y(y) = 7e^{-7(y/4-3/4)}/4$
$f_Y(y) = 7e^{21/4-7y/4}/4, \quad 3 < y < \infty$
$\square$

$\int\limits_3^\infty 7e^{21/4-7y/4}/4dy$
$-e^{21/4-7y/4}\Big|_3^\infty$
$= 0 + e^0 = 1$
$\square$

c

$Y = X^2$ and $f_X(X) = 30x^2(1-x)^2, \quad 0 < x < 1$

Y is increasing on the interval $(0, 1)$, so we will not need to split the pdf into a piecewise

$f_Y(y) = f_X(g^{-1}(y))|\frac{d}{dy}g^{-1}(y)|$

$g(x) = x^2$
$Y = g(X) = X^2$
$g^{-1}(y) = \sqrt{y}$
$\frac{d}{dy}g^{-1}(y) = y^{-1/2}/2$

$f_Y(y) = f_X(\sqrt{y})|y^{-1/2}/2|$
$f_Y(y) = 30\sqrt{y}^2(1-\sqrt{y})^2|y^{-1/2}/2|$
$f_Y(y) = 15\sqrt{y}(1-\sqrt{y})^2$
$f_Y(y) = 15\sqrt{y}(1-2\sqrt{y}+y)$
$f_Y(y) = 15(\sqrt{y}-2y+y^{3/2})$
$\square$

$\int\limits_0^1 15(\sqrt{y}-2y+y^{3/2})dy$
$15(2y^{3/2}/3 - y^2 + 2y^{5/2}/5)\Big|_0^1$
$= 15(2/3 - 1 + 2/5)$
$= 15(10/15 - 15/15 + 6/16)$
$= 1$
$\square$

2.3

$f_X(x) = (2/3)^x/3, \quad x = 0, 1, 2, \ldots$
$f_Y(y) = f_X(g^{-1}(y))|\frac{d}{dy}g^{-1}(y)|$

$g(x) = x/(x+1)$
$yx + y - x = 0$

$x(y-1) = -y$
$g^{-1}(y) = y/(1-y)$
$\frac{d}{dy}g^{-1}(y) = ((1)(1-y) - (-1)(y))/(1-y)^2$
$\frac{d}{dy}g^{-1}(y) = 1/(1-y)^2$

$f_Y(y) = f_X(y/(1-y))|1/(1-y)^2|, \quad y = 0, 1/2, 2/3, 3/4, 4/5, \ldots$
$f_Y(y) = (2/3)^{y/(1-y)}/3(1-y)^2, \quad y = 0, 1/2, 2/3, 3/4, 4/5, \ldots$

2.6

a

$f_X(x) = e^{-|x|}/2, \quad -\infty < x < \infty; Y = |X|^3$
$f_Y(y) = f_X(g^{-1}(y))|\frac{d}{dy}g^{-1}(y)|$

$g(x) = |x|^3$
$g^{-1}(y) = \pm\sqrt[3]{y}, \quad 0 \leq y$
$\frac{d}{dy}g^{-1}(y) = \pm y^{-2/3}/3$

$f_Y(y) = f_X(\pm\sqrt[3]{y})(y^{-2/3}/3)$
$f_Y(y) = y^{-2/3}e^{-|\pm\sqrt[3]{y}|}/6$
$\square$

$\int\limits_0^\infty y^{-2/3}e^{-|\pm\sqrt[3]{y}|}/6dy$
$2\int\limits_0^\infty y^{-2/3}e^{-\sqrt[3]{y}}/6dy$
$2(-e^{-\sqrt[3]{y}}/2)\Big|_0^\infty$
$-e^{-\sqrt[3]{y}}\Big|_0^\infty$
$= 0 - -1$
1
$\square$

b

$f_X(x) = 3(x+1)^2/8, \quad -1 < x < 1; Y = 1 - X^2$
$f_Y(y) = f_X(g^{-1}(y))|\frac{d}{dy}g^{-1}(y)|$

$g(x) = 1 - x^2$
$g^{-1}(y) = \pm\sqrt{1-y}, \quad 0 < y < 1$
$\frac{d}{dy}g^{-1}(y) = \mp(1-y)^{-1/2}/2$

$f_Y(y) = f_X(\pm\sqrt{1-y})|\mp(1-y)^{-1/2}/2|$
$f_Y(y) = 3((\pm\sqrt{1-y}) + 1)^2((1-y)^{-1/2}/2)/8$
$\square$

$$\int_0^1 f_Y(y)dy = \begin{cases} \int_0^1 ((-\sqrt{1-y})+1)^3/8dy & \text{for x>0} \\ \int_1^0 ((+\sqrt{1-y})+1)^3/8dy & \text{for x<0} \end{cases}$$
$= 1/8 - 0 + 1 - 1/8$
$= 1$
$\square$

c

$f_X(x) = 3(x+1)^2/8, \quad -1 < x < 1; Y = 1 - X^2 \text{ if } X \leq 0; Y = 1 - X \text{ if } X > 0$
$f_Y(y) = f_X(g^{-1}(y))|\frac{d}{dy}g^{-1}(y)|$

$$g(x) = \begin{cases} 1 - x^2 & x \leq 0 \\ 1 - x & x > 0 \end{cases}$$
$$g^{-1}(x) = \begin{cases} -\sqrt{1-y} & x \leq 0; 0 \leq y < 1 \\ 1 - y & x > 0; 1 > y > 0 \end{cases}$$
$$\frac{d}{dy}g^{-1}(x) = \begin{cases} (1-y)^{-1/2}/2 & x \leq 0; 0 \leq y < 1 \\ -1 & x > 0; 1 > y > 0 \end{cases}$$

$f_Y(y) = f_X(g^{-1}(y))|\frac{d}{dy}g^{-1}(y)|$
$$g^{-1}(x) = \begin{cases} f_X(-\sqrt{1-y})|(1-y)^{-1/2}/2| & x \leq 0; 0 \leq y < 1 \\ f_X(1-y)|-1| & x > 0; 1 > y > 0 \end{cases}$$
$$g^{-1}(x) = \begin{cases} 3(-\sqrt{1-y}+1)^2|(1-y)^{-1/2}/2|/8 & x \leq 0; 0 \leq y < 1 \\ 3(1-y+1)^2/8 & x > 0; 1 > y > 0 \end{cases}$$
$$g^{-1}(x) = \begin{cases} 3(-\sqrt{1-y}+1)^2(1-y)^{-1/2}/16 & x \leq 0; 0 \leq y < 1 \\ 3(2-y)^2/8 & x > 0; 1 > y > 0 \end{cases}$$
$\square$

$$\int f_Y(y)dy = \begin{cases} \int 3(-\sqrt{1-y}+1)^2(1-y)^{-1/2}/16dy & x \leq 0; 0 \leq y < 1 \\ \int 3(2-y)^2/8dy & x > 0; 1 > y > 0 \end{cases}$$
$$\int f_Y(y)dy = \begin{cases} (-\sqrt{1-y}+1)^3/8 & x \leq 0; 0 \leq y < 1 \\ (2-y)^3/8 & x > 0; 1 > y > 0 \end{cases}$$
$$\int_0^1 f_Y(y)dy = \begin{cases} (-\sqrt{1-y}+1)^3/8\Big|_0^1 & x \leq 0; 0 \leq y < 1 \\ (2-y)^3/8\Big|_1^0 & x > 0; 1 > y > 0 \end{cases}$$
$= 1/8 - 0 + 1 - 1/8$
$= 1$
$\square$

2.9

$$f_X(x) = \begin{cases} (x-1)/2 & 1 < x < 3 \\ 0 & \text{otherwise} \end{cases}$$
$$f_Y(y) = \begin{cases} 1 & 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$F_X(x) = \begin{cases} 0 & x \leq 1 \\ (x-1)^2/4 & 1 < x < 3 \\ 1 & 3 \leq x \end{cases}$$

$$F_Y(y) = \begin{cases} 0 & y \leq 0 \\ y & 0 < y < 1 \\ 1 & 1 \leq y \end{cases}$$

$$F_Y(y) = F_X(g^{-1}(y))$$
$$F_Y(g(x)) = F_X(x)$$

$$g(x) = (x-1)^2/4$$
$$g^{-1}(y) = 2\sqrt{y} + 1$$

$$u(X) = (X-1)^2/4$$
$\square$