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PHYS 122 HW 5

1

Dielectric interface

The xy plane is the interface between two essentially infinite dielectric slabs. Each dielectric is uniformly polarized. $\vec{P} = P_1\hat{k}$ for $z < 0$ and $\vec{P} = P_2\hat{k}$ for $z > 0$.

a

Determine σ, the surface density of bound charge at the interface.

Give your answer in terms of P_1 and P_2.

✓ **Answer**

$\sigma_{bound} = \hat{n} \cdot \vec{P}_1 - \hat{n} \cdot \vec{P}_2$
$\sigma_{bound} = \hat{n} \cdot \hat{k}P_1 - \hat{n} \cdot \hat{k}P_2$
$\sigma_{bound} = P_1 - P_2$
$\square$

b

Assume that the dielectrics are both linear. The one below the xy plane has susceptibility χ_1; the one above, χ_2. Determine E_1 and E_2 the electric fields inside the two dielectrics.

Give your answer in terms of P_1, P_2, χ_1, χ_2, ϵ_0 and $\hat{k}$.

✓ **Answer**

$\vec{E}_1 = \frac{\hat{k}P_1}{\chi_1\epsilon_0}$
$\vec{E}_2 = \frac{\hat{k}P_2}{\chi_2\epsilon_0}$
$\square$

c

What is the total charge density σ_{tot} of the interface?

Give your answer in terms of P_1, P_2, χ_1, χ_2, ϵ_0 and $\hat{k}$.

✓ **Answer**

By Gauss law:

$\sigma = \epsilon_0 \hat{n} \cdot (\vec{E}_1 - \vec{E}_2)$

$\sigma = \frac{P_1}{\chi_1} - \frac{P_2}{\chi_2}$

☐

d

Now assume that there is no free charge at the interface. Determine P_2 in terms of P_1, χ_1 and χ_2.

✓ **Answer**

$\sigma_{bound} = P_1 - P_2$

$\sigma = \frac{P_1}{\chi_1} - \frac{P_2}{\chi_2}$

$\sigma_{free} = \sigma - \sigma_{bound}$

$= \frac{P_1}{\chi_1} - \frac{P_2}{\chi_2} - (P_1 - P_2)$

$= \frac{P_1 - \chi_1}{\chi_1} - \frac{P_2 - \chi_2}{\chi_2}$

$= 0$

$\frac{P_1 - \chi_1}{\chi_1} = \frac{P_2 - \chi_2}{\chi_2}$

$\frac{\chi_2(P_1 - \chi_1)}{\chi_1} = P_2 - \chi_2$

$P_2 = \frac{\chi_2(P_1 - \chi_1)}{\chi_1} - \chi_2$

☐

2

Charged dielectric sphere

A dielectric sphere of radius a has free charge Q uniformly distributed over its interior.

a

What is the total charge of the sphere?

✓ **Answer**

Q

b

In what direction do you expect the electric field to point?

✓ **Answer**

Away from the sphere, assuming the sphere has a positive charge

c

Use Gauss's law to determine the electric field external to the sphere for $r > a$ (where r denotes distance from the center of the sphere).

✓ **Answer**

$$\Phi = \frac{Q}{\epsilon_0}$$
$$E = \frac{Q}{\epsilon_0 4\pi r^2}\hat{r}$$

d

In the interior of the sphere let us assume that the electric field is given by $\vec{E} = E\hat{r}$.

Here $\hat{r}$ is a unit vector that points directly away from the center of the sphere. E might vary with r, the distance from the center.

i

Assume the dielectric is linear with susceptibility χ. What is the polarization inside the sphere?

Give your answer in terms of E, $\hat{r}$, ϵ_0, and χ.

✓ **Answer**

$$\vec{E} = E\hat{r}$$
$$\vec{P} = \chi\epsilon_0\vec{E}$$

$\vec{P} = \chi\epsilon_0 E\hat{r}$

□

ii

Consider a Gaussian surface of radius $r < a$. How much bound charge does the surface contain?

Give your answer in terms of E, r, ϵ_0 and χ.

✓ **Answer**

$Q_{bound} = -\oint da\ \hat{n} \cdot \vec{P}$
$\vec{P} = \chi\epsilon_0 E\hat{r}$

$Q_{bound} = -\oint da\ \hat{n} \cdot \chi\epsilon_0 E\hat{r}$
$Q_{bound} = -\oint da\ \chi\epsilon_0 E$
$Q_{bound} = -4\pi a^2 \chi\epsilon_0 E$
□

iii

How much free charge does the Gaussian surface of part (ii) contain?

Give your answer in terms of Q, r and a.

✓ **Answer**

As the charge is uniformly distributed across the volume,
$Q_{free} = \frac{Qa^3}{r^3}$
□

iv

Now apply Gauss's law to the surface to determine E in terms of Q, r, a, ϵ_0 and χ. Thus we have now determined the electric field both inside and outside the dielectric sphere.

✓ **Answer**

$Q_{tot} = Q_{free} + Q_{bound}$
$= \epsilon_0 \oint da\ \hat{n} \cdot \vec{E}$
$= \epsilon_0 \oint da\ \hat{n} \cdot E\hat{r}$

$= \epsilon_0 \oint da\ E$
$= 4\pi a^2 \epsilon_0 E$

$Q_{free} = \frac{Qa^3}{r^3}$
$Q_{bound} = -4\pi a^2 \chi \epsilon_0 E$

$\frac{Qa^3}{r^3} - 4\pi a^2 \chi \epsilon_0 E = 4\pi a^2 \epsilon_0 E$
$\frac{Qa^3}{r^3} = 4\pi a^2 \epsilon_0 E(1+\chi)$
$E = \frac{Qa}{4r^3 \pi \epsilon_0 (1+\chi)}$
$\square$

v

Now determine the polarization $\vec{P}$ inside the sphere.

Give your answer in terms of Q, r, a, ϵ_0 and χ.

Hint: use the results of parts (i) and (iv)

✓ Answer

$\vec{P} = \chi \epsilon_0 E \hat{r}$
$E = \frac{Qa}{4r^3 \pi \epsilon_0 (1+\chi)}$

$\vec{P} = \frac{Qa\chi}{4r^3 \pi (1+\chi)} \hat{r}$

vi

What is the polarization $\vec{P}$ outside the sphere?

✓ Answer

Assuming that the sphere is surrounded by empty space,
$\vec{P} = \vec{0}$
$\square$

vii

What is the surface density σ_b of bound charge on the surface of the dielectric sphere?

Give your answer in terms of Q, ϵ_0, a and χ.

✓ **Answer**

$\sigma_{bound} = \hat{n} \cdot \vec{P}_1 - \hat{n} \cdot \vec{P}_2$
$\vec{P}_1 = \vec{P}$
$\vec{P}_2 = \vec{0}$

$\vec{P} = \frac{Qa\chi}{4r^3\pi(1+\chi)}\hat{r}$
$r = a$
$\vec{P} = \frac{Q\chi}{4a^2\pi(1+\chi)}\hat{r}$
$\sigma_{bound} = \frac{Q\chi}{4a^2\pi(1+\chi)}$
$\square$

3

Spherical capacitor

A conducting sphere of radius a is surrounded by a conducting shell (inner radius b, outer radius c). The sphere has a charge Q; the shell, a charge $-Q$. The space between the sphere and the shell is filled with a linear dielectric with susceptibility χ.

Assume the dielectric contains no free charge.

a

What is the total charge of the system (sphere, shell and dielectric)? What is the electric field external to the shell (i.e. for $r > c$ where r denotes distance from the center). Justify your answer.

✓ **Answer**

Since the dielectric has no free charge,
$Q_{tot} = Q - Q + 0$

$Q_{tot} = 0$
$\square$

b

What is the electric field in the interior of the conducting sphere? Where does the charge Q reside?

✓ **Answer**

Perfect conductors have no electric field within them

$\vec{E} = \vec{0}$

Q resides on the surface of the sphere.

c

What is the electric field within the shell (for $b < r < c$)?

✓ **Answer**

Perfect conductors have no electric field within them

$\vec{E} = \vec{0}$

d

Where does the charge on the shell reside? On the inner surface? The outer surface? Over both? Justify your answer.

✓ **Answer**

On the inner surface.
As the shell must have $\vec{E} = \vec{0}$ within, this means that any Gaussian surface cutting through just the shell must have a flux of 0.

Therefore, any Gaussian surface cutting through the shell must enclose no charge.

Therefore, a charge of $-Q$ must be present on the inner surface of the shell, to cancel out the Q of the inner sphere.

e

Now assume that the electric field in the dielectric region is $\vec{E} = E\hat{r}$ where $\hat{r}$ is a unit vector that points radially outward, away from the center. E might depend on r.

i

Determine the polarization $\vec{P}$ inside the dielectric. Give your answer in terms of E, χ, ϵ_0 and $\hat{r}$
.

✓ **Answer**

$\vec{P} = \chi\epsilon_0\vec{E}$
$\vec{P} = \chi\epsilon_0 E\hat{r}$
$\square$

ii

Consider a spherical Gaussian surface of radius r with $a < r < b$. How much bound charge does it enclose? Give your answer in terms of χ, E, r and ϵ_0.

✓ **Answer**

$Q_{bound} = -\oint da\ \hat{n}\cdot\vec{P}$
$Q_{bound} = -\oint da\ \hat{n}\cdot\chi\epsilon_0 E\hat{r}$
$Q_{bound} = -\oint da\ \chi\epsilon_0 E$
$Q_{bound} = -4\pi r^2\chi\epsilon_0 E$
$\square$

iii

How much total charge does the Gaussian surface enclose? Give your answer in terms of χ, E, r, ϵ_0 and Q.

✓ **Answer**

$Q_{tot} = Q_{free} + Q_{bound} = \epsilon_0\oint da\ \hat{n}\cdot\vec{E}$
$Q_{bound} = -4\pi r^2\chi\epsilon_0 E$
$Q_{free} = Q$
$Q_{tot} = Q - 4\pi r^2\chi\epsilon_0 E$
$\square$

iv

Use Gauss's law to determine the electric field $\vec{E}$. Give your answer in terms of χ, ϵ_0, r and Q.

✓ **Answer**

$Q_{tot} = Q - 4\pi r^2\chi\epsilon_0 E$
$Q - 4\pi r^2\chi\epsilon_0 E = \epsilon_0\oint da\ \hat{n}\cdot\vec{E}$

$Q - 4\pi r^2 \chi E = 4\pi r^2 E$
$Q = 4\pi r^2 E(1+\chi)$
$E = \frac{Q}{4\pi r^2(1+\chi)}$
$\vec{E} = \frac{Q}{4\pi r^2(1+\chi)}\hat{r}$
$\square$

v

Determine the potential difference $V_2 - V_1$ where V_2 is the potential of the conducting shell and V_1 is the potential of the conducting sphere.

✓ **Answer**

$\vec{E} = \frac{Q}{4\pi r^2(1+\chi)}\hat{r}$
$V_b - V_a = -\int\limits_a^b \vec{E} \cdot d\hat{r}$
$V_b - V_a = -\int\limits_a^b \frac{Q}{4\pi r^2(1+\chi)}\, dr$
$V_b - V_a = -\frac{Q}{4\pi(1+\chi)} \int\limits_a^b \frac{1}{r^2}\, dr$
$V_b - V_a = \frac{Q}{4\pi(1+\chi)} \left(\frac{1}{r}\right)\Big|_{r=a}^{b}$
$V_b - V_a = \frac{Q}{4\pi(1+\chi)} \left(\frac{1}{b} - \frac{1}{a}\right)$
$\square$

f

Capacitance

A capacitor is a device that consists of two conducting “electrodes”. Usually one electrode has a positive charge; the other has an equal and opposite negative charge. The capacitance of the capacitor is defined as $C = \frac{Q}{V}$ where Q is the charge on the positive electrode and $V = V_+ - V_-$ is the potential difference between the two electrodes.

i

Calculate the capacitance of the spherical capacitor analyzed in parts (a)-(e). Give your answer in terms of a, b, ϵ_0 and χ

✓ **Answer**

$$V_b - V_a = \frac{Q}{4\pi(1+\chi)}\left(\frac{1}{b} - \frac{1}{a}\right)$$
$$C = \frac{Q}{V}$$

$$C = 4\pi(1+\chi)\frac{ab}{a-b}$$
$\square$

ii*

You should find that $C \propto (1+\chi)$. In the absence of a dielectric filling $\chi = 0$. Thus filling the space between the electrodes with dielectric enhances the capacitance by a factor of $1+\chi$.

4

Capacitor and dielectrics

A parallel plate capacitor with plate area A and plate separation d is connected to a battery of voltage V_0. It is initially empty and then a dielectric slab of thickness t (where $t < d$) and dielectric constant κ (or χ) is inserted between the plates such that it covers the entire plate area but doesn't completely fill the space between the plates.

a

Find the capacitance of the capacitor when it is empty.

✓ **Answer**

$$C = \frac{\epsilon_0 A}{d}$$
$\square$

b

Find the capacitance of the capacitor after the dielectric is inserted.

✓ **Answer**

$$C_1 = (\kappa\epsilon_0)\frac{A}{t}$$
$$C_2 = \frac{\epsilon_0 A}{d-t}$$
$$C = \frac{1}{\frac{1}{C_1}+\frac{1}{C_2}}$$
$$C = \left(\frac{d-t}{\epsilon_0 A} + \frac{t}{A\kappa\epsilon_0}\right)^{-1}$$

c

If the battery remains connected while the dielectric is inserted, by what factor does the energy stored in the capacitor change? Check your results for $t = d$ and $t = 0$ to see if your factor has the expected behavior.

Answer

By a factor of κ

This lines up with substituting $t = d$ or $t = 0$ into the final equation