1

a

$$D = \sum_{n=1}^{N} \sum_{k=1}^{K} r_{n,k} \|\vec{x}_n - \vec{\mu}_k\|^2$$
$$D = \sum_{n=1}^{N} \sum_{k=1}^{K} \sum_{i=1}^{I} r_{n,k} (x_{ni} - \mu_{ki})^2$$

$$\frac{\partial D}{\partial \mu_{ki}} = \sum_{n=1}^{N} -2 r_{n,k} (x_{ni} - \mu_{ki}) = 0$$

$$\sum_{n=1}^{N} r_{n,k} x_{ni} - r_{n,k} \mu_{ki} = 0$$
$$\sum_{n=1}^{N} r_{n,k} x_{ni} - \sum_{n=1}^{N} r_{n,k} \mu_{ki} = 0$$
$$\sum_{n=1}^{N} r_{n,k} x_{ni} = \sum_{n=1}^{N} r_{n,k} \mu_{ki}$$
$$\sum_{n=1}^{N} r_{n,k} x_{ni} = \mu_{ki} \sum_{n=1}^{N} r_{n,k}$$
$$\frac{\sum_{n=1}^{N} r_{n,k} x_{ni}}{\sum_{n=1}^{N} r_{n,k}} = \mu_{ki}$$

b

$$D = \sum_{n=1}^{N} \sum_{k=1}^{K} r_{n,k} \|\vec{x}_n - \vec{\mu}_k\|^2$$

$$\frac{\partial D}{\partial \vec{\mu}_k} = -2 \sum_{n=1}^{N} r_{n,k} (\vec{x}_n - \vec{\mu}_k) = \vec{0}$$

$$\sum_{n=1}^{N} r_{n,k} (\vec{x}_n - \vec{\mu}_k) = \vec{0}$$
$$\sum_{n=1}^{N} r_{n,k} \vec{x}_n = \sum_{n=1}^{N} r_{n,k} \vec{\mu}_k$$

$$\sum_{n=1}^{N} r_{n,k}\vec{x}_n = \vec{\mu}_k \sum_{n=1}^{N} r_{n,k}$$

$$\frac{\sum_{n=1}^{N} r_{n,k}\vec{x}_n}{\sum_{n=1}^{N} r_{n,k}} = \vec{\mu}_k$$

2

a

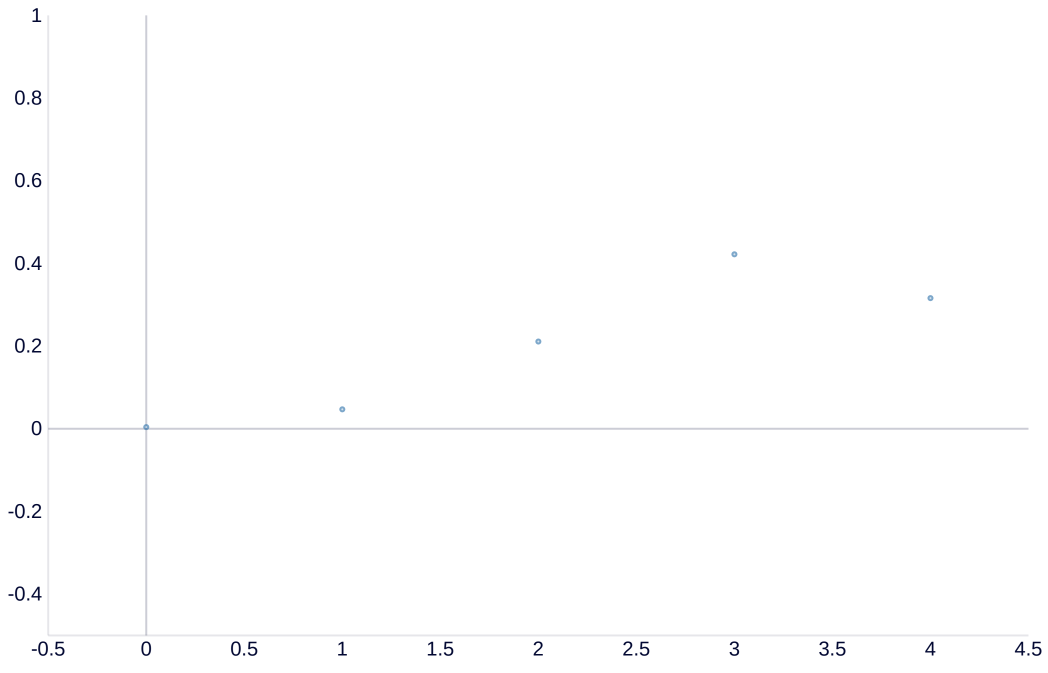
$$P(\theta|y,n) = P(y|\theta,n)P(\theta|n)/P(y|n)$$

$$P(\theta|y,n) = \binom{n}{y}\theta^y(1-\theta)^{n-y}\frac{1}{1}/\frac{1}{1+n}$$

$$P(\theta|y,n) = (1+n)\binom{n}{y}\theta^y(1-\theta)^{n-y}$$

b

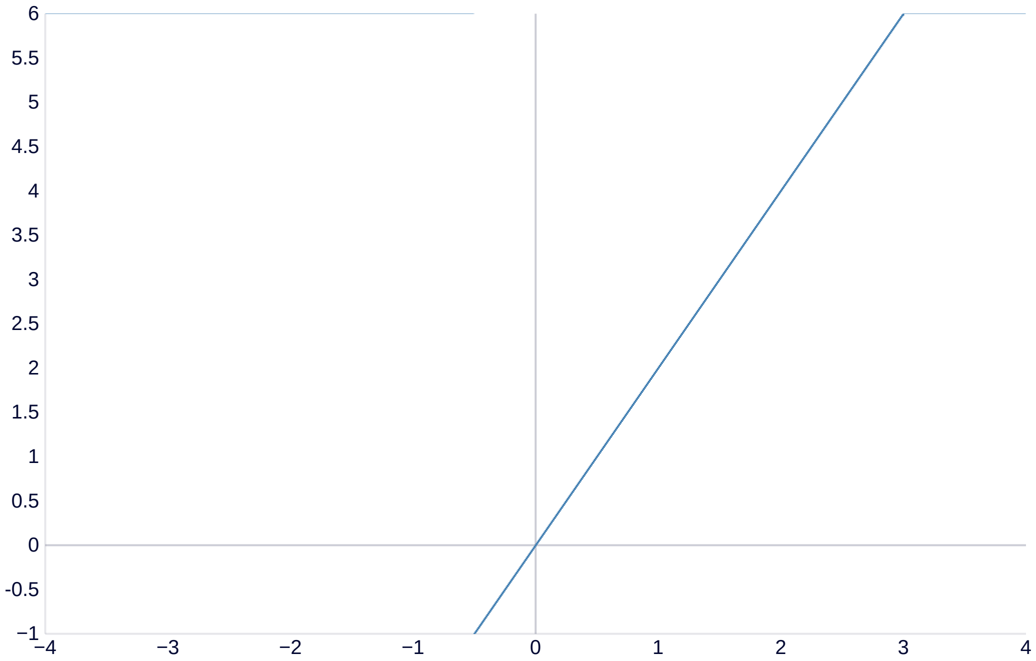
with $n = 4$, $\theta = 3/4$

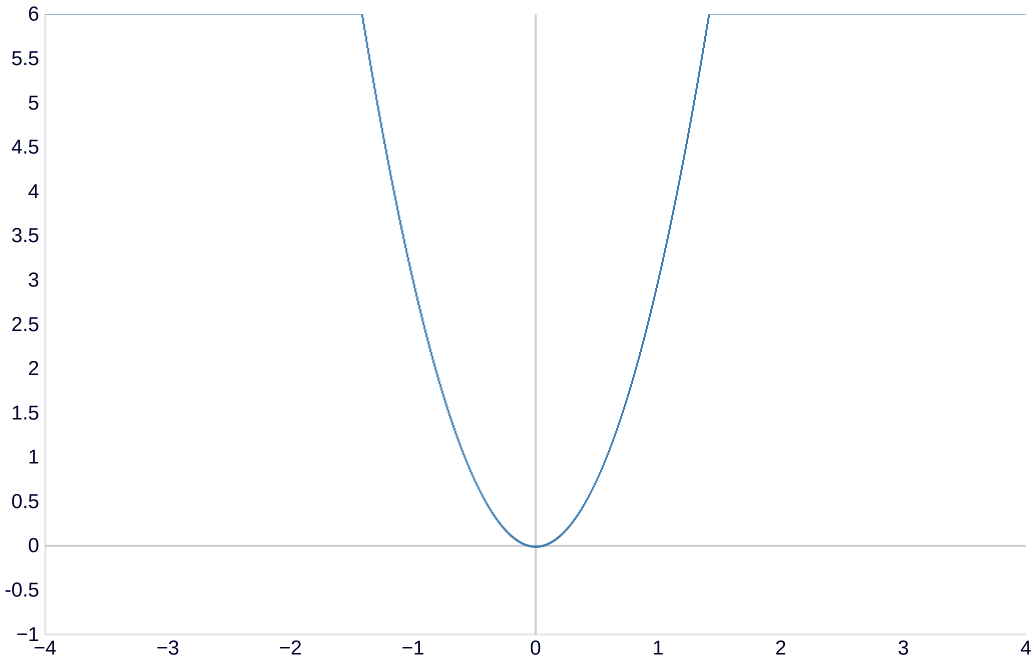


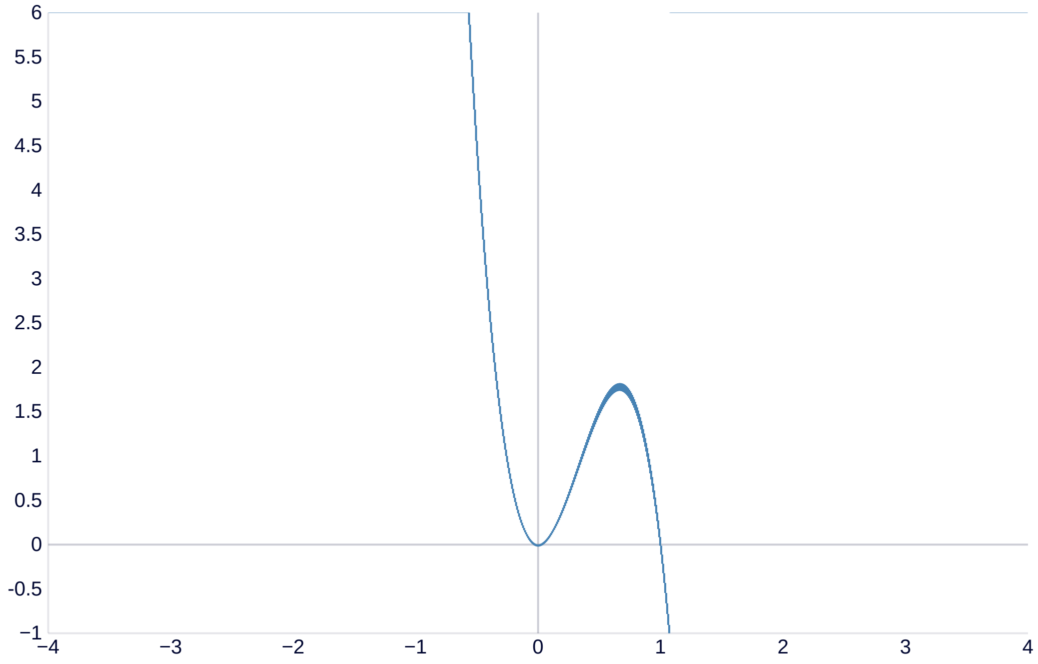
c

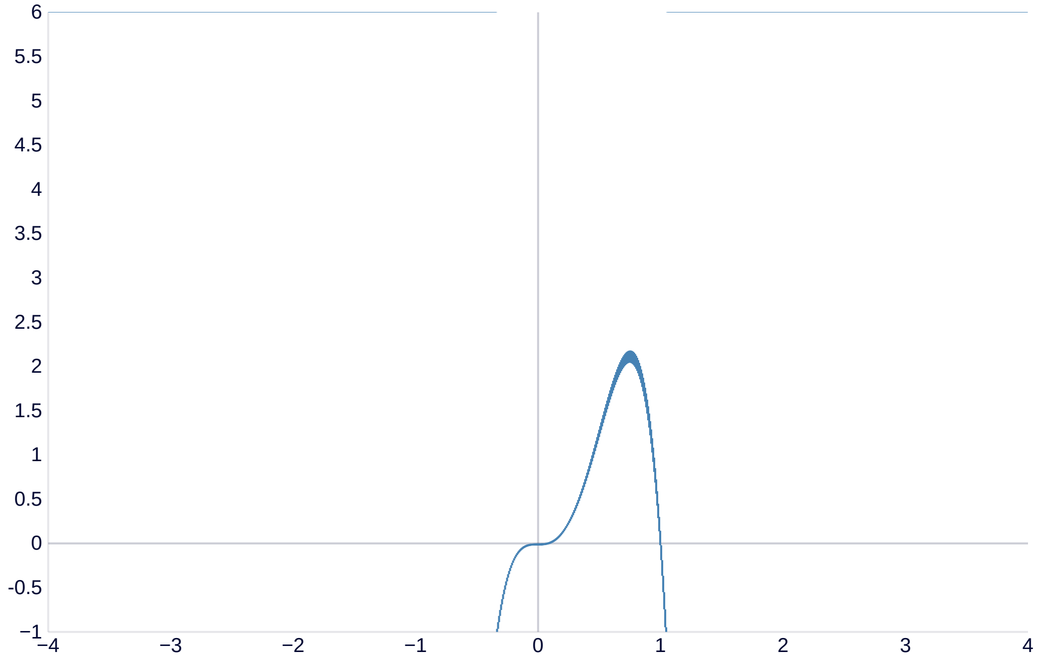
1. $y = 1$, $n = 1$
2. $y = 2$, $n = 2$
3. $y = 2$, $n = 3$

4. $y = 3$, $n = 4$


6
5.5
5
4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
-0.5
−1
−4
−3
−2
−1
0
1
2
3
4


6
5.5
5
4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
-0.5
−1
−4
−3
−2
−1
0
1
2
3
4





3

$$f(x|\mu, \sigma^2) = \frac{1}{(2\pi\sigma^2)^{1/2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$$

$$f(\vec{x}|\mu, \sigma^2) = \prod_{n=0}^{N} f(\vec{x}_n|\mu, \sigma^2)$$

$$f(\vec{x}|\mu, \sigma^2) = \prod_{n=0}^{N} \frac{1}{(2\pi\sigma^2)^{1/2}} e^{-\frac{1}{2\sigma^2}(\vec{x}_n-\mu)^2}$$

$$\ln f = \ln\left(\prod_{n=0}^{N} \frac{1}{(2\pi\sigma^2)^{1/2}} e^{-\frac{1}{2\sigma^2}(\vec{x}_n-\mu)^2}\right)$$

$$\ln f = \sum_{n=0}^{N} \ln\left(\frac{1}{(2\pi\sigma^2)^{1/2}} e^{-\frac{1}{2\sigma^2}(\vec{x}_n-\mu)^2}\right)$$

$$\ln f = \sum_{n=0}^{N} -\tfrac{1}{2}\ln(2\pi\sigma^2) - \tfrac{1}{2\sigma^2}(\vec{x}_n - \mu)^2$$

a

$$0 = \frac{\partial \ln f}{\partial \mu} = \frac{\partial}{\partial \mu}\sum_{n=0}^{N} -\tfrac{1}{2}\ln(2\pi\sigma^2) - \tfrac{1}{2\sigma^2}(\vec{x}_n - \mu)^2$$

$$= -\tfrac{1}{2\sigma^2}\frac{\partial}{\partial \mu}\sum_{n=0}^{N}(\vec{x}_n - \mu)^2$$

$$= -\tfrac{1}{2\sigma^2}\sum_{n=0}^{N} -2(\vec{x}_n - \mu)$$

$$= \tfrac{1}{\sigma^2}\sum_{n=0}^{N}(\vec{x}_n - \mu)$$

$$\implies 0 = \sum_{n=0}^{N}\vec{x}_n - \sum_{n=0}^{N}\mu$$

$$\implies 0 = \sum_{n=0}^{N}\vec{x}_n - N\mu$$

$$\implies \mu = \tfrac{1}{N}\sum_{n=0}^{N}\vec{x}_n$$

$\square$

b

$$0 = \frac{\partial \ln f}{\partial \sigma} = \frac{\partial}{\partial \sigma}\sum_{n=0}^{N} -\tfrac{1}{2}\ln(2\pi\sigma^2) - \tfrac{1}{2\sigma^2}(\vec{x}_n - \mu)^2$$

$$= \sum_{n=0}^{N}\frac{\partial}{\partial \sigma}\frac{1}{2}\ln(2\pi\sigma^2) + \sum_{n=0}^{N}\frac{\partial}{\partial \sigma}\frac{1}{2\sigma^2}(\vec{x}_n - \mu)^2$$

$$= \sum_{n=0}^{N}\frac{1}{2}\frac{4\pi\sigma}{2\pi\sigma^2} + \sum_{n=0}^{N}\frac{-2}{2\sigma^3}(\vec{x}_n - \mu)^2$$

$$= \sum_{n=0}^{N}\frac{1}{\sigma} - \sum_{n=0}^{N}\frac{1}{\sigma^3}(\vec{x}_n - \mu)^2$$

$$= \frac{N}{\sigma} - \frac{1}{\sigma^3}\sum_{n=0}^{N}(\vec{x}_n - \mu)^2$$

$$\implies \frac{N}{\sigma} = \frac{1}{\sigma^3}\sum_{n=0}^{N}(\vec{x}_n - \mu)^2$$

$$\implies N\sigma^2 = \sum_{n=0}^{N}(\vec{x}_n - \mu)^2$$

$$\implies \sigma^2 = \frac{1}{N}\sum_{n=0}^{N}(\vec{x}_n - \mu)^2$$

$\square$

4

a

$X \in \{C_1, C_2\}$

$P(X = C_1) = 2P(X = C_2)$

$$1 = \sum_{x \in \{C_1, C_2\}} P(X = x)$$
$$= P(X = C_1) + P(X = C_2)$$
$$= 3P(X = C_2)$$
$$\implies \begin{cases} P(X = C_1) = \frac{2}{3} \\ P(X = C_2) = \frac{1}{3} \end{cases}$$

	C_1	C_2
X	$\frac{2}{3}$	$\frac{1}{3}$

$\square$

b

Assuming $\mu_1 < \mu_2$,

$$E = P(X_1 > \theta \cap X \in C_1) + P(X_2 < \theta \cap X \in C_2)$$
$$E = P(X_1 > \theta)P(X \in C_1) + P(X_2 < \theta)P(X \in C_2)$$

$$X_n \sim N(\mu_n, \sigma_n)$$
$$P(X_1 > \theta) = 1 - F_{X_1}(\theta)$$
$$P(X_2 < \theta) = F_{X_2}(\theta)$$

$$F_N(x) = \Phi(x)$$
$$F_{X_n}(x) = F_N(\frac{x-\mu}{\sigma}) = \Phi(\frac{x-\mu_n}{\sigma_n})$$

$$E = \frac{2}{3}(1 - \Phi(\frac{\theta-\mu_1}{\sigma_1})) + \frac{1}{3}(\Phi(\frac{\theta-\mu_2}{\sigma_2}))$$
$$= 2/3 - 2\Phi(\frac{\theta-\mu_1}{\sigma_1})/3 + \Phi(\frac{\theta-\mu_2}{\sigma_2})/3$$

$\square$

c

Let $N(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$
$\frac{d\Phi}{dz} = N(z)$

$0 = \frac{\partial E}{\partial \theta} = 2/3 - 2\Phi(\frac{\theta - \mu_1}{\sigma_1})/3 + \Phi(\frac{\theta - \mu_2}{\sigma_2})/3$
$= N(\frac{\theta - \mu_2}{\sigma_2})/3\sigma_2 - 2N(\frac{\theta - \mu_1}{\sigma_1})/3\sigma_1$

$\implies N(\frac{\theta-\mu_2}{\sigma_2})/\sigma_2 = 2N(\frac{\theta-\mu_1}{\sigma_1})/\sigma_1$
$\implies \frac{1}{\sigma_2} e^{-\frac{1}{2\sigma_2^2}(x-\mu_2)^2} = \frac{1}{\sigma_1} e^{-\frac{1}{2\sigma_1^2}(x-\mu_1)^2}$
$\implies \ln(\frac{\sigma_1}{\sigma_2}) = \frac{1}{2\sigma_2^2}(\theta - \mu_2)^2 - \frac{1}{2\sigma_1^2}(\theta - \mu_1)^2$
$\implies \ln(\frac{\sigma_1}{\sigma_2}) = \frac{1}{2\sigma_2^2}(\theta^2 - \mu_2\theta + \mu_2^2) - \frac{1}{2\sigma_1^2}(\theta^2 - \mu_1\theta + \mu_1^2)$
$\implies 0 = (\frac{1}{2\sigma_2^2} - \frac{1}{2\sigma_1^2})\theta^2 + (\frac{1}{2\sigma_1^2}\mu_1 - \frac{1}{2\sigma_2^2}\mu_2)\theta + (\frac{1}{2\sigma_2^2}\mu_2^2 - \frac{1}{2\sigma_1^2}\mu_1^2 - \ln(\frac{\sigma_1}{\sigma_2}))$

Where
$A = \frac{1}{2\sigma_2^2} - \frac{1}{2\sigma_1^2}$
$B = \frac{1}{2\sigma_1^2}\mu_1 - \frac{1}{2\sigma_2^2}\mu_2$
$C = \frac{1}{2\sigma_2^2}\mu_2^2 - \frac{1}{2\sigma_1^2}\mu_1^2 - \ln(\frac{\sigma_1}{\sigma_2})$

$\theta = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$
$\square$