4.5

14

If today is Tuesday, what day of the week will it be 1,000 days from today?

✓ **Answer**

If the zeroth day of the week is Saturday, Tuesday will be the 3rd day of the week

Since the day of week is cyclical every 7 days, we can use $\mod 7$

$3 + 1000 \mod 7 = 2$
$\square$

The second day of the week in this case would be Monday, which is 1000 days after Tuesday.

21

Suppose b is any integer. If $b \mod 12 = 5$, what is $8b \mod 12$? In other words, if division of b by 12 gives a remainder of 5, what is the remainder when $8b$ is divided by 12? Your solution should show that you obtain the same answer no matter what integer you start with.

✓ **Answer**

$\exists N \in \mathbf{Z}$
$b = 12N + 5$ Congruence Relation
$8b = 12(8N) + 40$
$8b = 12(8N + 3) + 4$
$8N + 3 = M \in \mathbf{Z}$
$8b = 12M + 4$
$8b \mod 12 = 4$ Congruence Relation
$\square$

24

Prove that for all integers m and n, if $m \mod 5 = 2$ and $n \mod 5 = 1$ then $mn \mod 5 = 2$
.

✓ **Answer**

$\exists M, N \in \mathbf{Z}$
$m = 5N + 2$ Congruence Relation
$n = 5M + 1$ Congruence Relation
$mn = (5N + 2)(5M + 1)$
$mn = 5(5NM + N + 2M) + 2$
$5NM + N + 2M = P \in \mathbf{Z}$
$mn = 5P + 2$
$mn \mod 5 = 2$ Congruence Relation
$\square$

38

Prove the statement

For every integer m, $m^2 = 5k$, or $m^2 = 5k + 1$, or $m^2 = 5k + 4$ for some integer k.

Answer

$\exists B \in [0, 4] \in \mathbf{Z}$
$\exists N \in \mathbf{Z}$
$m \in 5N + B$
$m^2 \in 5(5N^2 + 2B) + B^2$
$B^2 \in \{0, 1, 4, 16, 25\}$
$\exists M \in \mathbf{Z}$
$B^2 + 5M \in \{0, 1, 4\} = B'^2$
$m^2 \in 5(5N^2 + 2B + M) + B'^2$
$m^2 \mod 5 \in B'^2$
$m^2 \mod 5 \in \{0, 1, 4\}$
$\square$

$\exists k \in \mathbf{Z}$
$\therefore m^2 \in \{5k, 5k + 1, 5k + 4\}$ Congruence Relation

4.6

7

If k is an integer, what is $\lfloor k + \frac{1}{2} \rfloor$? Why?

Answer

If k is an integer, then
$k \leq k + \frac{1}{2} < k + 1$
$\therefore \lfloor k + \frac{1}{2} \rfloor = k$

19-20

Some of the statements in 19-20 are true and some are
false. Prove each true statement and find a counterexample for each false statement, but do not use theorem 4.6.1 in your proofs.

19

For every real number x, $\lceil x - 1 \rceil = \lceil x \rceil - 1$

✓ **Answer**

$\exists n \in \mathbf{Z}$
$n = \lceil x \rceil$
$n - 1 < x \leq n$
$n - 2 \leq x - 1 < n - 1$
$n - 2 \in \mathbf{Z}$
$\lceil x - 1 \rceil = n - 1 = \lceil x \rceil - 1$
$\square$

20

For all real numbers x and y, $\lceil xy \rceil = \lceil x \rceil \cdot \lceil y \rceil$

✓ **Answer**

False.
Let $x = 5.5$, $y = 5.5$
$\lceil 30.25 \rceil \stackrel{?}{=} \lceil 5.5 \rceil \lceil 5.5 \rceil$
$31 \stackrel{?}{=} (6)(6)$
$31 \neq 36$
$\square$

4.7

4

Use proof by contradiction to show that for every integer m, $7m + 4$ is not divisible by 7.

✓ **Answer**

$\forall m \in \mathbf{Z}$
If $7|7m + 4$
$7|7m$ as $m \in \mathbf{Z}$
$7|4 \equiv \mathbf{c}$
$\square$

6

Carefully formulate the negations of the statement. Then prove it by contradiction.

There is no greatest negative real number.

✓ **Answer**

There is a greatest negative real number.
Let n be the greatest negative real number.
Let $m = \frac{n}{2}$
m is negative.
$n < m$ as both n and m are negative

If n were the greatest negative real number, then m could not exist.
m exists.
$\square$

17

Prove the statement by contradiction

For all prime numbers a, b, c, $a^2 + b^2 \neq c^2$

✓ **Answer**

There exists prime numbers a, b, c such that $a^2 + b^2 = c^2$
$a^2 = c^2 - b^2$
$a^2 = (c + b)(c - b)$
If a, b, c were prime,
Case 1: then $c + b = c - b = a$
$\begin{cases} a = c \\ b = 0 \end{cases}$
b cannot be 0 and prime.

Case 2: then $\begin{cases} c+b=1 \\ c-b=a^2 \end{cases}$

Since $a, b, c \in \mathbf{Z}^+$

$\begin{cases} b=1 \\ c=0 \end{cases}$ or $\begin{cases} c=1 \\ b=0 \end{cases}$

b, c cannot be 0 and prime

Case 3: then $\begin{cases} c-b=1 \\ c+b=a^2 \end{cases}$

$c=b+1$

$2b+1=a^2$

$2b=(a+1)(a-1)$

Case 3a:

$\begin{cases} a+1=2 \\ b=a-1 \end{cases}$

a cannot be 1 and prime.

Case 3b:

$\begin{cases} a-1=2 \\ b=a+1 \end{cases}$

$\begin{cases} a=3 \\ b=4 \end{cases}$

b cannot be 4 and prime.

Case 3c:

$\begin{cases} a+1=1 \\ 2b=a-1 \end{cases}$

a cannot be 0 and prime.

Case 3d:

$\begin{cases} a-1=1 \\ 2b=a+1 \end{cases}$

$\begin{cases} a=2 \\ 2b=3 \end{cases}$

b cannot be 1.5 and prime.

$\square$

4.8

14

Determine whether the statement is true or false. Prove it if it is true and disprove it if it is false.

The sum of any two positive irrational numbers is irrational.

✓ Answer

$\forall a, b \in \mathbf{I}$, $a + b \in \mathbf{I}$

False.
$\exists a, b \in \mathbf{I}$, $a + b \notin \mathbf{I}$
Let $A = \lceil a \rceil \in \mathbf{Z}$
$b = A - a \in \mathbf{I}$ by sum of rational and irrational numbers
$a + b = A \in \mathbf{Z} \notin \mathbf{I}$
$\square$

23

Prove that for any integer a, $9 \nmid (a^2 - 3)$.

✓ Answer

$\forall a \in \mathbf{Z}$
$\exists N \in \mathbf{Z}$
$\exists B \in \mathbf{Z} \in [0, 8]$
$a = 9N + B$
$a^2 = 9(9N + 2B) + B^2$
$B^2 \in \{0, 1, 4, 9, 16, 25, 36, 49, 64\}$
$\exists M \in \mathbf{Z} \in [0, 8]$
$B^2 \in 9M + \{0, 1, 4, 7\} = B'^2$
$a^2 = 9(9N + 2B + M) + B'^2$
$B'^2 - 3 \in 9M + \{1, 4, 6, 7\} = B''^2$
$a^2 - 3 = 9(9N + 2B + M') + B''^2$
$a^2 - 3 \mod 9 \in \{1, 4, 6, 7\}$
$0 \notin \{1, 4, 6, 7\}$
$9 \nmid (a^2 - 3)$
$\square$

4.10

15-16

Use the euclidean algorithm to hand-calculate the greatest common divisors of each of the pairs of integers in 15–16.

15

832 and 10933

✓ **Answer**

$A = 10933,\ B = 832$
$A \mod B = 117$
$A = 832,\ B = 117$
$A \mod B = 13$
$A = 117,\ B = 13$
$A \mod B = 0$
$A = 13,\ B = 0$

$\gcd(10933, 832) = 13$
$\square$

16

4131 and 2431

✓ **Answer**

$A = 4131,\ B = 2431$
$A \mod B = 1700$
$A = 2431,\ B = 1700$
$A \mod B = 731$
$A = 1700,\ B = 731$
$A \mod B = 238$
$A = 731,\ B = 238$
$A \mod B = 17$
$A = 238,\ B = 17$
$A \mod B = 0$
$A = 17,\ B = 0$

$\gcd(4131, 2431) = 17$
$\square$

5.1

6

Write the first four terms of the sequence

$f_n = \left\lfloor \frac{n}{4} \right\rfloor \cdot 4$ for every integer $n \geq 1$

✓ **Answer**

$f_1 = 0$
$f_2 = 0$
$f_3 = 0$
$f_4 = 4$
$\square$

60

Rewrite as a single summation or product

$$2 \cdot \sum_{k=1}^{n} (3k^2 + 4) + 5 \cdot \sum_{k=1}^{n} (2k^2 - 1)$$

Answer

$$= \sum_{k=1}^{n} (6k^2 + 8) + \sum_{k=1}^{n} (10k^2 - 5)$$
$$= \sum_{k=1}^{n} (16k^2 + 3)$$
$\square$