PHYS 122 HW 10

2

Electromagnetic waves. An electromagnetic plane wave propagates along the positive x-axis. The electric and magnetic fields are given by

$\vec{E} = \epsilon \cos(kx - \omega t)\hat{j}$
$\vec{B} = \frac{\epsilon}{c}\cos(kx - \omega t)\hat{k}$

Here $c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$ is the speed of light in vacuum. Note that the electric and magnetic field amplitudes are related by the ratio c, as required by Maxwell's equations in a vacuum. The frequency ω and the wave-vector k are related via $\omega = kc$.

a

Calculate u, the energy density of the field. Also calculate $\langle u \rangle$, the time averaged energy density defined as

$$\langle u \rangle = \frac{1}{T}\int_0^T dt\ u$$

✓ Answer

$u = \frac{\epsilon_0}{2}E^2 + \frac{1}{2\mu_0}B^2$
$u = \frac{\epsilon_0}{2}(\epsilon \cos(kx - \omega t))^2 + \frac{1}{2\mu_0}(\frac{\epsilon}{c}\cos(kx - \omega t))^2$
$u = \frac{\epsilon_0}{2}\epsilon^2 \cos^2(kx - \omega t) + \frac{1}{2\mu_0}\frac{\epsilon^2}{c^2}\cos^2(kx - \omega t)$
$u = \frac{\epsilon_0}{2}\epsilon^2 \cos^2(kx - \omega t) + \frac{\epsilon_0}{2}\epsilon^2 \cos^2(kx - \omega t)$
$u = \epsilon_0 \epsilon^2 \cos^2(kx - \omega t)$

$\langle u \rangle = \frac{1}{T}\int_0^T dt\ \epsilon_0 \epsilon^2 \cos^2(kx - \omega t)$
$\langle u \rangle = \frac{1}{2}\epsilon_0 \epsilon^2$

b

Calculate the Poynting vector $\vec{S}$ and the time averaged Poynting vector $\langle \vec{S} \rangle$. Recall that the Poynting vector corresponds to the energy per unit area delivered by the wave to a surface perpendicular to the direction of wave propagation, $\hat{i}$. The magnitude S is synonymous with the intensity of the wave sometimes denoted I.

✓ Answer

$\vec{S} = \frac{1}{\mu_0}\vec{E} \times \vec{B}$
$\vec{S} = \frac{1}{\mu_0}\epsilon\cos(kx - \omega t)\frac{\epsilon}{c}\cos(kx - \omega t)\ \hat{i}$
$\vec{S} = \frac{\epsilon^2}{c\mu_0}\cos^2(kx - \omega t)\ \hat{i}$
$\vec{S} = c\epsilon_0\epsilon^2\cos^2(kx - \omega t)\ \hat{i}$

$\langle\vec{S}\rangle = \frac{1}{T}\int_0^T dt\ c\epsilon_0\epsilon^2\cos^2(kx - \omega t)\hat{i}$
$\langle\vec{S}\rangle = \frac{c\epsilon_0\epsilon^2}{2}\hat{i}$

c

Calculate the momentum density Π and the time averaged momentum density $\langle\Pi\rangle$.

✓ **Answer**

$\vec{\Pi} = \frac{\vec{S}}{c^2}$
$\vec{\Pi} = \frac{\epsilon_0\epsilon^2}{c}\cos^2(kx - \omega t)\ \hat{i}$

$\langle\vec{\Pi}\rangle = \frac{\epsilon_0\epsilon^2}{2c}\hat{i}$

d

Calculate the time averaged radiation pressure force exerted by the wave on a surface perpendicular to $\hat{i}$ and with area A assuming that the surface is:

i

Perfectly absorbing

✓ **Answer**

$P = \frac{\langle S\rangle}{c}$
$F = PA = \frac{\langle S\rangle A}{c}$
$F = \frac{\epsilon_0\epsilon^2 A}{2}\hat{i}$

ii

Perfectly reflecting

✓ **Answer**

$P = \frac{2\langle S\rangle}{c}$
$F = PA = \frac{2\langle S\rangle A}{c}$
$F = \epsilon_0 \epsilon^2 A\hat{i}$

e

Dimensional analysis

i

A professor points out that the time averaged intensity $I = \langle S\rangle$ is related to the energy density by $I = \langle u\rangle c$. However, somewhat later he tells the class that $I = \langle u\rangle/c$. Use dimensional analysis to figure out which one he really meant.

Answer

$I \in \frac{J}{sm^2}$

$\langle u\rangle c \in \frac{J}{sm^2}$
$\langle u\rangle/c \in \frac{Js}{m^4}$

$\therefore I = \langle u\rangle c$

ii

The professor similarly manages to confuse the issue of whether $\langle \Pi\rangle = \langle u\rangle c$ or $\langle \Pi\rangle = \langle u\rangle/c$. Use dimensional analysis to determine which one is right.

Answer

$\langle \Pi\rangle \in \frac{Js}{m^4}$

$\langle u\rangle c \in \frac{J}{sm^2}$
$\langle u\rangle/c \in \frac{Js}{m^4}$

$\therefore \langle \Pi\rangle = \langle u\rangle/c$

f

Sunlight has a time averaged intensity $I = 1.36\ kW/m^2$ in the neighborhood of the Earth (it's a lot weaker near Pluto).

i

Calculate the electric field amplitude E and the magnetic field amplitude E/c for a plane wave with the same intensity as sunlight. Give your answer in terms of V/m and T respectively.

✓ **Answer**

$I = \frac{c\epsilon_0 E^2}{2}$
$E = \sqrt{\frac{2I}{c\epsilon_0}}$
$E = \sqrt{\frac{2(1360)}{299792458(8.854\times10^{-12})}}$
$E = 1.01 \times 10^3 \ \frac{V}{m}$

$\frac{E}{c} = \frac{1.01\times10^3}{299792458}$
$\frac{E}{c} = \frac{1.01\times10^3}{299792458}$
$\frac{E}{c} = 3.38 \times 10^{-6} \ T$

ii

Calculate the time averaged energy density of sunlight. Give your answer in J/m^3.

✓ **Answer**

$\langle u \rangle = \frac{I}{c}$
$\langle u \rangle = \frac{1.36\times10^3}{299792458}$
$\langle u \rangle = 4.54 \times 10^{-6} \ \frac{J}{m^3}$

iii

Calculate the time averaged momentum density of sunlight. Give your answer in kg/m^2s.

✓ **Answer**

$\langle \Pi \rangle = \frac{I}{c^2}$
$\langle \Pi \rangle = \frac{1.36\times10^3}{299792458^2}$
$\langle \Pi \rangle = 1.51 \times 10^{-14} \ \frac{kg}{m^2 s}$

iv

Calculate the radiation pressure force on a perfectly reflecting solar sail of area equal to one square mile. Give your answer in N.

✓ **Answer**

$F = 2\langle u \rangle A$
$F = 2\frac{I}{c}A$

$A = 1\ mi^2 = 2.589988 \times 10^6\ m^2$
$F = 23.5\ N$

4

Dynamics of a Comet's Tail: Cometary nuclei are mountain sized globs of dust and ice. As the nucleus approaches the sun, the surface ices vaporize and tiny particles of dust and ice are blown off to form the brilliant cometary tail.

a

Let L denote the total power radiated by the sun. Calculate I, the intensity of sunlight, at a distance R from the sun.

Hint: L is the total energy per second radiated by the sun. The intensity is the energy per second per unit area. To calculate the intensity imagine that the sun is surrounded by a sphere of radius R. How much energy passes this sphere per second? How much energy passes the sphere per second per unit area?

✓ **Answer**
$I = \frac{L}{4\pi R^2}$

b

Model the tail particles as dark spheres of radius r which completely absorb all incident solar radiation (together with the field momentum it carries). Calculate the magnitude of the force exerted by solar radiation on the tail particles. Give your answer in terms of L, R, r, c.

Hint: The force is the radiation pressure of sunlight multiplied by the cross section area of the tail particles.

✓ **Answer**
$A = \pi r^2$
$F = \frac{IA}{c}$
$F = \frac{Lr^2}{4cR^2}$

c

Calculate the radius of a dust particle for which radiation pressure forces precisely balance the sun's gravitational attraction. Give your answer in terms of G, M, ρ, c, L. Where ρ is the density of silicate dust and M is the mass of the sun.

✓ Answer

$F_G = G\frac{m_1 m_2}{R^2}$
$F_G = G\frac{4M\rho\pi r^3}{3R^2}$
$\frac{Lr^2}{4cR^2} = G\frac{4M\rho\pi r^3}{3R^2}$
$r = \frac{3L}{16GM\rho\pi c}$

d

Particles of radius smaller than the critical value calculated in part (c) will be blown outward by radiation pressure. Calculate the critical radius in μm. Take $\rho = 2.5 \times 10^3\ kg/m^3$ and $L = 4 \times 10^{26}\ W$. Also $M = 2 \times 10^{30}\ kg$, $c = 3.0 \times 10^8\ m/s$, and $G = 6.67 \times 10^{-11}$ S.I. units.

✓ Answer

$r = \frac{3L}{16GM\rho\pi c}$
$r = 0.24\ \mu m$

5

Inside an infinitely long cylinder of radius R, there is a uniform electric field that is changing overtime as

$$\vec{E}(t) = E_0 \sin(\omega t)\hat{i}$$

a

Use Ampere-Maxwell's Law to calculate the magnetic field at the surface of the cylinder (distance R from the axis). Give magnitude and direction (seen from the top).

✓ Answer

$\oint d\vec{s} \cdot \vec{B} = \mu_0 I + \mu_0\epsilon_0 \frac{d}{dt}(\int \vec{E} \cdot \hat{n}\ da)$
$\oint d\vec{s} \cdot \vec{B} = \mu_0 I + \mu_0\epsilon_0 \frac{d}{dt}(\int E_0 \sin(\omega t)\ da)$
$\oint d\vec{s} \cdot \vec{B} = \mu_0 I + \mu_0\epsilon_0 \frac{d}{dt}\pi R^2 E_0 \sin(\omega t)$
$\oint d\vec{s} \cdot \vec{B} = \mu_0 I + \mu_0\epsilon_0 \pi\omega R^2 E_0 \cos(\omega t)$
$2\pi R\vec{B} = \mu_0\epsilon_0\pi\omega R^2 E_0 \cos(\omega t)$
$\vec{B} = \frac{1}{2}\mu_0\epsilon_0\omega R E_0 \cos(\omega t)$ counter clockwise

b

Repeat for a point that is at a distance $2R$ from the axis of the cylinder.

✓ **Answer**

$4\pi R\vec{B} = \mu_0\epsilon_0\pi\omega R^2 E_0 \cos(\omega t)$

$\vec{B} = \frac{1}{4}\mu_0\epsilon_0\omega R E_0 \cos(\omega t)$ counter clockwise