7

PHYS 122 HW 7

1

Axial current on a cylindrical surface

A current i flows along the surface of a cylinder parallel to the axis of the cylinder. The radius of the cylinder is a and its length is presumed to be infinite.

a

Current density: What is the current per unit length, K? (Hint: think about the units that current per unit length must have.) Give your answer in terms of i and a.

> ✓ **Answer**
>
> $K = \frac{i}{2\pi a}$

b*

Field direction: At any point there are three possible directions along which the magnetic field can point in principle. These directions are “radial” (perpendicular to the axis of the cylinder and outwards), “axial” (along the axis of the cylinder) or “azimuthal” (perpendicular to the other two directions). The unit vectors along these directions are denoted $\hat{e}_r, \hat{e}_z = \hat{k}, \hat{e}_\theta$ respectively.

c

Gaussian analysis: A professor assumes that the magnetic field due to the cylinder described above points in the radial direction. Explain why this assumption is untenable.

Hint: Consider a suitable Gaussian surface and apply Gauss’s law for magnetism.

> ✓ **Answer**
>
> With a Gaussian surface being a circle perpendicular to the height of the cylinder, we know that
>
> $\Phi_B = 0$ by the Gauss Law of Magnetism.
>
> If the magnetic field were to be radial, by symmetry, the flux of this circle must be

$2\pi r B$, which is non-zero unless $r = 0$, which is not true

d

Amperian analysis: Now assume that the magnetic field points in the azimuthal direction. Consider the outer Amperian loop shown in the figure.

i

What is the current enclosed by the loop?

✓ **Answer**

i

ii

What is the integral of $\vec{B}$ around the loop? Give your answer in terms of B, the magnitude of B at a distance r from the axis, a and r.

✓ **Answer**

$\oint d\vec{l} \cdot \vec{B} = 2\pi r B(r)$

iii

Apply Ampere's law to determine B in terms of μ_0, i, a and r.

✓ **Answer**

$\oint d\vec{l} \cdot \vec{B} = \mu_0 i$
$\mu_0 i = 2\pi r B(r)$
$B(r) = \frac{\mu_0 i}{2\pi r}$
$\square$

e

Repeat part (d) for the inner Amperian loop shown in the figure.

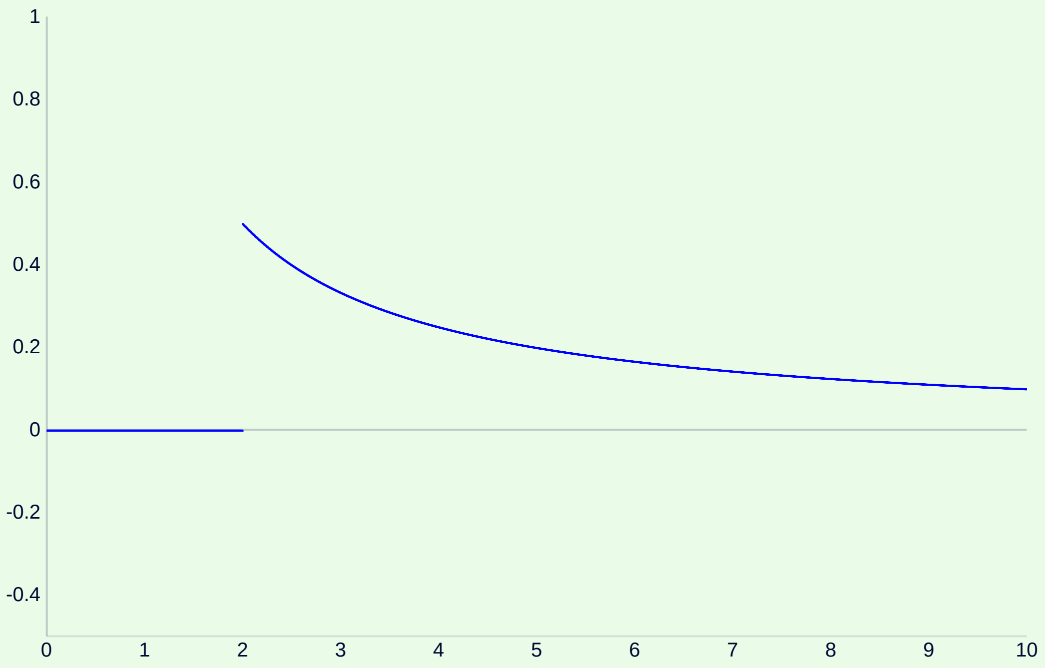
✓ **Answer**

$$i = 0$$
$$B(r) = 0$$

f

Make a plot of B (the magnitude of the magnetic field) as a function of r (the distance from the axis).

✓ Answer



2

Circulating current around a cylindrical surface

Current circulates around the circumference of a cylinder. The current per unit length is K. The radius of the cylinder is a; its length is assumed to be infinite

a*

Direction of field: As in the previous problem we can rule out a radial component for the magnetic field using Gauss's law for magnetism. We can also rule out that it has an azimuthal component by using Ampere's law as in the previous problem. Thus the magnetic field must point along the z-axis. We also assume that the magnetic field is zero outside the cylinder.

b

Consider the Amperian loop shown in the figure.

i

What is the integral of the magnetic field around the loop? Give your answer in terms of l, w and $\vec{B}$. Here $\vec{B} = B\hat{k}$ is the magnetic field along the inner vertical edge of the Amperian loop.

✓ **Answer**

$\oint d\vec{l} \cdot \vec{B} = 2Bl$

ii

What is the current that passes through the loop? Give your answer in terms of K, l and w.

✓ **Answer**

Kl

iii

Apply Ampere's law to determine B in terms of μ_0 and K.

✓ **Answer**

$B = \frac{\mu_0 K}{2}$

c*

Realizations: One way to realize a cylindrical sheath with circulating current is to stack rings. If N rings form a stack of height L and each ring has a circulating current i, then the current per unit length $K = N\frac{i}{L}$. More common than a stack of rings is to coil a wire into a helix (a corkscrew shape). Such coils are called solenoids and are important components of circuits, transformers, generators and machines. If the solenoid with current i has n turns per unit length it can be modeled as a cylindrical sheath of current with current density $K = ni$. When a strong magnetic field is applied to a type II super-conductor vortices (such as those depicted in fig 1) form. A vortex is a whirlpool of current with a magnetic field along its axis. The geometry analyzed here is a crude caricature of a vortex.

3

Cylindrical wire and wire with a cylindrical bore.

A cylindrical wire of radius a carries a current i out of the page. The current is uniformly distributed over the cross section of the wire

a

Calculate j, the current density, defined as the current per unit area of cross section.

Answer

$j = \frac{i}{\pi a^2}$

b

Apply Ampere's law to the outer loop shown in the figure in order to calculate the magnetic field $\vec{B}$ outside the wire (for $r > a$ where r is the distance from the axis of the wire).

Answer

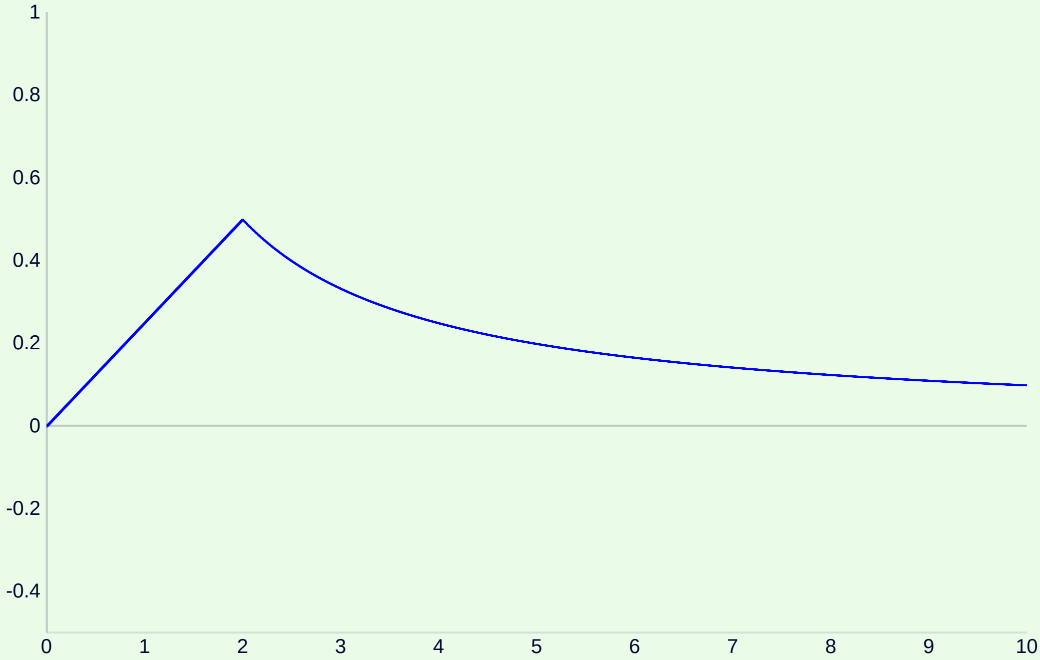
$\oint d\vec{l} \cdot \vec{B} = 2\pi r B(r)$
$= \mu_0 i$
$B(r) = \frac{\mu_0 i}{2\pi r}$
$\square$

c

Apply Ampere's law to the inner loop shown in the figure in order to calculate the magnetic field $\vec{B}$ inside the wire (for $0 < r < a$).

Answer

$\oint d\vec{l} \cdot \vec{B} = 2\pi r B(r)$
$= \frac{\mu_0 r^2 i}{a^2}$
$B(r) = \frac{\mu_0 r i}{a^2 2\pi}$
$\square$



d

Now consider a cylindrical wire in which a cylindrical hole has been excavated. The wire has a radius a; the cylindrical bore has a radius $\frac{a}{2}$. As shown in the figure the axis of the bore is parallel to the axis of the wire but offset from it. Assume that the wire carries a current i out of the page and that the current is uniformly distributed over the crescent shaped cross section of the wire (shown shaded in the figure).

Repeat part (a) for this wire.

✓ **Answer**

$j = \frac{4i}{3\pi a^2}$

e

Determine the magnetic field along the y-axis for

✓ **Answer**

Due to the unbored cylinder:
$I = \frac{4i}{3}$

$r > a$:
$2\pi r B(r) = \frac{\mu_0 4i}{3}$
$B(r) = \frac{2\mu_0 i}{3\pi r}$

$r < a$:

$I = \frac{4ir^2}{3a^2}$

$2\pi r B(r) = \frac{\mu_0 4ir^2}{3a^2}$

$B(r) = \frac{2\mu_0 ir}{3\pi a^2}$

$$B(r) = \frac{2\mu_0 i(\min(r,a))^2}{3\pi a^2 r}$$

Due to the inverse of the bore:

$I = -\frac{i}{3}$

$r > \frac{a}{2}$:

$2\pi r B(r) = -\frac{\mu_0 i}{3}$

$B(r) = -\frac{\mu_0 i}{6\pi r}$

$r < \frac{a}{2}$:

$I = -\frac{4ir^2}{3a^2}$

$2\pi r B(r) = -\frac{\mu_0 4ir^2}{3a^2}$

$B(r) = -\frac{2\mu_0 ir}{3\pi a^2}$

$$B(r) = -\frac{2\mu_0 i\left(\min\left(r,\frac{a}{2}\right)\right)^2}{3\pi a^2 r}$$

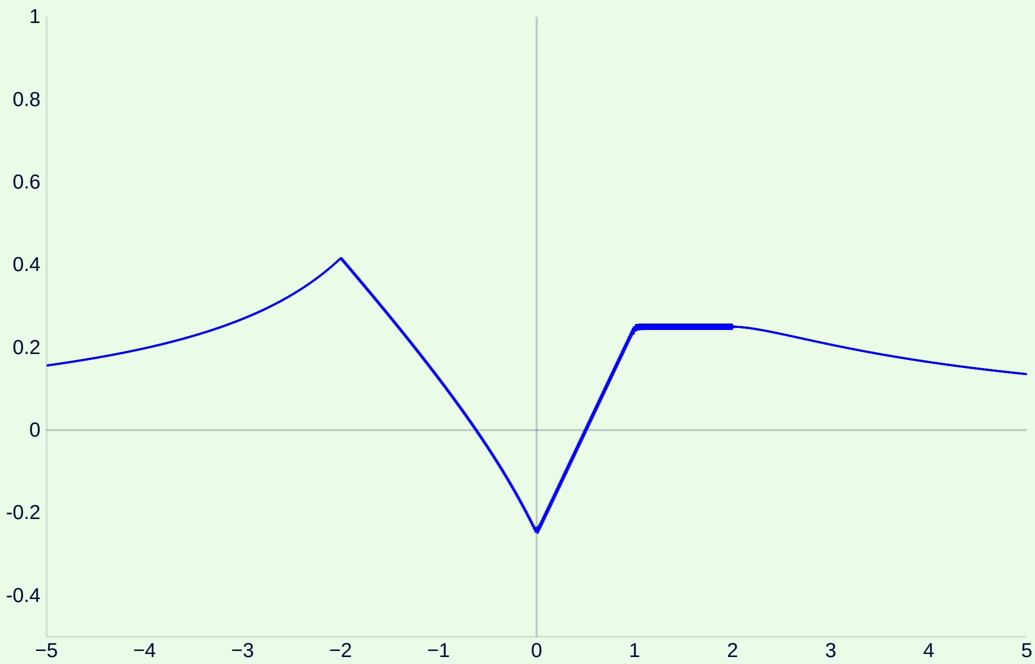
Superposition of both effects:

$$B(r_1, r_2) = \frac{2\mu_0 i(\min(r_1,a))^2}{3\pi a^2 r_1} - \frac{2\mu_0 i\left(\min\left(r_2,\frac{a}{2}\right)\right)^2}{3\pi a^2 r_2}$$

Where

$r_1 = \sqrt{x^2 + y^2}$

$r_2 = \sqrt{x^2 + \left(y - \frac{a}{2}\right)^2}$



i

$y > a$.

✓ **Answer**

$$B(r_1, r_2) = \frac{2\mu_0 i(\min(r_1, a))^2}{3\pi a^2 r_1} - \frac{2\mu_0 i\left(\min\left(r_2, \frac{a}{2}\right)\right)^2}{3\pi a^2 r_2}$$
$$r_1 = |y|$$
$$r_2 = |y - \tfrac{a}{2}|$$

$$B(r_1, r_2) = \frac{2\mu_0 i}{3\pi y} - \frac{2\mu_0 i}{3\pi\left(y - \frac{a}{2}\right)}$$

ii

$a > y > \frac{a}{2}$.

✓ **Answer**

$$B(r_1, r_2) = \frac{2\mu_0 i(\min(r_1, a))^2}{3\pi a^2 r_1} - \frac{2\mu_0 i\left(\min\left(r_2, \frac{a}{2}\right)\right)^2}{3\pi a^2 r_2}$$
$$r_1 = |y|$$
$$r_2 = |y - \tfrac{a}{2}|$$

$$B(r_1, r_2) = \frac{2\mu_0 i y}{3\pi a^2} - \frac{2\mu_0 i\left(y - \frac{a}{2}\right)}{3\pi a^2}$$

iii

$\frac{a}{2} > y > 0$.

✓ **Answer**

$$B(r_1, r_2) = \frac{2\mu_0 i(\min(r_1, a))^2}{3\pi a^2 r_1} - \frac{2\mu_0 i\left(\min\left(r_2, \frac{a}{2}\right)\right)^2}{3\pi a^2 r_2}$$
$$r_1 = |y|$$
$$r_2 = |y - \tfrac{a}{2}|$$

$$B(r_1, r_2) = \frac{2\mu_0 i y}{3\pi a^2} - \frac{2\mu_0 i\left(\frac{a}{2} - y\right)}{3\pi a^2}$$

iv

$0 > y > -a$.

✓ **Answer**

$$B(r_1, r_2) = \frac{2\mu_0 i(\min(r_1,a))^2}{3\pi a^2 r_1} - \frac{2\mu_0 i\left(\min\left(r_2, \frac{a}{2}\right)\right)^2}{3\pi a^2 r_2}$$
$$r_1 = |y|$$
$$r_2 = |y - \tfrac{a}{2}|$$

$$B(r_1, r_2) = -\frac{2\mu_0 i y}{3\pi a^2} + \frac{\mu_0 i}{6\pi y}$$

v

$-a > y$

Answer

$$B(r_1, r_2) = \frac{2\mu_0 i(\min(r_1,a))^2}{3\pi a^2 r_1} - \frac{2\mu_0 i\left(\min\left(r_2, \frac{a}{2}\right)\right)^2}{3\pi a^2 r_2}$$
$$r_1 = |y|$$
$$r_2 = |y - \tfrac{a}{2}|$$

$$B(r_1, r_2) = -\frac{2\mu_0 i}{3\pi y} + \frac{\mu_0 i}{6\pi y}$$