6

PHYS 122 HW 6

1

Units and dimensional analysis. (Recap)

Dimensional analysis is concerned with checking the consistency of units in an equation. Dimensional considerations put surprisingly tight constraints on the equations we can sensibly write down. This makes dimensional analysis a remarkably powerful tool as illustrated below.

a

In the SI system the units of all quantities can be expressed as a combination of kg, m, s and C. Express the units of the following quantities in these terms.

Hint: For each quantity think of some defining equation in which it appears and in which you know the units of all but the quantity of interest. For example use the force law to get the units of the electric field, Coulomb's force law to get units of the permittivity etc.

i

Electric field, E.

✓ **Answer**

$\vec{F} = q\vec{E}$
$\vec{E} \in \frac{N}{C}$
$= kg\, m\, s^{-3}\, A^{-1}$
$\square$

ii

Permittivity of free space, ϵ_0.

✓ **Answer**

$\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}$
$\epsilon_0 \in C^2\, m^{-2}\, N^{-1}$

$= A^2\ kg^{-1}\ m^{-3}\ s^4$
$\square$

iii

Electrostatic potential, ϕ.

✓ **Answer**

$\phi \in V$
$= kg\, m^2\ s^{-3}\ A^{-1}$
$\square$

iv

Magnetic field, B.

✓ **Answer**

$\vec{B} \in T$
$= kg\ s^{-2}\ A^{-1}$
$\square$

v

Electric current, i.

✓ **Answer**

$i \in A$
$= A$
$\square$

vi

Permeability of free space, μ_0.

✓ **Answer**

$$\frac{F}{L} = \frac{\mu_0 I_1 I_2}{2\pi r}$$
$$\mu_0 \in \frac{N}{A^2}$$
$$= kg\ m\ s^{-2}\ A^{-2}$$

vii

The electric dipole moment, p.

✓ Answer

$$\vec{p} = q\vec{d}$$
$$\vec{p} \in C\ m$$
$$= A\ s\ m$$

viii

Energy.

✓ Answer

$$E \in J$$
$$= kg\ m^2\ s^{-2}$$

b

Name that unit: The units of some of the quantities listed above have names. Here is a list of unit names. Match the named unit to the corresponding physical quantity.

i

Volt

✓ Answer

Electrostatic Potential

ii

Ampere

✓ Answer

Current

iii

Tesla

✓ **Answer**

Magnetic Field

iv

Joule

✓ **Answer**

Energy

v

eV

✓ **Answer**

Energy

c

Comparing units

i

Explain why $1N/C = 1V/m$.

[Hint: Reduce both to a combination of kg, m, s and C.]

✓ **Answer**

$\frac{N}{C} = kg\, m\, s^{-3}\, A^{-1}$
$\frac{V}{m} = kg\, m\, s^{-3}\, A^{-1}$

$\frac{N}{C} = \frac{V}{m}$
$\square$

ii

Which quantity in part (a) has units of N/C or equivalently V/m?

✓ **Answer**

Electric Field

iii

Let E denote the units of electric field and B denote the units of magnetic field. By what factor do the two sets of units differ? In other words, determine the ratio E/B.

✓ **Answer**

$E = kg\, m\, s^{-3}\, A^{-1}$
$B = kg\, s^{-2}\, A^{-1}$
$\frac{E}{B} = m\, s^{-1}$
$\square$

d

Dimensional analysis: Magnetic energy density

A professor tells his class that just as the energy density in a static electric field is given by $\frac{1}{2}\epsilon_0 E^2$, so also the energy density in a static magnetic field is given by $\frac{1}{2}\mu_0 B^2$. However in his written lecture notes the professor writes that the magnetic energy density is given by $\frac{1}{2\mu_0}B^2$.

Either the professor made a careless mistake in class or there is a typo in his lecture notes. Use dimensional analysis to determine which of these expressions is correct.

✓ **Answer**

$\frac{1}{2}\epsilon_0 E^2 \in kg\, m^{-1}\, s^{-2}$
$B^2 \in kg^2\, s^{-4}\, A^{-2}$
$\frac{1}{2\mu_0}B^2 \in kg\, m^{-1}\, s^{-2}$
$\square$

Class is correct

e

Dimensional analysis and the speed of light

i

Form a combination of the quantities ϵ_0 and μ_0 that has units of speed.

✓ **Answer**

$\frac{1}{\sqrt{\epsilon_0 \mu_0}}$

ii

Calculate the numerical value of this speed in m/s.
$\epsilon_0 = 8.85 \times 10^{-12}$ S.I. units
$\mu_0 = 4\pi \times 10^{-7}$ S.I. units

✓ **Answer**

$299863380 \ \frac{m}{s}$

2

Torque on a loop: A square loop of side l lies in the horizontal $x - y$ plane as shown in the figure. A current i flows counterclockwise around the loop. The z-axis points vertically out of the page.

a

Suppose that a vertical magnetic field $\vec{B} = B_z \hat{k}$ is applied.

i

Determine the magnetic force on each segment of the loop.

✓ **Answer**

$d\vec{F} = i \, d\vec{l} \times \vec{B}$
$\vec{F} = i \int d\vec{l} \times \vec{B}$
$\vec{F} = i \int d\vec{l} \times B_z \hat{k}$

$\vec{F} = iB_z \int d\vec{l} \times \hat{k}$
$\vec{F} = \frac{1}{2} iB_z l^2 (\hat{l} \times \hat{k})$

Where $(\hat{l} \times \hat{k})$ will point perpendicular to the edges of the square, away from the center of the square.

ii

Assuming the force on each segment acts at the center of the segment, compute the magnetic torque on each segment about the center of the square loop. (Recall that the torque τ due to a force F applied at position r is given by $\tau = r \times F$).

✓ **Answer**

Since $\vec{r}$ and $\vec{F}$ point the same direction,
$\vec{\tau} = \vec{0}$ as the cross product evaluates to 0

iii

What is the net torque on the loop?

✓ **Answer**

$\vec{0}$ the sum of 0s is 0

b

Suppose a horizontal magnetic field $\vec{B} = B_z \hat{i}$ is applied instead. Repeat parts (i), (ii) and (iii) above for this circumstance.

✓ **Answer**

$\vec{F} = \frac{1}{2} iB_z l^2 (\hat{l} \times \hat{i})$
$\vec{\tau} = \vec{r} \times \vec{F}$

Right: $\vec{\tau} = \vec{r} \times \vec{F} = \frac{1}{2} iB_z l^2 \hat{j}$
Left: $\vec{\tau} = \vec{r} \times \vec{F} = \frac{1}{2} iB_z l^2 \hat{j}$
The top and bottom have no net force.

$\vec{\tau}_{net} = iB_z l^2 \hat{j}$
□

c

Define the magnetic moment of the loop as a vector $\vec{m}$ that has magnitude ia where i is the current and a is the area of the loop. The direction of $\vec{m}$ is perpendicular to the loop and out of the page; it is determined by a right hand rule (point fingers along current, then thumb points along $\vec{m}$). Verify that the results of parts (a) and (b) are consistent with the general formula $\vec{\tau}_{net} = \vec{m} \times \vec{B}$ where τ_{net} is the net torque on the loop and $\vec{B}$ is the applied magnetic field.

Answer

$\vec{m} = ia\hat{p}$
$\vec{m} = il^2\hat{p}$
$\vec{m} = il^2\hat{p}$
$(a) \implies \vec{\tau} = \vec{0}$
$(b) \implies \vec{\tau} = iB_z l^2\hat{j}$

Yes, it is the same

3

Circular loop. A circular loop of wire of radius a lies in the $x - y$ plane centered about the origin. A current i flows through the wire counter-clockwise as seen from above.

a

Magnetic dipole moment: What is $\vec{m}$, the magnetic dipole moment of the loop? Give your answer in terms of $i, a, \hat{i}, \hat{j}, \hat{k}$.

The magnetic dipole moment of a flat current loop is a vector. The magnitude of the dipole moment is equal to iA where i is the current and A is the area of the loop. The direction of the dipole moment is perpendicular to the loop in the direction that the right thumb points when the right hand fingers are curled around the loop in the direction of the current flow.

Answer

$\vec{m} = i\pi a^2\hat{k}$

b

Axial field: Determine the magnetic field at a point P that lies at an elevation z above the center of the circle using the Biot-Savart law. To this end consider the infinitesimal segment dl of the ring shown in the figure.

i

Write down the vector $d\vec{l}$ in terms of $a, d\theta, \theta, \hat{i}, \hat{j}$

✓ **Answer**

$\vec{l} = ae^{i\theta}$
$\frac{d\vec{l}}{d\theta} = iae^{i\theta}$
$d\vec{l} = a(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j})d\theta$

ii

Determine $\vec{r}$, the displacement vector from the segment dl to P in terms of $a, \theta, z, \hat{i}, \hat{j}, \hat{k}$

Also for later convenience determine r.

✓ **Answer**

$\vec{r} = -a\cos\theta\,\hat{i} - a\sin\theta\,\hat{j} + z\hat{k}$
$r = \sqrt{a^2 + z^2}$

iii

Evaluate $d\vec{l} \times \vec{r}$. Give your answer in terms of $a, z, \theta, d\theta, \hat{i}, \hat{j}, \hat{k}$.

✓ **Answer**

$d\vec{l} = a(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j})d\theta$
$\vec{r} = -a\cos\theta\,\hat{i} - a\sin\theta\,\hat{j} + z\hat{k}$
$d\vec{l} \times \vec{r} = (az\cos\theta\;d\theta)\hat{i} + (az\sin\theta\;d\theta)\hat{j} + (a^2\sin^2\theta\;d\theta + a^2\cos^2\theta\;d\theta)\hat{k}$
$d\vec{l} \times \vec{r} = (az\cos\theta\;d\theta)\hat{i} + (az\sin\theta\;d\theta)\hat{j} + (a^2\;d\theta)\hat{k}$

iv

Recall that according to the Biot-Savart law the magnetic field due to an infinitesimal segment is given by

$d\vec{B} = \frac{\mu_0 i}{4\pi}\frac{d\vec{l}\times\vec{r}}{r^3}$

Here $\vec{r}$ represents the vector from the position of the segment $d\vec{l}$ to the point P and r is the magnitude of $\vec{r}$.

Use the equation and your results of parts (ii) and (iii) to determine $d\vec{B}$, the field produced by the infinitesimal segment $d\vec{l}$.

✓ **Answer**

$$d\vec{B} = \frac{\mu_0 i}{4\pi} \frac{(az\cos\theta\ d\theta)\hat{i} + (az\sin\theta\ d\theta)\hat{j} + (a^2\ d\theta)\hat{k}}{\left(\sqrt{a^2+z^2}\right)^3}$$

v

Integrate the result of part (iv) to determine the total magnetic field at the point P.

✓ **Answer**

$$d\vec{B} = \frac{\mu_0 i}{4\pi} \frac{(az\cos\theta\ d\theta)\hat{i} + (az\sin\theta\ d\theta)\hat{j} + (a^2\ d\theta)\hat{k}}{\left(\sqrt{a^2+z^2}\right)^3}$$
$$\vec{B} = \int_{\theta=0}^{2\pi} \frac{\mu_0 i}{4\pi\left(\sqrt{a^2+z^2}\right)^3}(a^2\ d\theta)\hat{k}$$
$$\vec{B} = \frac{\mu_0 i a^2}{2\left(\sqrt{a^2+z^2}\right)^3}\hat{k}$$

vi

Simplify your result in the limit that $z \gg a$. Verify that the magnetic field is given approximately by

$$\vec{B} = \frac{\mu_0}{2\pi}\frac{\vec{m}}{z^3}$$

✓ **Answer**

$$\vec{m} = i\pi a^2\hat{k}$$
$$\vec{B} = \frac{\mu_0 i a^2}{2z^3}\hat{k}$$
$$\vec{B} = \frac{\mu_0 \vec{m}}{2\pi z^3}\hat{k}$$

c

Now let us determine the magnetic field at the point Q on the x-axis at distance x. To this end consider the infinitesimal segment $d\vec{l}$ of the ring shown in the figure.

i*

You have already determined $d\vec{l}$ in part (b-i).

✓ **Answer**

$$d\vec{l} = a(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j})d\theta$$

ii

Determine $\vec{r}$, the displacement from the segment $d\vec{l}$ to Q. Give your answer in terms of $a, \theta, d\theta, x, \hat{i}, \hat{j}, \hat{k}$.

✓ **Answer**

$$\vec{r} = (-a\cos\theta + x)\,\hat{i} - a\sin\theta\,\hat{j}$$
$$r = \sqrt{(a\cos\theta + x)^2 + a^2\sin^2\theta}$$

iii

Evaluate $d\vec{l} \times \vec{r}$. Give your answer in terms of $a, \theta, d\theta, x, \hat{i}, \hat{j}, \hat{k}$.

✓ **Answer**

$$d\vec{l} \times \vec{r} = (a^2 d\theta + xa\cos\theta\,d\theta)\hat{k}$$

iv

Recall that according to Biot-Savart the magnetic field due to an infinitesimal segment is given by eq (5) with $\vec{r}$ the displacement from $d\vec{l}$ to Q and r the magnitude of $\vec{r}$. Use eq (5) and your results from parts (ii) and (iii) to determine $d\vec{B}$, the field produced by the infinitesimal segment $d\vec{l}$.

✓ **Answer**

$$d\vec{B} = \frac{\mu_0 i}{4\pi}\frac{(a^2 d\theta + xa\cos\theta\,d\theta)\hat{k}}{\left((a\cos\theta+x)^2 + a^2\sin^2\theta\right)^{3/2}}$$
$$d\vec{B} = \frac{\mu_0 ia}{4\pi}d\theta\frac{(a + x\cos\theta)\hat{k}}{\left((a\cos\theta+x)^2 + a^2\sin^2\theta\right)^{3/2}}$$
$$d\vec{B} = \frac{\mu_0 ia}{4\pi}\hat{k}\,d\theta\frac{a + x\cos\theta}{(a^2 + 2ax\cos\theta + x^2)^{3/2}}$$

v

It follows from part (iv) that the total field is given by the integral

$$\vec{B} = -\frac{\mu_0 i a}{4\pi x^2}\hat{k}\int_0^{2\pi} d\theta \frac{\cos\theta - \frac{a}{x}}{\left(1 - 2\left(\frac{a}{x}\right)\cos\theta + \left(\frac{a}{x}\right)^2\right)^{3/2}}$$

This integral cannot be evaluated exactly. Further calculation has to be carried out numerically. However at large distances, $x \gg a$, the integrand can be approximated as

$$\frac{\cos\theta - \frac{a}{x}}{\left(1 - 2\left(\frac{a}{x}\right)\cos\theta + \left(\frac{a}{x}\right)^2\right)^{3/2}} \approx -\cos\theta + \left(3\cos^2\theta - 1\right)\frac{a}{x} + \dots$$

With this approximation it is now possible to evaluate the integral. Verify that the magnetic field is approximately given by

$$\vec{B} = -\frac{\mu_0 \vec{m}}{4\pi x^3}$$

Answer

$$\vec{B} = -\frac{\mu_0 i a}{4\pi x^2}\hat{k}\int_0^{2\pi} d\theta \frac{\cos\theta - \frac{a}{x}}{\left(1 - 2\left(\frac{a}{x}\right)\cos\theta + \left(\frac{a}{x}\right)^2\right)^{3/2}}$$

$$\vec{B} = -\frac{\mu_0 i a}{4\pi x^2}\hat{k}\int_0^{2\pi} d\theta - \cos\theta + \left(3\cos^2\theta - 1\right)\frac{a}{x}$$

$$\vec{B} = -\frac{\mu_0 i a^2}{4\pi x^3}\hat{k}\int_0^{2\pi} d\theta \left(3\cos^2\theta - 1\right)$$

$$\vec{B} = -\frac{\mu_0 i a^2}{4 x^3}\hat{k}$$

$$\vec{m} = i\pi a^2 \hat{k}$$

$$\vec{B} = -\frac{\mu_0 \vec{m}}{4\pi x^3}$$