4.22

When

$U = g_1(X, Y)$
$V = g_2(X, Y)$
$X = h_1(U, V)$
$Y = h_2(U, V)$

$$f_{U,V}(u, v) = f_{X,Y}(h_1(U, V), h_2(U, V))|J(h)|$$

$U = aX + b$
$V = cY + d$
$X = \frac{U-b}{a}$
$Y = \frac{V-d}{c}$

$$J = \begin{vmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{c} \end{vmatrix} = \frac{1}{ac}$$

$$f_{U,V}(u, v) = \frac{1}{ac} f(\frac{u-b}{a}, \frac{v-d}{c})$$

4.23

$f_{X,Y}(x, y) = \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} x^{\alpha-1}(1-x)^{\beta-1} y^{\alpha+\beta-1}(1-y)^{\gamma-1}$
$0 < x < 1, \quad 0 < y < 1$

a

$U = XY$
$V = Y$
$X = \frac{U}{V}$
$Y = V$

$$0 < U < V < 1$$

$$f_{U,V}(u, v) = J f_{X,Y}(\frac{u}{v}, v)$$

$$J = \begin{vmatrix} \frac{1}{V} & \frac{-U}{V^2} \\ 0 & 1 \end{vmatrix} = \frac{1}{V}$$

$$f_{U,V}(u,v) = \frac{1}{v}\frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}\left(\frac{u}{v}\right)^{\alpha-1}\left(1-\left(\frac{u}{v}\right)\right)^{\beta-1}v^{\alpha+\beta-1}(1-v)^{\gamma-1}$$

$$f_U(u) = \int_u^1 \frac{1}{v}\frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}\left(\frac{u}{v}\right)^{\alpha-1}\left(1-\left(\frac{u}{v}\right)\right)^{\beta-1}v^{\alpha+\beta-1}(1-v)^{\gamma-1}\,dv$$

$$f_U(u) = \int_u^1 v^{-1}\frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}u^{\alpha-1}v^{1-\alpha}\left(\frac{v-u}{v}\right)^{\beta-1}v^{\alpha+\beta-1}(1-v)^{\gamma-1}\,dv$$

$$f_U(u) = \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}u^{\alpha-1}\int_u^1 \left(\frac{v-u}{v}\right)^{\beta-1}v^{\beta-1}(1-v)^{\gamma-1}\,dv$$

$$f_U(u) = \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}u^{\alpha-1}\int_u^1 (v-u)^{\beta-1}(1-v)^{\gamma-1}\,dv$$

$$y = \frac{1-v}{1-u} \quad dy = \frac{-dv}{1-u}$$

$$f_U(u) = \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}u^{\alpha-1}(1-u)^{\beta-1}(1-u)^{\gamma-1}(1-u)\int_u^1 (1-y)^{\beta-1}(y)^{\gamma-1}\,dv$$

$$f_U(u) = \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta+\gamma)}u^{\alpha-1}(1-u)^{\beta+\gamma-1}$$

$$\rho = \beta+\gamma$$

$$f_U(u) = \frac{\Gamma(\alpha+\rho)}{\Gamma(\alpha)\Gamma(\rho)}u^{\alpha-1}(1-u)^{\rho-1}$$

$$\sim \mathrm{Beta}(\alpha,\rho) = \mathrm{Beta}(\alpha,\beta+\gamma)$$

b

$$U = XY$$
$$V = \frac{X}{Y}$$
$$X = \sqrt{UV}$$
$$Y = \sqrt{\frac{U}{V}}$$

$$U < V < \frac{1}{U} \quad 0 < U < 1$$

$$f_{U,V}(u,v) = Jf_{X,Y}\left(\sqrt{UV}, \sqrt{\frac{U}{V}}\right)$$

$$J = \begin{vmatrix} \frac{V}{2\sqrt{UV}} & \frac{U}{2\sqrt{UV}} \\ \frac{1}{2V\sqrt{\frac{U}{V}}} & \frac{-U}{2V^2\sqrt{\frac{U}{V}}} \end{vmatrix} = \frac{-1}{2V}$$

$$f_{U,V}(u,v) = \frac{1}{2V}\frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}\sqrt{UV}^{\alpha-1}(1-\sqrt{UV})^{\beta-1}\sqrt{\frac{U}{V}}^{\alpha+\beta-1}(1-\sqrt{\frac{U}{V}})^{\gamma-1}$$
$$f_U(u) = \int_u^{1/u} \frac{1}{2V}\frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}\sqrt{UV}^{\alpha-1}(1-\sqrt{UV})^{\beta-1}\sqrt{\frac{U}{V}}^{\alpha+\beta-1}(1-\sqrt{\frac{U}{V}})^{\gamma-1}\,dv$$
$$f_U(u) = \int_u^{1/u} \frac{\sqrt{U}}{2V^{3/2}}\frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}(\sqrt{\frac{U}{V}}\sqrt{UV})^{\alpha-1}(\sqrt{\frac{U}{V}}(1-\sqrt{UV}))^{\beta-1}(1-\sqrt{\frac{U}{V}})^{\gamma-1}\,dv$$
$$f_U(u) = \int_u^{1/u} \frac{\sqrt{U}}{2V^{3/2}}\frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}U^{\alpha-1}(\sqrt{\frac{U}{V}}-U)^{\beta-1}(1-\sqrt{\frac{U}{V}})^{\gamma-1}\,dv$$
$$f_U(u) = \int_u^{1/u} \frac{\sqrt{U}}{2V^{3/2}}\frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}U^{\alpha-1}(1-U)^{\beta+\gamma-2}\left(\frac{\sqrt{\frac{U}{V}}-U}{1-U}\right)^{\beta-1}\left(\frac{1-\sqrt{\frac{U}{V}}}{1-U}\right)^{\gamma-1}\,dv$$

$$Z = \frac{\sqrt{\frac{U}{V}}-U}{1-U}$$
$$dZ = \frac{\sqrt{U}}{1-U}\frac{-1}{2V^{3/2}}$$

$$f_U(u) = \int_0^1 \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}U^{\alpha-1}(1-U)^{\beta+\gamma-1}(Z)^{\beta-1}(1-Z)^{\gamma-1}\,dz$$
$$f_U(u) = \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta+\gamma)}U^{\alpha-1}(1-U)^{\beta+\gamma-1}\int_0^1 \frac{\Gamma(\beta+\gamma)}{\Gamma(\beta)\Gamma(\gamma)}(Z)^{\beta-1}(1-Z)^{\gamma-1}\,dz$$
$$f_U(u) = \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta+\gamma)}U^{\alpha-1}(1-U)^{\beta+\gamma-1}$$

$$U \sim \text{Beta}(\alpha, \beta+\gamma)$$

4.24

$$X \sim \text{Gamma}(r,1)$$
$$Y \sim \text{Gamma}(s,1)$$

$$Z_1 = X+Y \quad Z_2 = \frac{X}{X+Y}$$
$$X = Z_1 Z_2$$
$$Y = Z_1 - Z_1 Z_2$$

$$0 < Z_1$$
$$0 < Z_2 < 1$$

$$J = \begin{vmatrix} Z_2 & Z_1 \\ 1-Z_2 & -Z_1 \end{vmatrix} = -Z_1$$

$$f_{X,Y}(x,y) = \frac{x^{r-1}e^{-x}}{\Gamma(r)}\frac{y^{s-1}e^{-y}}{\Gamma(s)}$$

$$f_{Z_1,Z_2}(z_1,z_2) = \frac{(z_1z_2)^{r-1}e^{-z_1z_2}}{\Gamma(r)}\frac{(z_1-z_1z_2)^{s-1}e^{-z_1+z_1z_2}}{\Gamma(s)}z_1$$
$$f_{Z_1,Z_2}(z_1,z_2) = \frac{1}{\Gamma(r)\Gamma(s)}z_1^{r+s-1}e^{-z_1}z_2^{r-1}(1-z_2)^{s-1}$$
$$f_{Z_1,Z_2}(z_1,z_2) = (\frac{1}{\Gamma(r+s)}z_1^{r+s-1}e^{-z_1})(\frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)}z_2^{r-1}(1-z_2)^{s-1})$$

Which can be defined in functions one of only z_1 and one of only z_2

$$f_{Z_1}(z_1) = \frac{1}{\Gamma(r+s)}z_1^{r+s-1}e^{-z_1}$$
$$f_{Z_2}(z_2) = \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)}z_2^{r-1}(1-z_2)^{s-1}$$

Such that

$$f_{Z_1}(z_1)f_{Z_2}(z_2) = f_{Z_1,Z_2}(z_1,z_2)$$

Which proves that Z_1 and Z_2 are independent and are of the distributions

$$Z_1 \sim \text{Gamma}(r+s,1)$$
$$Z_2 \sim \text{Beta}(r,s)$$

4.32

a

$$Y|\Lambda \sim \text{Poisson}(\Lambda)$$
$$\Lambda \sim \text{Gamma}(\alpha,\beta)$$

$$0 < \Lambda$$
$$Y \in \{0,1,2,3,\dots\}$$

$$f_{Y,\Lambda}(y,\lambda) = \frac{e^{-\lambda}\lambda^y}{y!}\frac{\lambda^{\alpha-1}e^{-\lambda/\beta}}{\Gamma(\alpha)\beta^\alpha}$$

$$f_Y(y) = \int_0^\infty \frac{e^{-\lambda}\lambda^y}{y!}\frac{\lambda^{\alpha-1}e^{-\lambda/\beta}}{\Gamma(\alpha)\beta^\alpha}\,d\lambda$$
$$f_Y(y) = \frac{1}{\Gamma(\alpha)y!\beta^\alpha}\int_0^\infty e^{-\lambda}\lambda^y\lambda^{\alpha-1}e^{-\lambda/\beta}\,d\lambda$$
$$f_Y(y) = \frac{1}{\Gamma(\alpha)y!\beta^\alpha}\int_0^\infty \lambda^{y+\alpha-1}e^{-\lambda(1+\beta)/\beta}\,d\lambda$$

$$\bar{\alpha} = y+\alpha$$
$$\bar{\beta} = \frac{\beta}{1+\beta}$$

$$f_Y(y) = \frac{1}{\Gamma(\alpha)y!\beta^\alpha} \int\limits_0^\infty \lambda^{\bar{\alpha}-1} e^{-\lambda/\bar{\beta}}\, d\lambda$$

$$f_Y(y) = \frac{1}{\Gamma(\alpha)y!\beta^\alpha} \Gamma(\bar{\alpha})\bar{\beta}^{\bar{\alpha}} \int\limits_0^\infty \frac{1}{\Gamma(\bar{\alpha})\bar{\beta}^{\bar{\alpha}}} \lambda^{\bar{\alpha}-1} e^{-\lambda/\bar{\beta}}\, d\lambda$$

$$f_Y(y) = \frac{1}{\Gamma(\alpha)y!\beta^\alpha} \Gamma(\bar{\alpha})\bar{\beta}^{\bar{\alpha}}$$

$$f_Y(y) = \frac{1}{\Gamma(\alpha)y!\beta^\alpha} \Gamma(y+\alpha) \left(\frac{\beta}{1+\beta}\right)^{y+\alpha}$$

If α is an integer

$$f_Y(y) = \frac{(y+\alpha-1)!}{(\alpha-1)!y!} \left(\frac{\beta}{1+\beta}\right)^y \left(\frac{1}{1+\beta}\right)^\alpha$$

$$f_Y(y) = \binom{y+\alpha-1}{y} \left(\frac{\beta}{1+\beta}\right)^y \left(\frac{1}{1+\beta}\right)^\alpha$$

$$\bar{p} = \frac{1}{1+\beta}$$

$$f_Y(y) = \binom{y+\alpha-1}{y} (1-\bar{p})^y (\bar{p})^\alpha$$

$$Y \sim \mathrm{NegBinom}(\alpha, \frac{1}{1+\beta})$$

EY

$$\mu = \frac{\alpha\left(\frac{\beta}{1+\beta}\right)}{\frac{1}{1+\beta}} = \alpha\beta$$

$Var(Y)$

$$Var(Y) = \frac{\mu}{\frac{1}{1+\beta}} = \alpha\beta(1+\beta)$$