HW 2

1.14

Assume that $P(A) = 0.4$ and $P(B) = 0.7$. Making no further assumptions on A and B, show that $P(A \cap B)$ satisfies $0.1 \leq P(A \cap B) \leq 0.4$.

✓ **Answer** ⌄

$P(A) + P(B) - P(A \cap B) = P(A \cup B)$

$P(A \cup B) \leq 1$
$P(A) + P(B) - P(A \cup B) = P(A \cap B) \geq 0.1$

$P(A \cup B) \geq P(B) > P(A)$
$P(A) + P(B) - P(A \cap B) \geq P(B)$
$P(A) \geq P(A \cap B)$
$0.4 \geq P(A \cap B)$

$0.1 \leq P(A \cap B) \leq 0.4$

1.18

The statement `SOME DOGS ARE BROWN` has 16 letters. Choose one of the 16 letters uniformly at random. Let X denote the length of the word containing the chosen letter. Determine the possible values and probability mass function of X.

✓ **Answer**

$\Omega = \{3, 4, 5\}$

Length of word	Possibilities
3	3
4	8
5	5

$P(\{3\}) = \frac{3}{16}$
$P(\{4\}) = \frac{8}{16}$
$P(\{5\}) = \frac{5}{16}$

1.26

10 men and 5 women are meeting in a conference room. Four people are chosen at random from the 15 to form a committee.

a

What is the probability that the committee consists of 2 men and 2 women?

✓ **Answer**

There are $4C15$ possible combinations of 4 people.

There are $2C10 \cdot 2C5$ possible selections of 2 men and 2 women.

This gives us the probability of $\frac{450}{1365} = \frac{30}{91}$

b

Among the 15 is a couple, Bob and Jane. What is the probability that Bob and Jane both end up on the committee?

✓ **Answer**

There are $13C2$ possible combinations for the remaining 2 seats.

Making the probability of $\frac{78}{1365} = \frac{2}{35}$

c

What is the probability that Bob ends up on the committee but Jane does not?

✓ **Answer**

There are $13C3$ possible combinations for the remaining 3 seats, excluding Jane.

Making the probability of $\frac{286}{1365} = \frac{22}{105}$

1.28

We have an urn with m green balls and n yellow balls. Two balls are drawn at random. What is the probability that the two balls have the same color?

a

Assume that the balls are sampled without replacement.

✓ **Answer**

There are $(m+n)C2$ possible drawings of two balls without replacement.

There are $mC2$ ways to pull two green balls and $nC2$ ways to pull two yellow balls.

Therefore, there is a $\frac{mC2+nC2}{(m+n)C2} = \frac{m(m-1)+n(n-1)}{(m+n)(m+n-1)}$ probability of drawing a pair of the same color.

b

Assume that the balls are sampled with replacement.

✓ **Answer**

There are $(m+n)^2$ possible drawings of two balls with replacement.

There are m^2 ways to pull two green balls and n^2 ways to pull two yellow balls.

Therefore, there is a $\frac{m^2n^2}{(m+n)^2}$ probability of drawing a pair of the same color.

c

When is the answer to part (b) larger than the answer to part (a)? Justify your answer. Can you give an intuitive explanation for what the calculation tells you?

✓ **Answer**

$\frac{m(m-1)+n(n-1)}{(m+n)(m+n-1)} \stackrel{?}{=} \frac{m^2n^2}{(m+n)^2}$

$\frac{m(m-1)+n(n-1)}{m+n-1} \stackrel{?}{=} \frac{m^2n^2}{m+n}$

$\frac{m^2-m+n^2-n}{m+n-1} \stackrel{?}{=} \frac{m^2n^2}{m+n}$

$m^3 - m^2 + mn^2 - mn + nm^2 - nm + n^3 - n^2 \stackrel{?}{=} m^3n^2 + m^2n^3 - m^2n^2$

$m^3 + mn^2 + nm^2 + n^3 + m^2n^2 \stackrel{?}{=} m^3n^2 + m^2n^3 + n^2 + 2nm + m^2$

$(m^2+n^2)(m+n) + m^2n^2 \stackrel{?}{=} m^3n^2 + m^2n^3 + n^2 + 2nm + m^2$

Let j be the greater of m, n, and k be the other

$(j^2+k^2)(j+k) + j^2k^2 \stackrel{?}{=} j^3k^2 + j^2k^3 + j^2 + 2jk + k^2$

$j^2(j+k) + k^2(j^2+j+k) \stackrel{?}{=} j^3k^2 + j^2k^3 + j^2 + 2jk + k^2$

Case $j, k \geq 2$

$\implies k^2(j^2+j+k) \leq k^2j^3$

$\implies j^2(j+k) < j^2k^3$

Case $j = 2, k = 1$
$4(2+1) + 1(4+2+1) = 14 < 21 = 8+4+4+4+1$

Case $j, k = 1$
$2 + 3 = 5 < 6$

$\therefore j^2(j+k) + k^2(j^2+j+k) < j^3k^2 + j^2k^3 + j^2 + 2jk + k^2$
$\implies \frac{m(m-1)+n(n-1)}{(m+n)(m+n-1)} < \frac{m^2n^2}{(m+n)^2}$

Meaning it is always more likely to pull a pair of the same color when replacing the ball than when you don't replace the ball.

1.33

You roll a fair die 5 times. What is the probability of seeing a “full house” in the sense of seeing three rolls of one type, and two rolls of another different type? Note that we do not allow for all five rolls to be of the same type.

Answer

There are $6P2$ ways of picking the two numbers that consist the full house.

There is exactly 1 combination to roll the full house after deciding those two numbers, with $5C3$ different ways to roll that combination

There are 6^5 different possible rolls, making the probability of rolling a full house

$\frac{25}{648}$

1.41

Imagine a game of three players where exactly one player wins in the end and all players have equal chances of being the winner. The game is repeated four times. Find the probability that there is at least one person who wins no games.

Hint. Let $A_i = \text{person } i \text{ wins no games}$ and utilize the inclusion-exclusion formula.

Answer

The only way everyone wins at least one game is if two people win once and one person wins twice.

There are $3^4 = 81$ different outcomes of the situation, with each being equally probable.

There are $3C1$ possible selections for the double winner.
There are $4P(2,1,1)$ possible permutations of rolling that configuration,

Leading to $3 \cdot 4 \cdot 3 = 36$ unfavorable situations.

Thus $P(\text{someone wins no games}) = \frac{45}{81} = \frac{5}{9}$

1.44

Two fair dice are rolled. Let X be the maximum of the two numbers and Y the minimum of the two numbers on the dice.

a

Find the possible values of X and the possible values of Y.

✓ **Answer**

$X, Y \in [\![1, 6]\!]$

b

Find the probabilities $P(X \leq k)$ for all integers k. Find the probability mass function of X.

Hint. Noticing that $P(X = k) = P(X \leq k) - P(X \leq k - 1)$ can save you some work.

✓ **Answer**

There are a total of 6^2 possible outcomes.

$P(X \leq 1) = \frac{1^2}{6^2}$
$P(X \leq 2) = \frac{2^2}{6^2}$
$P(X \leq 3) = \frac{3^2}{6^2}$
$P(X \leq 4) = \frac{4^2}{6^2}$
$P(X \leq 5) = \frac{5^2}{6^2}$
$P(X \leq 6) = \frac{6^1}{6^2}$

$P(X = k) = \frac{2k-1}{6^2}$
$P(X \geq k) = \frac{6^2-(k-1)^2}{6^2}$

c

Find the probability mass function of Y.

✓ **Answer**

There are a total of 6^2 possible outcomes.

$P(Y \geq 6) = \frac{1^1}{6^2}$
$P(Y \geq 5) = \frac{2^2}{6^2}$

$P(Y \geq 4) = \frac{3^2}{6^2}$
$P(Y \geq 3) = \frac{4^2}{6^2}$
$P(Y \geq 2) = \frac{5^2}{6^2}$
$P(Y \geq 1) = \frac{6^2}{6^2}$

$P(Y = k) = \frac{13-2k}{6^2}$
$P(Y \leq k) = \frac{6^2-(6-k)^2}{6^2}$