4.43

a

$$\text{Cov}(X_1+X_2, X_2+X_3) = E((X_1+X_2)(X_2+X_3)) - E(X_1+X_2)E(X_2+X_3)$$
$$= E(X_1X_2) + E(X_1X_3) + E(X_2X_2) + E(X_2X_3) - (2\mu)(2\mu)$$
$$= +E(X_1)E(X_2) + E(X_1)E(X_3) + \text{Cov}(X_2, X_2) + E(X_2)E(X_3) + E(X_2)E(X_3) - 4$$
$$= \mu^2 + \mu^2 + \sigma^2 + \mu^2 + \mu^2 - 4\mu^2$$
$$= \sigma^2$$

b

$$\text{Cov}(X_1+X_2, X_1-X_2) = E((X_1+X_2)(X_1-X_2)) - E(X_1+X_2)E(X_1-X_2)$$
$$= E(X_1X_1 - X_1X_2 + X_2X_1 - X_2X_2)$$
$$= E(X_1X_1) - E(X_2X_2)$$
$$= E(X_1X_1) - E(X_2X_2)$$
$$= \sigma^2 - \sigma^2$$
$$= 0$$

4.45

a

$$f(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right)\right)$$

$$z_X = \frac{x-\mu_X}{\sigma_X}$$
$$z_Y = \frac{y-\mu_Y}{\sigma_Y}$$

$$f(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left(-\frac{z_X^2 - 2\rho z_X z_Y + z_Y^2}{2(1-\rho^2)}\right)$$
$$f(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left(-\frac{(z_X-\rho z_Y)^2 + z_Y^2(1-\rho^2)}{2(1-\rho^2)}\right)$$
$$f(x,y) = \frac{1}{\sqrt{2\pi}\sigma_Y\sqrt{1-\rho^2}} \exp\left(-\frac{z_Y^2}{2}\right) \frac{1}{\sqrt{2\pi}\sigma_X} \exp\left(-\frac{(z_X-\rho z_Y)^2}{2(1-\rho^2)}\right)$$

$$f_Y(u) = \int_{-\infty}^{\infty} f(x,y)\, dx$$
$$f_Y(u) = \frac{1}{\sqrt{2\pi}\sigma_Y\sqrt{1-\rho^2}} \exp\left(-\frac{z_Y^2}{2}\right) \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_X} \exp\left(-\frac{(z_X-\rho z_Y)^2}{2(1-\rho^2)}\right) dx$$

$dx_X \sigma_X = dx$

$f_Y(u) = \frac{1}{\sqrt{2\pi}\sigma_Y} \exp\left(-\frac{z_Y^2}{2}\right) \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}} \exp\left(-\frac{(z_X-\rho z_Y)^2}{2\sqrt{1-\rho^2}^2}\right) dz$

$f_Y(u) = \frac{1}{\sqrt{2\pi}\sigma_Y} \exp\left(-\frac{z_Y^2}{2}\right)$

$Y \sim \mathrm{N}(\mu_Y, \sigma_Y^2)$

The exact same is true for X, just replacing $_X$ variables with $_Y$

b

$Y|X = \frac{f_{XY}}{f_X}$

$f(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left(-\frac{z_X^2 - 2\rho z_X z_Y + z_Y^2}{2(1-\rho^2)}\right)$

$f_X(x) = \frac{1}{\sigma_X\sqrt{2\pi}} \exp\left(-\frac{z_X^2}{2}\right)$

$f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi}\sigma_Y\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(\rho^2\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right)\right)$

$f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi}\sigma_Y\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(\left(\left(\frac{y-\mu_Y}{\sigma_Y}\right) - \rho\left(\frac{x-\mu_x}{\sigma_X}\right)\right)^2\right)\right)$

$f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi}\sigma_Y\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2\sigma_Y^2(1-\rho^2)}\left(y - \left(\mu_Y + \frac{\rho\sigma_Y(x-\mu_x)}{\sigma_X}\right)\right)^2\right)$

$\sim \mathrm{N}(\mu_Y + \rho(\sigma_Y/\sigma_X)(x-\mu_x), \sigma_Y^2(1-\rho^2))$

c

$U = aX + bY$

$V = Y$

$X = \frac{U-bV}{a}$

$Y = V$

$J = \begin{vmatrix} \frac{1}{a} & -\frac{b}{a} \\ 0 & 1 \end{vmatrix} = \frac{1}{a}$

$z_X = \frac{\frac{u-bv}{a} - \mu_X}{\sigma_X} = \frac{u-(bv+a\mu_X)}{a\sigma_X}$

$f(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left(-\frac{z_X^2 - 2\rho z_X z_Y + z_Y^2}{2(1-\rho^2)}\right)$

$f_{UV}(u,v) = \frac{1}{2a\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left(-\frac{(\frac{u-(bv+a\mu_X)}{a\sigma_X})^2 - 2\rho(\frac{u-(bv+a\mu_X)}{a\sigma_X})z_V + z_V^2}{2(1-\rho^2)}\right)$

Let

$A = (\frac{u-(bv+a\mu_X)}{a\sigma_X})^2 - 2\rho(\frac{u-(bv+a\mu_X)}{a\sigma_X})z_V + z_V^2$

$= \left(\frac{u-(bv+a\mu_X)}{a\sigma_X}\right)^2 - 2\rho\left(\frac{u-(bv+a\mu_X)}{a\sigma_X}\right)\left(\frac{v-\mu_Y}{\sigma_Y}\right) + \left(\frac{v-\mu_Y}{\sigma_Y}\right)^2$

$= \frac{1}{a^2\sigma_X^2\sigma_Y^2}(\sigma_Y^2(u^2 - 2buv - 2au\mu_X + b^2v^2 + 2bva\mu_X + a^2\mu_X^2) - 2\rho a\sigma_X\sigma_Y \left(uv - u\mu_Y -\right.$

$= (\sigma_Y^2 b^2 + 2\rho ab\sigma_X\sigma_Y + a^2\sigma_X^2)v^2 + (-2\sigma_Y^2 bu + 2\sigma_Y^2 ab\mu_X - 2\rho a\sigma_X\sigma_Y u + 2\rho a\sigma_X\sigma_Y a\mu$
$+\sigma_Y^2(u^2 - 2au\mu_X + a^2\mu_X^2) - 2\rho a\sigma_X\sigma_Y(-u\mu_Y + \mu_Y a\mu_X) + a^2\sigma_X^2\mu_Y^2$

5.3

$Y_i = \text{Bernoulli}(1 - F(\mu))$

$\sum_{i=1}^{n} Y_i \sim \text{Binomial}(n, 1 - F(\mu))$

5.5

$\bar{X} = \frac{1}{n}(X_1 + \ldots + X_n)$

$\bar{X} = \frac{1}{n}Y$

$f_{\bar{X}} = nf_Y(nx)$

$f_{\bar{X}} = nf_{X_1+\ldots+X_n}(nx)$

5.6

a

$U = X + Y$

$V = Y$

$X = U - V$

$Y = V$

$J = 1$

$f_{UV}(u, v) = f_{XY}(u - v, v)$

$f_U(u) = \int_{-\infty}^{\infty} f_X(u - v)f_Y(v)\, dv$

b

$$U = XY$$
$$V = Y$$

$$X = \frac{U}{V}$$
$$Y = V$$

$$J = \frac{1}{V}$$

$$f_{UV}(u, v) = |\frac{1}{v}| f_{XY}\left(\frac{u}{v}, v\right)$$
$$f_U(u) = \int\limits_{-\infty}^{\infty} |\frac{1}{v}| f_X\left(\frac{u}{v}\right) f_Y(v)\, dv$$

C

$$U = \frac{X}{Y}$$
$$V = Y$$

$$X = UV$$
$$Y = V$$

$$J = V$$

$$f_{UV}(u, v) = |v| f_{XY}(uv, v)$$
$$f_U(u) = |v| f_X(uv) f_Y(v)\, dv$$