3.22

b

$$X \sim \mathrm{Binom}(r,p) \ P(X=x) = inom{n}{x} p^x (1-p)^{n-x}$$

E(X)

$$egin{aligned} E(X) &= \sum_{x \in X} x P(X = x) \ E(X) &= \sum_{x=1}^n x inom{n}{x} p^x (1-p)^{n-x} \ E(X) &= \sum_{x=1}^n n p rac{(n-1)!}{(x-1)!(n-x)!} p^{x-1} (1-p)^{n-x} \ E(X) &= n p \sum_{x=1}^n inom{n-1}{x-1} p^{x-1} (1-p)^{n-x} \ ext{Let } ilde{x} = x-1 \qquad ilde{n} = n-1 \ E(X) &= n p \sum_{ ilde{x}=0}^{ ilde{n}} inom{ ilde{n}}{ ilde{x}} p^{ ilde{x}} (1-p)^{ ilde{n}- ilde{x}} \ E(X) &= n p \end{aligned}$$

$E(X^2)$

$$egin{aligned} E(X^2) &= \sum_{x \in X} x^2 P(X = x) \ E(X^2) &= \sum_{x=1}^n x^2 inom{n}{x} p^x (1-p)^{n-x} \ E(X^2) &= np \sum_{ ilde{x}=0}^{ ilde{n}} (ilde{x}+1) inom{ ilde{n}}{ ilde{x}} p^{ ilde{x}} (1-p)^{ ilde{n}- ilde{x}} \ E(X^2) &= np (1+\sum_{ ilde{x}=0}^{ ilde{n}} ilde{x} inom{ ilde{n}}{ ilde{x}} p^{ ilde{x}} (1-p)^{ ilde{n}- ilde{x}}) \ E(X^2) &= np (1+ ilde{n}p) \ E(X^2) &= np (1+(n-1)p) \ E(X^2) &= np + n^2 p^2 - np^2 \end{aligned}$$

Var(X)

$${
m Var}(X) = E(X^2) - (E(X))^2 \ {
m Var}(X) = np + n^2p^2 - np^2 - (np)^2$$

$$egin{aligned} \operatorname{Var}(X) &= np - np^2 \ \operatorname{Var}(X) &= np(1-p) \ \Box \end{aligned}$$

$$X \sim \mathrm{Gamma}(lpha,eta) \ f(x) = rac{x^{lpha-1}e^{-x/eta}}{\Gamma(lpha)eta^lpha}$$

E(X)

$$egin{aligned} E(X) &= \int\limits_{-\infty}^{\infty} x f(x) dx \ E(X) &= \int\limits_{0}^{\infty} x rac{x^{lpha-1}e^{-x/eta}}{\Gamma(lpha)eta^{lpha}} dx \ E(X) &= \int\limits_{0}^{\infty} lpha eta rac{x^{lpha}e^{-x/eta}}{\Gamma(lpha+1)eta^{lpha+1}} dx \ E(X) &= lpha eta \int\limits_{0}^{\infty} f(x|lpha+1,eta) dx \ E(X) &= lpha eta \end{aligned}$$

$E(X^2)$

$$egin{align} E(X^2) &= \int\limits_{-\infty}^{\infty} x^2 f(x) dx \ E(X) &= \int\limits_{0}^{\infty} x^2 rac{x^{lpha-1}e^{-x/eta}}{\Gamma(lpha)eta^{lpha}} dx \ E(X) &= lpha eta \int\limits_{0}^{\infty} x rac{x^{lpha}e^{-x/eta}}{\Gamma(lpha+1)eta^{lpha+1}} dx \ E(X) &= lpha eta \int\limits_{0}^{\infty} (lpha+1)eta rac{x^{lpha+1}e^{-x/eta}}{\Gamma(lpha+2)eta^{lpha+2}} dx \ E(X) &= lpha (lpha+1)eta^2 \int\limits_{0}^{\infty} f(x|lpha+2,eta) dx \ E(X) &= lpha (lpha+1)eta^2 \ E(X) &= lpha^2eta^2 + lpha eta^2 \ \end{align}$$

Var(X)

$$egin{split} \operatorname{Var}(X) &= E(X^2) - (E(X))^2 \ \operatorname{Var}(X) &= lpha^2 eta^2 + lpha eta^2 - (lpha eta)^2 \ \operatorname{Var}(X) &= lpha eta^2 \end{split}$$

$$X \sim ext{DExp}(\mu, \sigma) \ f(x) = rac{e^{-|x-\mu|/\sigma}}{2\sigma}$$

E(X)

$$E(X) = \int\limits_{-\infty}^{\infty} x f(x) dx$$
 $E(X) = rac{1}{2\sigma} \int\limits_{-\infty}^{\infty} x e^{-|x-\mu|/\sigma} dx$
 $E(X) = rac{1}{2\sigma} (\int\limits_{-\infty}^{\infty} x e^{-|x-\mu|/\sigma} dx + \int\limits_{\mu}^{\infty} x e^{-|x-\mu|/\sigma} dx)$
 $E(X) = rac{1}{2\sigma} (\int\limits_{-\infty}^{\mu} x e^{(x-\mu)/\sigma} dx + \int\limits_{\mu}^{\infty} x e^{(\mu-x)/\sigma} dx)$
 $\text{let } u = (x-\mu)/\sigma \text{ and } du = dx/\sigma$
 $\text{let } v = (\mu-x)/\sigma \text{ and } dv = -dx/\sigma$
 $E(X) = rac{1}{2\sigma} (\int\limits_{-\infty}^{0} \sigma(\sigma u + \mu) e^u du - \int\limits_{-\infty}^{0} \sigma(\sigma v - \mu) e^v dv)$
 $E(X) = rac{1}{2} (\sigma \int\limits_{-\infty}^{0} u e^u du + \mu \int\limits_{-\infty}^{0} e^u du - \sigma \int\limits_{-\infty}^{0} v e^v dv + \mu \int\limits_{-\infty}^{0} e^v dv)$
 $E(X) = \mu \int\limits_{0}^{\infty} e^u du$

$$E(X) = \mu \int\limits_{-\infty}^{0} e^{u} du$$

$$E(X) = \muigg(e^uigg|_{-\infty}^0igg)$$

$$E(X) = \mu(1)$$

$$E(X) = \mu$$

$E(X^2)$

$$egin{aligned} E(X^2) &= \int\limits_{-\infty}^{\infty} x^2 f(x) dx \ E(X^2) &= rac{1}{2\sigma} \int\limits_{-\infty}^{\infty} x^2 e^{-|x-\mu|/\sigma} dx \ E(X^2) &= rac{1}{2\sigma} (\int\limits_{-\infty}^{\mu} x^2 e^{-|x-\mu|/\sigma} dx + \int\limits_{\mu}^{\infty} x^2 e^{-|x-\mu|/\sigma} dx) \ E(X^2) &= rac{1}{2\sigma} (\int\limits_{-\infty}^{\mu} x^2 e^{(x-\mu)/\sigma} dx + \int\limits_{\mu}^{\infty} x^2 e^{(\mu-x)/\sigma} dx) \ ext{let } u &= (x-\mu)/\sigma ext{ and } du &= dx/\sigma \ ext{let } v &= (\mu-x)/\sigma ext{ and } dv &= -dx/\sigma \ E(X^2) &= rac{1}{2\sigma} (\int\limits_{-\infty}^{0} \sigma(\sigma u + \mu)^2 e^u du + \int\limits_{-\infty}^{0} \sigma(\sigma v - \mu)^2 e^v dv) \end{aligned}$$

$$egin{align} E(X^2) &= rac{1}{2} (\int\limits_{-\infty}^{0} (\sigma u + \mu)^2 e^u du + \int\limits_{-\infty}^{0} (\sigma v - \mu)^2 e^v dv) \ E(X^2) &= \int\limits_{-\infty}^{0} (\sigma^2 u^2 + \mu^2) e^u du \ E(X^2) &= \sigma^2 \int\limits_{-\infty}^{0} u^2 e^u du + \mu^2 \int\limits_{-\infty}^{0} e^u du \ E(X^2) &= \sigma^2 \int\limits_{-\infty}^{0} u^2 e^u du + \mu^2 \ \end{array}$$

u	dv
u^2	e^u
2u	e^u
2	e^u
0	e^u

$$egin{split} E(X^2) &= \sigma^2igg((u^2-2u+2)e^uigg|_{-\infty}^0igg) + \mu^2 \ E(X^2) &= \sigma^2(2) + \mu^2 \ E(X^2) &= 2\sigma^2 + \mu^2 \end{split}$$

Var(X)

$$egin{aligned} \operatorname{Var}(X) &= E(X^2) - (E(X))^2 \ \operatorname{Var}(X) &= 2\sigma^2 + \mu^2 - (\mu)^2 \ \operatorname{Var}(X) &= 2\sigma^2 \end{aligned}$$

3.24

C

$$egin{aligned} X &\sim \mathrm{Gamma}(a,b) \ Y &= 1/X \ g(x) &= 1/x \ g^{-1}(y) &= 1/y \end{aligned} \ F_Y(x) &= F_X(y^{-1}(y)) \ f_Y(y) &= f_X(g^{-1}(y))g^{-1'}(y) \ f_X(x) &= rac{x^{lpha-1}e^{-x/eta}}{\Gamma(lpha)eta^lpha} \ g^{-1'}(y) &= -1/y^2 \end{aligned}$$

$$egin{aligned} f_Y(y) &= rac{(1/y)^{lpha-1}e^{-(1/y)/eta}}{y^2\Gamma(lpha)eta^lpha} \ f_Y(y) &= rac{y^{-lpha+1}e^{-(1/y)/eta}}{y^2\Gamma(lpha)eta^lpha} \ f_Y(y) &= rac{y^{-lpha-1}e^{-1/yeta}}{\Gamma(lpha)eta^lpha} \ f_Y(y) &= rac{e^{-1/yeta}}{\Gamma(lpha)eta^lpha y^{lpha+1}} \sim ext{IGamma}(lpha,eta) \end{aligned}$$

Which is the inverse Gamma distribution's PDF

If we use the substitution $\tilde{\beta}=1/\beta$ we get the more common form of the inverse Gamma.

$$f_Y(y) = rac{ ilde{eta}^lpha e^{- ilde{eta}/y}}{\Gamma(lpha)y^{lpha+1}} \sim ext{IGamma}(lpha, 1/eta)$$

3.28

b

$$X \sim \operatorname{Gamma}(\alpha, \beta)$$

$$f(x)=rac{x^{lpha-1}e^{-x/eta}}{\Gamma(lpha)eta^lpha}$$

$$f(x) = rac{1}{\Gamma(lpha)} e^{(lpha-1) \ln x - x/eta - lpha \ln eta}$$

C

$$X \sim \mathrm{Beta}(lpha,eta)$$

$$f(x)=rac{x^{lpha-1}(1-x)^{eta-1}}{B(lpha,eta)}$$

$$f_X(x)=rac{1}{B(lpha,eta)}e^{(lpha-1)\ln(x)(eta-1)\ln(1-x)}$$

d

$$X \sim \operatorname{Poisson}(\lambda)$$

$$P(X=x)=rac{e^{-\lambda}\lambda^x}{x!}$$

$$P(X=x) = rac{1}{x!}e^{-\lambda}e^{x\ln(\lambda)}$$

3.33

b

 $X \sim ext{Normal}(heta, a heta^2|a)$

$$f(x)=rac{e^{-(x-\mu)^2/(2\sigma^2)}}{\sqrt{2\pi}\sigma}=rac{e^{-(x- heta)^2/(2a heta^2)}}{\sqrt{2\pi a} heta}$$

$$f(x)=rac{e^{-(x- heta)^2/(2a heta^2)}}{\sqrt{2\pi a} heta}$$

$$f(x)=rac{e^{-(x- heta)^2/(2a heta^2)}}{\sqrt{2\pi a} heta} \ f(x)=rac{1}{\sqrt{2\pi a} heta}e^{-(x- heta)^2/(2a heta^2)}$$

$$f(x)=rac{\sqrt{2\pi a heta}}{\sqrt{2\pi a heta}}e^{-(x^2-2x heta+ heta^2)/(2a heta^2)}$$

$$f(x) = rac{1}{\sqrt{2\pi a} heta}e^{-x^2rac{1}{2a heta^2} + xrac{1}{a heta} - rac{1}{2a}}$$

$$f(x)=rac{e^{-1/2a}}{\sqrt{2\pi a}}rac{1}{ heta}e^{-x^2rac{1}{2a heta^2}+xrac{1}{a heta}}$$

$$f(x)=rac{1}{\sqrt{2e^{1/a}\pi a}}rac{1}{ heta}e^{-x^2rac{1}{2a heta^2}+xrac{1}{a heta}}$$

 $(\theta,a\theta^2)$ makes a parabola

$$f(x) = ax^2$$

