walk of length k from v_1 to v_j to create a walk of length $k+1$ from v_i to v_j with v_1 as its second vertex. Thus, by the multiplication rule,

$$a_{i1}b_{1j} = \begin{bmatrix} \text{the number of walks of length } k+1 \text{ from} \\ v_i \text{ to } v_j \text{ that have } v_1 \text{ as their second vertex} \end{bmatrix}.$$

More generally, for each integer $r = 1, 2, \ldots, m$,

$$a_{ir}b_{rj} = \begin{bmatrix} \text{the number of walks of length } k+1 \text{ from} \\ v_i \text{ to } v_j \text{ that have } v_r \text{ as their second vertex} \end{bmatrix}.$$

Because every walk of length $k+1$ from v_i to v_j must have one of the vertices $v_1, v_2, \ldots, v_m$ as its second vertex, the total number of walks of length $k+1$ from v_i to v_j equals the sum in (10.2.1), which equals the ijth entry of $\mathbf{A}^{k+1}$. Hence

$$\text{the } ij\text{th entry of } \mathbf{A}^{k+1} = \text{the number of walks of length } k+1 \text{ from } v_i \text{ to } v_j$$

[as was to be shown].

[Since both the basis step and the inductive step have been proved, the sentence P(n) is true for every integer $n \geq 1$.]

TEST YOURSELF

1. In the adjacency matrix for a directed graph, the entry in the ith row and jth column is ______.

2. In the adjacency matrix for an undirected graph, the entry in the ith row and jth column is ______.

3. An $n \times n$ square matrix is called symmetric if, and only if, for all integers i and j from 1 to n, the entry in row ______ and column ______ equals the entry in row ______ and column ______.

4. The ijth entry in the product of two matrices $\mathbf{A}$ and $\mathbf{B}$ is obtained by multiplying row ______ of $\mathbf{A}$ by row ______ of $\mathbf{B}$.

5. In an $n \times n$ identity matrix, the entries on the main diagonal are all ______ and the off-diagonal entries are all ______.

6. If G is a graph with vertices $v_1, v_2, \ldots, v_m$ and $\mathbf{A}$ is the adjacency matrix of G, then for each positive integer n and for all integers i and j with $i, j = 1, 2, \ldots, m$, the ijth entry of $\mathbf{A}^n =$ ______.

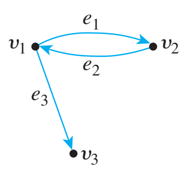
EXERCISE SET 10.2

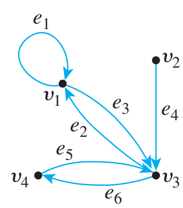
1. Find real numbers a, b, and c such that the following are true.

   a. $\begin{bmatrix} a+b & a-c \\ c & b-a \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 3 \end{bmatrix}$

   b. $\begin{bmatrix} 2a & b+c \\ c-a & 2b-a \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 1 & -2 \end{bmatrix}$

2. Find the adjacency matrices for the following directed graphs.

   a. 

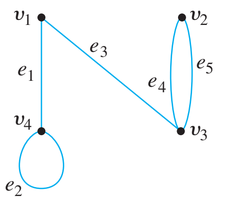
   b. 

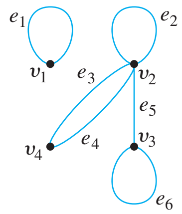
3. Find directed graphs that have the following adjacency matrices:

   a. $\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$

   b. $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

4. Find adjacency matrices for the following (undirected) graphs.

a. 

b. 

c. K_4, the complete graph on four vertices

d. $K_{2,3}$, the complete bipartite graph on (2, 3) vertices

5. Find graphs that have the following adjacency matrices.

a. $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}$ **b.** $\begin{bmatrix} 0 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

6. The following are adjacency matrices for graphs. In each case determine whether the graph is connected by analyzing the matrix without drawing the graph.

a. $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ **b.** $\begin{bmatrix} 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

7. Suppose that for every positive integer i, all the entries in the ith row and ith column of the adjacency matrix of a graph are 0. What can you conclude about the graph?

8. Find each of the following products.

a. $[2 \quad -1]\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ **b.** $[4 \quad -1 \quad 7]\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

9. Find each of the following products.

a. $\begin{bmatrix} 3 & 0 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 1 & -1 & 4 \\ 0 & 2 & 1 \end{bmatrix}$

b. $\begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 5 & -4 \\ -2 & 2 \end{bmatrix}$

c. $\begin{bmatrix} -1 \\ 2 \end{bmatrix}[2 \quad 3]$

d. $\begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}^2$

10. Let $\mathbf{A} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & -2 & 1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} -2 & 0 \\ 1 & 3 \end{bmatrix}$, and $\mathbf{C} = \begin{bmatrix} 0 & -2 \\ 3 & 1 \\ 1 & 0 \end{bmatrix}$.

For each of the following, determine whether the indicated product exists, and compute it if it does.

a. **AB** **b.** **BA** **c.** $\mathbf{A}^2$ **d.** **BC** **e.** **CB**
f. $\mathbf{B}^2$ **g.** $\mathbf{B}^3$ **h.** $\mathbf{C}^2$ **i.** **AC** **j.** **CA**

11. Give an example different from that in the text to show that matrix multiplication is not commutative. That is, find 2×2 matrices **A** and **B** such that **AB** and **BA** both exist but $\mathbf{AB} \neq \mathbf{BA}$.

12. Let O denote the matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Find 2×2 matrices **A** and **B** such that $\mathbf{A} \neq \text{O}$ and $\mathbf{B} \neq \text{O}$ but $\mathbf{AB} = \text{O}$.

13. Let O denote the matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Find 2×2 matrices **A** and **B** such that $\mathbf{A} \neq \mathbf{B}$, $\mathbf{B} \neq \text{O}$, and $\mathbf{AB} \neq \text{O}$, but $\mathbf{BA} = \text{O}$.

In 14–18, assume the entries of all matrices are real numbers.

H **14.** Prove that if **I** is the $m \times m$ identity matrix and **A** is any $m \times n$ matrix, then $\mathbf{IA} = \mathbf{A}$.

15. Prove that if **A** is an $m \times m$ symmetric matrix, then $\mathbf{A}^2$ is symmetric.

16. Prove that matrix multiplication is associative: If **A**, **B**, and **C** are any $m \times k$, $k \times r$, and $r \times n$ matrices, respectively, then $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$. (*Hint*: Summation notation is helpful.)

17. Use mathematical induction and the result of exercise 16 to prove that if **A** is any $m \times m$ matrix, then $\mathbf{A}^n\mathbf{A} = \mathbf{A}\mathbf{A}^n$ for each integer $n \geq 1$.

18. Use mathematical induction to prove that if **A** is an $m \times m$ symmetric matrix, then for any integer $n \geq 1$, $\mathbf{A}^n$ is also symmetric.