1

Chris is on a skateboard that is traveling at a constant horizontal speed $v_x = 4.35\ m/s$. They hold a ball in their hand at a height of $h = 1.22\ m$ above the ground and then they toss the ball “up” (as seen in their frame) so that they catch it back in their hand exactly $T = 2.85s$ later.

a

Relative to Chris’ hand, what is the speed of the ball as it it travels upward? Explain your answer.

✓ **Answer**

Conservation of energy
$v_i = -v_f$

The ball only has the force of gravity acting upon it.
$a_b = -g$

Definition of velocity given acceleration
$v(t) = v_0 + at$
$v(t) = v_i - gt$

$-v_i = v_f = v(T) = v_i - gT$
$v_i = \frac{gT}{2}$

$v(t) = \frac{gT}{2} - gt$

$v(t) = \frac{9.8 \cdot 2.85}{2} - 9.8t\ \frac{m}{s}$
$v(t) = 13.965 - 9.8t\ \frac{m}{s}$
Where $13.965\ \frac{m}{s}$ is the initial vertical upwards velocity

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$v(t) = \frac{gT}{2} - gt$
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Where $13.965\ \frac{m}{s}$ is the initial vertical upwards velocity
☐

b

Relative to the fixed ground, what is the maximum height of the ball in the air? Explain your answer.

✓ **Answer**

There will be maxima and minima in the ball's vertical position when the velocity is equal to 0.

$$v(t_{max}) = \frac{gT}{2} - gt_{max}$$
$$gt_{max} = \frac{gT}{2}$$
$$t_{max} = \frac{T}{2}$$

Definition of position given velocity

$$y(t) = y_0 + \int_0^t v(s)\, ds$$
$$y(t) = y_0 + \frac{gTt}{2} - \frac{gt^2}{2}$$

$$y(t_{max}) = h + \frac{gTt_{max}}{2} - \frac{gt_{max}^2}{2}$$
$$y(t_{max}) = h + \frac{gT^2}{4} - \frac{gT^2}{8}$$
$$y(t_{max}) = h + \frac{gT^2}{8}$$

$$y(t_{max}) = 1.22 + \frac{9.8(2.85)^2}{8}\ m$$
$$y(t_{max}) = 11.17\ m$$

$$y(t_{max}) = y_0 + \frac{gT^2}{8}$$
$$y(t_{max}) = 11.17\ m$$
$$\square$$

c

Suppose there is a tiny motionless bug on the ground immediately below Chris' hand at the instant when they toss the ball upward. Suppose we define standard unit vectors $\hat{\imath}$ and $\hat{\jmath}$ corresponding to the frame of reference for the bug. Write down vector expressions for the position, velocity, and acceleration of the ball $(\vec{r}, \vec{v}, \vec{a})$ as a function of time, t as seen by the bug at the instant of time just before the ball is caught by Chris. Explain your answer.

✓ **Answer**

The vectors from the hand to the ball can be given by (as calculated previously):

$\vec{p}(t) = \langle 0, \frac{gTt}{2} - \frac{gt^2}{2} \rangle$
$\vec{v}(t) = \langle 0, \frac{gT}{2} - gt \rangle$
$\vec{a}(t) = \langle 0, -g \rangle$

And the vectors from the bug to the hand can be given by

$\vec{a}(t) = \langle 0, 0 \rangle$ as neither object is accelerating
$\vec{v}(t) = \langle v_x, 0 \rangle$ movement of the skateboard
$\vec{p}(t) = \langle tv_x, h \rangle$ integral of velocity with initial displacement

The overall vectors can be given by the sum:

$\vec{p}(t) = \langle tv_x, h + \frac{gTt}{2} - \frac{gt^2}{2} \rangle$
$\vec{v}(t) = \langle v_x, \frac{gT}{2} - gt \rangle$
$\vec{a}(t) = \langle 0, -g \rangle$

$\vec{p}(t) = \langle 4.35t, 1.22 + 13.965t - 4.9t^2 \rangle$
$\vec{v}(t) = \langle 4.35, 13.965 - 9.8t \rangle$
$\vec{a}(t) = \langle 0, -9.8 \rangle$

$\vec{p}(t) = \langle tv_x, h + \frac{gTt}{2} - \frac{gt^2}{2} \rangle$
$\vec{v}(t) = \langle v_x, \frac{gT}{2} - gt \rangle$
$\vec{a}(t) = \langle 0, -g \rangle$

$\vec{p}(t) = \langle 4.35t, 1.22 + 13.965t - 4.9t^2 \rangle$
$\vec{v}(t) = \langle 4.35, 13.965 - 9.8t \rangle$
$\vec{a}(t) = \langle 0, -9.8 \rangle$

Where $\langle a, b \rangle = a\hat{\imath} + b\hat{\jmath}$
$\square$

2

Leonard Nimoy stands at the edge of a cliff at a given elevation h above the still waters of Lake Erie as shown in the figure. He throws a tennis ball at a given speed v_0 at a given angle of θ above the horizontal direction towards the Lake. Then he throws a second tennis ball at the same speed v_0 but downwards at an given angle ϕ below the horizontal. Here assume $\theta > \phi$.

a

What is the maximum vertical distance relative to the surface of the Lake that the first tennis ball will achieve? Give you answer in terms of the given parameters (not time). Explain how you got you answer in one or two sentences.

✓ **Answer**

We can find the max height using the conservation of energy.
$K_i = mv_0^2$ Where m is the mass of the tennis ball
$U_i = mgh$ No gravitational potential at the start

$K_f = mv_f^2$
$U_f = mgh_f$

$K_i + U_i + W = K_f + U_f$ No work other than gravity is being done, so $W = 0$

$mv_0^2 + mgh = mv_f^2 + mgh_f$
$v_0^2 + gh = v_f^2 + gh_f$

When the ball is at its max height, its vertical velocity will be zero.
Additionally, we know nothing influences the horizontal velocity, so it must remain constant

$\vec{v}_0 = v_0\langle\cos\theta, \sin\theta\rangle$
$\vec{v}_f = v_0\langle\cos\theta, 0\rangle$

$v_0^2 + gh = (v_0\cos\theta)^2 + gh_f$
$h_f = \frac{v_0^2 - (v_0\cos\theta)^2}{g} + h$
$h_f = \frac{v_0^2(1 - \cos^2\theta)}{g} + h$
$h_f = \frac{(v_0\sin\theta)^2}{g} + h$

$h_f = \frac{(v_0\sin\theta)^2}{g} + h$
□

b

Write down three vector expressions – one for the position, one for the velocity and one for the acceleration – corresponding to the motion of the first tennis ball just at the instant it achieves maximum vertical distance relative to the surface of the Lake. Your answer should be expressed in terms of given parameters only; note that if you have an expression that is a function of time t you are not done. Your expression should include relevant Cartesian basis unit-vectors. Explain how you got you answer in one or two sentences.

✓ **Answer**

Since the only force acting upon the tennis ball is the force of gravity.
$\vec{a} = \langle 0, -g \rangle$

We have already calculated the velocity in part (a).
$\vec{v} = \langle v_0 \cos\theta, 0 \rangle$

As for the position, we already know the max height
$h_y = \frac{(v_0 \sin\theta)^2}{g} + h$

We can calculate the time it took to the peak by using the velocity acceleration relationship.
$\vec{v}(t) = \vec{v}_0 + \vec{a}t$
$v_0\langle \cos\theta, 0 \rangle = v_0\langle \cos\theta, \sin\theta \rangle + \langle 0, -g \rangle t$
$0 = v_0 \sin\theta - gt$
$t = \frac{v_0 \sin\theta}{g}$

We can then use the velocity position relation to find the x position.
$x = x_0 + v_x t$
$x = \frac{v_0^2 \sin\theta \cos\theta}{g}$

$\vec{p} = \langle \frac{v_0^2 \sin\theta \cos\theta}{g}, \frac{(v_0 \sin\theta)^2}{g} + h \rangle$

$\vec{a} = \langle 0, -g \rangle$
$\vec{v} = \langle v_0 \cos\theta, 0 \rangle$
$\vec{p} = \langle \frac{v_0^2 \sin\theta \cos\theta}{g}, \frac{(v_0 \sin\theta)^2}{g} + h \rangle$

Where $\langle a, b \rangle = a\hat{x} + b\hat{y}$
□

C

The first tennis ball makes a parabolic curved path as it travels through the air. Calculate the radius of curvature on that path at the single point corresponding to when the tennis ball has maximum vertical distance relative to the surface of the Lake. Your answer should be expressed in terms of given parameters only; note that if you have an expression of time t you are not done. Explain how you got your answer in one or two sentences.

✓ **Answer**

We can use the $\frac{v^2}{a} = R$ formula to obtain the radius of curvature.

$v = v_0 \cos\theta$
$a = g$
$R = \frac{(v_0 \cos\theta)^2}{g}$

$R = \frac{(v_0 \cos\theta)^2}{g}$
$\square$

d

Which ball (the first ball or the second ball) will have a greater speed in the instant just prior to impact with the Lake? Explain your answer in one or two sentences.

✓ **Answer**

They will have the same speed due to the conservation of energy. They both begin with the same kinetic and gravitational potential energy, and both end with the same gravitational potential energy (height), so must also both end with the same kinetic energy (speed).

3

You are standing a given horizontal distance D from the base of the Leaning Tower of Pisa which has a given height h. Assume that your height and the diameter of the tower and the deflection of the tower are all negligible compared to the height of the tower. Neglect air resistance.
At what speed v_0 and at what angle θ relative to the horizontal should you throw the ball so that the ball just barely clears the top of the tower at its maximum height, as shown? Give your answers in terms of given parameters only. Explain your work.

✓ **Answer**

We can calculate the vertical speed by using the law of conservation of energy.

We initially start with no gravitational potential and end with only horizontal speed.

$\vec{v}_i = v_0 \langle \cos\theta, \sin\theta \rangle$
$\vec{v}_f = v_0 \langle \cos\theta, 0 \rangle$

$K_i = \frac{1}{2}mv_0^2$
$K_f = \frac{1}{2}mv_0^2\cos^2\theta$
$U_i = 0$
$U_f = mgh$

$v_0^2 = v_0^2\cos^2\theta + 2gh$
$v_0^2(1-\cos^2\theta) = 2gh$
$v_0^2\sin^2\theta = 2gh$

We can also use the velocity acceleration relation to find the time it took to get to the top.

$v_y(t) = v_{y0} + a_y t$
$0 = v_0\sin\theta - gt$
$t = \frac{v_0\sin\theta}{g}$

We also can use the position velocity relation to relate the distance to the tower to horizontal velocity.

$x(t) = x_0 + v_x t$
$D = \frac{v_0^2\sin\theta\cos\theta}{g}$

We now have this system of equations:

$$\begin{cases} v_0^2\sin^2\theta = 2gh \\ D = \frac{v_0^2\sin\theta\cos\theta}{g} \end{cases}$$
$$\begin{cases} v_0^2\sin^2\theta = 2gh \\ D = \frac{2h\cos\theta}{\sin\theta} \end{cases}$$
$$\begin{cases} \tan\theta = \frac{2h}{D} \\ v_0 = \frac{\sqrt{2gh}}{\sin\theta} \end{cases}$$
$$\begin{cases} \theta = \arctan\frac{2h}{D} \\ v_0 = \frac{\sqrt{2gh}}{\sin\arctan\frac{2h}{D}} \end{cases}$$

$$\begin{cases} \theta = \arctan\frac{2h}{D} \\ v_0 = \frac{\sqrt{2gh}}{\sin\arctan\frac{2h}{D}} \end{cases}$$
□

4

Suppose a particle has a position which is defined by a position vector expression as follows:

$\vec{r}(t) \equiv R\cos(\omega t)\hat{\imath} + R\sin(\omega t)\hat{\jmath}$
Prove that this represents Uniform Circular Motion. In other words, prove that the following

a

Prove that this particle travels on a path that is a circle by demonstrating that the particle is always found at a distance R from the origin.

✓ **Answer**

Taking the formula for $\vec{r}$ and converting it into standard vector notation,

$\vec{r}(t) = R\langle\cos(\omega t), \sin(\omega t)\rangle$
$\|\vec{r}(t)\| = \sqrt{R^2(\cos^2(\omega t) + \sin^2(\omega t))}$
$\|\vec{r}(t)\| = R$
Where R is a given constant.

$\therefore \|\vec{r}(t)\| = R$
$\square$

b

Prove that this particle has a constant speed v, where $v = \omega R$.

✓ **Answer**

Velocity is defined as the derivative of position.
$\vec{v}(t) = \frac{d}{dt}\vec{r}(t)$
$\vec{v}(t) = R\omega\langle-\sin(\omega t), \cos(\omega t)\rangle$
$\|\vec{v}(t)\| = \sqrt{(R\omega)^2(\cos^2(\omega t) + \sin^2(\omega t))}$
$\|\vec{v}(t)\| = R\omega$
Where both R, ω are given constants

$\therefore \|\vec{v}(t)\| = R\omega$
$\square$

c

Prove that the acceleration vector always has fixed magnitude $\frac{v2}{R}$ and points toward the center of the circular path.

Answer

Acceleration is defined as the derivative of velocity.
$\vec{a}(t) = \frac{d}{dt}\vec{v}(t)$
$\vec{a}(t) = R\omega^2\langle -\cos(\omega t), -\sin(\omega t)\rangle$
$\|\vec{a}(t)\| = \sqrt{(R\omega^2)^2(\cos^2(\omega t) + \sin^2(\omega t))}$
$\|\vec{a}(t)\| = R\omega^2$
Where both R, ω are given constants

Additionally we can also prove that it points towards the center of the path as the unit vector of acceleration and position are directly anti-parallel. We can also prove that acceleration and velocity are always perpendicular.

$\mathbf{e}_{\vec{r}} = \langle\cos(\omega t), \sin(\omega t)\rangle$
$\mathbf{e}_{\vec{a}} = -\langle\cos(\omega t), \sin(\omega t)\rangle$
$\mathbf{e}_{\vec{a}} = -\mathbf{e}_{\vec{r}}$

As $\vec{r}$ points from the center of the circle to the position, $\vec{a}$ being its opposite means it points from the position to the center of the circle.

To prove orthogonality of $\vec{a}$ and $\vec{v}$ we can use the dot product. The vectors being non-zero and orthogonal will make their dot product zero.

$\vec{v} \cdot \vec{a} = R\omega\langle -\sin(\omega t), \cos(\omega t)\rangle \cdot R\omega^2\langle -\cos(\omega t), -\sin(\omega t)\rangle$
$\vec{v} \cdot \vec{a} = R^2\omega^3(\langle -\sin(\omega t), \cos(\omega t)\rangle \cdot \langle -\cos(\omega t), -\sin(\omega t)\rangle)$
$\vec{v} \cdot \vec{a} = R^2\omega^3(\sin(\omega t)\cos(\omega t) - \cos(\omega t)\sin(\omega t))$
$\vec{v} \cdot \vec{a} = R^2\omega^3(0)$
$\vec{v} \cdot \vec{a} = 0$

$\therefore \|\vec{a}(t)\| = R\omega^2$
$\mathbf{e}_{\vec{a}} = -\mathbf{e}_{\vec{r}}$
$\vec{v} \cdot \vec{a} = 0$
$\square$

5

A particle moves on a 'spirograph' pattern as shown above which is defined by the following parameterization:

$\vec{r} = [P\cos(\Omega t) + Q\cos(\omega t)]\hat{i} + [P\sin(\Omega t) + Q\sin(\omega t)]\hat{j}$

Where $P = 4\ m$, $Q = 1\ m$, $\Omega = 2\pi\ \frac{rad}{s}$, and $\omega = 30\pi\ \frac{rad}{s}$.

The plot shown corresponds to the path of the particle for $0 < t < 1\ s$. Note that Point A as shown corresponds to the position at both $t = 0$ and $t = 1\ s$.

a

What is value of the time t corresponding to the position of the particle at point B as shown above? Explain or justify your answer. Hint: you cannot "solve for time" here. A better strategy is to guess the answer and then prove your guess is correct by calculating the position at that time.

Answer

We actually can solve for time.
We know the x position will be 0 and the y velocity will be 0, and that the y position is positive.

Putting $\vec{r}$ into standard vector notation.
$\vec{r} = \langle P\cos(\Omega t) + Q\cos(\omega t), P\sin(\Omega t) + Q\sin(\omega t)\rangle$

We also know that the function is periodic upon intervals of $1\ s$ by the coefficients of the trig functions.
$0 \leq t \leq 1$

With velocity as the derivative of position,
$\vec{v} = \langle -P\Omega\sin(\Omega t) - Q\omega\sin(\omega t), P\Omega\cos(\Omega t) + Q\omega\cos(\omega t)\rangle$

$$\begin{cases} P\cos(\Omega t) + Q\cos(\omega t) = 0 \\ P\Omega\cos(\Omega t) + Q\omega\cos(\omega t) = 0 \\ P\sin(\Omega t) + Q\sin(\omega t) > 0 \end{cases}$$
$$\begin{cases} P(\Omega - \omega)\cos(\Omega t) = 0 \\ Q(\Omega - \omega)\cos(\omega t) = 0 \\ P\sin(\Omega t) + Q\sin(\omega t) > 0 \end{cases}$$
$$\begin{cases} \cos(\Omega t) = 0 \\ \cos(\omega t) = 0 \\ P\sin(\Omega t) + Q\sin(\omega t) > 0 \end{cases}$$
$$\begin{cases} \Omega t = \frac{\pi}{2} + \pi C \\ \omega t = \frac{\pi}{2} + \pi D \\ P\sin(\Omega t) + Q\sin(\omega t) > 0 \end{cases}$$

$$\begin{cases} \frac{\Omega}{\pi}t = 0.5 + C \\ \frac{\omega}{\pi}t = 0.5 + D \\ P\sin(\Omega t) + Q\sin(\omega t) > 0 \end{cases}$$

$$\begin{cases} 2t = 0.5 + C \\ 30t = 0.5 + D \\ P\sin(\Omega t) + Q\sin(\omega t) > 0 \end{cases}$$
$$\begin{cases} t \in \{0.25, 0.75\} \\ t \in \left\{\frac{0.5}{30}, \ldots, 0.25, \ldots, 0.75, \ldots\right\} \\ P\sin(\Omega t) + Q\sin(\omega t) > 0 \end{cases}$$
$$\begin{cases} t \in \{0.25, 0.75\} \\ P\sin(\Omega t) + Q\sin(\omega t) > 0 \end{cases}$$
$$t = 0.25\ s$$

$$\begin{cases} \frac{\Omega}{\pi} t = 0.5 + C \\ \frac{\omega}{\pi} t = 0.5 + D \\ P\sin(\Omega t) + Q\sin(\omega t) > 0 \end{cases}$$
$$t = 0.25\ s$$
$$\square$$

b

What is the speed of the particle at point B above? Explain your answer.

✓ **Answer**

Using our earlier definition of the velocity,

$$\vec{v} = \langle -P\Omega\sin(\Omega t) - Q\omega\sin(\omega t), P\Omega\cos(\Omega t) + Q\omega\cos(\omega t) \rangle$$

$$s = ||\vec{v}||$$

$$\vec{v} = \left\langle -8\pi\sin\left(\frac{\pi}{2}\right) - 30\pi\sin\left(\frac{15\pi}{2}\right), 8\pi\cos\left(\frac{\pi}{2}\right) + 30\pi\cos\left(\frac{15\pi}{2}\right) \right\rangle$$
$$s = \sqrt{(-8\pi\sin\left(\frac{\pi}{2}\right) - 30\pi\sin\left(\frac{15\pi}{2}\right))^2 + (8\pi\cos\left(\frac{\pi}{2}\right) + 30\pi\cos\left(\frac{15\pi}{2}\right))^2}$$
$$s = 22\pi\ \frac{m}{s}$$

$$s = 22\pi\ \frac{m}{s}$$
$$\square$$

c

Write down a vector expression for the total acceleration of the particle at at point B. Explain your answer.

✓ **Answer**

We can derive acceleration from velocity
$\vec{a} = \langle -P\Omega^2 \cos(\Omega t) - Q\omega^2 \cos(\omega t), -P\Omega^2 \sin(\Omega t) - Q\omega^2 \sin(\omega t) \rangle$

$\vec{a} = \langle 0, -4(2\pi)^2 + (30\pi)^2 \rangle$
$\vec{a} = \langle 0, (900 - 16)\pi^2 \rangle$
$\vec{a} = \langle 0, 884\pi^2 \rangle$

$\vec{a} = \langle 0, 884\pi^2 \rangle \ \frac{m}{s^2}$
$\square$

d

Use your answers to parts (b) and (c) above to calculate the radius of curvature of the path at point B. Also try to estimate the radius of curvature directly from the graph. Do you get the same approximate result? Explain.

✓ **Answer**

$r = \frac{v^2}{a}$
$r = \frac{484}{884}$

$r = \frac{484}{884}\ m \approx 0.5475\ m$
I would have guessed somewhere between $0.5\ m$ and $1\ m$, so this is fairly accurate.
$\square$