Analysis of LTICs

Standard Form

LTICs are able to be formatted in the form:

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{dy(t)}{dt} + a_N y(t)$$
$$= b_{N-M} \frac{d^M x(t)}{dt^M} + b_{N-M+1} \frac{d^{M-1} x(t)}{dt^{M-1}} + \cdots + b_{N-1} \frac{dx(t)}{dt} + b_N x(t)$$

They can also be represented in differential operator form:

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) y(t)$$
$$= (b_{N-M} D^M + b_{N-M+1} D^{M-1} + \cdots + b_{N-1} D + b_N) x(t)$$

In Matrix form, the formula appears completely simple:

$$\vec{D}'_N y(t) \vec{a} = \vec{D}'_M x(t) \vec{b}$$

Where $\vec{D}_n$ is the vector of increasing-order derivative operators from 0 up to order n.

$\vec{D}'_N \vec{a}$ is typically referred to as $Q(D)$ and $\vec{D}'_M \vec{b}$ as $P(D)$

Properties of this Form

When $M > N$, this LTIC displays unwanted properties, including being unstable.

Total Response Decomposition

Total Response = Zero-Input Response + Zero-State Response

$$y(t) = y_{zir}(t) + y_{zsr}(t)$$

This is due to the fact that LTICs are linear, and thus satisfy the superposition principle.

$$\vec{D}'_N y_0(t) \vec{a} = 0$$
$$\vec{D}'_N y(t) \vec{a} = \vec{D}'_M x(t) \vec{b}$$
$$\implies \vec{D}'_N (y(t) + y_0(t)) \vec{a} = \vec{D}'_M x(t) \vec{b}$$

Zero Input Response (ZIR)

The zero input response is the response of y when the input $x(t) = 0$.

The ZIR is typically represented with the subscript of $_0$ or $_{zir}$.

This results in the following expression for the system:
$$\vec{D}'_N y_0(t) \vec{a} = 0$$

Since multiple derivatives of our equation must sum to 0 over all t, we can safely assume the form of $y_0(t)$ to be of the form $ce^{\lambda t}$.

This now produces the equation:

$$c\vec{a}'\vec{\lambda}_N e^{\lambda t} = 0$$

Where $\vec{\lambda}_n$ represents the vector of increasing powers of the constant λ, from λ^0 to λ^n

Ignoring trivial solutions for λ provides:

$$\vec{a}'\vec{\lambda}_N = 0 = Q(\lambda)$$

This formula here is also known as the characteristic equation of the system, as it finds all possible λ.

This is then factorized to find the roots.

$$\vec{a}'\vec{\lambda}_N = \prod_{n=1}^{N}(\lambda - \lambda_n)$$

We now call the vector of all λ_n, $\vec{\lambda}_N^*$

$$\vec{a}'\vec{\lambda}_N = \prod \lambda - \vec{\lambda}_N^*$$

The resulting general solution will be of the following form, assuming no repeated values in $\vec{\lambda}_N^*$

$$y_0(t) = \vec{c}' e^{t\vec{\lambda}_N^*}$$

Where $\vec{c}$ are arbitrary constants determined by the original constraints of the problem.

Repeated Roots

Since the prior solution required our λ to be unique, the solution does not work for repeated roots.

Let $\vec{\lambda}_N^{**}$ be the unique values of $\vec{\lambda}_N^*$
The following is a re-writing of the original general solution as similar λ's constants combine linearly.

$$y_0(t) = \sum_{\lambda \in \vec{\lambda}_N^{**}} c_\lambda e^{t\lambda}$$

However, the solution to the differential equation $(D-\lambda)^n y = 0$ has the general form of $\vec{c}'\vec{t}_\lambda e^{\lambda t}$

Where $\vec{t}_\lambda$ is a vector from $t^0 \ldots t^{n-1}$ where n is the number of times λ is repeated.

Substituting this general solution into our original solution yields

$$y_0(t) = \sum_{\lambda \in \vec{\lambda}_N^{**}} \vec{c}'_\lambda \vec{t}_\lambda e^{\lambda t}$$

In general this can be treated as:

$$y_0(t) = \vec{c}'(\vec{t} \odot e^{t\vec{\lambda}_N^*})$$

Where $\vec{t}$ is the vector containing the correspondent t^n for each duplicate in λ

Imaginary Roots

There is nothing special about imaginary roots. In real systems, if imaginary roots are present, they must appear in conjugate pairs.

Unit Impulse Response

The solving process is mainly similar to solving for the ZIR.

First, obtain our $\vec{\lambda}_N^*$:

$$\vec{a}'\vec{\lambda}_N = \prod \lambda - \vec{\lambda}_N^*$$

Our general solution to $h(t)$ will be of the form:

$$h(t) = b_0\delta(t) + u(t) \sum_{\lambda \in \vec{\lambda}_N^{**}} \vec{c}'_\lambda \vec{t}_\lambda e^{\lambda t}$$ or $$h(t) = b_0\delta(t) + \vec{c}'(\vec{t} \odot e^{t\vec{\lambda}_N^*})u(t)$$

Where $u(t)$ is the unit step function.

Substituting back into our original system:

$$\vec{D}'_N y(t)\vec{a} = \vec{D}'_M x(t)\vec{b} \implies \vec{D}'_N h(t)\vec{a} = \vec{D}'_M \delta(t)\vec{b}$$

Where $h(t)$ is the Unit Response

The derivatives of $h(t)$ around $t = 0$ will look similar to:

$$\dot{h}(t) = \vec{c}'(\vec{t} \odot \vec{\lambda}_N^* \odot e^{t\vec{\lambda}_N^*})\delta(t) \implies \dot{h}(t) = K_1\delta(t) \quad \text{around } 0$$
$\ddot{h}(t) = K_1\dot{\delta}(t) + K_2\delta(t)$ by taking the derivative of the previous equation, and adding a term to account for the possibility of $\dot{h}(t)$ being discontinuous at $t = 0$
This process repeats for further derivatives of $h(t)$

This also means that $K_1 = h(0^+)$, $K_2 = \dot{h}(0^+)$, etc.

Around 0, we may simplify our system:

$$\vec{a}'\text{Toeplitz}(\vec{K}) = \vec{b}$$

Which allows us to solve for $\vec{K}$
This also means that $\vec{D}'_N h(0^+)\vec{1} = \vec{K}$

Substituting in our original system,

$\vec{D}'_N \vec{c}'(\vec{t} \odot e^{t\vec{\lambda}^*_N})\vec{1} = \vec{K}$

Which allows us to solve for $\vec{c}$

Thus finalizing our solution for $h(t)$ as:

$h(t) = b_0\delta(t) + \vec{c}'(\vec{t} \odot e^{t\vec{\lambda}^*_N})u(t)$