4

PHYS 122 HW 4

1

Energy of a dipole

a*

Dipole moment

An ideal electric dipole moment is a pair of equal and opposite charges $+q$ and $-q$ separated by a distance a. The "dipole moment" of the pair is a vector $\vec{p}$ defined as $\vec{p} = q\vec{r}$ where $\vec{r}$ is the displacement vector from the negative to the positive charge. The magnitude of the dipole moment is qa.

b

Force on a dipole

Consider a dipole moment immersed in a uniform electric field $\vec{E}$. What is the total force on the dipole?

Answer

0 as whatever force effects one will have the opposite effect on the other.

c

Torque on a dipole

Assume the negative charge is at the origin and the position of the positive charge is $\vec{r}$. Assume that the dipole is immersed in a uniform electric field $\vec{E}$. What is the torque on the dipole? Give your answer in terms of $\vec{p}$ and $\vec{E}$.

Answer

$$F_- = -q\vec{E}$$
$$F_+ = q\vec{E}$$

$\tau_- = \frac{q}{2}\vec{r} \times \vec{E}$
$\tau_+ = \frac{q}{2}\vec{r} \times \vec{E}$

$\tau = q\vec{r} \times \vec{E}$
$\tau = \vec{p} \times \vec{E}$

d

Electrostatic Energy

Again assume that the negative charge is at the origin and the position of the positive charge is $\vec{r}$.

Assume that the dipole is immersed in a uniform electric field $\vec{E}$. What is the electrostatic potential energy of the dipole? Give your answer in terms of $\vec{p}$ and $\vec{E}$.

Hint: Recall that for a charge Q at position $\vec{R}$ in a uniform electric field $\vec{E}\hat{k}$ the electrostatic energy is $-Q\vec{E} \cdot \vec{R}$

✓ Answer

$U = q\vec{E} \cdot \vec{r} + 0$
$U = \vec{E} \cdot \vec{p}$

2

Equipotential Lines: Equipotentials for two point charges

a

Two point charges Q are located at $(\pm R, 0, 0)$. Find the electric field at all points on the z-axis. For what value of z is the field maximum?

✓ Answer

$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$

$\vec{E}(z) = \frac{1}{2\pi\epsilon_0}\frac{zq}{(R^2+z^2)^{3/2}}\hat{k}$
$\vec{E}'(z) = \frac{(R^2+z^2)^{3/2} - 3z^2(R^2+z^2)^{1/2}}{(R^2+z^2)^3}$
$\vec{E}'(z) = 0$
$(R^2+z^2)^{3/2} = 3z^2(R^2+z^2)^{1/2}$

$R^2 + z^2 = 3z^2$

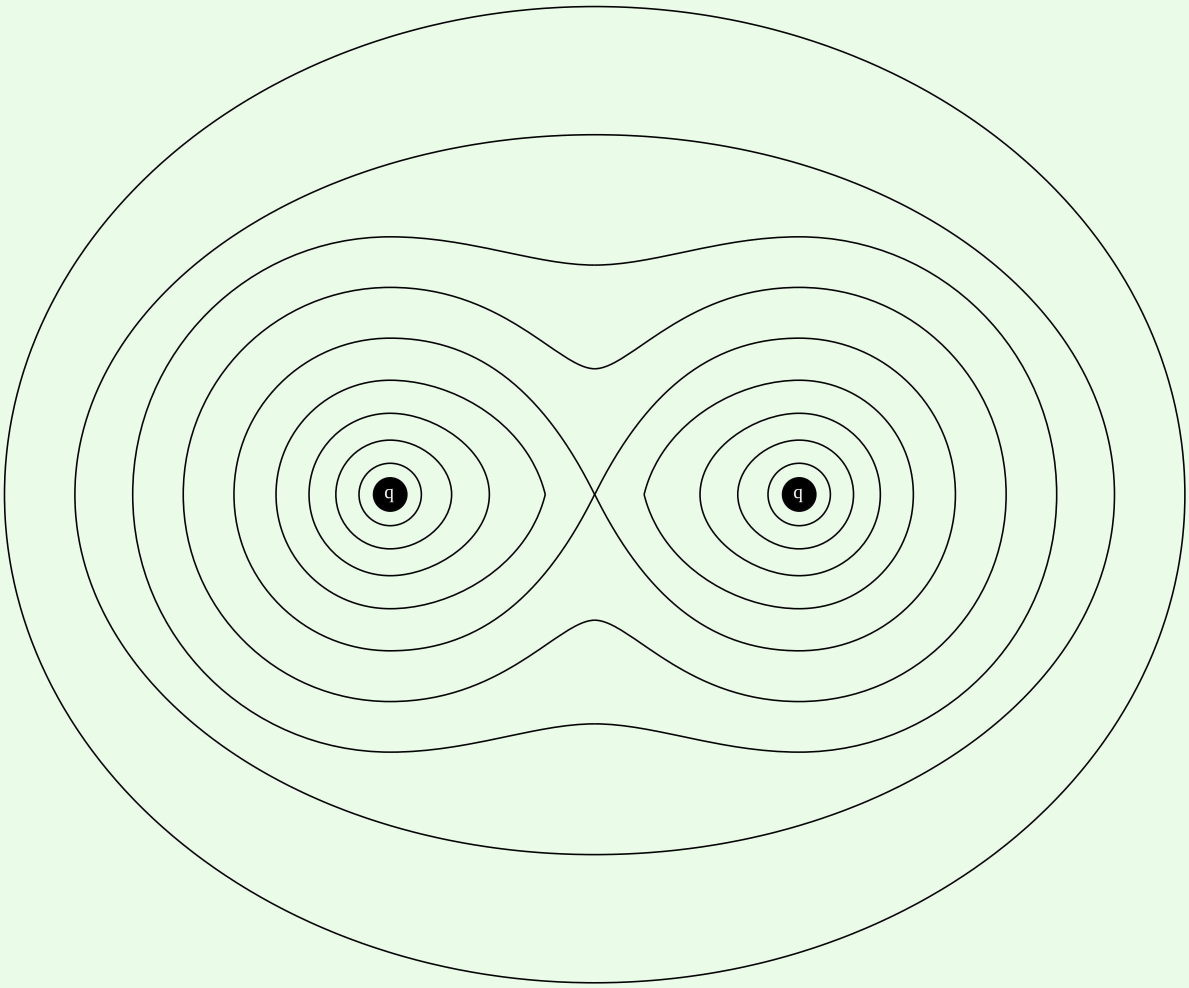
$z = \pm\sqrt{\frac{R^2}{2}}$

□

b

Make a rough sketch of the equipotential curves everywhere in space (or rather, everywhere in the xz plane). Be sure to indicate what the curves look like very close to and very far from the charges, and how the transition from close to far occurs.

✓ **Answer**



3

Dipole potential

Two particles of charge $+q$ and $-q$ respectively are placed on the z-axis symmetrically about the origin, each a distance a from the origin.

a

What is the dipole moment $\vec{p}$ of the two charges? Give your answer in terms of q, a, $\hat{i}$, $\hat{j}$ and $\hat{k}$

Hint: Recall that the dipole moment $\vec{p}$ is defined as $\vec{p} = q\vec{R}$ where $\vec{R}$ is the displacement vector from the negative charge to the positive

✓ **Answer**

$\vec{p} = q\vec{R}$
$\vec{p} = 2aq\hat{k}$
$\square$

b

What is the electrostatic potential at a point $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Hint: Use superposition

Give your answer in terms of x, y, z, a, q and ϵ_0

✓ **Answer**

$\phi = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$
$\phi_{e-} = \frac{1}{4\pi\epsilon_0}\frac{-q}{\sqrt{x^2+y^2+(z+a)^2}}$
$\phi_e = \frac{1}{4\pi\epsilon_0}\frac{q}{\sqrt{x^2+y^2+(z-a)^2}}$
$\phi(x,y,z) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{x^2+y^2+(z-a)^2}} - \frac{1}{\sqrt{x^2+y^2+(z+a)^2}}\right)$

c

Simplify your result in the limit that $r \gg a$
Here $r = \sqrt{x^2+y^2+z^2}$ is the magnitude of $\vec{r}$

You may use the approximation:

$\frac{1}{\sqrt{x^2+y^2+(z-a)^2}} \approx \frac{1}{r} + \frac{az}{r^3}$

✓ **Answer**

$$\phi(x,y,z) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{x^2+y^2+(z-a)^2}} - \frac{1}{\sqrt{x^2+y^2+(z+a)^2}}\right)$$
$$\phi(x,y,z) = \frac{q}{4\pi\epsilon_0}\left(\frac{2az}{r^3}\right)$$

d

Show that your result for part (c) is equivalent to

$\phi(\vec{r}) \approx \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\vec{r}}{r^3}$
The potential of an ideal dipole

✓ **Answer**

$$\vec{p}\cdot\vec{r} = q\langle 0,0,2a\rangle\cdot\vec{r}$$
$$= 2aqz$$

$$\frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\vec{r}}{r^3} = \frac{q}{4\pi\epsilon_0}\left(\frac{2az}{r^3}\right)$$
$\square$

4

Potential of a charged rod

A uniformly charged line segment of length $2l$ lies on the z-axis symmetrically about the origin. The charge per unit length is λ.

The infinitesimal segment $d\zeta$ is at an elevation ζ above the origin. The point P is at a distance r from the z-axis and at an elevation z above the origin.

a

What is the total charge of the line?

✓ **Answer**

$2l\lambda$

b

Consider the infinitesimal segment of length $d\zeta$ a distance ζ above the origin. What is the electrostatic potential due to this segment at the point P?

✓ **Answer**

$\phi = \frac{1}{4\pi\epsilon_0}\frac{\lambda d\zeta}{\sqrt{r^2+(\zeta-z)^2}}$

$\square$

c

Integrate the result of part (a) to obtain the electrostatic potential at P due to the entire line segment.

Helpful Integral:

$\int d\zeta \frac{1}{\sqrt{r^2+(\zeta-z)^2}} = \ln\left(\frac{\zeta-z+\sqrt{r^2+(\zeta-z)^2}}{r}\right)$

✓ **Answer**

$\phi = \int_{-l}^{l} \frac{1}{4\pi\epsilon_0}\frac{\lambda d\zeta}{\sqrt{r^2+(\zeta-z)^2}}$

$\frac{\lambda}{4\pi\epsilon_0}\ln\left(\frac{\zeta-z+\sqrt{r^2+(\zeta-z)^2}}{r}\right)\Big|_{-l}^{l}$

$\frac{\lambda}{4\pi\epsilon_0}\left(\ln\left(\frac{l-z+\sqrt{r^2+(l-z)^2}}{r}\right) - \ln\left(\frac{-l-z+\sqrt{r^2+(l+z)^2}}{r}\right)\right)$

$\frac{\lambda}{4\pi\epsilon_0}\ln\left(\frac{l-z+\sqrt{r^2+(l-z)^2}}{-l-z+\sqrt{r^2+(l+z)^2}}\right)$

d*

Optional. Very hard. Zero credit

Show that the equipotential surfaces are ellipsoids with the two ends of the rod the focii of the ellipsoids.

✓ **Answer**

$\frac{\lambda}{4\pi\epsilon_0}\ln\left(\frac{l-z+\sqrt{r^2+(l-z)^2}}{-l-z+\sqrt{r^2+(l+z)^2}}\right)$

In the rz-plane, an equipotential line would be of the form

$\frac{l-z+\sqrt{r^2+(l-z)^2}}{-l-z+\sqrt{r^2+(l+z)^2}} = c$

Where c is a constant.

Which is of the form

$\sqrt{r^2+(z-l)^2} + \sqrt{r^2+(z+l)^2} = c$

Which is the foci definition of an ellipses

5

Potential of a Disc

a

The Ring: A ring of radius r has a charge q.

The charge is distributed uniformly around the circumference of the ring. Determine the electrostatic potential ϕ at the point P that lies on the axis of the ring at a distance z from the center of the ring.

✓ Answer

$$\phi(z) = \frac{1}{4\pi\epsilon_0} \int\limits_0^{2\pi} \frac{1}{\sqrt{r^2+z^2}} \frac{q d\theta}{2\pi}$$
$$\phi(z) = \frac{1}{4\pi\epsilon_0} \frac{q}{\sqrt{r^2+z^2}}$$
$\square$

b

The Disc

Now consider a uniformly charged disc with a radius R and a surface charge per unit area σ. We wish to determine the potential at a point that lies on the axis of the disc at a distance z from the center of the disc.

i

To this end first consider the infinitesimal annulus of the inner radius r and outer radius $r + dr$. Determine the charge of this annulus. Determine the potential at P due to this infinitesimal annulus.

Hint: Use your result of part (a)

✓ Answer

$$A = \pi(r + dr)^2 - \pi r^2$$
$$= \pi(2rdr + dr^2)$$

$$\phi(z) = \frac{1}{2\epsilon_0} \frac{\sigma r dr}{\sqrt{r^2+z^2}}$$
$\square$

ii

Now integrate the result of part (i) to obtain the potential at P due to the disc.

Helpful integral:
$\int dr \frac{2r}{\sqrt{r^2+z^2}} = 2\sqrt{r^2+z^2}$

✓ **Answer**

$\phi(z) = \int\limits_0^R \frac{1}{2\epsilon_0} \frac{\sigma r dr}{\sqrt{r^2+z^2}}$
$\phi(z) = \frac{\sigma}{2\epsilon_0} \sqrt{r^2+z^2}\Big|_0^R$
$\phi(z) = \frac{\sigma}{2\epsilon_0} \left(\sqrt{R^2+z^2} - z\right)$
$\square$

iii

What form do you expect the potential to have for $z \gg R$.

Explain briefly. Does your result of part (ii) reduced to the expected result?

Helpful approximation: For $z \gg R$
$\sqrt{r^2+z^2} \approx z + \frac{R^2}{2z}$

✓ **Answer**

I expect it to go to 0 as $\sqrt{R^2+z^2} - z \to 0$

$\phi(z) = \frac{\sigma R^2}{4z\epsilon_0}$
$\square$

6

Spherical capacitor

A conducting sphere (radius a) is surrounded by a spherical conducting shell (inner radius b, outer radius c). The inner sphere has a charge $+q$; the outer shell has a charge $-q$. Throughout this problem r denotes distance from the center of the sphere.

a

What is the electric field inside the conducting sphere (for $r < a$)?

✓ **Answer**

$\vec{E} = \vec{0}$ inside perfect conductors always

b

Where is the charge located on the conducting sphere: in the interior or on the surface? What is the surface charge density?

✓ **Answer**

The surface, by Gauss law, for any shell smaller than the radius of the sphere, since $\vec{E} = \vec{0}$, it must enclose no charge.

c

Determine the electric field in the gap between the sphere and the shell (for $a < r < b$)

Hint: Apply Gauss's law to the inner Gaussian surface depicted in the figure.

✓ **Answer**

$Q_{enc} = q$
$\phi = \frac{q}{4\pi r^2 \epsilon_0}$

$\vec{E} = \frac{d\phi}{dr} = \frac{-q}{2\pi r^3 \epsilon_0}\hat{r}$
$\square$

d

What is the electric field within the conducting shell (for $b < r < c$)?

✓ **Answer**

$\vec{E} = \vec{0}$ inside perfect conductors always

e

Where is the charge located in the conducting shell? In the interior? On the inner surface? On the outer surface? Or divided in some proportion between the inner surface and the outer surface? Explain briefly.

Hint: Apply Gauss's law to the outer Gaussian surface depicted in the figure

✓ Answer

At the inner surface, because $\vec{E} = \vec{0}$ within the shell, so the charge enclosed by any Gausian surface within this shell must be 0

Since the sphere has charge of $+q$, and $q_{enc} = 0$ past any radius above the inner surface, the inner surface must have charge $-q$, thus the whole charge will be on the inner surface.

f

What is the potential difference between the outer shell and the inner sphere?

Hint: Recall that the potential difference between two points $\vec{r}_1$ and $\vec{r}_2$ is given by

$$V_1 - V_2 = -\int_{\vec{r}_1}^{\vec{r}_2} \vec{E} \cdot d\vec{l}$$

where the integral is evaluated along any path that runs from point 1 to point 2.

✓ Answer

$$V_{outer} - V_{sphere} = \int_{a\hat{r}}^{b\hat{r}} \frac{d}{dr}\left(\frac{q}{4\pi r^2 \epsilon_0}\right) \cdot d\vec{l} + \int_{b\hat{r}}^{c\hat{r}} 0 \cdot \vec{l}$$
$$\Delta V = \frac{q}{4\pi r^2 \epsilon_0}\bigg|_a^b$$
$$\Delta V = \frac{q}{4\pi \epsilon_0}\left(b^{-2} - a^{-2}\right)$$
$\square$

g

A pair of conductors with equal and opposite charge constitute a capacitor. The capacitance of the capacitor is defined as $C = \frac{q}{V}$. Here q is the charge on the positively charged conductor and V is the potential difference between the positively charged conductor and the negatively charged conductor. Calculate C for the spherical capacitor analyzed in this problem.

✓ Answer

$$q = +q$$
$$V = \frac{q}{4\pi \epsilon_0}\left(b^{-2} - a^{-2}\right)$$

$C = 4\pi\epsilon_0 \left(b^{-2} - a^{-2}\right)^{-1}$

$\square$