14.7

7

$f(x,y) = x^2 + y^2 - xy + x$

$\nabla f = \langle 2x - y + 1, 2y - x \rangle$

$\nabla f = \langle 0, 0 \rangle \iff (x,y) = (-2/3, -1/3)$

$$H(f) = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$
$$|H(f)| = 5$$

Since the determinant at $(-2/3, -1/3)$ is positive we can conclude it is a max/min. Since $H(f)_{xx}$ is positive, we can conclude that it is a minima.

9

$f(x,y) = x^3 + 2xy - 2y^2 - 10x$

$\nabla f = \langle 3x^2 + 2y - 10, 2x - 4y \rangle$

$$\nabla f = \langle 0, 0 \rangle \iff \begin{cases} 3x^2 + 2y - 10 = 0 \\ 2x - 4y = 0 \end{cases}$$
$$= \begin{cases} x = 2y \\ 6y^2 + y - 5 = 0 \end{cases}$$
$$= \begin{cases} x = 2y \\ (6y - 5)(y + 1) = 0 \end{cases}$$
$$= \begin{cases} x = \{5/3, -2\} \\ y = \{5/6, -1\} \end{cases}$$
$$\iff (x,y) \in \{(5/3, 5/6), (-2, -1)\}$$

$$H(f) = \begin{bmatrix} 6x & 2 \\ 2 & -4 \end{bmatrix}$$
$$|H(f)| = -24x - 4$$
$$|H(f(5/3, 5/6))| = -44$$
$$|H(f(-2, -1))| = 44$$

Since the determinant on $(5/3, 5/6)$ is negative, we can conclude that it is a saddle point. Since the determinant on $(-2, -1)$ is positive and $H(f(-2, -1))_{xx} = -12$ is negative, we can conclude it is a local maximum.

11

$f(x, y) = 4x - 3x^3 - 2xy^2$

$\nabla f = \langle 4 - 9x^2 - 2y^2, -4xy \rangle$

$$\nabla f = \langle 0, 0 \rangle \iff \begin{cases} 4 - 9x^2 - 2y^2 = 0 \\ 4xy = 0 \end{cases}$$
$$= \begin{cases} x = 0 \cup y = 0 \\ 2 = y^2 \cup 4/9 = x^2 \end{cases}$$
$$\iff (x, y) \in \{(0, \pm\sqrt{2}), (\pm 2/3, 0)\}$$

$$H(f) = \begin{bmatrix} -18x & -4y \\ -4y & -4x \end{bmatrix}$$
$$|H(f)| = 72x^2 - 16y^2$$

$$\Big|_{(0,\pm\sqrt{2})} = -32$$
$$\Big|_{(\pm 2/3,0)} = 48$$

Since the determinant on $(0, \pm\sqrt{2})$ is negative, we can conclude both points are saddle points. Since the determinant on $(\pm 2/3, 0)$ is positive we can conclude they are extrema. Since $H(f)_{xx}$ on $(2/3, 0)$ is negative and positive on $(-2/3, 0)$, they are a maximum and minimum respectively.

13

$f(x, y) = x^4 + y^4 - 4xy$

$\nabla f = \langle 4x^3 - 4y, 4y^3 - 4x \rangle$

$$\nabla f = \langle 0, 0 \rangle \iff \begin{cases} 4x^3 - 4y = 0 \\ 4y^3 - 4x = 0 \end{cases}$$
$$= \begin{cases} y = x^3 \\ x = y^3 \end{cases}$$
$$= \begin{cases} y = y^9 \\ x = x^9 \\ x = y^3 \end{cases}$$
$$\iff (x, y) \in \{(-1, -1), (0, 0), (1, 1)\}$$

$$H(f) = \begin{bmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{bmatrix}$$
$$|H(f)| = 144x^2y^2 - 16$$

$$\Big|_{(-1,-1)} = 128$$

$$\Big|_{(0,0)} = -16$$

$$\Big|_{(1,1)} = 128$$

Since the determinant on $\{(-1,-1),(1,1)\}$ is positive, they are extrema, and since $H(f)_{xx}$ for both are positive, they are both local minima. Since the determinant on $(0,0)$ is negative, it is a saddle point.

23

$f(x,y) = (x+3y)e^{y-x^2}$

$\nabla f = \langle (-2x(x+3y)+1)e^{y-x^2}, (x+3y+3)e^{y-x^2} \rangle$

$$\nabla f = \langle 0,0 \rangle \iff \begin{cases} (-2x(x+3y)+1)e^{y-x^2} = 0 \\ (x+3y+3)e^{y-x^2} = 0 \end{cases}$$
$$= \{e^{y-x^2} = 0 \text{ or } \begin{cases} -2x^2-6xy+1 = 0 \\ x+3y+3 = 0 \end{cases}$$
$$= \{y - x^2 = -\infty \text{ or } \begin{cases} -2(-3y-3)^2 - 6(-3y-3)y + 1 = 0 \\ x = -3y-3 \end{cases}$$
$$= \begin{cases} -17 - 18y = 0 \\ x = -3y-3 \end{cases}$$
$$= \begin{cases} y = -17/18 \\ x = -1/6 \end{cases}$$
$$\iff (x,y) = (-1/6, -17/18)$$

$$H(f) = \begin{bmatrix} (-2x(-2x(x+3y)+1)+(-4x-6y))e^{y-x^2} & (-2x(x+3y)+1-6x))e \\ (-2x(x+3y+3)+1)e^{y-x^2} & (x+3y+6)e^{y-x^2} \end{bmatrix}$$
$$= \begin{bmatrix} (4x^3+12x^2y-6x-6y)e^{y-x^2} & (-2x^2-6xy-6x+1))e^{y-x^2} \\ (-2x^2-6xy-6x+1)e^{y-x^2} & (x+3y+6)e^{y-x^2} \end{bmatrix}$$

$$|H(f)| = (4x^3+12x^2y-6x-6y)(x+3y+6)e^{2(y-x^2)} - (-2x^2-6xy-6x+1))^2 e$$

$$\Big|_{(-1/6,-17/18)} \approx 2.57520028958$$

Since the determinant at $(-1/6, -17/18)$ is positive and $H(f)_{xx} \approx 2.39552992222$ is also positive, it is a local minima.

29

$f(x,y) = x+y$

$0 \leq x \leq 1$
$0 \leq y \leq 1$

Since f is always increasing on both x and y, the bigger the value of x and the bigger the value of y, the bigger the value of f. 1 would maximize the value of both x and y as it is the maximum on the domain. 0 would minimize both x and y as it is the minimum.

Min: $(0, 0, 0)$
Max: $(1, 1, 2)$

37

$f(x, y) = xy$

$\nabla f = \langle y, x \rangle$

$\nabla f = \langle 0, 0 \rangle \iff (x, y) = (0, 0)$

$H(f) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
$|H(f)| = -1$

Since the determinant is negative, the point $(0, 0)$ is a saddle point. Since the edge of the domain is not included in the domain, we cannot check the edge for maxima, therefore the domain does not have any maxima.

39

$f(x, y) = x + y - x^2 - y^2 - xy$

a

$\nabla f = \langle 1 - 2x - y, 1 - 2y - x \rangle$
$\nabla f = \langle 0, 0 \rangle \iff (x, y) = (1/3, 1/3)$

$f(1/3, 1/3) = 1/3$

b

$y = 0$
$f(x) = x - x^2$
$f'(x) = 1 - 2x$

$f'(x) = 0 \iff x = 1/2$
$f(1/2, 0) = 1/4$

c

$y = 2$
$f(x) = -2 - x^2 - x$
$f'(x) = -2x - 1$
$f'(x) = 0 \iff x = -1/2$
$f(-1/2, 2) = -7/4$

$x = 0$
$f(y) = y - y^2$
$f'(y) = 1 - 2y$
$f'(y) = 0 \iff y = 1/2$
$f(0, 1/2) = 1/4$

$x = 2$
$f(y) = -2 - y^2 - y$
$f'(y) = -2y - 1$
$f'(y) = 0 \iff y = -1/2$
$f(2, -1/2) = -7/4$

d

The greatest is at $(1/3, 1/3, 1/3)$.

41

$f(x, y) = x^3 - 2y$
$0 \leq x \leq 1$
$0 \leq y \leq 1$

$\nabla f = \langle 3x^2, -2 \rangle$
$\nabla f = \langle 0, 0 \rangle \iff x = 0$

$x = 0$
$f(y) = -2y$
$f'(y) = -2$
$f'(y) \neq 0$

$x = 1$
$f(y) = 1 - 2y$
$f'(y) = -2$
$f'(y) \neq 0$

$y = 0$
$f(x) = x^3$
$f'(x) = 3x^2$
$f'(x) = 0 \iff x = 0$
$f(0,0) = 0$

$y = 1$
$f(x) = x^3 - 2$
$f'(x) = 3x^2$
$f'(x) = 0 \iff x = 0$
$f(0,1) = -2$

$f(1,0) = 1$
$f(1,1) = -1$

The maximum is $(1,0,1)$ and the minimum is $(0,1,-2)$

14.8

5

$f(x,y) = x^2 + y^2$
$g(x,y) = 2x + 3y - 6 = 0$

$\nabla f = \langle 2x, 2y \rangle = \lambda \nabla g = \lambda \langle 2, 3 \rangle$

$$= \begin{cases} 2x = 2\lambda \\ 2y = 3\lambda \\ 2x + 3y = 6 \end{cases}$$
$$= \begin{cases} x = \lambda \\ y = 3\lambda/2 \\ 13\lambda = 12 \end{cases}$$
$$= \begin{cases} x = 12/13 \\ y = 18/13 \\ \lambda = 12/13 \end{cases}$$

$(x,y) = (12/13, 18/13)$

$\text{Tangent} = \langle 2,3\rangle \times \langle 0,0,1\rangle = \langle 3,-2\rangle$
$\text{Derivative along g} = \langle 3,-2\rangle \cdot \langle 2x,2y\rangle = 6x-4y$
$\text{2nd der. along g} = \langle 3,-2\rangle \nabla(6x-4y) = \langle 3,-2\rangle \cdot \langle 6,-4\rangle = 26$

Since the 2nd derivative is positive, there is a minimum at $(12/13, 18/13, 36/13)$

7

$f(x,y) = xy$
$g(x,y) = 4x^2 + 9y^2 - 32 = 0$

$\nabla f = \langle y,x\rangle = \lambda \nabla g = \lambda\langle 8x, 18y\rangle$

$$= \begin{cases} y = 8\lambda x \\ x = 18\lambda y \\ 4x^2 + 9y^2 = 32 \end{cases} = \begin{cases} y = 8\lambda x \\ x = 18\lambda y \\ x = 144\lambda^2 x \\ \lambda = \pm 1/12 \\ 4x^2 + 9y^2 = 32 \end{cases} = \begin{cases} y = \pm 2x/3 \\ x = \pm 3y/2 \\ 8x^2 = 32 \end{cases} = \begin{cases} y = \pm 2x/3 \\ x = \pm 3y/2 \\ x = \pm 2 \\ y = \pm 4/3 \end{cases}$$

$\text{Tangent of g} = \langle 18y, -8x\rangle$

Point	Tangent	1st D.D.	Nabla of D.D.	2nd D.D.	Max/Min
$(-2,-4/3,8/3)$	$\langle -24,16\rangle$	$16x-24y$	$\langle 16,-24\rangle$	-768	Max
$(-2,4/3,-8/3)$	$\langle 24,16\rangle$	$16x+24y$	$\langle 16,24\rangle$	768	Min
$(2,-4/3,-8/3)$	$\langle -24,-16\rangle$	$-16x-24y$	$\langle -16,-24\rangle$	768	Min
$(2,4/3,8/3)$	$\langle 24,-16\rangle$	$24y-16x$	$\langle -16,24\rangle$	-768	Max

There are maxima at $(-2,-4/3,8/3)$ and $(2,4/3,8/3)$ and minima at $(-2,4/3,-8/3)$ and $(2,-4/3,-8/3)$

9

$f(x, y) = x^2 + y^2$
$g(x, y) = x^4 + y^4 - 1 = 0$

$\nabla f = \langle 2x, 2y \rangle = \lambda \nabla g = \lambda \langle 4x^3, 4y^3 \rangle$

$$= \begin{cases} 2x = \lambda 4x^3 \\ 2y = \lambda 4y^3 \\ x^4 + y^4 = 1 \end{cases}$$
$$= \begin{cases} 1 = \lambda 2x^2 \\ 1 = \lambda 2y^2 \\ x^4 + y^4 = 1 \end{cases}$$
or $(x, y) \in \{(0, \pm 1) \cup (\pm 1, 0)\}$
$$= \begin{cases} y = x = \pm\sqrt{1/2\lambda} \\ 1/\lambda^2 = 2 \end{cases}$$
$$= \begin{cases} y = x = \pm 1/\sqrt[4]{2} \\ \lambda = \pm\sqrt{1/2} \end{cases}$$

$f(\pm 1/\sqrt[4]{2}, \pm 1/\sqrt[4]{2}) = \sqrt{2}$
$f(\{(0, \pm 1) \cup (\pm 1, 0)\}) = 1$

Maximum: $\sqrt{2}$
Minimum: 1

13

$f(x, y, z) = xy + 2z$
$g(x) = x^2 + y^2 + z^2 - 36 = 0$

$\nabla f = \langle y, x, 2 \rangle = \lambda \nabla g = \lambda \langle 2x, 2y, 2z \rangle$

$$= \begin{cases} y = 2\lambda x \\ x = 2\lambda y \\ 2 = 2\lambda z \\ x^2 + y^2 + z^2 = 36 \end{cases}$$
$$= \begin{cases} y = x \\ \lambda = \pm 1/2 \\ z = \pm 2 \\ x^2 = 32/2 \end{cases}$$
$$= \begin{cases} \lambda = \pm 1/2 \\ z = \pm 2 \\ y = x = \pm 4 \end{cases}$$

$$\iff (x, y, z) = (\pm 4, \pm 4, \pm 2) | x = y$$

$f(4,4,2)=20$
$f(-4,-4,2)=20$
$f(4,4,-2)=-20$
$f(-4,-4,-2)=-20$

Maximums at $(4,4,2)=20$ and $(-4,-4,2)=20$.
Minimums at $(4,4,-2)=-20$ and $(-4,-4,-2)=-20$

21

$f(x,y)=x$
$g(x,y)=x^2+6y^2+3xy-40=0$

$\nabla f=\langle 1,0\rangle=\lambda\nabla g=\lambda\langle 2x+3y,12y+3x\rangle$

$$\begin{cases} x^2+6y^2+3xy=40 \\ \lambda 2x+\lambda 3y=1 \\ \lambda 12y+\lambda 3x=0 \end{cases}$$
$$\begin{cases} x^2+6y^2+3xy=40 \\ x=-3y/2+1/2\lambda \\ x=-4y \end{cases}$$
$$\begin{cases} y=\pm 2 \\ x=-3y/2+1/2\lambda \\ x=\mp 8 \end{cases}$$

$(x,y)=(8,-2)$ has the greatest x value

33

$$\begin{cases} V=54\pi \\ V=\pi r^2 h \\ S=2\pi rh+2\pi r^2 \end{cases}$$

$f(r,h)=2\pi rh+2\pi r^2$
$g(r,h)=\pi r^2 h-54\pi=0$

$\nabla f=\langle 2\pi h+4\pi r,2\pi r\rangle=\lambda\nabla g=\lambda\langle 2\pi rh,\pi r^2\rangle$

$$=\begin{cases} 2\pi h+4\pi r=\lambda 2\pi rh \\ 2\pi r=\lambda\pi r^2 \\ \pi r^2 h=54\pi \end{cases}$$
$$=\begin{cases} h+2r=\lambda rh \\ 2=\lambda r \\ \pi r^2 h=54\pi \end{cases}$$

$$= \begin{cases} h = 2r \\ 2 = \lambda r \\ r^3 = 27 \end{cases}$$

$$= \begin{cases} h = 6 \\ \lambda = 2/3 \\ r = 3 \end{cases}$$

$(r, h) = (3, 6)$

41

$f(x, y, z) = x^2 + y^2 + z^2$
$g(x, y, z) = x + y + z - 1 = 0$
$h(x, y, z) = x + 2y + 3z - 6 = 0$

$\nabla f = \langle 2x, 2y, 2z \rangle$
$\nabla g = \langle 1, 1, 1 \rangle$
$\nabla h = \langle 1, 2, 3 \rangle$

$\nabla f = \lambda \nabla g + \mu \nabla h$

$$= \begin{cases} 2x = \lambda + \mu \\ 2y = \lambda + 2\mu \\ 2z = \lambda + 3\mu \\ x + y + z = 1 \\ x + 2y + 3z = 6 \end{cases}$$

$$= \begin{cases} 2x = \lambda + \mu \\ 2y = \lambda + 2\mu \\ 2z = \lambda + 3\mu \\ 3\lambda + 6\mu = 2 \\ 3\lambda + 7\mu = 6 \end{cases}$$

$$= \begin{cases} x = -5/3 \\ y = 1/3 \\ z = 7/3 \\ \lambda = -22/3 \\ \mu = 4 \end{cases}$$

$f(-5/3, 1/3, 7/3) = 25/3$

53

a

$$f(x_1,\dots,x_n)=\prod_{j=1}^{n}x_j$$
$$g(x_1,\dots,x_n)=\sum_{j=1}^{n}x_j-B=0\quad B>0$$
$$h_j(x_1,\dots,x_n)=x_j\geq 0\quad \forall x_j\in\{x_1,\dots,x_n\}$$

$$\nabla f=\langle\tfrac{1}{x_1}\prod_{j=1}^{n}x_j,\dots,\tfrac{1}{x_n}\prod_{j=1}^{n}x_j\rangle$$
$$\nabla g=\langle 1,\dots,1\rangle$$

$$=\begin{cases}\langle 1/x_1,\dots,1/x_n\rangle\prod_{j=1}^{n}x_j=\lambda\langle 1,\dots,1\rangle\\ \sum_{j=1}^{n}x_j=B\qquad B>0\end{cases}$$
$$=\begin{cases}\langle 1/x_1,\dots,1/x_n\rangle=\frac{\lambda}{\prod_{j=1}^{n}x_j}\langle 1,\dots,1\rangle\\ \sum_{j=1}^{n}x_j\qquad B>0\end{cases}$$
$$=\begin{cases}\langle x_1,\dots,x_n\rangle=\frac{\prod_{j=1}^{n}x_j}{\lambda}\langle 1,\dots,1\rangle\\ \sum_{j=1}^{n}x_j=B\qquad B>0\end{cases}$$
$$=\begin{cases}x_i=\frac{\prod_{j=1}^{n}x_j}{\lambda}\qquad \forall x_i\in\{x_1,\dots,x_n\}\\ nx_i=B\qquad B>0\qquad \forall x_i\in\{x_1,\dots,x_n\}\end{cases}$$

$$=x_1=\ldots=x_n=B/n$$
$$(x_1,\dots,x_n)=(B/n,\dots,B/n)$$

The point $(x_1,\dots,x_n)=(B/n,\dots,B/n)$ maximizes f given the restraints g and h with a value of $(B/n)^n$

b

$$(a_1a_2\dots a_n)^{1/n}\leq\tfrac{a_1+\ldots+a_n}{n}$$

Let $B=\sum_{j=1}^{n}x_j$

Let $P=\prod_{j=1}^{n}x_j$

$$\implies P^{1/n}\leq B/n$$
$$\iff P\leq(B/n)^n$$

Given $a_j \geq 0 \qquad \forall a_j \in \{a_1, \ldots, a_n\}$
$\implies B \geq 0$

Since we know the maximum of P given $B \geq 0$ and
$a_j \geq 0 \qquad \forall a_j \in \{a_1, \ldots, a_n\}$ is $(B/n)^n$, any other combination of $a_1, \ldots, a_n$ will be less than the $(B/n)^n$.

Thus,

$P \leq (B/n)^n$

55 & 56

$$S(x_1, \ldots, x_n) = \sum_{j=1}^{n} x_j \ln x_j$$
$$g(x_1, \ldots, x_n) = -N + \sum_{j=1}^{n} x_j = 0$$
$$h(x_1, \ldots, x_n) = -E + \sum_{j=1}^{n} E_j x_j = 0$$

$$\nabla S_j = 1 + \ln x_j$$
$$\nabla g_j = 1$$
$$\nabla h_j = E_j$$

$$= \begin{cases} \ln e x_j = \lambda + \mu E_j \qquad \forall x_j \in \{x_1, \ldots, x_n\} \\ \sum_{j=1}^{n} x_j = N \\ \sum_{j=1}^{n} E_j x_j = E \end{cases}$$
$$= \begin{cases} x_j = e^{\lambda - 1} e^{\mu E_j} \qquad \forall x_j \in \{x_1, \ldots, x_n\} \\ e^{\lambda - 1} = \frac{N}{\sum_{j=1}^{n} e^{\mu E_j}} \\ \sum_{j=1}^{n} E_j e^{\mu E_j} = E / e^{\lambda - 1} \end{cases}$$
$$= \begin{cases} x_j = \frac{N}{\sum_{j=1}^{n} e^{\mu E_j}} e^{\mu E_j} \qquad \forall x_j \in \{x_1, \ldots, x_n\} \\ \sum_{j=1}^{n} E_j e^{\mu E_j} = \frac{E}{N} \sum_{j=1}^{n} e^{\mu E_j} \end{cases}$$

The first line proves the statement where A is the first item in the product.
$n = 3$ solves (55). And an arbitrary n solves (56)

> Just curious, is there a way to solve for μ?
>
> Let $F_j = e^{\mu E_j} \qquad \forall E_j \in \{E_1, \ldots, E_n\}$
>
> $$= \begin{cases} x_j = \frac{NF_j}{\sum\limits_{j=1}^{n} F_j} \qquad \forall x_j \in \{x_1, \ldots, x_n\} \\ \sum\limits_{j=1}^{n} E_j F_j = \sum\limits_{j=1}^{n} \frac{EF_j}{N} \end{cases}$$
>
> $$= \begin{cases} x_j = \frac{NF_j}{\sum\limits_{j=1}^{n} F_j} \qquad \forall x_j \in \{x_1, \ldots, x_n\} \\ \sum\limits_{j=1}^{n} E_j F_j - \frac{EF_j}{N} = 0 \end{cases}$$
>
> $$= \begin{cases} x_j = \frac{NF_j}{\sum\limits_{j=1}^{n} F_j} \qquad \forall x_j \in \{x_1, \ldots, x_n\} \\ \sum\limits_{j=1}^{n} F_j (E_j - \frac{E}{N}) = 0 \end{cases}$$