14.4

3

$P = (-1, 2)$
$f(x, y) = xy^2 + x^3y^2$
$f'_x(x, y) = y^2 + 3x^2y^2$
$f'_y(x, y) = 2xy + 2x^3y$

$f'_x(P) = 16$
$f'_y(P) = -8$
$f(P) = -8$

$A = -8 + 16(x + 1) - 8(y - 2)$
$A = 16x - 8y + 24$
$\square$

5

$P = (4, 1)$
$f(x, y) = x^2 + y^{-2}$
$f'_x(x, y) = 2x$
$f'_y(x, y) = -2y^{-3}$

$f'_x(P) = 8$
$f'_y(P) = -2$
$f(P) = 17$

$A = 17 + 8(x - 4) - 2(y - 1)$
$A = 8x - 2y - 13$
$\square$

7

$P = (2, 1)$
$F(r, s) = r^2s^{-1/2} + s^{-3}$
$F'_r(r, s) = 2rs^{-1/2}$
$F'_s(r, s) = -0.5r^2s^{-3/2} - 3s^{-4}$

$F'_r(P) = 4$
$F'_s(P) = -5$
$F(P) = 5$

$A = 5 + 4(r - 2) - 5(s - 1)$
$A = 4r - 5s + 2$
$\square$

13

$f(x,y)=3x^2-xy-y^2$
$f'_x(x,y)=6x-y$
$f'_y(x,y)=-x-2y$

$$\begin{cases}6x-y=0\\-x-2y=0\end{cases}$$
$$\begin{cases}6x-y=0\\-x-12x=0\end{cases}$$
$$\begin{cases}y=0\\x=0\end{cases}$$

$P=(0,0)$
$\square$

15

$P=(2,1)$
$f(x,y)=x^2y^3$
$f'_x(x,y)=2xy^3$
$f'_y(x,y)=3x^2y^2$

$L_{(a,b)}(x,y)=a^2b^3+2ab^3(x-a)+3a^2b^2(y-b)$
$L_P(x,y)=4+4(x-2)+12(y-1)$
$L_P(x,y)=4x+12y-16$
$\square$

$L_P(2.01,1.02)=4.28$
$L_P(1.97,1.01)=4$
$\square$

23

$f(2,4)=5$
$f'_x(2,4)=0.3$
$f'_y(2,4)=-0.2$

$L_{(2,4)}(x,y)=5+0.3(x-2)-0.2(y-4)$
$L_{(2,4)}(x,y)=0.3x-0.2y+5.2$
$\square$

$L_{(2,4)}(2.1,3.8)=5.07$
$\square$

43

a

$f(x,y) = 5x + 4y^2$
$f'_x(x,y) = 5$
$f'_y(x,y) = 8y$

$L_{(a,b)}(x,y) = 5a + 4b^2 + 5(x-a) + 8b(y-b)$
$L_{(2,1)}(x,y) = 14 + 5(x-2) + 8(y-1)$
$L_{(2,1)}(x,y) = 5x + 8y - 4$
$\square$

$L_{(2,1)}(x,y) + 4(y-1)^2 = 5x + 8y - 4 + 4y^2 - 8y + 4$
$= 5x + 4y^2$
$= f(x,y)$
$\square$

b

$\frac{4(y-1)^2}{\sqrt{(x-2)^2+(y-1)^2}}$

Since $(x-2)^2$ and $(y-1)^2$ are positive, the denominator is also positive.
Since $(y-1)^2$ is positive, so is the numerator and thus the whole fraction ≥ 0

$\frac{4(y-1)^2}{\sqrt{(x-2)^2+(y-1)^2}}$

$\sqrt{(x-2)^2+(y-1)^2} \geq \sqrt{(y-1)^2} = |y-1|^2$

$\frac{4(y-1)^2}{\sqrt{(x-2)^2+(y-1)^2}} \leq \frac{4(y-1)^2}{|y-1|} = 4|y-1|$
$\square$

c

$\lim_{(x,y)\to(a,b)} \frac{f(x,y)-L_{(a,b)}(x,y)}{\sqrt{(x-a)^2+(y-b)^2}}$
$\lim_{(x,y)\to(a,b)} \frac{(5x+4y^2)-(5a+4b^2+5(x-a)+8b(y-b))}{\sqrt{(x-a)^2+(y-b)^2}}$
$\lim_{(x,y)\to(a,b)} \frac{4y^2-8by+4b^2}{\sqrt{(x-a)^2+(y-b)^2}}$
$\lim_{(x,y)\to(a,b)} \frac{4(y-b)^2}{\sqrt{(x-a)^2+(y-b)^2}}$

The limit must ≥ 0 since the fraction is positive

$\lim_{(x,y)\to(a,b)} \frac{4(y-b)^2}{\sqrt{(x-a)^2+(y-b)^2}} \leq \lim_{(x,y)\to(a,b)} 4|y-b| = 0$

Thus, the limit must lie on the interval $[0,0]$, meaning it equals 0 everywhere.

Since the limit equals 0 for any (a,b), the function f is differentiable on any (a,b)
$\square$

14.5

9

$f(x,y) = 3x - 7y$
$\vec{r}(t) = \langle \cos t, \sin t \rangle$
$t = 0$

$\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}'(t)$

$\nabla f(\vec{r}(t)) = \langle 3, -7 \rangle$
$\vec{r}'(t) = \langle -\sin t, \cos t \rangle$

$\nabla f(\vec{r}(t)) \cdot \vec{r}'(t) = -3\sin t - 7\cos t$
$\nabla f(\vec{r}(0)) \cdot \vec{r}'(0) = -7$
$\square$

11

$f(x,y) = x^2 - 3xy$
$\vec{r}(t) = \langle \cos t, \sin t \rangle$
$t = 0$

$\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}'(t)$

$\nabla f(\vec{r}(t)) = \langle 2\cos t - 3\sin t, -3\cos t \rangle$
$\vec{r}'(t) = \langle -\sin t, \cos t \rangle$

$\nabla f(\vec{r}(0)) \cdot \vec{r}'(0) = 0 + -3$
$= -3$
$\square$

15

$f(x,y) = x - xy$
$\vec{r}(t) = \langle t^2, t^2 - 4t \rangle$
$t = 4$

$\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}'(t)$

$\nabla f(\vec{r}(t)) = \langle 1 - (t^2 - 4t), -(t^2) \rangle$
$\vec{r}'(t) = \langle 2t, 2t - 4 \rangle$

$\nabla f(\vec{r}(4)) \cdot \vec{r}'(4) = 1 * 8 + -16 * 4$
$= -56$
$\square$

17

$f(x,y) = \ln x + \ln y$
$\vec{r}(t) = \langle \cos t, t^2 \rangle$
$t = \frac{\pi}{4}$

$\nabla f(\vec{r}(0)) \cdot \vec{r}'(0) = 0 + -3$

$\nabla f(\vec{r}(t)) = \langle \frac{1}{\cos t}, \frac{1}{t^2} \rangle$
$\vec{r}'(t) = \langle -\sin t, 2t \rangle$

$\nabla f(\vec{r}(\frac{\pi}{4})) \cdot \vec{r}'(\frac{\pi}{4}) = -1 + \frac{8}{\pi}$
$= \frac{8}{\pi} - 1$
$\square$

21

$f(x,y) = x^2 + y^3$
$\vec{v} = \langle 4, 3 \rangle$
$P = (1, 2)$

$\nabla f(P) = \langle 2, 12 \rangle$
$\nabla f(P) \cdot \vec{v}/||\vec{v}|| = 8/5 + 36/5$
$= 44/5 = 8.8$

27

$f(x,y) = \ln(x^2 + y^2)$
$\vec{v} = 3i - 2j = \langle 3, -2 \rangle$
$P = (1, 0)$

$\nabla f(P) = \langle \frac{2x}{x^2+y^2}, \frac{2y}{x^2+y^2} \rangle \Big|_P$
$= \langle 2, 0 \rangle$

$\nabla f(P) \cdot \vec{v}/||\vec{v}|| = 6/\sqrt{13}$
$= \frac{6}{\sqrt{13}} = \frac{6\sqrt{13}}{13}$
$\square$

29

$g(x,y,z) = xe^{-yz}$
$\vec{v} = \langle 1, 1, 1 \rangle$
$P = (1, 2, 0)$

$\nabla g(P) = \langle e^{-yz}, -xze^{-yz}, -xye^{-yz} \rangle \Big|_P$
$= \langle 1, 0, -2 \rangle$

$\nabla g(P) \cdot \vec{v}/||\vec{v}|| = -1/\sqrt{3}$
$= \frac{-1}{\sqrt{3}} = \frac{-\sqrt{3}}{3}$

□

43

$\frac{x^2}{4}+\frac{y^2}{9}+z^2=1$

$f(x,y)=z=\pm\sqrt{1-\frac{x^2}{4}-\frac{y^2}{9}}$

$\nabla f((x,y,\pm\sqrt{1-\frac{x^2}{4}-\frac{y^2}{9}}))=\langle\mp\frac{x}{4\sqrt{1-\frac{x^2}{4}-\frac{y^2}{9}}},\mp\frac{y}{9\sqrt{1-\frac{x^2}{4}-\frac{y^2}{9}}}\rangle$

Meaning the z value is the opposite sign of the normal whilst x and y are the same

Which means the tangent plane at (x,y) is

$L_{(x,y)}(a,b)=f(x,y)+\nabla f_x(a-x)+\nabla f_y(b-y)$

Making the normal to the plane

$\vec{N}=\langle\pm\frac{x}{4\sqrt{1-\frac{x^2}{4}-\frac{y^2}{9}}},\pm\frac{y}{9\sqrt{1-\frac{x^2}{4}-\frac{y^2}{9}}},1\rangle$

$\vec{N}=\langle 1,1,-2\rangle$

$\begin{cases}\pm\frac{x}{4\sqrt{1-\frac{x^2}{4}-\frac{y^2}{9}}}=\frac{-1}{2}\\ \pm\frac{y}{9\sqrt{1-\frac{x^2}{4}-\frac{y^2}{9}}}=\frac{-1}{2}\end{cases}$

$=\begin{cases}x=\mp 2\sqrt{1-\frac{x^2}{4}-\frac{y^2}{9}}\\ y=\mp 4.5\sqrt{1-\frac{x^2}{4}-\frac{y^2}{9}}\end{cases}$

Which means that x and y are the same sign

$=\begin{cases}x^2=4(1-\frac{x^2}{4}-\frac{y^2}{9})\\ y^2=\frac{81}{4}(1-\frac{x^2}{4}-\frac{y^2}{9})\end{cases}$

$=\begin{cases}x^2=2-\frac{2y^2}{9}\\ y^2=\frac{81}{4}-\frac{81x^2}{16}-\frac{9y^2}{4}\end{cases}$

$=\begin{cases}x^2=2-\frac{2y^2}{9}\\ y^2=\frac{81}{4}-\frac{81(2-\frac{2y^2}{9})}{16}-\frac{9y^2}{4}\end{cases}$

$=\begin{cases}x^2=2-\frac{2y^2}{9}\\ y^2=\frac{81}{4}-\frac{81}{8}+\frac{9y^2}{8}-\frac{9y^2}{4}\end{cases}$

$=\begin{cases}x^2=2-\frac{2y^2}{9}\\ y^2=\frac{81}{8}-\frac{9y^2}{8}\end{cases}$

$=\begin{cases}x^2=2-\frac{2y^2}{9}\\ \frac{17}{8}y^2=\frac{81}{8}\end{cases}$

$=\begin{cases}x^2=2-\frac{2*\frac{81}{17}}{9}\\ y^2=\frac{81}{17}\end{cases}$

$=\begin{cases}x^2=\frac{16}{17}\\ y^2=\frac{81}{17}\end{cases}$

$(\pm\frac{4}{\sqrt{17}},\pm\frac{9}{\sqrt{17}})$

Where x and y are the same sign and z is the opposite sign

$(\pm\frac{4}{\sqrt{17}}, \pm\frac{9}{\sqrt{17}}, \mp\sqrt{1-\frac{x^2}{4}-\frac{y^2}{9}})$
$(\pm\frac{4}{\sqrt{17}}, \pm\frac{9}{\sqrt{17}}, \mp\sqrt{1-\frac{4}{17}-\frac{9}{17}})$
$(\pm\frac{4}{\sqrt{17}}, \pm\frac{9}{\sqrt{17}}, \mp\sqrt{\frac{4}{17}})$
$(\pm\frac{4}{\sqrt{17}}, \pm\frac{9}{\sqrt{17}}, \mp\frac{2}{\sqrt{17}})$

Therefore, the Normals are $\langle 1, 1, -2\rangle$ at
$(\frac{4}{\sqrt{17}}, \frac{9}{\sqrt{17}}, -\frac{2}{\sqrt{17}})$
and
$(-\frac{4}{\sqrt{17}}, -\frac{9}{\sqrt{17}}, \frac{2}{\sqrt{17}})$
$\square$

47

$xz + 2x^2y + y^2z^3 = 11$
$P = (2, 1, 1)$

$$\begin{cases} f_x x + z + 4xy + 3y^2z^2 f_x = 0 \\ f_y x + 2x^2 + 2z^3 y + 3y^2z^2 f_y = 0 \end{cases}$$

$$\begin{cases} 2f_{xP} + 1 + 8 + 3f_{xP} = 0 \\ 2f_{yP} + 8 + 2 + 3f_{yP} = 0 \end{cases}$$
$$\begin{cases} f_{xP} = \frac{-9}{5} \\ f_{yP} = \frac{-10}{5} \end{cases}$$
$\nabla f(P) = \langle -\frac{9}{5}, -2\rangle$

$L_P(x, y) = 1 - \frac{9}{5}(x-2) - 2(y-1)$
$L_P(x, y) = -\frac{9}{5}x - 2y + 33$
$\square$

14.6

3

$f(x, y, z) = xy + z^2$
$x = s^2$
$y = 2rs$
$z = r^2$

$f(s, r) = 2rs^3 + r^4$
$f_s'(s, r) = 6rs^2$
$f_r'(s, r) = 2s^3 + 4r^3$
$\square$

5

$g(\theta, \phi) = \tan(\theta + \phi)$
$\theta = xy$

$\phi = x + y$

$$g(x,y) = \tan(xy + x + y)$$
$$g'_x(x,y) = \sec^2(xy + x + y)(y + 1)$$
$$g'_y(x,y) = \sec^2(xy + x + y)(x + 1)$$
$\square$

11

$$f(x,y,z) = x^3 + yz^2$$
$$x = u^2 + v$$
$$y = u + v^2$$
$$z = uv$$
$$(u,v) = (-1,-1)$$

$$f(u,v) = (u^2 + v)^3 + (u + v^2)(uv)^2$$

$$f'_u(u,v) = 6u(u^2 + v)^2 + 2(u + v^2)(uv) + (uv)^2$$
$$\Big|_{(-1,-1)} = 1$$

$$f'_v(u,v) = 3(u^2 + v)^2 + 2(u + v^2)(uv) + 2v(uv)^2$$
$$\Big|_{(-1,-1)} = -2$$
$\square$

15

$$g(x,y) = x^2 - y^2$$
$$x = e^u \cos v$$
$$y = e^u \sin v$$
$$(u,v) = (0,1)$$

$$g(u,v) = e^{2u}\cos^2 v - e^{2u}\sin^2 v$$

$$g'_u(u,v) = 2e^{2u}\cos^2 v - 2e^{2u}\sin^2 v$$
$$g'_u(u,v) = 2e^{2u}(\cos(2v))$$
$$\Big|_{(0,1)} = 2\cos 2$$
$\square$

25

$$x = s + t$$
$$y = s - t$$
$$f(x,y)$$

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$
$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}$$

$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$
$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}$

$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}$
$\frac{\partial f}{\partial t}\frac{\partial f}{\partial s} = (\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y})(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y})$

$\frac{\partial f}{\partial t}\frac{\partial f}{\partial s} = (\frac{\partial f}{\partial x})^2 - (\frac{\partial f}{\partial y})^2$
$\square$

27

a

$F(x,y,z) = xz^2 + y^2z + xy - 1$

$F'_x(x,y,z) = z^2 + y$
$F'_y(x,y,z) = 2yz + x$
$F'_z(x,y,z) = 2xz + y^2$
$\square$

b

$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{z^2+y}{2xz+y^2}$
$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{2yz+x}{2xz+y^2}$
$\square$

35

$||\vec{r}|| = r = \sqrt{x^2 + y^2 + z^2}$
$\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle$

$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial F}\frac{\partial F}{\partial x}$

$f(x,y,z) = F(r)$
$\frac{\partial f}{\partial F} = 1$

$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial r}\frac{\partial r}{\partial x}$
$\frac{\partial f}{\partial x} = F'(r)\frac{\partial r}{\partial x}$

$\frac{\partial r}{\partial x} = \frac{1}{2\sqrt{x^2+y^2+z^2}}2x = \frac{x}{\sqrt{x^2+y^2+z^2}}$

$\frac{\partial f}{\partial x} = F'(r)\frac{x}{\sqrt{x^2+y^2+z^2}}$

By that same logic we can conclude

$\frac{\partial f}{\partial y} = F'(r)\frac{y}{\sqrt{x^2+y^2+z^2}}$
$\frac{\partial f}{\partial z} = F'(r)\frac{z}{\sqrt{x^2+y^2+z^2}}$

Which makes

$$\nabla f = \langle F'(r)\frac{x}{\sqrt{x^2+y^2+z^2}}, F'(r)\frac{y}{\sqrt{x^2+y^2+z^2}}, F'(r)\frac{z}{\sqrt{x^2+y^2+z^2}}\rangle$$
$$= F'(r)\langle x, y, z\rangle / \sqrt{x^2+y^2+z^2}$$

Since
$\vec{r} = \langle x, y, z\rangle$
$\|\vec{r}\| = \sqrt{x^2+y^2+z^2}$

$$\nabla f = F'(r)\vec{r}/\|\vec{r}\|$$
$$= F'(r)e_{\vec{r}}$$
$\square$