4.2

a

$E(ag_1(X,Y) + bg_2(X,Y) + c) = eE(g_1(X,Y)) + bE(g_2(X,Y)) + c$

$E(ag_1(X,Y) + bg_2(X,Y) + c) = \iint\limits_A (ag_1(x,y) + bg_2(x,y) + c)f(x,y)dx\ dy$
$= \iint\limits_A ag_1(x,y)f(x,y) + bg_2(x,y)f(x,y) + cf(x,y)dx\ dy$
$= \iint\limits_A ag_1(x,y)f(x,y)dx\ dy + \int\limits_A bg_2(x,y)f(x,y)dx\ dy + \iint\limits_A cf(x,y)dx\ dy$
$= a\iint\limits_A g_1(x,y)f(x,y)dx\ dy + b\iint\limits_A g_2(x,y)f(x,y)dx\ dy + c\iint\limits_A f(x,y)dx\ dy$
$= aE(g_1(X,Y)) + bE(g_2(X,Y)) + cE(1)$
$= aE(g_1(X,Y)) + bE(g_2(X,Y)) + c$
$\square$

b

$g_1(x,y) \geq 0 \implies E(g_1(X,Y)) \geq 0$

$E(g_1(X,Y)) = \iint\limits_A g_1(x,y)f(x,y)dx\ dy$

Let $P = g_1(x,y)f(x,y)$
$f(x,y) \geq$ since it is a pdf, and if $g_1(x,y) \geq 0$, then $P \geq 0$

$E(g_1(X,Y)) = \iint\limits_A Pdx\ dy$
Thus $\iint\limits_A Pdx\ dy \geq 0$

$g_1(x,y) \geq 0 \implies E(g_1(X,Y)) \geq 0$
$\square$

c

$g_1(x,y) \geq g_2(x,y)$ then $E(g_1(X,Y)) \geq E(g_2(X,Y))$

$$E(g_1(X,Y)) = \iint\limits_A g_1(x,y)f(x,y)dx\ dy$$
$$E(g_2(X,Y)) = \iint\limits_A g_2(x,y)f(x,y)dx\ dy$$

$g_1(x,y) \geq g_2(x,y)$
$\implies g_1(x,y) - g_2(x,y) \geq 0$
Let $S(x,y) = g_1(x,y) - g_2(x,y)$
$\implies S(x,y) \geq 0$

$$E(g_1(X,Y)) = \iint\limits_A (g_2(x,y) + S(x,y))f(x,y)dx\ dy$$
$$= \iint\limits_A g_2(x,y)f(x,y)dx\ dy + \iint\limits_A S(x,y)f(x,y)dx\ dy$$

By part (b), $\iint\limits_A S(x,y)f(x,y)dx\ dy \geq 0$ because $S(x,y) \geq 0$

$$E(g_1(X,Y)) \geq \iint\limits_A g_2(x,y)f(x,y)dx\ dy$$
$$E(g_1(X,Y)) \geq E(g_2(X,Y))$$
$\square$

d

$a \leq g_1(x,y) \leq b \implies a \leq E(g_1(X,Y)) \leq b$

Let $g_{1_a}(x,y) = g_1(x,y) - a$ and $g_{1_b}(x,y) = b - g_1(x,y)$
$0 \leq g_{1_{\{a,b\}}}(x,y) \leq b - a$

By part (b), $E(g_{1_{\{a,b\}}}(X,Y)) \geq 0$

$E(g_1(X,Y)) = E(g_{1_a}(X,Y) + a)$
$= a + E(g_{1_a}(X,Y)$

$E(g_1(X,Y)) = E(b - g_{1_b}(X,Y))$
$= b - E(g_{1_b}(X,Y)$

$a \leq E(g_{1_a}(X,Y)) + a$
$a \leq E(g_1(X,Y))$

$b \geq b - E(g_{1_b}(X,Y))$
$b \geq E(g_1(X,Y))$

$a \leq E(g_1(X,Y)) \leq b$
$\square$

4.4

$$f(x,y) = \begin{cases} C(x+2y) & 0 < y < 1 \text{ and } 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

a

$A = \{0 < y < 1 \cap 0 < x < 2\}$
$1 = \iint\limits_A C(x+2y)dx\,dy$
$\int\limits_0^1 \int\limits_0^2 C(x+2y)dx\,dy$
$\int\limits_0^1 C(x^2/2 + 2xy)\Big|_{x=0}^{2} dy$
$\int\limits_0^1 C(2+4y)dy$
$C(2y+2y^2)\Big|_{y=0}^{1}$
$C(2+2)$
$C = 1/4$
$\square$

b

$f_X(x) = \int\limits_{-\infty}^{\infty} f(x,y)dy$
$= \int\limits_{-\infty}^{\infty} I_A(x,y)(x+2y)/4dy$
$= \int\limits_0^1 (x+2y)/4dy$
$= (xy+y^2)/4\Big|_{y=0}^{1}$
$= (x+1)/4 \quad x \in A$
$= \begin{cases} (x+1)/4 & 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$
$\square$

c

$$F(x,y) = \int_{-\infty}^{y} \int_{-\infty}^{x} I_A(a,b) f(a,b)\, da\, db$$

$$= \int_{-\infty}^{y} \int_{-\infty}^{x} I_A(a,b)(a+2b)/4\, da\, db$$

$$= \begin{cases} 0 & x < 0 \cup y < 0 \\ \int_0^y \int_0^x (a+2b)/4\, da\, db & 0 \le x < 2 \cap 0 \le y < 1 \\ \int_0^y \int_0^2 (a+2b)/4\, da\, db & 2 \le x \cap 0 \le y < 1 \\ \int_0^1 \int_0^x (a+2b)/4\, da\, db & 0 \le x < 2 \cap 1 \le y \\ 1 & 2 \le x \cap 1 \le y \end{cases}$$

$$= \begin{cases} 0 & x < 0 \cup y < 0 \\ \int_0^y (a^2/2 + 2ba)/4 \Big|_0^x\, db & 0 \le x < 2 \cap 0 \le y < 1 \\ \int_0^y (a^2/2 + 2ba)/4 \Big|_0^2\, db & 2 \le x \cap 0 \le y < 1 \\ \int_0^1 (a^2/2 + 2ba)/4 \Big|_0^x\, db & 0 \le x < 2 \cap 1 \le y \\ 1 & 2 \le x \cap 1 \le y \end{cases}$$

$$= \begin{cases} 0 & x < 0 \cup y < 0 \\ \int_0^y (x^2/2 + 2bx)/4\, db & 0 \le x < 2 \cap 0 \le y < 1 \\ \int_0^y (2 + 4b)/4\, db & 2 \le x \cap 0 \le y < 1 \\ \int_0^1 (x^2/2 + 2bx)/4\, db & 0 \le x < 2 \cap 1 \le y \\ 1 & 2 \le x \cap 1 \le y \end{cases}$$

$$= \begin{cases} 0 & x < 0 \cup y < 0 \\ (bx^2/2 + b^2x)/4 \Big|_0^y & 0 \le x < 2 \cap 0 \le y < 1 \\ (2b + 2b^2)/4 \Big|_0^y & 2 \le x \cap 0 \le y < 1 \\ (bx^2/2 + b^2x)/4 \Big|_0^1 & 0 \le x < 2 \cap 1 \le y \\ 1 & 2 \le x \cap 1 \le y \end{cases}$$

$$= \begin{cases} 0 & x < 0 \cup y < 0 \\ (yx^2/2 + y^2x)/4 & 0 \le x < 2 \cap 0 \le y < 1 \\ (2y + 2y^2)/4 & 2 \le x \cap 0 \le y < 1 \\ (x^2/2 + x)/4 & 0 \le x < 2 \cap 1 \le y \\ 1 & 2 \le x \cap 1 \le y \end{cases}$$

$$= \begin{cases} 0 & x < 0 \cup y < 0 \\ \frac{1}{8}x^2y + \frac{1}{4}xy^2 & 0 \le x < 2 \cap 0 \le y < 1 \\ \frac{1}{2}y^2 + \frac{1}{2}y & 2 \le x \cap 0 \le y < 1 \\ \frac{1}{8}x^2 + \frac{1}{4}x & 0 \le x < 2 \cap 1 \le y \\ 1 & 2 \le x \cap 1 \le y \end{cases}$$

d

From part (4.4 > b)

$$f_X(x) = \begin{cases} (x+1)/4 & 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

$$Z = 9/(X+1)^2$$

$$F_Z(z) = F_X(Z^{-1}(z))$$

$$F_X(x) = \int_{-\infty}^{x} f_X(t)dt$$

$$= \begin{cases} 0 & x < 0 \\ (x^2 + 2x)/8 & 0 \le x < 2 \\ 1 & 2 \le x \end{cases}$$

$$X = \sqrt{9/Z} - 1 \quad Z \in (1, 9)$$

$$F_Z(z) = \begin{cases} 0 & z < 1 \\ 1 - ((\sqrt{9/z} - 1)^2 + 2(\sqrt{9/z} - 1))/8 & 1 \le z < 9 \\ 1 & 9 \le z \end{cases}$$

$$= \begin{cases} 0 & z < 1 \\ 1 - (9/z - 2\sqrt{9/z} + 1 + 2\sqrt{9/z} - 2)/8 & 1 \le z < 9 \\ 1 & 9 \le z \end{cases}$$

$$= \begin{cases} 0 & z < 1 \\ 1 - (9/z - 1)/8 & 1 \le z < 9 \\ 1 & 9 \le z \end{cases}$$

$$f_Z(z) = \frac{d}{dz}F_Z(z)$$

$$= \begin{cases} \frac{9}{8z^2} & 1 < z < 9 \\ 0 & \text{otherwise} \end{cases}$$

4.10

a

$P(Y = 3, X = 2) = 0$
$P(Y = 3) = 1/6 + 1/6 = 1/3$
$P(X = 2) = 1/6 + 1/3 = 1/2$

$0 \neq (1/3)(1/2)$
$P(Y = 3, X = 2) \neq P(Y = 3)P(X = 2)$

$\square$

b

$P(X = x) = [1/4, 1/2, 1/4]_x$
$P(Y = y) = [1/3, 1/3, 1/3]_y$

$$P(U = u, V = v) = P(X = u)P(Y = v)$$
$$= \begin{bmatrix} 1/12 & 1/6 & 1/12 \\ 1/12 & 1/6 & 1/12 \\ 1/12 & 1/6 & 1/12 \end{bmatrix}_{u,v}$$

Y\X	1	2	3
2	1/12	1/6	1/12
3	1/12	1/6	1/12
4	1/12	1/6	1/12