PHYS 122-119B Lab 6: LCR

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PHYS 122-119B
Station 32, Rockefeller 403
Lab 6: LCR (Damped and Forced Oscillator)
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1 Abstract

The purpose of this lab is to observe and model the oscillatory nature of the LC circuit, by observing its natural resonance frequency, then testing if forcing other frequencies on the circuit diminishes its output if not at our expected resonance frequency.

Results:

	$C = C_1$ $R = R_C$	$C = C_2$ $R = R_C$	$C = C_2$ $R = R_C + R_1$	$C = C_2$ $R = R_C + R_2$
$\tau_{experimental}$	0.00080 ± 0.00002	0.0008 ± 0.00001	0.0006 ± 0.00001	0.000195 ± 0.00000
$\omega'_{experimental}$	-23280 ± 20	-4950 ± 10	-4730 ± 10	-5979 ± 6
$\tau_{expected}$	0.00092 ± 0.00002	0.00092 ± 0.00002	0.00061 ± 0.00001	0.000260 ± 0.00000
$\omega'_{expected}$	22900 ± 300	5110 ± 70	5260 ± 70	6310 ± 70

Variable	**Value**
$Q_{u\,experimental}$	-3.05 ± 0.05
$\omega_{R\,experimental}$	-49.4 ± 0.1
$Q_{u\,expected}$	3.67 ± 0.06
$\omega_{R\,expected}$	48.9 ± 0.7

We fail to reject the null hypothesis that our experimental data follows the current accepted theory.

2 Theory

2.1 Constants

L	Inductance of a coil
Q	Charge within a capacitor
R	Resistance of a resistor
C	Capacitance of a capacitor
ω_R	Resonant frequency of an LC circuit
τ	Decay rate of an LC circuit
V_C	Voltage across a capacitor
Q_u	Quality factor of a response curve

2.2 Formulae

2.2.1 Damped Oscillator

Given the expected behavior of an inductor, capacitor, and capacitor, we may find that the charge within the capacitor follows the oscillatory differential equation below, with the capacitor having an initial charge of Q_0.

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = 0 \qquad \text{Expected behavior of LC} \tag{2.2.1.1}$$

$$Q = Q_0 e^{-t/\tau} \sin(\omega' t + \phi) \qquad \text{Gen. sol. of damped oscillator} \tag{2.2.1.2}$$

$$\tau = \frac{2L}{R} \qquad \text{Decay rate} \tag{2.2.1.3}$$

$$\omega = \frac{1}{\sqrt{LC}} \qquad \text{Freq. of undamped osci.} \tag{2.2.1.4}$$

$$\omega' = \sqrt{\omega^2 - \frac{1}{\tau^2}} \qquad \text{Freq. of damped osci.} \tag{2.2.1.5}$$

$$Q = CV_C \qquad \text{Charge in a capacitor} \tag{2.2.1.6}$$

$$V_C = \frac{Q_0}{C} e^{-t/\tau} \sin(\omega' t + \phi) \qquad \text{Gen. sol. of damped oscillator} \tag{2.2.1.7}$$

2.2.2 Forced Oscillator

Now, instead of just "releasing" the circuit after charging the capacitor, we instead force the circuit with a constant sine curve of amplitude V_m, and thus our differential equation will be updated to reflect that.

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = V_m \sin(\omega t) \qquad \text{Behavior of forced LC} \tag{2.2.2.1}$$

$$Q = \frac{V_m}{\sqrt{\left(\frac{1}{C} - L\omega^2\right)^2 + R^2\omega^2}}\sin(\omega t + \phi) \qquad \text{General solution} \tag{2.2.2.2}$$

$$\max_{\omega} Q = \frac{V_m}{R\omega}\sin(\omega t + \phi) \qquad \text{Maximize Q} \tag{2.2.2.3}$$

$$\max_{\omega} Q \implies \frac{1}{C} = L\omega^2 \qquad \text{Corresponding omega} \tag{2.2.2.4}$$

$$\omega = \frac{1}{\sqrt{LC}} = \omega_R \qquad \text{Max Q implies resonance} \tag{2.2.2.5}$$

$$I = \frac{dQ}{dt} \qquad \text{Definition of current} \tag{2.2.2.6}$$

$$I = \frac{V_m}{\sqrt{\left(\frac{1}{\omega C} - L\omega\right)^2 + R^2}}\sin(\omega t + \phi') \qquad \text{General solution} \tag{2.2.2.7}$$

$$\frac{V_R}{V_m} = \frac{R}{\sqrt{\left(\frac{1}{\omega C} - L\omega\right)^2 + R^2}} \qquad \text{Solve for gain} \tag{2.2.2.8}$$

Now, given the width $\Delta\omega$ at half the maximum of the curve, we may determine Q_u, the quality

$$Q_u = \frac{\omega_R L}{R} = \frac{1}{R}\sqrt{\frac{L}{C}} \quad \text{Definition of quality} \tag{2.2.2.9}$$

$$\frac{V_R}{V_m} = \frac{R}{\sqrt{\left(\frac{L}{\omega}(\omega_R^2 - \omega^2)\right)^2 + R^2}} \quad \text{In terms of resonant freq.} \tag{2.2.2.10}$$

$$\frac{V_R}{V_m} = \frac{\frac{R}{L}}{\sqrt{\left(\omega_R^2 - \omega^2\right)^2 + \frac{\omega_R^2\omega^2}{Q_u^2}}} \quad \text{In terms of Q} \tag{2.2.2.11}$$

$$\frac{V_R}{V_m} = \frac{\frac{RQ_u}{L\omega_R\omega}}{\sqrt{\left(Q_u\frac{\omega_R^2-\omega^2}{\omega_R\omega}\right)^2 + 1}} \quad \text{Simplify gain} \tag{2.2.2.12}$$

$$\frac{V_R}{V_m} = \frac{A}{\sqrt{\left(Q_u\frac{\omega_R^2-\omega^2}{\omega_R\omega}\right)^2 + 1}} \quad A \approx 1 \text{ around resonance} \tag{2.2.2.13}$$

$$\frac{1}{2} = \frac{\frac{R}{L}}{\sqrt{\left(\omega_R^2 - \omega^2\right)^2 + \frac{\omega_R^2\omega^2}{Q_u^2}}} \quad \text{Set half gain from 2.2.2.11} \tag{2.2.2.14}$$

$$\left(\omega_R^2 - \omega^2\right)^2 + \frac{\omega_R^2\omega^2}{Q_u^2} = 4\frac{\omega_R^2\omega^2}{Q_u^2} \quad \text{Using max I} \tag{2.2.2.15}$$

$$\left(\omega_R^2 - \omega^2\right)^2 = 3\frac{\omega_R^2\omega^2}{Q_u^2} \quad \text{Solving for Qu} \tag{2.2.2.16}$$

$$\omega_R^2 - \omega^2 = \sqrt{3}\frac{\omega_R\omega}{Q_u} \tag{2.2.2.17}$$

$$\frac{1}{2}\Delta_\omega = \frac{\sqrt{3}}{2}\frac{\omega_R}{Q_u} \quad \text{Approximate omega} \tag{2.2.2.18}$$

$$\frac{\Delta_\omega}{\omega_R} = \frac{\sqrt{3}}{Q_u} \quad \text{Solve for Qu} \tag{2.2.2.19}$$

$$\Delta_\omega = \frac{\sqrt{3}R}{L} \quad \text{From 2.2.2.9} \tag{2.2.2.20}$$

2.3 Error Propagation

2.3.1 Damped Oscillator

$$V_C = \frac{Q_0}{C} e^{-t/\tau} \sin(\omega' t + \phi) \qquad \text{From 2.2.1.7} \qquad (2.3.1.1)$$

$$\tau = \frac{2L}{R} \qquad \text{From 2.2.1.3} \qquad (2.3.1.2)$$

$$\delta_\tau = \tau \sqrt{\left(\frac{\delta_L}{L}\right)^2 + \left(\frac{\delta_R}{R}\right)^2} \qquad \text{Error prop.} \qquad (2.3.1.3)$$

$$\omega = \frac{1}{\sqrt{LC}} \qquad \text{From 2.2.1.4} \qquad (2.3.1.4)$$

$$\omega' = \sqrt{\omega^2 - \frac{1}{\tau^2}} \qquad \text{From 2.2.1.5} \qquad (2.3.1.5)$$

$$\omega' = \sqrt{\frac{1}{LC} - \frac{1}{\tau^2}} \qquad \text{Substitute} \qquad (2.3.1.6)$$

$$\delta_{\omega'} = \frac{1}{2\omega'} \sqrt{\left(\frac{\delta_L}{L^2 C}\right)^2 + \left(\frac{\delta_C}{LC^2}\right)^2 + \left(\frac{2\delta_\tau}{\tau^3}\right)^2} \qquad \text{Substitute} \qquad (2.3.1.7)$$

2.3.2 Forced Oscillator

$$\frac{V_R}{V_m} = \frac{A}{\sqrt{\left(Q \frac{\omega_R^2 - \omega^2}{\omega_R \omega}\right)^2 + 1}} \qquad \text{From 2.2.2.13} \qquad (2.3.2.1)$$

$$Q_u = \frac{1}{R} \sqrt{\frac{L}{C}} \qquad \text{From 2.2.2.9} \qquad (2.3.2.2)$$

$$\delta_{Q_u} = Q_u \sqrt{\left(\frac{\delta_R}{R}\right)^2 + \left(\frac{\delta_L}{2L}\right)^2 + \left(\frac{\delta_C}{2C}\right)^2} \qquad \text{Error Prop.} \qquad (2.3.2.3)$$

$$\omega_R = \frac{1}{\sqrt{LC}} \qquad \text{From 2.2.2.5} \qquad (2.3.2.4)$$

$$\delta_{\omega_R} = \omega_R \sqrt{\left(\frac{\delta_L}{2L}\right)^2 + \left(\frac{\delta_C}{2C}\right)^2} \qquad \text{Error Prop.} \qquad (2.3.2.5)$$

3 Procedure

3.1 Materials

1. $80 - 100\ mH$ inductor
2. 2x $1\ k\Omega$ resistors
3. $100\ \Omega$ resistor
4. $0.47\ \mu F$ capacitor
5. $0.022\ \mu F$ capacitor
6. DMM
7. Oscilloscope
8. Function generator

3.2 General Setup

3.2.1 Damped Oscillator

For each of the following RC combinations:

1. $C = 0.022\ \mu F, R = 0\ \Omega$
2. $C = 0.47\ \mu F, R = 0\ \Omega$
3. $C = 0.47\ \mu F, R = 100\ \Omega$
4. $C = 0.47\ \mu F, R = 500\ \Omega$

We performed the following:

1. Set up this diagram:

```
┏S1━━━V━┓
┃       ┃
┣S2━━━L━┫
┃       ┃
┣C━┳━━R━┛
┃  ┃
┗P━┛
```

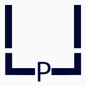
   I. $S1, S2$ are switches (open by default)
   II. V is our battery
   III. L is our induction coil
   IV. C is our capacitor
   V. R is our added resistance
   VI. P is our voltage probe
2. Charge the capacitor by activating $S1$
3. Begin collecting data with P
4. Deactivate $S1$ and activate $S2$ to allow the LCR circuit to enter damped oscillation
5. Fit the voltage vs. time data to eq. $2.2.1.7$
6. Check if our ω' and τ matches expectations
7. Classify if this system is over, under, or critically damped by the value of ω'

3.2.2 Forced Oscillator

We performed the following:

1. Set up this diagram:

```
┏F━━━C━┓
┃      ┃
┣R━┳━L━┛
```



  I. F is our function generator set to a sine wave with frequency $f = \frac{\omega}{2\pi}$ and amplitude V_m
  II. L is our induction coil
  III. C is our capacitor
  IV. R is our added resistance
  V. P is our voltage probe
2. Activate the function generator at various frequencies
3. Record the output amplitude of the sinusoidal on the probe
4. Fit our frequency vs. amplitude data with eq. $2.2.2.13$
5. Check if our fit values match the expected results

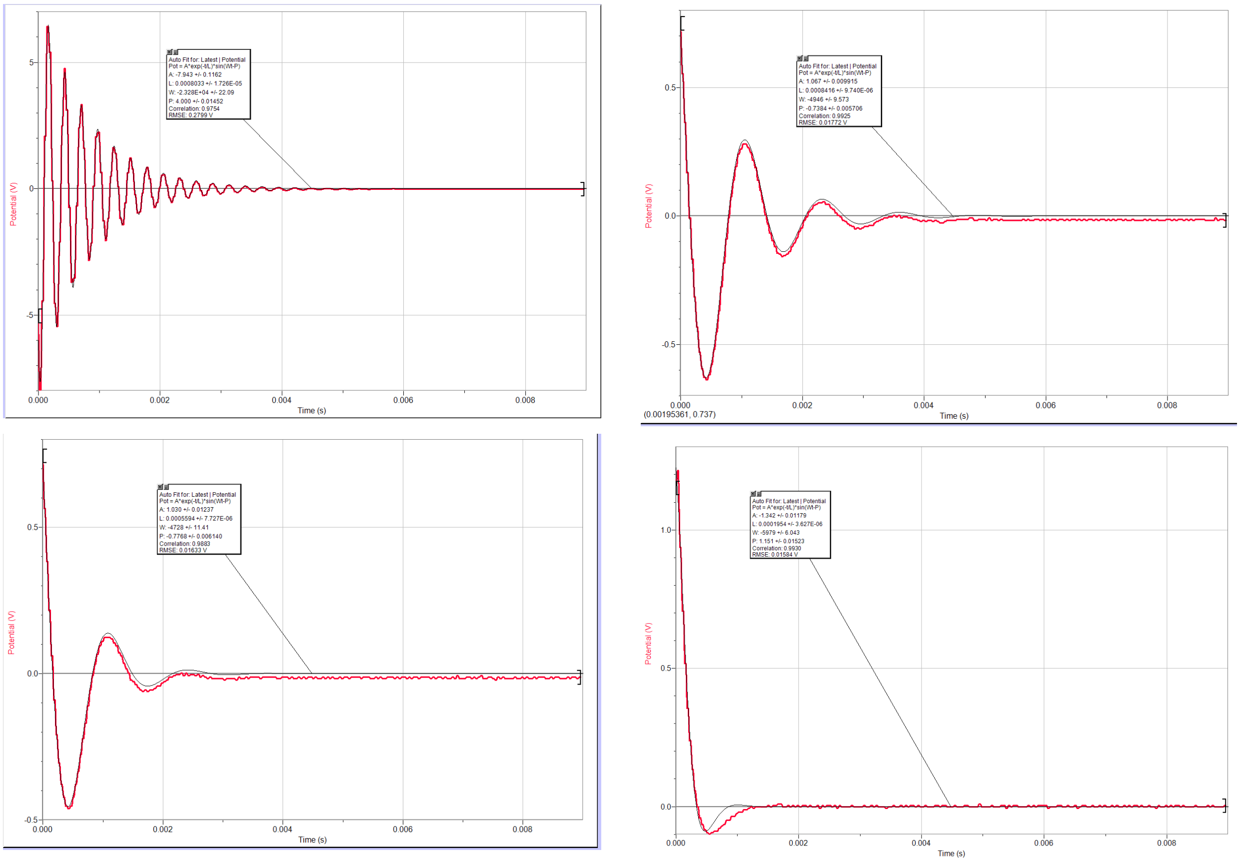
4 Analysis

4.1 Damped Oscillator

With the following constants:

$$
\begin{aligned}
R_1 &= 98.9 \pm 0.5\% \; \Omega \\
R_2 &= 0.49 \pm 0.5\% \; k\Omega \\
R_C &= 193.6 \pm 0.5\% \; \Omega \\
L &= 88.8 \pm 2\% \; mH \\
C_1 &= 21.6 \pm 2\% \; nF \\
C_2 &= 0.451 \pm 2\% \; \mu F
\end{aligned}
$$

With four trials with the following RC combinations, we fit our voltage vs. time data to eq. $2.2.1.7$ and obtained the following fit values



	$C = C_1$ $R = R_C$	$C = C_2$ $R = R_C$	$C = C_2$ $R = R_C + R_1$	$C = C_2$ $R = R_C + R_2$
$\frac{Q_0}{C}$ (V)	-7.9 ± 0.1	1.07 ± 0.01	1.03 ± 0.01	-1.34 ± 0.01
τ	0.00080 ± 0.00002	0.0008 ± 0.00001	0.0006 ± 0.00001	0.000195 ± 0.000004
ω'	-23280 ± 20	-4950 ± 10	-4730 ± 10	-5979 ± 6
ϕ	4.00 ± 0.01	-0.738 ± 0.006	-0.777 ± 0.006	1.15 ± 0.02

Using eq. 2.3.1.2, 2.3.1.3, 2.3.1.6, and 2.3.1.7, we obtain our expected results of:

	$C = C_1$ $R = R_C$	$C = C_2$ $R = R_C$	$C = C_2$ $R = R_C + R_1$	$C = C_2$ $R = R_C + R_2$
τ	0.00092 ± 0.00002	0.00092 ± 0.00002	0.00061 ± 0.00001	0.000260 ± 0.000005
ω'	22900 ± 300	5110 ± 70	5260 ± 70	6310 ± 70

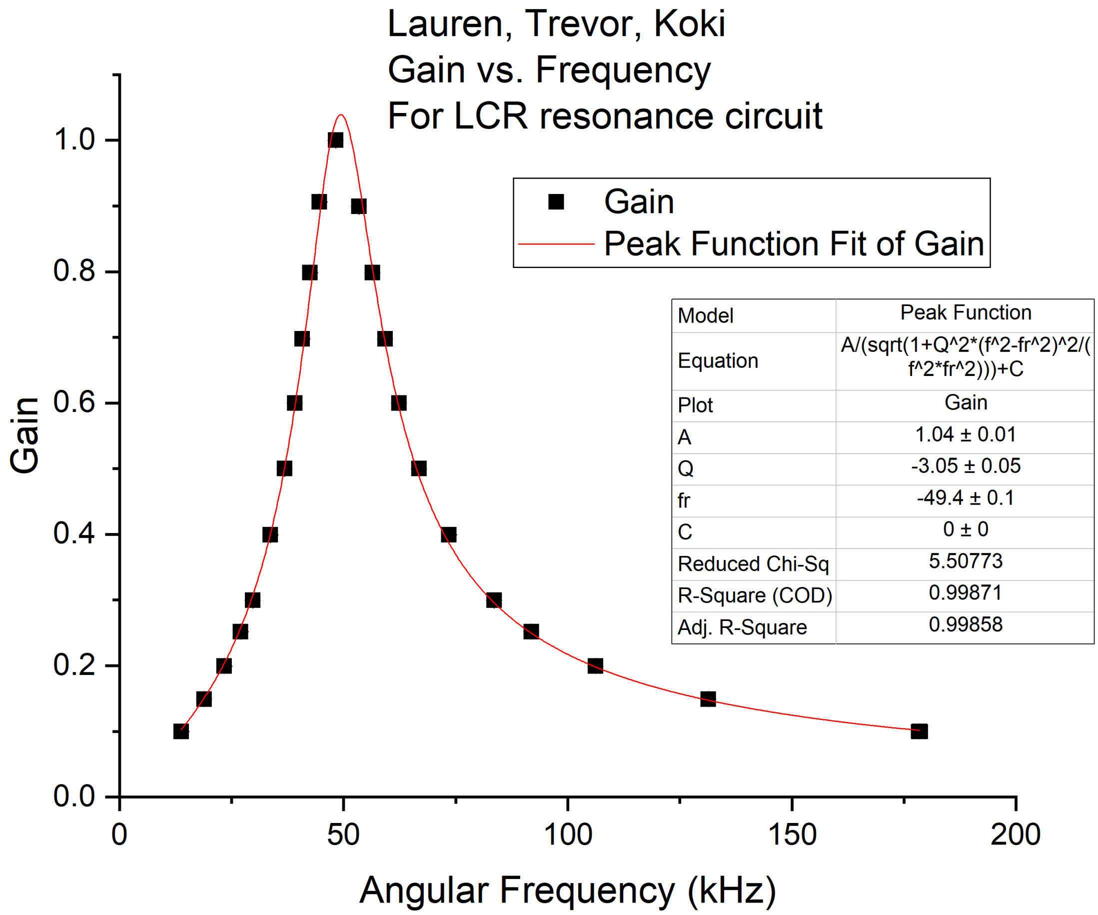
We find all our calculated values to be within around 3 to 6 standard deviations away from our experimental values, which is relatively close, and thus fail to reject the null hypothesis that our data matches the current expected values. Since all of our ω'^2 values are positive, all of these are underdamped.

4.2 Forced Oscillator

With the following constants:

$$
\begin{aligned}
R &= 0.99 \pm 0.5\%\ k\Omega \\
R_C &= 193.6 \pm 0.5\%\ \Omega \\
L &= 88.8 \pm 2\%\ mH \\
C &= 0.0047 \pm 2\%\ \mu F \\
V_{in} &= 16.000\ V
\end{aligned}
$$

After measuring the output amplitude at various frequencies, we obtain the following graph of gain $\frac{V_R}{V_m}$ vs. angual frequency ω.



This gives us the fit values fit to the equation of $2.2.2.13$:

Variable	Value
Q_u	-3.05 ± 0.05
ω_R	-49.4 ± 0.1

Using the eq. $2.3.2.2$, $2.3.2.3$, $2.3.2.4$, and $2.3.2.5$ we are able to calculate the expected values for Q_u and ω_R

Variable	Value
Q_u	3.67 ± 0.06
ω_R	48.9 ± 0.7

We find that our experimental values align fairly closely to our expected values, with our Q_u value being within 10 SD of our experimental value, and our ω_R being within 1 SD of our experimental value. Thus, we fail to reject the null hypothesis that our experiment follows the relevant theory.

5 Conclusion

In the purpose to observe and model the oscillatory nature of the LCR circuit, we fail to disprove the current theory of LCR circuits.

For the Damped Oscillator setup, we obtained the following values:

	$C = C_1$ $R = R_C$	$C = C_2$ $R = R_C$	$C = C_2$ $R = R_C + R_1$	$C = C_2$ $R = R_C + R_2$
$\tau_{experimental}$	0.00080 ± 0.00002	0.0008 ± 0.00001	0.0006 ± 0.00001	0.000195 ± 0.00000
$\omega'_{experimental}$	-23280 ± 20	-4950 ± 10	-4730 ± 10	-5979 ± 6
$\tau_{expected}$	0.00092 ± 0.00002	0.00092 ± 0.00002	0.00061 ± 0.00001	0.000260 ± 0.00000
$\omega'_{expected}$	22900 ± 300	5110 ± 70	5260 ± 70	6310 ± 70

For the Forced Oscillator setup, we obtained the following values:

Variable	Value
$Q_{u\,experimental}$	-3.05 ± 0.05
$\omega_{R\,experimental}$	-49.4 ± 0.1
$Q_{u\,expected}$	3.67 ± 0.06
$\omega_{R\,expected}$	48.9 ± 0.7

We find that all of our values are very close to the expected values, to within around 6 SD of the value. Thus, we fail to reject the null hypothesis that our experimental values follow the expected relations. We could possibly reduce error by being more careful to tare our tools before beginning the experiment, as well as taking significantly more data points, instead of merely 5 total. Overall, we find our data to be very close to what is expected.

6 Acknowledgements and Info

- Lab 6 - LCR
- 2024-11-18
- Station 32 Rockefeller 403
- PHYS 122-119B

Lab Partner: Lauren Lee, Koki Takizawa
Lab Manual: Lab 6 LCR PHYS 122

6.1 References

Driscoll, D., *General Physics 2: Electricity and Magnetism Lab Manual*, "Electric Potential and Fields".