HW 5

3.2

Suppose the random variable X has possible values $\{1, 2, 3, 4, 5, 6\}$ and probability mass function of the form $p(k) = ck$.

a

Find c.

✓ **Answer**

$\sum_{x \in X} p(x) = c(1 + 2 + 3 + 4 + 5 + 6) = 1$

$c = \frac{1}{21}$

b

Find the probability that X is odd.

✓ **Answer**

$P(x \text{ is odd}) = c(1 + 3 + 5) = \frac{9}{21} = \frac{3}{7}$

3.3

Let X be a continuous random variable with density function

$$f(x) = \begin{cases} 3e^{-3x} & x > 0 \\ 0 & \text{else} \end{cases}$$

a

Verify that f is a density function.

✓ **Answer**

$\int_{-\infty}^{\infty} f(x)\,dx = \int_{0}^{\infty} 3e^{-3x}\,dx = -e^{-3x}\Big|_{0}^{\infty}$

$= 0 - -1 = 1$

b

Calculate $P(-1 < X < 1)$.

✓ **Answer**

$\int_{-1}^{1} f(x)\,dx = -e^{-3x}\Big|_{0}^{1}$

$= 1 - e^{-3}$

c

Calculate $P(X < 5)$.

✓ **Answer**

$\int_{-\infty}^{5} f(x)\,dx = -e^{-3x}\Big|_{0}^{5}$

$= 1 - e^{-15}$

d

Calculate $P(2 < X < 4 | X < 5)$.

✓ **Answer**

$= \frac{P(2<X<4)}{P(X<5)} = \frac{\int_{2}^{4} f(x)\,dx}{\int_{-\infty}^{5} f(x)\,dx}$

$= \frac{e^{-6}-e^{-8}}{1-e^{-15}}$

3.7

Suppose that the continuous random variable X has cumulative distribution function given by

$$F(x) = \begin{cases} 0 & x < \sqrt{2} \\ x^2 - 2 & \sqrt{2} \leq x < \sqrt{3} \\ 1 & \sqrt{3} \leq x \end{cases}$$

a

Find the smallest interval $[a, b]$ such that of $P(a \leq X \leq b) = 1$

✓ **Answer**

$[a, b] = [\sqrt{2}, \sqrt{3}]$

b

Find $P(X = 1.6)$.

✓ **Answer**

$F(1.6) - F(1.6) = 0$

c

Find $P\left(1 \leq X \leq \frac{3}{2}\right)$.

✓ **Answer**

$F\left(\frac{3}{2}\right) - F(1) = \left(\frac{3}{2}\right)^2 - 2$

$= 0.25$

d

Find the probability density function of X.

✓ **Answer**

$f(x) = \frac{d}{dx}F(x)$

$$f(x) = \begin{cases} 0 & x < \sqrt{2} \\ 2x & \sqrt{2} \leq x < \sqrt{3} \\ 0 & \sqrt{3} \leq x \end{cases}$$

$F(x)$ is continuous, so no $\delta(x)$ will need to be added.

3.9

Let X be the random variable from Exercise 3.3.

$$f(x) = \begin{cases} 3e^{-3x} & x > 0 \\ 0 & \text{else} \end{cases}$$

a

Find the mean of X.

✓ **Answer**

$\int_0^\infty 3xe^{-3x}\,dx = -(x + \frac{1}{3})e^{-3x}\Big|_0^\infty$

$= \frac{1}{3}$

3.25

In each of the following cases find all values of b for which the given function is a probability density function.

b

$$h(x) = \begin{cases} \cos x & -b \leq x \leq b \\ 0 & \text{else} \end{cases}$$

✓ **Answer**

$\int_{-b}^{b} \cos x \, dx = 1$

$h(x)$ cannot be negative, so $b < \frac{\pi}{2}$

$\sin b = \frac{1}{2}$

$b = \frac{\pi}{6}$

3.32

Let X be a continuous random variable with density function

$$f_X(x) = \begin{cases} \frac{1}{2}x^{-3/2} & 1 < x < \infty \\ 0 & \text{else} \end{cases}$$

a

Find $P(X > 10)$.

✓ **Answer**

$\int_{10}^{\infty} f_X(x) \, dx = \int_{10}^{\infty} \frac{1}{2}x^{-3/2} \, dx$

$= -x^{-1/2}\Big|_{10}^{\infty}$

$= \frac{1}{\sqrt{10}}$

b

Find the cumulative distribution function F_X of X.

✓ **Answer**

$F_X(x) = -x^{-1/2} + C$
$F_X(1) = 0$
$F_X(x) = -x^{-1/2} + 1$

c

Find $E[X]$.

✓ **Answer**

$\int_1^\infty \frac{x}{2} x^{-3/2}$
$= \int_1^\infty \frac{1}{2} x^{-1/2}$
$= \sqrt{x}\Big|_1^\infty$
$E[X] = \infty$