Laplace Transforms

The Laplace transform compares how similar a function is to a standard function.

Its general form is

$$\int\limits_{-N}^{N} y(t)q(t)\ dt$$

For very large N
$q(t)$ is the standard function you are comparing to, often of the forms

- $\sin(\omega t)$
- e^{-zt} or e^{-st} and $e^{-i\omega t}$

For the purposes of differential equations, we will most commonly be using e^{-st} in the form

$$\int\limits_{0}^{\infty} y(t)e^{-st}\ dt$$

Properties of Laplace Transforms

- $L[\frac{dy}{dt}] = sL[y] - y(0)$
- $L[f+g] = L[f] + L[g]$
- $L[cf] = cL[f]$ for constant c
- $L^{-1}[F] = f \iff L[f] = F$
- $L[u_a(t)f(t-a)] = e^{-sa}F(s)$
- $L[e^{st}f(t)] = F(s-a)$

For the application of differential equations

It is often significantly easier to take the Laplace of both sides of the differential equation, then solve for $L[y]$ before inverting it to find y

Heaviside function

Turns "on" functions if you multiply them together

$$u_a(t) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t \geq a \end{cases}$$

Dirac-Delta function

This is the "derivative" of the Heaviside function.

$$\delta_a(t) = \begin{cases} \text{big enough that integrating this instant will increase the integral by } 1 & \text{if } t = a \\ 0 & \text{else} \end{cases}$$

Common transforms

- $L[e^{at}] = \frac{1}{s-a}$
- $L[\sin(\omega t)] = \frac{\omega}{s^2+\omega^2}$
- $L[e^{at}\sin(\omega t)] = \frac{\omega}{(s-a)^2+\omega^2}$
- $L[t\sin(\omega t)] = \frac{2\omega s}{(s^2+\omega^2)^2}$
- $L[u_a] = \frac{e^{-as}}{s}$
- $L[t^n] = \frac{n!}{s^{n+1}}$
- $L[\cos(\omega t)] = \frac{s}{s^2+\omega^2}$
- $L[e^{at}\cos(\omega t)] = \frac{s-a}{(s-a)^2+\omega^2}$
- $L[t\sin(\omega t)] = \frac{s^2-\omega^2}{(s^2+\omega^2)^2}$
- $L[\delta_a] = e^{-as}$