6.1

1

In each of (b)–(d), answer the following questions: Is $A \subseteq B$? Is $B \subseteq A$? Is either A or B a proper subset of the other?

b

$A = \{3, \sqrt{5^2 - 4^2}, 24 \mod 7\}$
$B = \{8 \mod 5\}$

✓ **Answer**

$A = \{3\}$
$B = \{3\}$

$A \subseteq B$
$B \subseteq A$
$A \equiv B$
$A \not\subset B$
$B \not\subset A$

d

$A = \{a, b, c\}$
$B = \{\{a\}, \{b\}, \{c\}\}$

✓ **Answer**

$A \nsubseteq B$
$B \nsubseteq A$

4

$A = \{n \in \mathbf{Z} | n = 5r, r \in \mathbf{Z}\}$
$B = \{m \in \mathbf{Z} | m = 20s, s \in \mathbf{Z}\}$
Prove or disprove each of the following statements.

a

$A \subseteq B$

✓ **Answer**

False, $5 \in A$, but $5 \notin B$

b

$B \subseteq A$

✓ **Answer**

$B = \{m \in \mathbf{Z} | m = 20s, s \in \mathbf{Z}\}$
$B = \{m \in \mathbf{Z} | m = 5(4s), s \in \mathbf{Z}\}$
Let $r = 4s \in \mathbf{Z}$
$B = \{m \in \mathbf{Z} | m = 5r, r = 4s, s \in \mathbf{Z}\}$
$A = \{m \in \mathbf{Z} | m = 5r, r \in \mathbf{Z}\}$
$B \subseteq A$

12

Let the universal set be $\mathbf{R}$, the set of all real numbers, and let
$A = \{x \in \mathbf{R} | -3 \leq x \leq 0\}$
$B = \{x \in \mathbf{R} | -1 < x < 2\}$
$C = \{x \in \mathbf{R} | 6 < x \leq 8\}$
Find each of the following:

a

$A \cup B$

✓ **Answer**

$A \cup B = \{x \in \mathbf{R} | -3 \leq x < 2\}$

b

$A \cap B$

✓ **Answer**

$A \cap B = \{x \in \mathbf{R} | -1 < x \leq 0\}$

g

$A^C \cap B^C$

✓ Answer

$$A^C = \{x \in \mathbf{R} | -3 > x \text{ or } x > 0\}$$
$$B^C = \{x \in \mathbf{R} | -1 \geq x \text{ or } x \geq 2\}$$
$$A^C \cap B^C = \{x \in \mathbf{R} | -3 > x \text{ or } x \geq 2\}$$

i

$(A \cup B)^C$

✓ Answer

$$A \cup B = \{x \in \mathbf{R} | -3 \leq x < 2\}$$
$$(A \cup B)^C = \{x \in \mathbf{R} | -3 > x \text{ or } x \geq 2\}$$

25

$$R_i = \left\{x \in \mathbf{R} | 1 \leq x \leq 1 + \frac{1}{i}\right\} = \left[1, 1 + \frac{1}{i}\right] \qquad : i \in \mathbf{Z}$$

a

$$\bigcup_{i=1}^{4} R_i = ?$$

✓ Answer

$[1, 2]$

c

Are $R_1, R_2, R_3, \ldots$ mutually disjoint? Explain.

✓ Answer

No

$R_1 = [1, 2]$

$R_2 = [1, 1.5]$
$R_3 = \left[1, \frac{4}{3}\right]$

All three of these sets have $\left[1, \frac{4}{3}\right]$ within them, therefore they are not disjoint

d

$$\bigcup_{i=1}^{n} R_i = ?$$

✓ **Answer**

$[1, 2]$

f

$$\bigcup_{i=1}^{\infty} R_i = ?$$

✓ **Answer**

$[1, 2]$

29

Let $\mathbf{R}$ be the set of all real numbers. Is $\{\mathbf{R}^+, \mathbf{R}^-, \{0\}\}$ a partition of $\mathbf{R}$? Explain your answer.

✓ **Answer**

Yes, because all numbers within $\mathbf{R}$ are within one of the elements of that partition.
In other words, all elements are mutually disjoint, and the sum of all the elements are equal to $\mathbf{R}$

6.2

Use an element argument to prove each statement in 17–18. Assume that all sets are subsets of a universal set $\mathbf{U}$.

17

For all sets A, B, C, if $A \subseteq B$ then $A \cup C \subseteq B \cup C$.

✓ **Answer**

$A \subseteq B$
$x \in A \to x \in B$
$B \subseteq B \cup C$
$x \in A \to x \in B \to x \in B \cup C$
$C \subseteq B \cup C$
$x \in C \to x \in B \cup C$

$x \notin B \cup C \to x \notin C \wedge x \notin A$
$x \in C \vee x \in A \to x \in B \cup C$
$x \in C \cup A \to x \in B \cup C$

$A \subseteq B \to A \cup C \subseteq B \cup C$

18

For all sets A and B, if $A \subseteq B$ then $B^C \subseteq A^C$

✓ **Answer**

$A \subseteq B$
$x \in A \to x \in B$
$x \notin B \to x \notin A$
$B^C \subseteq A^C$

25

Find the mistake in the following “proof” that for all sets A and B, $(A - B) \cup (A \cap B) \subseteq A$

“Proof: Suppose A and B are any sets, and suppose $x \in (A - B) \cup (A \cap B)$. If $x \in A$ then $x \in A - B$ and so, by definition of difference, $x \in A$ and $x \notin B$ In particular, $x \in A$, and, therefore $(A - B) \cup (A \cap B) \subseteq A$ by definition of subset.”

✓ **Answer**

$x \in A \nrightarrow x \in A - B$

42

For every positive integer n, if $A_1, A_2, A_3, \ldots$ and B are any sets, then

$$\bigcap_{i=1}^{n}(A_i - B) = \left(\bigcap_{i=1}^{n} A_i\right) - B$$

✓ **Answer**

$$\bigcap_{i=1}^{n}(A_i - B) = \bigcap_{i=1}^{n}(A_i \cap B^C) = \bigcap_{i=1}^{n}(A_i) \cap \bigcap_{i=1}^{n}(B^C)$$
$$= \bigcap_{i=1}^{n}(A_i) \cap B^C$$
$$\bigcap_{i=1}^{n}(A_i - B) = \left(\bigcap_{i=1}^{n} A_i\right) - B$$

9.2

11

c

How many bit strings of length 8 begin and end with a 1?

✓ **Answer**

$2^6 = 64$

17

c

How many integers from 1000 through 9999 have distinct digits?

✓ **Answer**

$10P4 - 9P3 = 4536$

e

What is the probability that a randomly chosen four-digit integer has distinct digits? has distinct digits and is odd?

✓ **Answer**

$\frac{4536}{9000}$

$(5 * 5 + 4 * 4) * 8 * 7 = 2296$
$\frac{2296}{9000}$

21

Suppose A is a set with m elements and B is a set with n elements.

a

How many relations are there from A to B? Explain.

✓ **Answer**

The Cardinality of the power set of the full relation of A and B,
As this gives us the length of the number of sets of any combination of all possible mappings from A to B.
$|P(A \times B)|$

b

How many functions are there from A to B? Explain.

✓ **Answer**

Each element is A must be mapped to one of the elements of B
$|A||B|$

c

What fraction of the relations from A to B are functions?

✓ **Answer**

$|P(A \times B)| = |P(|A||B|)| = 2^{|A||B|}$
$|A||B|$

$\frac{|A||B|}{2^{|A||B|}}$

31

d

If $p_1, p_2, \ldots, p_m$ are distinct prime numbers and $a_1, a_2, \ldots, a_m$ are positive integers, how many distinct positive divisors does $p_1^{a_1} p_2^{a_2} \ldots p_m^{a_m}$ have?

> **Answer**
>
> Each prime can have $[0, a_n]$ multiples, so for m primes, there are $\prod_{n=1}^{m} (a_n + 1)$ positive divisors

e

What is the smallest positive integer with exactly 12 divisors?

> **Answer**
>
> 12 must be the product of $n \in \mathbf{Z}$ integers, where each integer $m \geq 1$ must satisfy the restriction $\prod_{o=1}^{n} (m_o + 1)$
>
> 12 may be factorized as
> $\{\{1, 12\}, \{2, 6\}, \{3, 4\}\}$
>
> Therefore m may be
> $\{\{1, 5\}, \{2, 3\}\}$
>
> The smallest number with m as exponents for prime numbers would be
> $2^5 3 = 96$ or $2^3 3^2 = 72$
>
> 72 has $\{1, 2, 4, 8, 3, 6, 12, 24, 9, 18, 36, 72\}$ as positive divisors
>
> Therefore 72 is the smallest positive integer with 12 divisors

39

b

How many ways can six of the letters of the word $ALGORITHM$ be selected and written in a row?

> **Answer**
>
> There are 9 letters
> $9P6 = 60480$

d

How many ways can six of the letters of the word $ALGORITHM$ be selected and written in a row if the first two letters must be OR?

> ✓ **Answer**
>
> 7 letters left, choosing 4
> $7P4 = 840$

9.3

2

b

How many strings of hexadecimal digits consist of from two through five digits?

> ✓ **Answer**
>
> $16^5 - 16 = 1048560$

7

At a certain company, passwords must be from 3–5 symbols long and composed from the 26 uppercase letters of the Roman alphabet, the ten digits 0–9, and the 14 symbols !, @, #, $, %, ^, &, *, (,), -, +, {, and }.

c

How many passwords have at least one repeated symbol?

> ✓ **Answer**
>
> 50 possible characters
> $50^3 + 50^4 + 50^5 = 318875000$ possible passwords
> $50P3 + 50P4 + 50P5 = 259896000$ passwords with repeated symbols

d

What is the probability that a password chosen at random has at least one repeated symbol?

$\frac{259896000}{318875000}$

17

a

How many strings of four hexadecimal digits do not have any repeated digits?

Answer

$16P4 = 43680$

b

How many strings of four hexadecimal digits have at least one repeated digit?

Answer

$16^4 - 16P4 = 21856$

c

What is the probability that a randomly chosen string of four hexadecimal digits has at least one repeated digit?

Answer

$\frac{21856}{43680} = \frac{683}{1365}$

23

b

Suppose an integer from 1 through 1000 is chosen at random. Find the probability that the integer is a multiple of 4 or a multiple of 7.

Answer

There are 250 multiples of 4
There are 142 multiples of 7
There are 35 multiples of both at the same time
There is a probability of $\frac{357}{1000}$ to get a multiple of 4 or 7

c

How many integers from 1 through 1000 are neither multiples of 4 nor multiples of 7?

Answer

643