First-Order Systems

First-order systems have multiple related rates that tend to depend on each other for their change over time, for example a predator prey system.

Vectors and First-Order Systems

Take, for example, this predator-prey system:

$$\begin{cases} \frac{dR}{dt} = 2R - 1.2RF \\ \frac{dF}{dt} = -F + 0.9RF \end{cases}$$

This system can be plotted on an (R, F) plane as a vector or direction field to understand its movement.

Let $\vec{P}(t) = \langle R(t), F(t) \rangle$
$\frac{d\vec{P}}{dt} = \left\langle \frac{dR}{dt}, \frac{dF}{dt} \right\rangle$
Let $\vec{V}(\vec{P})$ be $\frac{d\vec{P}}{dt}$, our original system (in systems that do not depend on time)

This gives us $\frac{d\vec{P}}{dt} = \vec{V}(\vec{P})$

We can now understand that a vector field across all $\vec{P}$ will have a vector associated by $\vec{V}(\vec{P})$

Equilibrium Points

Equilibrium points in systems are when $\vec{V}(\vec{P}) = \vec{0}$, where $\vec{P}$ is the equilibrium point
This simply means that if the system were at this point, $\vec{P}$ would not move over time.

In predator-prey and many other relationships, $\vec{P} = \vec{0}$ will often be an equilibrium point.

Converting Second-Order Differential Equations into a First Order System

Given any second-order differential equation, we are able to assign a dummy variable to the first derivative of the variable of interest.

For example:

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = 0$$

$$\implies \begin{cases} \frac{dy}{dt} = v \\ \frac{dv}{dt} = -qy - pv \end{cases}$$

Linear second-order differential equations

For all second-order differential equations, we may skip converting to a system as a particular solution will always be of the form $e^{\lambda t}$ where λ can easily be solved in this case with the trace-determinant shortcut. The general solution may be easily obtained through the linearity principle:

$$\mathbf{Y} = k_1 e^{\lambda_1 t} + k_2 e^{\lambda_2 t}$$

Decoupled Systems

Fully decoupled systems are systems where the variables do not depend on each other, such as:

$$\begin{cases} \frac{dx}{dt} = -2x \\ \frac{dy}{dt} = -y \end{cases}$$

These can be solved independently in order to find a general solution of the system

Partially Decoupled Systems

Partially decoupled systems are when at least one variable does not depend on the rest of the system, such as:

$$\begin{cases} \frac{dx}{dt} = -2x + y \\ \frac{dy}{dt} = -y \end{cases}$$

These can be solved by solving the decoupled variables first

Euler's method

Given a system that can be represented as

$$\frac{d\vec{Y}}{dt} = \vec{V}(\vec{Y})$$

Pick a step size Δt

Choose a starting point $\vec{P} = \vec{Y}(0)$

We can approximate $\vec{Y}(\Delta t)$ by calculating $\vec{Y}(0) + \vec{V}(\vec{Y})\Delta t$
We can now repeat this process indefinitely.

Uniqueness

Autonomous system solutions cannot intersect unless they are the same solution.
This is true because if they did intersect and were different solutions, they would have a different derivative at the same point, which is not possible in an autonomous system.

"Fencing"

Periodic solutions separate their interior from their exterior as internal solutions may never cross out and vice versa.

Other solutions may also separate the space, like infinite lines or a combination of multiple solutions and static solutions.

The SIR Model

The SIR model is a ideal simplistic model of epidemic spread.

It categorizes people into Susceptible, Infected, or Recovered groups and follows a system as such:

$$\begin{cases} \frac{dS}{dt} = -\alpha SI \\ \frac{dI}{dt} = \alpha SI - \beta I \\ \frac{dR}{dt} = \beta I \end{cases}$$

Where:
$S + I + R = 1$
α is the contagion constant
β is the recovery constant

The Lorenz Attractor

Born out of weather predictions, Edward Lorenz created this simplified system to display the effects of "chaotic" systems

$$\begin{cases} \frac{dx}{dt} = \sigma(y - x) \\ \frac{dy}{dt} = \rho x - y - xz \\ \frac{dz}{dt} = -\beta z + xy \end{cases}$$

The parameters he chose in particular were

$$\begin{cases} \sigma = 10 \\ \beta = \frac{8}{3} \\ \rho = 28 \end{cases}$$

Three dimensional systems have much less restriction than those of two or one dimensional, as the sample space is significantly less likely to be partitioned by simple cases.