Linear Systems

The Harmonic Oscillator

A model for a mass on a spring

$$-ky - b\frac{dy}{dt} = m\frac{d^2y}{dt^2}$$
$$\implies \begin{cases} \frac{dy}{dt} = v \\ \frac{dv}{dt} = -\frac{k}{m}y - \frac{b}{m}v \end{cases}$$

Constant coefficient

$$\begin{cases} \frac{dx}{dt} = ax + by \\ \frac{dy}{dt} = cx + dy \end{cases}$$

Where a, b, c, d are constants

Matrix form

$$\mathbf{Y} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$
$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$$

Where the dimension of the system may be arbitrary

Equilibrium Points

$\mathbf{Y} = \vec{0}$
Will always be a solution to constant coefficient equations.

If the determinant $\det \mathbf{A} = 0$ there may be other equilibrium points

If $\mathbf{A} = \mathbf{0}$ then every point is an equilibrium point

Linearity Principle

1. If $\mathbf{Y}(t)$ is a solution, then for any arbitrary constant k, $k\mathbf{Y}(t)$ is also a solution
2. If $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ are solutions, then their sum is also a solution

Proof

This is because if

$$\begin{cases} \frac{d\mathbf{Y}_1}{dt} = \mathbf{A}\mathbf{Y}_1 \\ \frac{d\mathbf{Y}_2}{dt} = \mathbf{A}\mathbf{Y}_2 \end{cases}$$

then

$$k\frac{d\mathbf{Y}_1}{dt} = k\mathbf{A}\mathbf{Y}_1$$
$$\implies \frac{d\mathbf{k}Y_1}{dt} = \mathbf{A}(k\mathbf{Y}_1)$$

and

$$\frac{d\mathbf{Y}_1}{dt} + \frac{d\mathbf{Y}_2}{dt} = \mathbf{A}\mathbf{Y}_1 + \mathbf{A}\mathbf{Y}_1$$
$$\implies \frac{d(\mathbf{Y}_1+\mathbf{Y}_2)}{dt} = \mathbf{A}(\mathbf{Y}_1 + \mathbf{Y}_2)$$

General Solution given two independent solutions

Given two solutions $\mathbf{Y}_1$ and $\mathbf{Y}_2$
to the problem $\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y}$
with the initial value $\mathbf{Y}(0) = \vec{y}_0$

We will denote the initial values of $\mathbf{Y}_1$ and $\mathbf{Y}_2$ with $\vec{y}_1$ and $\vec{y}_2$ respectively

Via the linearity principle, we know k_1 and k_2 exist such that

$$\begin{cases} k_1\vec{y}_1 + k_2\vec{y}_2 = \vec{y}_0 \\ k_1\mathbf{Y}_1 + k_2\mathbf{Y}_2 = \mathbf{Y} \end{cases}$$

Let $\mathbf{K} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$
Let $\mathbf{Y}_0 = \begin{bmatrix} \vec{y}_1 & \vec{y}_2 \\ \vdots & \vdots \end{bmatrix}$
Let $\vec{\mathbf{Y}} = [\mathbf{Y}_1 \quad \mathbf{Y}_2]$

$$\implies \begin{cases} \mathbf{Y}_0\mathbf{K} = \vec{y}_0 \\ \vec{\mathbf{Y}}\mathbf{K} = \mathbf{Y} \end{cases}$$
$$\implies \begin{cases} \mathbf{K} = \mathbf{Y}_0^{-1}\vec{y}_0 \\ \vec{\mathbf{Y}}\mathbf{K} = \mathbf{Y} \end{cases}$$

Straight Line Solutions

There may exist straight line solutions to a linear system.

If a straight line solution exists, the following equation must be satisfied:

Given $\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y}$
$\mathbf{A}\vec{V} = \lambda\vec{V}$

Where λ is a constant

Finding eigenvalues and eigenvectors

We may rearrange this equation to be of the form

$(\mathbf{A} - \lambda\mathbf{I})\vec{V} = \vec{0}$

If $\vec{V} \neq \vec{0}$ then the first matrix must be degenerate, therefore

$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$

From this equation, we are able to solve for valid λs. These λs are called **eigenvalues**. This equation will be a quadratic called the characteristic polynomial.

The coefficients for this polynomial may be shortcutted with $a = 1$, $b = T$, $c = D$ where T is the trace of the matrix and D is the determinant.

The matrix $(\mathbf{A} - \lambda\mathbf{I})$ being degenerate, will have an entire line of valid solutions for $\vec{V}$ in the equation

$(\mathbf{A} - \lambda\mathbf{I})\vec{V} = \vec{0}$

All solutions for $\vec{V}$ for a particular λ will be multiples of itself. Any valid $\vec{V}$ for its associated λ is called an **eigenvector**

Specific solutions for distinct real non-zero eigenvalues

The solution associated with an eigenvalue and its associated eigenvector must satisfy the equation

$\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y}$

If we let the initial condition be $\vec{V}$, $\mathbf{Y}$ must always lie on a multiple of $\vec{V}$ as $\mathbf{A}\vec{V} = \lambda\vec{V}$ and $\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y}$

Therefore, for initial conditions of multiples of $\vec{V}$,

$\frac{d\mathbf{Y}}{dt} = \lambda\mathbf{Y}$
Where λ is the associated eigenvalue for the eigenvector $\vec{V}$

This differential equation can be easily solved for $\mathbf{Y}$ with an exponential

$\mathbf{Y}(t) = e^{\lambda t}\vec{V}$

For positive λs, the solution points away from the origin, while the opposite is true for negative λs. If both λs are positive, then the origin is a source; a sink if both are negative; or a saddle if they are of different signs.

If the eigenvectors are distinct and real, then there will be two distinct solutions for $\mathbf{Y}$ which may be combined with the linearity principle.

As such, the general solution will be:

$\mathbf{Y}(t) = k_1 e^{\lambda_1 t}\vec{V}_1 + k_2 e^{\lambda_2 t}\vec{V}_2$

Specific solutions for imaginary eigenvalues

Assuming the matrix $\mathbf{A}$ being real, the only time solutions for λ will be imaginary is when both solutions for λ are imaginary, due to $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ producing a quadratic of λ with real coefficients. Additionally, the solutions for λ will be conjugates of each other. This may be proved trivially.

However, due to λ being imaginary, the associated eigenvectors will also be imaginary, making the solution of $e^{\lambda t}\vec{V}$ not make sense. This $\mathbf{Y}(t)$ does satisfy the conditions however, just not on the real plane.

Using Euler's formula:
$e^{a+ib} = e^a(\cos b + i \sin b)$
We are able to split the solution into real and imaginary parts.

Let $a + ib = \lambda$, where a and b are real.
$e^{\lambda t}\vec{V} = \vec{V}e^{at}(\cos(bt) + i\sin(bt))$

After multiplying through with $\vec{V}$, the solution may be separated into its real and imaginary parts $\mathbf{Y}_{re}$ and $i\mathbf{Y}_{im}$

$\mathbf{Y}(t) = \mathbf{Y}_{re}(t) + i\mathbf{Y}_{im}(t)$

Since $\mathbf{Y}(t)$ is a valid solution,

$$\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y}$$
$$\frac{d\mathbf{Y}_{re} + i\mathbf{Y}_{im}}{dt} = \mathbf{A}(\mathbf{Y}_{re} + i\mathbf{Y}_{im})$$
$$\frac{d\mathbf{Y}_{re}}{dt} + i\frac{d\mathbf{Y}_{im}}{dt} = \mathbf{A}\mathbf{Y}_{re} + i\mathbf{A}\mathbf{Y}_{im}$$
$$\implies \begin{cases} \frac{d\mathbf{Y}_{re}}{dt} = \mathbf{A}\mathbf{Y}_{re} \\ \frac{d\mathbf{Y}_{im}}{dt} = \mathbf{A}\mathbf{Y}_{im} \end{cases}$$

As such, $\mathbf{Y}_{re}$ and $\mathbf{Y}_{im}$ are valid solutions for $\mathbf{Y}$

These solutions may be combined similarly to the real case to form a general solution: $\mathbf{Y}(t) = k_1\mathbf{Y}_{re} + k_2\mathbf{Y}_{im}$

As $\sin$ and $\cos$ are periodic and non-increasing nor decreasing, if a is positive, then the origin is a spiral source; a spiral sink if it were negative, and a center if $a = 0$, where all solutions are periodic ellipses.

Repeated Eigenvalues

Systems that have a non-zero repeated eigenvalue have solutions similar to other linear systems.

Similar to the other systems, a solution would be the particular solution given by the singular eigenvalue:

$$\mathbf{Y}_p(t) = ke^{\lambda t}\vec{V}$$

However, this solution only gives one dimension of freedom in the initial condition plane.

Assume that one of the non-diagonal attributes of the matrix $\mathbf{A}$ is 0. This implies that one of the differential equations within the system is decoupled from the rest of the system.

It can also be easily shown that the solution for a decoupled linear differential equation is of the form k_2e^{bt} where b is the coefficient of the linear equation. Additionally, k_2 is the initial condition of the decoupled equation.

Lastly, the other differential equation can also be easily shown to be of the form $k_1e^{at} + k_2te^{bt}$. This can easily be vectorized as $e^{\lambda t}\vec{V}_0 + te^{\lambda t}\vec{V}_1$. Where $\vec{V}_0$ is the initial condition. λ is the coefficient of the decoupled equation, or a calculated eigenvalue, it can be easily shown that these are equivalent. In the case of a decoupled equation.

$\vec{V}_1$ can be solved for like such:

$$\frac{d\mathbf{Y}}{dt} = \mathbf{AY}$$

$$\mathbf{Y} = e^{\lambda t}\vec{V}_0 + te^{\lambda t}\vec{V}_1$$

$$\lambda e^{\lambda t}\vec{V}_0 + (1 + \lambda t)e^{\lambda t}\vec{V}_1 = e^{\lambda t}\mathbf{A}\vec{V}_0 + te^{\lambda t}\mathbf{A}\vec{V}_1$$
$$\implies \begin{cases} \lambda\vec{V}_1 = \mathbf{A}\vec{V}_1 \\ \lambda\vec{V}_0 + \vec{V}_1 = \mathbf{A}\vec{V}_0 \end{cases}$$
$$\implies \begin{cases} \lambda\vec{V}_1 = \mathbf{A}\vec{V}_1 \\ \vec{V}_1 = (\mathbf{A} - \mathbf{I}\lambda)\vec{V}_0 \end{cases}$$

$\therefore \vec{V}_1$ is either an eigenvector or $\vec{0}$

If $\vec{V}_1 = \vec{0}$,
$\lambda \vec{V}_0 = \mathbf{A}\vec{V}_0$

Thus, if $\vec{V}_0$ is an eigenvector, $\vec{V}_1$ is $\vec{0}$, else $\vec{V}_1$ will be an eigenvector.

As such, the general solution will be

$$\begin{cases} \mathbf{Y} = e^{\lambda t}\vec{V}_0 + te^{\lambda t}\vec{V}_1 \\ \vec{V}_1 = (\mathbf{A} - \mathbf{I}\lambda)\vec{V}_0 \end{cases}$$

With $\vec{V}_0$ as the initial condition.

This turns out to be the general solution for any repeated eigenvalue differential system. I am unsure why and could not find a proof for this at the moment.

Repeated diagonal Matrices

A repeated diagonal matrix means that both equations are independent and as such all vectors are also eigenvectors.

The matrix takes the form of

$$\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$$

In this system, $\lambda = a$

The solution is $\mathbf{Y} = e^{\lambda t}\vec{V}_0$

Zero as an eigenvalue

Having zero as an eigenvalue immediate implies that the system is degenerate.

All associated eigenvectors for a zero eigenvalue are all equilibrium points.

All other points will approach the zero-eigenvector line

Damping

In the mass-spring harmonic oscillator equation:

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0$$

The oscillator may be categorized based on different characteristics:

- Undamped: $b = 0$

- Underdamped: $b^2 - 4km < 0$
- Critically damped: $b^2 - 4km = 0$
- Overdamped: $b^2 - 4km > 0$

An underdamped system will oscillate / orbit the origin.

Categorizing systems

Systems may be categorized via their eigenvalues or their determinant and traces.

Let $R = T^2 - 4D$

Where T is the trace and D is the determinant

	$T < 0$	$T = 0$	$T > 0$
$D > R$	Spiral Sink	Center	Spiral Source
$D = R$	One line sink	$\downarrow$	One line source
$R > D > 0$	Two line sink	All equilibrium	Two line sourrce
$D = 0$	Equilibrium line sink	$\uparrow$	Equilibrium line source
$D < 0$	$\rightarrow$	Saddle	$\leftarrow$

Examples

1 - Undampened Harmonic Oscillator

$\frac{d^2y}{dt^2} = -y$

Answer

$$\begin{cases} \frac{dy}{dt} = v \\ \frac{dv}{dt} = -y \end{cases}$$

$$\mathbf{Y}(t) = \begin{bmatrix} y(t) \\ v(t) \end{bmatrix}$$

$$\frac{d\mathbf{Y}}{dt} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mathbf{Y}$$

We can verify our guess of $y(t) = \sin(x)$

$$\frac{d\mathbf{Y}}{dt} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mathbf{Y}$$

$$\begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix} = \begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix}$$

We can verify another guess of $y(t) = \cos(x)$

$$\frac{d\mathbf{Y}}{dt} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\mathbf{Y}$$

$$\begin{bmatrix} -\sin(t) \\ -\cos(t) \end{bmatrix} = \begin{bmatrix} -\sin(t) \\ -\cos(t) \end{bmatrix}$$

And thus we have our general solution of

$$\mathbf{Y} = \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix}\mathbf{K}$$

Where $\mathbf{K}$ is any two-dimensional vector