4.22

When

$$U = g_1(X, Y)$$

$$V = g_2(X, Y)$$

$$X = h_1(U, V)$$

$$Y = h_2(U, V)$$

$$f_{U,V}(u,v) = f_{X,Y}(h_1(U,V),h_2(U,V)) |J(h)|$$

$$U = aX + b$$

$$V = cY + d$$

$$X = \frac{U-b}{a}$$

$$X=rac{U-b}{a} \ Y=rac{V-d}{c}$$

$$J = egin{bmatrix} rac{1}{a} & 0 \ 0 & rac{1}{c} \end{bmatrix} = rac{1}{ac}$$

$$f_{U,V}(u,v) = rac{1}{ac} f(rac{u-b}{a},rac{v-d}{c})$$

4.23

$$egin{aligned} f_{X,Y}(x,y) &= rac{\Gamma(lpha+eta+\gamma)}{\Gamma(lpha)\Gamma(eta)\Gamma(\gamma)} x^{lpha-1} (1-x)^{eta-1} y^{lpha+eta-1} (1-y)^{\gamma-1} \ 0 &< x < 1, \quad 0 < y < 1 \end{aligned}$$

a

$$U = XY$$

$$V = Y$$

$$X = \frac{U}{V}$$

$$Y = V$$

$$0 < U < V < 1$$

$$f_{U,V}(u,v) = J f_{X,Y}(rac{u}{v},v)$$

$$J=egin{bmatrix} rac{1}{V} & rac{-U}{V^2} \ 0 & 1 \end{bmatrix}=rac{1}{V}$$

$$f_{U,V}(u,v) = rac{1}{v} rac{\Gamma(lpha+eta+\gamma)}{\Gamma(lpha)\Gamma(eta)\Gamma(eta)} ig(rac{u}{v}ig)^{lpha-1} ig(1-ig(rac{u}{v}ig)ig)^{eta-1} v^{lpha+eta-1} (1-v)^{\gamma-1}$$

$$egin{aligned} f_U(u) &= \int\limits_u^1 rac{1}{v} rac{\Gamma(lpha+eta+\gamma)}{\Gamma(lpha)\Gamma(eta)\Gamma(\gamma)} ig(rac{u}{v}ig)^{lpha-1} ig(1-ig(rac{u}{v}ig)ig)^{eta-1} v^{lpha+eta-1} (1-v)^{\gamma-1} \ dv \ f_U(u) &= \int\limits_u^1 v^{-1} rac{\Gamma(lpha+eta+\gamma)}{\Gamma(lpha)\Gamma(eta)\Gamma(eta)} u^{lpha-1} v^{1-lpha} ig(rac{v-u}{v}ig)^{eta-1} v^{lpha+eta-1} (1-v)^{\gamma-1} \ dv \ f_U(u) &= rac{\Gamma(lpha+eta+\gamma)}{\Gamma(lpha)\Gamma(eta)\Gamma(eta)} u^{lpha-1} \int\limits_u^1 ig(rac{v-u}{v}ig)^{eta-1} v^{eta-1} (1-v)^{\gamma-1} \ dv \ f_U(u) &= rac{\Gamma(lpha+eta+\gamma)}{\Gamma(lpha)\Gamma(eta)\Gamma(eta)} u^{lpha-1} \int\limits_u^1 ig(v-u)^{eta-1} (1-v)^{\gamma-1} \ dv \end{aligned}$$

$$y=rac{1-v}{1-u}$$
 $dy=rac{-dv}{1-u}$

$$f_U(u) = rac{\Gamma(lpha+eta+\gamma)}{\Gamma(lpha)\Gamma(eta)\Gamma(eta)}u^{lpha-1}(1-u)^{eta-1}(1-u)^{\gamma-1}(1-u)\int\limits_u^1 (1-y)^{eta-1}(y)^{\gamma-1}\;dv \ f_U(u) = rac{\Gamma(lpha+eta+\gamma)}{\Gamma(lpha)\Gamma(eta+\gamma)}u^{lpha-1}(1-u)^{eta+\gamma-1}$$

$$\rho = \beta + \gamma$$

$$f_U(u) = rac{\Gamma(lpha +
ho)}{\Gamma(lpha)\Gamma(
ho)} u^{lpha - 1} (1 - u)^{
ho - 1}$$

$$\sim \mathrm{Beta}(lpha,
ho) = \mathrm{Beta}(lpha,eta+\gamma)$$

b

$$U = XY$$
 $V = \frac{X}{Y}$
 $X = \sqrt{UV}$
 $Y = \sqrt{\frac{U}{V}}$

$$U < V < \frac{1}{U}$$
 $0 < U < 1$

$$f_{U,V}(u,v) = J f_{X,Y}\left(\sqrt{UV},\sqrt{rac{U}{V}}
ight)$$

$$J = egin{bmatrix} rac{V}{2\sqrt{UV}} & rac{U}{2\sqrt{UV}} \ rac{1}{2V\sqrt{rac{U}{V}}} & rac{-U}{2V^2\sqrt{rac{U}{V}}} \end{bmatrix} = rac{-1}{2V}$$

$$\begin{split} f_{U,V}(u,v) &= \frac{1}{2V} \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \sqrt{UV}^{\alpha-1} (1-\sqrt{UV})^{\beta-1} \sqrt{\frac{U}{V}}^{\alpha+\beta-1} (1-\sqrt{\frac{U}{V}})^{\gamma-1} \\ f_{U}(u) &= \int\limits_{u}^{1/u} \frac{1}{2V} \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \sqrt{UV}^{\alpha-1} (1-\sqrt{UV})^{\beta-1} \sqrt{\frac{U}{V}}^{\alpha+\beta-1} (1-\sqrt{\frac{U}{V}})^{\gamma-1} \, dv \\ f_{U}(u) &= \int\limits_{u}^{1/u} \frac{\sqrt{U}}{2V^{3/2}} \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} (\sqrt{\frac{U}{V}} \sqrt{UV})^{\alpha-1} (\sqrt{\frac{U}{V}} (1-\sqrt{UV}))^{\beta-1} (1-\sqrt{\frac{U}{V}})^{\gamma-1} \, dv \\ f_{U}(u) &= \int\limits_{u}^{1/u} \frac{\sqrt{U}}{2V^{3/2}} \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} U^{\alpha-1} (\sqrt{\frac{U}{V}} - U)^{\beta-1} (1-\sqrt{\frac{U}{V}})^{\gamma-1} \, dv \\ f_{U}(u) &= \int\limits_{u}^{1/u} \frac{\sqrt{U}}{2V^{3/2}} \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} U^{\alpha-1} (1-U)^{\beta+\gamma-2} \left(\frac{\sqrt{\frac{U}{V}} - U}{1-U}\right)^{\beta-1} \left(\frac{1-\sqrt{\frac{U}{V}}}{1-U}\right)^{\gamma-1} \, dv \end{split}$$

$$Z=rac{\sqrt{rac{U}{V}}-U}{1-U}\ dZ=rac{\sqrt{U}}{1-U}rac{-1}{2V^{3/2}}$$

$$egin{aligned} f_U(u) &= \int\limits_0^1 rac{\Gamma(lpha+eta+\gamma)}{\Gamma(lpha)\Gamma(eta)\Gamma(eta)} U^{lpha-1} (1-U)^{eta+\gamma-1} (Z)^{eta-1} (1-Z)^{\gamma-1} \; dz \ f_U(u) &= rac{\Gamma(lpha+eta+\gamma)}{\Gamma(lpha)\Gamma(eta+\gamma)} U^{lpha-1} (1-U)^{eta+\gamma-1} \int\limits_0^1 rac{\Gamma(eta+\gamma)}{\Gamma(eta)\Gamma(eta)} (Z)^{eta-1} (1-Z)^{\gamma-1} \; dz \ f_U(u) &= rac{\Gamma(lpha+eta+\gamma)}{\Gamma(lpha)\Gamma(eta+\gamma)} U^{lpha-1} (1-U)^{eta+\gamma-1} \end{aligned}$$

$$U \sim \mathrm{Beta}(lpha, eta + \gamma)$$

4.24

$$X \sim \operatorname{Gamma}(r, 1)$$

$$Y \sim \mathrm{Gamma}(s,1)$$

$$Z_1 = X + Y$$
 $Z_2 = \frac{X}{X+Y}$

$$X = Z_1 Z_2$$

$$Y = Z_1 - Z_1 Z_2$$

$$0 < Z_1$$

$$0 < Z_2 < 1$$

$$J=egin{bmatrix} Z_2 & Z_1 \ 1-Z_2 & -Z_1 \end{bmatrix}=-Z_1$$

$$f_{X,Y}(x,y)=rac{x^{r-1}e^{-x}}{\Gamma(r)}rac{y^{s-1}e^{-y}}{\Gamma(s)}$$

$$egin{aligned} f_{Z_1,Z_2}(z_1,z_2) &= rac{(z_1z_2)^{r-1}e^{-z_1z_2}}{\Gamma(r)}rac{(z_1-z_1z_2)^{s-1}e^{-z_1+z_1z_2}}{\Gamma(s)}z_1 \ f_{Z_1,Z_2}(z_1,z_2) &= rac{1}{\Gamma(r)\Gamma(s)}z_1^{r+s-1}e^{-z_1}z_2^{r-1}(1-z_2)^{s-1} \ f_{Z_1,Z_2}(z_1,z_2) &= (rac{1}{\Gamma(r+s)}z_1^{r+s-1}e^{-z_1})(rac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)}z_2^{r-1}(1-z_2)^{s-1}) \end{aligned}$$

Which can be defined in functions one of only z_1 and one of only z_2

$$egin{aligned} f_{Z_1}(z_1) &= rac{1}{\Gamma(r+s)} z_1^{r+s-1} e^{-z_1} \ f_{Z_2}(z_2) &= rac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} z_2^{r-1} (1-z_2)^{s-1} \end{aligned}$$

Such that

$$f_{Z_1}(z_1)f_{Z_2}(z_2)=f_{Z_1,Z_2}(z_1,z_2)$$

Which proves that Z_1 and Z_2 are independent and are of the distributions

$$Z_1 \sim ext{Gamma}(r+s,1) \ Z_2 \sim ext{Beta}(r,s)$$

4.32

a

$$Y|\Lambda \sim \mathrm{Poisson}(\Lambda)$$

$$\Lambda \sim \operatorname{Gamma}(\alpha, \beta)$$

$$0<\Lambda$$

$$Y \in \{0,1,2,3,\dots\}$$

$$f_{Y,\Lambda}(y,\lambda) = rac{e^{-\lambda}\lambda^y}{y!} rac{\lambda^{lpha-1}e^{-\lambda/eta}}{\Gamma(lpha)eta^lpha}$$

$$f_Y(y) = \int\limits_0^\infty rac{e^{-\lambda}\lambda^y}{y!} rac{\lambda^{lpha-1}e^{-\lambda/eta}}{\Gamma(lpha)eta^lpha} \ d\lambda$$

$$f_Y(y) = rac{1}{\Gamma(lpha)y!eta^lpha}\int\limits_0^\infty e^{-\lambda}\lambda^y\lambda^{lpha-1}e^{-\lambda/eta}\ d\lambda$$

$$f_Y(y) = rac{1}{\Gamma(lpha)y!eta^lpha}\int\limits_0^\infty \lambda^{y+lpha-1}e^{-\lambda(1+eta)/eta}\ d\lambda$$

$$\bar{\alpha} = y + \alpha$$

$$\bar{eta} = rac{eta}{1+eta}$$

$$egin{aligned} f_Y(y) &= rac{1}{\Gamma(lpha)y!eta^lpha} \int\limits_0^\infty \lambda^{arlpha-1} e^{-\lambda/areta} \, d\lambda \ f_Y(y) &= rac{1}{\Gamma(lpha)y!eta^lpha} \Gamma(arlpha)areta^{arlpha} \int\limits_0^\infty rac{1}{\Gamma(arlpha)areta^{arlpha}} \lambda^{arlpha-1} e^{-\lambda/areta} \, d\lambda \ f_Y(y) &= rac{1}{\Gamma(lpha)y!eta^lpha} \Gamma(arlpha)areta^{arlpha} \ f_Y(y) &= rac{1}{\Gamma(lpha)y!eta^lpha} \Gamma(y+lpha) \Big(rac{eta}{1+eta}\Big)^{y+lpha} \end{aligned}$$

If α is an integer

$$f_Y(y) = rac{(y+lpha-1)!}{(lpha-1)!y!} \Big(rac{eta}{1+eta}\Big)^y \Big(rac{1}{1+eta}\Big)^lpha \ f_Y(y) = inom{y+lpha-1}{y} \Big(rac{eta}{1+eta}\Big)^y \Big(rac{1}{1+eta}\Big)^lpha$$

$$ar{p}=rac{1}{1+eta}$$

$$f_Y(y) = inom{y+lpha-1}{y}(1-ar{p})^y(ar{p})^lpha$$

$$Y \sim \mathrm{NegBinom}(lpha, rac{1}{1+eta})$$

EY

$$\mu=rac{lpha\left(rac{eta}{1+eta}
ight)}{rac{1}{1+eta}}=lphaeta$$

Var(Y)

$$Var(Y)=rac{\mu}{rac{1}{1+eta}}=lphaeta(1+eta)$$