3.22

b

$X \sim \text{Binom}(r, p)$
$P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$

$E(X)$

$E(X) = \sum_{x \in X} x P(X = x)$
$E(X) = \sum_{x=1}^{n} x \binom{n}{x} p^x (1-p)^{n-x}$
$E(X) = \sum_{x=1}^{n} np \frac{(n-1)!}{(x-1)!(n-x)!} p^{x-1} (1-p)^{n-x}$
$E(X) = np \sum_{x=1}^{n} \binom{n-1}{x-1} p^{x-1} (1-p)^{n-x}$
Let $\tilde{x} = x - 1 \qquad \tilde{n} = n - 1$
$E(X) = np \sum_{\tilde{x}=0}^{\tilde{n}} \binom{\tilde{n}}{\tilde{x}} p^{\tilde{x}} (1-p)^{\tilde{n}-\tilde{x}}$
$E(X) = np$

$E(X^2)$

$E(X^2) = \sum_{x \in X} x^2 P(X = x)$
$E(X^2) = \sum_{x=1}^{n} x^2 \binom{n}{x} p^x (1-p)^{n-x}$
$E(X^2) = np \sum_{\tilde{x}=0}^{\tilde{n}} (\tilde{x} + 1) \binom{\tilde{n}}{\tilde{x}} p^{\tilde{x}} (1-p)^{\tilde{n}-\tilde{x}}$
$E(X^2) = np(1 + \sum_{\tilde{x}=0}^{\tilde{n}} \tilde{x} \binom{\tilde{n}}{\tilde{x}} p^{\tilde{x}} (1-p)^{\tilde{n}-\tilde{x}})$
$E(X^2) = np(1 + \tilde{n}p)$
$E(X^2) = np(1 + (n-1)p)$
$E(X^2) = np + n^2p^2 - np^2$

$\text{Var}(X)$

$\text{Var}(X) = E(X^2) - (E(X))^2$
$\text{Var}(X) = np + n^2p^2 - np^2 - (np)^2$

$\text{Var}(X) = np - np^2$
$\text{Var}(X) = np(1-p)$
$\square$

C

$X \sim \text{Gamma}(\alpha, \beta)$
$f(x) = \frac{x^{\alpha-1}e^{-x/\beta}}{\Gamma(\alpha)\beta^\alpha}$

$E(X)$

$E(X) = \int\limits_{-\infty}^{\infty} xf(x)dx$
$E(X) = \int\limits_{0}^{\infty} x\frac{x^{\alpha-1}e^{-x/\beta}}{\Gamma(\alpha)\beta^\alpha}dx$
$E(X) = \int\limits_{0}^{\infty} \alpha\beta\frac{x^{\alpha}e^{-x/\beta}}{\Gamma(\alpha+1)\beta^{\alpha+1}}dx$
$E(X) = \alpha\beta\int\limits_{0}^{\infty} f(x|\alpha+1,\beta)dx$
$E(X) = \alpha\beta$

$E(X^2)$

$E(X^2) = \int\limits_{-\infty}^{\infty} x^2f(x)dx$
$E(X) = \int\limits_{0}^{\infty} x^2\frac{x^{\alpha-1}e^{-x/\beta}}{\Gamma(\alpha)\beta^\alpha}dx$
$E(X) = \alpha\beta\int\limits_{0}^{\infty} x\frac{x^{\alpha}e^{-x/\beta}}{\Gamma(\alpha+1)\beta^{\alpha+1}}dx$
$E(X) = \alpha\beta\int\limits_{0}^{\infty} (\alpha+1)\beta\frac{x^{\alpha+1}e^{-x/\beta}}{\Gamma(\alpha+2)\beta^{\alpha+2}}dx$
$E(X) = \alpha(\alpha+1)\beta^2\int\limits_{0}^{\infty} f(x|\alpha+2,\beta)dx$
$E(X) = \alpha(\alpha+1)\beta^2$
$E(X) = \alpha^2\beta^2 + \alpha\beta^2$

$\text{Var}(X)$

$\text{Var}(X) = E(X^2) - (E(X))^2$
$\text{Var}(X) = \alpha^2\beta^2 + \alpha\beta^2 - (\alpha\beta)^2$
$\text{Var}(X) = \alpha\beta^2$
$\square$

e

$X \sim \text{DExp}(\mu, \sigma)$
$f(x) = \frac{e^{-|x-\mu|/\sigma}}{2\sigma}$

$E(X)$

$E(X) = \int\limits_{-\infty}^{\infty} xf(x)dx$
$E(X) = \frac{1}{2\sigma} \int\limits_{-\infty}^{\infty} xe^{-|x-\mu|/\sigma}dx$
$E(X) = \frac{1}{2\sigma}(\int\limits_{-\infty}^{\mu} xe^{-|x-\mu|/\sigma}dx + \int\limits_{\mu}^{\infty} xe^{-|x-\mu|/\sigma}dx)$
$E(X) = \frac{1}{2\sigma}(\int\limits_{-\infty}^{\mu} xe^{(x-\mu)/\sigma}dx + \int\limits_{\mu}^{\infty} xe^{(\mu-x)/\sigma}dx)$
let $u = (x-\mu)/\sigma$ and $du = dx/\sigma$
let $v = (\mu - x)/\sigma$ and $dv = -dx/\sigma$
$E(X) = \frac{1}{2\sigma}(\int\limits_{-\infty}^{0} \sigma(\sigma u + \mu)e^{u}du - \int\limits_{-\infty}^{0} \sigma(\sigma v - \mu)e^{v}dv)$
$E(X) = \frac{1}{2}(\sigma \int\limits_{-\infty}^{0} ue^{u}du + \mu \int\limits_{-\infty}^{0} e^{u}du - \sigma \int\limits_{-\infty}^{0} ve^{v}dv + \mu \int\limits_{-\infty}^{0} e^{v}dv)$
$E(X) = \mu \int\limits_{-\infty}^{0} e^{u}du$
$E(X) = \mu\left(e^{u}\Big|_{-\infty}^{0} \right)$
$E(X) = \mu(1)$
$E(X) = \mu$

$E(X^2)$

$E(X^2) = \int\limits_{-\infty}^{\infty} x^2f(x)dx$
$E(X^2) = \frac{1}{2\sigma} \int\limits_{-\infty}^{\infty} x^2e^{-|x-\mu|/\sigma}dx$
$E(X^2) = \frac{1}{2\sigma}(\int\limits_{-\infty}^{\mu} x^2e^{-|x-\mu|/\sigma}dx + \int\limits_{\mu}^{\infty} x^2e^{-|x-\mu|/\sigma}dx)$
$E(X^2) = \frac{1}{2\sigma}(\int\limits_{-\infty}^{\mu} x^2e^{(x-\mu)/\sigma}dx + \int\limits_{\mu}^{\infty} x^2e^{(\mu-x)/\sigma}dx)$
let $u = (x-\mu)/\sigma$ and $du = dx/\sigma$
let $v = (\mu - x)/\sigma$ and $dv = -dx/\sigma$
$E(X^2) = \frac{1}{2\sigma}(\int\limits_{-\infty}^{0} \sigma(\sigma u + \mu)^2e^{u}du + \int\limits_{-\infty}^{0} \sigma(\sigma v - \mu)^2e^{v}dv)$

$$E(X^2) = \frac{1}{2}\left(\int_{-\infty}^{0} (\sigma u + \mu)^2 e^u du + \int_{-\infty}^{0} (\sigma v - \mu)^2 e^v dv \right)$$
$$E(X^2) = \int_{-\infty}^{0} (\sigma^2 u^2 + \mu^2) e^u du$$
$$E(X^2) = \sigma^2 \int_{-\infty}^{0} u^2 e^u du + \mu^2 \int_{-\infty}^{0} e^u du$$
$$E(X^2) = \sigma^2 \int_{-\infty}^{0} u^2 e^u du + \mu^2$$

u	dv
u^2	e^u
$2u$	e^u
2	e^u
0	e^u

$$E(X^2) = \sigma^2 \left((u^2 - 2u + 2)e^u \Big|_{-\infty}^{0} \right) + \mu^2$$
$$E(X^2) = \sigma^2(2) + \mu^2$$
$$E(X^2) = 2\sigma^2 + \mu^2$$

$\operatorname{Var}(X)$

$$\operatorname{Var}(X) = E(X^2) - (E(X))^2$$
$$\operatorname{Var}(X) = 2\sigma^2 + \mu^2 - (\mu)^2$$
$$\operatorname{Var}(X) = 2\sigma^2$$
$\square$

3.24

c

$$X \sim \text{Gamma}(a, b)$$
$$Y = 1/X$$
$$g(x) = 1/x$$
$$g^{-1}(y) = 1/y$$

$$F_Y(x) = F_X(y^{-1}(y))$$
$$f_Y(y) = f_X(g^{-1}(y)) g^{-1\prime}(y)$$

$$f_X(x) = \frac{x^{\alpha-1} e^{-x/\beta}}{\Gamma(\alpha)\beta^\alpha}$$
$$g^{-1\prime}(y) = -1/y^2$$

$$f_Y(y) = \frac{(1/y)^{\alpha-1} e^{-(1/y)/\beta}}{y^2 \Gamma(\alpha)\beta^\alpha}$$
$$f_Y(y) = \frac{y^{-\alpha+1} e^{-(1/y)/\beta}}{y^2 \Gamma(\alpha)\beta^\alpha}$$
$$f_Y(y) = \frac{y^{-\alpha-1} e^{-1/y\beta}}{\Gamma(\alpha)\beta^\alpha}$$
$$f_Y(y) = \frac{e^{-1/y\beta}}{\Gamma(\alpha)\beta^\alpha y^{\alpha+1}} \sim \text{IGamma}(\alpha, \beta)$$

Which is the inverse Gamma distribution's PDF

If we use the substitution $\tilde{\beta} = 1/\beta$ we get the more common form of the inverse Gamma.

$$f_Y(y) = \frac{\tilde{\beta}^\alpha e^{-\tilde{\beta}/y}}{\Gamma(\alpha) y^{\alpha+1}} \sim \text{IGamma}(\alpha, 1/\beta)$$
$\square$

3.28

b

$$X \sim \text{Gamma}(\alpha, \beta)$$

$$f(x) = \frac{x^{\alpha-1} e^{-x/\beta}}{\Gamma(\alpha)\beta^\alpha}$$

$$f(x) = \frac{1}{\Gamma(\alpha)} e^{(\alpha-1)\ln x - x/\beta - \alpha \ln \beta}$$

c

$$X \sim \text{Beta}(\alpha, \beta)$$

$$f(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}$$

$$f_X(x) = \frac{1}{B(\alpha,\beta)} e^{(\alpha-1)\ln(x)(\beta-1)\ln(1-x)}$$

d

$$X \sim \text{Poisson}(\lambda)$$
$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}$$

$$P(X = x) = \frac{1}{x!} e^{-\lambda} e^{x \ln(\lambda)}$$

3.33

b

i

$$X \sim \mathrm{Normal}(\theta, a\theta^2|a)$$

$$f(x) = \frac{e^{-(x-\mu)^2/(2\sigma^2)}}{\sqrt{2\pi}\sigma} = \frac{e^{-(x-\theta)^2/(2a\theta^2)}}{\sqrt{2\pi a}\theta}$$

$$f(x) = \frac{e^{-(x-\theta)^2/(2a\theta^2)}}{\sqrt{2\pi a}\theta}$$
$$f(x) = \frac{1}{\sqrt{2\pi a}\theta} e^{-(x-\theta)^2/(2a\theta^2)}$$
$$f(x) = \frac{1}{\sqrt{2\pi a}\theta} e^{-(x^2-2x\theta+\theta^2)/(2a\theta^2)}$$
$$f(x) = \frac{1}{\sqrt{2\pi a}\theta} e^{-x^2\frac{1}{2a\theta^2}+x\frac{1}{a\theta}-\frac{1}{2a}}$$
$$f(x) = \frac{e^{-1/2a}}{\sqrt{2\pi a}} \frac{1}{\theta} e^{-x^2\frac{1}{2a\theta^2}+x\frac{1}{a\theta}}$$
$$f(x) = \frac{1}{\sqrt{2e^{1/a}\pi a}} \frac{1}{\theta} e^{-x^2\frac{1}{2a\theta^2}+x\frac{1}{a\theta}}$$

ii

$(\theta, a\theta^2)$ makes a parabola

iii

$$f(x) = ax^2$$