4.1

9-11

Prove the statements in 9–11.

9

There is a real number x such that $x > 1$ and $2^x > x^{10}$.

✓ **Answer**

This is true iff there is an x that satisfies the constraint.
This x exists as:
$x \in (1, 1.078)$
True by proof by existence

11

There is an integer n such that $2n^2 - 5n + 2$ is prime.

✓ **Answer**

$n = 0$ satisfies the constraint and thus proves that the statement is true at least once.
True by proof by existence

25

The statement is true. For each, (a) rewrite the statement with the quantification implicit as If ____, then ____, and (b) write the first sentence of a proof (the “starting point”) and the last sentence of a proof (the “conclusion to be shown”).
(Note that you do not need to understand the statements in order to be able to do these exercises.)

For all integers m and n, if $mn = 1$ then $m = n = 1$ or $m = n = -1$.

✓ **Answer**

If the product of any two integers is 1, then the two integers must both be 1 or both -1

Proof:
Suppose m and n are both integers such that $mn = 1$

Conclusion:
$m = n = 1$ or $m = n = -1$

4.2

13

Prove that the statement is false

There exists an integer n such that $6n^2 + 27$ is prime.

Answer

Suppose there is an integer n such that $6n^2 + 27$ is prime.
$6n^2 + 27$ is prime,
therefore $3(2n^2 + 9)$ should also be prime
By the definition of a prime, it should not be composable of multiple different non-1 integers.
$6n^2 + 27$ could be composed into 3 and $2n^2 + 9$
$3 \in \mathbf{Z}$
$2n^2$ is positive when $n \in \mathbf{Z}$
$2n^2 + 9 \in \mathbf{Z}$
Also when $n \in \mathbf{Z}$, as $2n^2 + 9 \geq 9$ therefore $3 \neq 2n^2 + 9$

Thus, $6n^2 + 27$ may be composed into 3 and $2n^2 + 9$ both of which are distinct integers, making it non-prime
Proof by contradiction.

27

Determine whether the statement is true or false. Justify your answer with a proof or a counterexample, as appropriate. In each case use only the definitions of the terms and the assumptions listed on page 161, not any previously established properties.

The difference of any two odd integers is even.

Answer

Given odd integers n, m
Integers k, l exist such that

> $n = 2k + 1$ and $m = 2l + 1$ by definition of odd

$m - n = 2l - 1 - (2k - 1)$ By substitution
$m - n = 2l - 2k$ By algebraic laws
$m - n = 2(l - k)$ By distributive property of integers
Integer j exists such that $j = l - k$

$m-n=2j$ By substitution

Thus $m-n$ is even

4.3

2-7

The numbers in 2–7 are rational. Write each number as a ratio of two integers.

2

4.6037

✓ **Answer**

$4.7037=\frac{47037}{10000}$

7

$52.4\overline{6721}$

✓ **Answer**

$52.4\overline{6721}=52.4+0.0\overline{6721}$
$=\frac{524}{10}+\frac{6721}{99990}$
$=\frac{5246197}{99990}$

4.4

5

Give a reason for your answer in each of 1–13. assume that all variables represent integers.

Is $6m(2m+10)$ divisible by 4?

✓ **Answer**

Yes

Proof:

Given $m \in \mathbf{Z}$

Let $n=6m(2m+10)$

$n=6m(2m+10)=4(3m(m+5))$ By algebraic laws

$3m(m+5)$ Is an integer as integers are closed on addition and multiplication

Let $l=3m(m+5) \in \mathbf{Z}$

$n = 4l$ where $l \in \mathbf{Z}$
$4 \neq 0$, $l \in \mathbf{Z}$, and $6m(2m+10) = 4l \iff 4|6m(2m+10)$

16

Prove the statement directly from the definition of divisibility.

For all integers a, b, c if $a|b$ then $a|c$ then $a|(b-c)$.

✓ **Answer**

Given: $a, b, c \in \mathbf{Z}$, $a|b$, $a|c$
Proof:
$a|b \iff \exists k \in \mathbf{Z}, a \neq 0, ak = b$
$a|c \iff \exists l \in \mathbf{Z}, a \neq 0, al = c$
$ak = b, al = c \iff ak - al = b - c$ Algebraic laws
$\iff a(k-l) = b - c$
Let $j = k - l$, $j \in \mathbf{Z}$ Integers are closed on subtraction
$j \in \mathbf{Z}, a \neq 0, aj = b - c \iff a|b-c$
Thus $a|b-c$

21

For the statement, determine whether the statement is true or false. Prove the statement directly from the definitions if it is true, and give a counterexample if it is false.

The product of any two even integers is a multiple of 4.

✓ **Answer**

Given: Even integers m, n
Proof:
$\exists k \in \mathbf{Z}, 2k = m$ by def of even
$\exists l \in \mathbf{Z}, 2l = n$ by def of even
$mn = (2k)(2l)$ by substitution
$mn = 4kl$ by substitution
Let $j = kl$, $j \in \mathbf{Z}$ integers are closed on multiplication
$mn = 4j, 4 \neq 0, j \in \mathbf{Z} \iff 4|mn$
Thus $4|mn$

35

Two athletes run a circular track at a steady pace so that the first completes one round in 8 minutes and the second in 10 minutes. If they both start from the same spot at 4 p.m., when will be the first time they return to the start together?

Answer

If both runners start at the same time, then runner one cross the start whenever time $8|t$ and the second whenever $10|t$ where t is the number of minutes since they started running
If runner a, b are crossing the start, then $10|t$ and $8|t$
Time k will be first time they both cross the start $\iff 10|k, 8|k$ and there is no such time l such that $l < k, 10|l, 8|l$

Assume $k = 40$
$10|40, 8|40$
For any $0 < l < 40$
If $10|l$, $l \in \{10, 20, 30\}$
If $8|l$, $l \in \{8, 16, 24, 32\}$ by brute force
$l \in \{10, 20, 30\}, \{8, 16, 24, 32\} \iff l \in \varnothing$

Thus, $k = 40$ minutes after 4 p.m., or 4:40 p.m.