Forcing and Resonance

Forced Harmonic Resonators

It takes the form of:

$$m\frac{d^2y}{dt^2}+b\frac{dy}{dt}+ky=f(t)$$

or more commonly as:

$$\frac{d^2y}{dt^2}+p\frac{dy}{dt}+qy=g(t)$$

This is **second-order**, **linear**, **nonhomogeneous** (does have constants on the right side), **constant-coeffecient**, **nonautonomous**

$$\frac{d^2y}{dt^2}+p\frac{dy}{dt}+qy=0$$

Is referred to as the associated **homogenous** equation, as it has nothing on the right side.

Extended Linearity Principle

Let $y_p(t), y_q(t)$ be particular solutions of the nonhomogeneous system and $k_1y_1(t)+k_2y_2(t)$ be the general solution of the homogeneous system.

1. $k_1y_1(t)+k_2y_2(t)+y_p(t)$
   Is the general solution to the nonhomogeneous equation
2. $y_p(t)-y_q(t)$
   Is a solution to the homogeneous equation

Proof:

$$\frac{d^2y}{dt}(y_h+y_p)+p\frac{dy}{dt}(y_h+y_p)+q(y_h+y_p)$$
$$=0+g(t)=g(t)$$

$$\frac{d^2y}{dt}(y_p-y_q)+p\frac{dy}{dt}(y_p-y_q)+q(y_p-y_q)$$
$$=g(t)-g(t)=0$$

Solving for particular solutions of nonhomogeneous equations

Typically, the easiest way to solve for a particular solution is to guess and check based on what $g(t)$ is.

Exponentials

For example, if $g(t) = ae^{bt}$ we can guess $y(t) = ke^{bt}$ as a solution, and solve for k

If b happens to be an eigenvalue, we can guess $y(t) = kte^{bt}$ as the remaining e^{bt} terms will be absorbed by the homogeneous equation

Sinusoidals

For $g(t)$ being any sinusoidal we can structure our guess as $k_1 \sin(at) + k_2 \cos(at)$

For sinusoidals, we may also replace the sinusoidals with an e^{iat} then take the imaginary portion of the solution for $\sin$ or the real portion for $\cos$

For solutions of the form ae^{it}, we may solve for a simple solution by solving this equation

$ae^{it} = |a|e^{i(\theta+t)}$

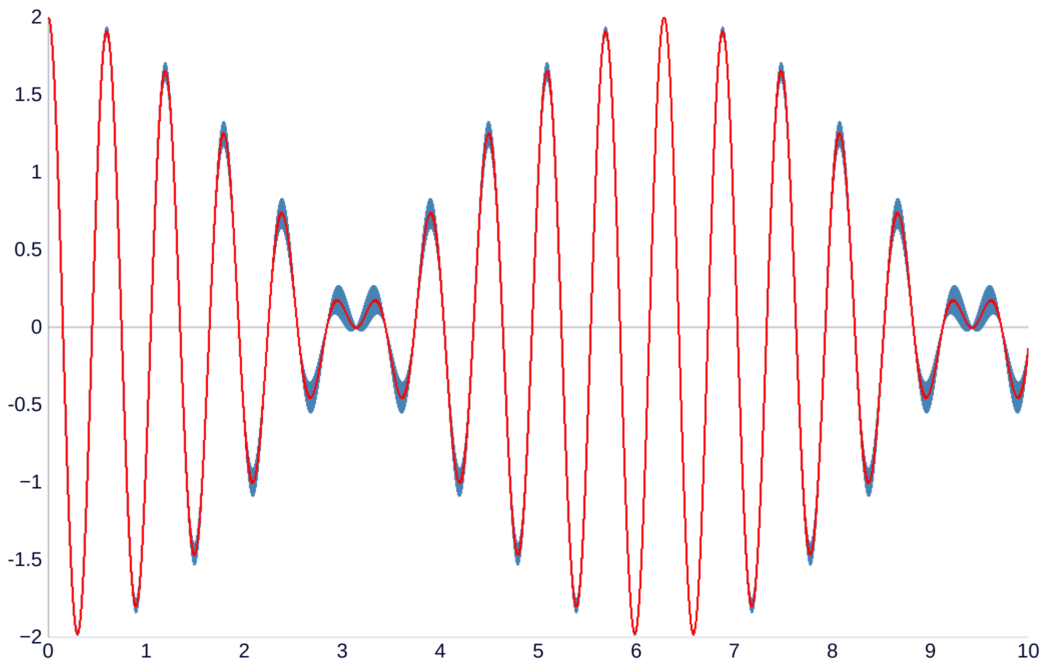
for θ, which happens to be the phase angle, which can be found from the initial complex a with $\arctan$

Resonance

Resonance occurs when a sinusoidal nonhomogeneous equation's period lines up (or closely lines up) with the natural frequency of the homogenous equation.

Beating

Beating occurs if the periods closely line up, creating a graph that looks like a sinusoidal within a low frequency sinusoidal.



The frequency of the beating can be solved for through complexification

$\cos(at) - \cos(bt)$
$= (e^{iat} - e^{ibt})_{re}$

For the sake of simplicity, we will take the real portion at the end

Let
$\alpha = \frac{a+b}{2}$
$\beta = \frac{a-b}{2}$

$= e^{i(\alpha+\beta)t} - e^{i(\alpha-\beta)t}$
$= e^{i\alpha t}(e^{i\beta t} - e^{-i\beta t})$
$= e^{i\alpha t}(2\cos(\beta t))$
$\implies 2\cos(\alpha t)\cos(\beta t)$

Solving the resonant case

When solving resonant cases, we should be using a guess of $k_1 t \sin(at) + k_2 t \cos(at)$ or te^{iat}

This will eventually lead to a solution that shows a linearly increasing sinusoidal.

Finding amplitude the end behavior of a damped equation

With the form:

$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = \cos(\omega t)$
$\implies \frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = e^{i\omega t}$

We guess $ae^{i\omega t}$

$= -a\omega^2 e^{i\omega t} + p(ai\omega e^{i\omega t}) + q(ae^{i\omega t})$
$a(-\omega^2 + q - pi\omega) = 1$
$a = \frac{1}{-\omega^2+q-ip\omega}$

Thus $ae^{i\omega t}$ will have the amplitude of $|a|$

$|a| = \frac{1}{\sqrt{(q-\omega^2)^2+(p\omega)^2}}$

And the phase we tend to choose is

$\phi = \arctan(\frac{-p\omega}{q-\omega^2})$

Additional parameters

If the phase θ is a parameter, we simply shift the phase of the particular solution of nonhomogeneous equation.

This can be easily proven with τ-sub

If the amplitude F is a parameter, the solving for k does not change, and will simply be a multiple of a solution of $F = 1$