5.2

3

For each positive integer n, let $P(n)$ be the formula

$1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$

a

Write $P(1)$. Is $P(1)$ true?

Answer

$1^2 = \frac{1(2)(3)}{6}$
$1 = 1$
True

b

Write $P(k)$

Answer

$1^2 + 2^2 + \ldots + k^2 = \frac{k(k+1)(2k+1)}{6}$

c

Write $P(k+1)$

Answer

$1^2 + 2^2 + \ldots + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}$

d

In a proof by mathematical induction that the formula holds for every integer $n \geq 1$, what must be shown in the inductive step?

Answer

We must show that given:
$1^2+2^2+...+k^2=\frac{k(k+1)(2k+1)}{6}$,
$1^2+2^2+...+(k+1)^2=\frac{(k+1)(k+2)(2k+3)}{6}$
Is true.

7

Prove the statement using mathematical induction. Do not derive them from theorem 5.2.1 or theorem 5.2.2.

For every integer $n \geq 1$
$1+6+11+16+...+(5n-4)=\frac{n(5n-3)}{2}$

✓ **Answer**

Let $P(n) \iff 1+6+11+16+...+(5n-4)=\frac{n(5n-3)}{2}$

$P(1) \iff 1=\frac{2}{2}=1$
$P(1)=\mathbf{t}$

Given $P(k) \iff 1+6+11+16+...+(5k-4)=\frac{k(5k-3)}{2}$
We must prove $P(k) \to P(k+1)$
$1+6+11+16+...+(5k-4)+(5k+1)=1+6+11+16+...+(5(k+1)-4)$
$1+6+11+16+...+(5(k+1)-4)=\frac{k(5k-3)}{2}+5k+1$
$=\frac{(k+1)(5(k+1)-3)}{2}$
$1+6+11+16+...+(5(k+1)-4)=\frac{(k+1)(5(k+1)-3)}{2}$
$\therefore P(k) \to P(k+1)$

$P(1), P(k) \to P(k+1)$
$\therefore \forall n \geq 1 \in \mathbf{Z}: P(n)$
$\square$

12

Prove the statement using mathematical induction.

$\frac{1}{1 \cdot 2}+\frac{1}{2 \cdot 3}+...+\frac{1}{n(n+1)}=\frac{n}{n+1}$ for every integer $n \geq 1$

✓ **Answer**

Let $P(n) \iff \frac{1}{1 \cdot 2}+\frac{1}{2 \cdot 3}+...+\frac{1}{n(n+1)}=\frac{n}{n+1}$

$P(1) \iff \frac{1}{2}=\frac{1}{2} \iff \mathbf{t}$

Given $P(k)$,

$\frac{1}{1\cdot2}+\frac{1}{2\cdot3}+\ldots+\frac{1}{k(k+1)}=\frac{k}{k+1}$

$\frac{1}{1\cdot2}+\frac{1}{2\cdot3}+\ldots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}=\frac{k}{k+1}+\frac{1}{(k+1)(k+2)}$

$\frac{k}{k+1}+\frac{1}{(k+1)(k+2)}=\frac{(k+2)k+1}{(k+1)(k+2)}$

$=\frac{k^2+2k+1}{(k+1)(k+2)}$

$=\frac{k+1}{k+1+1}$

$\therefore P(k)\to P(k+1)$

$P(1), P(k)\to P(k+1)$

$\therefore \forall n\geq 1\in\mathbf{Z}: P(n)$

□

29

Use the formula for the sum of the first n integers and/or the formula for the sum of a geometric sequence to evaluate the sum or to write it in closed form.

$1-2+2^2-2^3+\ldots+(-1)^n2^n$ where n is any positive integer

✓ **Answer**

$1-2+2^2-2^3+\ldots+(-1)^n2^n=\sum_{k=0}^{n}(-2)^k$

$\sum_{k=0}^{n}(-2)^k=\frac{1-(-2)^n}{3}$

□

5.3

4

For each positive integer n, let $P(n)$ be the sentence that describes the following divisibility property:

5^n-1 is divisible by 4.

a

Write $P(0)$. Is $P(0)$ true?

✓ **Answer**

$P(0) = 0|4$
Yes, it is true

b

Write $P(k)$

✓ **Answer**

$5^k - 1|4$

c

Write $P(k+1)$

✓ **Answer**

$5^{k+1} - 1|4$

d

In a proof by mathematical induction that this divisibility property holds for every integer $n \geq 0$, what must be shown in the inductive step?

✓ **Answer**

We must show that given $P(k)$, $P(k+1)$

$\exists n \in \mathbf{Z}$
$5^k - 1 = 4n$ Properties of divisibility
$5^k = 4n + 1$
$5^{k+1} = 20n + 5$
$5^{k+1} = 4(5n+1) + 1$
$5^{k+1} - 1 = 4(5n+1)$
$m = 5n + 1 : m \in \mathbf{Z}$ Integers are closed on multiplication and addition
$5^{k+1} - 1 = 4m$
$5^{k+1} - 1|4$ Properties of divisibility
$\therefore P(0), P(k) \to P(k+1)$
$\square$

7

For each positive integer n, let $P(n)$ be the sentence

In any round-robin tournament involving n teams, the teams can be labeled $T_1, T_2, T_3, \ldots, T_n$, so that T_i beats T_{i+1} for every $i = 1, 2, \ldots, n$.

a

Write $P(2)$. Is $P(2)$ true?

Answer

We can have two teams labeled:
T_1, T_2
We can organize these teams in such a way that:
T_i beats T_{i+1} for every i
The winner of the only match will be T_1
True.

b

Write $P(k)$.

Answer

In any round-robin tournament involving k teams, the teams can be labeled $T_1, T_2, T_3, \ldots, T_k$, so that T_i beats T_{i+1} for every $i = 1, 2, \ldots, k$.

c

Write $P(k+1)$

Answer

In any round-robin tournament involving $k+1$ teams, the teams can be labeled $T_1, T_2, T_3, \ldots, T_{k+1}$, so that T_i beats T_{i+1} for every $i = 1, 2, \ldots, k+1$.

d

In a proof by mathematical induction that $P(n)$ is true for each integer $n \geq 2$, what must be shown in the inductive step?

Answer

That given that:
"In any round-robin tournament involving k teams, the teams can be labeled $T_1, T_2, T_3, \ldots, T_k$, so that T_i beats T_{i+1} for every $i = 1, 2, \ldots, k$",
"In any round-robin tournament involving $k+1$ teams, the teams can be labeled $T_1, T_2, T_3, \ldots, T_{k+1}$, so that T_i beats T_{i+1} for every $i = 1, 2, \ldots, k+1$"
must be true.

15

Prove the statement by mathematical induction

$n(n^2+5)$ is divisible by 6, for each integer $n \geq 0$

✓ **Answer**

Let $P(n)$ be the statement $n(n^2+5)|6$
$P(0) = 0|6$, is true

Given $P(k) = k(k^2+5)|6$,
$\exists m \in \mathbf{Z}$
$k(k^2+5) = 6m$
$k(k^2+5) + 2k^2 + k = 6m + 2k^2 + k$
$k((k+1)^2+5) = 6m + 2k^2 + k$
$(k+1)((k+1)^2+5) = 6m + 2k^2 + k + ((k+1)^2+5)$
$(k+1)((k+1)^2+5) = 6m + 3k(k+1) + 6$
$(k+1)((k+1)^2+5) = 6(m+1) + 3k(k+1)$

Via modulus math, k or $k+1$ must $|2$ as it covers all integer cases mod 2 given $k \in \mathbf{Z}$
$k(k+1)|2$ as one of the factors must $|2$ and both factors are integers

$(k+1)((k+1)^2+5) = 6(m+1) + 3k(k+1)$
$m+1 \in \mathbf{Z} \rightarrow 6(m+1)|6$
$k(k+1)|2 \rightarrow 3k(k+1)|6$
$\rightarrow 6(m+1) + 3k(k+1)|6$
$\iff (k+1)((k+1)^2+5)|6$

$\therefore P(k) \rightarrow P(k+1)$

$P(0), P(k) \rightarrow P(k+1)$
$\therefore \forall n \geq 0 \in \mathbf{Z} : P(n)$

5.4

10

The introductory example solved with ordinary mathematical induction in Section 5.3 can also be solved using strong mathematical induction. Let $P(n)$ be “any n¢ can be obtained using a combination of 3¢ and 5¢ coins.” Use strong mathematical induction to prove that $P(n)$ is true for every integer $n \geq 8$.

✓ **Answer**

$P(8) = 3 + 5$
$P(9) = 3 + 3 + 3$
$P(10) = 5 + 5$

Given $\forall n > k \geq 8 : P(k)$
$P(n)$ is true if $P(n-5)$ or $P(n-3)$ as it can be factored adding a 3 or 5 coin.
$\therefore P(n) \to P(n+3)$

$P(8), P(9), P(10), P(n) \to P(n+3)$
$\therefore \forall n \geq 8 \in \mathbf{Z} : P(n)$

13

Use strong mathematical induction to prove the existence part of the unique factorization of integers theorem (Theorem 4.4.5). In other words, prove that every integer greater than 1 is either a prime number or a product of prime numbers.

✓ **Answer**

Let $Q(n)$ be the statement that $n \in \mathbf{Z}$ is prime
Let $P(n)$ be the statement that $n \in \mathbf{Z}$ is prime or can be represented by a product of prime numbers

Assuming $\forall k \in \mathbf{Z} : 2 \leq k < n, P(k)$
Either n is divisible by any number, or n is not divisible by any number

If n is divisible by any number $m > 1$,
$n|m \to m < n$
$m < n \to P(m)$
$\frac{n}{m} < n \to P\left(\frac{n}{m}\right)$
Since n can be composited into m, $\frac{n}{m}$, n will have all the factors of m and $\frac{n}{m}$ as multiplication is communicative, any satisfactory m will ultimately give the same factors, thus $P(n)$

If n is not divisible by any number
n is prime and can only be factorized as itself, thus $P(n)$

$\square$

16

Use strong mathematical induction to prove that for every integer $n \geq 2$, if n is even, then any sum of n odd integers is even, and if n is odd, then any sum of n odd integers is odd.

✓ **Answer**

$\forall P_1, P_2, \ldots, P_n : P = [P_1, \ldots, P_n]$ where $\forall k : P_k$ is odd
Let S_P represent $\sum_{k=1}^{n} P_k$
$\sum_{k=1}^{n} P'_k \in \mathbf{Z}$ as integers are closed on addition
$\forall n, \exists P'_n : P_n = 2P'_n + 1$ as P_k are odd
Let $P' = [P'_1, \ldots, P'_n]$
Thus, $S_P = n + 2\sum_{k=1}^{n} P'_k$

If n is even, then
$\exists m \in \mathbf{Z} : n = 2m$
$S_P = 2(m + \sum_{k=1}^{n} P'_k)$
$\therefore S_P|2$ and S_P is even

If n is odd, then
$\exists m \in \mathbf{Z} : n = 2m + 1$
$S_P = 2(m + \sum_{k=1}^{n} P'_k) + 1$
$\therefore S_P \mod 2 = 1$ and S_P is odd

$\square$

17

Compute $4^1, 4^2, 4^3, 4^4, 4^5, 4^6, 4^7, 4^8$. Make a conjecture about the units digit of 4^n where n is a positive integer. Use strong mathematical induction to prove your conjecture.

✓ **Answer**

Let U_n be the units digit of $4^n : \forall n \in \mathbf{Z}^+$
$U_n = 5 + (-1)^n \in \{4, 6\}$
Let $P(n)$ be that statement above.

$P(1)$ is that the units digit of 4^1 is $5 - 1 = 4$, which is true.

Assuming $\forall k \in \mathbf{Z}^+ : 1 < k < n \to P(k)$
$P(n-1) \to U_{n-1} \in \{4, 6\}$
$\exists M \in \mathbf{Z}^+ : 4^{n-1} = 10M + U_{n-1}$ by definition of the unit digit of base 10

If $U_{n-1} = 4$
$4^n = 4^{n-1}4 = 10(4M) + 4U_{n-1}$
$4U_{n-1} = 16$
$4^n = 10(4M+1) + 6$
$4^n \mod 10 = 6$
$\therefore U_n = 6$

$U_{n-1} < 5 \to (-1)^{n-1} < 0$
$\to e^{i\pi(n-1)} < 0$
$n-1$ is cyclical every 2
$e^{i\pi} = -1, e^0 = 1$
$\to n-1 \mod 2 = 1$
$n-1+1 \mod 2 = 0$
$n \mod 2 = 0$
$\to e^{i\pi n} = 1$
$\to (-1)^n = 1$
$\therefore U_n = 5 + (-1)^n$

If $U_{n-1} = 6$
$4^n = 4^{n-1}4 = 10(4M) + 4U_{n-1}$
$4U_{n-1} = 24$
$4^n = 10(4M+2) + 4$
$4^n \mod 10 = 4$
$\therefore U_n = 4$

$U_{n-1} > 5 \to (-1)^{n-1} > 0$
$\to e^{i\pi(n-1)} > 0$
$n-1$ is cyclical every 2
$e^{i\pi} = -1, e^0 = 1$
$\to n-1 \mod 2 = 0$
$n-1+1 \mod 2 = 1$
$n \mod 2 = 1$
$\to e^{i\pi n} = -1$
$\to (-1)^n = -1$
$\therefore U_n = 5 + (-1)^n$

$\square$

5.6

4

Find the first four terms of the sequence

$d_k = k(d_{k-1})^2$, for every integer $k \geq 1$
$d_0 = 3$

✓ Answer

$d_0 = 3$
$d_1 = 1(3)^2 = 9$
$d_2 = 2(9)^2 = 162$
$d_3 = 3(162)^2 = 78732$
$d_4 = 4(78732)^2 = 24794911296$

12

Let $s_0, s_1, s_2, \ldots$ be defined by the formula $s_n = \frac{(-1)^n}{n!}$ for every integer $n \geq 0$. Show that this sequence satisfies the following recurrence relation for every integer $k \geq 1$:

$s_k = \frac{-s_{k-1}}{k}$

✓ Answer

$s_n = \frac{(-1)^n}{n!}$

$s_{n-1} = \frac{(-1)^{n-1}}{(n-1)!}$
$-s_{n-1} = \frac{(-1)^n}{(n-1)!}$
$-\frac{s_{n-1}}{n} = \frac{(-1)^n}{n(n-1)!}$
$-\frac{s_{n-1}}{n} = \frac{(-1)^n}{n!}$
$-\frac{s_{n-1}}{n} = s_n$
$\square$

38

Compound Interest: Suppose a certain amount of money is deposited in an account paying 3% annual interest compounded monthly. For each positive integer n, let $S_n =$ the amount on deposit at the end of the nth month, and let S_0 be the initial amount deposited.

a

Find a recurrence relation for $S_0, S_1, S_2, \ldots$, assuming no additional deposits or withdrawals during the year. Justify your answer.

✓ **Answer**

$S_n = 1.03S_{n-1}$

Since each month, the money will grow by 3%, we can add 3% of the original back to the account, which is equivalent to multiplying the previous balance by 1.03

Additionally, we can represent this as
$S_n = S_0 1.03^n$

b

If $S_0 =$ $10,000, find the amount of money on deposit at the end of one year.

✓ **Answer**

$S_{12} = 1.03^{12} 10000 = 14257.6$

c

Find the APY for the account.

✓ **Answer**

$APY = 1.03^{12} = 42.576\%$
APY is just the percentage increase over 12 months