HW 1

1.1

We roll a fair die twice. Describe a sample space Ω and a probability measure P to model this experiment. Let A be the event that the second roll is larger than the first. Find the probability $P(A)$ that the event A occurs.

Answer

Let $\alpha = \{1, 2, 3, 4, 5, 6\}$
Let $\Omega = (i, j) : i, j \in \alpha$
Since $\forall i \in \alpha : P(\{i\}) = \frac{1}{6}$ for a fair die,
Let $P(\{(i, j)\}) = \frac{1}{36} : (i, j) \in \Omega$

First Roll	$P(A)$
1	$5/6$
2	$4/6$
3	$3/6$
4	$2/6$
5	$1/6$
6	$0/6$

With each of the states of the first roll having a $\frac{1}{6}$ chance of occurring, the probability of the second roll being larger must be $\frac{15}{36} = \frac{5}{12}$.

$P(A) = \frac{5}{12}$
$\square$

1.5

In one type of state lottery 5 distinct numbers are picked from $1, 2, 3, \ldots, 40$ uniformly at random.

a

Describe a sample space Ω and a probability measure P to model this experiment.

✓ **Answer**

Assuming replacement of the number, and independence in drawings

Let $\alpha = \{1, 2, 3, \ldots, 40\}$
Let $\Omega = \alpha^5$
Let A be when exactly 3 of the 5 numbers are even

Where $\forall \omega \in \Omega : P(\{\omega\}) = \frac{1}{40^5}$

What is the probability that out of the 5 picked numbers exactly three will be even?

✓ **Answer**

$\forall a \in \alpha : P(a) = \frac{1}{40}$
Let $\alpha' = \{a \text{ is even}, a \text{ is odd}\}$
$P(a \text{ is even}) = P(a \text{ is odd}) = \frac{1}{2}$ in α' since $\frac{20}{40}$ possibilities are even.

Let $\Omega' = \alpha'^5$
Since the elements of α' are equally probable, an independent combination of samples from α' must also be equally probable.
$P(\{\omega\}) = \frac{1}{2^5} : \omega \in \Omega'$

There are $5C3 = 10$ possible samples $\omega \in \Omega'$ where 3 are even.
$P(A) = \frac{10}{32} = \frac{5}{16}$

$P(A) = \frac{5}{16}$
□

1.7

We have an urn with 3 green and 4 yellow balls. We draw 3 balls one by one without replacement.

a

Find the probability that the colors we see in order are green, yellow, green.

✓ **Answer**

$\frac{3}{7} \cdot \frac{4}{6} \cdot \frac{2}{5} = \frac{24}{210} = \frac{4}{35}$
□

b

Find the probability that our sample of 3 balls contains 2 green balls and one yellow ball.

✓ **Answer**

GYG: $\frac{3}{7} \cdot \frac{4}{6} \cdot \frac{2}{5} = \frac{24}{210} = \frac{4}{35}$
YGG: $\frac{4}{7} \cdot \frac{3}{6} \cdot \frac{2}{5} = \frac{24}{210} = \frac{4}{35}$
GGY: $\frac{3}{7} \cdot \frac{2}{6} \cdot \frac{4}{5} = \frac{24}{210} = \frac{4}{35}$

The order does not matter for picking the colors of 3 samples.
The probability of pulling any of those three combinations is $\frac{12}{35}$

$\frac{12}{35}$

1.10

We roll a fair die repeatedly until we see the number four appear and then we stop. The outcome of the experiment is the number of rolls.

a

Following Example 1.16 describe a sample space Ω and a probability measure P to model this situation.

✓ **Answer**

Let $\Omega = \{\infty, 1, 2, ...\}$
$\forall \omega \in \Omega : P(\{\omega\}) = \left(\frac{5}{6}\right)^{\omega-1} \cdot \frac{1}{6}$

b

Calculate the probability that the number four never appears.

✓ **Answer**

$\lim\limits_{\omega \to \infty} P(\{\omega\}) = \lim\limits_{\omega \to \infty} \left(\frac{5}{6}\right)^{\omega-1} \cdot \frac{1}{6} = 0$

$P(\{\infty\}) = 0$

1.12

We roll a fair die repeatedly until we see the number four appear and then we stop.

a

What is the probability that we need at most 3 rolls?

✓ **Answer**

The probability of needing 1, 2, or 3 rolls is equivalent to not needing more than rolls.

$P(\omega > 3) = \left(\frac{5}{6}\right)^3$
$P(\omega \leq 3) = 1 - \left(\frac{5}{6}\right)^3 = \frac{91}{216}$

$P(\omega \leq 3) = \frac{91}{216}$
$\square$

b

What is the probability that we needed an even number of die rolls?

✓ **Answer**

Since the number of rolls are disjoint, we may add the probabilities directly.

$P(\omega \text{ is even}) = P(\{2\}) + P(\{4\}) + P(\{6\}) + \ldots$
$= \frac{5}{36}\left(1 + \frac{25}{36} + (\frac{26}{36})^2 + \ldots\right)$
$= \frac{5}{11}$ By the geometric sum rule

$P(\omega \text{ is even}) = \frac{5}{11}$

1.20

We roll a fair die four times.

a

Describe the sample space Ω and the probability measure P that model this experiment. To describe P, give the value $P(\{\omega\})$ for each outcome $\omega \in \Omega$.

✓ **Answer**

Let $\alpha = \{1, 2, \ldots, 6\}$
Let $\Omega = \alpha^4$

$\forall \omega \in \Omega : P(\{\omega\}) = \frac{1}{6^4}$

b

Let A be the event that there are at least two fives among the four rolls. Let B be the event that there is at most one five among the four rolls. Find the probabilities $P(A)$ and $P(B)$ by finding the ratio of the number of favorable outcomes to the total, as in Fact 1.8.

✓ Answer

Number of 5s	Number of Combinations
0	$1 \cdot 5^4$
1	$4 \cdot 5^3$
2	$6 \cdot 5^2$
3	$4 \cdot 5^1$
4	1

$P(A) = \frac{171}{1296}$
$P(B) = \frac{1125}{1296}$
□

c

What is the set $A \cup B$? What equality should $P(A)$ and $P(B)$ satisfy? Check that your answers to part (b) satisfy this equality.

✓ Answer
Since A and B are complementary, $P(A) + P(B) = 1$

$P(A) + P(B) = 1$ as checked.
□

1.22

We pick a card uniformly at random from a standard deck of 52 cards. (If you are unfamiliar with the deck of 52 cards, see the description above Example C.19 in Appendix C.)

a

Describe the sample space Ω and the probability measure P that model this experiment.

✓ **Answer**

$\Omega = \{\text{Diamonds}, \text{Clubs}, \text{Hearts}, \text{Spades}\} \times \{\text{A}, 2, 3, 4, 5, 6, 7, 8, 9, 10, \text{J}, \text{Q}, \text{K}\}$
$\forall \omega \in \Omega : P(\{\omega\}) = \frac{1}{52}$

b

Give an example of an event in this probability space with probability $\frac{3}{52}$

✓ **Answer**

$P(\omega \in \text{Royal Spades}) : \omega \in \Omega$

c

Show that there is no event in this probability space with probability $\frac{1}{5}$.

✓ **Answer**

Since there are 52 equal probability cards, any subset $A \subset \Omega$ must contain an integer amount of elements less than or equal to 52.

We may sum over the cards in A as pulling any card is disjoint from pulling any other card.

$P(A) = \sum_{i=1}^{n} P(\{A_i\}) \implies P(A) = \frac{|A|}{52}$

$\frac{1}{5}$ is not a multiple of $\frac{1}{52}$ and thus there cannot exist a set A such that $\frac{|A|}{52} = \frac{1}{5}$.

1.34

Pick a uniformly chosen random point inside a unit square (a square of sidelength 1) and draw a circle of radius $1/3$ around the point. Find the probability that the circle lies entirely inside the square.

✓ **Answer**

We may divide the square into 9 equal squares, dividing by three vertically and horizontally.

Any circle centered in the center of the 9 squares will be entirely contained by the square, as the distance to its closest edge is at least $\frac{1}{3}$.

Any circle not centered on the center of the 9 squares will not be entirely contained, as the distance to the closest edge is less than $\frac{1}{3}$.

Since the area of each of the sub-squares are the same, they have the same probability of being sampled:

$\Omega = [\![1, 9]\!]$
$\forall \omega \in \Omega : P(\{\omega\}) = \frac{1}{9}$

$P(\{5\}) = \frac{1}{9}$