1

a

Let $M = \begin{bmatrix} 0.06 & 0.08 & 0.04 & 0.02 \\ 0.12 & 0.16 & 0.08 & 0.04 \\ 0.09 & 0.12 & 0.06 & 0.03 \\ 0.03 & 0.04 & 0.02 & 0.01 \end{bmatrix}$

Then $P(x = i \cup y = j) = M_{(j,i)}$

If this were a valid probability distribution, then the sum of the probabilities of all possible samples will add up to 1.

$$\begin{aligned} &0.06 + 0.08 + 0.04 + 0.02+ \\ &0.12 + 0.16 + 0.08 + 0.04+ \\ &0.09 + 0.12 + 0.06 + 0.03+ \\ &0.03 + 0.04 + 0.02 + 0.01 \\ &= 1 \end{aligned}$$

Therefore, this is a valid probability distribution

b

Let $\vec{A} = \begin{bmatrix} 0.2 \\ 0.4 \\ 0.3 \\ 0.1 \end{bmatrix}$ and $\vec{B} = \begin{bmatrix} 0.3 & 0.4 & 0.2 & 0.1 \end{bmatrix}$

Such that $M = \vec{A}\vec{B}$

The probability of getting a particular $P(x = i)$ would be equal to $\sum_{j=1}^{n} P(x = i \cap y_j)$ for partition y.

Since we decomposed the probability matrix into a vector multiplication, we can say that:

$$\begin{aligned} &P(x = i) = \vec{A}_i * \sum \vec{B} \\ &\sum \vec{B} = 1 \\ &P(x = i) = \vec{A}_i \end{aligned}$$

$$P(x = i) = \begin{bmatrix} 0.2 \\ 0.4 \\ 0.3 \\ 0.1 \end{bmatrix}_i$$

$\square$

c

Yes, since in part (b) we decomposed the probability matrix of x and y into two vectors, x and y are independent as the independence rule holds.

$P(X \cap Y) = P(X)P(Y)$

We already proved in part (b) $P(X) = \vec{A}_i$
A similar proof can prove that $P(Y) = \vec{B}_j$

Given $P(X \cap Y) = M_{(j,i)}$

Since $M_{(j,i)} = \vec{A}_i \vec{B}_j$ => $P(X \cap Y) = P(X)P(Y)$

2

a

$P(A|B \cap C) = P(B|A \cap C)$

$P(B \cap C) = P(A \cap C)$ By Conditional Probability Rule

$P(B|C) = P(A|C)$ Also by Conditional Probability Rule
$\square$

b

$P(A|B \cap C) = P(A)$
Implies Independence between A and $B \cap C$, but tells us nothing about the relationship between B and C

Let $P(A = i) = [0.15, 0.35, 0.5]_i$

Let $P(B = j \cap C = k) = \begin{bmatrix} 0.05 & 0.05 & 0.15 \\ 0.15 & 0.2 & 0.15 \\ 0.125 & 0.1 & 0.025 \end{bmatrix}_{(k,j)}$

For the sample $(i, j, k) = (1, 1, 1)$

$P(a|b \cap c) = 0.15$
$P(a) = 0.15$

$P(b|c) = 0.1538$
$P(b) = 0.25$

Which are very clearly not equal.

c

$P(A|B) = P(A)$

Let both A and B be independent coin flips.
Let C be the probability that there is one head in A and B

$P(A|B) = 0.5$
$P(A) = 0.5$

$P(A|B \cup C) = 0$
$P(A|C) = 0.5$

3

a

$$P(L) = \sum_{i=F}^{T} \sum_{j=F}^{T} P(L \cap G_i \cap V_j)$$
$$= 0.00882 + 0.0108 + 0.049 + 0.0016$$
$$= 0.07022$$

b

$$P(G) = \sum_{i=F}^{T} \sum_{j=F}^{T} P(L_i \cap G \cap V_j)$$
$$= 0.049 + 0.0004 + 0.049 + 0.0016$$
$$= 0.1$$

c

$$P(L|G) = P(L \cap G)/P(G) = \frac{1}{P(G)} \sum_{i=F}^{T} P(L \cap G \cap V_i)$$
$$= 10(0.049 + 0.0016)$$
$$= 0.506$$

d

$$P(G = T|L = T \cup V = T) = P(G \cap (L \cup V))/P(L \cup V)$$
$$= \frac{1}{P(L \cup V)} \sum_i P(G \cap (L \cup V)_i)$$
$$= \frac{1}{0.0072+0.0004+0.00882+0.0108+0.049+0.0016}(0.0004 + 0.049 + 0.0016)$$
$$= \frac{0.051}{0.07782}$$
$$= 0.6553585196607556$$

e

$$P(L^C|G) = P(L^C \cap G)/P(G)$$
$$= 10(0.049 + 0.0004)$$
$$= 0.494$$

For the probability that you do not get a leak (L^C), given you know there is glass on the tire (G), with no information on the valve.

4

	D	D^C
E	0.000198	0.000002
E^C	0.04999	0.94981

a

$P(E|D) = 0.000198/(0.000198 + 0.04999)$
$= 0.003945166175181318$

b

$P(D)/P(E \cup D) = 1/P(E|D)$
$= 253.4747474747475$

c

	D_2	D_2^C
E	0.00019	0.00001
E^C	0.09998	0.89982

$P(E|D \cup D_2) = P(D|E)P(D_2|E)P(E)/P(D)P(D_2)$
$= P(D \cup E)P(D_2 \cup E)/P(D)P(D_2)P(E)$
$= (0.000198 * 0.00019)/((0.000198 + 0.04999)(0.00019 + 0.09998)0.0002)$
$= 0.03741547236120847$

5

a

Yes, given only the data that it appears blue and that discrimination is 75% reliable, we can still conclude that it is most likely blue with low certainty. (75% chance it is blue, with 25% chance its green)

b

Yes, we can now calculate with more certainty.