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FORM FINDING AND OPTIMIZATION OF GRID SHELLS USING FORCE DENSITY METHOD AND DISCRETE AIRY STRESS FUNCTIONS

By
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ABSTRACT

Force Density Method is widely used method for form finding of tensile cable nets and grid shells. In this thesis, a design tool that utilizes force density method along with Genetic Algorithm have been formulated in Rhino-Grasshopper, a parametric environment, for form finding and optimization of grid shells. Parametric study of a 12m *12m rectangular grid shell for variations in structural weight, height and deflection has been done for various topologies, subdivisions, and force density values. Genetic Algorithm has been used for optimization of grid-shell to get minimum weight for prescribed grid shell heights. Moreover, Force Density Method along with Airy stress functions in discrete form have been utilized for form finding of grid shells of various geometries. Using force density values and grid shell topology the projection of discrete airy stress function in form of plane faced polyhedron, form finding of grid shell self-supporting in lateral direction has been done. Variations in structural parameters between grid shell forms obtained using stress polyhedron and forms obtained using uniform force densities have been studied.

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

LIST OF SYMBOLS

σ_{xx}	Normal stress along x-axis
σ_{yy}	Normal stress along y-direction
τ_{xy}	Shear stress in XY plane
φ	Airy stress function
N_n	No. of modes
N_b	No. of branches/members
N_f	No. of fixed nodes
N	No. of free nodes
Cs	Branch Node Connectivity Matrix
C	Branch Node Connectivity Matrix for free nodes
C_f	Branch Node Connectivity Matrix for fixed nodes
q	Force density of a member
Q	Force density matrix
U	Co-ordinate Difference Matrix along x direction
V	Co-ordinate Difference Matrix along y direction
W	Co-ordinate Difference Matrix along z direction
p_x	Nodal load along x-direction
p_y	Nodal load along y-direction
p_z	Nodal load along z-direction
s	Member force
L	Member length
$X_{f,}$	Fixed node coordinate vector along x-direction
Y_f	Fixed node coordinate vector along y-direction
Z_f	Fixed node coordinate vector along z-direction
X	Free node coordinate vector along x-direction
Y	Free node coordinate vector along y-direction
Z	Free node coordinate vector along z-direction

CHAPTER ONE: INTRODUCTION

1.1 Introduction

A structure in which the thickness is small compared to other dimensions is generally called a thin shell. Thin shells are often the priority choice for long span roofing structures throughout the world for their aesthetic beauty and efficient transfer of loads with low structural weights, deriving strength from their form and geometry. Generally, thin shells can be categorized into two branches, namely, 1. continuous shells -as the name suggests are composed of continuous shell surface throughout the area of load transfer and 2. Grid-shells: composed of discrete grid members as structural elements, often steel or timber. Images of some famous grid-shell and continuous shells are shown below. The thesis is focused on grid-shells.

Figure 1 Continuous Shell - L'Oceanografic, City of Arts and Sciences, Valencia, Spain

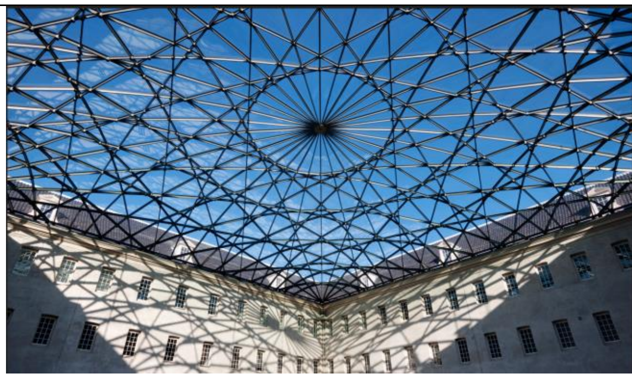


Figure 2 Grid Shell - Glass roof Dutch Maritime Museum at Amsterdam, Netherlands

In design of shell structures, the form and structural efficiency cannot be separated as the form regulates force flow and choice of a structurally efficient form can be realized in various forms. Shell geometry largely determine the characteristics and magnitude of forces arising in the members. The design of these structures in the usual workflow follows form design by architect, then structural design by structural engineer and construction by craftsmen. A major drawback of this workflow, in most cases, is that the structures are often uneconomical and sometimes unfeasible to construct, because of the fixed geometrical domain within which structural designers and craftsmen must work with. This drawback can be minimized by incorporating the basic structural and buildability constraints during initial design phase of the project. A well-conceived geometry and topology of structures can obviate a lot of design hurdles and reduce the project costs considerably. Here a convenient tool is proposed for the form finding of grid-shells, in a parametric environment (Rhino-grasshopper), addressing the structural requirements.

Shell geometries for design can be distinguished mainly in three types, namely Freeform shells where shapes are taken arbitrarily, Mathematical shells taken for their convenience in fabrication and analytical calculations and Form-found shells, mainly natural hanging shapes associated with funicular structures (Adriaenssens, et al., 2014). Form finding is a forward process in which parameters are explicitly/directly controlled to find an 'optimal' geometry of a structure which is in static equilibrium with a design loading (Adriaenssens, et al., 2014). In general, form found shells are structurally robust and optimized for design requirements, when compared to other shell geometries.

For shells, methods like Dynamic relaxation methods, hanging chain models, Force density methods are often used for funicular form finding where every member is either in compression or in tension. The major limitation of these methods is that, on their own they can only determine the compression or tension only forms, although structurally efficient. Finite Element Methods such as stiffness matrix methods, considering both tension-and-compression members, are often applied at later stages of design for verifications because of their higher computational demand during early stages of form finding (Veenendaal & Block, 2012). Also, modelling a fully formed 3D form is cumbersome often inviting various errors.

The force density method (Schek, 1974) was developed by H. J. Schek for finding equilibrium state of given pin-joint network consisting of cables or bars subjected to pre-stressing or any external loading with low computation costs. Preassigning force density, which is the force in member divided by its length, linearizes the geometrically non-linear equilibrium problem, thus finding the equilibrium shape in a direct way.

Force density variation with systematic manual input or assignment through results of plane faced airy stress polyhedron as in (Konstantatou, 2019) as well as assignments of various geometrical and boundary constraints can be done to obtain various funicular shapes, also tension and compression structures. A 2D self-stressed pinned structure in equilibrium, can be created from projection of a closed plane faced polyhedron, equivalent to 3D Airy stress Function, with member forces of the structure equal to the change in curvatures of two planes in Airy stress polyhedron, whose intersection form the member. Thus, these form finding methods mainly Force Density Method could be used in conjunction with Airy Stress Function to solve the general case of tension- and-compression grid-shells (Konstantatou, 2019).

Starting design from plane polyhedral Airy Stress Function ensures the equilibrium of structure which can be modified as per buildability and site-specific boundary constraints to determine the optimum form of grid shells. Furthermore, the structure is self-supporting i.e., reducing the lateral loads in the support, which could be extremely helpful when grid shells are to be constructed in conjunction with prebuilt or historic structures.

In this study, we employ Force Density Method (FDM), a method of form-finding, FEM for deflection and various load case analysis and genetic algorithm (GA) for design and optimization of grid-shells. Further, Discrete Airy Stress Function (DASF) in form of plane faced polyhedron is used in conjunction of FDM for form finding and rest of the process following the same as before. Here, a design tool is developed in widely used software- Rhino, that assist architects and engineers at early stages of design from form finding as well as discretization and optimization of grid shell, that would prove to be beneficial, minimizing the structural costs while satisfying architectural and structural demands.

Case studies of a 12m x 12m rectangular grid shell is done varying mainly parameters like force density, grid density and grid shapes regarding its effect on the structural height, structural weight and deflection. Optimization is done for the same shell using GA to determine shell configuration with minimum weight regardless of consideration for height and separately for shell to have a height equal to 1.5m. For the shell of same geometry but varying topology, Discrete Airy stress function in the form of plane faced polyhedron, henceforth referred as stress polyhedron, is used to determine force density input values to determine self-stressed grid shell structures. The results are compared with the case of grid shell obtained using FDM alone. Further, various self-supporting structures (in lateral direction) obtained using FDM and stress polyhedron are explored.

1.2 Problem Statement/Motivation

Although, having high efficiency in load transfer for their low self-weight and structural integrity, because of fabrication difficulties and computational hurdles, grid-shells are often disregarded in developing countries as Nepal. Geometrical methods of structural analysis and design of trusses and grid-shells not only allows for intuitive understanding of how loads are transferred within the structure, but it also simplifies the optimization ensuring the structural integrity. Thus, Geometrical methods are proposed for exploration to determine efficient structures for load transfers mainly in case of long span roofing using Discrete Airy Stress functions.

1.3 Purpose and Objectives of the Study

The general objective of this research is to explore the Form Finding and Optimization of grid-shells.

The specific objectives include:

- Determine the variation of grid shell weight, height, and deflection values for variation in grid density (shape), topology and force density values in a case study of a 12m*12m rectangular grid shell.
- Determine optimum grid shell for given limitations in weight, height, and deflection values.
- Study variations among grid shells determined using uniform force density and force densities from stress polyhedron.
- Develop a design tool that suggests structurally robust grid-shell forms at early phase of design.

1.4 Thesis Organization

The remainder of the thesis is organized as follows. The literature review on major concepts is presented in chapter 2 with some historical backgrounds. The methodology is presented in Chapter 3, which outline the formulation of force density method, demonstrates the fundamentals of Airy Stress Function and its representation of Reciprocal polyhedral in graphic statics, and optimization algorithm. The details of work procedure will also be presented in Chapter 3. In Chapter 4, results and discussions of case study of a rectangular plan grid shell form finding and design optimization as well will be presented, as well as the ability of design tool formulated. Chapter 5 presents the conclusions and further recommendations of the thesis.

CHAPTER TWO: LITERATURE REVIEW

2.1 General

Many research articles related to approaches for structural optimization and form finding of grid-shells are studied. Most notable ones are as follows:

2.2 Graphic Statics

Considerable work has been done by Maxwell in his graphical analysis of trusses (Maxwell J. , 1864, 1870) which expresses the reciprocity between form and force diagrams. Although being used throughout the 20th century though scarcely, Over the last few decades, graphic statics has seen renewed interest. Recent applications of graphic statics are underpinned by the computational and visual capabilities offered by contemporary computer aided design tools. These can be found among others in the design of compression-only or tension only spatial funicular structures by means of the Thrust Network Analysis (TNA) (Block & Ochsendorf, 2007) which combines 2D graphic statics with the force density method. (Beghini, et al., 2014) reestablish graphic statics as a structural optimization tool of discrete trusses exploring advantages of this method over conventional methods of analysis, design, and optimization.

2.3 Reciprocal diagrams and Airy Stress Functions

In 2D reciprocal diagrams, form edges map to force edges and form nodes to closed force polygons, obeying 3D duality. These reciprocal diagrams are projections of polyhedral Airy Stress functions which are reciprocal as well. Maxwell (Maxwell J. , 1870) said that this reciprocity between polyhedral can be obtained through a polar transformation, using paraboloid of revolution. The change of slope between adjacent faces of the Airy Stress function defines the axial force of the corresponding structural member of the 2D truss. (McRobie, et al., 2016) (Mitchell, et al., 2016). Moreover, the roles of reciprocal form and force diagrams are interchangeable an there are no distinctions between lines of action of the applied forces and structural members as the internal or external forces can be combined with the form diagram to make projection of a single polyhedron truss (McRobie, et al., 2016) (Mitchell, et al., 2016). Thus, this polyhedron can be seen as the Airy Stress function of an equivalent self-stressed truss. Further, (Miki, et al., 2020) investigate tension-compression mixed type shells by utilizing a NURBS-based iso-geometric form-finding approach that use Airy stress functions to expand the possible plan geometry.

2.4 Force Density Methods or Analysis methods

The fundamental rationale of the 3D hanging chain model has long been discussed, as a quotation from Robert Hooke in 1675 says "As hangs a flexible cable, so but inverted will stand the rigid arch." This principle describes the reciprocal relationship of a tension-only and a compression-only form in 2-dimensional (2D) space under the same loading case (Block & Ochsendorf, 2007). When extended to design in the 3-dimensional (3D) space, this rationale supports the use of the 3D hanging chain model for the form finding of a gravity-loaded structure (Jiang, 2015).

The force density method (Schek, 1974) or 'Stuttgart direct approach' was primarily used for form-finding and design explorations of prestressed spatial tension-only structures, such as hanging cable nets and membranes (Adriaenssens, et al., 2014). For this type of funicular structures, the form is interlinked to the internal forces and greatly affects the load-bearing behavior (Adriaenssens, et al., 2014).

Extension the potential and robustness of FDM has been done in form-finding of compression and tension structures such as tensegrities and suspended bars (Miki & Kawaguchi, 2010). The Discrete polyhedral stress function has been introduced (Fraternali, 2010) for the analysis of unreinforced masonry vaults and for the prediction of fracture-prone regions. Further, various studies for evaluation of grid shell performance and structural weight varying similar parameters have been done by various researchers, to name few, (Malek, 2012) using FEM for analysis, (Olsson, 2012) using SMART form, (Green & Lauri, 2017) Dynamic relaxation. (Konstantatou, 2019) combines the Discrete Airy Stress Function with Force density method as design exploration for grid shells containing both compression and tension.

Most of the research work are focused on implementing and extending the design and optimization of funicular form shells/grid-shells with very few focusing on both tension-compression grid-shells. Moreover, only a few research have been done combining the Discrete Airy Stress Method with Form finding method as FDM, although, leaving room for extended parametric study and application of this combination for design exploration. Thus, as an extension to the earlier works, it is proposed to combine Airy stress function method with FDM along with optimization methods to explore a robust design space.

Some literature reviews with their respective methodologies are tabulated below:

Table 1 Literature review with respective methodology

Literature Review	Methodology
(Maxwell J. , 1864; Maxwell J. , 1870)	Introduces Graphic Statics Method, Shows relation between 3d polyhedral Airy Stress Function and 2d trusses.
(Block & Ochsendorf, 2007)	Combination of 2d Graphic Statics with force density method for compression only or tension only spatial funicular structure.
(Beghini, et al., 2014)	Graphic Statics as analysis design and optimization of discrete 2D truss in modern context.
(Miki, et al., 2020)	Utilize NURBS based Iso-geometric analysis and Form finding approach using Airy Stress Functions
(Schek, 1974)	Introduction of FDM
(Jiang, 2015)	Compares Potential Energy Method -Dynamic Relaxation and FDM
(Adriaenssens, et al., 2014)	Form-finding and design exploration of pre-stressed spatial tension-only structures
(Konstantatou, 2019)	Combine Airy Stress Function and Force Density Method for design exploration of Grid Shells.
(Airy, 1862)	Introduce Airy Stress Function to calculate strains in frames.

CHAPTER THREE: METHODOLOGY

A method for structurally efficient form finding and optimization for grid-shells has been proposed. The visualization as well as computation are done in Rhino – Grasshopper Environment with additional use of Python programming language. Rhino- Grasshopper allows parametric workflow thus facilitating the study of variations of each parameter involved in the study.

FDM formulation has been done in Rhino using Grasshopper and Python for calculation and solving linear equations, while Galapagos plugin within grasshopper has been used for optimization by genetic algorithm. Karamba3D within Grasshopper has been used as a FEM analysis tool for further analysis and to determine deflection.

The initial design settings comprise of grid members, hinge nodes and nodal boundary conditions. Shell panels are considered for nodal load calculation, but not as structural members in the design. As loads are applied, the grid shell form will change so that an equilibrium state is achieved. The design goal is to find a grid form in equilibrium under a certain load case, where the members suffer no bending moment and no shear force. The state of zero bending moment and shear force is ensured by the assumptions that:

1. All grid members are hinge-connected.

2. For every grid member, loads are only applied at the end nodes.

Thus, the task left is to find the equilibrium form.

For the study, buckling has not been considered, connections are assumed to be pin jointed, connection costs been not taken for optimization and only statically uniformly distributed gravity load case has been done. However, Comparison of grid-shells of various geometries (spherical type, pyramidal and form found using FDM) is done for lateral load case, namely, static lateral wind loading in single direction. The comparison clearly shows the superiority of form found grid-shells over other types taken in the comparison. The results are presented in Table 8 and Table 9 in the Annex II.

3.2 Formulation of Force Density Method (FDM)

The FDM (Schek, 1974), or ‘Stuttgart direct approach’ was initially used for form-finding and design explorations of general networks of hanging cables, bars and membranes. The name is derived from ‘Force Density’, a parameter used, that represents the ratio of member forces by member length. Pre-assigning the Force density linearizes the non-linear geometry and static -equilibrium problem and its solution represent the final form.

The input of the FDM is 2D ground structure with force densities provided for each member and a list of fixed nodes. The output is co-ordinates (x, y and z) of each point in the ground structure. Changing the force density to positive or negative enables one to get a form with members in either tension or compression. Furthermore, fixed member lengths could also be assigned adding to more realistic design.

FDM offers features like (1) Manages equilibrium equations in totally direct way and is therefore especially suited for a funicular solution. (2) Equilibrium equations are linearized, which simplifies the numerical process (3) No member pre-sizing is required (4) The three equilibrium equations are uncoupled (Cercadillo-García & Fernández-Cabo, 2016).

The formulation of FDM is below with major reference to (Schek, 1974).

To begin with, we define a grid system with nodes from 1 to Nn and grid members (or branches) from 1 to N_b. To simplify, the calculation process, we define all the fixed nodes (N_f nodes, from N+1 to N+N_f) after all the free nodes (N nodes from 1 to N). Thus, we have $N_n = N+N_f$. For any grid member j, there are 2 corresponding nodes with number c(j) and d(j). With the information above, the branch-node matrix $\mathbf{C}s$ is defined by,

$$\mathbf{C}s(j,i)= \begin{cases} +1, \text{ for } c(j)=i \\ -1, \text{ for } d(j)=i \\ 0, \text{ otherwise} \end{cases} \qquad (3.2.1)$$

The matrix $\mathbf{C}_s$ has N_b rows and N_n columns. Note that the value of c(j) and d(j) for a grid member may interchange with each other. This interchange will affect $\mathbf{C}_s$ but will not influence the resultant linear system.

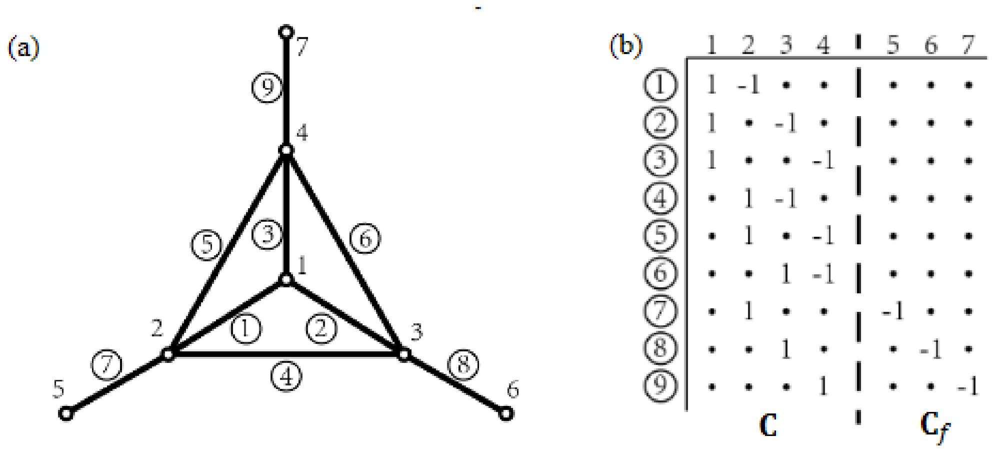


Figure 3 (a) A simple grid system. Nodes are denoted by plain numbers. Branches are denoted by circled numbers; (b) The connectivity matrix of the grid system

The branch-node matrix $\mathbf{C}_s$ is separated into $\mathbf{C}$ and $\mathbf{C}_f$, with $\mathbf{C}$ representing the portion for free nodes and $\mathbf{C}_f$ for the fixed nodes, as illustrated by Fig. 2.2. In a similar logic, $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$ and $\mathbf{X}_f$, $\mathbf{Y}_f$, $\mathbf{Z}_f$ correspond to the nodal coordinate vectors of free and fixed nodes in the x, y and z directions, respectively. Vectors holding the coordinate difference between 2 connected nodes are defined as u, v, and w. They are derived with branch-node matrices and nodal coordinates:

$$\begin{aligned}\mathbf{u}&=\mathbf{C}^*\mathbf{X}+\mathbf{C}_f\mathbf{X}_f\\ \mathbf{v}&=\mathbf{C}^*\mathbf{Y}+\mathbf{C}_f\mathbf{Y}_f\\ \mathbf{w}&=\mathbf{C}^*\mathbf{Z}+\mathbf{C}_f\mathbf{Z}_f\end{aligned} \qquad (3.2.2)$$

Let us define the member length vector as ℓ and the force density vector as q. The force density of the j^{th} grid members q_j is defined as its force-length ratios. Then define U, V, W, L, Q as the diagonal matrices of the vector u, v, w, $\boldsymbol{\ell}$ and q, respectively. Moreover, define s as the member internal force vector. To maintain force equilibrium at every node, the sum of internal forces should equal the external forces.

The equilibrium equations are formulated as follows:

$$\begin{aligned}\mathbf{C}^T\mathbf{U}\mathbf{L}^{-1}\mathbf{s}&=\mathbf{p}_x,\\ \mathbf{C}^T\mathbf{V}\mathbf{L}^{-1}\mathbf{s}&=\mathbf{p}_y,\\ \mathbf{C}^T\mathbf{W}\mathbf{L}^{-1}\mathbf{s}&=\mathbf{p}_z\end{aligned} \qquad (3.2.3)$$

$$\frac{\partial \mathrm{l}}{\partial \mathrm{x}} = \mathbf{C}^{\mathbf{T}}\mathbf{U}\mathbf{L}^{-\mathbf{1}}$$

$$\frac{\partial \mathrm{l}}{\partial \mathrm{y}} = \mathbf{C^T V L^{-1}} \quad (3.2.4)$$

$$\frac{\partial \mathrm{l}}{\partial \mathrm{z}} = \mathbf{C^T W L^{-1}}$$

In the equilibrium equations, representations of Jacobian matrices are utilized:

The force density vector $\mathbf{q}$ is obtained by:

$$\mathbf{q}=\mathbf{L}^{-1}\,\mathbf{s} \quad (3.2.5)$$

Where, $\mathbf{L}$ is the diagonal matrix of the member length vector $\boldsymbol{\ell}$, and $\mathbf{s}$ is the member force vector. With (3.2.5), we can then update (3.2.3) into:

$$\begin{aligned}\mathbf{C}^T\mathbf{Uq}&=\mathbf{p}_x\\ \mathbf{C}^T\mathbf{Vq}&=\mathbf{p}_y \quad (3.2.6)\\ \mathbf{C}^T\mathbf{Wq}&=\mathbf{p}_z\end{aligned}$$

By means of (3.2.2) and the following identities, we obtain:

$$\begin{aligned}\mathbf{Uq}&=\mathbf{Qu}\\ \mathbf{Vq}&=\mathbf{Qv} \quad (3.2.7)\\ \mathbf{Wq}&=\mathbf{Qw}\end{aligned}$$

The nodal force equilibrium equation systems in x, y and z directions are formulated as follows:

$$\begin{aligned}\mathbf{C}^T\mathbf{QCx}+\mathbf{C}^T\mathbf{QC}_f\mathbf{x}_f &=\mathbf{p}_x\\ \mathbf{C}^T\mathbf{QCy}+\mathbf{C}^T\mathbf{QC}_f\mathbf{y}_f &=\mathbf{p}_y \quad (3.2.8)\\ \mathbf{C}^T\mathbf{QCz}+\mathbf{C}^T\mathbf{QC}_f\mathbf{z}_f &=\mathbf{p}_Z\end{aligned}$$

For a simpler representation, we can write $\mathbf{D}=\mathbf{C}^T\mathbf{QC}$ and $\mathbf{D}_f=\mathbf{C}^T\mathbf{QC}_f$, so that (3.2.8) becomes:

$$\begin{aligned}\mathbf{DX}+\mathbf{D}_f\mathbf{X}_f &=\mathbf{p}_x\\ \mathbf{DY}+\mathbf{D}_f\mathbf{Y}_f &=\mathbf{p}_y \quad (3.2.9)\\ \mathbf{DZ}+\mathbf{D}_f\mathbf{Z}_f &=\mathbf{p}_z\end{aligned}$$

Simplifying we get,

$$\begin{aligned}\mathbf{X} &= \mathbf{D}^{-1}\,(\mathbf{p}_x - \mathbf{D}_f\,\mathbf{X}_f)\\ \mathbf{Y} &= \mathbf{D}^{-1}\,(\mathbf{p}_y - \mathbf{D}_f\,\mathbf{Y}_f) \quad (3.2.10)\\ \mathbf{Z} &= \mathbf{D}^{-1}\,(\mathbf{p}_z - \mathbf{D}_f\,\mathbf{Z}_f)\end{aligned}$$

Thus, member forces can be derived by:

$$\mathbf{s = Lq} \tag{3.2.11}$$

Given the external loads, the branch-node matrix, force densities and fixed degrees of freedom, we can determine a set of free nodal coordinates by solving the system in (3.2.11).

General workflow of FDM is shown in Figure 4. The input of the FDM functions are bar network or line mesh in 2D with their connectivity, force densities of each line/bar, nodal load at each node may it be in x, y or z direction, support points or fixed point and further constraints as mentioned in (Schek, 1974). The result of FDM will give nodal points, which are at equilibrium as per given input values. The bar nodes connected will give final structural layout in 3D. Forces in each member is calculated by product of force density with length of the member squared. Thus, form found structure need to be further analyzed for other load cases, global stability, buckling and other requirements.

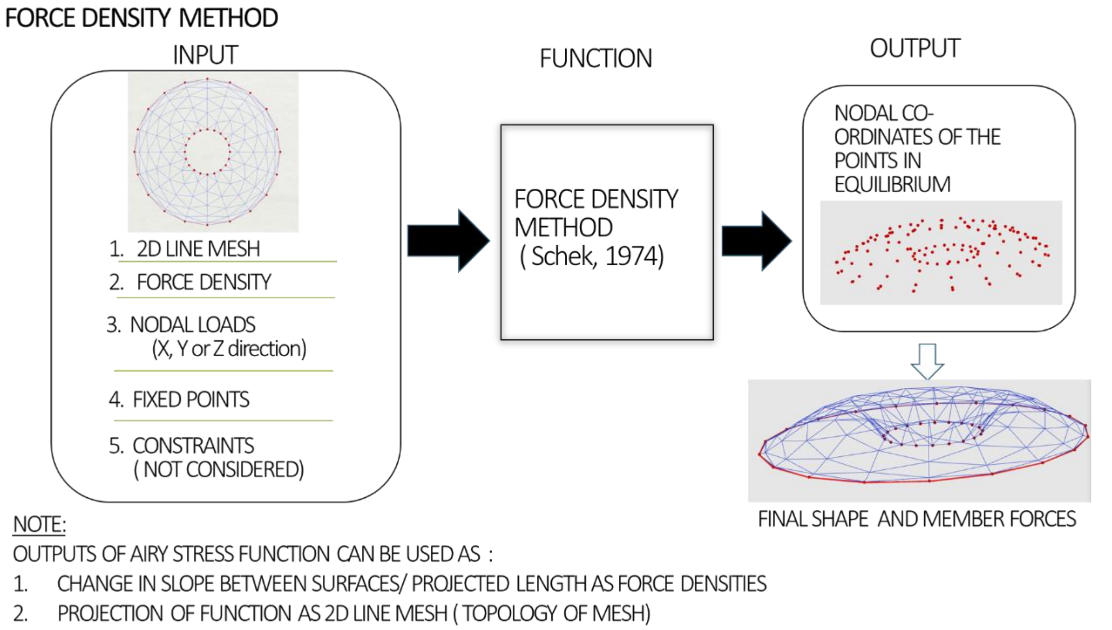


Figure 4 Force Density Method workflow

FDM FORMULATION VALIDATION

Validation of FDM formulation is done in two ways. Form finding of a pin connected line like bar network with constant force density is done. The resulting shape from FDM is that of a parabolic arch, same as a funicular form -catenary. Furthermore, a quadrilateral grid-shell of base dimension 4m *4m, with grid spacing 0.8m with quadrilateral panels has been analyzed for a constant force density and vertical point loading at grid joints. The resulting form of grid shell is analyzed in commercial design software ETABS v19 and also in utilized FEM plugin, Karamba3D, the resulting values of forces obtained from FDM and ETABS vary with 0.8% error, well within the acceptable limit.

STRUCTURAL WEIGHT AND TOTAL LOAD PATH

Grid shell members are considered to be fully stressed; thus, it is possible for representation of total structural weight by total load path of the structure. Load path for a member is given by its member length multiplied by force in the member.

Total weight of structure (W) = $\sum_1^n A_i L_i \rho_i$

where, n is the numbers of members, L is length and A, the area of each member, ρ density of the material used.

Considering each member to be fully stressed to stress capacity σ, Ai= Fi / σ,

Thus, W = $\sum_1^n$ Fi Li, (ρ_i/ σ). In the study, material grade is considered constant,

thus (ρ_i/ σ) = constant.

i.e. $W \propto \sum_1^n F_i L_i$, i.e., total load path of the structure. Hence, in the study, structural weight is represented by total load path of the structure where cross section of the members is not defined.

FORCE DENSITY

Shape analysis of tensile structures is a geometrically non–linear problem, the FDM linearizes the form–fitting equations analytically by using the force density ratio for each cable element, q = F/L, where F and L are the force and length of a cable element respectively (Southern, 2011). Same principle holds for compression - tension and compression only structures like grid-shells. Force density of a member during FDM is assigned as per its axial strength.

For form finding of a grid shell, force density values input in FDM, are directly dependent on the choices of materials and section sizes available. For similar sectional sizes available, for material like wood, force density values assigned are lower, given their lower compressive strength, whereas higher values of force densities can be assigned for steel, given its higher material strength. Here, effect of variation of force density on shell weight and height is studied for force density ranging from 20KNm^{-1} to 60KNm^{-1}.

GRID SHELL LOADING

Loads are applied as uniformly distributed static loads at member ends only. Uniformly distributed static load of 5KNm^{-2} is taken, which is applied to the nodes as per their tributary area. It is observed that for lower values of force densities in all topologies of grid shells considered, surface area of grid shell after form finding is significantly greater than that of original projected plan. Thus, multiple iterations are required in form finding to get actual loading after the shell assumes its shape.

Figure 5 shows, loading error for quadrangular grid shell for various force density and 6 subdivisions. Figure 6 shows loading error vs no. of iterations for various grid shell topologies at force density 20KNm^{-1} and 6 subdivisions. It has been observed that for all grid topologies considered maximum of three iterations suffice to get loading values within error of 5% for force density values higher than 20 KNm^{-1}. Thus, three iterations of FDM are done to get the loading on nodal points.

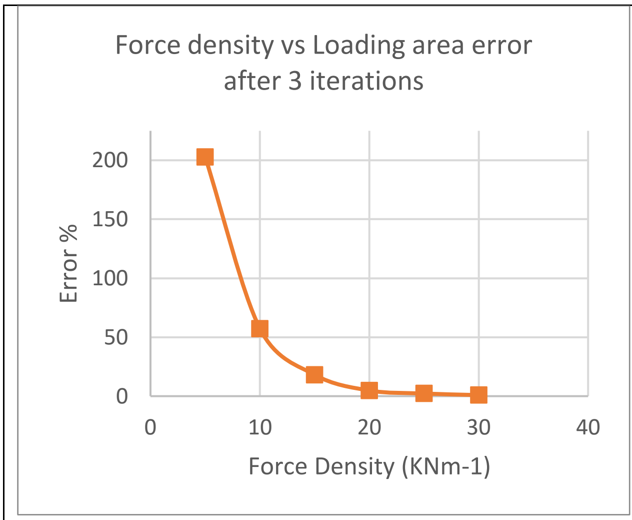


Figure 5 Loading error for quadrangular grid shell for various force density and 6 subdivisions.

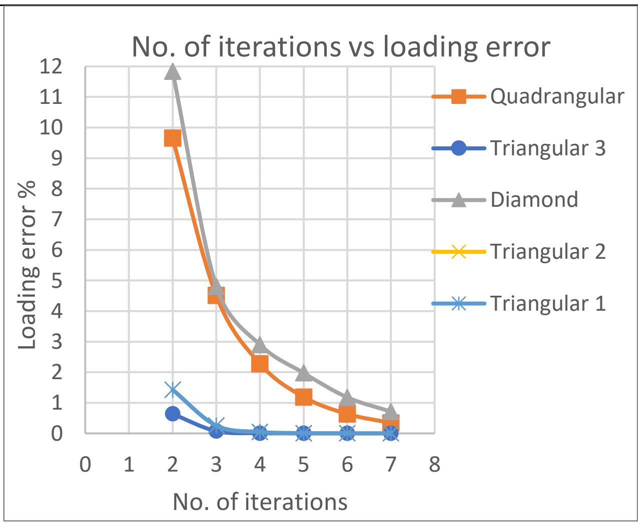


Figure 6 Loading error vs no. of iterations for various grid shell topologies at force density 20KNm^{-1} and 6 subdivisions.

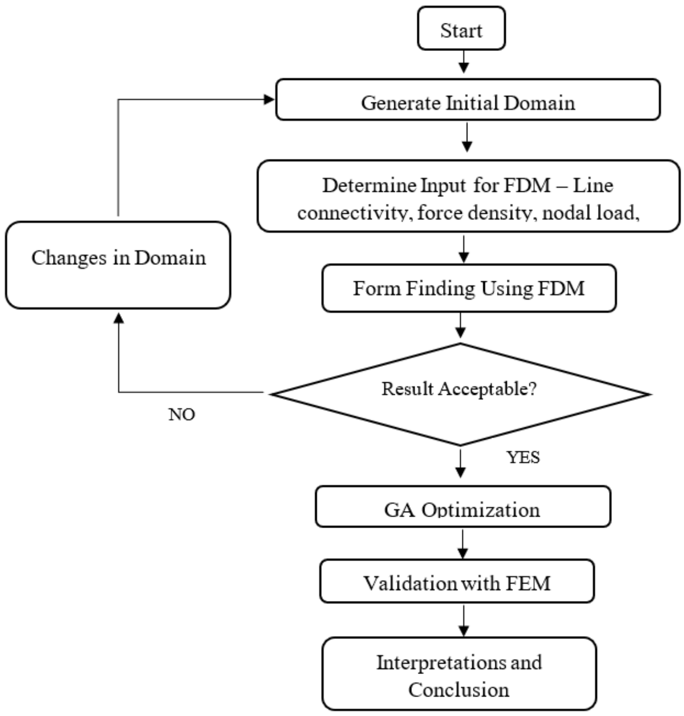


Figure 7 Flow Chart of Methodology for grid shell design and optimization using FDM.

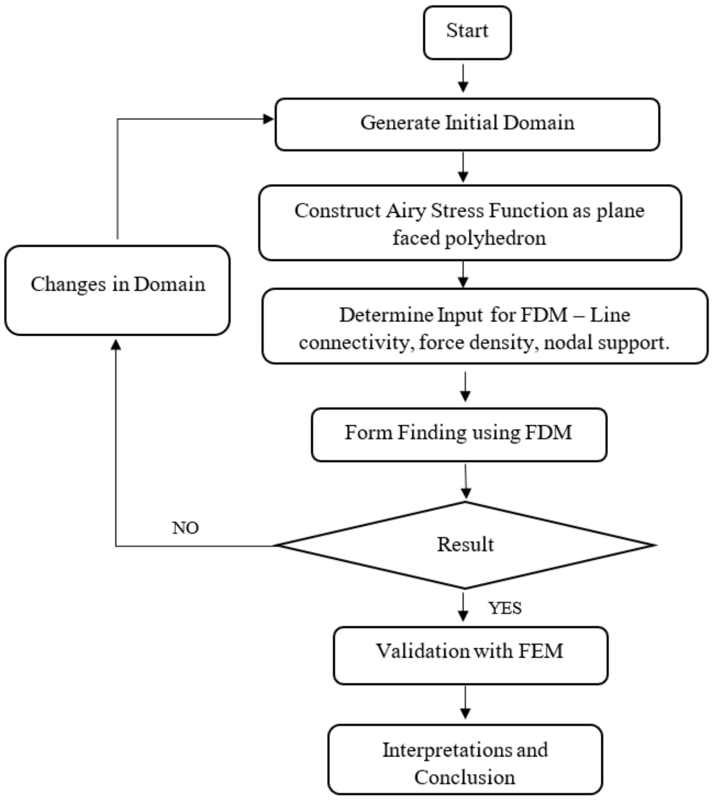


Figure 8 Flow Chart of Methodology for grid shell design and optimization using FDM and Airy Stress Function

3.1 Airy Stress Function and Form Polyhedron

Airy Stress Function, introduced by George Biddell Airy (Airy, 1862) describes the inner stresses in two-dimensional continuous structures. The stress function satisfies the governing equilibrium equations. Airy stress function can be regarded as a 3D surface $\varphi(x, y)$ over 2D xy-plane that contains the structure, the differentiation of which give the stresses in the plane as seen in equations below:

$$\sigma_{xx} = \frac{\partial^2 \varphi}{\partial y^2}$$
$$\sigma_{yy} = \frac{\partial^2 \varphi}{\partial x^2} \qquad (3.1.1)$$
$$\tau_{xy} = -\frac{\partial^2 \varphi}{\partial x \partial y}$$

For polyhedral stress functions, all curvature is concentrated at the edges of the plane faces, such that the stress field is zero everywhere except along a set of discrete lines, these being the force-carrying bars of the truss (Williams & McRobie, 2016). As a result, the Airy stress function can be used for studying the static equilibrium of pin-jointed frameworks (i.e., trusses). In this case, the surface φ(x, y) is not a continuously smooth surface but a plane-faced polyhedral surface in which the curvature is zero in the planar faces that are adjacent to each polyhedral edge while the curvature change is concentrated along the edges.
In fact, 2D trusses can be seen as projections of 3D polyhedral Airy stress functions, in which the bars of the 2D structures are the projections of the polyhedral edges of the 3D stress function. The relation between Airy stress functions and equilibrium of pin-jointed frameworks was also referred by James Clerk Maxwell in his articles on graphic statics (Maxwell J. , 1870).

The angle between two adjacent faces in the 3D polyhedral Airy stress function gives the internal axial force of the corresponding bar in the 2D structure. Additionally, the local convexity of the function, which determines whether the edge of the polyhedron is convex or concave, defines if the corresponding bar in the structure is in compression or in tension respectively.

In figure 7, φ represents the continuous airy stress function φ(x,y) in xy plane for body with force P acting along x direction over length 2B. The normal stress along x-direction is given by partial double differentiation of stress function along y which is equal to P/b. Assumed continuous airy stress function satisfies and represents the stress condition of taken case.

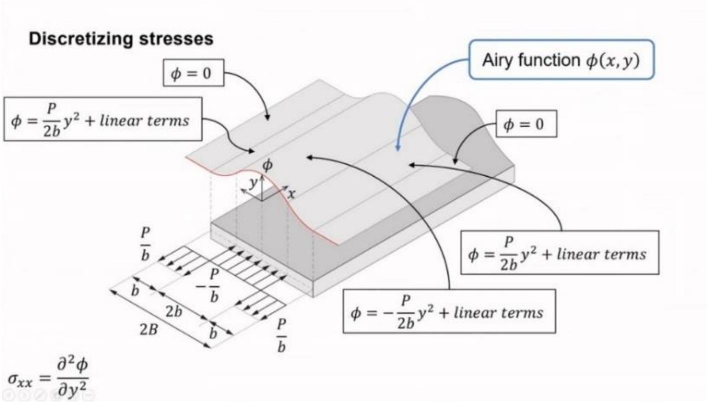


Figure 9 Continuous Airy Stress Function over continuous surface

In figure 8, φ represents the discretized form of airy stress function φ(x,y) as shown in figure 6. The faces of the stress function are plane, and curvature is changed only at the kinks where the total stresses are concentrated, alike to bar trusses. Assumed discrete airy stress function satisfies and represents the stress condition of taken case.

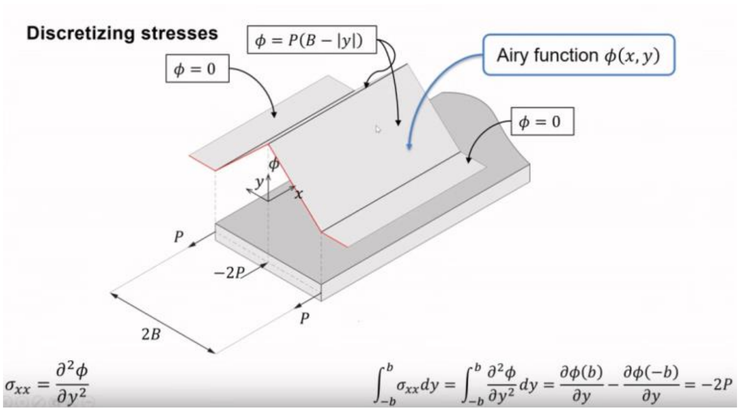


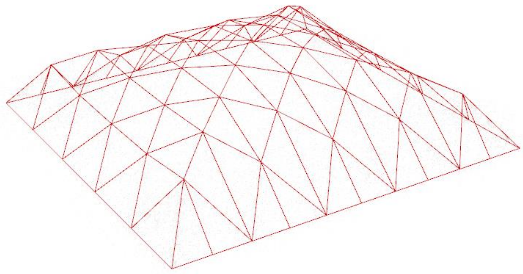
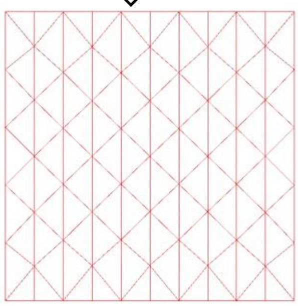
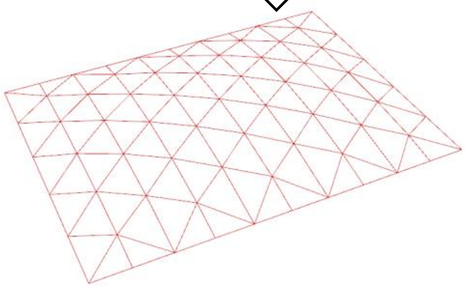
Figure 10 Discrete Airy Stress Function at intersection of two planes

AIRY STRESS FORMULATION FOR GRID-SHELLS

Any plane faced polyhedron represents an airy stress function of a potential self-stressed pin jointed member network may it be trusses, grid shells, tensegrity structures (Mitchell, et al., 2016). The projected network in 2d from a 3d plane faced polyhedron representing airy stress function is self-stressed i.e., that the forces in the network are in equilibrium in both plane directions (say x and y axis).

Taking the case of grid shell, a plane faced polyhedron as shown in Figure 11, represents an airy stress function. The 2D projection of stress polyhedron is a pin jointed bar network in 2D with same topology as of the stress polyhedron, shown in Figure 12. Member forces in the projected bar network can be calculated, in stress polyhedron, from the slope difference between adjacent faces of the edge, which projected is the bar network in 2D. Now, with member force and member length in 2D, one can determine the force density. FDM can be applied to the 2D network to generate grid shell with force density values obtained and chosen support and nodal loading condition. Thus, Airy stress functions can be used as topology and force density input of FDM producing a grid shell that is self-supporting in xy plane, shown in Figure 13. Form found shells in this way need to be checked further for global stability, performance in lateral loads and other parameters which is not in the scope of the study.

Table 2 Reciprocity of Airy Stress Function and Grid shell geometry.

Figures	Description
 *Figure 11 Airy Stress Function Represented by Plane faced polyhedron*	Discretized airy stress function in form of closed plane faced polyhedron. Note that the stress polyhedron is closed, hence boundary conditions satisfied and within the structure itself.
Projection	
 *Figure 12 Projection of stress polyhedron at xy plane*	Projected plan view of stress polyhedron at Figure 11. Force in a projected member of an edge in stress polyhedron is given by slope difference between adjacent faces. Force density = Member Force/ Member length.
FDM	
 *Figure 13 Grid shell form found using FDM*	Resulting grid shell geometry with support conditions at edges obtained with FDM for force density values obtained from stress polyhedron. The structure is in equilibrium and requires no horizontal support (self -supporting)

Self-stressed structures are highly valuable in cases where constructions are required to be done for historic or masonry structures which are strong in vertical direction but have little lateral load carrying capacity.

Airy stress function in form of plane faced polyhedron and the resulting grid shell are reciprocal in nature i.e., the resulting grid shell can be used as an airy stress function to generate grid shell with form like initially taken plane faced polyhedron representing airy stress i.e., the process followed in Table 2 can be reversed taking stress polyhedron as shown in Figure 13 to generate grid shell geometry resembling to Figure 11. Thus, representing reciprocal nature as mentioned by Maxwell (1870).

3.3 Genetic Algorithm

Genetic Algorithms (GAs), based on the principle of evolution is a stochastic method. In this method, best individuals, as per their fitness (here minimum structural weight) are chosen from a random population of individuals (here grid shells) with various gene sets (here topology, force density and subdivisions) are chosen for reproduction and with specific crossing techniques, solutions are combined to bring new offspring and in that way for a new generation (Dimcic & Knippers, 2011). Crossing methods are programmed to ensure conservation of good genes and addition of mutation algorithms enable random alteration of genes thus enabling convergence towards best fit solution. New generations are produced until a satisfactory result is found. In our case, the satisfactory result is statically stable grid-shells with minimum weight for various height configurations. More on application of GAs can be found in (Goldberg & Holland, 1988).

This process is very similar to natural selection in the real world since Galapagos iterates, or breeds, multiple generations of solutions until it finds what it believes to be the fittest solution. The image to the left shows the basic genetic algorithm process.

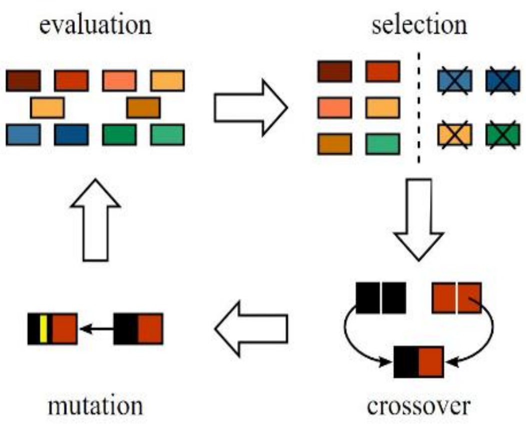


Figure 14 General overview of Evolutionary solver mechanism

Two cases are examined for optimization, one being minimum grid shell weight without any regards for height and second minimum grid shell weight for height equal to 1.5m. For weight minimization with limit on height of 1.5m, the objective function for minimization is set as:

$$\text{Minimize, } W = \sum_{1}^{n} F_iL_i * (h-1.5)^2,$$

where, F_i = force in individual member

L_i = member length

h= grid-shell height obtained

n = no. of members.

In this thesis, we will be using Galapagos Solver, in-built plugin in Rhino-Grasshopper

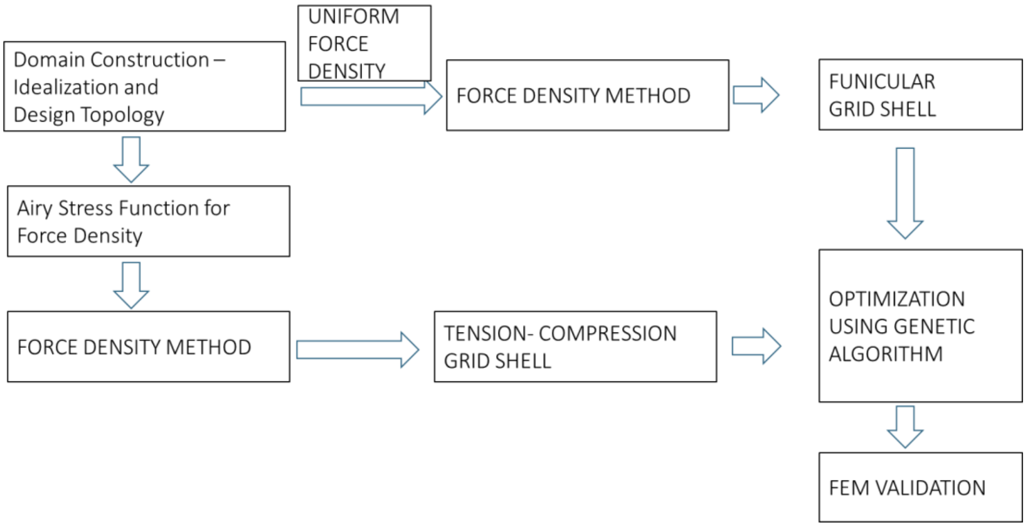


Figure 15 Basic Workflow for grid shell design using FDM only and FDM with Airy Stress Function.

CHAPTER FOUR: RESULTS AND DISCUSSIONS

Results and discussions are divided into three parts.

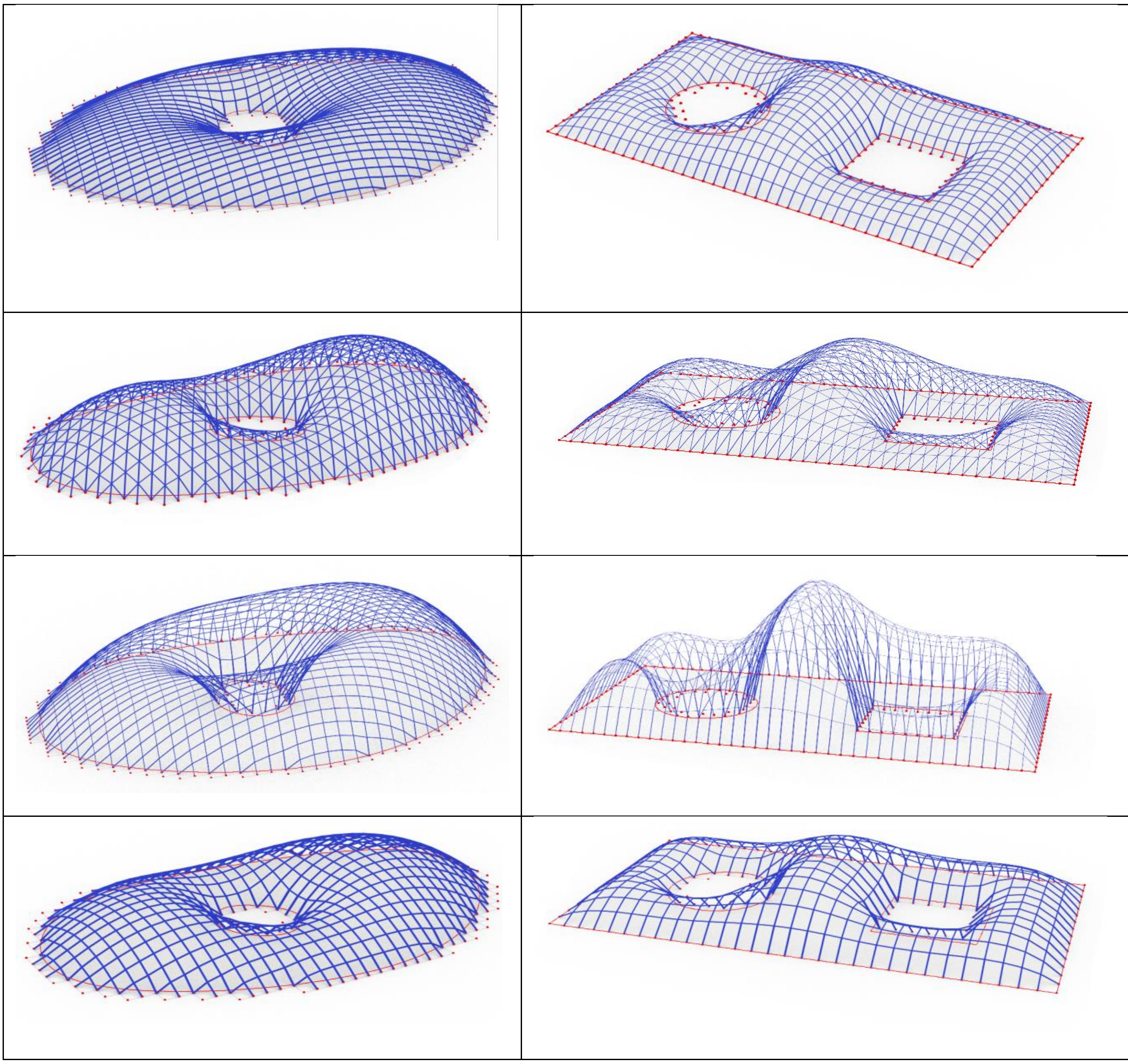
1. Form finding of arbitrary shell boundary shapes - the general ability of design tool formulated is shown.
2. Study of variation of weight, height, and deflection of a 12m * 12m grid shell – Parametric variation of shell height, weight and deflections are studied for mentioned grid shell with variations in topology, grid density and force density.
3. Design of self-supporting grid shell using Airy stress function – Airy stress function is used in conjunction with force density method to generate self-supporting (laterally) grid-shell forms. The resulting shapes are compared to the grid-shells, form found using FDM method only.

4.1 Form Finding Shells with irregular boundary shapes

Grid shells allows for compatibility and efficiency of load transfer even in areas with highly irregular geometrical plans. This ability to span over areas with irregular boundaries and support conditions is an advantage not provided by other spanning or roofing structures. At early phases of design, determination of grid shell geometry is crucial as it dictates the rest of the design. A design tool has been formulated that determines shell geometry in static equilibrium as per the conditions of grid densities, topologies preferred and construction material (represented by force density). Following are some shell geometries for irregular plan and support configurations obtained using FDM formulation in Rhino-Grasshopper.

Table 3 shows various grid shell geometries obtained varying the parameters in FDM for two types of arbitrarily chosen grid-shell plan geometries. Boundaries are represented by red lines and supports as red dots and resulting grid shell by blue lines.

Table 3 Variations of grid shell geometries obtained for boundary (red line) and support (red dot) for various grid densities, force densities and topologies.



First row shows a grid shell obtained with arbitrary shape, force density values and grid densities. In second row, the grid mesh is triangulated. In third row, force density values are reduced by half. In third row grid density is decreased from the first by seventy five percent. The resulting grid shells are in static equilibrium and further analysis for global stability, other load cases need to be performed for finalization of design. However, starting from form found geometry as above in comparison to arbitrary shapes take, would certainly be beneficial to obtain economic and structurally sound design.

4.2 Parametric variation in weight, height and deflection using FDM

Case study of a 12m x 12m span covering grid shell with pin support on all the edges is done. Variation of structural weight, represented by total load path, corresponding height, and maximum deflection of grid shell as per variation in parameters listed below are studied:

- Topology – quadrilateral grids (quadrangular and diamond shaped) and triangular grid of 3 types as shown in Figure 16.
- Grid density or number of subdivisions of span ranging from (6 to 24)
- Force density ranging from (20 KNm−1 to 60 KNm−1)

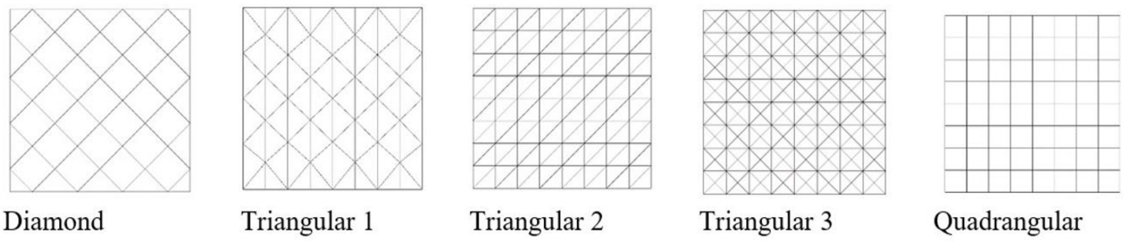


Figure 16 Grid shell typologies considered in study.

Table 4 shows variation of total structural weight, grid shell height and maximum deflection for five different grid topology types, keeping no. of subdivisions equal to 8 and constant force density of 25KNm−1. From the table among considered grid types, quadrangular has lowest structural weight (57%) of highest which is triangular type 1 and 3. At the same time, triangular type 3 has the lowest grid shell height (37%) of the diamond type. From table 1, among other grid types, quadrangular and diamond types have lower structural weight, because of lower value of load path, as fewer members transfer the forces to the support.

Table 4 Grid shell height, weight and deflection variation for variation in topology for 8 no. of divisions and force density 25KNm-^{1}

Grid Type	Load path (KNm)	Normalized Load path	Shell height (m)	Normalized shell height	Deflection (mm)	Normalized shell deflection
Diamond	8617.82	0.61	3.31	1.00	2.686	0.63
Triangular 1	13962.97	1.00	1.54	0.46	2.815	0.66
Triangular 2	13706.90	0.98	1.53	0.46	4.282	1.00
Triangular 3	14017.73	1.00	1.24	0.37	2.497	0.58

Quadrilateral	7984.91	0.57	3.26	0.98	1.142	0.27

Further, it has been seen that for Triangular 2 type topology grid shell attains the maximum deflection and Quadrangular type having the minimum i.e.,27% of the maximum value for taken conditions of 8 numbers of subdivisions and 25KNm^{-1} uniform force density. Besides the maximum and minimum deflection values rest have similar magnitude. The co-relation of deflection with topology is not found to be direct and further study is required.

From Figure 17 it is observed that keeping force densities constant for grid shell topologies, increase in grid density i.e. no. of subdivisions reduces the shell height considerably at lower values of subdivisions whereas, the changes are less pronounced at higher values. From figure it is seen that, quadrilateral type grid topologies i.e., quadrangular and diamond types exhibit similar variation, and all of the triangular topologies have alike variation in height for variations in grid density with constant force density.

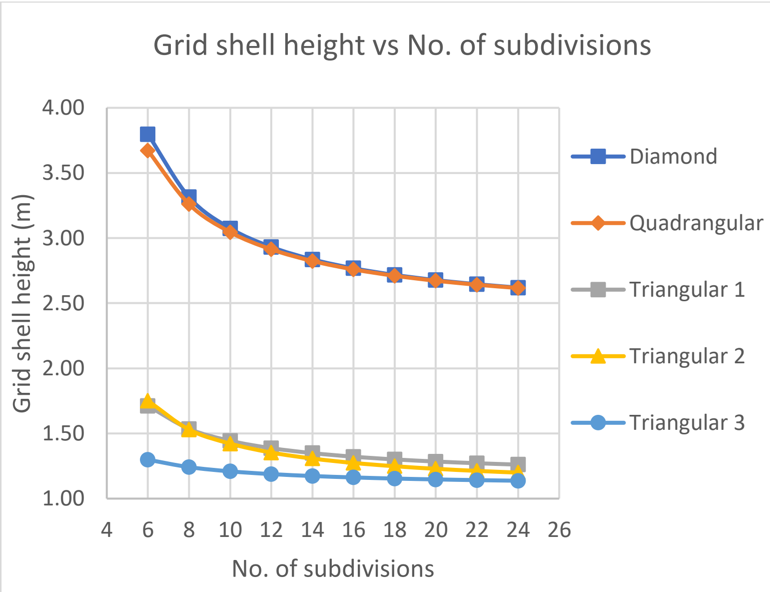


Figure 17 Variation of grid shell height as per grid density for force density of 25KNm-1.

Figure 18 shows that increasing sub-divisions increase grid shell structural weight for triangular type grid shells. In quadrangular type, grid shell weight decreases at lower values with rise in subdivisions assume almost constant value. In diamond type grid shell, it is seen that increasing subdivision decreases shell weight considerably at lower values and assumes almost constant value at higher subdivisions alike quadrangular type.

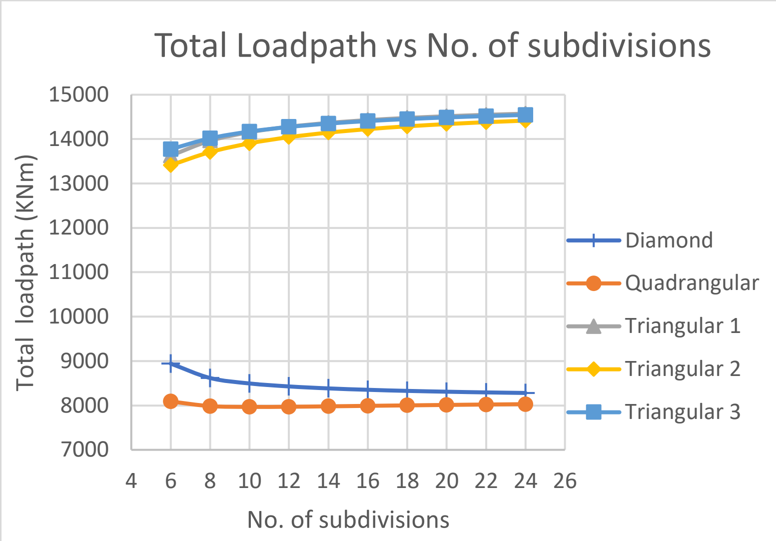


Figure 18 Variation of grid shell structural weight as per grid density for force density of 25KNm-1.

Figure 19 shows variation of grid shell deflection for various shell subdivisions at constant for density value of $25KNm^{-1}$. It is observed that triangular 2 type grid shells have highest deflection while quadrangular type has the lowest. For rest of topologies the deflections variation with subdivisions are alike.

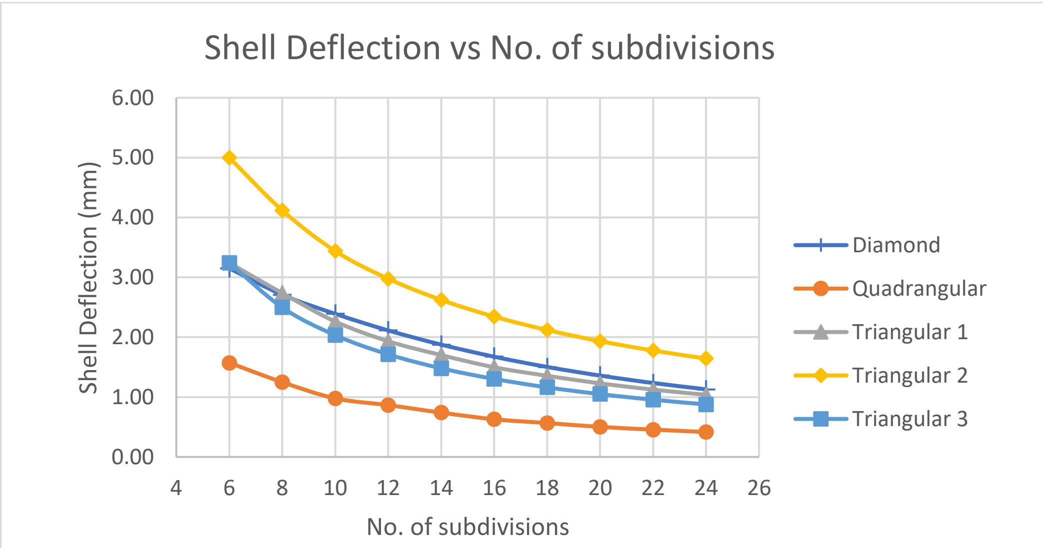


Figure 19 Variation of grid shell deflection with numbers of subdivision for force density of 25 KNm^{-1}

For all grid shells, variations of height, weight and deflections with subdivisions are more pronounced at lower values of subdivisions but at higher values the grid shell weight variation with subdivisions is subdued. This decreased variation in height, weight, and deflection of grid shell as grid density is increased can be explained by the fact that as we approach higher no. of subdivisions the grid shell assumes almost continuous surface with regular curvature. In lower subdivisions adjacent panels are highly irregular and even the small increment in subdivision makes more impact in making shell more regular. However, at higher subdivisions grid shell is almost regular and changes in subdivision have little or no impact in overall structural characteristic.

In next case, for a 12m x 12m grid shell with 12 numbers of subdivisions at support and various topologies, variation of grid shell height and structural weight is determined with respect to force density. Figure 20 shows that at lower values of force densities structural height reduce highly with increase in force density whereas for higher values of force densities, the rate of change of structural height is decreased highly assuming almost linear variation. Further, Figure 21 shows linear variation of grid shell structural weight with force density for triangular type grid shells. This linearity in quadrilateral type topologies is observed at higher values of force density with slight kink at the lower value.

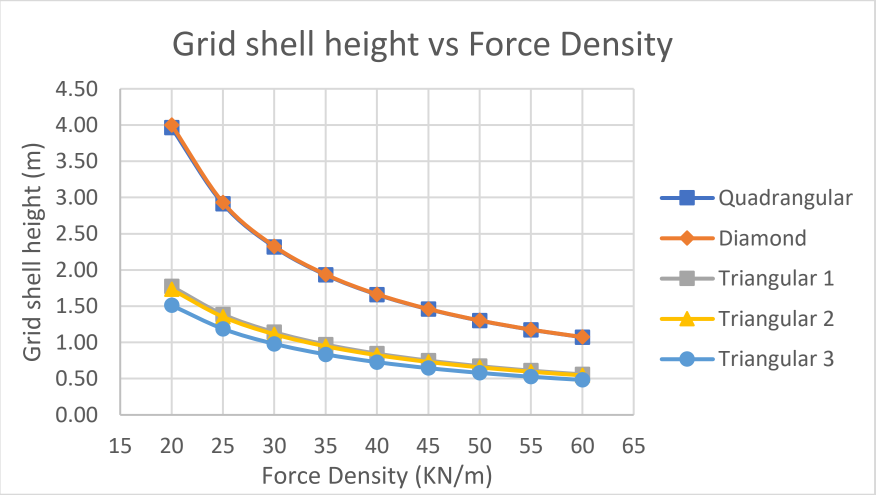


Figure 20 Variation of grid shell height as per force density for 12 subdivisions

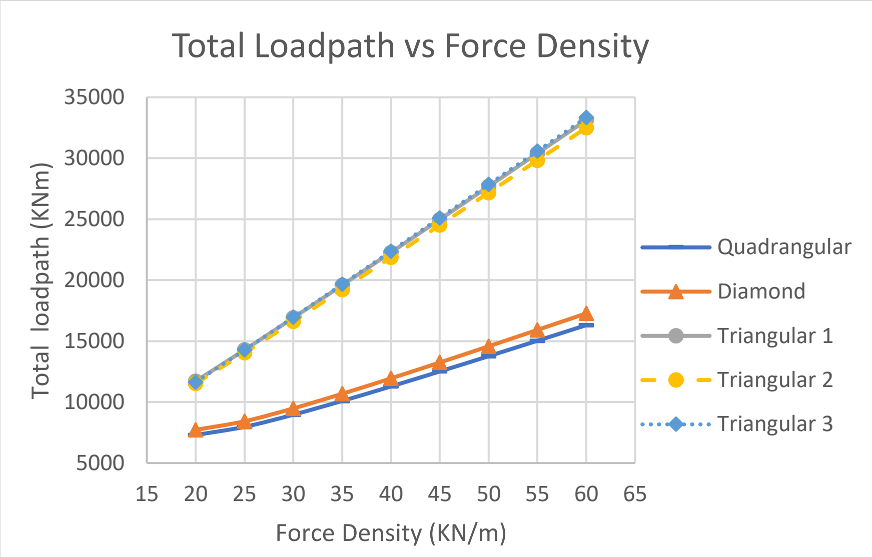


Figure 21 Variation of grid shell structural weight as per force density for 12 subdivisions

The relation of force density and maximum deflection for 12 subdivisions and various topologies is shown in Figure 22. It has been observed that, increasing the force density increases the shell deflection. This is more pronounced for Triangular 2 type grid shell and less for triangular 3 type topology, while rest showing similar variation. The increasing amount of deflection at higher force density can be attributed to the observation that as force density values are increased there is subsequent decrease in shell height.

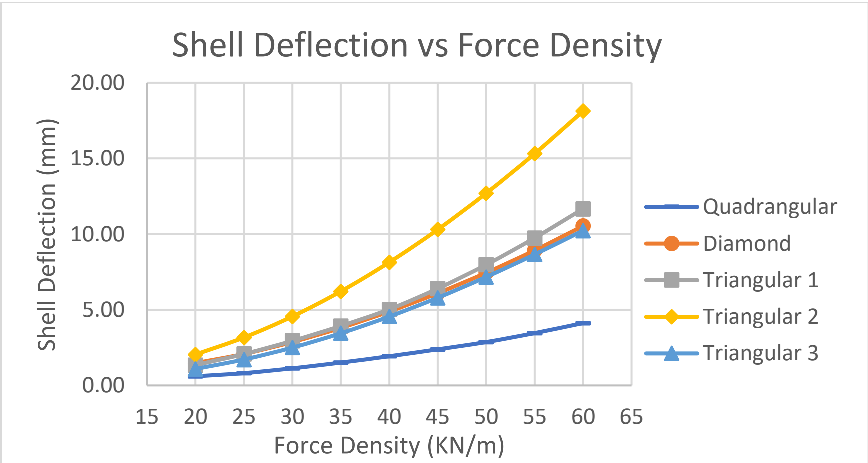


Figure 22 Variation of grid shell structural weight as per force density for 12 subdivisions

OPTIMIZATION FOR WEIGHT AND HEIGHT

The results of the case study show opposite variation of structural weight and height, for each of the parametric variation. Thus, optimization seems logical for the determination of case that satisfy both parameters. Results from genetic algorithm shows the value of optimum value of grid shell height and structural weight for given parameters as seen in Table 5.

Table 5 GA Optimization result for structural weight and height

S.no.	Cases	Grid type	No. of subdivisions	Total loadpath (KNm)	Structural height (m)	Force density (KNm^{-1})
1	Minimize Structural weight	Quadrangular	24	7136.3	3.51	20
2	Height =1.5m	Triangular 3	14	11700.76	1.49	20

From optimization of the grid shell for minimum values of structural weight and height it is observed that quadrangular type grid shell has lowest structural weight when no height limitation is imposed. The grid shell structure obtained from the optimization are presented in the Figure 23. Furthermore, grid type topology of triangular 3 has lowest structural weight with total load path of 11700.76 KNm when height limitation of 1.5m is imposed. The grid shell obtained is presented in Figure 24.

It has been observed that quadrilateral type grid shells, in comparison to triangular type grid shell topologies, have lower structural weight when no height limitations are imposed. However, given the non-planarity of panels, the fabrication costs are higher. Thus, further studies can be done, to obtain planar faced grid shells and then compare the results with triangular grid types in which planarity is inherent.

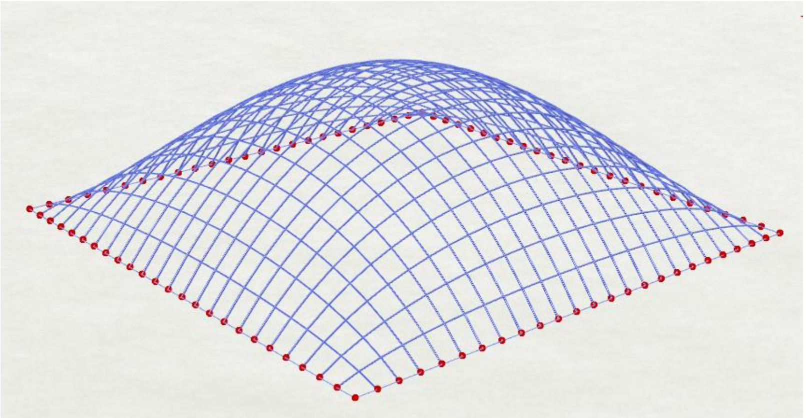


Figure 23 Grid shell form obtained for minimum structural weight as per case 1.

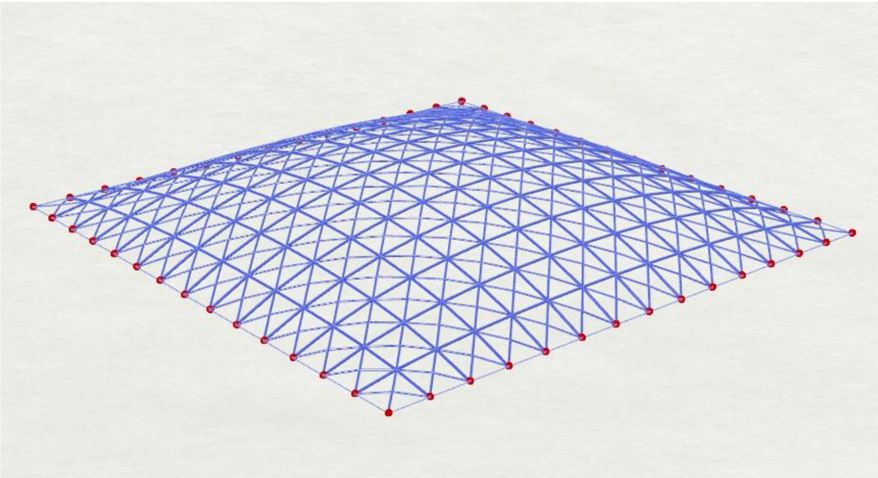


Figure 24 Grid shell form obtained for grid shell height of 1.5m as per case 2.

4.3 Discrete Airy Stress Function and FDM

Two self-stressed grid-shells whose forms are determined using airy stress function and FDM are studied and presented below with comparison of grid shells with similar nature form found using uniform force density for all members. The results and steps followed are presented below.

4.3.1 Rectangular Plan Grid Shell

Steps followed for comparison between grid shell geometries obtained from FDM alone and FDM in conjunction with stress polyhedron are given below:

1. Construct Airy Stress Function (Plane faced polyhedron)
   1.1 Take grid shell plan with appropriate meshing.
   1.2 Determine the grid shell form using FDM with support along the boundary, resulting grid shell is taken as stress polyhedron.

2. Determine Force Density values of each member from plane faced polyhedron.
3. Take projection of Plane faced polyhedron to get bar network (input for FDM) and determine slope difference between two plane faces (values representing force densities).
4. Determine grid shell form with supports as required using bar network, support condition and force density values. In this case support are taken to be at corners only.

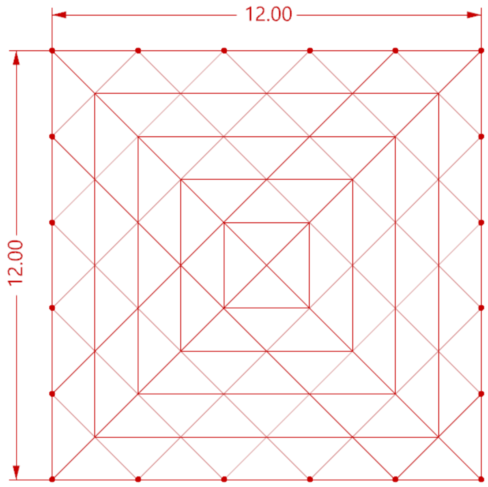


Figure 25 Grid shell plan with supports and member connectivity

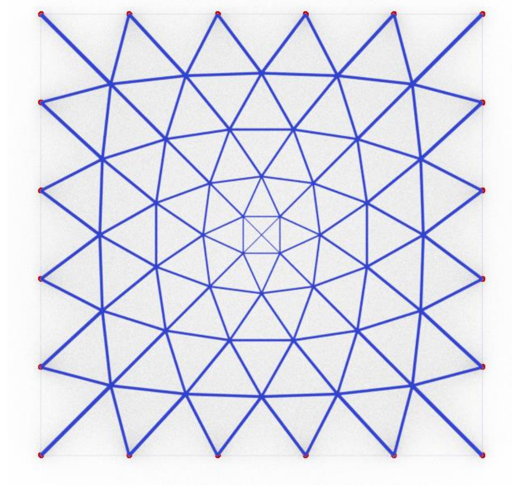


Figure 26 Plan view of Grid shell after form-finding with FDM (FD = 15 KNm-1)

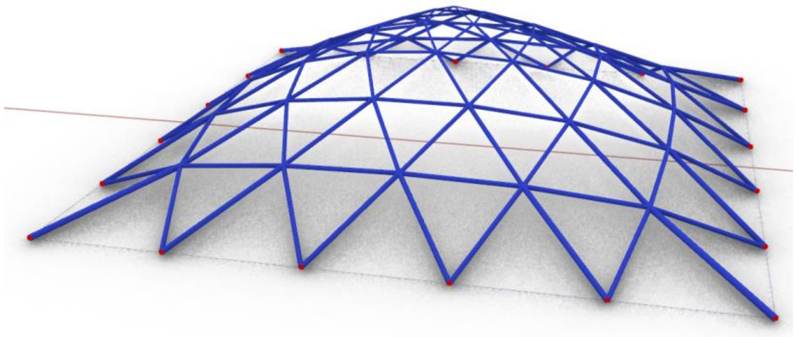


Figure 27 Perspective view of Grid shell after form-finding with FDM (FD = 15 KNm^{-1})

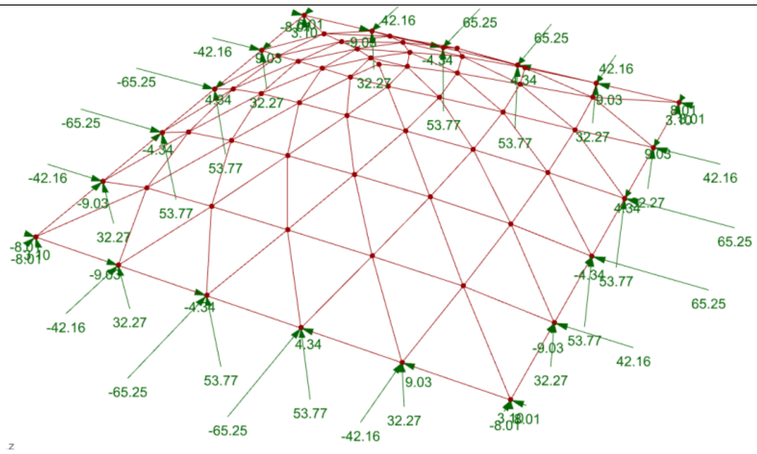


Figure 28 Perspective view of Grid shell support reaction (Reaction units in KN)

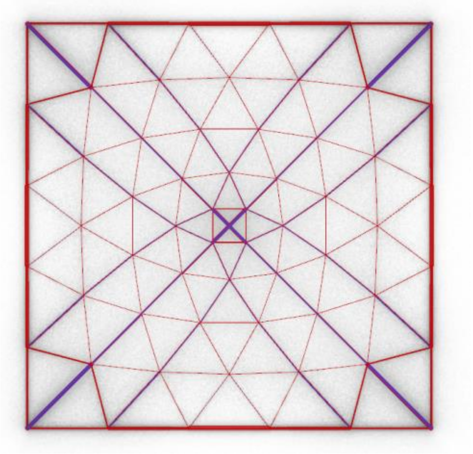


Figure 29 Plan view of Grid shell after form-finding with force density values input from Airy Stress Function taken to be grid shell form at Figure 24

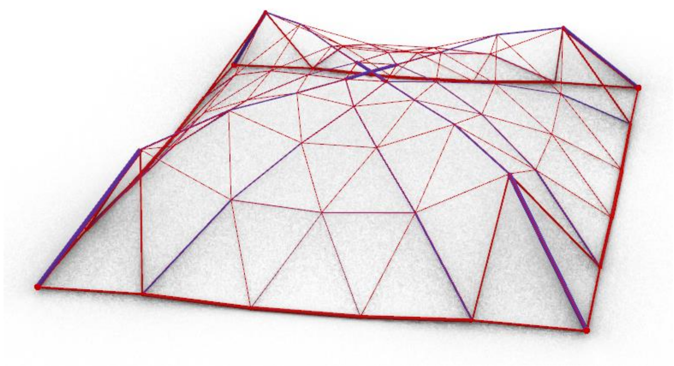


Figure 30 Figure – Perspective view Grid shell after form-finding with force density values input from Airy Stress Function taken to be grid shell form at Figure 24

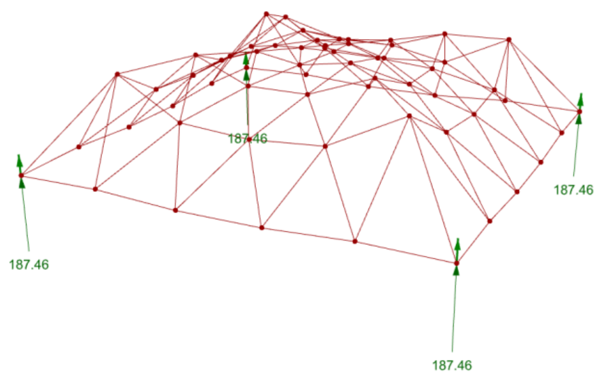
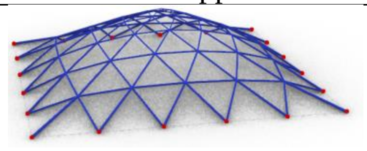
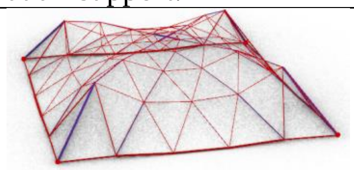


Figure 31 Perspective view Grid shell and support reactions after form-finding with force density values input from Airy Stress Function taken to be grid shell form at Figure 24

Table 6 presents comparison between basic parameters of grid shell geometries obtained using FDM alone and FDM in conjunction with Airy Stress function. Comparison between grid shells with same height is taken.

Note that there are not any horizontal reactions for vertical loading in the supports in case II. Thus, roof isolation could be achieved with limited load transferred to the substructure. Further study needs to be done for lateral load cases of wind and earthquake to finalize the design.

Table 6 Comparison of basic parameters using FDM and FDM along with stress polyhedron for rectangular plan grid shell.

Parameter		**Case I - FDM**	**Case II - FDM + AIRY**
Structural weight	Kg	1149.25	1852.6
Shell height	M	2.89	2.85
Vertical deflection	mm	5	98
Fundamental time-period	S	0.042	0.27
Force Type		Compression only	Both Compression and Tension
Minimum force in member	KN	9.86	-9.83 (Compression)
Maximum force in member	KN	44.1	-261.53 (Compression)
Members used		ISNB40M, ISNB50M	ISNB40M, ISNB50M, ISNB80M, ISNB110M
Reaction Type		Reaction along all three axes at each support.	Reaction only along z axis at all support.
Grid shell Geometry			

4.3.2 Circular Plan Grid Shell

Steps followed for comparison between grid shell geometries obtained from FDM alone and FDM in conjunction with stress polyhedron for a circular plan grid shell with opening in middle are given below:

1. Construct Airy Stress Function (Plane faced polyhedron). In this case done revolving a circular arc about z axis.
2. Determine Force Density values of each member from plane faced polyhedron.
3. Take projection of Plane faced polyhedron to get bar network (input for FDM) and determine slope difference between two plane faces (values representing force densities).
4. Determine grid shell form with supports as required using bar network, support condition and force density values. In this case support are taken to be at nodes at inner and outer boundaries of the area considered.

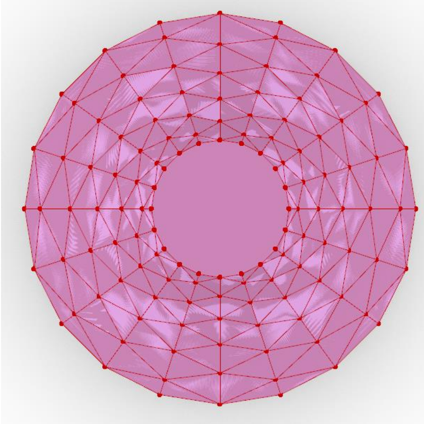


Figure 32 Plan view of stress polyhedron for circular type grid shell

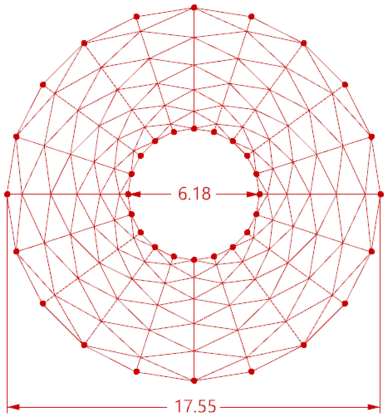


Figure 33 Projection of stress polyhedron of circular type grid shell representing base geometry for FDM

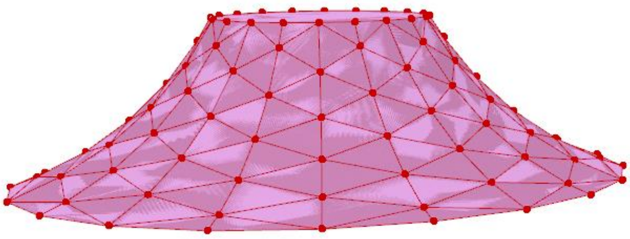


Figure 34 3D view of stress polyhedron for circular type grid shell

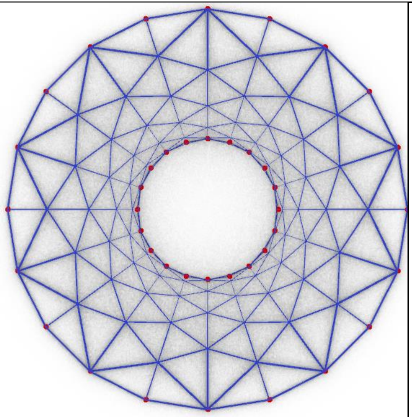
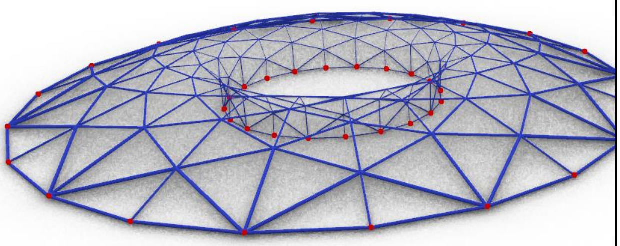
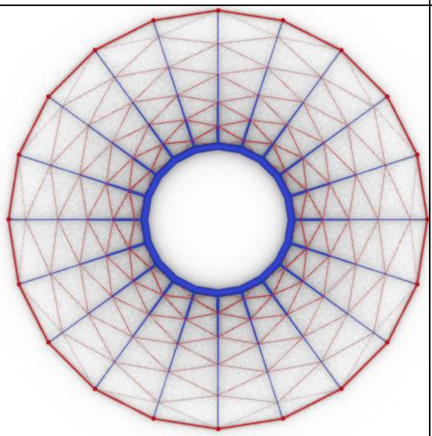
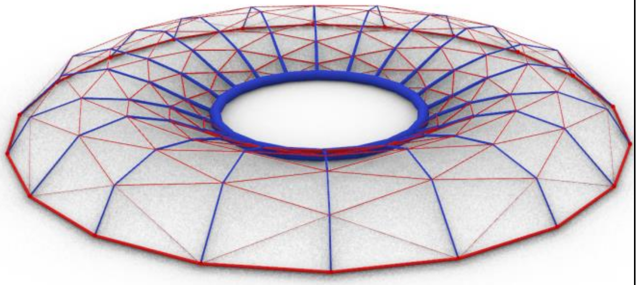
 *Figure 35 Plan view of grid shell geometry obtained from FDM using uniform force density.*	 *Figure 36 3D view of grid shell geometry obtained from FDM using uniform force density.*
 *Figure 37 Plan view of grid shell geometry obtained from FDM using force density values from stress polyhedron.*	 *Figure 38 3D view of grid shell geometry obtained from FDM using force density values from stress polyhedron.*

Table 7 compares two grid-shell parameters of similar height and base dimensions. The resulting structural weight is found to be similar. The deflection for grid shell form found using stress polyhedron and FDM is found to be higher, although within allowable limit. Grid shell form obtained using stress polyhedron and FDM is self-supporting in lateral direction, thus, limiting the forces in supports. However, to achieve this, the grid-shell undergoes in tension-compression in contrast to only compressive force in grid-shell obtained using uniform force density alone.

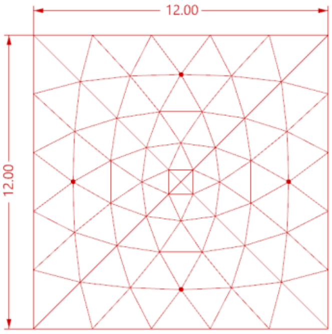
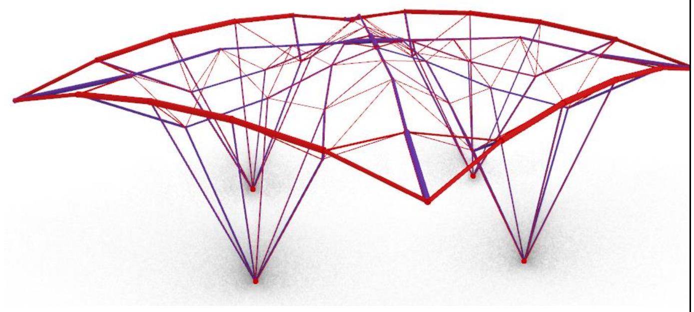
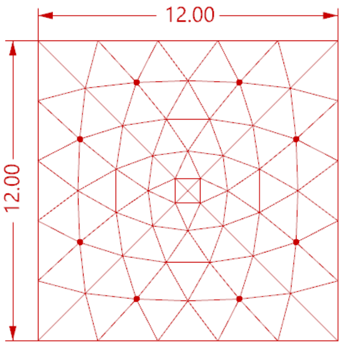
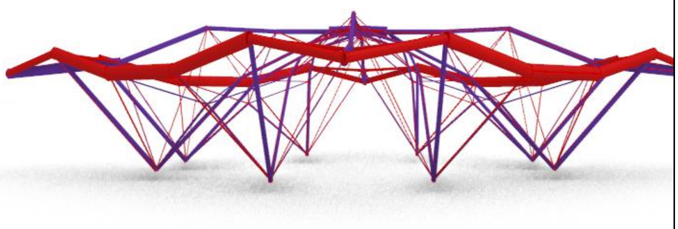
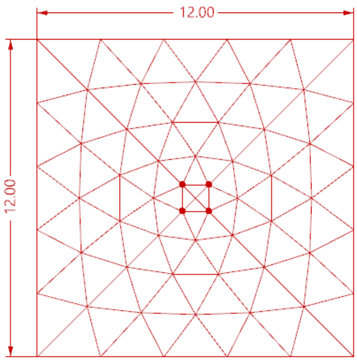
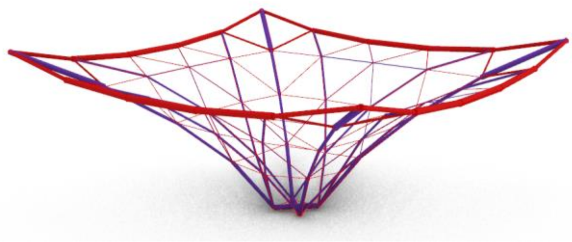
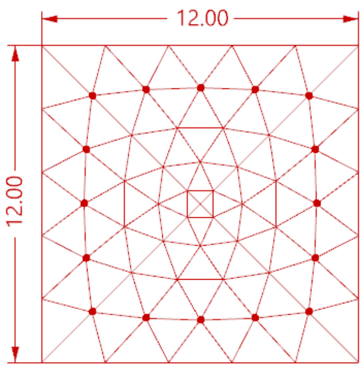
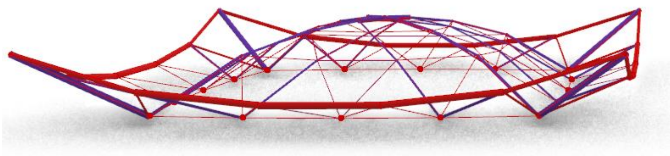
Table 7 Comparison of basic parameters using FDM and FDM along with stress polyhedron for circular plan grid shell

Parameter		**CASE I (FDM)**	**CASE II (FDM + AIRY)**
Structural weight	Kg	2102	2120
Shell height	M	1.39	1.37
Vertical deflection	mm	16	38
Fundamental time period	S	0.053	0.047
Force Type		Compression only	Both Compression and Tension
Maximum Compressive force in member	KN	46.7	168.69
Maximum Tensile force in member	KN	-	38.12 (Compression)
Members used		ISNB40M, ISNB50M	ISNB40M, ISNB65M
Reaction Type		Reaction along all three axes at each support.	Reaction only along z axis at all support.

In case of circular grid-shell with oculus in middle, the basic parameters of two grid-shells for static gravity loads are similar, however, self-supporting behavior is the major advantage of grid-shell in case II because it would reduce strength required at supports thus making overall design more economic.

4.3.3 Structures obtained by variation of support condition

Generation of several structures in equilibrium is possible by varying support condition in the same airy stress polyhedron as shown in Figure 25. The support conditions are varied in each row and force density values scaled by same value. Note that the structure is in static equilibrium with vertical members shown blue in compression and edge (peripheral) members at top in red in tension. It presents abundant design freedom, with possibility of numerous explorations of structures at early design phase, at the same time satisfying basic structural constraints, enabling more economic and optimized structures.

Member network in 2D as projection of plane faced polyhedron. Support is shown as dot.	Compression -Tension Pin joined structure obtained with given airy stress function and support condition.
	
	
	
	

CHAPTER FIVE: CONCLUSION

In this study, FDM along with discrete form of airy stress function has been utilized for design and optimization of grid-shells. A design tool for form finding and optimization at early design stage is created and, using the same, case study of a 12m x 12m rectangular plan grid shell has been done for various topologies, grid densities and force density values.

The FDM formulation has been validated against FEM model with errors within acceptable limits. Furthermore, GA has been employed to determine the optimum structural weight and height within the given limits. It has been observed, that for cases with no height limitation of grid shells, quadrangular followed by diamond shape grid type among other grid types have lowest structural weight whereas, triangular grid shapes generally have lowest height, although having higher structural weight. Optimization is generally case specific and required for each individual case. However, the study provides general outline to achieve economy in design.

Discrete Airy Stress Functions in form of plane faced polyhedrons have been used to generate grid shell structures self-supported in lateral directions for various support conditions. It has been found that, use of airy stress function can be done to generate self-supporting grid shell structures with structural performances similar to funicular structures. Even though, such structures have generally higher material requirements, they can be used in cases where roofing substructures are unable to sustain lateral loads such as retrofitting works in masonry or historic structures.

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ANNEX I

Force Density Method Formulation in Python using gh PythonRemote

```
##### Import libraries
import ghpythonremote
import scriptcontext as sc
from ghpythonlib.treehelpers import list_to_tree
ssa = sc.sticky['scipy.sparse']
sci = sc.sticky['scipy']
np =sc.sticky['numpy']

lines = np.array(x)
lines = np.around(lines,decimals =4)
no_lines = len(lines)  # no_lines means no. of lines

##########get unique sets of points save them in matrix u and get indices in indices
lines = np.reshape(lines,(no_lines*2,2))
u,indices = np.unique(lines,return_inverse = True,axis = 0)
unqpt= np.unique(indices,return_inverse = False)
u_pt_len = len(indices)

############## CREATE CONNNECTIVITY MATRIX.
row = np.repeat(np.arange(no_lines),repeats =2 )
col = indices
data = np.tile([1,-1],no_lines)
cols = len(u)
C = ssa.csc_matrix( (data,(row,col)), shape = (no_lines,cols) ).todense()
CON = C.copy()

##############GET FIXED POINT INDEX FROM LIST OF UNIQUE POINTS
fxpt = np.array(fxpt)
fxpt = np.around(fxpt,decimals =4)
fxpt_id = np.lexsort((fxpt[:,1],fxpt[:,0]))
fxpt = fxpt[fxpt_id]
res = (u[:, None] == fxpt).all(-1).any(-1)
fxpt_indx = np.where(res)
rpt_indx = np.where(np.invert(res))


#############create Q and Qf matrix for fixed point and free point - connectivity matrices.
Qf = C[:,fxpt_indx]
Cf = np.reshape(Qf,(Qf.shape[0],Qf.shape[2]))
Qr = C[:,rpt_indx]
Cr = np.reshape(Qr,(Qr.shape[0],Qr.shape[2]))

##########CREATE DIAGONAL FORCE DENSITY MATRIX
q=np.array(q)
q = np.around(q,decimals =4)
Q=np.diag(q)


#################CREATE POINT LOAD VECTORS
p_x = np.zeros((len(u)-len(fxpt),1))
p_y = np.zeros((len(u)-len(fxpt),1))
p_z = np.tile(load,len(u)-len(fxpt))
p_z = np.reshape(p_z,(p_z.shape[0],1))

##########CREATE FIXED POINT CO-ORDINATE VECTORS
x_f = fxpt[:,0]
x_f = np.reshape(x_f,(x_f.shape[0],))
y_f = fxpt[:,1]
y_f = np.reshape(y_f,(y_f.shape[0],))
z_f=  np.full((x_f.shape[0],), 0)
```

```
############CREATE D AND D_F
D = Cr.T.dot(Q).dot(Cr)
D_f = Cr.T.dot(Q).dot(Cf)

#GET THE CO-ORDINATES
x_ound = np.linalg.inv(D).dot(p_x - np.reshape(D_f.dot(x_f),(D_f.dot(x_f).shape[1],1)))
y_ound = np.linalg.inv(D).dot(p_y - np.reshape(D_f.dot(y_f),(D_f.dot(y_f).shape[1],1)))
z_found = np.linalg.inv(D).dot(p_z - np.reshape(D_f.dot(z_f),(D_f.dot(z_f).shape[1],1)))


############## EXPORT TO GRASSHOPPER FROM NUMPY ARRAY.
xx = np.ndarray.flatten(np.array(x_found))
xx = ghpythonremote.obtain(xx.tolist())
xx =list_to_tree(xx,source=[0,])

yy = np.ndarray.flatten(np.array(y_found))
yy = ghpythonremote.obtain(yy.tolist())
yy =list_to_tree(yy,source=[0,])

zz = np.ndarray.flatten(np.array(z_found))
zz = ghpythonremote.obtain(zz.tolist())
zz =list_to_tree(zz,source=[0,])


xxx = np.ndarray.flatten(np.array(x_ound))
xxx = ghpythonremote.obtain(xxx.tolist())
xxx =list_to_tree(xxx,source=[0,])

yyy = np.ndarray.flatten(np.array(y_ound))
yyy = ghpythonremote.obtain(yyy.tolist())
yyy =list_to_tree(yyy,source=[0,])
```

Mesh construction as per obtained points.

```
##### Import libraries
import rhinoscriptsyntax as rs
import ghpythonremote
import scriptcontext as sc
from ghpythonlib.treehelpers import list_to_tree
ssa = sc.sticky['scipy.sparse']
sci = sc.sticky['scipy']
np =sc.sticky['numpy']

########Convert arrays to numpy arrays
xx =np.array(x)
x=np.reshape(x,(len(x),1))
y = np.array(y)
y=np.reshape(y,(len(y),1))
z = np.array(z)
z = np.around(z,decimals=4)
z=np.reshape(z,(len(z),1))

###########Combine fix and free points
xedo = np.zeros((len(fxpt),1))
fxpt = np.hstack((fxpt,xedo))
fxpt_id = np.lexsort((fxpt[:,1],fxpt[:,0]))
fxpt = fxpt[fxpt_id]
xyz=np.concatenate((x,y,z),axis=1)

pt_num = len(fxpt)+len(x)
allpt = np.zeros((pt_num,3))
allpt[fxpt_indx] = fxpt
allpt[rpt_indx]=xyz

xedo = np.zeros((len(u),1))
u = np.hstack((u,xedo))
```

```
con =CON.copy()
cone =CON.copy()

con[con==1] =0
con[con==-1] = 1
spt = np.dot(con,allpt)

cone[cone==-1] =0
ept = np.dot(cone,allpt)

#PUSH DATA TO GRASSHOPPER TREE
sptx = ghpythonremote.obtain(spt[0:,0].tolist())
sptx =list_to_tree(sptx,source=[0,])

spty = ghpythonremote.obtain(spt[0:,1].tolist())
spty =list_to_tree(spty,source=[0,])

sptz = ghpythonremote.obtain(spt[0:,2].tolist())
sptz =list_to_tree(sptz,source=[0,])

eptx = ghpythonremote.obtain(ept[0:,0].tolist())
eptx =list_to_tree(eptx,source=[0,])

epty = ghpythonremote.obtain(ept[0:,1].tolist())
epty =list_to_tree(epty,source=[0,])

eptz = ghpythonremote.obtain(ept[0:,2].tolist())
eptz =list_to_tree(eptz,source=[0,])
```

ANNEX II: LATERAL LOADING IN VARIOUS SHELL GEOMETRIES

The comparison of effect of lateral loading (uniform static wind load) in deflection and total mass of the structure required to resist lateral and wind loading for various shell geometries are shown in table below.

Wind load intensity taken = 2 KNm^{-2}

Gravity loading = 5 KNm^{-2}

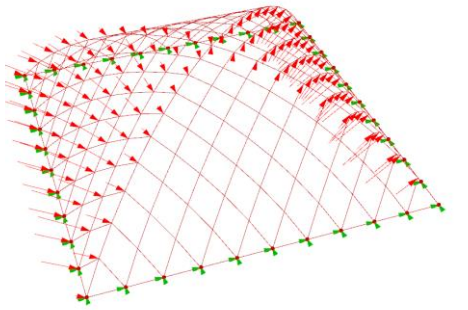
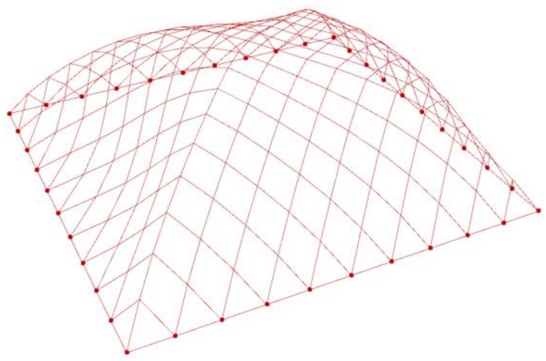
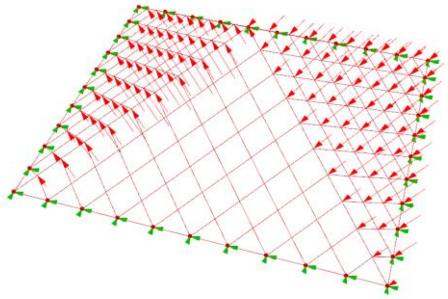
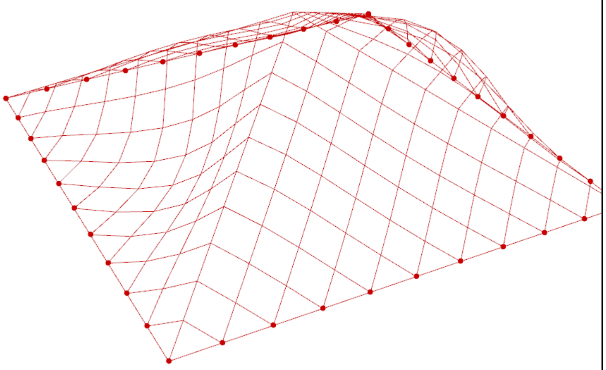
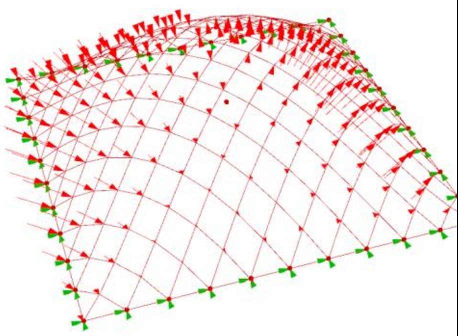
Grid shell Plan Dimensions = 12 m * 12m with 20 subdivisions in each side.

Grid shell Height taken = 3.5m

Table 8 Deflection and mass comparision table for lateral load in grid shells.

Grid shell type	Max deflection (mm)			Mass (kg)
	x	y	Z	
Spherical	2	0.4	1.5	1720.87
Pyramid	74	11.7	11.4	2608.08
Form found using FDM	1	0.4	0.8	1565.6

Table 9 Lateral load effects on various grid shell geometries

Shell and Loading type	Deflected Shape
 *Figure 39 Grid shell geometry formed lofting edges to circular arc*	
 *Figure 40 Grid shell constructed with pyramidal geometry*	
 *Figure 41 Form found grid shell*